MATH 400: CALCULUS I - DIFFERENTIAL CALCULUS

Roy Simpson

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Math 400: Calculus I - Differential Calculus

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Preface for the Instructor

Prerequisites

Most sections have a list of Core and Operational Prerequisites. The Core Prerequisites consist of material from previous math courses that students *must* be proficient with to understand the core concepts in the section. The Operational Prerequisites consist of material (again, from previous math courses) that students *must* be proficient with to understand the examples and do the homework. The line between these two might be a bit blurry at times—you can treat them simply as prerequisites.

These prerequisites come from the Cosumnes River College Corequisite Codex, and when a prerequisite topic is listed, a link is provided to the relevant section within the Codex so that the student can revisit that skill, if needed. **If you are teaching a Differential Calculus course with support**, you can use these prerequisites to help you figure out *when* to review certain topics (this is also known as, "just in time remediation").

Once a topic has been listed as a prerequisite, it is never listed again. That is, if factoring trinomials is mentioned as a prerequisite in one section, then it will not be mentioned in any future section *even if it is needed in that future section*. As such, it is to your benefit to pay attention to those prerequisites!

If you prefer to know which topics are required for the *entire course* ahead of time, please refer to the list below:

- Sets and Numbers
  - Sets of Numbers
  - Inequalities and Inequality Notation
  - Intervals and Interval Notation
- Arithmetic
  - Mathematical Operations — Addition, Subtraction, Multiplication, and Division
  - Mathematical Operations — Powers and Square Roots
  - Simplifying Fractions and Mixed Numbers
  - Multiplying and Dividing Fractions
  - Adding and Subtracting Fractions
  - Mathematical Operations — Higher-Index Roots
  - The Absolute Value of a Number
  - Percents
  - Measurement, Approximation, Rounding, and Significant Figures
- Geometry
  - The Pythagorean Theorem, Distance Formula, and Midpoint Formula
  - Similar Triangles
  - Circles — Properties, Area, and Circumference
  - Area and Perimeter of Triangles and Quadrilaterals
  - Volumes and Surface Areas of Common Shapes
- Factoring Techniques
  - Greatest Common Factor
  - Factoring Trinomials
  - Factoring Difference of Squares
  - Factoring Sums and Differences of Cubes
  - Factoring Expressions Involving Negative Rational Exponents
- Simplifying Expressions
  - Algebraic Vocabulary and Evaluating Expressions
  - Laws of Exponents — Product and Quotient Rules
  - Laws of Exponents — Power Rules
  - Laws of Exponents — Zero and Negative Exponent Rules
  - Adding and Subtracting Polynomials
  - Multiplying Polynomials and the Distributive Property
  - Simplifying Ratios of Polynomials
  - Dividing Polynomials — Long Division

    - Adding and Subtracting Rational Expressions
    - Simplifying Compound Rational Expressions
    - Simplifying Products of Radical Expressions
    - Rationalizing Denominators and Numerators; Simplifying Quotients of Radical Expressions
    - Simplifying Expressions Involving Positive Rational Exponents
    - Simplifying Expressions Involving Negative Rational Exponents
- Solving Equations
    - Solving Literal Equations
    - Solving Linear Equations — One or Two-Steps
    - Solving Linear Equations — Multistep with Variables on Both Sides
    - Solving Quadratic Equations by Factoring
    - Solving Quadratic Equations Using Extraction of Roots
    - Solving Quadratic Equations Using the Quadratic Formula
    - Solving Radical Equations
    - Solving Exponential Equations
    - Solving Logarithmic Equations
- Solving Inequalities
    - Solving Linear Inequalities
    - Solving Absolute Value Inequalities
    - Solving Quadratic Inequalities
- Graphs
    - The Cartesian Coordinate System and the Quadrants
    - Closed- and Open-Circle Notations
    - Reading Values from Graphs
- An Overview of Functions
    - The Definition of a Function
    - Evaluating Functions
    - The Domain and Range of a Function
    - Functional Arithmetic and Domain
    - The Difference Quotient
    - Function Composition and Domain
    - Inverse Functions
    - Restricting Domains and Forcing Invertibility
- Graphing Functions
    - Intercepts and Base Graphs
    - Symmetry
    - Intervals of Increase and Decrease
    - Local/Relative and Global/Absolute Extrema
    - Piecewise Functions
- Linear Functions
    - The Slope of a Line
    - Equations of Lines — Slope-Intercept Form
    - Equations of Lines — Point-Slope Form
    - Linear Models
    - The Average Rate of Change of a Function
    - Parallel and Perpendicular Lines
- Polynomial Functions
    - The Basics of Polynomial Functions
    - The Binomial Theorem
- Rational Functions
    - Asymptotes
- Exponential Functions
    - The Basics of Exponential Functions
    - Graphing Exponential Functions
- Logarithmic Functions
    - The Basics of Logarithmic Functions
    - Laws of Logarithms — Power Law
    - Laws of Logarithms — Product and Quotient Laws

    - Graphing Logarithmic Functions
- Absolute Value Functions
    - The Concept of Absolute Value
    - The Absolute Value Function
- Conic Sections
    - Equations and Graphs
- Sequences and Series
    - The Basics of Sequences
    - Summations and Their Properties
- Trigonometry
    - Evaluating Trigonometric Functions at Special Angles
    - Evaluating Trigonometric Functions Using Technology
    - Trigonometric Identities — The Reciprocal and Ratio Identities
    - Trigonometric Identities — The Pythagorean Identities
    - Trigonometric Functions — Right Triangle Definition
    - Converting Between Degrees and Radians
    - Trigonometric Functions — Unit Circle Definition
    - Sector Area
    - Base Graphs of the Trigonometric Functions
    - Trigonometric Identities — Sum and Difference of Angles Identities
    - Trigonometric Identities — Half-Angle and Power Reduction Identities
    - The Inverse Trigonometric Functions
    - Solving Basic Trigonometric Equations
    - Law of Cosines
- Miscellaneous Applications
    - Rates, Unit Rates, and Uniform Motion (Distance, Rate, Time)

Changelog

With a living OER text like this one, it is imperative that you check the changelog each semester. This is because feedback is likely incorporated into the text during semester breaks (and, at times, during the semester for minor fixes).

2026 Spring

I was on sabbatical Spring 2026 and revamped all of the OER textbooks at CRC during this time. The following is a coarse list of updates:

- Removed the "per-section" A.I. tutors (LibreTexts is performing a system-wide implementation of the A.I. tutors I began in Fall 2025)
- Removed the table at the top of each section that contained the Learning Objectives, Prerequisites, and the Hawk A.I.
- Instituted a "tableless" Section Preview area at the top of each section
- Wrote a "Note to the Instructor" in each section describing typical pitfalls and needs for the newer instructor
- Re-analyzed the needed prerequisites on a per-section basis and added these as either Core or Operational Prerequisites in the Section Preview area
- Cleaned any broken code in the text
- Cleaned any broken math in the text (not the homework areas... yet)
- Standardized styles throughout the text (e.g., figure caption placement, font, font weight, and font style)

2025 Fall

- Started including A.I. tutors in each section (this is a big project and will take until Summer 2026 to complete)
- Recoded the prerequisite topics to appear in the lecture part of the relevant section

2025 Summer

- Moved Chapter 1 (Review of Algebra and Trigonometry) to the Appendix
    - **Purpose:** As we have *everyone* landing in Math 400, a single chapter of review at the beginning is not viable. I still chose to keep it as an Appendix until a Departmental vote on what to do with it.
- Moved "Hyperbolic Functions" from 1.7 to "2.5: An Interlude—Hyperbolic Functions"
    - **Purpose:** While I prefer early hyperbolics (and still think covering them early would help reinforce Trigonometry skills), the truth is that incoming skill sets are so lacking that reserving the introduction to

hyperbolics until they are absolutely needed is preferred.
- Made sweeping changes to 1.1 through 1.10 to not include hyperbolic functions
- Split the new 1.1 (formerly "2.1: An Introduction to Limits") into two sections—"1.1: Numerical Investigation of Limits" and "1.2: Graphical Investigation of Limits"
  - **Purpose:** I love the logical split of first introducing limits from a calculator (numerical) perspective and *then*, in a separate section, approaching the same concepts from a graphical perspective. The split is also kinder to those who need the separation to conceptualize the content.
- Split the new 1.3 (formerly "2.2: Limit Laws") into two sections—"1.3: Limit Laws and the Squeeze Theorem" and "1.4: Algebraic Manipulations and the Limit Laws"
  - **Purpose:** 1.3 focuses solely on understanding the limit laws and the Squeeze Theorem. Students deal with limits without algebraically manipulations. 1.4 then focuses on the idea that we are using algebraic manipulations to investigate the limit of a *similar* function that is only different at $x = a$. This reinforces the key idea of locally equivalent functions.
- Split the new 1.5 (formerly "2.4: The Precise Definition of a Limit") into two sections—"1.6: The Precise Definition of a Limit" (focused on the finite limit at a finite number) and "1.7: The Precise Definitions of Limits Involving Infinity"
  - **Purpose:** This split is meant to rebalance the load of learning the precise definitions all at once.
- Included the derivatives of the hyperbolic functions in the new 2.6 (formerly "3.5: Derivatives of Exponential Functions") to create "2.6: Derivatives of Exponential and Hyperbolic Functions"
  - **Purpose:** This makes logical sense. Since hyperbolic functions are simply linear combinations of exponential functions, these derivatives fit nicely here.
- Made sweeping changes to sections 2.1 through 2.4 to not include derivatives of exponential functions
- Changed the end of Chapter 2 to be "2.7: Implicit Differentiation and the Derivatives of the Inverse Trigonometric and Hyperbolic Functions", "2.8: Derivatives of Logarithms and Logarithmic Differentiation," and then the traditional "2.9: Related Rates" followed by "2.10: Linear Approximations and Differentials"
  - **Purpose:** Prior to these changes, implicit differentiation was its own section, followed by derivatives of inverse trigonometric and hyperbolic functions, then following by derivatives of logarithms, and finally logarithmic differentiation. As such, these were tiny sections. I just felt that they were too small to be single sections, so I combined a few of them.
- Changed some of the section names in Chapter 3 (Applications of Derivatives) and changed all "l'Hospital's" to "L'Hôpital's"

2025 Winter
- Moved derivatives of exponential functions out of "3.1: Derivatives of Polynomial and Exponential Functions" and into "3.5: Derivatives of Exponential Functions"
- Included non-repeating prerequisite topics for each section (i.e., once a topic is mentioned as a prerequisite for a section, it is not mentioned later in the text even if it is prerequisite for that later material)
- Embedded lecture videos (if available) for most sections
- Included a section summary for each section

2023 - 2024

Massive (untracked) edits and rewrites throughout. Rewrote nearly half the text to improve rigor and logic.

2022

Forked OpenStax Calculus Volume I by Herman and Strang

Roadmap

There are still quite a few changes I want to implement before I consider this text "perfect."

1. Although there is a MyOpenMath course associated with this textbook, I would like to go through, section-by-section, making sure every concept within a section has a robust set of associated MyOpenMath problems. To be honest, this is almost the case at this time; however, a final check is on my "Roadmap" list.
2. Improve the *quality* of the MyOpenMath problems so that more require the use of prerequisite skills.
3. Embed MyOpenMath problems in the textbook for all Checkpoint problems. This allows a more dynamic text and gives the instructor the opportunity to use Checkpoint problems as lectures examples "on the fly" while not doing the exact problem in the student's version of the textbook.
4. Create and embed Desmos animations and digital manipulatables whenever appropriate. Some of the early material already

has this, but it would be great to have more.

5. Embed a list of "vetted" lecture videos for the topics in each section. While there are lecture videos already embedded, these are mine and I am not sure everyone likes them (they are also for the Stewart Calculus textbook, so they sometimes might feel slightly off).
6. Transition all of the homework sections in the textbook to embedded MyOpenMath problems. I still need to think about this one. Instructors typically assign "handwritten" homework because they are not comfortable with technology, or they have some pedagogical reason to not want to use online homework. As such, they conceivably want static homework in the textbook. I *can* embed specific randomizations of problems within this textbook—I just need to do that.
7. Completely align the MyOpenMath homework and the textbook homework. As it stands, there are some problems in the textbook homework that are not in the MyOpenMath homework, and vice versa. I just need the time and energy to align these.

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Detailed Licensing

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Preface to the Student

Welcome

Welcome to Calculus! This text covers a single-semester college-level course in Differential Calculus (which your college probably calls Calculus I). While I could wax poetic about the beauty of this subject, I really should focus on getting you the knowledge you need to start this course off on a good foot.

The Need to Read

It's no secret that many students make it through Algebra without cracking open their textbook, and some even manage to pass Precalculus or College Algebra using the same technique. Calculus, on the other hand, is a completely different beast. You will need to take advantage of every resource available to wrap your mind around the concepts of this course. Foremost among those resources is this textbook! Do not expect to read it like a book and devour 40 pages in an evening's study session. This material is dense and requires contemplation. Expect to read paragraphs several times before being able to completely understand what's going on. This "constructive struggle" is beneficial in the long run - trust me.

Horse... THEN Cart

Many students try their hand at the homework and only reference the text when they are completely befuddled. This is known as "placing the cart before the horse." If you think of the "cart" as the knowledge you need to attain in the section and the "horse" as the knowledge you currently have, then your initial situation in any section is a cart-load of information you need to figure out, and a small horse (your current understanding of the material in the current section) to pull that cart. With this poorly-designed analogy in mind, the order in which you should do things in a calculus course is as follows:

1. **Grease the Wheels:** Perform a cursory read-through of the section before attending the lecture. Don't expect to understand everything, but try your best to get a conceptual understanding of the goal of the section. Doing this cursory read-through is "greasing the wheels" to make the classroom experience much more enjoyable (yes, I said it!) and understandable. It also moves a small amount of that cart of unattained knowledge to the horse (making the horse a bit stronger and the cart a bit lighter).
2. **Push-Start the Cart:** Attend the lecture and **ask questions** as your professor gets to moments that confused you. The lecture should "push the cart" to help you get the momentum so you can start *really* learning the topic. During this phase, your professor is helping you lighten the load off your unattained knowledge cart and, simultaneously, making your horse stronger (filling in your knowledge gaps).
3. **Use the Momentum to Move Forward:** After lecture, thoroughly read the section, taking notes as you do so. You should *not* do this in a single sitting. Rather, dedicate an hour or so to reading the text and taking notes, give yourself a break, study another subject or go to work, then come back to the textbook and spend another hour reading and taking notes. Repeat this process for a couple of days until you have successfully finished the section. Make sure you try the Examples in the text *prior* to reading the worked-out solutions. During this step, the amount of unattained knowledge should dwindle to very little and your actual knowledge of the material in the section should grow.
4. **Let the Horse Pull the Cart:** Once you have greased the wheels, push-start the cart, and used the momentum to move forward, you THEN let the horse do all the work. That is, you let your knowledge carry you into the homework.

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Templates

The following content is reused throughout the course and is held here for simplicity.

Core Prerequisites

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

Operational Prerequisites

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

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1: LEARNING LIMITS

The idea of a limit is central to all of calculus. We begin this chapter by examining why limits are so important. Then, we go on to describe how to find the limit of a function at a given point. Not all functions have limits at all points, and we discuss what this means and how we can tell if a function does or does not have a limit at a particular value. This chapter has been created in an informal, intuitive fashion, but this is not always enough if we need to prove a mathematical statement involving limits. The last section of this chapter presents the more precise definition of a limit and shows how to prove whether a function has a limit.

1.1: Numerical Investigation of Limits

1.1.1: Resources and Key Concepts

1.1.2: Homework

1.1.3: Additional Exercises

1.2: Graphical Investigation of Limits

1.2.1: Resources and Key Concepts

1.2.2: Homework

1.2.3: Additional Exercises

1.3: Limit Laws and the Squeeze Theorem

1.3.1: Resources and Key Concepts

1.3.2: Homework

1.3.3: Additional Exercises

1.4: Algebraic Manipulations and the Limit Laws

1.4.1: Resources and Key Concepts

1.4.2: Homework

1.4.3: Additional Exercises

1.5: Limits at Infinity

1.5.1: Resources and Key Concepts

1.5.2: Homework

1.5.3: Additional Exercises

1.6: The Precise Definition of a Limit

1.6.1: Resources and Key Concepts

1.6.2: Homework

1.6.3: Additional Exercises

1.7: The Precise Definitions of Limits Involving Infinity

1.7.1: Resources and Key Concepts

1.7.2: Homework

1.7.3: Additional Exercises

1.8: Continuity

1.8.1: Resources and Key Concepts

1.8.2: Homework

1.8.3: Additional Exercises

1.9: Defining the Derivative

1.9.1: Resources and Key Concepts

1.9.2: Homework

1.9.3: Additional Exercises

1.10: The Derivative as a Function

1.10.1: Resources and Key Concepts

1.10.2: Homework

1.10.3: Additional Exercises

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1.1: Numerical Investigation of Limits

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

The concept of a limit (or limiting process) has been around for thousands of years. Early mathematicians used limits to obtain more accurate approximations of areas of circles. Yet, the formal definition of a limit—as we know and understand it today—did not appear until the late 19th century. Therefore, we begin our quest to understand limits, as our mathematical ancestors did, using an intuitive approach. Only after we have had time to play with this intuitive approach do we examine the formal definition of a limit.

Finite Limit at a Finite Number

Let's consider how the function $f(x) = \frac{x^2-4}{x-2}$ behaves around $x = 2$. Building a quick table of values, we get the following.

Table 1.1.1***:*** *Investigating the behavior of* $f(x) = \frac{x^2-4}{x-2}$ *near* $x = 2$

x	$f(x)$	x	$f(x)$
1.5	3.5	2.5	4.5
1.9	3.9	2.1	4.1
1.99	3.99	2.01	4.01
1.999	3.999	2.001	4.001

As the values of x approach 2 (from both sides of 2), the values of $y = f(x)$ *seem* to be approaching 4. If correct, we would then say that the **limit** of $f(x)$ as x approaches 2 is 4. Symbolically, we express this limit as

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4.$$

From this very brief informal look at one limit, let's start to develop an intuitive definition of the **finite limit at a finite number**.

We can think of the limit of a function at a finite number a as being the single finite number L (the **finite limit**) that the function values approach as the x-values approach a, provided such a real number L exists. Stated more carefully, we have the following intuitive, but not precise, definition.

Definition (Intuitive): Finite Limit at a Finite Number

Let $f(x)$ be a function defined at all values in an open interval containing a finite number a, with the possible exception of a itself, and let L be a finite real number. If the values of the function $f(x)$ approach the real number L as the values of x ($\neq a$) approach the number a, then we say that the **finite limit** of $f(x)$ as x approaches the finite number a is L. (Said more succinctly, as x gets closer to a, $f(x)$ gets closer to L.) Symbolically, we express this idea as

$$\lim_{x \to a} f(x) = L.$$

Often, this is simply referred to as a **limit** (rather than a finite limit at a finite number).

Understanding the concept of a limit is so important in Calculus that we will spend quite some time getting comfortable with it before finally (several sections from now) going deep into the precise definition of a limit.

Caution: $x \neq a$

The intuitive definition of a finite limit at a finite number states that x need not equal a. This is critical to understand! The x -values are allowed to get very close to a but will never become a. In a very real sense, the limit,

$$\lim_{x \to a} f(x) = L,$$

is concerned with the function values as x gets closer and closer to a; however, it is *not* asking, "What is $f(a)$?" In fact, it could be the case that $f(a)$ does not exist or $L \neq f(a)$.

In this section, we explore how to *estimate* the value of a limit using a combination of tables and computational technology. We will refer to this method of investigation as a **numerical approximation of the limit**. This can also be rephrased as, "investigating the limit **numerically**." In practice, using a numerical approximation to state the value of a limit presents three issues:

1. it can lead to incorrect presumptions about the true value of the limit,
2. as Calculus is the study of the infinite and infinitesimal, and technology uses finite precision arithmetic, limit computations attempted via technology can lead to massively incorrect values, and
3. using technology or tables to gain an approximation of the value of a limit is *not* Calculus—it is button pushing. Your task in Calculus is to gain a demonstrable understanding of the fundamental concepts of Calculus, and this includes how to work with limits without touching technology or building tables.

These warnings stated, we still take the time to investigate limits numerically in order to gain a stronger intuitive understanding of limits.

Using Tables to Estimate Limits

To begin our investigation of evaluating $\lim_{x \to a} f(x)$ numerically, we (again) build a table of function values. Choosing two sets of x-values—one set approaching a from "below" (these are x-values less than a), and another set of values approaching a from "above" (these are x-values greater than a). Doing so yields something like Table 1.1.2.

Table* 1.1.2*: Typical table investigating the behavior of f near $x = a$

x	$f(x)$	x	$f(x)$
$a - 0.1$	$f(a - 0.1)$	$a + 0.1$	$f(a + 0.1)$
$a - 0.01$	$f(a - 0.01)$	$a + 0.01$	$f(a + 0.01)$
$a - 0.001$	$f(a - 0.001)$	$a + 0.001$	$f(a + 0.001)$
$a - 0.0001$	$f(a - 0.0001)$	$a + 0.0001$	$f(a + 0.0001)$
Use additional values as necessary.		Use additional values as necessary.	

Next, we look at the values in each of the $f(x)$ columns and determine whether the values *seem* to be approaching a single value

as we move down the column. In our columns, we look at the sequence of function values $f(a-0.1)$, $f(a-0.01)$, $f(a-0.001)$, $f(a-0.0001)$, and so on, and the sequence of function values $f(a+0.1)$, $f(a+0.01)$, $f(a+0.001)$, $f(a+0.0001)$, and so on.

Caution: Be prepared to try "wiggle" sizes other than powers of ten

We have chosen the x-values $a \pm 0.1,\ a \pm 0.01,\ a \pm 0.001,\ a \pm 0.0001$, and so forth. These "wiggle" sizes (± 0.1, ± 0.01, ± 0.001, and ± 0.0001) will work *most* of the time; however, there will be occasions where we might need to modify our choice of "wiggle" sizes to something less predictable.

If both function value columns approach a common, L, we are inclined to think that

$$\lim_{x \to a} f(x) = L.$$

If unsure of our result, we could evaluate the given function at values closer (but not equal) to a; however, due to finite precision arithmetic, the technology we use will round our input to a if our "wiggle" from a is too small.

For example, when we were trying to evaluate $\lim_{x \to 2} \dfrac{x^2-4}{x-2}$ using a table of values, Desmos returns Table 1.1.3.

***Table** 1.1.3**:** Showcasing how technology can fail us*

x	$f(x) = \frac{x^2-4}{x-2}$	x	$f(x) = \frac{x^2-4}{x-2}$
1.9	3.9	2.1	4.1
1.9999	3.9999	2.0001	4.0001
1.9999999	3.9999999	2.0000001	4.0000001
1.9999999999999999	**undefined**	2.0000000000000001	**undefined**

In this case, the "wiggle" sizes we are choosing are too small for Desmos to handle.

It's time we practice some of these new skills.

Example 1.1.1

Evaluate

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$$

using a table of function values.

Solution available at the end of this section.

Before moving to another example, it's time I caution you about trigonometric functions and Calculus.

Caution: Calculus Does Not Work in Degrees

For reasons we will discuss later, when working with trigonometric functions in Calculus, you *must* work in radians—not degrees.

Example 1.1.2

Evaluate

$$\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}$$

using a table of function values.

Solution available at the end of this section.

Checkpoint 1.1.2

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Two Important Limits

Generating a table of function values provides useful insight into the value of a function's limit at a given point. However, this technique relies too much on technology and the hope that your technology is not "lying" to you. We eventually need to develop alternative methods of evaluating limits. These new methods are more algebraic in nature, and we explore them later in this chapter. At this point, we introduce two particular limits foundational to the techniques to come.

Theorem: Two Important Limits

Let a be a real number and c be a constant.

1. $\lim_{x\to a} x = a$ (**Limit of the Identity Function**)
2. $\lim_{x\to a} c = c$ (**Limit of a Constant**)

We can make the following observations about these two limits.

1. For the first limit, observe that as x approaches a, so does $f(x)$, because $f(x) = x$. Consequently, $\lim_{x\to a} x = a$.
2. For the second limit, consider Table 1.1.6.

***Table** 1.1.6**:** Investigating what a constant does as we change x*

x	$f(x)=c$	x	$f(x)=c$
$a-0.1$	c	$a+0.1$	c
$a-0.01$	c	$a+0.01$	c
$a-0.001$	c	$a+0.001$	c
$a-0.0001$	c	$a+0.0001$	c

Observe that for all values of x, the value of $f(x)$ remains constant. We have no choice but to conclude $\lim_{x\to a} c = c$.

Caution: Observations Are Not Proofs

As you move through this course (and toward higher-level courses in Mathematics), you must have a clear understanding between an observation and a proof. What we have observed above (that $\lim_{x\to a} c$ looks to be c) is **not a proof**. We will learn how to prove these types of statements later.

The Existence of a Limit

As we consider the limit in the following example, keep in mind that for a function's limit to exist at a point, the function values must approach a single real number. If the function values do not approach a single value, then the limit does not exist.

Example 1.1.3

Evaluate

$$\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$$

using a table of values.

Solution available at the end of this section.

Notation: DNE

When a limit does not become a single value (whether finite or infinity), it is an agreed mathematical convention to write **DNE**, which tells the reader the limit **does not exist**.

Checkpoint 1.1.3

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One-Sided Limits

Sometimes, indicating that the limit of a function fails to exist at a number does not provide us with enough information about the behavior of the function near that particular number. To see this, consider the function

$$g(x) = \frac{|x-2|}{x-2}.$$

***Table** 1.1.8: Investigating the behavior of $\frac{|x-2|}{x-2}$ as $x \to 2$*

x	$g(x) = \frac{\lvert x-2\rvert}{x-2}$	x	$g(x) = \frac{\lvert x-2\rvert}{x-2}$
1.9	-1	2.1	1
1.99	-1	2.01	1
1.999	-1	2.001	1
1.9999	-1	2.0001	1

As we pick values of x close to 2, $g(x)$ does not approach a single value, so the limit as x approaches 2 does not exist. That is,

$$\lim_{x\to 2} g(x) \text{ DNE}.$$

However, this statement alone does not give us a complete picture of the behavior of the function around $x = 2$. To provide a more accurate description, we introduce the idea of a **one-sided limit** (also known as a **single-sided limit**).

For all values to the left of 2 (or the "below" 2), $g(x) = -1$. Thus, as x approaches 2 from the left, $g(x)$ approaches -1.

Mathematically, we say that the limit as x approaches 2 from the left is -1. Symbolically, we express this idea as

$$\lim_{x \to 2^-} g(x) = -1.$$

Similarly, as x approaches 2 from the right (or from "above"), $g(x)$ approaches 1. Symbolically, we express this idea as

$$\lim_{x \to 2^+} g(x) = 1.$$

We can now present an informal definition of one-sided limits.

Definition: One-sided Limits

Limit from the left: Let $f(x)$ be a function defined at all values in an open interval of the form (z, a), and let L be a real number. If the values of the function $f(x)$ approach the real number L as the values of x (where \(x

Limit from the right: Let $f(x)$ be a function defined at all values in an open interval of the form (a, c), and let L be a real number. If the values of the function $f(x)$ approach the real number L as the values of x (where $x > a$) approach the number a, then we say that L is the limit of $f(x)$ as x approaches a from the right. Symbolically, we express this idea as

$$\lim_{x \to a^+} f(x) = L.$$

Example 1.1.4

For the function

$$f(x) = \begin{cases} x + 1, & \text{if } x < 2 \\ x^2 - 4, & \text{if } x \geq 2 \end{cases},$$

evaluate each of the following limits.

a. $\lim_{x \to 2^-} f(x)$
b. $\lim_{x \to 2^+} f(x)$

Solutions available at the end of this section.

Checkpoint 1.1.4

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Let's now consider the relationship between the limit of a function at a point and the limits from the right and left at that point. If the limit from the right and the limit from the left share a common value, then that is the limit of the function at that point. Similarly, suppose the limit from the left and the right take on different values. In that case, the limit of the function does not exist. These conclusions are summarized in the following theorem.

Theorem: Relating One-Sided and Two-Sided Limits

Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number, ∞, or $-\infty$. Then,

$$\lim_{x \to a} f(x) = L$$

if and only if

$$\lim_{x\to a^-} f(x) = L \quad \text{and} \quad \lim_{x\to a^+} f(x) = L.$$

Infinite Limit at a Finite Number

Evaluating the limit of a function at a point, or evaluating the limit of a function from the right and left at a point, helps us to characterize the behavior of a function around a given value. As we shall see, we can also describe the behavior of functions that do not have finite limits.

Consider the function $h(x) = \dfrac{1}{(x-2)^2}$. Let's build a table of values to help us determine the value of $\lim_{x\to 2} h(x)$.

Table 1.1.10***:*** *Investigating the values of* $\frac{1}{(x-2)^2}$ *as* $x \to 2$

x	$h(x) = \frac{1}{(x-2)^2}$	x	$h(x) = \frac{1}{(x-2)^2}$
1.9	100	2.1	100
1.99	10000	2.01	10000
1.999	1000000	2.001	1000000
1.9999	10^8	2.0001	10^8

We can clearly see that as the values of x approach 2, the values of $h(x) = \frac{1}{(x-2)^2}$ become larger and larger (in fact, the values approach infinity). Mathematically, we say that the limit of $h(x)$ as x approaches 2 is positive infinity. Symbolically, we express this idea as

$$\lim_{x\to 2} h(x) = +\infty.$$

More generally, we define (again, in an intuitive, but not precise way) an **infinite limit at a finite number** as follows.

Definitions (Intuitive): Infinite Limits at Finite Numbers

Infinite limits from the left: Let $f(x)$ be a function defined at all values in an open interval of the form (b, a).

i. If the values of $f(x)$ increase without bound as the values of x (where \(x
ii. If the values of $f(x)$ decrease without bound as the values of x (where \(x

Infinite limits from the right: Let $f(x)$ be a function defined at all values in an open interval of the form (a, c).

i. If the values of $f(x)$ increase without bound as the values of x (where $x > a$) approach the number a, then we say that the limit as x approaches a from the right is positive infinity and we write
$$\lim_{x\to a^+} f(x) = +\infty.$$
ii. If the values of $f(x)$ decrease without bound as the values of x (where $x > a$) approach the number a, then we say that the limit as x approaches a from the right is negative infinity and we write
$$\lim_{x\to a^+} f(x) = -\infty.$$

Two-sided infinite limit: Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a

i. If the values of $f(x)$ increase without bound as the values of x (where $x \neq a$) approach the number a, then we say that the limit as x approaches a is positive infinity and we write
$$\lim_{x\to a} f(x) = +\infty.$$
ii. If the values of $f(x)$ decrease without bound as the values of x (where $x \neq a$) approach the number a, then we say that the limit as x approaches a is negative infinity and we write

$$\lim_{x\to a} f(x) = -\infty.$$

It is important to understand that when we write statements such as $\lim_{x\to a} f(x) = \infty$ or $\lim_{x\to a} f(x) = -\infty$, we are describing the behavior of the function. We are not asserting that the limit exists. Recall that for the limit of a function f to exist at a, the function values must approach a finite, real number L as x approaches a. That said, if $\lim_{x\to a} f(x) = \infty$, we always write $\lim_{x\to a} f(x) = \infty$ rather than saying the limit does not exist.[1]

Example 1.1.5

Evaluate each of the following limits, if possible.

a. $\lim_{x\to 0^-} \frac{1}{x}$
b. $\lim_{x\to 0^+} \frac{1}{x}$
c. $\lim_{x\to 0} \frac{1}{x}$

Solutions available at the end of this section.

Checkpoint 1.1.5

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A Final Warning About Using Tables

Up to this point, the only way you know how to evaluate the limit of a function is by building a table of values to estimate the limit. Although you will be allowed to while doing the homework in this section, you will rarely (if ever) use a table of values to approximate the value of a limit as we move forward. Moreover, consider the following warning.

Caution: Using Tables for Limits is a Terrible Idea

Outside of the homework in this section, attempting to estimate a limit using a table of values is horrendously bad. Again, read the following sentence carefully:

Beyond this section, and unless specifically asked to do so, you are to never use a table of values to estimate a limit.

To illustrate why this warning is so important, consider the following example.

Example 1.1.6

Evaluate

$$\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$$

using a table of functional values.

Solution available at the end of this section.

Hidden Resources: Borrowing Technology

Having a computer is a requirement for any college class these days; however, what do you do if you don't have a computer? Cosumnes River College (and the Los Rios Community College District) has pulled together a great list of resources for students needing a computer. Whether you want to borrow a Chromebook from the CRC library for the semester or you prefer to purchase a discounted computer, the CRC Library has all the information you need!

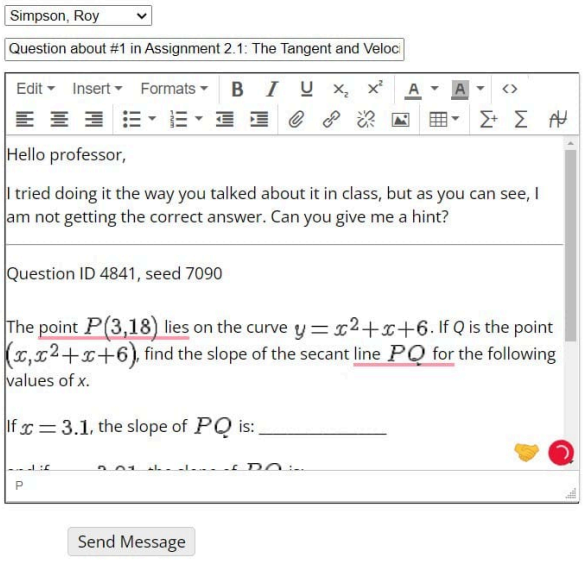
MyOpenMath Advice: "Message Instructor" Within Questions

What is it?

If you are stuck on a homework question, and if your professor has set up a discussion area on Canvas, post your question to that area first; however, not all professors have enabled a discussions area. In this case, you can ask your professor for help directly by clicking on the "Message instructor" link at the bottom of the question.

Why use it?

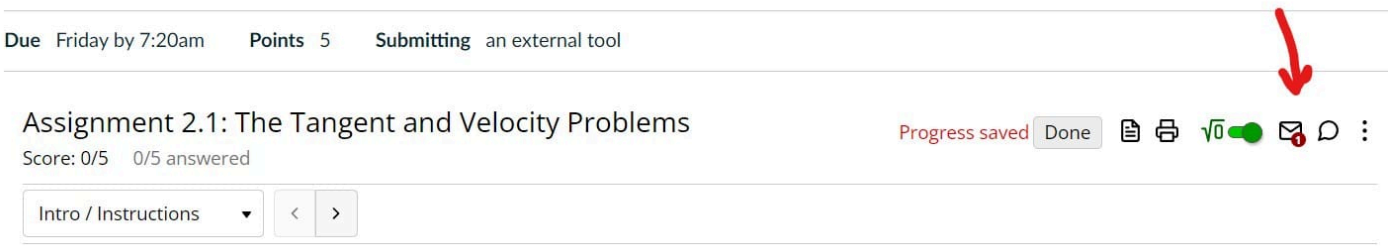
The nice thing about using the "Message instructor" link is that it sends *your version* of the question you are asking about directly to your professor. This helps them better understand your question. Be sure to include some information as to what might be causing your confusion.



How do you know when your professor has responded?

This is the only tricky part. When your professor responds, you will *not* get an email notification. This is a weakness of the MyOpenMath embed into Canvas (the messaging systems do not work well together); however, when your professor responds, you will see a message flag within *any* homework assignment you open.

Assignment 2.1: The Tangent and Velocity Problems



Just click on the envelope and you will see your professor's response.

Answers

Section Preview

This section should be used to introduce the intuitive idea of a limit at a finite number by way of building tables and using non-graphing technology (graphs are saved for the next section). Note that limits at infinity are avoided at this point because we are using computational technology to approximate limit values. Single-sided limits are introduced (again, intuitively) and the "DNE" notation is presented. Warnings should abound that this is the weakest form of limit approximation.

Section Preview

- **Sets and Numbers**
  - Sets of Numbers: The limit L is defined as a *finite real number*, and the infinite-limit section contrasts this with $\pm\infty$; a student must understand the real number system (and that infinity is not a real number) to grasp when a limit exists versus describes unbounded behavior.
  - Inequalities and Inequality Notation: Every one-sided limit definition is framed with $x < a$ (from the left) and $x > a$ (from the right), and the piecewise function in Example 4 is split by the conditions $x < 2$ and $x \geq 2$; reading these requires fluency with inequality symbols.
  - Intervals and Interval Notation: Each definition box describes the function on an open interval, e.g. "an open interval containing a," (z, a), (a, c), and (b, a); the student must read open-interval notation to understand where the function must be defined for the limit to be discussed.
- **An Overview of Functions**
  - Evaluating Functions: The entire limit concept is expressed in function notation ($\lim_{x\to a} f(x) = L$, the question "What is $f(a)$?"); a student cannot read the notation or follow the definitions without understanding what $f(x)$ means.
  - The Domain and Range of a Function: The central Caution ("$x \neq a$") hinges on the fact that f need only be defined on an interval *with the possible exception of* a *itself*, so $f(a)$ may not exist; understanding domain is what makes this idea coherent.
- **Trigonometry**
  - Converting Between Degrees and Radians: The Caution "Calculus Does Not Work in Degrees" requires the student to already understand radian measure so they can interpret $\lim_{\theta\to 0} \frac{\sin\theta}{\theta}$ (Example 1) and the other trig limits correctly.

Section Preview

- **Arithmetic**
  - Mathematical Operations — Powers and Square Roots: Building the tables requires evaluating squares (Table 1, $\frac{x^2-4}{x-2}$) and square roots (Example 2, $\frac{\sqrt{x}-2}{x-4}$ at $x = 3.9, 3.99, \ldots$).
  - The Absolute Value of a Number: The one-sided-limit example $g(x) = \frac{|x-2|}{x-2}$ (Table 8) requires evaluating absolute values numerically to see why $g = -1$ on the left and $+1$ on the right.
- **An Overview of Functions**
  - Evaluating Functions: Every table in the section (Tables 1, 4, 5, 7, 8, 10, 11, 12) is produced by substituting specific x-values into $f(x)$; this substitution skill is the mechanical core of the numerical method.
- **Graphing Functions**
  - Piecewise Functions: Example 4 requires selecting the correct branch ($x + 1$ for $x < 2$ versus $x^2 - 4$ for $x \geq 2$) when filling in Table 9 to compute each one-sided limit.
- **Trigonometry**
  - Evaluating Trigonometric Functions Using Technology: Examples 1, 3, and 6 require evaluating $\sin\theta$, $\sin(1/x)$, and $\sin(\pi/x)$ at small inputs with technology (in radian mode) to populate the tables.

Section Preview

- Describe the limit of a function and use proper limit notation.
- Use a table of values to estimate the limit of a function or to identify when the limit does not exist.
- Define and investigate one-sided limits.
- Explain the relationship between one-sided and two-sided limits.
- Using correct notation, describe an infinite limit.
- Explain why investigating limits using technology is inherently flawed.

Example 1.1.1

We have calculated some values of $f(\theta) = \frac{\sin\theta}{\theta}$ for θ near to 0 in Table 1.1.4.

Table 1.1.4***:*** *Investigating the values of* $\frac{\sin(\theta)}{\theta}$ *as* θ *approaches* 0

θ	$\frac{\sin\theta}{\theta}$	θ	$\frac{\sin\theta}{\theta}$
-0.1	0.99833417	0.1	0.99833417

θ	$\frac{\sin\theta}{\theta}$	θ	$\frac{\sin\theta}{\theta}$
-0.01	0.99998333	0.01	0.99998333
-0.001	0.99999983	0.001	0.99999983
-0.0001	1	0.0001	1

As we read down each $\frac{\sin\theta}{\theta}$ column, we see that the values appear to be approaching 1. In fact, Desmos (the technology I used to build this table of values) is confidently stating that the function evaluated at $\theta = \pm 0.001$ is *exactly* 1; however, this is not true. In fact, the value of $\frac{\sin(\pm 0.001)}{\pm 0.001}$ is approximately 0.9999998333. Nevertheless, it is *somewhat* reasonable to conclude that

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

based on this table of values.

Example 1.1.2

As before, we use a table to list values of the function as x gets close to 4 (from both sides of 4).

Table 1.1.5***:*** *Investigating the behavior of* $\frac{\sqrt{x}-2}{x-4}$ *as* $x \to 4$

x	$\frac{\sqrt{x}-2}{x-4}$	x	$\frac{\sqrt{x}-2}{x-4}$
3.9	0.251582341869	4.1	0.248456731317
3.99	0.25015644562	4.01	0.24984394501
3.999	0.250015627	4.001	0.249984377
3.9999	0.250001563	4.0001	0.249998438
3.99999	0.25000016	4.00001	0.24999984

After inspecting this table, the function values for $x < 4$ appear to decrease toward 0.25. Likewise, the function values for $x > 4$ appear to be increasing toward 0.25. We conclude that

$$\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4} = 0.25.$$

Example 1.1.3

Table 1.1.7 lists values for the function $\sin(1/x)$ as x approaches 0 from both the left and the right.

Table 1.1.7***:*** *Investigating the behavior of* $\sin\left(\frac{1}{x}\right)$ *as* $x \to 0$

x	$\sin(1/x)$	x	$\sin(1/x)$
-0.1	0.544021110889	0.1	-0.544021110889
-0.01	0.50636564111	0.01	-0.50636564111
-0.001	-0.8268795405312	0.001	0.8268795405312
-0.0001	0.305614388888	0.0001	-0.305614388888
-0.00001	-0.035748797987	0.00001	0.035748797987
-0.000001	0.349993504187	0.000001	-0.349993504187

After examining the table of function values, we can see that the y-values do not seem to approach any one single value. It appears the limit **does not exist**. Before drawing this conclusion, let's take a more systematic approach. Take the following sequence of x-values approaching 0:

$$\frac{2}{\pi},\ \frac{2}{3\pi},\ \frac{2}{5\pi},\ \frac{2}{7\pi},\ \frac{2}{9\pi},\ \frac{2}{11\pi},\ \ldots.$$

The corresponding y-values are

$$1,\ -1,\ 1,\ -1,\ 1,\ -1,\ \ldots.$$

At this point, we conclude that

$$\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$$

does not exist. Mathematicians frequently abbreviate "does not exist" as **DNE**. Thus, we would write

$$\lim_{x\to 0}\sin\left(\frac{1}{x}\right) \quad \text{DNE}.$$

Example 1.1.4

We can use tables of values again. Observe in Table 1.1.9, for values of x less than 2, we use $f(x) = x + 1$, and for values of x greater than 2, we use $f(x) = x^2 - 4$.

***Table** 1.1.9: Investigating the one-sided behavior of a piecewise-defined function*

x	$f(x) = x + 1$	x	$f(x) = x^2 - 4$
1.9	2.9	2.1	0.41
1.99	2.99	2.01	0.0401
1.999	2.999	2.001	0.004001
1.9999	2.9999	2.0001	0.00040001
1.99999	2.99999	2.00001	0.0000400001

Based on this table, we can conclude the following.

$$\lim_{x\to 2^-} f(x) = 3 \quad \text{and} \quad \lim_{x\to 2^+} f(x) = 0.$$

Therefore, the (two-sided) limit of $f(x)$ does not exist at $x = 2$.

Example 1.1.5

Begin by constructing a table of function values.

***Table** 1.1.11: Investigating the value of $\frac{1}{x}$ as $x \to 0$*

x	$\frac{1}{x}$	x	$\frac{1}{x}$
-0.1	-10	0.1	10
-0.01	-100	0.01	100
-0.001	-1000	0.001	1000
-0.0001	-10000	0.0001	10000
-0.00001	-100000	0.00001	100000
-0.000001	-1000000	0.000001	1000000

The values of $1/x$ decrease without bound as x approaches 0 from the left. We conclude that

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

The values of $1/x$ increase without bound as x approaches 0 from the right. We conclude that

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty.$$

Since $\lim_{x\to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x\to 0^+} \frac{1}{x} = +\infty$ have different values, we conclude that

$$\lim_{x\to 0} \frac{1}{x} \quad \text{DNE}.$$

Example 1.1.6

We have calculated the values of $f(x) = \sin\left(\frac{\pi}{x}\right)$ for the values of x listed in Table 1.1.12.

***Table** 1.1.12: Showcasing why you cannot trust tables of values (in general)*

x	$\sin\left(\frac{\pi}{x}\right)$	x	$\sin\left(\frac{\pi}{x}\right)$	x	$\sin\left(\frac{\pi}{x}\right)$	x	$\sin\left(\frac{\pi}{x}\right)$
0.1	0	0.3	-0.8660254	0.6	-0.8660254	0.7	-0.97492791
0.01	0	0.03	-0.8660254	0.06	0.8660254	0.07	0.78183148
0.001	0	0.003	-0.8660254	0.006	0.8660254	0.007	0.43388374
0.0001	0	0.0003	-0.8660254	0.0006	0.8660254	0.0007	0.97492791

The values chosen for x in each column approach 0; however, having the last digit as 1 tempts us into thinking the limit of this function is 0. Having the final digit as 3, makes -0.8660254 seem like the limiting value. With a 6 as the final digit, we *think* the limit is 0.8660254. Finally, the 7 as the final digit leaves us completely confused. Is there an actual value for the limit? We will have to wait until the next section to see...

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1.1.1: Resources and Key Concepts

Resources

Videos

- Limit of a function
  - Limits - The Basic Concepts of Limits - Finite Limits at Finite Points
- One-sided limits
  - Limits - The Basic Concepts of Limits - Single-Sided Limits

Key Concepts

Definitions

- **Finite Limit at a Finite Number (Intuitive Definition)**: Let $f(x)$ be a function defined at all values in an open interval containing a finite number a, with the possible exception of a itself, and let L be a finite real number. If the values of the function $f(x)$ approach the real number L as the values of x (where $x \neq a$) approach the number a, then we say that the finite limit of $f(x)$ as x approaches the finite number a is L. Symbolically, $\lim\limits_{x \to a} f(x) = L$.
- **DNE (Notation)**: An abbreviation for "does not exist," used to indicate that a limit does not approach a single value (whether finite or infinity).
- **Limit from the left**: Let $f(x)$ be a function defined at all values in an open interval of the form (z, a), and let L be a real number. If the values of $f(x)$ approach L as the values of x (where $x < a$) approach a, then L is the limit of $f(x)$ as x approaches a from the left, written $\lim\limits_{x \to a^-} f(x) = L$.
- **Limit from the right**: Let $f(x)$ be a function defined at all values in an open interval of the form (a, c), and let L be a real number. If the values of $f(x)$ approach L as the values of x (where $x > a$) approach a, then L is the limit of $f(x)$ as x approaches a from the right, written $\lim\limits_{x \to a^+} f(x) = L$.
- **Infinite limits from the left**: If the values of $f(x)$ increase without bound as x approaches a from the left ($x < a$), the limit is positive infinity ($\lim\limits_{x \to a^-} f(x) = +\infty$). If $f(x)$ decrease without bound, the limit is negative infinity ($\lim\limits_{x \to a^-} f(x) = -\infty$).
- **Infinite limits from the right**: If the values of $f(x)$ increase without bound as x approaches a from the right ($x > a$), the limit is positive infinity ($\lim\limits_{x \to a^+} f(x) = +\infty$). If $f(x)$ decrease without bound, the limit is negative infinity ($\lim\limits_{x \to a^+} f(x) = -\infty$).
- **Two-sided infinite limit**: If the values of $f(x)$ increase without bound as x (where $x \neq a$) approaches a, the limit is positive infinity ($\lim\limits_{x \to a} f(x) = +\infty$). If $f(x)$ decrease without bound, the limit is negative infinity ($\lim\limits_{x \to a} f(x) = -\infty$).

Theorems

- **Theorem: Two Important Limits**

  Let a be a real number and c be a constant.

  1. $\lim\limits_{x \to a} x = a$ (Limit of the Identity Function)
  2. $\lim\limits_{x \to a} c = c$ (Limit of a Constant)
- **Theorem: Relating One-Sided and Two-Sided Limits**

  Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number, ∞, or $-\infty$. Then,

  $$\lim_{x \to a} f(x) = L$$

  if and only if

$$\lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L.$$

Common Mistakes

- **Caution:** $x \neq a$: The intuitive definition of a finite limit at a finite number states that x need not equal a. The limit is concerned with function values as x gets closer to a, not what $f(a)$ is. It could be that $f(a)$ does not exist or $L \neq f(a)$.
- **Caution: Be prepared to try "wiggle" sizes other than powers of ten**: While choosing x-values like $a \pm 0.1, a \pm 0.01$, etc., often works, one may need to modify the choice of "wiggle" sizes (the difference from a) on some occasions to better observe the limiting behavior.
- **Caution: Calculus Does Not Work in Degrees**: When working with trigonometric functions in Calculus, angles must be expressed in radians, not degrees.
- **Caution: Observations Are Not Proofs**: Observing a trend in a table of function values that suggests a limit (e.g., that $\lim_{x \to a} c$ looks to be c) is not a mathematical proof. Formal proofs are required for such statements.
- **Caution: Using Tables for Limits is a Terrible Idea**: Attempting to estimate a limit using a table of values outside of introductory exercises in this section is highly discouraged. This method can be misleading due to finite precision arithmetic or by not choosing appropriate x-values.

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1.1.2: Homework

Note

The following exercises are meant as hand-in homework (if assigned by your instructor). Typed-in answers here will not be saved or scored.

Basic Skills

Exercise 1

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Exercise 2

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Exercise 3

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Exercise 4

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Exercise 5

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Exercise 6

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Exercise 7

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Exercise 8

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Exercise 9

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Exercise 10

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Exercise 11

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Exercise 12

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Exercise 13

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Exercise 14

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Synthesis Questions

Exercise 15

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1.1.3: Additional Exercises

Reading Questions

1. What is the intuitive meaning of the statement $\lim_{x\to a} f(x) = L$?
2. When evaluating $\lim_{x\to a} f(x)$ numerically using a table, why is it important that x approaches a but $x \neq a$?
3. Describe the general process of using a table of values to estimate the limit of a function.
4. What are the three issues the text mentions regarding the use of numerical approximation (tables/technology) to state the value of a limit?
5. What does the notation $\lim_{x\to a^-} f(x)$ represent?
6. What does the notation $\lim_{x\to a^+} f(x)$ represent?
7. How is the existence and value of a two-sided limit, $\lim_{x\to a} f(x)$, related to its corresponding one-sided limits?
8. What does it mean for a function to have an infinite limit, such as $\lim_{x\to a} f(x) = \infty$? According to the text, does this mean the limit "exists" in the same way a finite limit does?
9. What does "DNE" signify when discussing limits?
10. Why is it crucial to use radians instead of degrees when dealing with trigonometric functions in calculus limits?
11. State the "Limit of the Identity Function" and the "Limit of a Constant" as presented in the "Two Important Limits" theorem.
12. What is the primary warning given at the end of the section regarding the use of tables to estimate limits in future calculus work?

Homework

For exercises 1 - 2, consider the function $f(x) = \dfrac{x^2 - 1}{|x - 1|}$.

1) Complete the following table for the function. Round your solutions to four decimal places.

x	$f(x)$	x	$f(x)$
0.9	a.	1.1	e.
0.99	b.	1.01	f.
0.999	c.	1.001	g.
0.9999	d.	1.0001	h.

2) What do your results in the preceding exercise indicate about the two-sided limit $\lim_{x\to 1} f(x)$? Explain your response.

For exercises 3 - 5, consider the function $f(x) = (1 + x)^{1/x}$.

3) Make a table showing the values of f for $x = -0.01,\ -0.001,\ -0.0001,\ -0.00001$ and for $x = 0.01,\ 0.001,\ 0.0001,\ 0.00001$. Round your solutions to five decimal places.

x	$f(x)$	x	$f(x)$
-0.01	a.	0.01	e.
-0.001	b.	0.001	f.
-0.0001	c.	0.0001	g.
-0.00001	d.	0.00001	h.

4) What does the table of values in the preceding exercise indicate about the function $f(x) = (1 + x)^{1/x}$?

5) To which mathematical constant do the values in the preceding exercise appear to be approaching? This is the actual limit here.

In exercises 6 - 8, use the given values to set up a table to evaluate the limits. Round your solutions to eight decimal places.

6) $\lim_{x\to 0}\dfrac{\sin 2x}{x}$; $\pm 0.1,\ \pm 0.01,\ \pm 0.001,\ \pm .0001$

x	$\frac{\sin 2x}{x}$	x	$\frac{\sin 2x}{x}$
-0.1	a.	0.1	e.
-0.01	b.	0.01	f.
-0.001	c.	0.001	g.
-0.0001	d.	0.0001	h.

7) $\lim_{x\to 0}\dfrac{\sin 3x}{x}\ \pm 0.1,\ \pm 0.01,\ \pm 0.001,\ \pm 0.0001$

x	$\frac{\sin 3x}{x}$	x	$\frac{\sin 3x}{x}$
-0.1	a.	0.1	e.
-0.01	b.	0.01	f.
-0.001	c.	0.001	g.
-0.0001	d.	0.0001	h.

8) Use the preceding two exercises to conjecture (guess) the value of the following limit: $\lim_{x\to 0}\dfrac{\sin ax}{x}$ for a, a positive real value.

In exercises 9 - 14, set up a table of values to find the indicated limit. Round to eight significant digits.

9) $\lim_{x\to 2}\dfrac{x^2-4}{x^2+x-6}$

x	$\frac{x^2-4}{x^2+x-6}$	x	$\frac{x^2-4}{x^2+x-6}$
1.9	a.	2.1	e.
1.99	b.	2.01	f.
1.999	c.	2.001	g.
1.9999	d.	2.0001	h.

10) $\lim_{x\to 1}(1-2x)$

x	$1-2x$	x	$1-2x$
0.9	a.	1.1	e.
0.99	b.	1.01	f.
0.999	c.	1.001	g.
0.9999	d.	1.0001	h.

11) $\lim_{x\to 0}\dfrac{5}{1-e^{1/x}}$

x	$\frac{5}{1-e^{1/x}}$	x	$\frac{5}{1-e^{1/x}}$
-0.1	a.	0.1	e.
-0.01	b.	0.01	f.
-0.001	c.	0.001	g.
-0.0001	d.	0.0001	h.

12) $\lim_{z\to 0} \dfrac{z-1}{z^2(z+3)}$

z	$\frac{z-1}{z^2(z+3)}$	z	$\frac{z-1}{z^2(z+3)}$
-0.1	a.	0.1	e.
-0.01	b.	0.01	f.
-0.001	c.	0.001	g.
-0.0001	d.	0.0001	h.

13) $\lim_{t\to 0^+} \dfrac{\cos t}{t}$

t	$\frac{\cos t}{t}$
0.1	a.
0.01	b.
0.001	c.
0.0001	d.

14) $\lim_{x\to 2} \dfrac{1-\frac{2}{x}}{x^2-4}$

x	$\frac{1-\frac{2}{x}}{x^2-4}$	x	$\frac{1-\frac{2}{x}}{x^2-4}$
1.9	a.	2.1	e.
1.99	b.	2.01	f.
1.999	c.	2.001	g.
1.9999	d.	2.0001	h.

In exercises 15 - 16, set up a table of values and round to eight significant digits. Based on the table of values, make a guess about what the limit is. Then, use a calculator to graph the function and determine the limit. Was the conjecture correct? If not, why does the method of tables fail?

15) $\lim_{\theta\to 0} \sin\left(\dfrac{\pi}{\theta}\right)$

θ	$\sin\left(\frac{\pi}{\theta}\right)$	θ	$\sin\left(\frac{\pi}{\theta}\right)$
-0.1	a.	0.1	e.
-0.01	b.	0.01	f.
-0.001	c.	0.001	g.
-0.0001	d.	0.0001	h.

16) $\lim_{\alpha \to 0^+} \frac{1}{\alpha} \cos\left(\frac{\pi}{\alpha}\right)$

a	$\frac{1}{\alpha}\cos\left(\frac{\pi}{\alpha}\right)$
0.1	a.
0.01	b.
0.001	c.
0.0001	d.

17) A track coach uses a camera with a fast shutter to estimate the position of a runner with respect to time. A table of the values of position of the athlete versus time is given here, where x is the position in meters of the runner and t is time in seconds. What is $\lim_{t \to 2} x(t)$? What does it mean physically?

$t(sec)$	$x(m)$
1.75	4.5
1.95	6.1
1.99	6.42
2.01	6.58
2.05	6.9
2.25	8.5

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1.2: Graphical Investigation of Limits

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

We began our exploration of limits by numerically investigating the functions $f(x) = \frac{x^2-4}{x-2}$, $g(x) = \frac{|x-2|}{x-2}$, and $h(x) = \frac{1}{(x-2)^2}$. We now continue this investigation from a new perspective by looking at the graphs of these function, which are shown in Figure $1.2.1$. In particular, let's focus our attention on the behavior of each graph at and around $x = 2$.

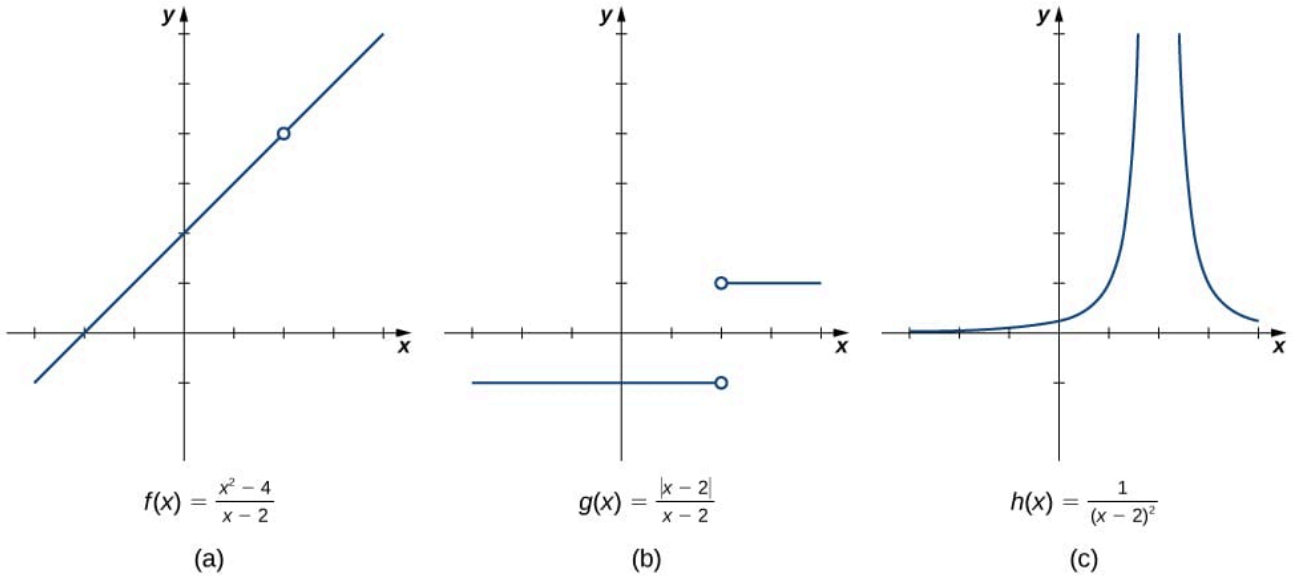

Figure $1.2.1$**:** *These graphs show the behavior of three different functions around* $x = 2$.

Each of the three functions is undefined at $x = 2$, but if we make this statement and no other, we give an incomplete picture of how each function behaves in the vicinity of $x = 2$. To express the behavior of each graph in the vicinity of 2 more thoroughly, we need to use the concept of a limit.

Finite Limit at a Finite Number - The Graphical Approach

Let's first take a closer look at Figure $1.2.1a$ and how the function $f(x) = \frac{x^2-4}{x-2}$ behaves around $x = 2$. As the values of x approach 2 from either side of 2, the values of $y = f(x)$ approach 4. As we did when investigating limits numerically, we say that the limit of $f(x)$ as x approaches 2 is 4. Again, we express this symbolically as

$$\lim_{x \to 2} f(x) = 4.$$

With graphical explorations of limits, it's even more important to repeat our caution from last section.

Caution: $x \neq a$

The intuitive definition of a finite limit at a finite number states that x need not equal a. This is critical to understand. The x -values are allowed to get very close to a but will never become a. In a very real sense, the limit,

$$\lim_{x \to a} f(x) = L,$$

is concerned with the function values as x gets closer and closer to a; however, it is *not* asking, "What is $f(a)$?" In fact, it could be the case that $f(a)$ does not exist or $L \neq f(a)$.

Interactive Element: A Limit at a Hole

The following Interactive Element might help you visualize how a limit can exist even where the function is undefined.

Interact: Drag the red point along the curve toward $x = 2$ from the left, then from the right.

Observation: As $x \to 2$ from either side, y approaches 4, so $\lim_{x \to 2} f(x) = 4$ — even though the open circle shows $f(2)$ is undefined.

We mentioned the issues with investigating limits numerically in the previous section. Graphical investigations are only slightly less dangerous. In practice, using a graph to state the value of a limit presents three issues:

1. if not given a graph, then it could be arduous (or impossible) to recreate the graph of the function, and
2. graphs can be visually "tricky," leading you to assume a function behavior that is not actually present.

These warnings stated, we still take the time to investigate limits graphically in order to gain an intuitive understanding of limits.

Using Graphs to Estimate Limits

Example 1.2.1

We built a table in the previous section to conjecture that

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1.$$

Verify the value of this limit by graphing $y = \frac{\sin \theta}{\theta}$ near to $x = 0$.

Solution available at the end of this section.

There are a few things we need to observe about Example 1.2.1.

First, recall the following caution from the last section.

Caution: Calculus Does Not Work in Degrees

For reasons we will discuss later, when working with trigonometric functions in Calculus, you *must* work in radians—not degrees.

If you attempted Example 1.2.1, but had a different graph, it is highly likely your calculator was set to degree mode. Do yourself a favor and set it to radian mode for the rest of Calculus—you will have one less thing to worry about as you adventure through the

Calculus series.

The second observation also centers on something I included in Figure $1.2.2$ that your graph would likely not have included—the open circle at $(0, 1)$. How did my graph show that and your didn't? Well, truthfully, my graph looked like your to begin with. That is, it didn't show that the given function wasn't defined at $\theta = 0$. This is why graphs can be deceiving! Luckily, I only wanted to know the behavior of the graph *near to* $\theta = 0$. Since the function had the obvious domain issue at $\theta = 0$, I manually placed an open circle on the graph at $(0, 1)$.

Our next observation is more of a question: We can see in Figure $1.2.2$ that the function, $f(\theta) = \frac{\sin\theta}{\theta}$, is not defined at $\theta = 0$. Why didn't we then say that the limit didn't exist?

Recall that $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta}$, is concerned with θ *approaching* 0—not *becoming* 0. Thus, the function not existing at $\theta = 0$ is not an issue. This is a common theme in Calculus and you must immediately come to terms with the reality that limits are unconcerned with the value of the function at the x-value they are approaching.

The final observation is yet another warning from the previous section—it *looks* like $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$; however, this is not a proof that the value of the limit is 1. For now, we only have two methods to evaluate limits (tables and graphs), so we will have to cross our fingers and hope we are not making too many assumptions. Luckily, we will eventually have analytical methods to determine the value of a limit precisely without the need for technology, tables, or graphs.

Example $1.2.2$

Verify

$$\lim_{x\to 4} \frac{\sqrt{x} - 2}{x - 4} = 0.25$$

using a graph (this is another example from last section).

Solution available at the end of this section.

Checkpoint $1.2.2$

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At this point, we see from Examples $1.2.1$ and $1.2.2$ that it may be just as easy, if not easier, to estimate a limit of a function by inspecting its graph (rather than building a table of function values). In Example $1.2.3$, we evaluate a new limit (not from the previous section) exclusively by looking at a graph rather than by using a table of function values.

Example $1.2.3$

For $g(x)$ shown in Figure $1.2.4$ below, evaluate

$$\lim_{x\to -1} g(x).$$

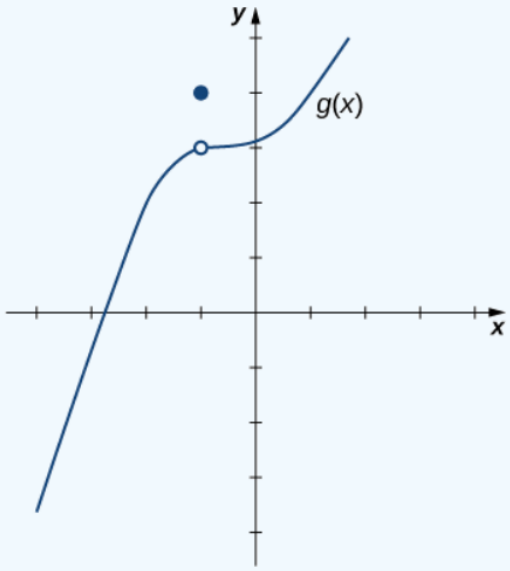

Figure 1.2.4

Solution available at the end of this section.

Checkpoint 1.2.3

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Revisiting the Existence of a Limit

The next example graphically verifies a result from the previous section.

Example 1.2.4

Evaluate

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$$

using a graph.

Solution available at the end of this section.

Checkpoint 1.2.4

Use a graph to evaluate

$$\lim_{x \to 2} \frac{|\, x^2 - 4 \,|}{x - 2},$$

if possible.

Answer available at the end of this section.

Revisiting One-Sided Limits

As will all limits in this section, one-sided limits are much easier when you have the graph of the function at hand.

Example 1.2.5

For the function

$$f(x) = \begin{cases} x+1, & \text{if } x < 2 \\ x^2 - 4, & \text{if } x \geq 2 \end{cases},$$

evaluate each of the following limits by creating a graph of the function.

a. $\lim_{x \to 2^-} f(x)$
b. $\lim_{x \to 2^+} f(x)$

Solutions available at the end of this section.

Interactive Element: When One-Sided Limits Disagree

The following Interactive Element might help you visualize why a limit fails to exist at a jump.

Interact: Drag the red point toward $x = 2$ from the left, then approach $x = 2$ again from the right.

Observation: From the left, $y \to 3$; from the right, $y \to 0$. Because the two one-sided limits disagree, $\lim_{x \to 2} f(x)$ does not exist.

Revisiting Infinite Limits at Finite Numbers

Example 1.2.6

Reevaluate each of the following limits from the previous section using a graph.

a. $\lim_{x \to 0^-} \frac{1}{x}$
b. $\lim_{x \to 0^+} \frac{1}{x}$
c. $\lim_{x \to 0} \frac{1}{x}$

Solutions available at the end of this section.

We finally reach a moment in this section where we do something more than we did last section. The definition of an infinite limit at a finite number and some inspiration from Figure 1.2.7 allow us to formalize a concept from past courses.

Definition: Vertical Asymptote

The function $f(x)$ is said to have a **vertical asymptote** at $x = a$ if any of the following are true.

$$\begin{aligned}\lim_{x\to a^-} f(x) &= +\infty\\ \lim_{x\to a^-} f(x) &= -\infty\\ \lim_{x\to a^+} f(x) &= +\infty\\ \lim_{x\to a^+} f(x) &= -\infty\\ \lim_{x\to a} f(x) &= +\infty\\ \lim_{x\to a} f(x) &= -\infty\end{aligned}$$

Interactive Element: A Vertical Asymptote as an Infinite Limit

The following Interactive Element might help you visualize how an infinite limit produces a vertical asymptote.

Interact: Drag the orange point toward $x = 2$ from the left, then from the right, and watch its height.

Observation: As $x \to 2$ from either side, y grows without bound, so $\lim_{x\to 2} \frac{1}{(x-2)^2} = +\infty$. An infinite limit like this is exactly what it means for the graph to have a vertical asymptote at $x = 2$.

Checkpoint 1.2.6

If possible, use a graph to evaluate each of the following limits and state whether the function has a vertical asymptote at $x = 0$.

a. $\lim_{x\to 0^-} \frac{1}{x^2}$
b. $\lim_{x\to 0^+} \frac{1}{x^2}$
c. $\lim_{x\to 0} \frac{1}{x^2}$

Answers available at the end of this section.

Conceptual Investigation of an Infinite Limit

When given a function that has a vertical asymptote at $x = a$, a conceptual investigation of the limit as x approaches a can often be done. This type of investigation relies heavily on your mastery of the prerequisite knowledge to Calculus—specifically, domains of functions, vertical asymptotes, and the graphs of logarithmic and trigonometric functions.

Before getting too deep into conceptual investigations of infinite limits, we need to understand the nature of a ratio when its denominator is approaching zero.

***Table** 1.2.1**:** A comparison of ratios with denominators approaching zero*

$f(x) = \frac{1}{x}$	$g(x) = \frac{13}{x}$	$h(x) = -\frac{2}{x^2}$
$\frac{1}{0.1} = 10$	$\frac{13}{-0.1} = -130$	$-\frac{2}{(-0.1)^2} = -200$
$\frac{1}{0.01} = 100$	$\frac{13}{-0.01} = -1300$	$-\frac{2}{(-0.01)^2} = -20000$
$\frac{1}{0.001} = 1000$	$\frac{13}{-0.001} = -13000$	$-\frac{2}{(-0.001)^2} = -2000000$
$\frac{1}{0.0001} = 10000$	$\frac{13}{-0.0001} = -130000$	$-\frac{2}{(-0.0001)^2} = -200000000$

The denominators are approaching zero in all three functions from Table 1.2.1. In the first and third columns, the denominators approach 0^+ (zero from the right). In the second column, the denominators approach 0^- (zero from the left). Notice, in all

three cases, that the smaller the denominator gets, the larger the value of the fraction. Understanding this property is vital to conceptually investigating infinite limits at finite numbers.

Theorem

Let $f(x) = \frac{N(x)}{D(x)}$, where $\lim_{x\to a} N(x)$ is a finite, nonzero constant, and $D(x)$ is approaching 0 as x approaches a. Then

$$\lim_{x\to a} \frac{N(x)}{D(x)}$$

is either ∞ or $-\infty$. The sign is determined by comparing the signs of the numerator and denominator of the ratio as x is approaching a.

Let's take a look at how this conceptual investigation should be done in practice.

Example 1.2.7

Evaluate each of the following limits conceptually.

a. $\lim_{x\to 7^+} \frac{2}{x-7}$
b. $\lim_{x\to 7^-} \frac{2}{x-7}$
c. $\lim_{x\to 7} \frac{2}{x-7}$
d. $\lim_{x\to 7^-} \frac{5-x}{(x-7)^2}$
e. $\lim_{x\to (\pi/2)^-} \ln(\cot(x))$

Solutions available at the end of this section.

Example $1.2.7d$ is often very challenging for students. This is because it requires a lot of visualization from your prerequisite courses. If you have difficulty visualizing the graphs of the trigonometric or logarithmic functions, you should review the material from the Appendix of this textbook. Understanding and quickly recalling the domains and graphs of those functions is a fundamental skill needed to survive Calculus.

The notation used in Example $1.2.7$ is very important to understand and use. For example, in Example $1.2.7d$, it would be totally incorrect to write

$$\lim_{x\to 7^-} \frac{5-x}{(x-7)^2} = \frac{-2}{0^+} = -\infty.$$

Why is this incorrect?

When you write that something is equal to $\frac{-2}{0^+}$, $\frac{-2}{0^-}$, or $\frac{-2}{0}$, you are stating to the world that such a number exists, which it obviously does not!

"But you know what I mean!"

Yes, your professor knows what you mean, but it is not their job to bend the rules of Mathematics to your will—that's not how Mathematics works. You must be able to show that you can embrace learning something new and bending to *try* to understand something that you never knew before. Trust me when I say—there are abstract and important reasons for all the subtle nuances your professor enforces, and those who don't enforce these nuances are wounding your learning experience.

Caution: Never Write That a Limit is Equal to an Indeterminate Form

You will often work with limits that *tend to* forms of $\frac{k}{0^+}$, $\frac{0}{0}$, $\frac{k}{\infty}$, $\frac{\infty}{\infty}$, and so on. These are known a **indeterminate forms** because we cannot easily determine their values within resorting to Calculus. Since these are what the limit is *tending* toward, but these are not numbers, these cannot be what the limit *equals*. Therefore, never write

$$\lim_{x\to a} f(x) = \frac{k}{0^+} \quad \text{or} \quad \frac{0}{0} \quad \text{or} \quad \frac{k}{\infty} \quad \text{or} \quad \frac{\infty}{\infty}.$$

Instead, show that you understand these are *tendancies* and use notation like

$$\lim_{x\to a} f(x) \begin{array}{l} \nearrow\, k \\ \searrow\, 0^+ \end{array}$$

or

$$\lim_{x\to a} f(x) \begin{array}{l} \nearrow\, 0 \\ \searrow\, 0 \end{array}$$

or

$$\lim_{x\to a} f(x) \begin{array}{l} \nearrow\, k \\ \searrow\, \infty \end{array}$$

or

$$\lim_{x\to a} f(x) \begin{array}{l} \nearrow\, \infty \\ \searrow\, 0^+ \end{array}$$

Checkpoint 1.2.7

Evaluate each of the following limits.

a. $\lim\limits_{x\to 2^-} \dfrac{1}{(x-2)^3}$
b. $\lim\limits_{x\to 2^+} \dfrac{1}{(x-2)^3}$
c. $\lim\limits_{x\to 2} \dfrac{1}{(x-2)^3}$
d. $\lim\limits_{x\to(\pi/2)^+} \sec(x)$

Answers available at the end of this section.

Example 1.2.8

Albert Einstein showed that a limit exists to how fast any object can travel. Given Einstein's equation for the mass of a moving object

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}},$$

what is the value of this bound?

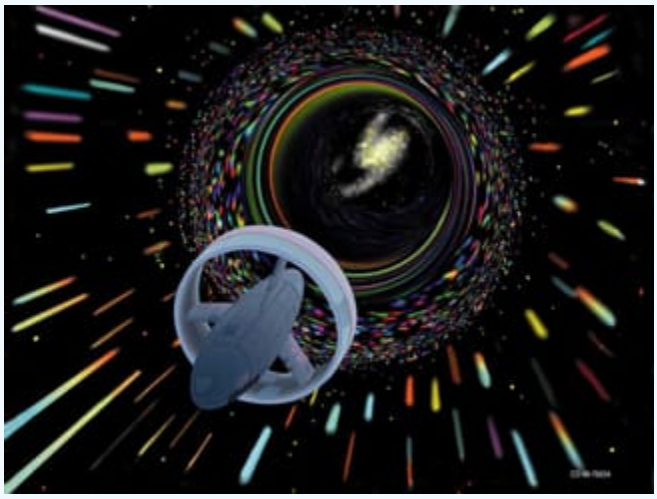

Figure 1.2.8: *(Credit: NASA)*

Solution available at the end of this section.

Revisiting Evaluation of Limits from Graphs

Let's finish this section by revisiting the graphical investigation of limits.

Interactive Element: Investigating Limits Graphically

Let $f(x)$ be the function graphed below. Use this Interactive Element to answer the questions.

Interact: Move the point to help investigate limits.

Questions: Use the graph to estimate the value of each limit.

a. $\lim_{x \to -2^-} f(x)$
b. $\lim_{x \to -2^+} f(x)$
c. $\lim_{x \to -2} f(x)$
d. $\lim_{x \to 0^-} f(x)$
e. $\lim_{x \to 0^+} f(x)$
f. $\lim_{x \to 0} f(x)$
g. $\lim_{x \to 1} f(x)$
h. $\lim_{x \to 2^-} f(x)$
i. $\lim_{x \to 2^+} f(x)$

j. $\lim_{x\to 2} f(x)$
k. $f(2)$
l. $\lim_{x\to 3^-} f(x)$
m. $\lim_{x\to 3^+} f(x)$
n. $\lim_{x\to 3} f(x)$

Answers available at the end of this section.

In the following example, we use our knowledge of various types of limits to analyze the behavior of a function at several different points.

Example 1.2.9

Use the graph of $f(x)$ in Figure $1.2.9$ to determine each of the following values:

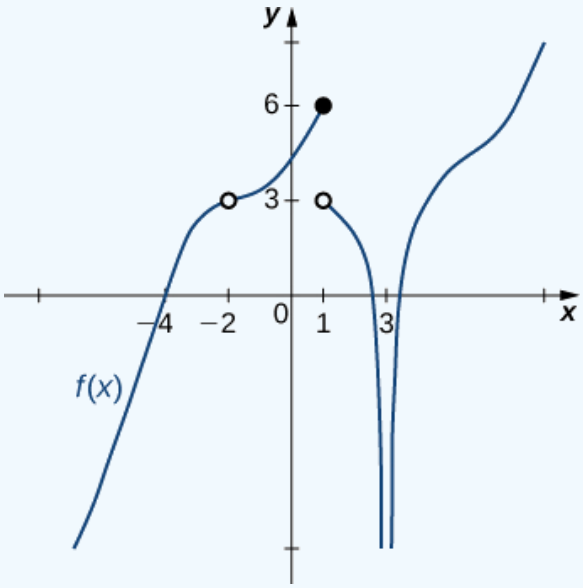

Figure $1.2.9$**:** *The graph shows* $f(x)$.

a. $\lim_{x\to -4^-} f(x)$; $\lim_{x\to -4^+} f(x)$; $\lim_{x\to -4} f(x)$; $f(-4)$
b. $\lim_{x\to -2^-} f(x)$; $\lim_{x\to -2^+} f(x)$; $\lim_{x\to -2} f(x)$; $f(-2)$
c. $\lim_{x\to 1^-} f(x)$; $\lim_{x\to 1^+} f(x)$; $\lim_{x\to 1} f(x)$; $f(1)$
d. $\lim_{x\to 3^-} f(x)$; $\lim_{x\to 3^+} f(x)$; $\lim_{x\to 3} f(x)$; $f(3)$

Solution available at the end of this section.

Checkpoint 1.2.9

Evaluate

$$\lim_{x\to 1} f(x)$$

for $f(x)$ shown below.

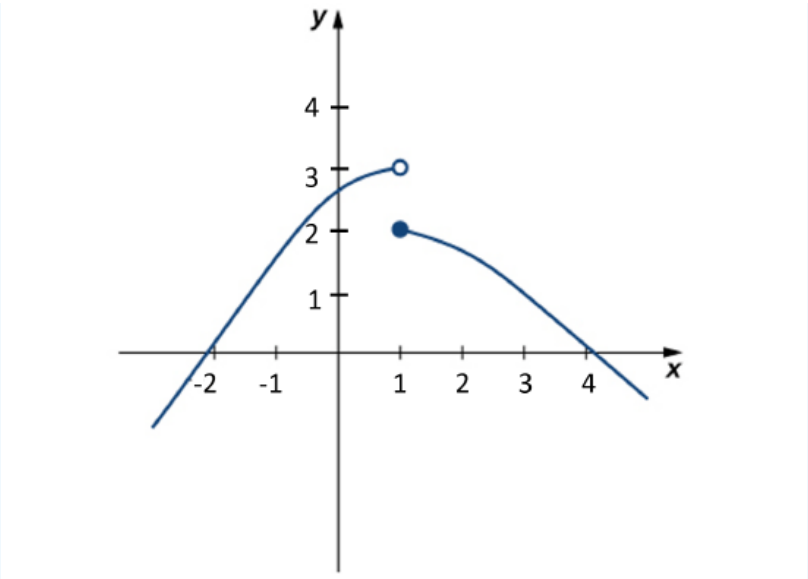

Figure 1.2.10**:** *The graph of a piecewise function* f.

Hint available at the end of this section.

In the previous section, we reached a point with the following example where we thought it was the case that its limit did not exist. However, it was difficult to say why at the time. We now lean on graphing to help us out.

Example 1.2.10

Evaluate

$$\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right)$$

using a graph.

Solution available at the end of this section.

Time-Sensitive Advice: HawkConnect

What is HawkConnect?

In the past, navigating through all the student resources (EOPS, TRIO, MI CASA, etc.) required you to go to individual sites and apply individually for each program. HawkConnect now unifies the entire process into a single, guided application process.

To learn more about HawkConnect and to apply, go to the CRC Support Services website.

Success in Mathematics: Show Your Work

Many mathematics exams include the statement, "Show Your Work" (if it's not explicitly written, then it is implied). The exact meaning of these words can change from one professor to another; however, they usually mean that you need to show more than just the answer to a problem. You need to explain or show how you found the answer. Your responsibility is to learn what "Show Your Work" means to your instructor.

Why do you need to show your work?

Your *authentic* work (e.g., without use of A.I., tutors, or any other external help) demonstrates what you know. It shows what you understand and what you do not understand. By looking at your work and the work of others, an instructor can help you improve your understanding.

Everyone makes mistakes. To learn from these mistakes, you need to be able to find them. If you show your work, you can go back and find mistakes. This helps you identify misunderstandings and bad habits. Without your work, all you can do is start over.

Finally, although you may be able to do a problem "in your head," this will not always be possible. The problems get more challenging and involve more detail as we move forward. Showing your work now develops the habits and procedures you will need to do math in the future.

Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. A student did a homework exercise. She checked her answer with the back of the book. It was wrong. So, she erased her work and started over. Why should she not erase her work?
2. A student did a homework exercise. He checked his answer with the back of the book. It was wrong. He marked the problem on his homework with an asterisk. He went on with the assignment. He never went back to the problem. Why is it essential for this student to finish the problem?

If you are unsure about the answers to either of these questions, feel free to ask your instructor - that's what they are there for!

Answers

Section Preview

This section is essentially a repeat of the previous section but from the graphical perspective. It will be important to let the student know that this method of investigating limits is only slightly better than numerical investigations.

Section Preview

- **Graphs**
  - The Cartesian Coordinate System and the Quadrants: Every limit estimate is read off the coordinate plane (ordered pairs, behavior across the quadrants); without a grasp of the coordinate system, none of the graphical investigations are interpretable.
  - Closed- and Open-Circle Notations: The graphs distinguish excluded points (open circles at $(0,1)$, $(2,4)$, $(1,3)$) from included ones (closed circle at $(1,6)$); understanding inclusion vs. exclusion is precisely what lets a student see that a limit can exist where $f(a)$ is undefined or different.
  - Reading Values from Graphs: The section's whole method is estimating a limit by reading the y-values a graph approaches; the student must already understand that a graph encodes input–output data to be extracted.
- **Rational Functions**
  - Asymptotes: The Vertical Asymptote definition explicitly "formalize[s] a concept from past courses," and the conceptual investigation "relies heavily on ... vertical asymptotes"; the student must arrive already knowing what a vertical asymptote is.
- **Logarithmic Functions**
  - Graphing Logarithmic Functions: Example 7e reasons that as $\cot(x) \to 0^+$, $\ln(\cot(x)) \to -\infty$, which requires already knowing the logarithmic graph's vertical-asymptote behavior near 0^+; the text directs struggling students to review log graphs.
- **Trigonometry**
  - Base Graphs of the Trigonometric Functions: Examples 1, 4, 7e, and 10 hinge on knowing trig-function behavior ($\sin$ oscillating, $\cot \to 0^+$ near $\pi/2$, $\sec$ blowing up near $\pi/2$); the section explicitly cites "the graphs of ... trigonometric functions" as required prior knowledge.

Section Preview

- **Arithmetic**
  - Mathematical Operations — Addition, Subtraction, Multiplication, and Division: The conceptual investigation of infinite limits (Table 1, Example 7) requires determining the sign of a ratio as the denominator approaches zero (e.g., $+/+ \to +\infty$, $+/- \to -\infty$, $-/+ \to -\infty$), which is signed-number division reasoning.
- **Trigonometry**
  - Trigonometric Identities — The Reciprocal and Ratio Identities: Checkpoint 7d rewrites $\sec(x)$ as $1/\cos(x)$ (and Example 7e works with $\cot(x)$) to assess one-sided behavior; the reciprocal and ratio identities are the operational tool for that rewrite.

Section Preview

- Describe the limit of a function and use proper limit notation.
- Use a graph to estimate a function's limit or identify when the limit does not exist.
- Investigate one-sided limits.
- Using correct notation, describe an infinite limit.
- Define a vertical asymptote in terms of limits.
- Determine the value of a limit near a vertical asymptote conceptually.

Example $1.2.1$

A Desmos-generated graph of $f(\theta) = \frac{\sin\theta}{\theta}$ is shown in Figure $1.2.2$, and it supports our estimate. We can see that the function values look to be tending to a value of 1 as $x \to 0$ from both the left and the right.

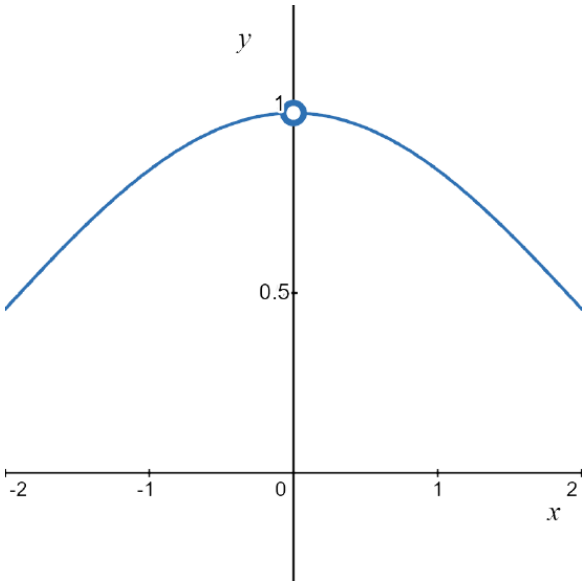

Figure 1.2.2**:** *The graph of* $f(t) = \frac{\sin\theta}{\theta}$ *confirms the estimate from the previous section.*

Example 1.2.2

The graph of $f(x) = \frac{\sqrt{x}-2}{x-4}$, shown in Figure 1.2.3, verifies our result from the previous section.

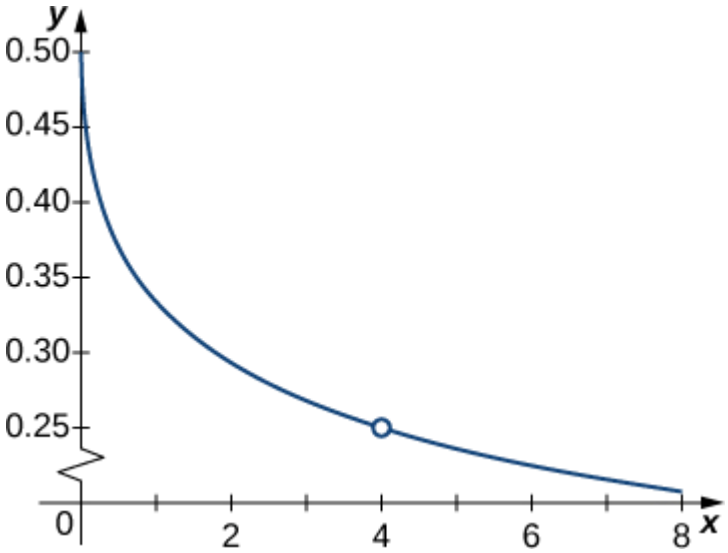

Figure 1.2.3**:** *The graph of* $\frac{\sqrt{x}-2}{x-4}$ *verifies the estimate from the previous section.*

Example 1.2.3

Despite the fact that $g(-1) = 4$, as the x-values approach -1 from either side, the corresponding values of g approach 3. Therefore,

$$\lim_{x\to -1} g(x) = 3.$$

Note that we can determine this limit without even knowing the algebraic expression of the function.

Example 1.2.4

The graph of $f(x) = \sin(1/x)$ is shown in Figure 1.2.5, and it gives a clearer picture of the behavior of $\sin(1/x)$ as x approaches 0. You can see that $\sin(1/x)$ oscillates ever more wildly between -1 and 1 as x approaches 0. Since this means the limit does not become a single value (finite or infinite), we can confidently *assume* that the limit DNE.

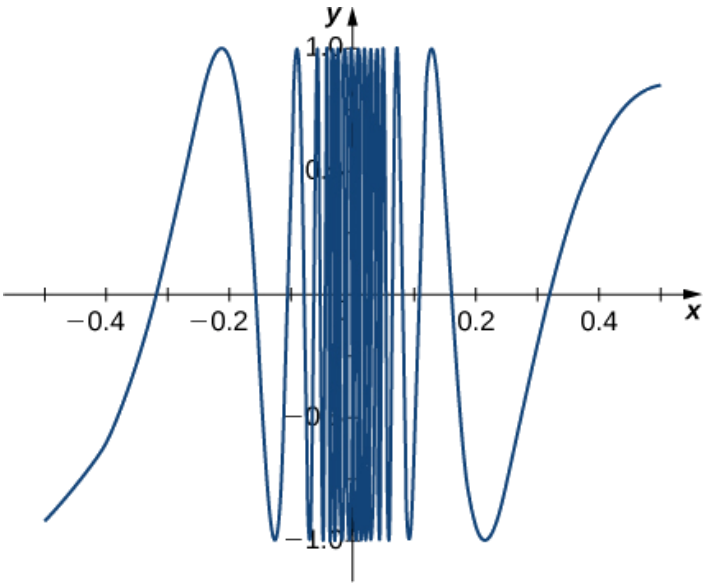

Figure 1.2.5**:** *The graph of* $f(x) = \sin(1/x)$ *oscillates rapidly between* -1 *and* 1 *as* x *approaches* 0.

Checkpoint 1.2.4

$\lim\limits_{x\to 2} \dfrac{|\, x^2 - 4\,|}{x-2}$ does not exist.

Example 1.2.5

Figure 1.2.6 shows a graph of $f(x)$. We can see that as $x \to 2^-$, the function values are climbing to $y = 3$, and as $x \to 2^+$, $f(x) \to 0$. Therefore,

$$\lim_{x\to 2^-} f(x) = 3 \quad \text{and} \quad \lim_{x\to 2^+} f(x) = 0.$$

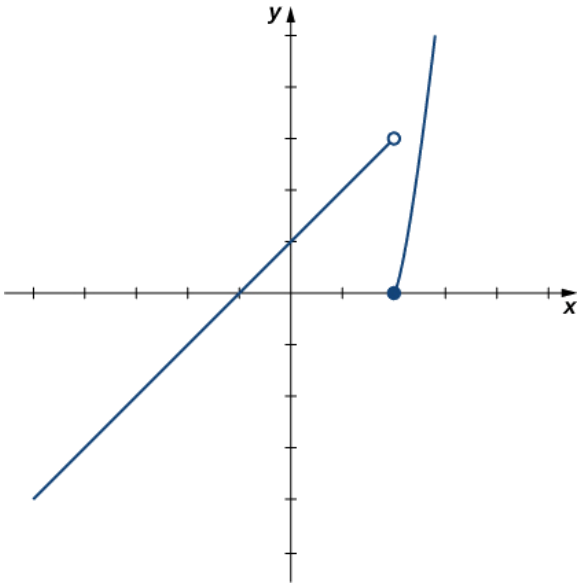

Figure 1.2.6**:** *The graph of* $f(x) = \begin{cases} x+1, & \text{if } x < 2 \\ x^2 - 4, & \text{if } x \geq 2 \end{cases}$ *has a break at* $x = 2$.

Example 1.2.6

The graph of $f(x) = 1/x$ is shown in Figure 1.2.7 below. From this graph, we can see that $\lim\limits_{x\to 0^-} \dfrac{1}{x} = -\infty$, $\lim\limits_{x\to 0^+} \dfrac{1}{x} = \infty$, and, because the left- and right-hand limits do not match, $\lim\limits_{x\to 0} \dfrac{1}{x}$ DNE.

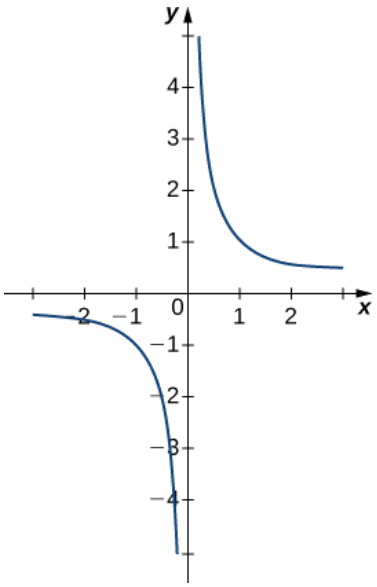

Figure 1.2.7**:** *The graph of $f(x) = 1/x$ confirms that the limit as x approaches 0 does not exist.*

Checkpoint 1.2.6

a. $\lim\limits_{x\to 0^-} \dfrac{1}{x^2} = +\infty$
b. $\lim\limits_{x\to 0^+} \dfrac{1}{x^2} = +\infty$
c. $\lim\limits_{x\to 0} \dfrac{1}{x^2} = +\infty$

From the fact that *any* of these limits (at the finite number $x = 0$) is infinite, we can state that $f(x)$ has a vertical asymptote at $x = 0$.

Example 1.2.7

In each of these solutions, pay close attention to how things are justified. You need to, at minimum, show this level of rigor in your classwork.

a. **Numerator:** As $x \to 7^+$, $x = 2$ (the numerator is *always* 2).
Denominator: As $x \to 7^+$, $(x - 7) \to 0^+$.
Thus,

$$\lim_{x\to 7^+} \frac{2}{x-7} \quad = \quad \infty \qquad \left(\frac{2}{0^+}\right)$$

Note: The numerator is approaching a *positive* constant and the denominator is approaching 0 from the *positive* side. The ratio of positive numbers is positive. This is why the result is *positive* ∞.

b. **Numerator:** As $x \to 7^+$, $x = 2$ (the numerator is *always* 2).
Denominator: As $x \to 7^-$, $(x - 7) \to 0^-$.
Thus,

$$\lim_{x\to 7^-} \frac{2}{x-7} \quad = \quad -\infty \qquad \left(\frac{2}{0^-}\right)$$

Note: The numerator is approaching a *positive* constant and the denominator is approaching 0 from the *negative* side. The ratio of a positive number and a negative number is negative. This is why the result is *negative* ∞. Since $\lim\limits_{x\to 7^+} \dfrac{2}{x-7} \neq \lim\limits_{x\to 7^+} \dfrac{2}{x-7}$, the *overalllimit* $\lim\limits_{x\to 7} \dfrac{2}{x-7}$ DNE (does not exist).

c. **Numerator:** As $x \to 7^-$, $(5 - x) \to -2^+$.
Denominator: As $x \to 7^-$, $(x - 7) \to 0^-$, but when we square this, it approaches 0^+.
Thus,

$$\lim_{x\to 7^-} \frac{5-x}{(x-7)^2} \quad = \quad -\infty \qquad \left(\frac{-2}{0^+}\right)$$

Note: The numerator is approaching a *negative* constant and the denominator is approaching 0 from the *positive* side. The ratio of a negative number and a positive number is negative. This is why the result is *negative* ∞.

d. **Innermost function:** From Trigonometry, we know that as $x \to (\pi/2)^-$, $\cot(x) \to 0^+$.
Outermost function: From Algebra, we know that as $\cot(x) \to 0^+$, $\ln(\cot(x)) \to -\infty$.

Thus,

$$\lim_{x\to(\pi/2)^-} \ln(\cot(x)) = -\infty$$

Checkpoint 1.2.7

a. $\lim_{x\to 2^-} \frac{1}{(x-2)^3} = -\infty$ ($1 \nearrow$, $0^- \searrow$)

b. $\lim_{x\to 2^+} \frac{1}{(x-2)^3} = \infty$ ($1 \nearrow$, $0^+ \searrow$)

c. $\lim_{x\to 2} \frac{1}{(x-2)^3}$ DNE

d. $\lim_{x\to(\pi/2)^+} \sec(x) = \lim_{x\to(\pi/2)^+} \frac{1}{\cos(x)} = \infty$ ($1 \nearrow$, $0^+ \searrow$)

Example 1.2.8

Our starting point is Einstein's equation for the mass of a moving object,

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}},$$

where m_0 is the object's mass at rest, v is its speed, and c is the speed of light. To see how the mass changes at high speeds, we start evaluating

$$\lim_{v\to c^-} m = \lim_{v\to c^-} \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}.$$

Note that we are considering the velocity as it approaches the speed of light *from below*. This is because nothing can move faster than the speed of light. This one-sidedness of the limit is *incredibly* important.

As $v \to c^-$, $v^2 \to (c^2)^-$.

However, as $v^2 \to (c^2)^-$, $\frac{v^2}{c^2} \to \frac{(c^2)^-}{c^2} \to 1^-$.

But as $\frac{v^2}{c^2} \to 1^-$, $1-\frac{v^2}{c^2} \to 1-1^- \to 0^+$.

This is important because, as $1-\frac{v^2}{c^2} \to 0^+$, $\sqrt{1-\frac{v^2}{c^2}} \to 0^+$. Had we allowed $v \to c^+$, we would have had a negative under the square root!

Finally, as $\sqrt{1-\frac{v^2}{c^2}} \to 0^+$, $\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \to \infty$.

Thus, as an object's velocity approaches the speed of light, it becomes infinitely massive. In other words, the function has a vertical asymptote at $v/c = 1$. We can try a few values of this ratio to test this idea.

Table 1.2.2

v/c	$\sqrt{1-\frac{v^2}{c^2}}$	m/m_o
0.99	0.1411	7.089
0.999	0.0447	22.37
0.9999	0.0141	70.7

Interactive Element: Investigating Limits Graphically

a. $\lim_{x\to -2^-} f(x) = 0$
b. $\lim_{x\to -2^+} f(x) = 2$
c. $\lim_{x\to -2} f(x)$ DNE
d. $\lim_{x\to 0^-} f(x) = -1$
e. $\lim_{x\to 0^+} f(x) = -1$
f. $\lim_{x\to 0} f(x) = -1$
g. $\lim_{x\to 1} f(x)$ DNE (the left-hand limit DNE)
h. $\lim_{x\to 2^-} f(x) \approx 0.84$
i. $\lim_{x\to 2^+} f(x) \approx 0.84$
j. $\lim_{x\to 2} f(x) \approx 0.84$

k. $f(2) = 2$ (this showcases that a limit need *not* equal the value of the function at the point)
l. $\lim_{x \to 3^-} f(x) = \infty$
m. $\lim_{x \to 3^+} f(x) = -\infty$
n. $\lim_{x \to 3} f(x)$ DNE

Example 1.2.9

Using the definitions above and the graph for reference, we arrive at the following values:

a. $\lim_{x \to -4^-} f(x) = 0$; $\lim_{x \to -4^+} f(x) = 0$; $\lim_{x \to -4} f(x) = 0$; $f(-4) = 0$
b. $\lim_{x \to -2^-} f(x) = 3$; $\lim_{x \to -2^+} f(x) = 3$; $\lim_{x \to -2} f(x) = 3$; $f(-2)$ is undefined
c. $\lim_{x \to 1^-} f(x) = 6$; $\lim_{x \to 1^+} f(x) = 3$; $\lim_{x \to 1} f(x)$ DNE; $f(1) = 6$
d. $\lim_{x \to 3^-} f(x) = -\infty$; $\lim_{x \to 3^+} f(x) = -\infty$; $\lim_{x \to 3} f(x) = -\infty$; $f(3)$ is undefined

Checkpoint 1.2.9

Compare the limit from the right with the limit from the left.

$\lim_{x \to 1} f(x)$ does not exist

Example 1.2.10

Looking at a graph of this function, as x approaches 0, it seems like the function bounces around infinitely many times between -1 and 1. Since we cannot lock down the value of the function to a single, finite number as x tends to 0, the limit does not exist.

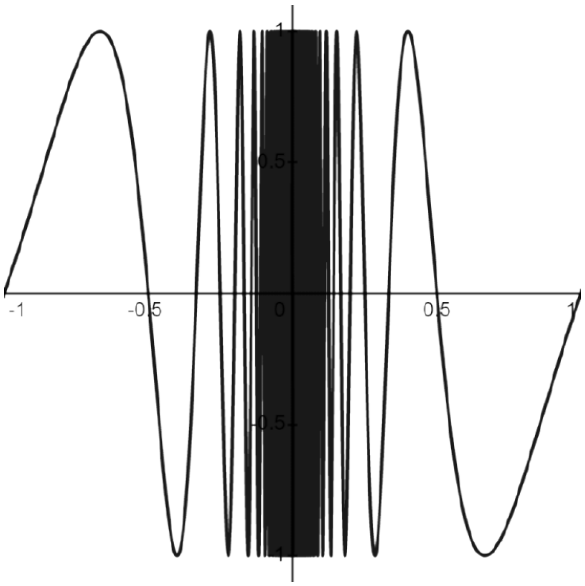

Figure 1.2.11**:** *The graph of $f(x) = \sin(\pi/x)$ shows a that the values of f varying wildly as x gets close to 0.*

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1.2.1: Resources and Key Concepts

Resources

Videos

- Graphical limits
  - Limits - The Basic Concepts of Limits - Finite Limits at Finite Points
- One-sided limits from graph
  - Limits - The Basic Concepts of Limits - Single-Sided Limits
- Conceptual evaluation of limits
  - Limits - The Basic Concepts of Limits - Investigating Limits Conceptually

Key Concepts

Definitions

- **Vertical Asymptote**: The function $f(x)$ is said to have a vertical asymptote at $x = a$ if $\lim_{x \to a^-} f(x) = \pm\infty$, or $\lim_{x \to a^+} f(x) = \pm\infty$, or $\lim_{x \to a} f(x) = \pm\infty$.

Theorems

- **Theorem (Conceptual Investigation of an Infinite Limit)**: Let $f(x) = \frac{N(x)}{D(x)}$, where $\lim_{x \to a} N(x)$ is a finite, nonzero constant, and $D(x)$ is approaching 0 as x approaches a. Then $\lim_{x \to a} \dfrac{N(x)}{D(x)}$ is either ∞ or $-\infty$. The sign is determined by comparing the signs of the numerator and denominator of the ratio as x is approaching a.

Common Mistakes

- **Caution:** $x \neq a$: The intuitive definition of a finite limit at a finite number states that x need not equal a. The limit is concerned with the function values as x gets closer to a, not what $f(a)$ is. This is especially clear graphically where there might be an open circle at $x = a$.
- **Caution: Issues with Graphical Limit Investigations**:
  - If not provided with a graph, creating an accurate one can be difficult or impossible.
  - Graphs can be visually "tricky," potentially hiding details like an open circle (where the function is undefined) or leading to incorrect assumptions about function behavior if not interpreted carefully. (e.g., the open circle at $(0, 1)$ for $f(\theta) = \sin\theta/\theta$).
- **Caution: Calculus Does Not Work in Degrees**: When working with trigonometric functions in Calculus, angles must be expressed in radians, not degrees. This affects the appearance of their graphs and thus limit investigations.
- **Caution: Limits are Unconcerned with** $f(a)$: The value of $\lim_{x \to a} f(x)$ depends on the function's behavior near a, not *at* a. The function might be undefined at a, or $f(a)$ might be different from the limit.
- **Caution: Graphical Observation is Not Proof**: Observing that a graph appears to approach a certain value (e.g., $\lim_{\theta \to 0} \dfrac{\sin\theta}{\theta}$ appears to be 1 from its graph) is an estimation, not a formal proof.
- **Caution: Never Write That a Limit is Equal to an Indeterminate Form**: When conceptually evaluating limits that tend to forms like $\frac{k}{0}$ (where $k \neq 0$), do not write that the limit *equals* $\frac{k}{0^+}$ or $\frac{k}{0^-}$. These expressions ($\frac{k}{0^+}$, etc.) describe tendencies, not actual numbers. Use arrow notation to show the behavior of the numerator and denominator separately, then conclude the limit (e.g., $\lim_{x \to 7^+} \dfrac{2}{x-7}$. Numerator $\to 2$, Denominator $(x-7) \to 0^+$. Thus, $\lim_{x \to 7^+} \dfrac{2}{x-7} = \infty$).

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1.2.2: Homework

Note

The following exercises are meant as hand-in homework (if assigned by your instructor). Typed-in answers here will not be saved or scored.

Basic Skills

Exercise 1

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Exercise 2

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Exercise 3

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Exercise 11

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Exercise 13

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Exercise 14

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Exercise 24

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Exercise 25

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Exercise 26

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Exercise 27

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Basic Skills - Applications

Exercise 28

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Synthesis Questions - Applications

Exercise 29

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1.2.3: Additional Exercises

Reading Questions

1. How does using a graph help in estimating $\lim_{x\to a} f(x)$?
2. What are two potential pitfalls when trying to determine a limit solely from a graph?
3. If a graph of $f(x)$ has an open circle at the point (a, L) but the curve smoothly approaches this point from both the left and the right, what is $\lim_{x\to a} f(x)$? What can be said about $f(a)$?
4. Explain how to find $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ by looking at a graph.
5. If $\lim_{x\to a^-} f(x) = L_1$ and $\lim_{x\to a^+} f(x) = L_2$, where $L_1 \neq L_2$, what can you conclude about $\lim_{x\to a} f(x)$ based on its graph?
6. What does the graph of a function typically show near a value $x = a$ if $\lim_{x\to a} f(x) = \infty$?
7. According to the definition provided in this section, what conditions involving limits must be met for a function $f(x)$ to have a vertical asymptote at $x = a$?
8. When evaluating a limit like $\lim_{x\to a} \frac{N(x)}{D(x)}$ where $N(x) \to k \neq 0$ and $D(x) \to 0$, what information about $N(x)$ and $D(x)$ as $x \to a$ is needed to determine if the limit is ∞ or $-\infty$?
9. Why is it incorrect to write $\lim_{x\to c} f(x) = \frac{k}{0}$? What notation should be used to show the conceptual steps?
10. In Example 7, part d, $\lim_{x\to(\pi/2)^-} \ln(\cot x)$, describe the step-by-step thought process for evaluating this limit conceptually by considering the behavior of the inner function $\cot x$ and then the outer function $\ln(u)$.
11. How does Figure 6 (the graph of $f(x) = \sin(1/x)$) visually demonstrate that $\lim_{x\to 0} \sin(1/x)$ does not exist?
12. What does the graph in Figure 10 illustrate about the relationship between $f(c)$ and $\lim_{x\to c} f(x)$ at various points c?

Homework

In exercises 1 - 2, set up a table of values and round to eight significant digits. Based on the table of values, make a guess about what the limit is. Then, use a calculator to graph the function and determine the limit. Was the conjecture correct? If not, why does the method of tables fail?

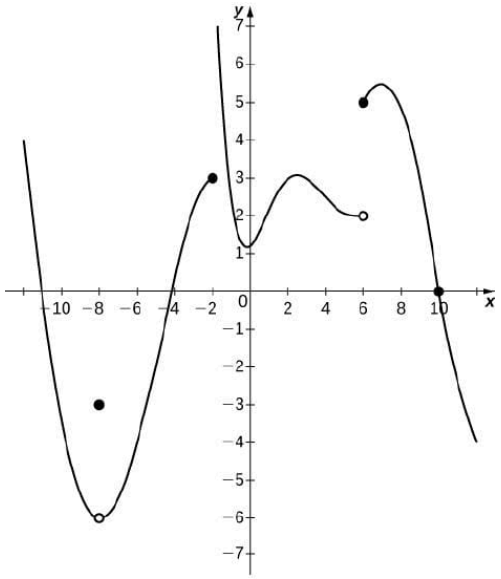
1) $\lim_{\theta\to 0} \sin\left(\frac{\pi}{\theta}\right)$

θ	$\sin\left(\frac{\pi}{\theta}\right)$	θ	$\sin\left(\frac{\pi}{\theta}\right)$
-0.1	a.	0.1	e.
-0.01	b.	0.01	f.
-0.001	c.	0.001	g.
-0.0001	d.	0.0001	h.

2) $\lim_{\alpha\to 0^+} \frac{1}{\alpha}\cos\left(\frac{\pi}{\alpha}\right)$

α	$\frac{1}{\alpha}\cos\left(\frac{\pi}{\alpha}\right)$
0.1	a.
0.01	b.
0.001	c.
0.0001	d.

In exercises 3 - 6, consider the graph of the function $y = f(x)$ shown here. Which of the statements about $y = f(x)$ are true and which are false? Explain why a statement is false.



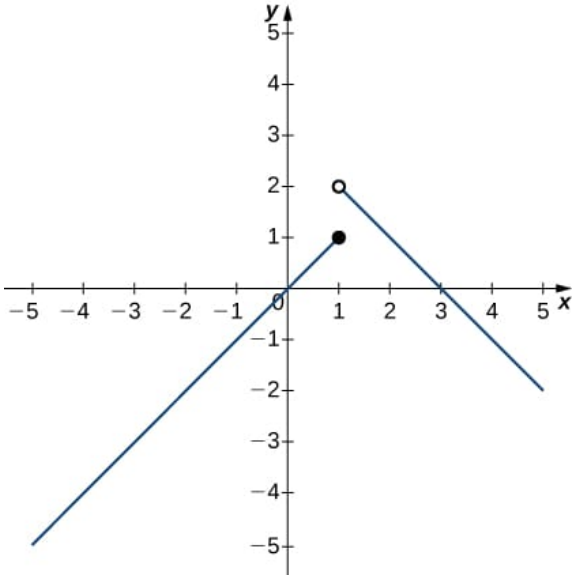
3) $\lim_{x \to 10} f(x) = 0$

4) $\lim_{x \to -2^+} f(x) = 3$

5) $\lim_{x \to -8} f(x) = f(-8)$

6) $\lim_{x \to 6} f(x) = 5$

In exercises 7 - 11, use the following graph of the function $y = f(x)$ to find the values, if possible. Estimate when necessary.



7) $\lim_{x \to 1^-} f(x)$

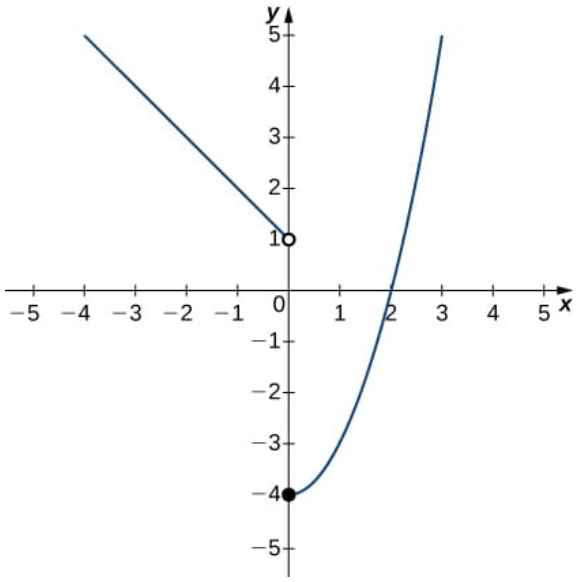
8) $\lim_{x \to 1^+} f(x)$

9) $\lim_{x \to 1} f(x)$

10) $\lim_{x \to 2} f(x)$

11) $f(1)$

In exercises 12 - 15, use the graph of the function $y = f(x)$ shown here to find the values, if possible. Estimate when necessary.



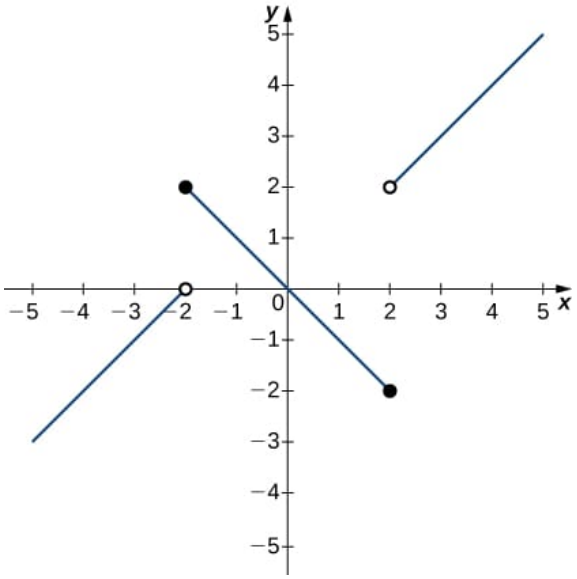
12) $\lim_{x \to 0^-} f(x)$

13) $\lim_{x \to 0^+} f(x)$

14) $\lim_{x \to 0} f(x)$

15) $\lim_{x \to 2} f(x)$

In exercises 16 - 21, use the graph of the function $y = f(x)$ shown here to find the values, if possible. Estimate when necessary.



16) $\lim_{x \to -2^-} f(x)$

17) $\lim_{x \to -2^+} f(x)$

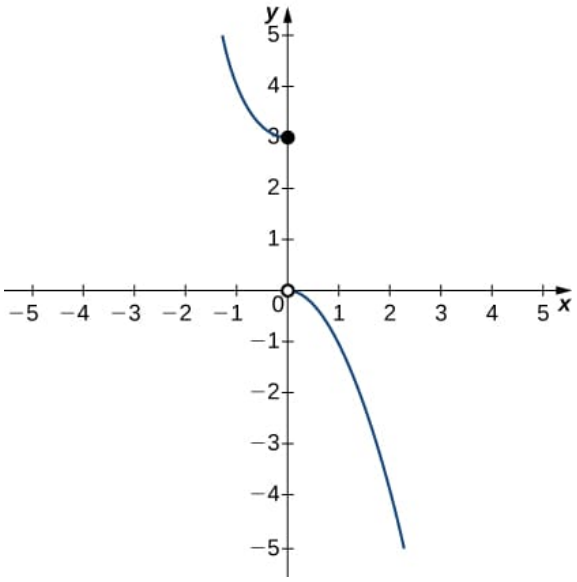
18) $\lim_{x \to -2} f(x)$

19) $\lim_{x \to 2^-} f(x)$

20) $\lim_{x \to 2^+} f(x)$

21) $\lim_{x \to 2} f(x)$

In exercises 22 - 24, use the graph of the function $y = g(x)$ shown here to find the values, if possible. Estimate when necessary.

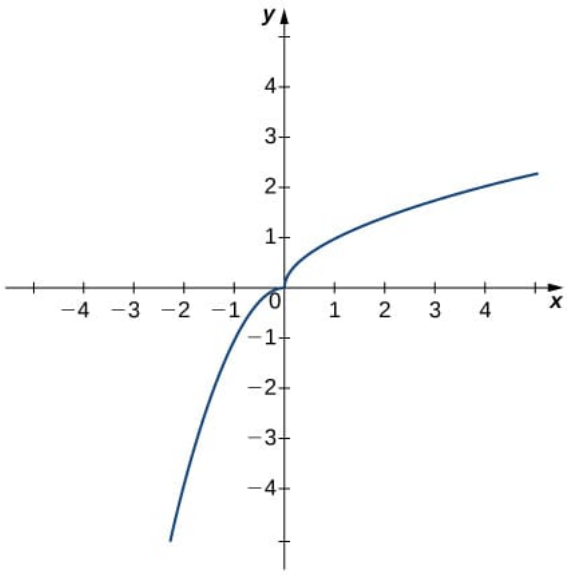


22) $\lim_{x \to 0^-} g(x)$

23) $\lim_{x \to 0^+} g(x)$

24) $\lim_{x \to 0} g(x)$

In exercises 25 - 27, use the graph of the function $y = h(x)$ shown here to find the values, if possible. Estimate when necessary.

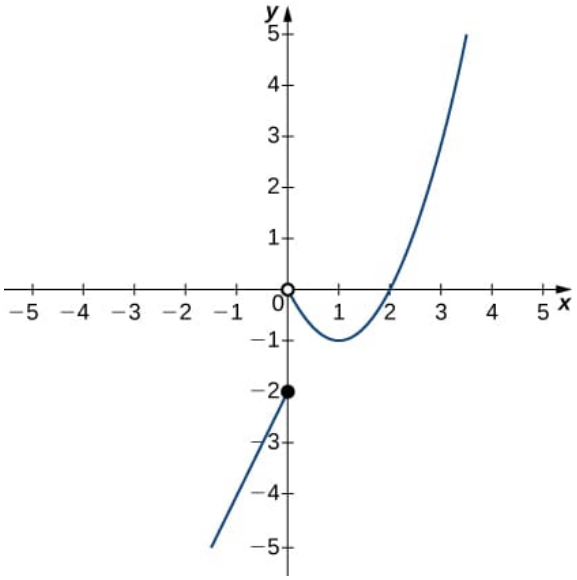


25) $\lim_{x \to 0^-} h(x)$

26) $\lim_{x \to 0^+} h(x)$

27) $\lim_{x \to 0} h(x)$

In exercises 28 - 32, use the graph of the function $y = f(x)$ shown here to find the values, if possible. Estimate when necessary.



28) $\lim_{x \to 0^-} f(x)$

29) $\lim_{x\to 0^+} f(x)$

30) $\lim_{x\to 0} f(x)$

31) $\lim_{x\to 1} f(x)$

32) $\lim_{x\to 2} f(x)$

In exercises 33 - 37, sketch the graph of a function with the given properties.

33)

$$\lim_{x\to 2} f(x) = 1,$$
$$\lim_{x\to 4^-} f(x) = 3,$$
$$\lim_{x\to 4^+} f(x) = 6,$$
$$\text{and } x = 4 \text{ is not defined.}$$

34)

$$\lim_{x\to -\infty} f(x) = 0,$$
$$\lim_{x\to -1^-} f(x) = -\infty,$$
$$\lim_{x\to -1^+} f(x) = \infty,$$
$$\lim_{x\to 0} f(x) = f(0),$$
$$f(0) = 1,$$
$$\text{and } \lim_{x\to \infty} f(x) = -\infty.$$

35)

$$\lim_{x\to -\infty} f(x) = 2,$$
$$\lim_{x\to 3^-} f(x) = -\infty,$$
$$\lim_{x\to 3^+} f(x) = \infty,$$
$$\lim_{x\to \infty} f(x) = 2,$$
$$\text{and } f(0) = -\frac{1}{3}.$$

36)

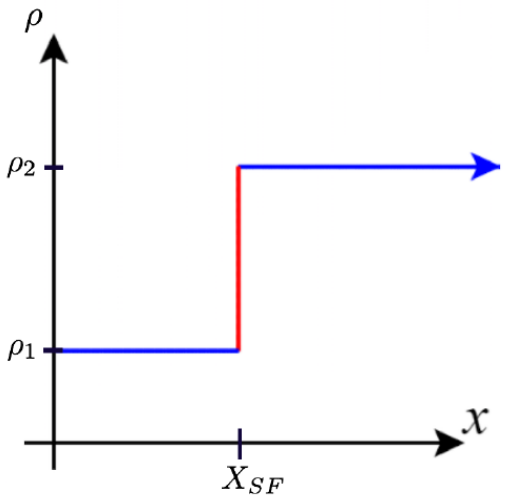
$$\lim_{x\to -\infty} f(x) = 2,$$
$$\lim_{x\to -2} f(x) = -\infty,$$
$$\lim_{x\to \infty} f(x) = 2,$$
$$\text{and } f(0) = 0.$$

37)

$$\lim_{x\to -\infty} f(x) = 0,$$
$$\lim_{x\to -1^-} f(x) = \infty,$$
$$\lim_{x\to -1^+} f(x) = -\infty,$$
$$f(0) = -1,$$
$$\lim_{x\to 1^-} f(x) = -\infty,$$

$$\lim_{x \to 1^+} f(x) = \infty,$$
$$\text{and } \lim_{x \to \infty} f(x) = 0.$$

38) Shock waves arise in many physical applications, ranging from supernovas to detonation waves. A graph of the density of a shock wave with respect to distance, x, is shown here. We are mainly interested in the location of the front of the shock, labeled X_{SF} in the diagram.



a. Evaluate $\lim_{x \to X_{SF}^+} \rho(x)$.

b. Evaluate $\lim_{x \to X_{SF}^-} \rho(x)$.

c. Evaluate $\lim_{x \to X_{SF}} \rho(x)$. Explain the physical meanings behind your answers.

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1.3: Limit Laws and the Squeeze Theorem

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

In the previous section, we evaluated limits by looking at graphs or by constructing a table of values. In this section, we establish laws for calculating limits and learn how to apply these laws. In the Student Project at the end of this section, you have the opportunity to use the Limit Laws to derive the formula for the area of a circle by adapting a method devised by the Greek mathematician Archimedes.

We begin by restating two useful limit results from the previous section. These two results, which we will call the Basic Limit Laws, together with the Limit Laws we will be introducing momentarily, serve as a foundation for calculating many limits.

Corollary: Basic Limit Laws

For any real number a and any constant c,

I. $\lim\limits_{x \to a} x = a$
II. $\lim\limits_{x \to a} c = c$

Example 1.3.1: Using the Basic Limit Laws

Evaluate each of the following limits.

a. $\lim\limits_{x \to 2} x$
b. $\lim\limits_{x \to 2} 5$

Solutions available at the end of this section.

Evaluating Finite Limits with the Limit Laws

We now take a look at the **Limit Laws**, the individual properties of limits. The proofs that these laws hold are omitted at this time; however, we will prove some of these once we have formalized the definition of a limit.

Theorem: Limit Laws

Let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing a. Assume that L and M are real numbers such that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Let c be a constant. Then, each of the following statements holds:

- **Sum Law for Limits**:
$$\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$$
- **Difference Law for Limits**:
$$\lim_{x \to a}(f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M$$
- **Constant Multiple Law for Limits**:
$$\lim_{x \to a} cf(x) = c \cdot \lim_{x \to a} f(x) = cL$$
- **Product Law for Limits**:
$$\lim_{x \to a}(f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L \cdot M$$
- **Quotient Law for Limits**:
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}$$
for $M \neq 0$.
- **Power Law for Limits**:
$$\lim_{x \to a} \big(f(x)\big)^n = \big(\lim_{x \to a} f(x)\big)^n = L^n$$
for every positive integer n.
- **Root Law for Limits**:
$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L}$$
for all L if n is odd and for $L \geq 0$ if n is even.

Before we go any further, a cautionary statement is required.

Caution: The Limit Laws Require Limits to Exist

A common mistake is to think that the Limit Laws apply to every limit encountered by the student. To be able to use the Limit Laws, both of the component limits

$$\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x)$$

must exist and be **finite**.

Example 1.3.2: Evaluating a Limit Using the Limit Laws

Use the Limit Laws to evaluate

$$\lim_{x \to -3}(4x + 2).$$

Solution available at the end of this section.

Example 1.3.3: Evaluating a Limit Using the Limit Laws

Use the Limit Laws to evaluate

$$\lim_{x\to 2}\frac{2x^2-3x+1}{x^3+4}.$$

Solution available at the end of this section.

Checkpoint 1.3.3

Use the Limit Laws to evaluate $\lim_{x\to 6}\left[(2x-1)\sqrt{x+4}\right]$. In each step, indicate the limit law applied.

Answer available at the end of this section.

Limits of Polynomial and Rational Functions

By now you have probably noticed that, in each of the previous examples, it has been the case that $\lim_{x\to a} f(x) = f(a)$. This is not always true; however, we will focus on such functions for the rest of this section. Specifically, it holds for all polynomials for any choice of a and for all rational functions at all values of a in the domain of the rational function.

Theorem: Direct Substitution Property (for polynomial and rational functions)

Let $p(x)$ and $q(x)$ be polynomial functions. Let a be a real number. Then,

$$\lim_{x\to a} p(x) = p(a)$$

and

$$\lim_{x\to a}\frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$$

when $q(a)\neq 0$.

Proof available at the end of this section.

Example 1.3.4: Using the Direct Substitution Property

Evaluate the $\lim_{x\to 3}\dfrac{2x^2-3x+1}{5x+4}$.

Solution available at the end of this section.

Checkpoint 1.3.4

Evaluate $\lim_{x\to -2}(3x^3-2x+7)$.

Answer available at the end of this section.

Critical Mistake: Misapplying the Direct Substitution Property

Question: Evaluate $\lim_{x\to a}\dfrac{p(x)}{q(x)}$, where $\lim_{x\to a} q(x)=0$.

Student Answer: $\frac{p(a)}{0}=\text{DNE}$

Mistake: The Direct Substitution Property (for rational functions) can only be applied if the limit in the denominator is **nonzero**. Your answer will only be "DNE" if you have evaluated both left- and right-hand limits and found them to not be equal.

Revisiting One-Sided Limits

Simple modifications to the Limit Laws allow us to apply them to one-sided limits. For example, to apply the Limit Laws to a limit of the form $\lim_{x\to a^-} h(x)$, we require the function $h(x)$ to be defined over an open interval of the form (b,a); for a limit of the form $\lim_{x\to a^+} h(x)$, we require the function $h(x)$ to be defined over an open interval of the form (a,c). Example 1.3.5 illustrates this point.

Example 1.3.5: Evaluating a One-Sided Limit Using the Limit Laws

Evaluate each of the following limits, if possible.

a. $\lim_{x\to 3^-}\sqrt{x-3}$
b. $\lim_{x\to 3^+}\sqrt{x-3}$

Solutions available at the end of this section.

In Example 1.3.6, we look at the one-sided limits of a piecewise-defined function and use these limits to draw a conclusion about a two-sided limit of the same function.

Example 1.3.6: Evaluating a Two-Sided Limit Using the Limit Laws

For

$$f(x)=\begin{cases}4x-3, & \text{if } x<2\\ (x-3)^2, & \text{if } x\geq 2\end{cases},$$

evaluate each of the following limits:

a. $\lim_{x\to 2^-} f(x)$
b. $\lim_{x\to 2^+} f(x)$
c. $\lim_{x\to 2} f(x)$

Solutions available at the end of this section.

Checkpoint 1.3.6

Graph

$$f(x)=\begin{cases}-x-2, & \text{if } x<-1\\ 2, & \text{if } x=-1\\ x^3, & \text{if } x>-1\end{cases}$$

and evaluate $\lim_{x\to -1^-} f(x)$.

Answer available at the end of this section.

Breaking the (Limit) Laws

As stated earlier, the Limit Laws can only be applied when the component limits are finite. A very common mistake made early in Calculus is for a student to think that they can apply the Limit Laws to any limit—again, this is not true.

Example 1.3.7: Breaking the (Limit) Laws

Evaluate each of the following limits.

a. $\lim\limits_{x\to 3^+}\left((x-3)\cdot\dfrac{1}{x-3}\right)$

b. $\lim\limits_{x\to 3^-}\left((x-3)^2\cdot\dfrac{1}{x-3}\right)$

c. $\lim\limits_{x\to 3^+}\left((x-3)\cdot\dfrac{1}{(x-3)^3}\right)$

Solutions available at the end of this section.

Pay close attention to Example 1.3.7. It should serve as a warning that anytime you reach an "answer" of the form $0\cdot\infty$, you are wrong.

The Squeeze Theorem

The techniques we have developed thus far work well for algebraic functions. However, we still need to evaluate the limits of elementary trigonometric functions. The next theorem, called the **Squeeze Theorem**, proves very useful for establishing basic trigonometric limits. It also helps us evaluate very abstract and theoretical limits for functions that cannot be described using simple functions from our Algebra courses. This theorem allows us to calculate limits by "squeezing" a function, with a limit at a point a that is unknown, between two functions having a common known limit at a. Figure 1.3.4 illustrates this idea.

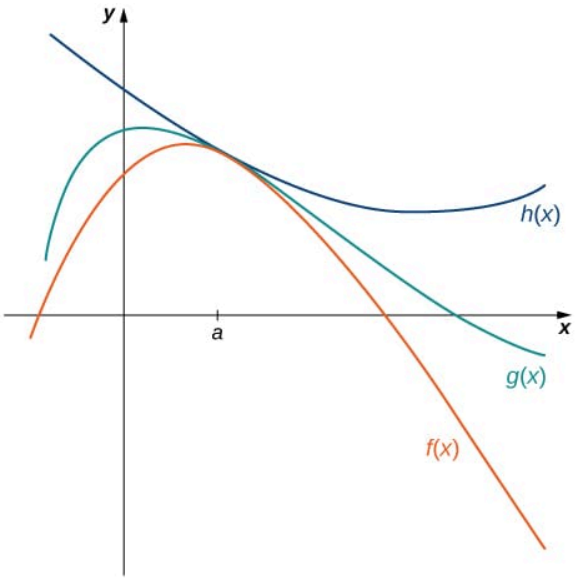

Figure 1.3.4: *The Squeeze Theorem applies when* $f(x)\le g(x)\le h(x)$ *and* $\lim\limits_{x\to a}f(x)=\lim\limits_{x\to a}h(x)$.

Theorem: Squeeze Theorem

Let $f(x)$,$g(x)$, and $h(x)$ be defined for all $x \neq a$ over an open interval containing a. If

$$f(x) \leq g(x) \leq h(x)$$

for all $x \neq a$ in an open interval containing a and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$$

where L is a real number, then $\lim_{x \to a} g(x) = L$.

Example 1.3.8: Applying the Squeeze Theorem

Evaluate $\lim_{x \to 0} (x \cos(x))$.

Solution available at the end of this section.

Interactive Element: The Squeeze in Action

The following Interactive Element might help you visualize how a function trapped between two others is forced to their common limit.

Interact: Drag the red point along the purple curve $g(x) = x \cos x$ toward the origin.

Observation: The bounds $y = x$ and $y = -x$ pinch shut at the origin, so $g(x)$ is squeezed to 0: $\lim_{x \to 0} x \cos x = 0$.

Checkpoint 1.3.8

Use the Squeeze Theorem to evaluate $\lim_{x \to 0} x^2 \sin \frac{1}{x}$.

Hint available at the end of this section.

Answer available at the end of this section.

Limits Involving the Sine and Cosine

We now use the Squeeze Theorem to tackle several fundamental limits presented as a theorem here. Although this discussion is lengthy, these limits prove invaluable for the development of the material involving trigonometric functions for the rest of Calculus.

Theorem

$$\lim_{\theta \to 0} (\sin(\theta)) = 0$$

and

$$\lim_{\theta \to 0} (\cos(\theta)) = 1$$

Proof available at the end of this section.

Caution: Calculus Requires Radian Measure

The theorem just introduced is the reason we stated way back in the first section that radians must be used when working in Calculus. The proof of both limits require radian measure. As such, all of the Calculus we build surrounding the trigonometric functions relies on us using radian measure. That is, Calculus and degree measure do not get along!

We now take a look at two limits that play important roles in later chapters.

Theorem

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

and

$$\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0$$

Proof available at the end of this section.

Example 1.3.9: Evaluating an Important Trigonometric Limit

Evaluate $\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta}$.

Solution available at the end of this section.

Example 1.3.10: Evaluating Finite Limits of Trigonometric Functions at Finite Points

Evaluate the following limit.

$$\lim_{\theta \to 0} \frac{7\theta}{\sin(5\theta)}$$

Solution available at the end of this section.

Checkpoint 1.3.10

Evaluate $\lim_{\theta \to 0} \frac{1 - \cos \theta}{\sin \theta}$.

Answer available at the end of this section.

Student Project

Deriving the Formula for the Area of a Circle

Some of the geometric formulas we take for granted today were first derived by methods that anticipate some of the methods of calculus. The Greek mathematician *Archimedes* (ca. 287-212; BCE) was particularly inventive, using polygons inscribed within circles to approximate the area of the circle as the number of sides of the polygon increased. He never came up with the idea of a limit. Still, we can use this idea to see what his geometric constructions could have predicted about the limit.

We can estimate the area of a circle by computing the area of an inscribed regular polygon. Think of the regular polygon as being made up of n triangles. By taking the limit as the vertex angle of these triangles goes to zero, you can obtain the area of the circle. To see this, carry out the following steps:

1. Express the height h and the base b of the isosceles triangle in Figure $1.3.8$ in terms of θ and r.

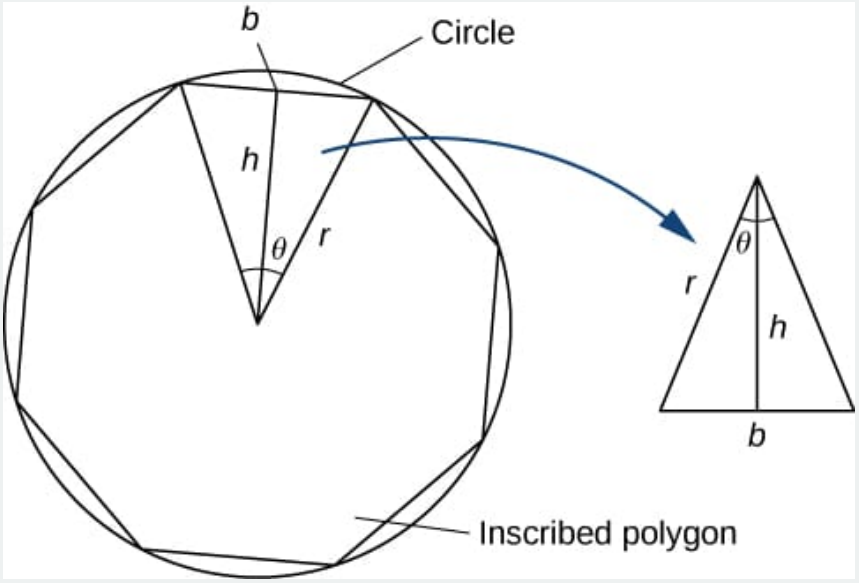

Figure $1.3.8$

2. Using the expressions that you obtained in step 1, express the area of the isosceles triangle in terms of θ and r. (Substitute $\frac{1}{2}\sin\theta$ for $\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$ in your expression.)
3. If an n-sided regular polygon is inscribed in a circle of radius r, find a relationship between θ and n. Solve this for n. Keep in mind there are 2π radians in a circle. (Use radians, not degrees.)
4. Find an expression for the area of the n-sided polygon in terms of r and θ.
5. To find a formula for the circle's area, find the expression's limit in step 4 as θ goes to zero. (Hint: $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1$).

Interactive Element: Archimedes' Polygon → Circle

The following Interactive Element might help you visualize how an inscribed polygon's area approaches the area of the circle as the number of sides grows.

Interact: Drag the red handle along the track to increase the number of sides n.

Observation: As n increases, the inscribed polygon's area approaches π, the area of the unit circle — exactly the limit $\lim\limits_{\theta\to 0}$ computed in the project.

The technique of estimating areas of regions by using polygons is revisited later in this course.

Time-Sensitive Advice: "Full Refund" and "No Notation" Drop Deadlines

"Full Refund" Drop Deadline

Within the Los Rios Community College District, the last day you can drop a class for a full refund is the Friday of the second week of class. Any drops processed after that point will only receive a *partial* refund.

"No Notation" Drop Deadline

The Sunday following the "full refund" drop deadline is the last day to drop without a "W" (withdrawal) notation on your transcript. You can only take a course, and receive a notation, a maximum of three times within a given community college district in California - this includes notations of D, F, *and* W.

Once you withdraw (W) or fail (D or F) a course three times within the Los Rios Community College District (ARC, CRC, FLC, or SCC), you are *never* allowed to take that course again at any of the colleges within the District. If you find yourself in that unfortunate position, you would need to take this course outside the Los Rios Community College District.

Take a moment of self-reflection to determine if you feel ready to continue this journey for the rest of this long semester. If not, consider dropping by this course by the "full refund" deadline or the "no notation" deadline, at the latest.

Success in College: What is Tutoring?

Tutoring is an age-old practice. The dictionary definition describes a tutor as a person who gives individual or, in some cases, small group instruction. The purpose of tutoring is to help students help themselves or assist or guide them to the point at which they become independent learners and thus no longer need a tutor.

Content knowledge is an essential ingredient for a tutor; however, to be truly effective, a tutor must combine content knowledge with empathy, honesty, and humor. Empathy requires tutors to "read" their students' emotional states, attitudes, and perceptions. Empathy is the ability to see others from their frame of reference and communicate this understanding to the person involved. To establish a supportive relationship with their students, tutors must be open and honest. Students are often reluctant to talk with a stranger about their academic problems. If a tutor is perceived as genuine and strongly desires to listen, students will be more willing to open up and discuss their problems. Humor can also play an essential part in a tutoring session. Humor can reduce tension. Shared laughter is a powerful way to reinforce learning, and humor can set students at ease and increase rapport. Tutors can also use humor to compliment, guide, or positively provide feedback.

In addition, a successful tutor demonstrates a caring attitude. Caring consists of being punctual and organized for the tutoring session, establishing a learning relationship with the student, developing unique teaching strategies, and becoming familiar with the learning process. Ultimately, tutoring is sharing yourself with another student in a way that makes a difference in both your lives.

There are many benefits to tutoring.

- Heightens a sense of competency/adequacy in conforming to a new role.
- Encourages higher levels of thinking.
- Permits more advanced students to study below-level material without embarrassment.
- Increases motivation to learn to maintain the new role.
- Increases ability to manage own learning and study strategies.
- Increases subject-specific knowledge.
- Increases related general knowledge.
- Increases understanding of the subject area.
- Improves attitude toward the subject area.
- Provides more empathy with students.

Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. Does your college have a place such as a tutoring center where you can get help?
2. If so, what are the hours that this tutoring center is open?
3. If you are unsure about the answers to either of these questions, feel free to ask your professor - that's what they are there for!

Answers

Section Preview

The Limit Laws are introduced, but algebraic manipulations are avoided (for now). Focus on making the students comfortable with the meticulous, step-by-step use of evaluating limits using the Limit Laws. Care should be taken to repeatedly inform the students that these laws only apply when the component limits exist and are finite. The Squeeze Theorem is introduced as an alternative to evaluating a limit when all other methods fail. Finally, take the time in class to derive the limit theorems involving the sine. These proofs should convey the tone of the course—that students should be attentive and prerequisite skills are of utmost importance.

Section Preview

- **Polynomial Functions**
  - The Basics of Polynomial Functions: The Direct Substitution Property and its proof are stated for polynomial and rational functions; recognizing polynomial structure ($c_n x^n + \cdots + c_0$) is needed to follow why the property holds.
- **Trigonometry**
  - Trigonometric Functions — Unit Circle Definition: The proofs of the fundamental trigonometric limits read $\sin\theta$ and $\cos\theta$ as the y- and x-coordinates of a point on the unit circle.

Section Preview

- **Arithmetic**
  - Multiplying and Dividing Fractions: Evaluating rational-function limits to values like $\frac{1}{4}$ (Example 3) and $\frac{10}{19}$ (Example 4), and managing the compound fraction in Example 10, require fraction arithmetic.
- **Simplifying Expressions**
  - Simplifying Ratios of Polynomials: The "Breaking the (Limit) Laws" examples require canceling common factors (e.g., $(x-3)^2 \cdot \frac{1}{x-3} = x-3$) before a limit can be evaluated.
  - Rationalizing Denominators and Numerators; Simplifying Quotients of Radical Expressions: Example 9 multiplies by the conjugate $\frac{1+\cos\theta}{1+\cos\theta}$ to transform $\frac{1-\cos\theta}{\theta}$; the conjugate maneuver is the operational heart of that computation.
- **Solving Inequalities**
  - Solving Linear Inequalities: The Squeeze proofs divide $\sin\theta < \theta < \tan\theta$ by $\sin\theta$, and Example 8 multiplies $-1 \le \cos x \le 1$ by x, flipping the signs when $x < 0$.
- **Trigonometry**
  - Trigonometric Identities — The Pythagorean Identities: The cosine proof uses $\cos\theta = \sqrt{1-\sin^2\theta}$, and Example 9 replaces $1-\cos^2\theta$ with $\sin^2\theta$.

Section Preview

- Describe the Limit Laws and understand when they can be used (and, importantly, when they cannot).
- Use the Limit Laws to evaluate the limit of a function.
- Use the Direct Substitution Property to evaluate the limit of a polynomial or rational function.
- Evaluate the limit of a function by using the Squeeze Theorem, when appropriate.
- Evaluate limits involving ratios of sine or cosine and angles.

Example 1.3.1: Using the Basic Limit Laws

a. The limit of x as x approaches a is a: $\lim\limits_{x\to 2} x = 2$.

b. The limit of a constant is that constant: $\lim\limits_{x\to 2} 5 = 5$.

Example 1.3.2: Evaluating a Limit Using the Limit Laws

Let's apply the Limit Laws one step at a time to be sure we understand how they work. We need to remember that, at each application of a limit law, new limits must exist for the limit law to be applied.

$$\begin{aligned}
\lim_{x\to -3}(4x+2) &= \lim_{x\to -3} 4x + \lim_{x\to -3} 2 && \text{(Sum Law)}\\
&= 4\cdot\lim_{x\to -3} x + \lim_{x\to -3} 2 && \text{(Constant Multiple Law)}\\
&= 4\cdot(-3)+2 && \text{(Basic Limit Laws)}\\
&= -10
\end{aligned}$$

Example 1.3.3: Evaluating a Limit Using the Limit Laws

To find this limit, we need to apply the Limit Laws several times. Again, we must remember that as we rewrite the limit in terms of other limits, each new limit must exist for the limit law to be applied.

$$\begin{aligned}
\lim_{x\to 2}\frac{2x^2-3x+1}{x^3+4} &= \frac{\lim\limits_{x\to 2}(2x^2-3x+1)}{\lim\limits_{x\to 2}(x^3+4)} && \text{(Quotient Law)}\\
&= \frac{2\cdot\lim\limits_{x\to 2}x^2 - 3\cdot\lim\limits_{x\to 2}x + \lim\limits_{x\to 2}1}{\lim\limits_{x\to 2}x^3 + \lim\limits_{x\to 2}4} && \text{(Sum and Constant Multiple Laws)}\\
&= \frac{2\cdot\left(\lim\limits_{x\to 2}x\right)^2 - 3\cdot\lim\limits_{x\to 2}x + \lim\limits_{x\to 2}1}{\left(\lim\limits_{x\to 2}x\right)^3 + \lim\limits_{x\to 2}4} && \text{(Power Law)}\\
&= \frac{2(4)-3(2)+1}{(2)^3+4} && \text{(Basic Limit Laws)}\\
&= \frac{1}{4}
\end{aligned}$$

Checkpoint 1.3.3

$11\sqrt{10}$

Theorem: Direct Substitution Property (for polynomial and rational functions)

Let

$$p(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0.$$

Then by applying the Sum, Constant Multiple, and Power Laws, we end up with

$$\begin{aligned}
\lim_{x\to a} p(x) &= \lim_{x\to a}(c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0) \\
&= \lim_{x\to a}(c_n x^n) + \lim_{x\to a}\left(c_{n-1}x^{n-1}\right) + \cdots + \lim_{x\to a}(c_1 x) + \lim_{x\to a} c_0 && \text{(Sum Law)} \\
&= c_n \lim_{x\to a} x^n + c_{n-1}\lim_{x\to a} x^{n-1} + \cdots + c_1 \lim_{x\to a} x + c_0 && \text{(Constant Multiple Law)} \\
&= c_n \left(\lim_{x\to a} x\right)^n + c_{n-1}\left(\lim_{x\to a} x\right)^{n-1} + \cdots + c_1\left(\lim_{x\to a} x\right) + c_0 && \text{(Power Law)} \\
&= c_n a^n + c_{n-1}a^{n-1} + \cdots + c_1 a + c_0 \\
&= p(a)
\end{aligned}$$

It now follows from the Quotient Law that if $p(x)$ and $q(x)$ are polynomials for which $q(a) \neq 0$, then

$$\lim_{x\to a}\frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}.$$

Example 1.3.4**: Using the Direct Substitution Property**

Since 3 is in the domain of the rational function $f(x) = \dfrac{2x^2 - 3x + 1}{5x + 4}$, we can calculate the limit by substituting 3 for x into the function. Thus,

$$\lim_{x\to 3}\frac{2x^2 - 3x + 1}{5x + 4} = \frac{10}{19}.$$

Checkpoint 1.3.4

-13

Example 1.3.5**: Evaluating a One-Sided Limit Using the Limit Laws**

Figure 1.3.1 illustrates the function $f(x) = \sqrt{x-3}$ and aids in our understanding of these limits.

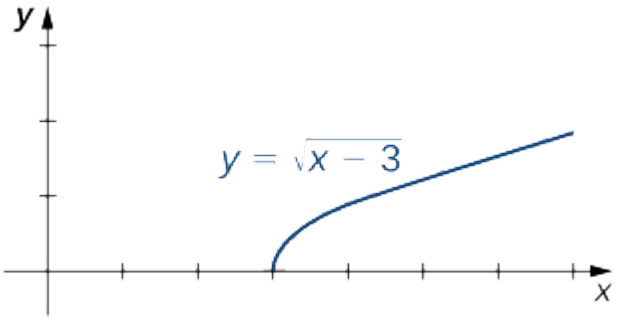

Figure 1.3.1**:** *The graph shows the function* $f(x) = \sqrt{x-3}$.

a. The function $f(x) = \sqrt{x-3}$ is defined over the interval $[3, \infty)$. Since this function is not defined to the left of 3, we cannot apply the Limit Laws to compute $\lim_{x\to 3^-}\sqrt{x-3}$. In fact, since $f(x) = \sqrt{x-3}$ is undefined to the left of 3, $\lim_{x\to 3^-}\sqrt{x-3}$ does not exist.

b. Since $f(x) = \sqrt{x-3}$ is defined for $x > 3$, the Limit Laws apply to $\lim_{x\to 3^+}\sqrt{x-3}$. By applying these Limit Laws we obtain $\lim_{x\to 3^+}\sqrt{x-3} = 0$.

Example 1.3.6**: Evaluating a Two-Sided Limit Using the Limit Laws**

Figure 1.3.2 illustrates the function $f(x)$ and aids in our understanding of these limits.

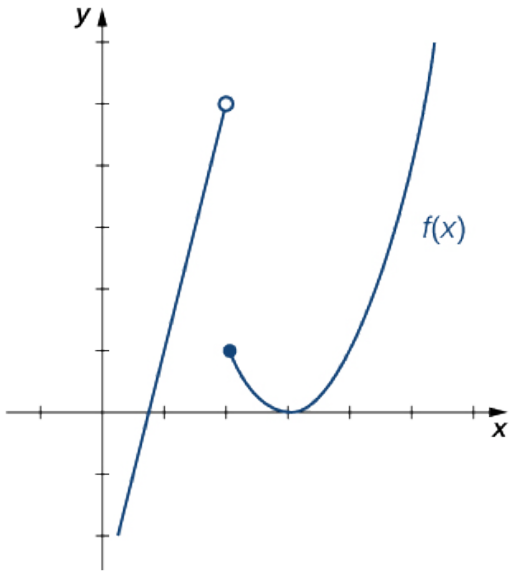

Figure 1.3.2**:** *This graph shows a function* $f(x)$.

a. Since $f(x) = 4x - 3$ for all x in $(-\infty, 2)$, replace $f(x)$ in the limit with $4x - 3$ and apply the Limit Laws:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (4x - 3) = 5$$

b. Since $f(x) = (x - 3)^2$ for all x in $(2, +\infty)$, replace $f(x)$ in the limit with $(x - 3)^2$ and apply the Limit Laws:

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} (x - 3)^2 = 1.$$

c. Since $\lim_{x \to 2^-} f(x) = 5$ and $\lim_{x \to 2^+} f(x) = 1$, we conclude that $\lim_{x \to 2} f(x)$ does not exist.

Checkpoint 1.3.6

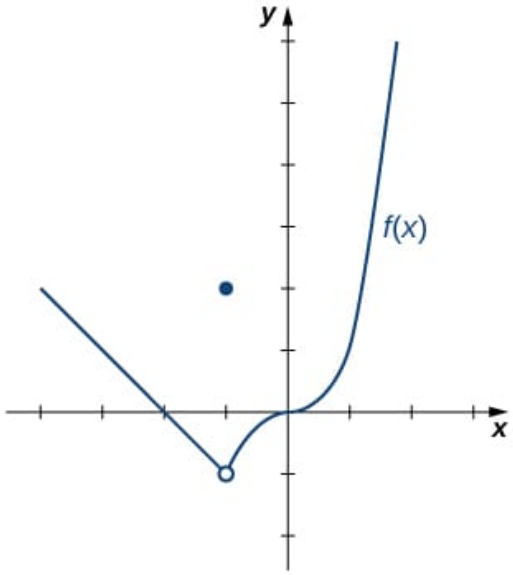

Figure 1.3.3**:** $\lim_{x \to -1^-} f(x) = -1$

Example 1.3.7**: Breaking the (Limit) Laws**

a. If we were to try to apply the Product Law for Limits, we would arrive at

$$\lim_{x \to 3^+} (x - 3) \cdot \lim_{x \to 3^+} \frac{1}{x - 3} = 0 \cdot \infty = 0 \text{ or is it } \infty?$$

This is wrong on several levels.
First, what is the product of something *approaching* 0 and something *approaching* ∞? This is a *key* difference between Algebra and Calculus. In Algebra, you are consistently dealing with *finite* values and, as such, the rules of Arithmetic apply neatly. In Calculus, you are dealing with levels of infinity (and the infinitesimal). These concepts do *not* obey the traditional rules of Arithmetic.
The second issue is that we are breaking the Limit Laws! Remember, **you can only use the Limit Laws if the component limits exist and are finite**.
So, how would you deal with the given limit? We begin by performing some very simple Algebra.

$$\lim_{x \to 3^+} \left((x - 3) \cdot \frac{1}{x - 3} \right) = \lim_{x \to 3^+} \left(\cancel{(x - 3)} \cdot \frac{1}{\cancel{(x - 3)}} \right) = \lim_{x \to 3^+} 1 = 1$$

b. Trying to apply the Product Law for Limits, we would get

$$\lim_{x\to 3^-}(x-3)^2\cdot\lim_{x\to 3^-}\frac{1}{x-3}=0\cdot-\infty=0 \text{ or is it } -\infty?$$

Just like part a, we are improperly trying to apply the Limit Laws in a situation where they are not applicable. That second limit, approaching $-\infty$, prevents us from using the Limit Laws. Instead, the proper way to evaluate this limit is to use some simple Algebra first.

$$\lim_{x\to 3^-}\left((x-3)^2\cdot\frac{1}{x-3}\right)=\lim_{x\to 3^-}(x-3)=0$$

c. Again, incorrectly applying the Product Law for Limits, we would get

$$\lim_{x\to 3^+}(x-3)\cdot\lim_{x\to 3^+}\frac{1}{(x-3)^3}=0\cdot\infty=0 \text{ or is it } \infty?$$

As in the previous two examples, we are breaking the Limit Laws because one of the component limits is not finite. We should have performed some simple Algebra first.

$$\lim_{x\to 3^+}\left((x-3)\cdot\frac{1}{(x-3)^3}\right)=\lim_{x\to 3^+}\left(\frac{1}{\cancel{(x-3)^2}}\right)=\infty$$

Example 1.3.8**: Applying the Squeeze Theorem**

If we were to try to evaluate this limit using the Limit Laws, we would arrive at

$$\lim_{x\to 0}(x\cos(x))=\lim_{x\to 0}(x)\cdot\lim_{x\to 0}\cos(x)=0\cdot\ldots \text{ ummm... what is this?}$$

I get it, you are going to say, "But, I know that $\cos(x)$ approaches 1 as x approaches 0!" However, our theory up to this point only includes algebraic functions (specifically, polynomial and rational functions). Therefore, our "intuition" of what this limit will be is not allowed.

We have the limit of the product of two functions. We know the limit of one of those functions; however, if we don't know the limit of the other function, or if we know the limit of the other function does not exist, the Squeeze Theorem can be very helpful if you can bound the function with the unknown or undefined limit.

In this case, we don't (officially) know the limit of the cosine as x is approaching 1. So, we build a bound for this function.

$$-1\le\cos(x)\le 1$$

We now rebuild the original limit by multiplying all three "sides" of this inequality by x. Before we do so, it is important to note two things: $x\neq 0$ and we don't know the sign of x. The first statement is just peripherally important so we know that we are not getting something trivial like $0\le 0\le 0$. The second statement is just for completeness. If $x>0$, then we would get

$$-x\le x\cos(x)\le x$$

If $x<0$, then the inequality signs switch direction to

$$-x\ge x\cos(x)\ge x$$

In either case, since $\lim_{x\to 0}(-x)=0=\lim_{x\to 0}(x)$, from the Squeeze Theorem, we obtain $\lim_{x\to 0}(x\cos(x))=0$. The graphs of $f(x)=-x,\ \ g(x)=x\cos x$, and $h(x)=x$ are shown in Figure 1.3.5.

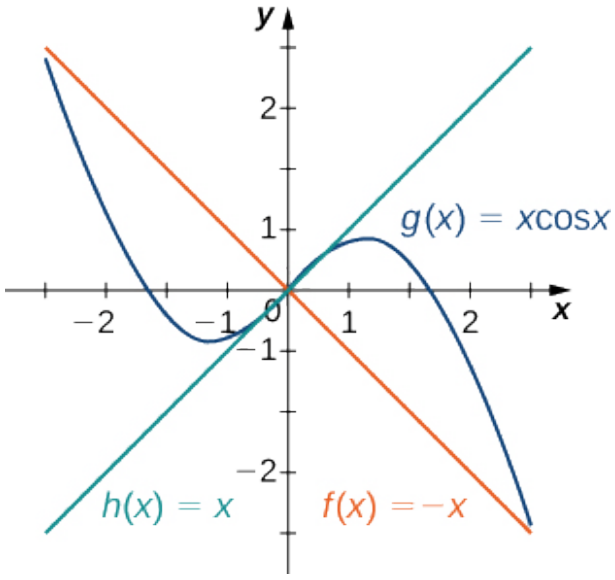

Figure 1.3.5**:** *The graphs of $f(x),\ g(x)$, and $h(x)$ are shown around the point $x=0$.*

Checkpoint 1.3.8

Use the fact that $-x^2\le x^2\sin(1/x)\le x^2$ to help you find two functions such that $x^2\sin(1/x)$ is squeezed between them.

Checkpoint 1.3.8

0

Theorem

The first of these limits is $\lim_{\theta\to 0}(\sin(\theta))$. Consider the unit circle shown in Figure 1.3.6. In the figure, we see that $\sin(\theta)$ is the y-coordinate on the unit circle, and it corresponds to the line segment shown in blue. The radian measure of angle θ is the length of the arc it subtends on the unit circle. Therefore, we see that for 0

$< \theta < \frac{\pi}{2}$, we have $0 < \sin(\theta) < \theta$.

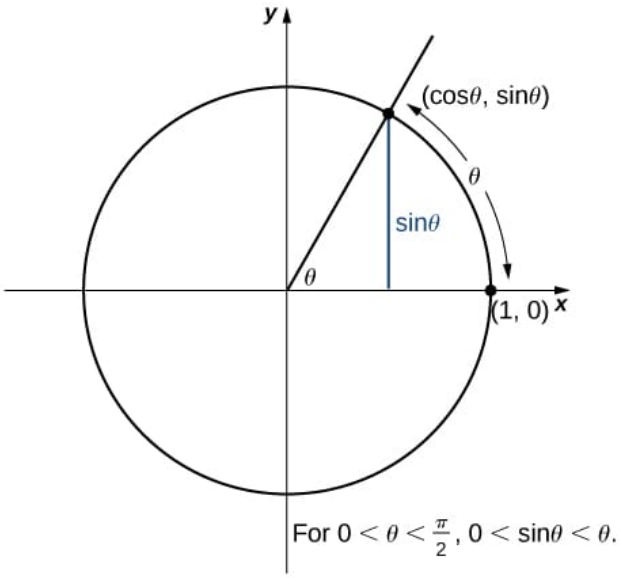

Figure 1.3.6**:** *The sine function is shown as a line on the unit circle.*

Because $\lim_{\theta \to 0^+} 0 = 0$ and $\lim_{\theta \to 0^+} \theta = 0$, by using the Squeeze Theorem we conclude that

$$\lim_{\theta \to 0^+} (\sin(\theta)) = 0.$$

To see that $\lim_{\theta \to 0^-} (\sin(\theta)) = 0$ as well, observe that for $-\frac{\pi}{2} < \theta < 0, 0 < -\theta < \frac{\pi}{2}$ and hence, $0 < \sin(-\theta) < -\theta$. Consequently, $0 < -\sin(\theta) < -\theta$. It follows that $0 > \sin(\theta) > \theta$. An application of the Squeeze Theorem produces the desired limit. Thus, since $\lim_{\theta \to 0^+} \sin(\theta) = 0$ and $\lim_{\theta \to 0^-} \sin(\theta) = 0$,

$$\lim_{\theta \to 0} \sin(\theta) = 0$$

Next, using the identity $\cos(\theta) = \sqrt{1 - \sin^2(\theta)}$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, we see that

$$\lim_{\theta \to 0} (\cos(\theta)) = \lim_{\theta \to 0} \sqrt{1 - \sin^2(\theta)} = 1.$$

Theorem

To prove the first of these limits, we use the unit circle in Figure 1.3.7. Notice that this figure adds one additional triangle to Figure 1.3.7 from the proof of the previous theorem. We see that the length of the side opposite angle θ in this new triangle is $\tan(\theta)$. Thus, we see that for $0 < \theta < \frac{\pi}{2}$, we have $\sin(\theta) < \theta < \tan(\theta)$.

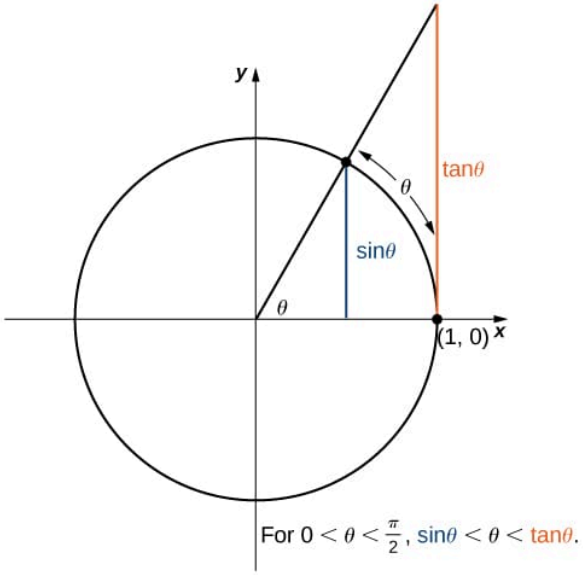

Figure 1.3.7**:** *The sine and tangent functions are shown as lines on the unit circle.*

By dividing by $\sin(\theta)$ in all parts of the inequality, we obtain

$$1 < \frac{\theta}{\sin(\theta)} < \frac{1}{\cos(\theta)}.$$

Equivalently, we have

$$1 > \frac{\sin(\theta)}{\theta} > \cos(\theta).$$

Since $\lim_{\theta \to 0^+} 1 = 1 = \lim_{\theta \to 0^+} \cos(\theta)$, we conclude that $\lim_{\theta \to 0^+} \frac{\sin(\theta)}{\theta} = 1$, by the Squeeze Theorem. By applying a manipulation similar to that used in demonstrating that $\lim_{\theta \to 0^-} (\sin(\theta)) = 0$, we can show that $\lim_{\theta \to 0^-} \frac{\sin(\theta)}{\theta} = 1$. Thus,

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1.$$

In Example 1.3.9, we use this limit to establish $\lim_{\theta \to 0} \frac{1 - \cos(\theta)}{\theta} = 0$. This limit will also prove useful in later chapters.

Example 1.3.9**: Evaluating an Important Trigonometric Limit**

In the first step, we multiply by the conjugate so that we can use a trigonometric identity to convert the cosine in the numerator to a sine:

$$\begin{aligned}
\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} &= \lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} \cdot \frac{1 + \cos\theta}{1 + \cos\theta} \\
&= \lim_{\theta \to 0} \frac{1 - \cos^2\theta}{\theta(1 + \cos\theta)} \\
&= \lim_{\theta \to 0} \frac{\sin^2\theta}{\theta(1 + \cos\theta)} \\
&= \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{1 + \cos\theta} \\
&= \left(\lim_{\theta \to 0} \frac{\sin\theta}{\theta}\right) \cdot \left(\lim_{\theta \to 0} \frac{\sin\theta}{1 + \cos\theta}\right) \\
&= 1 \cdot \frac{0}{2} = 0.
\end{aligned}$$

Therefore,

$$\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = 0.$$

Example 1.3.10**: Evaluating Finite Limits of Trigonometric Functions at Finite Points**

$$\begin{aligned}
\lim_{\theta \to 0} \frac{7\theta}{\sin(5\theta)} &= \lim_{\theta \to 0} \frac{\frac{7}{5}}{\frac{\sin(5\theta)}{5\theta}} \\
&= \frac{\lim_{\theta \to 0} \frac{7}{5}}{\lim_{\theta \to 0} \frac{\sin(5\theta)}{5\theta}} && \text{(Limit Law)} \\
&= \frac{\frac{7}{5}}{\lim_{\theta \to 0} \frac{\sin(5\theta)}{5\theta}} && \text{(Limit Law)} \\
&= \frac{7}{5} \cdot \frac{1}{\lim_{x \to 0} \frac{\sin(x)}{x}} && \text{(Let } x = 5\theta\text{. Then } x \to 0 \text{ as } \theta \to 0\text{)} \\
&\overset{\text{D.S.}}{=} \frac{7}{5} \cdot \frac{1}{1} && \text{(Direct Substitution)} \\
&= \frac{7}{5}
\end{aligned}$$

Checkpoint 1.3.10

0

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1.3.1: Resources and Key Concepts

Resources

Videos

- Limit Laws
  - Limits - Limit Laws - Concepts, Theorems, and Proof
- Evaluating limits using the Limit Laws
  - Limits - Limit Laws - Evaluating Limits Using the Limit Laws
- Power Law for limits
  - Proofs in Differential Calculus: Limits of Powers
- Direct Substitution Property
  - Limits - Limit Laws - Direct Substitution: Theorem, Proof, and Example
- Squeeze Theorem
  - Limits - The Squeeze Theorem - Concept, Theorem, and Examples
  - Limits - The Squeeze Theorem - Concept, Theorem, and Examples: Extra Example
- Limit of sin(x) as x approaches 0
  - Proofs in Differential Calculus - $\lim_{x\to 0}\dfrac{\sin x}{x}=1$
  - Derivatives of Trigonometric Functions - Using the Lemmas to Evaluate Limits
- Limit of cos(x) as x approaches 0
  - Proofs in Differential Calculus - $\lim_{x\to 0}\dfrac{\cos(x)-1}{x}=0$

Key Concepts

Theorems

- **Corollary: Basic Limit Laws**: For any real number a and any constant c,
  I. $\lim_{x\to a} x = a$
  II. $\lim_{x\to a} c = c$
- **Limit Laws**: Let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing a. Assume that L and M are real numbers such that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Let c be a constant. Then:
  - **Sum Law for Limits**: $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M$
  - **Difference Law for Limits**: $\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L - M$
  - **Constant Multiple Law for Limits**: $\lim_{x\to a} cf(x) = c\cdot\lim_{x\to a} f(x) = cL$
  - **Product Law for Limits**: $\lim_{x\to a}(f(x)\cdot g(x)) = \left(\lim_{x\to a} f(x)\right)\cdot\left(\lim_{x\to a} g(x)\right) = L\cdot M$
  - **Quotient Law for Limits**: $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{L}{M}$, for $M \neq 0$.
  - **Power Law for Limits**: $\lim_{x\to a}(f(x))^n = \left(\lim_{x\to a} f(x)\right)^n = L^n$, for every positive integer n.
  - **Root Law for Limits**: $\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$, for all L if n is odd and for $L \geq 0$ if n is even.
- **Theorem: Direct Substitution Property (for polynomial and rational functions)**: Let $p(x)$ and $q(x)$ be polynomial functions. Let a be a real number. Then, $\lim_{x\to a} p(x) = p(a)$, and $\lim_{x\to a}\dfrac{p(x)}{q(x)} = \dfrac{p(a)}{q(a)}$ when $q(a) \neq 0$.
- **One-Sided Limits (application of Limit Laws)**: Simple modifications to the Limit Laws allow them to be applied to one-sided limits. For $\lim_{x\to a^-} h(x)$, $h(x)$ must be defined over an open interval of the form (b, a). For $\lim_{x\to a^+} h(x)$, $h(x)$ must be defined over an open interval of the form (a, c).
- **Theorem: Squeeze Theorem**: Let $f(x)$, $g(x)$, and $h(x)$ be defined for all $x \neq a$ over an open interval containing a. If $f(x)$

$\leq g(x) \leq h(x)$ for all $x \neq a$ in an open interval containing a and $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$ where L is a real number, then $\lim_{x \to a} g(x) = L$.

- **Theorem: Limits Involving Sine and Cosine**:
  - $\lim_{\theta \to 0}(\sin(\theta)) = 0$
  - $\lim_{\theta \to 0}(\cos(\theta)) = 1$
- **Theorem: Important Trigonometric Limits**:
  - $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$
  - $\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0$

Common Mistakes

- **The Limit Laws Require Limits to Exist**: Thinking that the Limit Laws apply to every limit encountered. To use the Limit Laws, both of the component limits (e.g., $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$) must exist and be finite.
- **Misapplying the Direct Substitution Property**: The Direct Substitution Property for rational functions $\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$ can only be applied if the limit in the denominator $q(a)$ is nonzero. An answer will only be "DNE" if both left- and right-hand limits have been evaluated and found not to be equal (or if one or both tend to infinity differently).
- **Incorrectly Applying Limit Laws to Non-Finite Limits (Breaking the Limit Laws)**: Applying Limit Laws when one or more component limits are infinite. For example, if $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$, one cannot use the Product Law to conclude $\lim_{x \to a} f(x)g(x) = 0 \cdot \infty$. Algebraic simplification is needed first.
- **Using Degree Measure in Calculus for Trigonometric Functions**: The proofs of trigonometric limits (e.g., $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$) rely on radian measure. Calculus and degree measure do not get along for trigonometric functions.

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1.3.2: Homework

Note

The following exercises are meant as hand-in homework (if assigned by your instructor). Typed-in answers here will not be saved or scored.

Basic Skills

Exercise 1

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Exercise 2

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Exercise 4

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Exercise 5

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Exercise 7

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Exercise 8

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Exercise 9

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Exercise 10

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Exercise 11

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Exercise 12

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Synthesis Questions

Exercise 13

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Exercise 14

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Exercise 15

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Exercise 16

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Exercise 17

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Exercise 18

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Proofs

Exercise 19

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Exercise 20

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Exercise 21

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Exercise 22

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1.3.3: Additional Exercises

Reading Questions

1. What are the two Basic Limit Laws for $\lim_{x\to a} x$ and $\lim_{x\to a} c$?
2. State the Sum Law for Limits. What condition must be met for it to apply?
3. Under what condition can the Quotient Law for Limits be applied?
4. When can the Direct Substitution Property be used to evaluate the limit of a polynomial function?
5. When can the Direct Substitution Property be used to evaluate the limit of a rational function $\frac{p(x)}{q(x)}$ at $x = a$? What must be true about $q(a)$?
6. If direct substitution into a rational function yields $\frac{0}{0}$, what kind of algebraic manipulation might be useful before re-evaluating the limit?
7. How do the Limit Laws apply to one-sided limits? What condition must be met for the function's domain?
8. Explain the Squeeze Theorem in your own words. What three conditions must be met to use it to find $\lim_{x\to a} g(x)$?
9. What is the value of $\lim_{\theta\to 0} \sin(\theta)$?
10. What is the value of $\lim_{\theta\to 0} \cos(\theta)$?
11. What is the important limit $\lim_{\theta\to 0} \dfrac{\sin\theta}{\theta}$?
12. What is the important limit $\lim_{\theta\to 0} \dfrac{1-\cos\theta}{\theta}$?
13. Why is it crucial to use radian measure (not degrees) when working with limits of trigonometric functions in calculus?
14. What does it mean if applying Limit Laws leads to an indeterminate form like $0 \cdot \infty$? What should be done instead?

Homework

In exercises 1 - 4, use the Limit Laws to evaluate each limit. Justify each step by indicating the appropriate Limit Law(s).

1) $\lim_{x\to 0} (4x^2 - 2x + 3)$

2) $\lim_{x\to 1} \dfrac{x^3 + 3x^2 + 5}{4 - 7x}$

3) $\lim_{x\to -2} \sqrt{x^2 - 6x + 3}$

4) $\lim_{x\to -1} (9x + 1)^2$

In exercises 5 - 8, use the Direct Substitution Property to evaluate the limit of each function.

5) $\lim_{x\to 7} x^2$

6) $\lim_{x\to -2} (4x^2 - 1)$

7) $\lim_{x\to 0} \dfrac{1}{1 + \sin x}$

8) $\lim_{x\to 1} \dfrac{2 - 7x}{x + 6}$

In exercises 9 - 16, assume that $\lim_{x\to 6} f(x) = 4$, $\lim_{x\to 6} g(x) = 9$, and $\lim_{x\to 6} h(x) = 6$. Use these three facts and the Limit Laws to evaluate each limit.

9) $\lim_{x\to 6} 2f(x)g(x)$

10) $\lim_{x\to 6} \dfrac{g(x)-1}{f(x)}$

11) $\lim_{x\to 6} \left(f(x)+\dfrac{1}{3}g(x)\right)$

12) $\lim_{x\to 6} \dfrac{\left(h(x)\right)^3}{2}$

13) $\lim_{x\to 6} \sqrt{g(x)-f(x)}$

14) $\lim_{x\to 6} x\cdot h(x)$

15) $\lim_{x\to 6}[(x+1)\cdot f(x)]$

16) $\lim_{x\to 6}(f(x)\cdot g(x)-h(x))$

In exercises 17 - 19, evaluate the limits using the given piecewise-defined function and the Limit Laws.

17) $f(x)=\begin{cases} x^2, & x\le 3 \\ x+4, & x>3 \end{cases}$

a. $\lim_{x\to 3^-} f(x)$

b. $\lim_{x\to 3^+} f(x)$

18) $g(x)=\begin{cases} x^3-1, & x\le 0 \\ 1, & x>0 \end{cases}$

a. $\lim_{x\to 0^-} g(x)$

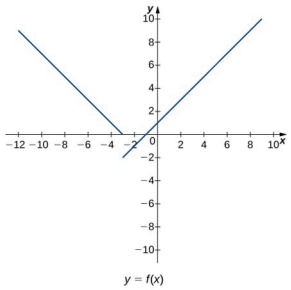
b. $\lim_{x\to 0^+} g(x)$

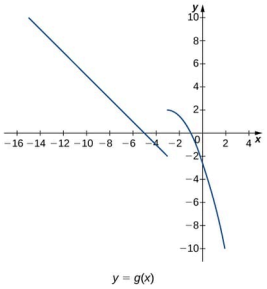
19) $h(x)=\begin{cases} x^2-2x+1, & x<2 \\ 3-x, & x\ge 2 \end{cases}$

a. $\lim_{x\to 2^-} h(x)$

b. $\lim_{x\to 2^+} h(x)$

In exercises 20 - 27, use the following graphs and the Limit Laws to evaluate each limit.





20) $\lim_{x \to -3^+} (f(x) + g(x))$

21) $\lim_{x \to -3^-} (f(x) - 3g(x))$

22) $\lim_{x \to 0} \dfrac{f(x)g(x)}{3}$

23) $\lim_{x \to -5} \dfrac{2 + g(x)}{f(x)}$

24) $\lim_{x \to 1} (f(x))^2$

25) $\lim_{x \to 1} \sqrt[3]{f(x) - g(x)}$

26) $\lim_{x \to -7} (x \cdot g(x))$

27) $\lim_{x \to -9} [x \cdot f(x) + 2 \cdot g(x)]$

For exercises 28 - 30, evaluate the limit using the Squeeze Theorem. Graph the functions $f(x)$, $g(x)$, and $h(x)$ when possible.

28) True or False? If $2x - 1 \leq g(x) \leq x^2 - 2x + 3$, then $\lim_{x \to 2} g(x) = 0$.

29) $\lim_{\theta \to 0} \theta^2 \cos\left(\dfrac{1}{\theta}\right)$

30) $\lim_{x \to 0} f(x)$, where $f(x) = \begin{cases} 0, & x \text{ rational} \\ x^2, & x \text{ irrrational} \end{cases}$

31) In physics, the magnitude of an electric field generated by a point charge at a distance r in vacuum is governed by Coulomb's law: $E(r) = \dfrac{q}{4\pi\epsilon_0 r^2}$, where E represents the magnitude of the electric field, q is the charge of the particle, r is the distance between the particle and where the strength of the field is measured, and $\dfrac{1}{4\pi\epsilon_0}$ is Coulomb's constant: $8.988 \times 109N \cdot m^2/C^2$.

a. Use a graphing calculator to graph $E(r)$ given that the charge of the particle is $q = 10^{-10}$.

b. Evaluate $\lim_{r \to 0^+} E(r)$. What is the physical meaning of this quantity? Is it physically relevant? Why are you evaluating from

the right?

32) The density of an object is given by its mass divided by its volume: $\rho = m/V$.

a. Use a calculator to plot the volume as a function of density $(V = m/\rho)$, assuming you are examining something of mass 8 kg $(m = 8)$.

b. Evaluate $\lim_{x \to 0^+} V(\rho)$ and explain the physical meaning.

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1.4: Algebraic Manipulations and the Limit Laws

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Most limits encountered in Calculus cannot be immediately evaluated using the Direct Substitution Property. There is often some bit of required manipulation to transform the function within the limit to something more amenable to our needs. It is this action—manipulating the function within the limit—that is the focus of this section. Before we begin, it is important to introduce (or reintroduce, if you have seen it before) the **Mathematical Mantra**.

The Mathematical Mantra

The following "mantra" is very helpful throughout all levels of Mathematics, and significantly more so in Calculus.

> Perform mathematics in the order you learned mathematics. That is, perform
>
> Arithmetic before Algebra,
>
> Algebra before Trigonometry,
>
> Trigonometry before Precalculus, and
>
> Precalculus before Differential Calculus.
>
> Always be willing to ask yourself, "Is there something I learned previously that can simplify my work here?"

We will reference the Mathematical Mantra throughout the rest of Calculus.

Caution: The Mathematical Mantra is a Rule of Thumb

The Mathematical Mantra is a "rule of thumb" and, as such, there are times when it is not applicable. In such cases, I will endeavor to point out why it does not apply.

Locally Equivalent Functions

As we have seen, we may easily evaluate the limits of polynomials and limits of some (but not all) rational functions using the Direct Substitution Property (for polynomial and rational functions). However, as we saw in the introductory section on limits, it

is certainly possible for $\lim_{x\to a} f(x)$ to exist when $f(a)$ is undefined. The following theorem allows us to evaluate many limits of this type.

Theorem: Locally Equivalent Functions Have Equivalent Limits

If for all $x \neq a$ we have $f(x) = g(x)$ over some open interval containing a, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x),$$

provided these limits exist.

This unlocks our ability to evaluate limits we otherwise would never be able to determine.

To understand this idea better, consider the limit $\lim_{x\to 1} \dfrac{x^2-1}{x-1}$. The function

$$f(x) = \frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1}$$

and the function $g(x) = x + 1$ are identical for all values of $x \neq 1$. The graphs of these two functions are shown in Figure $1.4.1$.

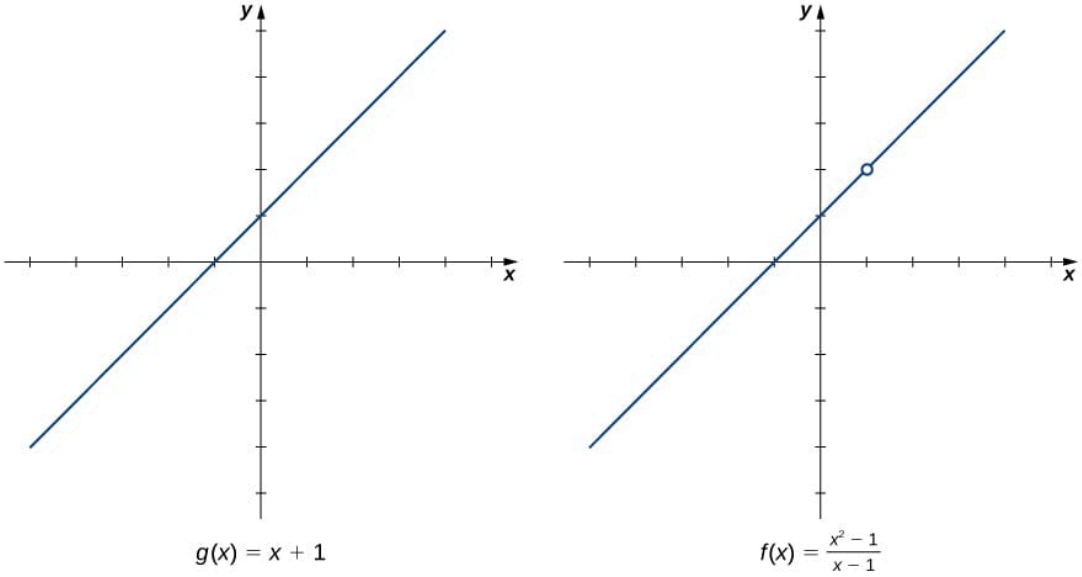

Figure $1.4.1$**:** *The graphs of $f(x)$ and $g(x)$ are identical for all $x \neq 1$. Their limits at 1 are equal.*

We see that

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1} (x+1) = 2.$$

The limit has the form $\lim_{x\to a} \dfrac{f(x)}{g(x)}$, where $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$. (In this case, we say that $\frac{f(x)}{g(x)}$ has the indeterminate form $\frac{0}{0}$.) When evaluating limits of this type, it is imperative to follow the Mathematical Mantra.

Locally Equivalent Functions - Found by Factoring

We spent years in Algebra learning how to factor, but never were told why (unless you had an excellent instructor). The truth is, factoring is necessary in Calculus. It allows us to create the "possible" from something that seems impossible. All of our previous factoring techniques—factoring differences of squares, sums of cubes, differences of cubes, and trinomials, to name the most popular—will be needed at one point or another. In fact, we might even need to dust off the old Rational Roots Theorem at times!

Example 1.4.1: Evaluating a Limit by Factoring and Canceling

Evaluate $\lim\limits_{x\to 3}\dfrac{x^2-3x}{2x^2-5x-3}$.

Solution available at the end of this section.

Example 1.4.1 uses the fact that $f(x)=\frac{x^2-3x}{2x^2-5x-3}$ is *identical* to the function $y=\frac{x}{2x+1}$ as long as $x\neq 3$. Therefore, around $x=3$, but not at $x=3$, these two functions are locally equivalent. Remember, a limit as $x\to 3$ is asking us to investigate the function as x gets as close as we want to 3, without ever becoming 3.

Caution: Limits Require... Limits

A widespread mistake is not using limit notation when working with limits. Be aware that limit notation is *incredibly* important in Calculus and must be used until you finish evaluating the limit. Example 1.4.1 illustrates the proper use of limit notation. The following, however, would be *absolutely* incorrect:

$$\begin{aligned}\lim_{x\to 3}\frac{x^2-3x}{2x^2-5x-3} &= \lim_{x\to 3}\frac{x(x-3)}{(x-3)(2x+1)}\\ &= \frac{x\,\cancel{(x-3)}}{\cancel{(x-3)}\,(2x+1)}\\ &= \frac{x}{2x+1}\\ &\overset{D.S.}{=} \frac{3}{7}\end{aligned}$$

Checkpoint 1.4.1

Evaluate $\lim\limits_{x\to -3}\dfrac{x^2+4x+3}{x^2-9}$.

Answer available at the end of this section.

Locally Equivalent Functions - Found by Conjugate Multiplication

Another manipulation that is commonly needed in Calculus can occur when your limit involves a sum or difference of expressions where at least one is a radical. We need to reach into our bag of Algebra tools to remember what to do in these situations.

Example 1.4.2: Evaluating a Limit by Multiplying by a Conjugate

Recall from the previous section that $\lim\limits_{\theta\to 0}\cos\theta=1$. Use this to evaluate the following limit.

$$\lim_{t\to 0}\frac{\sqrt{\cos(t)+3}-2}{\cos(t)-1}$$

Solution available at the end of this section.

Caution: Conjugate Multiplication and Distribution

When multiplying the numerator and denominator of an expression by a conjugate (of either the numerator or the denominator), it is advisable to only perform distribution on the conjugate factors.

For example, consider the following two limits

$$\lim_{x\to -10}\frac{\sqrt{110+x}-10}{x+10} \qquad \text{and} \qquad \lim_{\theta\to 0}\frac{\sin^3(\theta)}{1-\cos(\theta)}$$

Both are indeterminate of the form $0/0$. Multiplying the numerator and denominator of the first limit by $\sqrt{110+x}+10$ and the numerator and denominator of the second by $1+\cos(\theta)$, we get

$$\lim_{x\to -10}\left(\frac{\sqrt{110+x}-10}{x+10}\cdot\frac{\sqrt{110+x}+10}{\sqrt{110+x}+10}\right) \qquad \text{and} \qquad \lim_{\theta\to 0}\left(\frac{\sin^3(\theta)}{1-\cos(\theta)}\cdot\frac{1+\cos(\theta)}{1+\cos(\theta)}\right)$$

By distributing the *conjugate factors* and leaving the non-conjugate factors as an undistributed product, we get

$$\lim_{x\to -10}\left(\frac{110+x-100}{(x+10)\left(\sqrt{110+x}+10\right)}\right) \qquad \text{and} \qquad \lim_{\theta\to 0}\left(\frac{\sin^3(\theta)\,(1+\cos(\theta))}{1-\cos^2(\theta)}\right)$$

Simplifying...

$$\lim_{x\to -10}\left(\frac{10+x}{(x+10)\left(\sqrt{110+x}+10\right)}\right) \qquad \text{and} \qquad \lim_{\theta\to 0}\left(\frac{\sin^3(\theta)\,(1+\cos(\theta))}{\sin^2(\theta)}\right)$$

... something beautiful occurs!

$$\lim_{x\to -10}\left(\frac{\cancel{(10+x)}}{\cancel{(x+10)}\left(\sqrt{110+x}+10\right)}\right)\overset{D.S.}{=}\frac{1}{20} \qquad \text{and} \qquad \lim_{\theta\to 0}\left(\sin(\theta)\,(1+\cos(\theta))\right)\overset{D.S.}{=}0$$

(Note: I did a little bit of "hand-waving" on the Direct Substitution for the trigonometric expression. Technically, we have not learned the Direct Substitution Property for trigonometric functions. Still, the point of the example here was to showcase when to distribute conjugates and when not to.)

Checkpoint 1.4.2

Evaluate $\lim_{x\to 5}\dfrac{\sqrt{x-1}-2}{x-5}$.

Answer available at the end of this section.

Locally Equivalent Functions - Found by Simplifying Compound Rational Expressions

Another common algebraic manipulation that occurs throughout Calculus is the need to simplify a compound rational expression.

Example 1.4.3: Evaluating a Limit by Simplifying a Complex Fraction

Evaluate $\lim_{x\to 1}\dfrac{\frac{1}{x+1}-\frac{1}{2}}{x-1}$.

Solution available at the end of this section.

Checkpoint 1.4.3

Evaluate $\lim_{x\to -3} \dfrac{\frac{1}{x+2}+1}{x+3}$.

Answer available at the end of this section.

Locally Equivalent Functions - Found by Using the Definition of the Absolute Value Function

Knowing the true definition of the absolute value (the piecewise definition) is going to be critical in Calculus. Recall,

$$|\blacksquare| = \begin{cases} \blacksquare, & \text{if } \blacksquare \geq 0 \\ -\blacksquare, & \text{if } \blacksquare < 0 \end{cases}$$

Example 1.4.4: Evaluating a Limit Involving Absolute Values

Evaluate

$$\lim_{x\to(-5/3)^-} \frac{|3x+5|}{6x^2+7x-5}.$$

Solution available at the end of this section.

Interactive Element: The Absolute-Value Jump at $x = -\frac{5}{3}$

The following Interactive Element might help you visualize why the two-sided limit fails and why the *side* of approach decides the answer.

Interact: Drag the red point along the curve toward $x = -\frac{5}{3}$ — first from the left, then from the right.

Observation: From the left the point climbs toward $\frac{3}{13}$; from the right it drops toward $-\frac{3}{13}$. They differ because $|3x+5|$ switches to $-(3x+5)$ exactly at $x = -\frac{5}{3}$, so the left-hand limit is $\frac{3}{13}$.

Despite having written down the precise definition before Example 1.4.4, a cautionary statement is worthwhile.

Critical Mistake: Not Using the True Definition of the Absolute Value

Example 1.4.4 should remind you of the fact that $|f(x)|$ is not *necessarily* equivalent to $f(x)$. This is a critical prerequisite skill for Calculus and it bears repeating here:

$$|f(x)| = \begin{cases} f(x), & \text{if } f(x) \geq 0 \\ -f(x), & \text{if } f(x) < 0 \end{cases}$$

Knowing the piecewise definition of the absolute value function and using it to manipulate expressions involving absolute values is a key fundamental component to your success in Calculus. Theoretically, lacking this *single* prerequisite skill will not prevent you from succeeding in Calculus; however, in practice, a student who cannot master the understanding of this concept *rarely* (if ever) passes a Calculus course.

Locally Equivalent Functions - Found by Simplifying Rational Expressions

Example 1.4.5 does not fall neatly into any of the patterns established in the previous examples. However, we can still use these

same techniques with a little creativity.

Example 1.4.5: Evaluating a Limit When the Limit Laws Do Not Apply

Evaluate

$$\lim_{x\to 0}\left(\frac{1}{x}+\frac{5}{x(x-5)}\right).$$

Solution available at the end of this section.

Checkpoint 1.4.5

Evaluate $\lim_{x\to 3}\left(\frac{1}{x-3}-\frac{4}{x^2-2x-3}\right)$.

Answer available at the end of this section.

Armed with an ability to use your prerequisite Algebra skills, you should be well on your way to success in Calculus!

MyOpenMath Advice: Check For Feedback

When grading your uploaded (handwritten) work, your professor might have left feedback. Sometimes this feedback can be as simple as, "Please read the written solution," but it can be as robust as a breakdown of your mistake and how you can avoid such a mistake in the future. To check a past due assignment for any feedback, go to the Grades area in Canvas, click on the assignment in question, and you should be able to see any feedback there. If you cannot see feedback, please let your professor know.

Success in College: Meeting With Your Professor

Meeting with a faculty member might be intimidating. Students are encouraged to use the following guidelines to make the faculty meeting productive. These strategies were developed by the Retention Office at U. of Toledo (OH) and first appeared in National On-Campus Report's sister publication, Recruitment and Retention in Higher Education.

Step 1: Make an appointment

- Review the syllabus for the professor's office hours and make an appointment.
- If you're uncertain of your professor's office hours, ask.
- If your schedule conflicts with the office hours, tell your professor why you can't make the scheduled office hours. Then tell them about your specific concern.
- Set up a specific time to meet with your professor. Be sure to indicate the amount of time you will need.

Step 2: Establish rapport

- Be on time for your appointment.
- When you arrive, be pleasant, smile, introduce yourself again (include your name and class), and shake hands if appropriate.

Step 3: Present your concern

- Focus on the specific questions you've identified as problematic.
- Have your problem(s) written out so that the professor can see where your difficulties are.

Step 4: Provide background information

- Briefly tell your professor about your high school background and preparation for the course – if relevant to solving the problem.
- Explain the study strategies you've used to understand the material.

Step 5: Redirect for clarification

- If the professor's explanation isn't clear, redirect their attention to the specific point where you became confused.
- "Talk through" your problem.

Step 6: Summarize Resolution of the Problem

- "I was missing this step in... I need to apply this formula..."

Step 7: Thank your Professor

- Ask to come back if necessary.

Armed with this information, think about the following questions:

1. How would you make an appointment to talk to your professor if your schedule prevents you from visiting during office hours? If it is not possible to make such an appointment or your professor does not have office hours, think about another way you can get extra help.
2. Some professors prefer you talk to them using their title and last name. For example, you might say Dr. Rodriguez or Ms. Smith. Others prefer that you use just their first name. When asking your professor a question, what name or title do they prefer that you use?

If you are unsure about the answers to either of these questions, feel free to ask your professor - that's what they are there for!

Answers

Section Preview

This is the moment where you can *really* distinguish between the value of a function at a point and the value *near* to that point. The Locally Equivalent Functions material should really drive the point home that, when Direct Substitution fails us (which it often will), we can lean on a "locally-equivalent" function to evaluate. The "Mathematical Mantra" is introduced, and you should repeat this frequently throughout the course. There is a *lot* of old Algebra in this section, so you might need to remind students of factoring techniques. Using the piecewise definition of the absolute value function is also a critical topic!

Section Preview

- **Absolute Value Functions**
  - The Concept of Absolute Value: Example 4 and the "Critical Mistake" callout hinge on grasping that $|f(x)|$ is *not* necessarily $f(x)$—the conceptual core of the absolute-value limit work.

Section Preview

- **Factoring Techniques**
  - Greatest Common Factor: Example 1 factors the numerator $x^2-3x=x(x-3)$ by pulling out the GCF.
  - Factoring Trinomials: Example 1 $(2x^2-5x-3)$, Example 4 $(6x^2+7x-5)$, and Checkpoint 1 (x^2+4x+3) all require factoring trinomials with and without leading coefficients.
  - Factoring Difference of Squares: Checkpoint 1 factors $x^2-9=(x-3)(x+3)$ in the denominator.
  - Factoring Sums and Differences of Cubes: This topic is needed for several examples from this point forward.
- **Simplifying Expressions**
  - Adding and Subtracting Rational Expressions: Example 5 combines $\frac{1}{x}+\frac{5}{x(x-5)}$ and Checkpoint 5 combines two rational terms over a common denominator before the limit can be taken.
  - Simplifying Compound Rational Expressions: Example 3 simplifies the complex fraction $\frac{\frac{1}{x+1}-\frac{1}{2}}{x-1}$.

Section Preview

- Evaluate the limit of a function by first performing algebraic manipulations (factoring, conjugate multiplication, combining fractions, etc.).
- Evaluate the limit of a function involving absolute values using the definition of an absolute value function.

Example $1.4.1$: Evaluating a Limit by Factoring and Canceling

The function $f(x)=\frac{x^2-3x}{2x^2-5x-3}$ is undefined for $x=3$. In fact, if we substitute 3 into the function we find this limit is indeterminate of the form $0/0$. This means that we must follow the Mathematical Mantra and perform any Arithmetic, Algebra, and then Trigonometry before attempting to apply Calculus here.

From Algebra, we can see that factoring in both the numerator and denominator is possible.

$$\begin{aligned}
\lim_{x\to 3}\frac{x^2-3x}{2x^2-5x-3} &= \lim_{x\to 3}\frac{x(x-3)}{(x-3)(2x+1)} \\
&= \lim_{x\to 3}\frac{x\,\cancel{(x-3)}}{\cancel{(x-3)}\,(2x+1)} \\
&= \lim_{x\to 3}\frac{x}{2x+1} \\
&\overset{D.S.}{=} \frac{3}{7} && \text{(Direct Substitution)}
\end{aligned}$$

Checkpoint 1.4.1

$\frac{1}{3}$

Example 1.4.2**: Evaluating a Limit by Multiplying by a Conjugate**

As t approaches 0, both the numerator and denominator of $f(x)=\frac{\sqrt{\cos(t)+3}-2}{\cos(t)-1}$ approach 0. Therefore, this limit is indeterminate of the form $0/0$. This means that we must follow the Mathematical Mantra and perform any Arithmetic, Algebra, and then Trigonometry (generally, in that order) before attempting to apply Calculus here.

The numerator, having two terms where one involves a square root, is enticing. Let's try multiplying the numerator and denominator by the conjugate of the numerator.

$$\begin{aligned}
\lim_{t\to 0}\frac{\sqrt{\cos(t)+3}-2}{\cos(t)-1} &= \lim_{t\to 0}\left(\frac{\sqrt{\cos(t)+3}-2}{\cos(t)-1}\cdot\frac{\sqrt{\cos(t)+3}+2}{\sqrt{\cos(t)+3}+2}\right) \\
&= \lim_{t\to 0}\left(\frac{\cos(t)+3-4}{(\cos(t)-1)\left(\sqrt{\cos(t)+3}+2\right)}\right) && \text{(distributing}^{*}\text{ (see Note below))} \\
&= \lim_{t\to 0}\left(\frac{\cos(t)-1}{(\cos(t)-1)\left(\sqrt{\cos(t)+3}+2\right)}\right) \\
&= \lim_{t\to 0}\left(\frac{\cancel{(\cos(t)-1)}}{\cancel{(\cos(t)-1)}\left(\sqrt{\cos(t)+3}+2\right)}\right) \\
&= \lim_{t\to 0}\left(\frac{1}{\sqrt{\cos(t)+3}+2}\right) \\
&= \frac{\lim\limits_{t\to 0}1}{\lim\limits_{t\to 0}\left(\sqrt{\cos(t)+3}+2\right)} && \text{(Quotient Law)} \\
&= \frac{1}{\lim\limits_{t\to 0}\sqrt{\cos(t)+3}+\lim\limits_{t\to 0}2} && \text{(Sum Law and Basic Limit Laws)} \\
&= \frac{1}{\sqrt{\lim\limits_{t\to 0}(\cos(t)+3)}+2} && \text{(Root Law and Basic Limit Laws)} \\
&= \frac{1}{\sqrt{\lim\limits_{t\to 0}\cos(t)+\lim\limits_{t\to 0}3}+2} && \text{(Sum Law)} \\
&= \frac{1}{\sqrt{1+3}+2} && \text{(Basic Limit Laws)} \\
&= \frac{1}{4}
\end{aligned}$$

Checkpoint 1.4.2

$\frac{1}{4}$

Example 1.4.3**: Evaluating a Limit by Simplifying a Complex Fraction**

As $x\to 1$, the function $f(x)=\frac{\frac{1}{x+1}-\frac{1}{2}}{x-1}$ approaches $0/0$. This means that we must follow the Mathematical Mantra and perform any Arithmetic, Algebra, and then Trigonometry (generally, in this order) before attempting to apply Calculus.

From Algebra, we know that any compound fraction like this should be simplified.

$$\begin{aligned}
\lim_{x\to 1}\frac{\frac{1}{x+1}-\frac{1}{2}}{x-1} &= \lim_{x\to 1}\frac{2-(x+1)}{2(x-1)(x+1)} \\
&= \lim_{x\to 1}\frac{1-x}{2(x-1)(x+1)} \\
&= \lim_{x\to 1}\frac{-(x-1)}{2(x-1)(x+1)} \\
&= \lim_{x\to 1}\frac{-\cancel{(x-1)}}{2\cancel{(x-1)}(x+1)} \\
&= \lim_{x\to 1}\frac{-1}{2(x+1)} \\
&\overset{D.S.}{=} -\frac{1}{4} \qquad \text{(Direct Substitution)}
\end{aligned}$$

Checkpoint 1.4.3

-1

Example 1.4.4**: Evaluating a Limit Involving Absolute Values**

As $x\to(-5/3)^-$, both the numerator and denominator of the limit tend to 0. Therefore, this limit is indeterminate of the form $\frac{0}{0}$. Following the Mathematical Mantra, we clean things up first by performing some Algebra.

Our prerequisite factoring skills allow us to factor the denominator to the following:

$$6x^2+7x-5=(3x+5)(2x-1).$$

Therefore,

$$\lim_{x\to(-5/3)^-}\frac{|3x+5|}{6x^2+7x-5}=\lim_{x\to(-5/3)^-}\frac{|3x+5|}{(3x+5)(2x-1)}.$$

We know that $|3x+5|$ is not the same as $3x+5$, so we must use the definition of the absolute value to rewrite $|3x+5|$.

$$|3x+5|=\begin{cases}3x+5 & \text{, if } 3x+5\geq 0 \implies x\geq -\frac{5}{3}\\ -(3x+5) & \text{, if } 3x+5<0 \implies x<-\frac{5}{3}\end{cases}$$

Since x is approaching $-5/3$ from the left, $x<-5/3$. Therefore, $|3x+5|=-(3x+5)$. Hence,

$$\begin{aligned}
\lim_{x\to(-5/3)^-}\frac{|3x+5|}{6x^2+7x-5} &= \lim_{x\to(-5/3)^-}\frac{|3x+5|}{(3x+5)(2x-1)} \\
&= \lim_{x\to(-5/3)^-}\frac{-(3x+5)}{(3x+5)(2x-1)} \qquad \text{(using the definition of the absolute value)} \\
&= \lim_{x\to(-5/3)^-}\frac{-\cancel{(3x+5)}}{\cancel{(3x+5)}(2x-1)} \\
&= \lim_{x\to(-5/3)^-}\frac{-1}{2x-1} \\
&\overset{D.S.}{=} \frac{-1}{2(-5/3)-1} \qquad \text{(Direct Substitution)} \\
&= \frac{3}{13}
\end{aligned}$$

Example 1.4.5**: Evaluating a Limit When the Limit Laws Do Not Apply**

Both $1/x$ and $5/x(x-5)$ fail to have a limit at zero. Since neither function has a zero limit, we cannot apply the Sum Law for limits; we must use a different strategy. In this case, we find the limit by performing addition and then applying one of our previous strategies. Observe that

$$\frac{1}{x}+\frac{5}{x(x-5)}=\frac{x-5+5}{x(x-5)}=\frac{x}{x(x-5)}.$$

Thus,

$$\lim_{x\to 0}\left(\frac{1}{x}+\frac{5}{x(x-5)}\right)=\lim_{x\to 0}\frac{x}{x(x-5)}=\lim_{x\to 0}\frac{1}{x-5}=-\frac{1}{5}.$$

Checkpoint 1.4.5

$\frac{1}{4}$

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1.4.1: Resources and Key Concepts

Resources

Videos

- Algebraic manipulation for limits
  - Limits - Limit Laws - Evaluating Limits Using Algebraic Manipulations
- Evaluating limits with absolute value
  - Limits - Limit Laws - Evaluating Limits Involving Absolute Values
  - Limits - Limit Laws - Evaluating Limits Involving Absolute Values: Extra Example

Key Concepts

Definitions

- **Mathematical Mantra**: A guiding principle stating: "Perform mathematics in the order you learned mathematics. That is, perform Arithmetic before Algebra, Algebra before Trigonometry, Trigonometry before Precalculus, and Precalculus before Differential Calculus." It also encourages asking, "Is there something I learned previously that can simplify my work here?"
- **Indeterminate Form** $\frac{0}{0}$: A situation arising when direct substitution into a limit of a quotient $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ yields $\frac{0}{0}$, indicating that further algebraic manipulation is required to evaluate the limit.

Theorems

- **Theorem: Locally Equivalent Functions Have Equivalent Limits**: If for all $x \neq a$ we have $f(x) = g(x)$ over some open interval containing a, then $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x)$, provided these limits exist.

Common Mistakes

- **The Mathematical Mantra is a Rule of Thumb**: The Mathematical Mantra is a "rule of thumb" and, as such, there are times when it is not applicable.
- **Limits Require... Limits (Improper Limit Notation)**: Not using limit notation when working with limits. Limit notation is incredibly important in Calculus and must be used until the limit is fully evaluated.
- **Conjugate Multiplication and Distribution**: When multiplying the numerator and denominator of an expression by a conjugate, it is advisable to only perform distribution on the conjugate factors, leaving the non-conjugate factors as an undistributed product, as this often leads to simplification.
- **Not Using the True Definition of the Absolute Value**: Forgetting or misapplying the piecewise definition of the absolute value function ($|f(x)| = \{f(x) \text{ if } f(x) \geq 0, -f(x) \text{ if } f(x) < 0\}$). It's crucial to remember that $|f(x)|$ is not necessarily equivalent to $f(x)$.

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1.4.2: Homework

Note

The following exercises are meant as hand-in homework (if assigned by your instructor). Typed-in answers here will not be saved or scored.

Basic Skills

Exercise 1

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Exercise 2

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Exercise 3

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Exercise 4

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Exercise 10

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Exercise 11

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Exercise 13

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Exercise 14

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Exercise 30

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Challenge Problems

Exercise 31

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1.4.3: Additional Exercises

Reading Questions

1. What is the "Mathematical Mantra" introduced in this section?
2. According to the theorem on locally equivalent functions, if $f(x) = g(x)$ for all $x \neq a$ in an open interval containing a, what can be said about $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$?
3. If direct substitution into $\lim_{x\to a} \frac{N(x)}{D(x)}$ results in the indeterminate form $\frac{0}{0}$, what general strategy does this section suggest?
4. Name three algebraic techniques discussed in this section for manipulating a function to evaluate a limit when direct substitution fails.
5. When evaluating a limit by factoring and canceling, why is it permissible to cancel the common factor even though it makes the original function undefined at that point?
6. When using conjugate multiplication to evaluate a limit, which parts of the resulting expression is it generally advisable to distribute, and which parts to leave undistributed? Why?
7. What is the piecewise definition of the absolute value function $|u|$?
8. How does the piecewise definition of the absolute value function help in evaluating limits like $\lim_{x\to(-5/3)^-} \frac{|3x+5|}{6x^2+7x-5}$?
9. In Example 5, $\lim_{x\to 0}\left(\frac{1}{x} + \frac{5}{x(x-5)}\right)$, why can't the Sum Law for limits be applied directly? What algebraic step is performed first?
10. When working with limits, until what point in the calculation should the "lim" notation be written?

Homework

In exercises 1 - 10, use the Direct Substitution Property to show that each limit leads to the indeterminate form $0/0$. Then, evaluate the limit analytically.

1) $\lim_{x\to 4} \frac{x^2-16}{x-4}$

Answer

$$\text{When } x = 4, \quad \frac{x^2-16}{x-4} = \frac{16-16}{4-4} = \frac{0}{0};$$

$$\text{then, } \lim_{x\to 4} \frac{x^2-16}{x-4} = \lim_{x\to 4} \frac{(x+4)(x-4)}{x-4} = \lim_{x\to 4}(x+4) = 4+4 = 8$$

2) $\lim_{x\to 2} \frac{x-2}{x^2-2x}$

3) $\lim_{x\to 6} \frac{3x-18}{2x-12}$

Answer

$$\text{When } x = 6, \quad \frac{3x-18}{2x-12} = \frac{18-18}{12-12} = \frac{0}{0};$$

$$\text{then, } \lim_{x\to 6} \frac{3x-18}{2x-12} = \lim_{x\to 6} \frac{3(x-6)}{2(x-6)} = \lim_{x\to 6} \frac{3}{2} = \frac{3}{2}$$

4) $\lim_{h\to 0} \frac{(1+h)^2-1}{h}$

5) $\lim_{t\to 9} \frac{t-9}{\sqrt{t}-3}$

Answer

When $t = 9$, $\dfrac{t-9}{\sqrt{t}-3} = \dfrac{9-9}{3-3} = \dfrac{0}{0}$;

then, $\lim\limits_{t\to 9} \dfrac{t-9}{\sqrt{t}-3} = \lim\limits_{t\to 9} \dfrac{t-9}{\sqrt{t}-3}\dfrac{\sqrt{t}+3}{\sqrt{t}+3} = \lim\limits_{t\to 9} \dfrac{(t-9)(\sqrt{t}+3)}{t-9} = \lim\limits_{t\to 9}(\sqrt{t}+3) = \sqrt{9}+3 = 6$

6) $\lim\limits_{h\to 0} \dfrac{\dfrac{1}{a+h} - \dfrac{1}{a}}{h}$, where a is a real-valued constant

7) $\lim\limits_{\theta\to\pi} \dfrac{\sin\theta}{\tan\theta}$

Answer

When $\theta = \pi$, $\dfrac{\sin\theta}{\tan\theta} = \dfrac{\sin\pi}{\tan\pi} = \dfrac{0}{0}$;

then, $\lim\limits_{\theta\to\pi} \dfrac{\sin\theta}{\tan\theta} = \lim\limits_{\theta\to\pi} \dfrac{\sin\theta}{\frac{\sin\theta}{\cos\theta}} = \lim\limits_{\theta\to\pi} \cos\theta = \cos\pi = -1$

8) $\lim\limits_{x\to 1} \dfrac{x^3-1}{x^2-1}$

9) $\lim\limits_{x\to 1/2} \dfrac{2x^2+3x-2}{2x-1}$

Answer

When $x = 1/2$, $\dfrac{2x^2+3x-2}{2x-1} = \dfrac{\frac{1}{2}+\frac{3}{2}-2}{1-1} = \dfrac{0}{0}$;

then, $\lim\limits_{x\to 1/2} \dfrac{2x^2+3x-2}{2x-1} = \lim\limits_{x\to 1/2} \dfrac{(2x-1)(x+2)}{2x-1} = \lim\limits_{x\to 1/2}(x+2) = \dfrac{1}{2}+2 = \dfrac{5}{2}$

10) $\lim\limits_{x\to -3} \dfrac{\sqrt{x+4}-1}{x+3}$

In exercises 11 - 14, use the Direct Substitution Property to obtain an undefined expression, simplify the function, and determine the limit.

11) $\lim\limits_{x\to -2^-} \dfrac{2x^2+7x-4}{x^2+x-2}$

Answer

$-\infty$

12) $\lim\limits_{x\to -2^+} \dfrac{2x^2+7x-4}{x^2+x-2}$

13) $\lim\limits_{x\to 1^-} \dfrac{2x^2+7x-4}{x^2+x-2}$

Answer

$-\infty$

14) $\lim\limits_{x\to 1^+} \dfrac{2x^2+7x-4}{x^2+x-2}$

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1.5: Limits at Infinity

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

In this section, we continue delving into limits by considering limits as x tends to $\pm\infty$. These are known collectively as **limits at infinity**.

Finite Limits at Infinity and Horizontal Asymptotes

Recall that $\lim\limits_{x\to a} f(x) = L$ means $f(x)$ becomes arbitrarily close to L as long as x is sufficiently close to a. We can extend this idea to limits at infinity. For example, consider the function $f(x) = 2 + \frac{1}{x}$. As can be seen graphically in Figure $1.5.1$ and numerically in Table $1.5.1$, as the values of x get larger, the values of $f(x)$ approach 2. We say the limit as x approaches ∞ of $f(x)$ is 2 and write $\lim\limits_{x\to\infty} f(x) = 2$. Similarly, for $x < 0$, as the values $|x|$ get larger, the values of $f(x)$ approach 2. We say the limit as x approaches $-\infty$ of $f(x)$ is 2 and write $\lim\limits_{x\to-\infty} f(x) = 2$.

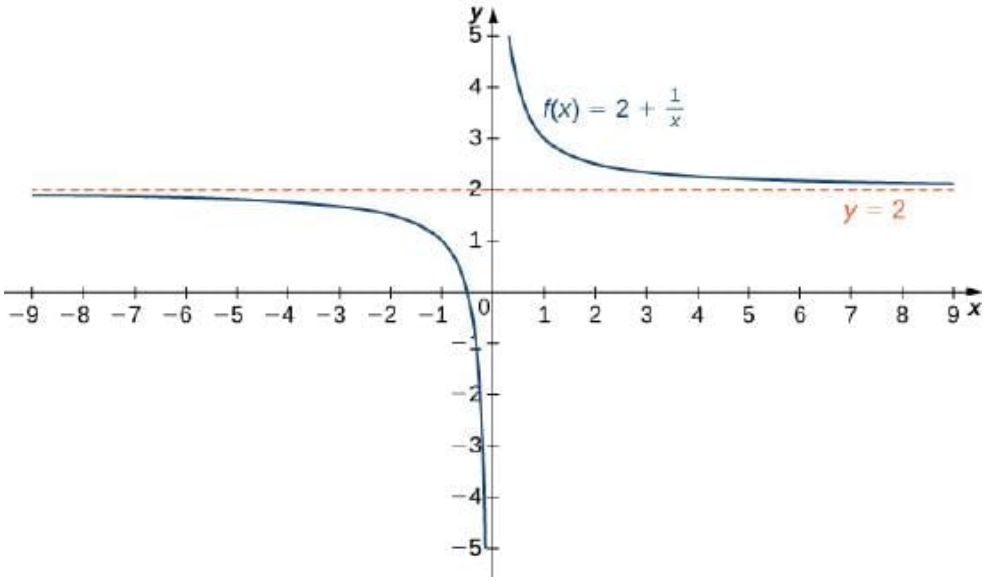

Figure $1.5.1$**:** *The function approaches the asymptote* $y = 2$ *as* x *approaches* $\pm\infty$.

Table $1.5.1$***:*** *Values of a function* f *as* $x \to \pm\infty$

x	10	100	1000	10000
$2+\frac{1}{x}$	2.1	2.01	2.001	2.0001
x	-10	-100	-1000	-10000
$2+\frac{1}{x}$	1.9	1.99	1.999	1.9999

More generally, for any function f, we say the limit as $x \to \infty$ of $f(x)$ is L if $f(x)$ becomes arbitrarily close to L as long as x is sufficiently large. In that case, we write $\lim\limits_{x\to\infty} f(x) = L$. Similarly, we say the limit as $x \to -\infty$ of $f(x)$ is L if $f(x)$ becomes arbitrarily close to L as long as $x < 0$ and $|x|$ is sufficiently large. In that case, we write $\lim\limits_{x\to-\infty} f(x) = L$. We now look at the definition of a function having a limit at infinity.

Definition: Finite Limit at Infinity (Informal)

If the values of $f(x)$ become arbitrarily close to the finite value L as x becomes sufficiently large, we say the function f has a **finite limit at infinity** and write

$$\lim_{x\to\infty} f(x) = L.$$

If the values of $f(x)$ become arbitrarily close to the finite value L for $x < 0$ as $|x|$ becomes sufficiently large, we say that the function f has a finite limit at negative infinity and write

$$\lim_{x\to-\infty} f(x) = L.$$

If the values $f(x)$ are getting arbitrarily close to some finite value L as $x \to \infty$ or $x \to -\infty$, the graph of f approaches the line $y = L$. In that case, the line $y = L$ is a horizontal asymptote of f (Figure $1.5.2$). For example, for the function $f(x) = \dfrac{1}{x}$, since $\lim\limits_{x\to\infty} f(x) = 0$, the line $y = 0$ is a horizontal asymptote of $f(x) = \dfrac{1}{x}$.

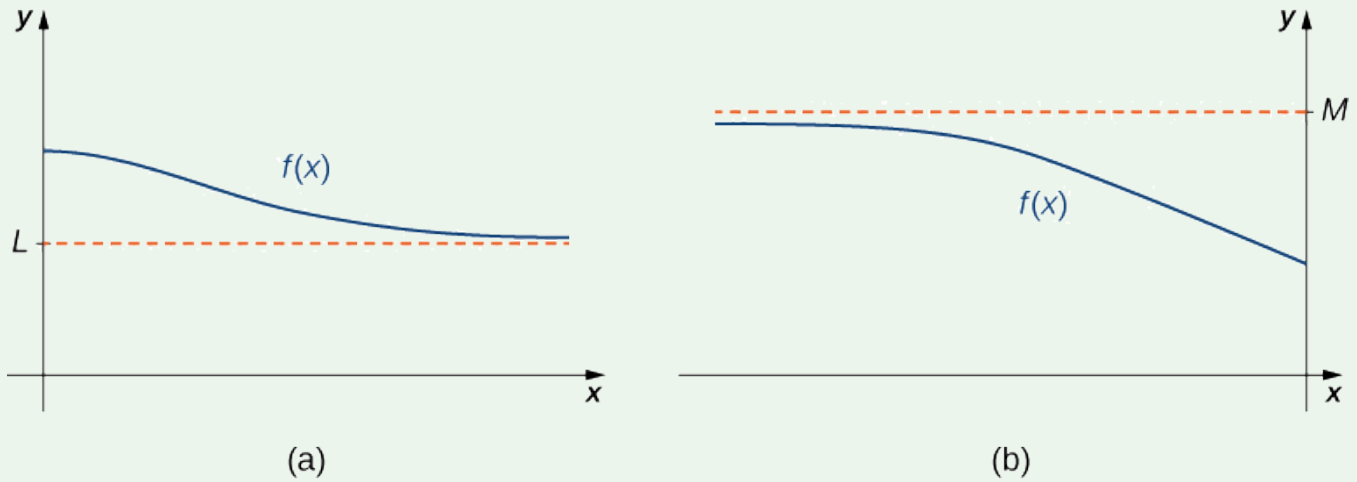

Figure $1.5.2$**:** *(a) As* $x \to \infty$*, the values of* f *are getting arbitrarily close to* L*. The line* $y = L$ *is a horizontal asymptote of* f*. (b) As* $x \to -\infty$*, the values of* f *are getting arbitrarily close to* M*. The line* $y = M$ *is a horizontal asymptote of* f*.*

Definition: Horizontal Asymptote

If $\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$, we say the line $y = L$ is a **horizontal asymptote** of f.

Interactive Element: Approaching a Horizontal Asymptote

The following Interactive Element might help you visualize how a function approaches a horizontal asymptote as $x \to \pm\infty$.

Interact: Drag the orange point along the curve toward large positive x, then toward the left branch for large negative x.

Observation: As $|x|$ grows, $f(x) = 2 + \frac{1}{x}$ gets arbitrarily close to 2—from above on the right, from below on the left—yet never equals it. The line $y = 2$ is a horizontal asymptote.

As a very important aside, we know from Algebra that a function cannot intersect a vertical asymptote; however, a function *can* cross a horizontal asymptote. In fact, a function may cross a horizontal asymptote an unlimited number of times. For example, the function $f(x) = \dfrac{\cos x}{x} + 1$ shown in Figure $1.5.3$ intersects the horizontal asymptote $y = 1$ an infinite number of times as it oscillates around the asymptote with ever-decreasing amplitude.

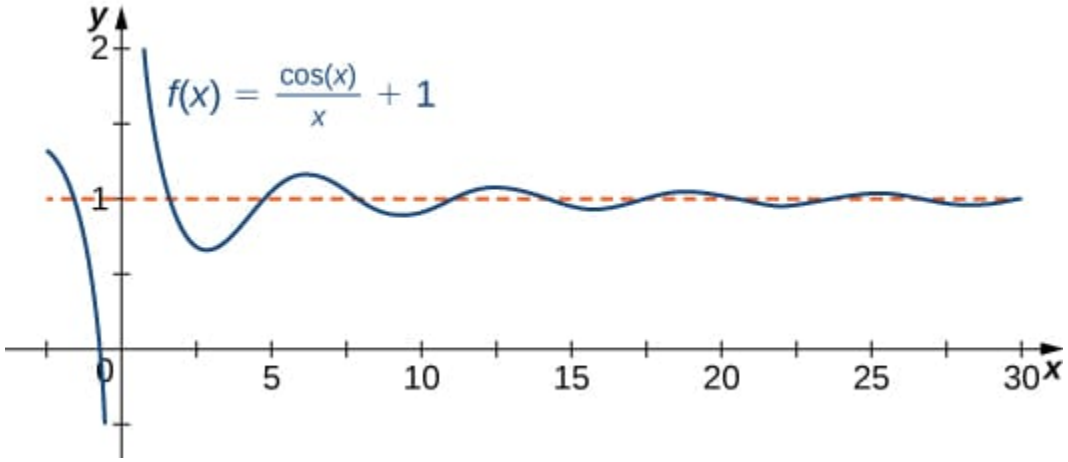

Figure $1.5.3$**:** *The graph of* $f(x) = \frac{\cos(x)}{x} + 1$ *crosses its horizontal asymptote* $y = 1$ *an infinite number of times.*

Revisiting the Limit Laws

Now that we have an informal definition of a finite limit at infinity, it's time we begin evaluating some limits *without* resorting to building tables of values. Before we do, however, we need some theory. The Limit Laws introduced earlier in this chapter still apply as long as the limit values are finite. These are restated here for your convenience.

Theorem: Limit Laws for Limits at Infinity

Let $f(x)$ and $g(x)$ be defined for all $x > a$, where a is a real number. Assume that L and M are real numbers such that $\lim\limits_{x\to\infty} f(x) = L$ and $\lim\limits_{x\to\infty} g(x) = M$. Let c be a constant. Then, each of the following statements holds:

- **Sum and Difference Laws for Limits**:
$$\lim_{x\to\infty} (f(x) \pm g(x)) = \lim_{x\to\infty} f(x) \pm \lim_{x\to\infty} g(x) = L \pm M$$
- **Constant Multiple Law for Limits**:
$$\lim_{x\to\infty} cf(x) = c \cdot \lim_{x\to\infty} f(x) = cL$$
- **Product Law for Limits**:
$$\lim_{x\to\infty} (f(x) \cdot g(x)) = \lim_{x\to\infty} f(x) \cdot \lim_{x\to\infty} g(x) = L \cdot M$$
- **Quotient Law for Limits**:

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to\infty}f(x)}{\lim\limits_{x\to\infty}g(x)}=\frac{L}{M}$$

for $M\neq 0$.

- **Power Law for Limits**:

$$\lim_{x\to\infty}\big(f(x)\big)^n=\big(\lim_{x\to\infty}f(x)\big)^n=L^n$$

for every positive integer n.

- **Root Law for Limits**:

$$\lim_{x\to\infty}\sqrt[n]{f(x)}=\sqrt[n]{\lim_{x\to\infty}f(x)}=\sqrt[n]{L}$$

for all L if n is odd and for $L\geq 0$ if n is even.

Each of these Limit Laws can be adjusted for $x\to-\infty$ as long as $f(x)$ and $g(x)$ are defined for all $x<a$, where a is a real number.

We add to our theory arsenal a small, but powerful lemma.

Lemma

$$\lim_{x\to\infty}x=\infty,$$
$$\lim_{x\to-\infty}x=-\infty,$$

and

$$\lim_{x\to\pm\infty}\frac{1}{x}=0.$$

While the results of this lemma are intuitively obvious, we will still prove them once we reach the precise definition of a limit later in this chapter. For now, this lemma allows us to prove the following theorem.

Theorem

If $p>0$ is a rational number, then

$$\lim_{x\to\infty}\frac{1}{x^p}=0.$$

If $p>0$ is a rational number such that x^p is defined for all x, then

$$\lim_{x\to-\infty}\frac{1}{x^p}=0.$$

Proof available at the end of this section.

Evaluating Finite Limits at Infinity

Example 1.5.1: Computing a Finite Limit at Infinity

Evaluate

$$\lim_{x\to\infty}\left(5-\frac{2}{x^2}\right)$$

and

$$\lim_{x\to-\infty}\left(5-\frac{2}{x^2}\right).$$

Determine the horizontal asymptote(s) for this function.

Solution available at the end of this section.

Checkpoint 1.5.1

Evaluate $\lim_{x\to-\infty}\left(3+\frac{4}{x}\right)$ and $\lim_{x\to\infty}\left(3+\frac{4}{x}\right)$. Determine the horizontal asymptotes of $f(x)=3+\frac{4}{x}$, if any.

Answer available at the end of this section.

In Example 1.5.1, there was no need to perform any initial manipulations using Arithmetic, Algebra, or Trigonometry; however, finite limits at infinity are often disguised as indeterminate forms (e.g., where the limit approaches $\frac{\infty}{\infty}$ or $\infty-\infty$). For these cases, we should keep in mind the Mathematical Mantra when evaluating finite limits at infinity.

Example 1.5.2: Evaluating Finite Limits at Infinity Having Indeterminate Forms

Evaluate each of the following limits.

a. $\lim_{x\to\infty}\dfrac{\pi x^5+x^3-1}{3x^5+x+2}$

b. $\lim_{x\to\infty}\dfrac{\sqrt{1-x+x^2}}{3-x}$

c. $\lim_{x\to-\infty}\dfrac{\sqrt{1-x+x^2}}{3-x}$

d. $\lim_{x\to-\infty}\left(\sqrt{1-x+9x^2}+3x\right)$

Solutions available at the end of this section.

As we have encountered in previous sections, conceptual investigations of limits based on our prerequisite knowledge of base graphs is a valid technique. The following example (and subsequent theorem) showcases such an investigation.

Example 1.5.3: Evaluating Finite Limits at Infinity Conceptually

Evaluate $\lim_{x\to\infty}\tan^{-1}(x)$ and $\lim_{x\to-\infty}\tan^{-1}(x)$. Determine the horizontal asymptote(s) for $\arctan(x)$.

Solution available at the end of this section.

This limit can be useful enough that it is good to formalize it in a theorem along with some other common limits.

Theorem

$$\begin{aligned}\lim_{x\to\infty}\tan^{-1}(x) &= \frac{\pi}{2}\\ \lim_{x\to-\infty}\tan^{-1}(x) &= -\frac{\pi}{2}\\ \lim_{x\to\infty}b^{-x} &= 0 \qquad \text{where } b>1.\end{aligned}$$

Revisiting the Squeeze Theorem

We have already seen that a lot of our original theory of finite limits at finite numbers can be applied to finite limits at infinity. We round out this collection of common theorems by revisiting the Squeeze Theorem.

Theorem: The Squeeze Theorem for Limits at Infinity

Let $f(x), g(x)$, and $h(x)$ be defined for all $x > a$, where a is a real number. If

$$f(x) \leq g(x) \leq h(x)$$

for all $x > a$ and

$$\lim_{x\to\infty} f(x) = L = \lim_{x\to\infty} h(x)$$

where L is a real number, then $\lim_{x\to\infty} g(x) = L$.

Likewise, let $f(x), g(x)$, and $h(x)$ be defined for all $x < a$, where a is a real number. If

$$f(x) \leq g(x) \leq h(x)$$

for all $x < a$ and

$$\lim_{x\to-\infty} f(x) = L = \lim_{x\to-\infty} h(x)$$

where L is a real number, then $\lim_{x\to-\infty} g(x) = L$.

Example 1.5.4

Evaluate $\lim_{x\to\infty} \frac{\sin(x)}{x}$ and $\lim_{x\to-\infty} \frac{\sin(x)}{x}$. Determine the horizontal asymptote(s) for $f(x) = \frac{\sin(x)}{x}$.

Solution available at the end of this section.

Interactive Element: The Squeeze Theorem at Infinity

The following Interactive Element might help you visualize how the Squeeze Theorem forces $\frac{\sin x}{x} \to 0$ as $x \to \pm\infty$.

Interact: Drag the orange point along $y = \frac{\sin x}{x}$ toward larger $|x|$.

Observation: The bounds $y = \pm\frac{1}{|x|}$ squeeze toward 0, forcing $\frac{\sin x}{x} \to 0$—even though $\frac{\sin x}{x}$ keeps oscillating and crossing the x-axis forever.

Infinite Limits at Infinity

Sometimes the values of a function f become arbitrarily large as $x \to \infty$ (or as $x \to -\infty$). In this case, we write $\lim_{x\to\infty} f(x) = \infty$ (or $\lim_{x\to-\infty} f(x) = \infty$). On the other hand, if the values of f are negative but become arbitrarily large in magnitude as $x \to \infty$ (or as $x \to -\infty$), we write $\lim_{x\to\infty} f(x) = -\infty$ (or $\lim_{x\to-\infty} f(x) = -\infty$).

For example, consider the function $f(x) = x^3$. As seen in Table 1.5.2 and Figure 1.5.8, as $x \to \infty$ the values $f(x)$ become arbitrarily large. Therefore, $\lim_{x\to\infty} x^3 = \infty$. On the other hand, as $x \to -\infty$, the values of $f(x) = x^3$ are negative but become arbitrarily large in magnitude. Consequently, $\lim_{x\to-\infty} x^3 = -\infty$.

Table $1.5.2$***:*** *Values of a power function as* $x \to \pm\infty$

x	10	20	50	100	1000
x^3	1000	8000	125000	1000000	1000000000
x	-10	-20	-50	-100	-1000
x^3	-1000	-8000	-125000	-1000000	-1000000000

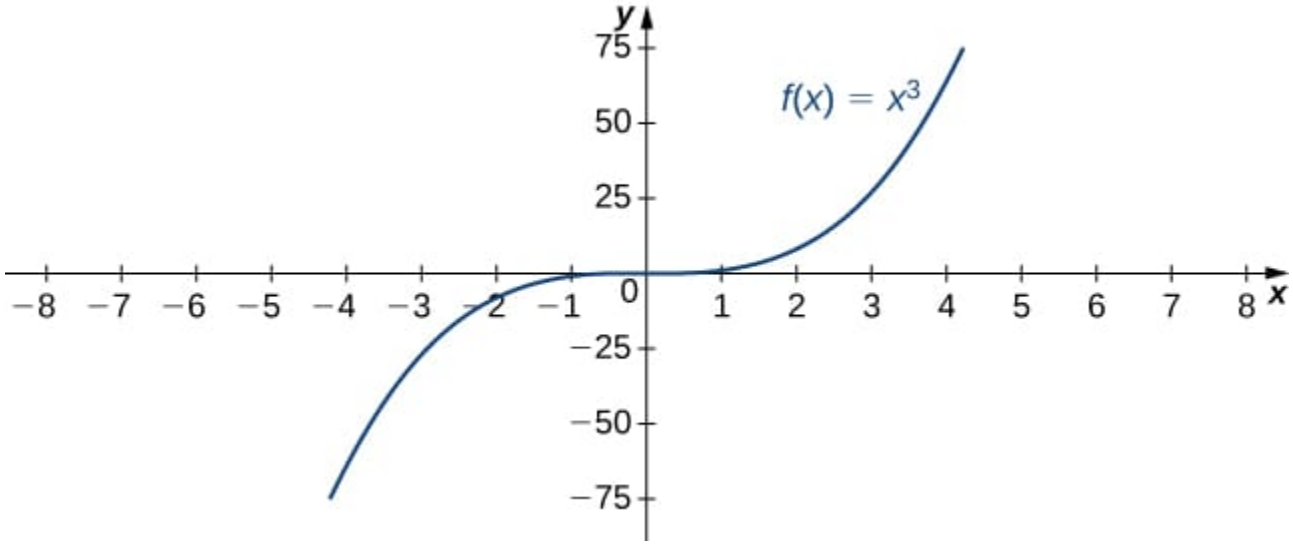

Figure $1.5.8$**:** *For this function, the functional values approach* $\pm$*infinity as* $x \to \pm\infty$.

Definition: Infinite Limit at Infinity (Informal)

We say a function f has an infinite limit at infinity and write

$$\lim_{x\to\infty} f(x) = \infty.$$

if $f(x)$ becomes arbitrarily large for x sufficiently large. We say a function has a negative infinite limit at infinity and write

$$\lim_{x\to\infty} f(x) = -\infty.$$

if $f(x) < 0$ and $|f(x)|$ becomes arbitrarily large for x sufficiently large. Similarly, we can define infinite limits as $x \to -\infty$.

In general, infinite limits at infinity require fewer prerequisite manipulations to evaluate than other types of limits and require more conceptual investigation. The following theorem (which we will prove parts of in the next section) contains some common infinite limits at infinity.

Theorem

- $\lim_{x\to\infty} x = \infty$
- $\lim_{x\to-\infty} x = -\infty$
- If $p > 0$, $\lim_{x\to\infty} x^p = \infty$
- If $b > 1$, $\lim_{x\to\infty} b^x = \infty$.

Example $1.5.5$: Evaluating an Infinite Limit at Infinity

Evaluate the limit.

$$\lim_{x \to -\infty} \frac{3 + x^6}{x^4 + 8}$$

Solution available at the end of this section.

Hidden Resources: Tutoring Services

Technically, this is not a "hidden resource," per se, but the tutoring services at CRC are a bit spread out between programs. We have the Tutoring Center, Math Center at CRC, Math Center at Elk Grove, Reading and Writing Center, Accounting Lab, Embedded Tutoring (Supplemental Instruction), and NetTutor. If you're an athlete, you might have additional access to tutors through the athletics program. Needless to say, being overwhelmed by the number of overlapping tutoring services is completely understandable. Luckily, CRC has a single webpage with all of the links to these services. Check out the details on this CRC webpage.

Hidden Resources: Student Health and Wellness Services

Cosumnes River College has amazing support services for students in need of emotional and/or mental health services. Our Student Health and Wellness Services include the following:

Physical Health Services (no cost)

- Health and illness assessments
- First aid for minor injuries & illness
- Over-the-counter medications
- Party Safe Kits - Narcan, fentanyl & drink spike test strips
- Sexual health education
- Condoms (internal & external) & lubricant
- Opill (birth control)
- Emergency contraception (Plan B)
- STI testing
- Pregnancy testing
- Flu A/B & Covid testing
- Flu vaccine
- Strep throat testing
- TB risk assessments
- Health requirements for academic programs

Physical Health Services (low cost)

- Immunizations/titers
- TB skin test

Mental Health Services (no cost)

- **Togetherall:** This is a safe online space to connect anonymously with other students to get and give support.
- **Therapy:** Meet with a Los Rios Student Health and Wellness Center (SHWC) therapist or advocate. Support is available in person and virtually.
- **BetterMynd:** Online Therapy: Talk to an online licensed therapist during the day, evenings or weekends. Or call 24/7 support line: (844) BTR-MYND.

For more information, please see this page on the CRC website.

Answers

Section Preview

We now turn our attention to limits at infinity and horizontal asymptotes. We still have not introduced the precise definition of limits, so all of the content is from a conceptual standpoint.

Section Preview

- **Exponential Functions**
  - The Basics of Exponential Functions: The theorems $\lim_{x\to\infty} b^x = \infty$ (for $b > 1$) and $\lim_{x\to\infty} b^{-x} = 0$ are stated without proof, relying on prior knowledge of exponential end behavior and the role of the base b (including the $y = 0$ asymptote).
- **Trigonometry**
  - The Inverse Trigonometric Functions: Example 3 and the subsequent theorem evaluate $\tan^{-1}(x) = \arctan(x)$; students must understand the inverse tangent and its principal-value range to interpret $\lim_{x\to\pm\infty} \tan^{-1}(x) = \pm\frac{\pi}{2}$.

Section Preview

- **Simplifying Expressions**
  - Laws of Exponents — Product and Quotient Rules: Example 2a and Example 5 divide numerator and denominator by x^p, simplifying terms such as $\frac{x^3}{x^5} = \frac{1}{x^2}$ via the quotient rule so that each $\frac{1}{x^p}$ term can be sent to 0.
  - Simplifying Products of Radical Expressions: Examples 2b–d split $\sqrt{x^2(\cdots)}$ into $\sqrt{x^2}\ \sqrt{\cdots}$ using the product rule for radicals before extracting $\sqrt{x^2} = |x|$.

Section Preview

- Calculate the limit of a function as x increases or decreases without bound.
- Define a horizontal asymptote in terms of a finite limit at infinity.
- Evaluate a finite limit at infinity by initially performing algebraic manipulations.
- Conceptually investigate an infinite limit at infinity.
- Describe when the Limit Laws cannot be applied.

Theorem

Let p be a rational number. The only constraint on using the Limit Laws is that the values of the limits must be finite. Therefore, a is allowed to be $\pm\infty$. Thus,

$$\begin{aligned}
\lim_{x\to\infty} \frac{1}{x^p} &= \lim_{x\to\infty} \left(\frac{1}{x}\right)^p \\
&= \left(\lim_{x\to\infty} \frac{1}{x}\right)^p && \text{(Power and Root Limit Laws)} \\
&= 0^p && \text{(Lemma)} \\
&= 0
\end{aligned}$$

If we allow $x \to -\infty$, then additional restrictions on p must be made. Specifically, since p is a rational number, it cannot be equivalent to a simplified fraction with an even denominator. Otherwise, the even denominator would imply an even-indexed root of x, and this would result in imaginary numbers. Hence, as long as x^p is defined,

$$\lim_{x\to-\infty} \frac{1}{x^p} = 0$$

by a similar derivation.

Example $1.5.1$: Computing a Finite Limit at Infinity

Using the Limit Laws, we have

$$\lim_{x\to\infty} \left(5 - \frac{2}{x^2}\right) = \lim_{x\to\infty} 5 - 2\left(\lim_{x\to\infty} \frac{1}{x}\right)^2 = 5 - 2\cdot 0 = 5.$$

Similarly, $\lim_{x\to-\infty} \left(5 - \frac{2}{x^2}\right) = 5$. Therefore, $f(x) = 5 - \frac{2}{x^2}$ has a horizontal asymptote of $y = 5$ and f approaches this horizontal asymptote as $x \to \pm\infty$ as shown in the following graph.

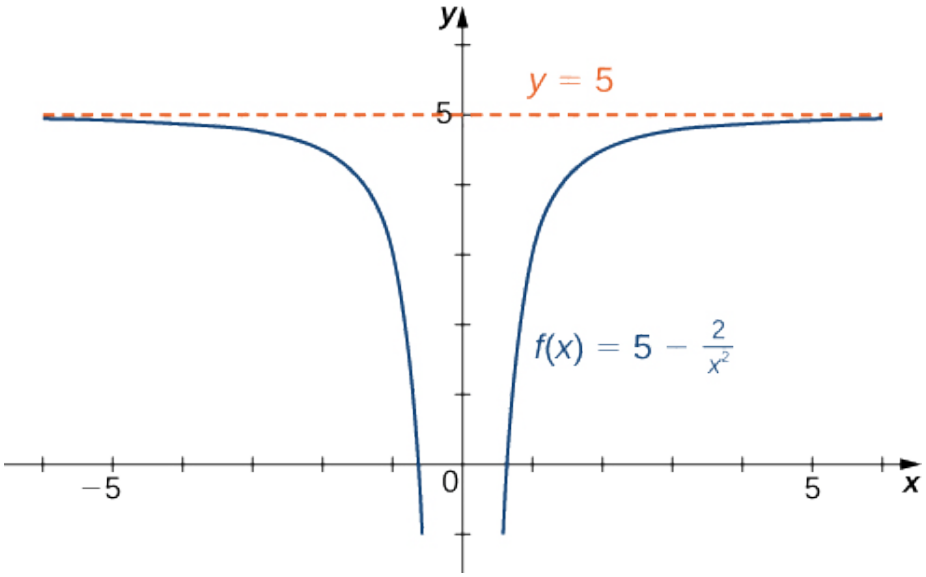

Figure 1.5.4**:** *This function approaches a horizontal asymptote as* $x \to \pm\infty$.

Checkpoint 1.5.1

Both limits are 3. The line $y = 3$ is a horizontal asymptote.

Example 1.5.2**: Evaluating Finite Limits at Infinity Having Indeterminate Forms**

a. A quick investigation of this limit shows that

$$\lim_{x\to\infty} \frac{\pi x^5 + x^3 - 1}{3x^5 + x + 2}$$

∞ ↗ ↘ ∞

Therefore, the limit is indeterminate of the form ∞/∞. What can we do? Following a creative use of the Mathematical Mantra, we could begin by dividing both numerator and denominator by x^p, where p is the largest existing power of x within the function. Doing so (and using the fact that $\lim_{x\to\infty} \frac{1}{x^p} = 0$ when $p > 0$), the limit becomes

$$\lim_{x\to\infty} \frac{\pi x^5 + x^3 - 1}{3x^5 + x + 2} = \lim_{x\to\infty} \frac{\pi + \frac{1}{x^2} - \frac{1}{x^5}}{3 + \frac{1}{x^4} + \frac{2}{x^5}} = \frac{\pi + 0 - 0}{3 + 0 + 0} = \frac{\pi}{3}$$

b. This limit is even more confounding than the previous one when you consider that, as $x \to \infty$, the numerator tends to the square root of $-\infty + \infty$ while the denominator tends to $-\infty$. There are so many issues preventing us from considering the Limit Laws. First, what is $-\infty + \infty$? Is it 0? Is it $-\infty$ (and, if so, does the world blow up because we are trying to take the square root of a negative number)? Is it $+\infty$? Second, *if* we can determine what the numerator is, will dividing by a value approaching $-\infty$ cause issues? As in the previous example, following the Mathematical Mantra and using a little Algebra at the beginning can go a long way here.

$$\begin{aligned}
\lim_{x\to\infty}\frac{\sqrt{1-x+x^2}}{3-x} &= \lim_{x\to\infty}\frac{\sqrt{x^2\left(\frac{1}{x^2}-\frac{1}{x}+1\right)}}{3-x}\\
&= \lim_{x\to\infty}\frac{\sqrt[2]{x^2}\sqrt{\frac{1}{x^2}-\frac{1}{x}+1}}{3-x}\\
&= \lim_{x\to\infty}\frac{|x|\sqrt{\frac{1}{x^2}-\frac{1}{x}+1}}{3-x} && \left(\sqrt[n]{x^n}=\begin{cases}x & \text{if } n \text{ is odd}\\ |x| & \text{if } n \text{ is even}\end{cases}\right)\\
&= \lim_{x\to\infty}\frac{x\sqrt{\frac{1}{x^2}-\frac{1}{x}+1}}{3-x}\\
&= \lim_{x\to\infty}\frac{\sqrt{\frac{1}{x^2}-\frac{1}{x}+1}}{\frac{3}{x}-1}\\
&= \frac{\sqrt{0-0+1}}{0-1} && \left(\lim_{x\to\infty}\frac{1}{x^p}=0,\ \text{when } p>0\right)\\
&= -1
\end{aligned}$$

c. This is similar to part b; however, the limit approaching $-\infty$ requires some special attention.

$$\begin{aligned}
\lim_{x\to-\infty}\frac{\sqrt{1-x+x^2}}{3-x} &= \lim_{x\to-\infty}\frac{\sqrt{x^2\left(\frac{1}{x^2}-\frac{1}{x}+1\right)}}{3-x}\\
&= \lim_{x\to-\infty}\frac{\sqrt[2]{x^2}\sqrt{\frac{1}{x^2}-\frac{1}{x}+1}}{3-x}\\
&= \lim_{x\to-\infty}\frac{|x|\sqrt{\frac{1}{x^2}-\frac{1}{x}+1}}{3-x} && \left(\sqrt[n]{x^n}=\begin{cases}x & \text{if } n \text{ is odd}\\ |x| & \text{if } n \text{ is even}\end{cases}\right)\\
&= \lim_{x\to-\infty}\frac{-x\sqrt{\frac{1}{x^2}-\frac{1}{x}+1}}{3-x} && (x\to-\infty, \text{ so } x<0.\ \text{Thus, } |x|=-x.)\\
&= \lim_{x\to-\infty}\frac{-\sqrt{\frac{1}{x^2}-\frac{1}{x}+1}}{\frac{3}{x}-1}\\
&= \frac{-\sqrt{0-0+1}}{0-1} && \left(\lim_{x\to\infty}\frac{1}{x^p}=0,\ \text{when } p>0\right)\\
&= 1
\end{aligned}$$

d. It is tempting with this final example to think it has the form $\infty+\infty$; however, x is approaching **negative** ∞. As such, the second term within the limit is approaching $-\infty$, and this limit is genuinely of the indeterminate form $\infty-\infty$. Again, the Mathematical Mantra comes to the rescue!

$$\begin{aligned}
\lim_{x\to-\infty}\left(\sqrt{1-x+9x^2}+3x\right) &= \lim_{x\to-\infty}\left(\sqrt{1-x+9x^2}+3x\right)\left(\frac{\sqrt{1-x+9x^2}-3x}{\sqrt{1-x+9x^2}-3x}\right)\\
&= \lim_{x\to-\infty}\frac{1-x+9x^2-9x^2}{\sqrt{1-x+9x^2}-3x}\\
&= \lim_{x\to-\infty}\frac{1-x}{\sqrt{1-x+9x^2}-3x}\\
&= \lim_{x\to-\infty}\frac{1-x}{\sqrt[3]{x^2}\sqrt{\frac{1}{x^2}-\frac{1}{x}+9}-3x}\\
&= \lim_{x\to-\infty}\frac{1-x}{|x|\sqrt{\frac{1}{x^2}-\frac{1}{x}+9}-3x} && \left(\sqrt[n]{x^n}=\begin{cases}x & \text{if } n \text{ is odd}\\ |x| & \text{if } n \text{ is even}\end{cases}\right)\\
&= \lim_{x\to-\infty}\frac{1-x}{-x\sqrt{\frac{1}{x^2}-\frac{1}{x}+9}-3x} && \left(\lim_{x\to\infty}\frac{1}{x^p}=0,\ \text{when } p>0\right)\\
&= \lim_{x\to-\infty}\frac{\frac{1}{x}-1}{-\sqrt{\frac{1}{x^2}-\frac{1}{x}+9}-3}\\
&= \frac{0-1}{-\sqrt{0-0+9}-3} && \left(\lim_{x\to\infty}\frac{1}{x^p}=0,\ \text{when } p>0\right)\\
&= \frac{-1}{-6}\\
&= \frac{1}{6}
\end{aligned}$$

Example 1.5.3**: Evaluating Finite Limits at Infinity Conceptually**

To evaluate $\lim_{x\to\infty}\tan^{-1}(x)$ and $\lim_{x\to-\infty}\tan^{-1}(x)$, we first consider the graph of $y=\tan(x)$ over the interval $(-\frac{\pi}{2},\frac{\pi}{2})$ as shown in the following graph.

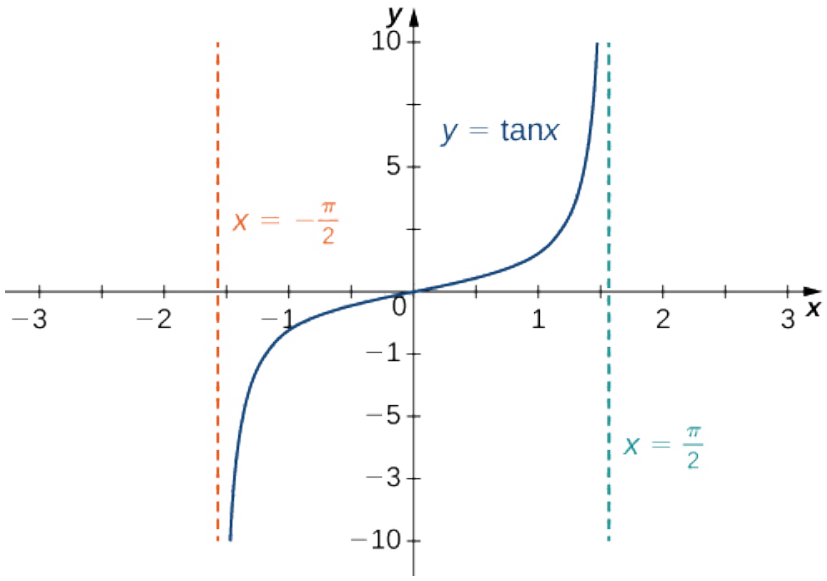

Figure 1.5.5**:** *The graph of $y=\tan x$ has vertical asymptotes at $x=\pm\frac{\pi}{2}$*

Since

$$\lim_{x\to\frac{\pi}{2}^-}\tan(x)=\infty,$$

it follows that

$$\lim_{x\to\infty}\tan^{-1}(x)=\frac{\pi}{2}.$$

Similarly, since

$$\lim_{x\to-\frac{\pi}{2}^+}\tan(x)=-\infty,$$

it follows that

$$\lim_{x\to-\infty}\tan^{-1}(x)=-\frac{\pi}{2}.$$

As a result, $y=\frac{\pi}{2}$ and $y=-\frac{\pi}{2}$ are horizontal asymptotes of $f(x)=\tan^{-1}(x)$ as shown in the following graph.

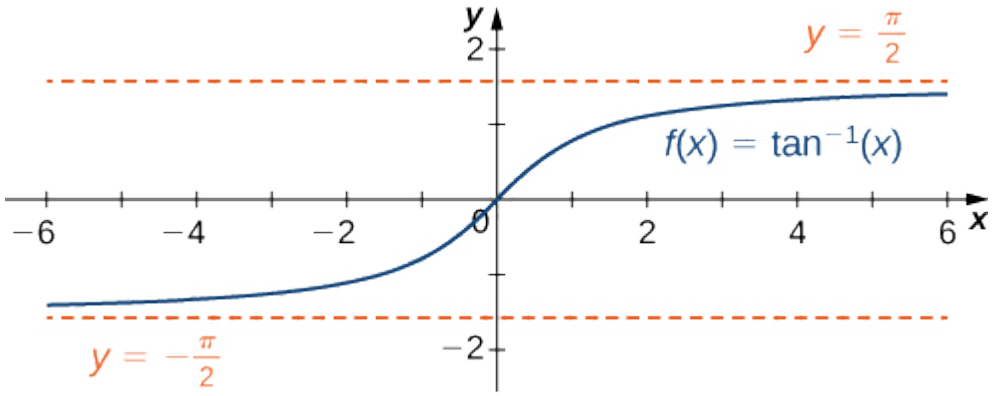

Figure 1.5.6**:** *This function has two horizontal asymptotes.*

Example 1.5.4

Since $-1\le\sin x\le 1$ for all x, we have

$$\frac{-1}{x}\le\frac{\sin x}{x}\le\frac{1}{x}$$

for all $x\neq 0$. Also, since

$$\lim_{x\to\infty}\frac{-1}{x}=0=\lim_{x\to\infty}\frac{1}{x},$$

we can apply the Squeeze Theorem to conclude that

$$\lim_{x\to\infty}\frac{\sin x}{x}=0.$$

Similarly,

$$\lim_{x\to-\infty}\frac{\sin x}{x}=0.$$

Thus, $f(x)=\frac{\sin x}{x}$ has a horizontal asymptote of $y=0$ and $f(x)$ approaches this horizontal asymptote as $x\to\pm\infty$ as shown in the following graph.

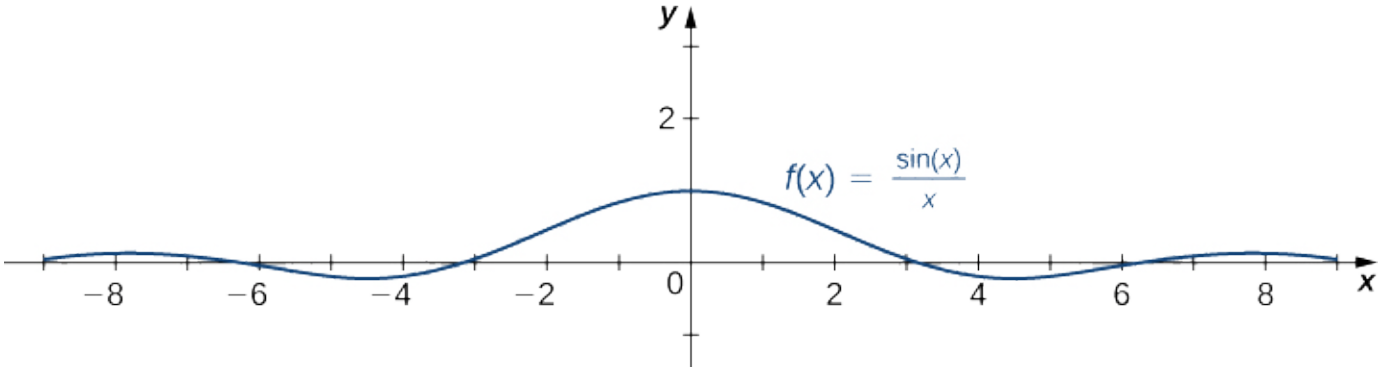

Figure 1.5.7**:** *This function crosses its horizontal asymptote multiple times.*

Example 1.5.5**: Evaluating an Infinite Limit at Infinity**

$$\lim_{x\to-\infty}\frac{3+x^6}{x^4+8}=\lim_{x\to-\infty}\frac{\frac{3}{x^6}+1}{\frac{1}{x^2}+\frac{8}{x^6}}\qquad(\text{dividing numerator and denominator by } x^6)$$

Since $3/x^6\to 0$ as $x\to-\infty$, we know the numerator is approaching 1. Moreover, as $x\to-\infty$,

$$\frac{1}{x^2}+\frac{8}{x^6}\to 0^+ + 0^+\to 0^+$$

Hence,

$$\lim_{x\to-\infty}\frac{3+x^6}{x^4+8}=\infty.$$

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1.5.1: Resources and Key Concepts

Resources

Videos

- Finite limits at infinity
  - Limits at Infinity - Evaluating Finite Limits at Infinity

Key Concepts

Definitions

- **Limits at Infinity (Informal)**:
  - **Finite Limit at Infinity**: If the values of $f(x)$ become arbitrarily close to the finite value L as x becomes sufficiently large, we say the function f has a finite limit at infinity and write $\lim_{x\to\infty} f(x) = L$.
  - **Finite Limit at Negative Infinity**: If the values of $f(x)$ become arbitrarily close to the finite value L for $x < 0$ as $|x|$ becomes sufficiently large, we say the function f has a finite limit at negative infinity and write $\lim_{x\to-\infty} f(x) = L$.
- **Horizontal Asymptote**: If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, where L is a finite number, we say the line $y = L$ is a horizontal asymptote of f.
- **Infinite Limit at Infinity (Informal)**:
  - We say a function f has an infinite limit at infinity and write $\lim_{x\to\infty} f(x) = \infty$ if $f(x)$ becomes arbitrarily large for x sufficiently large.
  - We say a function f has a negative infinite limit at infinity and write $\lim_{x\to\infty} f(x) = -\infty$ if $f(x) < 0$ and $|f(x)|$ becomes arbitrarily large for x sufficiently large. (Similar definitions apply for $x \to -\infty$).

Theorems

- **Limit Laws for Limits at Infinity**: Let $f(x)$ and $g(x)$ be defined for all $x > a$, where a is a real number. Assume that L and M are real numbers such that $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to\infty} g(x) = M$. Let c be a constant. Then the Sum, Difference, Constant Multiple, Product, Quotient (for $M \neq 0$), Power, and Root Laws hold for limits at infinity. (These can be adjusted for $x \to -\infty$).
- **Theorem (Limits of Powers of $1/x$)**: If $p > 0$ is a rational number, then $\lim_{x\to\infty} \frac{1}{x^p} = 0$. If $p > 0$ is a rational number such that x^p is defined for all x, then $\lim_{x\to-\infty} \frac{1}{x^p} = 0$.
- **Theorem (Common Limits at Infinity for Inverse Tangent and Hyperbolic Tangent)**:
  - $\lim_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2}$
  - $\lim_{x\to-\infty} \tan^{-1}(x) = -\frac{\pi}{2}$
  - $\lim_{x\to\infty} b^{-x} = 0$ where $b > 1$.
- **Theorem: The Squeeze Theorem for Limits at Infinity**:
  - Let $f(x), g(x)$, and $h(x)$ be defined for all $x > a$. If $f(x) \leq g(x) \leq h(x)$ for all $x > a$ and $\lim_{x\to\infty} f(x) = L = \lim_{x\to\infty} h(x)$, then $\lim_{x\to\infty} g(x) = L$.
  - (A similar statement holds for $x \to -\infty$).
- **Theorem (Common Infinite Limits at Infinity)**:
  - $\lim_{x\to\infty} x = \infty$
  - $\lim_{x\to-\infty} x = -\infty$
  - If $p > 0$, $\lim_{x\to\infty} x^p = \infty$
  - If $b > 1$, $\lim_{x\to\infty} b^x = \infty$

Common Mistakes

- **Assuming a function cannot cross its horizontal asymptote**: While a function cannot intersect a vertical asymptote, it can cross a horizontal asymptote, even an unlimited number of times (as shown with $f(x) = \frac{\cos x}{x} + 1$ in Figure 3).
- **Incorrectly handling $\sqrt{x^2}$ when $x \to -\infty$**: Forgetting that $\sqrt{x^2} = |x|$, and thus if $x < 0$ (as when $x \to -\infty$), then $\sqrt{x^2} = -x$. This is crucial in Example 2c.
- **Misinterpreting indeterminate forms like $\infty - \infty$**: Assuming that $\infty - \infty = 0$. This is an indeterminate form, and algebraic manipulation (like using conjugates in Example 2d) is required.

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1.5.2: Homework

Note

The following exercises are meant as hand-in homework (if assigned by your instructor). Typed-in answers here will not be saved or scored.

Basic Skills

Exercise 1

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Exercise 2

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Exercise 3

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Exercise 4

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Exercise 5

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Exercise 6

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Exercise 10

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Exercise 11

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Exercise 13

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Exercise 14

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Exercise 16

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Exercise 17

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Exercise 21

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Exercise 24

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Exercise 25

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Exercise 26

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Exercise 27

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Exercise 28

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Exercise 29

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Synthesis Questions

Exercise 30

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Exercise 31

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Exercise 32

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Exercise 33

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Exercise 34

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1.5.3: Additional Exercises

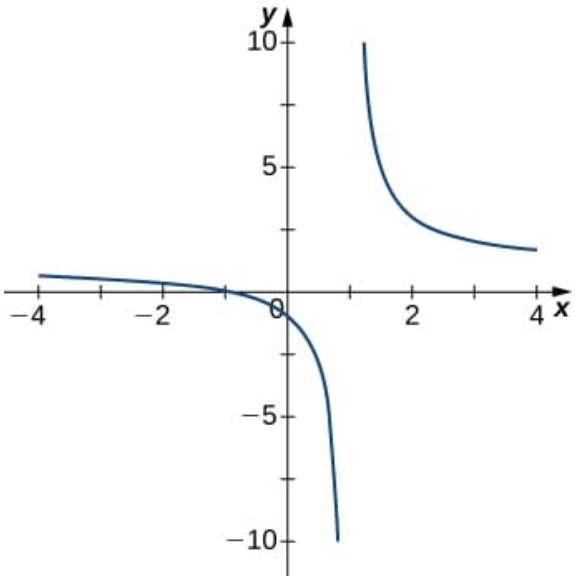
Reading Questions

1. What does it mean for a function $f(x)$ to have a finite limit L as x approaches infinity?
2. How is a horizontal asymptote $y = L$ defined in terms of limits at infinity?
3. Can a function's graph cross its horizontal asymptote? If so, provide an example or describe one from the text.
4. State the Product Law for Limits at Infinity. What conditions must be met for it to apply?
5. What is the value of $\lim_{x\to\infty} \frac{1}{x^p}$ if $p > 0$? Does this also hold for $x \to -\infty$? Are there any additional conditions for the $x \to -\infty$ case?
6. When evaluating the limit of a rational function at infinity, what is the common first algebraic step described in Example 2a?
7. In Example 2b, when simplifying $\lim_{x\to\infty} \frac{\sqrt{1-x+x^2}}{3-x}$, how is $\sqrt{x^2(\frac{1}{x^2} - \frac{1}{x} + 1)}$ simplified when $x \to \infty$?
8. How does the simplification in the previous question change if $x \to -\infty$, as in Example 2c? What does $\sqrt{x^2}$ become?
9. What indeterminate form is encountered in Example 2d, $\lim_{x\to-\infty} (\sqrt{1-x+9x^2} + 3x)$? What algebraic technique is used to resolve it?
10. What are the values of $\lim_{x\to\infty} \tan^{-1}(x)$ and $\lim_{x\to-\infty} \tan^{-1}(x)$?
11. State the Squeeze Theorem for Limits at Infinity.
12. What does it mean if $\lim_{x\to\infty} f(x) = \infty$?
13. When evaluating an infinite limit at infinity for a rational function, such as $\lim_{x\to-\infty} \frac{3+x^6}{x^4+8}$ in Example 1.5.5, what was the general strategy used after dividing by the highest power of x from the original expression?

Homework

For exercises 1 - 5, examine the graphs. Identify where the vertical asymptotes are located and use limit notation in your answer.

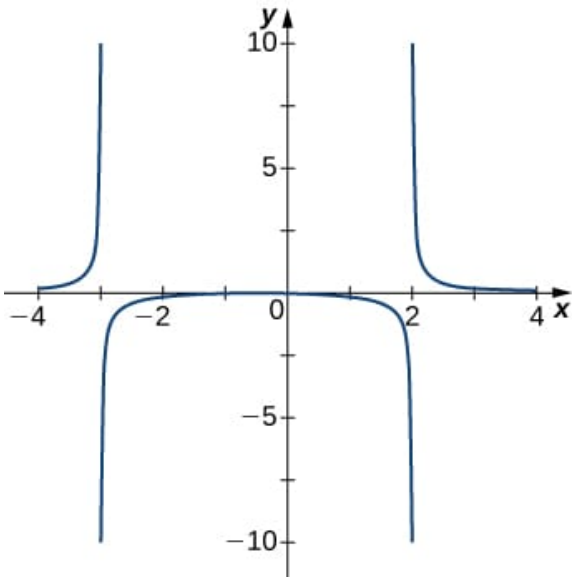
1)



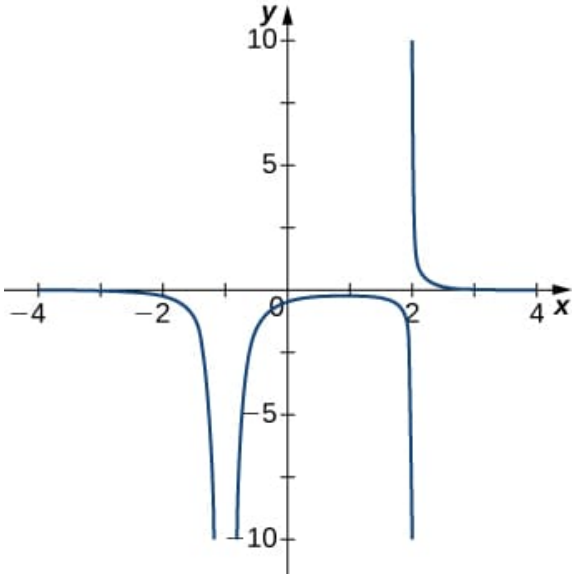
Answer

$x = 1$

2)



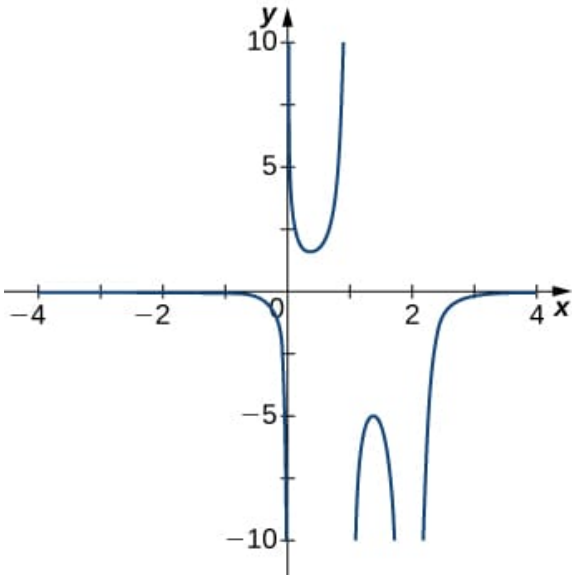
3)



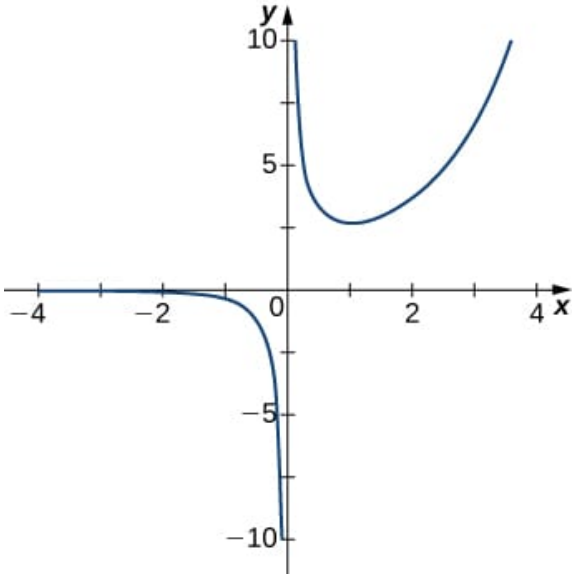
Answer

$x = -1, \;\; x = 2$

4)



5)



Answer

$x = 0$

For the functions $f(x)$ in exercises 6 - 10, determine whether there is an asymptote at $x = a$. Justify your answer using limits and without using graphing technology.

6) $f(x) = \dfrac{x+1}{x^2+5x+4}, \quad a = -1$

7) $f(x) = \dfrac{x}{x-2}, \quad a = 2$

Answer

Yes, there is a vertical asymptote at $x = 2$.

8) $f(x) = (x+2)^{3/2}, \quad a = -2$

9) $f(x) = (x-1)^{-1/3}, \quad a = 1$

Answer

Yes, there is vertical asymptote at $x = 1$.

10) $f(x) = 1 + x^{-2/5}, \quad a = 1$

In exercises 11 - 20, evaluate the limit.

11) $\lim\limits_{x\to\infty} \dfrac{1}{3x+6}$

Answer

$$\lim_{x\to\infty} \frac{1}{3x+6} = 0$$

12) $\lim\limits_{x\to\infty} \dfrac{2x-5}{4x}$

13) $\lim\limits_{x\to\infty} \dfrac{x^2-2x+5}{x+2}$

Answer

$$\lim_{x\to\infty} \frac{x^2-2x+5}{x+2} = \infty$$

14) $\lim\limits_{x\to-\infty} \dfrac{3x^3-2x}{x^2+2x+8}$

15) $\lim\limits_{x\to-\infty} \dfrac{x^4-4x^3+1}{2-2x^2-7x^4}$

Answer

$$\lim_{x\to-\infty} \frac{x^4-4x^3+1}{2-2x^2-7x^4} = -\frac{1}{7}$$

16) $\lim\limits_{x\to\infty} \dfrac{3x}{\sqrt{x^2+1}}$

17) $\lim\limits_{x\to-\infty} \dfrac{\sqrt{4x^2-1}}{x+2}$

Answer

$$\lim_{x\to-\infty} \frac{\sqrt{4x^2-1}}{x+2} = -2$$

18) $\lim\limits_{x\to\infty} \dfrac{4x}{\sqrt{x^2-1}}$

19) $\lim\limits_{x\to-\infty} \dfrac{4x}{\sqrt{x^2-1}}$

Answer

$$\lim_{x\to-\infty} \frac{4x}{\sqrt{x^2-1}} = -4$$

20) $\lim\limits_{x\to\infty} \dfrac{2\sqrt{x}}{x-\sqrt{x}+1}$

For exercises 21 - 25, find the horizontal and vertical asymptotes.

21) $f(x) = x - \dfrac{9}{x}$

Answer

Horizontal: none,

Vertical: $x = 0$

22) $f(x) = \dfrac{1}{1 - x^2}$

23) $f(x) = \dfrac{x^3}{4 - x^2}$

Answer

Horizontal: none,
Vertical: $x = \pm 2$

24) $f(x) = \dfrac{x^2 + 3}{x^2 + 1}$

25) $f(x) = \sin(x)\sin(2x)$

Answer

Horizontal: none,
Vertical: none

26) $f(x) = \cos x + \cos(3x) + \cos(5x)$

27) $f(x) = \dfrac{x\sin(x)}{x^2 - 1}$

Answer

Horizontal: $y = 0$,
Vertical: $x = \pm 1$

28) $f(x) = \dfrac{x}{\sin(x)}$

29) $f(x) = \dfrac{1}{x^3 + x^2}$

Answer

Horizontal: $y = 0$,
Vertical: $x = 0$ and $x = -1$

30) $f(x) = \dfrac{1}{x - 1} - 2x$

31) $f(x) = \dfrac{x^3 + 1}{x^3 - 1}$

Answer

Horizontal: $y = 1$,
Vertical: $x = 1$

32) $f(x) = \dfrac{\sin x + \cos x}{\sin x - \cos x}$

33) $f(x) = x - \sin x$

Answer

Horizontal: none,
Vertical: none

34) $f(x) = \dfrac{1}{x} - \sqrt{x}$

For exercises 35 - 38, construct a function $f(x)$ that has the given asymptotes.

35) $x = 1$ and $y = 2$

Answer

Answers will vary, for example: $y = \dfrac{2x}{x-1}$

36) $x = 1$ and $y = 0$

37) $y = 4, \quad x = -1$

Answer

Answers will vary, for example: $y = \dfrac{4x}{x+1}$

38) $x = 0$

In exercises 39 - 43, graph the function on a graphing calculator on the window $x = [-5, 5]$ and estimate the horizontal asymptote or limit. Then, calculate the actual horizontal asymptote or limit.

39) $f(x) = \dfrac{1}{x+10}$

Answer

$\lim\limits_{x\to\infty} \dfrac{1}{x+10} = 0$ so f has a horizontal asymptote of $y = 0$.

40) $f(x) = \dfrac{x+1}{x^2+7x+6}$

41) $\lim\limits_{x\to-\infty} x^2 + 10x + 25$

Answer

$\lim\limits_{x\to-\infty} x^2 + 10x + 25 = \infty$

42) $\lim\limits_{x\to-\infty} \dfrac{x+2}{x^2+7x+6}$

43) $\lim\limits_{x\to\infty} \dfrac{3x+2}{x+5}$

Answer

$\lim\limits_{x\to\infty} \dfrac{3x+2}{x+5} = 3$ so this function has a horizontal asymptote of $y = 3$.

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1.6: The Precise Definition of a Limit

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

By now, you have progressed from the very informal definition of a limit in the introduction of this chapter to the intuitive understanding of a limit. At this point, you should have a solid intuitive sense of what the limit of a function means and how you can find it. This section converts this intuitive idea of a limit into a formal definition using precise mathematical language. The formal definition of a limit is one of the most challenging definitions you will encounter in Calculus; however, it is well worth any effort you make to reconcile it with your intuitive notion of a limit. Understanding this definition is the key that opens the door to a better understanding of calculus.

Foreword: A Lesson in Patience

I have written this section to step you *slowly* through the concept, visualization, and understanding of the precise definition of a finite limit at a finite number (in addition to the other limits presented at the end of this section). Reading everything presented here, in the order it is presented, might seem exhaustingly lengthy; however, the presentation in this section is meticulously crafted to get you to a level of understanding necessary to succeed in calculus.

Quantifying Closeness

Before stating the formal definition of a limit, we must introduce a few preliminary ideas. Recall that the distance between two points a and b on a number line is given by $|a-b|$. From this point forward, we will constantly be using this interpretation.

- The statement $|f(x)-L|<\epsilon$ may be interpreted as: The distance between $f(x)$ and L is less than ϵ.
- The statement $0<|x-a|<\delta$ may be interpreted as: $x\neq a$ and the distance between x and a is less than δ.

In calculus, δ and ϵ (the Greek lowercase letters delta and epsilon, respectively) are inevitably going to be *very* small positive numbers. In fact, in most cases, they will be infinitesimally small (i.e., both $\to 0^+$).

It is also essential to look at the following equivalences for absolute value:

- The statement $|f(x)-L|<\epsilon$ is equivalent to the statement $L-\epsilon<f(x)<L+\epsilon$.
- The statement $0<|x-a|<\delta$ is equivalent to the statement $a-\delta<x<a+\delta$, where $x\neq a$.

With these concepts of distances between values in hand, we need to introduce some language commonly used to discuss the precise definition of a limit.

Definition: δ-Neighborhood Centered at a

A **δ-neighborhood centered at** a is defined to be the interval $(a-\delta, a) \cup (a, a+\delta)$ on the x-axis.

Definition: ϵ-Neighborhood Centered at L

An **ϵ-neighborhood centered at** L is defined to be the interval $(L-\epsilon, L+\epsilon)$ on the y-axis.

Do you see a difference between these two definitions? A δ-neighborhood *never* includes its center ($x = a$), while an ϵ-neighborhood *always* includes its center ($y = L$). Some people (myself included) call a δ-neighborhood a **punctured neighborhood**.

Interactive Element: Visualizing δ- and ϵ-Neighborhoods

The following Interactive Element might help you visualize critical characteristics of δ- and ϵ-neighborhoods.

Interact: Move the sliders to adjust $x = a$, δ, $y = L$, and ϵ.

Observation: Neighborhoods are symmetric about their centers, and the δ-neighborhood *never* includes its center.

The next example illustrates how this idea of "closeness" might be linked to the idea of a limit.

Example 1.6.1

Using the concepts of "closeness" presented here (specifically, δ- and ϵ-neighborhoods), create a possible interpretation of

$$\lim_{x \to 1} (5x + 1) = 6.$$

Solution available at the end of this section.

Take note of the imprecise language in the solution of Example 1.6.1—this is purposeful. We have not yet developed a definition of the limit, so we are trying to guess our way to a strong, infallible definition.

The ϵ Gauntlet

Our interpretation of the meaning of the limit from Example 1.6.1,

$$\lim_{x \to 1} (5x + 1) = 6,$$

was, "all x-values in a δ-neighborhood centered at $x = 1$ lead to function values in an ϵ-neighborhood centered at $y = 6$." Not only is this imprecise, but it is actually *backward* . For now, let's focus on thinking of our interpretation *in the correct order*.

When handed the limit

$$\lim_{x \to 1} (5x + 1) = 6,$$

you are essentially being told that, no matter what sized ϵ-neighborhood someone creates about $y = 6$, you *should* be able to find a small enough δ-neighborhood about $x = 1$ so that the function, $f(x) = 5x + 1$, is very close to the supposed limit 6 whenever x is close to 1. What do we mean by "close?" Hopefully, as long as our value of x is within the δ-neighborhood, then the corresponding value of $f(x)$ should be within the *given* ϵ-neighborhood.

At this point, it's understandable if you had to read those last two paragraphs a few times. If you did, then good for you—you are practicing becoming a better critical thinker! Let's reward ourselves with a visualization that will help us grasp this idea of "ϵ before δ."

Interactive Element: Getting Used to ϵ Before δ

Interact: Grab the point on the x-axis at $x = 1 + \delta$ and move it right and left.

Observation: We have been given a *static* ϵ-neighborhood about $y = 6$. Our job is to find a small enough δ-neighborhood about $x = 1$ so that $f(x)$ is within the ϵ-neighborhood for all values of x in our δ-neighborhood.

The act of being handed a static ϵ and challenged to find a corresponding δ is called the ϵ **gauntlet**. I call it this because it's as if someone is throwing a gauntlet at your feet and challenging you to find an appropriate δ-neighborhood—if you dare!

Example 1.6.2: Finding δ Given ϵ

Consider the limit

$$\lim_{x \to 1} (5x + 1) = 6.$$

Find an appropriate δ so that all the function values, $f(x) = 5x + 1$, lie within $\epsilon = 0.001$ units of $y = 6$ whenever x is within δ units of $x = 1$.

Solution available at the end of this section.

The Precise Definition of a Finite Limit at a Finite Number

With these clarifications, we can state the formal $\epsilon - \delta$ **definition of a finite limit at a finite number**.

Definition: Finite Limit at a Finite Number (Precise Definition)

Let $f(x)$ be defined for all $x \neq a$ over an open interval containing a, and let L be a real number. We say

$$\lim_{x \to a} f(x) = L$$

if for every $\epsilon > 0$, there exists a $\delta > 0$, such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

Stated with symbolic logic,

$$\lim_{x \to a} f(x) = L$$

means

$$\forall \epsilon > 0, \quad \exists \delta > 0 \ni$$
$$0 < |x - a| < \delta \implies |f(x) - L| < \epsilon.$$

You are probably not used to symbolic logic, so let's define those symbols.

Symbol	Meaning
$\forall$	" for all" or "for each" or "for every"
$\exists$	"there exists" or "there is"
	"implies" or "then"
$\ni$	"such that" or "so that"

Caution

The symbol $\ni$ is used frequently in symbolic logic for "such that"; however, many mathematicians do not use it for that purpose. Check with your instructor to see what logic symbol they prefer (many avoid the confusion entirely and use "s.t.").

Critically Analyzing the Structure of the Precise Definition of a Finite Limit at a Finite Number

This definition may seem rather complex from a mathematical point of view. Still, it becomes easier to understand if we break it down phrase by phrase. The statement itself involves something called a **universal quantifier** (for every $\epsilon > 0$, or $\forall \epsilon > 0$), an **existential quantifier** (there exists a $\delta > 0$, or $\exists \delta > 0$), and, last, a **conditional statement** (if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$). Let's look at Table $1.6.1$, which breaks down the definition and translates each part.

Table $1.6.1$

Definition	Symbolic Logic	Translation
For every $\epsilon > 0$,	$\forall \epsilon > 0$	For every positive distance ϵ from L,
there exists a $\delta > 0$,	$\exists \delta > 0$	there is a positive distance δ from a,
such that	$\ni$	such that
if $0 < \|x - a\| < \delta$, then $\|f(x) - L\| < \epsilon$.	$0 < \|x - a\| < \delta \implies \|f(x) - L\| < \epsilon$	if x is *within* δ units of a and $x \neq a$, then $f(x)$ is *within* ϵ units of L.

Graphically Interpreting the Precise Definition of a Finite Limit at a Finite Number

We can get a better handle on this definition by looking at the definition geometrically. Figure $1.6.1$ shows possible values of δ for various choices of $\epsilon > 0$ for a given function $f(x)$, a number a, and a limit L at a. Notice that as we choose smaller values of ϵ (the distance between the function and the limit), we can always find a δ small enough so that if we have chosen an x value within δ of a, then the value of $f(x)$ is within ϵ of the limit L.

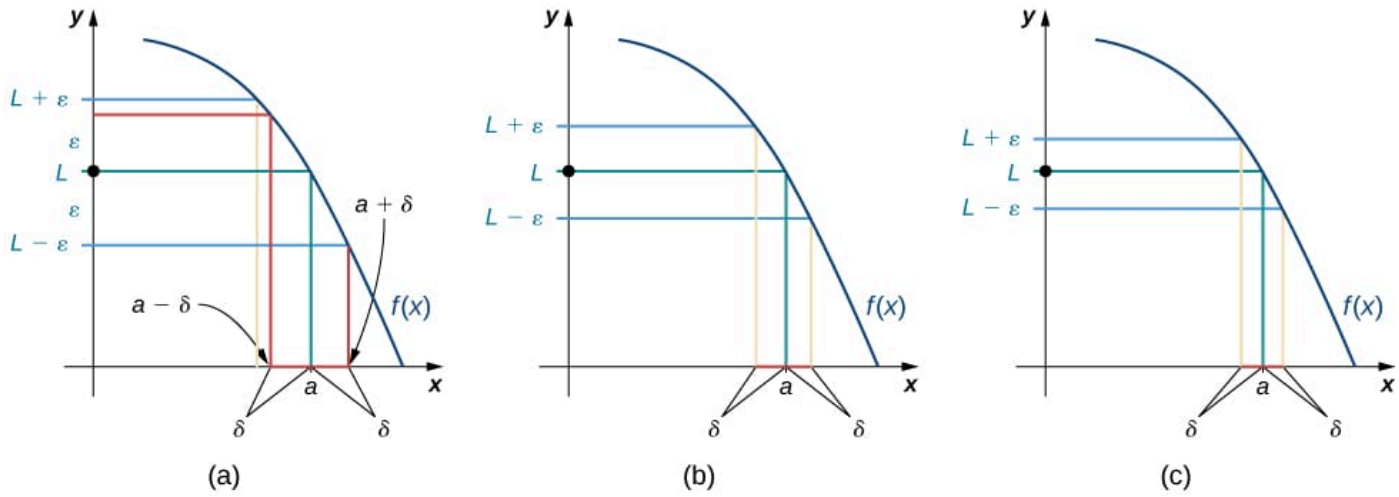


Figure $1.6.1$**:** *These graphs show possible values of δ, given successively smaller choices of ϵ.*

Interactive Element: For Every ϵ, a δ Exists

The following Interactive Element might help you visualize how a smaller ϵ always forces a smaller δ, yet a workable δ always exists.

Interact: Drag the bright point on the vertical line above $(4, 2)$ up or down to change ϵ.

Observation: Every time you shrink ϵ, a smaller δ-neighborhood still works — the curve over that δ-strip stays inside the ϵ-band. A δ always exists.

The Importance of Scratch Work

Your instructor has likely presented clean, concise, and well-polished proofs in class. Did she do so out of thin air? Is she just a genius that can throw together any logical argument without flaw? Nope!

Before delving into examples, it is *incredibly* important to understand that mathematicians (and, therefore, you) do not magically write clean proofs from step 1 to step 100 without doing a bit (or quite a bit) of scratch work. **Scratch work** is where the *real* magic happens. It is where you allow yourself to think creatively, experiment with options, and discover pathways to the proof without fear of judgment and nasty comments from someone looking over your shoulder. On the other hand, the proof is a cleaned-up version of the scratch work with logically deductive steps and *no flaws*.

To understand how scratch work is done (and *why* we need it), you need first to have a solid understanding of the structure of a theorem. Theorems come in many forms, but the form that is relevant to our current task is the "If..., then..." statement. An "If..., then..." statement is a theorem (or conjecture) that is either

a. written as "If P is true, then Q is true," where P and Q are simple statements, or
b. is written in a form that can be rewritten as the form above.

For an example of each, consider the following:

a. "If it's raining, then it's cloudy."
b. "It's only raining when it is cloudy."

These two conjectures are the same; however, in mathematics, it's often easier to rewrite conjectures (or theorems) in the first, "If..., then..." format.

The "If" part of a theorem is known as the **antecedent**, and the "then" part is called the **consequent**. In the example above, the antecedent is "it's raining," and the consequent is "it's cloudy." These parts of a theorem will be important as we move forward.

Once we have rewritten the theorem in "If..., then..." format, it can be visualized as a 1000-piece puzzle in a box. Conceptually, the theorem can be thought of as follows:

> **If** you can fit all the pieces together correctly, **then** the result will look like the picture on the box.

For reference, the antecedent is "you can fit all the pieces of the puzzle together correctly." The consequent is "the result will precisely match the picture on the box."

The scratch work in the puzzle-building process is constantly referencing the picture on the box (the consequent) to see what you want to end up with. You have a blue corner piece in your hand, and looking at the box, you see the only corner that is blue is the upper-right corner. So you place it there. You notice on the box that a creepy clown with red hair is not too far down from that corner (important: all clowns are creepy). Therefore, you start gathering all the puzzle pieces that look to have red hair. And on the process goes.

This is akin to working backward or reverse engineering, the puzzle-solving process, and this is precisely what we do in scratch work.

> Scratch work often starts by assuming we have arrived at the consequent. We then work backward to see how we got to the consequent. We hope this will eventually, with some possible conditions, allow us to work back to the antecedent.

We ask ourselves, "If I arrived at the picture on the box, what must be true about the last piece I fit into the puzzle? And what does that mean about the piece I fit in before that?" And so on.

In each proof throughout this section, I will demonstrate what good scratch work in mathematics looks like. Please pay close

attention to that *and* the proof structure.

Proving Finite Limits at Finite Numbers (symmetric δ-neighborhood)

We will take our time going through this first example. Still, as we progress, the amount of "discussion" within examples will decrease.

Example $1.6.3$: Proving a Statement about the Limit of a Linear Function

Prove

$$\lim_{x\to 1}(5x+1)=6.$$

Solution available at the end of this section.

Before moving on to the following example, let's take some lessons from Example $1.6.3$.

1. We always need scratch work when attempting a proof. Not doing so is akin to not looking at the cover of the puzzle box before attempting to piece a puzzle together.
2. The scratch work starts with the consequent and works backward to (hopefully) arrive at the antecedent.
3. We have no control over what ϵ they hand us, but δ will almost always be a function of ϵ (the single exception is when proving the limit of a constant function).
4. The structure of our proof will be closely aligned with the language of the precise definition of a limit with the exception that we will not be saying, "there exists a $\delta>0$," but rather, "let $\delta=\ldots$" because we found a workable δ in our scratch work.

We streamline our language and work as we progress into the next example. Additionally, we write our scratch work off to the side so that the reader can immediately see our proof (the scratch work is provided for you, the student, to demonstrate what proper scratch work looks like).

Example $1.6.4$: Proving a Statement about the Limit of a Nonlinear Function (symmetric δ)

Prove

$$\lim_{x\to -2/3}(9x^2+12x-7)=-11.$$

Solution available at the end of this section.

Let's add to our list of lessons by looking at the previous example.

5. We have *some* flexibility in the language used during the proof. You don't always have to write, "$\forall\epsilon>0$." Instead, you could use language.

Checkpoint $1.6.4$

Prove

$$\lim_{x\to 2}(3x-2)=4$$

Hint available at the end of this section.

Proving Finite Limits at Finite Numbers (non-symmetric δ-neighborhoods)

In Examples $1.6.3$ and $1.6.4$, the proofs were fairly straightforward, since the δ-neighborhoods were symmetric about $x=a$.

However, what happens if the δ-neighborhood is not symmetric about $x = a$? What does that question even mean?

Consider the function

$$f(x) = x^2$$

as $x \to 2$. From Figure 1.6.2, we can see that, for a given $\epsilon > 0$, we create an interval of sufficient x-values on the x-axis whose function values land in the interval $(4 - \epsilon, 4 + \epsilon)$.

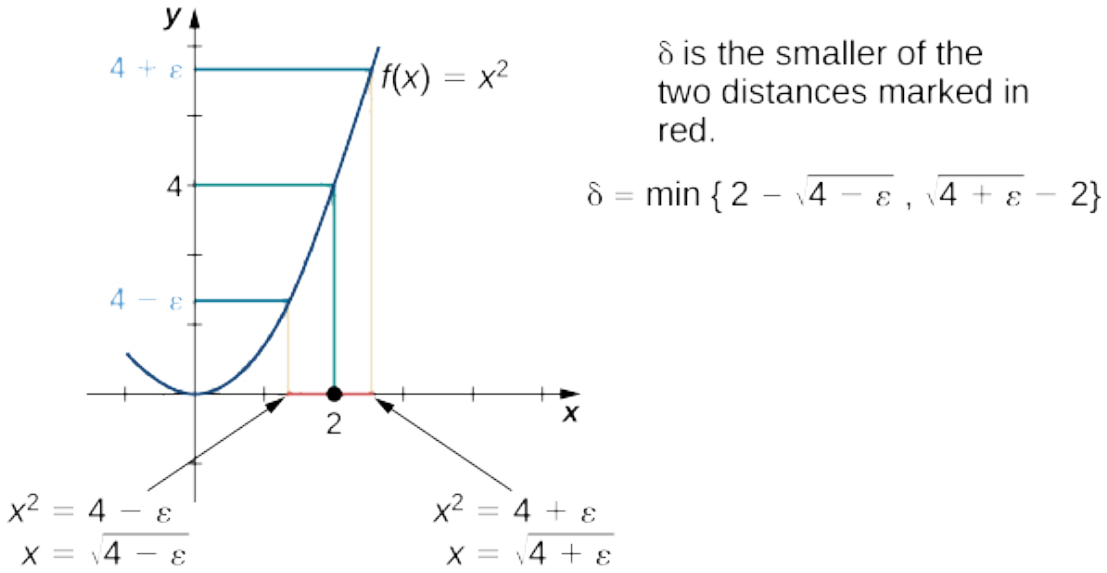


Figure 1.6.2**:** *This graph shows how we find δ geometrically for a given ϵ given $f(x) = x^2$.*

The size of the interval of acceptable values *to the left* of $x = 2$ looks to be slightly greater than the size of the interval of acceptable values to the right of $x = 2$. If we let δ_L be the size of the interval to the left and δ_R be the size of the interval to the right, then as long as $x \in (2 - \delta_L, 2) \cup (2, 2 + \delta_R)$, we know $f(x) = x^2 \in (4 - \epsilon, 4 + \epsilon)$.

However, δ- and ϵ- neighborhoods are defined to be *symmetric* about their centers. We have an issue because it looks as though our δ-neighborhood is not symmetric about $x = 2$ (as an aside, the ϵ-neighborhoods will *always* be symmetric about their centers, so you don't have to worry about those). What do we do?

If given two choices for δ, always take the smaller. For example, if we performed the computations given $\epsilon = 2$ (a gratuitously large ϵ-neighborhood), we would find $\delta_L \approx 0.586$ and $\delta_R \approx 0.449$. This means that for $x \in (1.414, 2) \cup (2, 2.449)$, $f(x) \in (2, 6)$.

If all of the x-values in the interval $(1.414, 2) \cup (2, 2.449)$ have this property, then all of the x-values in a smaller subinterval of $(1.414, 2) \cup (2, 2.449)$ must also share this same property. Therefore, if we "tighten up" the interval to be symmetric about $x = 2$ using the minimum of δ_L and δ_R, the resulting set of x-values should still map to function values within the given ϵ-neighborhood about $y = 4$.

$$\delta_L \approx 2 - 1.414 = 0.586 \quad \text{and} \quad \delta_R \approx 2.449 - 2 = 0.449.$$

Hence, we choose δ to be

$$\min\{\delta_L, \delta_R\} \approx \min\{0.586, 0.449\} = 0.449.$$

Interactive Element: Non-Symmetric δ-Neighborhoods

The following Interactive Element might help you visualize why δ_L and δ_R are usually different, and why we keep the smaller one.

Interact: Drag the bright point on the vertical line above $(2, 4)$ up or down to change ϵ.

Observation: The left and right half-widths respond by different amounts — the curve is steeper on the right — so $\delta = \min\{\delta_L, \delta_R\}$ is always the smaller of the two. In this case, δ_L is the distance from $x = 2$ to the *gray* vertical line and δ_R is the distance between $x = 2$ and the rightmost red vertical line. Since $\delta_R < \delta_L$, we choose $\delta = \delta_R$. This creates the symmetric δ-neighborhood centered at $x = 2$ represented by the red vertical lines.

Example 1.6.5: Determining an Appropriately-Sized δ-Neighborhood Geometrically

While investigating the following limit, someone states that you couldn't possibly find an appropriate δ-neighborhood given $\epsilon = 0.5$. With the use of graphing technology, prove them wrong.

$$\lim_{x \to -2} \left(e^{x+2} + 3 \sin\left(\frac{\pi}{12} x \right) + 2 \right) = 1.5$$

Solution available at the end of this section.

The geometric investigation we just performed provides invaluable insight into the formal definition of the limit. However, an algebraic approach is the primary tool used to prove statements about limits. In Example 1.6.6, we take on a purely algebraic approach to prove a limit involving non-symmetric values of δ.

Example 1.6.6: Proving a Statement about the Limit of a Nonlinear Function (non-symmetric δ)

Prove that

$$\lim_{x \to -1} (x^2 - 2x + 3) = 6.$$

Solution available at the end of this section.

Checkpoint 1.6.6

Prove that $\lim_{x \to 1} x^2 = 1$.

Answer available at the end of this section.

You will find that, in general, the more complex a function, the more likely it is that the algebraic approach is the easiest to apply. The algebraic approach is also more useful in proving statements about limits.

Proving Limit Laws

We now demonstrate how to use the $\epsilon - \delta$ definition of a limit to construct a rigorous proof of one of the Limit Laws. The **Triangle Inequality** is used at a key point of the proof, so we first review this key property of absolute value.

Theorem: Triangle Inequality

If a and b are any real numbers, then $|a + b| \leq |a| + |b|$.

Proof of the Sum Law

Recall: If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then $\lim_{x \to a} (f(x) + g(x)) = L + M$.

Proof available at the end of this section.

Showing a Limit Does Not Exist

We now explore what it means for a limit not to exist. The limit $\lim_{x\to a} f(x)$ does not exist if there is no real number L for which $\lim_{x\to a} f(x) = L$. Thus, for all real numbers L, $\lim_{x\to a} f(x) \neq L$. To understand what this means, we look at each part of the definition of $\lim_{x\to a} f(x) = L$ together with its opposite. A translation of the definition is given in Table $1.6.2$.

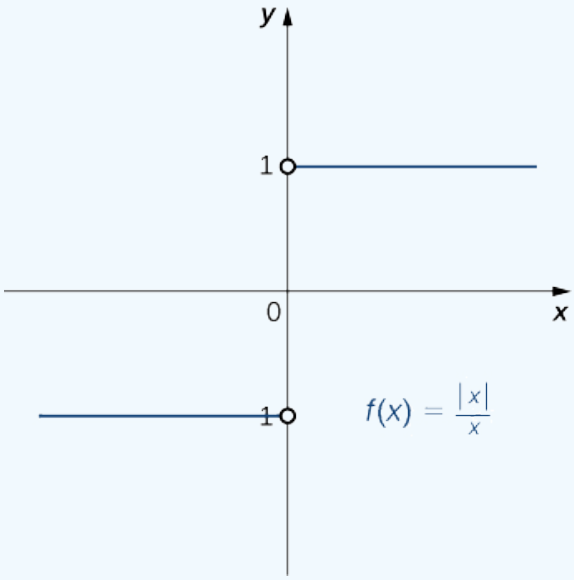
***Table* $1.6.2$*:** . Translation of the Definition of $\lim_{x\to a} f(x) = L$ and its Opposite*

Definition	Opposite
For every $\epsilon > 0$,	**There exists** $\epsilon > 0$ so that
there exists a $\delta > 0$, so that	**for every** $\delta > 0$,
if $0 < \lvert x - a\rvert < \delta$, then $\lvert f(x) - L\rvert < \epsilon$.	there is an x satisfying $0 < \lvert x - a\rvert < \delta$ so that $\lvert f(x) - L\rvert \geq \epsilon$.

Finally, we may state what it means for a limit not to exist. The limit $\lim_{x\to a} f(x)$ does not exist if for every real number L, there exists a real number $\epsilon > 0$ so that for all $\delta > 0$, there is an x satisfying $0 < \lvert x - a\rvert < \delta$, so that $\lvert f(x) - L\rvert \geq \epsilon$. Let's apply this in Example $1.6.7$ to show that a limit does not exist.

Example $1.6.7$: Showing That a Limit Does Not Exist

Show that $\lim_{x\to 0} \dfrac{|x|}{x}$ does not exist. The graph of $f(x) = |x|/x$ is shown here:



Solution available at the end of this section.

Interactive Element: A Limit That Fails to Exist

The following Interactive Element might help you visualize why $\lim_{x\to 0} \dfrac{|x|}{x}$ does not exist.

Interact: Drag the bright point up or down to move the candidate limit L; drag the point on the x-axis to shrink δ.

Observation: The $\epsilon = \frac{1}{2}$ band is too narrow to contain both branch values ($+1$ and -1) at once, so no choice of L works — and shrinking δ never removes either branch.

One-Sided Limits

Just as we first gained an intuitive understanding of limits and then moved on to a more rigorous definition, we now revisit one-sided limits. To do this, we modify the $\epsilon - \delta$ definition of a limit to give formal $\epsilon - \delta$ definitions for limits from the right and left at a point. These definitions only require slight modifications from the definition of the limit. In the definition of the limit from the right, the inequality $0 < x - a < \delta$ replaces $0 < |x - a| < \delta$, which ensures that we only consider values of x that are greater than (to the right of) a. Similarly, in the definition of the limit from the left, the inequality $-\delta < x - a < 0$ replaces $0 < |x - a| < \delta$, which ensures that we only consider values of x that are less than (to the left of) a.

Definition: One-Sided Finite Limits at Finite Numbers (Precise Definition)

Limit from the Right: Let $f(x)$ be defined over an open interval of the form (a, b) where \(a0\), there exists a $\delta > 0$, such that if \(0

Limit from the Left: Let $f(x)$ be defined over an open interval of the form (b, c) where \(b0\),there exists a $\delta > 0$ such that if \(- \delta

Example 1.6.8: Proving a Statement about a Limit From the Right

Prove that

$$\lim_{x \to 4^+} \sqrt{x - 4} = 0.$$

Solution available at the end of this section.

Checkpoint 1.6.8

Find δ corresponding to ϵ for a proof that $\lim_{x \to 1^-} \sqrt{1 - x} = 0$.

Answer available at the end of this section.

MyOpenMath Advice: Latepasses and File Uploads

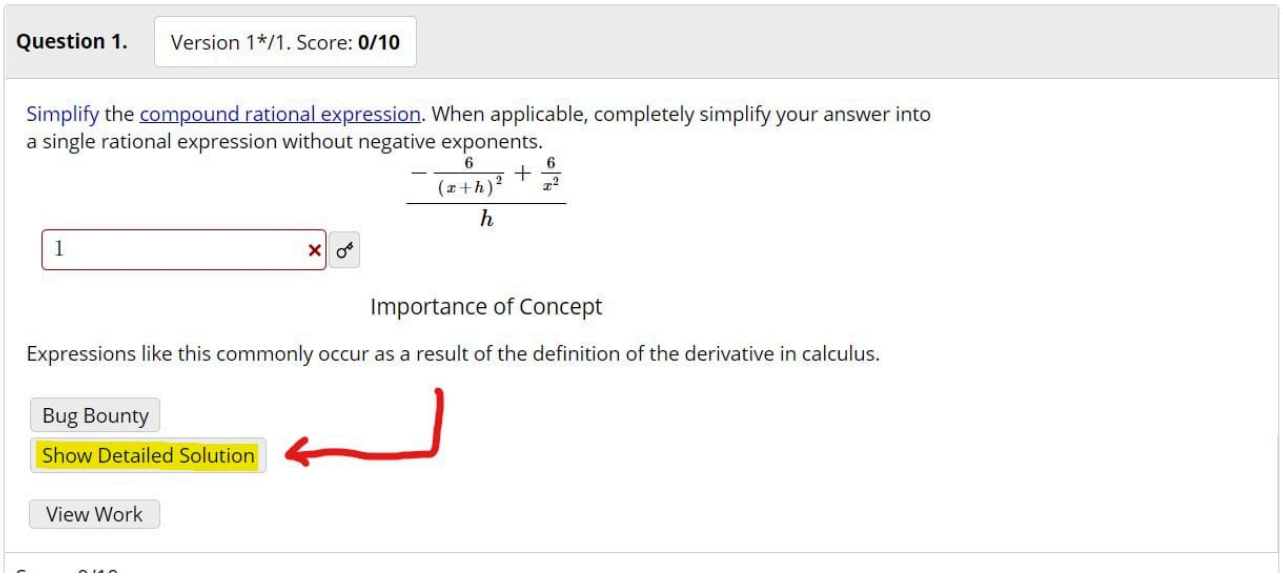
Sometimes, life can get overwhelmingly busy, and missing a due date happens. In those instances, you can use a Latepass to extend the original due date of the homework (see the syllabus for how long a single Latepass extends the original due date by).

Warning: Hand-Written Solutions

Read the syllabus to find out what happens to the questions that must be graded by hand (file uploads) - most professors are not likely to grade the late submitted hand-written homework solutions due to time constraints and fairness to other students.

MyOpenMath Advice: Reading Solutions to Exercises

Some problems have complete worked-out solutions. If your feedback has a comment like, "Please see the solutions," then you should probably look through these solutions. You can see solutions by going into the Grades area, finding the assignment, and clicking "Show Detailed Solution" on an exercise with a solution. Solutions are generally not shown until after the due date has passed.



I usually leave a comment of the form, "Please see the written solutions," if there were too many mistakes for me to cover by typing or if I could not follow your logic.

- Professor Simpson, Cosumnes River College

Answers

Section Preview

This is where the magic happens—you are likely going to spend around a week (at least) on this material. **It is NOT optional**. Use the digital manipulations in this section to help students visualize the $\epsilon-\delta$ proof. Additionally, students should be introduced to symbolic logic here (for use throughout the rest of Calculus).

Section Preview

- **Solving Inequalities**
  - Solving Absolute Value Inequalities: The "essential equivalences" $|f(x)-L|<\epsilon \leftrightarrow L-\epsilon<f(x)<L+\epsilon$ and $0<|x-a|<\delta \leftrightarrow a-\delta<x<a+\delta$ are exactly the splitting of an absolute-value inequality into a compound statement; the student must already know this to translate the definition into neighborhoods.

Section Preview

- **Simplifying Expressions**
  - Multiplying Polynomials and the Distributive Property: Example $1.6.4$ expands $(3x+2)^2=9x^2+12x+4$, and the proofs repeatedly distribute/factor a constant (e.g., $|5x-5|=|5(x-1)|=5|x-1|$).
- **Solving Equations**
  - Solving Linear Equations — One or Two-Steps: Example $1.6.2$ solves $5.999=5x+1$ and $6.001=5x+1$ to locate the neighborhood endpoints $x=0.9998,\ 1.0002$.

Section Preview

- Describe the precise definition of a limit for finite limits at finite points.
- Apply the epsilon-delta definition to find the limit of a function.
- Prove finite limits using the precise definition of a limit.
- Describe the epsilon-delta definitions of one-sided limits.
- Use the epsilon-delta definition to prove the Limit Laws.

Example $1.6.1$

Building the logic: Since $f(x)=5x+1$ is approaching 6 as x is approaching 1, it *seems* like we are considering a group of x-values near 1 whose corresponding values on the function $f(x)$ are approaching 6.

Tying this to intervals: In terms of intervals, it *appears to be the case* that we are saying all of the x-values in an interval (neighborhood) around $x=1$ lead to y-values in an interval (neighborhood) around $y=6$.

Tightening up the language: *Perhaps* this means that all the x-values in a δ-neighborhood centered at $x=1$ lead to function values in an ϵ-neighborhood centered at $y=6$.

Example $1.6.2$**: Finding** δ **Given** ϵ

The ϵ gauntlet has been thrown! Someone has issued the challenge and said that you could not find a δ-neighborhood about $x=1$ such that all of the values of the corresponding function land in their very small ϵ-neighborhood about $y=6$. The size of their given ϵ neighborhood is 0.001.

The given ϵ-neighborhood rises from

$$y=6-\epsilon=6-0.001=5.999$$

to

$$y=6+\epsilon=6+0.001=6.001.$$

Solving $5.999 = 5x + 1$ and $6.001 = 5x + 1$, we get $x = 0.9998$ and $x = 1.0002$, respectively. This means that as long as $x \in (0.9998, 1) \cup (1, 1.0002)$, $f(x) \in (5.999, 6.001)$.

To compute the value of δ, we consider the "length" of each subinterval on the x-axis. The first is $(0.9998, 1)$. This has a length of 0.0002. Likewise, the subinterval $(1, 1.0002)$ has a length of 0.0002. Thus, we choose $\delta = 0.0002$.

Whenever x is within $\delta = 0.0002$ units of 1 on the x-axis, the corresponding function value, $f(x)$, must therefore be within $\epsilon = 0.001$ units of $y = 6$.

A visual to confirm this is in Figure $1.6.1$ below

Interactive Element: Finding δ Given ϵ

Interact: Drag either point to confirm the computed δ-neighborhood.

Example $1.6.3$: Proving a Statement about the Limit of a Linear Function

A crucial initial step is to rewrite this limit as the precise definition of a limit.

$$\lim_{x \to 1} (5x + 1) = 6$$

means that

$$\forall \epsilon > 0, \quad \exists \delta > 0 \quad \ni$$
$$0 < |x - 1| < \delta \implies |(5x + 1) - 6| < \epsilon.$$

Read alternatively: No matter what $\epsilon > 0$ that someone hands us, it is our job to find a $\delta > 0$ (that almost always relies on the value of ϵ we were given) so that

If $0 < |x - 1| < \delta$, **then** we can guarantee that $|(5x + 1) - 6| < \epsilon$.

SCRATCH WORK

Just like we would reference the puzzle box to see where we want to arrive, we will look at the consequent of our "If..., then..." statement to see where we need our proof to finish.

In our proof, we want to arrive at

$$|(5x + 1) - 6| < \epsilon \quad \text{(consequent)}$$

given that

$$0 < |x - 1| < \delta. \quad \text{(antecedent)}$$

Working backward from the consequent and trying to arrive at the antecedent, we get

$$\begin{aligned} |(5x + 1) - 6| < \epsilon \quad &\implies \quad |5x - 5| < \epsilon \\ &\implies \quad |5(x - 1)| < \epsilon \\ &\implies \quad |5| \cdot |x - 1| < \epsilon \\ &\implies \quad 5 \cdot |x - 1| < \epsilon \\ &\implies \quad |x - 1| < \frac{\epsilon}{5} \end{aligned}$$

So we can see that if we want to arrive at the consequent ($|(5x + 1) - 6| < \epsilon$), a superb choice for δ in our antecedent ($0 < |x - 1| < \delta$) is $\epsilon/5$. We now know how to "correctly put this puzzle together."

Note that the proof structure should be closely aligned with the structure of the precise definition of the limit. That is, it should flow in the following direction:

$$\forall \epsilon > 0, \quad \exists \delta > 0 \quad \ni$$
$$0 < |x - 1| < \delta \implies |(5x + 1) - 6| < \epsilon.$$

PROOF

$\forall \epsilon > 0$, choose $\delta = \frac{\epsilon}{5}$. Therefore, if $0 < |x - 1| < \delta$, then

$$\begin{aligned} |x - 1| &< \frac{\epsilon}{5} \quad \text{(Substitution)} \\ \implies \quad 5|x - 1| &< \epsilon \\ \implies \quad |5x - 5| &< \epsilon \\ \implies \quad |(5x + 1) - 6| &< \epsilon \\ \implies \quad |f(x) - L| &< \epsilon \quad \text{(Substitution)} \end{aligned}$$

Thus, we have proved the limit

$$\lim_{x \to 1} (5x + 1) = 6.$$

Example $1.6.4$: Proving a Statement about the Limit of a Nonlinear Function (symmetric δ)

PROOF

For any given positive value of ϵ, let $\delta = \frac{\sqrt{\epsilon}}{3}$. If it is the case that $0 < \left|x - \left(-\frac{2}{3}\right)\right| < \delta$, then

$$\begin{aligned}
\left|x - \left(-\frac{2}{3}\right)\right| &< \frac{\sqrt{\epsilon}}{3} \\
\implies \left|3\left(x + \frac{2}{3}\right)\right| &< \sqrt{\epsilon} \\
\implies |3x + 2| &< \sqrt{\epsilon} \\
\implies |(3x + 2)^2| &< \epsilon \\
\implies |9x^2 + 12x + 4| &< \epsilon \\
\implies |(9x^2 + 12x - 7) - (-11)| &< \epsilon \\
\implies |f(x) - L| &< \epsilon
\end{aligned}$$

Thus, we have proved the limit

$$\lim_{x \to -2/3} (9x^2 + 12x - 7) = -11.$$

Checkpoint 1.6.4

You should eventually arrive at $\delta = \frac{\epsilon}{3}$.

Example 1.6.5**: Determining an Appropriately-Sized** δ**-Neighborhood Geometrically**

This style of problem displays whether or not someone truly understands the *language translation* of the precise definition of a finite limit at a finite number. Remember, this limit is equivalent to saying,

$$\forall \epsilon > 0, \quad \exists \delta > 0 \ni$$
$$0 < |x + 2| < \delta \implies \left|\left(e^{x+2} + 3\sin\left(\frac{\pi}{12}x\right) + 2\right) - 1.5\right| < \epsilon.$$

However, reading this in language and allowing $\epsilon = 0.5$, we can rephrase this as follows: Given $\epsilon = 0.5$, we should be able to find a positive distance, δ, from $x = -2$, such that if x is within δ units of -2, then $f(x)$ is within 0.5 units of 1.5.

Since we are allowed graphing technology, let's use Desmos to graph $f(x) = e^{x+2} + 3\sin\left(\frac{\pi}{12}x\right) + 2$, $y = 1.5$, and the edges of our provided ϵ-neighborhood $y = 1.5 \pm 0.5$.

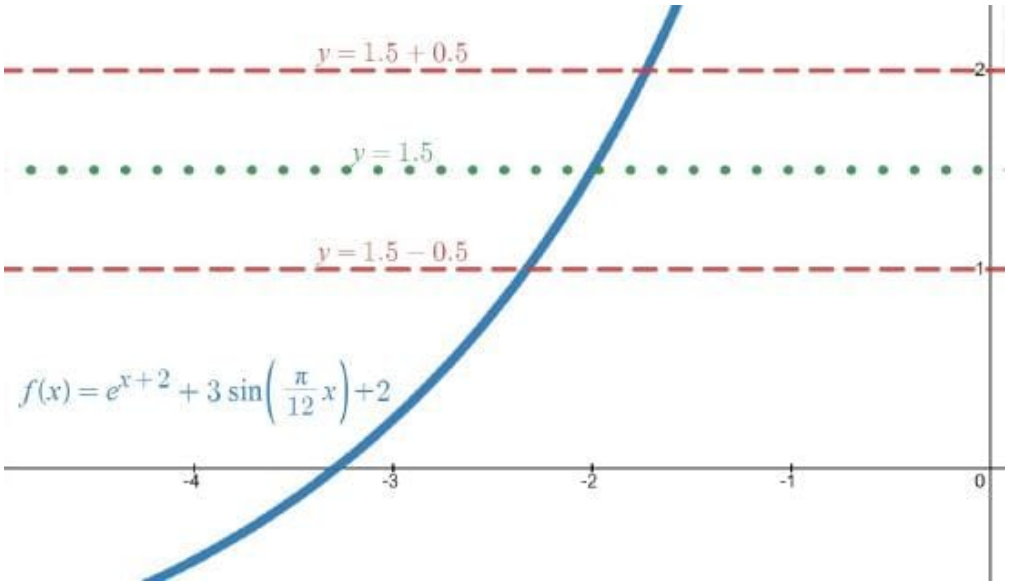


Figure 1.6.3

It is now our job to find an appropriately-sized δ-neighborhood about $x = -2$ so that all x-values from this δ-neighborhood lead to function values in the interval $(1.5 - 0.5, 1.5 + 0.5)$. Desmos automatically computes intersections for us, so all we need to do is click on the points where the function meets the upper and lower horizontal lines (the red-dashed lines).

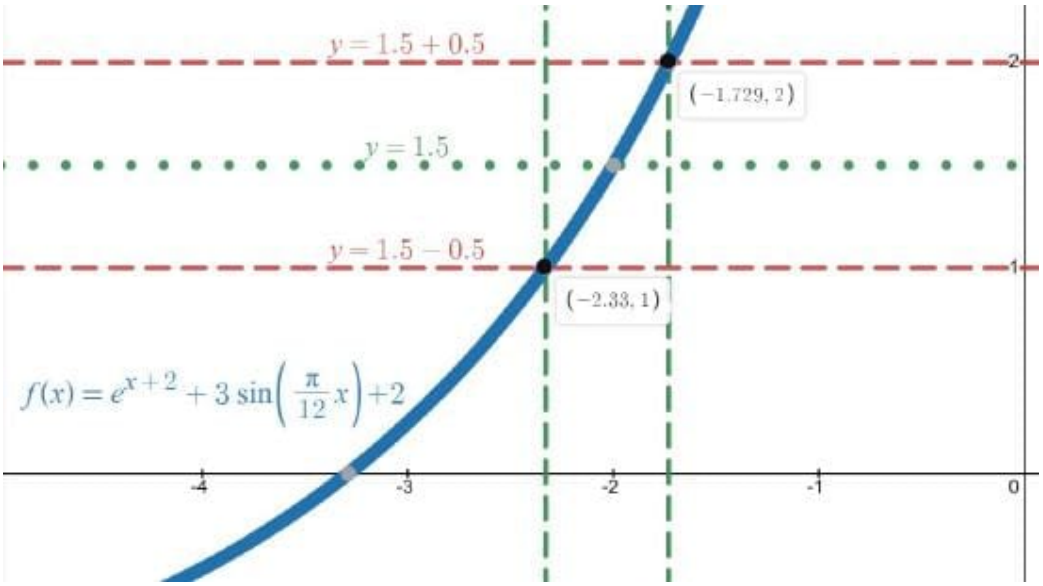


Figure 1.6.4

I also graph the vertical lines at these intersection points to help our visual further. Therefore, as long as $x \in (-2.33, -2) \cup (-2, -1.729)$, the function values look to satisfy the requirement by our challenger. That is, on the interval $(-2.33, -2) \cup (-2, -1.729)$, $|f(x) - 1.5| < 0.5$.

So how large should our δ-neighborhood be?

The size of the interval to the left of -2 is approximately $-2 - (-2.33) = 0.33$, and the size of the interval to the right of -2 is approximately $-1.729 - (-2) = 0.271$. I say "approximately" because Desmos only returns a maximum of three digits on their graphs.

When given the option of two values for δ, we always choose the smaller. Thus, we should choose $\delta = 0.271$ (due to rounding errors, we might want to choose δ *slightly* smaller—say, 0.270).

Example 1.6.6**: Proving a Statement about the Limit of a Nonlinear Function (non-symmetric** δ**)**

PROOF

$\forall \epsilon > 0$, I will choose $\delta = \min\left\{1, \frac{\epsilon}{5}\right\}$. If $0 < |x-(-1)| < \delta$, then

$$|x+1| < 1 \text{ and } |x+1| < \frac{\epsilon}{5}$$

Since $|x+1| < 1$,

$$\begin{aligned} -2 < x < 0 \quad &\Longrightarrow \quad -5 < x-3 < -3 \\ &\Longrightarrow \quad 5 > |x-3| > 3 \\ &\Longrightarrow \quad 5 > |x-3| \end{aligned}$$

Additionally, since $|x+1| < \frac{\epsilon}{5}$,

$$\begin{aligned} 5|x+1| < \epsilon \quad &\Longrightarrow \quad |x-3| \cdot |x+1| < 5|x+1| < \epsilon \\ &\Longrightarrow \quad |x-3| \cdot |x+1| < \epsilon \\ &\Longrightarrow \quad |x^2 - 2x - 3| < \epsilon \\ &\Longrightarrow \quad |(x^2 - 2x + 3) - 6| < \epsilon \\ &\Longrightarrow \quad |f(x) - L| < \epsilon \end{aligned}$$

Thus, we have proved the limit

$$\lim_{x \to -1} (x^2 - 2x + 3) = 6.$$

Checkpoint 1.6.6

$| x^2 - 1 | = |x-1| \cdot |x+1| < \epsilon/3 \cdot 3 = \epsilon$

Proof of the Sum Law

Let $\epsilon > 0$. Choose $\delta_1 > 0$ so that if $0 < |x-a| < \delta_1$, then $|f(x) - L| < \epsilon/2$, and choose $\delta_2 > 0$ so that if $0 < |x-a| < \delta_2$, then $|g(x) - M| < \epsilon/2$. Let $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |x-a| < \delta$, then

$$0 < |x-a| < \delta_1 \text{ and } 0 < |x-a| < \delta_2.$$

Hence,

$$\begin{aligned} |(f(x) + g(x)) - (L+M)| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

Q.E.D.

Example 1.6.7**: Showing That a Limit Does Not Exist**

Suppose that L is a candidate for a limit. Choose $\epsilon = 1/2$.

Let $\delta > 0$. Either $L \geq 0$ or $L < 0$. If $L \geq 0$, then let $x = -\delta/2$.

Thus,

$$|x - 0| = | -\frac{\delta}{2} - 0 | = \frac{\delta}{2} < \delta$$

and

$$\left| \frac{| -\frac{\delta}{2} |}{-\frac{\delta}{2}} - L \right| = | -1 - L| = L + 1 \geq 1 > \frac{1}{2} = \epsilon.$$

On the other hand, if $L < 0$, then let $x = \delta/2$. Thus,

$$|x - 0| = | \frac{\delta}{2} - 0 | = \frac{\delta}{2} < \delta$$

and

$$\left| \frac{|\frac{\delta}{2}|}{\frac{\delta}{2}} - L \right| = |1 - L| = |L| + 1 \geq 1 > \frac{1}{2} = \epsilon.$$

Thus, for any value of L, $\lim_{x \to 0} \frac{|x|}{x} \neq L$.

Example $1.6.8$**: Proving a Statement about a Limit From the Right**

PROOF

For $\epsilon > 0$, let $\delta = \epsilon^2$. If $0 < x - 4 < \delta$, then

$$\begin{aligned} & \sqrt{x-4} && < && \epsilon \\ \implies & |\sqrt{x-4} - 0| && < && \epsilon \\ \implies & |f(x) - L| && < && \epsilon \end{aligned}$$

Thus, we have proved the limit

$$\lim_{x \to 4^+} \sqrt{x-4} = 0.$$

Checkpoint $1.6.8$

$\delta = \epsilon^2$

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1.6.1: Resources and Key Concepts

Resources

Videos

- Precise definition of a limit
  - Limits - The Precise Definition of a Finite Limit at a Finite Point - The Concept
- Epsilon-delta definition
  - Limits - The Precise Definition of a Finite Limit at a Finite Point - The Actual Definition
- $\epsilon - \delta$ proofs
  - Limits - The Precise Definition of a Finite Limit at a Finite Point - Proofs on Linear Functions
  - Limits - The Precise Definition of a Finite Limit at a Finite Point - Finding δ Given ϵ
  - Limits - The Precise Definition of a Finite Limit at a Finite Point - Proofs on Nonlinear Functions
  - Limits - The Precise Definition of a Finite Limit at a Finite Point - Proof: Functions w/Same Limits
  - Limits - The Precise Definition of a Finite Limit at a Finite Point - Example: sin(x)/x
  - Limits - The Precise Definition of a Finite Limit at a Finite Point - Example: Disproving a Limit
- Proving limit laws
  - Limits - The Precise Definition of a Finite Limit at a Finite Point - Proof of the Product Limit Law
- One-sided limits (precise definition)
  - Limits - The Precise Definition of a Finite Limit at a Finite Point - Single-Sided Limits

Key Concepts

Definitions

- **δ-Neighborhood Centered at a**: The interval $(a-\delta, a) \cup (a, a+\delta)$ on the x-axis. (Also referred to as a **punctured open neighborhood**).
- **ϵ-Neighborhood Centered at L**: The interval $(L-\epsilon, L+\epsilon)$ on the y-axis.
- **Finite Limit at a Finite Number (Precise Definition)**: Let $f(x)$ be defined for all $x \neq a$ over an open interval containing a, and let L be a real number. We say $\lim_{x \to a} f(x) = L$ if for every $\epsilon > 0$, there exists a $\delta > 0$, such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.
  - Symbolic Logic Statement: $\forall \epsilon > 0, \exists \delta > 0 \ni 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$.
- **Antecedent (in an "If..., then..." statement)**: The "If" part of the statement.
- **Consequent (in an "If..., then..." statement)**: The "then" part of the statement.
- **One-Sided Finite Limits at Finite Numbers (Precise Definition)**:
  - **Limit from the Right**: Let $f(x)$ be defined over an open interval of the form (a, b) where $a < b$. Then $\lim_{x \to a^+} f(x) = L$ if for every $\epsilon > 0$, there exists a $\delta > 0$, such that if $0 < x - a < \delta$, then $|f(x) - L| < \epsilon$.
  - **Limit from the Left**: Let $f(x)$ be defined over an open interval of the form (b, c) where $b < c$. Then $\lim_{x \to c^-} f(x) = L$ if for every $\epsilon > 0$, there exists a $\delta > 0$, such that if $-\delta < x - c < 0$, then $|f(x) - L| < \epsilon$.

Theorems

- **Triangle Inequality**: If a and b are any real numbers, then $|a + b| \leq |a| + |b|$.

Common Mistakes

- **Symbolic Logic for "Such That"**: The symbol $\ni$ is used frequently in symbolic logic for "such that"; however, many mathematicians do not use it for that purpose. Students should check with their instructor to see what logic symbol they prefer (many avoid the confusion entirely and use "s.t.").

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1.6.2: Homework

Note

The following exercises are meant as hand-in homework (if assigned by your instructor). Typed-in answers here will not be saved or scored.

Concept Check

Exercise 1

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Basic Skills

Exercise 2

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Exercise 3

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Exercise 4

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Exercise 5

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Exercise 6

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Exercise 7

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Exercise 8

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Exercise 9

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Exercise 10

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Exercise 11

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Synthesis Questions - Applications

Exercise 12

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Proofs

Exercise 13

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Exercise 14

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Exercise 15

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Exercise 16

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Exercise 17

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Exercise 18

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Exercise 19

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Exercise 20

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1.6.3: Additional Exercises

Reading Questions

1. What does the statement $|f(x) - L| < \epsilon$ mean in terms of distance?
2. What does the statement $0 < |x - a| < \delta$ mean in terms of distance and the value of x relative to a?
3. What is a δ-neighborhood centered at a? Does it include a?
4. What is an ϵ-neighborhood centered at L? Does it include L?
5. State the precise ($\epsilon - \delta$) definition of a finite limit at a finite number: $\lim_{x \to a} f(x) = L$.
6. In the $\epsilon - \delta$ definition, which quantity is chosen first, ϵ or δ? Which one typically depends on the other?
7. What is the role of "scratch work" in constructing an $\epsilon - \delta$ proof? Which part of the definition (antecedent or consequent) does scratch work typically start with?
8. When proving a limit for a non-linear function where the δ-neighborhood might not be symmetric (e.g., $f(x) = x^2$ near $x = 2$), if you find two possible values δ_L (from the left) and δ_R (from the right), how do you choose the final δ?
9. How is the precise definition of a limit from the right, $\lim_{x \to a^+} f(x) = L$, different from the definition of a two-sided limit? Specifically, how does the condition on $x - a$ change?
10. What is the Triangle Inequality?
11. When proving that a limit does not exist, what needs to be shown regarding ϵ, δ, and L?

Homework

In exercises 1 - 4, write the appropriate $\epsilon - \delta$ definition for each of the given statements.

1) $\lim_{x \to a} f(x) = N$

2) $\lim_{t \to b} g(t) = M$

Answer

For every $\epsilon > 0$, there exists a $\delta > 0$, so that if $0 < |t - b| < \delta$, then $|g(t) - M| < \epsilon$

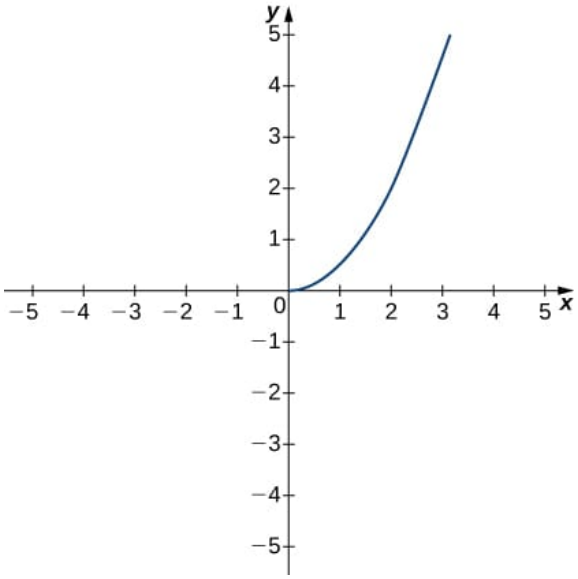
3) $\lim_{x \to c} h(x) = L$

4) $\lim_{x \to a} \phi(x) = A$

Answer

For every $\epsilon > 0$, there exists a $\delta > 0$, so that if $0 < |x - a| < \delta$, then $|\phi(x) - A| < \epsilon$

The following graph of the function f satisfies $\lim_{x \to 2} f(x) = 2$. In the following exercises, determine a value of $\delta > 0$ that satisfies each statement.



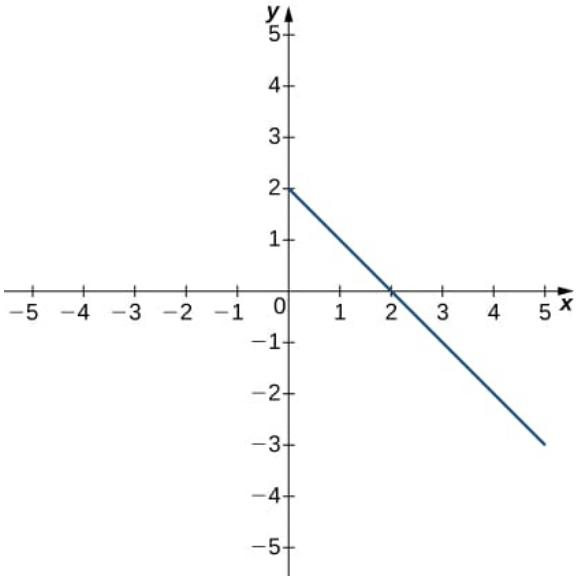
5) If $0 < |x - 2| < \delta$, then $|f(x) - 2| < 1$.

6) If $0 < |x - 2| < \delta$, then $|f(x) - 2| < 0.5$.

Answer

$\delta \leq 0.25$

The following graph of the function f satisfies $\lim_{x \to 3} f(x) = -1$. In the following exercises, determine a value of $\delta > 0$ that satisfies each statement.



7) If $0 < |x - 3| < \delta$, then $|f(x) + 1| < 1$.

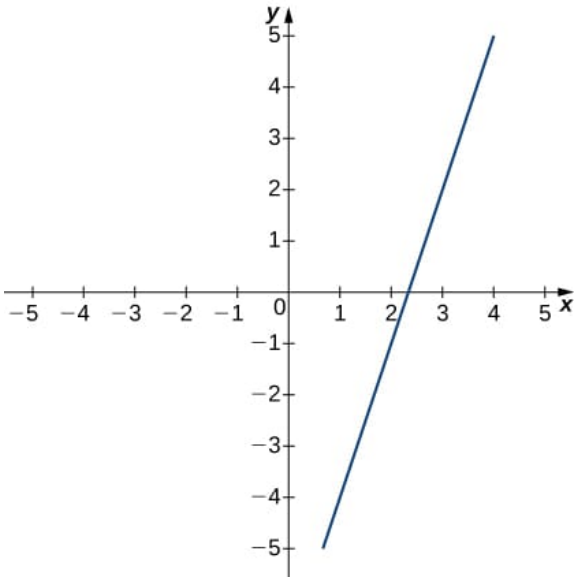
8) If $0 < |x - 3| < \delta$, then $|f(x) + 1| < 2$.

Answer

$\delta \leq 2$

The following graph of the function f satisfies $\lim_{x \to 3} f(x) = 2$. In the following exercises, for each value of ϵ, find a value of

$\delta > 0$ **such that the precise definition of limit holds true.**



9) $\epsilon = 1.5$

10) $\epsilon = 3$

Answer

$\delta \leq 1$

In exercises 11 - 12, use graphing technology to find a number δ such that the statements hold true.

11) $\left|\sin(2x) - \frac{1}{2}\right| < 0.1$, whenever $\left|x - \frac{\pi}{12}\right| < \delta$

12) $\left|\sqrt{x-4} - 2\right| < 0.1$, whenever $|x - 8| < \delta$

Answer

$\delta < 0.3900$

In exercises 13 - 17, use the precise definition of limit to prove the given limits.

13) $\lim\limits_{x \to 2} (5x + 8) = 18$

14) $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x - 3} = 6$

Answer

Let $\delta = \epsilon$. If $0 < |x - 3| < \epsilon$, then $\left|\dfrac{x^2 - 9}{x - 3} - 6\right| = \left|\dfrac{(x+3)(x-3)}{x-3} - 6\right| = |x + 3 - 6| = |x - 3| < \epsilon$.

15) $\lim\limits_{x \to 2} \dfrac{2x^2 - 3x - 2}{x - 2} = 5$

16) $\lim\limits_{x \to 0} x^4 = 0$

Answer

Let $\delta = \sqrt[4]{\epsilon}$. If $0 < |x| < \sqrt[4]{\epsilon}$, then $\left|x^4 - 0\right| = x^4 < \epsilon$.

17) $\lim\limits_{x \to 2} (x^2 + 2x) = 8$

In exercises 18 - 20, use the precise definition of limit to prove the given one-sided limits.

18) $\lim_{x\to 5^-} \sqrt{5-x} = 0$

Answer

Let $\delta = \epsilon^2$. If $-\epsilon^2 < x - 5 < 0$, we can multiply through by -1 to get $0 < 5 - x < \epsilon^2$.
Then $\left|\sqrt{5-x} - 0\right| = \sqrt{5-x} < \sqrt{\epsilon^2} = \epsilon$.

19) $\lim_{x\to 0^+} f(x) = -2$, where $f(x) = \begin{cases} 8x-3, & \text{if } x < 0 \\ 4x-2, & \text{if } x \geq 0 \end{cases}$.

20) $\lim_{x\to 1^-} f(x) = 3$, where $f(x) = \begin{cases} 5x-2, & \text{if } x < 1 \\ 7x-1, & \text{if } x \geq 1 \end{cases}$.

Answer

Let $\delta = \epsilon/5$. If $-\epsilon/5 < x - 1 < 0$, we can multiply through by -1 to get $0 < 1 - x < \epsilon/5$.
Then $|f(x) - 3| = |5x - 2 - 3| = |5x - 5| = 5(1 - x)$, since $x < 1$ here.
And $5(1 - x) < 5(\epsilon/5) = \epsilon$.

21) An engineer is using a machine to cut a flat square of Aerogel of area $144\ \text{cm}^2$. If there is a maximum error tolerance in the area of $8\ \text{cm}^2$, how accurately must the engineer cut on the side, assuming all sides have the same length? How do these numbers relate to δ, ϵ, a, and L?

Answer

$0.033\ \text{cm},\ \ \epsilon = 8,\ \ \delta = 0.33,\ \ a = 12,\ \ L = 144$

22) Use the precise definition of limit to prove that the following limit does not exist: $\lim_{x\to 1} \dfrac{|x-1|}{x-1}$.

23) Using precise definitions of limits, prove that $\lim_{x\to 0} f(x)$ does not exist, given that $f(x)$ is the ceiling function. (Hint: Try any $\delta < 1$.)

Answer

Answers may very.

24) Using precise definitions of limits, prove that $\lim_{x\to 0} f(x)$ does not exist: $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$. (Hint: Think about how you can always choose a rational number $0 < r < d$, but $|f(r) - 0| = 1$.)

25) Using precise definitions of limits, determine $\lim_{x\to 0} f(x)$ for $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$. (Hint: Break into two cases, x rational and x irrational.)

Answer

0

26) Using the function from the previous exercise, use the precise definition of limits to show that $\lim_{x\to a} f(x)$ does not exist for $a \neq 0$

For exercises 27 - 29, suppose that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ both exist. Use the precise definition of limits to prove the following limit laws:

27) $\lim_{x\to a}(f(x) - g(x)) = L - M$

Answer

$f(x) - g(x) = f(x) + (-1)g(x)$

28) $\lim_{x\to a}[cf(x)] = cL$ for any real constant c (Hint: Consider two cases: $c = 0$ and $c \neq 0$.)

29) $\lim_{x \to a}[f(x)g(x)] = LM$. (Hint: $|f(x)g(x) - LM| = |f(x)g(x) - f(x)M + f(x)M - LM| \leq |f(x)||g(x) - M| + |M||f(x) - L|$.)

Answer

Answers may vary.

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1.7: The Precise Definitions of Limits Involving Infinity

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

The previous section walked us slowly through the theory of finite limits at finite numbers. Instead of *evaluating* limits, we *proved* limits using $\epsilon - \delta$ proofs. We now turn our attention to limits involving infinity. There are three such limits:

1. infinite limits at finite numbers,
2. finite limits at infinity, and
3. infinite limits at infinity.

We treat each of these separately and pull all limit proof styles into a unified theory at the end of the section.

Infinite Limits at Finite Numbers

Previously, we learned how to *conceptually* investigate limits of the form

$$\lim_{x\to 1} \frac{1}{(x-1)^2}.$$

These types of limits are called **infinite limits at finite numbers** because their limits approach infinity as x approaches a finite number.

To investigate this limit, we stated that $(x-1)^2 \to 0^+$ as $x \to 1$. Thus, $\frac{1}{(x-1)^2} \to \frac{1}{0^+} \to \infty$.

At the time, these arguments were sufficient, if not precise; however, we now turn our attention to *proving*

$$\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty. \tag{1.7.1}$$

To understand the structure of the proof for infinite limits at finite numbers, we need to discuss the structure of the traditional $\epsilon - \delta$ proof.

Given $\lim\limits_{x\to a}(f(x)) = L$, where a and L are finite, it should now make sense that we *require* an ϵ-neighborhood about the *finite* number L, and a δ-neighborhood about the *finite* number a; however, in the limit 1.7.1, $L = \infty$ is *not* finite. Therefore, creating a small ϵ-neighborhood about L makes no sense. We cannot bound ∞ in a small ϵ-neighborhood. Thus, our precise definition needs some adjustment for this type of limit.

Figure 1.7.1 shows the graph of $f(x) = \frac{1}{(x-1)^2}$.

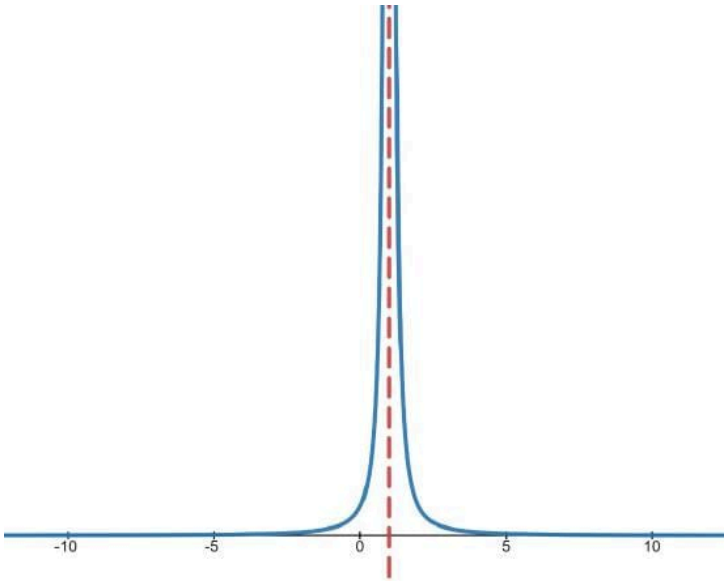

Figure 1.7.1**:** $f(x) = \frac{1}{(x-1)^2}$

While it's easy to *see* that the function values, $f(x)$, tend to ∞ as our δ-neighborhood centered at $x = 1$ gets smaller and smaller, this is *not* a proof. What we need is for someone to challenge us (as they did with the ϵ-gauntlet in the previous section) and say, "I bet you cannot find a small enough δ-neighborhood centered at $x = 1$ so that all the corresponding function values are above... oh, let's say a billion!" **And this is the key to the structure of the precise definition of an infinite limit at a finite point.** Instead of "for any given $\epsilon > 0$," we are faced with "for any given $N \ggg 0$," where the notation $\ggg$ means "much greater than." And instead of a small ϵ-neighborhood around L, we are challenged with getting the function values *above* (or below, if the limit is tending to $-\infty$) some massively large number N. Visually, it looks a bit like Figure 1.7.2.

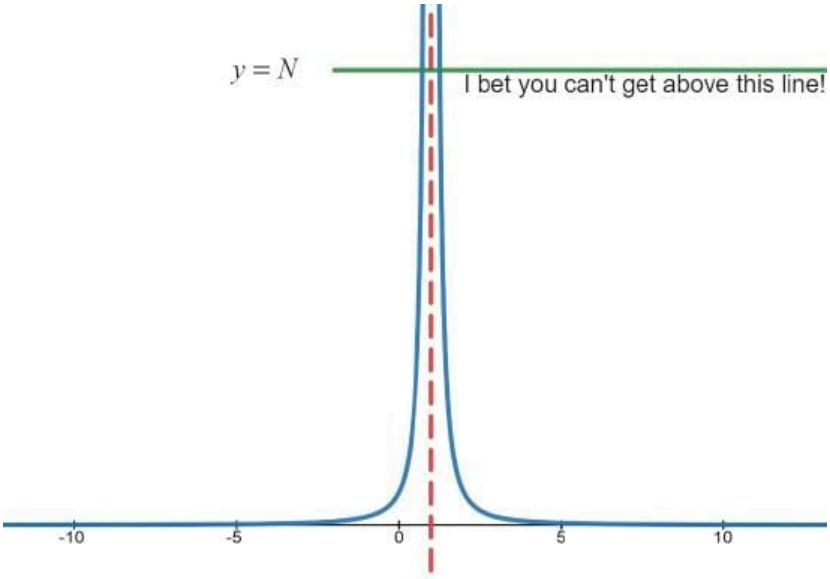

Figure 1.7.2

We can now formalize the precise definition of an infinite limit at a finite number.

Definition: Infinite Limit at a Finite Number (Precise Definition)

Let $f(x)$ be defined for all $x \neq a$ over an open interval containing a. Then we say

$$\lim_{x \to a} f(x) = \infty$$

if for every $N \ggg 0$, there exists a $\delta > 0$, such that if $0 < |x - a| < \delta$, then $f(x) > N$.

Stated with symbolic logic,

$$\lim_{x \to a} f(x) = \infty$$

means

$$\forall N \ggg 0, \quad \exists \delta > 0 \ni$$
$$0 < |x - a| < \delta \implies f(x) > N.$$

Additionally, we say

$$\lim_{x \to a} f(x) = -\infty$$

if for every $N \lll 0$, there exists a $\delta > 0$, such that if $0 < |x - a| < \delta$, then $f(x) < N$.

Stated with symbolic logic,

$$\lim_{x \to a} f(x) = -\infty$$

means

$$\forall N \lll 0, \quad \exists \delta > 0 \ni$$
$$0 < |x - a| < \delta \implies f(x) < N.$$

Let's see this in an example.

Example 1.7.1

Prove

$$\lim_{x \to 1} \frac{1}{(x-1)^2} = \infty.$$

Solution available at the end of this section.

Interactive Element: The N–δ Challenge — Infinite Limit at a Finite Number

The following Interactive Element might help you visualize how the δ-neighborhood responds when the challenge value N grows.

Interact: Drag the red point up and down to change the challenge height N.

Observation: No matter how high you push N, the shaded δ-neighborhood around $x = 1$ simply narrows, and inside it the curve still rises above $y = N$. A valid δ always exists, which is exactly what $\lim_{x \to 1} f(x) = \infty$ means.

A few tripping points commonly arise while working with this material. It's important to remember the following from your prerequisite courses:

1. An inequality switches directions if you multiply or divide by a negative.
2. Applying an even power to an expression results in a positive value (this can be seen between the first and second steps of the proof in Example 1.7.1).
3. $\sqrt{(\blacksquare)^2} = |\blacksquare|$. This fact is used in the second-to-last step of the Scratch Work.

We will add to this list as we move forward, but if you have solid prerequisites, you have nothing to worry about.

Vertical Asymptotes

You have encountered vertical asymptotes in your algebra courses as part of the rational and logarithmic functions material. Moreover, you worked with them in Trigonometry. We now formalize their definition using Calculus.

Definition: Vertical Asymptote

The vertical line $x = a$ is called a **vertical asymptote** of the function $f(x)$ if at least one of the following is true:

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty$$
$$\lim_{x\to a} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty$$

Example 1.7.2

Prove that $x = -2$ is a vertical asymptote of $f(x) = \ln(x+2)$.

Solution available at the end of this section.

Adding to that list of tripping points:

4. If $x < y$, then $e^x < e^y$ (this is a direct result of the fact that $y = e^x$ is an increasing function). This allows us to rewrite inequalities like $\ln(x+2) < N$ as $e^{\ln(x+2)} < e^N \implies x + 2 < e^N$.
5. Stating that $\delta = e^N$ might *seem* wrong to you because δ is typically *very* small; however, pay close attention to the value of N. In Example 1.7.2, $N \lll 0$. Therefore, e^N *is* very small.

Finite Limits at Infinity

Consider

$$\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Some algebraic manipulation is necessary to see how this limit behaves.

$$\begin{aligned}
\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} &= \lim_{x\to\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} && \left(\text{multiplying by } \frac{1/e^x}{1/e^x}\right) \\
&= \lim_{x\to\infty} \frac{1 - \frac{1}{e^{2x}}}{1 + \frac{1}{e^{2x}}} \\
&= \frac{1-0}{1+0} && \left(\frac{1}{e^{2x}} \to 0 \text{ as } x \to \infty\right) \\
&= 1
\end{aligned}$$

The second-to-last step in this derivation relied on *assumptions* that $\frac{1}{e^{2x}} \to 0$ as $x \to \infty$. In previous sections, we bolstered this assumption with a "hand-waving" argument, saying, "Well, it's *obvious* that the denominator of $\frac{1}{e^{2x}}$ goes to infinity as x goes to infinity. Therefore, 1 over something going to infinity will shrink to 0." However, this is not a rigorous mathematical justification.

So, how can we *prove*

$$\lim_{x\to\infty} \frac{1}{e^{kx}} = 0 \ \ (\text{for } k > 0), \tag{1.7.2}$$

which in turn will *prove*

$$\lim_{x\to\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}=1? \tag{1.7.3}$$

Limits like 1.7.2 and 1.7.3 are called **finite limits at infinity** because the limits become finite (0 in 1.7.2 and 1 in 1.7.3) as x approaches infinity.

To understand the proof structure for finite limits at infinity, we again need to modify the traditional $\epsilon-\delta$ proof. In 1.7.2, $L=0$ is finite, but $a=\infty$ is *not* finite. Therefore, creating a small δ-neighborhood about a makes no sense. Just as before, we cannot bound ∞ in a small δ-neighborhood.

Figure 1.7.3 shows the graph of $f(x)=\frac{1}{e^{2x}}$ (we will worry about $\frac{1}{e^{kx}}$ later).

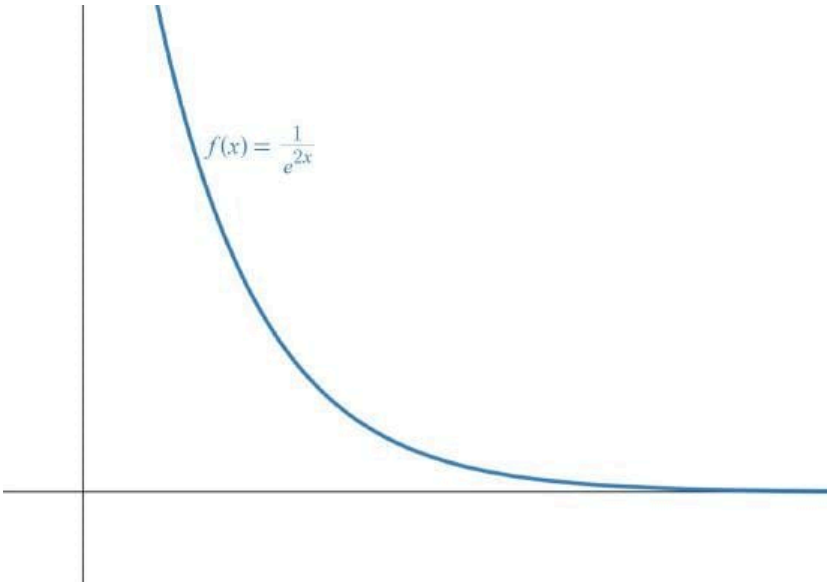

Figure 1.7.3**:** $f(x)=\frac{1}{e^{2x}}$

If the claim is that

$$\lim_{x\to\infty}\frac{1}{e^{2x}}=0,$$

then someone is going to throw the ϵ-gauntlet to challenge us to force the function to be bounded within the interval $(0-\epsilon,0+\epsilon)$, as in Figure 1.7.4.

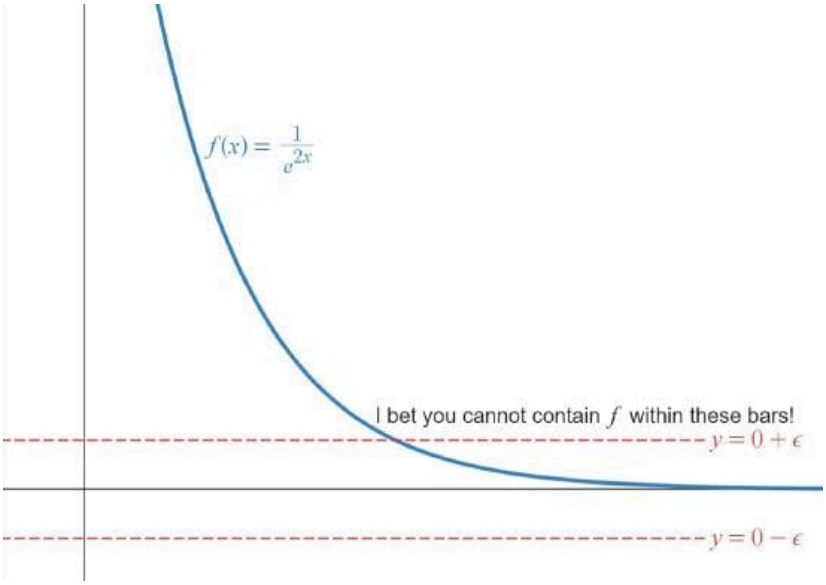

Figure 1.7.4

It's apparent from Figure $1.7.4$ that, as long as x is large enough, $-\epsilon < f(x) < \epsilon$. In fact, we see in Figure $1.7.5$, as long as $x > M$, $|f(x) - 0| < \epsilon$.

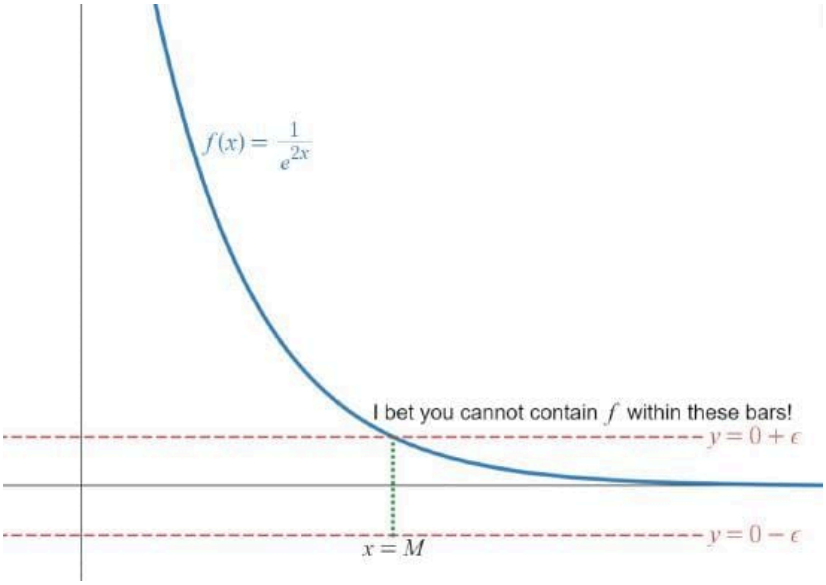

Figure $1.7.5$

This gives us great insight into the formal definition of finite limits at infinity.

Definition: Finite Limit at Infinity (Precise Definition)

Let $f(x)$ be defined for all $x > a$. Then we say

$$\lim_{x \to \infty} f(x) = L$$

if for every $\epsilon > 0$, there exists a number $M > 0$, such that if $x > M$, then $|f(x) - L| < \epsilon$.

Stated with symbolic logic,

$$\lim_{x \to \infty} f(x) = L$$

means

$$\forall \epsilon > 0, \quad \exists M > 0 \ni$$
$$x > M \implies |f(x) - L| < \epsilon.$$

Additionally, we say

$$\lim_{x \to -\infty} f(x) = L$$

if for every $\epsilon > 0$, there exists a number $M < 0$, such that if $x < M$, then $|f(x) - L| < \epsilon$.

Stated with symbolic logic,

$$\lim_{x \to -\infty} f(x) = L$$

means

$$\forall \epsilon > 0, \quad \exists M < 0 \ni$$
$$x < M \implies |f(x) - L| < \epsilon.$$

Interactive Element: The ϵ–M Challenge — Finite Limit at Infinity

The following Interactive Element might help you visualize how the threshold M responds as the ε-band tightens around the horizontal asymptote $y = 0$.

Interact: Drag the red point down to shrink the ε-band around $y = 0$.

Observation: As the band narrows, the threshold $x = M$ slides right, but past it the curve is always trapped inside the band. A valid M always exists, so $\lim_{x\to\infty} e^{-2x} = 0$.

We now introduce two very powerful theorems. The proofs of each of these theorems showcase how to use the precise definition of a finite limit at infinity.

Theorem

$$\lim_{x\to\infty} e^{-x} = 0$$

Proof available at the end of this section.

Again, adding to our list of tripping points:

6. $|e^{-x}| = e^{-x}$ because $e^{-x} > 0$.
7. In the proof of Theorem $1.7.1$, we found $M = -\ln(\epsilon)$; however, we also stated that $M > 0$. How can this be?
   Don't be fooled into thinking an "opposite sign" $(-)$ means that you're dealing with a negative number. In reality, ϵ is a *very* small positive number. Therefore, $\ln(\epsilon)$ is negative. Hence, $M = -\ln(\epsilon)$ is positive.

The previous theorem begets a simple corollary.

Corollary

If $k > 0$,

$$\lim_{x\to\infty} e^{-kx} = 0$$

Proof available at the end of this section.

The following theorem will be used throughout Calculus.

Theorem

If $p > 0$ is a rational number, then

$$\lim_{x\to\infty} \frac{1}{x^p} = 0.$$

If $p > 0$ is a rational number such that x^p is defined for all x, then

$$\lim_{x\to-\infty} \frac{1}{x^p} = 0.$$

Proof available at the end of this section.

The proof of the second half of this theorem is left as a homework exercise.

As before, the proof of this theorem adds to our list of tripping points.

8. If $x > 0$, then $\left|\frac{1}{x^p} - 0\right| = \frac{1}{x^p}$.

Horizontal Asymptotes

We have already introduced the definition of a horizontal asymptote; however, we repeat it here due to the connection with the finite limit at infinity.

Definition: Horizontal Asymptote

The vertical line $y = L$ is called a **horizontal asymptote** of the function $f(x)$ if at least one of the following is true:

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L$$

Infinite Limits at Infinity

Up to this point, we have allowed ourselves the luxury of *assuming*

$$\lim_{x\to\infty} 3^x = \infty; \tag{1.7.4}$$

however, it is at this moment that we need to *prove* this assumption is true. Limits of the form 1.7.4 are called **infinite limits at infinity** because the function tends to infinity (or negative infinity) and x tends to infinity (or negative infinity).

As with all our work in this section, developing the precise definition of an infinite limit at infinity requires adjusting the traditional $\epsilon - \delta$ definition of a limit.

Since both x and the function are approaching infinity in 1.7.4, it is impossible for us to say, "For any given $\epsilon > 0$, there exists a $\delta > 0$," because there is no way for us to contain infinity in a small interval (whether an ϵ- or δ-neighborhood).

So, how do we adjust the definition? Consider the following figure.

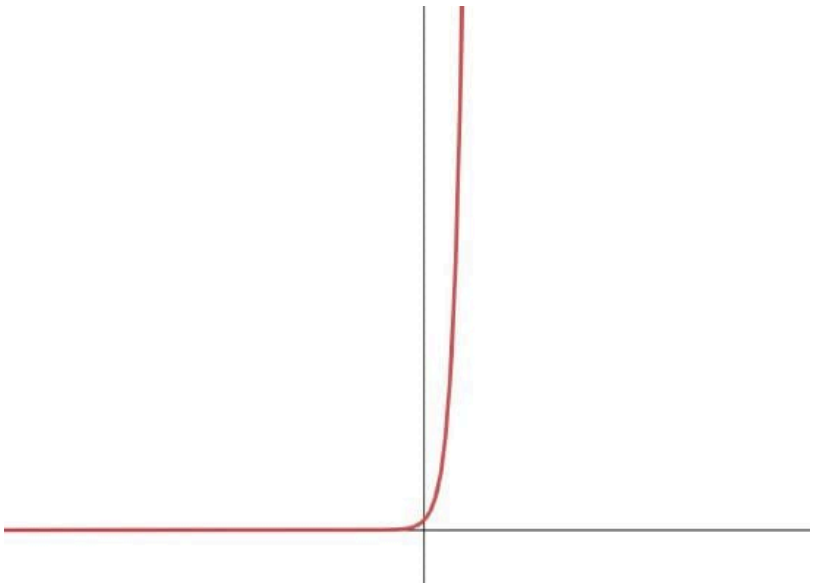

Figure 1.7.6: $f(x) = 3^x$

Instead of someone challenging us by throwing down an ϵ-gauntlet, they are going to challenge us to get our function values *above* a colossal number (see Figure 1.7.7.

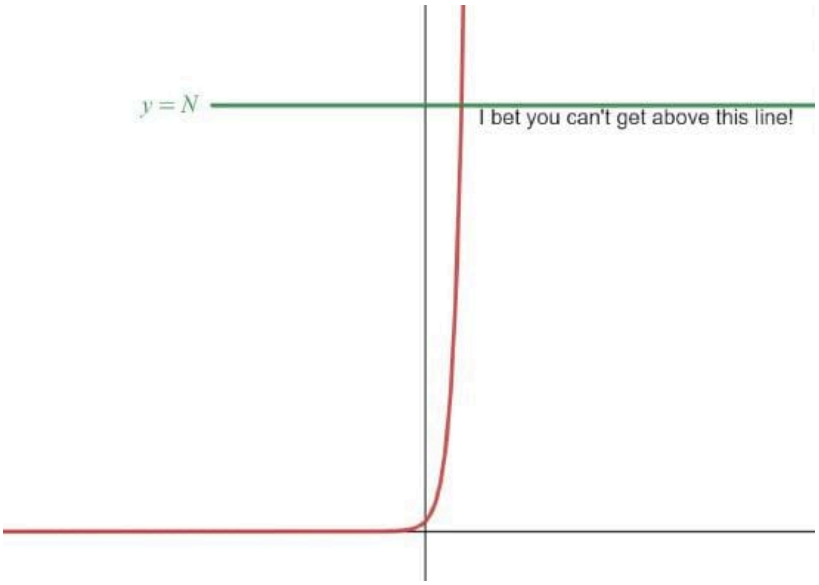

Figure 1.7.7

That is, someone is going to hand us $N \ggg 0$. Our job is to somehow find x-values whose function values are *above* the line $y = N$. Figure 1.7.8 shows graphically how this is done.

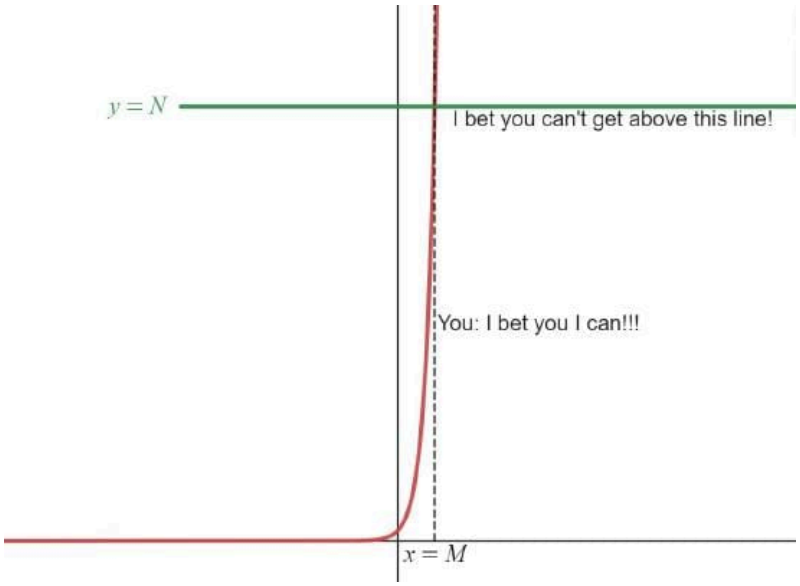

Figure 1.7.8

Figure 1.7.9 shows a zoomed-in view of Figure 1.7.8. As long as $x > M$, it looks as though $3^x > N$.

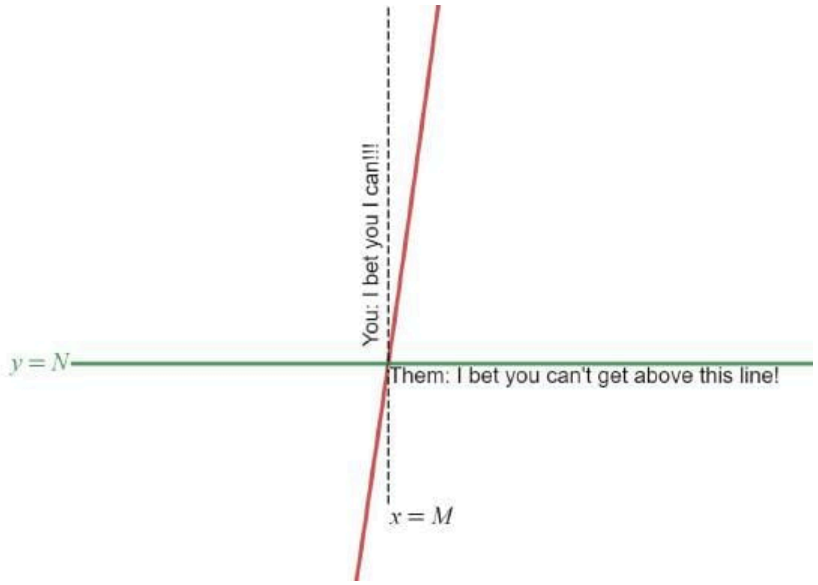

Figure 1.7.9

Using this same idea, let's write our final set of precise definitions for limits.

Definition: Infinite Limit at Infinity (Precise Definition)

As $x \to \infty$...

Let $f(x)$ be defined for all $x > a$. Then we say

$$\lim_{x\to\infty} f(x) = \infty$$

if for every $N \ggg 0$, there exists a number $M > 0$, such that if $x > M$, then $f(x) > N$.

Stated with symbolic logic,

$$\lim_{x\to\infty} f(x) = \infty$$

means

$$\forall N \ggg 0, \quad \exists M > 0 \ni$$
$$x > M \implies f(x) > N.$$

Additionally, we say

$$\lim_{x\to\infty} f(x) = -\infty$$

if for every $N \lll 0$, there exists a number $M > 0$, such that if $x > M$, then $f(x) < N$.

Stated with symbolic logic,

$$\lim_{x\to\infty} f(x) = -\infty$$

means

$$\forall N \lll 0, \quad \exists M > 0 \ni$$
$$x > M \implies f(x) < N.$$

Interactive Element: The N–M Challenge — Infinite Limit at Infinity

The following Interactive Element might help you visualize how the threshold M responds as the challenge height N rises.

Interact: Drag the red point up to raise the challenge height N.

Observation: The threshold $x = M$ meets the curve exactly where $3^x = N$; raise N and that meeting point climbs up the curve, pushing M right. Beyond M the curve always exceeds N, so $\lim_{x\to\infty} 3^x = \infty$.

Example 1.7.3

Prove

$$\lim_{x\to-\infty} x^3 = -\infty.$$

Solution available at the end of this section.

Checkpoint 1.7.3

Use the formal definition of infinite limit at infinity to prove that $\lim_{x\to\infty} 3x^2 = \infty$.

Answer available at the end of this section.

Unifying the Precise Definitions of Limits

The following table *might* help you understand when to use ϵ, δ, N, or M in your proofs.

	Finite Limit at a Finite Number
Limit	$\lim_{x\to a} f(x) = L$
Universal Quantifier	$\forall \epsilon > 0$
Existential Quantifier	$\exists \delta > 0$

I did not include the conditional statements in this list because each statement depends heavily on the type of limit in consideration. They each have the *basic* form

$$[\text{Universal Quantifier Statement}], [\text{Existential Quantifier Statement}] \ni$$
$$[\text{inequality involving } x \text{ and either } \delta \text{ or } M] \implies [\text{inequality involving } y \text{ and either } \epsilon \text{ or } N]$$

Success in College: Study Time

Believe it or not, there is a direct link between the number of units a class is worth and how many hours you should be studying

(outside of class). For each hour of class time, the California state education code suggests at least two hours of studying outside of class. If your math background is weak, you may need to spend more time studying.

At Cosumnes River College, each unit of a math course corresponds to one hour of in-class time per week. Thus, a five-unit math course requires five hours of in-class interaction per week and *ten* hours of study time outside of class per week.

One way to study is to do what you're doing now - read! Class time is an excellent opportunity to introduce new concepts and see how they fit with those previously learned; however, there is usually not enough time to address details with the depth that a textbook can. In this way, your professor and the textbook should be considered as a combined primary source of information in your course.

Here are some questions to consider regarding how you study.

1. Do you study at least two hours for every one hour of class time? If not, why?
2. How many hours do you dedicate to TV, YouTube, and social media every week (on average)?
3. How many hours do you dedicate to leisure activities per week (e.g., hanging out with friends, going to the movies, gaming)?
4. Would it be too difficult to trade a few hours for study time?

Success in College: I Understood It... Until I Left Class

You understand a concept when the professor does it in class, but you're lost when trying it on your own - this is very common. There are two general remedies to this problem.

The first revolves around how the information is delivered and how you engage with it. We tend to retain the least when only *listening* (maybe this is why those who passively sit in the back of a classroom and zone out end up having the highest failure rate). We gain a little more retention when we *see* someone do something (e.g., watching your professor, but not taking notes). A substantial leap in retention and understanding occurs when we *write* concepts down (e.g., taking notes during lectures). Finally, the biggest gains in understanding and retention occur when we *do* work on our own (e.g., practicing, doing homework, independently investigating a concept).

This kind of makes sense, right? This is why you see the most successful students in class attempting the examples before the professor starts them. It forces the information into their memory.

It gets even better if we involve "real-world" examples with manipulatives. Imagine learning about mathematical models that deal with finance (exponential and simple interest formulas) by *actually* investing. How cool would it be to build a fence that borders a river by using minimization techniques learned in Algebra? The problem with math and how it is taught is that we hear about the applications but nobody wants to go out and do them. If you had more physical or engaging examples, your chance of success would skyrocket. Unfortunately, there is only a limited amount of time in class, and this kind of teaching would require students to allot several more hours a day to their math class (however, the benefit would be enormous).

The second remedy is squarely on your shoulders. You will never master a skill by putting only half of your effort into it. To excel and enjoy mathematics, you must immerse yourself in it. Imagine yourself as a mathematician and look at the world differently. As you drive your car to work, make a mental graph of the distance you have traveled versus the time you have been driving (if you're in Calculus, then interpret the slope of the tangent line to this graph at any given point as the derivative). When you purchase items, do the math in your head... yes, even the tax! When you buy a car, make a function that describes the cost of owning the car! You'd be surprised at how helpful mathematics is.

In college, class time is limited. Often, class is over before students feel confident about new concepts. They leave class worried they won't be able to do the homework. These are just some suggestions, but others in your class might have some wonderful additions to this list. Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. How often does this happen to you (understanding in class but completely lost out of class)? When it does, what do you do about it?
2. How often do you think this happens to the other class members?

Answers

Section Preview

This section rounds out the precise definition of a limit by presenting the definitions involving infinities. As with the previous section, **this material is NOT optional**.

Section Preview

- **Logarithmic Functions**
  - The Basics of Logarithmic Functions: Reading $\ln(\epsilon) < 0$ for small ϵ (tripping point 7) and treating $\ln$ as the inverse of e^x require the conceptual relationship between logarithms and exponentials.

Section Preview

- **Arithmetic**
  - Mathematical Operations — Higher-Index Roots: Example 3 sets $M = \sqrt[3]{N}$ with $N \ll 0$, requiring evaluation of a cube root of a negative number.
- **Simplifying Expressions**
  - Laws of Exponents — Zero and Negative Exponent Rules: That same manipulation and Figure 3 rely on $e^{-x} = \frac{1}{e^x}$ and $e^{-2x} = \frac{1}{e^{2x}}$.
  - Simplifying Expressions Involving Positive Rational Exponents: The $\frac{1}{x^p}$ theorem (with p rational) sets $M = \frac{1}{e^{1/p}}$ and raises both sides to the p power, requiring fluency with rational-exponent expressions.
- **Solving Equations**
  - Solving Exponential Equations: The Theorem proof takes $\ln$ of both sides (e.g., $e^{-x} < \epsilon \implies -x < \ln(\epsilon)$)—the "take the log of both sides" maneuver.
  - Solving Logarithmic Equations: Example 2 exponentiates both sides ($\ln(x+2) < N \implies x + 2 < e^N$) to isolate the argument—the "exponentiate both sides" maneuver.

Section Preview

- Describe the precise definition of a limit for infinite limits, at finite points and infinity.
- Prove finite and infinite limits using the precise definition of a limit.

Example 1.7.1

PROOF

For $N \ggg 0$, let $\delta = \frac{1}{\sqrt{N}}$. If $0 < |x-1| < \delta$, then

$$\begin{aligned} |x-1| &< \frac{1}{\sqrt{N}} \\ \implies (x-1)^2 &< \frac{1}{N} \\ \implies \frac{1}{(x-1)^2} &> N \\ \implies f(x) &> N \end{aligned}$$

Thus, we have proved the limit

$$\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty.$$

Example 1.7.2

From algebra, we know the domain of $f(x) = \ln(x+2)$ is $(-2, \infty)$. Therefore, to prove $x = -2$ is a vertical asymptote, we only need to concern ourselves with the one-sided limit

$$\lim_{x\to -2^+} \ln(x+2).$$

We know that the graph of $\ln(x+2)$ tends to $-\infty$ as x approaches -2. Now we are asked to *prove* it. That is, we want to prove

$$\lim_{x\to -2^+} \ln(x+2) = -\infty.$$

PROOF

For $N \lll 0$, let $\delta = e^N$. If $0 < x - (-2) < \delta$, then

$$\begin{aligned} x + 2 &< e^N \\ \implies \ln(x+2) &< N \\ \implies f(x) &< N \end{aligned}$$

Thus, we have proved the limit

$$\lim_{x\to -2} \ln(x+2) = -\infty.$$

Therefore, $x = -2$ is a vertical asymptote of $f(x) = \ln(x+2)$.

Theorem

PROOF

For $\epsilon > 0$, let $M = -\ln(\epsilon)$. If $x > M$, then

$$\begin{aligned} & x > -\ln(\epsilon) \\ \Longrightarrow \quad & -x < \ln(\epsilon) \\ \Longrightarrow \quad & e^{-x} < \epsilon \\ \Longrightarrow \quad & |e^{-x} - 0| < \epsilon \\ \Longrightarrow \quad & |f(x) - L| < \epsilon \end{aligned}$$

Thus, we have proved the limit

$$\lim_{x \to \infty} e^{-x} = 0.$$

Corollary

$$\begin{aligned} \lim_{x \to \infty} e^{-kx} &= \lim_{x \to \infty} \left(e^{-x}\right)^k \\ &= \left(\lim_{x \to \infty} e^{-x}\right)^k \qquad \text{(Power Law of Limits)} \\ &= (0)^k \\ &= 0 \end{aligned}$$

Theorem

PROOF (for $\lim_{x \to \infty} \frac{1}{x^p} = 0$)

For $\epsilon > 0$, let $M = \frac{1}{\epsilon^{1/p}}$. If $x > M$, then

$$\begin{aligned} & x > \frac{1}{\epsilon^{1/p}} \\ \Longrightarrow \quad & x^p > \frac{1}{\epsilon} \\ \Longrightarrow \quad & \frac{1}{x^p} < \epsilon \\ \Longrightarrow \quad & \left|\frac{1}{x^p} - 0\right| < \epsilon \\ \Longrightarrow \quad & |f(x) - L| < \epsilon \end{aligned}$$

Thus, we have proved the limit

$$\lim_{x \to \infty} \frac{1}{x^p} = 0.$$

Example 1.7.3

PROOF

For $N \lll 0$, let $M = \sqrt[3]{N}$. If $x < M$, then

$$\begin{aligned} & x < \sqrt[3]{N} \\ \Longrightarrow \quad & x^3 < N \\ \Longrightarrow \quad & f(x) < N \end{aligned}$$

Thus, we have proved the limit

$$\lim_{x \to -\infty} x^3 = -\infty.$$

Checkpoint 1.7.3

Let $M > 0$. Let $N = \sqrt{\frac{M}{3}}$. Then, for all $x > N$, we have

$$3x^2 > 3N^2 = 3\left(\sqrt{\frac{M}{3}}\right)^2 = \frac{3M}{3} = M.$$

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1.7.1: Resources and Key Concepts

Resources

Videos

- Finite limits at infinity
  - Limits at Infinity - Evaluating Finite Limits at Infinity
- Proving infinite limits
  - Limits at Infinity - Proving Infinite Limits at Infinity
- Proving limits at infinity
  - Limits at Infinity - Proving Finite Limits at Infinity
- Calculus proofs for limits
  - Limits at Infinity - A Summary of Limit Proofs

Key Concepts

Definitions

- **Infinite Limit at a Finite Number (Precise Definition)**: Let $f(x)$ be defined for all $x \neq a$ over an open interval containing a.
  - We say $\lim_{x\to a} f(x) = \infty$ if for every $N \gg 0$, there exists a $\delta > 0$, such that if $0 < |x - a| < \delta$, then $f(x) > N$. (Symbolically: $\forall N \gg 0, \exists \delta > 0 \ni 0 < |x - a| < \delta \Rightarrow f(x) > N$)
  - We say $\lim_{x\to a} f(x) = -\infty$ if for every $N \ll 0$, there exists a $\delta > 0$, such that if $0 < |x - a| < \delta$, then $f(x) < N$. (Symbolically: $\forall N \ll 0, \exists \delta > 0 \ni 0 < |x - a| < \delta \Rightarrow f(x) < N$)
- **Vertical Asymptote**: The vertical line $x = a$ is called a vertical asymptote of the function $f(x)$ if at least one of the following is true: $\lim_{x\to a} f(x) = \infty$, $\lim_{x\to a} f(x) = -\infty$, $\lim_{x\to a^+} f(x) = \infty$, $\lim_{x\to a^+} f(x) = -\infty$, $\lim_{x\to a^-} f(x) = \infty$, or $\lim_{x\to a^-} f(x) = -\infty$.
- **Finite Limit at Infinity (Precise Definition)**:
  - Let $f(x)$ be defined for all $x > a$. We say $\lim_{x\to\infty} f(x) = L$ if for every $\epsilon > 0$, there exists a number $M > 0$, such that if $x > M$, then $|f(x) - L| < \epsilon$. (Symbolically: $\forall \epsilon > 0, \exists M > 0 \ni x > M \Rightarrow |f(x) - L| < \epsilon$)
  - Let $f(x)$ be defined for all $x < a$. We say $\lim_{x\to-\infty} f(x) = L$ if for every $\epsilon > 0$, there exists a number $M < 0$, such that if $x < M$, then $|f(x) - L| < \epsilon$. (Symbolically: $\forall \epsilon > 0, \exists M < 0 \ni x < M \Rightarrow |f(x) - L| < \epsilon$)
- **Horizontal Asymptote**: The line $y = L$ is called a horizontal asymptote of the function $f(x)$ if at least one of the following is true: $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$.
- **Infinite Limit at Infinity (Precise Definition)**:
  - **As** $x \to \infty$: Let $f(x)$ be defined for all $x > a$.
    - We say $\lim_{x\to\infty} f(x) = \infty$ if for every $N \gg 0$, there exists a number $M > 0$, such that if $x > M$, then $f(x) > N$.
    - We say $\lim_{x\to\infty} f(x) = -\infty$ if for every $N \ll 0$, there exists a number $M > 0$, such that if $x > M$, then $f(x) < N$.
  - **As** $x \to -\infty$: Let $f(x)$ be defined for all $x < a$.
    - We say $\lim_{x\to-\infty} f(x) = \infty$ if for every $N \gg 0$, there exists a number $M < 0$, such that if $x < M$, then $f(x) > N$.
    - We say $\lim_{x\to-\infty} f(x) = -\infty$ if for every $N \ll 0$, there exists a number $M < 0$, such that if $x < M$, then $f(x) < N$.

Theorems

- **Theorem**: $\lim_{x\to\infty} e^{-x} = 0$.
- **Corollary**: If $k > 0$, then $\lim_{x\to\infty} e^{-kx} = 0$.
- **Theorem**: If $p > 0$ is a rational number, then $\lim_{x\to\infty} \frac{1}{x^p} = 0$. (Also stated: If $p > 0$ is a rational number such that x^p is defined for all x, then $\lim_{x\to-\infty} \frac{1}{x^p} = 0$. The proof for $x \to -\infty$ is left as a homework exercise.)

Common Mistakes

- **Inequality Direction Switch**: Forgetting that an inequality switches directions if you multiply or divide by a negative number.
- **Even Powers and Positivity**: Forgetting that applying an even power to an expression results in a positive value (e.g., $(x-1)^2 > 0$).
- **Square Root of a Square**: Incorrectly simplifying $\sqrt{x^2}$ as x instead of $|x|$.
- **Exponential Function and Inequalities**: Misunderstanding how the increasing nature of $y = e^x$ affects inequalities: if $A < B$, then $e^A < e^B$.
- **Sign of e^N when $N \ll 0$**: Thinking e^N is large or negative when $N \ll 0$. For $N \ll 0$, e^N is a very small positive number.
- **Absolute Value of e^{-x}**: Forgetting that $|e^{-x}| = e^{-x}$ because $e^{-x} > 0$.
- **Sign of $M = -\ln(\epsilon)$**: Thinking $M = -\ln(\epsilon)$ is negative when ϵ is a very small positive number. In this case, $\ln(\epsilon)$ is negative, so $M = -\ln(\epsilon)$ is positive.
- **Absolute Value of $\frac{1}{x^p}$ for $x > 0$**: Forgetting that if $x > 0$, then $|\frac{1}{x^p}| = \frac{1}{x^p}$.

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1.7.2: Homework

Note

The following exercises are meant as hand-in homework (if assigned by your instructor). Typed-in answers here will not be saved or scored.

Proofs

Exercise 1

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Exercise 2

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Exercise 3

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Exercise 4

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Exercise 5

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Exercise 6

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Exercise 7

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Exercise 8

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1.7.3: Additional Exercises

Reading Questions

1. What is the key difference in the precise definition of an infinite limit at a finite point (e.g., $\lim_{x\to a} f(x) = \infty$) compared to a finite limit at a finite point, in terms of the "challenge" value (N vs. ϵ) and the condition on $f(x)$?
2. In the precise definition $\lim_{x\to a} f(x) = \infty$, what does $N \gg 0$ signify?
3. How is the definition of a vertical asymptote related to infinite limits at finite numbers?
4. When defining a finite limit at infinity (e.g., $\lim_{x\to\infty} f(x) = L$), what replaces the δ-neighborhood around a? What is the condition on x?
5. State the precise definition for $\lim_{x\to\infty} f(x) = L$.
6. State the precise definition for $\lim_{x\to-\infty} f(x) = L$.
7. How is the definition of a horizontal asymptote related to finite limits at infinity?
8. In the precise definition of an infinite limit at infinity (e.g., $\lim_{x\to\infty} f(x) = \infty$), what two quantities are we relating using N and M ?
9. State the precise definition for $\lim_{x\to\infty} f(x) = \infty$.
10. State the precise definition for $\lim_{x\to-\infty} f(x) = -\infty$.
11. What does the theorem regarding $\lim_{x\to\infty} \frac{1}{x^p}$ state for $p > 0$?
12. When proving $\lim_{x\to-2^+} \ln(x+2) = -\infty$, why is a one-sided limit considered?

Homework

In exercises 1 - 10, use the precise definitions to prove the given limits.

1) $\lim_{x\to 0} \frac{1}{x^2} = \infty$

2) $\lim_{x\to -1} \frac{3}{(x+1)^2} = \infty$

Answer

Let $\delta = \sqrt{\frac{3}{N}}$. If $0 < |x+1| < \sqrt{\frac{3}{N}}$, then $f(x) = \frac{3}{(x+1)^2} > N$.

3) $\lim_{x\to 2} -\frac{1}{(x-2)^2} = -\infty$

4) $\lim_{x\to \pi^+} \ln(x-\pi) = -\infty$

5) $\lim_{x\to\infty} \frac{2}{x^6} = 0$

6) $\lim_{x\to-\infty} -\frac{5}{x^3} = 0$

7) $\lim_{x\to\infty} ax^6 = \infty$, where $a > 0$

8) $\lim_{x\to-\infty} (-10 + 2x^5) = -\infty$

9) $\lim_{x\to-\infty} e^{-x} = \infty$

10) $\lim_{x\to-\infty} e^{x} = 0$

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1.8: Continuity

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

We begin our investigation of continuity by exploring what it means for a function to have *continuity at a point*. Intuitively, a function is continuous at a particular point if there is no break in its graph at that point; however, as often is the case, intuition has its flaws.

Continuity at a Point

Before we formally define what it means to be continuous at a point, let's consider various functions that fail to meet our intuitive notion of what it means to be continuous at a point. We then create a list of conditions that prevent such failures.

Our first function of interest is shown in Figure $1.8.1$. We see that the graph of $f(x)$ has a hole at a. In fact, $f(a)$ is undefined. At the very least, for $f(x)$ to be continuous at a, we need the following condition:

i. $f(a)$ is defined

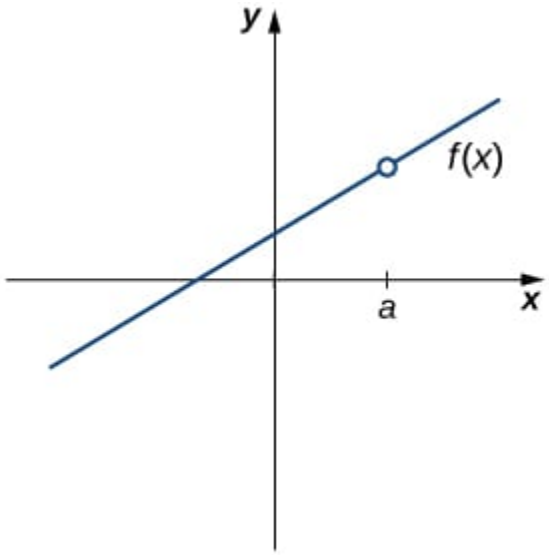

Figure $1.8.1$**:** *The function* $f(x)$ *is not continuous at* a *because* $f(a)$ *is undefined.*

However, as we see in Figure 1.8.2, this condition alone is insufficient to guarantee continuity at the point a. Although $f(a)$ is defined, the function has a gap at a. In this example, the gap exists because $\lim_{x \to a} f(x)$ does not exist. We must add another condition for continuity at a—namely,

ii. $\lim_{x \to a} f(x)$ **exists**

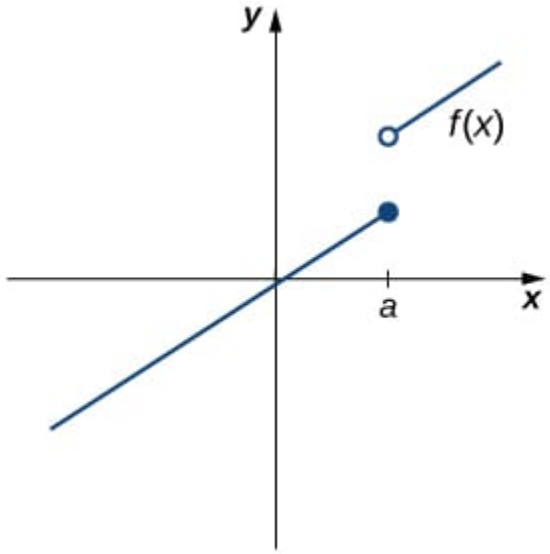

Figure 1.8.2**:** *The function $f(x)$ is not continuous at a because $\lim_{x \to a} f(x)$ does not exist.*

However, as we see in Figure 1.8.3, these two conditions by themselves do not guarantee continuity at a point. The function in this figure satisfies both of our first two conditions but is still not continuous at a. We must add a third condition to our list:

iii. $\lim_{x \to a} f(x) = f(a)$

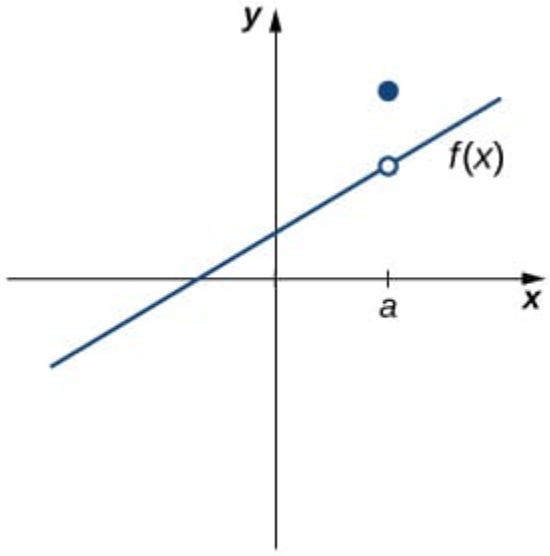

Figure 1.8.3**:** *The function $f(x)$ is not continuous at a because $\lim_{x \to a} f(x) \neq f(a)$.*

Now, we put our list of conditions together and form a definition of continuity at a point.

Definition: Continuous at a Point

A function $f(x)$ is said to be **continuous at a point** a if each of the following three conditions is satisfied:

i. $f(a)$ is defined
ii. $\lim_{x \to a} f(x)$ exists
iii. $\lim_{x \to a} f(x) = f(a)$

A function is **discontinuous at a point** $a$if it fails to be continuous at a.

Interactive Element: When Is a Function Continuous at a Point?

The following Interactive Element might help you visualize the third condition for continuity at a point.

Interact: Drag the red point up and down the line $x = 2$.

Observation: $f(2)$ is defined and the limit is 3 throughout, yet the graph is continuous only when the point lands in the hole — where $f(2) = 3 = \lim_{x \to 2} f(x)$. The value and the limit must be equal.

The next three examples demonstrate how to apply this definition to determine whether a function is continuous at a given point. These examples illustrate situations in which each of the conditions for continuity in the definition succeeds or fails.

Example $1.8.1A$: Determining Continuity at a Point, Condition 1

Using the definition, determine whether the function $f(x) = \frac{x^2-4}{x-2}$ is continuous at $x = 2$. Justify the conclusion.

Solution available at the end of this section.

Example $1.8.1B$: Determining Continuity at a Point, Condition 2

Using the definition, determine whether the function

$$f(x) = \begin{cases} -x^2 + 4, & \text{if } x \leq 3 \\ 4x - 8, & \text{if } x > 3 \end{cases}$$

is continuous at $x = 3$. Justify the conclusion.

Solution available at the end of this section.

Example $1.8.1C$: Determining Continuity at a Point, Condition 3

Using the definition, determine whether the function

$$f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$$

is continuous at $x = 0$.

Solution available at the end of this section.

Checkpoint 1.8.1

Using the definition, determine whether the function

$$f(x) = \begin{cases} 2x+1, & \text{if } x < 1 \\ 2, & \text{if } x = 1 \\ -x+4, & \text{if } x > 1 \end{cases}$$

is continuous at $x = 1$. If the function is not continuous at 1, indicate the condition for continuity at a point that fails to hold.

Answer available at the end of this section.

By applying the definition of continuity and previously established theorems concerning the evaluation of limits, we can state the following theorem.

Theorem: Continuity of Polynomials and Rational Functions

Polynomials and rational functions are continuous at every point in their domains.

Proof available at the end of this section.

We now apply this theorem to determine the points at which a given rational function is continuous.

Example 1.8.2: Continuity of a Rational Function

For what values of x is $f(x) = \frac{x+1}{x-5}$ continuous?

Solution available at the end of this section.

Checkpoint 1.8.2

For what values of x is $f(x) = 3x^4 - 4x^2$ continuous?

Answer available at the end of this section.

Types of Discontinuities

As we have seen in Example $1.8.1A$ and Example $1.8.1B$, discontinuities take on several different appearances. We classify the types of discontinuities we have seen thus far as removable, infinite, or jump discontinuities. Intuitively, a **removable discontinuity** is a discontinuity in which the graph has a hole. A **jump discontinuity** is a non-infinite discontinuity for which the sections of the function do not meet up. Finally, an **infinite discontinuity** is a discontinuity located at a vertical asymptote. Figure $1.8.6$ illustrates the differences in these types of discontinuities. Although these terms provide a handy way of describing three common types of discontinuities, remember that not all discontinuities fit neatly into these categories.

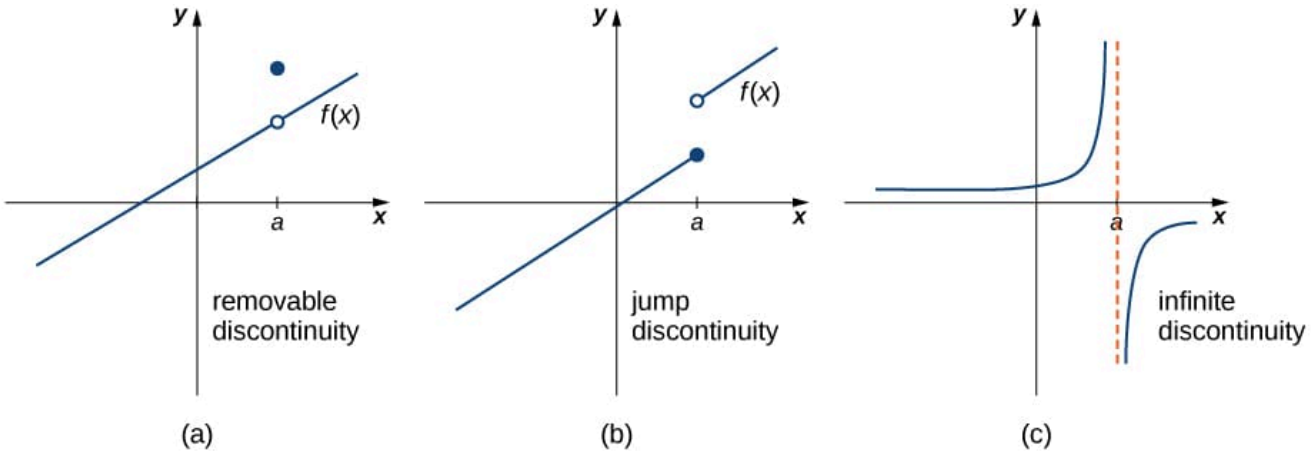

Figure 1.8.6**:** *Discontinuities are classified as (a) removable, (b) jump, or (c) infinite.*

These three discontinuities are formally defined as follows:

Definition: Types of Discontinuities

If $f(x)$ is discontinuous at a, then

i. f has a **removable discontinuity** at a if $\lim\limits_{x\to a} f(x)$ exists. (Note: When we state that $\lim\limits_{x\to a} f(x)$ exists, we mean that $\lim\limits_{x\to a} f(x) = L$, where L is a real number.)

ii. f has a **jump discontinuity** at a if $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ both exist, but $\lim\limits_{x\to a^-} f(x) \neq \lim\limits_{x\to a^+} f(x)$. (Note: When we state that $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ both exist, we mean that both are real-valued and that neither takes on the values $\pm\infty$.)

iii. f has an **infinite discontinuity** at a if $\lim\limits_{x\to a^-} f(x) = \pm\infty$ or $\lim\limits_{x\to a^+} f(x) = \pm\infty$.

Example 1.8.3: Classifying a Discontinuity

In Example 1.8.1A, we showed that $f(x) = \frac{x^2-4}{x-2}$ is discontinuous at $x = 2$. Classify this discontinuity as removable, jump, or infinite.

Solution available at the end of this section.

Example 1.8.4: Classifying a Discontinuity

In Example 1.8.1B, we showed that

$$f(x) = \begin{cases} -x^2 + 4, & \text{if } x \leq 3 \\ 4x - 8, & \text{if } x > 3 \end{cases}$$

is discontinuous at $x = 3$. Classify this discontinuity as removable, jump, or infinite.

Solution available at the end of this section.

Example 1.8.5: Classifying a Discontinuity

Determine whether $f(x) = \frac{x+2}{x+1}$ is continuous at -1. If the function is discontinuous at -1, classify the discontinuity as removable, jump, or infinite.

Solution available at the end of this section.

Checkpoint 1.8.5

For

$$f(x) = \begin{cases} x^2, & \text{if } x \neq 1 \\ 3, & \text{if } x = 1 \end{cases}$$

decide whether f is continuous at 1. If f is not continuous at 1, classify the discontinuity as removable, jump, or infinite.

Answer available at the end of this section.

Continuity over an Interval

Now that we have explored the concept of continuity at a point, we extend that idea to **continuity over an interval**. As we develop this idea for different types of intervals, it may be useful to keep in mind the intuitive (but flawed) idea that a function is continuous over an interval if we can use a pencil to trace the function between any two points in the interval without lifting the pencil from the paper. In preparation for defining continuity on an interval, we begin by defining what it means for a function to be continuous from the right at a point and continuous from the left at a point.

Definition: Continuity from the Right and from the Left

A function $f(x)$ is said to be **continuous from the right** at a if $\lim\limits_{x \to a^+} f(x) = f(a)$.

A function $f(x)$ is said to be **continuous from the left** at a if $\lim\limits_{x \to a^-} f(x) = f(a)$.

Definition: Continuity over Intervals

A function is **continuous over an open interval** if it is continuous at every point in the interval. A function $f(x)$ is **continuous over a closed interval** of the form $[a, b]$ if it is continuous at every point in (a, b) and is continuous from the right at a and is continuous from the left at b. Analogously, a function $f(x)$ is continuous over an interval of the form $(a, b]$ if it is continuous over (a, b) and is continuous from the left at b. Continuity over other types of intervals is defined similarly.

Requiring that $\lim\limits_{x \to a^+} f(x) = f(a)$ and $\lim\limits_{x \to b^-} f(x) = f(b)$ ensures that we can trace the graph of the function from the point $(a, f(a))$ to the point $(b, f(b))$ without lifting the pencil. If, for example, $\lim\limits_{x \to a^+} f(x) \neq f(a)$, we would need to lift our pencil to jump from $f(a)$ to the graph of the rest of the function over $(a, b]$.

Example 1.8.6: Continuity on an Interval

State the interval(s) over which the function $f(x) = \frac{x-1}{x^2+2x}$ is continuous.

Solution available at the end of this section.

Example 1.8.7: Continuity over an Interval

State the interval(s) over which the function $f(x) = \sqrt{4 - x^2}$ is continuous.

Solution available at the end of this section.

Checkpoint 1.8.7

State the interval(s) over which the function $f(x) = \sqrt{x+3}$ is continuous.

Answer available at the end of this section.

Continuity and Compositions

Theorem: Composite Function Theorem

If $f(x)$ is continuous at L and $\lim_{x \to a} g(x) = L$, then

$$\lim_{x \to a} f\big(g(x)\big) = f\Big(\lim_{x \to a} g(x)\Big) = f(L).$$

The Composite Function Theorem allows us to expand our ability to compute limits. In particular, this theorem ultimately allows us to demonstrate that trigonometric functions are continuous over their domains.

Before we move on to Example 1.8.8, recall that earlier, in the section on Limit Laws, we showed $\lim_{x \to 0} \cos x = 1 = \cos(0)$. Consequently, we know that $f(x) = \cos x$ is continuous at 0. In Example 1.8.8, we see how to combine this result with the Composite Function Theorem.

Example 1.8.8: Limit of a Composite Cosine Function

Evaluate $\lim_{x \to \pi/2} \cos\left(x - \frac{\pi}{2}\right)$.

Solution available at the end of this section.

Checkpoint 1.8.8

Evaluate $\lim_{x \to \pi} \sin(x - \pi)$.

Answer available at the end of this section.

The proof of the next theorem uses the Composite Function Theorem as well as the continuity of $f(x) = \sin x$ and $g(x) = \cos x$ at the point 0 to show that trigonometric functions are continuous over their entire domains.

Theorem: Continuity of Trigonometric Functions

Trigonometric functions are continuous over their entire domains.

Proof available at the end of this section.

As you can see, the Composite Function Theorem is invaluable in demonstrating the continuity of trigonometric functions. As we continue our study of calculus, we revisit this theorem many times.

The Intermediate Value Theorem

Functions that are continuous over intervals of the form $[a, b]$, where a and b are real numbers, exhibit many useful properties.

Throughout our study of calculus, we will encounter many powerful theorems concerning such functions. The first of these theorems is the **Intermediate Value Theorem.**

Theorem: Intermediate Value Theorem (IVT)

Let f be continuous over a closed, bounded interval $[a, b]$. If z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ satisfying $f(c) = z$.

See Figure 1.8.7 for a visualization of the IVT.

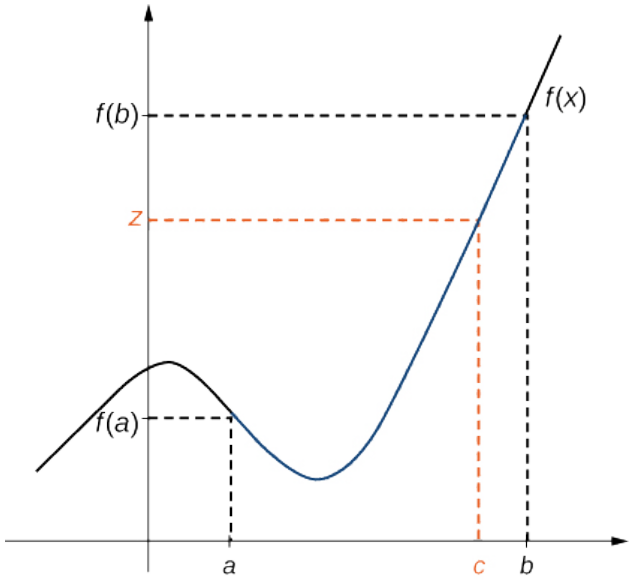

Figure 1.8.7: *There is a number $c \in [a, b]$ that satisfies $f(c) = z$.*

Interactive Element: The Intermediate Value Theorem

The following Interactive Element might help you visualize why a continuous function on a closed interval must hit every height between its endpoint values.

Interact: Drag the red point up and down the left edge to choose a target height z between $f(1) = 1$ and $f(5) = 5$.

Observation: No matter which z you pick in that band, the level line always meets the curve — there is always at least one c in $[1, 5]$ with $f(c) = z$.

Example 1.8.9: Application of the Intermediate Value Theorem

Show that $f(x) = x - \cos x$ has at least one zero.

Solution available at the end of this section.

Example 1.8.10: When Can You Apply the Intermediate Value Theorem?

If $f(x)$ is continuous over $[0, 2]$, $f(0) > 0$ and $f(2) > 0$, can we use the Intermediate Value Theorem to conclude that $f(x)$ has no zeros in the interval $[0, 2]$? Explain.

Solution available at the end of this section.

Example 1.8.11: When Can You Apply the Intermediate Value Theorem?

For $f(x) = 1/x, f(-1) = -1 < 0$ and $f(1) = 1 > 0$. Can we conclude that $f(x)$ has a zero in the interval $[-1, 1]$?

Solution available at the end of this section.

Checkpoint 1.8.11

Show that $f(x) = x^3 - x^2 - 3x + 1$ has a zero over the interval $[0, 1]$.

Answer available at the end of this section.

Success in College: Cheating and Consequences

The following is an excerpt from an article at WorldNetDaily.com by Dennis Prager.

> A report by Donald McCabe, professor of organization management at Rutgers University and founding president of the Center for Academic Integrity at Duke University, describes a pretty bad ethical situation in America's high schools. Seventy-five percent of students engage in "serious cheating" (such as repeated cheating and significant plagiarism).
>
> It gets worse.
>
> First, according to the professor, this epidemic is the culmination of three decades of increased cheating in high schools.
>
> Second, about half of all students (not just half of all cheaters) see nothing wrong with cheating.
>
> As one who has written and lectured on ethics for 30 years, during which time I, too, have spoken at high schools across America, I am certain that these statistics are accurate. I also believe that I know the major reasons.
>
> One reason is that parents emphasize many things over character.
>
> A second reason is the absurd emphasis on getting into the best college. This preoccupation on the part of parents (which often begins in preschool) has utterly distorted our children's (and high schools') priorities. Because students are taught that getting into the right college is the most important thing in their lives, they readily compromise their character to achieve this goal. And they could not care less about learning. Since the purpose of school is to get good grades, learning is incidental, and loving learning is a non-issue.
>
> A third and final reason is the death of the concept of sin, a product of the thoroughly secular education American high schools offer. As noted, far worse than the number of students who cheat is the number of students who think that there is nothing wrong with cheating. I can testify from my talks at high schools that the vast majority of students believe what they have been taught - that right and wrong - are entirely subjective concepts. There is no morality higher than one's own; therefore, right and wrong are defined by the individual, who can rationalize any behavior.

When students do not understand a concept, they need help. Instead, some students in this situation choose to cheat. In college, any form of cheating is "academic dishonesty."

1. Has your professor discussed or posted their cheating policy?
2. What happens if you cheat in this math class?

> To be clear, cheating - in any form - results in more than just a "zero" on that assignment. This includes, but is not limited to, using notes on exams, using a device that is not allowed during an exam, or using unapproved technology to help during an exam. If I suspect a student of cheating, I will have them meet me outside of class time in my office to discuss what happened and why it is considered cheating. They will receive a zero on the assignment/exam and will be reported to the Student Discipline Officer for further discussions. The Student Discipline Officer is involved for two reasons: to give the student tools to help them break the "cheating habit," and to create an official record within the

Los Rios Community College District to track that student's cheating offenses - if multiple offenses have occurred, the student could like be placed on academic probation or, in severe cases, completely expelled from the District.

- Professor Simpson, Cosumnes River College

Success in Mathematics: Holding Onto Old, Broken Techniques

It's common for students to learn a problem-solving technique years ago and not want to learn a different method because they are comfortable with their old habits. Unfortunately, we often fail to recognize that if our old, comfortable technique worked well, we should be much further along in our mathematical journey.

This adherence to comfort is widespread for all human beings in various situations. Whether it's a dead-end job, an abusive relationship, or a bad habit, we tend to ignore signals—remaining in a known situation rather than taking the chance to improve and evolve.

In general, when your professor gives you advice or a different approach to doing something, they are doing so from the lens of years of experience seeing students repeat old, broken techniques or habits that tend to lead students to make mistakes. Paying attention to these words of wisdom is critical.

EXAMPLE

Simplify the compound rational expression

$$\frac{-\frac{2}{(x+h)^2}+\frac{2}{x^2}}{h}$$

Students stuck in old habits tend to find the common denominator of the fractions in the numerator, make equivalent fractions, add fractions, divide the result by h, then simplify. This entire process looks like the following:

$$\begin{aligned}
\frac{-\frac{2}{(x+h)^2}+\frac{2}{x^2}}{h} &= \frac{-\frac{2x^2}{x^2(x+h)^2}+\frac{2(x+h)^2}{x^2(x+h)^2}}{h} \\
&= \frac{\frac{-2x^2+2(x+h)^2}{x^2(x+h)^2}}{h} \\
&= \frac{-2x^2+2(x+h)^2}{x^2(x+h)^2} \div h \\
&= \frac{-2x^2+2(x+h)^2}{x^2(x+h)^2} \cdot \frac{1}{h} \\
&= \frac{-2x^2+2(x+h)^2}{hx^2(x+h)^2} \\
&= \frac{-2x^2+2(x^2+2hx+h^2)}{hx^2(x+h)^2} \\
&= \frac{-2x^2+2x^2+4hx+2h^2}{hx^2(x+h)^2} \\
&= \frac{4hx+2h^2}{hx^2(x+h)^2} \\
&= \frac{h(4x+2h)}{hx^2(x+h)^2} \\
&= \frac{\cancel{h}(4x+2h)}{\cancel{h}x^2(x+h)^2} \\
&= \frac{(4x+2h)}{x^2(x+h)^2}
\end{aligned}$$

Compare this to the advice given by your math professor to begin by multiplying both numerator and denominator of the original expression by the LCD of all minor fractions within the compound rational expression.

$$\begin{aligned}
\frac{-\frac{2}{(x+h)^2}+\frac{2}{x^2}}{h} &= \frac{-2x^2+2(x+h)^2}{hx^2(x+h)^2} \\
&= \frac{-2x^2+2(x^2+2hx+h^2)}{hx^2(x+h)^2} \\
&= \frac{-2x^2+2x^2+4hx+2h^2}{hx^2(x+h)^2} \\
&= \frac{4hx+2h^2}{hx^2(x+h)^2} \\
&= \frac{h(4x+2h)}{hx^2(x+h)^2} \\
&= \frac{\cancel{h}(4x+2h)}{\cancel{h}x^2(x+h)^2} \\
&= \frac{(4x+2h)}{x^2(x+h)^2}
\end{aligned}$$

As you can see, the second method is cleaner and less complex.

The moral of the story is, when learning, try to be open to... learn.

Answers

Section Preview

Before we get to derivatives, we have one last limit-related concept—continuity. For students at this level, the most important result in this section is that, as long as we have a function that is continuous at $x = a$, we can simply substitute a when evaluating the limit of the function as x is approaching a. Don't skip over the Intermediate Value Theorem—it appears throughout the rest of the course!

Section Preview

- **An Overview of Functions**
  - Function Composition and Domain: The Composite Function Theorem and the proof of trigonometric continuity require understanding nested functions $f\big(g(x)\big)$; Example 8 treats $\cos\left(x - \frac{\pi}{2}\right)$ as a composition.

Section Preview

- **Trigonometry**
  - Evaluating Trigonometric Functions at Special Angles: The IVT application (Example 9) and Example 8 require exact values such as $\cos 0 = 1$, $\cos\frac{\pi}{2} = 0$, and $\sin\pi = 0$ (Checkpoint 8).
  - Trigonometric Identities — Sum and Difference of Angles Identities: The proof that trigonometric functions are continuous expands $\cos\big((x - a) + a\big)$ as $\cos(x - a)\cos a - \sin(x - a)\sin a$.

Section Preview

- Explain the three conditions for continuity at a point.
- Describe three kinds of discontinuities.
- Define continuity on an interval.
- Evaluate limits using the Generalized Direct Substitution Property.
- State the theorem for limits of composite functions.
- Understand and investigate uses of the Intermediate Value Theorem.

Example $1.8.1A$: Determining Continuity at a Point, Condition 1

Let's begin by trying to calculate $f(2)$. We can see that $f(2) = \frac{4-4}{2-2}$, which is undefined. Therefore, $f(x) = \frac{x^2-4}{x-2}$ is discontinuous at 2 because $f(2)$ is undefined. The graph of $f(x)$ is shown in Figure $1.8.4$.

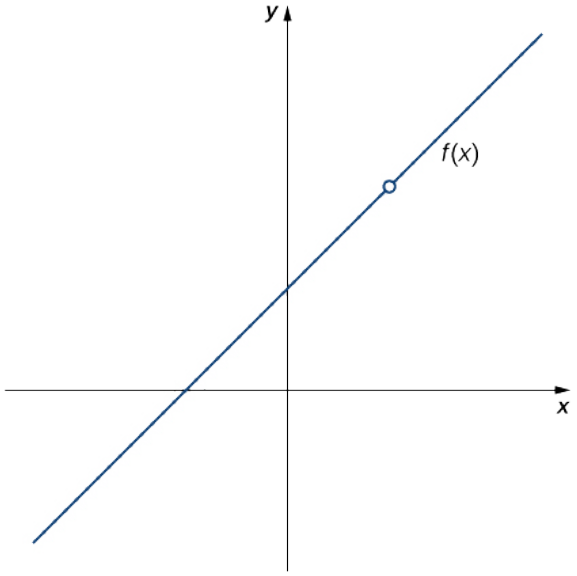

Figure $1.8.4$: *The function $f(x)$ is discontinuous at 2 because $f(2)$ is undefined.*

Example $1.8.1B$: Determining Continuity at a Point, Condition 2

Let's begin by trying to calculate $f(3)$.

$$f(3) = -(3^2) + 4 = -5.$$

Thus, $f(3)$ is defined. Next, we calculate $\lim\limits_{x\to 3} f(x)$. To do this, we must compute $\lim\limits_{x\to 3^-} f(x)$ and $\lim\limits_{x\to 3^+} f(x)$:

$$\lim_{x\to 3^-} f(x) = -(3^2) + 4 = -5$$

and

$$\lim_{x\to 3^+} f(x) = 4(3) - 8 = 4.$$

Therefore, $\lim\limits_{x\to 3} f(x)$ does not exist. Thus, $f(x)$ is not continuous at 3. The graph of $f(x)$ is shown in Figure $1.8.5$.

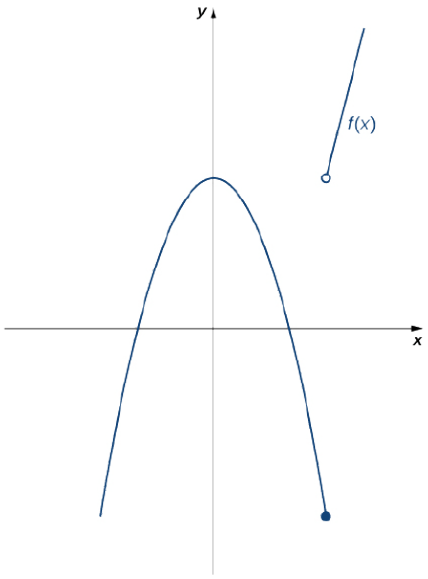

Figure 1.8.5**:** *The function $f(x)$ is not continuous at 3 because $\lim_{x\to 3} f(x)$ does not exist.*

Example $1.8.1C$**: Determining Continuity at a Point, Condition 3**

First, observe that

$$f(0) = 1.$$

Next,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Last, compare $f(0)$ and $\lim_{x\to 0} f(x)$. We see that

$$f(0) = 1 = \lim_{x\to 0} f(x).$$

Since all three of the conditions in the definition of continuity are satisfied, $f(x)$ is continuous at $x = 0$.

Checkpoint $1.8.1$

f is not continuous at 1 because $f(1) = 2 \neq 3 = \lim_{x\to 1} f(x)$.

Theorem: Continuity of Polynomials and Rational Functions

Previously, we showed that if $p(x)$ and $q(x)$ are polynomials, $\lim_{x\to a} p(x) = p(a)$ for every polynomial $p(x)$ and $\lim_{x\to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$ as long as $q(a) \neq 0$. Therefore, polynomials and rational functions are continuous on their domains.

Q.E.D.

Example $1.8.2$**: Continuity of a Rational Function**

The rational function $f(x) = \frac{x+1}{x-5}$ is continuous for every value of x except $x = 5$.

Checkpoint $1.8.2$

$f(x)$ is continuous at every real number.

Example $1.8.3$**: Classifying a Discontinuity**

To classify the discontinuity at 2 we must evaluate $\lim_{x\to 2} f(x)$:

$$\begin{aligned} \lim_{x\to 2} f(x) &= \lim_{x\to 2} \frac{x^2-4}{x-2} \\ &= \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} \\ &= \lim_{x\to 2} (x+2) \\ &= 4 \end{aligned}$$

Since f is discontinuous at 2 and $\lim_{x\to 2} f(x)$ exists, f has a removable discontinuity at $x = 2$.

Example $1.8.4$**: Classifying a Discontinuity**

Earlier, we showed that f is discontinuous at 3 because $\lim_{x\to 3} f(x)$ does not exist. However, since $\lim_{x\to 3^-} f(x) = -5$ and $\lim_{x\to 3^+} f(x) = 4$ both exist, we conclude that the function has a jump discontinuity at 3.

Example $1.8.5$**: Classifying a Discontinuity**

The function value $f(-1)$ is undefined. Therefore, the function is not continuous at -1. To determine the type of discontinuity, we must determine the limit at -1. We see that $\lim_{x\to -1^-} \frac{x+2}{x+1} = -\infty$ and $\lim_{x\to -1^+} \frac{x+2}{x+1} = +\infty$. Therefore, the function has an infinite discontinuity at -1.

Checkpoint 1.8.5

Discontinuous at 1; removable

Example 1.8.6**: Continuity on an Interval**

Since $f(x) = \frac{x-1}{x^2+2x}$ is a rational function, it is continuous at every point in its domain. The domain of $f(x)$ is the set $(-\infty,-2) \cup (-2,0) \cup (0,+\infty)$. Thus, $f(x)$ is continuous over each of the intervals $(-\infty,-2), (-2,0)$, and $(0,+\infty)$.

Example 1.8.7**: Continuity over an Interval**

From the Limit Laws, we know that $\lim_{x\to a} \sqrt{4-x^2} = \sqrt{4-a^2}$ for all values of a in $(-2,2)$. We also know that $\lim_{x\to -2^+} \sqrt{4-x^2} = 0$ exists and $\lim_{x\to 2^-} \sqrt{4-x^2} = 0$ exists. Therefore, $f(x)$ is continuous over the interval $[-2,2]$.

Checkpoint 1.8.7

$[-3,+\infty)$

Example 1.8.8**: Limit of a Composite Cosine Function**

The given function is a composite of $\cos x$ and $x - \frac{\pi}{2}$. Since $\lim_{x\to\pi/2} \left(x - \frac{\pi}{2}\right) = 0$ and $\cos x$ is continuous at 0, we may apply the Composite Function Theorem. Thus,

$$\lim_{x\to\pi/2} \cos\left(x - \frac{\pi}{2}\right) = \cos\left(\lim_{x\to\pi/2} \left(x - \frac{\pi}{2}\right)\right) = \cos(0) = 1.$$

Checkpoint 1.8.8

0

Theorem: Continuity of Trigonometric Functions

We begin by demonstrating that $\cos x$ is continuous at every real number. To do this, we must show that $\lim_{x\to a} \cos x = \cos a$ for all values of a.

$$\begin{aligned}
\lim_{x\to a} \cos x &= \lim_{x\to a} \cos((x-a)+a) && \text{(rewrite } x = x-a+a\text{)} \\
&= \lim_{x\to a} (\cos(x-a)\cos a - \sin(x-a)\sin a) && \text{(Sum and Difference of Angles Identities)} \\
&= \cos(\lim_{x\to a}(x-a))\cos a - \sin(\lim_{x\to a}(x-a))\sin a && \text{(since } \lim_{x\to a}(x-a) = 0, \text{ and } \sin x \text{ and } \cos x \text{ are continuous at } 0\text{)} \\
&= \cos(0)\cos a - \sin(0)\sin a \\
&= 1\cdot\cos a - 0\cdot\sin a \\
&= \cos a
\end{aligned}$$

The proof that $\sin x$ is continuous at every real number is analogous. Because the remaining trigonometric functions may be expressed in terms of $\sin x$ and $\cos x$, their continuity follows from the quotient limit law.

Q.E.D.

Example 1.8.9**: Application of the Intermediate Value Theorem**

Since $f(x) = x - \cos x$ is continuous over $(-\infty,+\infty)$, it is continuous over any closed interval of the form $[a,b]$. If you can find an interval $[a,b]$ such that $f(a)$ and $f(b)$ have opposite signs, you can use the Intermediate Value Theorem to conclude there must be a real number c in (a,b) that satisfies $f(c) = 0$. Note that

$$f(0) = 0 - \cos(0) = -1 < 0$$

and

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - \cos\frac{\pi}{2} = \frac{\pi}{2} > 0.$$

Using the Intermediate Value Theorem, we can see that there must be a real number c in $[0,\pi/2]$ that satisfies $f(c) = 0$. Therefore, $f(x) = x - \cos x$ has at least one zero.

Example 1.8.10**: When Can You Apply the Intermediate Value Theorem?**

No. The Intermediate Value Theorem only allows us to conclude that we can find a value between $f(0)$ and $f(2)$; it doesn't allow us to conclude that we can't find other values. To see this more clearly, consider the function $f(x) = (x-1)^2$. It satisfies $f(0) = 1 > 0, f(2) = 1 > 0$, and $f(1) = 0$.

Example 1.8.11**: When Can You Apply the Intermediate Value Theorem?**

No. The function is not continuous over $[-1,1]$. The Intermediate Value Theorem does not apply here.

Checkpoint 1.8.11

$f(0) = 1 > 0$, $f(1) = -2 < 0$; $f(x)$ is continuous over $[0,1]$. It must have a zero on this interval.

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1.8.1: Resources and Key Concepts

Resources

Videos

- Continuity
  - Limits - Continuity of Functions - Definition, Theory, and Examples
  - Limits - Continuity of Functions - Continuity on Intervals - Proving Arithmetic Retains Continuity
- Discontinuity
  - Limits - Continuity of Functions - Types of Discontinuities
- Continuity on an interval
  - Limits - Continuity of Functions - Continuity on Intervals
- Composite Function Theorem
  - Limits - Continuity of Functions - Continuity of Compositions - Proof: Limit(Comp) = Comp(Limit)
- Continuity of composite functions
  - Limits - Continuity of Functions - Continuity of Compositions
  - Limits - Continuity of Functions - Continuity of Compositions - Proof: Continuity of Compositions
- Intermediate Value Theorem
  - Limits - Continuity of Functions - The Intermediate Value Theorem
- Piecewise function continuity
  - Limits - Continuity of Functions - Example: Proving Continuity at 0 Without Using Squeeze Theorem

Key Concepts

Definitions

- **Continuous at a Point**: A function $f(x)$ is said to be continuous at a point a if each of the following three conditions is satisfied:
  i. $f(a)$ is defined
  ii. $\lim\limits_{x\to a} f(x)$ exists
  iii. $\lim\limits_{x\to a} f(x) = f(a)$
- **Discontinuous at a Point**: A function is discontinuous at a point a if it fails to be continuous at a.
- **Types of Discontinuities**: If $f(x)$ is discontinuous at a, then
  i. f has a **removable discontinuity** at a if $\lim\limits_{x\to a} f(x)$ exists. (Note: When we state that $\lim\limits_{x\to a} f(x)$ exists, we mean that $\lim\limits_{x\to a} f(x) = L$, where L is a real number.)
  ii. f has a **jump discontinuity** at a if $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ both exist, but $\lim\limits_{x\to a^-} f(x) \neq \lim\limits_{x\to a^+} f(x)$. (Note: When we state that $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ both exist, we mean that both are real-valued and that neither takes on the values $\pm\infty$.)
  iii. f has an **infinite discontinuity** at a if $\lim\limits_{x\to a^-} f(x) = \pm\infty$ or $\lim\limits_{x\to a^+} f(x) = \pm\infty$.
- **Continuity from the Right**: A function $f(x)$ is said to be continuous from the right at a if $\lim\limits_{x\to a^+} f(x) = f(a)$.
- **Continuity from the Left**: A function $f(x)$ is said to be continuous from the left at a if $\lim\limits_{x\to a^-} f(x) = f(a)$.
- **Continuity over Intervals**:
  - A function $f(x)$ is **continuous over an open interval** if it is continuous at every point in the interval.
  - A function $f(x)$ is **continuous over a closed interval** of the form $[a, b]$ if it is continuous at every point in (a, b) and is continuous from the right at a and is continuous from the left at b.

Theorems

- **Theorem: Continuity of Polynomials and Rational Functions**: Polynomials and rational functions are continuous at every

point in their domains.

- **Theorem: Composite Function Theorem**: If $f(x)$ is continuous at L and $\lim_{x \to a} g(x) = L$, then $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)) = f(L)$.
- **Theorem: Continuity of Trigonometric Functions**: Trigonometric functions are continuous over their entire domains.
- **Theorem: Intermediate Value Theorem (IVT)**: Let f be continuous over a closed, bounded interval $[a, b]$. If z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ satisfying $f(c) = z$.

Common Mistakes

- **Intuition Flaws for Continuity**: The intuitive notion of a function being continuous if its graph can be traced without lifting a pencil has flaws.
- **Misapplying the Intermediate Value Theorem**:
  - The IVT only allows us to conclude that we can find a value between $f(a)$ and $f(b)$; it doesn't allow us to conclude that we can't find other values (e.g., if $f(a) > 0$ and $f(b) > 0$, it doesn't mean there are no zeros in between).
  - The IVT does not apply if the function is not continuous over the interval.

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1.8.2: Homework

Note

The following exercises are meant as hand-in homework (if assigned by your instructor). Typed-in answers here will not be saved or scored.

Basic Skills

Exercise 1

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Exercise 2

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Exercise 9

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Exercise 23

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Synthesis Questions

Exercise 24

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Exercise 25

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Exercise 33

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Synthesis Questions - Applications

Exercise 34

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Proofs

Exercise 35

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Exercise 36

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Exercise 41

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Challenge Problems

Exercise 42

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1.8.3: Additional Exercises

Reading Questions

1. What are the three conditions that must be satisfied for a function $f(x)$ to be continuous at a point a?
2. If $\lim_{x \to a} f(x)$ exists but is not equal to $f(a)$ (assuming $f(a)$ is defined), what type of discontinuity does f have at a?
3. Describe a jump discontinuity in terms of one-sided limits.
4. What characterizes an infinite discontinuity at $x = a$ in terms of limits?
5. What does it mean for a function to be continuous from the left at a point a?
6. How is continuity over a closed interval $[a, b]$ defined, including the conditions at the endpoints a and b?
7. At what points are polynomial functions continuous?
8. At what points are rational functions continuous? What condition must be met for their denominators?
9. State the Composite Function Theorem as it applies to limits.
10. Are trigonometric functions like $\sin x$ and $\cos x$ continuous over their entire domains?
11. What are the conditions required to apply the Intermediate Value Theorem to a function f on an interval $[a, b]$?
12. If the conditions for the Intermediate Value Theorem are met, and z is a number between $f(a)$ and $f(b)$, what does the theorem guarantee?
13. If a function f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs, what specific conclusion can be drawn from the IVT?

Homework

For exercises 1 - 8, determine the point(s), if any, at which each function is discontinuous. Classify any discontinuity as *jump*, *removable*, *infinite*, or *other*.

1) $f(x) = \dfrac{1}{\sqrt{x}}$

Answer

The function is defined for all x in the interval $(0, \infty)$.

2) $f(x) = \dfrac{2}{x^2 + 1}$

3) $f(x) = \dfrac{x}{x^2 - x}$

Answer

Removable discontinuity at $x = 0$; infinite discontinuity at $x = 1$.

4) $g(t) = t^{-1} + 1$

5) $f(x) = \dfrac{5}{e^x - 2}$

Answer

Infinite discontinuity at $x = \ln 2$

6) $f(x) = \dfrac{|x - 2|}{x - 2}$

7) $H(x) = \tan 2x$

Answer

Infinite discontinuities at $x = \dfrac{(2k + 1)\pi}{4}$, for $k = 0, \ \pm 1, \ \pm 2, \ \pm 3, \ \ldots$

8) $f(t) = \dfrac{t+3}{t^2+5t+6}$

For exercises 9 - 14, decide if the function continuous at the given point. If it is discontinuous, what type of discontinuity is it?

9) $\dfrac{2x^2-5x+3}{x-1}$ at $x=1$

Answer

No. It is a removable discontinuity.

10) $h(\theta) = \dfrac{\sin\theta - \cos\theta}{\tan\theta}$ at $\theta = \pi$

11) $g(u) = \begin{cases} \dfrac{6u^2+u-2}{2u-1}, & \text{if } u \neq \frac{1}{2} \\ \frac{7}{2}, & \text{if } u = \frac{1}{2} \end{cases}$, at $u = \frac{1}{2}$

Answer

Yes. It is continuous.

12) $f(y) = \dfrac{\sin(\pi y)}{\tan(\pi y)}$, at $y=1$

13) $f(x) = \begin{cases} x^2 - e^x, & \text{if } x < 0 \\ x-1, & \text{if } x \geq 0 \end{cases}$, at $x=0$

Answer

Yes. It is continuous.

14) $f(x) = \begin{cases} x\sin(x), & \text{if } x \leq \pi \\ x\tan(x), & \text{if } x > \pi \end{cases}$, at $x=\pi$

In exercises 15 - 19, find the value(s) of k that makes each function continuous over the given interval.

15) \(f(x)=\begin{cases}3x+2, & \text{if }x

Answer

$k=-5$

16) $f(\theta) = \begin{cases} \sin\theta, & \text{if } 0 \leq \theta < \frac{\pi}{2} \\ \cos(\theta+k), & \text{if } \frac{\pi}{2} \leq \theta \leq \pi \end{cases}$

17) $f(x) = \begin{cases} \dfrac{x^2+3x+2}{x+2}, & \text{if } x \neq -2 \\ k, & \text{if } x = -2 \end{cases}$

Answer

$k=-1$

18) $f(x) = \begin{cases} e^{kx}, & \text{if } 0 \leq x < 4 \\ x+3, & \text{if } 4 \leq x \leq 8 \end{cases}$

19) \(f(x)=\begin{cases}\sqrt{kx}, & \text{if }0 \leq x \leq 3\x+1, & \text{if }3

Answer

$k = \frac{16}{3}$

In exercises 20 - 21, use the Intermediate Value Theorem (IVT).

20) Let $h(x) = \begin{cases} 3x^2-4, & \text{if } x \leq 2 \\ 5+4x, & \text{if } x > 2 \end{cases}$ Over the interval $[0,4]$, there is no value of x such that $h(x)=10$, although $h(0) < 10$ and $h(4)$

> 10. Explain why this does not contradict the IVT.

21) A particle moving along a line for time t has a position function $s(t)$, which is continuous. Assume $s(2) = 5$ and $s(5) = 2$. Another particle moves such that its position is given by $h(t) = s(t) - t$. Explain why there must be a value c for \(2

Answer

Since both s and $y = t$ are continuous everywhere, then $h(t) = s(t) - t$ is continuous everywhere and, in particular, it is continuous over the closed interval $[2, 5]$. Also, $h(2) = 3 > 0$ and $h(5) = -3 < 0$. Therefore, by the IVT, there is a value $x = c$ such that $h(c) = 0$.

22) Use the statement "The cosine of t is equal to t cubed."

a. Write a mathematical equation of the statement.

b. Prove that the equation in part a. has at least one real solution.

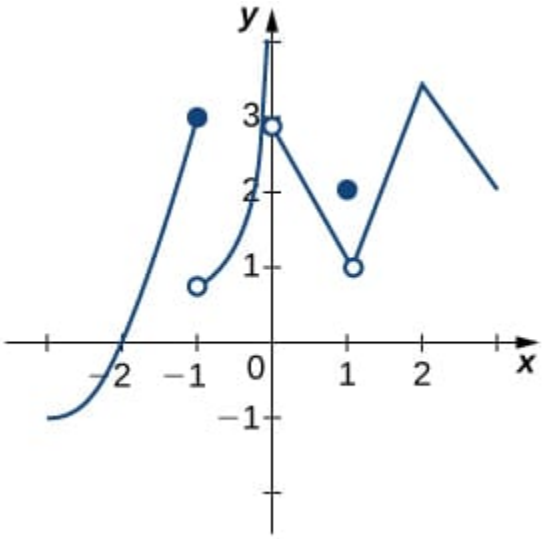
c. Use a calculator to find an interval of length 0.01 that contains a solution.

23) Apply the IVT to determine whether $2^x = x^3$ has a solution in one of the intervals $[1.25, 1.375]$ or $[1.375, 1.5]$. Briefly explain your response for each interval.

Answer

The function $f(x) = 2^x - x^3$ is continuous over the interval $[1.25, 1.375]$ and has opposite signs at the endpoints.

24) Consider the graph of the function $y = f(x)$ shown in the following graph.



a. Find all values for which the function is discontinuous.

b. For each value in part a., state why the formal definition of continuity does not apply.

c. Classify each discontinuity as either jump, removable, or infinite.

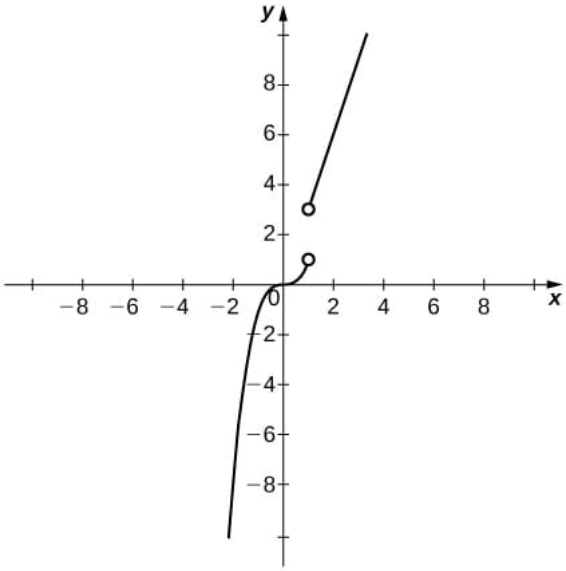
25) Let $f(x) = \begin{cases} 3x, & \text{if } x > 1 \\ x^3, & \text{if } x < 1 \end{cases}$.

a. Sketch the graph of f.

b. Is it possible to find a value k such that $f(1) = k$, which makes $f(x)$ continuous for all real numbers? Briefly explain.

Answer

a.



b. It is not possible to redefine $f(1)$ since the discontinuity is a jump discontinuity.

26) Let $f(x) = \dfrac{x^4 - 1}{x^2 - 1}$ for $x \neq -1, 1$.

a. Sketch the graph of f.

b. Is it possible to find values k_1 and k_2 such that $f(-1) = k$ and $f(1) = k_2$, and that makes $f(x)$ continuous for all real numbers? Briefly explain.

27) Sketch the graph of the function $y = f(x)$ with properties i. through vii.

i. The domain of f is $(-\infty, +\infty)$.

ii. f has an infinite discontinuity at $x = -6$.

iii. $f(-6) = 3$

iv. $\lim\limits_{x \to -3^-} f(x) = \lim\limits_{x \to -3^+} f(x) = 2$

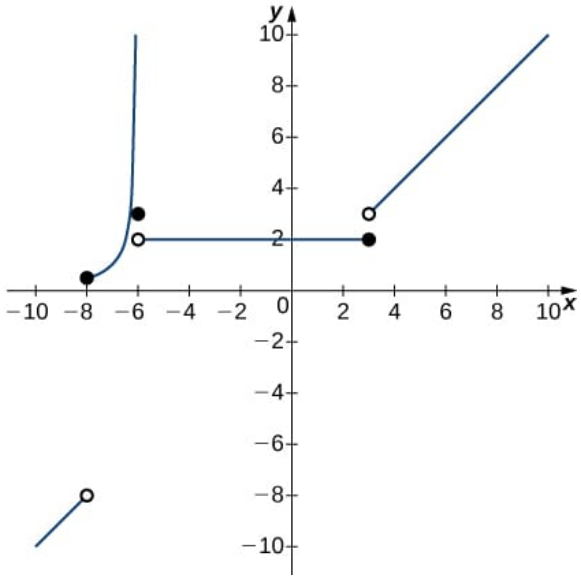
v. $f(-3) = 3$

vi. f is left continuous but not right continuous at $x = 3$.

vii. $\lim\limits_{x \to -\infty} f(x) = -\infty$ and $\lim\limits_{x \to +\infty} f(x) = +\infty$

Answer

Answers may vary; see the following example:



28) Sketch the graph of the function $y = f(x)$ with properties i. through iv.

i. The domain of f is $[0, 5]$.

ii. $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1^-} f(x)$ exist and are equal.

iii. $f(x)$ is left continuous but not continuous at $x = 2$, and right continuous but not continuous at $x = 3$.

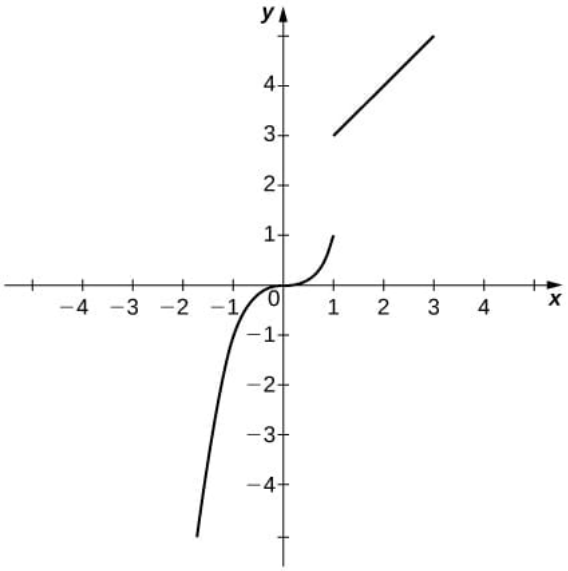
iv. $f(x)$ has a removable discontinuity at $x = 1$, a jump discontinuity at $x = 2$, and the following limits hold: $\lim_{x \to 3^-} f(x) = -\infty$ and $\lim_{x \to 3^+} f(x) = 2$.

In exercises 29 - 30, suppose $y = f(x)$ is defined for all x. For each description, sketch a graph with the indicated property.

29) Discontinuous at $x = 1$ with $\lim_{x \to -1} f(x) = -1$ and $\lim_{x \to 2} f(x) = 4$

Answer

Answers may vary; see the following example:



30) Discontinuous at $x = 2$ but continuous elsewhere with $\lim_{x \to 0} f(x) = \frac{1}{2}$

Determine whether each of the given statements is true. Justify your response with an explanation or counterexample.

31) $f(t) = \dfrac{2}{e^t - e^{-t}}$ is continuous everywhere.

Answer

False. It is continuous over $(-\infty, 0)$ \cup $(0, \infty)$.

32) If the left- and right-hand limits of $f(x)$ as $x \to a$ exist and are equal, then f cannot be discontinuous at $x = a$.

33) If a function is not continuous at a point, then it is not defined at that point.

Answer

False. Consider $f(x) = \begin{cases} x, & \text{if } x \neq 0 \\ 4, & \text{if } x = 0 \end{cases}$.

34) According to the IVT, $\cos x - \sin x - x = 2$ has a solution over the interval $[-1, 1]$.

35) If $f(x)$ is continuous such that $f(a)$ and $f(b)$ have opposite signs, then $f(x) = 0$ has exactly one solution in $[a, b]$.

Answer

False. Consider $f(x) = \cos(x)$ on $[-\pi, 2\pi]$.

36) The function $f(x) = \dfrac{x^2 - 4x + 3}{x^2 - 1}$ is continuous over the interval $[0, 3]$.

37) If $f(x)$ is continuous everywhere and $f(a), f(b) > 0$, then there is no root of $f(x)$ in the interval $[a, b]$.

Answer

False. The IVT does not work in reverse! Consider $(x - 1)^2$ over the interval $[-2, 2]$.

The following problems consider the scalar form of Coulomb's law, which describes the electrostatic force between two point charges, such as electrons. It is given by the equation $F(r) = k_e \dfrac{|q_1 q_2|}{r^2}$, where k_e is Coulomb's constant, q_i are the magnitudes of the charges of the two particles, and r is the distance between the two particles.

38) To simplify the calculation of a model with many interacting particles, after some threshold value $r = R$, we approximate F as zero.

a. Explain the physical reasoning behind this assumption.

b. What is the force equation?

c. Evaluate the force F using both Coulomb's law and our approximation, assuming two protons with a charge magnitude of 1.6022×10^{-19} **coulombs (C)**, and the Coulomb constant $k_e = 8.988 \times 10^9 Nm^2/C^2$ are 1 m apart. Also, assume $R < 1$ m. How much inaccuracy does our approximation generate? Is our approximation reasonable?

d. Is there any finite value of R for which this system remains continuous at R?

39) Instead of making the force 0 at R, we let the force be $10 - 20$ for $r \geq R$. Assume two protons, which have a magnitude of charge $1.6022 \times 10^{-19}\ C$, and the Coulomb constant $k_e = 8.988 \times 10^9\ \ Nm^2/C^2$. Is there a value R that can make this system continuous? If so, find it.

Answer

$R = 0.0001519$ m

Recall the discussion on spacecraft from the chapter opener. The following problems consider a rocket launch from Earth's surface. The force of gravity on the rocket is given by $F(d) = -mk/d^2$, where m is the mass of the rocket, d is the distance of the rocket from the center of Earth, and k is a constant.

40) Determine the value and units of k given that the mass of the rocket on Earth is 3 million kg. (Hint: The distance from the center of Earth to its surface is 6378 km.)

41) After a certain distance D has passed, the gravitational effect of Earth becomes quite negligible, so we can approximate the force function by \(F(d)=\begin{cases}–\dfrac{mk}{d^2}, & \text{if }d

Answer

$D = 63.78$ km

42) As the rocket travels away from Earth's surface, there is a distance D where the rocket sheds some of its mass, since it no longer needs the excess fuel storage. We can write this function as \(F(d)=\begin{cases} –\dfrac{m_1k}{d^2}, & \text{if }d

In Exercises 43 - 44, prove each function is continuous everywhere.

43) $f(\theta) = \sin\theta$

Answer

For all values of a, $f(a)$ is defined, $\lim_{\theta \to a} f(\theta)$ exists, and $\lim_{\theta \to a} f(\theta) = f(a)$. Therefore, $f(\theta)$ is continuous everywhere.

44) $g(x) = |x|$

45) Where is $f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ 1, & \text{if } x \text{ is rational} \end{cases}$ continuous?

Answer

Nowhere

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1.9: Defining the Derivative

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Now that we have both a conceptual understanding of a limit and the practical ability to compute limits, we have established the foundation for our study of Calculus, the branch of mathematics in which we compute derivatives and integrals. Most mathematicians and historians agree that calculus was developed independently by the Englishman Isaac **Newton (1643–1727)** and the German Gottfried **Leibniz (1646–1716)**, whose images appear in Figure 1.9.1. When we credit Newton and Leibniz with developing calculus, we are referring to the fact that Newton and Leibniz were the first to understand the relationship between the derivative and the integral. Both mathematicians benefited from the work of predecessors, such as Barrow, Fermat, and Cavalieri. The initial relationship between the two mathematicians appears to have been amicable; however, a bitter controversy erupted over whose work took precedence in later years. Although it seems likely that Newton did arrive at the ideas behind calculus first, we are indebted to Leibniz for the notation that we commonly use today.

Figure 1.9.1**:** *Newton and Leibniz are credited with developing calculus independently.*

Tangent Lines

We begin our study of differential calculus by revisiting the notion of secant and tangent lines. Recall that we used the slope of

a secant line to a function at a point $(a, f(a))$ to estimate the rate of change (which is the rate at which one variable changes in relation to another variable). We can obtain the slope of the secant by choosing a value of x near a and drawing a line through the points $(a, f(a))$ and $(x, f(x))$, as shown in Figure $1.9.2$. The slope of this line is given by an equation in the form of a difference quotient:

$$m_{sec} = \frac{f(x) - f(a)}{x - a}$$

We can also calculate the slope of a secant line to a function at a value a by using this equation and replacing x with $a + h$, where h is a value close to a. We can then calculate the slope of the line through the points $(a, f(a))$ and $(a + h, f(a + h))$. In this case, we find the secant line has a slope given by the following difference quotient with increment h:

$$m_{sec} = \frac{f(a+h) - f(a)}{a + h - a} = \frac{f(a+h) - f(a)}{h}$$

Definition: Difference Quotient

Let f be a function defined on an interval I containing a. If $x \neq a$ is in I, then

$$Q = \frac{f(x) - f(a)}{x - a}$$

is a **difference quotient**.

Also, if $h \neq 0$ is chosen so that $a + h$ is in I, then

$$Q = \frac{f(a+h) - f(a)}{h}$$

is a difference quotient with increment h.

These two expressions for calculating the slope of a secant line are illustrated in Figure $1.9.2$. We will see that each of these two methods for finding the slope of a secant line is of value. Depending on the setting, we can choose one or the other. The primary consideration in our choice depends on the ease of calculation.

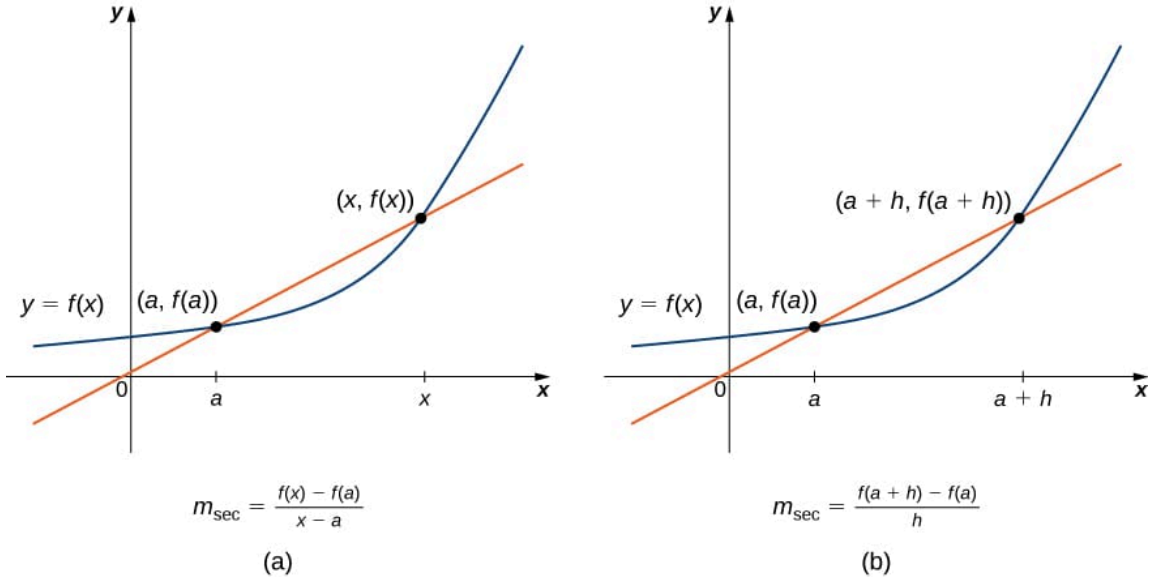

Figure $1.9.2$**:** *We can calculate the slope of a secant line in either of two ways.*

In Figure $1.9.3a$, we see that, as the values of x approach a, the slopes of the secant lines provide better estimates of the rate of change of the function at a. Furthermore, the secant lines approach the tangent line to the function at a, representing the limit of the secant lines. Similarly, Figure $1.9.3b$ shows that as the values of h get closer to 0, the secant lines also approach the tangent line. The slope of the tangent line at a is the rate of change of the function at a, as shown in Figure $1.9.3c$.

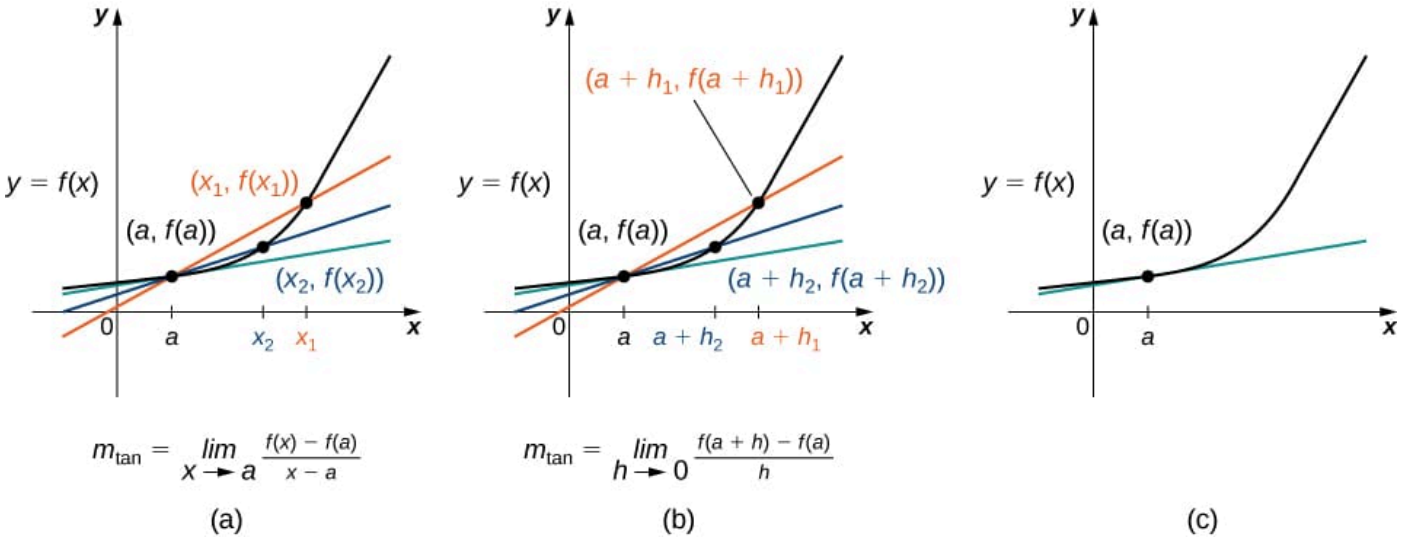

Figure 1.9.3**:** *The secant lines approach the tangent line (shown in green) as the second point approaches the first.*

Interactive Element: Secant Lines Approaching the Tangent Line

The following Interactive Element might help you visualize how the slope of a secant line approaches the slope of the tangent line as the two points merge.

Interact: Drag the orange point along the curve toward the black point of tangency at $(1, 1)$.

Observation: As the movable point approaches the point of tangency, the secant line pivots into the tangent line and its slope closes in on the derivative $f'(1) = 2$.

In Figure 1.9.4, we show the graph of $f(x) = \sqrt{x}$ and its tangent line at $(1, 1)$ in a series of tighter intervals about $x = 1$. As the intervals become narrower, the graph of the function and its tangent line appear to coincide, making the values on the tangent line a good approximation to the values of the function for choices of x close to 1. In fact, the graph of $f(x)$ itself appears to be **locally linear** near $x = 1$.

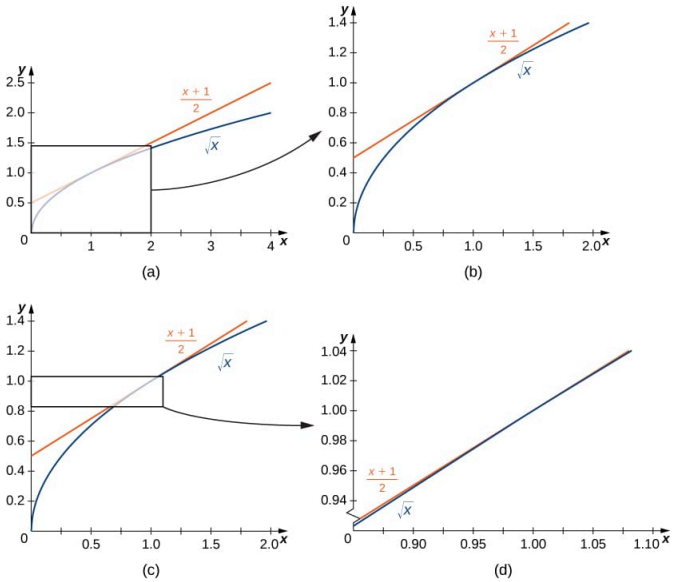

Figure 1.9.4**:** *For values of x close to 1, the graph of $f(x) = \sqrt{x}$ and its tangent line appear to coincide.*

Formally, we may define the tangent line to the graph of a function as follows.

Definition: Tangent Line

Let $f(x)$ be a function defined in an open interval containing a. The **tangent line** to $f(x)$ at a is the line passing through the point $(a, f(a))$ having slope

$$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{1.9.1}$$

provided this limit exists.

Equivalently, we may define the tangent line to $f(x)$ at a to be the line passing through the point $(a, f(a))$ having slope

$$m_{tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{1.9.2}$$

provided this limit exists.

Just as we have used two different expressions to define the slope of a secant line, we use two different forms to define the slope of the tangent line. In this text, we use both definitions. As before, the choice of definition will depend on the setting. Now that we have formally defined a tangent line to a function at a point, we can use this definition to find equations of tangent lines.

Example 1.9.1: Finding a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = x^2$ at $x = 3$.

Solution available at the end of this section.

Example 1.9.2: The Slope of a Tangent Line Revisited

Use Equation 1.9.2 to find the slope of the line tangent to the graph of $f(x) = x^2$ at $x = 3$.

Solution available at the end of this section.

Example 1.9.3: Finding the Equation of a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = 1/x$ at $x = 2$.

Solution available at the end of this section.

Checkpoint 1.9.3

Find the slope of the line tangent to the graph of $f(x) = \sqrt{x}$ at $x = 4$.

Answer available at the end of this section.

The Derivative of a Function at a Point

The type of limit we compute to find the slope of the line tangent to a function at a point occurs in many applications across many disciplines. These applications include velocity and acceleration in physics, marginal profit functions in business, and growth rates in biology. This limit occurs so frequently that we give this value a special name: the **derivative.** The process of finding a derivative is called **differentiation**.

Definition: Derivative

Let $f(x)$ be a function defined in an open interval containing a. The **derivative** of the function $f(x)$ at a, denoted by $f'(a)$, is defined by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{1.9.3}$$

provided this limit exists.

Alternatively, we may also define the derivative of $f(x)$ at a as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}. \tag{1.9.4}$$

Example 1.9.4: Finding a Derivative

For $f(x) = 3x^2 - 4x + 1$, find $f'(2)$ by using Equation 1.9.3.

Solution available at the end of this section.

Example 1.9.5: Revisiting the Derivative

For $f(x) = 3x^2 - 4x + 1$, find $f'(2)$ by using Equation 1.9.4.

Solution available at the end of this section.

Checkpoint 1.9.5

For $f(x) = x^2 + 3x + 2$, find $f'(1)$.

Answer available at the end of this section.

Velocities and Rates of Change

Now that we can evaluate a derivative, we can use it in velocity applications. Recall that if $s(t)$ is the position of an object moving along a coordinate axis, the **average velocity** of the object over a time interval $[a, t]$ if $t > a$ or $[t, a]$ if $t < a$ is given by the difference quotient

$$v_{avg} = \frac{s(t) - s(a)}{t - a}. \tag{1.9.5}$$

As the values of t approach a, the values of v_{avg} approach the value we call the **instantaneous velocity** at a. That is, instantaneous velocity at a, denoted $v(a)$, is given by

$$v(a) = s'(a) = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}. \tag{1.9.6}$$

To better understand the relationship between average velocity and instantaneous velocity, see Figure 1.9.7. In this figure, the slope of the tangent line (shown in red) is the instantaneous velocity of the object at time $t = a$ whose position at time t is given by the function $s(t)$. The slope of the secant line (shown in green) is the average velocity of the object over the time interval $[a, t]$.

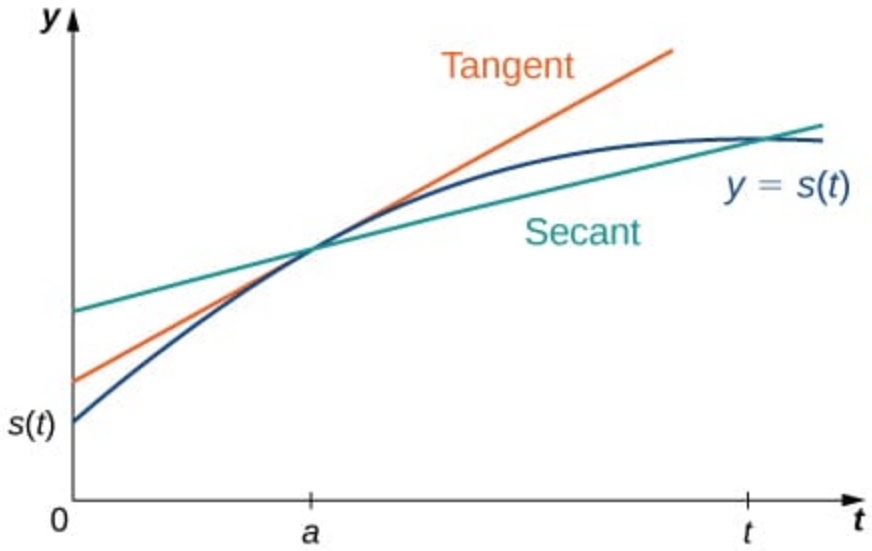

Figure 1.9.7**:** *The slope of the secant line is the average velocity over the interval* $[a, t]$*. The slope of the tangent line is the instantaneous velocity.*

We can use Equation 1.9.6 to calculate the instantaneous velocity, or we can estimate the velocity of a moving object by using a table of values. We can then confirm the estimate by using Equation 1.9.5.

Example 1.9.6: Estimating Velocity

A lead weight on a spring is oscillating up and down. Its position at time t with respect to a fixed horizontal line is given by $s(t) = \sin t$ (Figure 1.9.8). Use a table of values to estimate $v(0)$. Check the estimate by using Equation 1.9.6.

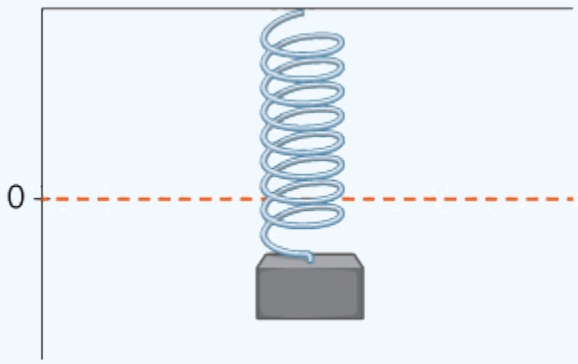

Figure 1.9.8**:** *A lead weight suspended from a spring in vertical oscillatory motion.*

Solution available at the end of this section.

Checkpoint 1.9.6

A rock is dropped from a height of 64 feet. Its height above ground at time t seconds later is given by $s(t) = -16t^2 + 64, \ \ 0 \leq t \leq 2$. Find its instantaneous velocity 1 second after it is dropped, using Equation 1.9.6.

Answer available at the end of this section.

As we have seen throughout this section, the slope of a tangent line to a function and instantaneous velocity are related concepts. Each is calculated by computing a derivative, and each measures the instantaneous rate of change of a function or the rate of change of a function at any point along the function.

Definition: Instantaneous Rate of Change

The **instantaneous rate of change** of a function $f(x)$ at a value a is its derivative $f'(a)$.

Example 1.9.7: Estimating Rate of Change of Velocity

Reaching a top speed of 270.49 mph, the Hennessey Venom GT is one of the fastest cars in the world. In tests it went from 0 to 60 mph in 3.05 seconds, from 0 to 100 mph in 5.88 seconds, from 0 to 200 mph in 14.51 seconds, and from 0 to 229.9 mph in 19.96 seconds. Use this data to conclude the rate of change of velocity (that is, its **acceleration**) as it approaches 229.9 mph. Does the rate at which the car accelerates appear to be increasing, decreasing, or constant?

Figure 1.9.9: *(credit: modification of work by Codex41, Flickr)*

Solution available at the end of this section.

Example 1.9.8: Rate of Change of Temperature

A homeowner sets the thermostat so that the temperature in the house begins to drop from $70^\circ F$ at 9 p.m., reaches a low of 60° during the night, and rises back to 70° by 7 a.m. the next morning. Suppose that the temperature in the house is given by $T(t) = 0.4t^2 - 4t + 70$ for $0 \le t \le 10$, where t is the number of hours past 9 p.m. Find the instantaneous rate of change of the temperature at midnight.

Solution available at the end of this section.

Example 1.9.9: Rate of Change of Profit

A toy company can sell x electronic gaming systems at a price of $p = -0.01x + 400$ dollars per gaming system. The cost of manufacturing x systems is given by $C(x) = 100x + 10000$ dollars. Find the rate of change of profit when $10,000$ games are produced. Should the toy company increase or decrease production?

Solution available at the end of this section.

Checkpoint 1.9.9

A coffee shop determines that the daily profit on scones obtained by charging s dollars per scone is $P(s) = -20s^2 + 150s - 10$. The coffee shop currently charges $\$3.25$ per scone. Find $P'(3.25)$, the rate of change of profit when the price is $\$3.25$, and decide whether or not the coffee shop should consider raising or lowering its prices on scones.

Answer available at the end of this section.

Hidden Resources: The Hawk Spot Food Pantry

The Hawk Spot Food Pantry is a program at CRC that gives food once per week to CRC students.

How Does CalFresh Work?

Details may change, so use the link below to verify what is stated here; however, you show your Student ID, fill out the intake form (on first visit), and pick-up your premade bag. Each student is allowed to be served once per week.

For more information, please see this page on the CRC website.

Hidden Resources: CalFresh

CalFresh (formerly known as Food Stamps) is a nutrition program that helps students and their families buy healthy foods.

How Does CalFresh Work?

CalFresh benefits are issued on an electronic benefit transfer (EBT) card that works like a debit card and can be used at most grocery stores and many farmers' markets to buy food. You can even use CalFresh to purchase food online through Amazon and Walmart.com.

If you are on a limited income and are struggling to buy the healthy foods that you need, then CalFresh may be able to help! Plus, CalFresh benefits do not impact your financial aid, and you do not need your parents' tax return information to apply.

For more information, please see this page on the CRC website.

Answers

Section Preview

This section focuses on tangent lines, their slopes, and the limit definition of a derivative at a number. Some instructors prefer to lump this (and the next) section in with the derivative material (Chapter 2); however, Chapter 2 is long enough as is and this material still focuses on limits, so it is included in Chapter 1.

Section Preview

- **An Overview of Functions**
  - The Difference Quotient: The derivative is defined as the limit of the difference quotient with increment h; the student must already grasp the difference quotient as an average rate of change before it is taken to a limit.
- **Linear Functions**
  - The Slope of a Line: The whole development rests on slope as a rate of change: the slope of a secant line, and the slope of the tangent line obtained as the limit of secant slopes.
  - The Average Rate of Change of a Function: The $\frac{f(x)-f(a)}{x-a}$ form of the difference quotient is exactly the average rate of change over $[a, x]$; the derivative is defined as its limit, so this concept is the conceptual spine of the section.

Section Preview

- **Linear Functions**
  - Equations of Lines — Point-Slope Form: Examples $1.9.1$ and $1.9.3$ build the tangent-line equation from a point and the computed slope ($y - 9 = 6(x - 3)$; $y = -\frac{1}{4}x + 1$).

Section Preview

- Identify the derivative as the limit of a difference quotient.
- Recognize the meaning of the tangent to a curve at a point.
- Calculate the slope and equation of a tangent line.
- Calculate the derivative of a given function at a point.
- Describe the velocity as a rate of change.
- Explain the difference between average velocity and instantaneous velocity.
- Estimate the derivative from a table of values.

Example $1.9.1$: Finding a Tangent Line

First, find the slope of the tangent line. In this example, use Equation $1.9.1$.

$$
\begin{aligned}
m_{tan} &= \lim_{x\to 3}\frac{f(x)-f(3)}{x-3} && \text{(definition of tangent slope)}\\
&= \lim_{x\to 3}\frac{x^2-9}{x-3} && \text{(substitute } f(x)=x^2 \text{ and } f(3)=9\text{)}\\
&= \lim_{x\to 3}\frac{(x-3)(x+3)}{x-3}\\
&= \lim_{x\to 3}\frac{\cancel{(x-3)}\,(x+3)}{\cancel{(x-3)}^{1}}\\
&= \lim_{x\to 3}(x+3)\\
&\overset{\text{D.S.}}{=} 6
\end{aligned}
$$

Next, find a point on the tangent line. Since the line is tangent to the graph of $f(x)$ at $x=3$, it passes through the point $(3, f(3))$. We have $f(3)=9$, so the tangent line passes through the point $(3,9)$.

Using the point-slope equation of the line with the slope $m=6$ and the point $(3,9)$, we obtain the line $y-9=6(x-3)$. Simplifying, we have $y=6x-9$. The graph of $f(x)=x^2$ and its tangent line at 3 are shown in Figure 1.9.5.

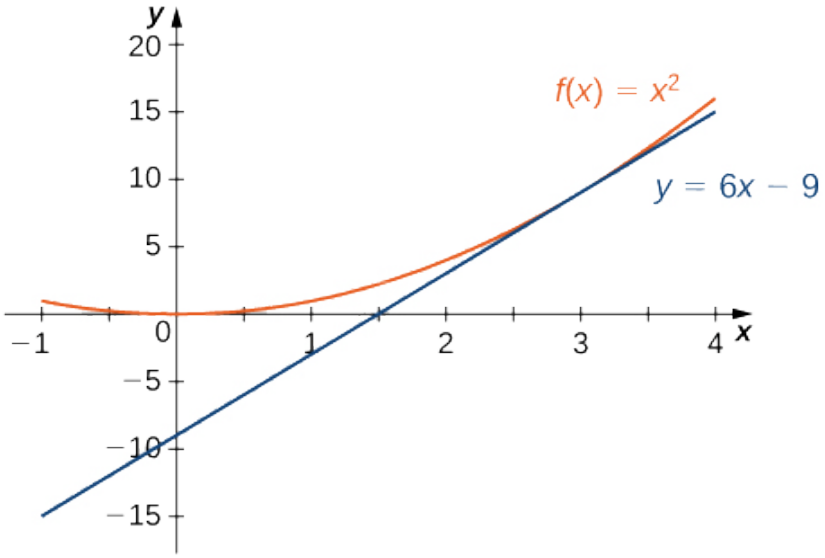

Figure 1.9.5**:** *The tangent line to* $f(x)$ *at* $x=3$.

Example 1.9.2**: The Slope of a Tangent Line Revisited**

The steps are very similar to Example 1.9.1. See Equation 1.9.2 for the definition.

$$
\begin{aligned}
m_{tan} &= \lim_{h\to 0}\frac{f(3+h)-f(3)}{h} && \text{(definition of tangent slope)}\\
&= \lim_{h\to 0}\frac{(3+h)^2-9}{h} && \text{(substitute } f(3+h)=(3+h)^2 \text{ and } f(3)=9\text{)}\\
&= \lim_{h\to 0}\frac{9+6h+h^2-9}{h}\\
&= \lim_{h\to 0}\frac{h(6+h)}{h}\\
&= \lim_{h\to 0}\frac{\cancel{h}\,(6+h)}{\cancel{h}^{1}}\\
&= \lim_{h\to 0}(6+h)\\
&\overset{\text{D.S.}}{=} 6
\end{aligned}
$$

We obtained the same value for the slope of the tangent line by using the other definition, demonstrating that the formulas can be interchanged.

Example 1.9.3: Finding the Equation of a Tangent Line

We can use Equation 1.9.1, but as we have seen, the results are the same if we use Equation 1.9.2.

$$\begin{aligned} m_{tan} &= \lim_{x\to 2} \frac{f(x)-f(2)}{x-2} \\ &= \lim_{x\to 2} \frac{\frac{1}{x}-\frac{1}{2}}{x-2} \\ &= \lim_{x\to 2} \frac{\frac{1}{x}-\frac{1}{2}}{x-2} \cdot \frac{2x}{2x} \\ &= \lim_{x\to 2} \frac{(2-x)}{(x-2)(2x)} \\ &= \lim_{x\to 2} \frac{-1}{2x} \\ &= -\frac{1}{4} \end{aligned}$$

We now know that the slope of the tangent line is $-\frac{1}{4}$. To find the equation of the tangent line, we also need a point on the line. We know that $f(2)=\frac{1}{2}$. Since the tangent line passes through the point $(2,\frac{1}{2})$, we can use the point-slope equation of a line to find the equation of the tangent line. Thus, the tangent line has the equation $y=-\frac{1}{4}x+1$. The graphs of $f(x)=\frac{1}{x}$ and $y=-\frac{1}{4}x+1$ are shown in Figure 1.9.6.

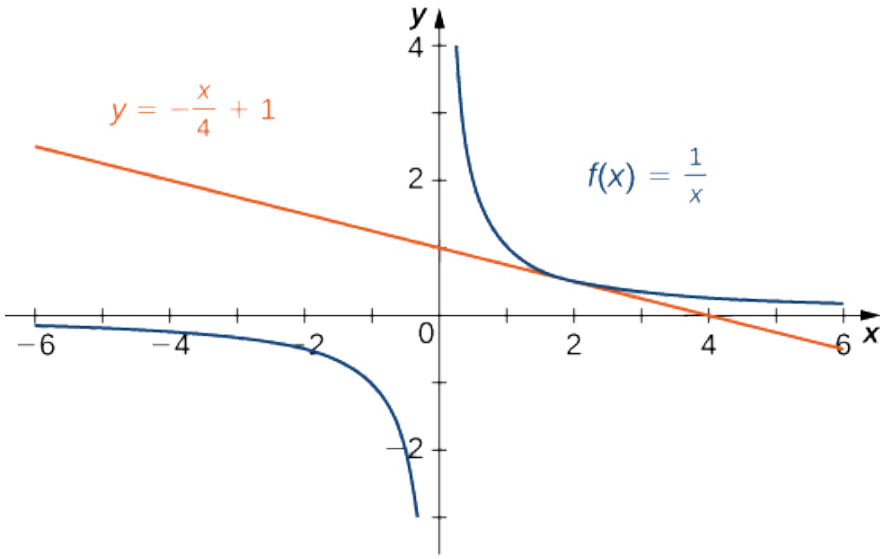

Figure 1.9.6: *The line is tangent to $f(x)$ at $x=2$.*

Checkpoint 1.9.3

$\frac{1}{4}$

Example 1.9.4: Finding a Derivative

Substitute the given function and value directly into the equation.

$$\begin{aligned} f'(2) &= \lim_{x\to 2} \frac{f(x)-f(2)}{x-2} && \text{(definition of the derivative)} \\ &= \lim_{x\to 2} \frac{(3x^2-4x+1)-5}{x-2} && \text{(substitute } f(x)=3x^2-4x+1 \text{ and } f(2)=5\text{)} \\ &= \lim_{x\to 2} \frac{(x-2)(3x+2)}{x-2} \\ &= \lim_{x\to 2} (3x+2) \\ &= 8 \end{aligned}$$

Example 1.9.5: Revisiting the Derivative

Using this equation, we can substitute two values of the function into the equation, and we should get the same value as in Example 1.9.4.

$$
\begin{aligned}
f'(2) &= \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} && \text{(definition of the derivative)}\\
&= \lim_{h\to 0}\frac{(3(2+h)^2-4(2+h)+1)-5}{h} && \text{(substituting } f(2)=5 \text{ and } f(2+h)=3(2+h)^2-4(2+h)+1)\\
&= \lim_{h\to 0}\frac{3(4+4h+h^2)-8-4h+1-5}{h} && \text{(expanding the numerator)}\\
&= \lim_{h\to 0}\frac{12+12h+3h^2-12-4h}{h}\\
&= \lim_{h\to 0}\frac{3h^2+8h}{h}\\
&= \lim_{h\to 0}\frac{h(3h+8)}{h} && \text{(factoring)}\\
&= \lim_{h\to 0}(3h+8) && \text{(canceling)}\\
&= 8
\end{aligned}
$$

The results are the same whether we use Equation 1.9.3 or Equation 1.9.4.

Checkpoint 1.9.5

$f'(1)=5$

Example 1.9.6**: Estimating Velocity**

We can estimate the instantaneous velocity at $t=0$ by computing a table of average velocities using values of t approaching 0, as shown in Table 1.9.1.

Table 1.9.1***:*** *Average velocities using values of t approaching* 0

t	$\frac{\sin t-\sin 0}{t-0}=\frac{\sin t}{t}$
-0.1	0.998334166
-0.01	0.9999833333
-0.001	0.999999833
0.001	0.999999833
0.01	0.9999833333
0.1	0.998334166

From the table, we see that the average velocity over the time interval $[-0.1,0]$ is 0.998334166, the average velocity over the time interval $[-0.01,0]$ is 0.9999833333, and so forth. Using this table of values, it appears that a reasonable estimate is $v(0)=1$.

By using Equation 1.9.6, we can see that

$$v(0)=s'(0)=\lim_{t\to 0}\frac{\sin t-\sin 0}{t-0}=\lim_{t\to 0}\frac{\sin t}{t}=1.$$

Thus, in fact, $v(0)=1$.

Checkpoint 1.9.6

-32 ft/s

Example 1.9.7**: Estimating Rate of Change of Velocity**

First observe that 60 mph = 88 ft/s, 100 mph ≈ 146.67 ft/s, 200 mph ≈ 293.33 ft/s, and 229.9 mph ≈ 337.19 ft/s. We can summarize the information in a table.

Table 1.9.2***:*** $v(t)$ *at different values of* t

t	$v(t)$
0	0
3.05	88
5.88	146.67
14.51	293.33
19.96	337.19

Now compute the average acceleration of the car in feet per second on intervals of the form $[t, 19.96]$ as t approaches 19.96, as shown in the following table.

Table $1.9.3$***:*** *Average acceleration*

t	$\dfrac{v(t)-v(19.96)}{t-19.96}=\dfrac{v(t)-337.19}{t-19.96}$
0.0	16.89
3.05	14.74
5.88	13.46
14.51	8.05

The rate at which the car is accelerating is decreasing as its velocity approaches 229.9 mph (337.19 ft/s).

Example $1.9.8$**: Rate of Change of Temperature**

Since midnight is 3 hours past 9 p.m., we want to compute $T'(3)$. Refer to Equation $1.9.3$.

$$\begin{aligned} T'(3) &= \lim_{t\to 3}\frac{T(t)-T(3)}{t-3} && \text{(definition of the derivative)} \\ &= \lim_{t\to 3}\frac{0.4t^2-4t+70-61.6}{t-3} && \text{(substituting } T(t)=0.4t^2-4t+70 \text{ and } T(3)=61.6) \\ &= \lim_{t\to 3}\frac{0.4t^2-4t+8.4}{t-3} \\ &= \lim_{t\to 3}\frac{0.4(t-3)(t-7)}{t-3} \\ &= \lim_{t\to 3}0.4(t-7) \\ &= -1.6 \end{aligned}$$

The instantaneous rate of change of the temperature at midnight is $-1.6^{\circ}F$ per hour.

Example $1.9.9$**: Rate of Change of Profit**

The profit $P(x)$ earned by producing x gaming systems is $R(x)-C(x)$, where $R(x)$ is the revenue obtained from the sale of x games. Since the company can sell x games at $p=-0.01x+400$ per game,

$$R(x)=xp=x(-0.01x+400)=-0.01x^2+400x.$$

Consequently,

$$P(x)=-0.01x^2+300x-10000.$$

Therefore, evaluating the rate of change of profit gives

$$\begin{aligned} P'(10000) &= \lim_{x\to 10000}\frac{P(x)-P(10000)}{x-10000} \\ &= \lim_{x\to 10000}\frac{-0.01x^2+300x-10000-1990000}{x-10000} \\ &= \lim_{x\to 10000}\frac{-0.01x^2+300x-2000000}{x-10000} \\ &= 100 \end{aligned}$$

Since the rate of change of profit $P'(10000)>0$ and $P(10000)>0$, the company should increase production.

Checkpoint $1.9.9$

$P'(3.25)=20>0$; raise prices

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1.9.1: Resources and Key Concepts

Resources

Videos

- Secant line
  - Limits - Secant and Tangent Lines - Secant Lines and Average Rates of Change: Definitions & Examples
  - Limits - Secant and Tangent Lines - Tangent Lines and Instantaneous Rates of Change: Definition & Examples
- Tangent line
  - Derivatives and Rates of Change - Recalling Tangent Lines
- Derivative at a point
  - Derivatives and Rates of Change - The Definition of the Derivative at a Point
  - Derivatives and Rates of Change - Finding the Derivative of a Function at a Point
  - Derivatives and Rates of Change - Finding the Derivative of a Function at an Arbitrary Point
  - Derivatives and Rates of Change - Interpreting Limits as Derivatives
- Average velocity
  - Limits - Secant and Tangent Lines - Average and Instantaneous Velocities
- Instantaneous velocity
  - Derivatives and Rates of Change - Rates of Change and Instantaneous Velocity
- Equation of tangent line
  - Derivatives and Rates of Change - Finding Equations of Tangent Lines

Key Concepts

Definitions

- **Difference Quotient**:
  - Given a function f defined on an interval I containing a, if $x \neq a$ is in I, then $Q = \frac{f(x)-f(a)}{x-a}$ is a difference quotient.
  - If $h \neq 0$ is chosen so that $a+h$ is in I, then $Q = \frac{f(a+h)-f(a)}{h}$ is a difference quotient with increment h.
- **Tangent Line**: Let $f(x)$ be a function defined in an open interval containing a. The tangent line to $f(x)$ at a is the line passing through the point $(a, f(a))$ having slope:
  - $m_{tan} = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$
  - or equivalently, $m_{tan} = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

  provided these limits exist.
- **Derivative**: Let $f(x)$ be a function defined in an open interval containing a. The derivative of the function $f(x)$ at a, denoted by $f'(a)$, is defined by:
  - $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$
  - or alternatively, $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

  provided this limit exists.
- **Differentiation**: The process of finding a derivative.
- **Average Velocity**: If $s(t)$ is the position of an object moving along a coordinate axis, the average velocity of the object over a time interval $[a, t]$ (or $[t, a]$) is given by the difference quotient $v_{avg} = \frac{s(t)-s(a)}{t-a}$.
- **Instantaneous Velocity**: As the values of t approach a, the values of v_{avg} approach the value we call the instantaneous velocity at a. Denoted $v(a)$, it is given by $v(a) = s'(a) = \lim\limits_{t \to a} \dfrac{s(t) - s(a)}{t - a}$.
- **Instantaneous Rate of Change**: The instantaneous rate of change of a function $f(x)$ at a value a is its derivative $f'(a)$.

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1.9.2: Homework

Note

The following exercises are meant as hand-in homework (if assigned by your instructor). Typed-in answers here will not be saved or scored.

Basic Skills

Exercise 1

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Exercise 2

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Exercise 3

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Exercise 4

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Exercise 5

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Exercise 6

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Exercise 7

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Exercise 8

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Exercise 9

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Exercise 10

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Exercise 11

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Exercise 12

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Exercise 13

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Exercise 14

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Exercise 15

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Exercise 16

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Basic Skills - Applications

Exercise 17

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Exercise 18

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Exercise 19

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Synthesis Questions

Exercise 20

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Exercise 21

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Exercise 22

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Exercise 23

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Exercise 24

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Exercise 25

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Exercise 26

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Exercise 27

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Exercise 28

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Exercise 29

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Synthesis Questions - Applications

Exercise 30

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Exercise 31

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1.9.3: Additional Exercises

Reading Questions

1. What are the two common forms of the difference quotient used to calculate the slope of a secant line?
2. How is the slope of a tangent line at a point $(a, f(a))$ defined using a limit? Provide both forms.
3. What is the definition of the derivative of a function $f(x)$ at a point a, denoted $f'(a)$? Provide both limit forms.
4. What does it mean if a function $f(x)$ is "locally linear" near $x = a$?
5. If $s(t)$ represents the position of an object at time t, how is the average velocity over the interval $[a, t]$ calculated?
6. How is the instantaneous velocity $v(a)$ at time a defined in terms of a derivative?
7. What does the derivative $f'(a)$ represent in terms of the instantaneous rate of change of the function f at a?
8. In Example 1, why was it necessary to factor $x^2 - 9$ before evaluating the limit for m_{tan}?
9. When calculating $f'(a)$ using the definition $\lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$, what is the typical algebraic goal before substituting $h = 0$?
10. How can a table of values for average velocities be used to estimate instantaneous velocity?

Homework

For exercises 1 - 10, use the equation $m_{\text{sec}} = \dfrac{f(x) - f(a)}{x - a}$ to find the slope of the secant line between the values x_1 and x_2 for each function $y = f(x)$.

1) $f(x) = 4x + 7; \quad x_1 = 2, \quad x_2 = 5$

Answer

$m_{\text{sec}} = 4$

2) $f(x) = 8x - 3; \quad x_1 = -1, \quad x_2 = 3$

3) $f(x) = x^2 + 2x + 1; \quad x_1 = 3, \quad x_2 = 3.5$

Answer

$m_{\text{sec}} = 8.5$

4) $f(x) = -x^2 + x + 2; \quad x_1 = 0.5, \quad x_2 = 1.5$

5) $f(x) = \dfrac{4}{3x - 1}; \quad x_1 = 1, \quad x_2 = 3$

Answer

$m_{\text{sec}} = -\frac{3}{4}$

6) $f(x) = \dfrac{x - 7}{2x + 1}; \quad x_1 = -2, \quad x_2 = 0$

7) $f(x) = \sqrt{x}; \quad x_1 = 1, \quad x_2 = 16$

Answer

$m_{\text{sec}} = 0.2$

8) $f(x) = \sqrt{x - 9}; \quad x_1 = 10, \quad x_2 = 13$

9) $f(x) = x^{1/3} + 1; \quad x_1 = 0, \quad x_2 = 8$

Answer

$m_{\text{sec}} = 0.25$

10) $f(x) = 6x^{2/3} + 2x^{1/3}; \quad x_1 = 1, \quad x_2 = 27$

For the functions in exercises 11 - 20,

a. use the equation $m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ to find the slope of the tangent line $m_{\tan} = f\,'(a)$, and

b. find the equation of the tangent line to f at $x = a$.

11) $f(x) = 3 - 4x, \quad a = 2$

Answer

a. $m_{\tan} = -4$
b. $y = -4x + 3$

12) $f(x) = \frac{x}{5} + 6, \quad a = -1$

13) $f(x) = x^2 + x, \quad a = 1$

Answer

a. $m_{\tan} = 3$
b. $y = 3x - 1$

14) $f(x) = 1 - x - x^2, \quad a = 0$

15) $f(x) = \frac{7}{x}, \quad a = 3$

Answer

a. $m_{\tan} = \frac{-7}{9}$
b. $y = \frac{-7}{9}x + \frac{14}{3}$

16) $f(x) = \sqrt{x+8}, \quad a = 1$

17) $f(x) = 2 - 3x^2, \quad a = -2$

Answer

a. $m_{\tan} = 12$
b. $y = 12x + 14$

18) $f(x) = \frac{-3}{x-1}, \quad a = 4$

19) $f(x) = \frac{2}{x+3}, \quad a = -4$

Answer

a. $m_{\tan} = -2$
b. $y = -2x - 10$

20) $f(x) = \frac{3}{x^2}, \quad a = 3$

For the functions $y = f(x)$ in exercises 21 - 30, find $f\,'(a)$ using the equation $f\,'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.

21) $f(x) = 5x + 4, \quad a = -1$

Answer

$f'(-1) = 5$

22) $f(x) = -7x + 1, \quad a = 3$

23) $f(x) = x^2 + 9x, \quad a = 2$

Answer

$f'(2) = 13$

24) $f(x) = 3x^2 - x + 2, \quad a = 1$

25) $f(x) = \sqrt{x}, \quad a = 4$

Answer

$f'(4) = \frac{1}{4}$

26) $f(x) = \sqrt{x-2}, \quad a = 6$

27) $f(x) = \dfrac{1}{x}, \quad a = 2$

Answer

$f'(2) = -\frac{1}{4}$

28) $f(x) = \dfrac{1}{x-3}, \quad a = -1$

29) $f(x) = \dfrac{1}{x^3}, \quad a = 1$

Answer

$f'(1) = -3$

30) $f(x) = \dfrac{1}{\sqrt{x}}, \quad a = 4$

For the following exercises, given the function $y = f(x)$,

a. find the slope of the secant line PQ for each point $Q(x, f(x))$ with x value given in the table.

b. Use the answers from a. to estimate the value of the slope of the tangent line at P.

c. Use the answer from b. to find the equation of the tangent line to f at point P.

31) [Technology Required] $f(x) = x^2 + 3x + 4, \quad P(1, 8)$ (Round to 6 decimal places.)

x	Slope m_{PQ}	x	Slope m_{PQ}
1.1	(i)	0.9	(vii)
1.01	(ii)	0.99	(viii)
1.001	(iii)	0.999	(ix)
1.0001	(iv)	0.9999	(x)
1.00001	(v)	0.99999	(xi)
1.000001	(vi)	0.999999	(xii)

Answer

$a. (i) 5.100000, (ii) 5.010000, (iii) 5.001000, (iv) 5.000100, (v) 5.000010, (vi) 5.000001, (vii) 4.900000, (viii) 4.990000, (ix) 4.999000, (x) 4.999900, (xi) 4.999990, (x) 4.999999$

b. $m_{\tan} = 5$

c. $y = 5x + 3$

32) [Technology Required] $f(x) = \dfrac{x+1}{x^2-1}, \quad P(0, -1)$

x	Slope m_{PQ}	x	Slope m_{PQ}
0.1	(i)	−0.1	(vii)
0.01	(ii)	−0.01	(viii)
0.001	(iii)	−0.001	(ix)
0.0001	(iv)	−0.0001	(x)
0.00001	(v)	−0.00001	(xi)
0.000001	(vi)	−0.000001	(xii)

33) [Technology Required] $f(x) = 10e^{0.5x}, \quad P(0, 10)$ (Round to 4 decimal places.)

x	Slope m_{PQ}
−0.1	(i)
−0.01	(ii)
−0.001	(iii)
−0.0001	(iv)
−0.00001	(v)
−0.000001	(vi)

Answer

a. $(i) 4.8771, \ (ii) 4.9875, \ (iii) 4.9988, \ (iv) 4.9999, \ (v) 4.9999, \ (vi) 4.9999$
b. $m_{\tan} = 5$
c. $y = 5x + 10$

34) [Technology Required] $f(x) = \tan(x), \quad P(\pi, 0)$

x	Slope m_{PQ}
3.1	(i)
3.14	(ii)
3.141	(iii)
3.1415	(iv)
3.14159	(v)
3.141592	(vi)

[Technology Required] For the following position functions $y = s(t)$, an object is moving along a straight line, where t is in seconds and s is in meters. Find

a. the simplified expression for the average velocity from $t = 2$ to $t = 2 + h$;

b. the average velocity between $t = 2$ and $t = 2 + h$, where $(i) \ h = 0.1, \ (ii) \ h = 0.01, \ (iii) \ h = 0.001$, and $(iv) \ h = 0.0001$; and

c. use the answer from a. to estimate the instantaneous velocity at $t = 2$ second.

35) $s(t) = \frac{1}{3}t + 5$

Answer

a. $\frac{1}{3}$;
b. $(i) \ \frac{1}{3}$ m/s, $(ii) \ \frac{1}{3}$ m/s, $(iii) \ \frac{1}{3}$ m/s, $(iv) \ \frac{1}{3}$ m/s;

c. $\frac{1}{3}$ m/s

36) $s(t) = t^2 - 2t$

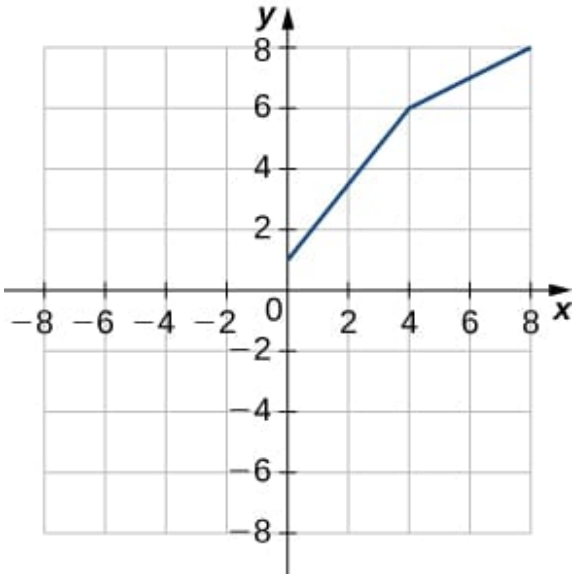
37) $s(t) = 2t^3 + 3$

Answer

a. $2(h^2 + 6h + 12)$;
b. $(i)\ \ 25.22$ m/s, $(ii)\ \ 24.12$ m/s, $(iii)\ \ 24.01$ m/s, $(iv)\ \ 24$ m/s;
c. 24 m/s

38) $s(t) = \dfrac{16}{t^2} - \dfrac{4}{t}$

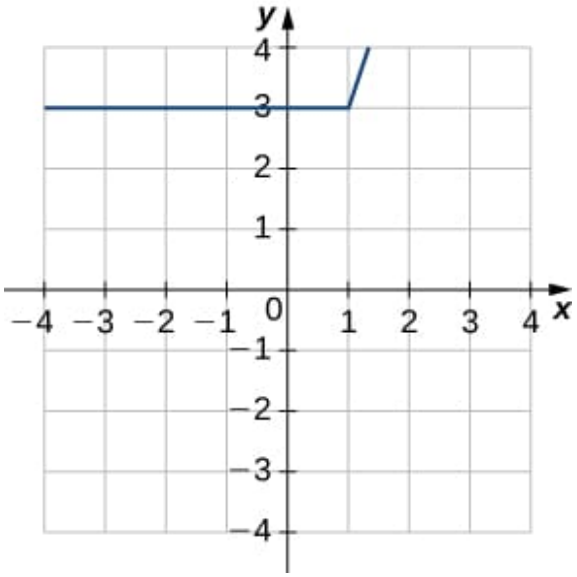
39) Use the following graph to evaluate a. $f\,'(1)$ and b. $f\,'(6)$.



Answer

a. 1.25; b. 0.5

40) Use the following graph to evaluate a. $f\,'(-3)$ and b. $f\,'(1.5)$.



For the following exercises, use the limit definition of derivative to show that the derivative does not exist at $x = a$ for each of the given functions.

41) $f(x) = x^{1/3}, \quad x = 0$

Answer

$$\lim_{x \to 0^-} \frac{x^{1/3} - 0}{x - 0} = \lim_{x \to 0^-} \frac{1}{x^{2/3}} = \infty$$

42) $f(x) = x^{2/3}, \quad x = 0$

43) $f(x) = \begin{cases} 1, & \text{if } x < 1 \\ x, & \text{if } x \geq 1 \end{cases}, \quad x = 1$

Answer

$$\lim_{x \to 1^-} \frac{1 - 1}{x - 1} = 0 \neq 1 = \lim_{x \to 1^+} \frac{x - 1}{x - 1}$$

44) $f(x) = \dfrac{|x|}{x}, \quad x = 0$

45) [Technology Required] The position in feet of a race car along a straight track after t seconds is modeled by the function $s(t) = 8t^2 - \frac{1}{16}t^3$.

a. Find the average velocity of the vehicle over the following time intervals to four decimal places:

i. $[4, 4.1]$

ii. $[4, 4.01]$

iii. $[4, 4.001]$

iv. $[4, 4.0001]$

b. Use a. to draw a conclusion about the instantaneous velocity of the vehicle at $t = 4$ seconds.

Answer

a. $(i) 61.7244 ft/s, \ (ii) 61.0725 ft/s, \ (iii) 61.0072 ft/s, \ (iv) 61.0007 ft/s$
b. At 4 seconds the race car is traveling at a rate/velocity of 61 ft/s.

46) [Technology Required] The distance in feet that a ball rolls down an incline is modeled by the function $s(t) = 14t^2$,

where t is seconds after the ball begins rolling.

a. Find the average velocity of the ball over the following time intervals:

i. [5, 5.1]

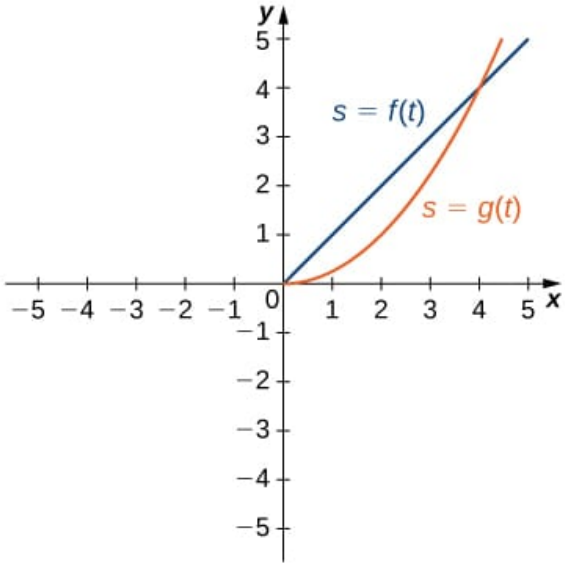
ii. [5, 5.01]

iii. [5, 5.001]

iv. [5, 5.0001]

b. Use the answers from a. to draw a conclusion about the instantaneous velocity of the ball at $t = 5$ seconds.

47) Two vehicles start out traveling side by side along a straight road. Their position functions, shown in the following graph, are given by $s = f(t)$ and $s = g(t)$, where s is measured in feet and t is measured in seconds.



a. Which vehicle has traveled farther at $t = 2$ seconds?

b. What is the approximate velocity of each vehicle at $t = 3$ seconds?

c. Which vehicle is traveling faster at $t = 4$ seconds?

d. What is true about the positions of the vehicles at $t = 4$ seconds?

Answer

a. The vehicle represented by $f(t)$, because it has traveled 2 feet, whereas $g(t)$ has traveled 1 foot.
b. The velocity of $f(t)$ is constant at 1 ft/s, while the velocity of $g(t)$ is approximately 2 ft/s.
c. The vehicle represented by $g(t)$, with a velocity of approximately 4 ft/s.
d. Both have traveled 4 feet in 4 seconds.

48) [Technology Required] The total cost $C(x)$, in hundreds of dollars, to produce x jars of mayonnaise is given by $C(x) = 0.000003x^3 + 4x + 300$.

a. Calculate the average cost per jar over the following intervals:

i. [100, 100.1]

ii. [100, 100.01]

iii. [100, 100.001]

iv. [100, 100.0001]

b. Use the answers from a. to estimate the average cost to produce 100 jars of mayonnaise.

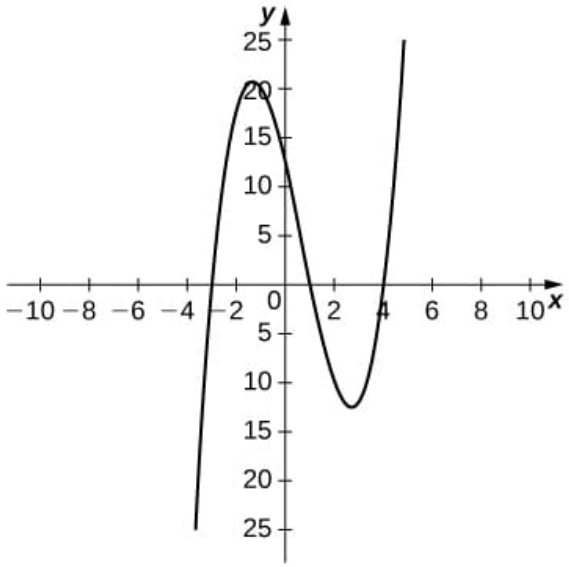
49) [Technology Required] For the function $f(x) = x^3 - 2x^2 - 11x + 12$, do the following.

a. Use a graphing calculator to graph f in an appropriate viewing window.

b. Use the ZOOM feature on the calculator to approximate the two values of $x = a$ for which $m_{tan} = f\,'(a) = 0$.

Answer

a.



b. $a \approx -1.361,\ \ 2.694$

50) [Technology Required] For the function $f(x) = \dfrac{x}{1 + x^2}$, do the following.

a. Use a graphing calculator to graph f in an appropriate viewing window.

b. Use the ZOOM feature on the calculator to approximate the values of $x = a$ for which $m_{\tan} = f\,'(a) = 0$.

51) Suppose that $N(x)$ computes the number of gallons of gas used by a vehicle traveling x miles. Suppose the vehicle gets 30 mpg.

a. Find a mathematical expression for $N(x)$.

b. What is $N(100)$? Explain the physical meaning.

c. What is $N\,'(100)$? Explain the physical meaning.

Answer

a. $N(x) = \dfrac{x}{30}$
b. ~ 3.3 gallons. When the vehicle travels 100 miles, it has used 3.3 gallons of gas.
c. $\frac{1}{30}$. The rate of gas consumption in gallons per mile that the vehicle is achieving after having traveled 100 miles.

52) [Technology Required] For the function $f(x) = x^4 - 5x^2 + 4$, do the following.

a. Use a graphing calculator to graph f in an appropriate viewing window.

b. Use the $nDeriv$ function, which numerically finds the derivative, on a graphing calculator to estimate $f\,'(-2)$, $f\,'(-0.5)$, $f\,'(1.7)$, and $f\,'(2.718)$.

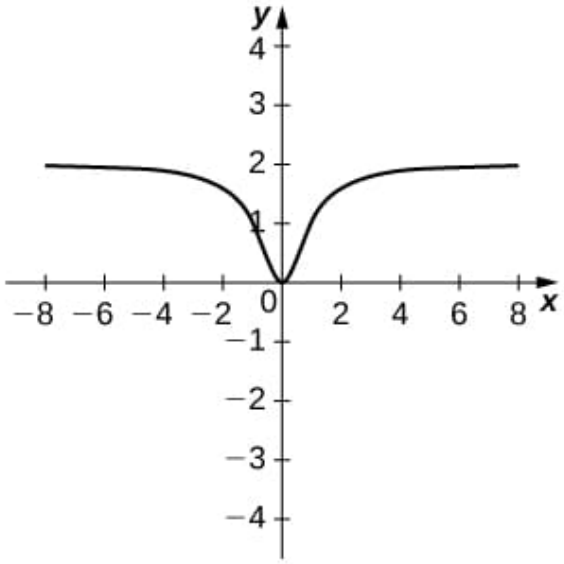
53) [Technology Required] For the function $f(x) = \dfrac{x^2}{x^2+1}$, do the following.

a. Use a graphing calculator to graph f in an appropriate viewing window.

b. Use the $nDeriv$ function on a graphing calculator to find $f\,'(-4)$, $f\,'(-2)$, $f\,'(2)$, and $f\,'(4)$.

Answer

a.



b. $-0.028, -0.16, 0.16, 0.028$

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1.10: The Derivative as a Function

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

As we have seen, the derivative of a function at a given point gives us the rate of change or slope of the tangent line to the function at that point. We obtain the velocity if we differentiate a position function at a given time. Knowing the function's derivative at every point produces valuable information about the function's behavior. However, finding the derivative at even a handful of values using the techniques of the preceding section would quickly become quite tedious. In this section, we define the derivative function and learn a process for finding it.

Derivative Functions

The derivative function gives the derivative of a function at each point in the domain of the original function for which the derivative is defined. We can formally define a derivative function as follows.

Definition: Derivative Function

Let f be a function. The **derivative function**, denoted by f', is the function whose domain consists of those values of x such that the following limit exists:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{1.10.1}$$

A function $f(x)$ is said to be **differentiable** at a if $f'(a)$ exists. More generally, a function is said to be **differentiable on** S if it is differentiable at every point in an open set S, and a differentiable function is one in which $f'(x)$ exists on its domain.

In the next few examples we use Equation 1.10.1 to find the derivative of a function.

Example 1.10.1: Finding the Derivative of a Square-Root Function

Find the derivative of $f(x) = \sqrt{x}$.

Solution available at the end of this section.

Example 1.10.2: Finding the Derivative of a Quadratic Function

Find the derivative of the function $f(x) = x^2 - 2x$.

Solution available at the end of this section.

Checkpoint 1.10.2

Find the derivative of $f(x) = x^2$.

Answer available at the end of this section.

Notations for the Derivative

There is no single uniform notation for differentiation. Instead, various notations for the derivative of a function or variable have been proposed by various mathematicians. The usefulness of each notation varies with the context, and it is sometimes advantageous to use more than one notation in a given context.

Lagrange's Notation: $f'(x)$

One of the most common modern notations for differentiation is named after Joseph Louis Lagrange, even though Euler invented it and Lagrange just popularized it. In Lagrange's notation, a prime mark denotes a derivative. If $y = f(x)$ is a function, then its derivative evaluated at x is written

$$f'(x) \quad (\text{a common alternative form is } y')$$

Lagrange's notation first appeared in print in 1749.

If the derivative is evaluated at $x = a$, then the value is written as $f'(a)$ or $y'(a)$.

Leibniz's Notation: $\dfrac{dy}{dx}$

The original notation employed by Gottfried Leibniz is still frequently used throughout mathematics. It is prevalent when the equation $y = f(x)$ is regarded as a functional relationship between dependent and independent variables y and x. Leibniz's notation makes this relationship explicit by writing the derivative as

$$\frac{dy}{dx} \quad \left(\text{common alternative forms are } \frac{df(x)}{dx}, \frac{df}{dx}(x), \text{ and } \frac{d}{dx}f(x)\right)$$

In Example 1.10.2, we could have expressed the derivative as $\frac{dy}{dx} = 2x - 2$. We could have conveyed the same information by writing $\frac{d}{dx}\left(x^2 - 2x\right) = 2x - 2$.

To understand this notation better, recall that the derivative of a function at a point is the limit of the slopes of secant lines as the secant lines approach the tangent line. The slopes of these secant lines are often expressed in the form $\frac{\Delta y}{\Delta x}$ where Δy is the difference in the y values corresponding to the difference in the x values, which are expressed as Δx (Figure 1.10.1). Thus the derivative, which can be thought of as the instantaneous rate of change of y with respect to x, is expressed as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

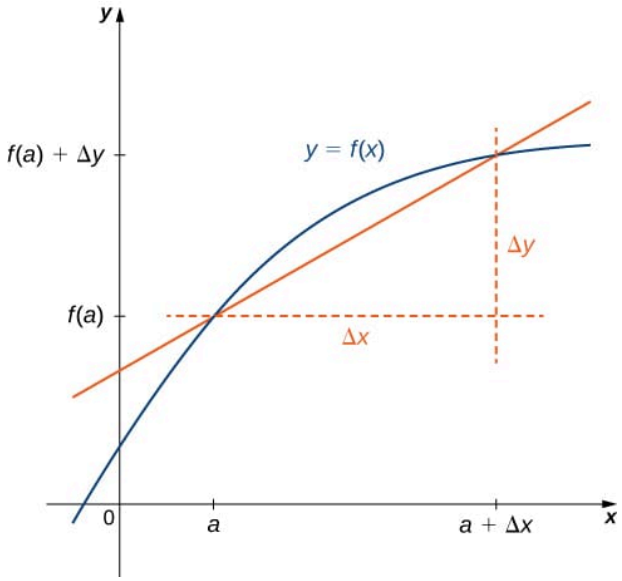

Figure 1.10.1**:** *The derivative is expressed as* $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$.

If the derivative is evaluated at $x = a$, then the value is written as $\left.\frac{dy}{dx}\right|_{x=a}$ or $\left.\frac{df}{dx}\right|_{x=a}$.

Caution: Careful Use of Leibniz's Notation

When using Leibniz's notation to find the derivative of $y = f(x)$ at $x = a$, it is incorrect to write $\frac{df(a)}{dx}$. The reason for this will be explored in the next chapter.

Euler's Notation: D

Leonhard Euler's notation uses a differential operator suggested by Louis François Antoine Arbogast, denoted as D. When applied to a function $f(x)$, it is defined by

$$(Df)(x)$$

Euler's notation is useful in a subject called differential equations; however, we will not be using it in this text.

Newton's Notation: $\dot{y}$

Isaac Newton's notation for differentiation (also called the dot notation, fluxions, or sometimes, crudely, the flyspeck notation for differentiation) places a dot over the dependent variable. That is if y is a function of t, then the derivative of y with respect to t is

$$\dot{y}$$

This notation is popular in physics and mathematical physics. It also appears in areas of mathematics connected with physics, such as differential equations. As with Euler's notation, we will not be using Newton's notation in this text.

Graphing a Derivative

We have already discussed how to graph a function, so given the equation of a function or the equation of a derivative function, we could graph it. Given both, we would expect to see a correspondence between the graphs of these two functions since $f'(x)$ gives the rate of change of a function $f(x)$ (or slope of the tangent line to $f(x)$).

In Example 1.10.1, we found that for $f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2\sqrt{x}}$. If we graph these functions on the same axes, as in Figure 1.10.2, we can use the graphs to understand the relationship between these two functions. First, we notice that $f(x)$ is increasing over its entire domain, which means that the slopes of its tangent lines at all points are positive. Consequently, we expect $f'(x) > 0$ for

all values of x in its domain. Furthermore, as x increases, the slopes of the tangent lines to $f(x)$ are decreasing. We expect to see a corresponding decrease in $f'(x)$. We also observe that $f(0)$ is undefined and that $\lim_{x \to 0^+} f'(x) = +\infty$, corresponding to a vertical tangent to $f(x)$ at 0.

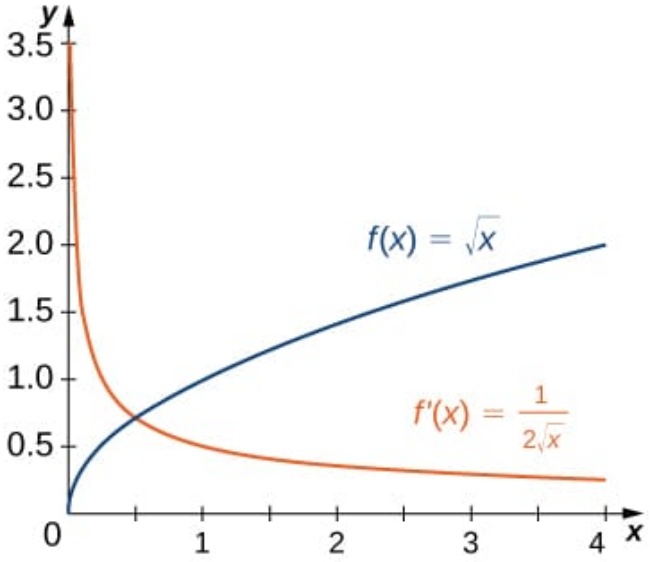

Figure $1.10.2$**:** *The derivative $f'(x)$ is positive everywhere because the function $f(x)$ is increasing.*

In Example $1.10.2$, we found that for $f(x) = x^2 - 2x$, $f'(x) = 2x - 2$. The graphs of these functions are shown in Figure $1.10.3$. Observe that $f(x)$ is decreasing for $x < 1$. For these same values of x, $f'(x) < 0$. For values of $x > 1$, $f(x)$ is increasing and $f'(x) > 0$. Also, $f(x)$ has a horizontal tangent at $x = 1$ and $f'(1) = 0$.

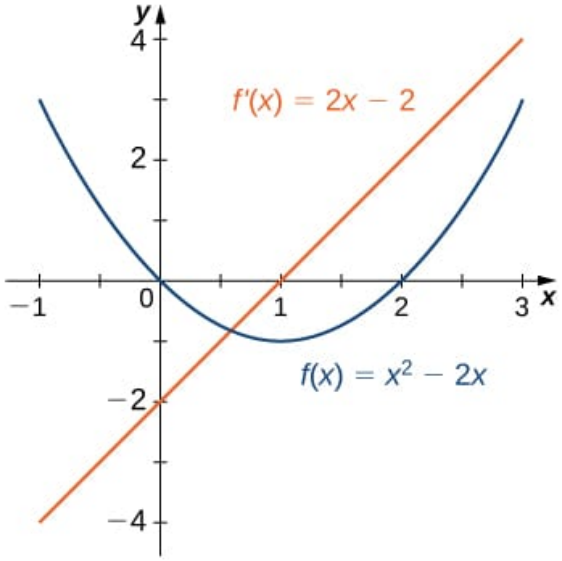

Figure $1.10.3$**:** *The derivative $f'(x) < 0$ where the function $f(x)$ is decreasing and $f'(x) > 0$ where $f(x)$ is increasing. The derivative is zero where the function has a horizontal tangent*

Interactive Element: Reading f' off the Slopes of f

The following Interactive Element might help you visualize how the graph of the derivative is built from the slopes of the original function.

Interact: Drag the orange point along the upper curve f; watch the red tangent rotate and the green point on f' ride up and down.

Observation: The height of the point on f' always equals the slope of the tangent to f at the same x. Where f has a horizontal tangent, f' meets the x-axis; where f rises, $f' > 0$; where f falls, $f' < 0$.

Example 1.10.3: Sketching a Derivative Using a Function

Use the following graph of $f(x)$ to sketch a graph of $f'(x)$.

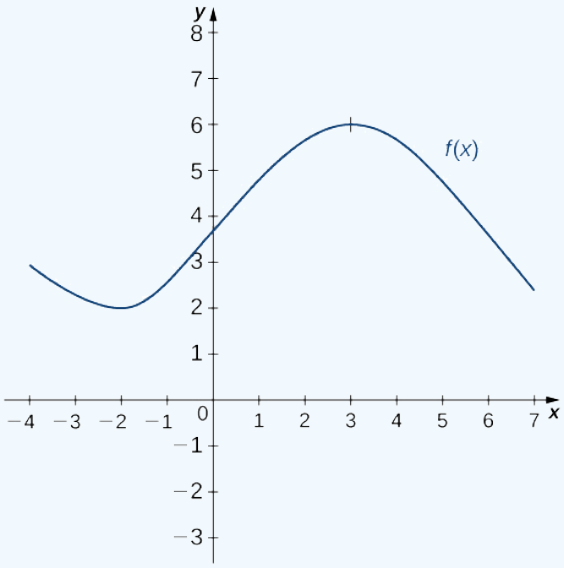

Figure 1.10.4

Solution available at the end of this section.

Checkpoint 1.10.3

Sketch the graph of $f(x) = x^2 - 4$. On what interval is the graph of $f'(x)$ above the x-axis?

Answer available at the end of this section.

Derivatives and Continuity

Now that we can graph a derivative let's examine the behavior of the graphs. First, we consider the relationship between differentiability and continuity. We will see that if a function is differentiable at a point, it must be continuous there; however, a continuous function need not be differentiable at that point. A function may be continuous at a point and fail to be differentiable at the point for one of several reasons.

Theorem: Differentiability Implies Continuity

Let $f(x)$ be a function and a be in its domain. If $f(x)$ is differentiable at a, then f is continuous at a.

Proof available at the end of this section.

We have just proven that differentiability implies continuity, but now we consider whether continuity implies differentiability. To

determine an answer to this question, we examine the function $f(x) = |x|$. This function is continuous everywhere; however, $f'(0)$ is undefined. This observation leads us to believe that continuity does not imply differentiability. Let's explore further. For $f(x) = |x|$,

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0} \frac{|x|}{x}.$$

This limit does not exist because

$$\lim_{x \to 0^-} \frac{|x|}{x} = -1 \quad \text{and} \quad \lim_{x \to 0^+} \frac{|x|}{x} = 1.$$

See Figure $1.10.6$.

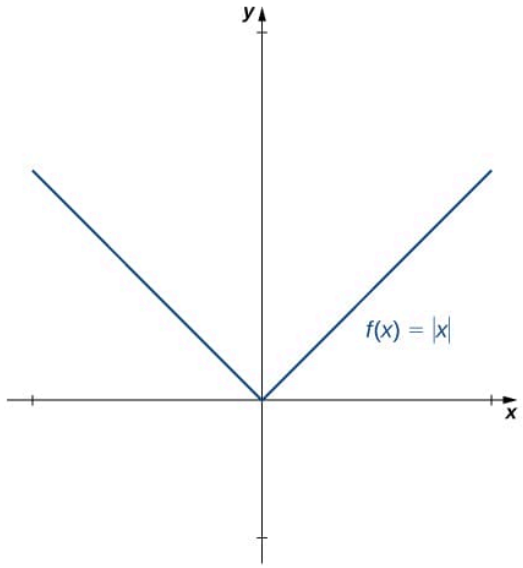

Figure $1.10.6$**:** *The function* $f(x) = |x|$ *is continuous at* 0 *but is not differentiable at* 0.

Interactive Element: Why $|x|$ Has No Tangent at the Corner

The following Interactive Element might help you visualize why a continuous function can still fail to be differentiable at a sharp corner.

Interact: Drag the orange point along $y = |x|$ from the right side across to the left side.

Observation: From the right the line has slope $+1$; from the left it has slope -1. No single straight line is tangent at the corner, so $f'(0)$ does not exist even though f is continuous there.

Let's consider some additional situations in which a continuous function fails to be differentiable. Consider the function $f(x) = \sqrt[3]{x}$:

$$f'(0) = \lim_{x \to 0} \frac{\sqrt[3]{x} - 0}{x - 0} = \lim_{x \to 0} \frac{1}{\sqrt[3]{x^2}} = +\infty.$$

Thus $f'(0)$ does not exist. A quick look at the graph of $f(x) = \sqrt[3]{x}$ clarifies the situation. The function has a vertical tangent line at 0 (Figure $1.10.7$).

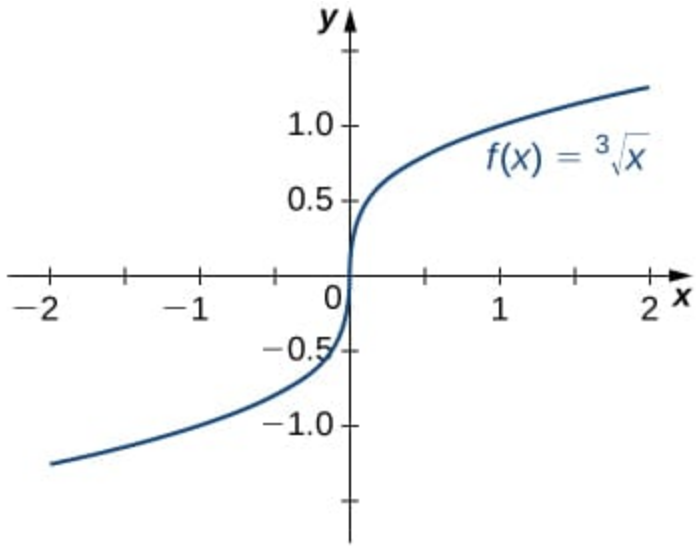

Figure 1.10.7**:** *The function* $f(x) = \sqrt[3]{x}$ *has a vertical tangent at* $x = 0$*. It is continuous at* 0 *but is not differentiable at* 0*.*

The function $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ also has a derivative that exhibits interesting behavior at 0.

We see that

$$f'(0) = \lim_{x \to 0} \frac{x \sin(1/x) - 0}{x - 0} = \lim_{x \to 0} \sin\left(\frac{1}{x}\right).$$

This limit does not exist, essentially because the slopes of the secant lines continuously change direction as they approach zero (Figure 1.10.8).

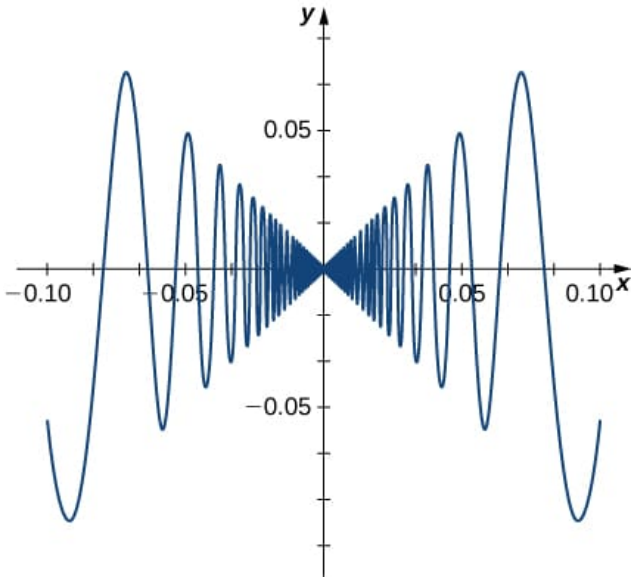

Figure 1.10.8**:** *The function* $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ *is not differentiable at* 0*.*

In summary:

- We observe that if a function is not continuous, it cannot be differentiable since every differentiable function must be continuous. However, if a function is continuous, it may still fail to be differentiable.
- We saw that $f(x) = |x|$ failed to be differentiable at 0 because the limits of the slopes of the tangent lines on the left and right were not the same. Visually, this resulted in a sharp corner on the function's graph at 0.. From this, we conclude that to be differentiable at a point, a function must be "smooth" at that point.

- As we saw in the example of $f(x) = \sqrt[3]{x}$, a function fails to be differentiable at a point where there is a vertical tangent line.
- As we saw with $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ a function may fail to be differentiable at a point in more complicated ways as well.

Example 1.10.4: A Piecewise Function that is Continuous and Differentiable

A toy company wants to design a track for a toy car that starts along a parabolic curve and then converts to a straight line (Figure 1.10.9). The function that describes the track is to have the form $f(x) = \begin{cases} \frac{1}{10}x^2 + bx + c, & \text{if } x < -10 \\ -\frac{1}{4}x + \frac{5}{2}, & \text{if } x \geq -10 \end{cases}$ where x and $f(x)$ are in inches. For the car to move smoothly along the track, the function $f(x)$ must be both continuous and differentiable at -10. Find values of b and c that make $f(x)$ both continuous and differentiable.

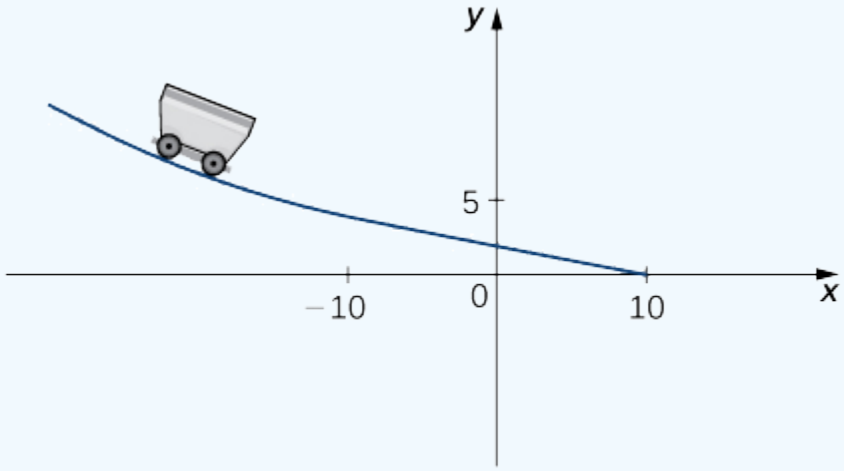

Figure 1.10.9: *The function must be continuous and differentiable for the car to move smoothly along the track.*

Solution available at the end of this section.

Checkpoint 1.10.4

Find values of a and b that make $f(x) = \begin{cases} ax + b, & \text{if } x < 3 \\ x^2, & \text{if } x \geq 3 \end{cases}$ both continuous and differentiable at 3.

Answer available at the end of this section.

Higher-Order Derivatives

The derivative of a function is itself a function, so we can find the derivative of a derivative. For example, the derivative of a position function is the rate of change of position or velocity. The derivative of velocity is the rate of change of velocity, which is acceleration. The new function obtained by differentiating the derivative is called the second derivative. Furthermore, we can continue taking derivatives to obtain the third, fourth, and so on. Collectively, these are referred to as **higher-order derivatives**. The notation for the higher-order derivatives of $y = f(x)$ can be expressed in any of the following forms:

$$\begin{gathered} f''(x),\ f'''(x),\ f^{(4)}(x),\ \ldots,\ f^{(n)}(x) \\ y''(x),\ y'''(x),\ y^{(4)}(x),\ \ldots,\ y^{(n)}(x) \\ \frac{d^2y}{dx^2},\ \frac{d^3y}{dy^3},\ \frac{d^4y}{dy^4},\ \ldots,\ \frac{d^ny}{dy^n}. \end{gathered}$$

It is interesting to note that the notation for $\frac{d^2y}{dx^2}$ may be viewed as an attempt to express $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ more compactly.

Analogously, $\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$.

Example 1.10.5: Finding a Second Derivative

For $f(x) = 2x^2 - 3x + 1$, find $f''(x)$.

Solution available at the end of this section.

Checkpoint 1.10.5

Find $f''(x)$ for $f(x) = x^2$.

Answer available at the end of this section.

Example 1.10.6: Finding Acceleration

The position of a particle along a coordinate axis at time t (in seconds) is given by $s(t) = 3t^2 - 4t + 1$ (in meters). Find the function that describes its acceleration at time t.

Solution available at the end of this section.

Checkpoint 1.10.6

For $s(t) = t^3$, find $a(t)$.

Answer available at the end of this section.

Success in Mathematics: Why Study Mathematics? (Part 1)

Until modern times, only a tiny percent of people had the opportunity to be "liberally educated." These people were not studying to learn a job or trade - they wanted to learn what other humans had thought and created. They were also trying to develop new knowledge. Menaechmus (350 B.C.E) was this kind of student. As he studied geometry, he discovered the properties of conic sections. Others built on his knowledge. Apollonius (350 B.C.E) wrote an eight-chapter book about conic sections.

Today we use our knowledge of conic sections to predict orbits, build telescopes, locate ships at sea, make "eyes" for robots, and collect solar energy. But we also study conic sections as an accomplishment of our human ancestors. In developing their equations, we also develop logic and creative problem-solving skills. Think about the football coach who has his team doing agility drills and running through tires. Why? When has a football team ever had to run through tires during a game? Similarly, with the right attitude, you can develop your ability to reason and think in your life outside of college by doing mathematics. You are responsible for recognizing that learning and studying may be valuable even when it is not immediately practical.

> One problem with teaching mathematics in the K-12 system - and I see it as a major difficulty - is that there is virtually nothing the pupils learn that has a non-trivial application in today's world. The most a teacher can tell a student who inquires, entirely reasonably, "How is this useful?" is that almost all mathematics finds uses, in many cases important ones, and that what they learn in school leads on to mathematics that definitely is used.
>
> - Keith Devlin, Standford University

By "non-trivial," Dr. Devlin means that many real-life applications require mathematics beyond what is studied in high school.

1. Younger students often ask, " When will I use this math?" Their teachers often reply, "You need to learn this math to go on to the next course. In college, you can take math that is very powerful for doing all kinds of practical things." Do you think this is the best reply to give to the student? Explain.
2. I've often pondered the best mixture of application and theory in a mathematics course. My love of mathematics lies in

theory; however, my research in mathematics is in applications. As a student, which do you prefer? Explain.

Success in College: Identifying Your Learning Style(s)

Visual/Verbal Learners

These people learn best when information is presented visually and in a written form. In a classroom setting, they prefer professors who use visual aids (i.e., whiteboard or PowerPoint presentation) to list the essential points of a lecture to provide them with an outline to follow. They benefit from information obtained from textbooks and class notes. These learners like to study by themselves in quiet environments. They visualize information in their "minds' eyes" to remember something. The online environment is especially appropriate for visual/verbal learners because most of the information for a course is presented in written form.

Visual/Nonverbal Learners

These people learn best when information is presented visually and in a picture or design format. In a classroom setting, they benefit from professors who supplement their lectures with materials such as film, video, maps, and diagrams. They relate well to information obtained from the images and charts in textbooks. They tend to prefer to work alone in quiet environments. They visualize an image of something in their mind when trying to remember it. These learners may also be artistic and enjoy visual art and design. The online environment is well suited for this learner because graphical representations of information can help them remember concepts and ideas. Graphical information can be presented using charts, tables, graphs, and images.

Auditory/Verbal Learners

These people learn best when information is presented aurally. They benefit from listening to lectures and participating in group discussions in a classroom setting. They also benefit from obtaining information from audio recordings. When trying to remember something, they often repeat it aloud and mentally "hear" how the professor explained the information to them. They learn best when interacting with others in a listening/speaking activity. Online learning environments can complement these learners' styles. Although most information is presented visually (either written or graphically), group participation and collaborative activities are accomplished well online. In addition, streaming audio and computer conferencing can be incorporated into an online course to best meet the learning style of these students.

Tactile/Kinesthetic Learners

These people learn best when doing a physical "hands-on" activity. In the classroom, they prefer to learn new materials in a lab setting where they can touch and manipulate materials. They learn best in physically active learning situations. They benefit from professors who use in-class demonstrations, hands-on learning experiences, and fieldwork outside the classroom. Online environments can provide learning opportunities for tactile/kinesthetic learners. Simulations with 3-Dimensional graphics can replicate physical demonstrations. Lab sessions can be conducted at predetermined locations or homes and then discussed online. Also, outside fieldwork can be incorporated into the coursework, with ample online discussion preceding and following the experience. Finally, the online environment is well suited for the presentation and discussion of either group or individual projects and activities.

Take the learning style survey at Learning Styles.

1. What was your most extreme score, and what was it about?
2. Of all the results, which surprised you the most?

Answers

Section Preview

In this section, focus on the limit definition of the derivative as a function (as opposed to at a number). Make sure that students learn how to graph the derivative given the graph of the original function and that they know (at the very least) Lagrange and Leibniz notations. Connect derivatives and continuity (differentiability implies continuity—not the other way around). Finally, introduce higher-order derivatives. **Derivative tricks and shortcuts are not to be used in this section!**

Section Preview

- **Graphing Functions**
  - Intervals of Increase and Decrease: Graphing the derivative rests on connecting where f increases or decreases to the sign of f' (Example 3).
- **Absolute Value Functions**
  - The Absolute Value Function — The corner of $f(x) = |x|$ at 0 is the section's canonical example of a continuous function that fails to be differentiable; recognizing the V-shape and its sharp vertex is conceptually central.

Section Preview

- **Simplifying Expressions**
  - Adding and Subtracting Polynomials: Forming $f(x+h) - f(x)$ requires distributing the subtraction and combining like terms (Examples 2, 5, and 6).

Section Preview

- Define the derivative function of a given function.
- Graph a derivative function from the graph of a given function.
- State the connection between derivatives and continuity.
- Describe three conditions when a function has no derivative.
- Explain the meaning of a higher-order derivative.

Example $1.10.1$**: Finding the Derivative of a Square-Root Function**

Start directly with the definition of the derivative function.

Substitute $f(x+h) = \sqrt{x+h}$ and $f(x) = \sqrt{x}$ into $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h\to 0} \frac{\left(\sqrt{x+h} - \sqrt{x}\right)}{h} \cdot \frac{\left(\sqrt{x+h} + \sqrt{x}\right)}{\left(\sqrt{x+h} + \sqrt{x}\right)} \\ &= \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)} \\ &= \lim_{h\to 0} \frac{\cancel{h}^{1}}{\cancel{h}\left(\sqrt{x+h} + \sqrt{x}\right)} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\ &\overset{\text{D.S.}}{=} \frac{1}{2\sqrt{x}} \end{aligned}$$

Example $1.10.2$**: Finding the Derivative of a Quadratic Function**

Follow the same procedure, but without multiplying by the conjugate.

Substitute $f(x+h) = (x+h)^2 - 2(x+h)$ and $f(x) = x^2 - 2x$ into $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{((x+h)^2 - 2(x+h)) - (x^2 - 2x)}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 2x - 2h - x^2 + 2x}{h} \\ &= \lim_{h\to 0} \frac{2xh - 2h + h^2}{h} \\ &= \lim_{h\to 0} \frac{h(2x - 2 + h)}{h} \\ &= \lim_{h\to 0} \frac{\cancel{h}^{1}(2x - 2 + h)}{\cancel{h}^{1}} \\ &= \lim_{h\to 0} (2x - 2 + h) \\ &\overset{\text{D.S.}}{=} 2x - 2 \end{aligned}$$

Checkpoint $1.10.2$

$f'(x) = 2x$

Example $1.10.3$**: Sketching a Derivative Using a Function**

The solution is shown in the following graph. Observe that $f(x)$ is increasing and $f'(x) > 0$ on $(-2, 3)$. Also, $f(x)$ is decreasing and $f'(x) < 0$ on $(-\infty, -2)$ and on

$(3, +\infty)$. Also note that $f(x)$ has horizontal tangents at -2 and 3, and $f'(-2) = 0$ and $f'(3) = 0$.

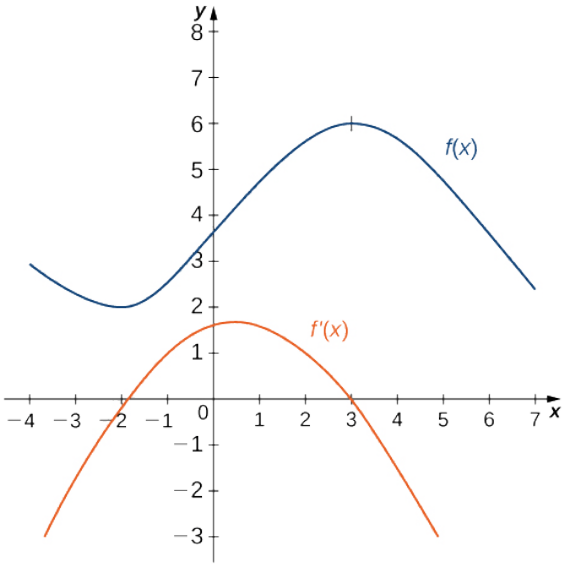

Figure 1.10.5

Checkpoint 1.10.3

$(0, +\infty)$

Theorem: Differentiability Implies Continuity

If $f(x)$ is differentiable at a, then $f'(a)$ exists and, if we let $h = x - a$, we have $x = a + h$, and as $h = x - a \to 0$, we can see that $x \to a$.

Then

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

can be rewritten as

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

We want to show that $f(x)$ is continuous at a by showing that $\lim_{x \to a} f(x) = f(a)$. Thus,

$$\begin{aligned} \lim_{x \to a} f(x) &= \lim_{x \to a} \big(f(x) - f(a) + f(a)\big) \\ &= \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \cdot (x - a) + f(a)\right) \\ &= \left(\lim_{x \to a} \frac{f(x) - f(a)}{x - a}\right) \cdot \left(\lim_{x \to a} (x - a)\right) + \lim_{x \to a} f(a) \\ &= f'(a) \cdot 0 + f(a) \\ &= f(a). \end{aligned}$$

Therefore, since $f(a)$ is defined and $\lim_{x \to a} f(x) = f(a)$, we conclude that f is continuous at a.

Q.E.D.

Example 1.10.4**: A Piecewise Function that is Continuous and Differentiable**

For the function to be continuous at $x = -10$, $\lim_{x \to 10^-} f(x) = f(-10)$. Thus, since

$$\lim_{x \to -10^-} f(x) = \frac{1}{10}(-10)^2 - 10b + c = 10 - 10b + c$$

and $f(-10) = 5$, we must have $10 - 10b + c = 5$. Equivalently, we have $c = 10b - 5$.

For the function to be differentiable at -10,

$$f'(10) = \lim_{x \to -10} \frac{f(x) - f(-10)}{x + 10}$$

must exist. Since $f(x)$ is defined using different rules on the right and the left, we must evaluate this limit from the right and the left and then set them equal to each other:

$$\begin{aligned}\lim_{x\to -10^-}\frac{f(x)-f(-10)}{x+10} &= \lim_{x\to -10^-}\frac{\frac{1}{10}x^2+bx+c-5}{x+10}\\ &= \lim_{x\to -10^-}\frac{\frac{1}{10}x^2+bx+(10b-5)-5}{x+10}\\ &= \lim_{x\to -10^-}\frac{x^2-100+10bx+100b}{10(x+10)}\\ &= \lim_{x\to -10^-}\frac{(x+10)(x-10+10b)}{10(x+10)}\\ &= b-2\end{aligned}$$

We also have

$$\begin{aligned}\lim_{x\to -10^+}\frac{f(x)-f(-10)}{x+10} &= \lim_{x\to -10^+}\frac{-\frac{1}{4}x+\frac{5}{2}-5}{x+10}\\ &= \lim_{x\to -10^+}\frac{-(x+10)}{4(x+10)}\\ &= -\frac{1}{4}\end{aligned}$$

This gives us $b-2=-\frac{1}{4}$. Thus $b=\frac{7}{4}$ and $c=10(\frac{7}{4})-5=\frac{25}{2}$.

Checkpoint 1.10.4

$a=6$ and $b=-9$

Example 1.10.5**: Finding a Second Derivative**

First find $f'(x)$.

Substitute $f(x)=2x^2-3x+1$ and $f(x+h)=2(x+h)^2-3(x+h)+1$ into $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

$$\begin{aligned}f'(x) &= \lim_{h\to 0}\frac{(2(x+h)^2-3(x+h)+1)-(2x^2-3x+1)}{h}\\ &= \lim_{h\to 0}\frac{4xh+2h^2-3h}{h}\\ &= \lim_{h\to 0}(4x+2h-3)\\ &\overset{\text{D.S.}}{=} 4x-3\end{aligned}$$

Next, find $f''(x)$ by taking the derivative of $f'(x)=4x-3$.

$$\begin{aligned}f''(x) &= \lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h}\\ &= \lim_{h\to 0}\frac{(4(x+h)-3)-(4x-3)}{h}\\ &= \lim_{h\to 0}4\\ &\overset{\text{D.S.}}{=} 4\end{aligned}$$

Checkpoint 1.10.5

$f''(x)=2$

Example 1.10.6**: Finding Acceleration**

Since $v(t)=s'(t)$ and $a(t)=v'(t)=s''(t)$, we begin by finding the derivative of $s(t)$:

$$\begin{aligned}s'(t) &= \lim_{h\to 0}\frac{s(t+h)-s(t)}{h}\\ &= \lim_{h\to 0}\frac{3(t+h)^2-4(t+h)+1-(3t^2-4t+1)}{h}\\ &= 6t-4\end{aligned}$$

Next,

$$\begin{aligned} s''(t) &= \lim_{h\to 0}\frac{s'(t+h)-s'(t)}{h} \\ &= \lim_{h\to 0}\frac{6(t+h)-4-(6t-4)}{h} \\ &= 6 \end{aligned}$$

Thus, $a = 6\ \text{m/s}^2$.

Checkpoint 1.10.6

$a(t) = 6t$

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1.10.1: Resources and Key Concepts

Resources

Videos

- Derivative function
  - The Derivative as a Function - The Derivative Function
- Differentiability
  - The Derivative as a Function - Differentiability: To Be or Not To Be
- Notation for derivatives
  - The Derivative as a Function - Derivative Notations
- Graphing derivatives
  - The Derivative as a Function - Graphing Derivatives and Finding Patterns
- Differentiability and continuity
  - Proofs in Differential Calculus - Differentiability Implies Continuity
- Higher-order derivatives
  - The Derivative as a Function - Higher-Order Derivatives

Key Concepts

Definitions

- **Derivative Function**: Let f be a function. The derivative function, denoted by f', is the function whose domain consists of those values of x such that the following limit exists: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.
- **Differentiable at** a: A function $f(x)$ is said to be differentiable at a if $f'(a)$ exists.
- **Differentiable on** S: A function is said to be differentiable on S if it is differentiable at every point in an open set S.
- **Differentiable Function**: A differentiable function is one in which $f'(x)$ exists on its domain.
- **Lagrange's Notation (for derivative)**: $f'(x)$ or y'. If evaluated at $x = a$, it's $f'(a)$ or $y'(a)$.
- **Leibniz's Notation (for derivative)**: $\frac{dy}{dx}$. Other forms include $\frac{df(x)}{dx}$, $\frac{df}{dx}$, $\frac{d}{dx}f(x)$. If evaluated at $x = a$, it's $\frac{dy}{dx}\Big|_{x=a}$ or $\frac{df}{dx}\Big|_{x=a}$.
- **Euler's Notation (for derivative)**: $(Df)(x)$, using the differential operator D.
- **Newton's Notation (for derivative)**: $\dot{y}$ (if y is a function of t).
- **Second Derivative**: The derivative of the derivative function.
- **Higher-Order Derivatives**: Derivatives beyond the first derivative (e.g., second, third, fourth, n^{th} derivatives). Notations include $f''(x), f'''(x), f^{(n)}(x)$, $y'', y''', y^{(n)}$, $\frac{d^2y}{dx^2}$, $\frac{d^ny}{dx^n}$.

Theorems

- **Theorem: Differentiability Implies Continuity**: Let $f(x)$ be a function and a be in its domain. If $f(x)$ is differentiable at a, then f is continuous at a.

Common Mistakes

- **Careful Use of Leibniz's Notation**: When using Leibniz's notation to find the derivative of $y = f(x)$ at $x = a$, it is incorrect to write $\frac{df(a)}{dx}$. The correct forms are $\frac{dy}{dx}\Big|_{x=a}$ or $\frac{df}{dx}\Big|_{x=a}$.
- **Continuity Does Not Imply Differentiability**: A function can be continuous at a point but not differentiable there. Reasons include:
  - A sharp corner (e.g., $f(x) = |x|$ at $x = 0$).
  - A vertical tangent line (e.g., $f(x) = \sqrt[3]{x}$ at $x = 0$).
  - Oscillating behavior (e.g., $f(x) = x\sin(1/x)$ for $x \neq 0$, $f(0) = 0$, at $x = 0$).

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1.10.2: Homework

Note

The following exercises are meant as hand-in homework (if assigned by your instructor). Typed-in answers here will not be saved or scored.

Basic Skills

Exercise 1

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Basic Skills - Applications

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Synthesis Questions

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Proofs

Exercise 55

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Exercise 56

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1.10.3: Additional Exercises

Reading Questions

1. How is the derivative function $f'(x)$ defined using a limit?
2. What does it mean for a function to be "differentiable at a point a"?
3. Name two common notations for the derivative of $y = f(x)$ besides $f'(x)$.
4. If $f(x) = \sqrt{x}$, what is $f'(x)$ according to Example 1?
5. What is the relationship between the sign of $f'(x)$ and whether $f(x)$ is increasing or decreasing?
6. If $f(x)$ has a horizontal tangent line at $x = a$, what is the value of $f'(a)$?
7. Does differentiability at a point imply continuity at that point? Explain the theorem.
8. Does continuity at a point imply differentiability at that point? Give an example of a function that is continuous at a point but not differentiable there.
9. List three reasons why a function might fail to be differentiable at a point where it is continuous.
10. What is a higher-order derivative? Give an example of notation for the second derivative.
11. If $s(t)$ is a position function, what do $s'(t)$ and $s''(t)$ represent physically?

Homework

For exercises 1 - 10, use the definition of a derivative to find $f'(x)$.

1) $f(x) = 6$

2) $f(x) = 2 - 3x$

Answer

$$f'(x) = -3$$

3) $f(x) = \dfrac{2x}{7} + 1$

4) $f(x) = 4x^2$

Answer

$$f'(x) = 8x$$

5) $f(x) = 5x - x^2$

6) $f(x) = \sqrt{2x}$

Answer

$$f'(x) = \frac{1}{\sqrt{2x}}$$

7) $f(x) = \sqrt{x - 6}$

8) $f(x) = \dfrac{9}{x}$

Answer

$$f'(x) = \frac{-9}{x^2}$$

9) $f(x) = x + \dfrac{1}{x}$

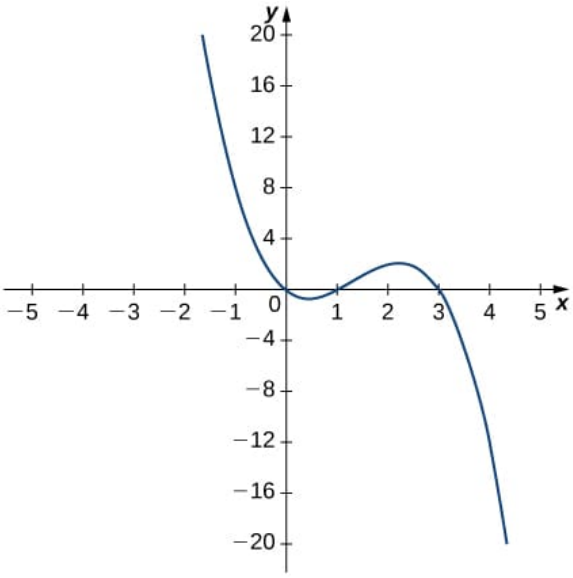
10) $f(x) = \dfrac{1}{\sqrt{x}}$

Answer

$$f'(x) = \frac{-1}{2x^{3/2}}$$

For the exercises 11 - 14, use the graph of $y = f(x)$ to sketch the graph of its derivative $f'(x)$.

11)



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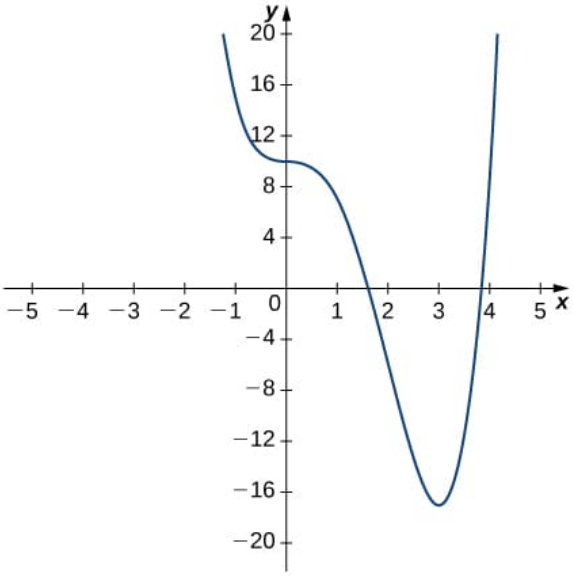
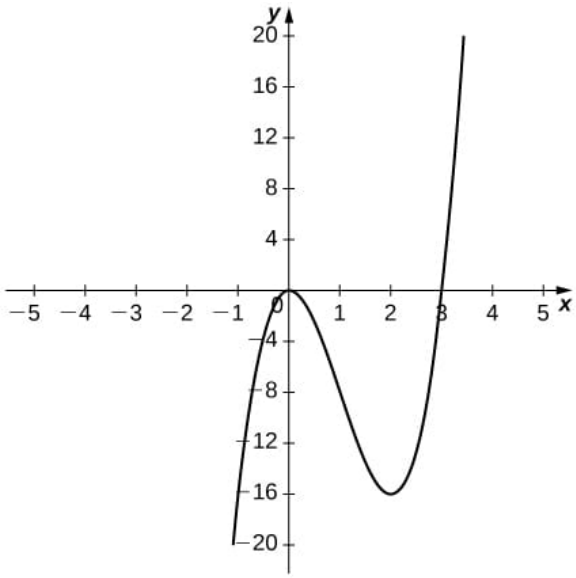


Answer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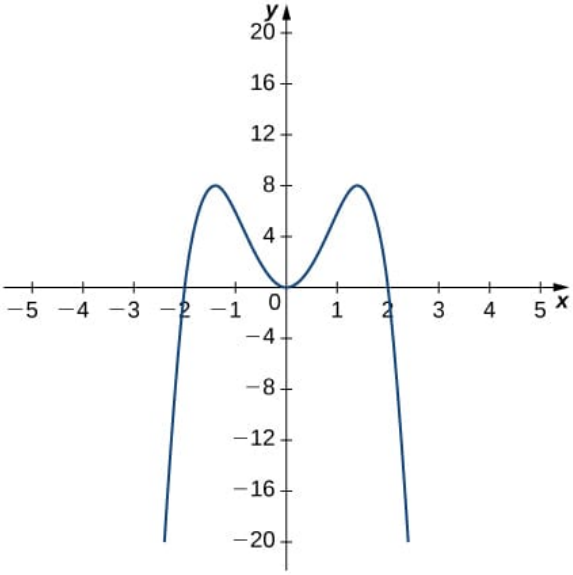
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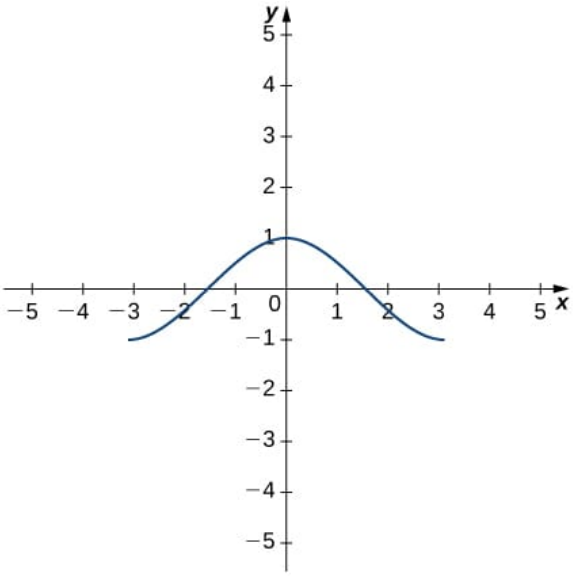
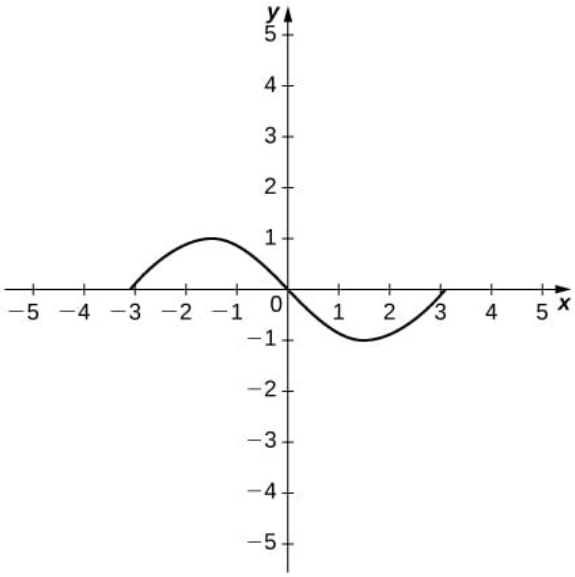


Answer



For exercises 15 - 20, the given limit represents the derivative of a function $y = f(x)$ at $x = a$. Find $f(x)$ and a.

15) $\lim\limits_{h \to 0} \dfrac{(1+h)^{2/3} - 1}{h}$

16) $\lim\limits_{h \to 0} \dfrac{[3(2+h)^2 + 2] - 14}{h}$

Answer

$f(x) = 3x^2 + 2, \quad a = 2$

17) $\lim\limits_{h \to 0} \dfrac{\cos(\pi + h) + 1}{h}$

18) $\lim\limits_{h \to 0} \dfrac{(2+h)^4 - 16}{h}$

Answer

$$f(x) = x^4, \quad a = 2$$

19) $\lim_{h \to 0} \dfrac{[2(3+h)^2 - (3+h)] - 15}{h}$

20) $\lim_{h \to 0} \dfrac{e^h - 1}{h}$

Answer

$$f(x) = e^x, \quad a = 0$$

For the functions in exercises 21 - 24,

a. sketch the graph and

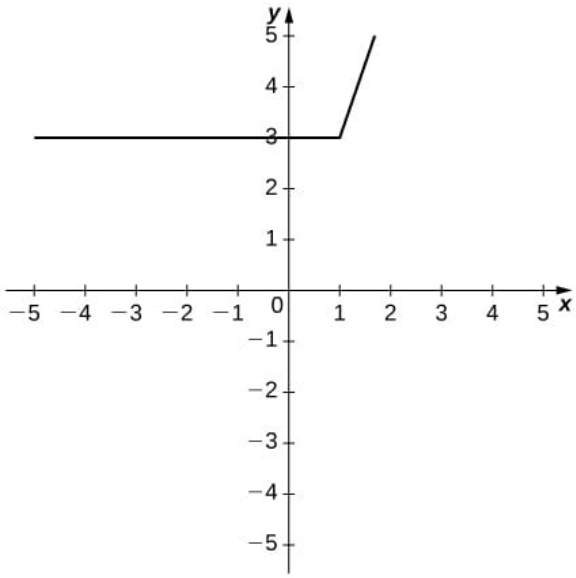
b. use the definition of a derivative to show that the function is not differentiable at $x = 1$.

21) $f(x) = \begin{cases} 2\sqrt{x}, & \text{if } 0 \le x \le 1 \\ 3x - 1, & \text{if } x > 1 \end{cases}$

22) $f(x) = \begin{cases} 3, & \text{if } x < 1 \\ 3x, & \text{if } x \ge 1 \end{cases}$

Answer

a.



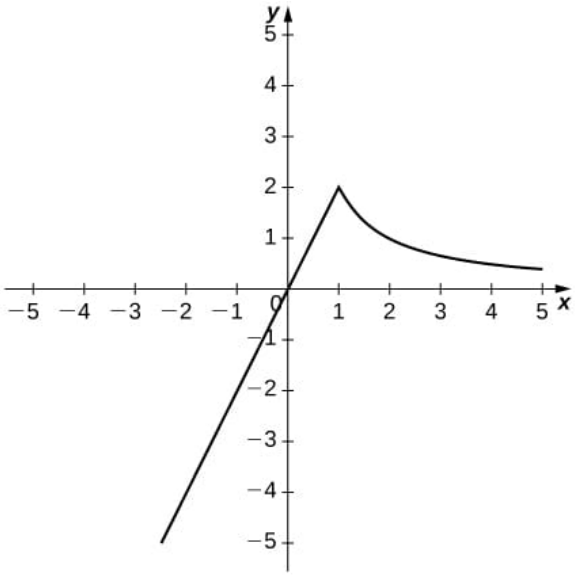
b. $\lim_{h \to 1^-} \dfrac{3-3}{h} \ne \lim_{h \to 1^+} \dfrac{3h}{h}$

23) $f(x) = \begin{cases} -x^2 + 2, & \text{if } x \le 1 \\ x, & \text{if } x > 1 \end{cases}$

24) $f(x) = \begin{cases} 2x, & \text{if } x \le 1 \\ \frac{2}{x}, & \text{if } x > 1 \end{cases}$

Answer

a.



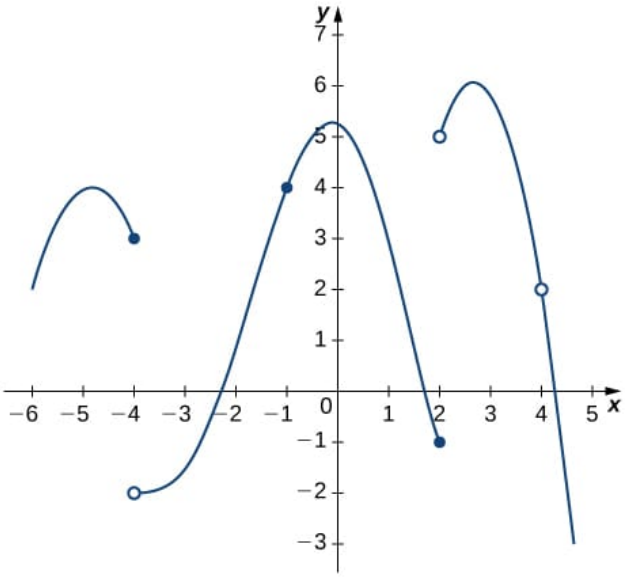
b. $\lim_{h\to 1^-} \frac{2h}{h} \neq \lim_{h\to 1^+} \frac{\frac{2}{x+h} - 2x}{h}$.

For the graphs in exercises 25 - 26,

a. determine for which values of $x = a$ the $\lim_{x\to a} f(x)$ exists but f is not continuous at $x = a$, and

b. determine for which values of $x = a$ the function is continuous but not differentiable at $x = a$.

25)



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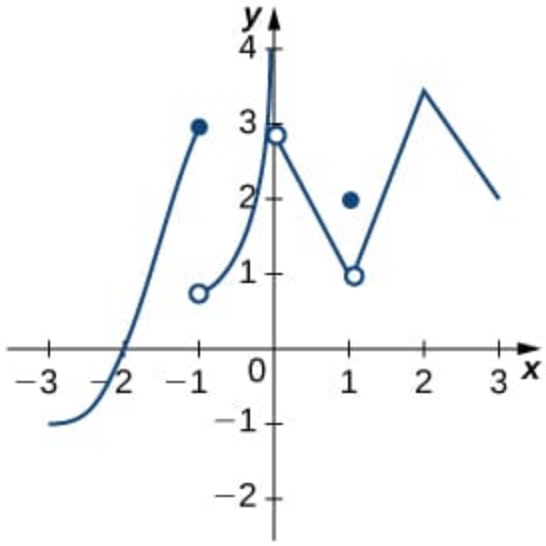
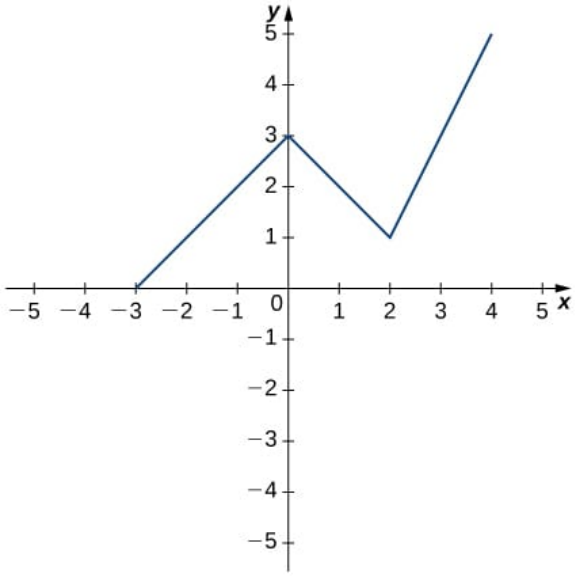


Answer

a. $x = 1$,
b. $x = 2$

27) Use the graph to evaluate a. $f\,'(-0.5)$, b. $f\,'(0)$, c. $f\,'(1)$, d. $f\,'(2)$, and e. $f\,'(3)$, if it exists.



For the functions in exercises 28 - 30, use $f''(x) = \lim_{h \to 0} \dfrac{f'(x+h) - f'(x)}{h}$ **to find** $f''(x)$.

28) $f(x) = 2 - 3x$

Answer

$f''(x) = 0$

29) $f(x) = 4x^2$

30) $f(x) = x + \dfrac{1}{x}$

Answer

$$f''(x) = \frac{2}{x^3}$$

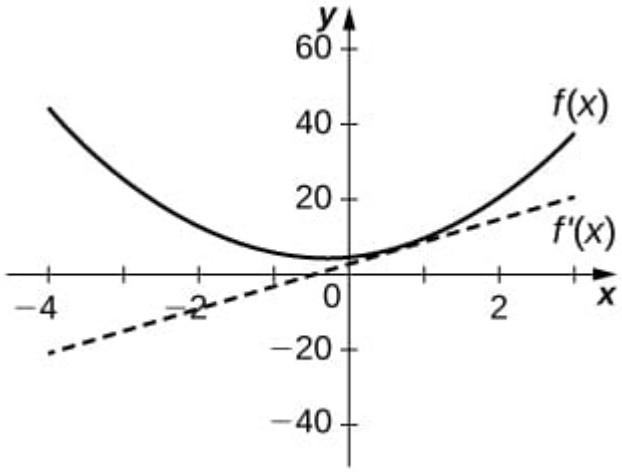
For exercises 31 - 36, use a calculator to graph $f(x)$. Determine the function $f'(x)$, then use a calculator to graph $f'(x)$.

31) [Technology Required] $f(x) = -\frac{5}{x}$

32) [Technology Required] $f(x) = 3x^2 + 2x + 4$.

Answer

$$f'(x) = 6x + 2$$

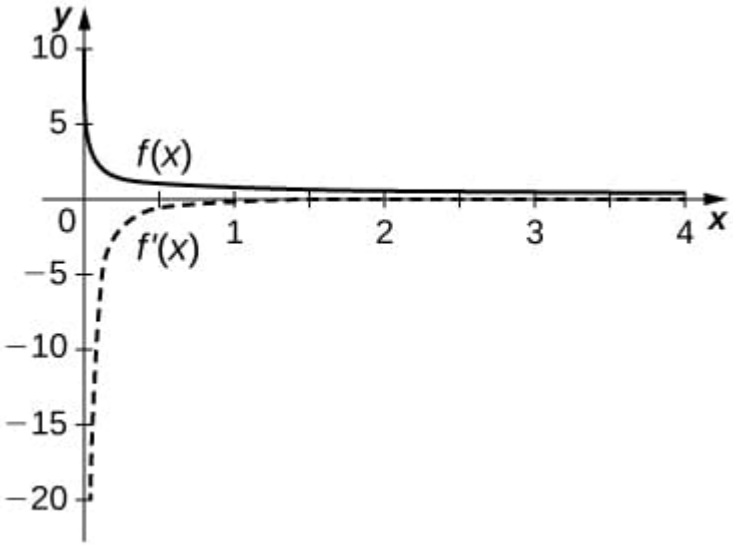


33) [Technology Required] $f(x) = \sqrt{x} + 3x$

34) [Technology Required] $f(x) = \frac{1}{\sqrt{2x}}$

Answer

$$f'(x) = -\frac{1}{(2x)^{3/2}}$$

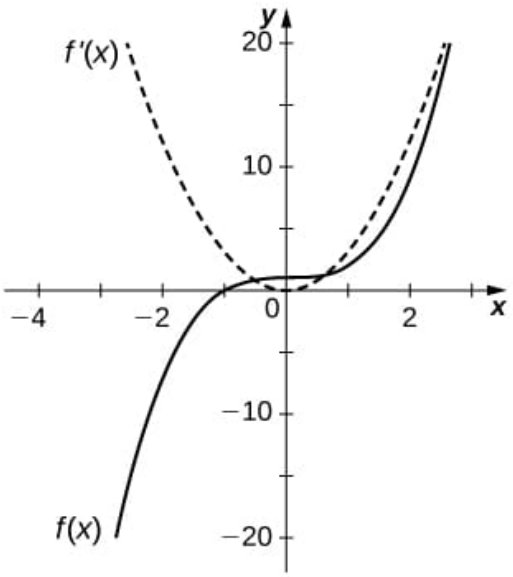


35) [Technology Required] $f(x) = 1 + x + \dfrac{1}{x}$

36) [Technology Required] $f(x) = x^3 + 1$

Answer

$f\,'(x) = 3x^2$



For exercises 37 - 42, describe what the two expressions represent in terms of each of the given situations. Be sure to include units.

a. $\dfrac{f(x+h) - f(x)}{h}$

b. $f\,'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

37) $P(x)$ denotes the population of a city at time x in years.

38) $C(x)$ denotes the total amount of money (in thousands of dollars) spent on concessions by x customers at an amusement park.

Answer

a. Average rate at which customers spent on concessions in thousands per customer.

b. Rate (in thousands per customer) at which x customers spent money on concessions in thousands per customer.

39) $R(x)$ denotes the total cost (in thousands of dollars) of manufacturing x clock radios

40) $g(x)$ denotes the grade (in percentage points) received on a test, given x hours of studying.

Answer

a. Average grade received on the test with an average study time between two values.

b. Rate (in percentage points per hour) at which the grade on the test increased or decreased for a given average study time of x hours.

41) $B(x)$denotes the cost (in dollars) of a sociology textbook at university bookstores in the United States in x years since 1990.

42) $p(x)$ denotes atmospheric pressure at an altitude of x feet.

Answer

a. Average change of atmospheric pressure between two different altitudes.

b. Rate (torr per foot) at which atmospheric pressure is increasing or decreasing at x feet.

43) Sketch the graph of a function $y = f(x)$ with all of the following properties:

a. $f\,'(x) > 0$ for $-2 \leq x < 1$

b. $f\,'(2) = 0$

c. $f\,'(x) > 0$ for $x > 2$

d. $f(2) = 2$ and $f(0) = 1$

e. $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} f(x) = \infty$

f. $f\,'(1)$ does not exist.

44) Suppose temperature T in degrees Fahrenheit at a height x in feet above the ground is given by $y = T(x)$.

a. Give a physical interpretation, with units, of $T\,'(x)$.

b. If we know that $T\,'(1000) = -0.1$, explain the physical meaning.

Answer

a. The rate (in degrees per foot) at which temperature is increasing or decreasing for a given height x.

b. The rate of change of temperature as altitude changes at 1000 feet is -0.1 degrees per foot.

45) Suppose the total profit of a company is $y = P(x)$ thousand dollars when x units of an item are sold.

a. What does $\dfrac{P(b) - P(a)}{b - a}$ for \(0

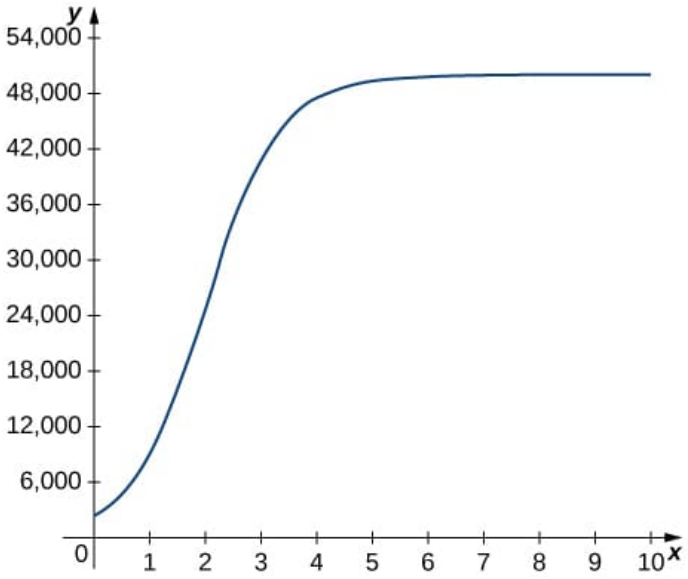
b. What does $P\,'(x)$ measure, and what are the units?

c. Suppose that $P\,'(30) = 5$, what is the approximate change in profit if the number of items sold increases from 30 to 31?

46) The graph in the following figure models the number of people $N(t)$ who have come down with the flu t weeks after its initial outbreak in a town with a population of 50,000 citizens.

a. Describe what $N\,'(t)$ represents and how it behaves as t increases.

b. What does the derivative tell us about how this town is affected by the flu outbreak?



Answer

a. The rate at which the number of people who have come down with the flu is changing t weeks after the initial outbreak.
b. The rate is increasing sharply up to the third week, at which point it slows down and then becomes constant.

For exercises 47 - 52, use the following table, which shows the height h of the Saturn V rocket for the Apollo 11 mission t seconds after launch.

Time (seconds)	Height (meters)
0	0
1	2
2	4
3	13
4	25
5	32

47) What is the physical meaning of $h\,'(t)$? What are the units?

48) [Technology Required] Construct a table of values for $h\,'(t)$ and graph both $h(t)$ and $h\,'(t)$ on the same graph. (Hint: for interior points, estimate both the left limit and right limit and average them.)

Answer

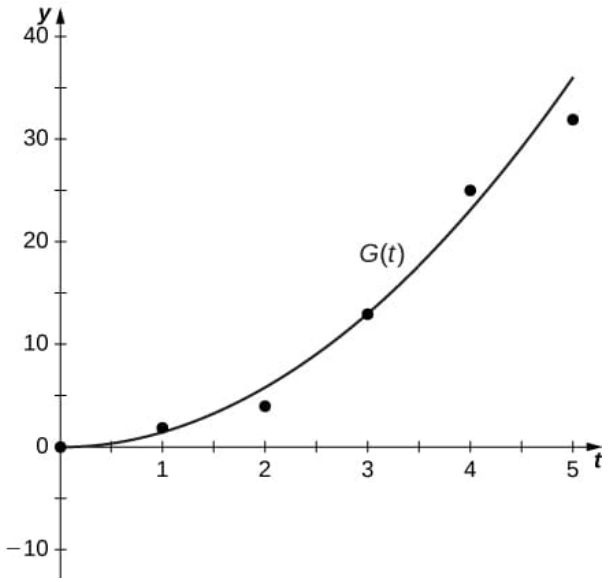
Time (seconds)	$h\,'(t)$ (m/s)
0	2
1	2
2	5.5
3	10.5
4	9.5
5	7

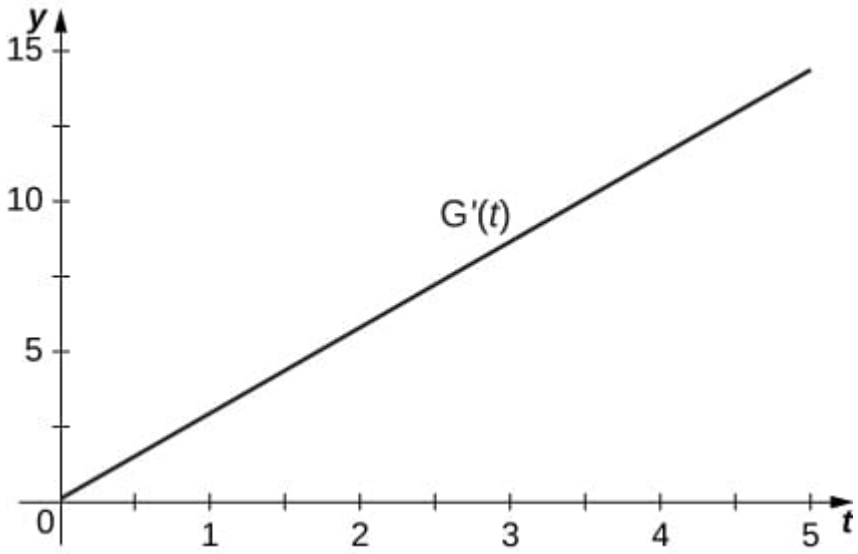
49) [Technology Required] The best linear fit to the data is given by $H(t) = 7.229t - 4.905$, where H is the height of the rocket (in meters) and t is the time elapsed since takeoff. From this equation, determine $H'(t)$. Graph $H(t$ with the given data and, on a separate coordinate plane, graph $H'(t)$.

50) [Technology Required] The best quadratic fit to the data is given by $G(t) = 1.429t^2 + 0.0857t - 0.1429$, where G is the height of the rocket (in meters) and t is the time elapsed since takeoff. From this equation, determine $G'(t)$. Graph $G(t)$ with the given data and, on a separate coordinate plane, graph $G'(t)$.

Answer

$$G'(t) = 2.858t + 0.0857$$





51) [Technology Required] The best cubic fit to the data is given by $F(t) = 0.2037t^3 + 2.956t^2 - 2.705t + 0.4683$, where F is the height of the rocket (in m) and t is the time elapsed since take off. From this equation, determine $F'(t)$. Graph $F(t)$ with the given data and, on a separate coordinate plane, graph $F'(t)$. Does the linear, quadratic, or cubic function fit the data best?

52) Using the best linear, quadratic, and cubic fits to the data, determine what $H''(t),\ G''(t)$ and $F''(t)$ are. What are the physical meanings of $H''(t),\ G''(t)$ and $F''(t)$, and what are their units?

Answer

$H''(t) = 0,\ \ G''(t) = 2.858$ and $f''(t) = 1.222t + 5.912$ represent the acceleration of the rocket, with units of meters per second squared $(\mathrm{m/s^2})$.

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2: DISCOVERING DERIVATIVES

Calculating velocity and changes in velocity are important uses of calculus, but it is far more widespread than that. Calculus is important in all branches of mathematics, science, and engineering, and it is critical to analysis in business and health as well. In this chapter, we explore one of the main tools of calculus, the derivative, and show convenient ways to calculate derivatives. We apply these rules to a variety of functions in this chapter so that we can then explore applications of these techniques.

2.1: Derivatives of Polynomial Functions

2.1.1: Resources and Key Concepts

2.1.2: Homework

2.1.3: Additional Exercises

2.2: Differentiation Techniques - The Product and Quotient Rules

2.2.1: Resources and Key Concepts

2.2.2: Homework

2.2.3: Additional Exercises

2.3: Derivatives of Trigonometric Functions

2.3.1: Resources and Key Concepts

2.3.2: Homework

2.3.3: Additional Exercises

2.4: Differentiation Techniques - The Chain Rule

2.4.1: Resources and Key Concepts

2.4.2: Homework

2.4.3: Additional Exercises

2.5: An Interlude - Hyperbolic Functions

2.5.1: Resources and Key Concepts

2.5.2: Homework

2.5.3: Additional Exercises

2.6: Derivatives of Exponential and Hyperbolic Functions

2.6.1: Resources and Key Concepts

2.6.2: Homework

2.6.3: Additional Exercises

2.7: Implicit Differentiation and the Derivatives of the Inverse Trigonometric and Hyperbolic Functions

2.7.1: Resources and Key Concepts

2.7.2: Homework

2.7.3: Additional Exercises

2.8: Derivatives of Logarithms and Logarithmic Differentiation

2.8.1: Resources and Key Concepts

2.8.2: Homework

2.8.3: Additional Exercises

2.9: Related Rates

2.9.1: Resources and Key Concepts

2.9.2: Homework

2.9.3: Additional Exercises

2.10: Linear Approximations and Differentials

2.10.1: Resources and Key Concepts

2.10.2: Homework

2.10.3: Additional Exercises

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2.1: Derivatives of Polynomial Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Finding derivatives of functions by using the definition of the derivative can be a lengthy and, for certain functions, a rather challenging process. For example, previously we found that

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$$

by using a process that involved multiplying an expression by a conjugate before evaluating a limit.

The process that we could use to evaluate $\frac{d}{dx}\left(\sqrt[3]{x}\right)$ using the definition, while similar, is more complicated.

In this section, we develop rules for finding derivatives that allow us to bypass this process. We begin with the basics.

The Basic Rules

The functions $f(x) = c$ and $g(x) = x^n$ where n is a positive integer are the building blocks from which all polynomials and rational functions are constructed. To find derivatives of polynomials and rational functions efficiently without resorting to the limit definition of the derivative, we must first develop formulas for differentiating these basic functions.

The Constant Rule

We first apply the limit definition of the derivative to find the derivative of the constant function, $f(x) = c$. For this function, both $f(x) = c$ and $f(x + h) = c$, so we obtain the following result:

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{c-c}{h} \\
&= \lim_{h\to 0}\frac{0}{h} \\
&= \lim_{h\to 0} 0 && \text{(Before evaluating the limit, perform any simple arithmetic. Zero over something nonzero is} \\
&= 0.
\end{aligned}
$$

The rule for differentiating constant functions is called the **Constant Rule**. It states that the derivative of a constant function is zero; that is, since a constant function is a horizontal line, the slope, or the rate of change, of a constant function is 0. We restate this rule in the following theorem.

Theorem: The Constant Rule

Let c be a constant. If $f(x) = c$, then $f'(x) = 0$.

Alternatively, we may express this rule as

$$\frac{d}{dx}(c) = 0.$$

Example 2.1.1: Applying the Constant Rule

Find the derivative of $f(x) = 8$.

Solution available at the end of this section.

Checkpoint 2.1.1

Find the derivative of $g(x) = -3$.

Answer available at the end of this section.

The Power Rule

During our exploration of the limit definition of the derivative, we showed that

$$\frac{d}{dx}\left(x^2\right) = 2x \quad \text{and} \quad \frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}x^{-1/2}.$$

At this point, you might see a pattern beginning to develop for derivatives of the form $\frac{d}{dx}\left(x^n\right)$. We continue our examination of derivative formulas by differentiating power functions of the form $f(x) = x^n$ where n is a positive integer. We develop formulas for derivatives of this type of function in stages, beginning with positive integer powers. Before stating and proving the general rule for derivatives of functions of this form, we take a look at a specific case, $\frac{d}{dx}(x^3)$, as the technique used in this case is essentially the same as the technique used to prove the general case.

Example 2.1.2: Differentiating x^3

Find $\frac{d}{dx}\left(x^3\right)$.

Solution available at the end of this section.

Checkpoint 2.1.2

Find $\frac{d}{dx}(x^4)$.

Answer available at the end of this section.

As we shall see, the procedure for finding the derivative of the general form $f(x) = x^n$ is very similar. Although it is often unwise to draw general conclusions from specific examples, we note that when we differentiate $f(x) = x^3$, the derivative becomes the product of the original power and x to a power one less than it was before. A lot of students think of it as follows:

> When you take the derivative of x^n, you move the power to the front and subtract 1 from the exponent.

The following theorem states that the **Power Rule** holds for all positive integer powers of x.

Theorem: The Power Rule (positive integer exponents only)

Let n be a positive integer. If $f(x) = x^n$, then

$$f'(x) = nx^{n-1}.$$

Alternatively, we may express this rule as

$$\frac{d}{dx}(x^n) = nx^{n-1.}$$

Proof available at the end of this section.

Example 2.1.3: Applying the Power Rule

Find the derivative of the function $f(x) = x^{10}$ by applying the Power Rule.

Solution available at the end of this section.

Checkpoint 2.1.3

Find the derivative of $f(x) = x^7$.

Answer available at the end of this section.

This theorem holds for *all* powers (not just positive integers). Despite being unable to prove it at this point in the course, we allow ourselves the luxury of using it.

Theorem: The Power Rule (General Version)

Let $n \in \mathbb{R}$. If $f(x) = x^n$,then

$$\frac{d}{dx}(x^n) = nx^{n-1.}$$

Caution

As a reminder, we have yet to prove the previous theorem. We are allowing ourselves the ability to use it for now. Still, I promise this fact will be proved rigorously when we have better mathematical "technology."

Example 2.1.4: Using the Extended Power Rule

Find $\frac{d}{dx}(x^{-4})$.

Solution available at the end of this section.

Example 2.1.5: Using the Extended Power Rule and the Constant Multiple Rule

Use the extended Power Rule and the Constant Multiple Rule to find $f(x) = \frac{6}{x^2}$.

Solution available at the end of this section.

Checkpoint 2.1.5

Find the derivative of $g(x) = \frac{1}{x^7}$ using the extended Power Rule.

Answer available at the end of this section.

Example 2.1.6: Applying the General Power Rule

Differentiate each of the following functions.

a. $f(x) = \frac{1}{x^{18}}$
b. $g(x) = \sqrt[5]{x^3}$

Solutions available at the end of this section.

Before stepping into the next subsection, a critical warning needs to be issued.

Caution

The Power Rule only applies to algebraic functions of the form $y = x^n$. It does not apply to functions that raise a constant to a variable power, such as $f(x) = 3^x$. That is,

$$\frac{d}{dx}(3^x) \neq x3^{x-1}.$$

The Sum, Difference, and Constant Multiple Rules

We find our next differentiation rules by looking at derivatives of sums, differences, and constant multiples of functions. Just as when we work with functions, there are rules that make it easier to find derivatives of functions that we add, subtract, or multiply by a constant. These rules are summarized in the following theorem.

Theorem: Sum, Difference, and Constant Multiple Rules

Let $f(x)$ and $g(x)$ be differentiable functions and k be a constant. Then, each of the following equations holds.

Sum Rule. The derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g.

$$\frac{d}{dx}\big(f(x)+g(x)\big)=\frac{d}{dx}\big(f(x)\big)+\frac{d}{dx}\big(g(x)\big);$$

that is,

$$\text{for } s(x)=f(x)+g(x), \quad s'(x)=f'(x)+g'(x).$$

Difference Rule. The derivative of the difference of a function f and a function g is the same as the difference of the derivative of f and the derivative of g :

$$\frac{d}{dx}(f(x)-g(x))=\frac{d}{dx}(f(x))-\frac{d}{dx}(g(x));$$

that is,

$$\text{for } d(x)=f(x)-g(x), \quad d'(x)=f'(x)-g'(x).$$

Constant Multiple Rule. The derivative of a constant k multiplied by a function f is the same as the constant multiplied by the derivative:

$$\frac{d}{dx}\big(kf(x)\big)=k\frac{d}{dx}\big(f(x)\big);$$

that is,

$$\text{for } m(x)=kf(x), \quad m'(x)=kf'(x).$$

Proof available at the end of this section.

Example 2.1.7: Applying the Constant Multiple Rule

Find the derivative of $g(x)=3x^2$ and compare it to the derivative of $f(x)=x^2$.

Solution available at the end of this section.

Example 2.1.8: Applying Basic Derivative Rules

Find the derivative of $f(x)=2x^5+7$.

Solution available at the end of this section.

Checkpoint 2.1.8

Find the derivative of $f(x)=2x^3-6x^2+3$.

Answer available at the end of this section.

The following example illustrates that we can use several rules within a single problem. It also emphasizes the need to pay attention to the Mathematical Mantra.

Example 2.1.9

Compute the derivative of $g(x) = x^2 + \frac{3x^3 - 4x + \sqrt{x}}{x} + x\left(x^2 - 1\right)^2$.

Solution available at the end of this section.

Synthesis Topic: Tangent Lines and Their Equations

Example 2.1.10: Finding the Equation of a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = x^2 - 4x + 6$ at $x = 1$.

Solution available at the end of this section.

Checkpoint 2.1.10

Find the equation of the line tangent to the graph of $f(x) = 3x^2 - 11$ at $x = 2$. Use the point-slope form.

Answer available at the end of this section.

Example 2.1.11: Determining Where a Function Has a Horizontal Tangent

Find the x-value(s) where the graph of $f(x) = 2x^3 + 6x^2 - 90x + 21$ has horizontal tangents.

Solution available at the end of this section.

Checkpoint 2.1.11

Find the values of x for which the line tangent to the graph of $f(x) = 4x^2 - 3x + 2$ has a tangent line parallel to the line $y = 2x + 3$.

Answer available at the end of this section.

Interactive Element: Reading a Derivative as a Slope

The following Interactive Element might help you visualize how the derivative of a polynomial gives the slope of its tangent line at every point.

Interact: Drag the orange point along the curve $f(x) = 2x^3 - 6x^2 + 3$; the red line stays tangent as you move it.

Observation: The tangent's slope at $x = a$ always equals $f'(a) = 6a^2 - 12a$, and the line lies flat exactly at $x = 0$ and $x = 2$ — the two places where solving $f'(x) = 0$ gives a horizontal tangent.

Example 2.1.12

For what values a and b is the line $4x + y = b$ tangent to the curve $y = ax^2$ when $x = -4$?

Solution available at the end of this section.

Applications of Differentiation: Physics

As we have mentioned previously, if $s(t)$ is the position of a particle at time t, then its velocity is

$$v(t) = \frac{ds}{dt}$$

and its acceleration is

$$a(t) = \frac{dv}{dt} = \frac{ds^2}{dt^2}.$$

Example 2.1.13: Ballistic Motion

A rocket is fired vertically upward. Its height in meters above the ground t seconds after launch is given by

$$h(t) = -4.9t^2 + 154t + 388.$$

a. How high was the rocket when it was launched?
b. How high is the rocket after 6 seconds?
c. What is the velocity of the rocket after 6 seconds?
d. What is the acceleration of the rocket after 6 seconds?

Solutions available at the end of this section.

Applications of Differentiation: Business

Most individuals taking this course are moving into a STEM field—not business; however, it is always nice to know what the news channels imply when they mention revenue, cost, profit, marginal cost, etc. To simplify the conversation, I have decided to forgo any lengthy discussion and jump right to the point.

- Revenue is the product of the price of the item and the number of items sold. Mathematically, $R(n) = p(n) \cdot n$, where $p(n)$ is the price-demand function (the price the company charges when the market demands n items) and n is the number of items sold.
- Profit is the difference between the revenue from sales and the cost of producing an item. Mathematically, $P(n) = R(n) - C(n)$, where $R(n)$ is the revenue function and $C(n)$ is the cost function.
- Marginal cost is the change in the total cost that arises when the quantity produced is incremented. That is, it is the change in cost with respect to the number of items produced. Mathematically, $M_C(n) = \frac{dC}{dn}$, where $C(n)$ is the cost function (i.e., the cost of producing n items).
- Marginal profit is the change in the total profit that arises when the quantity produced is incremented. That is, it is the change in profit with respect to the number of items produced. Mathematically, $M_P(n) = \frac{dP}{dn}$, where $P(n)$ is the profit function.
- Marginal revenue is the change in the total revenue that arises when the quantity produced is incremented. That is, it is the change in revenue with respect to the number of items produced. Mathematically, $M_R(n) = \frac{dR}{dn}$, where $R(n)$ is the revenue function.

Thus, when you hear "marginal [insert business phrasing here]," it means that you need to take the derivative of that "business phrasing." The result will represent the cost, profit, revenue, etcetera associated with one additional item being produced.

Example 2.1.14

Suppose a product's revenue function is given by $R(q) = -2q^2 + 800q$, where $R(q)$ is in dollars and q is units sold. Also, its cost function is given by $C(q) = 153q + 40,000$, where C is in dollars and q is units produced. Find an expression for the item's marginal profit function ($MP(q)$) and use it to determine and interpret $MP(220)$.

Solution available at the end of this section.

Your Turn

Formula One Grandstands

Formula One car races can be exhilarating to watch and attract a lot of spectators. Formula One track designers must ensure sufficient grandstand space around the track to accommodate these viewers. However, car racing can be dangerous, and safety considerations are paramount. The grandstands must be placed where spectators will not be in danger should a driver lose control of a car (Figure 2.1.2).

Figure 2.1.2**:** *The grandstand next to a straightaway of the Circuit de Barcelona-Catalunya race track, where the spectators are not in danger.*

Safety is especially a concern on turns. If a driver does not slow down enough before entering the turn, the car may slide off the racetrack. Normally, this results in a wider turn, which slows the driver down. But if the driver loses control completely, the car may fly off the track entirely, on a path tangent to the curve of the racetrack.

Suppose you are designing a new Formula One track. One section of the track can be modeled by the function $f(x) = x^3 + 3x^2 + x$ (Figure 2.1.3). The current plan calls for grandstands to be built along the first straightaway and around a portion of the first curve. The plans call for the front corner of the grandstand to be located at the point $(-1.9, 2.8)$. We want to determine whether this location puts the spectators in danger if a driver loses control of the car.

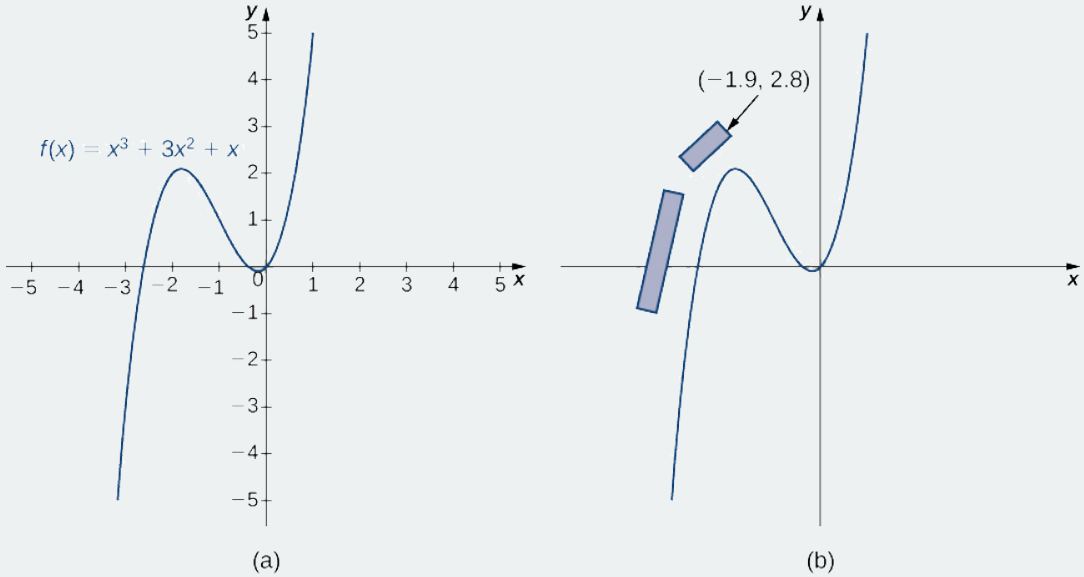

Figure $2.1.3a$*: One section of the racetrack can be modeled by the function* $f(x) = x^3 + 3x^2 + x$.
Figure $2.1.3b$*: The front corner of the grandstand is located at* $(-1.9, 2.8)$.

a. Physicists have determined that drivers are most likely to lose control of their cars as they are coming into a turn at the point where the slope of the tangent line is 1. Find the (x, y) coordinates of this point near the turn.
b. Find the equation of the tangent line to the curve at this point.
c. To determine whether the spectators are in danger in this scenario, find the x-coordinate of the point where the tangent line crosses the line $y = 2.8$. Is this point safely to the right of the grandstand? Or are the spectators in danger?
d. What if a driver loses control earlier than the physicists project? Suppose a driver loses control at the point $(-2.5, 0.625)$. What is the slope of the tangent line at this point?
e. If a driver loses control as described in part d, are the spectators safe?
f. Should you proceed with the current design for the grandstand, or should the grandstands be moved?

Success in Mathematics: Math Anxiety (Part 1)

The following is an excerpt of an article written at MathAcademy.com.

> A famous stage actress was once asked if she had ever suffered from stage fright and, if so, how she had gotten over it. She laughed at the interviewer's naive assumption that she must not feel that kind of anxiety since she was an accomplished actress. She assured him that she had always had stage fright and had never gotten over it. Instead, despite it, she had learned to walk on stage and perform.
>
> Like stage fright, math anxiety can be a disabling condition, causing humiliation, resentment, and even panic.
>
> Even though we all suffer from math anxiety to some degree – just as anyone feels at least a little nervous when speaking to an audience – for some of us, it is a severe problem. This burden interferes with our lives, preventing us from achieving our goals. The first step, without which no further progress is possible, is to recognize that math anxiety is an emotional response. (In fact, severe math anxiety is a learned emotional response.) As with any strong emotional reaction, there are constructive and unconstructive ways to manage math anxiety. Unconstructive (and damaging) ways include rationalization, suppression, and denial.
>
> By "rationalization," we mean finding reasons why it is okay and perhaps even inevitable – and therefore justified – for you to have this reaction. The myths of "men are better at math than women" or "math is for logical people and not creative people" are examples of rationalizations. While they may make you feel better (or at least less bad) about having math anxiety, they will do nothing to lessen it or to help you get it under control. Therefore, rationalization is unconstructive.
>
> "Suppression" means being aware of the anxiety – but trying very hard not to feel it. I have found that students very

commonly attempt this, and some severe self-criticism accompanies it. Students feel that they shouldn't feel this anxiety, that it's a weakness they should overcome by brute force if necessary. When this effort doesn't succeed (as invariably it doesn't), the self-criticism becomes ever harsher, leading to a deep sense of frustration and often a severe loss of self-esteem – mainly if the stakes for a student are high, as when their career or personal goals are riding on a successful outcome in a math class, or when parental disapproval is a factor. Consequently, suppression of math anxiety is not only unconstructive but can be damaging.

Finally, there is denial. People using this approach probably aren't likely to see this essay, much less read it, for they carefully construct their lives to avoid all mathematics as much as possible. They choose college majors and careers that don't require math and let the bank or their spouse balance the checkbook. This approach has the advantage that feelings of frustration and anxiety about math are mostly avoided. However, their lives are drastically constrained, for, in our society, fewer than 25% of all careers are, so to speak, "math-free." Thus, their choices of personal and professional goals are severely limited. (Incidentally, most math-free jobs are low-status and low-pay.)

People in denial about mathematics miss out on something else, too, for the student of mathematics learns to see aspects of the structure and beauty of our world that can be seen in no other way and to which the "innumerate" necessarily remain forever blind. It would be a lot like never hearing music or never seeing colors. (I understand that some people have these disabilities but didn't choose to have them.)

Okay, so what is the constructive way to manage math anxiety? I call it "taking possession." It involves identifying the sources of math anxiety in one's life, accepting those feelings without self-criticism, and then learning strategies for disarming math anxiety's influence on one's future study of mathematics.

Begin by understanding that your feelings of math anxiety are not uncommon and do not indicate that there is anything wrong with you or inferior about your ability to learn math. For some, this can be hard to accept, but it is worth trying to accept – since, after all, it happens to be true. This can be made more accessible by exploring your own "math history." Think back across your career as a math student, and identify those experiences which have contributed most to your feelings of frustration about math. For some, this will be a memory of a humiliating experience in school, such as being made to stand at the blackboard and embarrassed in front of one's peers. For others, it may involve interaction with a parent. Whatever the principle episodes are, recall them as vividly as you can. Then, write them down. This is important. After you have written the episode on a sheet(s) of paper, write down your reaction to the episode, both at the time and how it makes you feel to recall it now.

After completing this exercise, take a fresh sheet of paper and try to sum up in a few words what your feelings about math are at this point in your life, together with the reason or reasons you wish to succeed at math. This, too, is important. Not until after we lay out for ourselves consciously and deliberately what our feelings and desires are towards mathematics will it become possible to take possession of our feelings of math anxiety and become free to implement strategies for coping with those feelings.

At this point, sharing your memories, feelings, and goals with others can be enormously helpful. In a math class I teach for art majors, I hand out a questionnaire early in the semester asking students to do precisely what is described above. After spending about twenty minutes writing down their recollections and goals, I lead them in a classroom discussion on math anxiety. This process of dialogue and sharing – though it may seem just a bit on the goopy side – invariably brings out of each student their barriers to math, often helping them become completely conscious of these barriers for the first time. Just as important, it helps all my students understand that the negative experiences they have had, and their reactions to them, are shared one way or another by almost everyone else in the room.

If you do not have the opportunity to engage in a group discussion in a classroom setting, find friends or relatives you trust to respect your feelings and induce them to talk about their own math anxiety experiences and listen to yours.

Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. Are there counselors at your school that are available to help students who are struggling with math anxiety? If so, where are they located, and how to make an appointment to see one?
2. Estimate the percent of students in your class you think are negatively affected by math anxiety.

If you are unsure about the answers to either of these questions, feel free to ask your instructor - that's what they are there for!

Miscellaneous Topics: Thinking About Mathematics (Part 1)

1. What is it about math that you feel is difficult?
2. Would you think it an odd statement if someone told you that they suck at reading and could barely read a comic book? Why do you think people are more apt to proclaim that they suck at math?

Answers

Section Preview

The purpose of this section is to introduce the student to the Power Rule and the other general rules of differentiation (e.g., the Sum Rule). You should make sure to cover examples requiring equations of tangent lines, as well as some Physics examples.

Section Preview

- **Polynomial Functions**
  - The Binomial Theorem: The proof of the positive-integer Power Rule expands $(x+h)^n$ using binomial coefficients $\binom{n}{k}$; understanding the binomial expansion is required to follow the derivation.

Section Preview

- **Solving Equations**
  - Solving Quadratic Equations by Factoring: Finding horizontal tangents in Example 11 requires solving $f'(x)=0$ by factoring and applying the Zero Product Property to obtain $x=3$ and $x=-5$.
- **Linear Functions**
  - Equations of Lines — Slope-Intercept Form: Example 10 writes the tangent line as $y=-2x+5$, and Example 12 rewrites $4x+y=b$ as $y=-4x+b$ to read its slope.
  - Parallel and Perpendicular Lines: Checkpoint 11 asks where the tangent line is parallel to $y=2x+3$, which requires the equal-slope condition for parallel lines.

Section Preview

- State the constant, constant multiple, and Power Rules.
- Apply the Sum and Difference Rules to combine derivatives.
- Extend the Power Rule to functions with negative and rational exponents.
- Combine the differentiation rules to find the derivative of a polynomial or rational function.
- Use derivatives of polynomials for applications in the sciences, engineering, and business.
- Combine previous knowledge of tangent lines with derivatives of polynomials.

Example 2.1.1: Applying the Constant Rule

This is just a one-step application of the rule: $f'(x)=0$.

Checkpoint 2.1.1

0

Example 2.1.2: Differentiating x^3

$$\begin{aligned}
\frac{d}{dx}\left(x^3\right) &= \lim_{h\to 0}\frac{(x+h)^3-x^3}{h} \\
&= \lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h} && \text{(distributing)} \\
&= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h} && \text{(combining like terms)} \\
&= \lim_{h\to 0}\frac{h(3x^2+3xh+h^2)}{h} && \text{(factoring)} \\
&= \lim_{h\to 0}\frac{\cancel{h}(3x^2+3xh+h^2)}{\cancel{h}^1} && \text{(canceling like factors)} \\
&= \lim_{h\to 0}(3x^2+3xh+h^2) \\
&\overset{\text{D.S.}}{=} 3x^2
\end{aligned}$$

Checkpoint 2.1.2

$\frac{d}{dx}\left(x^4\right)=4x^3$

Theorem: The Power Rule (positive integer exponents only)

For $f(x)=x^n$ where n is a positive integer, we have

$$f'(x)=\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}.$$

Since

$$(x+h)^n = x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + ... + nxh^{n-1} + h^n,$$

we see that

$$(x+h)^n - x^n = nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + ... + nxh^{n-1} + h^n.$$

Next, divide both sides by h:

$$\frac{(x+h)^n - x^n}{h} = \frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + ... + nxh^{n-1} + h^n}{h}.$$

Thus,

$$\frac{(x+h)^n - x^n}{h} = nx^{n-1} + \binom{n}{2}x^{n-2}h + \binom{n}{3}x^{n-3}h^2 + ... + nxh^{n-2} + h^{n-1}.$$

Finally,

$$\begin{aligned} f'(x) &= \lim_{h\to 0}(nx^{n-1} + \binom{n}{2}x^{n-2}h + \binom{n}{3}x^{n-3}h^2 + ... + nxh^{n-2} + h^{n-1}) \\ &= nx^{n-1}. \end{aligned}$$

Q.E.D.

Example 2.1.3**: Applying the Power Rule**

Using the Power Rule with $n = 10$, we obtain

$$f'(x) = 10x^{10-1} = 10x^9.$$

Checkpoint 2.1.3

$f'(x) = 7x^6$

Example 2.1.4**: Using the Extended Power Rule**

By applying the extended Power Rule with $k = -4$, we obtain

$$\frac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5}.$$

Example 2.1.5**: Using the Extended Power Rule and the Constant Multiple Rule**

It may seem tempting to use the Quotient Rule to find this derivative, and it would undoubtedly be correct. However, it is far easier to differentiate this function by first rewriting it as $f(x) = 6x^{-2}$.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{6}{x^2}\right) \\ &= \frac{d}{dx}\left(6x^{-2}\right) && \text{(Laws of Exponents)} \\ &= 6\frac{d}{dx}\left(x^{-2}\right) && \text{(Constant Multiple Rule)} \\ &= 6(-2x^{-3}) && \text{(Power Rule)} \\ &= -12x^{-3} && \text{(simplifying)} \end{aligned}$$

Checkpoint 2.1.5

$g'(x) = -7x^{-8}$.

Example 2.1.6**: Applying the General Power Rule**

a. Following the Mathematical Mantra, we handle any Algebra before performing Calculus. This function can be rewritten as $f(x) = x^{-18}$. Applying the General Power Rule, we get

$$\frac{d}{dx}[f(x)] = -18x^{-18-1} = -18x^{-19}$$

b. Again, following the Mathematical Mantra, we simplify this function first.

$$\frac{d}{dx}\left(\sqrt[5]{x^3}\right) = \frac{d}{dx}\left(x^{3/5}\right) = \frac{3}{5}x^{3/5-1} = \frac{3}{5}x^{-2/5}$$

Theorem: Sum, Difference, and Constant Multiple Rules

We provide only the proof of the Sum Rule here. The rest follow similarly.

For differentiable functions $f(x)$ and $g(x)$, we set $s(x) = f(x) + g(x)$. Using the limit definition of the derivative we have

$$s'(x) = \lim_{h\to 0}\frac{s(x+h) - s(x)}{h}.$$

By substituting $s(x+h) = f(x+h) + g(x+h)$ and $s(x) = f(x) + g(x)$, we obtain

$$s'(x) = \lim_{h\to 0}\frac{\big(f(x+h) + g(x+h)\big) - \big(f(x) + g(x)\big)}{h}.$$

Rearranging and regrouping the terms, we have

$$s'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right).$$

We now apply the Sum Law for limits and the definition of the derivative to obtain

$$s'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f'(x) + g'(x).$$

Q.E.D.

Example 2.1.7**: Applying the Constant Multiple Rule**

We use the Power Rule directly:

$$g'(x) = \frac{d}{dx}(3x^2) = 3\frac{d}{dx}(x^2) = 3(2x) = 6x.$$

Since $f(x) = x^2$ has derivative $f'(x) = 2x$, we see that the derivative of $g(x)$ is 3 times the derivative of $f(x)$. This relationship is illustrated in Figure 2.1.1.

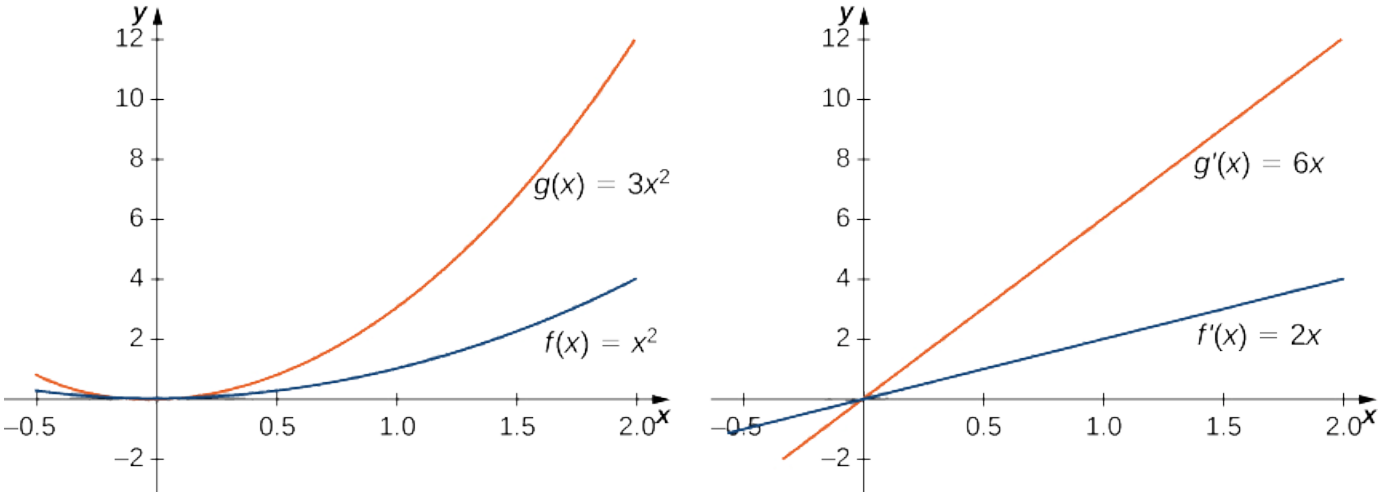

Figure 2.1.1**:** *The derivative of $g(x)$ is 3 times the derivative of $f(x)$.*

Example 2.1.8**: Applying Basic Derivative Rules**

We begin by applying the rule for differentiating the sum of two functions, followed by the rules for differentiating constant multiples of functions and the rule for differentiating powers. To better understand the sequence in which the differentiation rules are applied, we use Leibniz notation throughout the solution:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(2x^5 + 7\right) \\ &= \frac{d}{dx}(2x^5) + \frac{d}{dx}(7) && \text{(Sum Rule)} \\ &= 2\frac{d}{dx}(x^5) + \frac{d}{dx}(7) && \text{(Constant Multiple Rule)} \\ &= 2(5x^4) + 0 && \text{(Power and Constant Rules)} \\ &= 10x^4 && \text{(simplifying)} \end{aligned}$$

Checkpoint 2.1.8

$f'(x) = 6x^2 - 12x$.

Example 2.1.9

$$\begin{aligned} \frac{d}{dx}\left(x^2 + \frac{3x^3 - 4x + \sqrt{x}}{x} + x\left(x^2 - 1\right)^2\right) &= \frac{d}{dx}\left(x^2 + 3x^2 - 4 + \frac{1}{\sqrt{x}} + x\left(x^4 - 2x^2 + 1\right)\right) \\ &= \frac{d}{dx}\left(x^2 + 3x^2 - 4 + \frac{1}{\sqrt{x}} + x^5 - 2x^3 + x\right) \\ &= \frac{d}{dx}\left(x^5 - 2x^3 + 4x^2 + x + x^{-1/2} - 4\right) \\ &= 5x^4 - 6x^2 + 8x + 1 - \frac{1}{2}x^{-3/2} \end{aligned}$$

Example 2.1.10**: Finding the Equation of a Tangent Line**

To find the equation of the tangent line, we need a point and a slope. To find the point, compute

$$f(1) = 1^2 - 4(1) + 6 = 3.$$

This gives us the point $(1, 3)$. Since the slope of the tangent line at 1 is $f'(1)$, we must first find $f'(x)$. Using the definition of a derivative, we have

$$f'(x) = 2x - 4$$

so the slope of the tangent line is $f'(1) = -2$. Using the point-slope formula, we see that the equation of the tangent line is

$$y - 3 = -2(x - 1).$$

Putting the equation of the line in slope-intercept form, we obtain

$$y = -2x + 5.$$

Checkpoint 2.1.10

$y = 12x - 23$

Example 2.1.11**: Determining Where a Function Has a Horizontal Tangent**

To find the values of x for which $f(x)$ has a horizontal tangent line, we must solve $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 6x^2 + 12x - 90 \\ &= 6\left(x^2 + 2x - 15\right) \\ &= 6\left(x - 3\right)\left(x + 5\right) \end{aligned}$$

Hence, f has horizontal tangent lines at $x = 3$ and $x = -5$.

Checkpoint 2.1.11

$\frac{5}{8}$

Example 2.1.12

Careful reading is important in Calculus. We know that, for some special values of a and b, the line $4x + y = b$ will be tangent to $y = ax^2$ at $x = -4$. This fact implies **two** important pieces of information:

1. the slope of the line and the slope of the tangent line to the quadratic function will match at $x = -4$, and
2. the line and the parabola will share a common point at $x = -4$.

Matching Slopes

To find the slope of the line, we place it in slope-intercept form.

$$y = -4x + b$$

Thus, the slope of the line is -4.To find the slope of the tangent line to $y = ax^2$, we first compute the derivative.

$$y' = 2ax$$

Evaluating this at $x = -4$, we get

$$y'(-4) = -8a.$$

Since the slope of the tangent line to the quadratic and the slope of the line are supposed to match at $x = -4$,

$$-4 = -8a \implies a = \frac{1}{2}.$$

Common Point

Since the line and the quadratic meet at $x = -4$, their y-values will be the same. That is, at $x = -4$, $-4x + b = ax^2$. Therefore,

$$\begin{aligned} -4(-4) + b &= a(-4)^2 \implies 16 + b = \frac{1}{2}(16) \\ &\implies 16 + b = 8 \\ &\implies b = -8 \end{aligned}$$

Thus, for the line and quadratic to meet and be tangent at $x = -4$, we require $a = \frac{1}{2}$ and $b = -8$.

Example 2.1.13**: Ballistic Motion**

a. The initial height is given by $h(0) = 388$. Therefore, the initial height of the rocket is 388 meters.
b. $h(6) = 1,135.6$. Therefore, the height of the rocket 6 seconds after launch is $1,135.6$ meters.
c. To compute the velocity, we need to find the derivative of $h(t)$.

$$v(t) = h'(t) = -9.8t + 154$$

Evaluating h' at 6, we get

$$v(6) = h'(6) = -9.8(6) + 154 = 95.2.$$

This means that the velocity of the rocket exactly 6 seconds after launch is 95.2 meters per second. Since this velocity is positive, the rocket is still climbing upward.

d. To compute the acceleration, we need to find the derivative of the velocity.

$$a(t) = v'(t) = -9.8$$

Hence, the acceleration is *always* -9.8 meters per second squared. That is, the rocket is *decelerating*at 9.8 meters per second squared the entire time. While this idea of "slowing down" makes sense on the rise, how can we make sense of it as the rocket begins to fall? Well, the negative acceleration implies the rocket's velocity is decreasing as the rocket rises. At some point, the velocity would have to hit 0. At that moment, the rocket begins its descent back to Earth. The velocity becomes negative as the distance between the rocket and the Earth decreases, and the rate of this descent increases (in the negative direction) until the rocket crashes into the ground. The negative acceleration acted to slow the ascent and speed the descent.

Example 2.1.14

Marginal profit is the derivative of the profit function, which they did not *explicitly* hand us; however, we know that profit is revenue minus cost. Thus,

$$P(q) = R(q) - C(q) = -2q^2 + 800q - (153q + 40,000) = -2q^2 + 647q - 40,000.$$

Therefore, the marginal profit function is

$$MP(q) = P'(q) = -4q + 647.$$

Hence,

$$MP(220) = -233.$$

This means that if the company produces and sells 220 units, its profit will drop by $233 if it sells another unit. This is usually tough for students to understand at first, so let's take a moment to dive a little deeper. The derivative, $\frac{dP}{dq}$, is the rate of change of profit with respect to the number of units produced (and sold). Since $\frac{dP}{dq}(220) = -233 = \frac{-233}{1}$, we can interpret this as *losing* $233 in profit per 1 unit produced and sold when we have produced and sold *exactly* 220 units.

It's important to note that marginal profit is *not* the profit. In fact, the profit if the company produces and sells 220 units is $P(220) = 5540$. That is, the company is *still* making a profit of $5,540 at that moment, but producing more units will eat into those profits.

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2.1.1: Resources and Key Concepts

Resources

Videos

- Derivative rules
  - Proofs in Differential Calculus - The Derivative of x is 1
  - Proofs in Differential Calculus - The Derivative of cx is c
- Constant Rule
  - Proofs in Differential Calculus - Derivatives of Constants are Zero
- Power Rule
  - Proofs in Differential Calculus - The Derivative of x^n is nx^{n-1}
- Sum Rule for derivatives
  - Proofs in Differential Calculus - The Derivative of a sum is the sum of a derivative
- Derivatives of polynomials
  - The Theory Behind Derivatives of Polynomials
  - Computing Derivatives of Polynomials
  - Derivatives of Polynomials - Beyond the Basics
  - Synthesis Questions
- Tangent and normal lines
  - Tangent and Normal Lines to Polynomials

Key Concepts

Theorems

- **Theorem: The Constant Rule**: Let c be a constant. If $f(x) = c$, then $f'(x) = 0$. (Alternatively, $\frac{d}{dx}(c) = 0$.)
- **Theorem: The Power Rule (positive integer exponents only)**: Let n be a positive integer. If $f(x) = x^n$, then $f'(x) = nx^{n-1}$. (Alternatively, $\frac{d}{dx}(x^n) = nx^{n-1}$.)
- **Theorem: The Power Rule (General Version)**: Let $n \in \mathbb{R}$. If $f(x) = x^n$, then $\frac{d}{dx}(x^n) = nx^{n-1}$.
- **Theorem: Sum, Difference, and Constant Multiple Rules**: Let $f(x)$ and $g(x)$ be differentiable functions and k be a constant. Then, each of the following equations holds:
  - **Sum Rule**: $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$. (Alternatively, for $s(x) = f(x) + g(x)$, $s'(x) = f'(x) + g'(x)$.)
  - **Difference Rule**: $\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x))$. (Alternatively, for $d(x) = f(x) - g(x)$, $d'(x) = f'(x) - g'(x)$.)
  - **Constant Multiple Rule**: $\frac{d}{dx}(kf(x)) = k\frac{d}{dx}(f(x))$. (Alternatively, for $m(x) = kf(x)$, $m'(x) = kf'(x)$.)

Common Mistakes

- **Misapplying the Power Rule**: The Power Rule only applies to algebraic functions of the form $y = x^n$. It does not apply to functions that raise a constant to a variable power, such as $f(x) = 3^x$. That is, $\frac{d}{dx}(3^x) \neq x3^{x-1}$.

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2.1.2: Homework

Note

The following exercises are meant as hand-in homework (if assigned by your instructor). Typed-in answers here will not be saved or scored.

Concept Check

Exercise 1

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Basic Skills

Exercise 2

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Exercise 3

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Exercise 25

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Basic Skills - Applications

Exercise 26

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Exercise 27

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Exercise 31

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Synthesis Questions

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Synthesis Questions - Applications

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Exercise 45

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Challenge Problems

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Exercise 47

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2.1.3: Additional Exercises

Reading Questions

1. What is the derivative of a constant function, $f(x) = c$?
2. State the Power Rule for differentiating $f(x) = x^n$. Does this rule apply only to positive integer exponents n, or is it more general?
3. What is the derivative of $f(x) = x^5$?
4. What is the derivative of $g(x) = \frac{1}{x^3}$ using the Power Rule?
5. State the Sum Rule for derivatives. If you know the derivatives of $f(x)$ and $g(x)$, how do you find the derivative of $f(x) + g(x)$?
6. State the Difference Rule for derivatives.
7. State the Constant Multiple Rule. If you know the derivative of $f(x)$, how do you find the derivative of $k \cdot f(x)$, where k is a constant?
8. How would you find the derivative of $f(x) = 3x^4 - 7x^2 + 2x - 5$? Which rules would you apply?
9. If a function has a horizontal tangent line at a point, what is the value of its derivative at that point?
10. In business applications, what does "marginal cost" (or "marginal profit," "marginal revenue") generally refer to in terms of Calculus?
11. If $s(t)$ is the position of a particle at time t, what do $v(t) = s'(t)$ and $a(t) = v'(t) = s''(t)$ represent?

Homework

In exercises 1 - 10, find $f'(x)$ for each function.

1) $f(x) = x^7 + 10$

2) $f(x) = 5x^3 - x + 1$

Answer

$$f'(x) = 15x^2 - 1$$

3) $f(x) = 4x^2 - 7x$

4) $f(x) = 8x^4 + 9x^2 - 1$

Answer

$$f'(x) = 32x^3 + 18x$$

5) $f(x) = x^4 + 2x$

7) $f(x) = (x + 2)(2x^2 - 3)$

8) $f(x) = x^2\left(\frac{2}{x^2} + \frac{5}{x^3}\right)$

Answer

$$f'(x) = \frac{-5}{x^2}$$

9) $f(x) = \frac{x^3+2x^2-4}{3}$

10) $f(x) = \frac{4x^3-2x+1}{x^2}$

Answer

$$f'(x) = \frac{4x^4 + 2x^2 - 2x}{x^4}$$

[Technology Required] In exercises 11 - 13, find the equation of the tangent line $T(x)$ to the graph of the given function

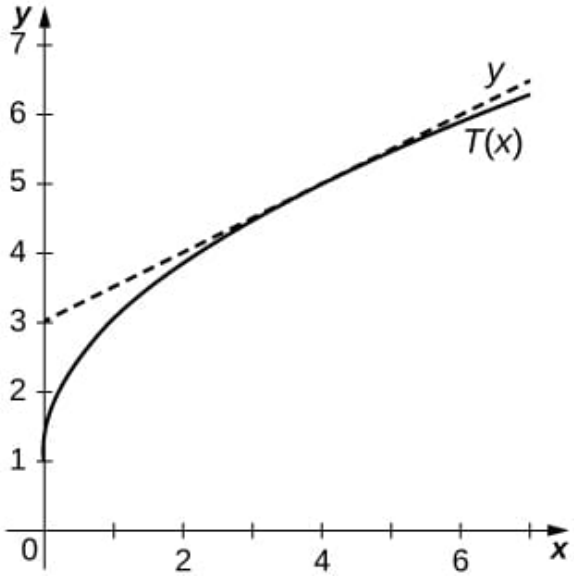
at the indicated point. Use graphing technology to graph the function and the tangent line.

11) $y = 3x^2 + 4x + 1$ at $(0, 1)$

12) $y = 2\sqrt{x} + 1$ at $(4, 5)$

Answer

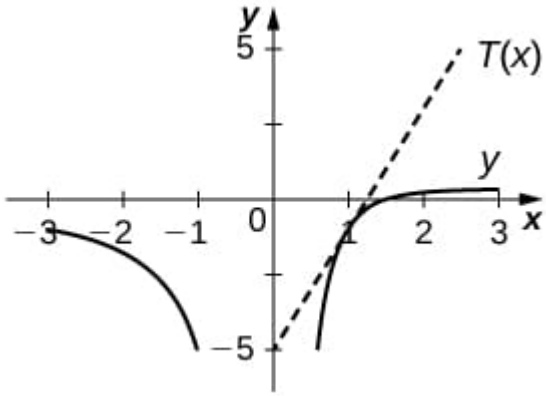
$T(x) = \frac{1}{2}x + 3$



13) $y = \frac{2}{x} - \frac{3}{x^2}$ at $(1, -1)$

Answer

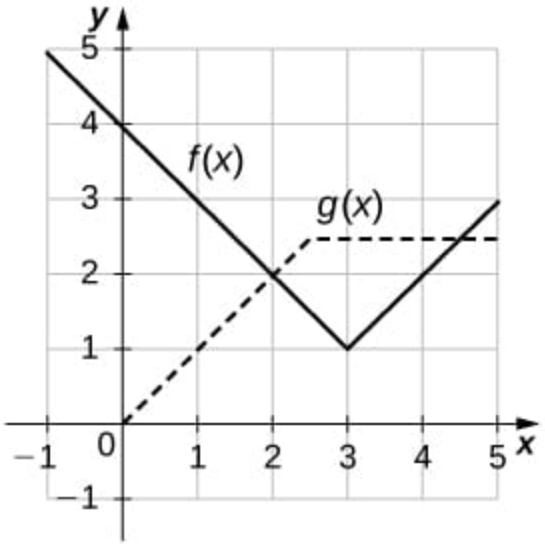
$T(x) = 4x - 5$



In exercise 14, assume that $f(x)$ and $g(x)$ are both differentiable functions for all x. Find the derivative of the function $h(x)$.

14) $h(x) = 4f(x) + \frac{g(x)}{7}$

In exercise 15, use the following figure to find the indicated derivatives, if they exist.



15) Let $h(x) = f(x) + g(x)$. Find

a) $h'(1)$,

b) $h'(3)$, and

c) $h'(4)$.

[Technology Required] In exercises 16 - 19,

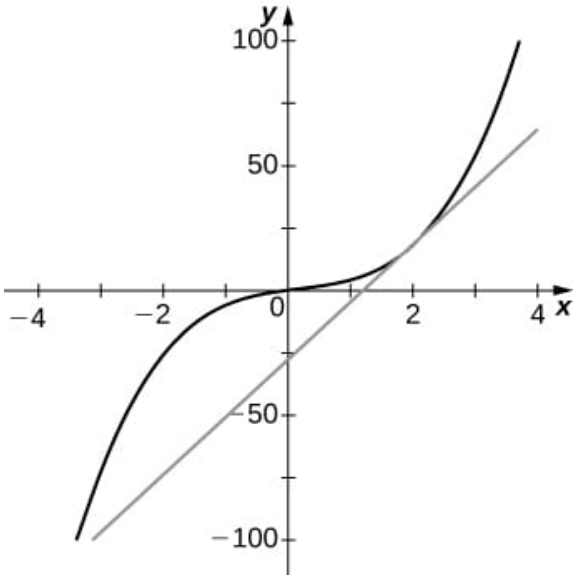
a) evaluate $f'(a)$, and

b) graph the function $f(x)$ and the tangent line at $x = a$.

16) $f(x) = 2x^3 + 3x - x^2, \quad a = 2$

Answer

a. 23
b. $y = 23x - 28$

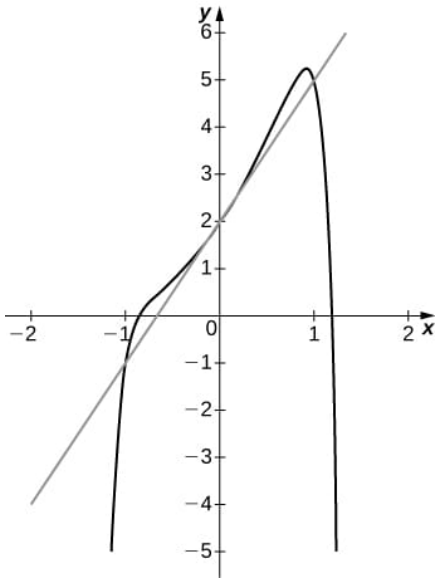


17) $f(x) = \frac{1}{x} - x^2, \quad a = 1$

18) $f(x) = x^2 - x^{12} + 3x + 2, \quad a = 0$

Answer

a. 3
b. $y = 3x + 2$



19) $f(x) = \frac{1}{x} - x^{2/3}, \quad a = -1$

In exercises 20 - 22, find the equation of the tangent line to the graph of the function at the given point.

20) $f(x) = 2x^3 + 4x^2 - 5x - 3$ at $x = -1$.

Answer

$y = -7x - 3$

21) $f(x) = x^2 + \frac{4}{x} - 10$ at $x = 8$.

22) $f(x) = (3x - x^2)(3 - x - x^2)$ at $x = 1$.

Answer

$y = -5x + 7$

23) Find the point on the graph of $f(x) = x^3$ such that the tangent line at that point has an x-intercept of $(6, 0)$.

24) Determine all points on the graph of $f(x) = x^3 + x^2 - x - 1$ for which the slope of the tangent line is

a. horizontal

b. –1.

25) Find a quadratic polynomial such that $f(1) = 5, \quad f'(1) = 3$ and $f''(1) = -6$.

Answer

$y = -3x^2 + 9x - 1$

Applications of the Derivative: Physics

Note

- Velocity is the change in position with respect to time. That is, $v(t) = \frac{ds}{dt}$, where $s(t)$ is the position function.
- Acceleration is the change in velocity with respect to time. That is, $a(t) = \frac{dv}{dt} = \frac{ds^2}{dt^2}$.

In exercises 26 - 27, the given functions represent the position of a particle traveling along a horizontal line.

a. Find the velocity and acceleration functions.

b. Determine the time intervals when the object is slowing down or speeding up.

26) $s(t) = 2t^3 - 3t^2 - 12t + 8$

27) $s(t) = 2t^3 - 15t^2 + 36t - 10$

Answer

a. $v(t) = 6t^2 - 30t + 36, \quad a(t) = 12t - 30$;
b. speeds up for $(2, 2.5) \cup (3, \infty)$, slows down for $(0, 2) \cup (2.5, 3)$

28) A rocket is fired vertically upward from the ground. The distance s in feet that the rocket travels from the ground after t seconds is given by $s(t) = -16t^2 + 560t$.

a. Find the velocity of the rocket 3 seconds after being fired.

b. Find the acceleration of the rocket 3 seconds after being fired.

29) A ball is thrown downward with a speed of 8 ft/s from the top of a 64-foot-tall building. After t seconds, its height above the ground is given by $s(t) = -16t^2 - 8t + 64$.

a. Determine how long it takes for the ball to hit the ground.

b. Determine the velocity of the ball when it hits the ground.

30) The position function $s(t) = t^2 - 3t - 4$ represents the position of the back of a car backing out of a driveway and then driving in a straight line, where s is in feet and t is in seconds. In this case, $s(t) = 0$ represents the time at which the back of the car is at the garage door, so $s(0) = -4$ is the starting position of the car, 4 feet inside the garage.

a. Determine the velocity of the car when $s(t) = 0$.

b. Determine the velocity of the car when $s(t) = 14$.

31) The position of a hummingbird flying along a straight line in t seconds is given by $s(t) = 3t^3 - 7t$ meters.

a. Determine the velocity of the bird at $t = 1$ sec.

b. Determine the acceleration of the bird at $t = 1$ sec.

c. Determine the acceleration of the bird when the velocity equals 0.

32) A potato is launched vertically upward with an initial velocity of 100 ft/s from a potato gun at the top of an 85-foot-tall building. The distance in feet that the potato travels from the ground after t seconds is given by $s(t) = -16t^2 + 100t + 85$.

a. Find the velocity of the potato after 0.5 s and 5.75 s.

b. Find the speed of the potato at 0.5 s and 5.75 s.

c. Determine when the potato reaches its maximum height.

d. Find the acceleration of the potato at 0.5 s and 1.5 s.

e. Determine how long the potato is in the air.

f. Determine the velocity of the potato upon hitting the ground.

Answer

a. 84 ft/s, –84 ft/s
b. 84 ft/s
c. $\frac{25}{8}$ s
d. $-32\ \text{ft/s}^2$ in both cases
e. $\frac{1}{8}(25+\sqrt{965})$ s
f. $-4\sqrt{965}$ ft/s

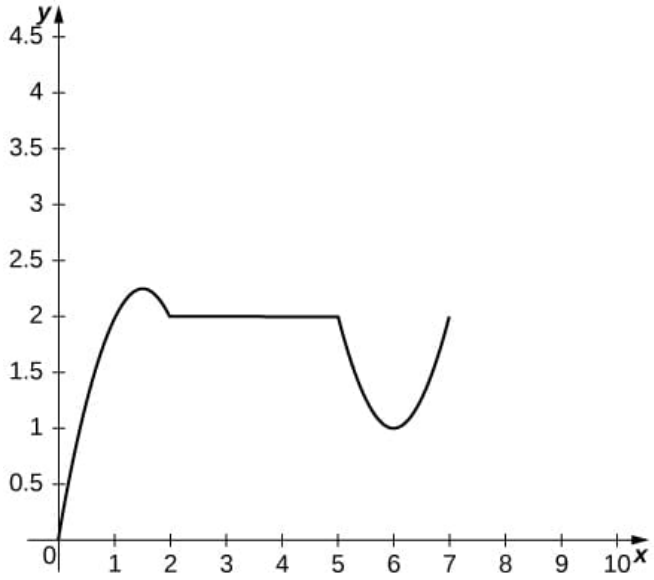
33) The position function $s(t)=t^3-8t$ gives the position in miles of a freight train where east is the positive direction and t is measured in hours.

a. Determine the direction the train is traveling when $s(t)=0$.

b. Determine the direction the train is traveling when $a(t)=0$.

c. Determine the time intervals when the train is slowing down or speeding up.

34) The following graph shows the position $y=s(t)$ of an object moving along a straight line.



a. Use the graph of the position function to determine the time intervals when the velocity is positive, negative, or zero.

b. Sketch the graph of the velocity function.

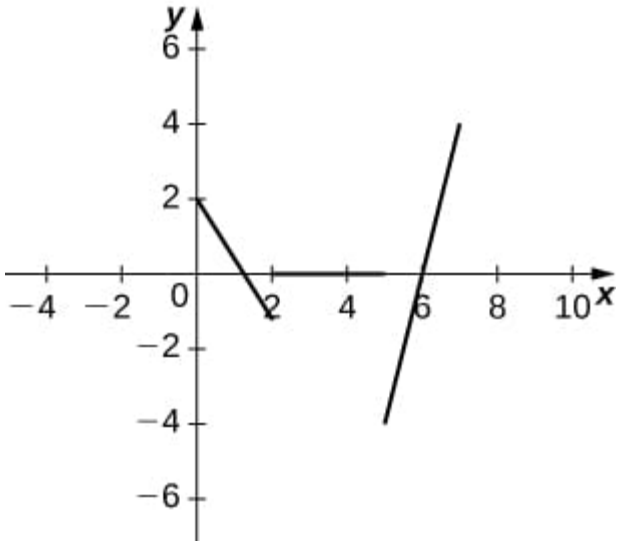
c. Use the graph of the velocity function to determine the time intervals when the acceleration is positive, negative, or zero.

d. Determine the time intervals when the object is speeding up or slowing down.

Answer

a. Velocity is positive on $(0,1.5)\cup(6,7)$, negative on $(1.5,2)\cup(5,6)$, and zero on $(2,5)$.

b.



c. Acceleration is positive on $(5, 7)$, negative on $(0, 2)$, and zero on $(2, 5)$.
d. The object is speeding up on $(6, 7) \cup (1.5, 2)$ and slowing down on $(0, 1.5) \cup (5, 6)$.

35) A car driving along a freeway with traffic has traveled $s(t) = t^3 - 6t^2 + 9t$ meters in t seconds.

a. Determine the time in seconds when the velocity of the car is 0.

b. Determine the acceleration of the car when the velocity is 0.

36) [Technology Required] According to Newton's law of universal gravitation, the force F between two bodies of constant mass m_1 and m_2 is given by the formula $F = \frac{Gm_1m_2}{d^2}$, where G is the gravitational constant and d is the distance between the bodies.

a. Suppose that G, m_1, and m_2 are constants. Find the rate of change of force F with respect to distance d.

b. Find the rate of change of force F with gravitational constant $G = 6.67 \times 10^{-11}\text{Nm}^2/\text{kg}^2$, on two bodies 10 meters apart, each with a mass of 1000 kilograms.

Answer

a. $F'(d) = \frac{-2Gm_1m_2}{d_3}$
b. -1.33×10^{-7} N/m

37) The centripetal force of an object of mass m is given by $F(r) = \frac{mv^2}{r}$, where v is the speed of rotation and r is the distance from the center of rotation.

a. Find the rate of change of centripetal force with respect to the distance from the center of rotation.

b. Find the rate of change of centripetal force of an object with mass 1000 kilograms, velocity of 13.89 m/s, and a distance from the center of rotation of 200 meters.

38) [Technology Required] Consider an astronaut on a large planet in another galaxy. To learn more about the composition of this planet, the astronaut drops an electronic sensor into a deep trench. The sensor transmits its vertical position every second in relation to the astronaut's position. The summary of the falling sensor data is displayed in the following table.

Time after dropping (s)	Position (m)
0	0
1	–1
2	–2

3	–5
4	–7
5	–14

a. Using a calculator or computer program, find the best-fit quadratic curve to the data.

b. Find the derivative of the position function and explain its physical meaning.

c. Find the second derivative of the position function and explain its physical meaning.

Answer

a. $p(t) = -0.6071x^2 + 0.4357x - 0.3571$
b. $p'(t) = -1.214x + 0.4357$. This is the velocity of the sensor.
c. $p''(t) = -1.214$. This is the acceleration of the sensor; it is a constant acceleration downward.

Applications of the Derivative: Business

Note

- Revenue is the product of the price of the item and the number of items sold. Mathematically, $R(n) = p(n) \cdot n$, where $p(n)$ is the price-demand function (the price the company charges when the market demands n items) and n is the number of items sold.
- Profit is the difference between the revenue from sales and the cost of producing an item. Mathematically, $P(n) = R(n) - C(n)$, where $R(n)$ is the revenue function and $C(n)$ is the cost function.
- Marginal cost is the change in the total cost that arises when the quantity produced is incremented. That is, it is the change in cost with respect to the number of items produced. Mathematically, $M_C(n) = \frac{dC}{dn}$, where $C(n)$ is the cost function (i.e., the cost of producing n items).
- Marginal profit is the change in the total profit that arises when the quantity produced is incremented. That is, it is the change in profit with respect to the number of items produced. Mathematically, $M_P(n) = \frac{dP}{dn}$, where $P(n)$ is the profit function.
- Marginal revenue is the change in the total revenue that arises when the quantity produced is incremented. That is, it is the change in revenue with respect to the number of items produced. Mathematically, $M_R(n) = \frac{dR}{dn}$, where $R(n)$ is the revenue function.

39) A book publisher has a cost function given by $C(x) = \frac{x^3+2x+3}{x^2}$, where x is the number of copies of a book in thousands and C is the cost, per book, measured in dollars. Evaluate $C'(2)$ and explain its meaning.

40) The cost function, in dollars, of a company that manufactures food processors is given by $C(x) = 200 + \frac{7}{x} + \frac{x}{27}$, where x is the number of food processors manufactured.

a. Find the marginal cost function.

b. Find the marginal cost of manufacturing 12 food processors.

c. Find the actual cost of manufacturing the thirteenth food processor.

41) The price p (in dollars) and the demand x for a certain digital clock radio is given by the price–demand function $p = 10 - 0.001x$.

a. Find the revenue function $R(x)$

b. Find the marginal revenue function.

c. Find the marginal revenue at $x = 2000$ and 5000.

Answer

a. $R(x) = 10x - 0.001x^2$
b. $R'(x) = 10 - 0.002x$
c. \$6 per item, \$0 per item

42) [Technology Required] A profit is earned when revenue exceeds cost. Suppose the profit function for a skateboard manufacturer is given by $P(x) = 30x - 0.3x^2 - 250$, where x is the number of skateboards sold.

a. Find the exact profit from the sale of the thirtieth skateboard.

b. Find the marginal profit function and use it to estimate the profit from the sale of the thirtieth skateboard.

43) [Technology Required] In general, the profit function is the difference between the revenue and cost functions: $P(x) = R(x) - C(x)$.

Suppose the price-demand and cost functions for the production of cordless drills is given respectively by $p = 143 - 0.03x$ and $C(x) = 75,000 + 65x$, where x is the number of cordless drills that are sold at a price of p dollars per drill and $C(x)$ is the cost of producing x cordless drills.

a. Find the marginal cost function.

b. Find the revenue and marginal revenue functions.

c. Find $R'(1000)$ and $R'(4000)$. Interpret the results.

d. Find the profit and marginal profit functions.

e. Find $P'(1000)$ and $P'(4000)$. Interpret the results.

Answer

a. $C'(x) = 65$
b. $R(x) = 143x - 0.03x^2, R'(x) = 143 - 0.06x$
c. $R'(1000) = 83, \quad R'(4000) = -97$. At a production level of 1000 cordless drills, revenue is increasing at a rate of \$83 per drill; at a production level of 4000 cordless drills, revenue is decreasing at a rate of \$97 per drill.
d. $P(x) = -0.03x^2 + 78x - 75000, \quad P'(x) = -0.06x + 78$
e. $P'(1000) = 18, \quad P'(4000) = -162$. At a production level of 1000 cordless drills, profit is increasing at a rate of \$18 per drill; at a production level of 4000 cordless drills, profit is decreasing at a rate of \$162 per drill.

Applications of the Derivative: Population Models

44) A small town in Ohio commissioned an actuarial firm to conduct a study that modeled the rate of change of the town's population. The study found that the town's population (measured in thousands of people) can be modeled by the function $P(t) = -\frac{1}{3}t^3 + 64t + 3000$, where t is measured in years.

a. Find the rate of change function $P'(t)$ of the population function.

b. Find $P'(1), \ P'(2), \ P'(3)$, and $P'(4)$. Interpret what the results mean for the town.

c. Find $P''(1), \ P''(2), \ P''(3)$, and $P''(4)$. Interpret what the results mean for the town's population.

[Technology Required] In Exercises 45 - 46, use the following table concerning the population (in millions) of London by decade in the 19th century.

Year Since 1800	Population (millions)
1	0.8975
11	1.040

21	1.264
31	1.516
41	1.661
51	2.000
61	2.634
71	3.272
81	3.911
91	4.422

Population of London Source: http://en.Wikipedia.org/wiki/Demographics_of_London

45)

a. Using a calculator or a computer program, find the best-fit linear function to measure the population.

b. Find the derivative of the equation in a. and explain its physical meaning.

c. Find the second derivative of the equation and explain its physical meaning.

Answer

a. $P(t) = 0.03983t + 0.4280$
b. $P'(t) = 0.03983$. The population is increasing.
c. $P''(t) = 0$. The rate at which the population is increasing is constant.

46)

a. Using a calculator or a computer program, find the best-fit quadratic curve through the data.

b. Find the derivative of the equation and explain its physical meaning.

c. Find the second derivative of the equation and explain its physical meaning.

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2.2: Differentiation Techniques - The Product and Quotient Rules

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

The Product Rule

Now that we have examined the basic rules, we can look at some of the more advanced ones. The first one examines the derivative of the product of two functions. Although it might be tempting to assume that the derivative of the product is the product of the derivatives, similar to the Sum and Difference Rules, the **Product Rule** does not follow this pattern. To see why we cannot use this pattern, consider the function $f(x) = x^2$, whose derivative is $f'(x) = 2x$ and not $\frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1$.

Theorem: The Product Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}(f(x)) \cdot g(x) + \frac{d}{dx}(g(x)) \cdot f(x).$$

That is,

$$\text{if } p(x) = f(x)g(x), \quad \text{then } p'(x) = f'(x)g(x) + g'(x)f(x).$$

This means that the derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function.

Proof available at the end of this section.

Example 2.2.1: Applying the Product Rule to Constant Functions

For $p(x) = f(x)g(x)$, use the Product Rule to find $p'(2)$ if $f(2) = 3$, $f'(2) = -4$, $g(2) = 1$, and $g'(2) = 6$.

Solution available at the end of this section.

Example 2.2.2: Applying the Product Rule to Binomials

For $p(x) = (x^2 + 2)(3x^3 - 5x)$, find $p'(x)$ by applying the Product Rule. Check the result by first finding the product and then differentiating.

Solution available at the end of this section.

Checkpoint 2.2.2

Use the Product Rule to obtain the derivative of $p(x) = 2x^5(4x^2 + x)$.

Answer available at the end of this section.

Before we move on, it is important to repeat a previous warning.

Caution: The Product Rule is NOT the Product of Derivatives!

$$\frac{d}{dx}\left(f(x) \cdot g(x)\right) \neq \frac{d}{dx}\left(f(x)\right) \cdot \frac{d}{dx}\left(g(x)\right)$$

Interactive Element: Why the Product Rule Has Two Terms

The following Interactive Element might help you visualize why the derivative of a product is $f'g + g'f$ and not $f'g'$.

Interact: Drag the large corner point toward the base to shrink the growth.

Observation: First, recall that $\frac{d}{dx}(f \cdot g)$ represents how quickly the product, $f \cdot g$, is changing at any given moment. Second, if the product, $f \cdot g$, is changing, there is an associated change in both f and g (these are Δf and Δg). Notice from the manipulable that, as the area shrinks, the corner, $\Delta f \cdot \Delta g$, vanishes far faster than the strips $f \cdot \Delta g$ and $g \cdot \Delta f$. This implies (but does not prove) that only the two strips contribute anything of note to the derivative. That is, to compute how quickly the product $f \cdot g$ is changing, we only need to consider the blue and green areas in the manipulable above. Hence, it seems that $\frac{d}{dx}(fg) \approx g \cdot \Delta f + f \cdot \Delta g$ and never just $\Delta f \cdot \Delta g$.

The Quotient Rule

Having developed and practiced the Product Rule, we now consider differentiating quotients of functions. As we see in the following theorem, the derivative of the quotient is not the quotient of the derivatives; rather, it is the derivative of the function in the numerator times the function in the denominator minus the derivative of the function in the denominator times the function in the numerator, all divided by the square of the function in the denominator. To better grasp why we cannot simply take the quotient of the derivatives, keep in mind that

$$\frac{d}{dx}(x^2) = 2x, \text{ not } \frac{\frac{d}{dx}(x^3)}{\frac{d}{dx}(x)} = \frac{3x^2}{1} = 3x^2.$$

Theorem: The Quotient Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{\frac{d}{dx}(f(x))\cdot g(x)-\frac{d}{dx}(g(x))\cdot f(x)}{\big(g(x)\big)^2}.$$

That is, if

$$q(x)=\frac{f(x)}{g(x)}$$

then

$$q'(x)=\frac{f'(x)g(x)-g'(x)f(x)}{\big(g(x)\big)^2}.$$

The proof of the Quotient Rule is very similar to the proof of the Product Rule, so it is omitted here. Instead, we apply this new rule to find derivatives in the next example.

Example 2.2.3: Applying the Quotient Rule

Find the derivative of each function.

a. $q(x)=\frac{5x^2}{4x+3}$
b. $s(t)=\frac{\sqrt[3]{t}}{t-5}$

Solutions available at the end of this section.

The previous example demonstrates the need to use our prerequisite algebra skills often when using the Quotient Rule. We will commonly have to factor, simplify, and multiply by fractions equivalent to 1 when using this rule.

Checkpoint 2.2.3

Find the derivative of $h(x)=\frac{3x+1}{4x-3}$.

Answer available at the end of this section.

As with the last topic, caution is merited here.

Caution: The Quotient Rule is NOT the Quotient of Derivatives!

$$\frac{d}{dx}\left(\frac{N(x)}{D(x)}\right)\neq\frac{\frac{d}{dx}\left(N(x)\right)}{\frac{d}{dx}\left(D(x)\right)}$$

Revisiting the Power Rule

Using the Quotient Rule to prove most of the General Power Rule is now possible. In the previous section, recall that we proved the Power Rule for *positive integers*. Still, we stated that we had to wait until later in the course to prove the Power Rule for the general case when the exponent is any real number. At this point, we can also add to our previous proof by extending the Power Rule to negative integers. This proof is provided for completeness.

Proof

Let $f(x) = x^k$. If k is a negative integer, we may set $n = -k$, so that n is a positive integer with $k = -n$. Since for each positive integer n,$x^{-n} = \frac{1}{x^n}$, we may now apply the Quotient Rule by setting $f(x) = 1$ and $g(x) = x^n$. In this case, $f'(x) = 0$ and $g'(x) = nx^{n-1}$. Thus,

$$\frac{d}{dx}(x^{-n}) = \frac{0(x^n) - 1(nx^{n-1})}{(x^n)^2}.$$

Simplifying, we see that

$$\begin{aligned} \frac{d}{dx}(x^{-n}) &= \frac{-nx^{n-1}}{x^{2n}} \\ &= -nx^{(n-1)-2n} \\ &= -nx^{-n-1}. \end{aligned}$$

Finally, observe that since $k = -n$, by substituting we have

$$\frac{d}{dx}(x^k) = kx^{k-1}.$$

Q.E.D.

Combining Differentiation Rules

As we have seen throughout the examples in this section, it seldom happens that we are called on to apply just one differentiation rule to find the derivative of a given function. By combining the differentiation rules, we may find the derivatives of any polynomial or rational function. Later on, we will encounter more complex combinations of differentiation rules. A good rule of thumb to use when applying several rules is to apply the rules in reverse of the order in which we would evaluate the function.

Example 2.2.4: Combining Differentiation Rules

For $k(x) = 3h(x) + x^2 g(x)$, find $k'(x)$.

Solution available at the end of this section.

Example 2.2.5: Extending the Product Rule

For $k(x) = f(x)g(x)h(x)$, express $k'(x)$ in terms of $f(x), g(x), h(x)$, and their derivatives.

Solution available at the end of this section.

Example 2.2.6: Combining the Quotient Rule and the Product Rule

For $h(x) = \frac{2x^3 k(x)}{3x+2}$, find $h'(x)$.

Solution available at the end of this section.

Success in College: Dealing With Failure (Part 1)

People sometimes do not achieve their goals. Athletes lose games. Students fail tests or classes. A job interview does not go well. It is your responsibility to learn how to cope with failure.

Former Chicago Bears football coach, Mike Ditka, said, "Success isn't permanent, and failure isn't fatal." This idea applies beyond the game of football. Failure on one test or even failing a class does not mean you cannot earn your degree. However, a continuing pattern of failures can prevent success.

When failure happens, you need to take action. Change the way you study. Redo the problems you missed on a test. Make an appointment with your professor to learn what you don't understand. If you are failing a course, meet with your advisor. Perhaps you should consider withdrawing. Make sure you know the effect of withdrawing or failing on any financial aid you receive.

Failure is not fun. It is tempting to skip class the day after a nasty test. It is easy to avoid talking to your professor about it. However, it is your responsibility to be honest about failure. Ignoring failure does not make it go away.

1. A class has four tests. A student taking this class fails one test with a score of 45%. The student's other test scores were 72%, 72%, and 79%. Find the student's test average for this class.
2. The effect of a bad test score depends on the number of tests in the class. It also depends on how much test scores contribute to the final course grade. If you had the same test average in this class as the student in the last question, could you still pass? Explain.

Hidden Resources: Microsoft Office 365

Los Rios students have free access to Microsoft Office 365 for as long as they remain enrolled! This is an incredible deal! A warning is necessary - do not use the included OneDrive storage as it gets deleted each semester. Other than that minor issue, this is a fantastic resource. Check out the details on this CRC webpage.

Answers

Section Preview

Since the students will only know how to take derivatives of polynomials and expressions with rational exponents, this is a fast section. A good chunk of your time here (other than going through examples with the class) should be spent on proving the Product Rule.

Section Preview

- **An Overview of Functions**
  - Function Arithmetic and Domain: The Product and Quotient Rules differentiate a *product* $f(x)g(x)$ and a *quotient* $\frac{f(x)}{g(x)}$ of functions; understanding what those combinations of functions mean is the conceptual basis for the entire section.

Section Preview

- **Factoring Techniques**
  - Factoring Expressions Involving Negative Rational Exponents: In Example 3(b), simplifying the Quotient-Rule result requires factoring/clearing the $t^{-2/3}$ term so the denominator becomes $3t^{2/3}(t-5)^2$.
- **Simplifying Expressions**
  - Simplifying Expressions Involving Negative Rational Exponents: Differentiating $t^{1/3}$ produces $\frac{1}{3}t^{-2/3}$, and simplifying the resulting Quotient-Rule expression in Example 3(b) requires manipulating negative rational exponents.

Section Preview

- Use the Product Rule for finding the derivative of a product of functions.
- Use the Quotient Rule for finding the derivative of a quotient of functions.
- Combine the differentiation rules to find the derivative of a polynomial or rational function.

Theorem: The Product Rule

We begin by assuming that $f(x)$ and $g(x)$ are differentiable functions. At a key point in this proof, we need to use the fact that since $g(x)$ is differentiable, it is also continuous. In particular, we use the fact that since $g(x)$ is continuous, $\lim_{h\to 0} g(x+h) = g(x)$.

By applying the limit definition of the derivative to $p(x) = f(x)g(x)$, we obtain

$$p'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

By adding and subtracting $f(x)g(x+h)$ in the numerator, we have

$$p'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}.$$

After breaking apart this quotient and applying the Sum Law for limits, the derivative becomes

$$p'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \lim_{h\to 0} \frac{f(x)g(x+h) - f(x)g(x)}{h}.$$

Rearranging, we obtain

$$\begin{aligned} p'(x) &= \lim_{h\to 0} \left(\frac{f(x+h) - f(x)}{h} \cdot g(x+h) \right) + \lim_{h\to 0} \left(\frac{g(x+h) - g(x)}{h} \cdot f(x) \right) \\ &= \left(\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \right) \cdot \left(\lim_{h\to 0} g(x+h) \right) + \left(\lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \right) \cdot f(x) \end{aligned}$$

By using the continuity of $g(x)$, the definition of the derivatives of $f(x)$ and $g(x)$, and applying the Limit Laws, we arrive at the Product Rule,

$$p'(x) = f'(x)g(x) + g'(x)f(x).$$

Q.E.D.

Example 2.2.1**: Applying the Product Rule to Constant Functions**

Since $p(x) = f(x)g(x)$, $p'(x) = f'(x)g(x) + g'(x)f(x)$, and hence

$$p'(2) = f'(2)g(2) + g'(2)f(2) = (-4)(1) + (6)(3) = 14.$$

Example 2.2.2**: Applying the Product Rule to Binomials**

If we set $f(x) = x^2 + 2$ and $g(x) = 3x^3 - 5x$, then $f'(x) = 2x$ and $g'(x) = 9x^2 - 5$. Thus,

$$p'(x) = f'(x)g(x) + g'(x)f(x) = (2x)(3x^3 - 5x) + (9x^2 - 5)(x^2 + 2).$$

Simplifying, we have

$$p'(x) = 15x^4 + 3x^2 - 10.$$

To check, we see that $p(x) = 3x^5 + x^3 - 10x$ and, consequently, $p'(x) = 15x^4 + 3x^2 - 10$.

Checkpoint 2.2.2

$p'(x) = 10x^4(4x^2 + x) + (8x + 1)(2x^5) = 56x^6 + 12x^5$.

Example 2.2.3**: Applying the Quotient Rule**

a. Let $f(x) = 5x^2$ and $g(x) = 4x + 3$. Thus, $f'(x) = 10x$ and $g'(x) = 4$. Substituting into the Quotient Rule, we have

$$q'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2} = \frac{10x(4x + 3) - 4(5x^2)}{(4x + 3)^2}.$$

Simplifying, we obtain

$$q'(x) = \frac{20x^2 + 30x}{(4x + 3)^2}.$$

b. It's best to rewrite s using rational exponents, where possible.

$$s(t) = \frac{t^{1/3}}{t - 5}$$

Now we can start to take the derivative of this quotient using the Quotient Rule. For reference, if we let $N(t)$ and $D(t)$ be the numerator and denominator, respectively, then

$$N(t) = t^{1/3} \quad \text{and} \quad D(t) = t - 5$$

$$N'(t) = \frac{1}{3}t^{-2/3} \quad \text{and} \quad D'(t) = 1$$

Therefore,

$$\begin{aligned} \frac{d}{dt}\left(s(t)\right) &= \frac{d}{dt}\left(\frac{N(t)}{D(t)}\right) \\ &= \frac{N'(t)D(t) - N(t)D'(t)}{[D(t)]^2} \qquad \text{(Quotient Rule)} \\ &= \frac{\left(\frac{1}{3}t^{-2/3}\right)(t-5) - \left(t^{1/3}\right)(1)}{(t-5)^2} \\ &= \frac{(t-5) - 3t}{3t^{2/3}(t-5)^2} \\ &= \frac{-2t - 5}{3t^{2/3}(t-5)^2} \end{aligned}$$

Checkpoint 2.2.3

$h'(x) = -\frac{13}{(4x-3)^2}$.

Example 2.2.4**: Combining Differentiation Rules**

Finding this derivative requires the Sum Rule, the Constant Multiple Rule, and the Product Rule.

$$\begin{aligned} k'(x) &= \frac{d}{dx}\big(3h(x)+x^2g(x)\big) \\ &= \frac{d}{dx}\big(3h(x)\big)+\frac{d}{dx}\big(x^2g(x)\big) && \text{(Sum Rule)} \\ &= 3\frac{d}{dx}\big(h(x)\big)+\Big(\frac{d}{dx}(x^2)g(x)+\frac{d}{dx}(g(x))x^2\Big) && \text{(Constant Multiple and Product Rules)} \\ &= 3h'(x)+2xg(x)+g'(x)x^2 \end{aligned}$$

Example 2.2.5**: Extending the Product Rule**

We can think of the function $k(x)$ as the product of the function $f(x)g(x)$ and the function $h(x)$. That is, $k(x)=(f(x)g(x))\cdot h(x)$. Thus,

$$\begin{aligned} k'(x) &= \frac{d}{dx}\big(f(x)g(x)\big)\cdot h(x)+\frac{d}{dx}\big(h(x)\big)\cdot\big(f(x)g(x)\big) && \text{(Product Rule)} \\ &= \big(f'(x)g(x)+g'(x)f(x)\big)h(x)+h'(x)f(x)g(x) && \text{(Product Rule)} \\ &= f'(x)g(x)h(x)+f(x)g'(x)h(x)+f(x)g(x)h'(x) \end{aligned}$$

Example 2.2.6**: Combining the Quotient Rule and the Product Rule**

This procedure is typical for finding the derivative of a rational function.

$$\begin{aligned} h'(x) &= \frac{\frac{d}{dx}(2x^3k(x))\cdot(3x+2)-\frac{d}{dx}(3x+2)\cdot(2x^3k(x))}{(3x+2)^2} && \text{(Quotient Rule)} \\ &= \frac{(6x^2k(x)+k'(x)\cdot 2x^3)(3x+2)-3(2x^3k(x))}{(3x+2)^2} && \text{(Product Rule)} \\ &= \frac{-6x^3k(x)+18x^3k(x)+12x^2k(x)+6x^4k'(x)+4x^3k'(x)}{(3x+2)^2} \end{aligned}$$

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2.2.1: Resources and Key Concepts

Resources

Videos

- Product Rule
  - Proofs in Differential Calculus: The Product Rule
- Quotient Rule
  - Proofs in Differential Calculus: The Quotient Rule
- Derivative of a product
  - Using the Product Rule
- Derivative of a quotient
  - Using the Quotient Rule
- Combining differentiation rules
  - Synthesis Questions

Key Concepts

Theorems

- **Theorem: The Product Rule**: Let $f(x)$ and $g(x)$ be differentiable functions. Then $\frac{d}{dx}(f(x)g(x)) = \left(\frac{d}{dx}(f(x))\right) \cdot g(x) + \left(\frac{d}{dx}(g(x))\right) \cdot f(x)$.
  - Alternatively, if $p(x) = f(x)g(x)$, then $p'(x) = f'(x)g(x) + g'(x)f(x)$.
- **Theorem: The Quotient Rule**: Let $f(x)$ and $g(x)$ be differentiable functions. Then $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\left(\frac{d}{dx}(f(x))\right)\cdot g(x) - \left(\frac{d}{dx}(g(x))\right)\cdot f(x)}{(g(x))^2}$.
  - Alternatively, if $q(x) = \frac{f(x)}{g(x)}$, then $q'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$.
- *(Proof of the Power Rule for negative integers is presented using the Quotient Rule).*

Common Mistakes

- **The Product Rule is NOT the Product of Derivatives!**: $\frac{d}{dx}(f(x) \cdot g(x)) \neq \left(\frac{d}{dx}(f(x))\right) \cdot \left(\frac{d}{dx}(g(x))\right)$.
- **The Quotient Rule is NOT the Quotient of Derivatives!**: $\frac{d}{dx}\left(\frac{N(x)}{D(x)}\right) \neq \frac{\frac{d}{dx}(N(x))}{\frac{d}{dx}(D(x))}$.

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2.2.2: Homework

Note

The following exercises are meant as hand-in homework (if assigned by your instructor). Typed-in answers here will not be saved or scored.

Basic Skills

Exercise 1

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Exercise 2

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Exercise 3

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View this Interactive Content online

Exercise 4

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Exercise 5

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Exercise 10

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Exercise 11

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Exercise 12

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Exercise 13

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Exercise 14

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Exercise 15

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Basic Skills - Applications

Exercise 16

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Synthesis Questions

Exercise 17

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Exercise 18

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Exercise 19

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Exercise 20

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Exercise 22

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Exercise 24

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Exercise 26

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Exercise 27

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Proofs and Problems that Require Explanations or Logical Justifications

Exercise 28

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Exercise 29

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2.2.3: Additional Exercises

Reading Questions

1. State the Product Rule for finding the derivative of $p(x) = f(x)g(x)$.
2. Is the derivative of a product of two functions equal to the product of their derivatives? Explain.
3. State the Quotient Rule for finding the derivative of $q(x) = \frac{f(x)}{g(x)}$.
4. Is the derivative of a quotient of two functions equal to the quotient of their derivatives? Explain.
5. When applying the Quotient Rule to a function like $s(t) = \frac{\sqrt[3]{t}}{t-5}$, what algebraic step is often helpful before starting the differentiation?
6. How can the Quotient Rule be used to prove the Power Rule for negative integer exponents, assuming the Power Rule for positive integers is known?
7. When finding the derivative of a function that involves sums, products, and quotients, what is a good rule of thumb for the order in which to apply the differentiation rules?
8. Given $k(x) = f(x)g(x)h(x)$, how can the Product Rule be extended or applied iteratively to find $k'(x)$?

Homework

In exercises 1 - 5, find $f'(x)$ for each function.

1) $f(x) = 3x\left(18x^4 + \dfrac{13}{x+1}\right)$

Answer

$$f'(x) = 270x^4 + \frac{39}{(x+1)^2}$$

2) $f(x) = (x+2)(2x^2-3)$

3) $f(x) = \dfrac{4x^3 - 2x + 1}{x^2}$

Answer

$$f'(x) = \frac{4x^4 + 2x^2 - 2x}{x^4}$$

4) $f(x) = \dfrac{x^2+4}{x^2-4}$

5) $f(x) = \dfrac{x+9}{x^2-7x+1}$

Answer

$$f'(x) = \frac{-x^2 - 18x + 64}{(x^2-7x+1)^2}$$

In exercise 6, find the equation of the tangent line $T(x)$ to the graph of the given function at the indicated point. Use a graphing calculator to graph the function and the tangent line.

6) [Technology Required] $y = \dfrac{2x}{x-1}$ at $(-1, 1)$

In exercises 7 - 10, assume that $f(x)$ and $g(x)$ are both differentiable functions for all x. Find the derivative of each of the functions $h(x)$.

7) $h(x) = 4f(x) + \dfrac{g(x)}{7}$

8) $h(x) = x^3 f(x)$

Answer

$$h'(x) = 3x^2 f(x) + x^3 f\,'(x)$$

9) $h(x) = \dfrac{f(x)g(x)}{2}$

10) $h(x) = \dfrac{3f(x)}{g(x)+2}$

Answer

$$h'(x) = \frac{3f'(x)(g(x)+2) - 3f(x)g'(x)}{(g(x)+2)^2}$$

For exercises 11 - 14, assume that $f(x)$ and $g(x)$ are both differentiable functions with values as given in the following table. Use the following table to calculate the following derivatives.

x	1	2	3	4
$f(x)$	3	5	−2	0
$g(x)$	2	3	−4	6
$f\,'(x)$	−1	7	8	−3
$g\,'(x)$	4	1	2	9

11) Find $h\,'(1)$ if $h(x) = xf(x) + 4g(x)$.

12) Find $h\,'(2)$ if $h(x) = \dfrac{f(x)}{g(x)}$.

Answer

$$h'(2) = \tfrac{16}{9}$$

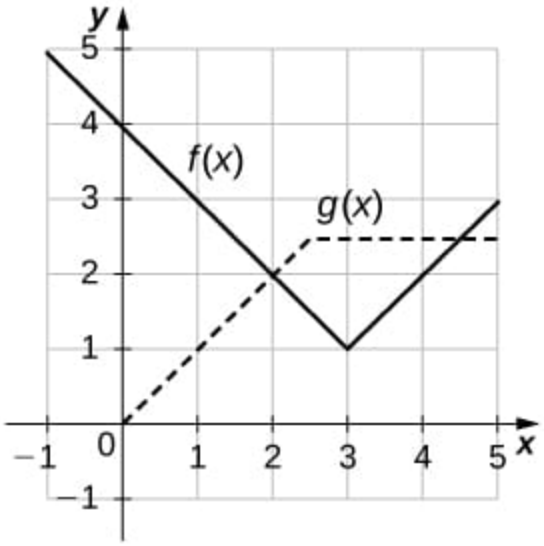
13) Find $h\,'(3)$ if $h(x) = 2x + f(x)g(x)$.

14) Find $h\,'(4)$ if $h(x) = \dfrac{1}{x} + \dfrac{g(x)}{f(x)}$.

Answer

$h'(4)$ is undefined.

In exercises 15 - 16, use the following figure to find the indicated derivatives, if they exist.



15) Let $h(x) = f(x)g(x)$. Find

a) $h\,'(1)$,

b) $h\,'(3)$, and

c) $h\,'(4)$.

Answer

a. $h'(1) = 2$,
b. $h'(3)$ does not exist,
c. $h'(4) = 2.5$

16) Let $h(x) = \dfrac{f(x)}{g(x)}$. Find

a) $h\,'(1)$,

b) $h\,'(3)$, and

c) $h\,'(4)$.

17) Find the equation of the tangent line to the graph of $f(x) = (3x - x^2)(3 - x - x^2)$ at $x = 1$.

Answer

$y = -5x + 7$

18) Find the equation of the line passing through the point $P(3, 3)$ and tangent to the graph of $f(x) = \dfrac{6}{x - 1}$.

Answer

$y = -\frac{3}{2}x + \frac{15}{2}$

19) [Technology Required] A herring swimming along a straight line has traveled $s(t) = \dfrac{t^2}{t^2 + 2}$ feet in t seconds. Determine the velocity of the herring when it has traveled 3 seconds.

Answer

$\frac{12}{121}$ or 0.0992 ft/s

20) The population in millions of arctic flounder in the Atlantic Ocean is modeled by the function $P(t) = \dfrac{8t+3}{0.2t^2+1}$, where t is measured in years.

a. Determine the initial flounder population.

b. Determine $P\,'(10)$ and briefly interpret the result.

21) [Technology Required] The concentration of antibiotic in the bloodstream t hours after being injected is given by the function $C(t) = \dfrac{2t^2+t}{t^3+50}$, where C is measured in milligrams per liter of blood.

a. Find the rate of change of $C(t)$.

b. Determine the rate of change for $t = 8, 12, 24$,and 36.

c. Briefly describe what seems to be occurring as the number of hours increases.

Answer

a. $\dfrac{-2t^4-2t^3+200t+50}{(t^3+50)^2}$
b. -0.02395 mg/L-hr, -0.01344 mg/L-hr, -0.003566 mg/L-hr, -0.001579 mg/L-hr
c. The rate at which the concentration of drug in the bloodstream decreases is slowing to 0 as time increases.

In exercise 22, the given function represents the position of a particle traveling along a horizontal line.

a. Find the velocity and acceleration functions.

b. Determine the time intervals when the object is slowing down or speeding up.

22) $s(t) = \dfrac{t}{1+t^2}$

23) [Technology Required] A culture of bacteria grows in number according to the function $N(t) = 3000(1 + \dfrac{4t}{t^2+100})$, where t is measured in hours.

a. Find the rate of change of the number of bacteria.

b. Find $N\,'(0),\ \ N\,'(10),\ \ N\,'(20)$, and $N\,'(30)$.

c. Interpret the results in (b).

d. Find $N''(0),\ \ N''(10),\ \ N''(20)$, and $N''(30)$. Interpret what the answers imply about the bacteria population growth.

Answer

a. $N\,'(t) = 3000\left(\dfrac{-4t^2+400}{(t^2+100)^2}\right)$
b. $120, 0, -14.4, -9.6$
c. The bacteria population increases from time 0 to 10 hours; afterwards, the bacteria population decreases.
d. $0, -6, 0.384, 0.432$. The rate at which the bacteria is increasing is decreasing during the first 10 hours. Afterwards, the bacteria population is decreasing at a decreasing rate.

The following problems deal with the Holling type I, II, and III equations. These equations describe the ecological event of growth of a predator population given the amount of prey available for consumption.

24) [Technology Required] The **Holling type I equation** is described by $f(x) = ax$, where x is the amount of prey available and $a > 0$ is the rate at which the predator meets the prey for consumption.

a. Graph the Holling type I equation, given $a = 0.5$.

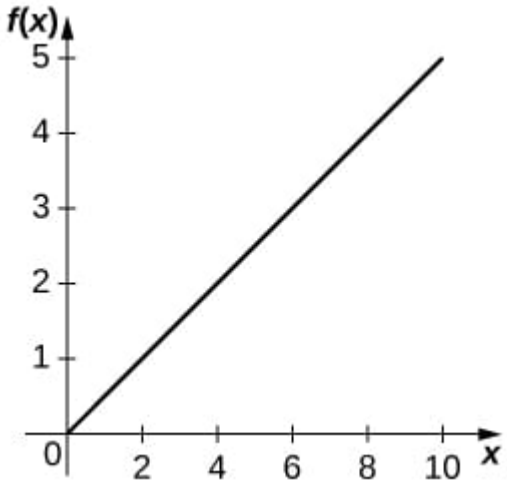
b. Determine the first derivative of the Holling type I equation and explain physically what the derivative implies.

c. Determine the second derivative of the Holling type I equation and explain physically what the derivative implies.

d. Using the interpretations from b. and c. explain why the Holling type I equation may not be realistic.

Answer

a.



b. $f'(x) = a$. The more increase in prey, the more growth for predators.
c. $f''(x) = 0$. As the amount of prey increases, the rate at which the predator population growth increases is constant.
d. This equation assumes that if there is more prey, the predator is able to increase consumption linearly. This assumption is unphysical because we would expect there to be some saturation point at which there is too much prey for the predator to consume adequately.

25) [Technology Required] The Holling type II equation is described by $f(x) = \frac{ax}{n+x}$, where x is the amount of prey available and $a > 0$ is the maximum consumption rate of the predator.

a. Graph the Holling type II equation given $a = 0.5$ and $n = 5$. What are the differences between the Holling type I and II equations?

b. Take the first derivative of the Holling type II equation and interpret the physical meaning of the derivative.

c. Show that $f(n) = \frac{1}{2}a$ and interpret the meaning of the parameter n.

d. Find and interpret the meaning of the second derivative. What makes the Holling type II function more realistic than the Holling type I function?

26) [Technology Required] The Holling type III equation is described by $f(x) = \frac{ax^2}{n^2+x^2}$, where x is the amount of prey available and $a > 0$ is the maximum consumption rate of the predator.

a. Graph the Holling type III equation given $a = 0.5$ and $n = 5$. What are the differences between the Holling type II and III equations?

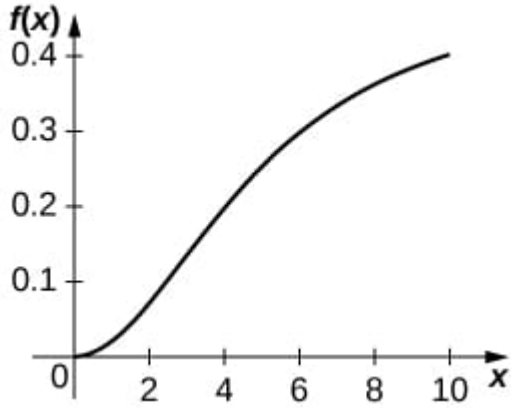
b. Take the first derivative of the Holling type III equation and interpret the physical meaning of the derivative.

c. Find and interpret the meaning of the second derivative (it may help to graph the second derivative).

d. What additional ecological phenomena does the Holling type III function describe compared with the Holling type II function?

Answer

a.



b. $f'(x) = \frac{2axn^2}{(n^2+x^2)^2}$. When the amount of prey increases, the predator growth increases.

c. $f''(x) = \frac{2an^2(n^2-3x^2)}{(n^2+x^2)^3}$. When the amount of prey is extremely small, the rate at which predator growth is increasing is increasing, but when the amount of prey reaches above a certain threshold, the rate at which predator growth is increasing begins to decrease.

d. At lower levels of prey, the prey is more easily able to avoid detection by the predator, so fewer prey individuals are consumed, resulting in less predator growth.

27) [Technology Required] The populations of the snowshoe hare (in thousands) and the lynx (in hundreds) collected over 7 years from 1937 to 1943 are shown in the following table. The snowshoe hare is the primary prey of the lynx.

Population of snowshoe hare (thousands)	Population of lynx (hundreds)
20	10
5	15
65	55
95	60

Snowshoe Hare and Lynx Populations Source: http://www.biotopics.co.uk/newgcse/predatorprey.html.

a. Graph the data points and determine which Holling-type function fits the data best.

b. Using the meanings of the parameters a and n, determine values for those parameters by examining a graph of the data. Recall that n measures what prey value results in the half-maximum of the predator value.

c. Plot the resulting Holling-type I, II, and III functions on top of the data. Was the result from part a. correct?

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2.3: Derivatives of Trigonometric Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

One of the most important types of motion in physics is simple harmonic motion, which is associated with such systems as an object with mass oscillating on a spring. Simple harmonic motion can be described by using either sine or cosine functions. In this section, we expand our knowledge of derivative formulas to include derivatives of these and other trigonometric functions. We begin with the derivatives of the sine and cosine functions and then use them to obtain formulas for the derivatives of the remaining four trigonometric functions. Calculating the derivatives of the sine and cosine functions will enable us to find the velocity and acceleration of simple harmonic motion.

Derivatives of the Sine and Cosine Functions

We begin our exploration of the derivative for the sine function by using the formula to make a reasonable guess at its derivative. Recall that for a function $f(x)$,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Consequently, for values of h very close to 0,

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}.$$

We see that by using $h = 0.01$,

$$\frac{d}{dx}(\sin x) \approx \frac{\sin(x + 0.01) - \sin x}{0.01}$$

By setting

$$D(x) = \frac{\sin(x + 0.01) - \sin x}{0.01}$$

and using a graphing utility, we can get a graph of an approximation to the derivative of $\sin x$ (Figure $2.3.1$).

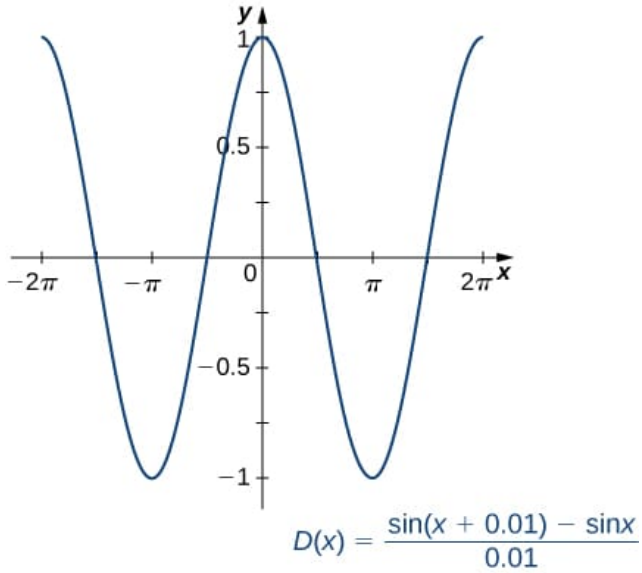

Figure 2.3.1**:** *The graph of the function $D(x)$ looks a lot like a cosine curve.*

Upon inspection, the graph of $D(x)$ appears to be very close to the graph of the cosine function. Indeed, we will show that

$$\frac{d}{dx}(\sin x) = \cos x.$$

If we were to follow the same steps to approximate the derivative of the cosine function, we would find that

$$\frac{d}{dx}(\cos x) = -\sin x.$$

Theorem: The Derivatives of $\sin(x)$ and $\cos(x)$

The derivative of the sine function is the cosine, and the derivative of the cosine function is the negative sine.

$$\frac{d}{dx}(\sin(x)) = \cos(x)$$

$$\frac{d}{dx}(\cos(x)) = -\sin(x)$$

Proof available at the end of this section.

Figure 2.3.3 shows the relationship between the graph of $f(x) = \sin x$ and its derivative $f'(x) = \cos x$. Notice that at the points where $f(x) = \sin x$ has a horizontal tangent, its derivative $f'(x) = \cos x$ takes on the value zero. We also see that where f$(x) = \sin x$ is increasing, $f'(x) = \cos x > 0$ and where $f(x) = \sin x$ is decreasing, $f'(x) = \cos x < 0$.

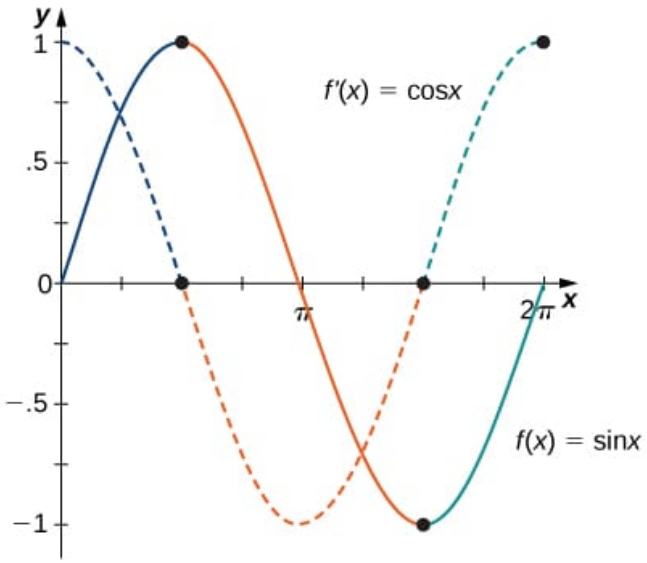

Figure 2.3.3**:** *Where $f(x)$ has a maximum or a minimum, $f'(x) = 0$. That is, $f'(x) = 0$ where $f(x)$ has a horizontal tangent. These points are noted with dots on the graphs.*

Interactive Element: The Slope of Sine Is Cosine

The following Interactive Element might help you visualize why the derivative of $\sin x$ is exactly $\cos x$.

Interact: Drag the large red point along the sine curve.

Observation: At every x, the slope of the red tangent line equals the height of the green point on $\cos x$. Where $\sin x$ has a horizontal tangent, $\cos x = 0$; where $\sin x$ rises, $\cos x > 0$; where it falls, $\cos x < 0$.

Example 2.3.1: Differentiating a Function Containing $\sin x$

Find the derivative of $f(x) = 5x^3 \sin x$.

Solution available at the end of this section.

Checkpoint 2.3.1

Find the derivative of $f(x) = \sin x \cos x$.

Answer available at the end of this section.

Example 2.3.2: Finding the Derivative of a Function Containing cos x

Find the derivative of $g(x) = \frac{\cos x}{4x^2}$.

Solution available at the end of this section.

Checkpoint 2.3.2

Find the derivative of $f(x) = \frac{x}{\cos x}$.

Answer available at the end of this section.

Example 2.3.3: An Application to Velocity

A particle moves along a coordinate axis in such a way that its position at time t is given by $s(t) = 2\sin t - t$ for $0 \le t \le 2\pi$. At what times is the particle at rest?

Solution available at the end of this section.

Checkpoint 2.3.3

A particle moves along a coordinate axis. Its position at time t is given by $s(t) = \sqrt{3}t + 2\cos t$ for $0 \le t \le 2\pi$. At what times is the particle at rest?

Answer available at the end of this section.

Derivatives of Other Trigonometric Functions

Since the remaining four trigonometric functions may be expressed as quotients involving sine, cosine, or both, we can use the Quotient Rule to find formulas for their derivatives.

Example 2.3.4: The Derivative of the Tangent Function

Find the derivative of $f(x) = \tan x$.

Solution available at the end of this section.

Checkpoint 2.3.4

Find the derivative of $f(x) = \cot x$.

Answer available at the end of this section.

The derivatives of the remaining trigonometric functions may be obtained by using similar techniques. We provide these formulas in the following theorem.

Theorem: Derivatives of $\tan(x)$, $\cot(x)$, $\sec(x)$, and $\csc(x)$

The derivatives of the remaining trigonometric functions (along with the sine and cosine) are as follows:

$$\frac{d}{dx}(\sin(x)) = \cos(x) \qquad \frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$$

$$\frac{d}{dx}(\cos(x)) = -\sin(x) \qquad \frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$$

$$\frac{d}{dx}(\tan(x)) = \sec^2(x) \qquad \frac{d}{dx}(\cot(x)) = -\csc^2(x)$$

Getting these formulas cemented in your memory as quickly and efficiently as possible is critical. The following might help a little:

A Trick to Remember

The derivatives of the **co**sine, **co**secant, and **co**tangent have a negative sign in their formulas (not to be construed as "they are always negative," which is not true).

Example 2.3.5: Finding the Equation of a Tangent Line

Find the equation of a line tangent to the graph of $f(x) = \cot x$ at $x = \frac{\pi}{4}$.

Solution available at the end of this section.

Example 2.3.6: Finding the Derivative of Trigonometric Functions

Find the derivative of $f(x) = \csc x + x \tan x$.

Solution available at the end of this section.

Checkpoint 2.3.6

Find the derivative of $f(x) = 2 \tan x - 3 \cot x$.

Answer available at the end of this section.

Example 2.3.7

Evaluate.

$$\frac{d}{dx}\left(\frac{\tan(x)}{1+\cos(x)}\right)$$

Solution available at the end of this section.

Checkpoint 2.3.7

Find the slope of the line tangent to the graph of $f(x) = \tan x$ at $x = \frac{\pi}{6}$.

Answer available at the end of this section.

Higher-Order Derivatives

The higher-order derivatives of $\sin x$ and $\cos x$ follow a repeating pattern. By following the pattern, we can find any higher-order derivative of $\sin x$ and $\cos x$.

Example 2.3.8: Finding Higher-Order Derivatives of $y = \sin x$

Find the first four derivatives of $y = \sin x$.

Solution available at the end of this section.

Checkpoint 2.3.8

For $y = \cos x$, find $\frac{d^4y}{dx^4}$.

Answer available at the end of this section.

Example 2.3.9: Using the Pattern for Higher-Order Derivatives of $y = \sin x$

Find $\frac{d^{74}}{dx^{74}}(\sin x)$.

Solution available at the end of this section.

Checkpoint 2.3.9

For $y = \sin x$, find $\frac{d^{59}}{dx^{59}}(\sin x)$.

Answer available at the end of this section.

Example 2.3.10: An Application to Acceleration

A particle moves along a coordinate axis in such a way that its position at time t is given by $s(t) = 2 - \sin t$. Find $v(\pi/4)$ and $a(\pi/4)$. Compare these values and decide whether the particle is speeding up or slowing down.

Solution available at the end of this section.

Checkpoint 2.3.10

A block attached to a spring is moving vertically. Its position at time t is given by $s(t) = 2\sin t$. Find $v\left(\frac{5\pi}{6}\right)$ and $a\left(\frac{5\pi}{6}\right)$. Compare these values and decide whether the block is speeding up or slowing down.

Answer available at the end of this section.

Success in College: Learning From Tests (Part 1)

1. A student scored 75% on their last math test. They tried to redo the questions and problems that they missed. Now they need to know if their new answers are correct. For your current math class, how could they find this out?
2. Look over your last math test. Did you miss any points because you did not show enough work? If so, explain why you didn't show the work on the test.

Success in College: Relying On Others and Fooling Yourself

It is a common mistake that students make:

- You do homework with your friend, and she constantly is the one with the correct answers. Instead of talking about the mistakes, you copy the answer and move on.
- Your classmates have formed a Discord, and you copy the solutions from one of them.
- You only have a couple hours before the homework is due and you use your preferred A.I. model to do most of the work, copying frantically to submit the work before it's too late.

Is this helping you learn? Do you truly think it is helping your grade?

If you answered yes, you're in the majority of students who have been fooled by this method of "get the right answer by any means and move on."

> When I was a student at UC Davis, I concentrated on understanding every concept that came my way. I studied with other students, but more often than not, they relied on my answers and explanations to succeed in the course. The problem is that they were just "parroting" my methods and not understanding them. In the end, most of them did poorly due to their lack of understanding of the material.
>
> - Professor Simpson, Cosumnes River College

This sort of "copying" also leads to another damaging process - a lack of self-confidence. Because they never solve anything by themselves, the students who copy other students' work end up freaking out more before exams and, as a result of their heightened state of panic and their fundamental lack of understanding of the material, they fail the exams.

Think about what kind of student you are, and, without fear of judgement, consider the following questions:

1. A student was having trouble with his homework. For the next few weeks, a friend helped him do his homework. The friend answered his questions and told him what to do when he was stuck on a problem. Why do you think this student may have trouble doing well on a test?
2. A student can do about 75% of the problems on most of her homework assignments without help. She thinks that this is good enough. She thinks she should be able to score about 75% on a test and pass the test. Do you think she is right?

Answers

Section Preview

This section introduces the student to the derivatives of all the trigonometric functions (and you should prove the derivative of either the sine or the cosine). This is a great opportunity to use the Product and Quotient Rules with new functions. Moreover, this is a *wonderful* opportunity to solve some trig equations when working with tangent lines.

Section Preview

- **Simplifying Expressions**
  - Laws of Exponents — Power Rules: Example 2 simplifies $(4x^2)^2 = 16x^4$ in the denominator of the Quotient Rule result.
- **Trigonometry**
  - Solving Basic Trigonometric Equations: Example 3 and Checkpoint 3 solve equations like $2\cos t - 1 = 0$ on $[0, 2\pi]$ to find when a particle is at rest.

Section Preview

- Compute the derivatives of the standard trigonometric functions.
- Calculate the higher-order derivatives of the sine and cosine.

Theorem: The Derivatives of $\sin(x)$ and $\cos(x)$

Because the proofs for $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$ use similar techniques, we provide only the proof for $\frac{d}{dx}(\sin x) = \cos x$. Before beginning, recall two important trigonometric limits:

$$\lim_{h\to 0} \frac{\sin h}{h} = 1 \quad \text{and} \quad \lim_{h\to 0} \frac{\cos h - 1}{h} = 0.$$

Before we get too far in the proof, it is important to remind ourselves that these limits were only proved using radian measure. As such, the remainder of this proof assumes radian measure! If we wish to work in degrees, we must adjust a lot of our work (as such, we will *not* work in degrees). The graphs of $y = \frac{\sin h}{h}$ and $y = \frac{\cos h - 1}{h}$ are shown in Figure 2.3.2.

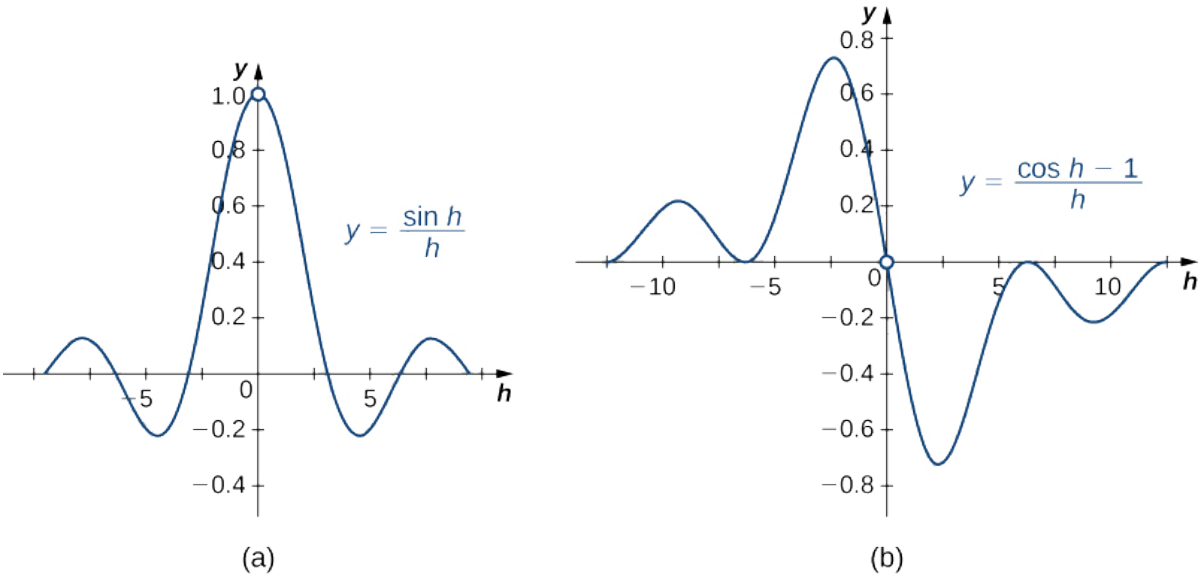

Figure 2.3.2**:** *These graphs show two important limits needed to establish the derivative formulas for the sine and cosine functions.*

We also recall the following trigonometric identity for the sine of the sum of two angles:

$$\sin(x+h) = \sin x \cos h + \cos x \sin h.$$

Now that we have gathered all the necessary equations and identities, we proceed with the proof.

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} && \text{(definition of the derivative)}\\
&= \lim_{h\to 0} \frac{\sin x\cos h + \cos x \sin h - \sin x}{h} && \text{(Sum of Angles Identity)}\\
&= \lim_{h\to 0} \left(\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h}\right) && \text{(regrouping)}\\
&= \lim_{h\to 0} \left(\sin x \left(\frac{\cos h - 1}{h}\right) + (\cos x)\left(\frac{\sin h}{h}\right)\right) && \text{(factoring)}\\
&= (\sin x)\lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right) + (\cos x)\lim_{h\to 0}\left(\frac{\sin h}{h}\right) && \text{(factoring)}\\
&= (\sin x)(0) + (\cos x)(1)\\
&= \cos x
\end{aligned}$$

Q.E.D.

Example 2.3.1**: Differentiating a Function Containing** $\sin x$

Using the Product Rule, we have

$$\begin{aligned}
f'(x) &= \frac{d}{dx}(5x^3)\cdot \sin x + \frac{d}{dx}(\sin x)\cdot 5x^3\\
&= 15x^2 \cdot \sin x + \cos x \cdot 5x^3.
\end{aligned}$$

After simplifying, we obtain

$$f'(x) = 15x^2 \sin x + 5x^3 \cos x.$$

Checkpoint 2.3.1

$$f'(x) = \cos^2 x - \sin^2 x$$

Example 2.3.2**: Finding the Derivative of a Function Containing cos x**

By applying the Quotient Rule, we have

$$g'(x) = \frac{(-\sin x)4x^2 - 8x(\cos x)}{(4x^2)^2}.$$

Simplifying, we obtain

$$g'(x) = \frac{-4x^2 \sin x - 8x\cos x}{16x^4} = \frac{-x\sin x - 2\cos x}{4x^3}.$$

Checkpoint 2.3.2

$f'(x) = \frac{\cos x + x \sin x}{\cos^2 x}$

Example 2.3.3**: An Application to Velocity**

To determine when the particle is at rest, set $s'(t) = v(t) = 0$. Begin by finding $s'(t)$. We obtain

$$s'(t) = 2\cos t - 1,$$

so we must solve

$$2\cos t - 1 = 0 \text{ for } 0 \le t \le 2\pi.$$

The solutions to this equation are $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$. Thus the particle is at rest at times $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$.

Checkpoint 2.3.3

$t = \frac{\pi}{3}, \quad t = \frac{2\pi}{3}$

Example 2.3.4**: The Derivative of the Tangent Function**

Start by expressing $\tan x$ as the quotient of $\sin x$ and $\cos x$:

$$f(x) = \tan x = \frac{\sin x}{\cos x}.$$

Now apply the Quotient Rule to obtain

$$f'(x) = \frac{\cos x \cos x - (-\sin x)\sin x}{(\cos x)^2}.$$

Simplifying, we obtain

$$f'(x) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}.$$

Recognizing that $\cos^2 x + \sin^2 x = 1$, by the Pythagorean Identity, we now have

$$f'(x) = \frac{1}{\cos^2 x}.$$

Finally, use the identity $\sec x = \frac{1}{\cos x}$ to obtain

$$f'(x) = \sec^2 x.$$

Checkpoint 2.3.4

$f'(x) = -\csc^2 x$

Example 2.3.5**: Finding the Equation of a Tangent Line**

To find the equation of the tangent line, we need a point and a slope at that point. To find the point, compute

$$f\left(\frac{\pi}{4}\right) = \cot\frac{\pi}{4} = 1.$$

Thus the tangent line passes through the point $(\frac{\pi}{4}, 1)$. Next, find the slope by finding the derivative of $f(x) = \cot x$ and evaluating it at $\frac{\pi}{4}$:

$$f'(x) = -\csc^2 x \quad \text{and} \quad f'\left(\frac{\pi}{4}\right) = -\csc^2\left(\frac{\pi}{4}\right) = -2.$$

Using the point-slope equation of the line, we obtain

$$y - 1 = -2\left(x - \frac{\pi}{4}\right)$$

or equivalently,

$$y = -2x + 1 + \frac{\pi}{2}.$$

Example 2.3.6**: Finding the Derivative of Trigonometric Functions**

To find this derivative, we must use both the Sum Rule and the Product Rule. Using the Sum Rule, we find

$$f'(x) = \frac{d}{dx}(\csc x) + \frac{d}{dx}(x\tan x).$$

In the first term, $\frac{d}{dx}(\csc x) = -\csc x \cot x$, and by applying the Product Rule to the second term we obtain

$$\frac{d}{dx}(x\tan x) = (1)(\tan x) + (\sec^2 x)(x).$$

Therefore, we have

$$f'(x) = -\csc x \cot x + \tan x + x\sec^2 x.$$

Checkpoint 2.3.6

$f'(x) = 2\sec^2 x + 3\csc^2 x$

Example 2.3.7

$$\frac{d}{dx}\left(\frac{\tan(x)}{1+\cos(x)}\right) = \frac{\frac{d}{dx}(\tan(x))[1+\cos(x)] - \tan(x)\frac{d}{dx}(1+\cos(x))}{[1+\cos(x)]^2} \qquad \text{(Quotient Rule)}$$

$$= \frac{\sec^2(x)[1+\cos(x)] - \tan(x)[-\sin(x)]}{[1+\cos(x)]^2}$$

$$= \frac{\sec^2(x)[1+\cos(x)] + \sin(x)\tan(x)}{[1+\cos(x)]^2}$$

The tricky thing about trigonometric functions is the number of identities and the ultimate desire to simplify answers. It's often best to perform a few extra steps to see if any algebra or Trigonometry could simplify this result further.

$$\begin{aligned}
\frac{\sec^2(x)\left[1+\cos(x)\right]+\sin(x)\tan(x)}{\left[1+\cos(x)\right]^2} &= \frac{\left[1+\cos(x)\right]+\cos(x)\sin^2(x)}{\cos^2(x)\left[1+\cos(x)\right]^2} && \left(\text{multiplying by } \frac{\cos^2(x)}{\cos^2(x)}\right)\\
&= \frac{\left[1+\cos(x)\right]+\cos(x)\left[1-\cos^2(x)\right]}{\cos^2(x)\left[1+\cos(x)\right]^2} && \text{(Pythagorean Identity)}\\
&= \frac{\left[1+\cos(x)\right]+\cos(x)\left[1-\cos(x)\right]\left[1+\cos(x)\right]}{\cos^2(x)\left[1+\cos(x)\right]^2} && \text{(Difference of Squares)}\\
&= \frac{\left[1+\cos(x)\right]\left(1+\cos(x)\left[1-\cos(x)\right]\right)}{\cos^2(x)\left[1+\cos(x)\right]^2} && \text{(factoring)}\\
&= \frac{\cancel{\left[1+\cos(x)\right]}\left(1+\cos(x)\left[1-\cos(x)\right]\right)}{\cos^2(x)\left[1+\cos(x)\right]^{\cancel{2}1} } && \text{(canceling)}\\
&= \frac{1+\cos(x)-\cos^2(x)}{\cos^2(x)\left[1+\cos(x)\right]} && \text{(distributing)}
\end{aligned}$$

As you can see, our efforts to arrive at a more "simplified" form have not paid off. Any of these answers are equivalent, and none are really "simpler" than others; however, it is *always* advisable to perform a few extra steps like this to see if your results simplify (they often will).

Checkpoint 2.3.7

$\frac{4}{3}$

Example 2.3.8**: Finding Higher-Order Derivatives of** $y=\sin x$

Each step in the chain is straightforward:

$$\begin{aligned}
y &= \sin x\\
\frac{dy}{dx} &= \cos x\\
\frac{d^2y}{dx^2} &= -\sin x\\
\frac{d^3y}{dx^3} &= -\cos x\\
\frac{d^4y}{dx^4} &= \sin x
\end{aligned}$$

Analysis

Once we recognize the pattern of derivatives, we can find any higher-order derivative by determining the step in the pattern to which it corresponds. For example, every fourth derivative of $\sin x$ equals $\sin x$, so

$$\frac{d^4}{dx^4}(\sin x)=\frac{d^8}{dx^8}(\sin x)=\frac{d^{12}}{dx^{12}}(\sin x)=\ldots=\frac{d^{4n}}{dx^{4n}}(\sin x)=\sin x$$
$$\frac{d^5}{dx^5}(\sin x)=\frac{d^9}{dx^9}(\sin x)=\frac{d^{13}}{dx^{13}}(\sin x)=\ldots=\frac{d^{4n+1}}{dx^{4n+1}}(\sin x)=\cos x.$$

Checkpoint 2.3.8

$\cos x$

Example 2.3.9**: Using the Pattern for Higher-Order Derivatives of** $y=\sin x$

We can see right away that for the 74th derivative of $\sin x$, $74=4(18)+2$, so

$$\frac{d^{74}}{dx^{74}}(\sin x)=\frac{d^{72+2}}{dx^{72+2}}(\sin x)=\frac{d^2}{dx^2}(\sin x)=-\sin x.$$

Checkpoint 2.3.9

$-\cos x$

Example 2.3.10**: An Application to Acceleration**

First find $v(t)=s'(t)$.

$$v(t)=s'(t)=-\cos t.$$

Thus,

$$v\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}=-\frac{\sqrt{2}}{2}.$$

Next, find $a(t)=v'(t)$. Thus, $a(t)=v'(t)=\sin t$ and we have

$$a\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

Since $v\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} < 0$ and $a\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} > 0$, we see that velocity and acceleration are acting in opposite directions; that is, the object is being accelerated in the direction opposite to the direction in which it is traveling. Consequently, the particle is slowing down.

Checkpoint 2.3.10

$v\left(\frac{5\pi}{6}\right) = -\sqrt{3} < 0$ and $a\left(\frac{5\pi}{6}\right) = -1 < 0$. The block is speeding up.

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2.3.1: Resources and Key Concepts

Resources

Videos

- Derivatives of trigonometric functions
  - Derivatives of Trigonometric Functions: Theorem and Proofs
  - Proofs in Differential Calculus - The Derivative of Sine is Cosine
  - Proofs in Differential Calculus - The Derivative of Tangent is Secant Squared
  - Computing Derivatives Involving Trigonometric Functions
  - Synthesis Questions

Key Concepts

Theorems

- **Theorem: The Derivatives of $\sin(x)$ and $\cos(x)$**
  - $\frac{d}{dx}(\sin(x)) = \cos(x)$
  - $\frac{d}{dx}(\cos(x)) = -\sin(x)$
- **Theorem: Derivatives of $\tan(x), \cot(x), \sec(x)$, and $\csc(x)$**
  - $\frac{d}{dx}(\tan(x)) = \sec^2(x)$
  - $\frac{d}{dx}(\cot(x)) = -\csc^2(x)$
  - $\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$
  - $\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$

Common Mistakes

- **Forgetting Radian Measure**: The proofs for the derivatives of trigonometric functions (especially $\sin x$ and $\cos x$, which rely on $\lim_{h\to 0}\frac{\sin h}{h} = 1$ and $\lim_{h\to 0}\frac{\cos h - 1}{h} = 0$) assume radian measure. Working in degrees would require significant adjustments.
- **Sign Errors in Derivatives**: Forgetting the negative signs in the derivatives of cosine, cosecant, and cotangent.
- **Oversimplification or Lack of Simplification**: Trigonometric expressions can often be simplified using identities. While not always leading to a "simpler" final form, it's advisable to check for possible simplifications after differentiation (see end of Example 7).

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2.3.2: Homework

Note

The following exercises are meant as hand-in homework (if assigned by your instructor). Typed-in answers here will not be saved or scored.

Basic Skills

Exercise 1

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Exercise 13

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Exercise 14

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Exercise 23

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Exercise 24

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Exercise 25

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Basic Skills - Applications

Exercise 26

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Synthesis Questions

Exercise 27

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Exercise 28

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Exercise 29

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Exercise 30

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Proofs and Problems that Require Explanations or Logical Justifications

Exercise 31

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2.3.3: Additional Exercises

Reading Questions

1. What is the derivative of $f(x) = \sin x$?
2. What is the derivative of $f(x) = \cos x$?
3. Which two trigonometric limits are crucial for proving the derivative of $\sin x$?
4. How is the Quotient Rule used to find the derivative of $f(x) = \tan x$?
5. What is the derivative of $f(x) = \tan x$?
6. What is the derivative of $f(x) = \cot x$?
7. What is the derivative of $f(x) = \sec x$?
8. What is the derivative of $f(x) = \csc x$?
9. Which three standard trigonometric functions have derivatives that include a negative sign?
10. If a particle's position is given by $s(t) = 2\sin t - t$, how would you find the times when the particle is at rest?
11. Describe the pattern of higher-order derivatives for $y = \sin x$. What is the fourth derivative of $\sin x$?
12. If a particle's velocity is $v(t)$ and its acceleration is $a(t)$, what does it mean if $v(t)$ and $a(t)$ have opposite signs at a particular time?

Homework

In exercises 1 - 10, find $\frac{dy}{dx}$ for the given functions.

1) $y = x^2 - \sec x + 1$

Answer

$$\frac{dy}{dx} = 2x - \sec x \tan x$$

2) $y = 3\csc x + \frac{5}{x}$

3) $y = x^2 \cot x$

Answer

$$\frac{dy}{dx} = 2x \cot x - x^2 \csc^2 x$$

4) $y = x - x^3 \sin x$

5) $y = \frac{\sec x}{x}$

Answer

$$\frac{dy}{dx} = \frac{x \sec x \tan x - \sec x}{x^2}$$

6) $y = \sin x \tan x$

7) $y = (x + \cos x)(1 - \sin x)$

Answer

$$\frac{dy}{dx} = (1 - \sin x)(1 - \sin x) - \cos x(x + \cos x)$$

8) $y = \frac{\tan x}{1 - \sec x}$

9) $y = \dfrac{1 - \cot x}{1 + \cot x}$

Answer

$$\frac{dy}{dx} = \frac{2\csc^2 x}{(1 + \cot x)^2}$$

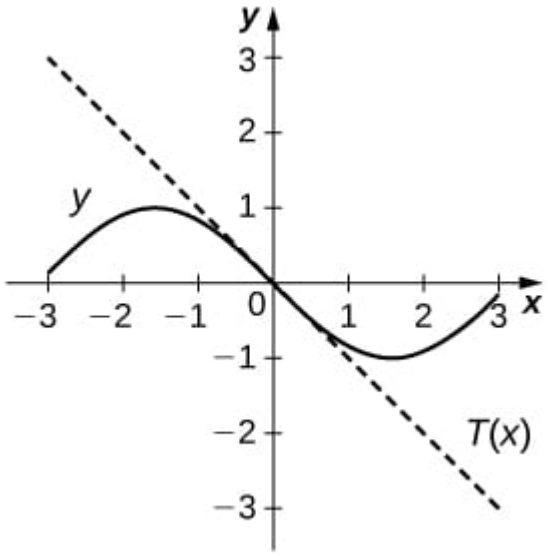
10) $y = (\cos x)(1 + \csc x)$

In exercises 11 - 16, find the equation of the tangent line to each of the given functions at the indicated values of x. Then use graphing technology to graph both the function and the tangent line to ensure the equation for the tangent line is correct.

11) $f(x) = -\sin x, \quad x = 0$

Answer

$y = -x$

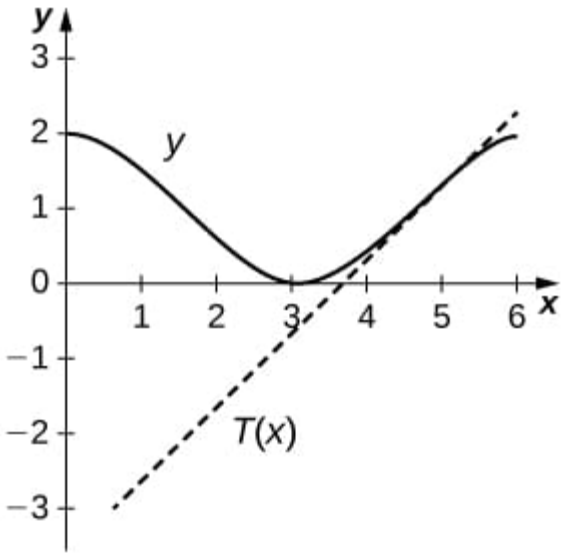


12) $f(x) = \csc x, \quad x = \frac{\pi}{2}$

13) $f(x) = 1 + \cos x, \quad x = \frac{3\pi}{2}$

Answer

$y = x + \frac{2-3\pi}{2}$

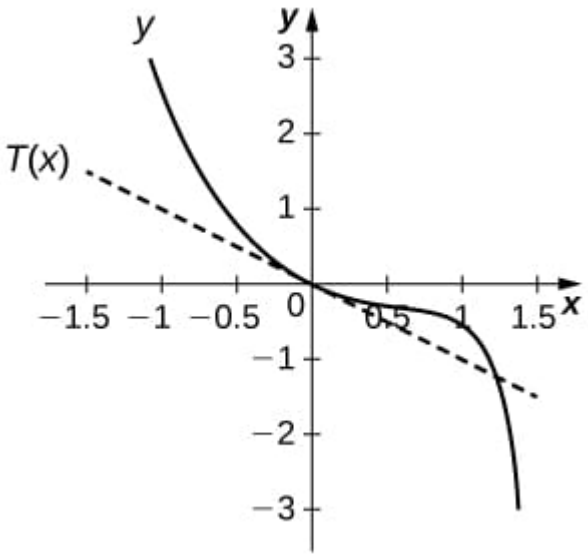


14) $f(x) = \sec x, \quad x = \frac{\pi}{4}$

15) $f(x) = x^2 - \tan x, \quad x = 0$

Answer

$y = -x$



16) $f(x) = 5 \cot x, \quad x = \frac{\pi}{4}$

In exercises 17 - 22, find $\frac{d^2y}{dx^2}$ for the given functions.

17) $y = x \sin x - \cos x$

Answer

$\frac{d^2y}{dx^2} = 3 \cos x - x \sin x$

18) $y = \sin x \cos x$

19) $y = x - \frac{1}{2}\sin x$

Answer

$$\frac{d^2y}{dx^2} = \frac{1}{2}\sin x$$

20) $y = \dfrac{1}{x} + \tan x$

21) $y = 2\csc x$

Answer

$$\frac{d^2y}{dx^2} = 2\csc(x)\left(\csc^2(x) + \cot^2(x)\right)$$

22) $y = \sec^2 x$

23) Find all x values on the graph of $f(x) = -3\sin x \cos x$ where the tangent line is horizontal.

Answer

$x = \dfrac{(2n+1)\pi}{4}$,where n is an integer

24) Find all x values on the graph of $f(x) = x - 2\cos x$ for \(0

25) Let $f(x) = \cot x$. Determine the points on the graph of f for \(0

Answer

$\left(\frac{\pi}{4}, 1\right), \quad \left(\frac{3\pi}{4}, -1\right), \quad \left(\frac{5\pi}{4}, 1\right), \quad \left(\frac{7\pi}{4}, -1\right)$

26) [Technology Required] A mass on a spring bounces up and down in simple harmonic motion, modeled by the function $s(t) = -6\cos t$ where s is measured in inches and t is measured in seconds. Find the rate at which the spring is oscillating at $t = 5$ s.

27) Let the position of a swinging pendulum in simple harmonic motion be given by $s(t) = a\cos t + b\sin t$. Find the constants a and b such that when the velocity is 3 cm/s, $s = 0$ and $t = 0$.

Answer

$a = 0, \quad b = 3$

28) After a diver jumps off a diving board, the edge of the board oscillates with position given by $s(t) = -5\cos t$ cm at t seconds after the jump.

a. Sketch one period of the position function for $t \geq 0$.

b. Find the velocity function.

c. Sketch one period of the velocity function for $t \geq 0$.

d. Determine the times when the velocity is 0 over one period.

e. Find the acceleration function.

f. Sketch one period of the acceleration function for $t \geq 0$.

29) The number of hamburgers sold at a fast-food restaurant in Pasadena, California, is given by $y = 10 + 5\sin x$ where y is the number of hamburgers sold and x represents the number of hours after the restaurant opened at 11 a.m. until 11 p.m., when the store closes. Find y' and determine the intervals where the number of burgers being sold is increasing.

Answer

$y' = 5\cos(x)$, increasing on $\left(0, \frac{\pi}{2}\right), \quad \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$, and $\left(\frac{7\pi}{2}, 12\right)$

30) [Technology Required] The amount of rainfall per month in Phoenix, Arizona, can be approximated by $y(t) = 0.5 + 0.3\cos t$, where t is months since January. Find y' and use a calculator to determine the intervals where the amount of rain falling is decreasing.

For exercises 31 - 33, use the quotient rule to derive the given equations.

31) $\frac{d}{dx}(\cot x) = -\csc^2 x$

32) $\frac{d}{dx}(\sec x) = \sec x \tan x$

33) $\frac{d}{dx}(\csc x) = -\csc x \cot x$

34) Use the definition of derivative and the identity $\cos(x+h) = \cos x \cos h - \sin x \sin h$ to prove that $\frac{d}{dx}(\cos x) = -\sin x$.

For exercises 35 - 39, find the requested higher-order derivative for the given functions.

35) $\frac{d^3y}{dx^3}$ of $y = 3\cos x$

Answer

$$\frac{d^3y}{dx^3} = 3\sin x$$

36) $\frac{d^2y}{dx^2}$ of $y = 3\sin x + x^2\cos x$

37) $\frac{d^4y}{dx^4}$ of $y = 5\cos x$

Answer

$$\frac{d^4y}{dx^4} = 5\cos x$$

38) $\frac{d^2y}{dx^2}$ of $y = \sec x + \cot x$

39) $\frac{d^3y}{dx^3}$ of $y = x^{10} - \sec x$

Answer

$$\frac{d^3y}{dx^3} = 720x^7 - 5\tan(x)\sec^3(x) - \tan^3(x)\sec(x)$$

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2.4: Differentiation Techniques - The Chain Rule

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

We have seen the techniques for differentiating basic functions ($x^n, \sin x, \cos x$, etc.) as well as sums, differences, products, quotients, and constant multiples of these functions. However, these techniques do not allow us to differentiate compositions of functions, such as $h(x) = \sin(x^3)$ or $k(x) = \sqrt{3x^2 + 1}$. In this section, we study the rule for finding the derivative of the composition of two or more functions.

Deriving the Chain Rule

When we have a function composed of two or more functions, we can use all the techniques we have already learned to differentiate it. However, using all those techniques to break down a function into simpler parts that we can differentiate can be challenging. Instead, we use the **Chain Rule**, which states that the derivative of a composite function is the derivative of the outer function evaluated at the inner function times the derivative of the inner function.

To put this rule into context, let's take a look at an example: $h(x) = \sin(x^3)$. We can think of the derivative of this function with respect to x as the rate of change of $\sin(x^3)$ relative to the change in x. Consequently, we want to know how $\sin(x^3)$ changes as x changes. We can think of this event as a chain reaction: As x changes, x^3 changes, which leads to a change in $\sin(x^3)$. This chain reaction hints at what is involved in computing the derivative of $\sin(x^3)$. First of all, a change in x forcing a change in x^3 suggests that somehow the derivative of x^3 is involved. In addition, the change in x^3 forcing a change in $\sin(x^3)$ suggests that the derivative of $\sin(u)$ with respect to u, where $u = x^3$, is also part of the final derivative.

We can take a more formal look at the derivative of $h(x) = \sin(x^3)$ by setting up the limit that would give us the derivative at a specific value a in the domain of $h(x) = \sin(x^3)$.

$$h'(a) = \lim_{x \to a} \frac{\sin(x^3) - \sin(a^3)}{x - a}$$

This expression does not seem particularly helpful; however, we can modify it by multiplying and dividing by the expression $x^3 - a^3$ to obtain

$$h'(a) = \lim_{x \to a} \frac{\sin(x^3) - \sin(a^3)}{x^3 - a^3} \cdot \frac{x^3 - a^3}{x - a}.$$

From the definition of the derivative, we can see that the second factor is the derivative of x^3 at $x = a$. That is,

$$\lim_{x \to a} \frac{x^3 - a^3}{x - a} = \frac{d}{dx}(x^3)\Big|_{x=a} = 3a^2.$$

However, recognizing that the first term is also a derivative might be a little more challenging. We can see this by letting $u = x^3$ and observing that as $x \to a, u \to a^3$:

$$\begin{aligned} \lim_{x \to a} \frac{\sin(x^3) - \sin(a^3)}{x^3 - a^3} &= \lim_{u \to a^3} \frac{\sin u - \sin(a^3)}{u - a^3} \\ &= \frac{d}{du}(\sin u)\Big|_{u=a^3} \\ &= \cos(a^3) \end{aligned}$$

Thus, $h'(a) = \cos(a^3) \cdot 3a^2$.

In other words, if $h(x) = \sin(x^3)$, then $h'(x) = \cos(x^3) \cdot 3x^2$. Thus, if we think of $h(x) = \sin(x^3)$ as the composition $(f \circ g)(x) = f(g(x))$ where $f(x) = \sin x$ and $g(x) = x^3$, then the derivative of $h(x) = \sin(x^3)$ is the product of the derivative of $g(x) = x^3$ and the derivative of the function $f(x) = \sin x$ evaluated at the function $g(x) = x^3$. At this point, we anticipate that for $h(x) = \sin(g(x))$, it is quite likely that $h'(x) = \cos(g(x))\, g'(x)$. As we determined above, this is the case for $h(x) = \sin(x^3)$.

Now that we have derived a special case of the Chain Rule, we state the general case and then apply it in a general form to other composite functions.

Theorem: The Chain Rule (Lagrange's Notation)

Let f and g be functions. For all x in the domain of g for which g is differentiable at x and f is differentiable at $g(x)$, the derivative of the composite function

$$h(x) = (f \circ g)(x) = f(g(x))$$

is given by

$$h'(x) = f'(g(x)) \cdot g'(x).$$

Alternatively, if y is a function of u, and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Proof (informal) available at the end of this section.

The Chain and Power Rules Combined

We can now apply the Chain Rule to composite functions, but note that we often need to use it with other rules. For example, to find derivatives of functions of the form $h(x) = (g(x))^n$, we need to use the Chain Rule combined with the Power Rule. To do so, we can think of $h(x) = (g(x))^n$ as $f(g(x))$ where $f(x) = x^n$. Then $f'(x) = nx^{n-1}$. Thus, $f'(g(x)) = n(g(x))^{n-1}$. This leads us to the derivative of a power function using the Chain Rule,

$$h'(x) = n(g(x))^{n-1} \cdot g'(x)$$

Theorem: Power Rule for Composition of Functions (General Power Rule)

For all values of x for which the derivative is defined, if

$$h(x) = (g(x))^n,$$

Then

$$h'(x) = n(g(x))^{n-1} \cdot g'(x). \tag{2.4.1}$$

Example 2.4.1: Using the Chain and Power Rules

Find the derivative of $h(x) = \frac{1}{(3x^2+1)^2}$.

Solution available at the end of this section.

Checkpoint 2.4.1

Find the derivative of $h(x) = (2x^3 + 2x - 1)^4$.

Answer available at the end of this section.

Example 2.4.2: Using the Chain and Power Rules with a Trigonometric Function

Find the derivative of $h(x) = \sin^3 x$.

Solution available at the end of this section.

Example 2.4.3: Finding the Equation of a Tangent Line

Find the equation of a line tangent to the graph of $h(x) = \frac{1}{(3x-5)^2}$ at $x = 2$.

Solution available at the end of this section.

Checkpoint 2.4.3

Find the equation of the line tangent to the graph of $f(x) = (x^2 - 2)^3$ at $x = -2$.

Answer available at the end of this section.

Combining the Chain Rule with Other Rules

Now that we can combine the Chain Rule and the Power Rule, we examine how to combine the Chain Rule with the other rules we have learned. In particular, we can use it with the formulas for the derivatives of trigonometric functions or with the Product Rule.

Example 2.4.4: Using the Chain Rule on a General Cosine Function

Find the derivative of $h(x) = \cos(g(x))$.

Solution available at the end of this section.

In the following example, we apply the rule that we have just derived.

Example 2.4.5: Using the Chain Rule on a Cosine Function

Find the derivative of $h(x) = \cos(5x^2)$.

Solution available at the end of this section.

Example 2.4.6: Using the Chain Rule on Another Trigonometric Function

Find the derivative of $h(x) = \sec(4x^5 + 2x)$.

Solution available at the end of this section.

Checkpoint 2.4.6

Find the derivative of $h(x) = \sin(7x + 2)$.

Answer available at the end of this section.

Example 2.4.7: Combining the Chain Rule with the Product Rule

Find the derivative of $h(x) = (2x+1)^5(3x-2)^7$.

Solution available at the end of this section.

Checkpoint 2.4.7

Find the derivative of $h(x) = \frac{x}{(2x+3)^3}$.

Answer available at the end of this section.

Composites of Three or More Functions

We can now combine the Chain Rule with other rules for differentiating functions, but when differentiating the composition of three or more functions, we need to apply the Chain Rule more than once. Let's look at this situation in general terms. We can generate a formula without remembering it, as we can apply the Chain Rule multiple times.

In general terms, first we let

$$k(x) = h\left(f\left(g(x)\right)\right).$$

Then, applying the Chain Rule once we obtain

$$k'(x) = \frac{d}{dx}\left(h\left(f\left(g(x)\right)\right)\right) = h'\left(f\left(g(x)\right)\right) \cdot \frac{d}{dx}\left(f\left(g(x)\right)\right).$$

Applying the Chain Rule again, we obtain

$$k'(x) = h'\left(f\left(g(x)\right)\right) \cdot f'\left(g(x)\right) \cdot g'(x).$$

Corollary: Chain Rule for a Composition of Three Functions

For all values of x for which the function is differentiable, if

$$k(x) = h\left(f\left(g(x)\right)\right),$$

then

$$k'(x) = h'\left(f\left(g(x)\right)\right) \cdot f'\left(g(x)\right) \cdot g'(x).$$

In other words, we are applying the Chain Rule twice.

Notice that the derivative of the composition of three functions has three parts. (Similarly, the derivative of the composition of four functions has four parts, and so on.) Also, *remember, we can always work from the outside in, taking one derivative at a time.*

Example 2.4.8: Differentiating a Composite of Three Functions

Find the derivative of $k(x) = \cos^4(7x^2+1)$.

Solution available at the end of this section.

Checkpoint 2.4.8

Find the derivative of $h(x) = \sin^6(x^3)$.

Answer available at the end of this section.

Example 2.4.9: Using the Chain Rule in a Velocity Problem

A particle moves along a coordinate axis. Its position at time t is given by $s(t) = \sin(2t) + \cos(3t)$. What is the velocity of the particle at time $t = \frac{\pi}{6}$?

Solution available at the end of this section.

Example 2.4.10: Using the Chain Rule with Functional Values

Let $h(x) = f(g(x))$. If $g(1) = 4, g'(1) = 3$, and $f'(4) = 7$, find $h'(1)$.

Solution available at the end of this section.

Checkpoint 2.4.10

Given $h(x) = f(g(x))$. If $g(2) = -3, g'(2) = 4$, and $f'(-3) = 7$, find $h'(2)$.

Answer available at the end of this section.

The Chain Rule Using Leibniz's Notation

As with other derivatives that we have seen, we can express the Chain Rule using Leibniz's notation. This notation for the Chain Rule is used heavily in physics applications.

For $h(x) = f(g(x))$, let $u = g(x)$ and $y = h(x) = f(u)$. Thus,

$$h'(x) = \frac{dy}{dx}$$

$$f'(g(x)) = f'(u) = \frac{dy}{du}$$

and

$$g'(x) = \frac{du}{dx}.$$

Consequently,

$$\frac{dy}{dx} = h'(x) = f'\left(g(x)\right) \cdot g'(x) = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Theorem: Chain Rule (Leibniz's Notation)

If y is a function of u, and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Example 2.4.11: Taking a Derivative Using Leibniz's Notation I

Find the derivative of $y = \left(\frac{x}{3x+2}\right)^5$.

Solution available at the end of this section.

Example 2.4.12: Taking a Derivative Using Leibniz's Notation II

Find the derivative of $y = \tan(4x^2 - 3x + 1)$.

Solution available at the end of this section.

Checkpoint 2.4.12

Use Leibniz's notation to find the derivative of $y = \cos(x^3)$. Ensure that the final answer is expressed entirely in terms of the variable x.

Answer available at the end of this section.

Success in Mathematics: Why Study Mathematics? (Part 2)

Tracy Moody graduated in 2006 with a bachelor's degree in mathematics. She then went to graduate school, studying to become a physical therapist. She writes,

> "Many people cannot seem to fathom how math and physical therapy overlap. Well, one of the obvious things is that physical therapists use statistics to see if new treatments work. However, the main thing I bring as a mathematician to my graduate studies is my ability to problem solve. A degree in mathematics taught me how to look at a problem from several angles. Furthermore, one of the main classes that carried over into my career choice is, believe it or not, real analysis. I can remember working with my fellow classmates on problems for hours. However, through this process, I learned the importance of perseverance, and I learned how to communicate the problem and solutions in several ways."
>
> *Source: www.maa.org, Career Profiles.*

1. Do you think studying mathematics has helped you become a better problem solver? Explain.
2. Do you think studying mathematics has helped you learn to "persevere," to keep working on a difficult problem even though you are struggling? Explain.

Success in College: Getting Help

Learning math can be a struggle. Sometimes your professor seems to be talking in a foreign language. Other students look like they get it. The professor keeps going. The stuff you don't understand keeps piling up.

Struggling in a college math class is expected. You must learn to work through your confusion. Your professor expects you to ask questions in class. If needed, you must also get help outside of class. Visit your professor during office hours. Ask if appointments are available outside of office hours. Use the math tutors on campus. It is your responsibility to get the help you need. You must work, ask questions, practice, and think until you get it. You have to take the time to get extra help.

Students who let frustration get to them fail. They put off getting help. They sometimes skip class. They forget that college is different than many high schools. In college, you can fail. If you do not learn enough, you cannot go on.

When successful students are confused, they get help. They work until they understand. You should expect to be confused at times. It would be best if you planned on needing extra time and energy to deal with it.

Here are some questions that may help you succeed in this class (and possibly many other classes as well):

1. Where is your professor's office located?
2. Your professor might keep regular office hours when they are available for help. When are your professor's office hours?

If you are unsure about the answers to either of these questions, feel free to ask your professor - that's what they are there for!

Answers

Section Preview

This is a standard Chain Rule section. As with everything up this point, derive theorems and tie in old Algebra and Trig as often as possible.

Section Preview

- State the Chain Rule using both Lagrange and Leibniz notations.
- Use the Chain Rule combined with the Power Rule.
- Apply the Chain Rule and the Product/Quotient Rules correctly when necessary.
- Describe the proof of the Chain Rule.

Theorem: The Chain Rule (Lagrange's Notation)

At this point, we present a very informal proof of the Chain Rule. For simplicity's sake, we ignore certain issues: For example, we assume that $g(x) \neq g(a)$ for $x \neq a$ in some open interval containing a. We begin by applying the limit definition of the derivative to the function $h(x)$ to obtain $h'(a)$:

$$h'(a) = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a}.$$

Rewriting, we obtain

$$h'(a) = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a}.$$

Although it is clear that

$$\lim_{x \to a} \frac{g(x) - g(a)}{x - a} = g'(a),$$

it is not obvious that

$$\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} = f'(g(a)).$$

To see that this is true, first recall that since g is differentiable at a, g is also continuous at a. Thus,

$$\lim_{x \to a} g(x) = g(a).$$

Next, make the substitution $y = g(x)$and $b = g(a)$ and use change of variables in the limit to obtain

$$\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} = \lim_{y \to b} \frac{f(y) - f(b)}{y - b} = f'(b) = f'(g(a)).$$

Finally,

$$h'(a) = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a} = f'(g(a)) \cdot g'(a).$$

Q.E.D.

Example 2.4.1**: Using the Chain and Power Rules**

First, rewrite $h(x) = \frac{1}{(3x^2+1)^2} = (3x^2 + 1)^{-2}$. Applying the Power Rule with $g(x) = 3x^2 + 1$, we have

$$h'(x) = -2(3x^2 + 1)^{-3} \cdot 6x.$$

Rewriting back to the original form gives us

$$h'(x) = \frac{-12x}{(3x^2 + 1)^3}.$$

Checkpoint 2.4.1

$h'(x) = 4(2x^3 + 2x - 1)^3(6x^2 + 2) = 8(3x^2 + 1)(2x^3 + 2x - 1)^3$

Example 2.4.2**: Using the Chain and Power Rules with a Trigonometric Function**

First recall that $\sin^3 x = (\sin x)^3$, so we can rewrite $h(x) = \sin^3 x$ as $h(x) = (\sin x)^3$.

Applying the Power Rule with $g(x) = \sin x$, we obtain

$$h'(x) = 3(\sin x)^2 \cos x = 3 \sin^2 x \cos x.$$

Example 2.4.3**: Finding the Equation of a Tangent Line**

Because we are finding an equation of a line, we need a point. The x-coordinate of the point is 2. To find the y-coordinate, substitute 2 into $h(x)$. Since $h(2) = \frac{1}{(3(2)-5)^2} = 1$, the point is $(2, 1)$.

For the slope, we need $h'(2)$. To find $h'(x)$, first we rewrite $h(x) = (3x - 5)^{-2}$ and apply the Power Rule to obtain

$$h'(x) = -2(3x - 5)^{-3}(3) = -6(3x - 5)^{-3}.$$

By substituting, we have $h'(2) = -6(3(2) - 5)^{-3} = -6$.

Therefore, the line has equation $y - 1 = -6(x - 2)$. Rewriting, the equation of the line is $y = -6x + 13$.

Checkpoint 2.4.3

$y = -48x - 88$

Example 2.4.4**: Using the Chain Rule on a General Cosine Function**

Think of $h(x) = \cos(g(x))$ as $f(g(x))$ where $f(x) = \cos x$. Since $f'(x) = -\sin x$, we have $f'(g(x)) = -\sin(g(x))$. Then we do the following calculation.

$$\begin{aligned} h'(x) &= f'(g(x)) \cdot g'(x) && \text{(Chain Rule)} \\ &= -\sin(g(x)) \cdot g'(x) && \text{(substituting } f'(g(x)) = -\sin(g(x))\text{)} \end{aligned}$$

Thus, the derivative of $h(x) = \cos(g(x))$ is given by $h'(x) = -\sin(g(x)) \cdot g'(x)$.

Example 2.4.5**: Using the Chain Rule on a Cosine Function**

Let $g(x) = 5x^2$. Then $g'(x) = 10x$. Using the result from the previous example,

$$h'(x) = -\sin(5x^2) \cdot 10x = -10x\sin(5x^2).$$

Example 2.4.6**: Using the Chain Rule on Another Trigonometric Function**

Apply the Chain Rule to $h(x) = \sec(g(x))$ to obtain

$$h'(x) = \sec(g(x))\tan(g(x)) \cdot g'(x).$$

In this problem, $g(x) = 4x^5 + 2x$, so we have $g'(x) = 20x^4 + 2$. Therefore, we obtain

$$h'(x) = \sec(4x^5 + 2x)\tan(4x^5 + 2x)(20x^4 + 2) = (20x^4 + 2)\sec(4x^5 + 2x)\tan(4x^5 + 2x).$$

Checkpoint 2.4.6

$h'(x) = 7\cos(7x + 2)$

Example 2.4.7**: Combining the Chain Rule with the Product Rule**

First apply the Product Rule, then apply the Chain Rule to each term of the product.

$$\begin{aligned} h'(x) &= \frac{d}{dx}\left((2x+1)^5\right) \cdot (3x-2)^7 + \frac{d}{dx}\left((3x-2)^7\right) \cdot (2x+1)^5 && \text{(Product Rule)} \\ &= 5(2x+1)^4 \cdot 2 \cdot (3x-2)^7 + 7(3x-2)^6 \cdot 3 \cdot (2x+1)^5 && \text{(Chain Rule)} \\ &= 10(2x+1)^4(3x-2)^7 + 21(3x-2)^6(2x+1)^5 && \text{(simplifying)} \\ &= (2x+1)^4(3x-2)^6(10(3x-2) + 21(2x+1)) && \text{(factoring)} \\ &= (2x+1)^4(3x-2)^6(72x+1) && \text{(simplifying)} \end{aligned}$$

Checkpoint 2.4.7

$h'(x) = \frac{3-4x}{(2x+3)^4}$

Example 2.4.8**: Differentiating a Composite of Three Functions**

First, rewrite $k(x)$ as

$$k(x) = \left(\cos(7x^2 + 1)\right)^4.$$

Then apply the Chain Rule several times.

$$\begin{aligned} k'(x) &= 4(\cos(7x^2+1))^3 \cdot \frac{d}{dx}\left(\cos(7x^2+1)\right) && \text{(Chain Rule)} \\ &= 4(\cos(7x^2+1))^3(-\sin(7x^2+1)) \cdot \frac{d}{dx}\left(7x^2+1\right) && \text{(Chain Rule)} \\ &= 4(\cos(7x^2+1))^3(-\sin(7x^2+1))(14x) && \text{(Chain Rule)} \\ &= -56x\sin(7x^2+1)\cos^3(7x^2+1) && \text{(simplifying)} \end{aligned}$$

Checkpoint 2.4.8

$h'(x) = 18x^2\sin^5(x^3)\cos(x^3)$

Example 2.4.9**: Using the Chain Rule in a Velocity Problem**

To find $v(t)$, the velocity of the particle at time t, we must differentiate $s(t)$. Thus,

$$v(t) = s'(t) = 2\cos(2t) - 3\sin(3t).$$

Example 2.4.10**: Using the Chain Rule with Functional Values**

Use the Chain Rule, then substitute.

$$\begin{aligned} h'(1) &= f'(g(1)) \cdot g'(1) && \text{(Chain Rule)} \\ &= f'(4) \cdot 3 && \text{(substituting } g(1) = 4 \text{ and } g'(1) = 3) \\ &= 7 \cdot 3 && \text{(substituting } f'(4) = 7) \\ &= 21 && \text{(simplifying)} \end{aligned}$$

Checkpoint 2.4.10

28

Example 2.4.11**: Taking a Derivative Using Leibniz's Notation I**

First, let $u = \frac{x}{3x+2}$. Thus, $y = u^5$. Next, find $\frac{du}{dx}$ and $\frac{dy}{du}$. Using the Quotient Rule,

$$\frac{du}{dx} = \frac{2}{(3x+2)^2}$$

and

$$\frac{dy}{du} = 5u^4,$$

Finally, we put it all together.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx} && \text{(Chain Rule)}\\
&= 5u^4\cdot\frac{2}{(3x+2)^2} && \left(\text{substituting } \frac{dy}{du}=5u^4 \text{ and } \frac{du}{dx}=\frac{2}{(3x+2)^2}\right)\\
&= 5\left(\frac{x}{3x+2}\right)^4\cdot\frac{2}{(3x+2)^2} && \left(\text{substituting } u=\frac{x}{3x+2}\right)\\
&= \frac{10x^4}{(3x+2)^6} && \text{(simplifying)}
\end{aligned}$$

It is important to remember that when using the Leibniz form of the Chain Rule, the final answer must be expressed entirely in terms of the original variable given in the problem.

Example 2.4.12**: Taking a Derivative Using Leibniz's Notation II**

First, let $u=4x^2-3x+1$. Then $y=\tan u$. Next, find $\frac{du}{dx}$ and $\frac{dy}{du}$:

$$\frac{du}{dx}=8x-3 \quad \text{and} \quad \frac{dy}{du}=\sec^2 u.$$

Finally, we put it all together.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx} && \text{(Chain Rule)}\\
&= \sec^2 u\cdot(8x-3) && \left(\frac{du}{dx}=8x-3 \text{ and } \frac{dy}{du}=\sec^2 u\right)\\
&= \sec^2(4x^2-3x+1)\cdot(8x-3) && \text{(substituting } u=4x^2-3x+1.)
\end{aligned}$$

Checkpoint 2.4.12

$\frac{dy}{dx}=-3x^2\sin(x^3)$.

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2.4.1: Resources and Key Concepts

Resources

Videos

- Chain Rule
  - The Chain Rule: Concept, Theorem, and Proof
  - Using the Chain Rule
  - Synthesis Questions

Key Concepts

Theorems

- **Theorem: The Chain Rule (Lagrange's Notation)**: Let f and g be functions. For all x in the domain of g for which g is differentiable at x and f is differentiable at $g(x)$, the derivative of the composite function $h(x) = (f \circ g)(x) = f(g(x))$ is given by $h'(x) = f'(g(x)) \cdot g'(x)$.
- **Theorem: Power Rule for Composition of Functions (General Power Rule)**: For all values of x for which the derivative is defined, if $h(x) = (g(x))^n$, then $h'(x) = n(g(x))^{n-1} \cdot g'(x)$.
- **Corollary: Chain Rule for a Composition of Three Functions**: If $k(x) = h(f(g(x)))$, then $k'(x) = h'(f(g(x))) \cdot f'(g(x)) \cdot g'(x)$, for all values of x for which the function is differentiable.
- **Theorem: Chain Rule (Leibniz's Notation)**: If y is a function of u, and u is a function of x, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Common Mistakes

- **Forgetting the "Inner" Derivative**: A common mistake when first learning the Chain Rule is to differentiate the "outer" function but forget to multiply by the derivative of the "inner" function. For example, incorrectly stating $\frac{d}{dx}(\sin(x^3)) = \cos(x^3)$ instead of $\cos(x^3) \cdot 3x^2$.
- **Incorrect Application to Products/Quotients**: Confusing the Chain Rule with the Product or Quotient Rule, or misapplying them in combination.
- **Leibniz Notation Final Answer**: When using the Leibniz form of the Chain Rule, the final answer must be expressed entirely in terms of the original independent variable (e.g., x), not intermediate variables (e.g., u).

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2.4.2: Homework

Note

The following exercises are meant as hand-in homework (if assigned by your instructor). Typed-in answers here will not be saved or scored.

Basic Skills

Exercise 1

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Exercise 2

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Exercise 10

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Basic Skills - Applications

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Synthesis Questions

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Proofs and Problems that Require Explanations or Logical Justifications

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Challenge Problems

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2.4.3: Additional Exercises

Reading Questions

1. State the Chain Rule using Lagrange's notation for a composite function $h(x) = f(g(x))$.
2. State the Chain Rule using Leibniz's notation if y is a function of u and u is a function of x.
3. What is the General Power Rule (Power Rule for Composition of Functions) for differentiating $h(x) = (g(x))^n$?
4. If $h(x) = \sin(x^3)$, identify the "outer" function $f(u)$ and the "inner" function $u = g(x)$. Then find $h'(x)$.
5. How is the Chain Rule applied to find the derivative of a composition of three functions, say $k(x) = h(f(g(x)))$?
6. When differentiating a function like $h(x) = \frac{1}{(3x^2+1)^2}$, what algebraic step is recommended before applying the General Power Rule?
7. In Example 7, $h(x) = (2x+1)^5(3x-2)^7$, which differentiation rule is applied first, and which rule is then applied to the terms resulting from the first rule?
8. When using Leibniz notation for the Chain Rule, like $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$, what must be done to the term $\frac{dy}{du}$ to express the final answer in terms of x?

Homework

In exercises 1 - 6, given $y = f(u)$ and $u = g(x)$, find $\frac{dy}{dx}$ by using Leibniz's notation for the chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

1) $y = 3u - 6, \quad u = 2x^2$

2) $y = 6u^3, \quad u = 7x - 4$

Answer

$$\frac{dy}{dx} = 18u^2 \cdot 7 = 18(7x-4)^2 \cdot 7 = 126(7x-4)^2$$

3) $y = \sin u, \quad u = 5x - 1$

4) $y = \cos u, \quad u = -\frac{x}{8}$

Answer

$$\frac{dy}{dx} = -\sin u \cdot \left(-\tfrac{1}{8}\right) = \tfrac{1}{8}\sin(-\tfrac{x}{8})$$

5) $y = \tan u, \quad u = 9x + 2$

6) $y = \sqrt{4u+3}, \quad u = x^2 - 6x$

Answer

$$\frac{dy}{dx} = \frac{8x-24}{2\sqrt{4u+3}} = \frac{4x-12}{\sqrt{4x^2-24x+3}}$$

For each of the following exercises,

a. decompose each function in the form $y = f(u)$ and $u = g(x)$, and

b. find $\frac{dy}{dx}$ as a function of x.

7) $y = (3x-2)^6$

8) $y = (3x^2+1)^3$

Answer

a. $f(u) = u^3, \quad u = 3x^2 + 1;$

b. $\dfrac{dy}{dx} = 18x(3x^2 + 1)^2$

9) $y = \sin^5(x)$

10) $y = \left(\dfrac{x}{7} + \dfrac{7}{x}\right)^7$

Answer

a. $f(u) = u^7, \quad u = \dfrac{x}{7} + \dfrac{7}{x};$

b. $\dfrac{dy}{dx} = 7\left(\dfrac{x}{7} + \dfrac{7}{x}\right)^6 \cdot \left(\dfrac{1}{7} - \dfrac{7}{x^2}\right)$

11) $y = \tan(\sec x)$

12) $y = \csc(\pi x + 1)$

Answer

a. $f(u) = \csc u, \quad u = \pi x + 1;$

b. $\dfrac{dy}{dx} = -\pi \csc(\pi x + 1) \cdot \cot(\pi x + 1)$

13) $y = \cot^2 x$

14) $y = -6\sin^{-3} x$

Answer

a. $f(u) = -6u^{-3}, \quad u = \sin x;$

b. $\dfrac{dy}{dx} = 18\sin^{-4} x \cdot \cos x$

In exercises 15 - 24, find $\dfrac{dy}{dx}$ for each function.

15) $y = (3x^2 + 3x - 1)^4$

16) $y = (5 - 2x)^{-2}$

Answer

$\dfrac{dy}{dx} = \dfrac{4}{(5 - 2x)^3}$

17) $y = \cos^3(\pi x)$

18) $y = (2x^3 - x^2 + 6x + 1)^3$

Answer

$\dfrac{dy}{dx} = 6(2x^3 - x^2 + 6x + 1)^2 \cdot (3x^2 - x + 3)$

19) $y = \dfrac{1}{\sin^2(x)}$

20) $y = \left(\tan x + \sin x\right)^{-3}$

Answer

$$\frac{dy}{dx} = -3\Big(\tan x + \sin x\Big)^{-4} \cdot (\sec^2 x + \cos x)$$

21) $y = x^2 \cos^4 x$

22) $y = \sin(\cos 7x)$

Answer

$$\frac{dy}{dx} = -7\cos(\cos 7x) \cdot \sin 7x$$

23) $y = \sqrt{6 + \sec \pi x^2}$

24) $y = \cot^3(4x + 1)$

Answer

$$\frac{dy}{dx} = -12\cot^2(4x + 1) \cdot \csc^2(4x + 1)$$

25) Let $y = \big[f(x)\big]^3$ and suppose that $f\,'(1) = 4$ and $\frac{dy}{dx} = 10$ for $x = 1$. Find $f(1)$.

26) Let $y = \Big(f(x) + 5x^2\Big)^4$ and suppose that $f(-1) = -4$ and $\frac{dy}{dx} = 3$ when $x = -1$. Find $f\,'(-1)$

Answer

$f\,'(-1) = 10\frac{3}{4}$

27) Let $y = (f(u) + 3x)^2$ and $u = x^3 - 2x$. If $f(4) = 6$ and $\frac{dy}{dx} = 18$ when $x = 2$, find $f\,'(4)$.

28) [Technology Required] Find the equation of the tangent line to $y = -\sin(\frac{x}{2})$ at the origin. Use a calculator to graph the function and the tangent line together.

Answer

$y = -\frac{1}{2}x$

29) [Technology Required] Find the equation of the tangent line to $y = \big(3x + \frac{1}{x}\big)^2$ at the point $(1, 16)$. Use a calculator to graph the function and the tangent line together.

30) Find the x -coordinates at which the tangent line to $y = \big(x - \frac{6}{x}\big)^8$ is horizontal.

Answer

$x = \pm\sqrt{6}$

31) [Technology Required] Find an equation of the line that is normal to $g(\theta) = \sin^2(\pi\theta)$ at the point $\big(\frac{1}{4}, \frac{1}{2}\big)$. Use a calculator to graph the function and the normal line together.

For exercises 32 - 39, use the information in the following table to find $h\,'(a)$ at the given value for a.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	5	0	2
1	1	–2	3	0
2	4	4	1	–1
3	3	–3	2	3

32) $h(x) = f\Big(g(x)\Big); \quad a = 0$

Answer

$h'(0) = 10$

33) $h(x) = g\big(f(x)\big); \quad a = 0$

34) $h(x) = \big(x^4 + g(x)\big)^{-2}; \quad a = 1$

Answer

$h'(1) = -\frac{1}{8}$

35) $h(x) = \left(\dfrac{f(x)}{g(x)}\right)^2; \quad a = 3$

36) $h(x) = f\big(x + f(x)\big); \quad a = 1$

Answer

$h'(1) = -4$

37) $h(x) = \big(1 + g(x)\big)^3; \quad a = 2$

38) $h(x) = g\big(2 + f(x^2)\big); \quad a = 1$

Answer

$h'(1) = -12$

39) $h(x) = f\big(g(\sin x)\big); \quad a = 0$

40) [Technology Required] The position function of a freight train is given by $s(t) = 100(t+1)^{-2}$, with s in meters and t in seconds. At time $t = 6$ s, find the train's

a. velocity and

b. acceleration.

c. Considering your results in parts a. and b., is the train speeding up or slowing down?

Answer

a. $v(6) = -\frac{200}{343}$ m/s,

b. $a(6) = \frac{600}{2401}$ $\mathrm{m/s^2}$,

c. The train is slowing down since velocity and acceleration have opposite signs.

41) [Technology Required] A mass hanging from a vertical spring is in simple harmonic motion as given by the following position function, where t is measured in seconds and s is in inches:

$$s(t) = -3\cos\left(\pi t + \frac{\pi}{4}\right).$$

a. Determine the position of the spring at $t = 1.5$ s.

b. Find the velocity of the spring at $t = 1.5$ s.

42) [Technology Required] The total cost to produce x boxes of Thin Mint Girl Scout cookies is C dollars, where $C = 0.0001x^3 - 0.02x^2 + 3x + 300$. In t weeks production is estimated to be $x = 1600 + 100t$ boxes.

a. Find the marginal cost $C\,'(x)$.

b. Use Leibniz's notation for the chain rule, $\dfrac{dC}{dt} = \dfrac{dC}{dx} \cdot \dfrac{dx}{dt}$, to find the rate with respect to time t that the cost is changing.

c. Use your result in part b. to determine how fast costs are increasing when $t = 2$ weeks. Include units with the answer.

Answer

a. $C'(x) = 0.0003x^2 - 0.04x + 3$

b. $\frac{dC}{dt} = 100 \cdot (0.0003x^2 - 0.04x + 3) = 100 \cdot (0.0003(1600 + 100t)^2 - 0.04(1600 + 100t) + 3) = 300t^2 + 9200t + 70700$

c. Approximately $90,300 per week

43) [Technology Required] The formula for the area of a circle is $A = \pi r^2$, where r is the radius of the circle. Suppose a circle is expanding, meaning that both the area A and the radius r (in inches) are expanding.

a. Suppose $r = 2 - \frac{100}{(t+7)^2}$ where t is time in seconds. Use the chain rule $\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$ to find the rate at which the area is expanding.

b. Use your result in part a. to find the rate at which the area is expanding at $t = 4$ s.

44) [Technology Required] The formula for the volume of a sphere is $S = \frac{4}{3}\pi r^3$, where r (in feet) is the radius of the sphere. Suppose a spherical snowball is melting in the sun.

a. Suppose $r = \frac{1}{(t+1)^2} - \frac{1}{12}$ where t is time in minutes. Use the chain rule $\frac{dS}{dt} = \frac{dS}{dr} \cdot \frac{dr}{dt}$ to find the rate at which the snowball is melting.

b. Use your result in part a. to find the rate at which the volume is changing at $t = 1$ min.

Answer

a. $\frac{dS}{dt} = -\frac{8\pi r^2}{(t+1)^3} = -\frac{8\pi\left(\frac{1}{(t+1)^2} - \frac{1}{12}\right)^2}{(t+1)^3}$

b. The volume is decreasing at a rate of $-\frac{\pi}{36}$ ft^3/min

45) [Technology Required] The daily temperature in degrees Fahrenheit of Phoenix in the summer can be modeled by the function $T(x) = 94 - 10\cos\left[\frac{\pi}{12}(x-2)\right]$, where x is hours after midnight. Find the rate at which the temperature is changing at 4 p.m.

46) [Technology Required] The depth (in feet) of water at a dock changes with the rise and fall of tides. The depth is modeled by the function $D(t) = 5\sin\left(\frac{\pi}{6}t - \frac{7\pi}{6}\right) + 8$, where t is the number of hours after midnight. Find the rate at which the depth is changing at 6 a.m.

Answer

2.3 ft/hr

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2.5: An Interlude - Hyperbolic Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Figure 2.5.1**:** *A.I.-generated image of hanging cables forming a curve called a catenary. These curves are modeled using a new (to you) family of functions called the hyperbolic functions.*

Unlike all the functions encountered in Calculus up to this point, the functions in this section should be unfamiliar to you. Here we define hyperbolic and inverse hyperbolic functions, which involve combinations of exponential and logarithmic functions. These provide a unique bridge between two groups of transcendental functions—exponential and trigonometric.

From Circular to Hyperbolic Functions

Before introducing the hyperbolic functions, it is worthwhile to review a particular feature of the trigonometric functions.

Most people refer to the sine, cosine, tangent, and their reciprocals as trigonometric functions; however, they are also known as **circular functions**. This is because their entire definition is based on the unit circle. Recall that the unit circle is defined by the relation

$$x^2 + y^2 = 1.$$

The graph of this relation can be seen in Figure 2.5.2 below.

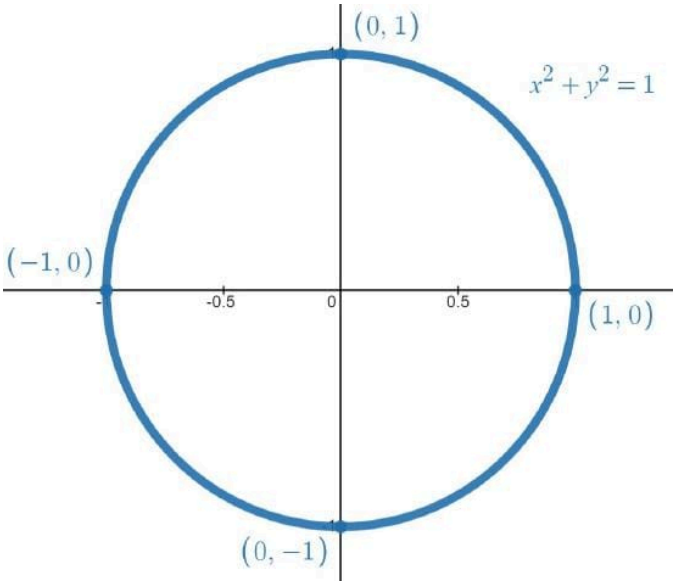

Figure 2.5.2**:** *The unit circle.*

During our journey through Trigonometry, we discovered that the length of the arc on the unit circle subtended by an angle θ (in radians) is $s = \theta$. Moreover, we derived the area of a sector formula to be

$$A = \frac{1}{2}r^2\theta.$$

Since the radius of the unit circle is $r = 1$ and $\theta = s$, we arrived at the fact that the area seen in Figure 2.5.3 is

$$A = \frac{s}{2}.$$

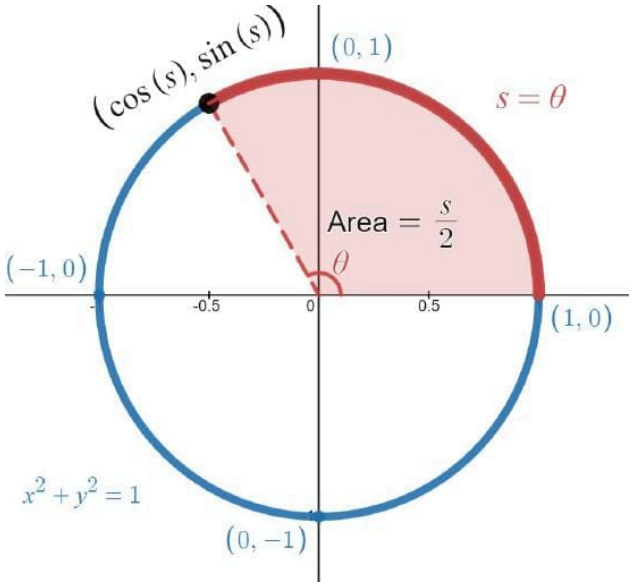

Figure 2.5.3**:** *The area of the sector of the unit circle having arc length s is $\frac{s}{2}$.*

Stated clearly, the area of the region bounded by the positive x-axis, the unit circle, and the line segment connecting the origin to the point $(\cos(s), \sin(s))$ is $s/2$.

In this section, we ask the following question:

> Could we develop a set of functions, let's call them $\cosh(s)$ and $\sinh(s)$, such that the area of the region bounded by

the positive x-axis, the unit hyperbola, and the line segment connecting the origin to the point $(\cosh(s), \sinh(s))$ is $s/2$?

Before we can answer this question, we need to define the unit hyperbola. For our purposes, the **unit hyperbola** will be defined by the relation

$$x^2 - y^2 = 1.$$

Furthermore, we restrict our work to the right branch of this hyperbola (see Figure 2.5.4).

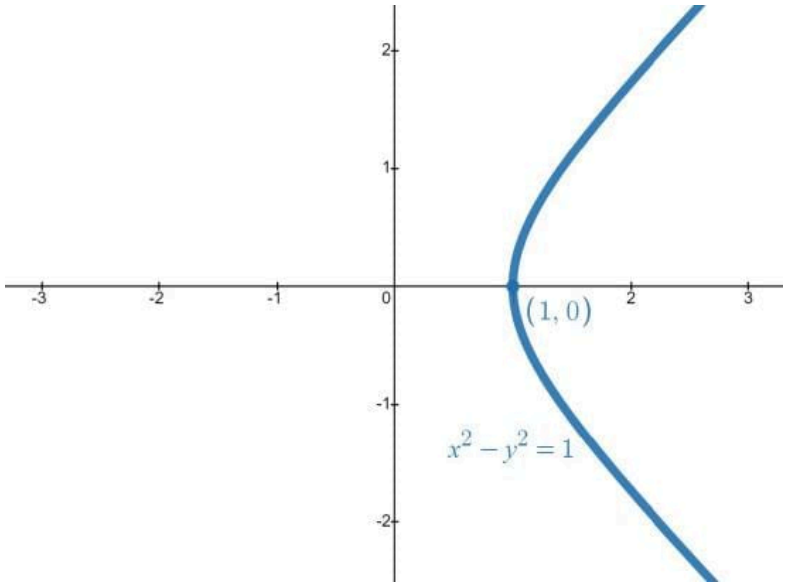

Figure 2.5.4: *The unit hyperbola (right branch only).*

We plot a point $(x, y) = (\cosh(s), \sinh(s))$ on this branch, highlight the arc length along the unit hyperbola to this point, draw a line segment connecting the origin to this point, and shade the bounded region (see Figure 2.5.5).

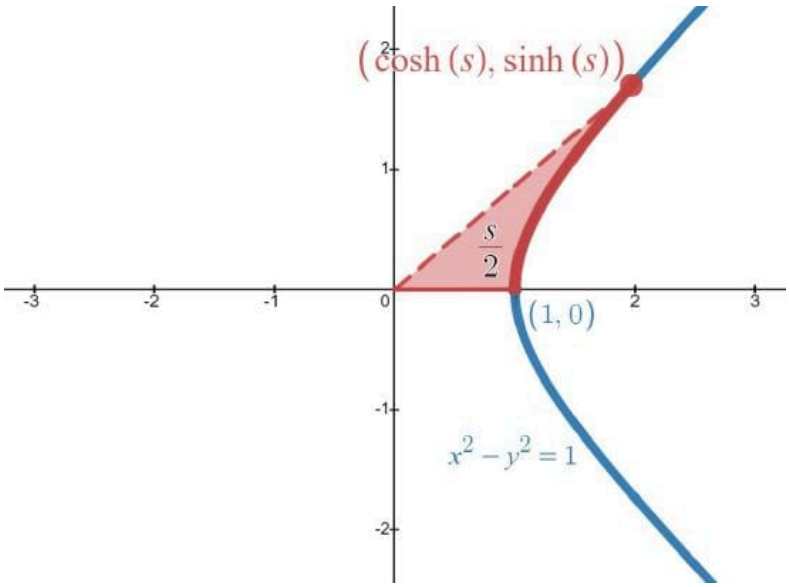

Figure 2.5.5: *We desire* $\cosh(s)$ *and* $\sinh(s)$ *so that the shaded region is* $s/2$.

We want to define these new functions, $\cosh(s)$ and $\sinh(s)$, so that the shaded region is $s/2$.

Caution: The Argument of a Hyperbolic Function is Not an Angle, Nor an Arc Length

It is *critical* to point out here that the highlighted arc length will *not* be s. In fact, the angle between the positive x-axis and the line segment connecting the origin to the point $(\cosh(s), \sinh(s))$ will also *not* be s. Our goal with this discussion is *not* to find the arc length *nor* to find the angle, but instead to create functions, $\cosh(s)$ and $\sinh(s)$, so that the area of the bounded region is $s/2$.

As you can see in Figure 2.5.5, we define the **hyperbolic cosine** to be the x-value of this terminal point, and the **hyperbolic sine** as the y-value. We denote these functions as $\cosh(s)$ and $\sinh(s)$, respectively.

Interactive Element: The Hyperbolic Sector — Defining $\cosh$ and $\sinh$ by Area

The following Interactive Element might help you visualize how $\cosh(t)$ and $\sinh(t)$ are defined by area on the unit hyperbola — the exact hyperbolic counterpart of the unit-circle sector reviewed above.

Interact: Drag the large point P; it rides along the right branch of the unit hyperbola $x^2 - y^2 = 1$.

Observation: Wherever P sits, the shaded region's area is exactly $t/2$, and the coordinates of P are exactly $(\cosh(t), \sinh(t))$. Notice it is the *area* — not the arc length and not the angle — that equals $t/2$.

Hyperbolic Functions

The previous discussion considered the hyperbolic cosine and hyperbolic sine as functions of s; however, most textbooks work exclusively with these as functions of t. Since s is a "dummy variable," we can replace it easily with t for the remainder of this section.

It turns out (and can be proven in Calculus II) that the hyperbolic cosine and the hyperbolic sine can be written in terms of specific combinations of e^t and e^{-t}. These functions arise naturally in various engineering and physics applications, including the study of water waves and vibrations of elastic membranes. Another common use, at least for the hyperbolic cosine, is the representation of a hanging chain or cable, also known as a **catenary** (Figure 2.5.6). If we introduce a coordinate system so that the low point of the chain lies along the y-axis, we can describe the height of the chain in terms of the hyperbolic cosine.

Figure 2.5.6: *The shape of a strand of silk in a spider's web can be described in terms of a hyperbolic function. The same shape applies to a chain or cable hanging from two supports with only its own weight. (credit: "Mtpaley", Wikimedia Commons)*

Theorem: Hyperbolic Functions

Let $\frac{t}{2}$ be the area of the region bounded by the arc along the right branch of the unit hyperbola $x^2 - y^2 = 1$ whose initial point is $(1, 0)$ and terminal point is (x, y), the x-axis, and the line segment connecting the origin to (x, y). Then, the **hyperbolic functions** in terms of twice this bounded area, are defined (and derived) as follows:[1]

Hyperbolic cosine

$$x = \cosh(t) = \frac{e^t + e^{-t}}{2}$$

Hyperbolic sine

$$y = \sinh(t) = \frac{e^t - e^{-t}}{2}$$

Hyperbolic tangent

$$\tanh(t) = \frac{\sinh(t)}{\cosh(t)} = \frac{e^t - e^{-t}}{e^t + e^{-t}}$$

Hyperbolic cosecant

$$\operatorname{csch}(t) = \frac{1}{\sinh(t)} = \frac{2}{e^t - e^{-t}}$$

Hyperbolic secant

$$\operatorname{sech}(t) = \frac{1}{\cosh(t)} = \frac{2}{e^t + e^{-t}}$$

Hyperbolic cotangent

$$\coth(t) = \frac{\cosh(t)}{\sinh(t)} = \frac{e^t + e^{-t}}{e^t - e^{-t}}$$

Despite writing this as a theorem, we have not *proven* that the hyperbolic functions are equal to these exponential forms. The reality is that you need Integral Calculus (Calculus II) to prove these relationships.

The name $\cosh$ rhymes with "gosh," whereas the name $\sinh$ is pronounced "sinch." $\tanh$, sech, csch, and $\coth$ are pronounced "tanch," "seech," "coseech," and "cotanch," respectively.

Example 2.5.1

Simplify $\sinh(5\ln(x))$.

Solution available at the end of this section.

Checkpoint 2.5.1

Simplify $\cosh(2\ln x)$.

Hint available at the end of this section.

Graphs of Hyperbolic Functions

To investigate the graphs of the hyperbolic functions, we need to start with a discussion related to Calculus—what happens to e^{-x} as $x \to \infty$? Drawing on our work with limits, we know that $e^{-x} \to 0$ as $x \to \infty$ and $e^x \to 0$ as $x \to -\infty$. Therefore,

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \to \frac{e^x}{2}, \text{ as } x \to \infty, \text{ and } \cosh(x) = \frac{e^x + e^{-x}}{2} \to \frac{e^x}{2}, \text{ as } x \to \infty.$$

We purposefully did not write these using true limit notation because we are trying to discover the end behavior of the graphs of these functions. As similar analysis shows that,

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \to \frac{-e^{-x}}{2}, \text{ as } x \to -\infty, \text{ and } \cosh(x) = \frac{e^x + e^{-x}}{2} \to \frac{e^{-x}}{2}, \text{ as } x \to -\infty.$$

Hence, using the graphs of $\frac{1}{2}e^x$, $\frac{1}{2}e^{-x}$, and $-\frac{1}{2}e^{-x}$ as guides, we can graph $\cosh(x)$ and $\sinh(x)$ (see Figure 2.5.7 below).

To graph $\tanh(x)$, we recognize the fact that $\tanh(0) = 0$. Moreover,

$$\begin{aligned}
\lim_{x\to\infty} \tanh(x) &= \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \\
&= \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \frac{e^{-x}}{e^{-x}} && \left(\text{multiplying numerator and denominator by } \frac{e^{-x}}{e^{-x}}\right) \\
&= \lim_{x\to\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} && \text{(simplifying)} \\
&= \frac{1 - \lim\limits_{x\to\infty} e^{-2x}}{1 + \lim\limits_{x\to\infty} e^{-2x}} && \text{(Limit Laws)} \\
&= \frac{1-0}{1+0} && \left(\lim_{x\to\infty} e^{-x} = 0\right) \\
&= 1
\end{aligned}$$

A similar argument shows $\lim\limits_{x\to-\infty} \tanh(x) = -1$.

The graphs of the other three hyperbolic functions can be sketched using the graphs of $\cosh(x)$, $\sinh(x)$, and $\tanh(x)$ (Figure 2.5.7).

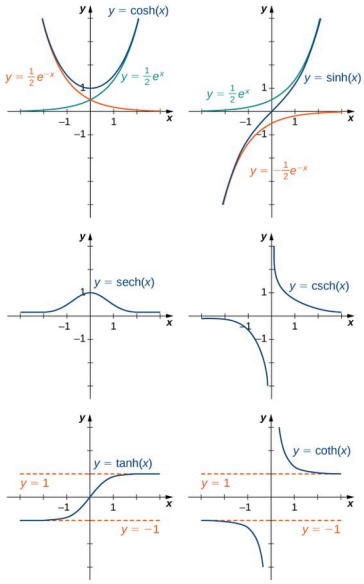

Figure 2.5.7**:** *The hyperbolic functions involve combinations of e^x and e^{-x}.*

Interactive Element: Building $\cosh$ and $\sinh$ from Exponential Guide Curves

The following Interactive Element might help you visualize how $\cosh(x)$ and $\sinh(x)$ are assembled from $\frac{1}{2}e^x$ and $\pm\frac{1}{2}e^{-x}$, and why both curves merge into $\frac{1}{2}e^x$ as x grows.

Interact: Drag the red point left and right along the x-axis.

Observation: At every x, $\cosh(x) = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$ sits the same small distance $\frac{1}{2}e^{-x}$ above the guide $\frac{1}{2}e^x$ that $\sinh(x) = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$ sits below it; as $x \to \infty$ that distance shrinks to 0, so both curves squeeze onto $\frac{1}{2}e^x$.

Hyperbolic Identities

Once you grasp that the hyperbolic functions are based on the unit hyperbola, $x^2 - y^2 = 1$, you immediately arrive at the first of many hyperbolic identities.

Theorem: The Fundamental Hyperbolic Identity

$$\cosh^2(t) - \sinh^2(t) = 1$$

Proof available at the end of this section.

The Fundamental Hyperbolic Identity is one of many identities involving the hyperbolic functions, *some* of which are listed next. The first four properties follow quickly from the definitions of hyperbolic sine and hyperbolic cosine. Except for some differences in signs, most of these properties are analogous to identities for trigonometric functions.

Theorem: Basic Hyperbolic Identities

- $\cosh(-t) = \cosh(t)$
- $\sinh(-t) = -\sinh(t)$
- $\cosh(t) + \sinh(t) = e^t$
- $\cosh(t) - \sinh(t) = e^{-t}$
- $1 - \tanh^2(t) = \operatorname{sech}^2(t)$
- $\coth^2(t) - 1 = \operatorname{csch}^2(t)$
- $\sinh(t \pm v) = \sinh(t)\cosh(v) \pm \cosh(t)\sinh(v)$
- $\cosh(t \pm v) = \cosh(t)\cosh(v) \pm \sinh(t)\sinh(v)$

Example 2.5.2

If $\sinh(t) = 3/4$, find the values of the remaining five hyperbolic functions.

Solution available at the end of this section.

Example 2.5.3

Without using $\cosh(t \pm v) = \cosh(t)\cosh(v) \pm \sinh(t)\sinh(v)$, prove

$$\cosh(2t) = \sinh^2(t) + \cosh^2(t).$$

Solution available at the end of this section.

Inverse Hyperbolic Functions

From the graphs of the hyperbolic functions, we see that all of them are one-to-one except $\cosh x$ and $\operatorname{sech} x$. If we restrict the domains of these two functions to the interval $[0, \infty)$, then all the hyperbolic functions are one-to-one, and we can define the **inverse hyperbolic functions**. Since the hyperbolic functions themselves involve exponential functions, it should make sense to the reader that the inverse hyperbolic functions involve logarithmic functions.

Theorem: Inverse Hyperbolic Functions

$$\sinh^{-1} x = \operatorname{arcsinh} x = \ln\left(x + \sqrt{x^2+1}\right) \qquad \cosh^{-1} x = \operatorname{arccosh} x = \ln\left(x + \sqrt{x^2-1}\right)$$

$$\tanh^{-1} x = \operatorname{arctanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad \coth^{-1} x = \operatorname{arccot} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$$

$$\operatorname{sech}^{-1} x = \operatorname{arcsech} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) \qquad \operatorname{csch}^{-1} x = \operatorname{arccsch} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)$$

Proof that $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$ *available at the end of this section.*

Caution

The independent variable x in the previous theorem is a *dummy variable*. It is *not* the x-coordinate of the terminal point of the arc of length t along the unit hyperbola.

Example 2.5.4

Evaluate each of the following expressions.

$$\sinh^{-1}(2) \quad \text{and} \quad \tanh^{-1}(1/4)$$

Solution available at the end of this section.

Checkpoint 2.5.4

Evaluate $\tanh^{-1}(1/2)$.

Hint available at the end of this section.

Success in Mathematics: Mathematics in Other Courses

In other courses, we use mathematics to understand and explore the big ideas of the subject. Professors may use different vocabulary, variables, and notations than you learned in your math classes. The mathematics is the same. The way of talking about it may be different.

Also, some courses will use formulas based on mathematics that you have not yet learned. You may not be ready to understand why these formulas work. Sometimes you have to accept that the formula is correct.

Making connections between different courses takes work. It would be best if you were willing to ask questions. You may need to ask your math instructor about topics in your other classes. You may need to ask your other professors why they use different words or notations. It is your responsibility to make these connections. Your ability to do critical thinking in other classes depends on it.

1. There are many formulas that we hear about in daily news or history. Of all these famous formulas, name one.
2. Can you describe the use of the formula you mentioned and what each variable stands for?

Hidden Resources: CRC Hub

As a college student, you have access to so many resources that it can be overwhelming. Luckily, the CRC Hub is your "one-stop

shop" for all local resources covering various topics (from healthcare to legal fees). Check out the CRC Hub here.

Answers

Section Preview

This definitely will be the first time students have seen hyperbolic functions. Do not skip this material as it appears throughout Calculus (and Differential Equations) from this point forward!

Section Preview

- **An Overview of Functions**
  - Inverse Functions: The "Inverse Hyperbolic Functions" portion of the section depends entirely on understanding what an inverse function is and the one-to-one requirement; students cannot make sense of $\sinh^{-1}$ or $\tanh^{-1}$ without this idea.
  - Restricting Domains and Forcing Invertibility: The text explicitly restricts $\cosh x$ and $\operatorname{sech} x$ to $[0, \infty)$ so they become one-to-one; recognizing why a domain restriction is needed to define an inverse is exactly this concept.
- **Exponential Functions**
  - Graphing Exponential Functions: The graphs of $\cosh x$ and $\sinh x$ are built directly from the graphs of $\frac{1}{2}e^x$, $\frac{1}{2}e^{-x}$, and $-\frac{1}{2}e^{-x}$ used as guide curves, which the "identify their graphs" objective requires students to recognize.
- **Trigonometry**
  - Sector Area: The geometric definition of the hyperbolic functions hinges on the bounded region having area $s/2$, which mirrors the unit-circle sector-area result $A = \frac{1}{2}r^2\theta$; the sector-area concept is the conceptual hinge of that derivation.

Section Preview

- **Arithmetic**
  - Adding and Subtracting Fractions: Example 2 computes $1 + \frac{9}{16} = \frac{25}{16}$; Example 4 forms $1 + \frac{1}{4} = \frac{5}{4}$ and $1 - \frac{1}{4} = \frac{3}{4}$ inside the $\tanh^{-1}$ evaluation.
- **Logarithmic Functions**
  - Laws of Logarithms — Power Law: Example 1 and Checkpoint 1 rewrite $5\ln x = \ln(x^5)$ and $2\ln x = \ln(x^2)$ before applying the inverse property.

Section Preview

- Identify the hyperbolic functions, their graphs, and basic identities.

Example 2.5.1

Using the definition of the $\sinh$ function, we get

$$\begin{aligned}
\sinh(5\ln(x)) &= \frac{e^{5\ln(x)} - e^{-5\ln(x)}}{2} && \text{(definition of Hyperbolic Functions)} \\
&= \frac{e^{\ln(x^5)} - e^{\ln(x^{-5})}}{2} && \text{(Laws of Logarithms)} \\
&= \frac{x^5 - x^{-5}}{2} && \text{(inverse property ofexponentialand logarithmic functions)}
\end{aligned}$$

Checkpoint 2.5.1

Use the definition of the $\cosh$ function and the power property of logarithm functions.

$(x^2 + x^{-2})/2$

Theorem: The Fundamental Hyperbolic Identity

Since $x = \cosh(t)$ and $y = \sinh(t)$ on the unit hyperbola $x^2 - y^2 = 1$, we substitute to arrive at $\cosh^2(t) - \sinh^2(t) = 1$.

It is also important to note that this can be proved using the exponential form of the hyperbolic functions.

Example 2.5.2

Using the identity $\cosh^2(t) - \sinh^2(t) = 1$,we see that

$$\cosh^2(t) = 1 + \left(\frac{3}{4}\right)^2 = \frac{25}{16}.$$

Since $\cosh x \geq 1$ for all x, we must have $\cosh x = 5/4$. Then, using the definitions for the other hyperbolic functions, we conclude that $\tanh x = 3/5, \operatorname{csch} x = 4/3, \operatorname{sech} x = 4/5$, and $\coth x = 5/3$.

Example 2.5.3

$$\begin{aligned}
\cosh^2(t) + \sinh^2(t) &= \left(\frac{e^t + e^{-t}}{2}\right)^2 + \left(\frac{e^t - e^{-t}}{2}\right)^2 && \text{(exponential form of hyperbolic functions)} \\
&= \frac{e^{2t} + 2 + e^{-2t}}{4} + \frac{e^{2t} - 2 + e^{-2t}}{4} && \text{(distributing andLaws of Exponents)} \\
&= \frac{2e^{2t} + 2e^{-2t}}{4} && \text{(combining like terms)} \\
&= \frac{\cancel{2}\left(e^{2t} + e^{-2t}\right)}{\cancel{2} \cdot 2} && \text{(factoring out the GCFand canceling like factors)} \\
&= \frac{e^{2t} + e^{-2t}}{2} && \text{(canceling like factors)} \\
&= \cosh(2t) && \text{(exponential form of hyperbolic functions)}
\end{aligned}$$

Theorem: Inverse Hyperbolic Functions

Suppose $y = \sinh^{-1} x$. Then, $x = \sinh y$ and, by the definition of the hyperbolic sine function, $x = \frac{e^y - e^{-y}}{2}$. Therefore,

$$e^y - 2x - e^{-y} = 0.$$

Multiplying this equation by e^y, we obtain

$$e^{2y} - 2xe^y - 1 = 0.$$

This can be solved like a quadratic equation, with the solution

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}.$$

Since $e^y > 0$,the only solution is the one with the positive sign. Applying the natural logarithm to both sides of the equation, we conclude that

$$y = \ln(x + \sqrt{x^2 + 1}).$$

Q.E.D.

Example 2.5.4

$$\sinh^{-1}(2) = \ln(2 + \sqrt{2^2 + 1}) = \ln(2 + \sqrt{5}) \approx 1.4436$$

$$\tanh^{-1}(1/4) = \frac{1}{2}\ln\left(\frac{1 + 1/4}{1 - 1/4}\right) = \frac{1}{2}\ln\left(\frac{5/4}{3/4}\right) = \frac{1}{2}\ln\left(\frac{5}{3}\right) \approx 0.2554$$

Checkpoint 2.5.4

Use the definition of $\tanh^{-1} x$ and simplify.

$\frac{1}{2}\ln(3) \approx 0.5493$.

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2.5.1: Resources and Key Concepts

Resources

Prerequisite Topics

The following is a list of prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Solving Equations**
  - Solving Exponential Equations: Implicit in finding inverse hyperbolic functions, as it involves solving for y in expressions like $x = \frac{e^y - e^{-y}}{2}$.
  - Solving Logarithmic Equations: Inverse hyperbolic functions are defined in terms of natural logarithms.
- **Exponential Functions**
  - Graphing Exponential Functions: Understanding the graphs of e^x and e^{-x} helps in visualizing the graphs of hyperbolic functions.
- **Logarithmic Functions**
  - Laws of Logarithms: Used in simplifying expressions involving hyperbolic functions when written in exponential form, particularly after applying logarithms (Example 1).
- **Trigonometry**
  - Proving Trigonometric Identities: The same skills you used to prove trigonometric identities in your Trigonometry course are required in this section.
  - Sector Area: The area of the sector subtended by the angle θ in a unit circle is our motivation to defining something similar to trigonometric functions when dealing with the unit hyperbola.

Videos

- Hyperbolic functions
  - An Introduction to Hyperbolic Functions
- Unit hyperbola
  - A Recreational Interlude: Euler's Formula and the Unit Circle
- Hyperbolic identities
  - Hyperbolic Identities
- Inverse hyperbolic functions
  - Inverse Hyperbolic Functions
- Logarithmic form of inverse hyperbolic functions
  - Proofs in Differential Calculus - $\operatorname{arcosh}(y) = \ln(y + \sqrt{y^2 - 1})$

Key Concepts

Definitions

- **Circular Functions**: An alternative name for trigonometric functions, emphasizing their definition based on the unit circle.
- **Unit Hyperbola**: For the purposes of this material, the right branch of the hyperbola defined by the relation $x^2 - y^2 = 1$.
- **Hyperbolic Cosine (geometric interpretation)**: If $s/2$ is the area of the region bounded by the positive x-axis, the unit hyperbola, and the line segment connecting the origin to the point (x, y) on the hyperbola, then $\cosh(s)$ is the x-value of this terminal point.
- **Hyperbolic Sine (geometric interpretation)**: If $s/2$ is the area of the region bounded by the positive x-axis, the unit hyperbola, and the line segment connecting the origin to the point (x, y) on the hyperbola, then $\sinh(s)$ is the y-value of this terminal point.
- **Hyperbolic Functions (Exponential Definitions)**:

- **Hyperbolic Cosine**: $\cosh(t) = \frac{e^t + e^{-t}}{2}$
- **Hyperbolic Sine**: $\sinh(t) = \frac{e^t - e^{-t}}{2}$
- **Hyperbolic Tangent**: $\tanh(t) = \frac{\sinh(t)}{\cosh(t)} = \frac{e^t - e^{-t}}{e^t + e^{-t}}$
- **Hyperbolic Cosecant**: $\operatorname{csch}(t) = \frac{1}{\sinh(t)} = \frac{2}{e^t - e^{-t}}$
- **Hyperbolic Secant**: $\operatorname{sech}(t) = \frac{1}{\cosh(t)} = \frac{2}{e^t + e^{-t}}$
- **Hyperbolic Cotangent**: $\coth(t) = \frac{\cosh(t)}{\sinh(t)} = \frac{e^t + e^{-t}}{e^t - e^{-t}}$

- **Catenary**: The shape of a hanging chain or cable, which can be represented by the hyperbolic cosine function.

Theorems

- **Theorem: Hyperbolic Functions (Exponential Forms)**: Lists the exponential definitions for $\cosh(t)$, $\sinh(t)$, $\tanh(t)$, $\operatorname{csch}(t)$, $\operatorname{sech}(t)$, and $\coth(t)$.
- **Theorem: The Fundamental Hyperbolic Identity**: $\cosh^2(t) - \sinh^2(t) = 1$.
- **Theorem: Basic Hyperbolic Identities**:
  1. $\cosh(-t) = \cosh(t)$
  2. $\sinh(-t) = -\sinh(t)$
  3. $\cosh(t) + \sinh(t) = e^t$
  4. $\cosh(t) - \sinh(t) = e^{-t}$
  5. $1 - \tanh^2(t) = \operatorname{sech}^2(t)$
  6. $\coth^2(t) - 1 = \operatorname{csch}^2(t)$
  7. $\sinh(t \pm v) = \sinh(t)\cosh(v) \pm \cosh(t)\sinh(v)$
  8. $\cosh(t \pm v) = \cosh(t)\cosh(v) \pm \sinh(t)\sinh(v)$
- **Inverse Hyperbolic Functions (Logarithmic Forms)**:
  - $\sinh^{-1}x = \operatorname{arsinh}\, x = \ln(x + \sqrt{x^2 + 1})$
  - $\cosh^{-1}x = \operatorname{arccosh}\, x = \ln(x + \sqrt{x^2 - 1})$ for $x \geq 1$
  - $\tanh^{-1}x = \operatorname{artanh}\, x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ for $|x| < 1$
  - $\coth^{-1}x = \operatorname{arccoth}\, x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$ for $|x| > 1$
  - $\operatorname{sech}^{-1}x = \operatorname{arcsech}\, x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$ for $0 < x \leq 1$
  - $\operatorname{csch}^{-1}x = \operatorname{arccsch}\, x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)$ for $x \neq 0$

Common Mistakes

- **The Argument of a Hyperbolic Function is Not an Angle, Nor an Arc Length**: It is critical to point out that the argument s in $\cosh(s)$ or $\sinh(s)$ (when related to the unit hyperbola area interpretation) is not an angle or a direct arc length along the hyperbola in the same way θ is for circular functions. The argument s is such that $s/2$ is an area.
- **Independent Variable as a Dummy Variable**: The independent variable x in the theorem for Inverse Hyperbolic Functions is a dummy variable. It is not the x-coordinate of the terminal point of the arc of length t along the unit hyperbola.

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2.5.2: Homework

Note

The following exercises are meant as hand-in homework (if assigned by your instructor). Typed-in answers here will not be saved or scored.

Basic Skills

Exercise 1

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Exercise 2

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Exercise 3

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Exercise 4

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Synthesis Questions

Exercise 5

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Exercise 6

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Exercise 7

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Proofs and Problems that Require Explanations or Logical Justifications

Exercise 8

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Exercise 9

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Exercise 10

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Exercise 11

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Exercise 12

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2.5.3: Additional Exercises

Reading Questions

1. What is the algebraic relation that defines the unit circle? What analogous relation defines the unit hyperbola (right branch)?
2. How are $\cosh(t)$ and $\sinh(t)$ defined in terms of exponential functions?
3. What is the Fundamental Hyperbolic Identity, and how does it relate to the Pythagorean identity for trigonometric functions?
4. Express $\tanh(t)$ in terms of $\sinh(t)$ and $\cosh(t)$, and then in terms of exponential functions.
5. Which hyperbolic function represents the shape of a catenary (a hanging cable)?
6. How does the graph of $\cosh(x)$ differ from $\sinh(x)$ in terms of symmetry and y-intercept?
7. What are the horizontal asymptotes of $y = \tanh(x)$?
8. Are all hyperbolic functions one-to-one? If not, which ones require domain restrictions to define their inverses, and what is a common restriction?
9. How is $\sinh^{-1}x$ defined in terms of a natural logarithm?
10. How is $\cosh^{-1}x$ defined in terms of a natural logarithm? What is the domain for this inverse function?
11. Is $\cosh(x)$ an even or odd function? How about $\sinh(x)$?

Homework

Evaluations

1. Evaluate the given function. Round your answers to 3 decimal places, if necessary.
   a. $\coth(-7)$
   b. $\operatorname{csch}(\ln(3))$
   c. $\cosh(3)$
   d. $\sinh(0)$
   e. $\tanh(1)$
   f. $\operatorname{sech}(\ln(7))$
   g. $\sinh^{-1}(3)$
   h. $\cosh^{-1}\left(\frac{3}{2}\right)$
   i. $\tanh^{-1}\left(\frac{1}{2}\right)$

Simplifications

2. Rewrite the following expressions in terms of exponentials and simplify.
   a. $2\cosh(\ln x)$
   b. $\cosh 4x + \sinh 4x$
   c. $\cosh 2x - \sinh 2x$
   d. $\ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x)$

Proofs

3. Prove the identity. Your work should be legible, and all your logic should be clear and justified.
$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$$
4. Prove the identity. Your work should be legible, and all your logic should be clear and justified.
$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$$
5. Prove the identity. Your work should be legible, and all your logic should be clear and justified.
$$\cosh(2x) = \cosh^2(x) + \sinh^2(x)$$
6. Prove the identity. Your work should be legible, and all your logic should be clear and justified.

$$\sinh(2x) = 2\sinh(x)\cosh(x)$$

7. Prove that $f(x) = \cosh(x)$ is an even function.
8. Prove that $f(x) = \sinh(x)$ is an odd function.
9. Prove the identitiy. Your work should be legible, and all your logic should be clear and justified.

$$\cosh(x) + \sinh(x))^n = \cosh(nx) + \sinh(nx)$$

10. Prove the expression for $\sinh^{-1}(x)$. Multiply $x = \sinh(y) = \dfrac{e^y - e^{-y}}{2}$ by $2e^y$ and solve for y. Does your expression match the textbook?
11. Prove the expression for $\cosh^{-1}(x)$. Multiply $x = \cosh(y) = \dfrac{e^y + e^{-y}}{2}$ by $2e^y$ and solve for y. Does your expression match the textbook?

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2.6: Derivatives of Exponential and Hyperbolic Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

So far, we have learned how to differentiate various functions, including polynomial, radical, rational, and trigonometric functions. In this section, we explore derivatives of exponential and hyperbolic functions.

Discovering e

In Calculus (and often in Precalculus or College Algebra), a common "beginning" limit to investigate numerically is

$$\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k. \tag{2.6.1}$$

Table 2.6.1 shows values of this limit for large values of k.

***Table** 2.6.1**:** Investigating a Famous Limit*

k	$\left(1+\frac{1}{k}\right)^k$
10	2.593742460**1**...
100	2.70**4**81382942153...
1000	2.71692393223552...
10000	2.71814592682436...
100000	2.71826823719753...
1000000	2.718280**4**6915643...
10000000	2.71828169398**0**37...
100000000	2.71828178639580...
1000000000	2.718282**0**3081451...

It *appears* this limit is approaching some value around $2.71828\ldots$. This exceptional value is a non-terminating, non-repeating number. That is, it is an irrational number—just like π. If we could assign importance to the value of this limit, it would be tied with the importance of the numbers 0, 1, π, and $i=\sqrt{-1}$.

This limit comes up so often in the sciences that we give it a unique name. Rather than calling it "the limit that approaches $2.718281828459045\ldots$," we call it e. As was mentioned, it *appears* that this limit approaches a constant; however, the proof of this fact is reserved for later in the course. For now, it's important to have the limit, 2.6.1, in hand for use.

An important alternative form of 2.6.1 is found by letting $n=\frac{1}{k}$. As $k\to\infty$, $n\to 0$ and we get the form

$$\lim_{n\to 0}(1+n)^{1/n}. \tag{2.6.2}$$

Let's formalize this definition before we move on.

Definition: e

The limit

$$\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = \lim_{n\to 0}(1+n)^{1/n}$$

is defined to be the irrational number e.

We will be using this unique number shortly; however, we need to discuss some Calculus before we do.

Derivatives Exponential Functions

Just as when we found the derivatives of other functions, we can find the derivatives of exponential functions using formulas. Recall that all non-transformed exponential functions have the basic form $B(x) = b^x$, where $b > 0$ and $b \neq 1$. Our initial goal is to prove that exponential functions are continuous everywhere. Once we have this, the proof of the derivative of an exponential function becomes trivial.

Proving that Exponential Functions are Continuous Everywhere

Believe it or not, we have had a long history with exponential functions. We began in Arithmetic with the definition of b^n, where $n \in \mathbb{N}$, as the product of b multiplied by itself n times. Later, in Algebra, we defined $b^0 = 1$, $b^{-n} = \frac{1}{b^n}$ for $n \in \mathbb{N}$, and $b^{s/t} = (\sqrt[t]{b})^s$ for positive integers s and t. These define exponential functions over the *rational numbers* but leave open the question of the value of b^r where r is an *irrational number*. That is, we know that $4^{3.14} = 4^{314/100}$ is a shortcut notation for $\sqrt[100]{4^{314}}$, but what if that exponent is not a terminating decimal approximation of π, but instead is π itself? What does 4^π mean?

By *assuming* the continuity of $B(x) = b^x$, $b > 0$, we may interpret b^r as $\lim_{x\to r} b^x$ where the values of x as we take the limit are rational. For example, we may view 4^π as the number satisfying

$$\begin{aligned}
4^3 &< 4^\pi < 4^4\\
4^{3.1} &< 4^\pi < 4^{3.2}\\
4^{3.14} &< 4^\pi < 4^{3.15}\\
4^{3.141} &< 4^\pi < 4^{3.142}\\
4^{3.1415} &< 4^\pi < 4^{3.1416}\\
&\vdots
\end{aligned}$$

As we see in Table 2.6.2, $4^\pi \approx 77.88$.

***Table* 2.6.2*: Investigating* 4^x *as* $x \to \pi$**

x	4^x	x	4^x
4^3	64	$4^{3.141593}$	77.8802710486
$4^{3.1}$	73.5166947198	$4^{3.1416}$	77.8810268071
$4^{3.14}$	77.7084726013	$4^{3.142}$	77.9242251944
$4^{3.141}$	77.8162741237	$4^{3.15}$	78.7932424541
$4^{3.1415}$	77.8702309526	$4^{3.2}$	84.4485062895
$4^{3.14159}$	77.8799471543	4^4	256

While we have stated (via the Generalized Direct Substitution Property) that all our basic functions (including exponentials) are continuous on their domains, we have yet to prove that result for exponential functions officially. To do so, we need to first prove that $B(x) = b^x$ is continuous at $x = 0$.

Suppose $x > 0$. $\forall \epsilon > 0$, if we choose $\delta = \log_b(1+\epsilon)$, then

$$\begin{aligned}
& 0 < |x-0| < \delta \\
\implies & x < \delta && (x > 0 \implies |x| = x) \\
\implies & x < \log_b(1+\epsilon) \\
\implies & b^x < 1+\epsilon \\
\implies & b^x - 1 < \epsilon \\
\implies & |b^x - 1| < \epsilon && (x > 0 \implies b^x > b^0 = 1 \implies b^x - 1 > 0 \implies b^x - 1 = |b^x - 1|)
\end{aligned}$$

Therefore, $\lim\limits_{x \to 0^+} b^x = 1 = b^0$.

Moreover, if $x < 0$, then $\forall \epsilon > 0$, we choose $\delta = -\log_b(1-\epsilon)$. Therefore,

$$\begin{aligned}
& 0 < |x-0| < \delta \\
\implies & -x < \delta && (x < 0 \implies |x| = -x) \\
\implies & -x < -\log_b(1-\epsilon) \\
\implies & x > \log_b(1-\epsilon) \\
\implies & b^x > 1-\epsilon \\
\implies & b^x - 1 > -\epsilon \\
\implies & -(b^x - 1) < \epsilon \\
\implies & |b^x - 1| < \epsilon && (x < 0 \implies b^x < b^0 = 1 \implies b^x - 1 < 0 \implies -(b^x - 1) = |b^x - 1|)
\end{aligned}$$

Therefore, $\lim\limits_{x \to 0^-} b^x = 1 = b^0$.

Since both one-sided limits are equal to b^0, $\lim\limits_{x \to 0} b^x = 1 = b^0$. Thus, $B(x) = b^x$ is continuous at $x = 0$.

We are now ready to prove that $B(x) = b^x$ is continuous everywhere.

Let $B(x) = b^x$ and let a be arbitrary. Then

$$\begin{aligned}
\lim_{x \to a} b^x &= \lim_{h \to 0} b^{a+h} && (\text{substituting } x = a+h \implies h = x-a \implies h \to 0 \text{ as } x \to a) \\
&= \lim_{h \to 0} \left(b^a \cdot b^h\right) \\
&= b^a \cdot \lim_{h \to 0} b^h && (\text{Constant Multiple Limit Law}) \\
&= b^a \cdot b^0 && (b^x \text{ is continuous at } 0) \\
&= b^a
\end{aligned}$$

Since a is arbitrary, this shows that the exponential function $B(x) = b^x$ is continuous everywhere.

The next theorem is a fascinating fact and is the student's favorite derivative. In words, the function $y = e^x$ is the only function (besides $y = 0$) whose derivative is itself!

Theorem: Derivative of e^x

$$\frac{d}{dx}\left(e^x\right) = e^x$$

Proof available at the end of this section.

Interactive Element: The Derivative of e^x

The following Interactive Element might help you visualize why the slope of $y = e^x$ at any point is equal to the height of the curve at that same point.

Interact: Drag the red point along the curve $y = e^x$.

Observation: The tangent line always crosses the x-axis exactly one unit to the left of the point, so over a run of 1 the rise equals the curve's height. The slope therefore always equals e^x—the value of the function itself.

Before going into examples, let's add a quick corollary.

Corollary: Derivative of b^x

$$\frac{d}{dx}\left(b^x\right) = b^x \ln\left(b\right)$$

Proof available at the end of this section.

Example 2.6.1: Derivative of an Exponential Function

Find the derivative of $f(x) = e^{\tan(2x)}$.

Solution available at the end of this section.

Example 2.6.2: Combining Differentiation Rules

Find the derivative of $y = \frac{e^{x^2}}{x}$.

Solution available at the end of this section.

Checkpoint 2.6.2

Find the derivative of $h(x) = xe^{2x}$.

Answer available at the end of this section.

Example 2.6.3: Applying the Natural Exponential Function

A colony of mosquitoes has an initial population of 1000. After t days, the population is given by $A(t) = 1000e^{0.3t}$. Show that the ratio of the rate of change of the population, $A'(t)$, to the population, $A(t)$ is constant.

Solution available at the end of this section.

Checkpoint 2.6.3

If $A(t) = 1000e^{0.3t}$ describes the mosquito population after t days, as in the preceding example, what is the rate of change of $A(t)$ after four days?

Answer available at the end of this section.

Example 2.6.4: Applying Derivative Formulas

Find the derivative of $h(x) = \frac{3^x}{3^x+2}$.

Solution available at the end of this section.

Checkpoint 2.6.4

Find the slope for the line tangent to $y = 3^x$ at $x = 2$.

Answer available at the end of this section.

Derivatives of the Hyperbolic Functions

Recall that the hyperbolic sine and hyperbolic cosine are defined as

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

and

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

The other hyperbolic functions are then defined in terms of $\sinh x$ and $\cosh x$. The graphs of the hyperbolic functions are shown in Figure 2.6.1.

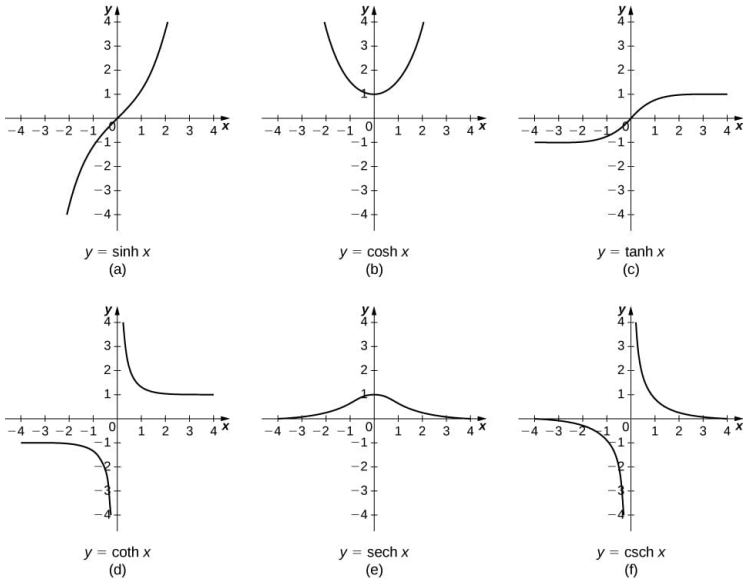

Figure 2.6.1: *Graphs of the hyperbolic functions.*

Developing differentiation formulas for the hyperbolic functions is easy. For example, looking at $\sinh x$ we have

$$\begin{aligned}\frac{d}{dx}(\sinh x) &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right)\\ &= \frac{1}{2}\left[\frac{d}{dx}(e^x) - \frac{d}{dx}(e^{-x})\right]\\ &= \frac{1}{2}[e^x + e^{-x}]\\ &= \cosh x.\end{aligned}$$

Similarly,

$$\frac{d}{dx}\cosh x = \sinh x.$$

We summarize the differentiation formulas for the hyperbolic functions in Table 2.6.3.

***Table** 2.6.3**:** Derivatives of the Hyperbolic Functions*

$f(x)$	$\frac{d}{dx}f(x)$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2\ x$
$\coth\ x$	$-\operatorname{csch}^2\ x$
$\operatorname{sech}\ x$	$-\operatorname{sech}\ x \tanh x$
$\operatorname{csch}\ x$	$-\operatorname{csch}\ x \coth x$

Let's take a moment to compare the derivatives of the hyperbolic functions with the derivatives of the standard trigonometric functions. There are a lot of similarities and differences. For example, the derivatives of the sine functions match:

$$\frac{d}{dx}\sin x = \cos x$$

and

$$\frac{d}{dx}\sinh x = \cosh x.$$

The derivatives of the cosine functions, however, differ in sign:

$$\frac{d}{dx}\cos x = -\sin x,$$

but

$$\frac{d}{dx}\cosh x = \sinh x.$$

Example 2.6.5

Evaluate the following derivatives:

a. $\frac{d}{dx}(\sinh(x^2))$
b. $\frac{d}{dx}(\cosh x)^2$

Solutions available at the end of this section.

Checkpoint 2.6.6

Evaluate the following derivatives:

a. $\frac{d}{dx}(\tanh(x^2+3x))$
b. $\frac{d}{dx}\left(\frac{1}{(\sinh x)^2}\right)$

Answer a available at the end of this section.

Success in College: Dealing With Failure (Part 2)

A student gets a test back from the professor. The test score is 55%. The professor has written at the top of the exam, "Make an

appointment to talk to me about this."

1. Why do you think that the professor wants to talk to the student?
2. Making an appointment to talk to your professor is usually an option in college. You can choose not to do this. Describe the reasons someone may use for making the appointment. Describe the reasons someone may use for not making the appointment.

Success in College: Memorization Learning Preferences

> Students may be inclined to approach their courses in one of three ways. Those with a reproducing orientation tend to take a surface approach to learning, relying on rote memorization and mechanical formula substitution and making little or no effort to understand the material being taught. Those with a meaning orientation tend to adopt a deep approach, probing and questioning and exploring the limits of applicability of new material. Those with an achieving orientation tend to use a strategic approach, doing whatever is necessary to get the highest grade they can, taking a surface approach if that suffices and a deep approach when necessary.
>
> Source: R. Felder, R. Brent , Journal of Engineering Education, 94 (1), 57-72 (2005).

1. Students have different motivations for going to math class. What do you think the main motivation is for students who have a reproducing orientation and who take a surface approach?
2. What do you think is the main motivation for students who have a meaning orientation and who take a deep approach?

Answers

Section Preview

A sloppy and superficial treatment of this material would come from a purely mechanical perspective—your students deserve better. Take the time to define e and derive the theory of continuity and derivatives of the base-e exponential function.

Section Preview

- **Arithmetic**
  - Simplifying Fractions and Mixed Numbers: In Example 3 the ratio $\frac{300e^{0.3t}}{1000e^{0.3t}}$ is reduced by canceling the common factor $e^{0.3t}$ and reducing $\frac{300}{1000}$ to 0.3.
- **Simplifying Expressions**
  - Algebraic Vocabulary and Evaluating Expressions: Checkpoint 3 evaluates $A'(4) = 300e^{1.2} \approx 996$, and Checkpoint 4 evaluates the derivative at $x = 2$ to obtain $9 \ln 3$; both require substituting a value into an expression and evaluating it.

Section Preview

- Find the derivative of exponential functions, both natural-based and non-natural-based.
- Find the derivative of logarithmic functions, both natural-based and non-natural-based.
- Apply the formulas for derivatives of the hyperbolic functions.
- Explore applications for the derivatives of exponential and hyperbolic functions in the sciences and engineering.

Theorem: Derivative of e^x

Using the limit definition of the derivative, we get the following:

$$\begin{aligned}
\frac{d}{dx}\left(e^x\right) &= \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} && \text{(definition of a derivative)}\\
&= \lim_{h\to 0}\frac{e^x\cdot e^h-e^x}{h}\\
&= \lim_{h\to 0}\frac{e^x\left(e^h-1\right)}{h}\\
&= e^x\cdot\lim_{h\to 0}\frac{e^h-1}{h} && \text{(since } e^x \text{ does not rely on } h\text{, it can be factored out)}
\end{aligned}$$

Recall that we defined e to be $\lim_{n\to 0}(1+n)^{1/n}$. If we let $n = e^h - 1$, then $n + 1 = e^h$. Since exponential functions are continuous everywhere,

$$\lim_{h\to 0} e^h = e^0 = 1.$$

This means that $n \to 0$ as $h \to 0$. Moreover, solving $n + 1 = e^h$ for h, we get $\ln(n + 1) = h$. Thus,

$$\begin{aligned}
\frac{d}{dx}\left(e^x\right) &= e^x \cdot \lim_{h\to 0}\frac{e^h-1}{h} \\
&= e^x \cdot \lim_{n\to 0}\frac{n}{\ln(1+n)} \\
&= e^x \cdot \lim_{n\to 0}\frac{1}{\frac{1}{n}\ln(1+n)} \\
&= e^x \cdot \lim_{n\to 0}\frac{1}{\ln(1+n)^{1/n}} \\
&= e^x \cdot \frac{1}{\lim\limits_{n\to 0}\ln(1+n)^{1/n}} && \text{(Limit Laws)} \\
&= e^x \cdot \frac{1}{\ln\left(\lim\limits_{n\to 0}(1+n)^{1/n}\right)} && \text{(Limit Laws)} \\
&= e^x \cdot \frac{1}{\ln(e)} && \text{(definition of } e\text{)} \\
&= e^x \cdot \frac{1}{1} \\
&= e^x
\end{aligned}$$

Q.E.D.

Corollary: Derivative of b^x

$$\begin{aligned}
\frac{d}{dx}\left(b^x\right) &= \frac{d}{dx}\left(e^{\ln\left(b^x\right)}\right) \\
&= \frac{d}{dx}\left(e^{x\ln(b)}\right) \\
&= e^{x\ln(b)}\cdot\ln(b) && \text{(Chain Rule)} \\
&= e^{\ln\left(b^x\right)}\cdot\ln(b) \\
&= b^x\cdot\ln(b)
\end{aligned}$$

Q.E.D.

Example 2.6.1**: Derivative of an Exponential Function**

Using the derivative formula and the Chain Rule,

$$f'(x) = e^{\tan(2x)}\frac{d}{dx}\left(\tan(2x)\right) = e^{\tan(2x)}\sec^2(2x)\cdot 2$$

Example 2.6.2**: Combining Differentiation Rules**

Use the derivative of the natural exponential function, the Quotient Rule, and the Chain Rule.

$$\begin{aligned}
y' &= \frac{\left(e^{x^2}\cdot 2\right)x\cdot x - 1\cdot e^{x^2}}{x^2} && \text{(Quotient Rule)} \\
&= \frac{e^{x^2}(2x^2-1)}{x^2}
\end{aligned}$$

Checkpoint 2.6.2

$h'(x) = e^{2x} + 2xe^{2x}$

Example 2.6.3**: Applying the Natural Exponential Function**

First find $A'(t)$. By using the Chain Rule, we have $A'(t) = 300e^{0.3t}$. Thus, the ratio of the rate of change of the population to the population is given by

$$\frac{A'(t)}{A(t)} = \frac{300e^{0.3t}}{1000e^{0.3t}} = 0.3.$$

The ratio of the rate of change of the population to the population is the constant 0.3.

Checkpoint 2.6.3

996

Example 2.6.4**: Applying Derivative Formulas**

$$h'(x) = \frac{3^x \ln 3(3^x + 2) - 3^x \ln 3(3^x)}{(3^x + 2)^2} \qquad \text{(Quotient Rule)}$$

$$= \frac{2 \cdot 3^x \ln 3}{(3x + 2)^2}$$

Checkpoint 2.6.4

$9 \ln(3)$

Example 2.6.5

Using the formulas in Table 2.6.3 and the Chain Rule, we get

a.

$$\frac{d}{dx}(\sinh(x^2)) = \cosh(x^2) \cdot 2x$$

b.

$$\frac{d}{dx}(\cosh x)^2 = 2 \cosh x \sinh x$$

Checkpoint 2.6.6

$\frac{d}{dx}(\tanh(x^2 + 3x)) = (\operatorname{sech}^2(x^2 + 3x))(2x + 3)$

$\frac{d}{dx}\left(\frac{1}{(\sinh x)^2}\right) = \frac{d}{dx}(\sinh x)^{-2} = -2(\sinh x)^{-3} \cosh x$

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2.6.1: Resources and Key Concepts

Resources

Videos

- Derivative of exponential functions
  - Derivatives of Exponential Functions: Theorem, Proof, and Examples
- Derivative of e^x
  - The Derivative of e^x
  - Proofs in Differential Calculus - The Derivative of e^x is e^x
- Derivative of b^x
  - Proofs in Differential Calculus - The Derivative of b^x is $b^x ln(b)$
- Derivative of hyperbolic functions
  - Derivatives of Hyperbolic Functions: Theorem, Proof, and Example
- Derivative of $\sinh(x)$
  - Proofs in Differential Calculus - The Derivative of $\sinh(x)$ is $\cosh(x)$

Key Concepts

Definitions

- **The number** e: The irrational number defined as the limit $\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = \lim_{n\to 0}(1+n)^{1/n}$.

Theorems

- **Theorem: Derivative of** e^x: $\frac{d}{dx}(e^x) = e^x$.
- **Corollary: Derivative of** b^x: $\frac{d}{dx}(b^x) = b^x \ln(b)$.
- **Derivatives of Hyperbolic Functions**:
  - $\frac{d}{dx}(\sinh x) = \cosh x$
  - $\frac{d}{dx}(\cosh x) = \sinh x$
  - $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$
  - $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$
  - $\frac{d}{dx}(\operatorname{sech}\ x) = -\operatorname{sech}\ x \tanh x$
  - $\frac{d}{dx}(\operatorname{csch}\ x) = -\operatorname{csch}\ x \coth x$

Common Mistakes

- **Confusing** $\frac{d}{dx}(e^x)$ **with** $\frac{d}{dx}(x^e)$: The rule for e^x (derivative is itself) is different from the Power Rule.
- **Forgetting** $\ln(b)$ **in** $\frac{d}{dx}(b^x)$: A common error is to write $\frac{d}{dx}(b^x) = b^x$ instead of $b^x \ln(b)$.
- **Sign Errors in Hyperbolic Derivatives**: Similar to trigonometric functions, some hyperbolic derivatives involve negative signs ($\coth x, \operatorname{sech}\ x, \operatorname{csch}\ x$). It's important to note that $\frac{d}{dx}(\cosh x) = \sinh x$ (positive), unlike $\frac{d}{dx}(\cos x) = -\sin x$.
- **Incorrectly Applying Chain Rule**: Forgetting to multiply by the derivative of the exponent when differentiating $e^{u(x)}$ or $b^{u(x)}$, or the argument of a hyperbolic function.

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2.6.2: Homework

Note

The following exercises are meant as hand-in homework (if assigned by your instructor). Typed-in answers here will not be saved or scored.

Basic Skills

Exercise 1

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Basic Skills - Applications

Exercise 40

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Exercise 41

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Synthesis Questions

Exercise 42

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Exercise 43

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Proofs and Problems that Require Explanations or Logical Justifications

Exercise 44

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Exercise 45

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2.6.3: Additional Exercises

Reading Questions

1. How is the number e defined using a limit?
2. What is the derivative of $f(x) = e^x$?
3. What is the derivative of $f(x) = b^x$, where b is a positive constant?
4. How is the Chain Rule applied to find the derivative of $f(x) = e^{g(x)}$?
5. What is the derivative of $f(x) = \sinh x$?
6. What is the derivative of $f(x) = \cosh x$? How does this compare to the derivative of $\cos x$?
7. What is the derivative of $f(x) = \tanh x$?
8. Which of the six hyperbolic functions have derivatives that include a negative sign?
9. How are the differentiation formulas for hyperbolic functions derived?
10. If a population is given by $A(t) = 1000e^{0.3t}$, what does the ratio $\frac{A'(t)}{A(t)}$ represent?

Homework

In exercises 1 - 16, find $f'(x)$ for each function.

1) $f(x) = x^2e^x$

Answer

$$f'(x) = 2xe^x + x^2e^x$$

2) $f(x) = \dfrac{e^{-x}}{x}$

3) $f(x) = \sqrt{e^{2x} + 2x}$

4) $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Answer

$$f'(x) = \frac{4}{(e^x + e^{-x})^2}$$

5) $f(x) = \dfrac{10^x}{\ln 10}$

6) $f(x) = 2^{4x} + 4x^2$

Answer

$$f'(x) = 2^{4x+2} \cdot \ln 2 + 8x$$

7) $f(x) = 3^{\sin 3x}$

8) $f(x) = x^\pi \cdot \pi^x$

Answer

$$f'(x) = \pi x^{\pi-1} \cdot \pi^x + x^\pi \cdot \pi^x \ln \pi$$

9) $f(x) = \cosh(3x + 1)$

Answer

$$f'(x) = 3\sinh(3x + 1)$$

10) $f(x) = \sinh(x^2)$

11) $f(x) = \dfrac{1}{\cosh(x)}$

Answer

$$f'(x) = -\tanh(x)\operatorname{sech}(x)$$

12) $f(x) = \cosh^2(x) + \sinh^2(x)$

Answer

$$f'(x) = 4\cosh(x)\sinh(x)$$

13) $f(x) = \cosh^2(x) - \sinh^2(x)$

14) $f(x) = \tanh(\sqrt{x^2+1})$

Answer

$$f'(x) = \frac{x\operatorname{sech}^2(\sqrt{x^2+1})}{\sqrt{x^2+1}}$$

15) $f(x) = \dfrac{1+\tanh(x)}{1-\tanh(x)}$

16) $f(x) = \sinh^6(x)$

Answer

$$f'(x) = 6\sinh^5(x)\cosh(x)$$

17) Show that $\cosh(x)$ and $\sinh(x)$ satisfy $y'' = y$.

Answer

Answers may vary

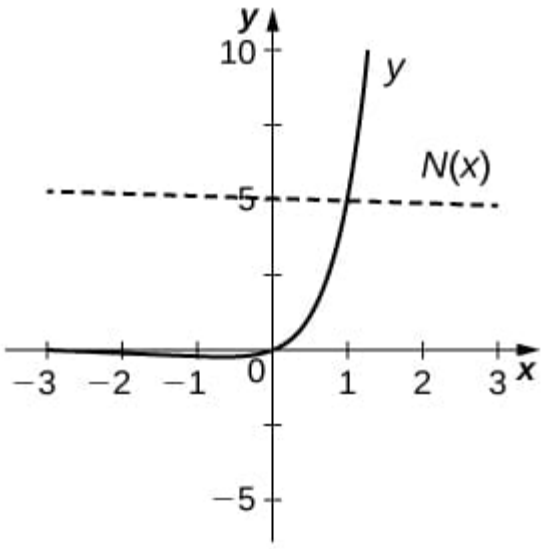
18) Use the quotient rule to verify that $\dfrac{d}{dx}\big(\tanh(x)\big) = \operatorname{sech}^2(x)$.

19) [Technology Required] Find an equation of the tangent line to the graph of $f(x) = 4xe^{(x^2-1)}$ at the point where $x = -1$. Graph both the function and the tangent line.

20) [Technology Required] Find the equation of the line that is normal to the graph of $f(x) = x \cdot 5^x$ at the point where $x = 1$. Graph both the function and the normal line.

Answer

$$y = \tfrac{-1}{5+5\ln 5}x + \left(5 + \tfrac{1}{5+5\ln 5}\right)$$



21) The formula $I(t) = \dfrac{\sin t}{e^t}$ is the formula for a decaying alternating current.

a. Complete the following table with the appropriate values.

t	$\frac{\sin t}{e^t}$
0	(i)
$\pi/2$	(ii)
π	(iii)
$3\pi/2$	(vi)
2π	(v)
2π	(vi)
3π	(vii)

b. Using only the values in the table, determine where the tangent line to the graph of $I(t)$ is horizontal.

22) [Technology Required] The population of Toledo, Ohio, in 2000 was approximately 500,000. Assume the population is increasing at a rate of 5% per year.

a. Write the exponential function that relates the total population as a function of t.

b. Use part a. to determine the rate at which the population is increasing in t years.

c. Use part b. to determine the rate at which the population is increasing in 10 years

Answer

a. $P = 500,000(1.05)^t$ individuals
b. $P'(t) = 24395 \cdot (1.05)^t$ individuals per year
c. $39,737$ individuals per year

23) [Technology Required] An isotope of the element erbium has a half-life of approximately 12 hours. Initially there are 9 grams of the isotope present.

a. Write the exponential function that relates the amount of substance remaining as a function of t, measured in hours.

b. Use a. to determine the rate at which the substance is decaying in t hours.

c. Use b. to determine the rate of decay at $t = 4$ hours.

24) [Technology Required] The number of cases of influenza in New York City from the beginning of 1960 to the beginning of 1964 is modeled by the function $N(t) = 5.3e^{0.093t^2 - 0.87t}, (0 \le t \le 4)$, where $N(t)$ gives the number of cases (in thousands) and t is measured in years, with $t = 0$ corresponding to the beginning of 1960.

a. Show work that evaluates $N(0)$ and $N(4)$. Briefly describe what these values indicate about the disease in New York City.

b. Show work that evaluates $N'(0)$ and $N'(3)$. Briefly describe what these values indicate about the disease in the United States.

Answer

a. At the beginning of 1960 there were 5.3 thousand cases of the disease in New York City. At the beginning of 1963 there were approximately 723 cases of the disease in the United States.
b. At the beginning of 1960 the number of cases of the disease was decreasing at rate of -4.611 thousand per year; at the beginning of 1963, the number of cases of the disease was decreasing at a rate of -0.2808 thousand per year.

25) [Technology Required] The relative rate of change of a differentiable function $y = f(x)$ is given by $\frac{100 \cdot f'(x)}{f(x)}$. One model for population growth is a Gompertz growth function, given by $P(x) = ae^{-b \cdot e^{-cx}}$ where a, b, and c are constants.

a. Find the relative rate of change formula for the generic Gompertz function.

b. Use part a. to find the relative rate of change of a population in $x = 20$ months when $a = 204,\ \ b = 0.0198$, and $c = 0.15$.

c. Briefly interpret what the result of part b. means.

For exercises 26 - 29, use the population of New York City from 1790 to 1860, given in the following table.

Year since 1790	Population
0	33,131
10	60,515
20	96,373
30	123,706
40	202,300
50	312,710
60	515,547
70	813,669

New York City Population Over Time Source: http://en.Wikipedia.org/wiki/Largest..._United_States

_by_population_by_decade

26) [Technology Required] Using a computer program or a calculator, fit a growth curve to the data of the form $p = ab^t$.

Answer

$p = 35741(1.045)^t$

27) [Technology Required] Using the exponential best fit for the data, write a table containing the derivatives evaluated at each year.

28) [Technology Required] Using the exponential best fit for the data, write a table containing the second derivatives evaluated at each year.

Answer

Year since 1790	P''
0	69.25
10	107.5
20	167.0
30	259.4
40	402.8
50	625.5
60	971.4
70	1508.5

29) [Technology Required] Using the tables of first and second derivatives and the best fit, answer the following questions:

a. Will the model be accurate in predicting the future population of New York City? Why or why not?

b. Estimate the population in 2010. Was the prediction correct from part a.?

30) [Technology Required] A chain hangs from two posts 2m apart to form a catenary described by the equation $y = 2\cosh(x/2) - 1$. Find the slope of the catenary at the left fence post.

Answer

-0.521095

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2.7: Implicit Differentiation and the Derivatives of the Inverse Trigonometric and Hyperbolic Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

We have already studied how to find equations of tangent lines to functions and the rate of change of a function at a specific point. In all these cases, we had the explicit equation for the function and differentiated these functions explicitly. Suppose instead that we want to determine the equation of a tangent line to an arbitrary curve or the rate of change of an arbitrary curve at a point. In this section, we solve these problems by finding the derivatives of functions that define y implicitly in terms of x.

Implicit Differentiation

In most discussions of math, if the dependent variable y is a function of the independent variable x, we express y in terms of x. If this is the case, we say that y is an **explicit function** of x. For example, when we write the equation $y = x^2 + 1$, we are defining y explicitly in terms of x. On the other hand, if the relationship between the function y and the variable x is expressed by an equation where y is not expressed entirely in terms of x, we say that the equation defines y **implicitly** in terms of x. For example, the equation $y - x^2 = 1$ defines the function $y = x^2 + 1$ implicitly.

Implicit differentiation allows us to find slopes of tangents to curves that are not functions (they fail the Vertical Line Test). We are using the idea that portions of y are functions that satisfy the given equation but that y is not a function of x.

Generally, an equation defines a function implicitly if the function satisfies that equation. An equation may define many different functions implicitly. For example, the functions

$$y = \sqrt{25 - x^2}$$

and

$$y = \begin{cases} \sqrt{25 - x^2}, & \text{if } -5 \le x < 0 \\ -\sqrt{25 - x^2}, & \text{if } 0 \le x \le 5 \end{cases}$$

which are illustrated in Figure 2.7.1, are just two of the many functions defined implicitly by the equation $x^2 + y^2 = 25$.

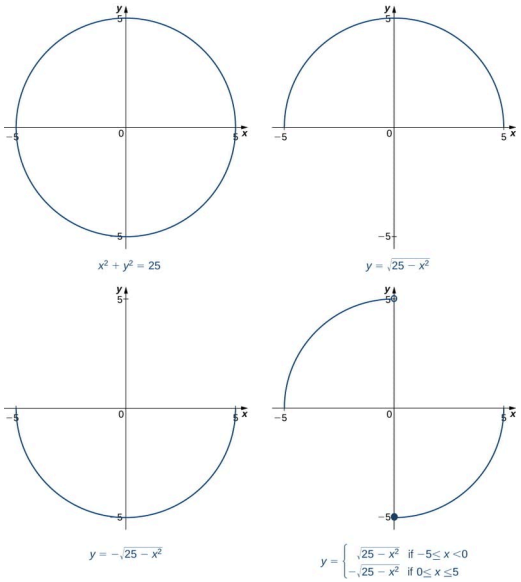

Figure 2.7.1**:** *The equation* $x^2 + y^2 = 25$ *defines many functions implicitly.*

If we want to find the slope of the line tangent to the graph of $x^2 + y^2 = 25$ at the point $(3, 4)$, we could evaluate the derivative of the function $y = \sqrt{25 - x^2}$ at $x = 3$. On the other hand, if we want the slope of the tangent line at the point $(3, -4)$, we could use the derivative of $y = -\sqrt{25 - x^2}$. However, it is not always easy to solve for a function defined implicitly by an equation. Fortunately, the technique of **implicit differentiation** allows us to find the derivative of an implicitly defined function without ever solving for the function explicitly.

A simplified explanation of implicit differentiation is that you take the derivatives of both sides of a given equation (whether *explicitly* solved for y or not) with respect to the independent variable and perform the Chain Rule whether or not it is necessary. For example, suppose $y = \sinh(x) - 2^x$. Then

$$\frac{d}{dx}(y) = \frac{d}{dx}(\sinh(x) - 2^x) \quad \Longrightarrow \quad \frac{dy}{dx} = \cosh(x)\frac{dx}{dx} - 2^x \ln(2)\frac{dx}{dx}$$

$$\Longrightarrow \quad \frac{dy}{dx} = \cosh(x) \cdot 1 - 2^x \ln(2) \cdot 1$$

$$\Longrightarrow \quad \frac{dy}{dx} = \cosh(x) - 2^x \ln(2)$$

While the results are no different than how we have performed differentiation so far, the *process* is important to understand. Suppose, instead, that we had the equation $\cos(x^2 + x) = \tanh(x) + 3y$ and were asked to find y'. We *could* start by solving for y, *or* we could take the derivative of both sides immediately as follows:

$$\frac{d}{dx}(\cos(x^2 + x)) = \frac{d}{dx}(\tanh(x) + 3y) \quad \Longrightarrow \quad -\sin(x^2 + x) \cdot \frac{d}{dx}(x^2 + x) = \operatorname{sech}^2(x)\frac{dx}{dx} + 3\frac{dy}{dx}$$

$$\Longrightarrow \quad -\sin(x^2 + x) \cdot (2x + 1) \cdot \frac{dx}{dx} = \operatorname{sech}^2(x) \cdot 1 + 3\frac{dy}{dx}$$

$$\Longrightarrow \quad -\sin(x^2 + x) \cdot (2x + 1) \cdot 1 = \operatorname{sech}^2(x) \cdot 1 + 3\frac{dy}{dx}$$

$$\Longrightarrow \quad \frac{-\sin(x^2 + x) \cdot (2x + 1) - \operatorname{sech}^2(x)}{3} = \frac{dy}{dx}$$

However, the *real* power of implicit differentiation comes into play when given the equation for an implicitly defined curve. For example, if $e^{xy} - \sqrt{x + y} = \cot(y)$, then the derivative of y with respect to x can be found by taking the derivative *implicitly*.

$$\frac{d}{dx}\left(e^{xy}-\sqrt{x+y}\right)=\frac{d}{dx}\left(\cot\left(y\right)\right) \quad \Longrightarrow \quad \frac{d}{dx}e^{xy}-\frac{d}{dx}\sqrt{x+y}=-\csc^2\left(y\right)\cdot\frac{dy}{dx}$$

$$\Longrightarrow \quad e^{xy}\cdot\frac{d}{dx}\left(xy\right)-\frac{1}{2\sqrt{x+y}}\cdot\frac{d}{dx}\left(x+y\right)=-\csc^2\left(y\right)\cdot\frac{dy}{dx}$$

$$\Longrightarrow \quad e^{xy}\cdot\left(\frac{dx}{dx}\cdot y+x\cdot\frac{dy}{dx}\right)-\frac{1}{2\sqrt{x+y}}\cdot\left(\frac{dx}{dx}+\frac{dy}{dx}\right)=-\csc^2\left(y\right)\cdot\frac{dy}{dx}$$

$$\Longrightarrow \quad e^{xy}\cdot\left(y+x\cdot\frac{dy}{dx}\right)-\frac{1}{2\sqrt{x+y}}\cdot\left(1+\frac{dy}{dx}\right)=-\csc^2\left(y\right)\cdot\frac{dy}{dx}$$

At this point, we would feel compelled to solve for $\frac{dy}{dx}$; however, let's hold off for now. It's enough at this point to understand how implicit differentiation works.

Example 2.7.1: Using Implicit Differentiation

Assuming that y is defined implicitly by the equation $x^2+y^2=25$, find $\frac{dy}{dx}$.

Solution available at the end of this section.

Example 2.7.2: Using Implicit Differentiation and the Product Rule

Assuming that y is defined implicitly by the equation $x^3\sin y+y=4x+3$, find $\frac{dy}{dx}$.

Solution available at the end of this section.

Example 2.7.3: Using Implicit Differentiation to Find a Second Derivative

Find $\frac{d^2y}{dx^2}$ if $x^2+y^2=25$.

Solution available at the end of this section.

Checkpoint 2.7.3

Find $\frac{dy}{dx}$ for y defined implicitly by the equation $4x^5+\tan y=y^2+5x$.

Answer available at the end of this section.

Finding Tangent Lines Implicitly

Now that we have seen the technique of implicit differentiation, we can apply it to the problem of finding equations of tangent lines to curves described by equations.

Interactive Element: Tangent Lines to an Implicit Curve

The following Interactive Element might help you visualize how the tangent line to the circle $x^2+y^2=25$ pivots as the point of tangency moves, and why its slope $-\frac{x}{y}$ depends on both coordinates.

Interact: Drag the orange point around the circle and watch the red tangent line pivot.

Observation: The slope equals $-\frac{x}{y}$: at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ it is -1, but at $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ it is $+1$ — the same x gives opposite slopes, which is why $\frac{dy}{dx}$ must depend on y as well as x. Where the circle is vertical (at $(\pm 5, 0)$, where $y = 0$) the slope is undefined.

Great Advice to Save Time and Prevent Mistakes

When asked to find the value of the derivative or the equation of the tangent line for an implicitly-defined curve at a given point, it's best to *not* solve for $\frac{dy}{dx}$ immediately after implicitly differentiating. Instead, once you have implicitly differentiated the relation, substitute the given values of x and y into the resulting equation. It is *at that point* that you should solve for $\frac{dy}{dx}$. Examples $2.7.4$ and $2.7.5$ illustrate this process.

Example $2.7.4$: Finding the Equation of a Tangent Line

Find the equation of the line tangent to the curve $\cosh(xy) + xy = \frac{5}{4} + \ln(2)$ at the point $(1, \ln(2))$.

Solution available at the end of this section.

Example $2.7.5$: Finding the Equation of the Tangent Line to a Curve

Find the equation of the line tangent to the graph of $y^3 + x^3 - 3xy = 0$ at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$ (Figure $2.7.3$). This curve is known as the folium (or leaf) of Descartes.

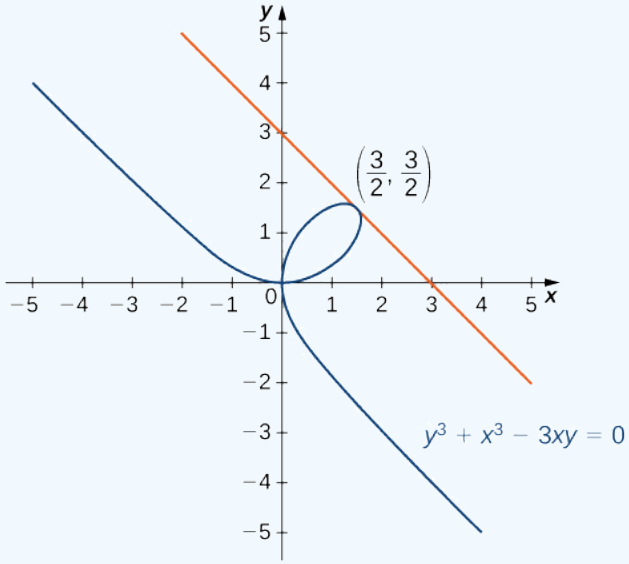

Figure $2.7.3$: *Finding the tangent line to the folium of Descartes at $\left(\frac{3}{2}, \frac{3}{2}\right)$.*

Solution available at the end of this section.

Example $2.7.6$: Applying Implicit Differentiation

In a simple video game, a rocket travels in an elliptical orbit whose path is described by the equation $4x^2 + 25y^2 = 100$. The rocket can fire missiles along lines tangent to its path. The object of the game is to destroy an incoming asteroid traveling along the positive x-axis toward $(0, 0)$. If the rocket fires a missile when it is located at $\left(3, \frac{8}{5}\right)$, where will it intersect the x-axis?

Solution available at the end of this section.

Checkpoint 2.7.6

Find the equation of the line tangent to the hyperbola $x^2 - y^2 = 16$ at the point $(5, 3)$.

Answer available at the end of this section.

Derivatives of Inverse Trigonometric Functions

We now focus on finding derivatives of inverse trigonometric functions. These derivatives will prove invaluable in the study of integration later in this text. The derivatives of inverse trigonometric functions are quite surprising in that their derivatives are algebraic functions. Previously, derivatives of algebraic functions have proven to be algebraic functions, and derivatives of trigonometric functions have been shown to be trigonometric functions. Here, for the first time, we see that the derivative of a function need not be of the same type as the original function.

Theorem: Derivatives of the Inverse Trigonometric Functions

$$\begin{aligned}
\frac{d}{dx}\left(\sin^{-1} x\right) &= \frac{1}{\sqrt{1-x^2}} & \frac{d}{dx}\left(\csc^{-1} x\right) &= -\frac{1}{x\sqrt{x^2-1}} \\
\frac{d}{dx}\left(\cos^{-1} x\right) &= -\frac{1}{\sqrt{1-x^2}} & \frac{d}{dx}\left(\sec^{-1} x\right) &= \frac{1}{x\sqrt{x^2-1}} \\
\frac{d}{dx}\left(\tan^{-1} x\right) &= \frac{1}{1+x^2} & \frac{d}{dx}\left(\cot^{-1} x\right) &= -\frac{1}{1+x^2}
\end{aligned}$$

Proof that $\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$ available at the end of this section.

Caution: Inverse Secant and Inverse Cosecant Ranges

As was mentioned in Chapter 1 of this textbook, the definition we use for the ranges of the secant and cosecant functions are *very* specific to this book. A lot of other texts use a different definition. I have chosen to restrict the ranges of the inverse cosecant to $(0, \pi/2] \cup (\pi, 3\pi/2]$ and the inverse secant to $[0, \pi/2) \cup [\pi, 3\pi/2)$.

This choice is made to avoid challenging issues in Calculus.

The proof we just did indicates the thought process you must go through when proving the derivative of an inverse trigonometric function. You must address the fact that the inverse function has a restricted range. The possible angles an inverse trigonometric function returns are always limited to two quadrants. For each quadrant, drawing a right triangle whose hypotenuse lies on the terminal side of an angle in that restricted quadrant is beneficial.

Example 2.7.7: Finding the Derivative of an Inverse Trigonometric Function

Find the derivative of $h(x) = \sin^{-1}(2x^3)$.

Solution available at the end of this section.

Checkpoint 2.7.7

Use the inverse function theorem to find the derivative of $g(x) = \tan^{-1} x$.

Answer available at the end of this section.

Example 2.7.8: Applying Differentiation Formulas to an Inverse Tangent Function

Find the derivative of $f(x) = \tan^{-1}(x^2)$.

Solution available at the end of this section.

Example 2.7.9: Applying Differentiation Formulas to an Inverse Cosecant Function

Find the derivative of $h(x) = x^2 \csc^{-1}(x)$.

Solution available at the end of this section.

Checkpoint 2.7.9

Find the derivative of $h(x) = \cos^{-1}(3x - 1)$.

Answer available at the end of this section.

Example 2.7.10: Applying the Inverse Tangent Function

The position of a particle at time t is given by $s(t) = \tan^{-1}\left(\frac{1}{t}\right)$ for $t \geq \frac{1}{2}$. Find the velocity of the particle at time $t = 1$.

Solution available at the end of this section.

Checkpoint 2.7.10

Find the equation of the line tangent to the graph of $f(x) = \sin^{-1} x$ at $x = 0$.

Answer available at the end of this section.

Derivatives of Inverse Hyperbolic Functions

Looking at the graphs of the hyperbolic functions, we see that with appropriate range restrictions, they all have inverses. Most of the necessary range restrictions can be discerned by closely examining the graphs. The domains and ranges of the inverse hyperbolic functions are summarized in Table 2.7.1.

***Table** 2.7.1: Domains and Ranges of the Inverse Hyperbolic Functions*

Function	Domain	Range
$\sinh^{-1} x$	$(-\infty, \infty)$	$(-\infty, \infty)$
$\cosh^{-1} x$	$(1, \infty)$	$[0, \infty)$
$\tanh^{-1} x$	$(-1, 1)$	$(-\infty, \infty)$

Function	Domain	Range
$\coth^{-1} x$	$(-\infty, 1) \cup (1, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$\operatorname{sech}^{-1} x$	$(0, 1)$	$[0, \infty)$
$\operatorname{csch}^{-1} x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$

The graphs of the inverse hyperbolic functions are shown in the following figure.

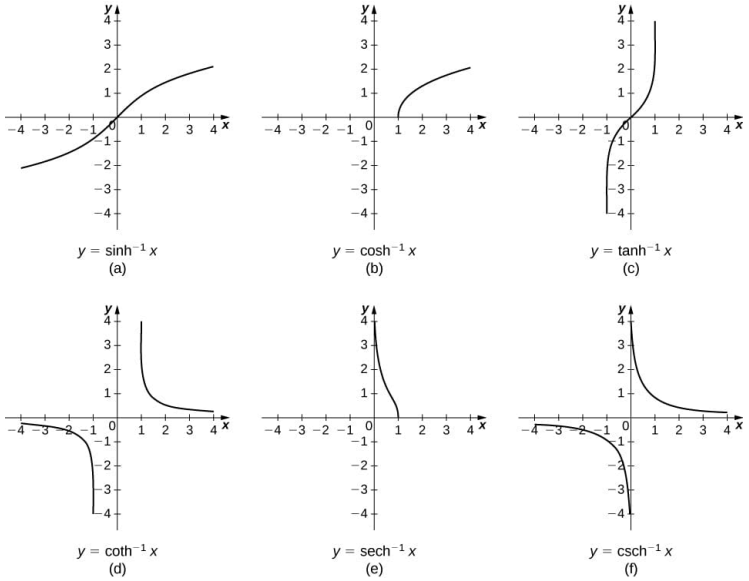

Figure 2.7.5**:** *Graphs of the inverse hyperbolic functions.*

To find the derivatives of the inverse functions, we use implicit differentiation. We have

$$\begin{aligned} y &= \sinh^{-1} x \\ \sinh y &= x \\ \frac{d}{dx} \sinh y &= \frac{d}{dx} x \\ \cosh y \frac{dy}{dx} &= 1. \end{aligned}$$

Recall that $\cosh^2 y - \sinh^2 y = 1$, so $\cosh y = \sqrt{1 + \sinh^2 y}$.Then,

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}}.$$

We can derive differentiation formulas for the other inverse hyperbolic functions similarly. These differentiation formulas are summarized in Table 2.7.2.

Table 2.7.2**:** *Derivatives of the Inverse Hyperbolic Functions*

$f(x)$	$\frac{d}{dx} f(x)$
$\sinh^{-1} x$	$\frac{1}{\sqrt{1+x^2}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$

$f(x)$	$\frac{d}{dx}f(x)$
$\coth^{-1}x$	$\frac{1}{1-x^2}$
$\operatorname{sech}^{-1}x$	$\frac{-1}{x\sqrt{1-x^2}}$
$\operatorname{csch}^{-1}x$	$\frac{-1}{\lvert x\rvert\sqrt{1+x^2}}$

Note that the derivatives of $\tanh^{-1}x$ and $\coth^{-1}x$ are the same.

Example 2.7.11: Differentiating Inverse Hyperbolic Functions

Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\sinh^{-1}\left(\frac{x}{3}\right)\right)$
b. $\frac{d}{dx}\left(\tanh^{-1}x\right)^2$

Solution available at the end of this section.

Checkpoint 2.7.11

Evaluate the following derivatives:

a. $\frac{d}{dx}(\cosh^{-1}(3x))$
b. $\frac{d}{dx}(\coth^{-1}x)^3$

Answer a available at the end of this section.

Success in College: Learning From Tests (Part 2)

1. A student earned 75% of the points on her last math test. On the first class day after the test, the class started a new chapter with new material. She does not understand several vital procedures from the last test. What do you think she should do about this situation?
2. A student usually finishes tests right as class ends. He doesn't have time to check for careless mistakes. He looks carefully at his corrected tests. He realizes that many of his errors are arithmetic or algebra mistakes. He knew how to do the problems; he didn't do them well enough. What do you think he should do?

Success in Mathematics: Math Anxiety (Part 2)

I'm going to keep this short (for a change). Please take this math anxiety test to help you better understand your math anxiety. While at that website, check out the hints on overcoming math anxiety.

Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. A suggestion for reducing math anxiety is to come to class prepared. If you are prepared, then you are ready to be successful. When you are successful, you start learning that you do not have to be anxious. What do you think it means to be prepared for your math class?
2. A suggestion for reducing math anxiety is to practice math every day. How many days a week do you do math? How long, on average, do you spend doing math on those days?

Answers

Section Preview

Implicit curves begin to take on a more important role as students progress through Calculus. As such, this material is critical. Moreover, implicit differentiation is crucial to implementing certain integration techniques (both in Differential and Integral Calculus). If you (the instructor) have not taught from the Stewart Calculus text, then you might not be used to the "re-definition" of the ranges for both the inverse secant and inverse cosecant functions. Trust me when I say that this "re-definition" is overwhelmingly important for trigonometric substitutions in Integral Calculus.

Section Preview

- **An Overview of Functions**
  - The Definition of a Function: The motivation for implicit differentiation rests on recognizing that a relation such as $x^2+y^2=25$ is not a function (it fails the Vertical Line Test) yet still defines y implicitly; the concept of function vs. relation is needed to understand why the technique is required.

Section Preview

- **Solving Equations**
  - Solving Literal Equations: The central operational maneuver of implicit differentiation is isolating $\frac{dy}{dx}$ while treating x and y as constants (Examples 1 and 2, Checkpoint 3).

Section Preview

- Use implicit differentiation to find the derivative given an implicitly defined relation between two variables.
- Use implicit differentiation to determine the equation of a tangent line to an implicitly-defined curve.
- Calculate the derivative of an inverse trigonometric function.
- Recognize the derivatives of the inverse hyperbolic functions.

Example 2.7.1: Using Implicit Differentiation

$$\begin{aligned}
& \frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(25) && \text{(differentiating implicitly)}\\
\Longrightarrow\ & \frac{d}{dx}(x^2)+\frac{d}{dx}(y^2) = 0 && \text{(Sum and Constant derivative rules)}\\
\Longrightarrow\ & 2x+2y\frac{dy}{dx} = 0 && \left(\frac{d}{dx}(x^2)=2x\frac{dx}{dx}=2x \text{ and } \frac{d}{dx}(y^2)=2y\frac{dy}{dx}\right)\\
\Longrightarrow\ & 2y\frac{dy}{dx} = -2x && \text{(grouping like terms)}\\
\Longrightarrow\ & \frac{dy}{dx} = -\frac{x}{y} && \text{(dividing both sides by } 2y)
\end{aligned}$$

Analysis

Note that the resulting expression for $\frac{dy}{dx}$ is in terms of both the independent variable x and the dependent variable y. Although in some cases it may be possible to express $\frac{dy}{dx}$ in terms of x only, it is generally not possible to do so.

Example 2.7.2: Using Implicit Differentiation and the Product Rule

$$\begin{aligned}
& \frac{d}{dx}(x^3\sin y+y) = \frac{d}{dx}(4x+3) && \text{(differentiating implicitly)}\\
\Longrightarrow\ & \frac{d}{dx}(x^3\sin y)+\frac{d}{dx}(y) = 4 && \left(\text{Sum Rule on the left, and } \frac{d}{dx}(4x+3)=4 \text{ on the right}\right)\\
\Longrightarrow\ & \left(\frac{d}{dx}(x^3)\cdot\sin y+\frac{d}{dx}(\sin y)\cdot x^3\right)+\frac{dy}{dx} = 4 && \text{(Product Rule)}\\
\Longrightarrow\ & 3x^2\sin y+(\cos y\frac{dy}{dx})\cdot x^3+\frac{dy}{dx} = 4 && \left(\frac{d}{dx}(x^3)=3x^2, \text{ and Chain Rule to obtain } \frac{d}{dx}(\sin y)=\cos y\frac{dy}{dx}\right)\\
\Longrightarrow\ & x^3\cos y\frac{dy}{dx}+\frac{dy}{dx} = 4-3x^2\sin y && \text{(grouping like terms)}\\
\Longrightarrow\ & \frac{dy}{dx}(x^3\cos y+1) = 4-3x^2\sin y && \text{(factoring)}\\
\Longrightarrow\ & \frac{dy}{dx} = \frac{4-3x^2\sin y}{x^3\cos y+1} && \left(\text{solving for } \frac{dy}{dx}\right)
\end{aligned}$$

Example 2.7.3: Using Implicit Differentiation to Find a Second Derivative

In Example 2.7.1, we showed that $\frac{dy}{dx}=-\frac{x}{y}$. We can take the derivative of both sides of this equation to find $\frac{d^2y}{dx^2}$.

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dy}\left(-\frac{x}{y}\right) && \text{(differentiating implicitly)}\\
&= -\frac{\left(1\cdot y - x\frac{dy}{dx}\right)}{y^2} && \text{(Quotient Rule)}\\
&= \frac{-y + x\frac{dy}{dx}}{y^2}\\
&= \frac{-y + x\left(-\frac{x}{y}\right)}{y^2} && \left(\text{substituting } \frac{dy}{dx} = -\frac{x}{y}.\right)\\
&= \frac{-y^2 - x^2}{y^3}
\end{aligned}$$

At this point we have found an expression for $\frac{d^2y}{dx^2}$. If we choose, we can simplify the expression further by recalling that $x^2 + y^2 = 25$ and making this substitution in the numerator to obtain $\frac{d^2y}{dx^2} = -\frac{25}{y^3}$.

Checkpoint 2.7.3

$$\frac{dy}{dx} = \frac{5 - 20x^4}{\sec^2 y - 2y}$$

Example 2.7.4**: Finding the Equation of a Tangent Line**

$$\begin{aligned}
&& \frac{d}{dx}\left(\cosh(xy) + xy\right) &= \frac{d}{dx}\left(\frac{5}{4} + \ln(2)\right)\\
\implies && \frac{d}{dx}\cosh(xy) + \frac{d}{dx}(xy) &= 0 && \text{(Sum and Constant Rules)}\\
\implies && \sinh(xy)\cdot\frac{d}{dx}(xy) + \frac{d}{dx}(xy) &= 0 && \text{(Chain Rule)}\\
\implies && \sinh(xy)\cdot\left(\frac{d}{dx}x\cdot y + x\cdot\frac{d}{dx}y\right) + \left(\frac{d}{dx}x\cdot y + x\cdot\frac{d}{dx}y\right) &= 0 && \text{(Product Rule)}\\
\implies && \sinh(xy)\cdot\left(y + x\cdot\frac{dy}{dx}\right) + \left(y + x\cdot\frac{dy}{dx}\right) &= 0 && \text{(Power Rule)}
\end{aligned}$$

At this point, it is important to remember the task at hand. We are asked to find the equation of the tangent line to the curve $\cosh(xy) + xy = \frac{5}{4} + \ln(2)$ at the point $(1, \ln(2))$. There is *no need* to explicitly solve for $\frac{dy}{dx}$ at this moment. Instead, let's substitute 1 for x and $\ln(2)$ for y.

$$\begin{aligned}
&& \sinh(\ln(2))\cdot\left(\ln(2) + \frac{dy}{dx}\right) + \left(\ln(2) + \frac{dy}{dx}\right) &= 0 && \text{(substituting)}\\
\implies && (\sinh(\ln(2)) + 1)\cdot\left(\ln(2) + \frac{dy}{dx}\right) &= 0\\
\implies && \left(\ln(2) + \frac{dy}{dx}\right) &= 0 && \text{(dividing both sides by a nonzero constant}^*\text{)}\\
\implies && \left.\frac{dy}{dx}\right|_{(1,\ln(2))} &= -\ln(2)
\end{aligned}$$

Now that we have the slope of the tangent line at the point $(1, \ln(2))$, we use the point-slope form of a line to get the equation of the tangent line.

$$y - \ln(2) = -\ln(2)(x - 1) \implies y = -\ln(2)x + 2\ln(2).$$

Figure 2.7.2 shows the curve and the tangent line.

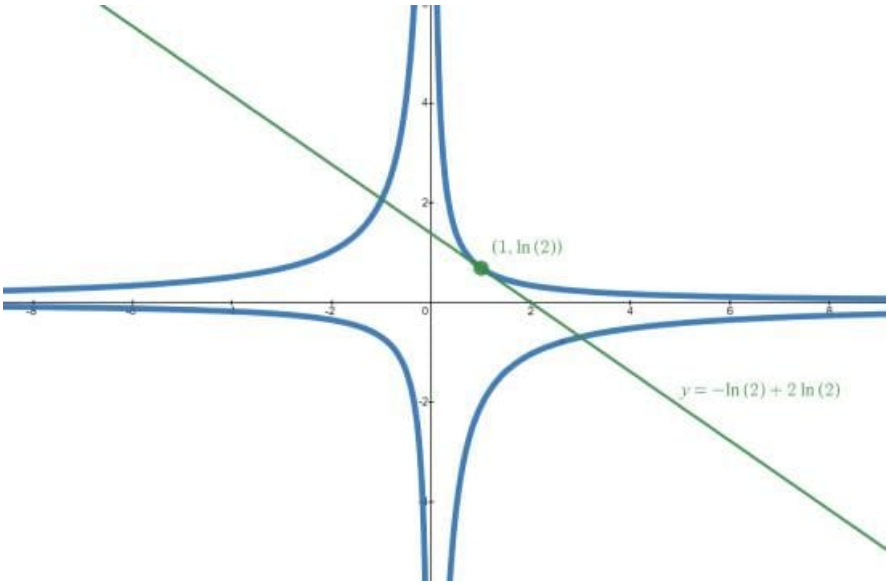

Figure 2.7.2

* I leave it to the reader to check that $\sinh(\ln(2)) + 1 \neq 0$.

Example 2.7.5**: Finding the Equation of the Tangent Line to a Curve**

Begin by implicitly differentiating both sides.

$$\begin{aligned} \frac{d}{dx}\left(y^3 + x^3 - 3xy\right) &= \frac{d}{dx}(0) \\ \Rightarrow \quad 3y^2\frac{dy}{dx} + 3x^2 - \left(3y + 3x\frac{dy}{dx}\right) &= 0 \\ \Rightarrow \quad 3y^2\frac{dy}{dx} + 3x^2 - 3y - 3x\frac{dy}{dx} &= 0 \end{aligned}$$

It is important to read what we are asked to do now. We are not being asked to solve for dy/dx, but instead to find the equation of the tangent line at $\left(\frac{3}{2}, \frac{3}{2}\right)$. As such, we let $x = \frac{3}{2}$ and $y = \frac{3}{2}$ in our implicitly-differentiated equation.

$$\begin{aligned} 3\left(\frac{3}{2}\right)^2\frac{dy}{dx} + 3\left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) - 3\left(\frac{3}{2}\right)\frac{dy}{dx} &= 0 \\ \Rightarrow \quad \frac{27}{4}\frac{dy}{dx} + \frac{27}{4} - \frac{9}{2} - \frac{9}{2}\frac{dy}{dx} &= 0 \\ \Rightarrow \quad 27\frac{dy}{dx} + 27 - 18 - 18\frac{dy}{dx} &= 0 \\ \Rightarrow \quad 9\frac{dy}{dx} + 9 &= 0 \\ \Rightarrow \quad \frac{dy}{dx} &= -1 \end{aligned}$$

Finally, substitute into the point-slope equation of the line to obtain

$$y = -x + 3.$$

Example 2.7.6**: Applying Implicit Differentiation**

To solve this problem, we must determine where the line tangent to the graph of $4x^2 + 25y^2 = 100$ at $\left(3, \frac{8}{5}\right)$ intersects the x-axis. Begin by finding $\frac{dy}{dx}$ implicitly. Differentiating, we have

$$8x + 50y\frac{dy}{dx} = 0.$$

Letting $x = 3$ and $y = \frac{8}{5}$, we get

$$8(3) + 50\left(\frac{8}{5}\right)\frac{dy}{dx} = 0 \Rightarrow 24 + 80\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{3}{10}.$$

The equation of the tangent line is $y = -\frac{3}{10}x + \frac{5}{2}$. To determine where the line intersects the x-axis, solve $0 = -\frac{3}{10}x + \frac{5}{2}$. The solution is $x = \frac{25}{3}$. The missile intersects the x-axis at the point $\left(\frac{25}{3}, 0\right)$.

Checkpoint 2.7.6

$y = \frac{5}{3}x - \frac{16}{3}$

Theorem: Derivatives of the Inverse Trigonometric Functions

A **critical** step in this proof is to recall that the ranges of the inverse trigonometric functions are restricted. For the arcsine, we have

$$-\frac{\pi}{2} \leq \sin^{-1}(x) \leq \frac{\pi}{2}.$$

Therefore,

$$y = \sin^{-1}(x), \text{ where } \sin(y) = x \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}.$$

An illustration of where the angle y lies and the right triangles formed from this arcsine can be seen in Figure 2.7.4.

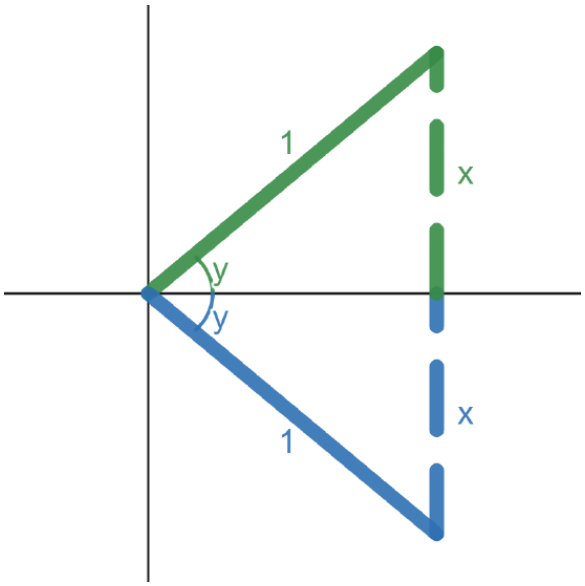

Figure 2.7.4**:** *A graph of where the arcsine returns and the corresponding triangles.*

From Figure 2.7.2, we can see that the length of the adjacent side, in either case, is $+\sqrt{1-x^2}$ (positive because these triangles both share adjacent sides along the positive x-axis).

If $\sin(y) = x$, then

$$\begin{aligned} \frac{d}{dx}(\sin(y)) &= \frac{d}{dx}(x) \implies \cos(y)\frac{dy}{dx} = 1 \\ \implies \frac{dy}{dx} &= \frac{1}{\cos(y)} \\ \implies &= \frac{1}{\sqrt{1-x^2}}. \end{aligned}$$

Q.E.D.

Example 2.7.7**: Finding the Derivative of an Inverse Trigonometric Function**

$$\begin{aligned} \frac{d}{dx}(\sin^{-1}(2x^3)) &= \frac{1}{\sqrt{1-(2x^3)^2}} \cdot \frac{d}{dx}(2x^3) \\ &= \frac{1}{\sqrt{1-4x^6}} \cdot 6x^2 \\ &= \frac{6x^2}{\sqrt{1-4x^6}} \end{aligned}$$

Checkpoint 2.7.7

$g'(x) = \frac{1}{1+x^2}$

Example 2.7.8**: Applying Differentiation Formulas to an Inverse Tangent Function**

$$\begin{aligned} \frac{d}{dx}(\tan^{-1}(x^2)) &= \frac{1}{1+(x^2)^2} \cdot \frac{d}{dx}(x^2) \\ &= \frac{2x}{1+x^4} \end{aligned}$$

Example 2.7.9**: Applying Differentiation Formulas to an Inverse Cosecant Function**

$$\frac{d}{dx}\left(x^2\csc^{-1}(x)\right) = 2x\csc^{-1}(x) - \frac{x^2}{x\sqrt{x^2-1}}$$

$$= 2x\csc^{-1}(x) - \frac{x}{\sqrt{x^2-1}}$$

Checkpoint 2.7.9

$h'(x) = \frac{-3}{\sqrt{6x-9x^2}}$

Example 2.7.10**: Applying the Inverse Tangent Function**

Begin by differentiating $s(t)$ in order to find $v(t)$. Thus,

$$v(t) = s'(t) = \frac{1}{1+\left(\frac{1}{t}\right)^2}\cdot\frac{-1}{t^2}.$$

Simplifying, we have

$$v(t) = -\frac{1}{t^2+1}.$$

Thus, $v(1) = -\frac{1}{2}$.

Checkpoint 2.7.10

$y = x$

Example 2.7.11**: Differentiating Inverse Hyperbolic Functions**

Using the formulas in Table 2.7.3 and the Chain Rule, we obtain the following results:

a. $\frac{d}{dx}(\sinh^{-1}(\frac{x}{3})) = \frac{1}{3\sqrt{1+\frac{x^2}{9}}} = \frac{1}{\sqrt{9+x^2}}$

b. $\frac{d}{dx}(\tanh^{-1}x)^2 = \frac{2(\tanh^{-1}x)}{1-x^2}$

Checkpoint 2.7.11

$\frac{d}{dx}(\cosh^{-1}(3x)) = \frac{3}{\sqrt{9x^2-1}}$

$\frac{d}{dx}(\coth^{-1}x)^3 = \frac{3(\coth^{-1}x)^2}{1-x^2}$

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2.7.1: Resources and Key Concepts

Resources

Prerequisite Topics

The following is a list of prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Solving Equations**
  - Solving Literal Equations: After differentiating implicitly, one needs to algebraically solve for $\frac{dy}{dx}$ (or y'), which is a literal equation solving skill.

Videos

- Implicit differentiation
  - Implicit Differentiation - Differentiating Implicitly
  - Synthesis Questions
- Derivative of inverse trigonometric functions
  - Derivatives of Inverse Trigonometric Functions: Theorem, Proof, and Example
- Derivative of inverse hyperbolic functions
  - Derivatives of Inverse Hyperbolic Functions: Theorem, Proof, and Example

Key Concepts

Definitions

- **Explicit Function**: A function where the dependent variable y is expressed directly in terms of the independent variable x (e.g., $y = x^2 + 1$).
- **Implicit Function**: A function where the relationship between x and y is defined by an equation where y is not expressed entirely in terms of x (e.g., $y - x^2 = 1$ or $x^2 + y^2 = 25$). An equation defines a function implicitly if the function satisfies that equation.
- **Implicit Differentiation**: A technique to find the derivative $\frac{dy}{dx}$ for a function defined implicitly, by differentiating both sides of the equation with respect to x and then algebraically solving for $\frac{dy}{dx}$.

Theorems

- **Theorem: Derivatives of the Inverse Trigonometric Functions**:
  - $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
  - $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$
  - $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
  - $\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$ (Note: text uses specific range for $\csc^{-1} x$)
  - $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$ (Note: text uses specific range for $\sec^{-1} x$)
  - $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$
- **Derivatives of the Inverse Hyperbolic Functions**:
  - $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$
  - $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$ for $x > 1$
  - $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$ for $|x| < 1$
  - $\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}$ for $|x| > 1$

- $\frac{d}{dx}(\text{sech}^{-1}x) = -\frac{1}{x\sqrt{1-x^2}}$ for $0 < x < 1$
- $\frac{d}{dx}(\text{csch}^{-1}x) = -\frac{1}{|x|\sqrt{1+x^2}}$ for $x \neq 0$

Common Mistakes

- **Forgetting $\frac{dy}{dx}$ (or y')**: When differentiating a term involving y with respect to x, a common error is to forget to multiply by $\frac{dy}{dx}$ as required by the Chain Rule. For example, incorrectly differentiating y^2 as $2y$ instead of $2y\frac{dy}{dx}$.
- **Algebraic Errors when Solving for $\frac{dy}{dx}$**: After differentiating both sides, errors can occur when isolating the $\frac{dy}{dx}$ term.
- **Solving for y Prematurely**: While sometimes possible, trying to solve the original equation for y explicitly before differentiating can be much harder or impossible, defeating the purpose of implicit differentiation.
- **Not Substituting Point Values Before Solving for $\frac{dy}{dx}$ (When a specific point is given)**: When asked for the slope at a specific point, it's often easier to substitute the x and y coordinates of the point into the differentiated equation *before* solving for $\frac{dy}{dx}$.
- **Forgetting the Chain Rule**: When differentiating functions like $\ln(u(x))$, $\sin^{-1}(u(x))$, or $\sinh^{-1}(u(x))$, it's crucial to multiply by $u'(x)$.
- **Ignoring Domain Restrictions**: Derivatives of inverse functions (trigonometric and hyperbolic) often have specific domains where they are valid, related to the ranges of the original functions and the domains of the inverse functions. For example, $\frac{d}{dx}(\cosh^{-1}x)$ is for $x > 1$.
- **Inverse Secant and Cosecant Range Ambiguity**: Different textbooks may use different range restrictions for $\sec^{-1}x$ and $\csc^{-1}x$, which can affect the sign of their derivatives or the form involving $|x|$. This textbook explicitly states its chosen ranges.
- **Confusing Derivatives of Similar Functions**: For example, mixing up the derivatives of $\tan^{-1}x$ and $\tanh^{-1}x$. Note that $\frac{d}{dx}(\tanh^{-1}x) = \frac{d}{dx}(\coth^{-1}x) = \frac{1}{1-x^2}$, but their domains differ.

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2.7.2: Homework

Note

The following exercises are meant as hand-in homework (if assigned by your instructor). Typed-in answers here will not be saved or scored.

Basic Skills

Exercise 1

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Basic Skills - Applications

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Synthesis Questions

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Synthesis Questions - Applications

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Proofs and Problems that Require Explanations or Logical Justifications

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2.7.3: Additional Exercises

Reading Questions

1. What is the difference between an explicit function and an implicit function? Give an example of each.
2. What is the main goal of implicit differentiation?
3. When using implicit differentiation to find $\frac{dy}{dx}$, what rule must be applied every time a term involving y is differentiated with respect to x?
4. If you differentiate the term y^3 with respect to x, what is the result?
5. If you differentiate the term xy with respect to x, what rule must you use and what is the result?
6. After differentiating both sides of an implicitly defined equation with respect to x, what is the next general step to find $\frac{dy}{dx}$?
7. In Example 1, where $x^2 + y^2 = 25$, what was $\frac{dy}{dx}$ found to be?
8. Is the expression for $\frac{dy}{dx}$ found through implicit differentiation typically in terms of x only, y only, or both x and y?
9. How can implicit differentiation be used to find the equation of a tangent line to a curve at a given point?
10. What is the "Great Advice to Save Time and Prevent Mistakes" regarding finding the slope at a specific point?
11. How can implicit differentiation be used to find a second derivative, $\frac{d^2y}{dx^2}$?
12. What is the derivative of $f(x) = \sin^{-1} x$? Are there any restrictions on x?
13. What is the derivative of $f(x) = \tan^{-1} x$?
14. The derivatives of inverse trigonometric functions are what type of functions (e.g., trigonometric, algebraic, exponential)?
15. What critical step involving function ranges must be recalled when proving the derivative of an inverse trigonometric function like $\arcsin x$?
16. What is the derivative of $f(x) = \sinh^{-1} x$?
17. What is the derivative of $f(x) = \tanh^{-1} x$?
18. Which two inverse hyperbolic functions have the same derivative formula, $\frac{1}{1-x^2}$, and how do their domains differ for this derivative?
19. How is implicit differentiation used to find the derivatives of inverse hyperbolic functions?

Homework

In exercises 1 - 10, use implicit differentiation to find $\frac{dy}{dx}$.

1) $x^2 - y^2 = 4$

2) $6x^2 + 3y^2 = 12$

Answer

$$\frac{dy}{dx} = \frac{-2x}{y}$$

3) $x^2 y = y - 7$

4) $3x^3 + 9xy^2 = 5x^3$

Answer

$$\frac{dy}{dx} = \frac{x}{3y} - \frac{y}{2x}$$

5) $xy - \cos(xy) = 1$

6) $y\sqrt{x+4} = xy + 8$

Answer

$$\frac{dy}{dx} = \frac{y - \dfrac{y}{2\sqrt{x+4}}}{\sqrt{x+4} - x}$$

7) $-xy - 2 = \frac{x}{7}$

8) $y\sin(xy) = y^2 + 2$

Answer

$$\frac{dy}{dx} = \frac{y^2\cos(xy)}{2y - \sin(xy) - xy\cos(xy)}$$

9) $(xy)^2 + 3x = y^2$

10) $x^3y + xy^3 = -8$

Answer

$$\frac{dy}{dx} = \frac{-3x^2y - y^3}{x^3 + 3xy^2}$$

In exercises 11 - 26, find the derivatives for the functions.

11) $y = \dfrac{1}{\tan^{-1}(x)}$

Answer

$$\frac{dy}{dx} = \frac{-1}{(1+x^2)(\tan^{-1}x)^2}$$

12) $y = x \cdot \csc^{-1}x$

13) $\tanh^{-1}(4x)$

Answer

$$\frac{4}{1 - 16x^2}$$

14) $\sinh^{-1}(x^2)$

15) $\sinh^{-1}(\cosh(x))$

Answer

$$\frac{\sinh(x)}{\sqrt{\cosh^2(x) + 1}}$$

16) $\cosh^{-1}(x^3)$

17) $\tanh^{-1}(\cos(x))$

Answer

$$-\csc(x)$$

18) $e^{\sinh^{-1}(x)}$

19) $y = \sin^{-1}(x^2)$

Answer

$$\frac{dy}{dx} = \frac{2x}{\sqrt{1 - x^4}}$$

20) $y = \sec^{-1}(-x)$

21) $y = \cos^{-1}(2x) \cdot \sin^{-1}(2x)$

22) $y = \cos^{-1}\left(\sqrt{x}\right)$

23) $y = \sec^{-1}\left(\frac{1}{x}\right)$

Answer

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

24) $y = \sqrt{\csc^{-1} x}$

25) $y = (1 + \tan^{-1} x)^3$

Answer

$$\frac{dy}{dx} = \frac{3(1+\tan^{-1} x)^2}{1+x^2}$$

26) $y = \cot^{-1}\sqrt{4-x^2}$

Answer

$$\frac{dy}{dx} = \frac{x}{(5-x^2)\sqrt{4-x^2}}$$

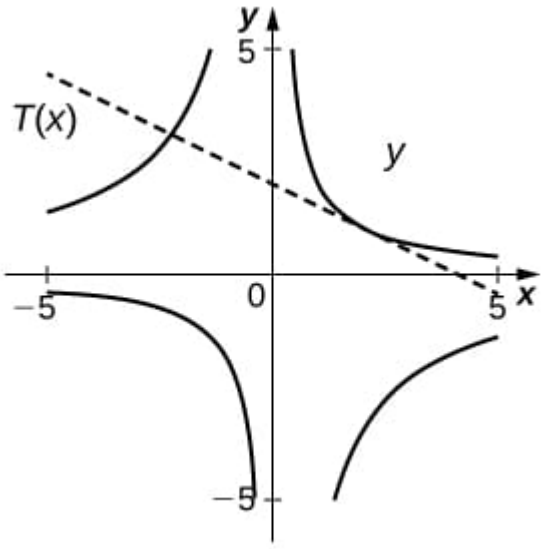
For exercises 27 - 32, find the equation of the tangent line to the graph of the given equation at the indicated point. Use a calculator or computer software to graph the function and the tangent line.

27) [Technology Required] $x^4y - xy^3 = -2, \quad (-1,-1)$

28) [Technology Required] $x^2y^2 + 5xy = 14, \quad (2,1)$

Answer

$y = -\frac{1}{2}x + 2$

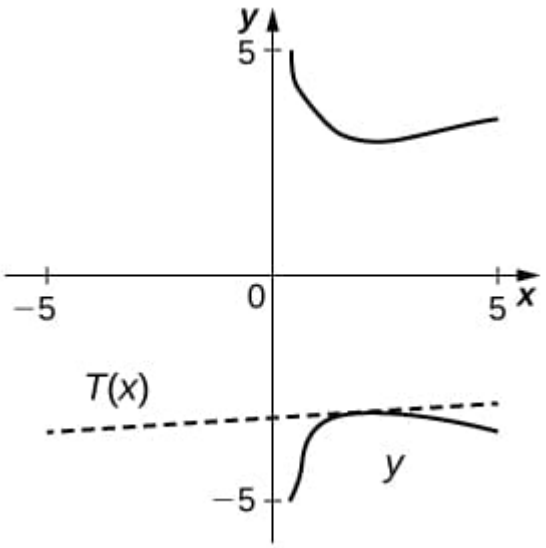


29) [Technology Required] $\tan(xy) = y, \quad \left(\frac{\pi}{4}, 1\right)$

30) [Technology Required] $xy^2 + \sin(\pi y) - 2x^2 = 10, \quad (2,-3)$

Answer

$y = \frac{1}{\pi+12}x - \frac{3\pi+38}{\pi+12}$

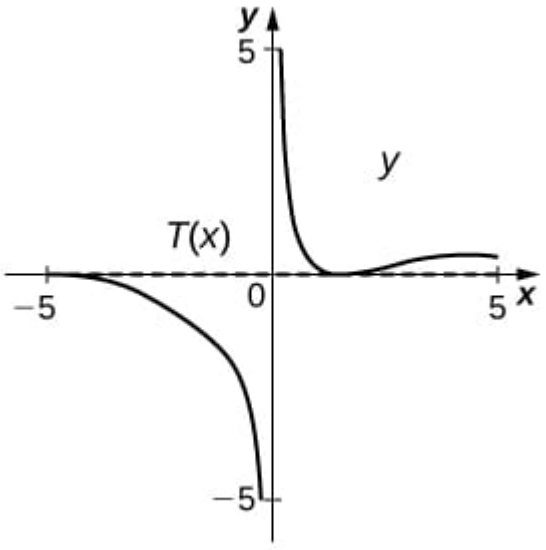


31) [Technology Required] $\frac{x}{y} + 5x - 7 = -\frac{3}{4}y, \quad (1, 2)$

32) [Technology Required] $xy + \sin(x) = 1, \quad (\frac{\pi}{2}, 0)$

Answer

$y = 0$



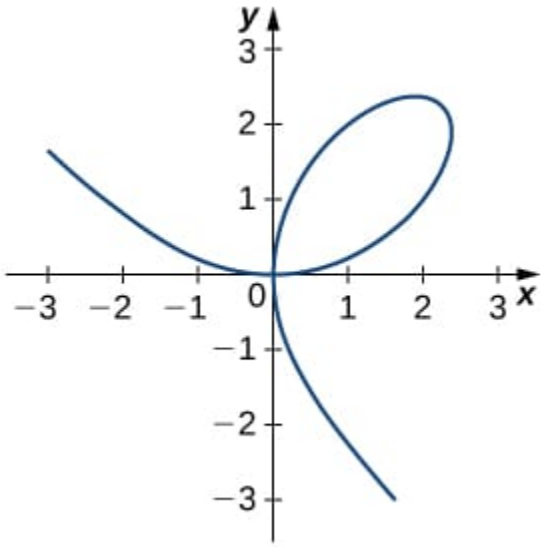
33) Find the equation of the tangent line to the graph of the equation $\sin^{-1} x + \sin^{-1} y = \frac{\pi}{6}$ at the point $(0, \frac{1}{2})$.

34) Find the equation of the tangent line to the graph of the equation $\tan^{-1}(x + y) = x^2 + \frac{\pi}{4}$ at the point $(0, 1)$.

Answer

$y = -x + 1$

35) [Technology Required] The graph of a folium of Descartes with equation $2x^3 + 2y^3 - 9xy = 0$ is given in the following graph.



a. Find the equation of the tangent line at the point $(2, 1)$. Graph the tangent line along with the folium.

b. Find the equation of the normal line to the tangent line in a. at the point $(2, 1)$.

36) For the equation $x^2 + 2xy - 3y^2 = 0$,

a. Find the equation of the normal to the tangent line at the point $(1, 1)$.

b. At what other point does the normal line in a. intersect the graph of the equation?

Answer

a. $y = -x + 2$
b. $(3, -1)$

37) Find all points on the graph of $y^3 - 27y = x^2 - 90$ at which the tangent line is vertical.

38) For the equation $x^2 + xy + y^2 = 7$,

a. Find the x-intercept(s).

b.Find the slope of the tangent line(s) at the x-intercept(s).

c. What does the value(s) in part b. indicate about the tangent line(s)?

Answer

a. $\left(\pm\sqrt{7}, 0\right)$
b. -2
c. They are parallel since the slope is the same at both intercepts.

39) Find y' and y'' for $x^2 + 6xy - 2y^2 = 3$.

40) [Technology Required] The number of cell phones produced when x dollars is spent on labor and y dollars is spent on capital invested by a manufacturer can be modeled by the equation $60x^{3/4}y^{1/4} = 3240$.

a. Find $\frac{dy}{dx}$ and evaluate at the point $(81, 16)$.

b. Interpret the result of a.

Answer

a. $\frac{dy}{dx} = -0.5926$
b. When \$81 is spent on labor and \$16 is spent on capital, the amount spent on capital is decreasing by \$0.5926 per \$1 spent on labor.

41) [Technology Required] The number of cars produced when x dollars is spent on labor and y dollars is spent on capital invested by a manufacturer can be modeled by the equation $30x^{1/3}y^{2/3} = 360$.

(Both $x$and y are measured in thousands of dollars.)

a. Find $\frac{dy}{dx}$ and evaluate at the point $(27, 8)$.

b. Interpret the result of part a.

42) The volume of a right circular cone of radius x and height y is given by $V = \frac{1}{3}\pi x^2 y$. Suppose that the volume of the cone is $85\pi \text{ cm}^3$. Find $\dfrac{dy}{dx}$ when $x = 4$ and $y = 16$.

Answer

$$\frac{dy}{dx} = -8$$

For exercises 43 - 44, consider a closed rectangular box with a square base with side x and height y.

43) Find an equation for the surface area of the rectangular box, $S(x, y)$.

44) If the surface area of the rectangular box is 78 square feet, find $\dfrac{dy}{dx}$ when $x = 3$ feet and $y = 5$ feet.

Answer

$$\frac{dy}{dx} = -2.67$$

In exercises 45 - 47, use implicit differentiation to determine y'.

45) $x = \sin y$

46) $x = \cos y$

Answer

$$y' = -\frac{1}{\sqrt{1 - x^2}}$$

47) $x = \tan y$

48) [Technology Required] The position of a moving hockey puck after t seconds is $s(t) = \tan^{-1} t$ where s is in meters.

a. Find the velocity of the hockey puck at any time t.

b. Find the acceleration of the puck at any time t.

c. Evaluate parts a. and b. for $t = 2,\ 4$,and 6 seconds.

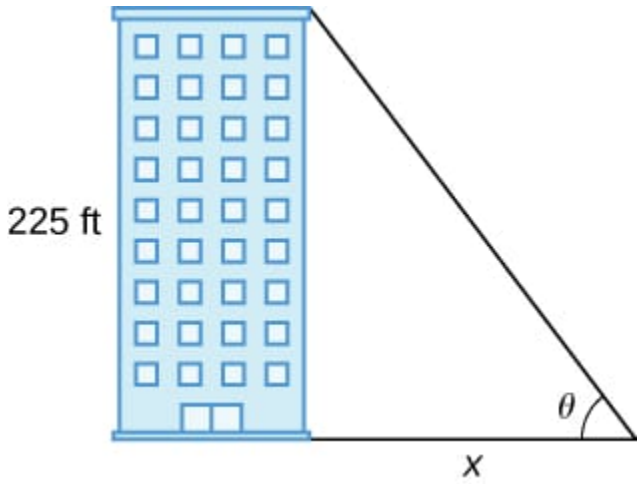
d. What conclusion can be drawn from the results in c.?

Answer

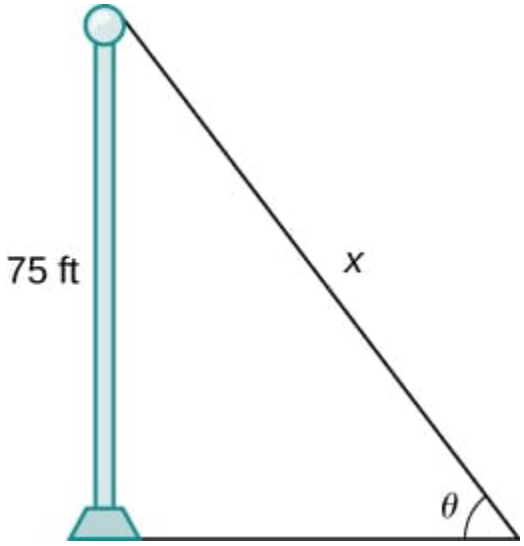
a. $v(t) = \dfrac{1}{1 + t^2}$
b. $a(t) = \dfrac{-2t}{(1 + t^2)^2}$
c. (a) $0.2,\ 0.06,\ 0.03$; (b) $-0.16,\ -0.028,\ -0.0088$

d. The hockey puck is decelerating/slowing down at 2, 4, and 6 seconds.

49) [Technology Required] A building that is 225 feet tall casts a shadow of various lengths x as the day goes by. An angle of elevation θ is formed by lines from the top and bottom of the building to the tip of the shadow, as seen in the following figure. Find the rate of change of the angle of elevation $\frac{d\theta}{dx}$ when $x = 272$ feet.



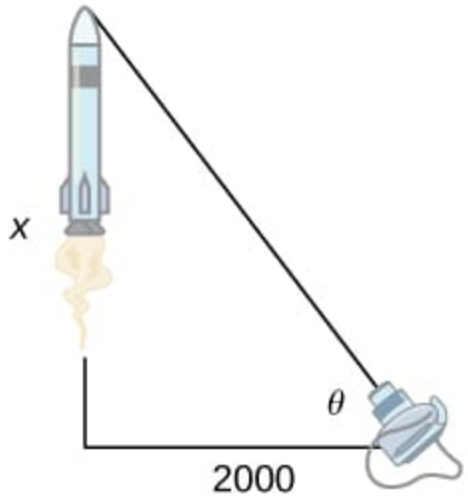
50) [Technology Required] A pole stands 75 feet tall. An angle θ is formed when wires of various lengths of x feet are attached from the ground to the top of the pole, as shown in the following figure. Find the rate of change of the angle $\frac{d\theta}{dx}$ when a wire of length 90 feet is attached.



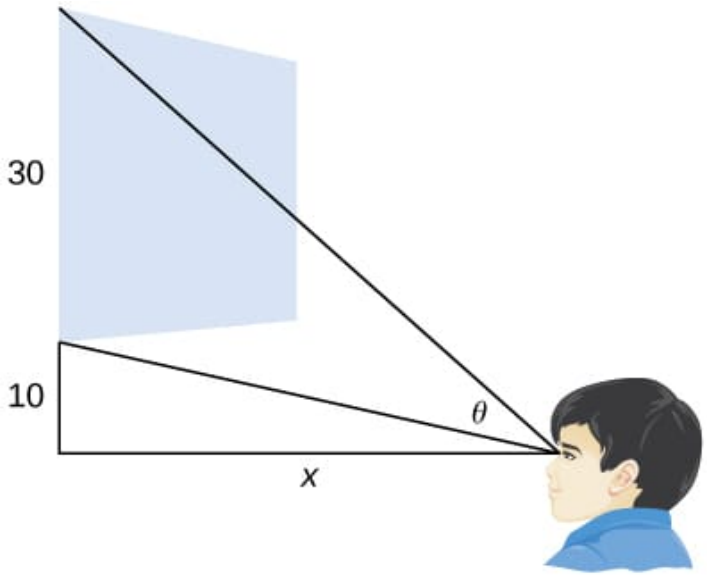
Answer

-0.0168 radians per foot

51) [Technology Required] A television camera at ground level is 2000 feet away from the launching pad of a space rocket that is set to take off vertically, as seen in the following figure. The angle of elevation of the camera can be found by $\theta = \tan^{-1}\left(\frac{x}{2000}\right)$, where x is the height of the rocket. Find the rate of change of the angle of elevation after launch when the camera and the rocket are 5000 feet apart.



52) [Technology Required] A local movie theater with a 30-foot-high screen that is 10 feet above a person's eye level when seated has a viewing angle θ (in radians) given by $\theta = \cot^{-1}\frac{x}{40} - \cot^{-1}\frac{x}{10}$, where x is the distance in feet away from the movie screen that the person is sitting, as shown in the following figure.



a. Find $\frac{d\theta}{dx}$.

b. Evaluate $\frac{d\theta}{dx}$ for $x = 5,\ 10,\ 15$, and 20.

c. Interpret the results in part b.

d. Evaluate $\frac{d\theta}{dx}$ for $x = 25,\ 30,\ 35$, and 40.

e. Interpret the results in part d. At what distance x should the person stand to maximize his or her viewing angle?

Answer

a. $\frac{d\theta}{dx} = \frac{10}{100 + x^2} - \frac{40}{1600 + x^2}$

b. $\frac{18}{325},\ \frac{9}{340},\ \frac{42}{4745},\ 0$

c. As a person moves farther away from the screen, the viewing angle is increasing, which implies that as he

or she moves farther away, his or her screen vision is widening. d. $-\frac{54}{12905}$, $-\frac{3}{500}$, $-\frac{198}{29945}$, $-\frac{9}{1360}$
e. As the person moves beyond 20 feet from the screen, the viewing angle is decreasing. The optimal distance the person should stand for maximizing the viewing angle is 20 feet.

53) Prove the formula for the derivative of $y = \sinh^{-1}(x)$ by differentiating $x = \sinh(y)$.

54) Prove the formula for the derivative of $y = \cosh^{-1}(x)$ by differentiating $x = \cosh(y)$.

55) Prove the formula for the derivative of $y = \operatorname{sech}^{-1}(x)$ by differentiating $x = \operatorname{sech}(y)$.

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2.8: Derivatives of Logarithms and Logarithmic Differentiation

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Now that we have the Chain Rule and implicit differentiation under our belts, we can explore the derivatives of logarithmic functions and the relationship between a function's derivative and its inverse's derivative. For functions whose derivatives we already know, we can use this relationship to find derivatives of inverses without using the limit definition of the derivative. In particular, we will apply the formula for derivatives of inverse functions to trigonometric functions. This formula may also be used to extend the Power Rule to rational exponents.

Derivative of the Logarithmic Function

Now that we have the derivative of the natural exponential function, we can use implicit differentiation to find the derivative of its inverse, the natural logarithmic function.

Theorem: The Derivative of the Natural Logarithmic Function

If $y = \ln x$, then

$$\frac{dy}{dx} = \frac{1}{x}.$$

Proof available at the end of this section.

The graph of $y = \ln x$ and its derivative $\frac{dy}{dx} = \frac{1}{x}$ are shown in Figure $2.8.1$.

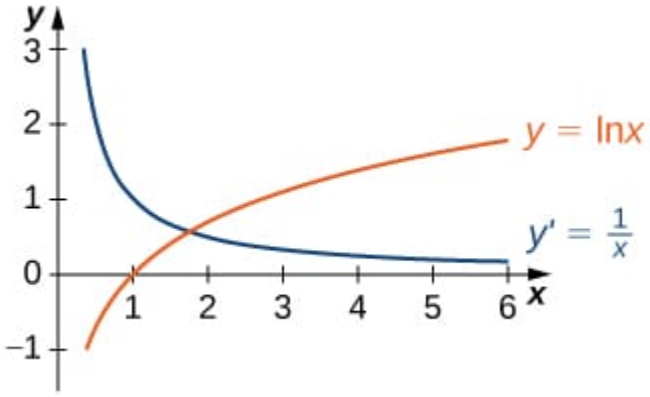

Figure $2.8.1$: *The function $y = \ln x$ is increasing on $(0, \infty)$. Its derivative $y' = \frac{1}{x}$ is greater than zero on $(0, \infty)$*

Interactive Element: The Slope of $\ln x$ Is $\frac{1}{x}$

The following Interactive Element might help you visualize why the derivative of $y = \ln x$ is $\frac{1}{x}$.

Interact: Drag the red point along the curve $y = \ln x$.

Observation: The slope of the tangent line always equals the height of the $\frac{1}{x}$ curve at the same x — large near 0 and flattening as x grows, yet always positive, so $\ln x$ is always increasing.

Example 2.8.1: Taking a Derivative of a Natural Logarithm

Find the derivative of $f(x) = \ln(x^3 + 3x - 4)$.

Solution available at the end of this section.

Example 2.8.2: Using Properties of Logarithms in a Derivative

Find the derivative of $f(x) = \ln\left(\frac{x^2 \sin x}{2x+1}\right)$.

Solution available at the end of this section.

Checkpoint 2.8.2

Differentiate: $f(x) = \ln(3x + 2)^5$.

Answer available at the end of this section.

Now that we can differentiate the natural logarithmic function, we can use this result to find the derivatives of $y = \log_b x$ for $b > 0$, $b \neq 1$.

Theorem: The General Derivative of a Logarithmic Function

$$\frac{d}{dx}\left(\log_b(x)\right) = \frac{1}{x \ln(b)}$$

Proof available at the end of this section.

Example 2.8.3: Finding the Slope of a Tangent Line

Find the slope of the line tangent to the graph of $y = \log_2(3x + 1)$ at $x = 1$.

Solution available at the end of this section.

Aside: Revisiting the Generalized Power Rule

We now have enough "mathematical prowess" to continue our proof of the Generalized Power Rule from earlier in this chapter. This time, we add the ability to take the derivative of $y = x^n$, where n is a rational number.

Proof

Let $y = x^n$, where n is a rational number. Then n can be written as p/q, for integers p and q.

$$
\begin{aligned}
y = x^n &\implies y = x^{p/q} \\
&\implies y^q = x^p \\
&\implies \frac{d}{dx}\left(y^q\right) = \frac{d}{dx}\left(x^p\right) \\
&\implies qy^{q-1}\frac{dy}{dx} = px^{p-1} \\
&\implies \frac{dy}{dx} = \frac{px^{p-1}}{qy^{q-1}} \\
&\implies \frac{dy}{dx} = \frac{p}{q}\cdot\frac{x^{p-1}}{y^{q-1}} \\
&\implies \frac{dy}{dx} = \frac{p}{q}\cdot\frac{x^{p-1}}{\left(x^{p/q}\right)^{q-1}} \\
&\implies \frac{dy}{dx} = \frac{p}{q}\cdot\frac{x^{p-1}}{x^{p-p/q}} \\
&\implies \frac{dy}{dx} = \frac{p}{q}\cdot x^{p/q-1} \\
&\implies \frac{dy}{dx} = nx^{n-1}
\end{aligned}
$$

Q.E.D.

Logarithmic Differentiation

At this point, we can take derivatives of functions of the form $y = (g(x))^n$ for certain values of n, as well as functions of the form $y = b^{g(x)}$, where $b > 0$ and $b \neq 1$. Unfortunately, we still do not know the derivatives of functions such as $y = x^x$ or $y = x^{\sin(x)}$. These functions require a technique called **logarithmic differentiation**, which allows us to differentiate any function of the form $h(x) = g(x)^{f(x)}$. It can also be used to convert a very complex differentiation problem into a simpler one, such as finding the derivative of $y = \frac{x\sqrt{2x+1}}{e^x \sin^3 x}$.

Example 2.8.4: Using Logarithmic Differentiation

Find the derivative of $y = (2x^4 + 1)^{\tan x}$.

Solution available at the end of this section.

Example 2.8.5: Simplifying a "Tough" Differentiation Problem

Find the derivative of $y = \frac{x\sqrt{2x+1}}{e^x \sin^3 x}$.

Solution available at the end of this section.

Checkpoint 2.8.5A

Use logarithmic differentiation to find the derivative of $y = x^x$.

Answer available at the end of this section.

Checkpoint 2.8.5B

Find the derivative of $y = (\tan x)^\pi$.

Answer available at the end of this section.

Miscellaneous Topics: Thinking About Mathematics (Part 2)

A significant concern in mathematics education is the trend that students rely on calculators more. An argument for calculator usage is that, in "real life," a person will always have access to a computer or a calculator. Therefore, forcing students to do problems "by hand" is useless.

1. Give an argument against high calculator usage in math classes.
2. Do you think that your professor relies too much on calculator usage in class? What about their use of technology? Would you like more or less?

Hidden Resources: Adobe Creative Cloud

The Adobe Creative Cloud is a collection of Adobe software products. In addition to offering the products, the Creative Cloud also offers cloud storage.

Apps Included in the Adobe Creative Suite are:

- Design Apps: Photoshop, InCopy, Spark, Illustrator, Adobe XD, Dimension, InDesign
- Web & UX Apps: Adobe XD, Spark, Dreamweaver, Animate, InCopy, InDesign, Photoshop
- Photography Apps: Photoshop, Lightroom
- Video & Audio Apps: Premiere Pro, Premiere Rush, After Effects, Spark, Audition

Typically, the Adobe Creative Suite costs $55/month, but Los Rios students have access to Adobe Creative Cloud for $39.99 per semester or $79.99 per year. Check out this page for more details.

Answers

Section Preview

The ultimate goal of this section (other than just making sure students understand logarithmic differentiation) is to make sure students can take the derivative of a function raised to a functional power.

Section Preview

- **Logarithmic Functions**
  - Laws of Logarithms — Product and Quotient Laws: Objective 2 (differentiating products and ratios) depends conceptually on understanding that logs

convert products to sums and quotients to differences; without this idea, the technique in Examples 2 and 5 has no rationale.

Section Preview

- Compute the derivative of a logarithmic function, both natural-based and non-natural-based.
- Use logarithmic differentiation to determine the derivative of products and ratios of functions.
- Use logarithmic differentiation to determine the derivative of functions to functional powers.

Theorem: The Derivative of the Natural Logarithmic Function

If $y = \ln x$, then $e^y = x$. Differentiating both sides of this equation results in the equation

$$e^y \frac{dy}{dx} = 1.$$

Solving for $\frac{dy}{dx}$ yields

$$\frac{dy}{dx} = \frac{1}{e^y}.$$

Finally, we substitute $x = e^y$ to obtain

$$\frac{dy}{dx} = \frac{1}{x}.$$

Q.E.D.

Example 2.8.1**: Taking a Derivative of a Natural Logarithm**

$$\begin{aligned} f'(x) &= \frac{1}{x^3+3x-4} \cdot \frac{d}{dx}\left(x^3+3x-4\right) \qquad \text{(Chain Rule)} \\ &= \frac{1}{x^3+3x-4} \cdot (3x^2+3) \\ &= \frac{3x^2+3}{x^3+3x-4} \\ &= \frac{3(x^2+1)}{x^3+3x-4} \end{aligned}$$

As an aside, cleaning up derivatives using our prerequisite Algebra is always a good idea. Sometimes, things clean up very nicely.

Example 2.8.2**: Using Properties of Logarithms in a Derivative**

At first glance, taking this derivative appears rather complicated. However, by using the properties of logarithms before finding the derivative, we can make the problem much simpler.

$$f(x) = \ln\left(\frac{x^2 \sin x}{2x+1}\right) = 2\ln x + \ln(\sin x) - \ln(2x+1)$$

Hence,

$$\begin{aligned} f'(x) &= \frac{2}{x} + \frac{1}{\sin x} \cdot \cos x - \frac{1}{2x+1} \cdot 2 \qquad \text{(derivative of the natural log and the Chain Rule)} \\ &= \frac{2}{x} + \cot x - \frac{2}{2x+1} \end{aligned}$$

Checkpoint 2.8.2

$f'(x) = \frac{15}{3x+2}$

Theorem: The General Derivative of a Logarithmic Function

If $y = \log_b x$, then $b^y = x$. It follows that $\ln(b^y) = \ln x$. Thus $y \ln b = \ln x$. Solving for y, we have $y = \frac{\ln x}{\ln b}$. Differentiating and keeping in mind that $\ln b$ is a constant, we see that

$$\frac{dy}{dx} = \frac{1}{x \ln b}.$$

Q.E.D.

Example 2.8.3**: Finding the Slope of a Tangent Line**

To find the slope, we must evaluate $\frac{dy}{dx}$ at $x = 1$.

$$\frac{dy}{dx} = \frac{3}{(3x+1)\ln 2}.$$

By evaluating the derivative at $x = 1$, we see that the tangent line has slope

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{3}{4\ln 2} = \frac{3}{\ln 16}.$$

Example 2.8.4**: Using Logarithmic Differentiation**

Use logarithmic differentiation to find this derivative.

$$\begin{aligned}
\ln(y) &= \ln\left(2x^4+1\right)^{\tan(x)} && \text{(taking the natural logarithm of both sides)}\\
\Longrightarrow \quad \ln(y) &= \tan(x)\ln\left(2x^4+1\right) && \text{(Properties of Logarithms)}\\
\Longrightarrow \quad \frac{1}{y}\frac{dy}{dx} &= \sec^2(x)\ln\left(2x^4+1\right)+\frac{8x^3}{2x^4+1}\cdot\tan(x) && \text{(implicitly differentiating)}\\
\Longrightarrow \quad \frac{dy}{dx} &= y\cdot\left(\sec^2(x)\ln\left(2x^4+1\right)+\frac{8x^3}{2x^4+1}\cdot\tan(x)\right)\\
\Longrightarrow \quad \frac{dy}{dx} &= \left(2x^4+1\right)^{\tan(x)}\left(\sec^2(x)\ln\left(2x^4+1\right)+\frac{8x^3}{2x^4+1}\cdot\tan(x)\right) && \left(\text{substituting } y=\left(2x^4+1\right)^{\tan(x)}.\right)
\end{aligned}$$

Example 2.8.5**: Simplifying a "Tough" Differentiation Problem**

This problem uses the Properties of Logarithms and the differentiation rules introduced in this chapter.

$$\begin{aligned}
\ln y &= \ln\frac{x\sqrt{2x+1}}{e^x\sin^3 x} && \text{(taking the natural logarithm of both sides)}\\
\Longrightarrow \quad \ln y &= \ln x+\tfrac{1}{2}\ln(2x+1)-x\ln e-3\ln\sin x && \text{(Properties of Logarithms)}\\
\Longrightarrow \quad \frac{1}{y}\frac{dy}{dx} &= \frac{1}{x}+\frac{1}{2x+1}-1-3\frac{\cos x}{\sin x} && \text{(implicitly differentiating)}\\
\Longrightarrow \quad \frac{dy}{dx} &= y\left(\frac{1}{x}+\frac{1}{2x+1}-1-3\cot x\right)\\
\Longrightarrow \quad \frac{dy}{dx} &= \frac{x\sqrt{2x+1}}{e^x\sin^3 x}\left(\frac{1}{x}+\frac{1}{2x+1}-1-3\cot x\right) && \left(\text{substituting } y=\frac{x\sqrt{2x+1}}{e^x\sin^3 x}.\right)
\end{aligned}$$

Checkpoint 2.8.5A

$\frac{dy}{dx}=x^x(1+\ln x)$

Checkpoint 2.8.5B

$y'=\pi(\tan x)^{\pi-1}\sec^2 x$

This page titled 2.8: Derivatives of Logarithms and Logarithmic Differentiation is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

2.8.1: Resources and Key Concepts

Resources

Videos

- Derivative of logarithmic functions
  - Logarithmic Derivatives: Theorems, Proofs, and Example
- Derivative of function to a functional power
  - Derivatives of Functions to Functional Powers
  - Synthesis Questions

Key Concepts

Definitions

- **Logarithmic Differentiation**: A technique used to differentiate functions, particularly those of the form $h(x) = g(x)^{f(x)}$ or complex products/quotients, by first taking the natural logarithm of both sides of $y = h(x)$, using properties of logarithms to simplify, then differentiating implicitly with respect to x, and finally solving for $\frac{dy}{dx}$.

Theorems

- **Theorem: The Derivative of the Natural Logarithmic Function**: If $y = \ln x$, then $\frac{dy}{dx} = \frac{1}{x}$.
- **Theorem: The General Derivative of a Logarithmic Function**: $\frac{d}{dx}(\log_b(x)) = \frac{1}{x \ln(b)}$.

Common Mistakes

- **Incorrectly Applying Logarithm Properties**: Errors in using logarithm properties (e.g., $\ln(A + B) \neq \ln A + \ln B$) before differentiation can lead to incorrect derivatives.
- **Errors in Applying Logarithm Properties**: Incorrectly expanding or simplifying the logarithmic expression (e.g., $\ln(A + B) \neq \ln A + \ln B$) before differentiation.
- **Forgetting to Differentiate Implicitly**: After taking $\ln y$, forgetting that its derivative with respect to x is $\frac{1}{y}\frac{dy}{dx}$, not just $\frac{1}{y}$.
- **Not Multiplying by y at the End**: After finding $\frac{1}{y}\frac{dy}{dx} = \text{expression}$, forgetting to multiply the "expression" by y (and substituting back the original function for y) to solve for $\frac{dy}{dx}$.
- **Applying Logarithmic Differentiation When it Cannot Be Applied**: While it might seem to be attractive to use logarithmic differentiation for any function that contains compositions of powers of rational functions, you must remain aware that the argument of a logarithm has to be positive. Therefore, using logarithmic differentiation to find the derivative of, for example, $y = \sqrt[3]{\frac{(x-2)^3}{x(x+8)^2}}$ would not work because the radicand *could* be negative (we can take odd roots of negative numbers).

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2.8.2: Homework

Note

The following exercises are meant as hand-in homework (if assigned by your instructor). Typed-in answers here will not be saved or scored.

Basic Skills

Exercise 1

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Synthesis Questions

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2.8.3: Additional Exercises

Reading Questions

1. What is the derivative of $f(x) = \ln x$?
2. How is implicit differentiation used to find the derivative of $y = \ln x$?
3. What is the derivative of $f(x) = \log_b x$? How does it relate to the derivative of $\ln x$?
4. Why is it often beneficial to use properties of logarithms before differentiating a function involving logarithms?
5. What is the primary purpose of logarithmic differentiation? For what types of functions is it particularly useful?
6. What is the first step in performing logarithmic differentiation for a function $y = f(x)$?
7. Which properties of logarithms are commonly used to simplify the expression after taking the natural logarithm of both sides? Give an example for $\ln(A \cdot B)$, $\ln(A/B)$, and $\ln(A^n)$.
8. After simplifying $\ln y$, what differentiation technique is applied to both sides of the equation with respect to x?
9. When differentiating $\ln y$ with respect to x, what is the result?
10. After differentiating both sides and obtaining an expression for $\frac{1}{y}\frac{dy}{dx}$, what is the final step to find $\frac{dy}{dx}$?
11. For the function $y = (2x^4 + 1)^{\tan x}$, what does $\ln y$ become after applying logarithm properties?
12. Can logarithmic differentiation be used for functions that could also be differentiated by the Product or Quotient Rule? What might be an advantage of using it in such cases?
13. When would you choose to use logarithmic differentiation over other known rules?

Homework

In exercises 1 - 12, find the derivatives for the functions.

1) $f(x) = \sinh(\ln(x))$

2) $\cos(\ln x)$

3) $f(x) = \ln(4x^3 + x)$

4) $f(x) = \log(\sec x)$

Answer

$$f'(x) = \frac{\tan x}{\ln 10}$$

5) $\ln(\tanh^{-1}(x))$

Answer

$$-\frac{1}{(x^2-1)\tanh^{-1}(x)}$$

6) $f(x) = \ln(\operatorname{sech}(x) + \tanh(x))$

7) $f(x) = e^{x^3 \ln x}$

Answer

$$f'(x) = e^{x^3 \ln x}\left(3x^2 \ln x + x^2\right)$$

8) $f(x) = x^2 \ln 9x$

9) $f(x) = 2^x \cdot \log_3 7^{x^2-4}$

Answer

$$f'(x) = 2^x \cdot \ln 2 \cdot \log_3 7^{x^2-4} + 2^x \cdot \frac{2x \ln 7}{\ln 3}$$

10) $\ln\left(\cos^2 x\right)$

11) $f(x) = \ln\sqrt{5x-7}$

12) $f(x) = \log_7\left(6x^4+3\right)^5$

Answer

$$f'(x) = \frac{5}{2(5x-7)}$$

13) [Technology Required] Find the equation of the tangent line to the graph of $x^3 - x\ln y + y^3 = 2x + 5$ at the point where $x = 2$. (Hint: Use implicit differentiation to find $\frac{dy}{dx}$.) Graph both the curve and the tangent line.

For exercises 14 - 21, use logarithmic differentiation to find $\frac{dy}{dx}$.

14) $y = x^{\sqrt{x}}$

15) $y = (\sin 2x)^{4x}$

Answer

$$\frac{dy}{dx} = (\sin 2x)^{4x}\left[4\cdot\ln(\sin 2x) + 8x\cdot\cot 2x\right]$$

16) $y = (\ln x)^{\ln x}$

17) $y = x^{\log_2 x}$

Answer

$$\frac{dy}{dx} = x^{\log_2 x}\cdot\frac{2\ln x}{x\ln 2}$$

18) $y = (x^2-1)^{\ln x}$

19) $y = x^{\cot x}$

Answer

$$\frac{dy}{dx} = x^{\cot x}\cdot\left[-\csc^2 x\cdot\ln x + \frac{\cot x}{x}\right]$$

20) $y = \dfrac{x+11}{\sqrt[3]{x^2-4}}$

21) $y = x^{-1/2}(x^2+3)^{2/3}(3x-4)^4$

Answer

$$\frac{dy}{dx} = x^{-1/2}(x^2+3)^{2/3}(3x-4)^4\cdot\left[\frac{-1}{2x} + \frac{4x}{3(x^2+3)} + \frac{12}{3x-4}\right]$$

22) Consider the function $y = x^{1/x}$ for $x > 0$.

a. Determine the points on the graph where the tangent line is horizontal.

b. Determine the points on the graph where $y' > 0$ and those where $y' < 0$.

Answer

a. $x = e \approx 2.718$
b. $y' > 0$ for $(0, e)$ and $y' < 0$ for (e, ∞).

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2.9: Related Rates

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

We have seen that for quantities that change over time, the rates at which these quantities change are given by derivatives. If two related quantities are changing over time, the rates at which the quantities change are related. For example, suppose a balloon is being filled with air. In that case, both the balloon's radius and the balloon's volume are increasing. In this section, we consider several problems in which two or more related quantities are changing, and we study how to determine the relationship between the rates of change of these quantities.

Setting up Related-Rates Problems

In many real-world applications, related quantities are changing with respect to time. For example, let's consider the balloon example again. The rate of change in the volume, V, is related to the rate of change in the radius, r. In this case, we say that $\frac{dV}{dt}$ and $\frac{dr}{dt}$ are **related rates** because V is related to r. Here, we study several examples of related quantities that are changing with respect to time, and we look at how to calculate one rate of change given another rate of change.

Example 2.9.1: Inflating a Balloon

A spherical balloon is being filled with air at the constant rate of $2\ \text{cm}^3/\text{sec}$ (Figure 2.9.1). How fast is the radius increasing when the radius is 3 cm?

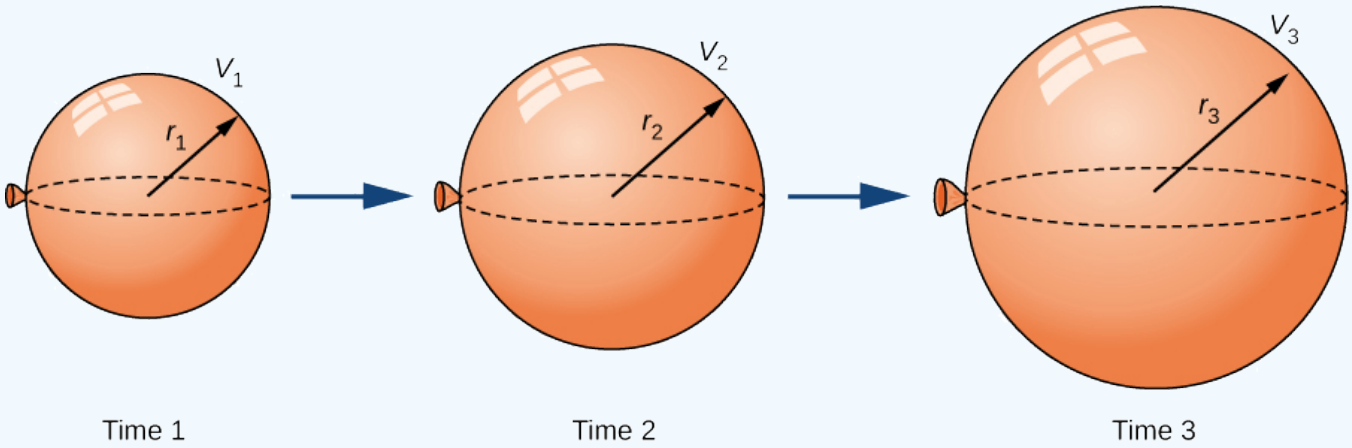


Figure 2.9.1: *As the balloon is filled with air, both the radius and the volume are increasing with respect to time.*

Solution available at the end of this section.

Checkpoint 2.9.1

What is the instantaneous rate of change of the radius when $r = 6$ cm?

Answer available at the end of this section.

While memorizing procedures is the absolute worst way to enjoy Mathematics, applications (also known as "word problems") tend to confound students so much that it is beneficial to develop a procedure for solving them. Let's outline a modified problem-solving strategy developed by Polya.

In 1945 George Polya published the book *How To Solve It* which quickly became his most prized publication. It sold over one million copies and has been translated into 17 languages. In this book, he identifies basic principles of problem-solving. I have modified these problem-solving strategies to fit Calculus best (we will see these again in a section titled Optimization Problems).

Polya's Modified Problem-Solving Process

1. **Read** the given problem.
2. **Understand** the given problem. This might require you to rephrase the problem in terms that you can understand or, more commonly, to draw a picture of the situation.
3. **Label** unknowns. All problems presented in mathematics have unknown quantities. In this step, you take the time to label these unknowns using variables. This step is often combined with the previous step (especially when drawing pictures).
4. List the **Givens** and **Wants**. This step is dedicated to making a table with one column dedicated to the given information and another to the wanted or desired information. Normally, we only list the given and desired *rates* in this table.
5. Create a **Master Equation** involving the variables within the given and wanted rate information. This is a critical step in all problem-solving processes involving Polya's method.
   Warning: The Master Equation should be a relation between two variables (those listed in the Givens and Wants table). If your Master Equation has more than two variables, you need to find a relation between the excess variables to reduce the number of final variables in your Master Equation to two.
6. Find the **Rate Equation** from the Master Equation. This almost always requires implicit differentiation.
7. **Substitute** in any constants given within the problem.
8. **Solve** the resulting equation for the desired "wanted" piece of information.

Note that when solving a related-rates problem, it is crucial not to **substitute** known values too soon. For example, suppose the value for a changing quantity is substituted into an equation before both sides are differentiated. In that case, the quantity will behave as a constant, and its derivative will not appear in the new equation found during the **rate equation** step. We examine this potential error in the following example.

Examples of Polya's Problem-Solving Process

Let's implement the described strategy to solve several related-rates problems. The first example involves a plane flying overhead. The relationship we are studying is between the speed of the aircraft and the rate at which the distance between the plane and a person on the ground is changing.

Example 2.9.2: An Airplane Flying at a Constant Elevation

An airplane is flying overhead at a constant elevation of 4000 ft. A man is viewing the plane from a position 3000 ft from the base of a radio tower. The airplane is flying horizontally away from the man. If the plane is flying at the rate of 600 ft/sec, at what rate does the distance between the man and the plane increase when the plane passes over the radio tower?

Solution available at the end of this section.

Interactive Element: The Distance-Rate Depends on Where the Plane Is

The following Interactive Element might help you visualize how the rate at which the plane's distance grows depends on the plane's position, even though the plane's own speed stays constant.

Interact: Drag the red plane along its flight path — each position is one frozen instant — and read the live value of $\frac{ds}{dt}$.

Observation: Directly overhead the distance barely changes, $\frac{ds}{dt} \approx 0$; far away $\frac{ds}{dt}$ climbs toward the plane's full 600 ft/sec. Line the plane up with the tower at $x = 3000$ to recover the 360 ft/sec from the example — the plane's speed never changes, but the rate the distance grows does.

Checkpoint $2.9.2$

What is the speed of the plane if the distance between the person and the plane is increasing at the rate of 300 ft/sec?

Answer available at the end of this section.

Example $2.9.3$: A Rocket Launch

A rocket is launched so that it rises vertically. A camera is positioned 5000 ft from the launch pad. When the rocket is 1000 ft above the launch pad, its velocity is 600 ft/sec.

Figure $2.9.3$: *(credit: modification of work by Steve Jurvetson, Wikimedia Commons)*

Find the necessary rate of change of the camera's angle as a function of time so that it stays focused on the rocket.

Solution available at the end of this section.

Checkpoint $2.9.3$

What rate of change is necessary for the elevation angle of the camera if the camera is placed on the ground at a distance of 4000 ft from the launch pad and the velocity of the rocket is 500 ft/sec when the rocket is 2000 ft off the ground?

Answer available at the end of this section.

In the following example, we consider water draining from a cone-shaped funnel. We compare the rate at which the level of water in the cone is decreasing with the rate at which the volume of water is decreasing.

Example 2.9.4: Water Draining from a Funnel

Water is draining from the bottom of a cone-shaped funnel at the rate of $0.03\ \text{ft}^3/\text{sec}$. The height of the funnel is 2 ft, and the radius at the top of the funnel is 1 ft. At what rate is the height of the water in the funnel changing when the height of the water is $\frac{1}{2}$ ft?

Solution available at the end of this section.

Checkpoint 2.9.4

At what rate is the height of the water changing when the height of the water is $\frac{1}{4}$ ft?

Answer available at the end of this section.

Success in College: Negative Talk

Sometimes math students use “negative self-talk” about math. They say, “I can’t do math” instead of, “I can do some math and, with hard work, I’ll be able to do more.” They say, “I should be able to work faster” instead of “I can work fast enough to pass the course.”

It has been proven that positive thinking is part of success. Being an optimist about math, but more importantly, about life, is a crucial building block of success.

A favorite quote of mine comes from Thomas Edison. When asked how he felt about so many failed tests before finally inventing the lightbulb, he responded, “I didn’t fail the tests; I just found 100 ways to do it wrong.”

Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. What effect(s) do you think negative self-talk can have on a student’s attitude about math? Why?
2. What effect do you think negative self-talk can have on a student’s performance in math? Why?

Success in Mathematics: Why Study Mathematics? (Part 3)

Matthew Jannsen graduated with a degree in mathematics in 2000. He went to law school and is now an attorney. He writes,

> "At this stage of my career I spend a majority of my time conducting legal research and drafting legal memoranda. Most people assume that mathematics would not be the most appropriate preparation for a person who makes their living engaging in these activities. I would disagree. In fact, I would suggest that my mathematics background has been one of my greatest assets throughout my law school and legal careers and will continue to help me develop as a lawyer throughout my career. In order to succeed as a math major, I always had to engage in logical thought... I constantly rely on the logical thinking skills that I developed as a math major in executing my tasks as an attorney and I am grateful for my mathematics education."
>
> *Source: http://www.maa.org/careers/janssen.html*

1. Describe, in detail, what you think it means to "think logically."
2. Is logical thought a vital skill in the career that you hope to have? Explain.

Answers

Section Preview

This material (and the optimization material in the next chapter) are *great* applications of differentiation. There aren't any new theorems or definitions here—just bending the mind to visualize *how* to apply differentiation in novel settings.

Section Preview

- **Geometry**
  - The Pythagorean Theorem, Distance Formula, and Midpoint Formula: In Examples $2.9.2$ (airplane) and $2.9.3$ (rocket), the master equation relating the changing quantities *is* the Pythagorean relationship $[x]^2 + 4000^2 = [s]^2$; the student must grasp this right-triangle relationship to model the geometry before differentiating.
  - Similar Triangles: Example $2.9.4$ (draining funnel) hinges on recognizing that the water's surface radius and its height form triangles similar to the funnel, so that $r = \frac{h}{2}$; this proportional reasoning is what lets the master equation be reduced to a single variable.
  - Volumes and Surface Areas of Common Shapes: The master equations in Examples $2.9.1$ and $2.9.4$ are built from the volume of a sphere $\left(V = \frac{4}{3}\pi r^3\right)$ and the volume of a cone $\left(V = \frac{1}{3}\pi r^2 h\right)$; the student must already know these geometric relationships to set the problems up.
- **Trigonometry**
  - Trigonometric Functions — Right Triangle Definition: Example $2.9.3$ forms its master equation $h = 5000\tan(\theta)$ from the right-triangle definition of tangent (opposite over adjacent) and later interprets $\sec(\theta)$ as hypotenuse over adjacent; understanding trig ratios as side ratios is conceptually required, and the source explicitly cues it with "Recall that..."
  - Law of Cosines: This is needed for examples within the homework.

Section Preview

- Express changing quantities in terms of derivatives.
- Find relationships among the derivatives in a given problem.
- Use the Chain Rule to find the rate of change of one quantity that depends on the rate of change of other quantities.
- Apply the modified version of Polya's Problem-Solving Strategy to solve related rate problems.

Example $2.9.1$: Inflating a Balloon

The volume of a sphere of radius r centimeters is

$$V = \frac{4}{3}\pi r^3 \text{ cm}^3.$$

Since the balloon is being filled with air, both the volume and the radius are functions of time. Therefore, t seconds after beginning to fill the balloon with air, the volume of air in the balloon is

$$V(t) = \frac{4}{3}\pi\big[r(t)\big]^3 \text{cm}^3.$$

Differentiating both sides of this equation with respect to time and applying the Chain Rule, we see that the rate of change in the volume is related to the rate of change in the radius by the equation

$$V'(t) = 4\pi\big[r(t)\big]^2 r'(t).$$

The balloon is being filled with air at the constant rate of $2\ \text{cm}^3/\text{sec}$, so $V'(t) = 2\ \text{cm}^3/\text{sec}$. Therefore,

$$2\,\text{cm}^3/\text{sec} = \left(4\pi\big[r(t)\big]^2\ \text{cm}^2\right) \cdot \left(r'(t)\ \text{cm/s}\right),$$

which implies

$$r'(t) = \frac{1}{2\pi\big[r(t)\big]^2}\ \text{cm/sec}.$$

When the radius $r = 3$ cm,

$$r'(t) = \frac{1}{18\pi}\ \text{cm/sec}.$$

Checkpoint $2.9.1$

$\frac{1}{72\pi}$ cm/sec, or approximately 0.0044 cm/sec

Example $2.9.2$: An Airplane Flying at a Constant Elevation

Read. This step is only listed so you can have a "win." All of us can *read* a problem, but *understanding* requires a bit more effort.

Understand and **Label.** Draw a picture, introducing variables to represent the quantities involved.

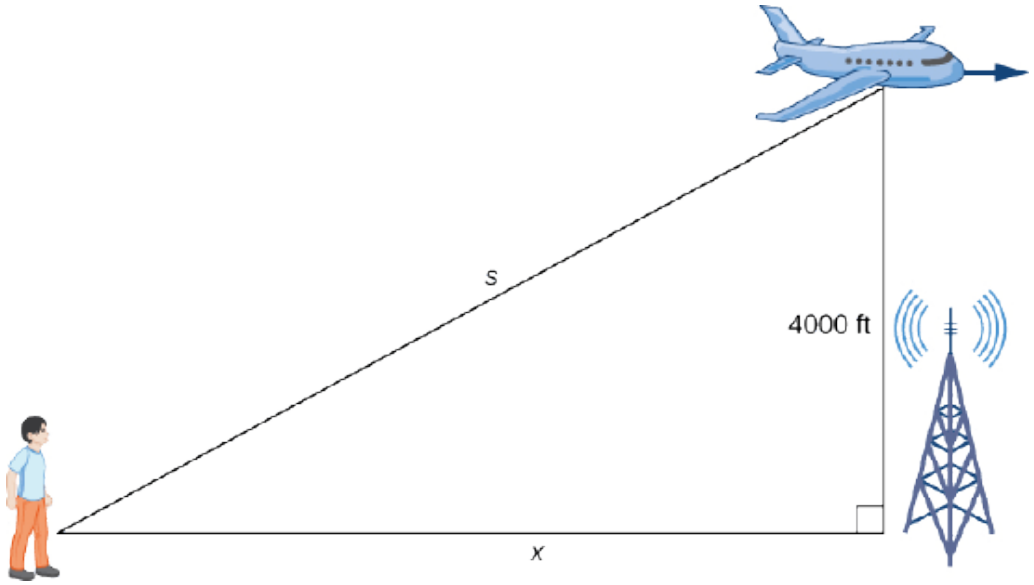


Figure 2.9.2**:** *An airplane is flying at a constant height of* 4000 *ft. The distances between the person and the airplane and the person and the place on the ground directly below the airplane are changing. We denote those quantities with the variables* s *and* x*, respectively.*

As shown, x denotes the distance between the man and the position on the ground directly below the airplane. The variable s denotes the distance between the man and the plane. Note that both x and s are functions of time. We do not introduce a variable for the plane's height because it remains at a constant elevation of 4000 ft. Since an object's height above the ground is measured as the shortest distance between the object and the ground, the line segment of length 4000 ft is perpendicular to the line segment of length x feet, creating a right triangle.

Givens and **Wants.**

Given	Want
$\frac{dx}{dt} = 600$ feet per second	$\frac{ds}{dt}\,\vert_{x=3000}$

Since x denotes the horizontal distance between the man and the point on the ground below the plane, dx/dt represents the speed of the plane. We are told the plane's speed is 600 ft/sec. Therefore, $\frac{dx}{dt} = 600$ ft/sec. Since we are asked to find the rate of change in the distance between the man and the plane when the plane is directly above the radio tower, we need to find ds/dt when $x = 3000$ ft.

As was mentioned in Polya's modified problem-solving process, we only list the given and desired *rate* information. That is, don't bother listing the facts that the height of the plane is always 4000 feet, and the distance from the tower to our position is always 3000 feet.

Master Equation. The variables within our table above are x and s. From the figure, we can use the Pythagorean Theorem to write an equation relating these two variables:$[x(t)]^2 + 4000^2 = [s(t)]^2$.

Rate Equation. Differentiating this equation with respect to time and using the fact that the derivative of a constant is zero, we arrive at the equation

$$x\frac{dx}{dt} = s\frac{ds}{dt}.$$

Substitute. When we "stop time" to do the problem, we know that $\frac{dx}{dt} = 600$ ft/sec and $x = 3000$ ft. Thus,

$$(3000)(600) = s\frac{ds}{dt}.$$

Solve. You can see that the only thing we are missing is s. However, we can use the Pythagorean Theorem to determine the distance s when $x = 3000$ ft and the height is 4000 ft. Solving the equation

$$3000^2 + 4000^2 = s^2$$

for s, we have $s = 5000$ ft at the time of interest. Using these values, we conclude that ds/dt is a solution of the equation

$$(3000)(600) = (5000) \cdot \frac{ds}{dt}.$$

Therefore,

$$\frac{ds}{dt} = \frac{3000 \cdot 600}{5000} = 360\,\text{ft/sec}.$$

Note: When solving related-rates problems, it is important not to substitute values for the variables too soon. For example, in the **Master Equation** step, we related the variable quantities $x(t)$ and $s(t)$ by the equation

$$[x(t)]^2 + 4000^2 = [s(t)]^2.$$

Since the plane remains at a constant height, it is not necessary to introduce a variable for the height, and we are allowed to use the constant 4000 to denote that quantity. However, the other two quantities are changing. If we mistakenly substituted $x(t) = 3000$ into the equation before differentiating, our equation would have been

$$3000^2 + 4000^2 = [s(t)]^2.$$

After differentiating, our equation would become

$$0 = s(t)\frac{ds}{dt}.$$

As a result, we would incorrectly conclude that $\frac{ds}{dt} = 0$.

Checkpoint 2.9.2

500 ft/sec

Example 2.9.3**: A Rocket Launch**

Read. Again, this is a freebie step. Make sure you read the problem thoroughly!

Understand and **Label.** Draw a picture introducing the variables.

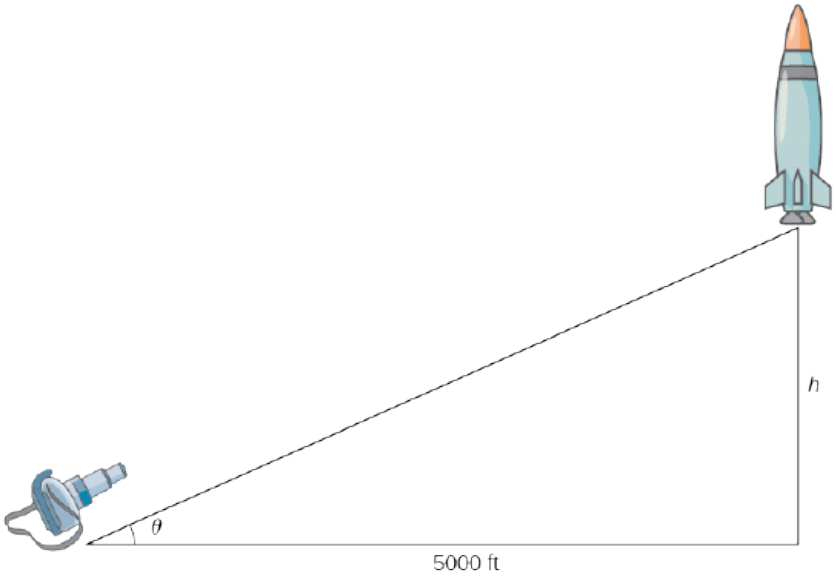


Figure 2.9.4**:** *A camera is positioned* 5000 *ft from the launch pad of the rocket. The rocket's height and the camera's angle are changing with respect to time. We denote those quantities with the variables* h *and* θ*, respectively.*

Let h denote the height of the rocket above the launch pad and θ be the angle between the camera lens and the ground.

Givens and **Wants.** We are trying to find the rate of change in the camera angle with respect to time when the rocket is 1000 ft off the ground. That is, we need to find $\frac{d\theta}{dt}$ when $h = 1000$ ft. At that time, we know the velocity of the rocket is $\frac{dh}{dt} = 600$ ft/sec.

Given	Want
$\frac{dh}{dt}\,\vert_{h=1000} = 600$ feet per second	$\frac{d\theta}{dt}\,\vert_{h=1000}$

Master Equation. Using the fact that we have drawn a right triangle, it is natural to think about trigonometric functions. Recall that $\tan(\theta)$ is the ratio of the length of the opposite side of the triangle to the length of the adjacent side. Thus, we have

$$\tan(\theta) = \frac{h}{5000}.$$

This gives us the equation

$$h = 5000\tan(\theta).$$

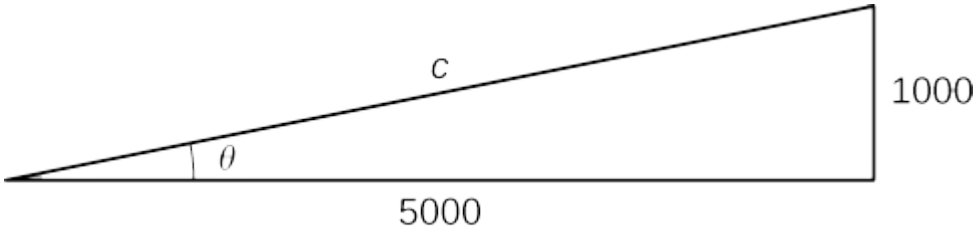
Rate Equation. Differentiating this equation with respect to time t, we obtain

$$\frac{dh}{dt} = 5000\sec^2(\theta)\frac{d\theta}{dt}.$$

Substitute. Letting $h = 1000$ and $\frac{dh}{dt} = 600$, we get

$$600 = 5000\sec^2(\theta)\frac{d\theta}{dt}.$$

Solve. We want to find $\frac{d\theta}{dt}$ when $h = 1000$ ft. We need to determine $\sec^2(\theta)$. Recall that $\sec(\theta)$ is the ratio of the length of the hypotenuse to the length of the adjacent side. We know the length of the adjacent side is 5000 ft. To determine the length of the hypotenuse, we use the Pythagorean Theorem, where the length of one leg is 5000 ft, the length of the other leg is $h = 1000$ ft, and the length of the hypotenuse is c feet as shown in the following figure.



We see that

$$1000^2 + 5000^2 = c^2$$

and we conclude that the hypotenuse is

$$c = 1000\sqrt{26}\,\text{ft}.$$

Therefore, when $h = 1000$, we have

$$\sec^2(\theta) = \left(\frac{1000\sqrt{26}}{5000}\right)^2 = \frac{26}{25}.$$

Doing a final substitution into our formula from the **Substitute** step, we arrive at

$$600 = 5000\left(\frac{26}{25}\right)\frac{d\theta}{dt}.$$

Therefore, $\frac{d\theta}{dt} = \frac{3}{26}$ rad/sec.

Checkpoint 2.9.3

$\frac{1}{10}$ rad/sec

Example 2.9.4**: Water Draining from a Funnel**

This time, we will shorthand our steps to mirror the pacing you should get used to.

Read. Done.

Understand and **Label.** See the picture below.

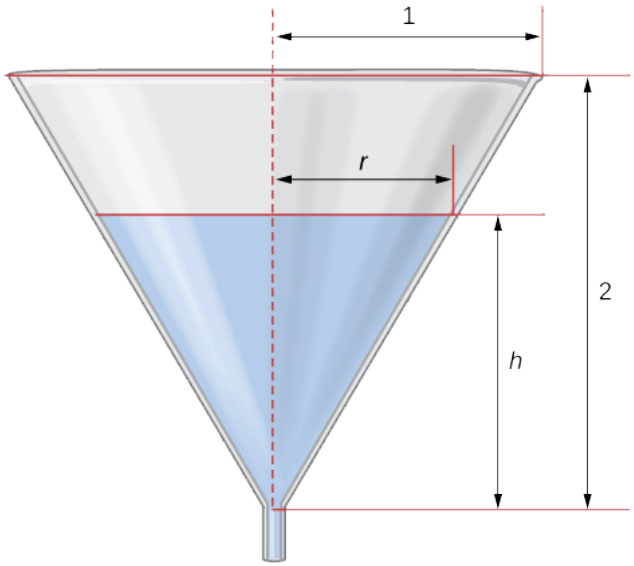


Figure 2.9.5**:** *Water is draining from a funnel of height 2 ft and radius 1 ft. The height of the water and the radius of the top layer of water change over time. We denote these quantities with the variables h and r.*

Let h denote the height of the water in the funnel, r denote the radius of the water at its surface, and V denote the volume of the water.

Givens and **Wants.**

Given	Want
$\frac{dV}{dt} = -0.03\ \text{ft}^3/\text{sec}$	$\frac{dh}{dt}\big\vert_{h=1/2}$

Master Equation.

$$V = \frac{1}{3}\pi r^2 h.$$

Caution: Substitute Out a Variable

Since the Master Equation involves more than two variables, we want to find a relation to rewrite r in terms of either h or V.

From the figure, we see that we have similar triangles. Therefore, the ratio of the sides in the two triangles is the same. Therefore, $\frac{r}{h} = \frac{1}{2}$ or $r = \frac{h}{2}$. Using this fact, the equation for volume can be simplified to

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3.$$

Rate Equation.

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}.$$

Substitute.

$$-0.03 = \frac{\pi}{4} \cdot \left(\frac{1}{2}\right)^2 \cdot \frac{dh}{dt},$$

Solve. The previous step implies

$$-0.03 = \frac{\pi}{16} \frac{dh}{dt}.$$

It follows that

$$\frac{dh}{dt} = -\frac{0.48}{\pi} \approx -0.153\,\text{ft/sec}.$$

Checkpoint 2.9.4

-0.61 ft/sec

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2.9.1: Resources and Key Concepts

Resources

Prerequisite Topics

The following is a list of prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Geometry**
  - The Pythagorean Theorem, Distance Formula, and Midpoint Formula: The Pythagorean Theorem is used to relate sides of right triangles in several examples (e.g., airplane problem Example 2, rocket launch Example 3).
  - Similar Triangles: Used to relate the radius and height of water in a conical funnel (Example 4).
  - Common Areas: Formula for the area of a circle (needed in the homework).
  - Common Volumes: Formula for the volume of a sphere (Example 1 and homework), volume of a cube (homework), volume of a cone (Example 4 and homework), volume of a cylinder (homework).
- **Solving Equations**
  - Solving Linear Equations: After differentiation and substitution, the final step is often solving a linear equation for the unknown rate.
- **Trigonometry**
  - Evaluating Trigonometric Functions Using Technology: Needed for almost any related rates problem involving trigonometric functions.
  - Trigonometric Functions - Right Triangle Definition: Used to relate an angle to sides of a right triangle, e.g., $\tan(\theta)$ in the rocket launch problem (Example 3).
  - Solving Right Triangles: This skill is needed whenever a related rates problem ends up with a right triangle.
  - Solving Simple Trigonometric Equations: Necessary when using related rates with trigonometric functions.
  - Law of Cosines: Required in the homework.

Videos

- Related rates
  - Example: Basic, Non-Application
  - Example: Two Objects Moving Away From Each Other (Pythagorean Master Equation)
  - Example: Two Objects Moving Towards Each Other (Pythagorean Master Equation)
  - Example: Rising Water Levels in a Tank
  - Example: Riding a Ferris Wheel (Trigonometric Master Equation)
  - Example: Changing Distance Between Hands of a Clock (Law of Cosines Master Equation)
- Polya's problem-solving strategy
  - Tactics

Key Concepts

Definitions

- **Related Rates**: If two or more related quantities are changing over time, the rates at which these quantities change are related. For example, if volume V is related to radius r, then $\frac{dV}{dt}$ and $\frac{dr}{dt}$ are related rates.
- **Polya's Modified Problem-Solving Process**: A structured approach for solving problems, adapted for related rates:
  1. Read the given problem.
  2. Understand the given problem (rephrase, draw a picture).
  3. Label unknowns (assign variables).
  4. List the Givens and Wants (especially rates).

5. Create a Master Equation relating variables (not rates yet).
6. Find the Rate Equation from the Master Equation (using implicit differentiation with respect to time).
7. Substitute in any constants given.
8. Solve for the desired "wanted" rate.

Common Mistakes

- **Substituting Known Values Too Soon**: Substituting a known value for a changing quantity into the Master Equation *before* differentiating with respect to time. This causes the quantity to be treated as a constant, and its derivative (rate of change) will incorrectly become zero and not appear in the Rate Equation.
- **Forgetting to Apply the Chain Rule**: When differentiating terms with respect to time, if a variable (like r or V) is a function of time, its derivative must include $\frac{dr}{dt}$ or $\frac{dV}{dt}$. For example, $\frac{d}{dt}(r^3) = 3r^2\frac{dr}{dt}$, not just $3r^2$.
- **Incorrect Master Equation**: Setting up an incorrect geometric or algebraic relationship between the variables.
- **Unit Inconsistencies**: Not ensuring all units are consistent before performing calculations (e.g., feet vs. inches, seconds vs. minutes).

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2.9.2: Homework

Note

The following exercises are meant as hand-in homework (if assigned by your instructor). Typed-in answers here will not be saved or scored.

Basic Skills

Exercise 1

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Exercise 2

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Exercise 4

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Exercise 5

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Basic Skills - Applications

Exercise 6

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Exercise 7

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Exercise 8

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Exercise 9

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Exercise 10

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Exercise 11

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Exercise 12

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Exercise 13

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Exercise 24

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Exercise 25

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Challenge Problems

Exercise 26

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2.9.3: Additional Exercises

Reading Questions

1. What does it mean for quantities to have "related rates"?
2. In the context of related rates, if a quantity Q is changing over time, how is its rate of change typically represented?
3. Outline the key steps in Polya's Modified Problem-Solving Process as applied to related rates problems.
4. What is a "Master Equation" in a related rates problem? What does it relate?
5. How is the "Rate Equation" obtained from the "Master Equation"? What differentiation technique is primarily used?
6. Why is it crucial *not* to substitute known values for changing quantities into the Master Equation before differentiating? What happens if you do?
7. In Example 2.10.1 (inflating a balloon), what was the Master Equation relating volume V and radius r? What was the resulting Rate Equation relating $\frac{dV}{dt}$ and $\frac{dr}{dt}$?
8. In Example 2.10.2 (airplane flying), what geometric theorem was used to establish the Master Equation?
9. When solving for a missing quantity (like s in Example 2.10.2) at the instant of interest, when should this substitution be made – before or after finding the Rate Equation?
10. In Example 2.10.4 (water draining from a funnel), why was it necessary to find a relationship between the radius r and height h of the water before differentiating the volume formula? What geometric concept was used?

Homework

In exercises 1 - 3, find the quantities for the given equation.

1) Find $\frac{dy}{dt}$ at $x = 1$ and $y = x^2 + 3$ if $\frac{dx}{dt} = 4$.

Answer

$$\frac{dy}{dt} = 8$$

2) Find $\frac{dx}{dt}$ at $x = -2$ and $y = 2x^2 + 1$ if $\frac{dy}{dt} = -1$.

3) Find $\frac{dz}{dt}$ at $(x, y) = (1, 3)$ and $z^2 = x^2 + y^2$ if $\frac{dx}{dt} = 4$ and $\frac{dy}{dt} = 3$.

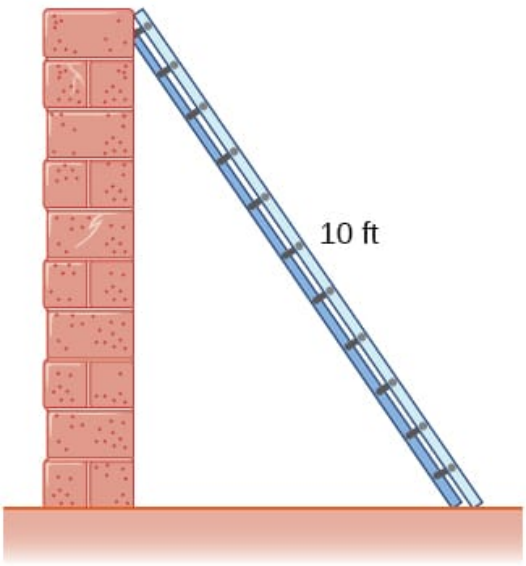
Answer

$$\frac{dz}{dt} = \frac{13}{\sqrt{10}}$$

In exercises 4 - 15, sketch the situation if necessary and used related rates to solve for the quantities.

4) If two electrical resistors are connected in parallel, the total resistance (measured in ohms, denoted by the Greek capital letter omega, ω) is given by the equation $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. If R_1 is increasing at a rate of $0.5\omega/\min$ and R_2 decreases at a rate of $1.1\omega/\min$, at what rate does the total resistance change when $R_1 = 20\omega$ and $R_2 = 50\omega/\min$?

5) A 10-ft ladder is leaning against a wall. If the top of the ladder slides down the wall at a rate of 2 ft/sec, how fast is the bottom moving along the ground when the bottom of the ladder is 5 ft from the wall?

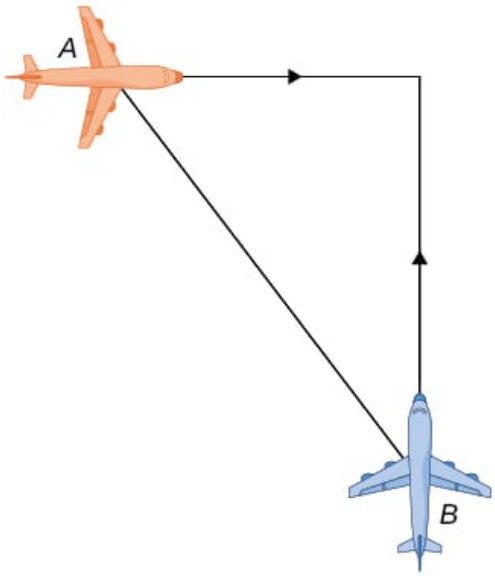


Answer

$2\sqrt{3}$ ft/sec

6) A 25-ft ladder is leaning against a wall. If we push the ladder toward the wall at a rate of 1 ft/sec, and the bottom of the ladder is initially 20 ft away from the wall, how fast does the ladder move up the wall 5 sec after we start pushing?

7) Two airplanes are flying in the air at the same height: airplane A is flying east at 250 mi/h and airplane B is flying north at 300 mi/h. If they are both heading to the same airport, located 30 miles east of airplane A and 40 miles north of airplane B, at what rate is the distance between the airplanes changing?



Answer

The distance is decreasing at 390 mi/h.

8) You and a friend are riding your bikes to a restaurant that you think is east; your friend thinks the restaurant is north. You both leave from the same point, with you riding at 16 mph east and your friend riding 12 mph north. After you traveled 4 mi, at what rate is the distance between you changing?

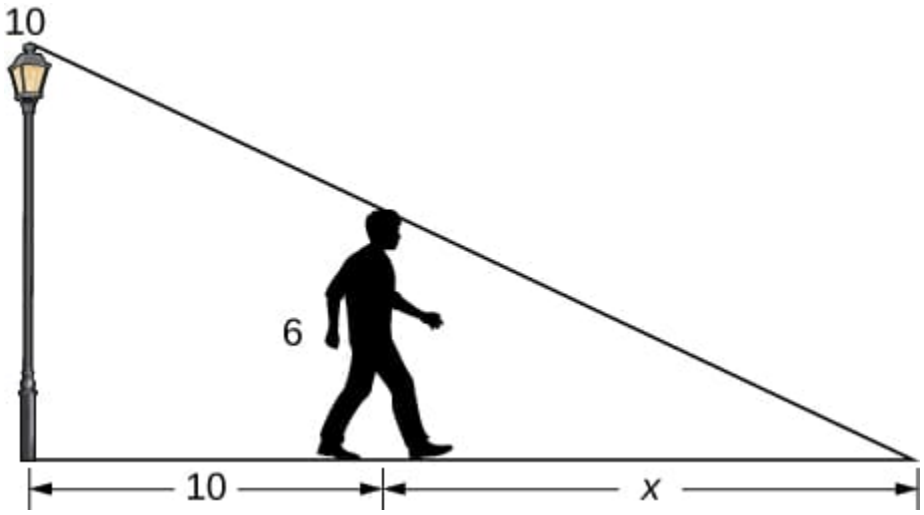
9) Two buses are driving along parallel freeways that are 5 mi apart, one heading east and the other heading west. Assuming that each bus drives a constant 55 mph, find the rate at which the distance between the buses is changing when they are 13 mi apart

(as the crow flies), heading toward each other.

Answer

The distance between them shrinks at a rate of $\frac{1320}{13} \approx 101.5$ mph.

10) A 6-ft-tall person walks away from a 10-ft lamppost at a constant rate of 3 ft/sec. What is the rate that the tip of the shadow moves away from the pole when the person is 10 ft away from the pole?



11) Using the previous problem, what is the rate at which the tip of the shadow moves away from the person when the person is 10 ft from the pole?

Answer

$\frac{9}{2}$ ft/sec

12) A 5-ft-tall person walks toward a wall at a rate of 2 ft/sec. A spotlight is located on the ground 40 ft from the wall. How fast does the height of the person's shadow on the wall change when the person is 10 ft from the wall?

13) Using the previous problem, what is the rate at which the shadow changes when the person is 10 ft from the wall, if the person is walking away from the wall at a rate of 2 ft/sec?

Answer

It grows at a rate $\frac{4}{9}$ ft/sec

14) A helicopter starting on the ground is rising directly into the air at a rate of 25 ft/sec. You are running on the ground starting directly under the helicopter at a rate of 10 ft/sec. Find the rate of change of the distance between the helicopter and yourself after 5 sec.

15) Using the previous problem, assuming the helicopter is again rising at a rate of 25 ft/sec and you are running on the ground starting directly under the helicopter at a rate of 10 ft/sec, what is the rate at which the distance between you and the helicopter is changing when the helicopter has risen to a height of 60 ft in the air, assuming that, initially, it was 30 ft above you?

Answer

The distance is increasing at $\frac{135\sqrt{26}}{26}$ ft/sec

In exercises 16 - 24, draw and label diagrams to help solve the related-rates problems.

16) The side of a cube increases at a rate of $\frac{1}{2}$ m/sec. Find the rate at which the volume of the cube increases when the side of the cube is 4 m.

17) The volume of a cube decreases at a rate of $10\ \text{m}^3$/sec. Find the rate at which the side of the cube changes when the side of the cube is 2 m.

Answer

$-\frac{5}{6}$ m/sec

18) The radius of a circle increases at a rate of 2 m/sec. Find the rate at which the area of the circle increases when the radius is 5 m.

19) The radius of a sphere decreases at a rate of 3 m/sec. Find the rate at which the surface area decreases when the radius is 10 m.

Answer

$240\pi \text{ m}^2/\text{sec}$

20) The radius of a sphere increases at a rate of 1 m/sec. Find the rate at which the volume increases when the radius is 20 m.

21) The radius of a sphere is increasing at a rate of 9 cm/sec. Find the radius of the sphere when the volume and the radius of the sphere are increasing at the same numerical rate.

Answer

$\frac{1}{2\sqrt{\pi}}$ cm

22) The base of a triangle is shrinking at a rate of 1 cm/min and the height of the triangle is increasing at a rate of 5 cm/min. Find the rate at which the area of the triangle changes when the height is 22 cm and the base is 10 cm.

23) A triangle has two constant sides of length 3 ft and 5 ft. The angle between these two sides is increasing at a rate of 0.1 rad/sec. Find the rate at which the area of the triangle is changing when the angle between the two sides is $\pi/6$.

Answer

The area is increasing at a rate $\frac{3\sqrt{3}}{8} \text{ ft}^2/\text{sec}$.

24) A triangle has a height that is increasing at a rate of 2 cm/sec and its area is increasing at a rate of $4 \text{ cm}^2/\text{sec}$. Find the rate at which the base of the triangle is changing when the height of the triangle is 4 cm and the area is 20 cm^2.

In exercises 25 - 27, consider an inverted right cone that is leaking water. (Inverted means the cone's point is facing down, like a funnel.) The dimensions of the conical tank are a height of 16 ft and a radius of 5 ft.

25) How fast does the depth of the water change when the water is 10 ft high if the cone leaks water at a rate of $10 \text{ ft}^3/\text{min}$?

Answer

The depth of the water decreases at $\frac{128}{125\pi}$ ft/min.

26) Find the rate at which the surface area of the water changes when the water is 10 ft high if the cone leaks water at a rate of $10 \text{ ft}^3/\text{min}$.

27) If the water level is decreasing at a rate of 3 in/min when the depth of the water is 8 ft, determine the rate at which water is leaking out of the cone.

Answer

The volume is decreasing at a rate of $\frac{25\pi}{16} \text{ ft}^3/\text{min}$.

28) A vertical cylinder is leaking water at a rate of $1 \text{ ft}^3/\text{sec}$. If the cylinder has a height of 10 ft and a radius of 1 ft, at what rate is the height of the water changing when the height is 6 ft?

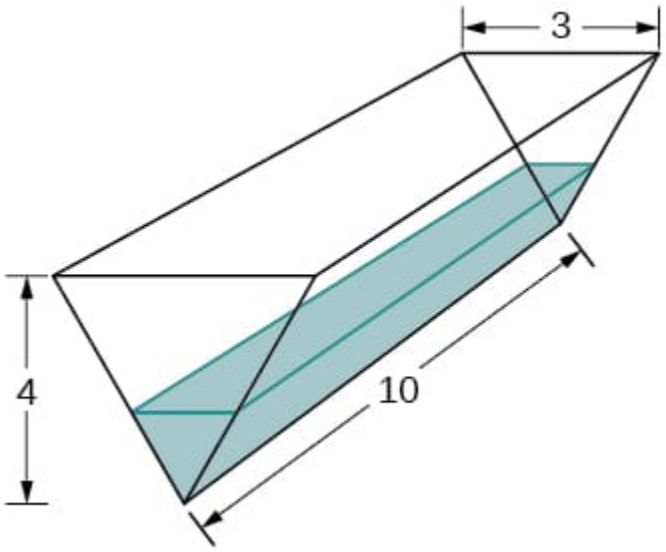
29) A cylinder is leaking water but you are unable to determine at what rate. The cylinder has a height of 2 m and a radius of 2 m. Find the rate at which the water is leaking out of the cylinder if the rate at which the height is decreasing is 10 cm/min when the height is 1 m.

Answer

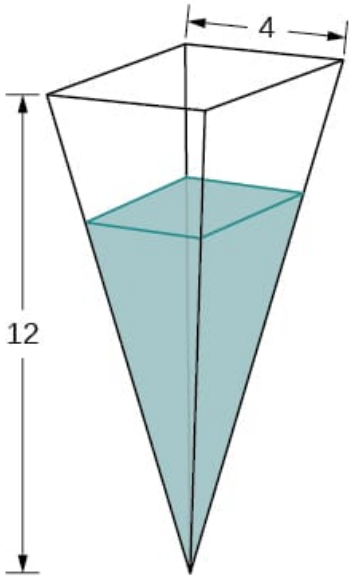
The water flows out at rate $\frac{2\pi}{5} \text{ m}^3/\text{min}$.

30) A trough has ends shaped like isosceles triangles, with width 3 m and height 4 m, and the trough is 10 m long. Water is being

pumped into the trough at a rate of $5\ \mathrm{m}^3/\mathrm{min}$. At what rate does the height of the water change when the water is 1 m deep?



31) A tank is shaped like an upside-down square pyramid, with base of 4 m by 4 m and a height of 12 m (see the following figure). How fast does the height increase when the water is 2 m deep if water is being pumped in at a rate of $\frac{2}{3}\ \mathrm{m}^3$/sec?



Answer

$\frac{3}{2}$ m/sec

For exercises 32 - 34, consider a pool shaped like the bottom half of a sphere, that is being filled at a rate of $25\ \mathrm{ft}^3$ /min. The radius of the pool is 10 ft.

32) Find the rate at which the depth of the water is changing when the water has a depth of 5 ft.

33) Find the rate at which the depth of the water is changing when the water has a depth of 1 ft.

Answer

$\frac{25}{19\pi}$ ft/min

34) If the height is increasing at a rate of 1 in/sec when the depth of the water is 2 ft, find the rate at which water is being pumped in.

35) Gravel is being unloaded from a truck and falls into a pile shaped like a cone at a rate of $10\ \text{ft}^3/\text{min}$. The radius of the cone base is three times the height of the cone. Find the rate at which the height of the gravel changes when the pile has a height of 5 ft.

Answer

$\frac{2}{45\pi}$ ft/min

36) Using a similar setup from the preceding problem, find the rate at which the gravel is being unloaded if the pile is 5 ft high and the height is increasing at a rate of 4 in/min.

In exercises 37 - 41, draw the situations and solve the related-rate problems.

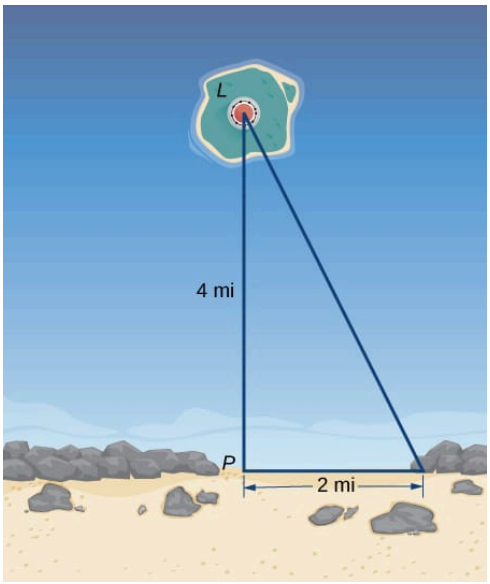
37) You are stationary on the ground and are watching a bird fly horizontally at a rate of 10 m/sec. The bird is located 40 m above your head. How fast does the angle of elevation change when the horizontal distance between you and the bird is 9 m?

Answer

The angle decreases at $\frac{400}{1681}$ rad/sec.

38) You stand 40 ft from a bottle rocket on the ground and watch as it takes off vertically into the air at a rate of 20 ft/sec. Find the rate at which the angle of elevation changes when the rocket is 30 ft in the air.

39) A lighthouse, L, is on an island 4 mi away from the closest point, P, on the beach (see the following image). If the lighthouse light rotates clockwise at a constant rate of 10 revolutions/min, how fast does the beam of light move across the beach 2 mi away from the closest point on the beach?



Answer

100π mi/min

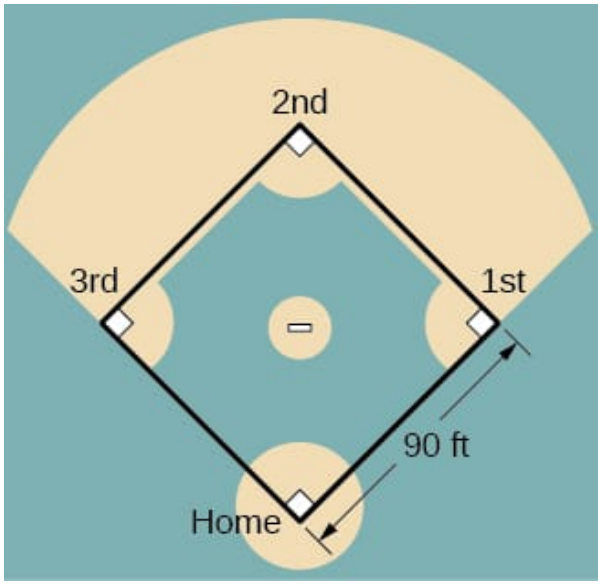
40) Using the same setup as the previous problem, determine at what rate the beam of light moves across the beach 1 mi away from the closest point on the beach.

41) You are walking to a bus stop at a right-angle corner. You move north at a rate of 2 m/sec and are 20 m south of the intersection. The bus travels west at a rate of 10 m/sec away from the intersection – you have missed the bus! What is the rate at which the angle between you and the bus is changing when you are 20 m south of the intersection and the bus is 10 m west of the intersection?

Answer

The angle is changing at a rate of $\frac{11}{25}$ rad/sec.

In exercises 42 - 45, refer to the figure of baseball diamond, which has sides of 90 ft.



42) A batter hits a ball toward third base at 75 ft/sec and runs toward first base at a rate of 24 ft/sec. At what rate does the distance between the ball and the batter change when 2 sec have passed?

43) A batter hits a ball toward second base at 80 ft/sec and runs toward first base at a rate of 30 ft/sec. At what rate does the distance between the ball and the batter change when the runner has covered one-third of the distance to first base? (Hint: Recall the law of cosines.)

Answer

The distance is increasing at a rate of 62.50 ft/sec.

44) A batter hits the ball and runs toward first base at a speed of 22 ft/sec. At what rate does the distance between the runner and second base change when the runner has run 30 ft?

45) Runners start at first and second base. When the baseball is hit, the runner at first base runs at a speed of 18 ft/sec toward second base and the runner at second base runs at a speed of 20 ft/sec toward third base. How fast is the distance between runners changing 1 sec after the ball is hit?

Answer

The distance is decreasing at a rate of 11.99 ft/sec.

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2.10: Linear Approximations and Differentials

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

We have just seen how derivatives allow us to compare related quantities that change over time. In this section, we examine another application of derivatives: the ability to approximate functions locally by linear functions. Linear functions are the most straightforward functions to work with, so they provide a valuable tool for approximating function values. In addition, the ideas presented in this section are generalized later in the text when we study how to approximate functions by higher-degree polynomials using Taylor Polynomial Expansions.

Linear Approximation of a Function at a Point

Consider a function f that is differentiable at a point $x = a$. The tangent line to f at the point $(a, f(a))$ is

$$y - f(a) = f'(a)(x - a) \implies y = f(a) + f'(a)(x - a).$$

For example, consider the function $f(x) = \frac{1}{x}$ at $a = 2$. Since f is differentiable at $x = 2$ and $f'(x) = -\frac{1}{x^2}$, we see that $f'(2) = -\frac{1}{4}$. Therefore, the tangent line to the graph of f at $a = 2$ is given by the equation

$$y = \frac{1}{2} - \frac{1}{4}(x - 2).$$

Figure $2.10.1a$ shows a graph of $f(x) = \frac{1}{x}$ along with the tangent line to f at $x = 2$. Note that for x near 2, the graph of the tangent line is close to the graph of f. As a result, we can use the equation of the tangent line to approximate $f(x)$ for x near 2. For example, if $x = 2.1$, the y value of the corresponding point on the tangent line is

$$y = \frac{1}{2} - \frac{1}{4}(2.1 - 2) = 0.475.$$

The actual value of $f(2.1)$ is given by

$$f(2.1) = \frac{1}{2.1} \approx 0.47619.$$

Therefore, the tangent line gives us a fairly good approximation of $f(2.1)$ (Figure $2.10.1b$). However, note that for values of x far from 2, the equation of the tangent line does not give us a good approximation. For example, if $x = 10$, the y-value of the corresponding point on the tangent line is

$$y = \frac{1}{2} - \frac{1}{4}(10 - 2) = \frac{1}{2} - 2 = -1.5,$$

whereas the value of the function at $x = 10$ is $f(10) = 0.1$.

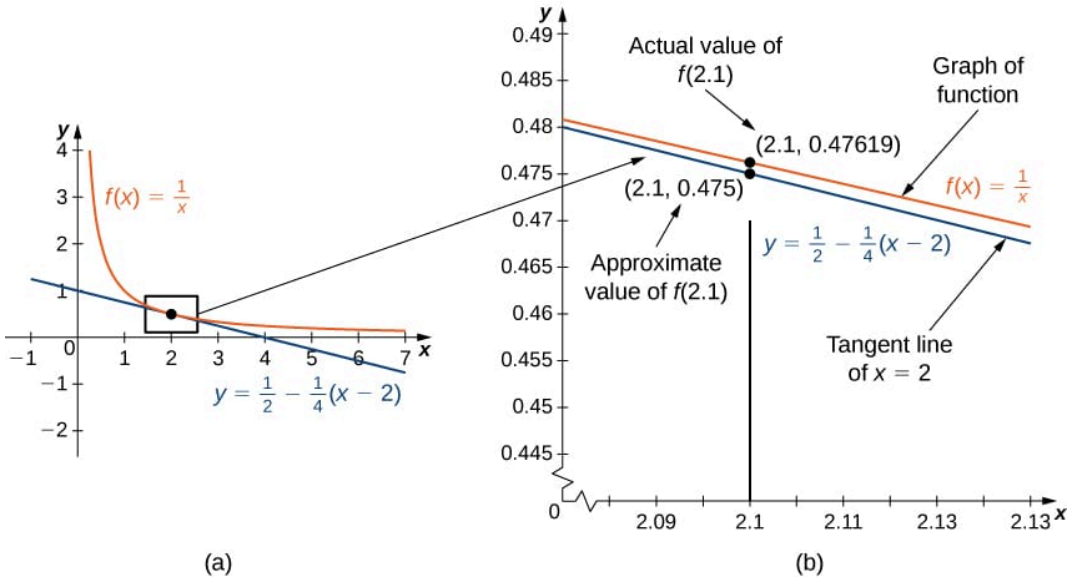

Figure 2.10.1*a*: *The tangent line to* $f(x) = 1/x$ *at* $x = 2$ *provides a good approximation to* f *for* x *near* 2.
Figure 2.10.1*b*: *At* $x = 2.1$, *the value of* y *on the tangent line to* $f(x) = 1/x$ *is* 0.475. *The actual value of* $f(2.1)$ *is* $1/2.1$, *which is approximately* 0.47619.

Interactive Element: How Good Is a Linear Approximation?

The following Interactive Element might help you visualize how the tangent-line approximation to $f(x) = \frac{1}{x}$ tracks the curve near the point of tangency and drifts away from it.

Interact: Drag the point of tangency along the curve to choose where the line touches, then drag the point on the x-axis to pick where you evaluate. The vertical segment is the approximation error $|f(x) - L(x)|$.

Observation: The error is nearly invisible when the evaluation point sits close to the point of tangency, and it grows quickly as that point moves away.

In general, for a differentiable function f, the equation of the tangent line to f at $x = a$ can be used to approximate $f(x)$ for x near a. Therefore, we can write

$$f(x) \approx f(a) + f'(a)(x - a) \text{ for } x \text{ near } a.$$

We call the linear function

$$L(x) = f(a) + f'(a)(x - a) \tag{2.10.1}$$

the **linear approximation,** or **tangent line approximation**, of f at $x = a$. This function L is also known as the **linearization** of f at $x = a$.

To show how beneficial the linear approximation can be, we look at how to find the linear approximation for $f(x) = \sqrt{x}$ at $x = 9$.

Example 2.10.1: Linear Approximation of $\sqrt{x}$

Find the linear approximation of $f(x) = \sqrt{x}$ at $x = 9$ and use the approximation to estimate $\sqrt{9.1}$.

Solution available at the end of this section.

Checkpoint 2.10.1

Find the local linear approximation to $f(x) = \sqrt[3]{x}$ at $x = 8$. Use it to approximate $\sqrt[3]{8.1}$ to five decimal places.

Answer available at the end of this section.

Example 2.10.2: Linear Approximation of $\sin x$

Find the linear approximation of $f(x) = \sin(x)$ at $x = \frac{\pi}{3}$ and use it to approximate $\sin(62^\circ)$.

Solution available at the end of this section.

Checkpoint 2.10.2

Find the linear approximation for $f(x) = \cos x$ at $x = \frac{\pi}{2}$.

Answer available at the end of this section.

Linear approximations may be used in estimating roots and powers. In the next example, we find the linear approximation for $f(x) = (1+x)^n$ at $x = 0$, which can be used to estimate roots and powers for real numbers near 1. The same idea can be extended to a function of the form $f(x) = (m+x)^n$ to estimate roots and powers near a different number m.

Example 2.10.3: Approximating Roots and Powers

Find the linear approximation of $f(x) = (1+x)^n$ at $x = 0$. Use this approximation to estimate $(1.01)^3$.

Solution available at the end of this section.

Common Mistake

In Example 2.10.3, we linearized the function $f(x) = (1+x)^3$ at $x = 0$ to approximate $(1.01)^3$. In this case, $(1.01)^3 = f(0.01) \approx L(0.01)$. It's common for students to make the mistake of thinking, to approximate $(1.01)^3$, we need to evaluate f (and, therefore, L) at 1.01; however, $f(x) = (1+x)^3$ already has the 1 (also known as the **whole part**) built-in. Therefore, we need to evaluate f (and, therefore, L) at the **decimal part**.

Checkpoint 2.10.3

Find the linear approximation of $f(x) = (1+x)^4$ at $x = 0$ without using the result from the preceding example.

Answer available at the end of this section.

Differentials

We have seen that linear approximations can be used to estimate function values. They can also be used to estimate the amount a function value changes due to a small change in the input. To discuss this more formally, we define a related concept: differentials. Differentials provide us with a way of estimating the amount a function changes due to a small change in input values.

When we first looked at derivatives, we used the Leibniz notation dy/dx to represent the derivative of y with respect to x. Although we used the expressions dy and dx in this notation, they did not have meaning on their own. Here, we see a meaning to the expressions dy and dx. Suppose $y = f(x)$ is a differentiable function. Let dx be an independent variable that can be assigned any nonzero real number, and define the dependent variable dy by

$$dy = f'(x)\,dx. \tag{2.10.2}$$

The expressions dy and dx are called **differentials**. It is important to notice that dy is a function of both x and dx.

Example 2.10.4: Computing Differentials

For each of the following functions, find dy and evaluate when $x = 3$ and $dx = 0.1$.

a. $y = x^2 + 2x$
b. $y = \cos x$

Solutions available at the end of this section.

Checkpoint 2.10.4

For $y = e^{x^2}$, find dy.

Answer available at the end of this section.

We now connect differentials to linear approximations. Differentials can be used to estimate the change in the value of a function resulting from a small change in input values. Consider a function f that is differentiable at point a. Suppose the input x changes by a small amount. We are interested in how much the output y changes. If x changes from a to $a + dx$, then the change in x is dx (also denoted Δx), and the change in y is given by

$$\Delta y = f(a + dx) - f(a).$$

Instead of calculating the exact change in y, however, it is often easier to approximate the change in y by using a linear approximation. For x near a, $f(x)$ can be approximated by the linear approximation (Equation 2.10.1)

$$L(x) = f(a) + f'(a)(x - a).$$

Therefore, if dx is small,

$$f(a + dx) \approx L(a + dx) = f(a) + f'(a)(a + dx - a).$$

That is,

$$f(a + dx) - f(a) \approx L(a + dx) - f(a) = f'(a)\, dx.$$

In other words, the actual change in the function f if x increases from a to $a + dx$ is approximately the difference between $L(a + dx)$ and $f(a)$, where $L(x)$ is the linear approximation of f at a. By definition of $L(x)$, this difference is equal to $f'(a)\ dx$. In summary,

$$\Delta y = f(a + dx) - f(a) \approx L(a + dx) - f(a) = f'(a)\, dx = dy.$$

Therefore, we can use the differential $dy = f'(a)\ dx$ to approximate the change in y if x increases from $x = a$ to $x = a + dx$. We can see this in the following graph.

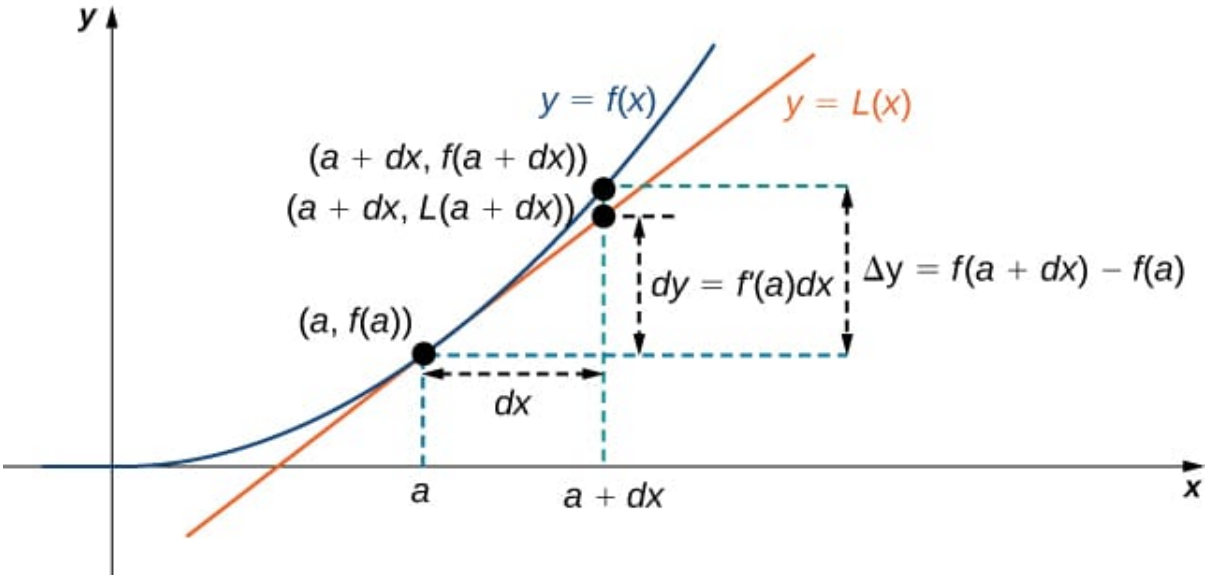

Figure 2.10.5**:** *The differential $dy = f'(a)\ dx$ is used to approximate the actual change in y if x increases from a to $a + dx$.*

Interactive Element: Differentials: dy versus Δy

The following Interactive Element might help you visualize how the differential $dy = f'(a)\ dx$ compares with the true change $\Delta y = f(a + dx) - f(a)$ as dx varies.

Interact: Drag the red point on the x-axis to change dx. The green segment is dy; the true change Δy reaches up to the curve, so Δy is the green segment plus the red gap.

Observation: For small dx the red gap almost disappears and $dy \approx \Delta y$; as dx grows, the gap widens and dy underestimates the true change.

We now look at how to use differentials to approximate the change in the value of the function that results from a small change in the value of the input. Note that the calculation with differentials is much simpler than calculating actual values of functions, and the result is very close to what we would obtain with a more exact calculation.

Example 2.10.5: Approximating Change with Differentials

Let $y = x^2 + 2x$. Compute Δy and dy at $x = 3$ if $dx = 0.1$.

Solution available at the end of this section.

Checkpoint 2.10.5

For $y = x^2 + 2x$, find Δy and dy at $x = 3$ if $dx = 0.2$.

Answer available at the end of this section.

A Technical Aside: The Happy Accident of Leibniz's Notation

Leibniz notation is handy throughout Calculus; however, it can be confusing for students when it comes to differentials. Visually, it looks like we can divide both sides of Equation 2.10.2 by dx to arrive at

$$\frac{dy}{dx} = f'(x),$$

however, this equivalence is a happy accident.

Technically speaking, $\frac{d}{dx}$ is an **operator**. That is, it is a set of instructions that accepts a function and returns the derivative of that function. For example, $\frac{d}{dx}(y) = \frac{dy}{dx}$ is the operator that returns the derivative of the function y with respect to x; however, to say that you can "pull apart" the operator and multiply both sides by dx to get $dy = f'(x)dx$ is an abuse of logic. Notationally, it works, but to be clear, it is not true that we are multiplying both sides by dx.

$$\frac{dy}{dx} \text{ is } \mathbf{defined} \text{ to be } f'(x).$$

Whereas,

$$dy \text{ is } \mathbf{defined} \text{ to be } f'(x)dx.$$

As a simple clarification, you have dealt with operators your entire life. For example, $+$ is an operator that accepts two numbers and returns the sum of those two numbers. Therefore, $3 + 2 = 5$ is true because the operator, $+$, returns the sum of 3 and 2 (which is 5). I hope you would agree that "pulling apart" the operator and dividing by a "piece" of the operator to get

$$3| - 2 = 5 \implies 3|2 = \frac{5}{-}$$

leads to a breakdown in logic (and notation).

All of this is to say that, technically, you should not tell someone, "I see you have $dy = f'(x)dx$. Just *divide both sides* by dx to get the derivative $\frac{dy}{dx} = f'(x)$." That statement is (again, very technically) an abuse of logic. Moreover, saying to your friend, "Oh, you have $\frac{dy}{dx} = f'(x)$? Great! Just *multiply both sides* by dx to get $dy = f'(x)dx$," is a similar abuse in logic.

It's okay to *silently* view the notation this way, but you should realize this result is just a happy accident of Leibniz's notation.

Calculating the Amount of Error

Any type of measurement is prone to a certain amount of error. In many applications, certain quantities are calculated based on measurements. For example, the area of a circle is calculated by measuring the radius of the circle. An error in the measurement of the radius leads to an error in the computed value of the area. Here, we examine this type of error and study how differentials can be used to estimate the error.

Consider a function f with an input that is a measured quantity. Suppose the exact value of the measured quantity is a, but the measured value is $a + dx$. We say the **measurement error** is dx (or Δx). As a result, an error occurs in the calculated quantity $f(x)$. This type of error is known as a **propagated error** and is given by

$$\Delta y = f(a + dx) - f(a).$$

Since all measurements are prone to some error, we do not know the exact value of a measured quantity, so we cannot calculate the propagated error exactly. However, given an estimate of the accuracy of a measurement, we can use differentials to approximate the propagated error Δy. Specifically, if f is a differentiable function at a, the propagated error is

$$\Delta y \approx dy = f'(a)\, dx.$$

Unfortunately, we do not know the exact value a. However, we can use the measured value $a + dx$, and estimate

$$\Delta y \approx dy \approx f'(a + dx)\, dx.$$

In the next example, we look at how differentials can be used to estimate the error in calculating the volume of a box if we assume the measurement of the side length is made with a certain amount of accuracy.

Example 2.10.6: Volume of a Cube

Suppose the side length of a cube is measured to be 5 cm, but we think this could be off by at most 0.1 cm.

a. Use differentials to estimate the error in the computed volume of the cube.

b. Compute the volume of the cube if the side length is (i) 4.9 cm and (ii) 5.1 cm to compare the estimated error with the actual potential error.

Solutions available at the end of this section.

Checkpoint 2.10.6

Estimate the error in the computed volume of a cube if the side length is measured to be 6 cm with an accuracy of 0.2 cm.

Answer available at the end of this section.

The measurement error $dx\ (=\Delta x)$ and the propagated error Δy are absolute errors. We are typically interested in the size of an error relative to the size of the quantity being measured or calculated. Given an absolute error Δq for a particular quantity, we define the **relative error** as $\frac{\Delta q}{q}$, where q is the actual value of the quantity. The **percentage error** is the relative error expressed as a percentage. For example, if we measure the height of a ladder to be 63 in. when the actual height is 62 in., the absolute error is 1 in., but the relative error is $\frac{1}{62}=0.016$, or 1.6%. By comparison, if we measure the width of a piece of cardboard to be 8.25 in. when the actual width is 8 in., our absolute error is $\frac{1}{4}$ in., whereas the relative error is $\frac{0.25}{8}=\frac{1}{32}$, or 3.1%. Therefore, the percentage error in the measurement of the cardboard is larger, even though 0.25 in. is less than 1 in.

Example 2.10.7: Relative and Percentage Error

An astronaut using a camera measures the radius of Earth as 4000 mi with an error of ± 80 mi. Let's use differentials to estimate the relative and percentage error of using this radius measurement to calculate the volume of Earth, assuming the planet is a perfect sphere.

Solution available at the end of this section.

Checkpoint 2.10.7

Determine the percentage error if the radius of Earth is measured to be 3950 mi with an error of ± 100 mi.

Answer available at the end of this section.

Miscellaneous Topics: Thinking About Mathematics (Part 3)

> When I would get frustrated in school, I often found it good to reflect upon some of my favorite teachers and think about how they would approach the subject.
>
> - Professor Simpson, Cosumnes River College

1. Who was your favorite teacher, and what subject did they teach?
2. Can you make any suggestions I can incorporate into the classroom that worked for that teacher?

Success in College: Self-Motivation Learning Preferences

Choose the statements that best describe you.

I come to class because I don't want to miss anything.	or	I come to class because it is required.

I would probably do less work in this class if I were graded pass/fail instead of a letter grade.	or	I would probably do the same amount of work in this class if I was graded pass/fail instead of with a letter grade.
I need frequent homework assignments to keep me working.	or	I do homework frequently to know what is going on in class.

1. Do you consider the choices above to be good or bad for your education?
2. Some professors provide detailed schedules with assignments and test dates at the beginning of the term. Others announce the homework assignments at each class meeting and test days during the term. Which do you prefer?

Answers

Section Preview

This section is often thought of as two separate topics—linear approximations and differentials (as the title of the section states); however, these are, in essence, the same topic. Despite being two sides of the same coin, knowing the difference will be important for the student. Linear approximations are a wonderful extension of applications of differentiation and can be used to preview the concept of Taylor polynomials (Integral Calculus). Differentials, on the other hand, should be stressed as infinitesimal changes. This concept will return in the chapter on integration.

Section Preview

- **Arithmetic**
  - Percents: Relative errors are reported as percentages (6%, 1.6%, 3.1%); converting a decimal or fraction to a percent is required for the percentage-error objective (Example 7, Checkpoint 7).
  - Measurement, Approximation, Rounding, and Significant Figures: Example 1 and Checkpoint 1 round approximations to a stated number of decimal places (four, then five); rounding to a specified place value is exercised throughout.

Section Preview

- Describe the linear approximation to a function at a point.
- Write the linearization of a given function.
- Draw a graph that illustrates the use of differentials to approximate the change in a quantity.
- Calculate the relative and percentage error when using a differential approximation.

Example $2.10.1$: Linear Approximation of $\sqrt{x}$

Since we are looking for the linear approximation at $x = 9$, using Equation $2.10.1$ we know the linear approximation is given by

$$L(x) = f(9) + f'(9)(x - 9).$$

We need to find $f(9)$ and $f'(9)$.

$$f(x) = \sqrt{x} \implies f(9) = \sqrt{9} = 3$$

and

$$f'(x) = \frac{1}{2\sqrt{x}} \implies f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}.$$

Therefore, the linear approximation is given by Figure $2.10.2$.

$$L(x) = 3 + \frac{1}{6}(x - 9)$$

Using the linear approximation, we can estimate $\sqrt{9.1}$ by writing

$$\sqrt{9.1} = f(9.1) \approx L(9.1) = 3 + \frac{1}{6}(9.1 - 9) \approx 3.0167.$$

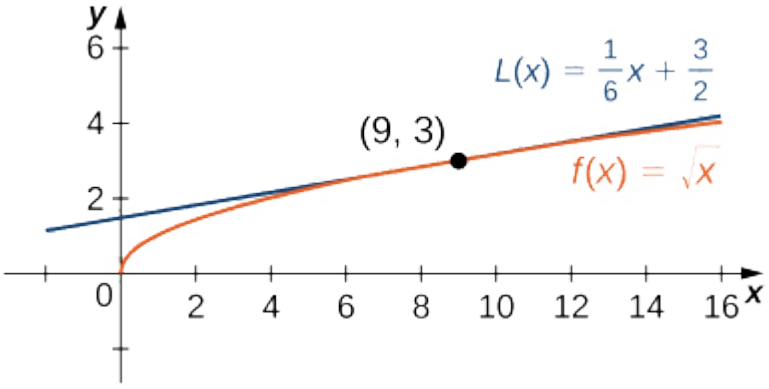

Figure $2.10.2$: *The local linear approximation to $f(x) = \sqrt{x}$ at $x = 9$ provides an approximation to f for x near 9.*

Analysis

Using a calculator, the value of $\sqrt{9.1}$ to four decimal places is 3.0166. The value given by the linear approximation, 3.0167, is very close to the value obtained with a calculator, so it appears that using this linear approximation is a good way to estimate $\sqrt{x}$, at least for x near 9. At the same time, it may seem odd to use a linear approximation when we can push a few buttons on a calculator to evaluate $\sqrt{9.1}$. However, how does the calculator evaluate $\sqrt{9.1}$? The calculator uses an approximation! Calculators and computers always use approximations to evaluate mathematical expressions; they just use higher-degree approximations.

Checkpoint 2.10.1

$L(x) = 2 + \frac{1}{12}(x - 8)$; 2.00833

Example 2.10.2**: Linear Approximation of** $\sin x$

First, we recall that degree measure and calculus do not work well together. Therefore, we trade out 60° for $\frac{\pi}{3}$ radians. The linear approximation is given by

$$L(x) = f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right).$$

We see that

$$f(x) = \sin(x) \implies f\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$
$$f'(x) = \cos(x) \implies f'\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

Therefore, the linear approximation of f at $x = \pi/3$ is given by Figure 2.10.3.

$$L(x) = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right)$$

To estimate $\sin(62^\circ)$ using L, we must first convert 62° to radians. We have $62^\circ = \frac{62\pi}{180}$ radians, so the estimate for $\sin(62^\circ)$ is given by

$$\sin(62^\circ) = f\left(\frac{62\pi}{180}\right) \approx L\left(\frac{62\pi}{180}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(\frac{62\pi}{180} - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(\frac{2\pi}{180}\right) = \frac{\sqrt{3}}{2} + \frac{\pi}{180} \approx 0.88348.$$

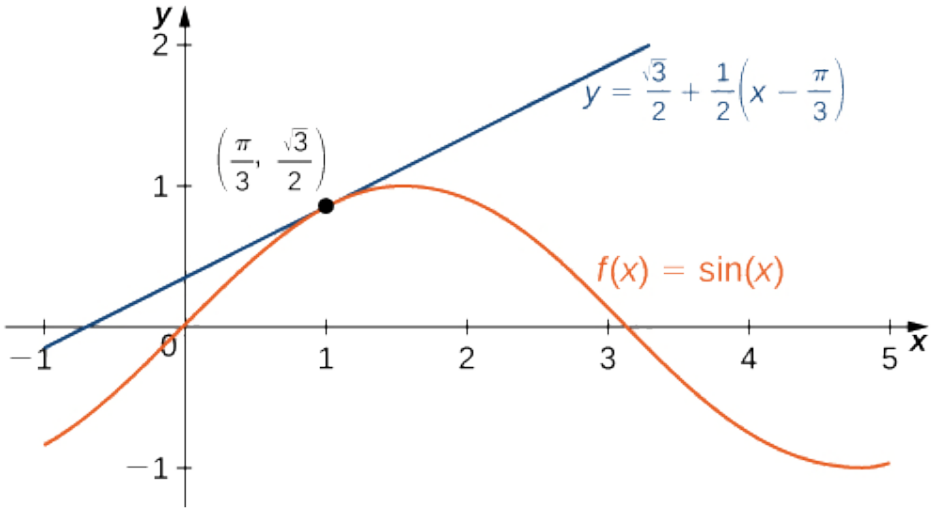

Figure 2.10.3**:** *The linear approximation to* $f(x) = \sin x$ *at* $x = \pi/3$ *provides an approximation to* $\sin x$ *for* x *near* $\pi/3$.

Checkpoint 2.10.2

$L(x) = -x + \frac{\pi}{2}$

Example 2.10.3**: Approximating Roots and Powers**

The linear approximation at $x = 0$ is given by

$$L(x) = f(0) + f'(0)(x - 0).$$

Because

$$f(x) = (1 + x)^n \implies f(0) = 1$$
$$f'(x) = n(1 + x)^{n-1} \implies f'(0) = n,$$

the linear approximation is given by Figure 2.10.4*a*.

$$L(x) = 1 + n(x - 0) = 1 + nx$$

We can approximate $(1.01)^3$ by evaluating $L(0.01)$ when $n = 3$. We conclude that

$$(1.01)^3 = f(1.01) \approx L(1.01) = 1 + 3(0.01) = 1.03.$$

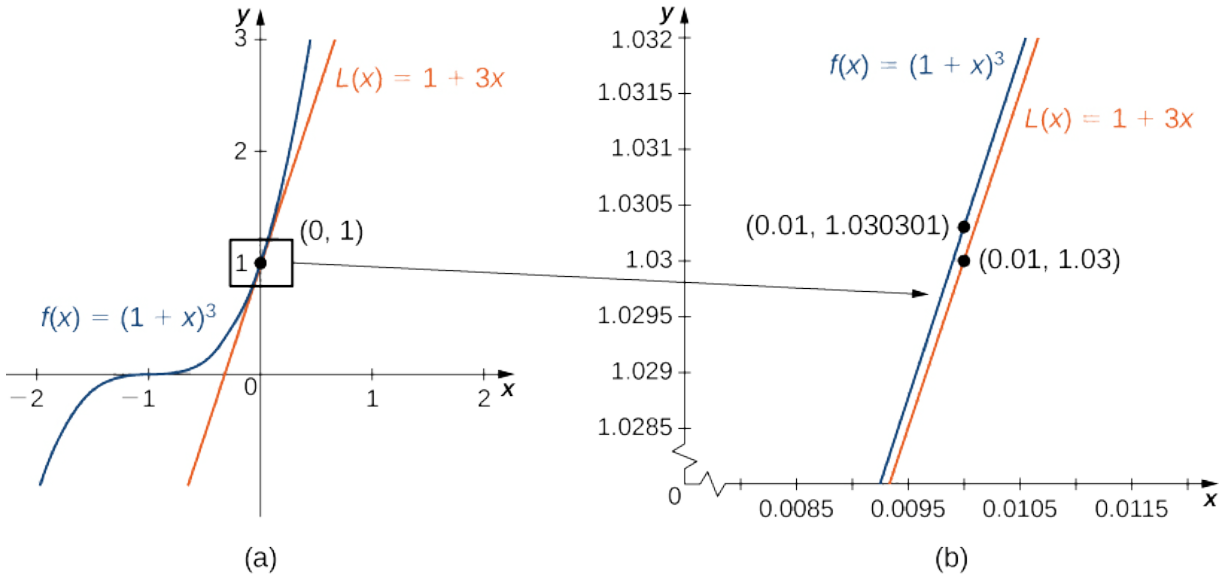

Figure 2.10.4*a*: *The linear approximation of* $f(x)$ *at* $x = 0$ *is* $L(x)$.
Figure 2.10.4*b*: *The actual value of* 1.01^3 *is* 1.030301. *The linear approximation of* $f(x)$ *at* $x = 0$ *estimates* 1.01^3 *to be* 1.03.

Checkpoint 2.10.3

$L(x) = 1 + 4x$

Example 2.10.4: **Computing Differentials**

The key step is calculating the derivative. We can obtain dy directly when we have that.

a. Since $f(x) = x^2 + 2x$, we know $f'(x) = 2x + 2$, and therefore

$$dy = (2x + 2)\,dx.$$

When $x = 3$ and $dx = 0.1$,

$$dy = (2 \cdot 3 + 2)(0.1) = 0.8.$$

b. Since $f(x) = \cos x, f'(x) = -\sin(x)$. This gives us

$$dy = -\sin x\,dx.$$

When $x = 3$ and $dx = 0.1$,

$$dy = -\sin(3)(0.1) = -0.1\sin(3).$$

Checkpoint 2.10.4

$dy = 2xe^{x^2}dx$

Example 2.10.5: **Approximating Change with Differentials**

The actual change in y if x changes from $x = 3$ to $x = 3.1$ is given by

$$\Delta y = f(3.1) - f(3) = [(3.1)^2 + 2(3.1)] - [3^2 + 2(3)] = 0.81.$$

The approximate change in y is given by $dy = f'(3)\ dx$. Since $f'(x) = 2x + 2$, we have

$$dy = f'(3)\,dx = (2(3) + 2)(0.1) = 0.8.$$

Checkpoint 2.10.5

$dy = 1.6,\ \ \Delta y = 1.64$

Example 2.10.6: **Volume of a Cube**

a. The measurement of the side length is accurate to within ± 0.1 cm. Therefore,

$$-0.1 \le dx \le 0.1.$$

The volume of a cube is given by $V = x^3$, which leads to

$$dV = 3x^2dx.$$

Using the measured side length of 5 cm, we can estimate that

$$-3(5)^2(0.1) \le dV \le 3(5)^2(0.1).$$

Therefore,

$$-7.5 \le dV \le 7.5.$$

That is, with our possible measurement error of ± 0.1 cm, our estimation of the volume could be off by at most ± 7.5 cm^3.

b. If the side length is actually 4.9 cm, then the volume of the cube is

$$V(4.9) = (4.9)^3 = 117.649\text{cm}^3.$$

If the side length is actually 5.1 cm, then the volume of the cube is

$$V(5.1) = (5.1)^3 = 132.651\text{cm}^3.$$

Therefore, the actual volume of the cube is between 117.649 and 132.651. Since the side length is measured to be 5 cm, the computed volume is $V(5) = 5^3 = 125$. Therefore, the error in the computed volume is

$$117.649 - 125 \leq \Delta V \leq 132.651 - 125.$$

That is,

$$-7.351 \leq \Delta V \leq 7.651.$$

We see the estimated error dV is relatively close to the actual potential error in the computed volume.

Checkpoint $2.10.6$

The volume measurement is accurate to within 21.6 cm^3.

Example $2.10.7$**: Relative and Percentage Error**

If the measurement of the radius is accurate to within ± 80, we have

$$-80 \leq dr \leq 80.$$

Since the volume of a sphere is given by $V = (\frac{4}{3})\pi r^3$, we have

$$dV = 4\pi r^2 dr.$$

Using the measured radius of 4000 mi, we can estimate

$$-4\pi(4000)^2(80) \leq dV \leq 4\pi(4000)^2(80).$$

To estimate the relative error, consider $\frac{dV}{V}$. Since we do not know the exact value of the volume V, we use the measured radius $r = 4000$ mi to estimate V. We obtain $V \approx (\frac{4}{3})\pi(4000)^3$. Therefore the relative error satisfies

$$\frac{-4\pi(4000)^2(80)}{4\pi(4000)^3/3} \leq \frac{dV}{V} \leq \frac{4\pi(4000)^2(80)}{4\pi(4000)^3/3},$$

which simplifies to

$$-0.06 \leq \frac{dV}{V} \leq 0.06.$$

The relative error is 0.06 and the percentage error is 6%.

Checkpoint $2.10.7$

7.6%

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2.10.1: Resources and Key Concepts

Resources

Prerequisite Topics

The following is a list of prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Arithmetic**
  - Percents: Used when calculating percentage error from relative error.

Videos

- Linear approximation
  - Linear Approximations
- Differentials
  - Differentials

Key Concepts

Definitions

- **Linear Approximation (or Tangent Line Approximation)**: For a function f differentiable at a, the linear approximation of f at a is the linear function $L(x) = f(a) + f'(a)(x - a)$. For x near a, $f(x) \approx L(x)$.
- **Linearization**: The function $L(x)$ is also known as the linearization of f at $x = a$.
- **Differentials**:
  - Let $y = f(x)$ be a differentiable function. The differential dx is an independent variable that can be assigned any nonzero real number.
  - The differential dy is defined as $dy = f'(x)\ dx$.
- **Measurement Error (dx or Δx)**: If the exact value of a measured quantity is a and the measured value is $a + dx$, then dx is the measurement error.
- **Propagated Error (Δy)**: The error that occurs in a calculated quantity $f(x)$ due to a measurement error dx in the input x. It is given by $\Delta y = f(a + dx) - f(a)$.
- **Approximation of Propagated Error using Differentials (dy)**: $\Delta y \approx dy = f'(a)\ dx$.
- **Relative Error**: If Δq is the absolute error in a quantity whose actual value is q, the relative error is $\frac{\Delta q}{q}$. (Can be approximated by $\frac{dq}{q}$).
- **Percentage Error**: The relative error expressed as a percentage.

Common Mistakes

- **Incorrect Evaluation Point for Linear Approximation**: When approximating $f(x)$ using $L(x) = f(a) + f'(a)(x - a)$ for x near a, if $x = a + \text{decimal part}$, students might incorrectly evaluate f and L at x instead of evaluating $f(a)$ and $f'(a)$ and then L at the "decimal part" relative to a. For example, to approximate $(1.01)^3$ using $f(x) = (1 + x)^3$ linearized at $a = 0$, one evaluates $L(0.01)$, not $L(1.01)$ directly if $L(x)$ was defined as $f(0) + f'(0)x$.
- **Confusing dy and Δy**: While dy approximates Δy, they are not identical. dy is the change along the tangent line, while Δy is the actual change in the function value.
- **Leibniz Notation as a Fraction**: Treating $\frac{dy}{dx}$ as a true fraction that can be "pulled apart" by multiplying by dx to get $dy = f'(x)\ dx$ is a notational convenience that works, but the underlying logic is based on $\frac{dy}{dx}$ as an operator.

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2.10.2: Homework

Note

The following exercises are meant as hand-in homework (if assigned by your instructor). Typed-in answers here will not be saved or scored.

Basic Skills

Exercise 1

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Exercise 2

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Exercise 3

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Exercise 4

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Exercise 5

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Exercise 6

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Exercise 7

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Exercise 8

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Exercise 9

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Exercise 10

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Exercise 11

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Exercise 12

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Exercise 13

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Exercise 14

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Exercise 15

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Exercise 16

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Basic Skills - Applications

Exercise 17

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Exercise 18

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Synthesis Questions

Exercise 19

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Synthesis Questions - Applications

Exercise 20

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2.10.3: Additional Exercises

Reading Questions

1. What is the formula for the linear approximation (or linearization) $L(x)$ of a function $f(x)$ at a point $x = a$?
2. What geometric line does the graph of $L(x)$ represent?
3. How are differentials dx and dy defined for a function $y = f(x)$?
4. What is the relationship between the differential dy and the actual change in y, denoted Δy, when x changes from a to $a + dx$?
5. How can differentials be used to approximate the change in a function's value due to a small change in its input?
6. What is meant by "measurement error" and "propagated error"?
7. How is relative error defined? How is percentage error related to relative error?
8. In Example 1, how was the linear approximation of $f(x) = \sqrt{x}$ at $x = 9$ used to estimate $\sqrt{9.1}$?
9. Why might one use a linear approximation to estimate a value like $\sqrt{9.1}$ when a calculator can provide a more precise value?
10. When using differentials to estimate the error in the volume of a cube (Example 6), if the side length is measured as x with a possible error of dx, what is the differential dV?

Homework

1) What is the linear approximation for any generic linear function $y = mx + b$?

2) Determine the necessary conditions such that the linear approximation function is constant. Use a graph to prove your result.

Answer

$f'(a) = 0$

3) Explain why the linear approximation becomes less accurate as you increase the distance between x and a. Use a graph to prove your argument.

4) When is the linear approximation exact?

Answer

The linear approximation exact when $y = f(x)$ is linear or constant.

In exercises 5 - 10, find the linear approximation $L(x)$ to $y = f(x)$ near $x = a$ for the function.

5) $f(x) = x + x^4, \quad a = 0$

6) $f(x) = \dfrac{1}{x}, \quad a = 2$

Answer

$L(x) = \frac{1}{2} - \frac{1}{4}(x - 2)$

7) $f(x) = \tan x, \quad a = \frac{\pi}{4}$

8) $f(x) = \sin x, \quad a = \frac{\pi}{2}$

Answer

$L(x) = 1$

9) $f(x) = x \sin x, \quad a = 2\pi$

10) $f(x) = \sin^2 x, \quad a = 0$

Answer

$L(x) = 0$

In exercises 11 - 16, compute the values given within 0.01 by deciding on the appropriate $f(x)$ and a, and evaluating $L(x) = f(a) + f'(a)(x - a)$. Check your answer using a calculator.

11) [Technology Required] $(2.001)^6$

12) [Technology Required] $\sin(0.02)$

Answer

$\sin(0.02) \approx 0.02$

13) [Technology Required] $\cos(0.03)$

14) [Technology Required] $(15.99)^{1/4}$

Answer

$(15.99)^{1/4} \approx 1.9996875$

15) [Technology Required] $\dfrac{1}{0.98}$

16) [Technology Required] $\sin(3.14)$

Answer

$\sin(3.14) \approx 0.001593$

In exercises 17 - 22, determine the appropriate $f(x)$ and a, and evaluate $L(x) = f(a) + f'(a)(x - a)$. Calculate the numerical error in the linear approximations that follow.

17) $(1.01)^3$

18) $\cos(0.01)$

Answer

$\cos(0.01) \approx L(0.01) = f(0) + f'(0)(0 - 0.01) = 1$; error, ≈ 0.00005

19) $(\sin(0.01))^2$

20) $(1.01)^{-3}$

Answer

$(1.01)^{-3} \approx L(1.01) = f(1) + f'(1)(1.01 - 1) = 0.97$; error, ≈ 0.0006

21) $\left(1 + \frac{1}{10}\right)^{10}$

22) $\sqrt{8.99}$

Answer

$\sqrt{8.99} \approx L(8.99) = f(9) + f'(9)(8.99 - 9) = 3 - \frac{1}{600}$; error, $\approx 4.632 \times 10^{-7}$

In exercises 23 - 26, find the differential of the function.

23) $y = 3x^4 + x^2 - 2x + 1$

24) $y = x \cos x$

Answer

$dy = (\cos x - x \sin x)\ dx$

25) $y = \sqrt{1+x}$

26) $y = \dfrac{x^2+2}{x-1}$

Answer

$$dy = (\frac{x^2-2x-2}{(x-1)^2})dx$$

In exercises 27 - 32, find the differential and evaluate for the given x and dx.

27) $y = 3x^2 - x + 6, \;\; x = 2, \;\; dx = 0.1$

28) $y = \dfrac{1}{x+1}, \;\; x = 1, \;\; dx = 0.25$

Answer

$$dy = -\frac{1}{(x+1)^2}dx, \quad dy = -\tfrac{1}{16}$$

29) $y = \tan x, \;\; x = 0, \;\; dx = \frac{\pi}{10}$

30) $y = \dfrac{3x^2+2}{\sqrt{x+1}}, \;\; x = 0, \;\; dx = 0.1$

Answer

$$dy = \frac{9x^2+12x-2}{2(x+1)^{3/2}}dx, \quad dy = -0.1$$

31) $y = \dfrac{\sin(2x)}{x}, \;\; x = \pi, \;\; dx = 0.25$

32) $y = x^3 + 2x + \dfrac{1}{x}, \;\; x = 1, \;\; dx = 0.05$

Answer

$$dy = \left(3x^2 + 2 - \frac{1}{x^2}\right)dx, \quad dy = 0.2$$

In exercises 33 - 38, find the change in volume dV or in surface area dA.

33) dV if the sides of a cube change from 10 to 10.1.

34) dA if the sides of a cube change from x to $x + dx$.

Answer

$$dA = 12x \;\; dx$$

35) dA if the radius of a sphere changes from r by dr.

36) dV if the radius of a sphere changes from r by dr.

Answer

$$dV = 4\pi r^2 dr$$

37) dV if a circular cylinder with $r = 2$ changes height from 3 cm to $3.05cm$.

38) dV if a circular cylinder of height 3 changes from $r = 2$ to $r = 1.9$ cm.

Answer

$$dV = -1.2\pi \text{ cm}^3$$

In exercises 39 - 41, use differentials to estimate the maximum and relative error when computing the surface area or volume.

39) A spherical golf ball is measured to have a radius of 5 mm, with a possible measurement error of 0.1 mm. What is the possible change in volume?

40) A pool has a rectangular base of 10 ft by 20 ft and a depth of 6 ft. What is the change in volume if you only fill it up to 5.5 ft?

Answer

-100 ft^3

41) An ice cream cone has height 4 in. and radius 1 in. If the cone is 0.1 in. thick, what is the difference between the volume of the cone, including the shell, and the volume of the ice cream you can fit inside the shell?

In exercises 42 - 44, confirm the approximations by using the linear approximation at $x = 0$.

42) $\sqrt{1-x} \approx 1 - \frac{1}{2}x$

43) $\dfrac{1}{\sqrt{1-x^2}} \approx 1$

44) $\sqrt{c^2 + x^2} \approx c$

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3: ADVANCING WITH APPLICATIONS

A rocket launch involves two related quantities that change over time. Being able to solve this type of problem is just one application of derivatives introduced in this chapter. We also look at how derivatives are used to find maximum and minimum values of functions. As a result, we will be able to solve applied optimization problems, such as maximizing revenue and minimizing surface area. In addition, we examine how derivatives are used to evaluate complicated limits, to approximate roots of functions, and to provide accurate graphs of functions.

3.1: Extrema

3.1.1: Resources and Key Concepts

3.1.2: Homework

3.2: The Mean Value Theorem

3.2.1: Resources and Key Concepts

3.2.2: Homework

3.3: Shapes of Curves

3.3.1: Resources and Key Concepts

3.3.2: Homework

3.4: l'Hôpital's Rule and Indeterminate Forms

3.4.1: Resources and Key Concepts

3.4.2: Homework

3.5: Graphing Using Calculus

3.5.1: Resources and Key Concepts

3.5.2: Homework

3.6: Optimization

3.6.1: Resources and Key Concepts

3.6.2: Homework

3.7: Newton’s Method

3.7.1: Resources and Key Concepts

3.7.2: Homework

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3.1: Extrema

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Given a particular function, we are often interested in determining the largest and smallest values. This information is important in creating accurate graphs. Finding a function's maximum and minimum values is also practical because this method can solve optimization problems, such as maximizing profit, minimizing the amount of material used in manufacturing an aluminum can, or finding the maximum height a rocket can reach. This section uses derivatives to find a function's largest and smallest values.

Absolute Extrema

Consider the function $f(x) = x^2 + 1$ over the interval $(-\infty, \infty)$. As $x \to \pm\infty, f(x) \to \infty$. Therefore, the function does not have a largest value. However, since $x^2 + 1 \geq 1$ for all real numbers x and $x^2 + 1 = 1$ when $x = 0$, the function has a smallest value, 1, when $x = 0$. We say that 1 is the **absolute minimum** of $f(x) = x^2 + 1$ and it occurs at $x = 0$. We say that $f(x) = x^2 + 1$ does not have an **absolute maximum** (Figure $3.1.1$).

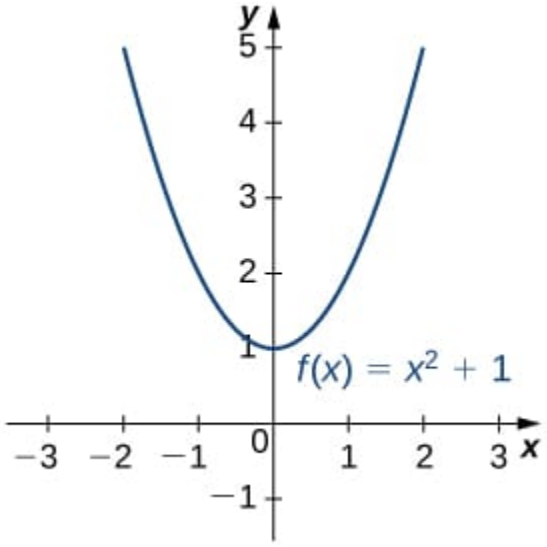

Figure $3.1.1$**:** *The given function has an absolute minimum of* 1 *at* $x = 0$*. The function does not have an absolute maximum.*

Definition: Absolute Extrema

Let f be a function defined over an interval I and let $c \in I$. We say f has an **absolute maximum** on I at c if $f(c) \geq f(x)$ for all $x \in I$. We say f has an **absolute minimum** on I at c if $f(c) \leq f(x)$ for all $x \in I$. If f has an absolute maximum on I at c or an absolute minimum on I at c, we say f has an **absolute extremum** on I at c.

Before proceeding, let's note two important issues regarding this definition. First, the term absolute here does not refer to absolute value. An absolute extremum may be positive, negative, or zero. Second, if a function f has an absolute extremum over an interval I at c, the absolute extremum is $f(c)$. The real number c is a point in the domain at which the absolute extremum occurs. For example, consider the function $f(x) = \frac{1}{x^2+1}$ over the interval $(-\infty, \infty)$. Since

$$f(0) = 1 \geq \frac{1}{x^2+1} = f(x)$$

for all real numbers x, we say f has an absolute maximum over $(-\infty, \infty)$ at $x = 0$. The absolute maximum is $f(0) = 1$. It occurs at $x = 0$, as shown in Figure $3.1.2b$.

A function may have both an absolute maximum and an absolute minimum, just one extremum, or neither. Figure $3.1.2$ shows several functions and some of the different possibilities regarding **absolute extrema** ("extrema" is the plural form of extremum). However, the following theorem, called the **Extreme Value Theorem**, guarantees that a continuous function f over a closed interval $[a, b]$ has both an absolute maximum and an absolute minimum.

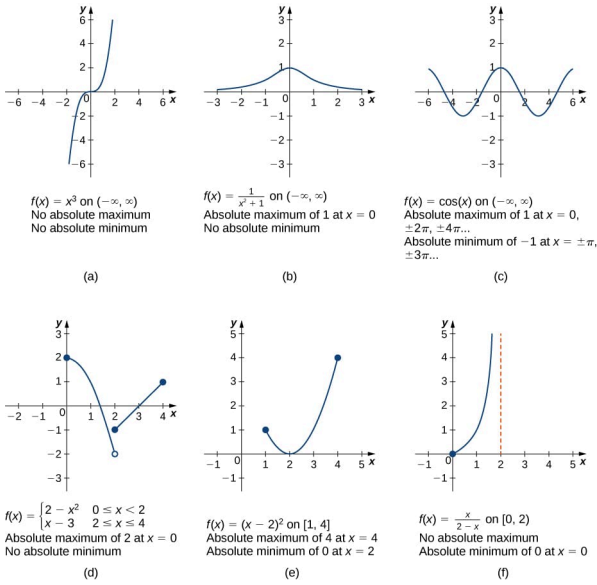

Figure $3.1.2$**:** *Graphs (a), (b), and (c) show several possibilities for absolute extrema for functions with a domain of* $(-\infty, \infty)$*. Graphs (d), (e), and (f) show several possibilities for absolute extrema for functions with a domain that is a bounded interval.*

Theorem: Extreme Value Theorem (EVT)

If f is a continuous function over the closed interval $[a, b]$, then there is a point in $[a, b]$ at which f has an absolute maximum over $[a, b]$ and there is a point in $[a, b]$ at which f has an absolute minimum over $[a, b]$.

The proof of the Extreme Value Theorem is beyond the scope of this text. Typically, it is proved in a course on Real Analysis. There are a couple of key points to note about the statement of this theorem. To apply the Extreme Value Theorem, the function must be continuous over a closed interval. Suppose the interval I is open, or the function has even one point of discontinuity. In that case, the function may not have an absolute maximum or absolute minimum over I.

For example, consider the functions shown in Figure $3.1.2d$, $3.1.2e$, and $3.1.2f$. All three of these functions are defined over closed intervals. However, the function in graph $3.1.2e$ is the only one with both an absolute maximum and an absolute minimum over its domain. The Extreme Value Theorem cannot be applied to the functions in graphs $3.1.2d$ and $3.1.2f$ because neither of these functions is continuous over a closed interval. Although the function in graph $3.1.2d$ is defined over the closed interval $[0, 4]$, the function is discontinuous at $x = 2$. The function has an absolute maximum over the closed interval $[0, 4]$ but does not have an absolute minimum. The function in graph $3.1.2f$ is continuous over the half-open interval $[0, 2)$, but is not defined at $x = 2$, and therefore is not continuous over a closed interval. The function has an absolute minimum over $[0, 2)$ but does not have an absolute maximum over $[0, 2)$. These two graphs illustrate why a function over a closed interval may fail to have an absolute maximum and/or minimum.

Caution: Continuity, Closed Intervals, and the EVT

The Extreme Value Theorem only applies to functions that are continuous over a closed interval $[a, b]$.

Before looking at how to find absolute extrema, let's examine the related concept of local extrema. This idea is useful in determining where absolute extrema occur.

Local Extrema and Critical Numbers

Consider the function f shown in Figure 3.1.3. The graph can be described as two mountains with a valley in the middle. The absolute maximum value of the function occurs at the higher peak, at $x = 2$. However, $x = 0$ is also a point of interest. Although $f(0)$ is not the largest value of f, the value $f(0)$ is larger than $f(x)$ for all x near 0. We say f has a **local maximum** (or **relative maximum**) at $x = 0$. Similarly, the function f does not have an absolute minimum, but it does have a **local minimum** (or **relative minimum**) at $x = 1$ because $f(1)$ is less than $f(x)$ for x near 1.

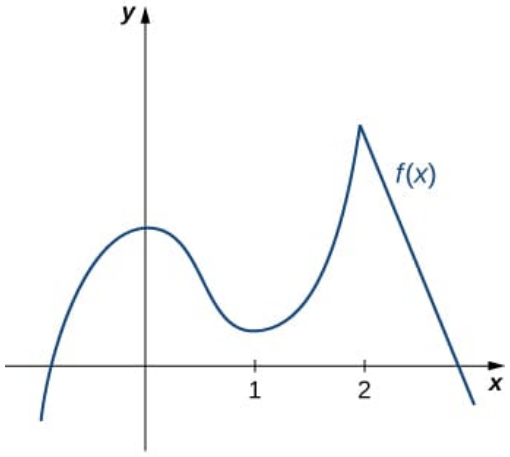

Figure 3.1.3**:** *This function f has two local maxima and one local minimum. The local maximum at $x = 2$ is also the absolute maximum.*

Definition: Local Extrema

A function f has a **local maximum** at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \geq f(x)$ for all $x \in I$. A function f has a **local minimum** at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \leq f(x)$ for all $x \in I$. A function f has a **local extremum** at c if f has a local maximum at c or f has a local minimum at c.

Note that if f has an absolute extremum at c and f is defined over an interval containing c, then $f(c)$ is also considered a local extremum.

Common Mistake

Local extrema require the function to be defined in a neighborhood containing c. That is, the function must be defined on both sides of c for the possibility of a local extremum at c. Hence, endpoints of a function's domain are never considered local extrema, even if they are absolute extrema.

Given the graph of a function f, it is sometimes easy to see where a local maximum or local minimum occurs. However, since the

interesting features on the graph of a function may not be visible because they occur at a tiny scale, we might miss out on seeing these local extrema. Also, we may not have a graph of the function. In these cases, how can we use a formula for a function to determine where these extrema occur?

To answer this question, let's look at Figure $3.1.3$ again. The local extrema occur at $x = 0$, $x = 1$, and $x = 2$. Notice that at $x = 0$ and $x = 1$, $f'(x) = 0$. At $x = 2$, the derivative $f'(x)$ does not exist, since the function f has a corner there. In fact, if f has a local extremum at a point $(c, f(c))$, the derivative $f'(c)$ must satisfy one of the following conditions: either $f'(c) = 0$ or $f'(c)$ is undefined. Such a value c is known as a **critical number**, and the corresponding point $(c, f(c))$ is called a **critical point**. Critical numbers (and, therefore, critical points) are important in finding extreme values for functions.

Definition: Critical Numbers and Critical Points

Let c be an interior point in the domain of f. We say that c is a **critical number** of f if $f'(c) = 0$ or $f'(c)$ is undefined. The corresponding point, $(c, f(c))$, is called a **critical point**.

Note: Critical Numbers, Points, and Values... Oh My!

On the Cartesian coordinate system, the notation $(a, f(a))$ is used to reference a **point**. It is definitely *not* a number *nor* is it a value. In fact, we refer to $f(a)$ as the **value** of the function f at the **number** a.

If you are okay with those previous statements, then you should be fine with the distinction between a **critical point**, $(a, f(a))$, the **critical number**, $x = a$, and the **critical value**, $f(a)$.

It is unfortunate that most authors blur the line between these definitions and call $x = a$ a critical point. In this text, I endeavor to be consistent with my language.

As mentioned earlier, if f has a local extremum at $x = c$, then c must be a critical number of f. This fact is known as **Fermat's Theorem.**

Theorem: Fermat's Theorem

If f has a local extremum at c and f is differentiable at c, then $f'(c) = 0$.

Proof available at the end of this section.

From Fermat's Theorem, we conclude the following corollary.

Corollary

If f has a local extremum at c, then either $f'(c) = 0$ or $f'(c)$ is undefined.

In other words, local extrema can only occur at critical points.

Interactive Element: Fermat's Theorem — Horizontal Tangents at Local Extrema

The following Interactive Element might help you visualize why a smooth local maximum or minimum forces $f'(c) = 0$.

Interact: Drag the point along the curve $y = x^3 - 3x$ toward the peak and the valley.

Observation: The tangent line becomes horizontal (slope 0) exactly at the local maximum and local minimum; a horizontal tangent marks a critical point—a candidate for a local extremum.

Caution: Critical Points Need Not Be Local Extremums

Fermat's Theorem does not claim that a function f *must* have a local extremum at a critical point. Rather, it states that critical points are candidates for local extrema.

The cautionary statement above needs some explanation. Consider the function $f(x) = x^3$. We have $f'(x) = 3x^2 = 0$ when $x = 0$. Therefore, $x = 0$ is a critical number. However, $f(x) = x^3$ is increasing over $(-\infty, \infty)$, and thus f does not have a local extremum at $x = 0$.

In Figure 3.1.4, we see several different possibilities for critical points. In some cases, the functions have local extrema at critical points. In contrast, in other cases, the functions do not. Note that these graphs do not show all possibilities for the behavior of a function at a critical point.

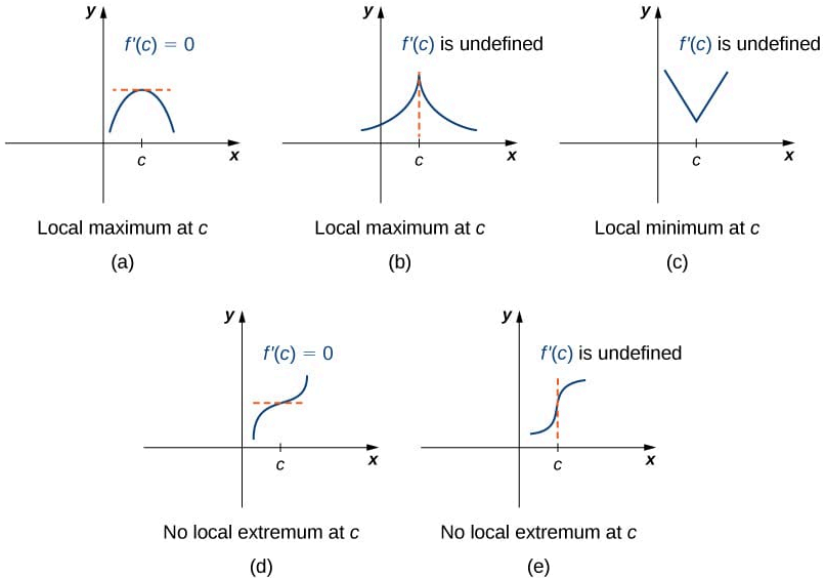

Figure 3.1.4$a - e$**:** *A function f has a critical number at c if $f'(c) = 0$ or $f'(c)$ is undefined. A function may or may not have a local extremum at a critical number.*

Later in this chapter, we look at analytical methods for determining whether a function has a local extremum at a critical point. For now, let's turn our attention to finding critical points. We will use graphical observations to determine whether a critical point is associated with a local extremum.

Example 3.1.1: Locating Critical Points

For each of the following functions, find all critical numbers. Use a graphing utility to determine whether the function has a local extremum at each critical number.

a. $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x$
b. $f(x) = (x^2 - 1)^3$
c. $f(x) = \frac{4x}{1+x^2}$

Solutions available at the end of this section.

Checkpoint 3.1.1

Find all critical numbers for $f(x) = x^3 - \frac{1}{2}x^2 - 2x + 1$.

Answer available at the end of this section.

Locating Absolute Extrema

The Extreme Value Theorem states that a continuous function over a closed interval attains an absolute maximum and minimum. Figure $3.1.2$ shows that one or both of these absolute extrema could occur at an endpoint. If an absolute extremum does not occur at an endpoint, however, it must occur at an **interior point**, in which case the absolute extremum is a local extremum. Therefore, by Fermat's Theorem, the interior point $(c, f(c))$ at which the local extremum occurs must be a critical point. We summarize this result in the following theorem.

Theorem: Location of Absolute Extrema

Let f be a continuous function over a closed interval I. The absolute maximum of f over I and the absolute minimum of f over I must occur at endpoints of I or at critical points of f in the interior of I.

Therefore, when finding the absolute extrema of a given function defined over a closed interval, we evaluate the function at the endpoints of the interval. These will be *candidates* for absolute extrema. In addition, we determine the critical points. The critical values (the values of the function at the critical numbers) will also be *candidates* for absolute extrema. The largest of these function values will be the absolute maximum of the function on the given interval, and the least will be the absolute minimum.

Interactive Element: Locating Absolute Extrema on a Closed Interval

The following Interactive Element might help you visualize how the absolute maximum and minimum of a continuous function on a closed interval $[a, b]$ are chosen from among the endpoints and the interior critical points.

Interact: Drag the red endpoints A and B along the curve to move and resize the interval $[a, b]$.

Observation: The absolute maximum (green ring) and absolute minimum (purple ring) always occur at an endpoint or at an interior critical point—and an interior critical point drops out of consideration as soon as it leaves the interval.

Now, let's use this strategy to find the absolute maximum and minimum values for continuous functions.

Example $3.1.2$: Locating Absolute Extrema

For each function, find the absolute maximum and minimum over the specified interval and state where those values occur.

a. $f(x) = -x^2 + 3x - 2$ over $[1, 3]$.
b. $f(x) = x^2 - 3x^{2/3}$ over $[0, 2]$.

Solutions available at the end of this section.

Checkpoint $3.1.2$

Find the absolute maximum and absolute minimum of $f(x) = x^2 - 4x + 3$ over the interval $[1, 4]$.

Answer available at the end of this section.

At this point, we know how to locate absolute extrema for continuous functions over closed intervals. We have also defined local extrema and determined that if a function f has a local extremum at a number c, then c must be a critical number of f. However, c being a critical number is not a sufficient condition for f to have a local extremum at c. Later in this chapter, we show how to determine whether a function has a local extremum at a critical point. First, however, we need to introduce the Mean Value

Theorem, which will help as we analyze the behavior of the graph of a function.

Time-Sensitive Advice: Drop Deadline

Within the Los Rios Community College District, the last day you can drop a class is the Monday (or, sometimes, the Tuesday) of the thirteenth week of the semester. After this date, you will no longer be able to drop a class for that semester and the only grades you can receive in the classes you remain in are A, B, C, D, or F.

Will you still get a "W?"

Dropping any class between the third week and this final drop deadline will result in a "W" notation on your record for that class.

What does this mean?

Once you withdraw (W) or fail (D or F) a course three times within the Los Rios Community College District (CRC, SCC, FLC, or ARC), you are never allowed to take that course again at any of the colleges within the district. Thus, if you wanted to continue through the math sequence, you would need to take this course outside the Los Rios Community College District.

Success in Mathematics: Math Anxiety (Part 3)

For some math students, anxiety changes their ability to show what they know. They have physical and emotional symptoms. You are responsible for recognizing if math anxiety is interfering with your success. Do you need to make some changes to reduce your math anxiety? Do you need help to do so?

Researchers have identified various events that seem to result in math anxiety. Perhaps you had negative experiences learning math. For you, learning math meant feeling dumb or embarrassed. Perhaps a teacher or parent who struggled with math has passed on their fears to you. Or you believe that your gender or ethnic heritage means you cannot do the math.

Some students learn to cope on their own with their math anxiety. They use relaxation techniques. They practice problems over and over. At test time, their anxiety does not block them from doing the work. Others find help from counselors at their college.

Math anxiety is common among college students. It is your responsibility to think about it and, if needed, get help so that you can cope with your fears.

Math anxiety affects students in physical ways such as increasing blood pressure and heart rate, breathing faster, excess sweating, dry mouth, restlessness, or nausea. Emotionally, it can cause fear, anger, nervousness, worry, and feelings of hopelessness. Students may have trouble concentrating, sitting still, reading, or keeping their legs from bouncing. They may feel desperate - like they need to run away or cry.

Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. Using a scale of 1 to 10, with 10 describing overwhelming anxiety that makes it extremely difficult to do math, rate your math anxiety. How does it affect you?
2. Most students have some anxiety about their classes. They worry about giving speeches, taking tests, writing a good paper, or completing all their work. Worrying can be motivating. It can help us get going and to get things done. Or, worrying can be a bad thing. It paralyzes us. In general, how does worrying affect you?

Answers

Section Preview

The material in this section is meant to introduce the student to extrema from a Calculus perspective.

Section Preview

- **Graphing Functions**
  - Local/Relative and Global/Absolute Extrema: This section formalizes, using derivatives, the graphical notion of peaks and valleys and the local-versus-

absolute distinction that students first meet in precalculus; that prior graphical intuition is the conceptual scaffold for the calculus definitions and for reading extrema off the graphs in Example 1.

Section Preview

- Define absolute extrema, local extrema, and critical numbers.
- Explain how to find the critical points of a function over a closed interval.
- Describe the Extreme Value Theorem and Fermat's Theorem.
- Describe how to use critical points to locate absolute extrema over a closed interval.

Theorem: Fermat's Theorem

Suppose f has a local extremum at c and f is differentiable at c. Without loss of generality, suppose the local extremum at c is a local maximum (the case in which f has a local minimum at c can be handled similarly). There then exists an open interval I such that $f(c) \geq f(x)$ for all $x \in I$. Since f is differentiable at c, we know that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

Since this limit exists, both one-sided limits also exist and are equal $f'(c)$. Therefore,

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c,} \tag{3.1.1}$$

and

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}.$$

Since $f(c)$ is a local maximum, we see that $f(x) - f(c) \leq 0$ for x near c. Therefore, for x near c (but $x > c$), we have $\frac{f(x)-f(c)}{x-c} \leq 0$. From Equation 3.1.1 we conclude that $f'(c) \leq 0$. Similarly, it can be shown that $f'(c) \geq 0$. Therefore, $f'(c) = 0$.

Q.E.D.

Example 3.1.1**: Locating Critical Points**

a. The derivative $f'(x) = x^2 - 5x + 4$ is defined for all real numbers x. Therefore, we only need to find the values for x where $f'(x) = 0$. Since $f'(x) = x^2 - 5x + 4 = (x - 4)(x - 1)$, the critical numbers are $x = 1$ and $x = 4$. From the graph of f in Figure 3.1.5, we see that f has a local maximum at $x = 1$ and a local minimum at $x = 4$.

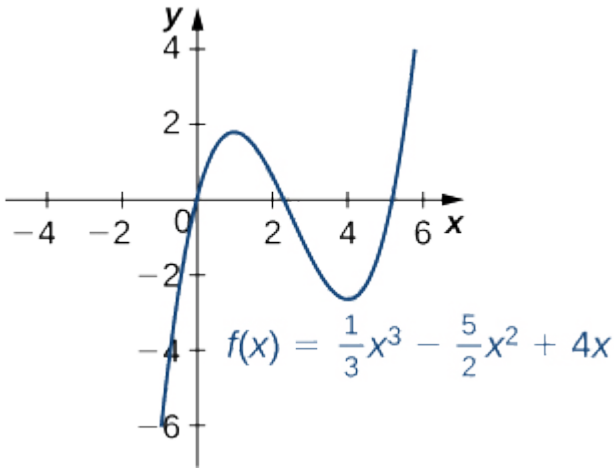

Figure 3.1.5**:** *This function has a local maximum and a local minimum.*

b. Using the Chain Rule, we see the derivative is

$$f'(x) = 3(x^2 - 1)^2(2x) = 6x(x^2 - 1)^2.$$

Therefore, f has critical points when $x = 0$ and when $x^2 - 1 = 0$. We conclude that the critical numbers are $x = 0$, and $x = \pm 1$. From the graph of f in Figure 3.1.6, we see that f has a local (and absolute) minimum at $x = 0$, but does not have a local extrema at $x = 1$ and $x = -1$.

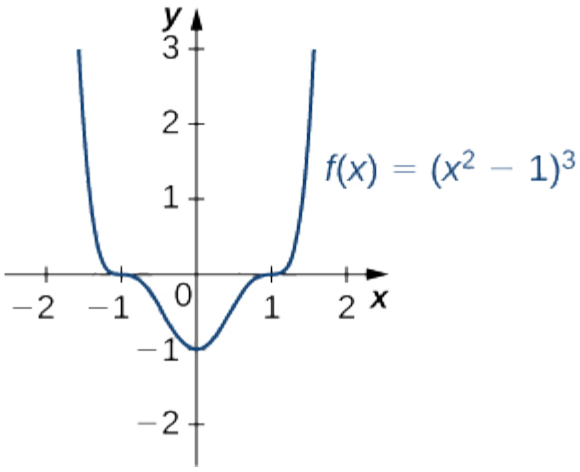

Figure 3.1.6**:** *This function has three critical numbers: $x = 0$, $x = 1$, and $x = -1$. The function has a local (and absolute) minimum at $x = 0$ but does not have extrema at the other two critical points.*

c. By the Quotient Rule, we see that the derivative is

$$f'(x) = \frac{4(1+x^2) - 4x(2x)}{(1+x^2)^2} = \frac{4-4x^2}{(1+x^2)^2}.$$

The derivative is defined everywhere. Therefore, we only need to find values for x where $f'(x) = 0$. Solving $f'(x) = 0$, we see that $4 - 4x^2 = 0$, which implies $x = \pm 1$. Therefore, the critical numbers are $x = \pm 1$. From the graph of f in Figure 3.1.7, we see that f has an absolute maximum at $x = 1$ and an absolute minimum at $x = -1$. Hence, f has a local maximum at $x = 1$ and a local minimum at $x = -1$. (Note that if f has an absolute extremum over an interval I at a point c that is not an endpoint of I, then f has a local extremum at c.)

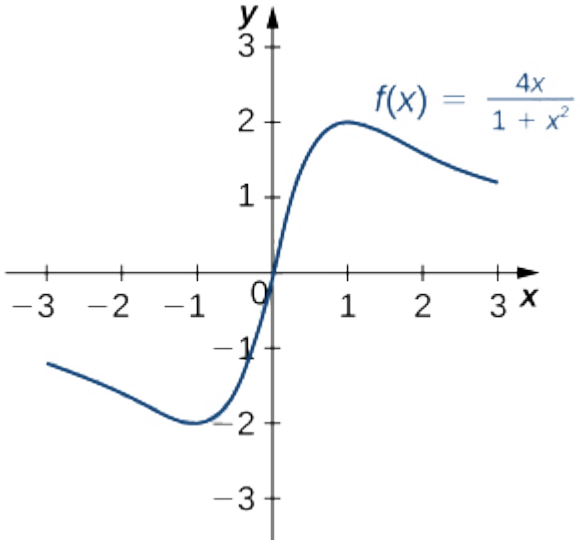

Figure 3.1.7**:** *This function has an absolute maximum and minimum.*

Checkpoint 3.1.1

$x = \frac{-2}{3}, x = 1$

Example 3.1.2**: Locating Absolute Extrema**

a. Evaluate f at the endpoints $x = 1$ and $x = 3$.

$$f(1) = 0 \quad \text{and} \quad f(3) = -2$$

Since $f'(x) = -2x + 3$, f' is defined for all real numbers x. Hence, there are no critical points where the derivative is undefined. It remains to check where $f'(x) = 0$. Since $f'(x) = -2x + 3 = 0$ at $x = \frac{3}{2}$ and $\frac{3}{2}$ is in the interval $[1, 3]$, $f(\frac{3}{2})$ is a *candidate* for an absolute extremum of f over $[1, 3]$. We evaluate $f(\frac{3}{2})$ and find

$$f\left(\frac{3}{2}\right) = \frac{1}{4}.$$

We then set up the following table to compare the functions values we found.

x	$f(x)$	Conclusion
1	0	
$\frac{3}{2}$	$\frac{1}{4}$	Absolute maximum

3	-2	Absolute minimum

From the table, we find that the absolute maximum of f over the interval $[1,3]$ is $\frac{1}{4}$, and it occurs at $x=\frac{3}{2}$. The absolute minimum of f over the interval $[1,3]$ is -2, and it occurs at $x=3$ as shown in Figure 3.1.8.

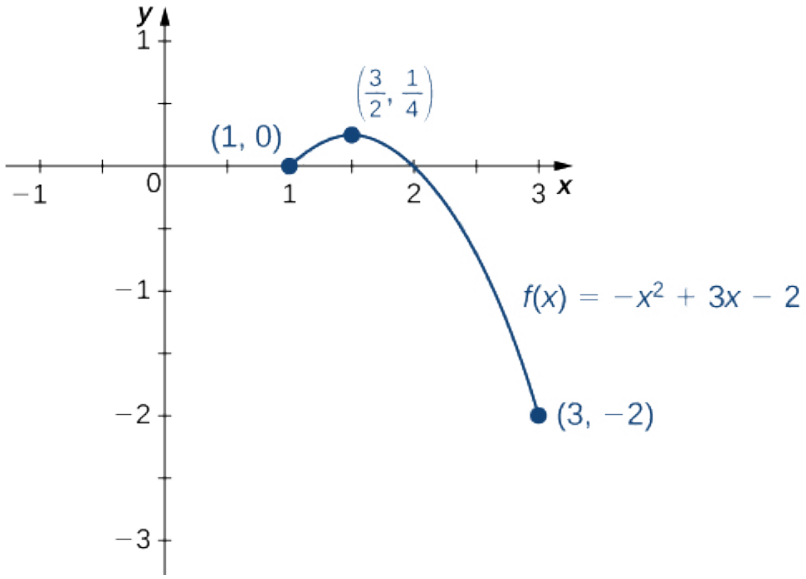

Figure 3.1.8**:** *This function has both an absolute maximum and an absolute minimum.*

b. Evaluate f at the endpoints $x=0$ and $x=2$.

$$f(0)=0 \quad \text{and} \quad f(2)=4-3\,(2)^{2/3}\approx -0.762$$

The derivative of f is given by

$$f'(x)=2x-\frac{2}{x^{1/3}}=\frac{2x^{4/3}-2}{x^{1/3}}$$

for $x\neq 0$. The derivative is zero when $2x^{4/3}-2=0$, which implies $x=\pm 1$. The derivative is undefined at $x=0$. Therefore, the critical numbers of f are $x=0$, 1, and -1. We already evaluated $f(0)$. The point $x=-1$ is not in the interval of interest, so we need only evaluate $f(1)$. We find that $f(1)=-2$. Finally, we compare the values in the following table.

x	$f(x)$	Conclusion
0	0	Absolute maximum
1	-2	Absolute minimum
2	-0.762	

We conclude that the absolute maximum of f over the interval $[0,2]$ is zero, and it occurs at $x=0$. The absolute minimum is -2, and it occurs at $x=1$ as shown in Figure 3.1.9.

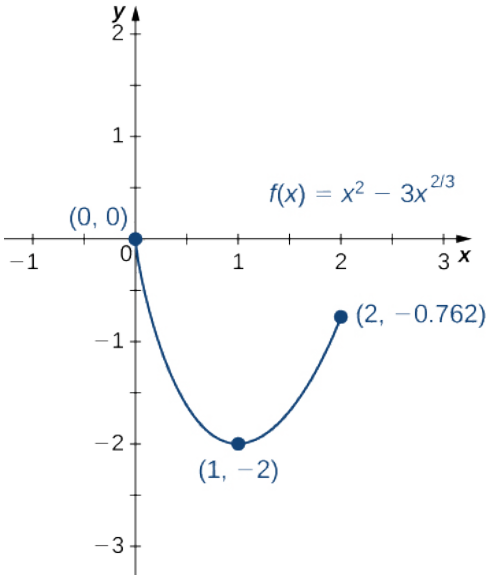

Figure 3.1.9**:** *This function has an absolute maximum at an endpoint of the interval.*

Checkpoint 3.1.2

The absolute maximum is 3 and it occurs at $x = 4$. The absolute minimum is -1 and it occurs at $x = 2$.

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3.1.1: Resources and Key Concepts

Resources

Prerequisite Topics

The following is a list of prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Solving Equations**
  - Solving Polynomial Equations: To find critical numbers, one often needs to solve $f'(x) = 0$, which can be a polynomial equation (e.g., $x^2 - 5x + 4 = 0$ in Example 1a).
  - Solving Radical Equations: May be needed if $f'(x)$ contains radicals.

Videos

- Extrema
  - Local Extrema, Critical Points, and Fermat's Interior Extremum Theorem
  - Absolute Extrema

Key Concepts

Definitions

- **Absolute Maximum**: Let f be a function defined over an interval I and let $c \in I$. We say f has an absolute maximum on I at c if $f(c) \geq f(x)$ for all $x \in I$. The value $f(c)$ is the absolute maximum value.
- **Absolute Minimum**: Let f be a function defined over an interval I and let $c \in I$. We say f has an absolute minimum on I at c if $f(c) \leq f(x)$ for all $x \in I$. The value $f(c)$ is the absolute minimum value.
- **Absolute Extremum**: If f has an absolute maximum on I at c or an absolute minimum on I at c, we say f has an absolute extremum on I at c. (Plural: extrema)
- **Local Maximum (Relative Maximum)**: A function f has a local maximum at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \geq f(x)$ for all $x \in I$.
- **Local Minimum (Relative Minimum)**: A function f has a local minimum at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \leq f(x)$ for all $x \in I$.
- **Local Extremum**: A function f has a local extremum at c if f has a local maximum at c or f has a local minimum at c.
- **Critical Number**: Let c be an interior point in the domain of f. We say that c is a critical number of f if $f'(c) = 0$ or $f'(c)$ is undefined.
- **Critical Point**: The point $(c, f(c))$ on the graph of f where c is a critical number.

Theorems

- **Extreme Value Theorem (EVT)**: If f is a continuous function over the closed interval $[a, b]$, then there is a point in $[a, b]$ at which f has an absolute maximum over $[a, b]$ and there is a point in $[a, b]$ at which f has an absolute minimum over $[a, b]$.
- **Fermat's Theorem**: If f has a local extremum at c and f is differentiable at c, then $f'(c) = 0$.
- **Corollary (to Fermat's Theorem)**: If f has a local extremum at c, then c must be a critical number of f (i.e., either $f'(c) = 0$ or $f'(c)$ is undefined).
- **Theorem: Location of Absolute Extrema**: Let f be a continuous function over a closed interval I. The absolute maximum of f over I and the absolute minimum of f over I must occur at endpoints of I or at critical points of f in the interior of I.

Common Mistakes

- **Misunderstanding "Absolute" in Absolute Extrema**: The term "absolute" here does not refer to absolute value; an absolute extremum can be positive, negative, or zero.
- **Confusing Location and Value of Extrema**: If a function has an absolute extremum at c, the extremum (value) is $f(c)$, while c is the point (number/location) in the domain where it occurs.
- **Applying EVT Incorrectly**: The Extreme Value Theorem only applies to functions that are continuous over a closed, bounded interval. If the interval is open or the function has discontinuities, an absolute maximum or minimum may not exist.
- **Considering Endpoints as Local Extrema**: Local extrema require the function to be defined in an open interval (a neighborhood) around the point. Endpoints of a function's domain are generally not considered local extrema, even if they are absolute extrema.
- **Assuming Critical Points Are Always Local Extrema**: Fermat's Theorem states that if a local extremum occurs at a differentiable point c, then $f'(c) = 0$. However, $f'(c) = 0$ or $f'(c)$ being undefined (i.e., c is a critical point) does not guarantee a local extremum at c. Critical points are only candidates.

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3.1.2: Homework

Reading Questions

1. What is the difference between an absolute maximum and a local maximum?
2. State the Extreme Value Theorem. What two conditions must a function f satisfy on an interval $[a, b]$ for this theorem to guarantee the existence of absolute extrema?
3. Can a function that is not continuous on a closed interval still have an absolute maximum and minimum on that interval? Can the EVT be used to guarantee it?
4. What is a critical number of a function f?
5. What is a critical point of a function f?
6. State Fermat's Theorem regarding local extrema and derivatives.
7. If $f'(c) = 0$, does f necessarily have a local extremum at c? Explain.
8. According to the "Location of Absolute Extrema" theorem, where should you look for the absolute maximum and minimum of a continuous function on a closed interval?
9. Describe the process for finding the absolute extrema of a continuous function on a closed interval $[a, b]$.
10. What does the term "relative extrema" mean, as clarified in the "Some Clarifications" box?

Homework

1) In Precalculus, you learned a formula for the position of the maximum or minimum of a quadratic equation $y = ax^2 + bx + c$, which was $m = -\frac{b}{2a}$. Prove this formula using calculus.

2) If you are finding an absolute minimum over an interval $[a, b]$, why do you need to check the endpoints? Draw a graph that supports your hypothesis.

Answer

On a closed interval, the endpoints often lie above or below any local (relative) extrema. Answers may vary for the graph.

3) If you are examining a function over an interval (a, b), for a and b finite, is it possible not to have an absolute maximum or absolute minimum?

4) When you are checking for critical points to locate the extrema of a function f, explain why you also need to determine points where $f'(x)$ is undefined. Draw a graph to support your explanation.

Answer

Points on the graph of f where there is a corner, a cusp, or a jump discontinuity or removable discontinuity can easily be absolute (or local) extrema of the function. Answers may vary for the graph.

5) Can you have a finite absolute maximum for $y = ax^2 + bx + c$ over $(-\infty, \infty)$? Explain why or why not using graphical arguments.

6) Can you have a finite absolute maximum for $y = ax^3 + bx^2 + cx + d$ over $(-\infty, \infty)$ assuming a is non-zero? Explain why or why not using graphical arguments.

Answer

No; answers will vary

7) Let m be the number of local minima and M be the number of local maxima. Can you create a function where $M > m + 2$? Draw a graph to support your explanation.

8) Is it possible to have more than one absolute maximum? Use a graphical argument to prove your hypothesis.

Answer

Since the absolute maximum is the function (output) value rather than the x value, the answer is no; answers

will vary

9) Is it possible to have no absolute minimum or maximum for a function? If so, construct such a function. If not, explain why this is not possible.

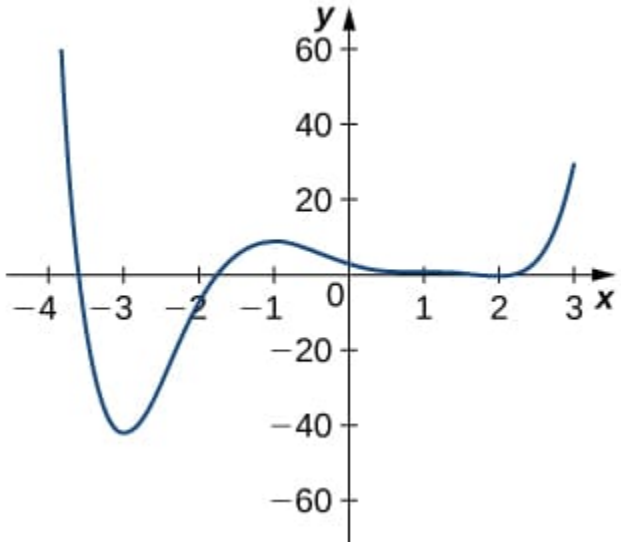
10) Graph the function $y = e^{ax}$. For which values of a, on any infinite domain, will you have an absolute minimum and absolute maximum?

Answer

When $a = 0$

In exercises 11 - 14, determine where the local and absolute maxima and minima occur on the graph given. Assume domains are closed intervals unless otherwise specified.

11)



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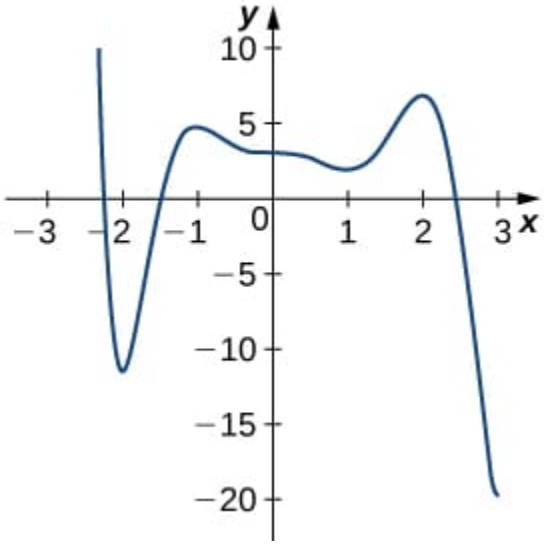
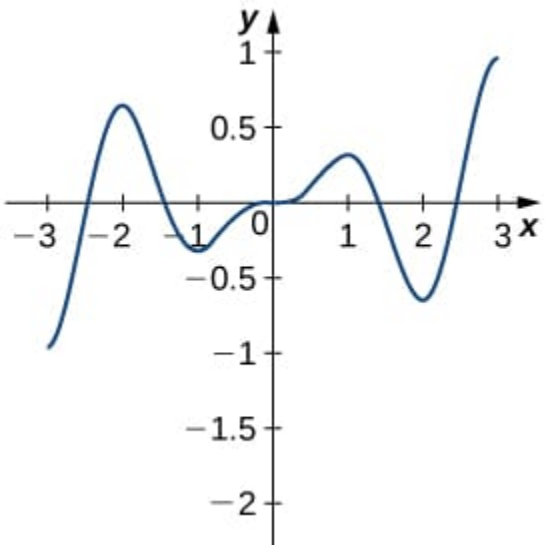


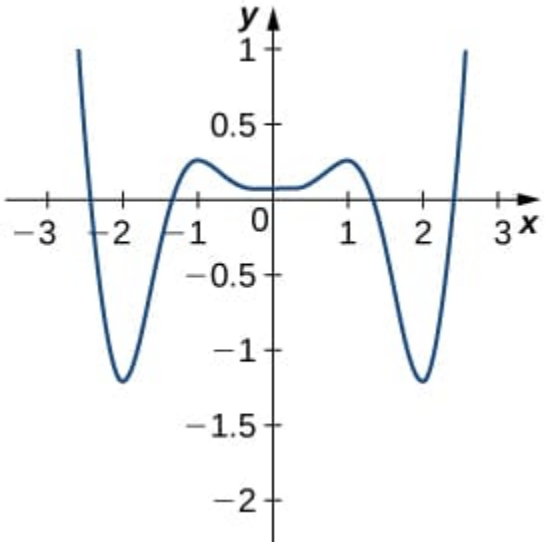
Answer

Absolute minimum at 3; Absolute maximum at −2.2; local minima at −2, 1; local maxima at −1, 2

13)



14)



Answer

Absolute minima at –2, 2; absolute maxima at –2.5, 2.5; local minimum at 0; local maxima at –1, 1

For exercises 15 - 18, draw graphs of $f(x)$, which is continuous, over the interval $[-4, 4]$ with the following properties:

15) Absolute maximum at $x = 2$ and absolute minima at $x = \pm 3$

16) Absolute minimum at $x = 1$ and absolute maximum at $x = 2$

Answer

Answers may vary.

17) Absolute maximum at $x = 4$, absolute minimum at $x = -1$, local maximum at $x = -2$, and a critical point that is not a maximum or minimum at $x = 2$

18) Absolute maxima at $x = 2$ and $x = -3$, local minimum at $x = 1$, and absolute minimum at $x = 4$

Answer

Answers may vary.

In exercises 19 - 28, find the critical points in the domains of the given functions.

19) $y = 4x^3 - 3x$

20) $y = 4\sqrt{x} - x^2$

Answer

$x = 1$

21) $y = \dfrac{1}{x-1}$

22) $y = \ln(x-2)$

Answer

None

23) $y = \tan(x)$

24) $y = \sqrt{4-x^2}$

Answer

$x = 0$

25) $y = x^{3/2} - 3x^{5/2}$

26) $y = \dfrac{x^2-1}{x^2+2x-3}$

Answer

None

27) $y = \sin^2(x)$

28) $y = x + \dfrac{1}{x}$

Answer

$x = -1$ and $x = 1$

In exercises 29 - 39, find the local and/or absolute maxima for the functions over the specified domain.

29) $f(x) = x^2 + 3$ over $[-1, 4]$

30) $y = x^2 + \dfrac{2}{x}$ over $[1, 4]$

Answer

Absolute maximum: $x = 4, y = \frac{33}{2}$; absolute minimum: $x = 1, y = 3$

31) $y = (x - x^2)^2$ over $[-1, 1]$

32) $y = \dfrac{1}{x - x^2}$ over $[0, 1]$

Answer

Absolute minimum: $x = \frac{1}{2}, y = 4$

33) $y = \sqrt{9-x}$ over $[1, 9]$

34) $y = x + \sin(x)$ over $[0, 2\pi]$

Answer

Absolute maximum: $x = 2\pi, y = 2\pi$; absolute minimum: $x = 0, y = 0$

35) $y = \dfrac{x}{1+x}$ over $[0, 100]$

36) $y = |x+1| + |x-1|$ over $[-3, 2]$

Answer

Absolute maximum: $x = -3, y = 6$; absolute minimum: $-1 \le x \le 1, y = 2$

37) $y = \sqrt{x} - \sqrt{x^3}$ over $[0, 4]$

38) $y = \sin x + \cos x$ over $[0, 2\pi]$

Answer

Absolute maximum: $x = \frac{\pi}{4}, y = \sqrt{2}$; absolute minimum: $x = \frac{5\pi}{4}, y = -\sqrt{2}$

39) $y = 4\sin\theta - 3\cos\theta$ over $[0, 2\pi]$

In exercises 40 - 45, find the local and absolute minima and maxima for the functions over $(-\infty, \infty)$.

40) $y = x^2 + 4x + 5$

Answer

Absolute minimum: $x = -2, y = 1$

41) $y = x^3 - 12x$

42) $y = 3x^4 + 8x^3 - 18x^2$

Answer

Absolute minimum: $x = -3, y = -135$; local maximum: $x = 0, y = 0$; local minimum: $x = 1, y = -7$

43) $y = x^3(1-x)^6$

44) $y = \dfrac{x^2 + x + 6}{x - 1}$

Answer

Local maximum: $x = 1 - 2\sqrt{2}, y = 3 - 4\sqrt{2}$; local minimum: $x = 1 + 2\sqrt{2}, y = 3 + 4\sqrt{2}$

45) $y = \dfrac{x^2 - 1}{x - 1}$

In exercises 46 - 50, use a calculator to graph the function and to estimate the absolute and local maxima and minima. Then, solve for them explicitly.

46) [Technology Required] $y = 3x\sqrt{1 - x^2}$

Answer

Absolute maximum: $x = \frac{\sqrt{2}}{2}, y = \frac{3}{2}$; absolute minimum: $x = -\frac{\sqrt{2}}{2}, y = -\frac{3}{2}$

47) [Technology Required] $y = x + \sin(x)$

48) [Technology Required] $y = 12x^5 + 45x^4 + 20x^3 - 90x^2 - 120x + 3$

Answer

Local maximum: $x = -2, y = 59$; local minimum: $x = 1, y = -130$

49) [Technology Required] $y = \dfrac{x^3 + 6x^2 - x - 30}{x - 2}$

50) [Technology Required] $y = \dfrac{\sqrt{4 - x^2}}{\sqrt{4 + x^2}}$

Answer

Absolute maximum: $x = 0, y = 1$; absolute minimum: $x = -2, 2, y = 0$

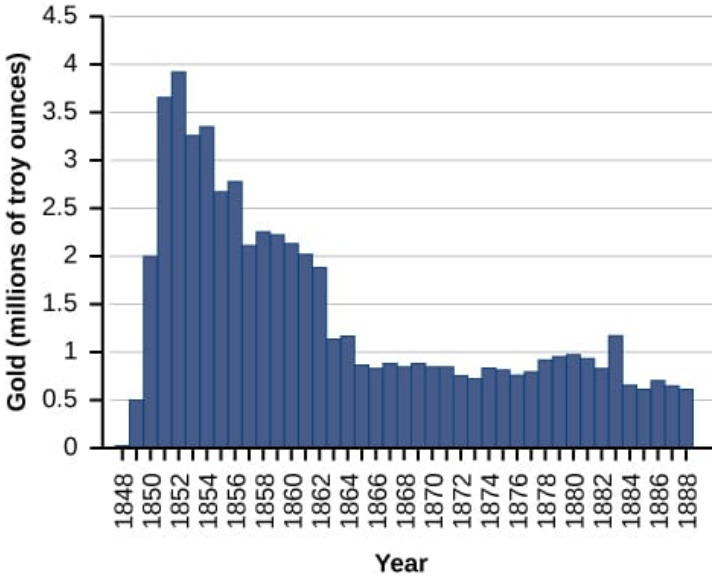
51) A company that produces cell phones has a cost function of $C = x^2 - 1200x + 36,400$, where C is cost in dollars and x is number of cell phones produced (in thousands). How many units of cell phone (in thousands) minimizes this cost function?

52) A ball is thrown into the air and its position is given by $h(t) = -4.9t^2 + 60t + 5m$. Find the height at which the ball stops ascending. How long after it is thrown does this happen?

Answer

$h = \frac{9245}{49}$ m, $t = \frac{300}{49}$ s

For exercises 53-54, consider the production of gold during the California gold rush (1848–1888). The production of gold can be modeled by $G(t) = \dfrac{(25t)}{(t^2 + 16)}$, where t is the number of years since the rush began $(0 \leq t \leq 40)$ and G is ounces of gold produced (in millions). A summary of the data is shown in the following figure.



53) Find when the maximum (local and global) gold production occurred, and the amount of gold produced during that maximum.

54) Find when the minimum (local and global) gold production occurred. What was the amount of gold produced during this minimum?

Answer

The global minimum was in 1848, when no gold was produced.

In exercises 55 & 56, find the critical points, maxima, and minima for the given piecewise functions.

55) \(y= \begin{cases} x^2–4x, & \text{if }0 \leq x \leq 1\x^2–4, & \text{if }1

56) $y = \begin{cases} x^2 + 1, & \text{if } x \leq 1 \\ x^2 - 4x + 5, & \text{if } x > 1 \end{cases}$

Answer

Absolute minima: $x = 0, x = 2, y = 1$; local maximum at $x = 1, y = 2$

In exercises 57 - 58, find the critical points of the following generic functions. Are they maxima, minima, or neither? State the necessary conditions.

57) $y = ax^2 + bx + c$, given that $a > 0$

58) $y = (x-1)^a$, given that $a > 1$

Answer

No maxima/minima if a is odd, minimum at $x = 1$ if a is even

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3.2: The Mean Value Theorem

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

The Mean Value Theorem is one of the most important theorems in Calculus. We look at some of its implications at the end of this section. First, let's start with a particular case of the Mean Value Theorem, called Rolle's Theorem.

Rolle's Theorem

Informally, **Rolle's Theorem** states that if the outputs of a differentiable function f are equal at the endpoints of an interval, then there must be an interior point c where $f'(c) = 0$. Figure $3.2.1$ illustrates this theorem.

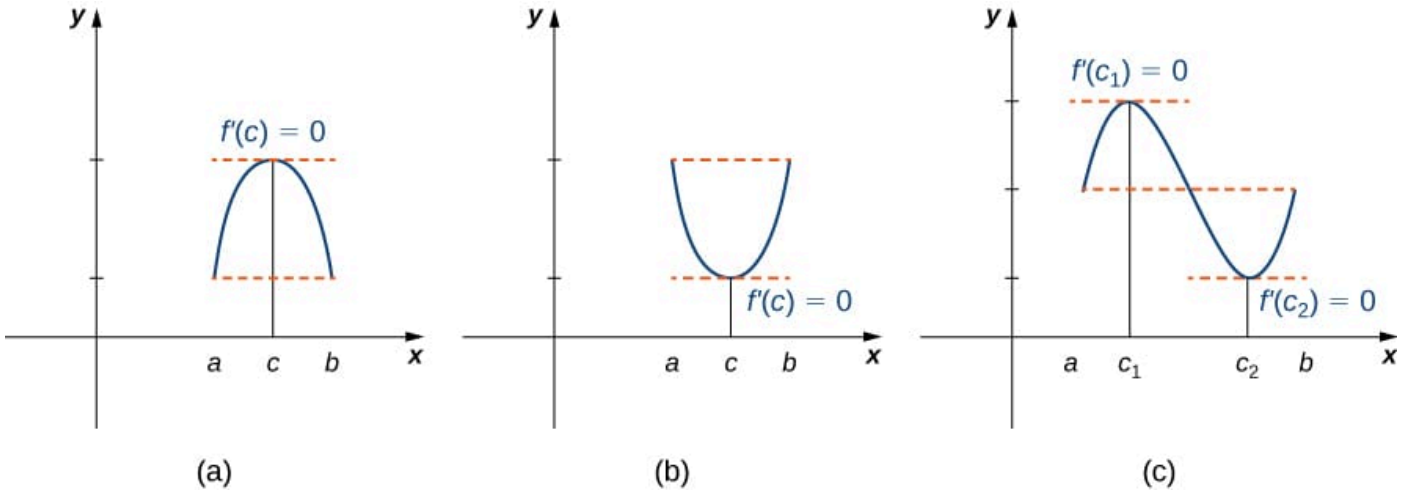

Figure $3.2.1$**:** *If a differentiable function f satisfies $f(a) = f(b)$, then its derivative must be zero at some point(s) between a and b.*

Theorem: Rolle's Theorem

Let f be a continuous function over the closed interval $[a, b]$ and differentiable over the open interval (a, b) such that $f(a) = f(b)$. There then exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

Proof available at the end of this section.

Caution: f Must Be Differentiable!

An important point about Rolle's Theorem is that the differentiability of the function f is critical. If f is not differentiable, even at a single point, the result may not hold. For example, the function $f(x) = |x| - 1$ is continuous over the closed interval $[-1, 1]$ and $f(-1) = 0 = f(1)$, but $f'(c) \neq 0$ for any $c \in (-1, 1)$ as shown Figure 3.2.2.

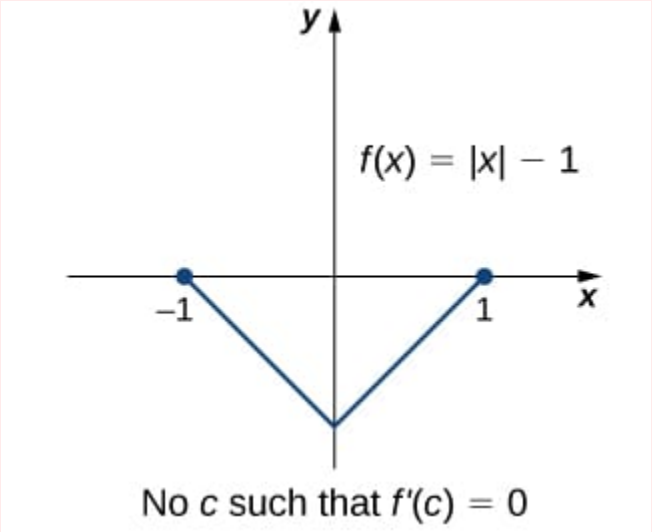

Figure 3.2.2: *Since $f(x) = |x| - 1$ is not differentiable at $x = 0$, the conditions of Rolle's Theorem are not satisfied. In fact, the conclusion does not hold here; there is no $c \in (-1, 1)$ such that $f'(c) = 0$.*

Let's now consider functions that satisfy the conditions of Rolle's Theorem and calculate explicitly the numbers c where $f'(c) = 0$.

Example 3.2.1

For each of the following functions, verify that the function satisfies the criteria stated in Rolle's Theorem and find all numbers c in the given interval where $f'(c) = 0$.

a. $f(x) = x^2 + 2x$ over $[-2, 0]$
b. $f(x) = x^3 - 4x$ over $[-2, 2]$

Solutions available at the end of this section.

Interactive Element: Rolle's Theorem: A Horizontal Tangent Inside the Interval

The following Interactive Element might help you visualize why level endpoints force a horizontal tangent somewhere inside the interval — and that there can be more than one.

Interact: Drag the red point along $f(x) = x^3 - 4x$ on $[-2, 2]$ until its tangent line (red) is horizontal, matching the level secant (blue).

Observation: Because $f(-2) = f(2)$, the tangent goes flat — $f'(c) = 0$ — at two interior points, $c = \pm\frac{2}{\sqrt{3}}$; Rolle's Theorem guarantees at least one.

Checkpoint 3.2.1

Verify that the function $f(x) = 2x^2 - 8x + 6$ defined over the interval $[1, 3]$ satisfies the conditions of Rolle's Theorem. Find all numbers c guaranteed by Rolle's Theorem.

Answer available at the end of this section.

A creative use of Rolle's Theorem is to prove a given equation has *exactly* one root. We did something almost like this when working through continuity in the limits unit. Recall, by the Intermediate Value Theorem, if f is a continuous function on a closed interval $[a, b]$, where the signs of $f(a)$ and $f(b)$ are opposite, there must exist some number $c \in (a, b)$ such that $f(c) = 0$. This was used to show the *existence* of a root to a given equation in the interval $[a, b]$. This proof is commonly called an **existence proof** because you are proving something (a root) exists.

We are now extending this concept to a **uniqueness proof**. We not only want to prove that there *exists* a root to a given equation, but we also want to prove that root is the only such root (i.e., that the root is *unique*).

Example 3.2.2

Prove the equation $2x = \cos(x)$ has only one solution.

Solution available at the end of this section.

The Mean Value Theorem and Its Meaning

Rolle's Theorem is a special case of the **Mean Value Theorem**. In Rolle's Theorem, we consider differentiable functions on closed intervals having the same function values at the endpoints of the interval. The Mean Value Theorem generalizes Rolle's Theorem by considering functions that do not necessarily have the same function values at the endpoints. Consequently, we can view the Mean Value Theorem as a slanted version of Rolle's Theorem (see Figure 3.2.5). The Mean Value Theorem states that if f is continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b), then there exists a point $c \in (a, b)$ such that the tangent line to the graph of f at c is parallel to the secant line connecting $(a, f(a))$ and $(b, f(b))$.

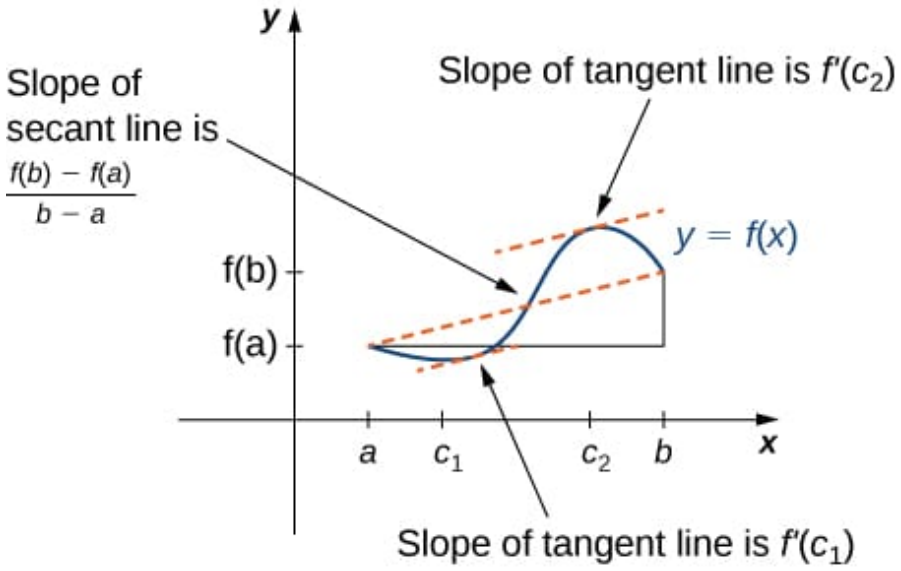

Figure 3.2.5: *The Mean Value Theorem says that for a function that meets its conditions, at some point, the tangent line has the same slope as the secant line between the ends. For this function, there are two values c_1 and c_2 such that the tangent line to f at c_1 and c_2 has the same slope as the secant line.*

Theorem: The Mean Value Theorem (MVT)

Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b). Then, there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof available at the end of this section.

As a notational point, the Mean Value Theorem is often written as the MVT. In the following example, we show how the MVT can be applied to the function $f(x) = \sqrt{x}$ over the interval $[0, 9]$. The method is the same for other functions, although sometimes with more interesting consequences.

Example 3.2.3

For $f(x) = \sqrt{x}$ over the interval $[0, 9]$, show that f satisfies the hypothesis of the Mean Value Theorem. Find the numbers c guaranteed by the Mean Value Theorem.

Solution available at the end of this section.

Interactive Element: The Mean Value Theorem: A Tangent Parallel to the Secant

The following Interactive Element might help you visualize how, somewhere inside the interval, the tangent line matches the slope of the secant line.

Interact: Drag the red point along the curve of $f(x) = \sqrt{x}$ until its tangent line (red) is parallel to the secant line (blue) joining $(0, 0)$ and $(9, 3)$.

Observation: There is exactly one interior point — at $c = \frac{9}{4}$ — where the tangent is parallel to the secant; that is the c the Mean Value Theorem guarantees.

One application that helps illustrate the Mean Value Theorem involves velocity. For example, suppose we drive a car for one hour down a straight road with an average velocity of 45 miles per hour. Let $s(t)$ and $v(t)$ denote the position and velocity of the car, respectively, for $0 \le t \le 1$ hour. Assuming that the position function, $s(t)$, is differentiable, we can apply the Mean Value Theorem to conclude that, at some time $c \in (0, 1)$, the speed of the car was precisely

$$v(c) = s'(c) = \frac{s(1) - s(0)}{1 - 0} = 45 \text{ mph}.$$

Example 3.2.4

If a rock is dropped from a height of 100 ft, its position t seconds after it is dropped until it hits the ground is given by the function $s(t) = -16t^2 + 100$.

a. Determine how long it takes before the rock hits the ground.
b. Find the average velocity v_{avg} of the rock for when the rock is released and the rock hits the ground.
c. Find the time t guaranteed by the Mean Value Theorem when the instantaneous velocity of the rock is v_{avg}.

Solutions available at the end of this section.

Checkpoint 3.2.4

Suppose a ball is dropped from a height of 200 ft. Its position at time t is $s(t) = -16t^2 + 200$. Find the time t when the instantaneous velocity of the ball equals its average velocity.

Answer available at the end of this section.

Corollaries of the Mean Value Theorem

Let's look at three Mean Value Theorem corollaries. These results have important consequences, which we will use in upcoming sections.

At this point, we know the derivative is zero if we are given a constant function. The Mean Value Theorem allows us to conclude that the converse is also true. In particular, if $f'(x) = 0$ for all x in some interval I, then $f(x)$ is constant over that interval. This result may seem intuitively obvious, but it has important implications that are not obvious, and we discuss them shortly.

Corollary: Functions with a Derivative of Zero

Let f be differentiable over an interval I. If $f'(x) = 0$ for all $x \in I$, then $f(x) =$ constant for all $x \in I$.

Proof available at the end of this section.

From this corollary, it follows that if two functions have the same derivative, they differ by, at most, a constant.

Corollary: Constant Difference Theorem

If f and g are differentiable over an interval I and $f'(x) = g'(x)$ for all $x \in I$, then $f(x) = g(x) + C$ for some constant C.

Proof available at the end of this section.

This corollary can be used to prove identities (like your trigonometric identities from your previous Trigonometry course).

Example 3.2.5

Prove the following identity.

$$2\sin^{-1}(x) = \cos^{-1}(1 - 2x^2), \text{ where } x \geq 0$$

Solution available at the end of this section.

A final corollary adds a unique use of the Mean Value Theorem to our toolbox. In my undergraduate studies, I had moments when I required an inequality to be true but had no idea how to show it was true. For example, there might have been a moment when I needed the inequality

$$\ln(x) \leq x - 1, \text{ for } x \geq 1$$

to be true, but I had no clue how to show the truthfulness of the statement. Many students would "hand-wave" the process and say, "This is true." However, at this level, that type of justification is insufficient. This is where the **Racetrack Principle** comes into play.

Corollary: The Racetrack Principle

If f and g are continuous for all $x \geq a$ and differentiable for all $x > a$, and if $f(a) = g(a)$ and $f'(x) \geq g'(x)$ for all $x > a$, then $f(x) \geq g(x)$ for all $x \geq a$.

Proof available at the end of this section.

This is called the Racetrack Principle because you can imagine the validity of this statement if you visualize a racetrack with two cars. Their positions (the functions) are continuous and differentiable (at least for $x > a$). This means that no car can teleport and radically change its position between moments in time, and their velocities always exist. These two conditions also allow us to immediately use the MVT if needed. Additionally, $f(a) = g(a)$ means that the two cars start the race at the same position. Finally,

$f'(x) \geq g'(x)$ means that car "f" is as fast or faster than car "g." It should then make sense that car "f" will always be in the lead (or at least neck-and-neck with car "g").

Let's do an example to see the power of the Racetrack Principle.

Example 3.2.6

Prove the inequality.

$$\ln(x) \leq x - 1, \text{ for } x \geq 1$$

Solution available at the end of this section.

Success in College: Dealing With Failure (Part 3)

Sometimes unexpected events happen during a term. You get sick and are in bed for a week. Someone that you need to help care for gets sick. You are in a car accident and lose your transportation. A daycare provider quits. A relationship falls apart. Your boss at work needs you to increase your hours. Because this happens, you don't have time to keep up with your courses. You miss class. You fail one or more tests.

1. A student in this situation decides to withdraw from the math class. She reasons that it is better to concentrate on passing her other classes. She will re-enroll in math the next term. Are there any adverse consequences of making this choice? Explain.
2. A student in this situation decides to keep coming to class and trying to do the homework. He plans to take the course over the next term but wants to learn as much as he can now. This should make the course easier next term. Are there any adverse consequences of making this choice? Explain.

Applications of Mathematics: Poker

So, where does mathematics apply in poker? Poker players can use mathematics in many different situations; however, it is most commonly used when a player is on a draw, such as a flush or straight draw. If an opponent makes a bet and you are on a draw, you are faced with the decision of whether or not you should call to try and complete the draw by the next card or fold and let your opponent take the pot. In this situation, a player with good knowledge of poker mathematics will always know whether or not to call, while a player who does not know mathematics will be unsure.

Players unfamiliar with poker odds will make an educated guess on whether or not they should call. If the bet is large, they may feel it is too expensive to try and catch the right card, but if the bet is small, they will be more inclined to call. On the other hand, a player that can use mathematics correctly will be able to work out the pot odds they are getting on the hand and act accordingly. Pot odds consider the amount your opponent has bet in relation to the pot size and the likelihood of completing your draw to inform you about whether or not you should call or fold.

1. Can you think of three other games that rely explicitly or implicitly on mathematics? Explain.
2. Oddly enough, most of us are superstitious about numbers. Some people even play their "favorite" numbers when they play the lottery. Suppose you pick your favorite numbers, and I pick the numbers 1, 2, 3, 4, 5, and 6. Who has better odds of winning? Why are you still likely to win more if the odds are the same?

Success in Mathematics: Why Study Mathematics? (Part 4)

Dr. Robert Cabin teaches biology in New York State. He writes,

> "... but many students are disillusioned by and uninterested in academics. They are on our campuses because they think they need a degree to get a good job, because their parents expect them to go to college, or because they just don't know

what else to do."

Source: Chronicle of Higher Education, October 2003.

1. Dr. Cabin states that many students are uninterested in academics. What do you think he means by academics? Explain.
2. Are you interested in academics? Explain.

Answers

Section Preview

The material in this section is likely the most theoretical in the chapter. The Mean Value Theorem is typically needed to prove other theorems from this point forward—stress that to your students as they should understand that college-level Calculus involves proof-building skills.

Section Preview

- **Solving Equations**
  - Solving Quadratic Equations Using Extraction of Roots: Example 1b ($3c^2 - 4 = 0$), Example 4a ($-16t^2 + 100 = 0$), and Checkpoint 4 isolate a squared term and apply $\pm\sqrt{k}$.

Section Preview

- Explain the meaning of Rolle's theorem
- Describe the significance of the Mean Value Theorem
- Use the Racetrack Principle to prove inequalities
- State two critical consequences of the Mean Value Theorem

Theorem: Rolle's Theorem

Let $k = f(a) = f(b)$. We consider three cases:

Case 1—$f(x) = k$ for all $x \in (a, b)$: If $f(x) = k$ for all $x \in (a, b)$, then $f'(x) = 0$ for all $x \in (a, b)$.

Case 2—There exists $x \in (a, b)$ such that $f(x) > k$: Since f is a continuous function over the closed, bounded interval $[a, b]$, by the Extreme Value Theorem, it has an absolute maximum. Also, since there is a point $x \in (a, b)$ such that $f(x) > k$, the absolute maximum is greater than k. Therefore, the absolute maximum does not occur at either endpoint. As a result, the absolute maximum must occur at an interior point $c \in (a, b)$. Because f has a maximum at an interior point c, and f is differentiable at c, by Fermat's Theorem, $f'(c) = 0$.

Case 3—There exists $x \in (a, b)$ such that \(f(x) The case when there exists a point $x \in (a, b)$ such that \(f(x)

Q.E.D.

Example 3.2.1

a. Since f is a polynomial, it is continuous and differentiable everywhere. In addition, $f(-2) = 0 = f(0)$. Therefore, f satisfies the criteria of Rolle's Theorem. We conclude that there exists at least one value $c \in (-2, 0)$ such that $f'(c) = 0$. Since $f'(x) = 2x + 2 = 2(x + 1)$, we see that $f'(c) = 2(c + 1) = 0$ implies $c = -1$ as shown in Figure 3.2.3.

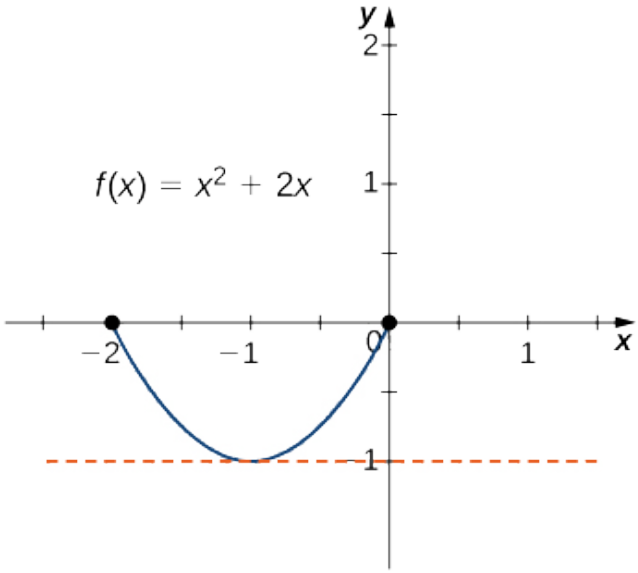

Figure 3.2.3: *This function is continuous and differentiable over $[-2, 0]$, $f'(c) = 0$ when $c = -1$.*

b. As in part a., f is a polynomial and is continuous and differentiable everywhere. Also, $f(-2) = 0 = f(2)$. That said, f satisfies the criteria of Rolle's Theorem. Differentiating, we find that $f'(x) = 3x^2 - 4$. Therefore, $f'(c) = 0$ when $x = \pm\frac{2}{\sqrt{3}}$. Both points are in the interval $[-2, 2]$, and, therefore, both points satisfy the conclusion of Rolle's Theorem as shown in Figure 3.2.4.

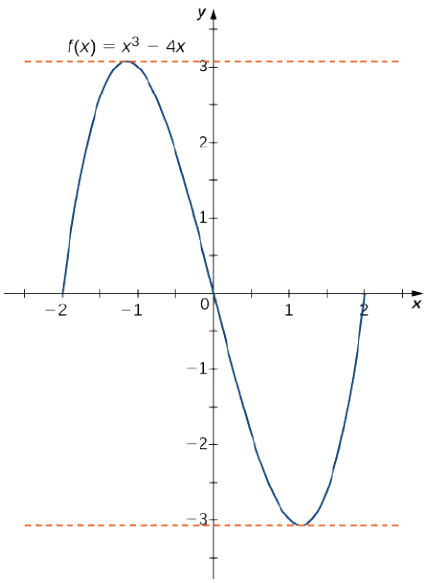

Figure 3.2.4: *For this polynomial over* $[-2,2], f'(c)=0$ ***at*** $x=\pm 2/\sqrt{3}$.

Checkpoint 3.2.1

$c=2$

Example 3.2.2

Existence. We begin our proof by showing that a solution to the equation exists.

Let $f(x)=2x-\cos(x)$**. Then** f **has a zero precisely when** $2x=\cos(x)$ **has a solution. Since** f **is the sum of a polynomial function and a cosine function, both of which are continuous on** $\mathbb{R}$**,** f **is continuous on** $\mathbb{R}$**. Moreover,** $f(0)=-1<0$ **and** $f(\pi/2)=\pi>0$**. By the Intermediate Value Theorem,** f **must have a zero on the interval** $(0,\pi/2)$**. Therefore, the given equation must have a solution on the interval** $(0,\pi/2)$**.**

Uniqueness. We continue our proof by showing that the solution on the interval $(0,\pi/2)$ **must be the *only* solution. To showcase this result, we argue using a proof by contradiction.**

Let's *assume* that the given equation has more than one solution. This means the function, f**, would have more than one zero. That is, there must exist two values,** c **and** d**, such that** $f(c)=0=f(d)$**, where** $c\neq d$**.**

We have the following three conditions met:

1. f **is continuous on the closed interval** $[c,d]$**,**
2. $f'(x)=2+\sin(x)$ **exists on the open interval** (c,d)**, and**
3. $f(c)=f(d)$**.**

Thus, by Rolle's Theorem, there must exist a number $k\in(c,d)$ **such that** $f'(k)=0$**.**

However, $f'(x)=2+\sin(x)\geq 1$ **for all** $x\in\mathbb{R}$**. This contradicts the statement that** $f'(k)=0$ **for some** $k\in(c,d)$**. Hence, our only *assumption*, that the given equation has more than one solution, must be incorrect. Since we have already shown that a solution exists, the equation must only have that one solution.**

Q.E.D.

Theorem: The Mean Value Theorem (MVT)

The proof follows from Rolle's Theorem by introducing an appropriate function that satisfies the criteria of Rolle's Theorem. Consider the line connecting $(a,f(a))$ **and** $(b,f(b))$**. Since the slope of that line is**

$$\frac{f(b)-f(a)}{b-a}$$

and the line passes through the point $(a,f(a))$**, the equation of that line can be written as**

$$y=\frac{f(b)-f(a)}{b-a}(x-a)+f(a).$$

Let $g(x)$ **denote the vertical difference between the point** $(x,f(x))$ **and the point** (x,y) **on that line. Therefore,**

$$g(x)=f(x)-\left[\frac{f(b)-f(a)}{b-a}(x-a)+f(a)\right].$$

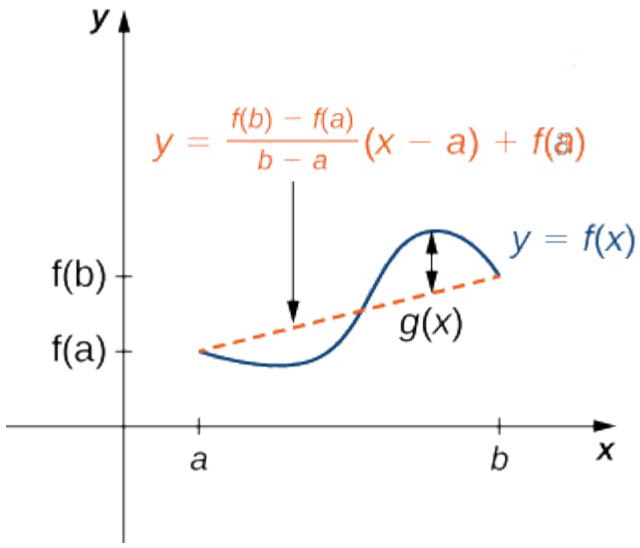

Figure 3.2.6: *The value $g(x)$ is the vertical difference between the point $(x, f(x))$ and the point (x, y) on the secant line connecting $(a, f(a))$ and $(b, f(b))$.*

Since the graph of f intersects the secant line when $x = a$ and $x = b$, we see that $g(a) = 0 = g(b)$. Since f is a differentiable function over (a, b), g is also a differentiable function over (a, b). Furthermore, since f is continuous over $[a, b]$, g is also continuous over $[a, b]$. Therefore, g satisfies the criteria of Rolle's Theorem. Consequently, there exists a point $c \in (a, b)$ such that $g'(c) = 0$. Since

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

we see that

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

Since $g'(c) = 0$, we conclude that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Q.E.D.

Example 3.2.3

We know that $f(x) = \sqrt{x}$ is continuous over $[0, 9]$ and differentiable over $(0, 9)$. Therefore, f satisfies the hypotheses of the Mean Value Theorem. As such, there must exist at least one value $c \in (0, 9)$ such that $f'(c)$ is equal to the slope of the line connecting $(0, f(0))$ and $(9, f(9))$ (see Figure 3.2.7). To determine the value(s) of c, first calculate the derivative of f.

$$f'(x) = \frac{1}{(2\sqrt{x})}.$$

The slope of the line connecting $(0, f(0))$ and $(9, f(9))$ is given by

$$\frac{f(9) - f(0)}{9 - 0} = \frac{\sqrt{9} - \sqrt{0}}{9 - 0} = \frac{3}{9} = \frac{1}{3}.$$

We want to find c such that $f'(c) = \frac{1}{3}$. That is, we want to find c such that

$$\frac{1}{2\sqrt{c}} = \frac{1}{3}.$$

Solving this equation for c, we obtain $c = \frac{9}{4}$. At this point, the slope of the tangent line equals the slope of the line joining the endpoints.

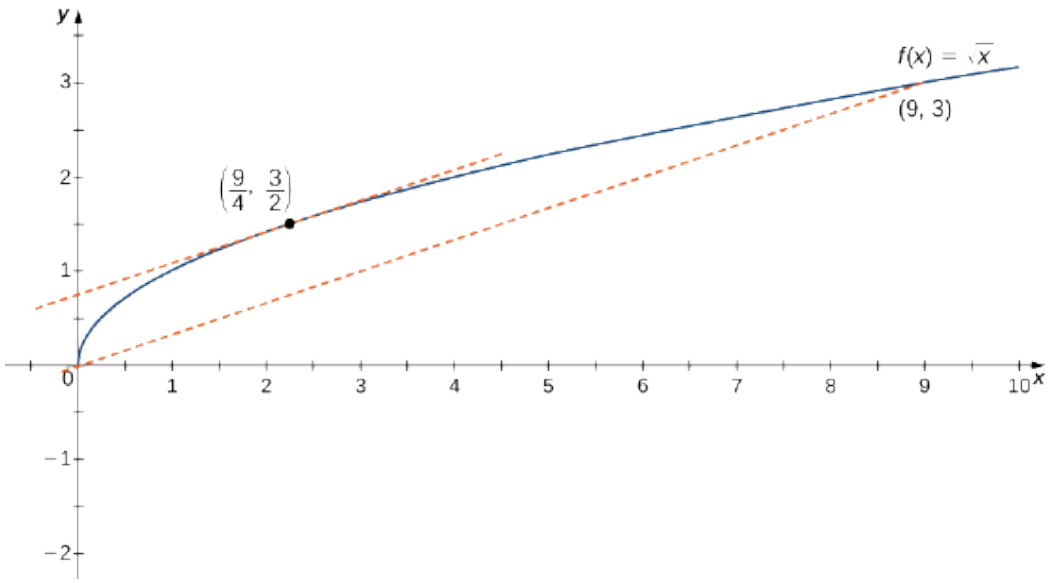

Figure 3.2.7**:** ***The slope of the tangent line at*** $c = 9/4$ ***is the same as the slope of the line segment connecting (0,0) and (9,3).***

Example 3.2.4

a. When the rock hits the ground, its position is $s(t) = 0$**. Solving the equation** $-16t^2 + 100 = 0$ **for** t**, we find that** $t = \pm\frac{5}{2}$ **seconds. Since we are only considering** $t \geq 0$**, the ball will hit the ground** $\frac{5}{2}$ **seconds after it is dropped.**

b. The average velocity is given by

$$v_{avg} = \frac{s(5/2) - s(0)}{5/2 - 0} = \frac{0 - 100}{5/2} = -40\,\text{ft/sec}.$$

c. The instantaneous velocity is given by the derivative of the position function. Therefore, we need to find a time t **such that** $v(t) = s'(t) = v_{avg} = -40$ **feet per second. Since** $s(t)$ **is continuous over the interval** $[0, \frac{5}{2}]$ **and differentiable over the interval** $(0, \frac{5}{2})$**, by the Mean Value Theorem, there is guaranteed to be a point** $c \in (0, \frac{5}{2})$ **such that**

$$s'(c) = \frac{s(5/2) - s(0)}{5/2 - 0} = -40.$$

Taking the derivative of the position function $s(t)$**, we find that** $s'(t) = -32t$**. Therefore, the equation reduces to** $s'(c) = -32c = -40$**. Solving this equation for** c**, we have** $c = \frac{5}{4}$**. Therefore,** $\frac{5}{4}$ **seconds after the rock is dropped, the instantaneous velocity equals the average velocity of the rock during its free fall:** -40 **feet per second.**

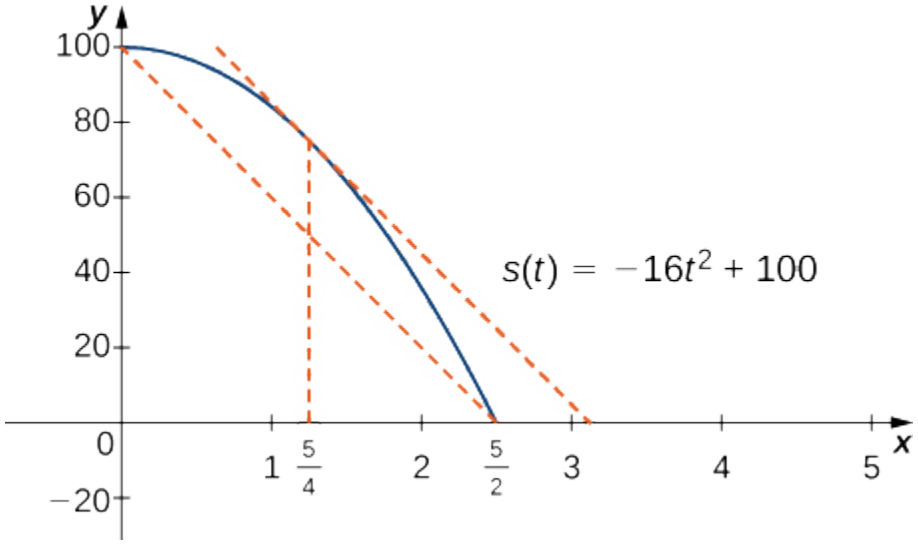

Figure 3.2.8**:** ***At time*** $t = 5/4$ ***sec, the velocity of the rock is equal to its average velocity from the time it is dropped until it hits the ground.***

Checkpoint 3.2.4

$\frac{5}{2\sqrt{2}}$ **seconds**

Corollary: Functions with a Derivative of Zero

Since f **is differentiable over** I**,** f **must be continuous over** I**. Suppose** $f(x)$ **is not constant for all** x **in** I**. Then there exist** $a, b \in I$**, where** $a \neq b$ **and** $f(a) \neq f(b)$**. Choose the notation so that \(a**

Q.E.D.

Corollary: Constant Difference Theorem

Let $h(x) = f(x) - g(x)$. Then, $h'(x) = f'(x) - g'(x) = 0$ for all $x \in I$. By the previous corollary, there is a constant C such that $h(x) = C$ for all $x \in I$. Therefore, $f(x) = g(x) + C$ for all $x \in I$.

Q.E.D.

Example 3.2.5

Let $f(x) = 2\sin^{-1}(x)$ and $g(x) = \cos^{-1}(1 - 2x^2)$. Note that the normal domain of f is $[-1, 1]$ and we could quickly determine the domain of g to also be $[-1, 1]$; however, we have been told that $x \geq 0$. Therefore, the domains of both functions are $[0, 1]$.

Computing the derivatives, we get

$$f'(x) = 2 \cdot \frac{1}{\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}}$$

and

$$\begin{aligned} g'(x) &= -\frac{1}{\sqrt{1-(1-2x^2)^2}} \cdot (-4x) \\ &= \frac{4x}{\sqrt{1-(1-4x^2+4x^4)}} \\ &= \frac{4x}{\sqrt{4x^2-4x^4}} \\ &= \frac{4x}{2x\sqrt{1-x^2}} \qquad (\text{since } x \geq 0) \\ &= \frac{2}{\sqrt{1-x^2}} \\ &= f'(x) \end{aligned}$$

Since $f'(x) = g'(x)$ for $x \in (0, 1)$, $f(x) = g(x) + C$ for all $x \in (0, 1)$ by the Constant Difference Theorem. That is, there exists C such that

$$2\sin^{-1}(x) = \cos^{-1}(1 - 2x^2) + C$$

Since this equation is true for all $x \in (0, 1)$, we can let x be any value in $(0, 1)$ to find the value of C. Let $x = \frac{1}{2}$. Then

$$2\sin^{-1}\left(\frac{1}{2}\right) = \cos^{-1}\left(1 - 2\left(\frac{1}{2}\right)^2\right) + C \implies 2\left(\frac{\pi}{6}\right) = \cos^{-1}\left(\frac{1}{2}\right) + C \implies \frac{\pi}{3} = \frac{\pi}{3} + C \implies C = 0.$$

To check the edges of the domain, let $x = 0$ and $x = 1$ in the original equation to get $0 = 0$ and $\pi = \pi$, respectively. Thus, the identity is true for all $x \in [0, 1]$.

Corollary: The Racetrack Principle

Let f and g be continuous functions on the interval $[a, \infty)$, and suppose both are differentiable on (a, ∞). Also, suppose $f(a) = g(a)$ and $f'(x) \geq g'(x)$ for all $x > a$.

Let $h(x) = f(x) - g(x)$. Then h is continuous on $[a, \infty)$ and differentiable on (a, ∞). Moreover, $h(a) = f(a) - g(a) = 0$ and $h'(x) = f'(x) - g'(x) \geq 0$ for all $x > a$.

If we let $b > a$ be arbitrary, then by the MVT there exists $c \in (a, b)$ such that

$$h'(c) = \frac{h(b) - h(a)}{b - a}.$$

However, we know that $h'(c) \geq 0$ (since $c > a$). Therefore,

$$h'(c) = \frac{h(b) - h(a)}{b - a} \geq 0.$$

Since $b > a$, we can multiply all three side of this inequality by $(b - a)$ without changing the inequality signs to get

$$h'(c)(b - a) = h(b) - h(a) \geq 0.$$

This means that

$$\begin{aligned} f(b) - g(b) - (f(a) - g(a)) &\geq 0 \qquad (\text{substituting } h(a) = f(a) - g(a)) \\ f(b) - g(b) - (0) &\geq 0 \qquad (\text{using the assumption that } f(a) = g(a)) \\ f(b) &\geq g(b) \end{aligned}$$

Since b was arbitrary, $f(x) \geq g(x)$ for all $x \geq a$.

Q.E.D.

Example 3.2.6

Let $f(x) = x - 1$ and $g(x) = \ln(x)$. Both functions are continuous and differentiable for all $x > 0$. Thus, they are continuous on $[1, \infty)$ and differentiable on $(1, \infty)$. Moreover, $f(1) = 0 = g(1)$. Finally,

$$f'(x) = 1 \text{ and } g'(x) = 1/x.$$

For $x \geq 1$, f' is always 1, but g' is *at most* 1. In fact, for $x > 1$, $g'(x) < 1$. Hence, $f'(x) \geq g'(x)$ for all $x \geq 1$. Therefore, $f(x) \geq g(x)$ for all $x \geq 1$, by the Racetrack Principle. That is,

$$x - 1 \geq \ln(x), \text{ for } x \geq 1.$$

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3.2.1: Resources and Key Concepts

Resources

Prerequisite Topics

The following is a list of prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Logic and Proofs**
  - Indirect Proofs - Contradiction: Used in the uniqueness proof in Example 2. The concept of a "proof by contradiction" is used quite extensively in Mathematics from this moment forward.

Videos

- Rolle's Theorem
  - Rolle's Theorem: Concept, Theorem, and Proof
  - Using Rolle's Theorem
- Mean Value Theorem (MVT)
  - The Mean Value Theorem for Derivatives: Concept, Theorem, and Proof
  - Using the Mean Value Theorem for Derivatives
- Constant Difference Theorem and the Racetrack Principle
  - Consequences of the Mean Value Theorem for Derivatives

Key Topics

Definitions

- **Existence Proof**: A proof that demonstrates something (e.g., a root of an equation) exists.
- **Uniqueness Proof**: A proof that not only shows existence but also that the item in question is the only one of its kind.

Theorems

- **Theorem: Rolle's Theorem**: Let f be a continuous function over the closed interval $[a, b]$ and differentiable over the open interval (a, b) such that $f(a) = f(b)$. There then exists at least one $c \in (a, b)$ such that $f'(c) = 0$.
- **Theorem: The Mean Value Theorem (MVT)**: Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b). Then, there exists at least one point $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.
- **Corollary: Functions with a Derivative of Zero**: Let f be differentiable over an interval I. If $f'(x) = 0$ for all $x \in I$, then $f(x) = \text{constant}$ for all $x \in I$.
- **Corollary: Constant Difference Theorem**: If f and g are differentiable over an interval I and $f'(x) = g'(x)$ for all $x \in I$, then $f(x) = g(x) + C$ for some constant C.
- **Corollary: The Racetrack Principle**: If f and g are continuous for all $x \geq a$ and differentiable for all $x > a$, and if $f(a) = g(a)$ and $f'(x) \geq g'(x)$ for all $x > a$, then $f(x) \geq g(x)$ for all $x \geq a$.

Common Mistakes

- **Ignoring Hypotheses for Rolle's Theorem/MVT**: Applying Rolle's Theorem or the Mean Value Theorem without verifying that the function is continuous on the closed interval and differentiable on the open interval.
- **Misinterpreting "Exactly One Root" Proofs**: Confusing the existence part (often shown by IVT) with the uniqueness part (often shown by Rolle's Theorem via contradiction).

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3.2.2: Homework

Reading Questions

1. What are the three conditions a function f must satisfy on an interval $[a, b]$ for Rolle's Theorem to apply?
2. If the conditions for Rolle's Theorem are met, what does the theorem conclude about the derivative $f'(c)$ for some c in (a, b)?
3. What are the two conditions a function f must satisfy on an interval $[a, b]$ for the Mean Value Theorem to apply?
4. If the conditions for the Mean Value Theorem are met, what does the theorem conclude about the derivative $f'(c)$ for some c in (a, b)? What is this value equal to in terms of $f(a)$, $f(b)$, a, and b?
5. Geometrically, what does the conclusion of the Mean Value Theorem mean about the tangent line at $x = c$ and the secant line connecting $(a, f(a))$ and $(b, f(b))$?
6. How can Rolle's Theorem be used to prove that an equation has at most one root in an interval (uniqueness)?
7. If $f'(x) = 0$ for all x in an interval I, what can be concluded about the function $f(x)$ on I?
8. If two functions $f(x)$ and $g(x)$ have the same derivative on an interval I (i.e., $f'(x) = g'(x)$ for all $x \in I$), how are $f(x)$ and $g(x)$ related?
9. What is the Racetrack Principle? What conditions must be met for it to apply?
10. In Example 4 (the falling rock), how was the Mean Value Theorem applied to find a time when the instantaneous velocity equaled the average velocity?

Homework

1)Why do you need continuity to apply the Mean Value Theorem? Construct a counterexample.

2) Why do you need differentiability to apply the Mean Value Theorem? Find a counterexample.

Answer

One example is $f(x) = |x| + 3, -2 \leq x \leq 2$

3) When are Rolle's theorem and the Mean Value Theorem equivalent?

4) If you have a function with a discontinuity, is it still possible to have $f'(c)(b - a) = f(b) - f(a)$? Draw such an example or prove why not.

Answer

Yes, but the Mean Value Theorem still does not apply

In exercises 5 - 9, determine over what intervals (if any) the Mean Value Theorem applies. Justify your answer.

5) $y = \sin(\pi x)$

6) $y = \dfrac{1}{x^3}$

Answer

$(-\infty, 0), (0, \infty)$

7) $y = \sqrt{4 - x^2}$

8) $y = \sqrt{x^2 - 4}$

Answer

$(-\infty, -2), (2, \infty)$

9) $y = \ln(3x - 5)$

In exercises 10 - 13, graph the functions on a calculator and draw the secant line that connects the endpoints. Estimate the number of points c such that $f'(c)(b-a) = f(b) - f(a)$.

10) [Technology Required] $y = 3x^3 + 2x + 1$ over $[-1, 1]$

Answer

2 points

11) [Technology Required] $y = \tan\left(\frac{\pi}{4}x\right)$ over $\left[-\frac{3}{2}, \frac{3}{2}\right]$

12) [Technology Required] $y = x^2\cos(\pi x)$ over $[-2, 2]$

Answer

5 points

13) [Technology Required] $y = x^6 - \frac{3}{4}x^5 - \frac{9}{8}x^4 + \frac{15}{16}x^3 + \frac{3}{32}x^2 + \frac{3}{16}x + \frac{1}{32}$ over $[-1, 1]$

In exercises 14 - 19, use the Mean Value Theorem and find all points \(0such that $f(2) - f(0) = f'(c)(2-0)$.

14) $f(x) = x^3$

Answer

$c = \frac{2\sqrt{3}}{3}$

15) $f(x) = \sin(\pi x)$

16) $f(x) = \cos(2\pi x)$

Answer

$c = \frac{1}{2}, 1, \frac{3}{2}$

17) $f(x) = 1 + x + x^2$

18) $f(x) = (x-1)^{10}$

Answer

$c = 1$

19) $f(x) = (x-1)^9$

In exercises 20 - 23, show there is no c such that $f(1) - f(-1) = f'(c)(2)$. Explain why the Mean Value Theorem does not apply over the interval $[-1, 1]$.

20) $f(x) = \left|x - \frac{1}{2}\right|$

Answer

Not differentiable

21) $f(x) = \dfrac{1}{x^2}$

22) $f(x) = \sqrt{|x|}$

Answer

Not differentiable

23) $f(x) = \lfloor x \rfloor$ (Hint: This is called the floor function and it is defined so that $f(x)$ is the largest integer less than or equal to x.)

In exercises 24 - 34, determine whether the Mean Value Theorem applies for the functions over the given interval $[a, b]$. Justify your answer.

24) $y = e^x$ over $[0, 1]$

Answer

Yes

25) $y = \ln(2x + 3)$ over $[-\frac{3}{2}, 0]$

26) $f(x) = \tan(2\pi x)$ over $[0, 2]$

Answer

The Mean Value Theorem does not apply since the function is discontinuous at $x = \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}$.

27) $y = \sqrt{9 - x^2}$ over $[-3, 3]$

28) $y = \dfrac{1}{|x + 1|}$ over $[0, 3]$

Answer

Yes

29) $y = x^3 + 2x + 1$ over $[0, 6]$

30) $y = \dfrac{x^2 + 3x + 2}{x}$ over $[-1, 1]$

Answer

The Mean Value Theorem does not apply; discontinuous at $x = 0$.

31) $y = \dfrac{x}{\sin(\pi x) + 1}$ over $[0, 1]$

32) $y = \ln(x + 1)$ over $[0, e - 1]$

Answer

Yes

33) $y = x \sin(\pi x)$ over $[0, 2]$

34) $y = 5 + |x|$ over $[-1, 1]$

Answer

The Mean Value Theorem does not apply; not differentiable at $x = 0$.

For exercises 35 - 37, consider the roots of each equation.

35) Show that the equation $y = x^3 + 3x^2 + 16$ has exactly one real root. What is it?

36) Find the conditions for exactly one root (double root) for the equation $y = x^2 + bx + c$

Answer

$b = \pm 2\sqrt{c}$

37) Find the conditions for $y = e^x - b$ to have one root. Is it possible to have more than one root?

In exercises 38 - 42, use a calculator to graph the function over the interval $[a, b]$ and graph the secant line from a to b. Use the calculator to estimate all values of c as guaranteed by the Mean Value Theorem. Then, find the exact value of c, if possible, or write the final equation and use a calculator to estimate to four digits.

38) [Technology Required] $y = \tan(\pi x)$ over $\left[-\frac{1}{4}, \frac{1}{4}\right]$

Answer

$c \approx \pm 0.1533$
$c = \pm \frac{1}{\pi} \cos^{-1}(\frac{\sqrt{\pi}}{2})$

39) [Technology Required] $y = \dfrac{1}{\sqrt{x+1}}$ over $[0, 3]$

40) [Technology Required] $y = |x^2 + 2x - 4|$ over $[-4, 0]$

Answer

The Mean Value Theorem does not apply.

41) [Technology Required] $y = x + \dfrac{1}{x}$ over $\left[\frac{1}{2}, 4\right]$

42) [Technology Required] $y = \sqrt{x+1} + \dfrac{1}{x^2}$ over $[3, 8]$

Answer

$\dfrac{1}{2\sqrt{c+1}} - \dfrac{2}{c^3} = \dfrac{521}{2880}$
$c \approx 3.133, 5.867$

43) At 10:17 a.m., you pass a police car at 55 mph that is stopped on the freeway. You pass a second police car at 55 mph at 10:53 a.m., which is located 39 mi from the first police car. If the speed limit is 60 mph, can the police cite you for speeding?

44) Two cars drive from one stoplight to the next, leaving at the same time and arriving at the same time. Is there ever a time when they are going the same speed? Prove or disprove.

Answer

Yes

45) Show that $y = \sec^2 x$ and $y = \tan^2 x$ have the same derivative. What can you say about $y = \sec^2 x - \tan^2 x$?

46) Show that $y = \csc^2 x$ and $y = \cot^2 x$ have the same derivative. What can you say about $y = \csc^2 x - \cot^2 x$?

Answer

It is constant.

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3.3: Shapes of Curves

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Earlier in this chapter we stated that if a function f has a local extremum at a point $(c, f(x))$, then c must be a critical number of f. However, a function is not guaranteed to have a local extremum at a critical number. For example, $f(x) = x^3$ has a critical point at $(0, 0)$ since $f'(x) = 3x^2$ is zero at $x = 0$, but f does not have a local extremum at $x = 0$. Using the results from the previous section, we can now determine whether a critical point of a function corresponds to a local extreme value. In this section, we also see how the second derivative provides information about the shape of a graph by describing whether the graph of a function curves upward or downward.

Increasing/Decreasing Functions

We begin this section by allowing for one final corollary from the Mean Value Theorem. This corollary discusses when a function is increasing and when it is decreasing. Recall that a function f is increasing over I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, whereas f is decreasing over I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$. Using the Mean Value Theorem, we can show that if the derivative of a function is positive, then the function is increasing; if the derivative is negative, then the function is decreasing (Figure $3.3.1$). We use this fact throughout the remainder of this chapter, where we show how to use the derivative of a function to locate the local maximum and minimum values of the function and how to determine the shape of the graph.

Corollary: Increasing and Decreasing Functions

Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b).

i. If $f'(x) > 0$ for all $x \in (a, b)$, then f is an increasing function over $[a, b]$.
ii. If $f'(x) < 0$ for all $x \in (a, b)$, then f is a decreasing function over $[a, b]$.

Proof (by contradiction) available at the end of this section.

Figure $3.3.1$ demonstrates this corollary perfectly.

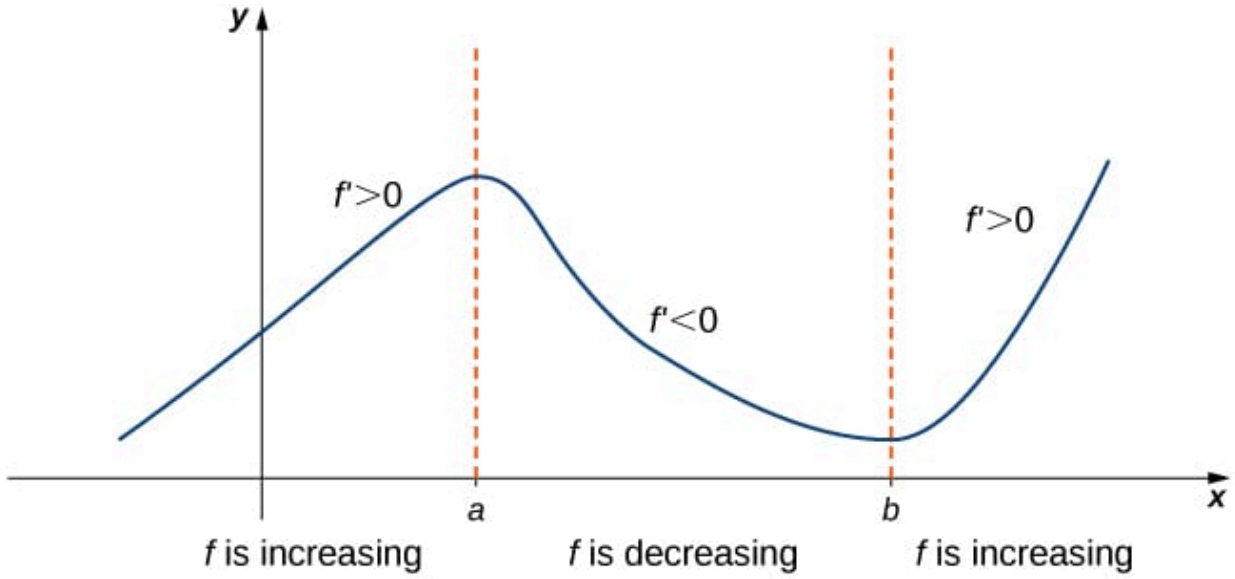

Figure 3.3.1**:** *If a function has a positive derivative over some interval I, then the function increases over that interval I; if the derivative is negative over some interval I, then the function decreases over that interval I.*

The First Derivative Test

The previous corollary shows that if the derivative of a function is positive over an interval I, then the function is increasing over I. On the other hand, if the function's derivative is negative over an interval I, then the function is decreasing over I, as shown in the following figure.

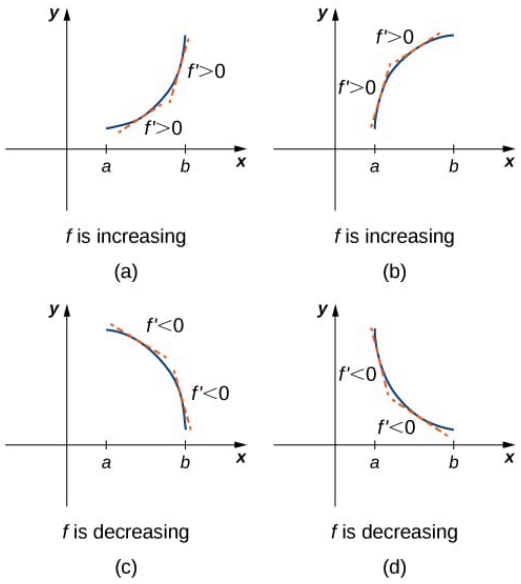

Figure 3.3.2**:** *Both functions are increasing over the interval (a, b). At each point x, the derivative $f'(x) > 0$. Both functions decrease over the interval (a, b). At each point x, the derivative $f'(x) < 0$.*

A *continuous* function f has a local maximum at point c if and only if f switches from increasing to decreasing at point c. Similarly, f has a local minimum at c if and only if f switches from decreasing to increasing at c. If f is a continuous function over an interval I containing c and differentiable over I, except possibly at c, the only way f can switch from increasing to decreasing (or vice versa) at point c is if f' changes sign as x increases through c. If f is differentiable at c, the only way that f' can change sign as x increases through c is if $f'(c) = 0$. Therefore, for a function f that is continuous over an interval I containing c and differentiable over I, except possibly at c, the only way f can switch from increasing to decreasing (or vice versa) is if $f'(c) = 0$ or $f'(c)$ is undefined. Consequently, to locate local extrema for a function f, we look for numbers c in the domain of f such that $f'(c) = 0$ or $f'(c)$ is undefined. Recall that such numbers are called critical numbers of f.

Note that f need not have a local extremum at a critical number. The critical points are *candidates* for local extrema only. In Figure $3.3.3$, we show that if a continuous function f has a local extremum, it must occur at a critical point. Still, it can be the case that a function does not have a local extremum at a critical point. We show that if f has a local extremum at a critical point, then the sign of f' switches as x increases through that point.

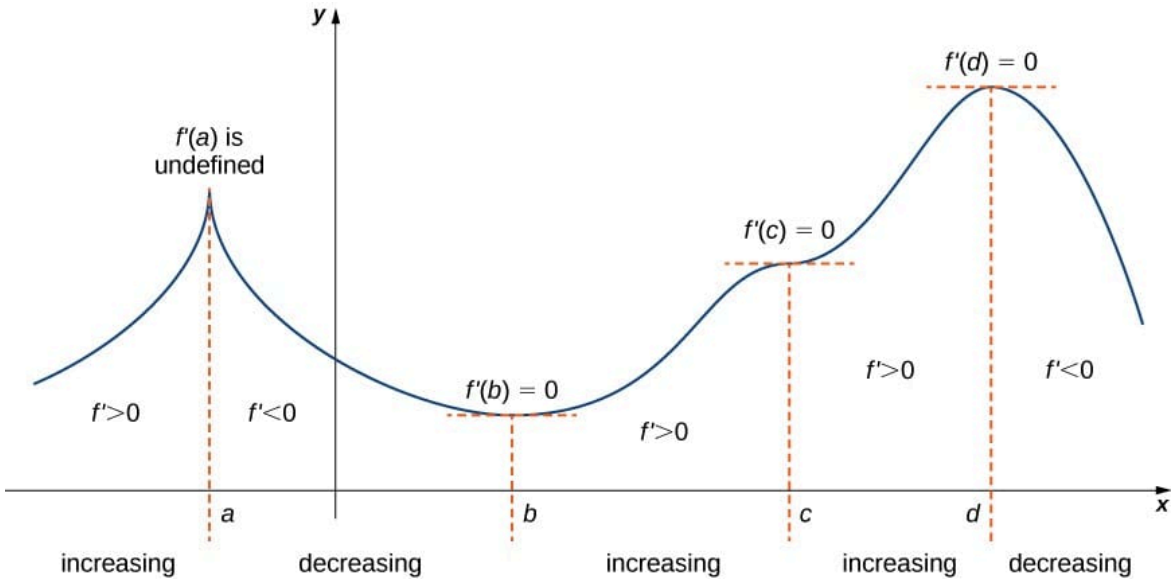

Figure $3.3.3$**:** *The function f has four critical points: a, b, c,and d. The function f has local maxima at a and d, and a local minimum at b. The function f does not have a local extremum at c. The sign of f' changes at all local extrema.*

Using Figure $3.3.3$, we summarize the main results regarding local extrema.

- If a continuous function f has a local extremum, it must occur at a critical number c.
- The function has a local extremum at the critical number c if and only if the derivative, f', switches sign as x increases through c.
- Therefore, to test whether a function has a local extremum at a critical number c, we must determine the sign of $f'(x)$ to the left and right of c.

This result is known as the **First Derivative Test**.

Theorem: First Derivative Test

Suppose that f is a continuous function over an interval I containing a critical number c. If f is differentiable over I, except possibly at c, then $f(c)$ satisfies one of the following descriptions:

i. If f' changes sign from positive when $x < c$ to negative when $x > c$, then $f(c)$ is a local maximum of f.
ii. If f' changes sign from negative when $x < c$ to positive when $x > c$, then $f(c)$ is a local minimum of f.
iii. If f' has the same sign for $x < c$ and $x > c$, then $f(c)$ is neither a local maximum nor a local minimum of f

Interactive Element: First Derivative Test — Reading the Sign of the Slope

The following Interactive Element might help you visualize how the sign of f' controls where a curve rises and falls, and why a local extremum occurs where the slope switches sign.

Interact: Drag the orange point along the curve and watch the tangent line.

Observation: The tangent tilts uphill (positive slope) where the curve is green and increasing, and downhill (negative slope) where it is red and decreasing; each local extreme value sits exactly where the tangent goes horizontal and f' changes sign.

Now let's look at how to use this strategy to locate all local extrema for particular functions.

Example 3.3.1

Use the First Derivative Test to find the location of all local extrema for $f(x) = x^3 - 3x^2 - 9x - 1$. Use graphing technology to confirm your results.

Solution available at the end of this section.

Checkpoint 3.3.1

Use the First Derivative Test to locate all local extrema for $f(x) = -x^3 + \frac{3}{2}x^2 + 18x$.

Answer available at the end of this section.

Example 3.3.2

Use the First Derivative Test to find the location of all local extrema for $f(x) = 5x^{1/3} - x^{5/3}$. Use graphing technology to confirm your results.

Solution available at the end of this section.

Checkpoint 3.3.2

Use the First Derivative Test to find all local extrema for $f(x) = \frac{3}{x-1}$.

Answer available at the end of this section.

Concavity and Points of Inflection

We now know how to determine where a function is increasing or decreasing. However, there is another issue to consider regarding the shape of the graph of a function. If the graph curves, does it curve upward or curve downward? This notion is called the **concavity** of the function.

Figure $3.3.6a$ shows a function f with a graph that curves upward. As x increases, the slope of the tangent line increases. Thus, since the derivative increases as x increases, f' is an increasing function. We say this function f is **concave up**. Figure $3.3.6b$ shows a function f that curves downward. As x increases, the slope of the tangent line decreases. Since the derivative decreases as x increases, f' is a decreasing function. We say this function f is **concave down**.

Definition: Concavity

Let f be a function that is differentiable over an open interval I. If f' is increasing over I, we say f is **concave up** over I. If f' is decreasing over I, we say f is **concave down** over I.

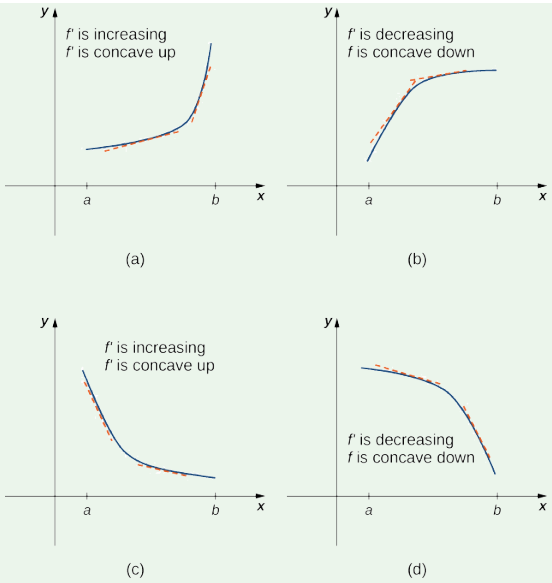

Figure 3.3.6**:** *(a), (c) Since f' is increasing over the interval (a, b), we say f is concave up over (a, b). (b), (d) Since f' is decreasing over the interval (a, b), we say f is concave down over (a, b).*

Note: Another Viewpoint

An equivalent way to define concavity is that a differentiable (and, therefore, continuous) function is concave up over an open interval I if it lies above all of its tangent lines on the interval. Likewise, the function is said to be concave down on I if it lies below all of its tangent lines on I.

In general, without having the graph of a function f, how can we determine its concavity? By definition, a function f is concave up if f' is increasing. From our previous corollary, we know that if f' is a differentiable function, then f' is increasing if its derivative $f''(x) > 0$. Therefore, a function f that is twice differentiable is concave up when $f''(x) > 0$. Similarly, a function f is concave down if f' is decreasing. We know that a differentiable function f' is decreasing if its derivative $f''(x) < 0$. Therefore, a twice-differentiable function f is concave down when $f''(x) < 0$. Applying this logic is known as the **Concavity Test**.

Theorem: Concavity Test

Let f be a function that is twice differentiable over an interval I.

i. If $f''(x) > 0$ for all $x \in I$, then f is concave up over I
ii. If $f''(x) < 0$ for all $x \in I$, then f is concave down over I.

We conclude that we can determine the concavity of a function f by looking at the second derivative of f. In addition, we observe that a function f can switch concavity (Figure 3.3.7). However, a continuous function can switch concavity only at a point x if $f''(x) = 0$ or $f''(x)$ is undefined. Consequently, to determine the intervals where a function f is concave up and concave down, we look for those values of x where $f''(x) = 0$ or $f''(x)$ is undefined. When we have determined these points, we divide the domain of f into smaller intervals and determine the sign of f'' over each of these smaller intervals. If f'' changes sign as we pass through a point x, then f changes concavity. It is important to remember that a function f may not change concavity at a point x even if $f''(x) = 0$ or $f''(x)$ is undefined. If, however, f does change concavity at a point a and f is continuous at a, we say the point $(a, f(a))$ is an **inflection point** of f.

Definition: Inflection Point

If f is continuous at a and f changes concavity at a, the point $(a, f(a))$ is an **inflection point** of f.

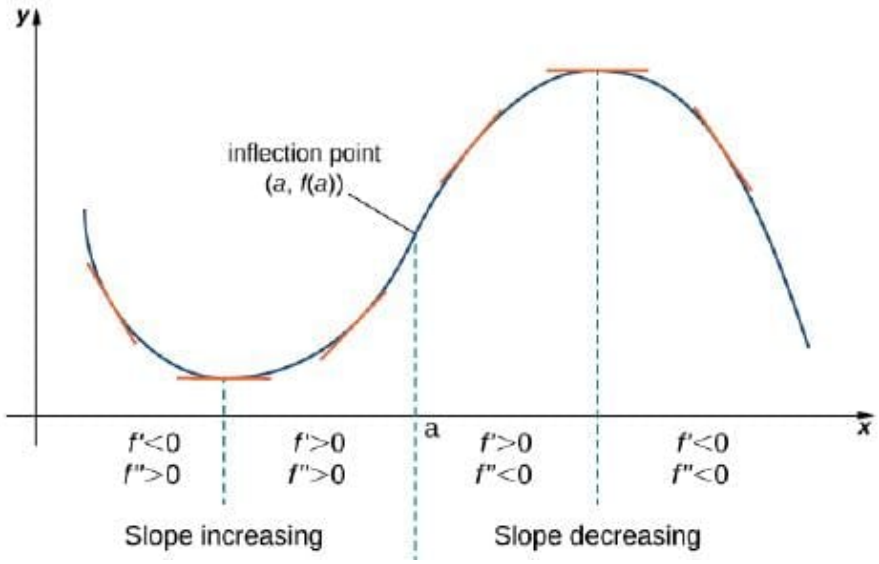

Figure 3.3.7**:** *Since $f''(x) > 0$ for $x < a$, the function f is concave up over the interval $(-\infty, a)$. Since $f''(x) < 0$ for $x > a$, the function f is concave down over the interval (a, ∞). The point $(a, f(a))$ is an inflection point of f.*

Interactive Element: Concavity and the Tangent Line

The following Interactive Element might help you visualize how concavity determines whether a curve lies above or below its tangent lines, and what happens at an inflection point.

Interact: Drag the orange point along the curve and watch how the tangent line sits relative to the curve.

Observation: Where the curve is concave up (blue), the tangent line lies below it; where it is concave down (purple), the tangent lies above it; at the inflection point the tangent crosses through the curve.

Example 3.3.3

For the function $f(x) = x^3 - 6x^2 + 9x + 30$, determine all intervals where f is concave up and all intervals where f is concave down. List all inflection points for f. Use a graphing utility to confirm your results.

Solution available at the end of this section.

Checkpoint 3.3.3

For $f(x) = -x^3 + \frac{3}{2}x^2 + 18x$, find all intervals where f is concave up and all intervals where f is concave down.

Answer available at the end of this section.

We now summarize, in Table 3.3.4, the information that the first and second derivatives of a function f provide about the graph of f and illustrate this information in Figure 3.3.9.

Table:* 3.3.4*: What Derivatives Tell Us about Graphs

Sign of f'	Sign of f''	Is f increasing or decreasing?	Concavity
Positive	Positive	Increasing	Concave up
Positive	Negative	Increasing	Concave down

Sign of f'	Sign of f''	Is f increasing or decreasing?	Concavity
Negative	Positive	Decreasing	Concave up
Negative	Negative	Decreasing	Concave down

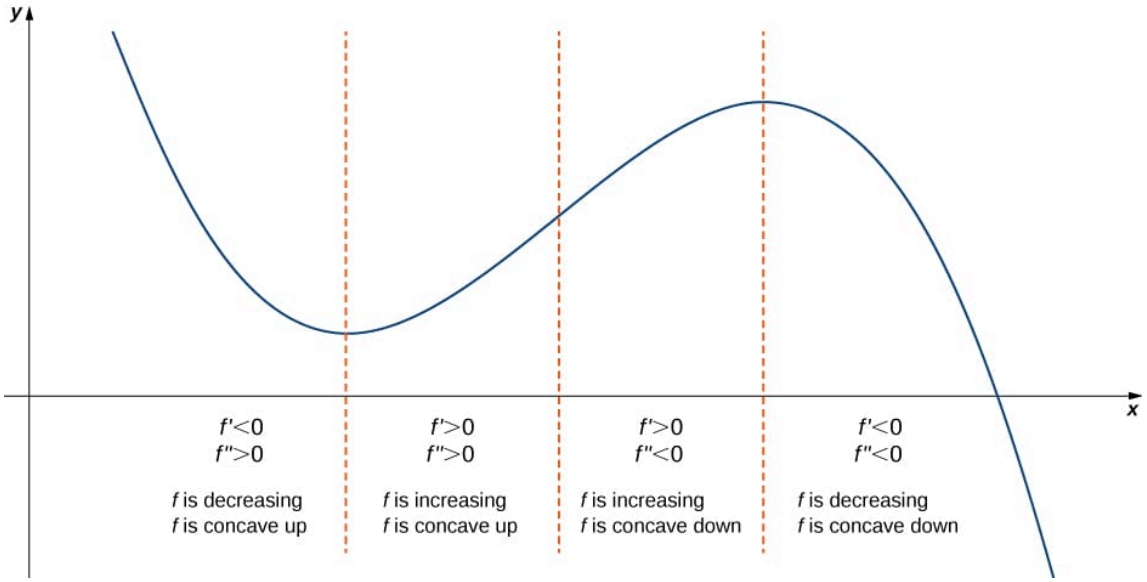

Figure 3.3.9**:** *Consider a twice-differentiable function f over an open interval I. If $f'(x) > 0$ for all $x \in I$, the function is increasing over I. If $f'(x) < 0$ for all $x \in I$, the function is decreasing over I. If $f''(x) > 0$ for all $x \in I$, the function is concave up. If $f''(x) < 0$ for all $x \in I$, the function is concave down on I.*

The Second Derivative Test

The First Derivative Test provides an analytical tool for finding local extrema. Still, the second derivative can also be used to locate extreme values. Using the second derivative can sometimes be a more straightforward method than using the first derivative.

We know that if a *continuous* function has a local extremum, it must occur at a critical point. However, a function does not need a local extremum at a critical point. Here, we examine how the **Second Derivative Test** can determine whether a function has a local extremum at a critical point. Let f be a twice-differentiable function such that $f'(a) = 0$ and f'' is continuous over an open interval I containing a. Suppose $f''(a) < 0$. Since f'' is continuous over I, $f''(x) < 0$ for all $x \in I$ (Figure 3.3.10). Then f' is a decreasing function over I. Since $f'(a) = 0$, we conclude that for all $x \in I$, $f'(x) > 0$ if \(xa\). Therefore, by the First Derivative Test, f has a local maximum at $x = a$.

On the other hand, suppose there exists a point b such that $f'(b) = 0$ but $f''(b) > 0$. Since f'' is continuous over an open interval I containing b, $f''(x) > 0$ for all $x \in I$ (Figure 3.3.10). Then f' is an increasing function over I. Since $f'(b) = 0$, we conclude that for all $x \in I$, $f'(x) < 0$ if \(x0\) if $x > b$. Therefore, by the First Derivative Test, f has a local minimum at $x = b$.

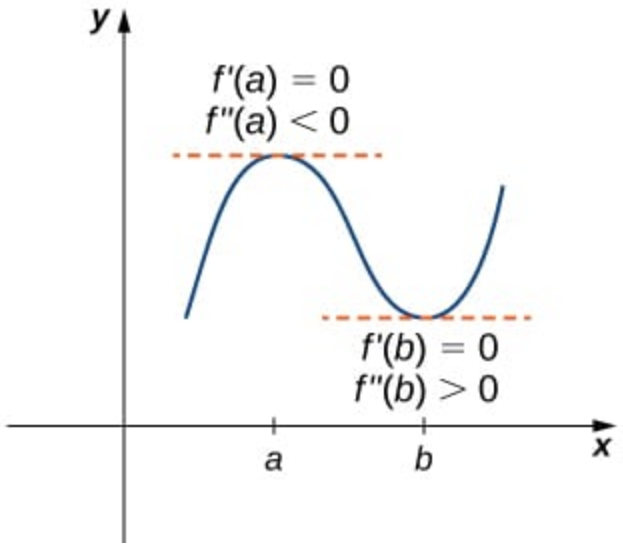

Figure 3.3.10**:** *Consider a twice-differentiable function f such that f'' is continuous. Since $f'(a) = 0$ and $f''(a) < 0$, there is an interval I containing a such that for all x in I, f is increasing if \(xa\). As a result, f has a local maximum at $x = a$. Since $f'(b) = 0$ and $f''(b) > 0$, there is an interval I containing b such that for all x in I, f is decreasing if \(xb\). As a result, f has a local minimum at $x = b$.*

Theorem: Second Derivative Test

Suppose $f'(c) = 0$ and f'' is continuous over an interval containing c.

i. If $f''(c) > 0$, then f has a local minimum at c.
ii. If $f''(c) < 0$, then f has a local maximum at c.
iii. If $f''(c) = 0$, then the test is inconclusive.

Note, when $f''(c) = 0$ (case iii), f may have a local maximum, local minimum, or neither at c. For example, the functions $f(x) = x^3$, $f(x) = x^4$, and $f(x) = -x^4$ all have critical points at $(0, 0)$. In each case, the second derivative is zero at $x = 0$. However, the function $f(x) = x^4$ has a local minimum at $x = 0$ whereas the function $f(x) = -x^4$ has a local maximum at $x = 0$, and the function $f(x) = x^3$ does not have a local extremum at $x = 0$.

Let's now look at how to use the Second Derivative Test to determine whether f has a local maximum or local minimum at a critical number c where $f'(c) = 0$.

Example 3.3.4

Use the second derivative to find the location of all local extrema for $f(x) = x^5 - 5x^3$.

Solution available at the end of this section.

Checkpoint 3.3.4

Consider the function $f(x) = x^3 - (\frac{3}{2})x^2 - 18x$. The points $c = 3,\ -2$ satisfy $f'(c) = 0$. Use the Second Derivative Test to determine whether f has a local maximum or local minimum at those points.

Answer available at the end of this section.

We have now developed the tools to determine where a function is increasing and decreasing and understand the graph's basic shape. The next section discusses what happens to a function as $x \to \pm\infty$. At that point, we have enough tools to provide accurate graphs of various functions.

Success in College: Learning From Tests (Part 3)

1. Look at your last test from this class. How many points did you miss because you could not remember something or did not know how to do the problem?
2. Knowing what you know now, do you think you could do better on this exam?

Hidden Resources: Kanopy

Did you know that as a Los Rios Community College student, you have free access to Kanopy?

What is Kanopy?

According to Wikipedia, Kanopy "is an on-demand streaming video platform for public libraries and universities that offers films and documentaries."

Why would you be interested?

There are a lot of great movies, documentaries, and educational videos on this service - and it's free for you because you are a student of the Los Rios Community College District.

Is it as good as Hulu or Netflix?

No, but it does have a lot of the same titles.

Why aren't any of the Great Courses available to us?

Not being able to access the Great Courses is a disappointment. That was one of the options Los Rios chose not to purchase; however, drown your sorrows in What We Do in the Shadows instead!

How do I sign up?

All you need to do is go to https://losrios.kanopy.com and start typing Los Rios when asked for your institution. That should be it.

Answers

Section Preview

If you consider the first section of the chapter an introduction to graphing with Calculus, this is the section that expects students to be proficient. Two sections from now will be the masterclass on graphing.

Section Preview

- Explain how the sign of the first derivative affects the shape of a function's graph.
- State the First Derivative Test for critical points.
- Use concavity and inflection points to explain how the sign of the second derivative affects the shape of a function's graph.
- Explain the Concavity Test for a function over an open interval.
- Explain the relationship between a function and its first and second derivatives.
- State the Second Derivative Test for local extrema.

Corollary: Increasing and Decreasing Functions

We will prove i. by contradiction; the proof of ii. is similar.

Assume f is *not* an increasing function on I. Then there exist a and b in I such that $a < b$, but $f(a) \geq f(b)$. Since f is a differentiable function over I, by the Mean Value Theorem there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Since $f(a) \geq f(b)$, we know that $f(b) - f(a) \leq 0$. Also, $a < b$ tells us that $b - a > 0$. We conclude that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \leq 0.$$

However, $f'(x) > 0$ for all $x \in I$. This is a contradiction. The only ***assumption*** we made was that f was *not* an increasing function on I. Therefore, f must be an increasing function over I.

Q.E.D.

Example 3.3.1

The derivative is $f'(x) = 3x^2 - 6x - 9$. To find the critical numbers, we need to find where $f'(x) = 0$. Factoring the polynomial, we conclude that the critical numbers must satisfy

$$3(x^2 - 2x - 3) = 3(x - 3)(x + 1) = 0.$$

Therefore, the critical numbers are $x = 3$ and $x = -1$. Now divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, -1)$, $(-1, 3)$, and $(3, \infty)$.

Since f' is a continuous function, to determine the sign of $f'(x)$ over each subinterval, it suffices to choose a point over each of the intervals $(-\infty, -1)$, $(-1, 3)$, and $(3, \infty)$, and determine the sign of f' at each of these points. For example, let's choose $x = -2$, $x = 0$, and $x = 4$ as test points.

Table: 3.3.1***: First Derivative Test for*** $f(x) = x^3 - 3x^2 - 9x - 1$**.**

Interval	Test Point	Sign of $f'(x) = 3(x - 3)(x + 1)$ at Test Point	Conclusion
$(-\infty, -1)$	$x = -2$	(+)(-)(-)=+	f is increasing.
$(-1, 3)$	$x = 0$	(+)(-)(+)=-	f is decreasing.
$(3, \infty)$	$x = 4$	(+)(+)(+)=+	f is increasing.

Since f' switches sign from positive to negative as x increases through -1, f has a local maximum at $x = -1$. Since f' switches sign from negative to positive as x increases through 3, f has a local minimum at $x = 3$. These analytical results agree with the following graph.

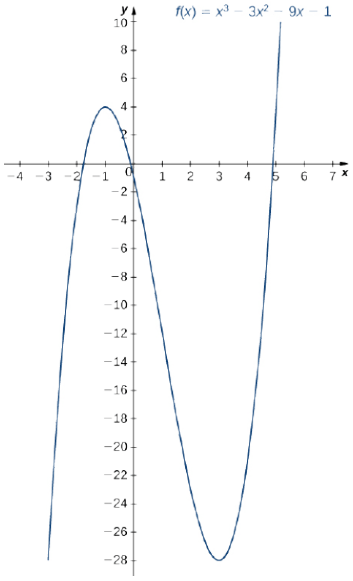

Figure 3.3.4**:** *The function f has a maximum at $x = -1$ and a minimum at $x = 3$*

Checkpoint 3.3.1

f has a local minimum at -2 and a local maximum at 3.

Example 3.3.2

The derivative is

$$f'(x) = \frac{5}{3}x^{-2/3} - \frac{5}{3}x^{2/3} = \frac{5}{3x^{2/3}} - \frac{5x^{2/3}}{3} = \frac{5 - 5x^{4/3}}{3x^{2/3}} = \frac{5(1 - x^{4/3})}{3x^{2/3}}.$$

The derivative $f'(x) = 0$ when $1 - x^{4/3} = 0$. Therefore, $f'(x) = 0$ at $x = \pm 1$. The derivative $f'(x)$ is undefined at $x = 0$. Therefore, we have three critical numbers: $x = 0$, $x = 1$, and $x = -1$. Consequently, divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, \infty)$.

Since f' is continuous over each subinterval, it suffices to choose a test point x in each of the intervals and determine the sign of f' at each of these points. The points $x = -2$, $x = -\frac{1}{2}$, $x = \frac{1}{2}$, and $x = 2$ are test points for these intervals.

Table: 3.3.2***: First Derivative Test for*** $f(x) = 5x^{1/3} - x^{5/3}$**.**

Interval	Test Point	Sign of $f'(x) = \frac{5(1-x^{4/3})}{3x^{2/3}}$ at Test Point	Conclusion
$(-\infty, -1)$	$x = -2$	$\frac{(+)(-)}{+} = -$	f is decreasing.
$(-1, 0)$	$x = -\frac{1}{2}$	$\frac{(+)(+)}{+} = +$	f is increasing.
$(0, 1)$	$x = \frac{1}{2}$	$\frac{(+)(+)}{+} = +$	f is increasing.

Interval	Test Point	Sign of $f'(x) = \frac{5(1-x^{4/3})}{3x^{2/3}}$ at Test Point	Conclusion
$(1, \infty)$	$x = 2$	$\frac{(+)(-)}{+} = -$	f is decreasing.

Since f is decreasing over the interval $(-\infty, -1)$ and increasing over the interval $(-1, 0)$, f has a local minimum at $x = -1$. Since f is increasing over the interval $(-1, 0)$ and the interval $(0, 1)$, f does not have a local extremum at $x = 0$. Since f is increasing over the interval $(0, 1)$ and decreasing over the interval $(1, \infty)$, f has a local maximum at $x = 1$. The analytical results agree with the following graph.

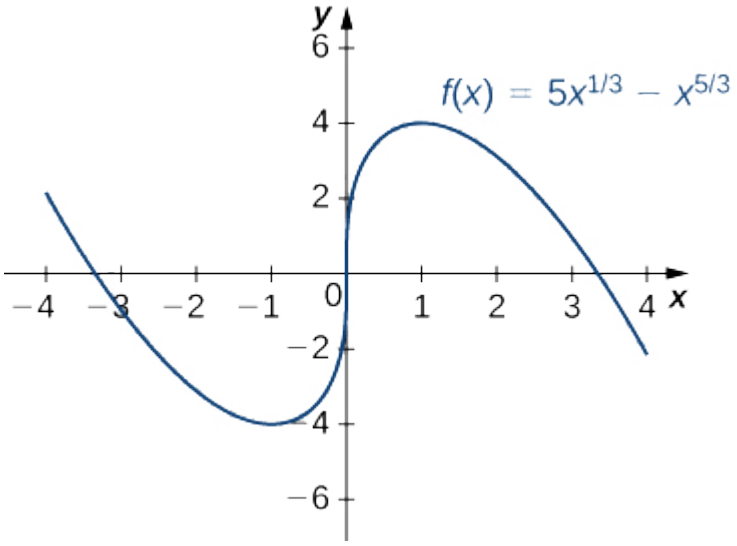

Figure 3.3.5**:** *The function f has a local minimum at $x = -1$ and a local maximum at $x = 1$*

Checkpoint 3.3.2

f has no local extrema because f' does not change sign at $x = 1$.

Example 3.3.3

To determine concavity, we need to find the second derivative $f''(x)$. The first derivative is $f'(x) = 3x^2 - 12x + 9$, so the second derivative is $f''(x) = 6x - 12$. If the function changes concavity, it occurs either when $f''(x) = 0$ or $f''(x)$ is undefined. Since f'' is defined for all real numbers x, we need only find where $f''(x) = 0$. Solving the equation $6x - 12 = 0$, we see that $x = 2$ is the only place where f could change concavity. We now test points over the intervals $(-\infty, 2)$ and $(2, \infty)$ to determine the concavity of f. The points $x = 0$ and $x = 3$ are test points for these intervals.

Table:* 3.3.3*: Test for Concavity for $f(x) = x^3 - 6x^2 + 9x + 30$**.**

Interval	Test Point	Sign of $f''(x) = 6x - 12$ at Test Point	Conclusion
$(-\infty, 2)$	$x = 0$	-	f is concave down
$(2, \infty)$	$x = 3$	+	f is concave up

We conclude that f is concave down over the interval $(-\infty, 2)$ and concave up over the interval $(2, \infty)$. Since f changes concavity at $x = 2$, the point $(2, f(2)) = (2, 32)$ is an inflection point. Figure 3.3.8 confirms the analytical results.

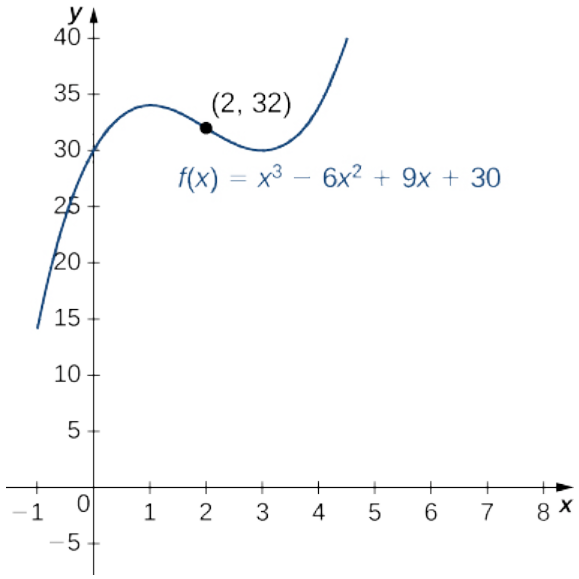

Figure 3.3.8**:** *The given function has a point of inflection at* $(2, 32)$ *where the graph changes concavity.*

Checkpoint 3.3.3

f is concave up over the interval $(-\infty, \frac{1}{2})$ and concave down over the interval $(\frac{1}{2}, \infty)$

Example 3.3.4

To apply the Second Derivative Test, we first need to find critical numbers c where $f'(c) = 0$. The derivative is $f'(x) = 5x^4 - 15x^2$. Therefore, $f'(x) = 5x^4 - 15x^2 = 5x^2(x^2 - 3) = 0$ when $x = 0, \ \pm\sqrt{3}$.

To determine whether f has a local extremum at any of these points, we need to evaluate the sign of f'' at these points. The second derivative is

$$f''(x) = 20x^3 - 30x = 10x(2x^2 - 3).$$

In the following table, we evaluate the second derivative at each critical point and use the Second Derivative Test to determine whether f has a local maximum or minimum at any of these points.

Table:* 3.3.5*: Second Derivative Test for $f(x) = x^5 - 5x^3$**.**

x	$f''(x)$	Conclusion
$-\sqrt{3}$	$-30\sqrt{3}$	Local maximum
0	0	Second Derivative Test is inconclusive
$\sqrt{3}$	$30\sqrt{3}$	Local minimum

By the Second Derivative Test, we conclude that f has a local maximum at $x = -\sqrt{3}$ and f has a local minimum at $x = \sqrt{3}$. The Second Derivative Test is inconclusive at $x = 0$. We apply the First Derivative Test to determine whether f has a local extremum at $x = 0$. To evaluate the sign of $f'(x) = 5x^2(x^2 - 3)$ for $x \in (-\sqrt{3}, 0)$ and $x \in (0, \sqrt{3})$, let $x = -1$ and $x = 1$ be the two test points. Since $f'(-1) < 0$ and $f'(1) < 0$, we conclude that f is decreasing on both intervals and, therefore, f does not have a local extremum at $x = 0$ as shown in the following graph.

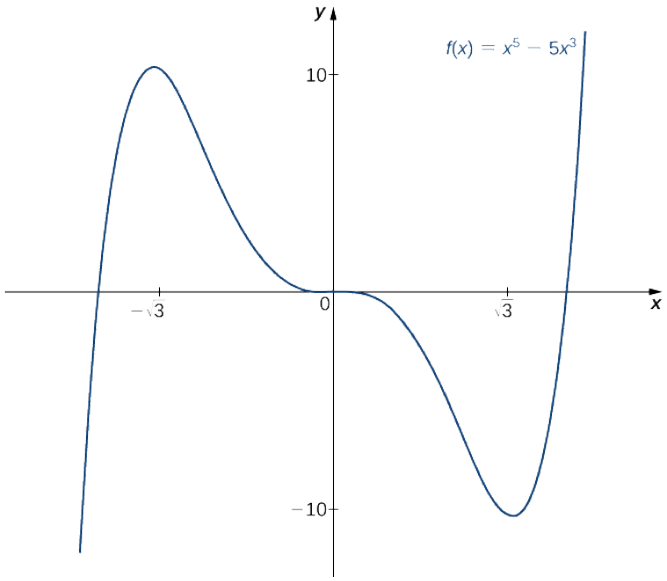

Figure 3.3.11**:** *The function f has a local maximum at $x = -\sqrt{3}$ and a local minimum at $x = \sqrt{3}$*

Checkpoint 3.3.4

f has a local maximum at -2 and a local minimum at 3.

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3.3.1: Resources and Key Concepts

Resources

Prerequisite Topics

The following is a list of prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Solving Inequalities**
  - Solving Polynomial Inequalities: To determine intervals of increase/decrease (sign of $f'(x)$) and intervals of concavity (sign of $f''(x)$), one often solves polynomial inequalities.

Videos

- Increasing and decreasing functions
  - The Calculus of Increasing and Decreasing Functions
- First Derivative Test
  - The First Derivative Test and Finding Local Extrema
- Second Derivative Test
  - Calculus of Concavity and the Second Derivative Test
- Calculus and graphing
  - Graphing Using Calculus (Part 1)
  - Graphing Using Calculus (Part 2)
  - Graphing Using Calculus (Part 3)
  - Graphing Using Calculus (Part 4)
  - Graphing Using Calculus (Part 5)

Key Concepts

Definitions

- **Increasing/Decreasing Functions (Calculus Definition based on Derivative)**:
  - If $f'(x) > 0$ for all x in an interval (a, b), then f is an increasing function over $[a, b]$.
  - If $f'(x) < 0$ for all x in an interval (a, b), then f is a decreasing function over $[a, b]$.
- **Concavity**: Let f be a function that is differentiable over an open interval I.
  - If f' is increasing over I, we say f is **concave up** over I.
  - If f' is decreasing over I, we say f is **concave down** over I.

  (Equivalent definition: A differentiable function is concave up over an open interval I if it lies above all of its tangent lines on the interval. It is concave down on I if it lies below all of its tangent lines on I.)
- **Inflection Point**: If f is continuous at a and f changes concavity at a, the point $(a, f(a))$ is an inflection point of f.

Theorems

- **Corollary: Increasing and Decreasing Functions**: Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b).
  i. If $f'(x) > 0$ for all $x \in (a, b)$, then f is an increasing function over $[a, b]$.
  ii. If $f'(x) < 0$ for all $x \in (a, b)$, then f is a decreasing function over $[a, b]$.
- **Theorem: First Derivative Test**: Suppose that f is a continuous function over an interval I containing a critical number c. If f is differentiable over I, except possibly at c, then $f(c)$ satisfies one of the following descriptions:

i. If f' changes sign from positive when \(xc\), then $f(c)$ is a local maximum of f.
ii. If f' changes sign from negative when \(xc\), then $f(c)$ is a local minimum of f.
iii. If f' has the same sign for \(xc\), then $f(c)$ is neither a local maximum nor a local minimum of f.

- **Theorem: Concavity Test**: Let f be a function that is twice differentiable over an interval I.
  i. If $f''(x) > 0$ for all $x \in I$, then f is concave up over I.
  ii. If $f''(x) < 0$ for all $x \in I$, then f is concave down over I.
- **Theorem: Second Derivative Test**: Suppose $f'(c) = 0$ and f'' is continuous over an interval containing c.
  i. If $f''(c) > 0$, then f has a local minimum at c.
  ii. If $f''(c) < 0$, then f has a local maximum at c.
  iii. If $f''(c) = 0$, then the test is inconclusive.

Common Mistakes

- **Assuming Critical Points are Always Extrema**: While local extrema must occur at critical points, not all critical points are local extrema. The function $f(x) = x^3$ has a critical point at $x = 0$ but no local extremum there.
- **Assuming $f''(c) = 0$ Implies Inflection Point**: A function may not change concavity at a point c even if $f''(c) = 0$ or $f''(c)$ is undefined. For example, $f(x) = x^4$ has $f''(0) = 0$, but it is concave up everywhere and has no inflection point at $x = 0$.
- **Second Derivative Test Inconclusive**: If $f''(c) = 0$ at a critical point c (where $f'(c) = 0$), the Second Derivative Test provides no information about whether there is a local extremum at c. The First Derivative Test must be used in such cases.

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3.3.2: Homework

Reading Questions

1. How does the sign of the first derivative, $f'(x)$, relate to whether a function $f(x)$ is increasing or decreasing on an interval?
2. What is the First Derivative Test used for? Describe how it determines if a critical point c corresponds to a local maximum, local minimum, or neither.
3. What does it mean for a function to be "concave up" on an interval? How does this relate to the behavior of $f'(x)$?
4. What does it mean for a function to be "concave down" on an interval? How does this relate to the behavior of $f'(x)$?
5. What is the Concavity Test? How does the sign of the second derivative, $f''(x)$, determine the concavity of $f(x)$?
6. What is an inflection point? What conditions must be met for $(a, f(a))$ to be an inflection point?
7. If $f''(a) = 0$, does $f(x)$ necessarily have an inflection point at $x = a$? Explain.
8. What is the Second Derivative Test used for?
9. Under what conditions does the Second Derivative Test determine that a critical point c (where $f'(c) = 0$) is a local maximum? A local minimum?
10. When is the Second Derivative Test inconclusive? What should be done in such a case?
11. Can a function have a local extremum at a point where its derivative is undefined? If so, give an example.

Homework

1) If c is a critical point of $f(x)$, when is there no local maximum or minimum at c? Explain.

2) For the function $y = x^3$, is $x = 0$ both an inflection point and a local maximum/minimum?

Answer

It is not a local maximum/minimum because f' does not change sign

3) For the function $y = x^3$, is $x = 0$ an inflection point?

4) Is it possible for a point c to be both an inflection point and a local extremum of a twice differentiable function?

Answer

No

5) Why do you need continuity for the first derivative test? Come up with an example.

6) Explain whether a concave-down function has to cross $y = 0$ for some value of x.

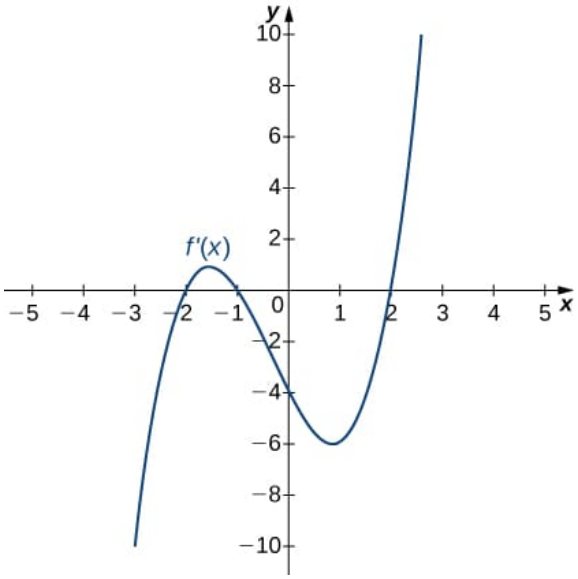
Answer

False; for example, $y = \sqrt{x}$.

7) Explain whether a polynomial of degree 2 can have an inflection point.

In exercises 8 - 12, analyze the graphs of f', then list all intervals where f is increasing or decreasing.

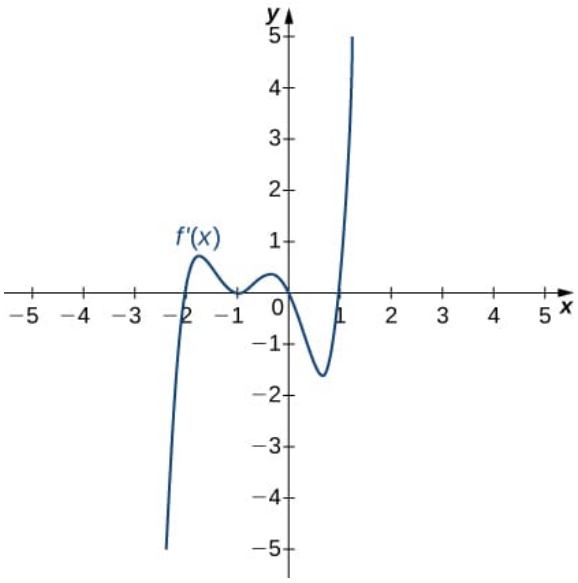
8)



Answer

Increasing for \(−22\);
Decreasing for $x < -2$ and \(−1

9)



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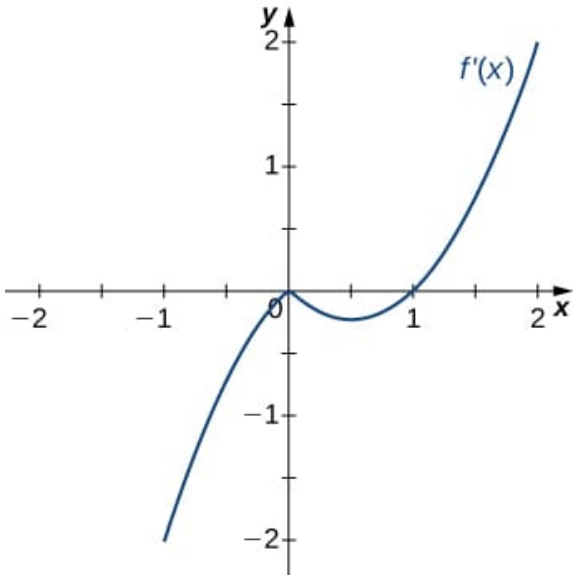
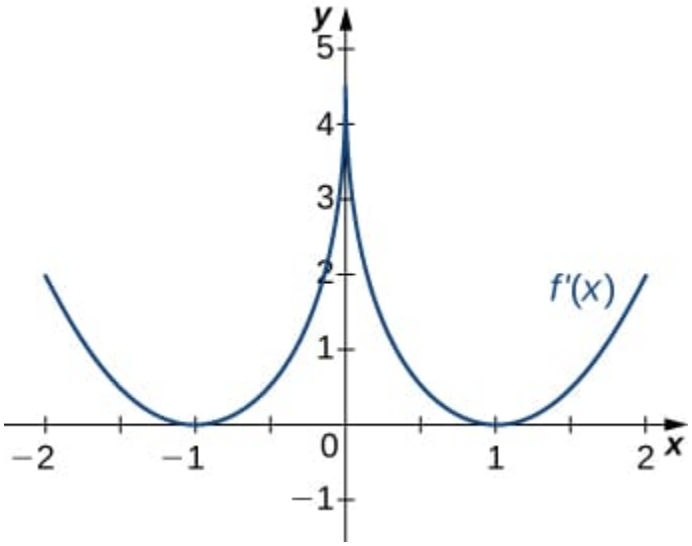


Answer

Decreasing for $x < 1$,
Increasing for $x > 1$

11)



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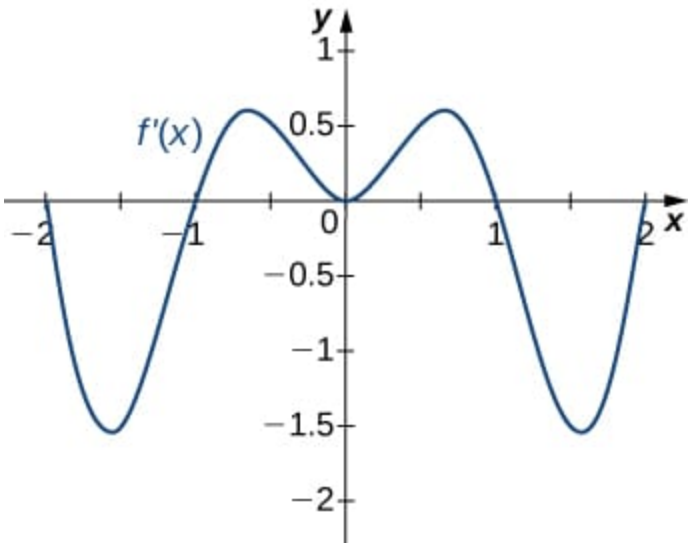


Answer

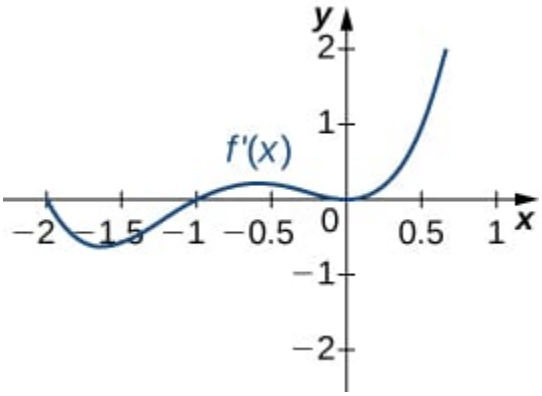
Decreasing for \(–2 Increasing for \(–12\)

In exercises 13 - 17, analyze the graphs of f', then list all intervals where

a. f is increasing and decreasing and

b. the minima and maxima are located.

13)



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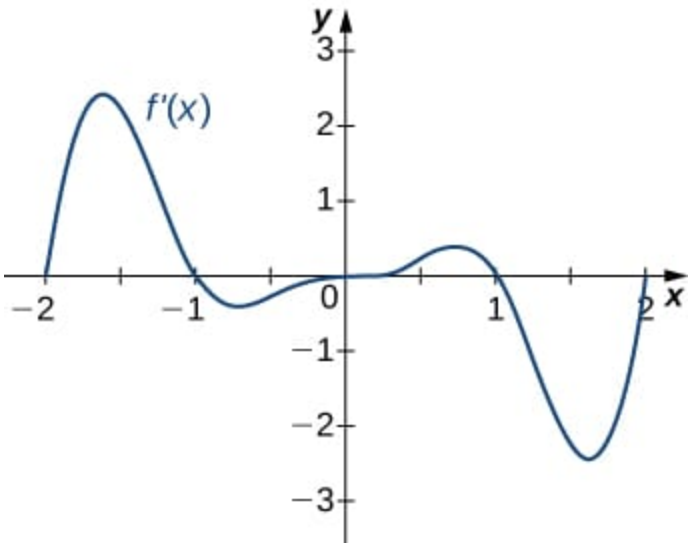
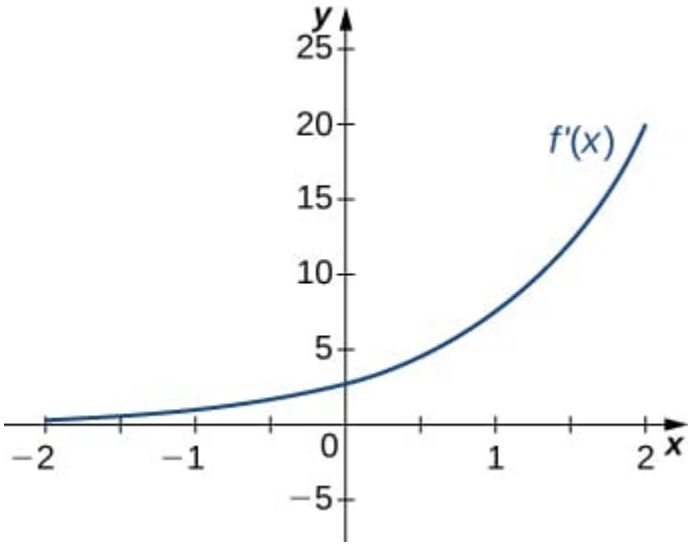


Answer

a. Increasing over \(−22\), Decreasing over \(x<−2, \;−1 b. Maxima at $x = -1$ and $x = 1$, Minima at $x = -2$ and $x = 0$ and $x = 2$

15)



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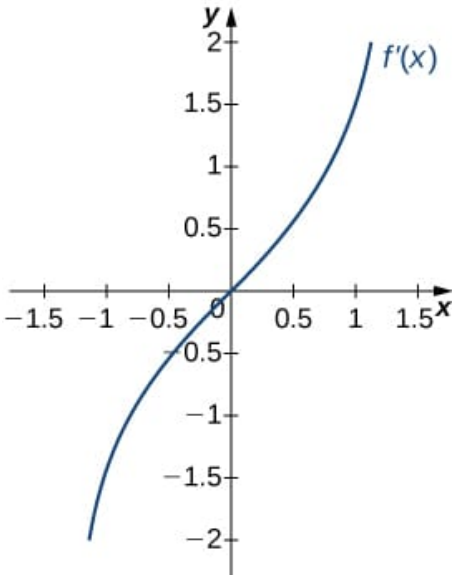
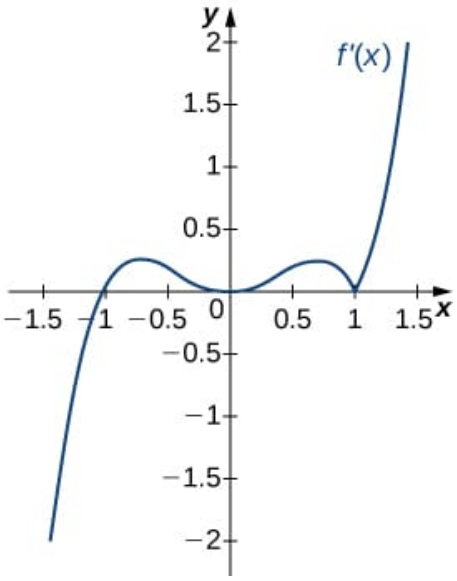


Answer

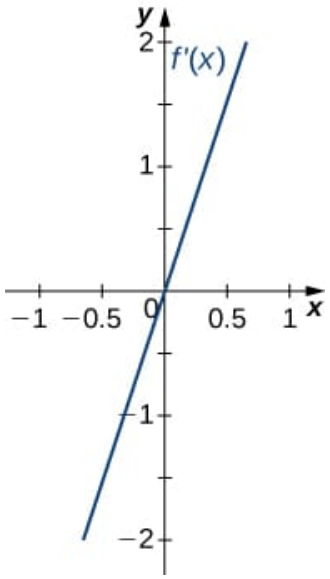
a. Increasing over $x > 0$, Decreasing over $x < 0$;
b. Minimum at $x = 0$

17)



In exercises 18 - 22, analyze the graphs of f', then list all inflection points and intervals f that are concave up and concave down.

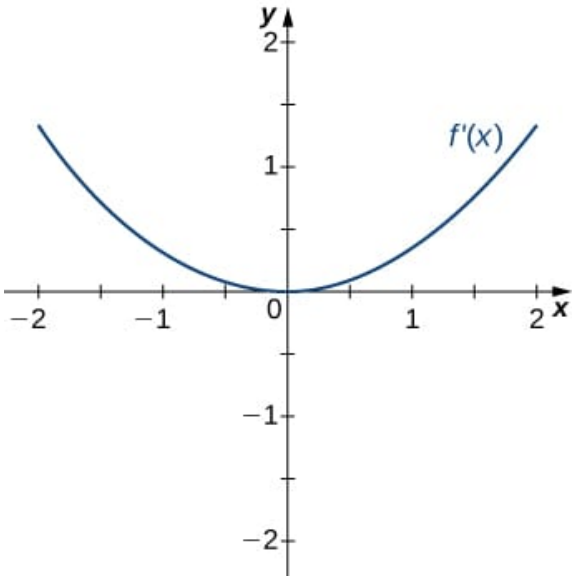
18)



Answer

Concave up for all x,
No inflection points

19)



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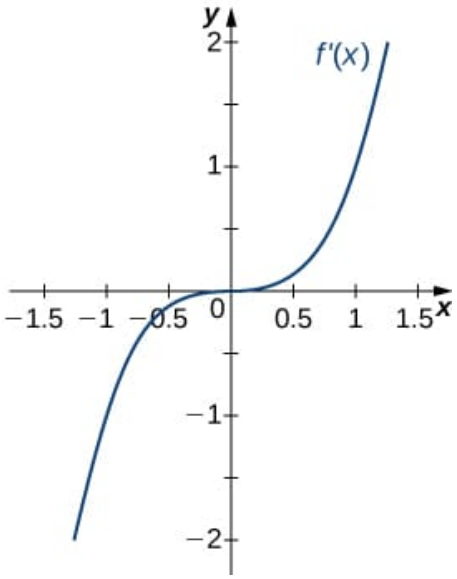
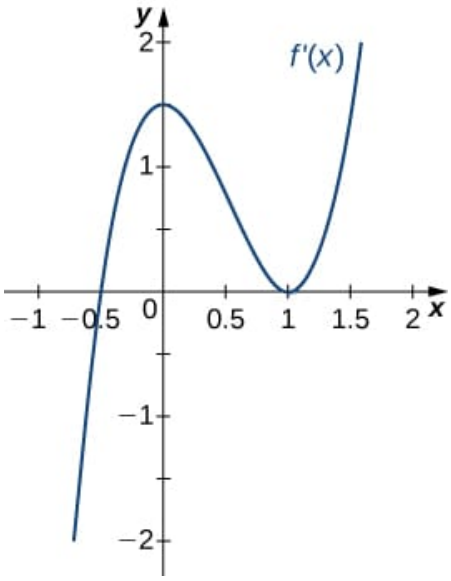


Answer

Concave up for all x,
No inflection points

21)



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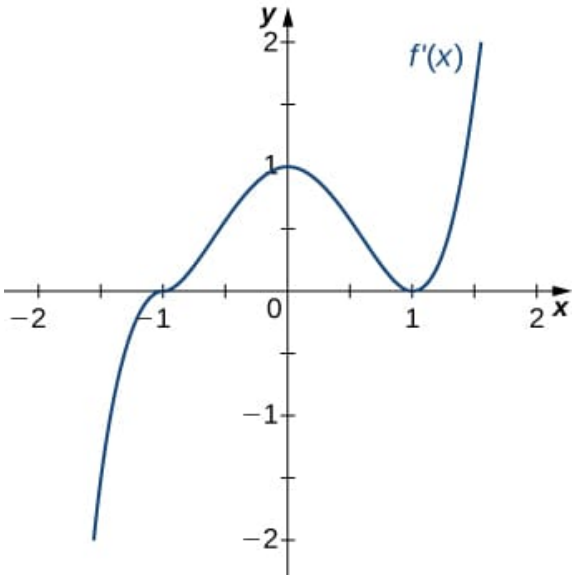

Answer

Concave up for $x < 0$ and $x > 1$,
Concave down for \(0 Inflection points at $x = 0$ and $x = 1$

For exercises 23 - 27, draw a graph that satisfies the given specifications for the domain $x = [-3, 3]$. The function does not have to be continuous or differentiable.

23) $f(x) > 0, \quad f'(x) > 0$ over $x > 1, \quad -3 < x < 0, \quad f'(x) = 0$ over="" $0 < x < 1$<="" p=""> </x<0,\;f^{\prime}(x)=0\)>

24) $f'(x) > 0$ over \(x>2,\;–3

Answer

Answers will vary

25) $f''(x) < 0$ over \(–10,\;–3

26) There is a local maximum at $x = 2$, local minimum at $x = 1$, and the graph is neither concave up nor concave down.

Answer

Answers will vary

27) There are local maxima at $x = \pm 1$, the function is concave up for all x, and the function remains positive for all x.

For the following exercises, determine

a. intervals where f is increasing or decreasing and

b. local minima and maxima of f.

28) $f(x) = \sin x + \sin^3 x$ over \(– \pi

Answer

a. Increasing over \(–\frac{ \pi }{2}\frac{ \pi }{2}\)

b. Local maximum at $x = \frac{\pi}{2}$; local minimum at $x = -\frac{\pi}{2}$

29) $f(x) = x^2 + \cos x$

For exercise 30, determine

a. intervals where f is concave up or concave down, and

b. the inflection points of f.

30) $f(x) = x^3 - 4x^2 + x + 2$

Answer

a. Concave up for $x > \frac{4}{3}$, concave down for $x < \frac{4}{3}$

b. Inflection point at $x = \frac{4}{3}$

For exercises 31 - 37, determine

a. intervals where f is increasing or decreasing,

b. local minima and maxima of f,

c. intervals where f is concave up and concave down, and

d. the inflection points of f.

31) $f(x) = x^2 - 6x$

32) $f(x) = x^3 - 6x^2$

Answer

a. Increasing over $x < 0$ and $x > 4$, decreasing over \(0 b. Maximum at $x = 0$, minimum at $x = 4$
c. Concave up for $x > 2$, concave down for $x < 2$
d. Inflection point at $x = 2$

33) $f(x) = x^4 - 6x^3$

34) $f(x) = x^{11} - 6x^{10}$

Answer

a. Increasing over $x < 0$ and $x > \frac{60}{11}$, decreasing over \(0 b. Maximum at $x = 0$, minimum at $x = \frac{60}{11}$
c. Concave down for $x < \frac{54}{11}$, concave up for $x > \frac{54}{11}$
d. Inflection point at $x = \frac{54}{11}$

35) $f(x) = x + x^2 - x^3$

36) $f(x) = x^2 + x + 1$

Answer

a. Increasing over $x > -\frac{1}{2}$, decreasing over $x < -\frac{1}{2}$
b. Minimum at $x = -\frac{1}{2}$
c. Concave up for all x
d. No inflection points

37) $f(x) = x^3 + x^4$

For exercises 38 - 47, determine

a. intervals where f is increasing or decreasing,

b. local minima and maxima of f,

c. intervals where f is concave up and concave down, and

d. the inflection points of f. Sketch the curve, then use a calculator to compare your answer. If you cannot determine the exact answer analytically, use a calculator.

38) [Technology Required] $f(x) = \sin(\pi x) - \cos(\pi x)$ over $x = [-1, 1]$

Answer

a. Increases over \(–\frac{1}{4}\frac{3}{4}\) and $x < -\frac{1}{4}$
b. Minimum at $x = -\frac{1}{4}$, maximum at $x = \frac{3}{4}$
c. Concave up for \(–\frac{3}{4}\frac{1}{4}\)
d. Inflection points at $x = -\frac{3}{4}, \quad x = \frac{1}{4}$

39) [Technology Required] $f(x) = x + \sin(2x)$ over $x = [-\frac{\pi}{2}, \frac{\pi}{2}]$

40) [Technology Required] $f(x) = \sin x + \tan x$ over $(-\frac{\pi}{2}, \frac{\pi}{2})$

Answer

a. Increasing for all x
b. No local minimum or maximum
c. Concave up for $x > 0$, concave down for $x < 0$
d. Inflection point at $x = 0$

41) [Technology Required] $f(x) = (x-2)^2(x-4)^2$

42) [Technology Required] $f(x) = \dfrac{1}{1-x}, \quad x \neq 1$

Answer

a. Increasing for all x where defined
b. No local minima or maxima
c. Concave up for $x < 1$; concave down for $x > 1$
d. No inflection points in domain

43) [Technology Required] $f(x) = \dfrac{\sin x}{x}$ over $x = [-2\pi, 0) \cup (0, 2\pi]$

44) $f(x) = \sin(x)e^x$ over $x = [-\pi, \pi]$

Answer

a. Increasing over \(–\frac{ \pi }{4}\frac{3 \pi }{4},\;x<–\frac{ \pi }{4}\)
b. Minimum at $x = -\frac{\pi}{4}$, maximum at $x = \frac{3\pi}{4}$
c. Concave up for \(–\frac{ \pi }{2}\frac{ \pi }{2}\)
d. Inflection points at $x = \pm\frac{\pi}{2}$

45) $f(x) = \ln x\sqrt{x}, \quad x > 0$

46) $f(x) = \frac{1}{4}\sqrt{x} + \frac{1}{x}, \quad x > 0$

Answer

a. Increasing over $x > 4$, decreasing over \(0 b. Minimum at $x = 4$
c. Concave up for \(08\sqrt[3]{2}\)
d. Inflection point at $x = 8\sqrt[3]{2}$

47) $f(x) = \dfrac{e^x}{x}, \quad x \neq 0$

In exercises 48 - 52, interpret the sentences in terms of f, f', and f''.

48) The population is growing more slowly. Here f is the population.

Answer

$f > 0, \quad f' > 0, \quad f'' < 0$

49) A bike accelerates faster, but a car goes faster. Here $f =$ Bike's position minus Car's position.

50) The airplane lands smoothly. Here f is the plane's altitude.

Answer

$f > 0, \quad f' < 0, \quad f'' > 0$

51) Stock prices are at their peak. Here $f$is the stock price.

52) The economy is picking up speed. Here f is a measure of the economy, such as GDP.

Answer

$f > 0, \quad f' > 0, \quad f'' > 0$

For exercises 53 - 57, consider a third-degree polynomial $f(x)$, which has the properties $f'(1) = 0$ and $f'(3) = 0$.

Determine whether the following statements are true or false. Justify your answer.

53) $f(x) = 0$ for some $1 \leq x \leq 3$.

54) $f''(x) = 0$ for some $1 \leq x \leq 3$.

Answer

True, by the Mean Value Theorem

55) There is no absolute maximum at $x = 3$.

56) If $f(x)$ has three roots, then it has 1 inflection point.

Answer

True, examine derivative

57) If $f(x)$ has one inflection point, then it has three real roots.

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3.4: l'Hôpital's Rule and Indeterminate Forms

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

This section examines a powerful tool for evaluating limits. This tool, **l'Hôpital's Rule**, uses derivatives to calculate limits. With this rule, we will be able to evaluate many limits we have not yet been able to determine. Instead of relying on numerical evidence to conjecture that a limit exists, we can show definitively that a limit exists and determine its exact value.

Applying l'Hôpital's Rule

l'Hôpital's Rule can be used to evaluate limits involving the quotient of two functions. Consider

$$\lim_{x\to a}\frac{f(x)}{g(x)}.$$

If $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2 \neq 0$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{L_1}{L_2}.$$

However, what happens if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$? We call this one of the **indeterminate forms,** of type $\frac{0}{0}$. This is considered an indeterminate form because we cannot *determine* the exact behavior of $\frac{f(x)}{g(x)}$ as $x \to a$ without further analysis. We have seen examples of this earlier in the text. For example, consider

$$\lim_{x\to 2}\frac{x^2-4}{x-2}$$

and

$$\lim_{x\to 0}\frac{\sin x}{x}.$$

For the first of these examples, we can evaluate the limit by factoring the numerator and writing

$$\lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2}\frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2}(x+2) = 2+2 = 4.$$

For $\lim_{x\to 0}\frac{\sin x}{x}$, we were able to show, using a geometric argument, that

$$\lim_{x\to 0}\frac{\sin x}{x} = 1.$$

Here, we use a different technique for evaluating limits such as these. Not only does this technique provide an easier way to evaluate these limits, but it also (and more importantly) provides us with a way to evaluate many other limits we could not previously calculate.

The idea behind l'Hôpital's Rule can be explained using local linear approximations. Consider two differentiable functions f and g such that $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$ and such that $g'(a) \neq 0$. For x near a,we can write

$$f(x) \approx f(a) + f'(a)(x-a)$$

and

$$g(x) \approx g(a) + g'(a)(x-a).$$

Therefore,

$$\frac{f(x)}{g(x)} \approx \frac{f(a)+f'(a)(x-a)}{g(a)+g'(a)(x-a)}.$$

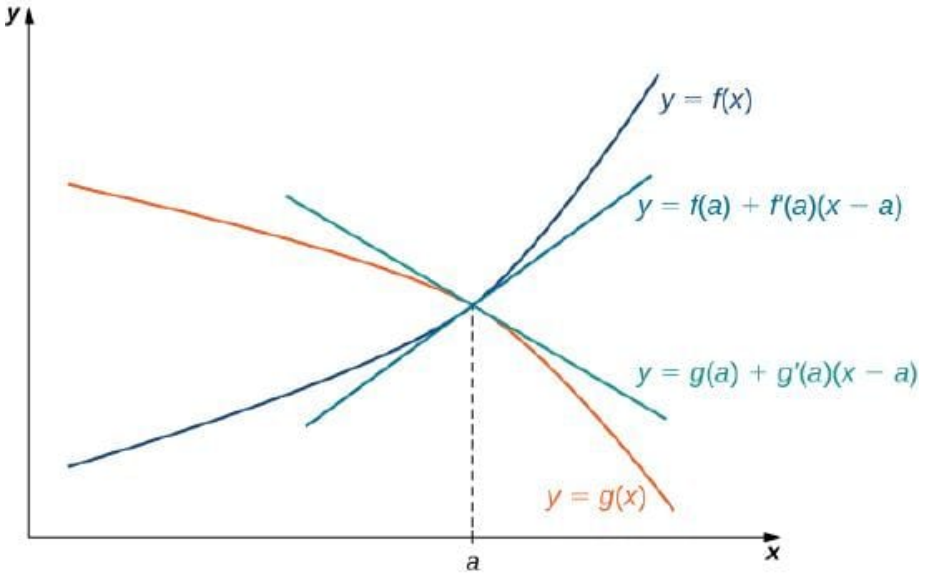

Figure 3.4.1**:** *If* $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$*, then the ratio* $f(x)/g(x)$ *is approximately equal to the ratio of their linear approximations near* a*.*

Since f is differentiable at a, then f is continuous at a, and therefore $f(a) = \lim_{x\to a} f(x) = 0$. Similarly, $g(a) = \lim_{x\to a} g(x) = 0$. If we also assume that f' and g' are continuous at $x = a$, then $f'(a) = \lim_{x\to a} f'(x)$ and $g'(a) = \lim_{x\to a} g'(x)$. Using these ideas, we conclude that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)(x-a)}{g'(x)(x-a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Note that the assumption that f' and g' are continuous at a and $g'(a) \neq 0$ can be loosened. We state l'Hôpital's Rule formally for the indeterminate form $\frac{0}{0}$.

Caution: Abusive Notation

Note that the notation $\frac{0}{0}$ does not mean we are actually dividing zero by zero. Rather, we are using the notation $\frac{0}{0}$ to *represent* a quotient of limits, each of which is zero.

Theorem: l'Hôpital's Rule $(0/0$ Case)

Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

assuming the limit on the right exists or is ∞ or $-\infty$. This result also holds if we consider one-sided limits, or if $a = \infty$ or $a = -\infty$.

Proof available at the end of this section.

Example 3.4.1

Evaluate each of the following limits by applying l'Hôpital's Rule.

a. $\lim_{x\to 0} \dfrac{1-\cos x}{x}$
b. $\lim_{x\to 1} \dfrac{\sin(\pi x)}{\ln x}$
c. $\lim_{x\to \infty} \dfrac{e^{1/x}-1}{1/x}$
d. $\lim_{x\to 0} \dfrac{\sin x - x}{x^2}$

Solutions available at the end of this section.

Checkpoint 3.4.1

Evaluate

$$\lim_{x\to 0} \frac{x}{\tan x}.$$

Answer available at the end of this section.

We can also use l'Hôpital's Rule to evaluate limits of quotients $\frac{f(x)}{g(x)}$ in which $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$. Limits of this form are classified as indeterminate forms of type $\frac{\infty}{\infty}$.

Caution: More Notational Abuse

Again, note that we are not actually dividing ∞ by ∞. Since ∞ is not a real number, that is impossible; rather, ∞/∞ is used to *represent* a quotient of limits, each of which is ∞ or $-\infty$.

Theorem: l'Hôpital's Rule (∞/∞ Case)

Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. Suppose $\lim_{x\to a} f(x) = \infty$ (or $-\infty$) and $\lim_{x\to a} g(x) = \infty$ (or $-\infty$). Then,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

assuming the limit on the right exists or is ∞ or $-\infty$. This result also holds if the limit is infinite, if $a = \infty$ or $-\infty$, or if the limit is one-sided.

Example 3.4.2

Evaluate each of the following limits by applying l'Hôpital's Rule.

a. $\lim_{x\to \infty} \dfrac{3x+5}{2x+1}$
b. $\lim_{x\to 0^+} \dfrac{\ln x}{\cot x}$

Solutions available at the end of this section.

Checkpoint 3.4.2

Evaluate

$$\lim_{x\to\infty}\frac{\ln x}{5x}.$$

Answer available at the end of this section.

As mentioned, l'Hôpital's Rule is a handy tool for evaluating limits. It is important to remember, however, that to apply l'Hôpital's Rule to a quotient $\frac{f(x)}{g(x)}$, it is essential that the limit of $\frac{f(x)}{g(x)}$ be of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Consider the following example.

Example 3.4.3

Consider $\lim_{x\to 1}\frac{x^2+5}{3x+4}$. Show that the limit cannot be evaluated by applying l'Hôpital's Rule.

Solution available at the end of this section.

Checkpoint 3.4.3

Explain why we cannot apply l'Hôpital's Rule to evaluate $\lim_{x\to 0^+}\frac{\cos x}{x}$. Evaluate $\lim_{x\to 0^+}\frac{\cos x}{x}$ by other means.

Answer available at the end of this section.

Other Indeterminate Forms

l'Hôpital's Rule is very useful for evaluating limits involving the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$. However, we can also use l'Hôpital's Rule to help evaluate limits involving other indeterminate forms that arise when evaluating limits. The expressions $0\cdot\infty$, $\infty-\infty$, 1^∞, ∞^0, and 0^0 are all considered indeterminate forms. These expressions are not real numbers. Rather, they represent forms that arise when trying to evaluate certain limits. Next, we realize why these are indeterminate forms and then understand how to use l'Hôpital's Rule in these cases. The key idea is that we must rewrite the indeterminate forms in such a way that we arrive at the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Indeterminate Form of Type $0\cdot\infty$

Suppose we want to evaluate $\lim_{x\to a}(f(x)\cdot g(x))$, where $f(x)\to 0$ and $g(x)\to\infty$ (or $-\infty$) as $x\to a$. Since one term in the product is approaching zero, but the other is becoming arbitrarily large (in magnitude), anything can happen to the product. We use the notation $0\cdot\infty$ to denote the form in this situation. The expression $0\cdot\infty$ is considered indeterminate because we cannot *determine* without further analysis the exact behavior of the product $f(x)g(x)$ as $x\to\infty$. For example, let n be a positive integer and consider

$$f(x)=\frac{1}{(x^n+1)}$$

and

$$g(x)=3x^2.$$

As $x\to\infty$, $f(x)\to 0$ and $g(x)\to\infty$. However, the limit as $x\to\infty$ of $f(x)g(x)=\frac{3x^2}{(x^n+1)}$ varies, depending on n. If $n=2$, then $\lim_{x\to\infty}f(x)g(x)=3$. If $n=1$, then $\lim_{x\to\infty}f(x)g(x)=\infty$. If $n=3$, then $\lim_{x\to\infty}f(x)g(x)=0$. Here, we consider another limit involving the indeterminate form $0\cdot\infty$ and show how to rewrite the function as a quotient to use l'Hôpital's Rule.

Interactive Element: Why $0\cdot\infty$ Is Indeterminate

The following Interactive Element might help you visualize why the form $0\cdot\infty$ does not determine a limit.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Desmos Interactive online

>

Interact: Drag the large point left and right to change the exponent n in the product $f(x)g(x) = \frac{1}{x^n+1} \cdot 3x^2$.

Observation: As n increases past 2, the limit of the product as $x \to \infty$ switches from ∞ to 3 to 0 — the same $0 \cdot \infty$ form, three different fates.

Example 3.4.4

Evaluate $\lim_{x \to 0^+} x \ln(x)$.

Solution available at the end of this section.

Checkpoint 3.4.4

Evaluate

$$\lim_{x \to 0} x \cot x.$$

Answer available at the end of this section.

Indeterminate Form of Type $\infty - \infty$

Another type of indeterminate form is $\infty - \infty$. Consider the following example. Let n be a positive integer and let $f(x) = 3x^n$ and $g(x) = 3x^2 + 5$. As $x \to \infty$, $f(x) \to \infty$ and $g(x) \to \infty$. We are interested in $\lim_{x \to \infty} (f(x) - g(x))$. Depending on whether $f(x)$ grows faster, $g(x)$ grows faster, or they grow at the same rate, as we see next, anything can happen in this limit. Since $f(x) \to \infty$ and $g(x) \to \infty$, we write $\infty - \infty$ to denote the form of this limit.

Caution: Yet More Notational Abuse

As with our other indeterminate forms, $\infty - \infty$ has no meaning on its own, and we must do more analysis to determine the value of the limit.

For example, suppose the exponent n in the function $f(x) = 3x^n$ is $n = 3$, then

$$\lim_{x \to \infty} (f(x) - g(x)) = \lim_{x \to \infty} (3x^3 - 3x^2 - 5) = \infty.$$

On the other hand, if $n = 2$, then

$$\lim_{x \to \infty} (f(x) - g(x)) = \lim_{x \to \infty} (3x^2 - 3x^2 - 5) = -5.$$

However, if $n = 1$, then

$$\lim_{x \to \infty} (f(x) - g(x)) = \lim_{x \to \infty} (3x - 3x^2 - 5) = -\infty.$$

Therefore, the limit cannot be determined by considering only $\infty - \infty$. Next, we see how to rewrite an expression involving the indeterminate form $\infty - \infty$ as a fraction to apply l'Hôpital's Rule.

Example 3.4.5

Evaluate

$$\lim_{x\to 0^+}\left(\frac{1}{x^2}-\frac{1}{\tan x}\right).$$

Solution available at the end of this section.

Checkpoint 3.4.5

Evaluate $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$.

Answer available at the end of this section.

Another indeterminate form that arises when evaluating limits involves exponents. The expressions 0^0, ∞^0, and 1^∞ are all indeterminate forms. These expressions are meaningless because we cannot evaluate them as we would evaluate an expression involving real numbers. Rather, these expressions represent forms that arise when finding limits. Now, we examine how l'Hôpital's Rule can be used to evaluate limits involving these indeterminate forms.

Since l'Hôpital's Rule applies to quotients, we use the natural logarithm function and its properties to reduce a problem evaluating a limit involving exponents to a related problem involving a limit of a quotient. For example, suppose we want to evaluate $\lim_{x\to a} f(x)^{g(x)}$ and we arrive at the indeterminate form ∞^0. We proceed as follows. Let

$$y=f(x)^{g(x)}.$$

Then,

$$\ln y=\ln(f(x)^{g(x)})=g(x)\ln(f(x)).$$

Therefore,

$$\lim_{x\to a}[\ln(y)]=\lim_{x\to a}[g(x)\ln(f(x))].$$

Since $\lim_{x\to a} f(x)=\infty$, we know that $\lim_{x\to a}\ln(f(x))=\infty$. Therefore, $\lim_{x\to a} g(x)\ln(f(x))$ is of the indeterminate form $0\cdot\infty$, and we can use the techniques discussed earlier to rewrite the expression $g(x)\ln(f(x))$ in a form so that we can apply l'Hôpital's Rule.

The indeterminate forms 0^0 and 1^∞ can be handled similarly.

Example 3.4.6

Evaluate

$$\lim_{x\to\infty} x^{1/x}.$$

Solution available at the end of this section.

Checkpoint 3.4.6

Evaluate

$$\lim_{x\to\infty} x^{1/\ln(x)}.$$

Answer available at the end of this section.

Example 3.4.7

Evaluate

$$\lim_{x\to 0^+} x^{\sin(x)}.$$

Solution available at the end of this section.

Checkpoint 3.4.7

Evaluate $\lim_{x\to 0^+} x^x$.

Answer available at the end of this section.

Growth Rates of Functions

We have shown how to use a function's first and second derivatives to describe the shape of a graph. To graph a function f defined on an unbounded domain, we also need to know the behavior of f as $x \to \pm\infty$.

Suppose the functions f and g both approach infinity as $x \to \infty$. Although the values of both functions become arbitrarily large as the values of x become sufficiently large, sometimes one function grows more quickly than the other.

We say g grows more rapidly than f as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = \infty \quad \text{or, equivalently,} \quad \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

On the other hand, if there exists a constant $M \neq 0$ such that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = M,$$

we say f and g grow at the same rate as $x \to \infty$.

Next, we see how to use l'Hôpital's Rule to compare the growth rates of power, exponential, and logarithmic functions.

Example 3.4.8

For each of the following pairs of functions, use l'Hôpital's Rule to evaluate

$$\lim_{x\to\infty} \frac{f(x)}{g(x)}.$$

a. $f(x) = x^2$ and $g(x) = e^x$
b. $f(x) = \ln(x)$ and $g(x) = x^2$

Solutions available at the end of this section.

Checkpoint 3.4.8

Compare the growth rates of x^{100} and 2^x.

Answer available at the end of this section.

Using the same ideas as in Example $3.4.8a$, it is not difficult to show that e^x grows more rapidly than x^p for any $p > 0$. In Figure

3.4.5 and Table 3.4.3, we compare e^x with x^3 and x^4 as $x \to \infty$.

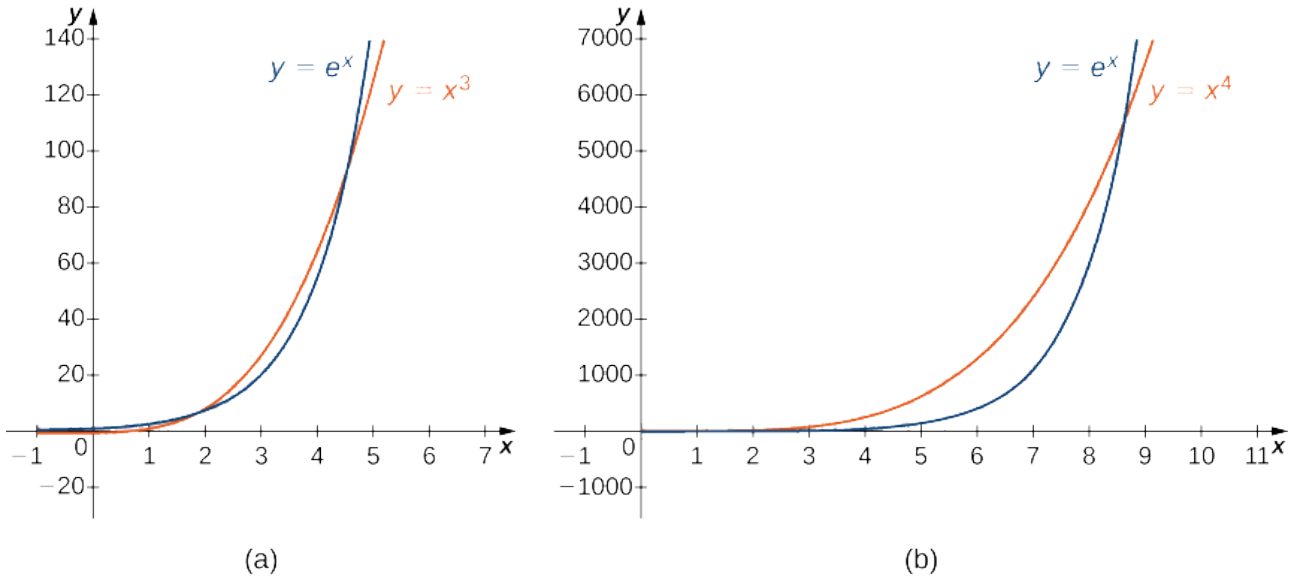

Figure 3.4.5**:** *The exponential function e^x grows faster than x^p for any $p > 0$. (a) A comparison of e^x with x^3. (b) A comparison of e^x with x^4.*

Table 3.4.3**:** *An exponential function grows at a faster rate than any power function.*

x	5	10	15	20
x^3	125	1000	3375	8000
x^4	625	10000	50625	160000
e^x	148	22026	3269017	485165195

Similarly, it is not difficult to show that x^p grows more rapidly than $\ln x$ for any $p > 0$. In Figure 3.4.6 and Table 3.4.4, we compare $\ln x$ with $\sqrt[3]{x}$ and $\sqrt{x}$.

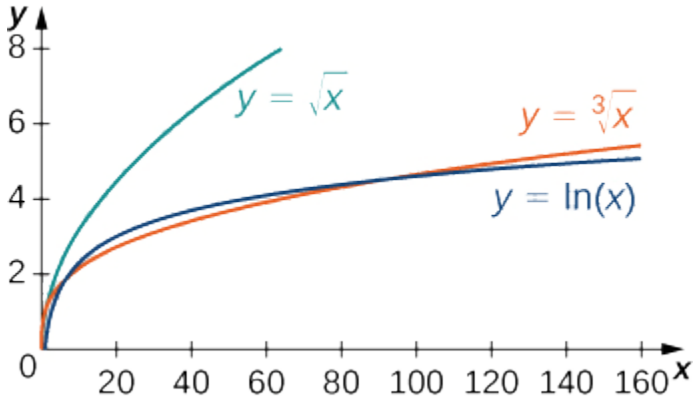

Figure 3.4.6**:** *The function $y = \ln(x)$ grows more slowly than x^p for any $p > 0$ as $x \to \infty$.*

Table 3.4.4**:** *A logarithmic function grows at a slower rate than any root function.*

x	10	100	1000	10000
$\ln(x)$	2.303	4.605	6.908	9.210
$\sqrt[3]{x}$	2.154	4.642	10	21.544
$\sqrt{x}$	3.162	10	31.623	100

Interactive Element: Exponential Growth Outruns Every Power

The following Interactive Element might help you visualize why e^x eventually outgrows x^p for every power p.

Interact: Drag the large point (the top of the hump) left and right to change the power p.

Observation: No matter how large you make p, the ratio $\frac{x^p}{e^x}$ still rises to a hump and then falls back toward 0 — so e^x grows faster than x^p for every $p > 0$.

Success in College: Environment and Learning Preferences

Throughout your academic career, you have completed many exams. While you might not have thought about it, you should be able to identify a strong correlation between the environment you study in and your performance on exams.

Learning preferences describe the situations in which you best learn math. Are you a visual, auditory, or kinesthetic learner? Should you take notes as you listen? Should you record your class? Should you trace the graphs in your book as you study?

Think about whether you learn better by working by yourself or with others. Do you prefer to work in quiet or with music in the background? Do you like to sit at a desk or recline in an easy chair? Do you learn best at a particular time of the day? When you study, does it help if you can move around? All of these are part of your learning preferences.

If you know your learning preferences, you can try to adjust your studying and class behavior to improve your work. It is your responsibility to think about and, if needed, make changes in how you are learning and studying.

Think about the following list. When you study, do you prefer...

- a quiet room, or background noise (music or TV)
- bright light shining on your work, or dim lighting
- sitting at a desk or table, or being relaxed on a couch or a bed
- eating or drinking while you learn, or only eating or drinking during breaks
- working hard for a reasonably long time or taking frequent breaks

1. Do you work in conditions that match these preferences when you do your math homework?
2. Do you work in conditions that match these preferences when you study for math tests?

Success in Mathematics: Sequential Versus Global Learners

> Sequential learners tend to think in a linear manner and are able to function with only partial understanding of material they have been taught. Global learners think in a systems-oriented manner, and may have trouble applying new material until they fully understand it and see how it relates to material they already know about and understand. Once they grasp the big picture, however, their holistic perspective enables them to see innovative solutions to problems that sequential learners might take much longer to reach.
>
> Source: R. Felder, R. Brent , Journal of Engineering Education, 94 (1), 57-72 (2005).

1. When learning math, do you think you are more of a sequential learner or a global learner?
2. Students often have different learning preferences for different subjects. When learning history or another subject in the humanities, do you think you are more a sequential learner or a global learner?

Applications of Mathematics: Game Theory Could Save the World!

The following is from The Telegraph (U.K. newspaper):

A team has been using "game theory" - where mathematics is used to capture how people deal with each other - to show that a classic problem that undermines the ability of individuals to cooperate can be overcome, if people are diverse enough, as is the case when it comes to the 6.5 billion citizens of planet Earth.

Working together for the common good is crucial for progress in any society - not least for effectively addressing big issues such as recycling and tackling climate change. But there is a basic problem with how to make the public share responsibility for common problems, such as climate change.

This was most vividly illustrated by Prof Garrett Hardin, of the University of California, Santa Barbara, in his highly influential 1968 paper 'The tragedy of the commons'.

He used the example of a public pasture. Each herdsman will keep adding cows to a common field, because the benefit of an additional cow goes exclusively to the herdsman. Because the cost of overgrazing is shared by all, the pasture will end up ruined.

Mathematical models have confirmed that this is a profound problem, whether for coming together to fight climate change or pay taxes, but today, in the journal Nature, hope for cooperation is raised by Prof Jorge Pacheco of the University of Lisbon, who did the work there with Ms Marta Santos and Dr Francisco Santos from the Free University of Brussels.

Existing mathematical models treat individuals as equivalent, ignoring real-life diversity and population structure. So the team made the mathematics more realistic in this respect.

The team shows that, contrary to expectations, the temptation to cheat ("Why shouldn't I let others pay more for being green?") declines as society becomes more diverse.

Another discovery is that diversity also plays an important role in wealth distribution.

"Cooperation blooms whenever the act of giving is more important than the amount given," said Prof Pacheco.

The most important factor, he said, seems to be that people actually contribute to the public good.

"In particular, the fact that the wealthy minority contributes stimulates the contribution from the rest, who take them as role models," he said.

Prof Pacheco recalled the words of Anna Dreber and Prof Martin Nowak from Harvard University, leading game theorists, who stated that "preserving the global climate is the biggest public good dilemma ever, the one we cannot afford to lose."

1. The basis of this article, game theory, is used to predict human behavior in many areas. Name one possible application of game theory.
2. It turns out that we, as humans, are reasonably predictable. We do things in patterns, and our behaviors can be modeled using complex functions. Do you believe there is a formula for how many heartbeats you will have (that may depend on how many cigarettes you smoke, steps you take, etc.)?

Answers

Section Preview

This is the typical l'Hôpital's Rule section found in many Differential Calculus texts. As usual, be thorough.

Section Preview

- Recognize when to apply l'Hôpital's Rule.
- Identify indeterminate forms produced by quotients, products, subtractions, and powers, and apply l'Hôpital's Rule in each case.

Theorem: l'Hôpital's Rule ($0/0$ Case)

We provide a proof of this theorem in the special case when f, g, f', and g' are all continuous over an open interval containing a. In that case, since $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$ and f and g are continuous at a, it follows that $f(a) = 0 = g(a)$. Therefore,

$$\begin{aligned}
\lim_{x\to a}\frac{f(x)}{g(x)} &= \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)} && (f(a)=0=g(a))\\
&= \lim_{x\to a}\frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} && \left(\text{multiplying numerator and denominator by } \tfrac{1}{x-a}\right)\\
&= \frac{\lim\limits_{x\to a}\frac{f(x)-f(a)}{x-a}}{\lim\limits_{x\to a}\frac{g(x)-g(a)}{x-a}} && (\text{Limit Laws})\\
&= \frac{f'(a)}{g'(a)} && (\text{definition of the derivative})\\
&= \frac{\lim\limits_{x\to a}f'(x)}{\lim\limits_{x\to a}g'(x)} && (\text{continuity of } f' \text{ and } g')\\
&= \lim_{x\to a}\frac{f'(x)}{g'(x)} && (\text{The limit of a quotient})
\end{aligned}$$

Note that l'Hôpital's Rule states we can calculate the limit of a quotient $\frac{f}{g}$ by considering the limit of the quotient of the derivatives $\frac{f'}{g'}$. It is essential to realize that we are not calculating the derivative of the quotient $\frac{f}{g}$.

Q.E.D.

Example 3.4.1

a. Since the numerator $1-\cos x\to 0$ and the denominator $x\to 0$, we can apply l'Hôpital's Rule to evaluate this limit. We have

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\cos x}{x} &\overset{\text{l'H}}{=} \lim_{x\to 0}\frac{\frac{d}{dx}(1-\cos x)}{\frac{d}{dx}(x)}\\
&= \lim_{x\to 0}\frac{\sin x}{1}\\
&= \frac{\lim\limits_{x\to 0}\sin x}{\lim\limits_{x\to 0}1}\\
&= \frac{0}{1}=0.
\end{aligned}$$

b. As $x\to 1$, the numerator $\sin(\pi x)\to 0$ and the denominator $\ln(x)\to 0$. Therefore, we can apply l'Hôpital's Rule. We obtain

$$\begin{aligned}
\lim_{x\to 1}\frac{\sin(\pi x)}{\ln x} &\overset{\text{l'H}}{=} \lim_{x\to 1}\frac{\pi\cos(\pi x)}{1/x}\\
&= \lim_{x\to 1}(\pi x)\cos(\pi x)\\
&= (\pi\cdot 1)(-1)\\
&= -\pi.
\end{aligned}$$

c. As $x\to\infty$, the numerator $e^{1/x}-1\to 0$ and the denominator $\frac{1}{x}\to 0$. Therefore, we can apply l'Hôpital's Rule. We obtain

$$\begin{aligned}
\lim_{x\to\infty}\frac{e^{1/x}-1}{\frac{1}{x}} &\overset{\text{l'H}}{=} \lim_{x\to\infty}\frac{e^{1/x}\left(\frac{-1}{x^2}\right)}{\left(\frac{-1}{x^2}\right)}\\
&= \lim_{x\to\infty}e^{1/x}\\
&= e^0\\
&= 1.
\end{aligned}$$

d. As $x\to 0$, both the numerator and denominator approach zero. Therefore, we can apply l'Hôpital's Rule. We obtain

$$\lim_{x\to 0}\frac{\sin x-x}{x^2}\overset{\text{l'H}}{=}\lim_{x\to 0}\frac{\cos x-1}{2x}.$$

Since the numerator and denominator of this new quotient both approach zero as $x\to 0$, we apply l'Hôpital's Rule again. In doing so, we see that

$$\lim_{x\to 0}\frac{\cos x-1}{2x}\overset{\text{l'H}}{=}\lim_{x\to 0}\frac{-\sin x}{2}=0.$$

Therefore, we conclude that

$$\lim_{x\to 0}\frac{\sin x-x}{x^2}=0.$$

Checkpoint 3.4.1

1

Example 3.4.2

a. Since $3x+5$ and $2x+1$ are first-degree polynomials with positive leading coefficients, $\lim\limits_{x\to\infty}(3x+5)=\infty$ and $\lim\limits_{x\to\infty}(2x+1)=\infty$. Therefore, we apply l'Hôpital's Rule and obtain

$$\lim_{x\to\infty}\frac{3x+5}{2x+1} \overset{\text{l'H}}{=} \lim_{x\to\infty}\frac{3}{2}$$
$$= \frac{3}{2}$$

Note that this limit can also be calculated without invoking l'Hôpital's Rule. Earlier in the textbook, we showed how to evaluate such a limit by dividing the numerator and denominator by the highest power of x in the denominator. In doing so, we saw that

$$\lim_{x\to\infty}\frac{3x+5}{2x+1} = \lim_{x\to\infty}\frac{3+5/x}{2+1/x} = \frac{3}{2}.$$

l'Hôpital's Rule provides us with an alternative means of evaluating this type of limit.

b. Here, $\lim_{x\to 0^+}\ln x = -\infty$ and $\lim_{x\to 0^+}\cot x = \infty$. Therefore, we can apply l'Hôpital's Rule and obtain

$$\lim_{x\to 0^+}\frac{\ln(x)}{\cot(x)} \overset{\text{l'H}}{=} \lim_{x\to 0^+}\frac{1/x}{-\csc^2(x)}$$
$$= \lim_{x\to 0^+}\frac{1}{-x\csc^2(x)}$$

Now as $x\to 0^+, \csc^2 x\to\infty$. Therefore, the first term in the denominator is approaching zero, and the second term is getting really large. In such a case, anything can happen with the product. Therefore, we cannot make any conclusion yet. To evaluate the limit, we use the definition of $\csc x$ to write

$$\lim_{x\to 0^+}\frac{1}{-x\csc^2 x} = \lim_{x\to 0^+}\frac{\sin^2 x}{-x}.$$

Now $\lim_{x\to 0^+}\sin^2 x = 0$ and $\lim_{x\to 0^+} -x = 0$, so we apply l'Hôpital's Rule again. We find

$$\lim_{x\to 0^+}\frac{\sin^2(x)}{-x} \overset{\text{l'H}}{=} \lim_{x\to 0^+}\frac{2\sin(x)\cos(x)}{-1}$$
$$= \frac{0}{-1}$$
$$= 0.$$

We conclude that

$$\lim_{x\to 0^+}\frac{\ln x}{\cot x} = 0.$$

Checkpoint 3.4.2

0

Example 3.4.3

Because the limits of the numerator and denominator are not zero or infinite, we cannot apply l'Hôpital's Rule. If we try to do so, we get

$$\frac{d}{dx}(x^2+5) = 2x$$

and

$$\frac{d}{dx}(3x+4) = 3.$$

At which point we would conclude erroneously that

$$\lim_{x\to 1}\frac{x^2+5}{3x+4} = \lim_{x\to 1}\frac{2x}{3} = \frac{2}{3}.$$

However, since $\lim_{x\to 1}(x^2+5) = 6$ and $\lim_{x\to 1}(3x+4) = 7$, we actually have

$$\lim_{x\to 1}\frac{x^2+5}{3x+4} = \frac{6}{7}.$$

We can conclude that

$$\lim_{x\to 1}\frac{x^2+5}{3x+4} \neq \lim_{x\to 1}\frac{\frac{d}{dx}(x^2+5)}{\frac{d}{dx}(3x+4)}.$$

Checkpoint 3.4.3

$\lim_{x\to 0^+}\cos x = 1$. Therefore, we cannot apply l'Hôpital's Rule. The limit of the quotient is ∞.

Example 3.4.4

We first note that, as $x\to 0^+$, $\ln(x)\to -\infty$. Therefore, this limit is indeterminate of the form $0\cdot\infty$.

Rewrite the function $x\ln(x)$ as a quotient to apply l'Hôpital's Rule. If we write

$$x\ln(x) = \frac{\ln(x)}{1/x},$$

we see that $\ln(x)\to -\infty$ as $x\to 0^+$ and $\frac{1}{x}\to\infty$ as $x\to 0^+$. Therefore, we can apply l'Hôpital's Rule and obtain

$$\lim_{x\to 0^+} \frac{\ln(x)}{1/x} \overset{\text{l'H}}{=} \lim_{x\to 0^+} \frac{\frac{d}{dx}(\ln(x))}{\frac{d}{dx}(1/x)}$$
$$= \lim_{x\to 0^+} \frac{1/x}{-1/x^2}$$
$$= \lim_{x\to 0^+} (-x)$$
$$= 0.$$

We conclude that

$$\lim_{x\to 0^+} x\ln(x) = 0.$$

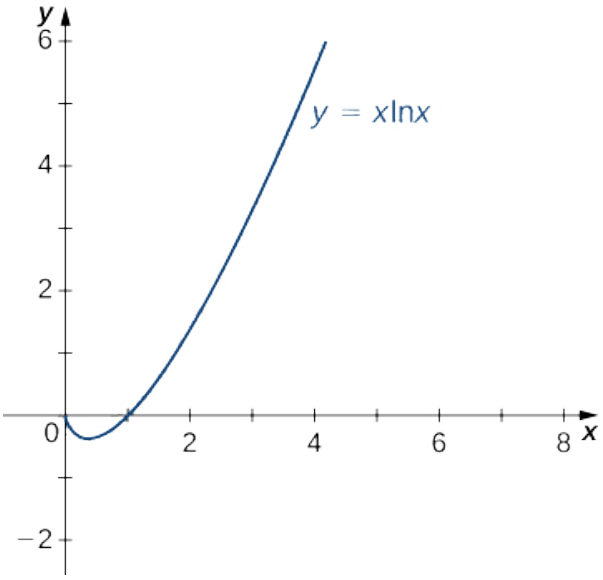

Figure 3.4.2**:** *Finding the limit at $x = 0$ of the function $f(x) = x\ln x$.*

Checkpoint 3.4.4

1

Example 3.4.5

As $x \to 0^+$, $\frac{1}{x^2} \to \infty$ and $\frac{1}{\tan(x)} \to \infty$. Therefore, this limit is of the form $\infty - \infty$.

By combining the fractions, we can write the function as a quotient. Since the least common denominator is $x^2\tan x$, we have

$$\frac{1}{x^2} - \frac{1}{\tan x} = \frac{(\tan x) - x^2}{x^2\tan x}.$$

As $x \to 0^+$, the numerator $\tan x - x^2 \to 0$ and the denominator $x^2\tan x \to 0$. Therefore, we can apply l'Hôpital's Rule. Taking the derivatives of the numerator and the denominator, we have

$$\lim_{x\to 0^+} \frac{(\tan x) - x^2}{x^2\tan x} \overset{\text{l'H}}{=} \lim_{x\to 0^+} \frac{(\sec^2 x) - 2x}{x^2\sec^2 x + 2x\tan x}.$$

As $x \to 0^+$, $(\sec^2 x) - 2x \to 1$ and $x^2\sec^2 x + 2x\tan x \to 0$. Since the denominator is positive as x approaches zero from the right, we conclude that

$$\lim_{x\to 0^+} \frac{(\sec^2 x) - 2x}{x^2\sec^2 x + 2x\tan x} = \infty.$$

Therefore,

$$\lim_{x\to 0^+} \left(\frac{1}{x^2} - \frac{1}{\tan x}\right) = \infty.$$

Checkpoint 3.4.5

∅

Example 3.4.6

As $x \to \infty$, $\frac{1}{x} \to 0$. Therefore, this limit is indeterminate of the form ∞^0.

Let $y = x^{1/x}$. Then,

$$\ln(x^{1/x}) = \frac{1}{x}\ln x = \frac{\ln x}{x}.$$

We need to evaluate $\lim_{x\to\infty} \frac{\ln x}{x}$, which is now of the form $\frac{\infty}{\infty}$. Applying l'Hôpital's Rule, we obtain

$$\begin{aligned}\lim_{x\to\infty}\ln(y) &= \lim_{x\to\infty}\frac{\ln(x)}{x}\\ &\overset{\text{l'H}}{=} \lim_{x\to\infty}\frac{1/x}{1}\\ &= 0.\end{aligned}$$

Therefore, $\lim_{x\to\infty}\ln y = 0$. Since the natural logarithm function is continuous, we conclude that

$$\ln\left(\lim_{x\to\infty} y\right) = 0,$$

which leads to

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} y = e^{\ln(\lim_{x\to\infty} y)} = e^0 = 1.$$

Hence,

$$\lim_{x\to\infty} x^{1/x} = 1.$$

Checkpoint 3.4.6

e

Example 3.4.7

This limit is definitely indeterminate of the form 0^0.

Let

$$y = x^{\sin(x)}.$$

Therefore,

$$\ln(y) = \ln\left(x^{\sin(x)}\right) = \sin(x)\ln(x).$$

We now evaluate $\lim_{x\to 0^+}\sin(x)\ln(x)$. Since $\lim_{x\to 0^+}\sin(x) = 0$ and $\lim_{x\to 0^+}\ln(x) = -\infty$, we have the indeterminate form $0\cdot\infty$. To apply l'Hôpital's Rule, we need to rewrite $\sin(x)\ln(x)$ as a fraction. We could write

$$\sin(x)\ln(x) = \frac{\sin(x)}{1/\ln(x)}$$

or

$$\sin(x)\ln(x) = \frac{\ln(x)}{1/\sin(x)} = \frac{\ln(x)}{\csc(x)}.$$

Let's consider the first option. In this case, applying l'Hôpital's Rule, we would obtain

$$\begin{aligned}\lim_{x\to 0^+}\sin(x)\ln(x) &= \lim_{x\to 0^+}\frac{\sin(x)}{1/\ln(x)}\\ &= \lim_{x\to 0^+}\frac{\cos(x)}{-1/(x(\ln(x))^2)}\\ &= \lim_{x\to 0^+}\left(-x(\ln(x))^2\cos(x)\right).\end{aligned}$$

Unfortunately, we not only have another expression involving the indeterminate form $0\cdot\infty$, but the new limit is even more complicated to evaluate than the one with which we started. Instead, we try the second option. By writing

$$\sin(x)\ln(x) = \frac{\ln(x)}{1/\sin(x)} = \frac{\ln(x)}{\csc(x)},$$

and applying l'Hôpital's Rule, we obtain

$$\begin{aligned}\lim_{x\to 0^+}\sin(x)\ln(x) &= \lim_{x\to 0^+}\frac{\ln(x)}{\csc(x)}\\ &\overset{\text{l'H}}{=} \lim_{x\to 0^+}\frac{1/x}{-\csc(x)\cot(x)}\\ &= \lim_{x\to 0^+}\frac{-1}{x\csc(x)\cot(x)}.\end{aligned}$$

Using the fact that $\csc(x) = \frac{1}{\sin(x)}$ and $\cot(x) = \frac{\cos(x)}{\sin(x)}$, we can rewrite the expression on the right-hand side as

$$\lim_{x\to 0^+}\frac{-\sin^2(x)}{x\cos(x)} = \lim_{x\to 0^+}\left[\frac{\sin(x)}{x}\cdot(-\tan(x))\right] = \left(\lim_{x\to 0^+}\frac{\sin(x)}{x}\right)\cdot\left(\lim_{x\to 0^+}(-\tan(x))\right) = 1\cdot 0 = 0.$$

We conclude that $\lim_{x\to 0^+}\ln(y) = 0$. Therefore, $\ln\left(\lim_{x\to 0^+} y\right) = 0$ and we have

$$\lim_{x\to 0^+} y = \lim_{x\to 0^+} x^{\sin(x)} = e^0 = 1.$$

Hence,

$$\lim_{x\to 0^+} x^{\sin(x)} = 1.$$

Checkpoint 3.4.7

1

Example 3.4.8

a. Since $\lim_{x\to\infty} x^2 = \infty$ and $\lim_{x\to\infty} e^x = \infty$, we can use l'Hôpital's Rule to evaluate $\lim_{x\to\infty}\left[\frac{x^2}{e^x}\right]$. We obtain

$$\lim_{x\to\infty}\frac{x^2}{e^x} \overset{\text{l'H}}{=} \lim_{x\to\infty}\frac{2x}{e^x}.$$

Since $\lim_{x\to\infty} 2x = \infty$ and $\lim_{x\to\infty} e^x = \infty$, we can apply l'Hôpital's Rule again. Since

$$\lim_{x\to\infty}\frac{2x}{e^x} \overset{\text{l'H}}{=} \lim_{x\to\infty}\frac{2}{e^x} = 0,$$

we conclude that

$$\lim_{x\to\infty}\frac{x^2}{e^x} = 0.$$

Therefore, e^x grows more rapidly than x^2 as $x \to \infty$ (See Figure 3.4.3 and Table 3.4.1)

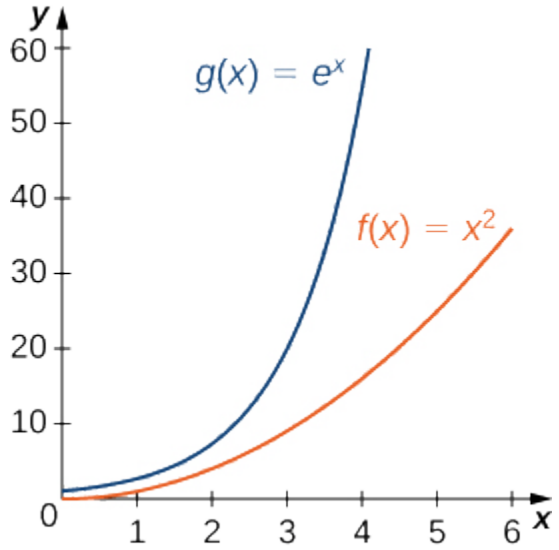

Figure 3.4.3**:** *An exponential function grows at a faster rate than a power function.*

Table 3.4.1***:*** *Growth rates of a power function and an exponential function.*

x	5	10	15	20
x^2	25	100	225	400
e^x	148	22026	3269017	485165195

b. Since $\lim_{x\to\infty} \ln x = \infty$ and $\lim_{x\to\infty} x^2 = \infty$, we can use l'Hôpital's Rule to evaluate $\lim_{x\to\infty}\frac{\ln x}{x^2}$. We obtain

$$\lim_{x\to\infty}\frac{\ln x}{x^2} \overset{\text{l'H}}{=} \lim_{x\to\infty}\frac{1/x}{2x} = \lim_{x\to\infty}\frac{1}{2x^2} = 0.$$

Thus, x^2 grows more rapidly than $\ln x$ as $x \to \infty$ (see Figure 3.4.4 and Table 3.4.2).

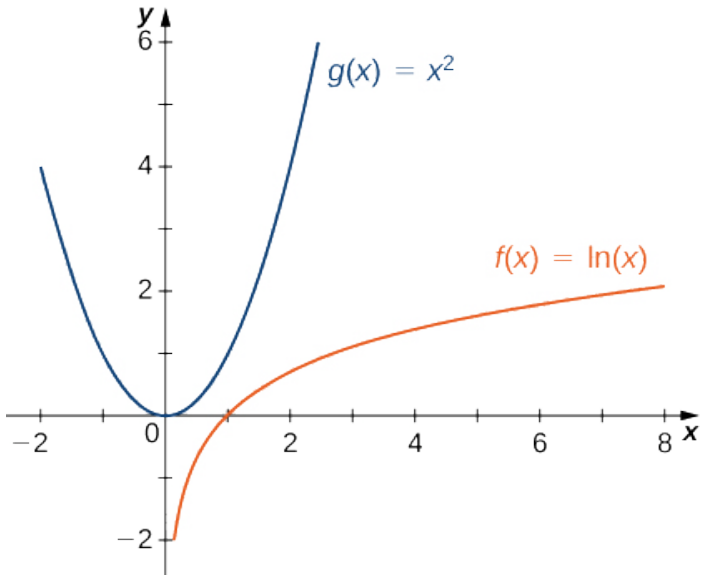

Figure 3.4.4**:** *A power function grows at a faster rate than a logarithmic function.*

Table 3.4.2***:*** *Growth rates of a power function and a logarithmic function.*

x	10	100	1000	10000
$\ln(x)$	2.303	4.605	6.908	9.210
x^2	100	10000	1000000	100000000

Checkpoint 3.4.8

The function 2^x grows faster than x^{100}.

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3.4.1: Resources and Key Concepts

Resources

Videos

- Indeterminate forms
  - Indeterminate Forms
- l'Hôpital's Rule
  - l'Hôpital's Rule: Concept and Theorem
- Indeterminate forms: $0/0$ and ∞/∞
  - Evaluating Limits Using L'Hôpital's Rule: $0/0$ and ∞/∞ Forms
- Indeterminate forms: $0 \cdot \infty$
  - Evaluating Limits Using L'Hôpital's Rule: $0 \cdot \infty$ Form
- Indeterminate forms: 0^0, 1^∞, and ∞^0
  - Evaluating Limits Using L'Hôpital's Rule: 0^0, 1^∞, and ∞^0 Forms
- Indeterminate forms: $\infty - \infty$
  - Evaluating Limits Using L'Hôpital's Rule: $\infty - \infty$ Form

Key Concepts

Definitions

- **Indeterminate Forms**: Expressions arising from limits whose value cannot be determined by direct substitution or simple arithmetic of limits. This section focuses on:
  - $\frac{0}{0}$: An indeterminate form where both the numerator and denominator of a quotient approach zero.
  - $\frac{\infty}{\infty}$: An indeterminate form where both the numerator and denominator of a quotient approach infinity (or negative infinity).
  - $0 \cdot \infty$: An indeterminate form arising from the product of a term approaching zero and a term approaching infinity.
  - $\infty - \infty$: An indeterminate form arising from the difference of two terms both approaching infinity.
  - $0^0, \infty^0, 1^\infty$: Indeterminate forms involving exponents.
- **Growth Rates of Functions**:
  - Function $g(x)$ **grows more rapidly** than $f(x)$ as $x \to \infty$ if $\lim\limits_{x\to\infty} \dfrac{g(x)}{f(x)} = \infty$ or, equivalently, $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = 0$.
  - Functions $f(x)$ and $g(x)$ **grow at the same rate** as $x \to \infty$ if there exists a constant $M \neq 0$ such that $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = M$.

Theorems

- **Theorem: L'Hôpital's Rule ($\frac{0}{0}$ Case)**: Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. If $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = 0$, then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$, assuming the limit on the right exists or is ∞ or $-\infty$. This result also holds for one-sided limits, or if $a = \infty$ or $a = -\infty$.
- **Theorem: L'Hôpital's Rule ($\frac{\infty}{\infty}$ Case)**: Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. Suppose $\lim\limits_{x\to a} f(x) = \infty$ (or $-\infty$) and $\lim\limits_{x\to a} g(x) = \infty$ (or $-\infty$). Then, $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$, assuming the limit on the right exists or is ∞ or $-\infty$. This result also holds if the limit is infinite, if $a = \infty$ or $a = -\infty$, or if the limit is one-sided.

Common Mistakes

- **Abusive Notation ($\frac{0}{0}$ or $\frac{\infty}{\infty}$)**: The notation $\frac{0}{0}$ or $\frac{\infty}{\infty}$ does not mean actual division by zero or division of infinities. It represents

a quotient of limits, where each limit individually evaluates to 0 or ∞.

- **Applying L'Hôpital's Rule Incorrectly**:
  - Applying the rule when the limit is not an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.
  - Differentiating the quotient $\frac{f(x)}{g(x)}$ using the Quotient Rule instead of taking the quotient of the derivatives, $\frac{f'(x)}{g'(x)}$.
- **Not Converting Other Indeterminate Forms**: Forgetting to algebraically manipulate indeterminate forms like $0 \cdot \infty$, $\infty - \infty$, 0^0, ∞^0, or 1^∞ into a $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form before attempting to apply L'Hôpital's Rule.
- **Errors in Algebraic Manipulation**: Mistakes made when rewriting expressions to fit the $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form, or when simplifying after applying L'Hôpital's Rule.
- **Stopping Too Early or Applying Too Many Times**: After applying L'Hôpital's Rule, one must re-evaluate the new limit. If it's still indeterminate and the conditions are met, the rule can be applied again. If it's determinate, the rule should not be applied further.

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3.4.2: Homework

Reading Questions

1. What are the two primary indeterminate forms for which L'Hôpital's Rule can be directly applied?
2. State L'Hôpital's Rule for the $\frac{0}{0}$ case. What conditions must be met by the functions f and g?
3. If $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$, what limit do you evaluate according to L'Hôpital's Rule?
4. Does L'Hôpital's Rule apply if $\lim\limits_{x\to a} f(x)=0$ but $\lim_{x\to a} g(x)=5$? Why or why not?
5. How can the indeterminate form $0\cdot\infty$ be rewritten so that L'Hôpital's Rule might be applicable?
6. How can the indeterminate form $\infty-\infty$ be rewritten so that L'Hôpital's Rule might be applicable?
7. What technique is used to handle indeterminate forms involving exponents, such as 0^0, ∞^0, or 1^∞, before L'Hôpital's Rule can be applied?
8. If $\lim\limits_{x\to a}\ln(y)=L$, what is $\lim\limits_{x\to a} y$?
9. When comparing the growth rates of two functions $f(x)$ and $g(x)$ that both approach ∞ as $x\to\infty$, what does it mean if $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)}=0$? Which function grows more rapidly?
10. Based on the examples, which type of function generally grows faster as $x\to\infty$: an exponential function (e^x), a power function ($x^p, p>0$), or a logarithmic function ($\ln x$)?

Homework

In exercises 1 - 6, evaluate the limit.

1) Evaluate the limit $\lim\limits_{x\to\infty}\dfrac{e^x}{x}$.

2) Evaluate the limit $\lim\limits_{x\to\infty}\dfrac{e^x}{x^k}$.

Answer

$$\lim_{x\to\infty}\frac{e^x}{x^k} \quad = \quad \infty$$

3) Evaluate the limit $\lim\limits_{x\to\infty}\dfrac{\ln x}{x^k}$.

4) Evaluate the limit $\lim\limits_{x\to a}\dfrac{x-a}{x^2-a^2}$.

Answer

$$\lim_{x\to a}\frac{x-a}{x^2-a^2} \quad = \quad \frac{1}{2a}$$

5. Evaluate the limit $\lim\limits_{x\to a}\dfrac{x-a}{x^3-a^3}$.

6. Evaluate the limit $\lim\limits_{x\to a}\dfrac{x-a}{x^n-a^n}$.

Answer

$$\lim_{x\to a}\frac{x-a}{x^n-a^n} \quad = \quad \frac{1}{na^{n-1}}$$

In exercises 7 - 11, determine whether you can apply l'Hôpital's Rule directly. Explain why or why not. Then, indicate if there is some way you can alter the limit so you can apply l'Hôpital's Rule.

7) $\lim_{x\to 0^+} x^2 \ln x$

8) $\lim_{x\to\infty} x^{1/x}$

Answer

Cannot apply directly; use logarithms

9) $\lim_{x\to 0} x^{2/x}$

10) $\lim_{x\to 0} \dfrac{x^2}{1/x}$

Answer

Cannot apply directly; rewrite as $\lim_{x\to 0} x^3$

11) $\lim_{x\to\infty} \dfrac{e^x}{x}$

In exercises 12 - 40, evaluate the limits with either l'Hôpital's Rule or previously learned methods.

12) $\lim_{x\to 3} \dfrac{x^2-9}{x-3}$

Answer

$$\lim_{x\to 3} \frac{x^2-9}{x-3} \quad = \quad 6$$

13) $\lim_{x\to 3} \dfrac{x^2-9}{x+3}$

14) $\lim_{x\to 0} \dfrac{(1+x)^{-2}-1}{x}$

Answer

$$\lim_{x\to 0} \frac{(1+x)^{-2}-1}{x} \quad = \quad -2$$

15) $\lim_{x\to \pi/2} \dfrac{\cos x}{\frac{\pi}{2}-x}$

16) $\lim_{x\to \pi} \dfrac{x-\pi}{\sin x}$

Answer

$$\lim_{x\to \pi} \frac{x-\pi}{\sin x} \quad = \quad -1$$

17) $\lim_{x\to 1} \dfrac{x-1}{\sin x}$

18) $\lim_{x\to 0} \dfrac{(1+x)^n-1}{x}$

Answer

$$\lim_{x\to 0} \frac{(1+x)^n-1}{x} \quad = \quad n$$

19) $\lim_{x\to 0} \dfrac{(1+x)^n-1-nx}{x^2}$

20) $\lim_{x\to 0} \dfrac{\sin x-\tan x}{x^3}$

Answer

$$\lim_{x\to 0}\frac{\sin x-\tan x}{x^3} \quad = \quad -\frac{1}{2}$$

21) $\lim_{x\to 0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$

22) $\lim_{x\to 0}\dfrac{e^x-x-1}{x^2}$

Answer

$$\lim_{x\to 0}\frac{e^x-x-1}{x^2} \quad = \quad \frac{1}{2}$$

23) $\lim_{x\to 0}\dfrac{\tan x}{\sqrt{x}}$

24) $\lim_{x\to 1}\dfrac{x-1}{\ln x}$

Answer

$$\lim_{x\to 1}\frac{x-1}{\ln x} \quad = \quad 1$$

25) $\lim_{x\to 0}\ (x+1)^{1/x}$

26) $\lim_{x\to 1}\dfrac{\sqrt{x}-\sqrt[3]{x}}{x-1}$

Answer

$$\lim_{x\to 1}\frac{\sqrt{x}-\sqrt[3]{x}}{x-1} \quad = \quad \frac{1}{6}$$

27) $\lim_{x\to 0^+} x^{2x}$

28) $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$

Answer

$$\lim_{x\to\infty} x\sin\left(\tfrac{1}{x}\right) \quad = \quad 1$$

29) $\lim_{x\to 0}\dfrac{\sin x-x}{x^2}$

30) $\lim_{x\to 0^+} x\ln\left(x^4\right)$

Answer

$$\lim_{x\to 0^+} x\ln\left(x^4\right) \quad = \quad 0$$

31) $\lim_{x\to\infty}(x-e^x)$

32) $\lim_{x\to\infty} x^2e^{-x}$

Answer

$$\lim_{x\to\infty} x^2e^{-x} \quad = \quad 0$$

33) $\lim_{x\to 0}\dfrac{3^x-2^x}{x}$

34) $\lim_{x\to 0}\dfrac{1+1/x}{1-1/x}$

Answer

$$\lim_{x\to 0}\frac{1+1/x}{1-1/x} \quad = \quad -1$$

35) $\lim_{x\to\pi/4}(1-\tan x)\cot x$

36) $\lim_{x\to\infty} xe^{1/x}$

Answer

$$\lim_{x\to\infty} xe^{1/x} \quad = \quad \infty$$

37) $\lim_{x\to 0} x^{1/\cos x}$

38) $\lim_{x\to 0^+} x^{1/x}$

Answer

$$\lim_{x\to 0^+} x^{1/x} \quad = \quad 0$$

39) $\lim_{x\to 0}\left(1-\frac{1}{x}\right)^x$

40) $\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x$

Answer

$$\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x \quad = \quad \frac{1}{e}$$

For exercises 41 - 50, use a calculator to graph the function and estimate the value of the limit, then use l'Hôpital's Rule to find the limit directly.

41) [Technology Required] $\lim_{x\to 0}\frac{e^x-1}{x}$

42) [Technology Required] $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$

Answer

$$\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) \quad = \quad 0$$

43) [Technology Required] $\lim_{x\to 1}\frac{x-1}{1-\cos(\pi x)}$

44) [Technology Required] $\lim_{x\to 1}\frac{e^{x-1}-1}{x-1}$

Answer

$$\lim_{x\to 1}\frac{e^{x-1}-1}{x-1} \quad = \quad 1$$

45) [Technology Required] $\lim_{x\to 1}\frac{(x-1)^2}{\ln x}$

46) [Technology Required] $\lim_{x\to\pi}\frac{1+\cos x}{\sin x}$

Answer

$$\lim_{x\to\pi}\frac{1+\cos x}{\sin x} \quad = \quad 0$$

47) [Technology Required] $\lim_{x\to 0}\left(\csc x-\frac{1}{x}\right)$

48) [Technology Required] $\lim_{x \to 0^+} \tan\left(x^x\right)$

Answer

$$\lim_{x \to 0^+} \tan\left(x^x\right) \quad = \quad \tan 1$$

49) [Technology Required] $\lim_{x \to 0^+} \dfrac{\ln x}{\sin x}$

50) [Technology Required] $\lim_{x \to 0} \dfrac{e^x - e^{-x}}{x}$

Answer

$$\lim_{x \to 0} \frac{e^x - e^{-x}}{x} \quad = \quad 2$$

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3.5: Graphing Using Calculus

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

We have gathered *almost* everything we need to graph a given function using Calculus accurately. In this section, we finalize the topic by focusing on the end behavior of the graph of a function (specifically, whether or not the function has an oblique asymptote). Once that topic is finished, we present a tactic for graphing a function using Calculus.

End Behavior

The behavior of a function as $x \to \pm\infty$ is called the function's **end behavior**, and finding the end behavior of a function is the precise reason we introduced ourselves to l'Hôpital's Rule. If f is a function of x, then as $x \to \infty$ or $x \to -\infty$, the function could exhibit one of the following types of behavior:

1. **No End Behavior:** A given function may have a restricted domain. For example, the function might be defined on the closed interval $[-1, 10]$. The function does not have an end behavior in these cases because it is not defined as $x \to \pm\infty$.
2. **Horizontal Asymptote(s):** If
$$\lim_{x\to\pm\infty} f(x) = L,$$
where L is finite, the function approaches a horizontal asymptote $y = L$ as $x \to \infty$ or as $x \to -\infty$ (or both).
3. **Slant Asymptote(s):** If
$$\lim_{x\to\pm\infty} f(x) = \pm\infty,$$
then there is a little more work we need to do to determine if the function is mimicking a linear or nonlinear function near the edges of the domain.
4. **Oscillatory Behavior:** If
$$\lim_{x\to\pm\infty} f(x) \text{ DNE and is not } \pm\infty,$$
then the function does not have any asymptotic end behavior; however, it could be the case that the function is periodic and is following an oscillatory pattern as its end behavior.

Notes About the End Behavior of a Function

- Each of the limits mentioned in the list above could have different "asymptote types" at each end of the function. That is, it's possible that $\lim_{x\to-\infty} f(x) = 5$ and $\lim_{x\to\infty} (f(x) = -\infty$ (or a constant not equal to 5). Do not assume that end behavior is symmetric.

- The list presented above is purposeful in its order.
  - When determining the end behavior of a given function, you immediately throw out the possibility for end behavior if your function has a restricted domain.
  - If the domain is unrestricted, you take limits as $x \to \pm\infty$.
    - If a limit is finite, then your function has a horizontal asymptote.
    - If the limit is infinite, then you have some extra work determining if there is an underlying linear or nonlinear function the original function is approaching as $x \to \pm\infty$.
    - If the limit DNE and is not infinite, then your function might have oscillatory behavior and, if true, you will need to find the period of the oscillation.

Since we have already covered the topic of horizontal asymptotes, the remainder of this subsection is dedicated to the discussion of **slant asymptotes**.

Definition: Slant Asymptote

The line $y = mx + b$ is called a **slant asymptote** of the function f if

$$\lim_{x\to\infty} (f(x) - y) = 0$$

or

$$\lim_{x\to-\infty} (f(x) - y) = 0.$$

Example 3.5.1

Determine the slant asymptote of $f(x) = 3x - 1 + e^{-2x}$

Solution available at the end of this section.

It is important to mention here that the function in Example 3.5.1 does not have a slant asymptote as $x \to -\infty$. This is because, as $x \to -\infty$, there is not a term in $f(x)$ that "disappears" to leave us with only a linear function.

Interactive Element: End Behavior Need Not Be Symmetric

The following Interactive Element might help you visualize how a function can have a slant asymptote at one end but not the other.

Interact: Drag the point along the curve and watch the colored gap between the curve and the line $y = 3x - 1$.

Observation: Drag to the right and the gap vanishes — $y = 3x - 1$ is a slant asymptote as $x \to \infty$. Drag to the left and the gap grows without bound — there is no slant asymptote as $x \to -\infty$.

Guidelines for Drawing the Graph of a Function

We now have enough analytical tools to draw graphs of various algebraic and transcendental functions. Before showing how to graph specific functions, let's look at a general strategy when graphing any function. Notice in the steps provided that I follow the Mathematical Mantra and step through the graphing process doing mathematics in the order we learned topics (except the domain, which I often do first despite having learned that skill *after* learning how to find intercepts).

Tactic: Drawing the Graph of a Function

Given a function f, use the following steps to sketch its graph by hand:

1. **Domain:** Determine the domain of the function.
2. **Intercepts:** Locate the x- and y-intercepts. Create a table of values to hold all the data you will collect.
3. **Symmetry:** Determine if f is odd, even, or neither. If it's neither odd nor even, determine if it is periodic. This step is *incredibly* useful because once you know a function is, for example, odd, you can focus your energy on gathering all the information about the function for $x \geq 0$. After you precisely graph the function for these x-values, you can use the symmetry of the function to graph the other half.
4. **Vertical Asymptotes:** Use Calculus to determine the behavior of the function near any vertical asymptote(s). This step will involve limits of the form
$$\lim_{x \to a^+} f(x) \text{ and } \lim_{x \to a^-} f(x).$$
5. **End Behavior:** Use Calculus to determine the end behavior of the function. This step will involve limits of the form
$$\lim_{x \to \infty} f(x) \text{ and } \lim_{x \to -\infty} f(x).$$
If either of these limits is a finite number L, then $y = L$ is a horizontal asymptote. If either of these limits is ∞ or $-\infty$, determine whether f has a slant asymptote.
6. **First Derivative Information:** Calculate f'. Find all critical points and determine the intervals where f is increasing and where f is decreasing. Determine whether f has any local extrema.
7. **Second Derivative Information:** Calculate f''. Determine the intervals where f is concave up and where f is concave down. Use this information to determine whether f has any inflection points. The second derivative can also be used as an alternate means to determine or verify that f has a local extremum at a critical point.
8. **Graph like a boss**

Graphing Examples

Now, let's use this strategy to graph several different functions. We start by graphing a polynomial function.

Example 3.5.2

Sketch a graph of $R(x) = \frac{x^2}{x^2+1}$.

Solution available at the end of this section.

Notice in Example 3.5.2 that we approximated the value of $\frac{1}{\sqrt{3}}$. When graphing by hand, it is sometimes necessary to use technology to get decimal approximations to place our points properly.

Example 3.5.3

Sketch the graph of $f(x) = x\sqrt{1-x^2}$.

Solution available at the end of this section.

Note: Tricks to Knowing When a Function Does Not Have a Vertical Asymptote

Both functions in Examples 3.5.2 and 3.5.3 were continuous on their domains. Moreover, the function's domain in Example 3.5.3 was a closed interval. When a function is continuous and has a domain of either $\mathbb{R}$ or a closed interval, you can guarantee that the function does not have any vertical asymptotes. This is because of the definition of a vertical asymptote from Chapter 2. $x = a$ is a vertical asymptote of f if any of the limits (single-sided or total) as $x \to a$ becomes infinite. For a continuous function, this condition can only occur at the edge of its domain and only if the domain doesn't include that edge.

Example 3.5.4

Sketch the graph of $f(x) = \frac{x^2}{x-1}$

Solution available at the end of this section.

Interactive Element: The Vanishing Gap

The following Interactive Element might help you visualize what it means for a line to be a slant asymptote of a curve.

Interact: Drag the point along the curve. The colored segment measures the vertical gap between the curve and the line $y = x + 1$.

Observation: As you drag the point far from $x = 1$ in either direction, the gap shrinks toward 0 — the curve approaches the line $y = x + 1$. Near $x = 1$, the curve races away along its vertical asymptote.

Checkpoint 3.5.4

Find the oblique asymptote for $f(x) = \frac{3x^3-2x+1}{2x^2-4}$.

Answer available at the end of this section.

Example 3.5.5

Sketch a graph of $f(x) = (x-1)^{2/3}$

Solution available at the end of this section.

Checkpoint 3.5.5

Consider the function $f(x) = 5 - x^{2/3}$. Determine the point on the graph where a cusp is located. Determine the end behavior of f.

Answer available at the end of this section.

Success in College: Learning From Tests (Part 4)

Tests and quizzes have two purposes. First, they assess your performance. Your test scores are part of a professor's information to assign you a grade for the course. Tests also allow you to see what you do not know. It is your responsibility to use tests to improve your skills and knowledge.

Often, a passing score on a test is 70%. However, the missing 30% is still critically important. When preparing for the next test, you must study your mistakes from the last test. Perhaps you will realize that you still do not understand an important concept. This concept will not disappear. Now is the time to get extra help. When it reappears, you will be ready. Perhaps you will realize that you are making many careless mistakes. Now is the time to plan strategies for checking your work on the next test. Perhaps you will realize that you made most of your mistakes on the last few questions of the exam. Now is the time to ponder how to pace yourself as you work on a test.

Learning from tests takes self-discipline. It is tempting to throw the test away or hide it in a folder. It is your responsibility to review your tests critically. Your ability to do well in the future depends on it.

1. Describe what you usually do with a returned math test.
2. Does what you do with a returned math test depend on whether you earned a good or poor grade on the test? Explain.

Miscellaneous Topics: What Are Your Top Degree Choices?

I have seen all the statistics regarding the correlation between income and education. In general, the higher your education, the higher your income.

1. Name three benefits, besides money, that a higher degree can give you.
2. What are your top three choices for degrees (e.g., English, Mathematics, Psychology)?

Answers

Section Preview

This is the masterclass of graphing using Calculus. The only thing that students have *not* been able to do up to this point is to truly investigate end behavior (via l'Hôpital's Rule). This allows us to introduce oblique asymptotes in an official context. I have made the mistake in the past of spending time on non-linear oblique asymptotes and can clearly state that, while interesting, it is a bit far afield for what we need. If you desire such instruction in your classroom, I recommend having it as an exploration, but not as an expectation.

Section Preview

- **Graphing Functions**
  - Symmetry: The even/odd classification in Examples 2, 3, and 5 is the conceptual tool that lets a student investigate only half the graph and then reflect; understanding what even and odd symmetry mean is needed to justify that shortcut.

Section Preview

- **Simplifying Expressions**
  - Dividing Polynomials — Long Division: Example 4 and Checkpoint 4 find the slant asymptote by polynomial long division ($\frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}$); the section treats the division as an assumed skill ("use long division of polynomials to write...").
- **Solving Inequalities**
  - Solving Quadratic Inequalities: Example 3's domain is obtained by solving $1 - x^2 \geq 0$ to get $x \in [-1, 1]$.

Section Preview

- Estimate the end behavior of a function as x increases or decreases without bound.
- Prove the equation of the slant asymptote to a function.
- Recognize an oblique asymptote on the graph of a function.
- Use the Guidelines to Graphing Using Calculus to analyze a function and its derivatives to draw its graph accurately.

Example 3.5.1

As $x \to \infty$, $e^{-2x} \to 0$ leaving us with $3x - 1$. Thus, our claim is that $y = 3x - 1$ is a slant asymptote for the function f as $x \to \infty$. To prove this statement, we use the definition.

$$\begin{aligned} \lim_{x\to\infty} [f(x) - y] &= \lim_{x\to\infty} \left(3x - 1 + e^{-2x} - (3x - 1)\right) \\ &= \lim_{x\to\infty} e^{-2x} \\ &= 0 \end{aligned}$$

Thus, we have proven that $y = 3x - 1$ is a slant asymptote of $f(x) = 3x - 1 + e^{-2x}$ as $x \to \infty$.

Example 3.5.2

Domain: Since $R(x)$ is a rational function and its denominator, $x^2 + 1$, is always greater than or equal to 1, the domain is $\mathbb{R} = (-\infty, \infty)$.

Intercepts: $(0, 0)$ is the only intercept.

Symmetry: $R(-x) = \frac{(-x)^2}{(-x)^2+1} = \frac{x^2}{x^2+1} = R(x)$. Therefore, R is an even function. As such, we will limit the rest of our investigation to the interval $[0, \infty)$ and reflect our graph across the y-axis when we are finished.

Vertical Asymptote(s): Since the domain of R is all real numbers and R is continuous, there is not going to be a vertical asymptote.

End Behavior:

$$\begin{aligned} \lim_{x\to\infty} R(x) &= \lim_{x\to\infty} \frac{x^2}{x^2 + 1} \\ &= \lim_{x\to\infty} \frac{1}{1 + \frac{1}{x^2}} \\ &= \frac{1}{1 + 0} \\ &= 1 \end{aligned}$$

Thus, R has a horizontal asymptote of $y = 1$ as $x \to \infty$.

First Derivative Information:

$$\begin{aligned} R'(x) &= \frac{2x(x^2+1)-x^2(2x)}{(x^2+1)^2} \\ &= \frac{2x(x^2+1-x^2)}{(x^2+1)^2} \\ &= \frac{2x}{(x^2+1)^2} \end{aligned}$$

$R'(x)=0$, when $x=0$. Otherwise, $R'(x)$ always exists. Thus, the only critical point of R is $(0,0)$.

For $x>0$, $R'(x)>0$. Therefore, R is increasing on the interval $(0,\infty)$. Since R is an even function, R will be decreasing for $x<0$. This means that $(0,0)$ is a local minimum. All of the information can be seen on the number line for R' in Figure $3.5.1A$ below.

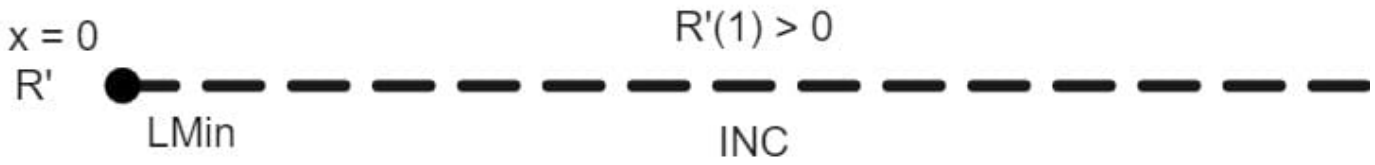

Figure $3.5.1A$**:** *The number line sign graph for* R'.

Second Derivative Information:

$$\begin{aligned} R''(x) &= \frac{2(x^2+1)^2-2x\cdot 2(x^2+1)\cdot 2x)}{(x^2+1)^4} \\ &= \frac{(x^2+1)\left(2(x^2+1)-8x^2\right)}{(x^2+1)^4} \\ &= \frac{-6x^2+2}{(x^2+1)^3} \\ &= \frac{-2(3x^2-1)}{(x^2+1)^3} \end{aligned}$$

$R''(x)=0$, when $x=\pm\frac{1}{\sqrt{3}}$. Otherwise, $R''(x)$ always exists.

All of the information for R'' can be seen on the number line for R'' in Figure $3.5.1B$ below (again, we only concern ourselves with x-values greater than 0 because R is even).

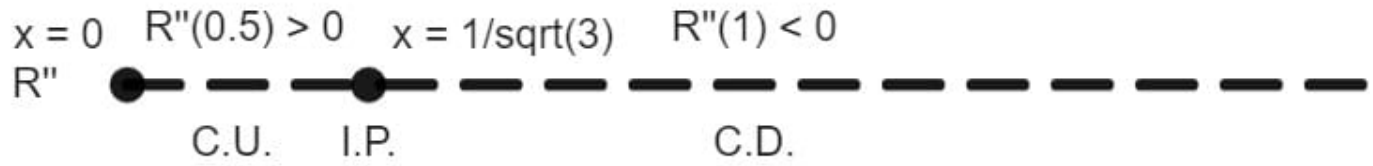

Figure $3.5.1B$**:** *The number line sign graph for* R''.

Graph Like a Boss: Before graphing, it's best to collect all the important information into one clean "pile." It's also **required** that you compute the function values for R wherever there are local extrema, inflection points, and intercepts.

x	$R(x)$	Note(s)
0	0	x- and y-intercept, LMin
$\frac{1}{\sqrt{3}}\approx 0.577$	$\frac{1}{4}$	I.P.

Symmetry: Even

Horizontal Asymptote: $y=1$ as $x\to\infty$

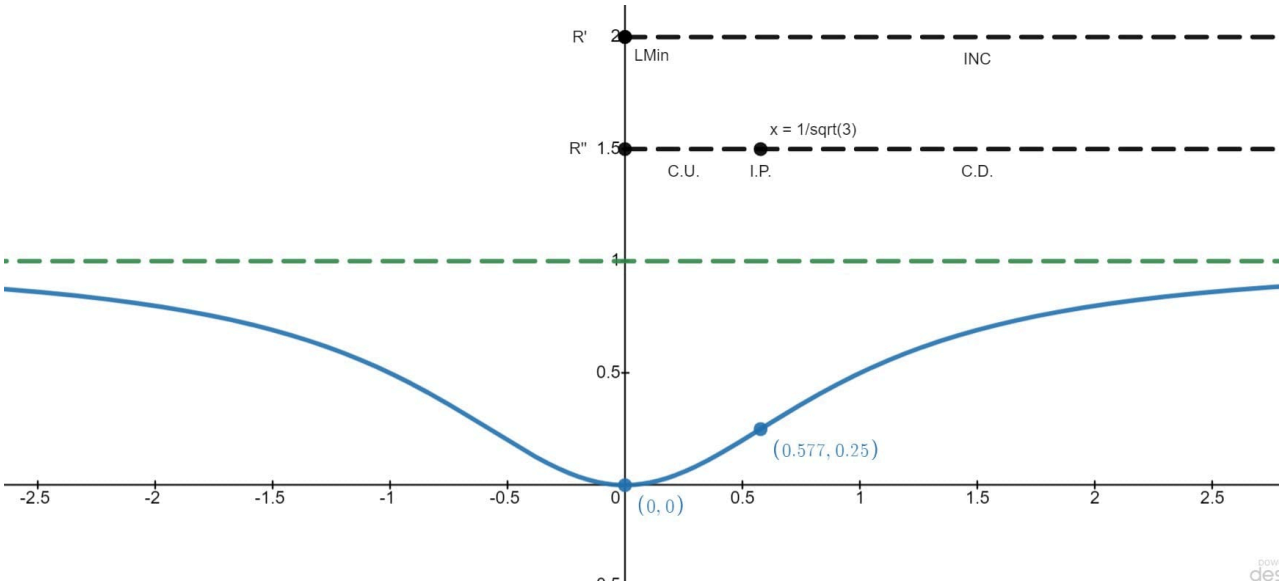

Figure 3.5.2

Example 3.5.3

Domain: We require $1 - x^2 \geq 0 \implies 1 \geq x^2 \implies x \in [-1, 1]$.

Intercepts: $(0, 0)$ and $(\pm 1, 0)$ are the intercepts.

Symmetry: $f(-x) = -x\sqrt{1-(-x)^2} = -x\sqrt{1-x^2} = -f(x)$. Therefore, f is an odd function. As such, we will limit the rest of our investigation to the interval $[0, 1]$ and reflect our graph about the origin when we are finished.

Vertical Asymptote(s): Since the domain of f is a closed interval and f is continuous, there is not going to be a vertical asymptote.

End Behavior: Since the domain of f does not extend to infinity (in either direction), there is no end behavior to speak of.

First Derivative Information:

$$\begin{aligned} f'(x) &= \sqrt{1-x^2} + x \cdot \frac{-2x}{2\sqrt{1-x^2}} \\ &= \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} \\ &= \frac{1-x^2-x^2}{\sqrt{1-x^2}} \\ &= \frac{1-2x^2}{\sqrt{1-x^2}} \end{aligned}$$

$f'(x) = 0$, when $x = \pm\frac{1}{\sqrt{2}}$, and $f'(x)$ does not exist at $x = \pm 1$. Since f is odd, we will only worry about $x = \frac{1}{\sqrt{2}}$ and $x = 1$. The corresponding critical points are $(1, 0)$ and $\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$. We first draw the number line.

Figure 3.5.3

Testing each interval (and only substituting these values into the numerator of f' because the denominator is always positive), we get $f'\left(\frac{1}{2}\right) > 0$ and $f'\left(\frac{3}{4}\right) < 0$. Therefore, f is increasing on $\left(0, \frac{1}{\sqrt{2}}\right)$ and decreasing on $\left(\frac{1}{\sqrt{2}}, 0\right)$. Hence, f has a local maximum at the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$. All of this information can be seen on the number line for f' in Figure 3.5.4A below.

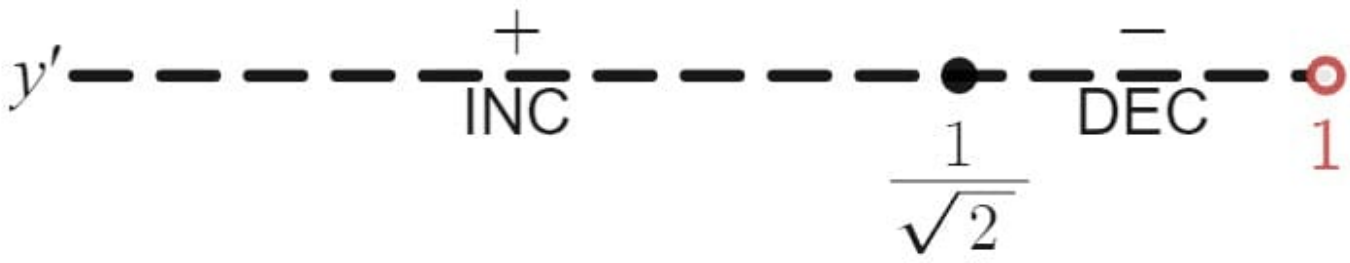

Figure 3.5.4*A*: *The number line sign graph for f'.*

Second Derivative Information:

$$\begin{aligned} f''(x) &= \frac{-4x\sqrt{1-x^2}-(1-2x^2)\cdot\frac{-x}{\sqrt{1-x^2}}}{1-x^2} \\ &= \frac{-4x\left(1-x^2\right)+x\left(1-2x^2\right)}{\left(1-x^2\right)^{3/2}} \\ &= \frac{x\left(2x^2-3\right)}{\left(1-x^2\right)^{3/2}} \end{aligned}$$

$f''(x)=0$, when $x=\pm\frac{\sqrt{6}}{2}$ and when $x=0$. Moreover, f'' does not exist when $x=\pm 1$. Again, since f is odd, we will only concern ourselves with $x=0$, $x=\frac{\sqrt{6}}{2}$, and $x=1$; **however**, $\frac{\sqrt{6}}{2}>1$ and, as such, is outside of the domain of f. Hence, we only need to worry about $x=0$ and $x=1$. Since $f''\left(\frac{1}{2}\right)<0$, f is concave down on $(0,1)$ (see Figure 3.5.4*B*).

Figure 3.5.4*B*: *The number line sign graph for f''.*

Graph Like a Boss:

x	$R(x)$	Note(s)
0	0	x- and y-intercept, I.P.
$\frac{1}{\sqrt{2}}\approx 0.707$	$\frac{1}{2}$	L.Max
1	0	x-intercept

Symmetry: Odd

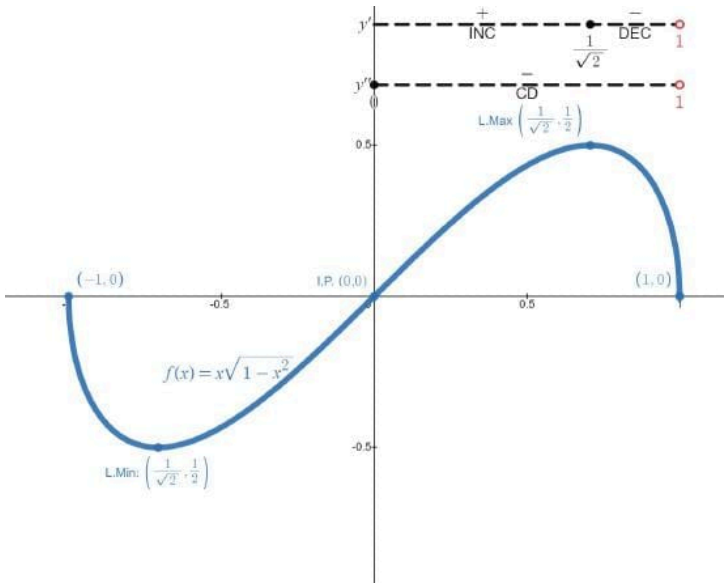

Figure 3.5.5

Example 3.5.4

Domain: The domain of f is the set of all real numbers x except $x = 1$.

Intercepts: We can see that when $x = 0,\ \ f(x) = 0$, so $(0, 0)$ is the only intercept.

Symmetry: Since $f(-x) = \frac{(-x)^2}{(-x)-1} = \frac{x^2}{-x-1} = -\frac{x^2}{x+1} \neq \pm f(x)$, f is neither odd nor even.

Vertical Asymptote(s): To check for vertical asymptotes, look at where the denominator is zero. Here, the denominator is zero at $x = 1$. Looking at both one-sided limits as $x \to 1$, we find

$$\lim_{x \to 1^+} \frac{x^2}{x-1} = \infty$$

and

$$\lim_{x \to 1^-} \frac{x^2}{x-1} = -\infty.$$

Therefore, $x = 1$ is a vertical asymptote, and we have determined the behavior of f as x approaches 1 from the right and the left.

End Behavior: Evaluate the limits at infinity. Since the degree of the numerator is one more than the degree of the denominator, f must have a slant asymptote. To find the slant asymptote, use long division of polynomials to write

$$f(x) = \frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}.$$

Since $\frac{1}{x-1} \to 0$ as $x \to \pm\infty$, $f(x)$ approaches the line $y = x + 1$ as $x \to \pm\infty$. The line $y = x + 1$ is a slant asymptote for f.

First Derivative Information: Calculate the first derivative:

$$f'(x) = \frac{(x-1)(2x) - x^2(1)}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2}.$$

We have $f'(x) = 0$ when $x^2 - 2x = x(x-2) = 0$. Therefore, $x = 0$ and $x = 2$ are critical points. Since f is undefined at $x = 1$, we need to divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 0), (0, 1), (1, 2)$, and $(2, \infty)$, and choose a test point from each interval to evaluate the sign of $f'(x)$ in each of these smaller intervals. Doing so, we get the following sign chart for f'.

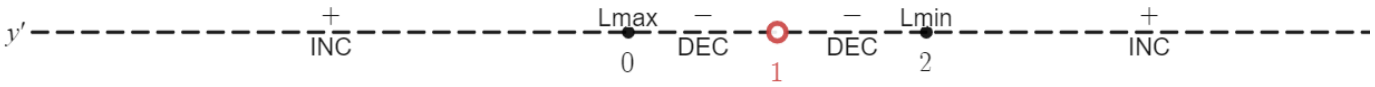

Figure 3.5.6

From this sign chart, we see that f has a local maximum at $x = 0$ and a local minimum at $x = 2$. The value of f at the local maximum is $f(0) = 0$ and the value of f at the local minimum is $f(2) = 4$. Therefore, $(0, 0)$ and $(2, 4)$ are important points on the graph.

Second Derivative Information: Calculate the second derivative:

$$\begin{aligned} f''(x) &= \frac{(x-1)^2(2x-2) - 2(x-1)(x^2-2x)}{(x-1)^4} \\ &= \frac{2(x-1)[(x-1)^2 - (x^2-2x)]}{(x-1)^4} \\ &= \frac{2[x^2 - 2x + 1 - x^2 + 2x]}{(x-1)^3} \\ &= \frac{2}{(x-1)^3}. \end{aligned}$$

We see that $f''(x)$ is never zero or undefined for x in the domain of f. Since f is undefined at $x = 1$, to check concavity we just divide the interval $(-\infty, \infty)$ into the two smaller intervals $(-\infty, 1)$ and $(1, \infty)$, and choose a test point from each interval to evaluate the sign of $f''(x)$ in each of these intervals.

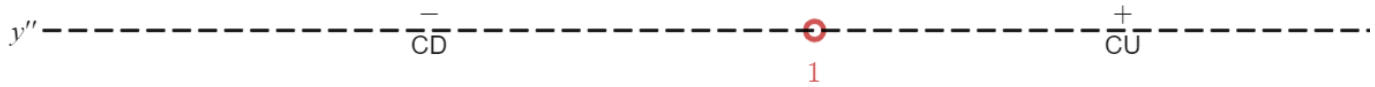

Figure 3.5.7

Graph Like a Boss: When graphing this function by hand, it's best first to plot all the key points ($(0, 0)$ and $(2, 4)$) as well as the slant asymptote. Once that is done, you can get a good idea of the scale of things. We arrive at the following graph for f from the information gathered.

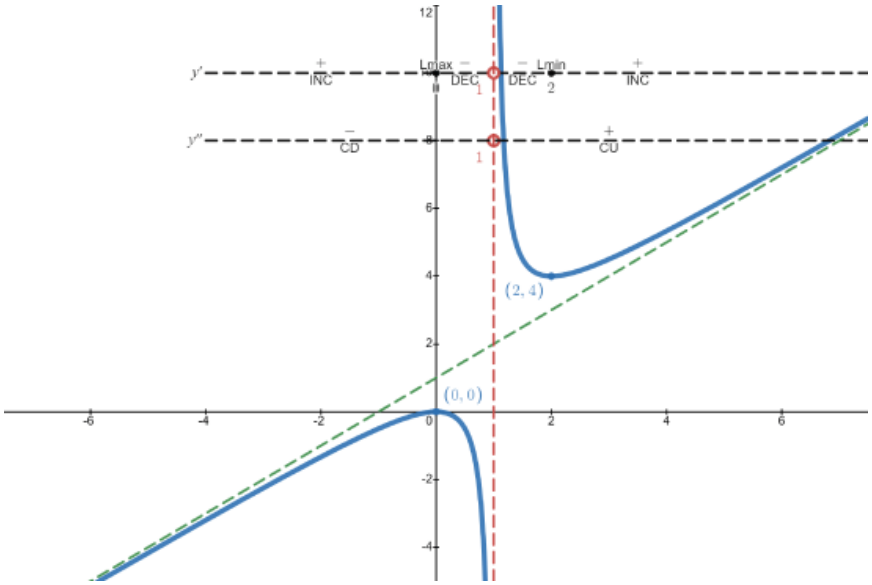

Figure 3.5.8

Checkpoint 3.5.4

$y = \frac{3}{2}x$

Example 3.5.5

Domain: Since the cube-root function is defined for all real numbers x and $(x-1)^{2/3} = (\sqrt[3]{x-1})^2$, the domain of f is all real numbers.

Intercepts: To find the y-intercept, evaluate $f(0)$. Since $f(0) = 1$, the y-intercept is $(0, 1)$. To find the x-intercept, solve $(x-1)^{2/3} = 0$. The solution of this equation is $x = 1$, so the x-intercept is $(1, 0)$.

Symmetry: Since

$$\begin{aligned} f(-x) &= (-x-1)^{2/3} \\ &= (x+1)^{2/3} \\ &\neq \pm f(x), \end{aligned}$$

we know that f is neither odd nor even.

Vertical Asymptote(s): The function has no vertical asymptotes.

End Behavior: Since $\lim_{x\to\pm\infty} (x-1)^{2/3} = \infty$, the function continues to grow without bound as $x \to \infty$ and $x \to -\infty$.

First Derivative Information: To determine where f is increasing or decreasing, calculate f'. We find

$$f'(x) = \frac{2}{3}(x-1)^{-1/3} = \frac{2}{3(x-1)^{1/3}}$$

This function is not zero anywhere, but it is undefined when $x = 1$. Therefore, the only critical point is $x = 1$. Divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 1)$ and $(1, \infty)$, and choose test points in each of these intervals to determine the sign of $f'(x)$ in each of these smaller intervals.

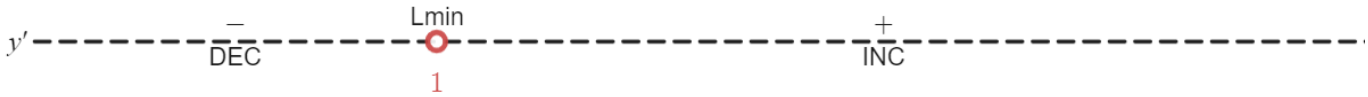

Figure 3.5.9

We conclude that f has a local minimum at $x = 1$. Evaluating f at $x = 1$, we find that the value of f at the local minimum is zero. Note that $f'(1)$ is undefined, so to determine the behavior of the function at this critical point, we need to examine $\lim_{x\to 1} f'(x)$. Looking at the one-sided limits, we have

$$\lim_{x\to 1^+} \frac{2}{3(x-1)^{1/3}} = \infty \text{ and } \lim_{x\to 1^-} \frac{2}{3(x-1)^{1/3}} = -\infty.$$

Therefore, f has a cusp at $x = 1$.

Second Derivative Information: To determine concavity, we calculate the second derivative of f :

$$f''(x) = -\frac{2}{9}(x-1)^{-4/3} = \frac{-2}{9(x-1)^{4/3}}.$$

We find that $f''(x)$ is defined for all x, but is undefined when $x = 1$. Therefore, divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 1)$ and $(1, \infty)$, and choose test points to evaluate the sign of $f''(x)$ in each of these intervals.

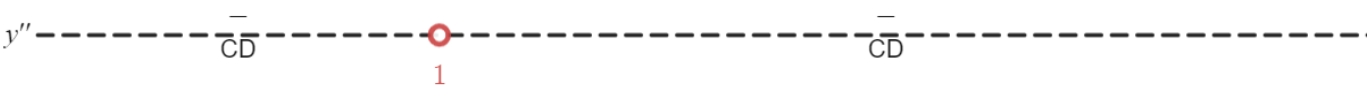

Figure 3.5.10

We conclude that f is concave down everywhere.

Graph Like a Boss:

Combining all this information, we arrive at the following graph for f.

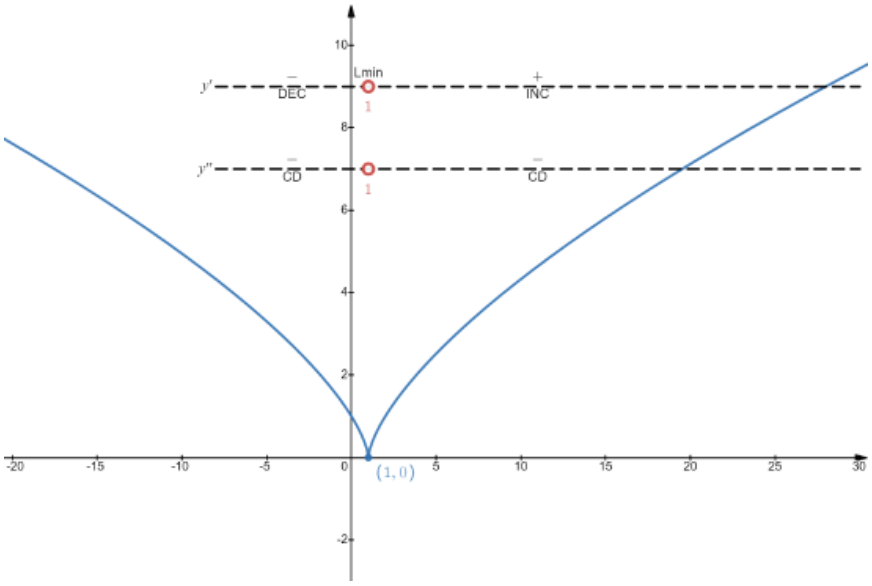

Figure 3.5.11

Checkpoint 3.5.5

The function f has a cusp at $(0, 5)$, since $\lim_{x \to 0^-} f'(x) = \infty$ and $\lim_{x \to 0^+} f'(x) = -\infty$. For end behavior, $\lim_{x \to \pm\infty} f(x) = -\infty$.

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3.5.1: Resources and Key Concepts

Resources

Prerequisite Topics

This page lists the prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Simplifying Expressions**
  - Dividing Polynomials - Long Division: Used to find slant asymptotes when the degree of the numerator is exactly one more than the degree of the denominator (Example 4).
- **Rational Functions**
  - Graphing Rational Functions: This section formalizes and expands on finding vertical, horizontal, and introduces slant (oblique) asymptotes using calculus (limits).

Videos

- Slant and oblique asymptotes
  - The Calculus of Slant Asymptotes
- Graphing with Calculus
  - Graphing Using Calculus (Part 6)
  - Graphing Using Calculus (Part 7)
  - Graphing Using Calculus (Part 8)
  - Graphing Using Calculus (Part 9)
  - Graphing Using Calculus (Part 10)
  - Graphing Using Calculus (Part 11)
  - Graphing Using Calculus (Part 12)

Key Concepts

Definitions

- **End Behavior**: The behavior of a function $f(x)$ as $x \to \pm\infty$. Types include:
  - **No End Behavior**: For functions with restricted domains.
  - **Horizontal Asymptote**: If $\lim_{x\to\pm\infty} f(x) = L$ (where L is finite).
  - **Slant Asymptote (Oblique Asymptote)**: The line $y = mx + b$ is a slant asymptote of the function f if $\lim_{x\to\infty}(f(x) - y) = 0$ or $\lim_{x\to-\infty}(f(x) - y) = 0$. This occurs when $\lim_{x\to\pm\infty} f(x) = \pm\infty$ and the function mimics a linear function.
- **Guidelines for Drawing the Graph of a Function (Tactic)**: A comprehensive list of steps to analyze a function for graphing:
  1. Domain
  2. Intercepts
  3. Symmetry (even, odd, periodic)
  4. Vertical Asymptotes (and behavior near them)
  5. End Behavior (Horizontal/Slant Asymptotes)
  6. First Derivative Information (critical points, increase/decrease, local extrema)
  7. Second Derivative Information (concavity, inflection points)
  8. Graph like a boss (plot key points, sketch curve)

Common Mistakes

- **Assuming Symmetric End Behavior**: The end behavior of a function as $x \to \infty$ can be different from its behavior as $x \to -\infty$. For example, a function might have a horizontal asymptote on one side and a slant asymptote or no asymptote on the other.
- **Incorrectly Identifying Slant Asymptotes**:
  - Assuming a slant asymptote exists when the degree of the numerator is not exactly one greater than the degree of the denominator (for rational functions).
  - Errors in polynomial long division when finding the equation of the slant asymptote.
- **Forgetting to Check All Steps in Graphing Strategy**: Overlooking aspects like domain restrictions, symmetry, or behavior near vertical asymptotes can lead to an inaccurate graph.
- **Misinterpreting Information from Derivatives**: Incorrectly relating the signs of f' and f'' to the shape of the graph.
- **Not Plotting Key Points Accurately**: Even with Calculus, plotting intercepts, critical points, and inflection points accurately is important for a good sketch. Decimal approximations might be needed.
- **Assuming No Vertical Asymptotes for Continuous Functions on Closed Intervals**: A function continuous on a closed interval $[a, b]$ or continuous on $\mathbb{R}$ will not have vertical asymptotes. Vertical asymptotes can only occur at the edge of a domain if the domain doesn't include that edge, or at points of infinite discontinuity within an open domain.

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3.5.2: Homework

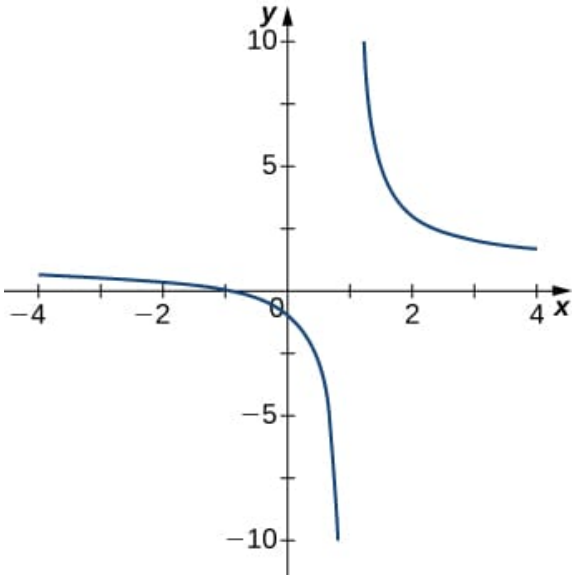
Reading Questions

1. What is meant by the "end behavior" of a function?
2. List the four types of end behavior a function might exhibit as $x \to \pm\infty$.
3. How is a horizontal asymptote defined using limits?
4. How is a slant (oblique) asymptote $y = mx + b$ defined using limits?
5. For a rational function, under what condition regarding the degrees of the numerator and denominator will it have a slant asymptote? How is the equation of the slant asymptote found?
6. Outline the 8-step "Tactic: Drawing the Graph of a Function."
7. Why is determining the domain of a function listed as the first step in the graphing strategy?
8. How can checking for symmetry (even, odd, or periodic) simplify the graphing process?
9. What Calculus tool is used to determine the behavior of a function near a vertical asymptote?
10. What information does the first derivative $f'(x)$ provide about the graph of $f(x)$?
11. What information does the second derivative $f''(x)$ provide about the graph of $f(x)$?
12. In Example 2 ($R(x) = \frac{x^2}{x^2+1}$), how was the end behavior determined?
13. In Example 5 ($f(x) = (x-1)^{2/3}$), what feature did the graph have at $x = 1$ where $f'(x)$ was undefined?

Homework

For exercises 1 - 5, examine the graphs. Identify where the vertical asymptotes are located.

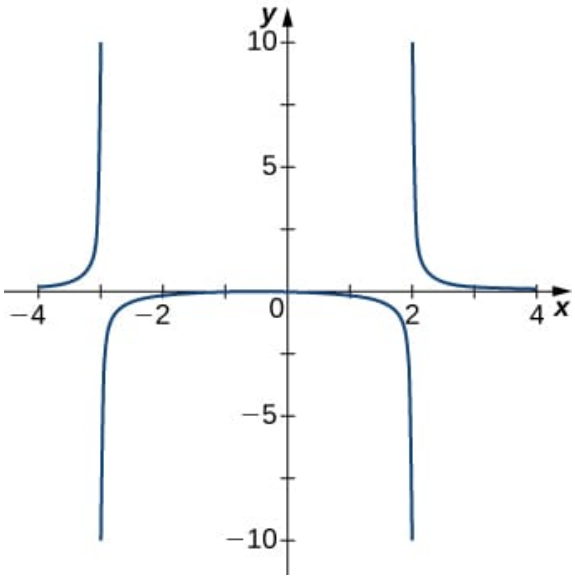
1)



Answer

$x = 1$

2)



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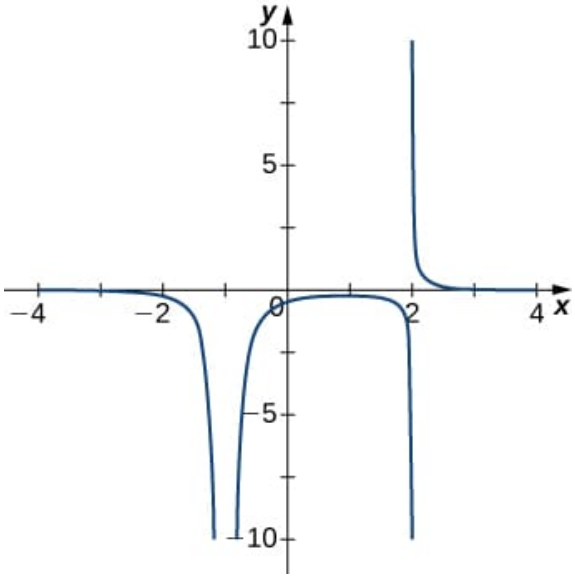
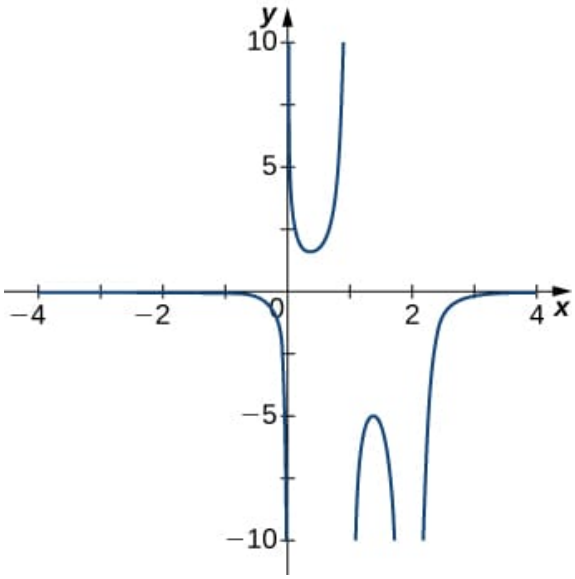


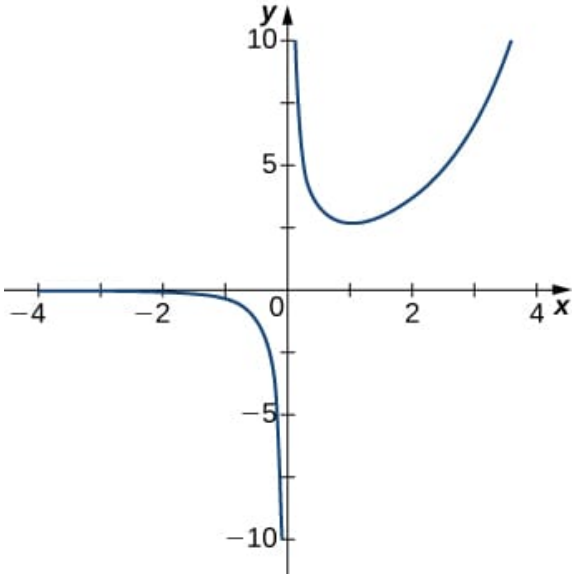
Answer

$x = -1, \quad x = 2$

4)



5)



Answer

$x = 0$

For the functions $f(x)$ in exercises 6 - 10, determine whether there is an asymptote at $x = a$. Justify your answer without graphing on a calculator.

6) $f(x) = \dfrac{x+1}{x^2+5x+4}, \quad a = -1$

7) $f(x) = \dfrac{x}{x-2}, \quad a = 2$

Answer

Yes, there is a vertical asymptote at $x = 2$.

8) $f(x) = (x+2)^{3/2}, \quad a = -2$

9) $f(x) = (x-1)^{-1/3}, \quad a = 1$

Answer

Yes, there is vertical asymptote at $x = 1$.

10) $f(x) = 1 + x^{-2/5}, \quad a = 1$

In exercises 11 - 20, evaluate the limit.

11) $\lim\limits_{x\to\infty} \dfrac{1}{3x+6}$

Answer

$$\lim_{x\to\infty} \frac{1}{3x+6} = 0$$

12) $\lim\limits_{x\to\infty} \dfrac{2x-5}{4x}$

13) $\lim\limits_{x\to\infty} \dfrac{x^2-2x+5}{x+2}$

Answer

$$\lim_{x\to\infty} \frac{x^2-2x+5}{x+2} = \infty$$

14) $\lim\limits_{x\to-\infty} \dfrac{3x^3-2x}{x^2+2x+8}$

15) $\lim\limits_{x\to-\infty} \dfrac{x^4-4x^3+1}{2-2x^2-7x^4}$

Answer

$$\lim_{x\to-\infty} \frac{x^4-4x^3+1}{2-2x^2-7x^4} = -\frac{1}{7}$$

16) $\lim\limits_{x\to\infty} \dfrac{3x}{\sqrt{x^2+1}}$

17) $\lim\limits_{x\to-\infty} \dfrac{\sqrt{4x^2-1}}{x+2}$

Answer

$$\lim_{x\to-\infty} \frac{\sqrt{4x^2-1}}{x+2} = -2$$

18) $\lim\limits_{x\to\infty} \dfrac{4x}{\sqrt{x^2-1}}$

19) $\lim\limits_{x\to-\infty} \dfrac{4x}{\sqrt{x^2-1}}$

Answer

$$\lim_{x\to-\infty} \frac{4x}{\sqrt{x^2-1}} = -4$$

20) $\lim\limits_{x\to\infty} \dfrac{2\sqrt{x}}{x-\sqrt{x}+1}$

For exercises 21 - 25, find the horizontal and vertical asymptotes.

21) $f(x) = x - \dfrac{9}{x}$

Answer

Horizontal: none,

Vertical: $x = 0$

22) $f(x) = \dfrac{1}{1 - x^2}$

23) $f(x) = \dfrac{x^3}{4 - x^2}$

Answer

Horizontal: none,
Vertical: $x = \pm 2$

24) $f(x) = \dfrac{x^2 + 3}{x^2 + 1}$

25) $f(x) = \sin(x)\sin(2x)$

Answer

Horizontal: none,
Vertical: none

26) $f(x) = \cos x + \cos(3x) + \cos(5x)$

27) $f(x) = \dfrac{x\sin(x)}{x^2 - 1}$

Answer

Horizontal: $y = 0$,
Vertical: $x = \pm 1$

28) $f(x) = \dfrac{x}{\sin(x)}$

29) $f(x) = \dfrac{1}{x^3 + x^2}$

Answer

Horizontal: $y = 0$,
Vertical: $x = 0$ and $x = -1$

30) $f(x) = \dfrac{1}{x - 1} - 2x$

31) $f(x) = \dfrac{x^3 + 1}{x^3 - 1}$

Answer

Horizontal: $y = 1$,
Vertical: $x = 1$

32) $f(x) = \dfrac{\sin x + \cos x}{\sin x - \cos x}$

33) $f(x) = x - \sin x$

Answer

Horizontal: none,
Vertical: none

34) $f(x) = \dfrac{1}{x} - \sqrt{x}$

For exercises 35 - 38, construct a function $f(x)$ that has the given asymptotes.

35) $x = 1$ and $y = 2$

Answer

Answers will vary, for example: $y = \dfrac{2x}{x-1}$

36) $x = 1$ and $y = 0$

37) $y = 4, \quad x = -1$

Answer

Answers will vary, for example: $y = \dfrac{4x}{x+1}$

38) $x = 0$

In exercises 39 - 43, graph the function on a graphing calculator on the window $x = [-5, 5]$ and estimate the horizontal asymptote or limit. Then, calculate the actual horizontal asymptote or limit.

39) [Technology Required] $f(x) = \dfrac{1}{x+10}$

Answer

$\lim\limits_{x\to\infty} \dfrac{1}{x+10} = 0$ so f has a horizontal asymptote of $y = 0$.

40) [Technology Required] $f(x) = \dfrac{x+1}{x^2+7x+6}$

41) [Technology Required] $\lim\limits_{x\to-\infty} x^2 + 10x + 25$

Answer

$\lim\limits_{x\to-\infty} x^2 + 10x + 25 = \infty$

42) [Technology Required] $\lim\limits_{x\to-\infty} \dfrac{x+2}{x^2+7x+6}$

43) [Technology Required] $\lim\limits_{x\to\infty} \dfrac{3x+2}{x+5}$

Answer

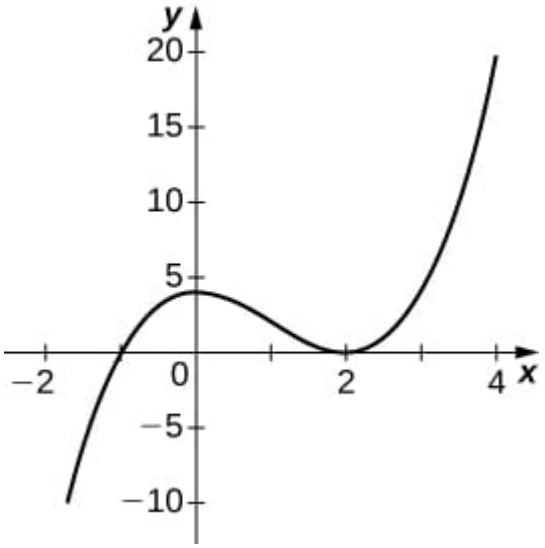
$\lim\limits_{x\to\infty} \dfrac{3x+2}{x+5} = 3$ so this function has a horizontal asymptote of $y = 3$.

In exercises 44 - 55, draw a graph of the functions without using a calculator. Be sure to notice all important features of the graph: local maxima and minima, inflection points, and asymptotic behavior.

44) $y = 3x^2 + 2x + 4$

45) $y = x^3 - 3x^2 + 4$

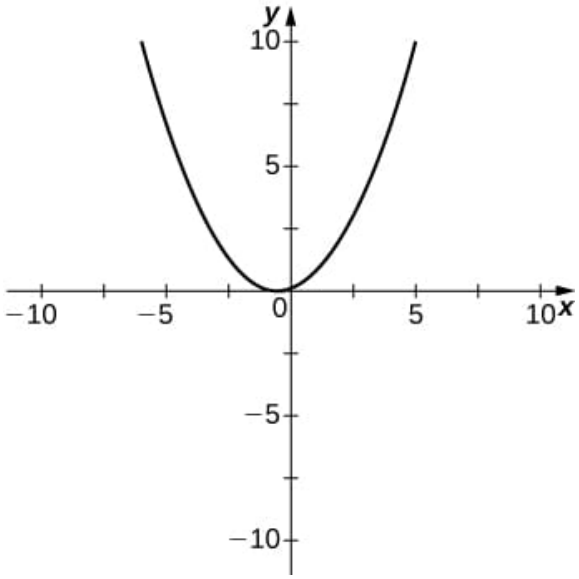
Answer



46) $y = \dfrac{2x+1}{x^2+6x+5}$

47) $y = \dfrac{x^3+4x^2+3x}{3x+9}$

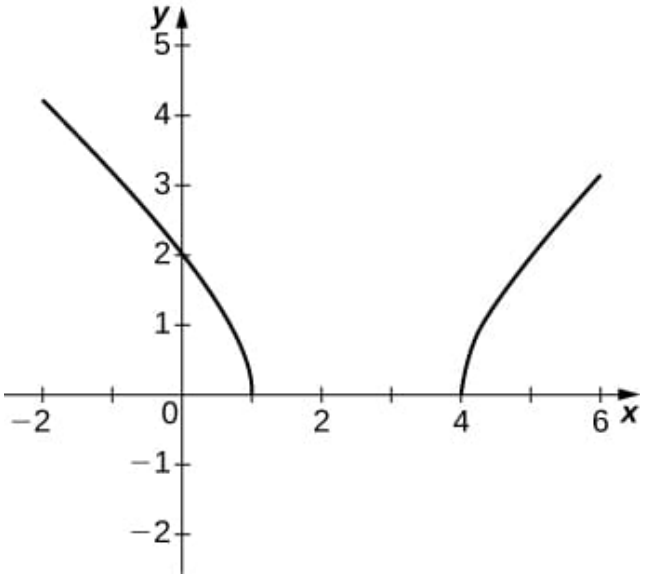
Answer



48) $y = \dfrac{x^2+x-2}{x^2-3x-4}$

49) $y = \sqrt{x^2-5x+4}$

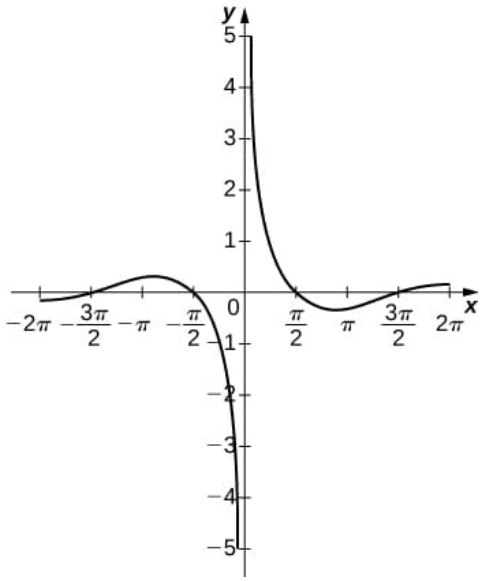
Answer



50) $y = 2x\sqrt{16 - x^2}$

51) $y = \dfrac{\cos x}{x}$, on $x = [-2\pi, 2\pi]$

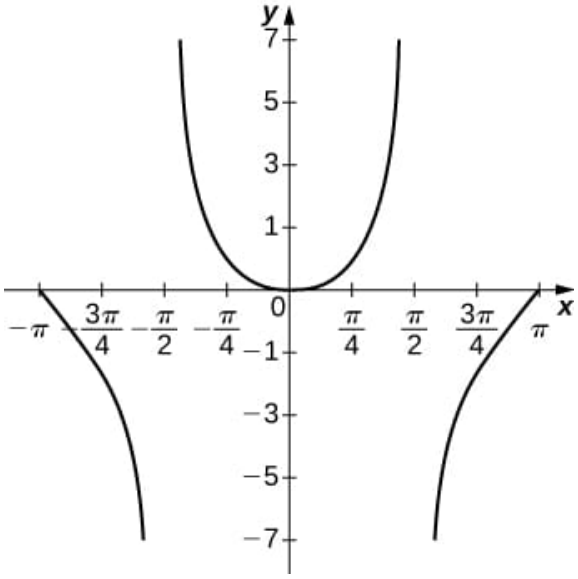
Answer



52) $y = e^x - x^3$

53) $y = x \tan x, \quad x = [-\pi, \pi]$

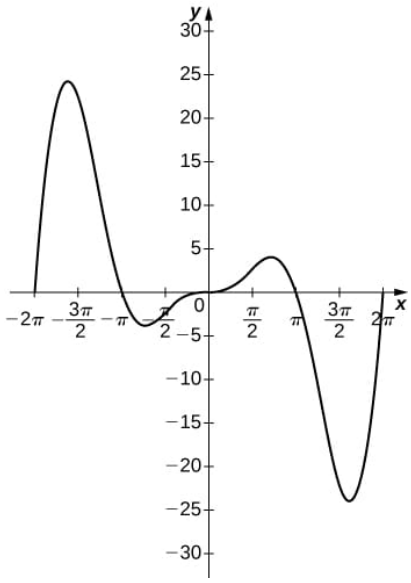
Answer



54) $y = x\ln(x), \quad x > 0$

55) $y = x^2\sin(x), \quad x = [-2\pi, 2\pi]$

Answer



56) For $f(x) = \dfrac{P(x)}{Q(x)}$ to have an asymptote at $y = 2$ then the polynomials $P(x)$ and $Q(x)$ must have what relation?

57) For $f(x) = \dfrac{P(x)}{Q(x)}$ to have an asymptote at $x = 0$, then the polynomials $P(x)$ and $Q(x)$. must have what relation?

Answer

$Q(x)$. must have have x^{k+1} as a factor, where $P(x)$ has x^k as a factor.

58) If $f\,'(x)$ has asymptotes at $y = 3$ and $x = 1$, then $f(x)$ has what asymptotes?

59) Both $f(x) = \dfrac{1}{x-1}$ and $g(x) = \dfrac{1}{(x-1)^2}$ have asymptotes at $x = 1$ and $y = 0$. What is the most obvious difference between

these two functions?

Answer

$$\lim_{x\to 1^-} f(x) = -\infty \text{ and } \lim_{x\to 1^-} g(x) = \infty$$

60) True or false: Every ratio of polynomials has vertical asymptotes.

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3.6: Optimization

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

One common application of Calculus is calculating a function's minimum or maximum value. For example, companies often want to minimize production costs or maximize revenue. In manufacturing, minimizing the amount of material used to package a product with a certain volume is often desirable. In this section, we show how to set up these minimization and maximization problems and solve them using the tools developed in this chapter.

Solving Optimization Problems over a Closed, Bounded Interval

The basic idea of the optimization problems that follow is the same—we have a particular quantity that we are interested in maximizing or minimizing. However, we also have some auxiliary conditions that need to be satisfied. For example, in Example $3.6.1$, we are interested in maximizing the area of a rectangular garden. Certainly, if we keep making the side lengths of the garden larger, the area will continue to become larger. However, what if we restrict how much fencing we can use for the perimeter? In this case, we cannot make the garden as large as we like. Let's look at how we can maximize the area of a rectangle subject to some constraint on the perimeter.

Example $3.6.1$

A rectangular garden is to be constructed using a rock wall on one side of the garden and wire fencing for the other three sides (Figure $3.6.1$). Given $100\ \text{ft}$ of wire fencing, determine the dimensions that would create a garden of maximum area. What is the maximum area?

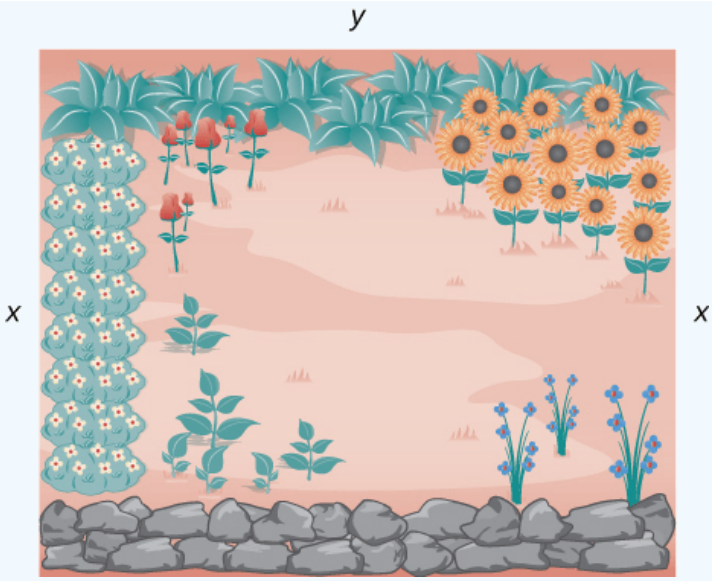

Figure $3.6.1$**:** *We want to determine the measurements* x *and* y *that will create a garden with a maximum area using* 100 ft *of fencing.*

Solution available at the end of this section.

Interactive Element: Maximizing the Garden's Area

The following Interactive Element might help you visualize how a fixed amount of fencing forces a tradeoff between the garden's width and height.

Interact: Drag the red corner point. The rectangle trades width for height while the total fencing stays 100 ft.

Observation: The area is largest at $x = 25$ ft (base 50 ft, height 25 ft), giving 1250 ft^2; pushing the shape taller or wider only shrinks the area.

Checkpoint $3.6.1$

Determine the maximum area if we want to make the same rectangular garden as in Figure $3.6.2$, but we have 200 ft of fencing.

Answer available at the end of this section.

Now let's look at a general strategy for solving optimization problems similar to Example $3.6.1$.

While memorizing procedures is the absolute worst way to enjoy Mathematics, applications (also known as "word problems") tend to confound students so much that it is beneficial to develop a procedure for solving them. We will borrow the strategy we used from the Related Rates material.

Polya's Modified Problem-Solving Process

1. **Read** the given problem.
2. **Understand** the given problem. This might require you to rephrase the problem in terms that you can understand or, more commonly, to draw a picture of the situation.
3. **Label** unknowns. All problems presented in mathematics have unknown quantities. In this step, you take the time to label these unknowns using variables. This step is often combined with the previous step (especially when drawing pictures).

4. **Identify** what needs to be optimized (e.g., area, cost, profit, angle, etc.).
5. Create a **Master Equation** for the quantity to be optimized. This is a critical step in all problem-solving processes involving Polya's method.
   Warning: The Master Equation should be a relation between two variables (the variable to be optimized and one other). If your Master Equation has more than two variables, you need to find a relation between the excess variables to reduce the number of final variables in your Master Equation to two.
6. Unlike in the Related Rates material, it will be useful in this step to identify the **realistic domain** for your Master Equation. If the realistic domain is a closed and bounded interval, we can invoke the Extreme Value Theorem. In Example $3.6.6$, we show what to do if the realistic domain is unbounded or open.
7. **Optimize** by locating the absolute maximum or minimum value of the Master Equation on the *realistic* domain.

Now let's apply this strategy to maximize the volume of an open-top box given a constraint on the amount of material to be used.

Example $3.6.2$

An open-top box is to be made from a 24 in. by 36 in. piece of cardboard by removing a square from each corner of the box and folding up the flaps on each side. What size square should be cut out of each corner to get a box with the maximum volume?

Solution available at the end of this section.

Checkpoint $3.6.2$

Suppose the dimensions of the cardboard in Example $3.6.2$ are 20 in. by 30 in. Let x be the side length of each square and write the volume of the open-top box as a function of x. Determine the domain of consideration for x.

Answer available at the end of this section.

Example $3.6.3$

An island is 2 mi due north of its closest point along a straight shoreline. A visitor stays at a cabin on the shore that is 6 mi west of that point. The visitor is planning to go from the cabin to the island. Suppose the visitor runs at a 8 mph rate and swims at a 3 mph rate. How far should the visitor run before swimming to minimize the time it takes to reach the island?

Solution available at the end of this section.

Interactive Element: The Fastest Route to the Island

The following Interactive Element might help you visualize how the total travel time depends on where the visitor stops running and starts swimming.

Interact: Drag the crossover point along the shore to change how far the visitor runs before swimming.

Observation: Total time is least when the visitor runs about 5.19 mi — not all the way to the closest point — and then swims; entering the water much earlier or later both take longer.

Checkpoint $3.6.3$

Suppose the island is 1 mi from shore, and the distance from the cabin to the point on the shore closest to the island is 15 mi. Suppose a visitor swims at the 2.5 mph rate and runs at a 6 mph rate. Let x denote the distance the visitor will run before swimming, and find a function for the time it takes the visitor to get from the cabin to the island.

Answer available at the end of this section.

In business, companies are interested in **maximizing revenue.** In the following example, we consider a scenario in which a company has collected data on how many cars it can lease, depending on the price it charges its customers to rent a car. Let's use these data to determine the price the company should charge to maximize the amount of money it brings in.

Example 3.6.4

Owners of a car rental company have determined that if they charge customers p dollars per day to rent a car, where $50 \leq p \leq 200$, the number of cars n they rent per day can be modeled by the linear function $n(p) = 1000 - 5p$. They will rent all their cars if they charge \$50 per day or less. They will not rent cars if they charge \$200 per day or more. Assuming the owners plan to charge customers between \$50 per day and \$200 per day to rent a car, how much should they charge to maximize their revenue?

Solution available at the end of this section.

Checkpoint 3.6.4

A car rental company charges its customers p dollars per day, where $60 \leq p \leq 150$. They determine that the number of cars rented per day can be modeled by the linear function $n(p) = 750 - 5p$. How much should the company charge each customer to maximize revenue?

Answer available at the end of this section.

Example 3.6.5

A rectangle is to be inscribed in the ellipse

$$\frac{x^2}{4} + y^2 = 1.$$

What should the rectangle's dimensions be to maximize its area? What is the maximum area?

Solution available at the end of this section.

Interactive Element: Largest Rectangle in an Ellipse

The following Interactive Element might help you visualize how the inscribed rectangle's area changes as its corner slides around the ellipse.

Interact: Drag the corner point along the ellipse.

Observation: The area is largest, 4, when the corner sits at $x = \sqrt{2},\ y = \frac{1}{\sqrt{2}}$; sliding the corner toward either axis collapses the rectangle to a sliver.

Checkpoint 3.6.5

Modify the area function A if the rectangle is to be inscribed in the unit circle $x^2 + y^2 = 1$. What is the domain of consideration?

Answer available at the end of this section.

Solving Optimization Problems when the Interval Is Not Closed or Is Unbounded

In the previous examples, we considered functions on closed, bounded domains. Consequently, by the Extreme Value Theorem, we were guaranteed that the functions had absolute extrema. Let's now consider functions for which the domain is neither closed nor bounded.

Many functions still have at least one absolute extrema, even if the domain is not closed or unbounded. For example, the function $f(x) = x^2 + 4$ over $(-\infty, \infty)$ has an absolute minimum of 4 at $x = 0$. Therefore, we can still consider functions over unbounded domains or open intervals and determine whether they have absolute extrema. In the next example, we try to minimize a function over an unbounded domain. Although the domain of consideration is $(0, \infty)$, the function has an absolute minimum.

In the following example, we look at constructing a box of least surface area with a prescribed volume. It is not difficult to show that for a closed-top box, by symmetry, among all boxes with a specified volume, a cube will have the smallest surface area. Consequently, we consider the modified problem of determining which open-topped box with a specified volume has the smallest surface area.

Example 3.6.6

A rectangular box with a square base, an open top, and a volume of $216\ \text{in}^3$ is to be constructed. What should the dimensions of the box be to minimize the surface area of the box? What is the minimum surface area?

Solution available at the end of this section.

Checkpoint 3.6.6

Consider the same open-top box, which is to have volume $216\ \text{in}^3$. Suppose the cost of the material for the base is $\$0.20/\text{in}^2$ and the cost of the material for the sides is $\$0.30/\text{in}^2$, and we are trying to minimize the cost of this box. Write the cost as a function of the side lengths of the base. (Let x be the side length of the base and y be the height of the box.)

Answer available at the end of this section.

Success in College: Affirmations

Do you ever tell yourself (or others) that you're not good at math? This is called negative self-talk. The more you talk that way, the more likely your subconscious will believe it - and you will do poorly in math.

You can counteract years of negative self-talk by telling yourself with conviction that you're good at math. It might seem strange to state that something that hasn't happened yet is true, but it works! Such statements are called affirmations.

There are four guiding principles for getting the most out of affirmations:

I. Say affirmations that imply that the desired event is currently happening. For example, say, "I am good at algebra," not "I will be good at algebra."
II. Say affirmations that you are making progress toward the desired result. For example, say, "I am good at trigonometry, and I continue to improve at it."
III. Say affirmations in the positive rather than the negative. For example, say "I attend each class," not "I don't cut classes."
IV. Say affirmations with conviction.

1. Do you practice affirmations? If so, how?
2. After reading this, will you try to practice affirmations? Where else in life could affirmations be put to work?

Success in Mathematics: Process-Oriented Learning Preferences

Choose the statements that best describe you.

I learn better if I know why I'm learning something.	or	I don't need to know why I'm doing something before learning how to do it.
I find it annoying to learn something I don't think I will use anywhere else.	or	I believe that learning new things is always worth it because it exercises my brain.
I naturally think in steps.	or	It is hard for me to learn something step by step.

1. Do you consider your choices above good or bad for your education?
2. In specific textbooks, many procedures are summarized as a list of steps. This list is shown in a separate box. Do you refer to these boxes when you do your homework? Why or why not?

Success in College: Groupwork Learning Preferences

Most students are very intimidated by working in groups. It usually takes quite a while for students to realize that studying together can be very rewarding.

Think about the following questions regarding learning or practicing complicated math.

- Do you prefer to do homework by yourself, or do you like to do homework with others?
- Do you like to do homework at school, or do you like to do homework at home?
- If you need help, do you prefer to get it from friends or a spouse, or do you prefer to get it from your professor or a tutor?
- Do you concentrate on learning what you don't know by yourself, or do you prefer to help others because it helps you learn it better?

1. Do you work in conditions that match these preferences when you do your math homework? Why or why not?
2. Spending time with old and new friends is an integral part of college. However, studying with friends is not always practical. Do you ever feel peer pressure to study in ways that are not the best for you?

Answers

Section Preview

This is the typical optimization material found in any Differential Calculus textbook. There are no special notes to the instructor for this material.

Section Preview

- **Geometry**
  - Area and Perimeter of Triangles and Quadrilaterals: Students must understand area and perimeter as quantities in order to build the master equations: the garden's area $A = xy$ under a perimeter constraint (Example 1) and the inscribed rectangle's area (Example 5).
- **Linear Functions**
  - Linear Models: Example 4 models the number of cars rented as the linear function $n(p) = 1000 - 5p$ and interprets it in context before forming the revenue function $R(p)$.
- **Miscellaneous Applications**
  - Rates, Unit Rates, and Uniform Motion (Distance, Rate, Time): Example 3 builds the time function from Distance $=$ Rate $\times$ Time, expressing the running and swimming times as $x/8$ and $y/3$.

Section Preview

- **Solving Equations**
  - Solving Quadratic Equations Using the Quadratic Formula: Example 2 solves $x^2 - 20x + 72 = 0$ to locate the critical points $10 \pm 2\sqrt{7}$.
  - Solving Radical Equations: Example 3 isolates and squares $3\sqrt{(6-x)^2+4} = 8(6-x)$, then verifies that the surviving solution satisfies the original equation.

Section Preview

- Use the Problem Solving Process to set up and solve optimization problems in several applied fields.

Example $3.6.1$

Let x denote the length of the side of the garden perpendicular to the rock wall and y denote the length of the side parallel to the rock wall. Then the area of the garden is

$$A = x \cdot y.$$

We want to find the maximum possible area subject to the constraint that the total fencing is 100 ft. From Figure $3.6.1$, the total amount of fencing used will be $2x$

$+ y$. Therefore, the constraint equation is

$$2x + y = 100.$$

Solving this equation for y, we have $y = 100 - 2x$. Thus, we can write the area as

$$A(x) = x \cdot (100 - 2x) = 100x - 2x^2.$$

Before trying to maximize the area function $A(x) = 100x - 2x^2$, we need to determine the domain under consideration. To construct a rectangular garden, we certainly need the lengths of both sides to be positive. Therefore, we need $x > 0$ and $y > 0$. Since $y = 100 - 2x$, if $y > 0$, then $x < 50$. Therefore, we are trying to determine the maximum value of $A(x)$ for x over the open interval $(0, 50)$. We do not know that a function necessarily has a maximum value over an open interval. However, we know a continuous function has an absolute maximum (and absolute minimum) over a closed interval. Therefore, let's consider the function $A(x) = 100x - 2x^2$ over the closed interval $[0, 50]$. Suppose the maximum value occurs at an interior point. In that case, we have found the value x in the open interval $(0, 50)$ that maximizes the area of the garden.

Therefore, we consider the following problem:

Maximize $A(x) = 100x - 2x^2$ over the interval $[0, 50]$.

As mentioned earlier, since A is a continuous function on a closed, bounded interval, by the Extreme Value Theorem, it has a maximum and a minimum. These extreme values occur either at endpoints or critical points. At the endpoints, $A(x) = 0$. Since the area is positive for all x in the open interval $(0, 50)$, the maximum must occur at a critical point. Differentiating the function $A(x)$, we obtain

$$A'(x) = 100 - 4x.$$

Therefore, the only critical point is $x = 25$ (Figure 3.6.2). We conclude that the maximum area must occur when $x = 25$.

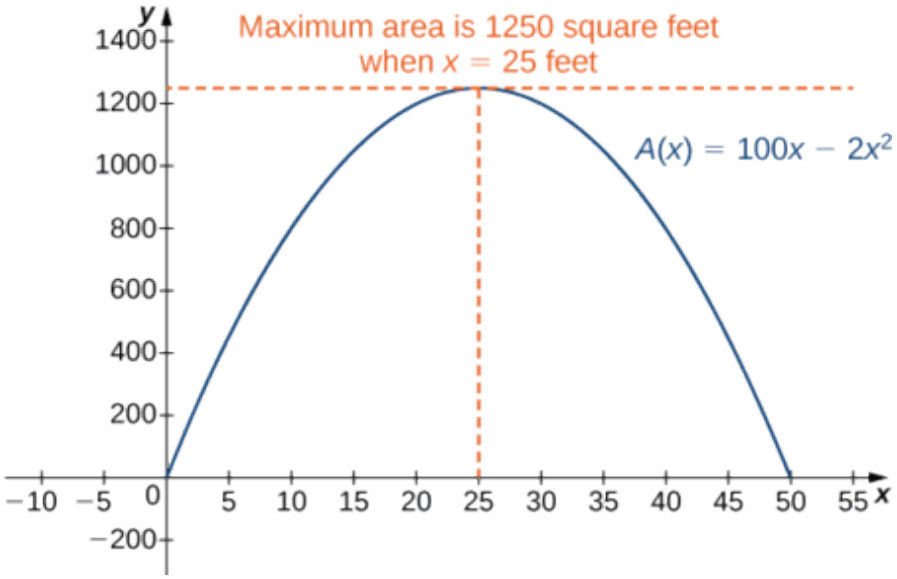

Figure 3.6.2**:** *To maximize the area of the garden, we need to find the maximum value of the function* $A(x) = 100x - 2x^2$.

Then we have $y = 100 - 2x = 100 - 2(25) = 50$. To maximize the area of the garden, let $x = 25$ ft and $y = 50$ ft. The area of this garden is 1250 ft^2.

Checkpoint 3.6.1

The maximum area is 5000 ft^2.

Example 3.6.2

Read: Just as in the Related Rates material, I include this step so you have a "win."

Understand and Label: Drawing a picture of the situation (and appropriately labeling things), we let x be the side length of the square to be removed from each corner (Figure 3.6.3). Then, the remaining four flaps can be folded to form an open-top box. Let V be the volume of the resulting box.

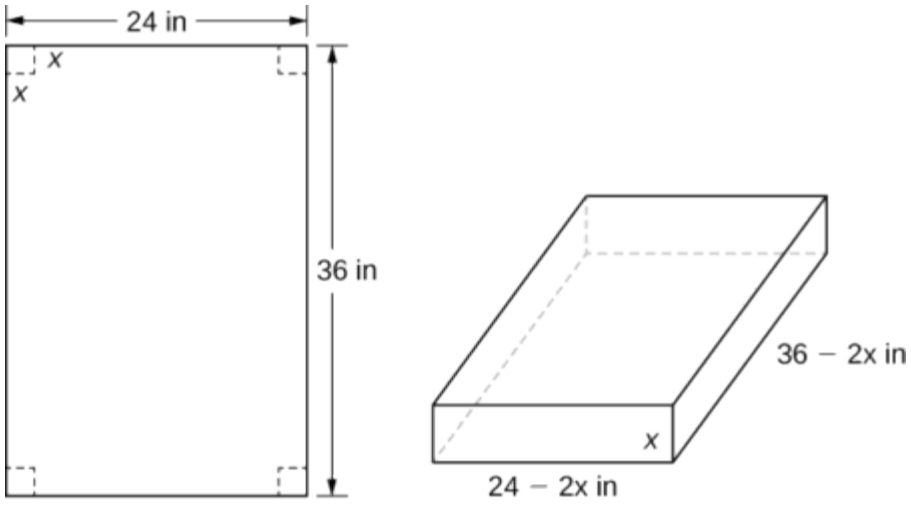

Figure 3.6.3**:** *A square with side length x inches is removed from each corner of the piece of cardboard. The remaining flaps are folded to form an open-top box.*

Identify: We are trying to maximize the volume of a box. Therefore, the problem is to maximize V.

Master Equation: The volume of a box is

$$V = L \cdot W \cdot H,$$

where L, W, and H are the length, width, and height, respectively. From our drawing, this means

$$L = 36 - 2x, W = 24 - 2x, \text{ and } H = x.$$

Therefore,

$$V = (36 - 2x)(24 - 2x)x.$$

Realistic Domain: To determine the domain of consideration, let's examine Figure 3.6.3. Certainly, we need $x > 0$. Furthermore, the square's side length must be less than half the length of the shorter side, 24 in.; otherwise, one of the flaps would be completely cut off. Therefore, we are trying to determine whether there is a maximum volume of the box for x over the open interval $(0, 12)$. Since V is a continuous function over the closed interval $[0, 12]$, we know V will have an absolute maximum over the closed interval. Therefore, we consider V over the closed interval $[0, 12]$ and check whether the absolute maximum occurs at an interior point.

Optimize: Since $V(x)$ is a continuous function over the closed, bounded interval $[0, 12]$, V must have an absolute maximum (and an absolute minimum). Since $V(x) = 0$ at the endpoints and $V(x) > 0$ for \(0

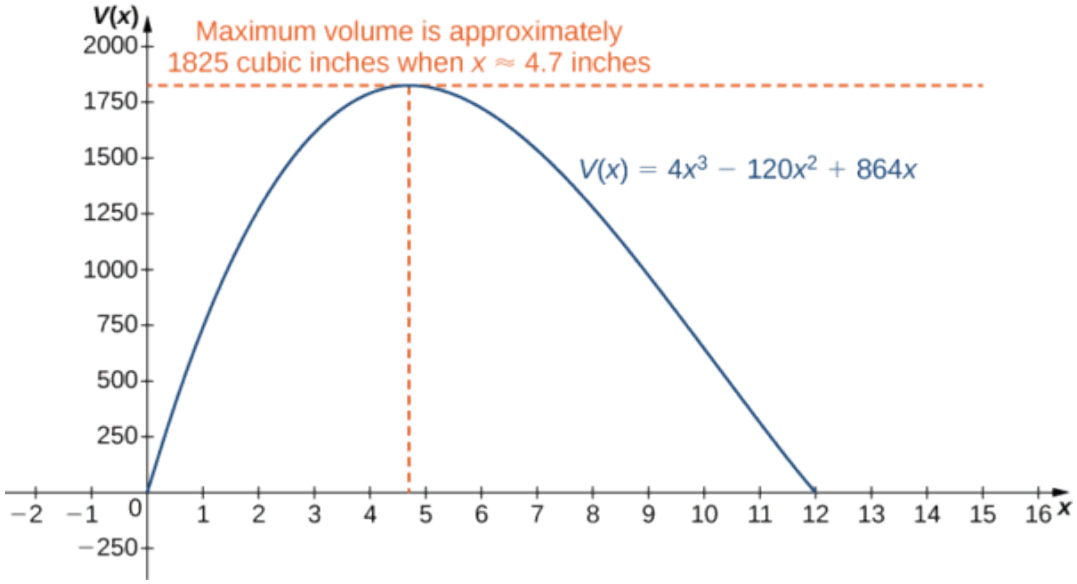

Figure 3.6.4**:** *Maximizing the box's volume leads to finding the maximum value of a cubic polynomial.*

Checkpoint 3.6.2

$V(x) = x(20 - 2x)(30 - 2x)$. The domain is $[0, 10]$.

Example 3.6.3

Read, Understand, and Label: Let x be the distance running and let y be the distance swimming (Figure 3.6.5). Let T be the time it takes to get from the cabin to the island.

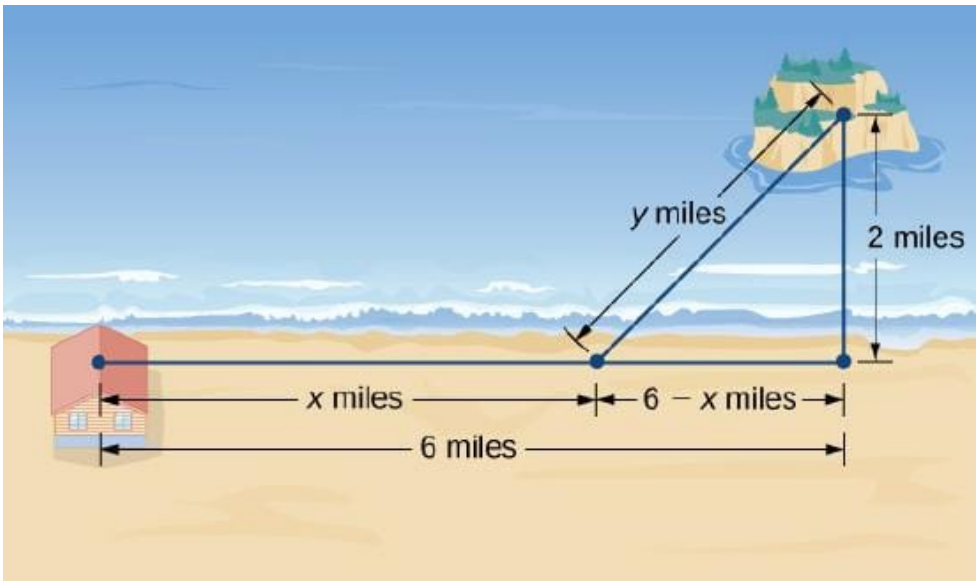

Figure 3.6.5**:** *How can we choose x and y to minimize the travel time from the cabin to the island?*

Identify: The problem is to minimize the total time it takes to reach the island, T.

Master Equation: To find the time spent traveling from the cabin to the island, add the time spent running and the time spent swimming. Since Distance = Rate \times Time $(D = R \times T)$, the time spent running is

$$T_{running} = \frac{D_{running}}{R_{running}} = \frac{x}{8},$$

and the time spent swimming is

$$T_{swimming} = \frac{D_{swimming}}{R_{swimming}} = \frac{y}{3}.$$

Therefore, the total time spent traveling is

$$T = \frac{x}{8} + \frac{y}{3}.$$

From Figure 3.6.5, the line segment of y miles forms the hypotenuse of a right triangle with legs of length 2 mi and $6 - x$ mi. Therefore, by the Pythagorean Theorem, $2^2 + (6 - x)^2 = y^2$, and we obtain $y = \sqrt{(6 - x)^2 + 4}$. Thus, the total time spent traveling is given by the function

$$T(x) = \frac{x}{8} + \frac{\sqrt{(6 - x)^2 + 4}}{3}.$$

Realistic Domain: From Figure 3.6.5, we see that $0 \le x \le 6$. Therefore, $[0, 6]$ is the domain of consideration.

Optimize: Since $T(x)$ is a continuous function over a closed, bounded interval, it has a maximum and a minimum. Let's begin by looking for any critical points of T over the interval $[0, 6]$. The derivative is

$$\begin{aligned} T'(x) &= \frac{1}{8} - \frac{1}{2}\frac{[(6 - x)^2 + 4]^{-1/2}}{3} \cdot 2(6 - x) \\ &= \frac{1}{8} - \frac{(6 - x)}{3\sqrt{(6 - x)^2 + 4}} \end{aligned}$$

If $T'(x) = 0$, then

$$\frac{1}{8} = \frac{6 - x}{3\sqrt{(6 - x)^2 + 4}} \tag{3.6.1}$$

Therefore,

$$3\sqrt{(6 - x)^2 + 4} = 8(6 - x). \tag{3.6.2}$$

Squaring both sides of this equation, we see that if x satisfies this equation, then x must satisfy

$$9[(6 - x)^2 + 4] = 64(6 - x)^2,$$

which implies

$$55(6 - x)^2 = 36.$$

We conclude that if x is a critical point, then x satisfies

$$(x - 6)^2 = \frac{36}{55}.$$

[Note that since we are squaring, $(x - 6)^2 = (6 - x)^2$.]

Therefore, the possibilities for critical points are

$$x = 6 \pm \frac{6}{\sqrt{55}}.$$

Since $x = 6 + 6/\sqrt{55}$ is not in the domain, it is not a possibility for a critical point. On the other hand, $x = 6 - 6/\sqrt{55}$ is in the domain. Since we squared both sides of Equation 3.6.2 to arrive at the possible critical points, it remains to verify that $x = 6 - 6/\sqrt{55}$ satisfies Equation 3.6.1. Since $x = 6 - 6/\sqrt{55}$ does satisfy that equation, we conclude that $x = 6 - 6/\sqrt{55}$ is a critical point, and it is the only one. To justify that the time is minimized for this value of x, we need to check the values of $T(x)$ at the endpoints $x = 0$ and $x = 6$ and compare them with the value of $T(x)$ at the critical point $x = 6 - 6/\sqrt{55}$. We find that $T(0) \approx 2.108$ h and $T(6) \approx 1.417$ h, whereas

$$T(6 - 6/\sqrt{55}) \approx 1.368\,\text{h}.$$

Therefore, we conclude that T has a local minimum at $x \approx 5.19$ mi.

Checkpoint 3.6.3

$T(x) = \frac{x}{6} + \frac{\sqrt{(15-x)^2+1}}{2.5}$

Example 3.6.4

Read, Understand, and Label: There is not really a point in drawing anything. Let p be the price charged per car per day, and let n be the number of cars rented daily. Let R be the revenue per day.

Identify: The problem is to maximize R.

Master Equation: The revenue (per day) is equal to the number of cars rented per day times the price charged per car per day—that is, $R = n \times p$.

Since the number of cars rented per day is modeled by the linear function $n(p) = 1000 - 5p$, the revenue R can be represented by the function

$$\begin{aligned} R(p) &= n \times p \\ &= (1000 - 5p)p \\ &= -5p^2 + 1000p. \end{aligned}$$

Realistic Domain: Since the owners plan to charge between \$50 per car per day and \$200 per car per day, the problem is to find the maximum revenue $R(p)$ for p in the closed interval $[50, 200]$.

Optimize: Since R is a continuous function over the closed, bounded interval $[50, 200]$, it has an absolute maximum (and an absolute minimum) in that interval. To find the maximum value, look for critical points. The derivative is $R'(p) = -10p + 1000$. Therefore, the critical point is $p = 100$. When $p = 100, R(100) = \$50,000$. When $p = 50, R(p) = \$37,500$. When $p = 200, R(p) = \$0$.

Therefore, the absolute maximum occurs at $p = \$100$. The car rental company should charge \$100 per day per car to maximize revenue, as shown in the following figure.

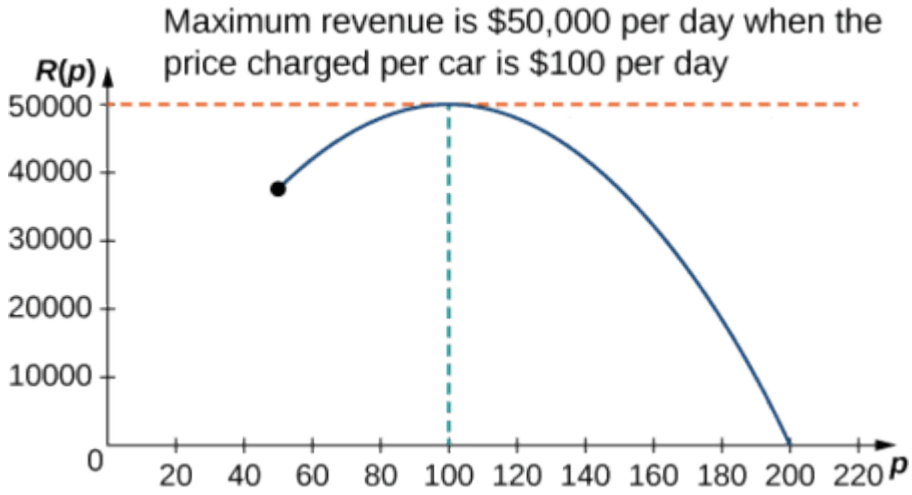

Figure 3.6.6**:** *To maximize revenue, a car rental company has to balance the rental price against the number of cars people will rent at that price.*

Checkpoint 3.6.4

The company should charge \$75 per car per day.

Example 3.6.5

Read, Understand, and Label: For a rectangle to be inscribed in the ellipse, the sides of the rectangle must be parallel to the axes. Let L be the length of the rectangle and W be its width. Let A be the area of the rectangle.

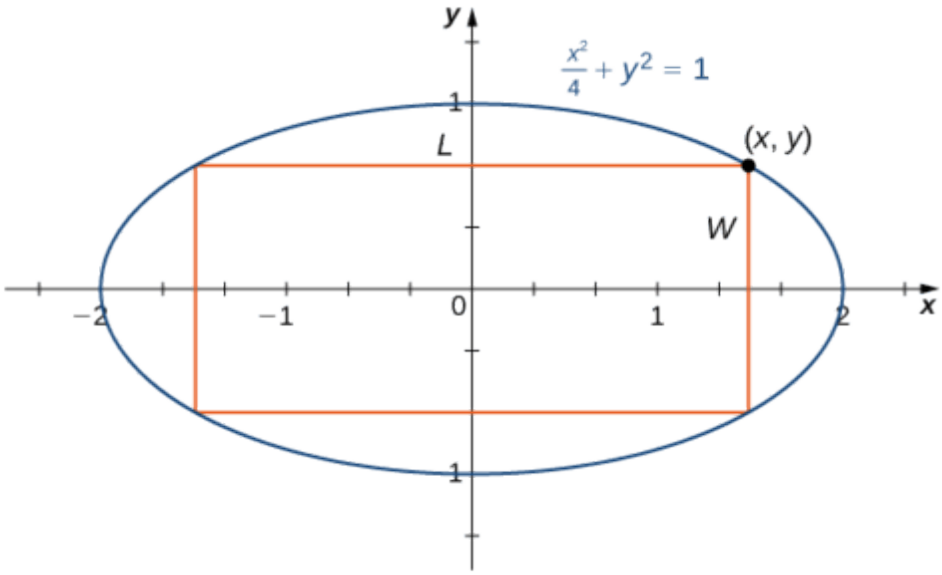

Figure 3.6.7**:** *We want to maximize the area of a rectangle inscribed in an ellipse.*

Identify: The problem is to maximize A.

Master Equation: The area of the rectangle is $A = LW$.

Let (x, y) be the corner of the rectangle that lies in the first quadrant, as shown in Figure 3.6.7. We can write length $L = 2x$ and width $W = 2y$. Since $\frac{x^2}{4} + y^2 = 1$ and $y > 0$, we have $y = \sqrt{1 - \frac{x^2}{4}}$. Therefore, the area is

$$A = LW = (2x)(2y) = 4x\sqrt{1 - \frac{x^2}{4}} = 2x\sqrt{4 - x^2}$$

Realistic Domain: From Figure 3.6.7, we see that to inscribe a rectangle in the ellipse, the x-coordinate of the corner in the first quadrant must satisfy \(0

Optimize: As mentioned earlier, $A(x)$ is a continuous function over the closed, bounded interval $[0, 2]$. Therefore, it has an absolute maximum (and absolute minimum). At the endpoints $x = 0$ and $x = 2$, $A(x) = 0$. For \(00\).

Therefore, the maximum must occur at a critical point. Taking the derivative of $A(x)$, we obtain

$$\begin{aligned} A'(x) &= 2\sqrt{4 - x^2} + 2x \cdot \frac{1}{2\sqrt{4 - x^2}}(-2x) \\ &= 2\sqrt{4 - x^2} - \frac{2x^2}{\sqrt{4 - x^2}} \\ &= \frac{8 - 4x^2}{\sqrt{4 - x^2}}. \end{aligned}$$

To find critical points, we need to find where $A'(x) = 0$. We can see that if x is a solution of

$$\frac{8 - 4x^2}{\sqrt{4 - x^2}} = 0, \tag{3.6.3}$$

then x must satisfy

$$8 - 4x^2 = 0.$$

Therefore, $x^2 = 2$. Thus, $x = \pm\sqrt{2}$ are the possible solutions of Equation 3.6.3. Since we are considering x over the interval $[0, 2]$, $x = \sqrt{2}$ is a possibility for a critical point, but $x = -\sqrt{2}$ is not. Therefore, we check whether $\sqrt{2}$ is a solution of Equation 3.6.3. Since $x = \sqrt{2}$ is a solution of Equation 3.6.3, we conclude that $\sqrt{2}$ is the only critical point of $A(x)$ in the interval $[0, 2]$.

Therefore, $A(x)$ must have an absolute maximum at the critical point $x = \sqrt{2}$. To determine the rectangle's dimensions, we need to find the length L and the width W. If $x = \sqrt{2}$ then

$$y = \sqrt{1 - \frac{(\sqrt{2})^2}{4}} = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}.$$

Therefore, the dimensions of the rectangle are $L = 2x = 2\sqrt{2}$ and $W = 2y = \frac{2}{\sqrt{2}} = \sqrt{2}$. The area of this rectangle is $A = LW = (2\sqrt{2})(\sqrt{2}) = 4$.

Checkpoint 3.6.5

$A(x) = 4x\sqrt{1 - x^2}$. The domain of consideration is $[0, 1]$.

Example 3.6.6

Read, Understand, and Label: Draw a rectangular box and introduce the variable x to represent the length of each side of the square base; let y represent the height of the box. Let S denote the surface area of the open-top box.

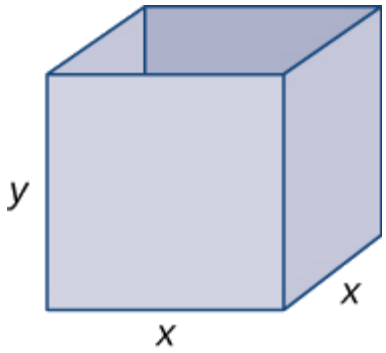

Figure 3.6.8**:** *We want to minimize the surface area of a square-based box with a given volume.*

Identify: We need to minimize the surface area. Therefore, we need to minimize S.

Master Equation: Since the box has an open top, we need only determine the area of the four vertical sides and the base. The area of each of the four vertical sides is $x \cdot y$. The area of the base is x^2. Therefore, the surface area of the box is

$$S = 4xy + x^2.$$

Since the volume of this box is x^2y and the volume is given as $216\ \text{in}^3$, the constraint equation is

$$x^2y = 216.$$

Solving the constraint equation for y, we have $y = \frac{216}{x^2}$. Therefore, we can write the surface area as a function of x only:

$$S(x) = 4x\left(\frac{216}{x^2}\right) + x^2.$$

Therefore, $S(x) = \frac{864}{x} + x^2$.

Realistic Domain: Since we are requiring that $x^2y = 216$, we cannot have $x = 0$. Therefore, we need $x > 0$. On the other hand, x is allowed to have any positive value. Note that as x becomes large, the height of the box y becomes correspondingly small so that $x^2y = 216$. Similarly, as x becomes small, the height of the box becomes correspondingly large. We conclude that the domain is the open, unbounded interval $(0, \infty)$. Unlike the previous examples, we cannot reduce our problem to looking for an absolute maximum or minimum over a closed, bounded interval. However, in the next step, we discover why this function must have an absolute minimum over the interval $(0, \infty)$.

Optimize: Note that as $x \to 0^+,\ S(x) \to \infty$. Also, as $x \to \infty,\ S(x) \to \infty$. Since S is a continuous function that approaches infinity at the ends, it must have an absolute minimum at some $x \in (0, \infty)$. This minimum must occur at a critical point of S. The derivative is

$$S'(x) = -\frac{864}{x^2} + 2x.$$

Therefore, $S'(x) = 0$ when $2x = \frac{864}{x^2}$. Solving this equation for x, we obtain $x^3 = 432$, so $x = \sqrt[3]{432} = 6\sqrt[3]{2}$. Since this is the only critical point of S, the absolute minimum must occur at $x = 6\sqrt[3]{2}$ (see Figure 3.6.9).

When $x = 6\sqrt[3]{2}$, $y = \frac{216}{(6\sqrt[3]{2})^2} = 3\sqrt[3]{2}$ in. Therefore, the dimensions of the box should be $x = 6\sqrt[3]{2}$ in. and $y = 3\sqrt[3]{2}$ in. With these dimensions, the surface area is

$$S(6\sqrt[3]{2}) = \frac{864}{6\sqrt[3]{2}} + (6\sqrt[3]{2})^2 = 108\sqrt[3]{4}\,\text{in}^2$$

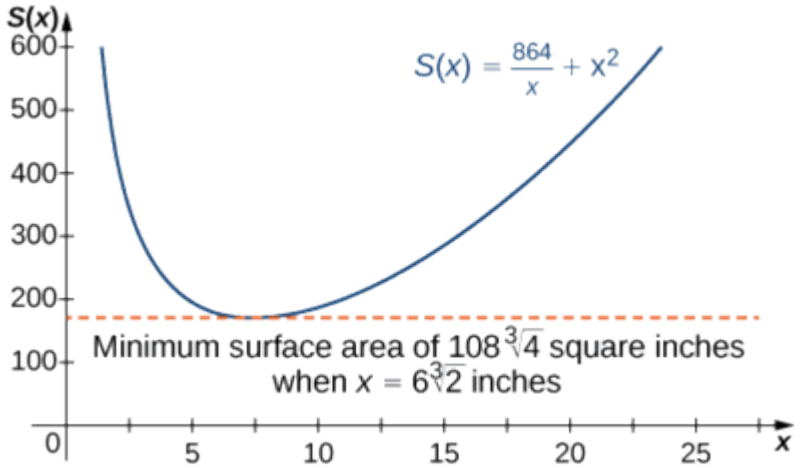

Figure 3.6.9**:** *We can use a graph to determine the dimensions of a box given the volume and the minimum surface area.*

Checkpoint 3.6.6

$c(x) = \frac{259.2}{x} + 0.2x^2$ dollars

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3.6.1: Resources and Key Concepts

Resources

Videos

- Polya's problem-solving strategy
  - Tactic for Setting Up and Solving Optimization Problems in Calculus
- Optimization Problems
  - Minimizing Distance
  - Maximizing Areas
  - Maximizing Volume
  - Minimizing Time
  - Maximizing an Angle
  - Maximizing Revenue the Non-Calculus Way

Key Concepts

Definitions

- **Optimization Problems**: Problems that involve finding the maximum or minimum value of a particular quantity, often subject to certain auxiliary conditions or constraints.
- **Constraint Equation**: An equation that represents an auxiliary condition that must be satisfied in an optimization problem (e.g., fixed perimeter, fixed volume).
- **Master Equation (Objective Function)**: The equation for the quantity that is to be maximized or minimized.
- **Realistic Domain**: The set of possible values for the independent variable in the master equation that make sense in the context of the real-world problem.

Common Mistakes

- **Incorrectly Setting Up the Master Equation or Constraint Equation**: Errors in translating the word problem into mathematical equations.
- **Not Considering the Realistic Domain**: Failing to identify the appropriate domain for the variable being optimized, which can affect whether endpoints need to be checked or if the Extreme Value Theorem applies.
- **Errors in Differentiation**: Mistakes when finding the derivative of the master equation.
- **Errors in Solving for Critical Points**: Algebraic mistakes when solving $f'(x) = 0$.
- **Only Finding Local Extrema Instead of Absolute Extrema**: Forgetting to check endpoints of a closed interval or failing to analyze the behavior of the function over its entire realistic domain to ensure the found extremum is absolute.
- **Not Answering the Question Asked**: For example, finding the value of x that optimizes a quantity but not finding the maximum/minimum value itself, or not finding all required dimensions.
- **Assuming a Critical Point is the Required Extremum Without Verification**: Not using the First or Second Derivative Test, or endpoint evaluation, to confirm that a critical point yields a maximum when a maximum is sought (or a minimum when a minimum is sought).

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3.6.2: Homework

Reading Questions

1. What is the general goal of an optimization problem?
2. What is a "master equation" in the context of Polya's problem-solving process for optimization?
3. What is a "constraint equation," and how is it typically used in optimization problems?
4. Why is it important to determine the "realistic domain" of the function being optimized?
5. If the realistic domain for an optimization problem is a closed, bounded interval and the function is continuous, which theorem guarantees the existence of an absolute maximum and minimum? Where can these occur?
6. How are critical points used in finding absolute extrema on a closed, bounded interval?
7. In Example 1 (maximizing garden area), what was the master equation and what was the constraint equation?
8. In Example 2 (maximizing box volume), how was the realistic domain for x (the side length of the cut square) determined?
9. If the domain of an optimization problem is an open or unbounded interval (as in Example 6, minimizing surface area of a box), how might you justify that a critical point yields an absolute extremum?
10. What does it mean if the derivative of the function to be optimized is zero at a certain point? Does this automatically mean it's a maximum or minimum?

Homework

For exercises 1 - 4, answer by proof, counterexample, or explanation.

1) When you find the maximum for an optimization problem, why do you need to check the sign of the derivative around the critical points?

Answer

The critical points can be the minima, maxima, or neither.

2) Why do you need to check the endpoints for optimization problems?

3) True or False. For every continuous nonlinear function, you can find the value x that maximizes the function.

Answer

False; $y = -x^2$ has a minimum only

4) True or False. For every continuous non-constant function on a closed, finite domain, there exists at least one x that minimizes or maximizes the function.

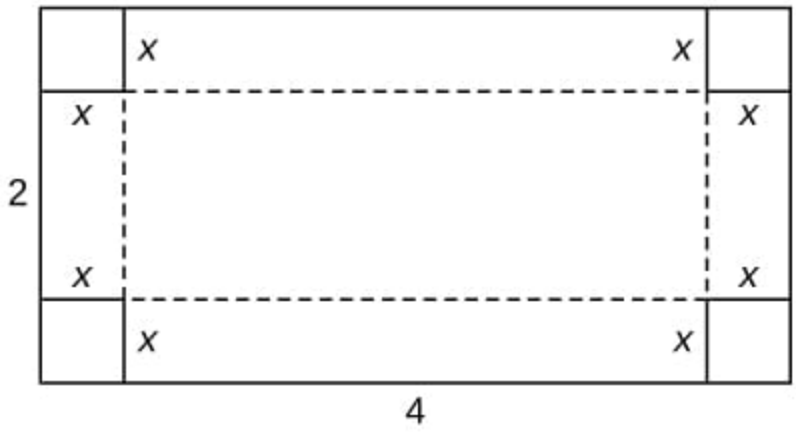
In exercises 5 - 8, set up and evaluate each optimization problem.

5) To carry a suitcase on an airplane, the $\text{length} + \text{width} + \text{height}$ of the box must be less than or equal to 62 in. Assuming the height is fixed, show that the maximum volume is $V = h\left(31 - \frac{1}{2}h\right)^2$. What height allows you to have the largest volume?

Answer

$h = \frac{62}{3}$ in.

6) You are constructing a cardboard box with the dimensions 2 m by 4 m. You then cut equal-size squares from each corner so you may fold the edges. What are the dimensions of the box with the largest volume?



7) Find the positive integer that minimizes the sum of the number and its reciprocal.

Answer

1

8) Find two positive integers such that their sum is 10, and minimize and maximize the sum of their squares.

In exercises 9 - 11, consider the construction of a pen to enclose an area.

9) You have $400\ \text{ft}$ of fencing to construct a rectangular pen for cattle. What are the dimensions of the pen that maximize the area?

Answer

$100\ \text{ft}$ by $100\ \text{ft}$

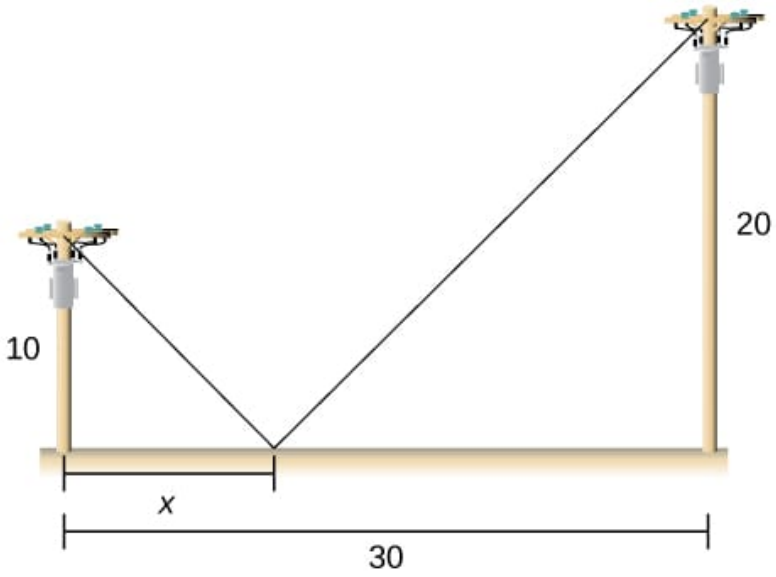
10) You have $800\ \text{ft}$ of fencing to make a pen for hogs. If you have a river on one side of your property, what is the dimension of the rectangular pen that maximizes the area?

11) You need to construct a fence around an area of $1600\ \text{ft}^2$. What are the dimensions of the rectangular pen to minimize the amount of material needed?

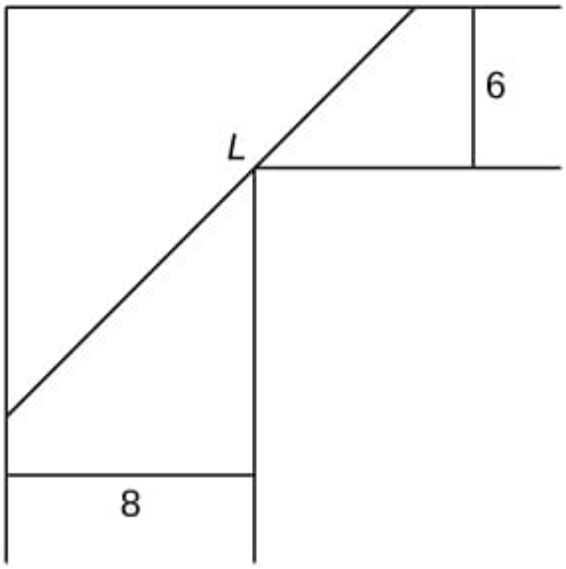
Answer

$40\ \text{ft}$ by $40\ \text{ft}$

12) Two poles are connected by a wire that is also connected to the ground. The first pole is $20\ \text{ft}$ tall and the second pole is $10\ \text{ft}$ tall. There is a distance of $30\ \text{ft}$ between the two poles. Where should the wire be anchored to the ground to minimize the amount of wire needed?



13) [Technology Required] You are moving into a new apartment and notice there is a corner where the hallway narrows from 8 ft to 6 ft. What is the length of the longest item that can be carried horizontally around the corner?



Answer

19.73 ft

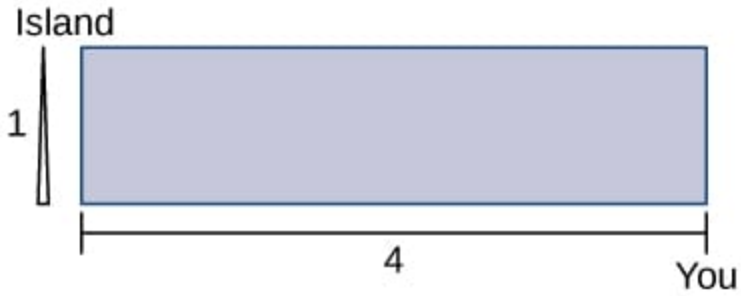
14) A patient's pulse measures 70 bpm, 80 bpm, then 120 bpm. To determine an accurate measurement of pulse, the doctor wants to know what value minimizes the expression $(x-70)^2+(x-80)^2+(x-120)^2$? What value minimizes it?

15) In the previous problem, assume the patient was nervous during the third measurement, so we only weight that value half as much as the others. What is the value that minimizes $(x-70)^2+(x-80)^2+\frac{1}{2}(x-120)^2$?

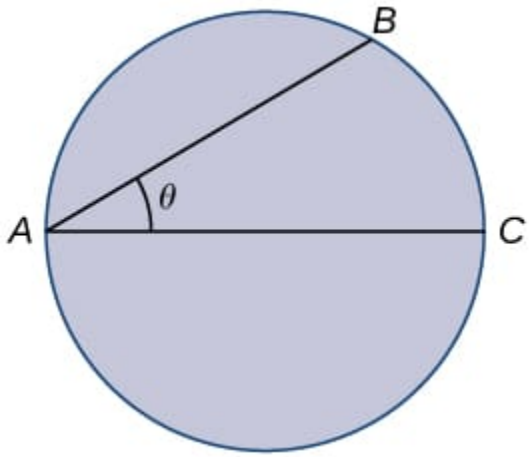
Answer

84 bpm

16) You can run at a speed of 6 mph and swim at a speed of 3 mph and are located on the shore, 4 miles east of an island that is 1 mile north of the shoreline. How far should you run west to minimize the time needed to reach the island?



For exercises 17 - 19, consider a lifeguard at a circular pool with diameter 40 m. He must reach someone who is drowning on the exact opposite side of the pool, at position C. The lifeguard swims with a speed v and runs around the pool at speed $w = 3v$.



17) Find a function that measures the total amount of time it takes to reach the drowning person as a function of the swim angle, θ.

Answer

$$T(\theta) = \frac{40\theta}{3v} + \frac{40\cos\theta}{v}$$

18) Find at what angle θ the lifeguard should swim to reach the drowning person in the least amount of time.

19) A truck uses gas as $g(v) = av + \dfrac{b}{v}$, where v represents the speed of the truck and g represents the gallons of fuel per mile. At what speed is fuel consumption minimized?

Answer

$$v = \sqrt{\frac{b}{a}}$$

For exercises 20 - 21, consider a limousine that gets $m(v) = \frac{120-2v}{5}$ $\mathrm{mi/gal}$ at speed v, the chauffeur costs $\$15/\mathrm{h}$, and gas is $\$3.50/\mathrm{gal}$.

20) Find the cost per mile at speed v.

21) Find the cheapest driving speed.

Answer

approximately 34.02 mph

For exercises 22 - 24, consider a pizzeria that sell pizzas for a revenue of $R(x) = ax$ and costs $C(x) = b + cx + dx^2$, where x represents the number of pizzas.

22) Find the profit function for the number of pizzas. How many pizzas gives the largest profit per pizza?

23) Assume that $R(x) = 10x$ and $C(x) = 2x + x^2$. How many pizzas sold maximizes the profit?

Answer

Selling 4 pizzas will maximize the profit.

24) Assume that $R(x) = 15x$, and $C(x) = 60 + 3x + \frac{1}{2}x^2$. How many pizzas sold maximizes the profit?

For exercises 25 - 26, consider a wire 4 ft long cut into two pieces. One piece forms a circle with radius r and the other forms a square of side x.

25) Choose x to maximize the sum of their areas.

Answer

$x = 0$

26) Choose x to minimize the sum of their areas.

For exercises 27 - 30, consider two nonnegative numbers x and y such that $x + y = 10$. Maximize and minimize the quantities.

27) xy

Answer

Maximal: $x = 5,\ \ y = 5$;
Minimal: $x = 0,\ \ y = 10$ and $y = 0,\ \ x = 10$

28 x^2y^2

29) $y - \dfrac{1}{x}$

Answer

Maximal: $x = 1,\ \ y = 9$;
Minimal: none

30) $x^2 - y$

In exercises 31 - 36, draw the given optimization problem and solve.

31) Find the volume of the largest right circular cylinder that fits in a sphere of radius 1.

Answer

$V = \frac{4\pi}{3\sqrt{3}}$

32) Find the volume of the largest right cone that fits in a sphere of radius 1.

33) Find the area of the largest rectangle that fits into the triangle with sides $x = 0,\ \ y = 0$ and $\dfrac{x}{4} + \dfrac{y}{6} = 1$.

Answer

$A = 6$

34) Find the largest volume of a cylinder that fits into a cone that has base radius R and height h.

35) Find the dimensions of the closed cylinder volume $V = 16\pi$ that has the least amount of surface area.

Answer

$r = 2,\ \ h = 4$

36) Find the dimensions of a right cone with surface area $S = 4\pi$ that has the largest volume.

For exercises 37 - 40, consider the points on the graphs of the given equations. Use a calculator to graph the functions.

37) [Technology Required] Where is the line $y = 5 - 2x$ closest to the origin?

Answer

$(2, 1)$

38) [Technology Required] Where is the line $y = 5 - 2x$ closest to point $(1, 1)$?

39) [Technology Required] Where is the parabola $y = x^2$ closest to point $(2, 0)$?

Answer

$(0.8351, 0.6974)$

40) [Technology Required] Where is the parabola $y = x^2$ closest to point $(0, 3)$?

In exercises 41 - 45, set up, but do not evaluate, each optimization problem.

41) A window is composed of a semicircle placed on top of a rectangle. If you have $20\ \text{ft}$ of window-framing materials for the outer frame, what is the maximum size of the window you can create? Use r to represent the radius of the semicircle.

Answer

$A = 20r - 2r^2 - \frac{1}{2}\pi r^2$

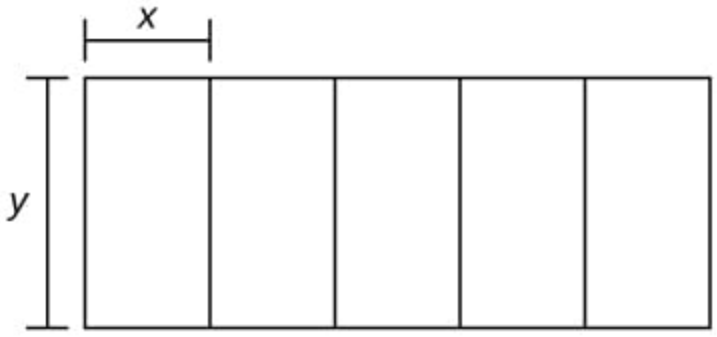
42) You have a garden row of 20 watermelon plants that produce an average of 30 watermelons apiece. For any additional watermelon plants planted, the output per watermelon plant drops by one watermelon. How many extra watermelon plants should you plant?

43) You are constructing a box for your cat to sleep in. The plush material for the square bottom of the box costs $\$5/\text{ft}^2$ and the material for the sides costs $\$2/\text{ft}^2$. You need a box with volume $4\ \text{ft}^3$. Find the dimensions of the box that minimize cost. Use x to represent the length of the side of the box.

Answer

$C(x) = 5x^2 + \dfrac{32}{x}$

44) You are building five identical pens adjacent to each other with a total area of $1000\ \text{m}^2$, as shown in the following figure. What dimensions should you use to minimize the amount of fencing?



45) You are the manager of an apartment complex with 50 units. When you set rent at $\$800/\text{month}$, all apartments are rented. As you increase rent by $\$25/\text{month}$, one fewer apartment is rented. Maintenance costs run $\$50/\text{month}$ for each occupied unit. What is the rent that maximizes the total amount of profit?

Answer

$$P(x) = (50 - x)(800 + 25x - 50)$$

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3.7: Newton's Method

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

In many areas of Pure and Applied Mathematics, we are interested in finding solutions to an equation of the form $f(x) = 0$. For most functions, however, it is difficult—if not impossible—to calculate their zeros explicitly. In this section, we look at a technique that provides a very efficient way of approximating the zeros of functions. This technique uses tangent line approximations, which most calculators and computers use to find zeros.

Describing Newton's Method

Consider the task of finding the solutions of $f(x) = 0$. If f is the first-degree polynomial $f(x) = ax + b$, then the solution of $f(x) = 0$ is given by the formula $x = -\frac{b}{a}$. If f is the second-degree polynomial $f(x) = ax^2 + bx + c$, the solutions of $f(x) = 0$ can be found by using the Quadratic Formula. However, for polynomials of degree 3 or more, finding roots of f becomes more complicated. Although formulas exist for third- and fourth-degree polynomials, they are quite complicated. Also, if f is a polynomial of degree 5 or greater, it is known that no such formulas exist. For example, consider the function

$$f(x) = x^5 + 8x^4 + 4x^3 - 2x - 7.$$

No formula exists that allows us to find the solutions of $f(x) = 0$. Similar difficulties exist for non-polynomial functions. For example, consider finding solutions of $\tan(x) - x = 0$. No simple formula exists for the solutions of this equation. We can use Newton's Method to approximate the roots in such cases.

Newton's Method makes use of the following idea to approximate the solutions of $f(x) = 0$. By sketching a graph of f, we can estimate a root of $f(x) = 0$. Let's call this estimate x_0. We then draw the tangent line to f at x_0. If $f'(x_0) \neq 0$, this tangent line intersects the x-axis at some point $(x_1, 0)$. Now, let x_1 be the next approximation to the actual root. Typically, x_1 is closer than x_0 to an actual root. Next, we draw the tangent line to f at x_1. If $f'(x_1) \neq 0$, this tangent line also intersects the x-axis, producing another approximation, x_2. We continue in this way, deriving a list of approximations: x_0, x_1, x_2, ... Typically, the numbers x_0, x_1, x_2, ... quickly approach an actual root x^*, as shown in the following figure.

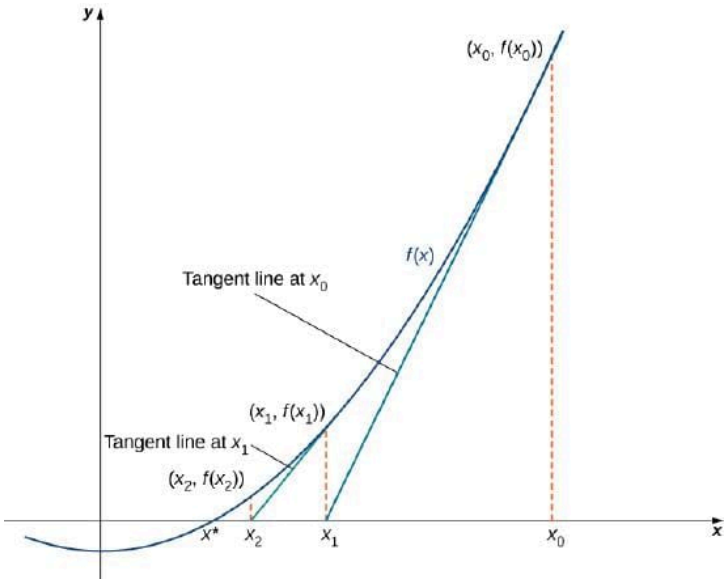

Figure 3.7.1**:** *The approximations* $x_0,\ x_1,\ x_2,\ \ldots$ *approach the actual root* x^**. The approximations are derived by looking at tangent lines to the graph of* f*.*

Now let's look at how to calculate the approximations x_0, x_1, x_2, ... If x_0 is our first approximation, the approximation x_1 is defined by letting $(x_1, 0)$ be the x-intercept of the tangent line to f at x_0. The equation of this tangent line is given by

$$y = f(x_0) + f'(x_0)(x - x_0).$$

Therefore, x_1 must satisfy

$$f(x_0) + f'(x_0)(x_1 - x_0) = 0.$$

Solving this equation for x_1, we conclude that

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Similarly, the point $(x_2, 0)$ is the x-intercept of the tangent line to f at x_1. Therefore, x_2 satisfies the equation

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

In general, for $n > 0, x_n$ satisfies

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}. \tag{3.7.1}$$

Next, we see how to make use of this technique to approximate the root of the polynomial $f(x) = x^3 - 3x + 1$.

Example 3.7.1

Use Newton's Method to approximate a root of $f(x) = x^3 - 3x + 1$ in the interval $[1, 2]$. Let $x_0 = 2$ and find x_1, x_2, x_3, x_4, and x_5.

Solution available at the end of this section.

Checkpoint 3.7.1

Letting $x_0 = 0$, let's use Newton's Method to approximate the root of $f(x) = x^3 - 3x + 1$ over the interval $[0, 1]$ by calculating x_1 and x_2.

Answer available at the end of this section.

Interactive Element: Newton's Method — the Tangent-Line Cascade

The following Interactive Element might help you visualize how each tangent line's x-intercept becomes the next approximation x_n, and how much the starting guess x_0 matters.

Interact: Drag the orange point x_0 along the x-axis to choose a starting guess for a root of $f(x) = x^3 - 3x + 1$.

Observation: For most starting guesses, the intercepts $x_1,\ x_2,\ x_3$ close in on a root within a few steps. Start near a spot where the curve levels off (where $f'(x_0) = 0$) and the next guess is flung far away; start near a different root and the cascade may settle there instead.

Newton's Method can also approximate square roots. Here we show how to approximate $\sqrt{2}$. This method can be modified to approximate the square root of any positive number.

Example 3.7.2

Use Newton's Method to approximate $\sqrt{2}$ (Figure 3.7.3). Let $f(x) = x^2 - 2$, let $x_0 = 2$, and calculate x_1, x_2, x_3, x_4, and x_5. (We note that since $f(x) = x^2 - 2$ has a zero at $\sqrt{2}$, the initial value $x_0 = 2$ is a reasonable choice to approximate $\sqrt{2}$).

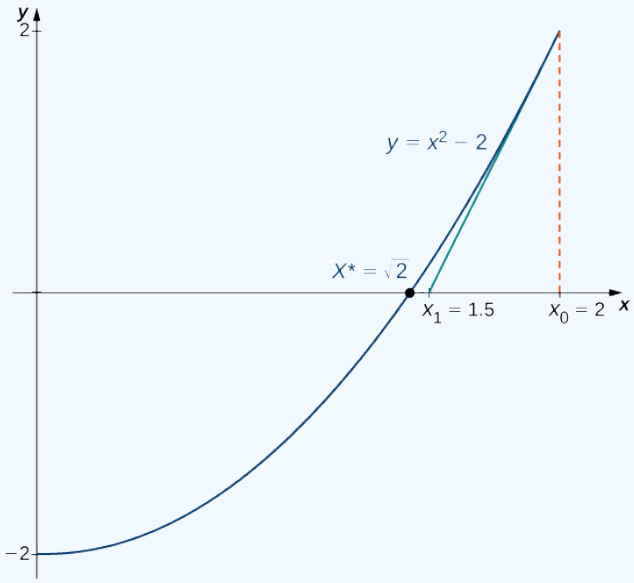

Figure 3.7.3: *We can use Newton's Method to find $\sqrt{2}$.*

Solution available at the end of this section.

Checkpoint 3.7.2

Use Newton's Method to approximate $\sqrt{3}$ by letting $f(x) = x^2 - 3$ and $x_0 = 3$. Find x_1 and x_2.

Answer available at the end of this section.

When using Newton's Method, each approximation after the initial guess is defined in terms of the previous approximation by using the same formula. In particular, by defining the function $F(x) = x - \left[\frac{f(x)}{f'(x)}\right]$, we can rewrite Equation 3.7.1 as $x_n = F(x_{n-1})$. This type of process, where each x_n is defined in terms of x_{n-1} by repeating the same function, is an example of an **iterative process**. Shortly, we examine other iterative processes. First, let's look at why Newton's Method could fail to find a root.

Failures of Newton's Method

Typically, Newton's Method finds roots reasonably quickly. However, things can go wrong. Some reasons why Newton's Method might fail include the following:

1. At one of the approximations x_n, the derivative f' is zero (or very close to 0) at x_n, but $f(x_n) \neq 0$. As a result, the tangent line of f at x_n does not intersect the x-axis (or, if it does, it is far away from the intended root). Therefore, we cannot continue the iterative process.
2. The approximations x_0, x_1, x_2, ... may approach a different root. If the function f has more than one root, it is possible that our approximations do not approach the one for which we are looking but approach a different root (see Figure $3.7.4$). This event often occurs when we do not choose the approximation x_0 close enough to the desired root.
3. The approximations may fail to approach a root entirely. In Example $3.7.3$, we provide an example of a function and an initial guess x_0 such that the successive approximations never approach a root because the successive approximations continue to alternate back and forth between two values.

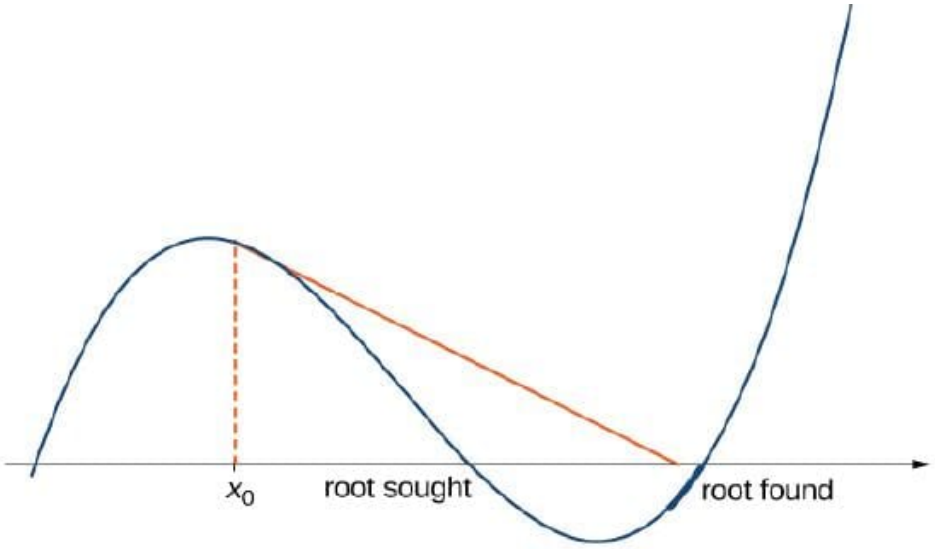

Figure $3.7.4$**:** *If the initial guess x_0 is too far from the root sought, it may lead to approximations that approach a different root.*

Example $3.7.3$

Consider the function $f(x) = x^3 - 2x + 2$. Let $x_0 = 0$. Show that the sequence x_1, x_2, ... fails to approach a root of f.

Solution available at the end of this section.

Checkpoint $3.7.3$

For $f(x) = x^3 - 2x + 2$, let $x_0 = -1.5$ and find x_1 and x_2.

Answer available at the end of this section.

From Example $3.7.3$, we see that Newton's Method does not always work. However, when it does work, the sequence of approximations approaches the root very quickly. Discussions of how quickly the sequence of approximations approach a root found using Newton's Method are included in texts on Numerical Analysis.

Other Iterative Processes

As mentioned earlier, Newton's Method is an iterative process. We now look at an example of a different kind of iterative process.

Consider a function F and an initial number x_0. Define the subsequent numbers x_n by the formula $x_n = F(x_{n-1})$. This process is an **iterative process** that creates a list of numbers x_0, x_1, x_2, ..., x_n, ... This list of numbers may approach a finite number x^* as n gets larger, or it may not. In Example 3.7.4, we see an example of a function F and an initial guess x_0 such that the resulting list of numbers approaches a finite value.

Example 3.7.4

Let $F(x) = \frac{1}{2}x + 4$ and let $x_0 = 0$. For all $n \geq 1$, let $x_n = F(x_{n-1})$. Find the values x_1, x_2, x_3, x_4, and x_5. Make a conjecture about what happens to this list of numbers x_1, x_2, ..., x_n, ... as $n \to \infty$. If the list of numbers x_1, x_2, ... approaches a finite number x^*, then x^* satisfies $x^* = F(x^*)$, and x^* is called a fixed point of F.

Solution available at the end of this section.

Interactive Element: Iterating to a Fixed Point

The following Interactive Element might help you visualize why the list $x_0,\ x_1,\ x_2, \ldots$ marches to the value where F crosses the line $y = x$.

Interact: Drag the orange point x_0 along the x-axis to change where the staircase starts.

Observation: Wherever you start, the staircase climbs (or drops) to the same corner—the point where $F(x) = \frac{1}{2}x + 4$ meets $y = x$, namely $x^* = 8$. That crossing is the fixed point $x^* = F(x^*)$.

Checkpoint 3.7.4

Consider the function $F(x) = \frac{1}{3}x + 6$. Let $x_0 = 0$ and let $x_n = F(x_{n-1})$ for $n \geq 2$. Find $x_1,\ x_2,\ x_3,\ x_4,\ x_5$. Make a conjecture about what happens to the list of numbers $x_1,\ x_2,\ x_3,\ \ldots\ x_n,\ \ldots$ as $n \to \infty$.

Answer available at the end of this section.

Iterative Processes and Chaos

Iterative processes can yield some fascinating behavior. In this section, we have seen several examples of iterative processes that converge to a fixed point. We also saw in Example 3.7.3 that the iterative process bounced back and forth between two values. We call this kind of behavior a 2-cycle. Iterative processes can converge to cycles with various periodicities, such as 2-cycles, 4-cycles (where the iterative process repeats a sequence of four values), 8-cycles, and so on.

Some iterative processes yield what mathematicians call chaos. In this case, the iterative process randomly jumps from value to value and never converges nor settles into a cycle. Although a complete exploration of **chaos** is beyond the scope of this text, in this project we look at one of the key properties of a chaotic iterative process: sensitive dependence on initial conditions. This property refers to the concept that small changes in initial conditions can generate drastically different behavior in the iterative process.

Probably the best-known example of chaos is the **Mandelbrot set** (see Figure 3.7.7), named after Benoit Mandelbrot (1924–2010), who investigated its properties and helped popularize the field of chaos theory. The Mandelbrot set is usually generated by computer and shows fascinating details on enlargement, including self-replication of the set. Several colorized versions of the set have been shown in museums and can be found online and in popular books on the subject.

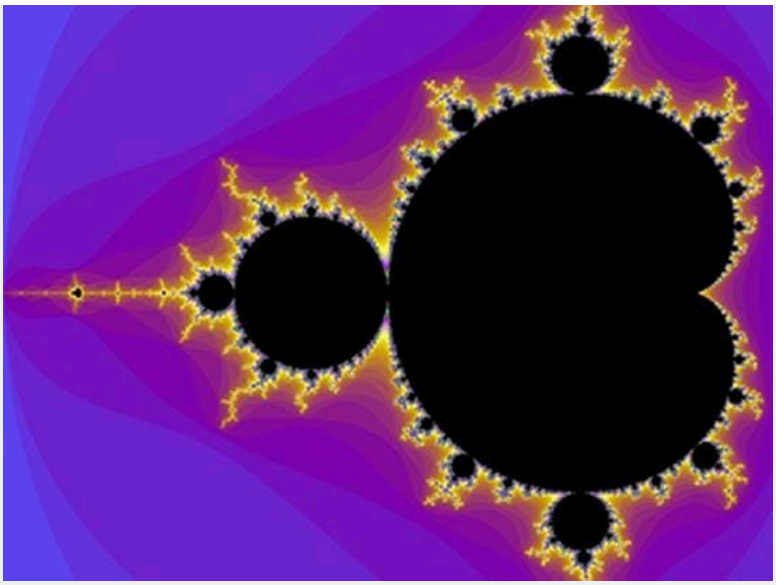

Figure 3.7.7**:** *The Mandelbrot set is a well-known example of a set of points generated by the iterative chaotic behavior of a relatively simple function.*

In this project we use the logistic map

$$f(x) = rx(1 - x)$$

where $x \in [0, 1]$ and $r > 0$ as the function in our iterative process. The logistic map is a deceptively simple function. Still, depending on the value of r, the resulting iterative process displays some fascinating behavior. It can lead to fixed points, cycles, and even chaos.

To visualize the long-term behavior of the iterative process associated with the logistic map, we will use a cobweb diagram tool. As we did with the iterative process we examined earlier in this section, we first draw a vertical line from the point $(x_0, 0)$ to the point $(x_0, f(x_0)) = (x_0, x_1)$. We then draw a horizontal line from that point to the point (x_1, x_1), then draw a vertical line to $(x_1, f(x_1)) = (x_1, x_2)$, and continue the process until the long-term behavior of the system becomes apparent. Figure 3.7.8 shows the long-term behavior of the logistic map when $r = 3.55$ and $x_0 = 0.2$. (The first 100 iterations are not plotted.) The long-term behavior of this iterative process is an 8-cycle.

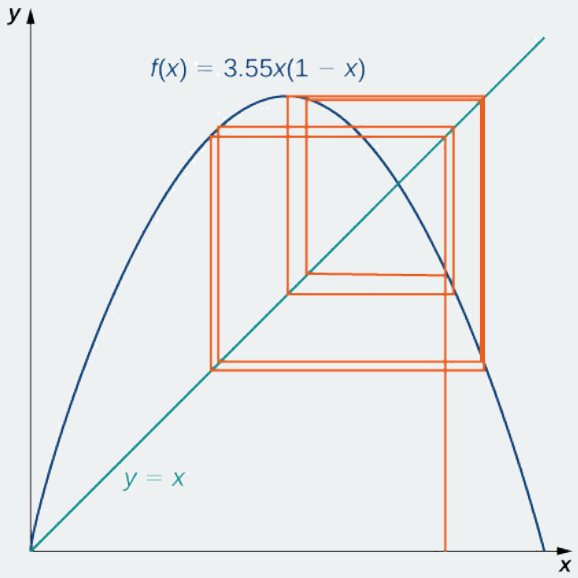

Figure 3.7.8**:** *A cobweb diagram for $f(x) = 3.55x(1 - x)$ is presented here. The sequence of values results in an 8-cycle.*

Interactive Element: The Logistic Map — From a Fixed Point to Chaos

The following Interactive Element might help you visualize how a single parameter r turns the iteration $x_n = rx_{n-1}(1 - x_{n-1})$ into a fixed point, a cycle, or chaos.

Interact: Drag the green point at the top of the curve up or down to change r (the curve's height is $r/4$); drag the orange point to set the starting value x_0.

Observation: For small r the cobweb spirals into a single point. Raise r and it splits into a 2-cycle, then a 4-cycle; near $r = 4$ it scatters and never settles. At $r = 4$, nudging x_0 by a hair sends the path somewhere completely different.

1. Let $r = 0.5$ and choose $x_0 = 0.2$. Either by hand or by using a computer, calculate the first 10 values in the sequence. Does the sequence appear to converge? If so, to what value? Does it result in a cycle? If so, what kind of cycle (for example, 2-cycle, 4-cycle.)?
2. What happens when $r = 2$?
3. For $r = 3.2$ and $r = 3.5$, calculate the first 100 sequence values. Generate a cobweb diagram for each iterative process. (Several free applets that generate cobweb diagrams for the logistic map are available online.) What is the long-term behavior in each of these cases?
4. Now let $r = 4$. Calculate the first 100 sequence values and generate a cobweb diagram. What is the long-term behavior in this case?
5. Repeat the process for $r = 4$, but let $x_0 = 0.201$. How does this behavior compare with the behavior for $x_0 = 0.2$?

Miscellaneous Topics: How Does Math Relate to Your Field?

1. Please tell me what degree you are currently pursuing.
2. Name at least five things in your chosen field with which mathematics can help.

Success in College: To Be Smart or Not To Be Smart... Is This Your Question?

> When I attended CRC, I had a computer science professor who had some great sayings. One of my favorites occurred when a student raised her hand and said, in a sheepish voice, "This may be a stupid question, but..."
>
> The brilliantly opinionated professor immediately interrupted the student and stated, in a voice beaming with inspiring confidence, "There are no stupid questions... only stupid people who don't ask them."
>
> "That is so true," I thought to myself. Here we are, learning, experiencing, and becoming better citizens of this great wide world. Still, most students refuse to involve themselves in their education by remaining silent when they should be speaking up. This single abrasive comment by the professor summed up the majority of my fellow students!
>
> I immediately swore to myself not to be shy anymore about asking questions. If I annoy my neighbor by slowing the lecture, so be it! If I look like I am lost in front of others, that's a small price to pay to gain understanding.
>
> And so it went... I carried this habit throughout the universities I attended, in every course, with every professor. As time went on, and I was in classes with more capable students because those earlier students fell by the wayside, I found I was surrounded by those with the same habit. It turned out that one of the (many) keys to success on the road to becoming a critical thinker was being engaged and asking questions.
>
> - Professor Simpson, Cosumnes River College

So what about you? Are you willing to let go of that social stigma that asking questions makes you look "dumb?" Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. In some high schools, it is not "socially acceptable" to act as if you care about classwork. Students do not ask for help in front of others. This attitude may or may not continue in beginning college classes, even for older students. Is this a typical attitude at your college?
2. Some professors check for understanding by calling on students in class. This way, the professor knows if more examples or discussions are needed. Some students have no problem answering questions in front of the class. Other students struggle to answer in front of the class even if they think they know the correct answer. How do you react when a professor asks you a question in front of the class? Why?

Answers

Section Preview

This material, while often considered optional, is quick to cover and adds a lot for those students who are considering entering into Computer Science. It's a great preview of iterative schemes. I *highly* recommend you cover it, if you have the time.

Section Preview

- **Graphing Functions**
  - Intercepts and Base Graphs: The whole premise is finding a root of f—that is, solving $f(x) = 0$ / locating an x-intercept; students must understand that each x_n is the x-intercept of a tangent line.
- **Sequences and Series**
  - The Basics of Sequences: An iterative process is a recursively-defined list $x_n = F(x_{n-1})$; familiarity with recursive sequence definitions grounds the idea of generating each term from the previous one.

Section Preview

- **Solving Equations**
  - Solving Linear Equations — Multistep with Variables on Both Sides: Locating the fixed point in Example 4 requires solving $x = \frac{1}{2}x + 4$ (yielding $x = 8$); Checkpoint 4 similarly solves $x = \frac{1}{3}x + 6$.

Section Preview

- Describe the steps of Newton's Method.
- Explain what an iterative process means.
- Recognize when Newton's Method does not work.
- Apply iterative processes to various situations.

Example 3.7.1

From Figure 3.7.2, we see that f has one root over the interval $[1, 2]$. Therefore, $x_0 = 2$ seems like a reasonable first approximation. To find the next approximation, we use Equation 3.7.1. Since $f(x) = x^3 - 3x + 1$, the derivative is $f'(x) = 3x^2 - 3$. Using Equation 3.7.1 with $n = 1$ (and a calculator that displays 10 digits), we obtain

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{3}{9} \approx 1.666666667.$$

To find the next approximation, x_2, we use Equation 3.7.1 with $n = 2$ and the value of x_1 stored on the calculator. We find that

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 1.548611111.$$

Continuing in this way, we obtain the following results:

- $x_1 \approx 1.666666667$
- $x_2 \approx 1.548611111$
- $x_3 \approx 1.532390162$
- $x_4 \approx 1.532088989$
- $x_5 \approx 1.532088886$
- $x_6 \approx 1.532088886$.

We note that we obtained the same value for x_5 and x_6. Therefore, any subsequent application of Newton's Method will likely give the same value for x_n.

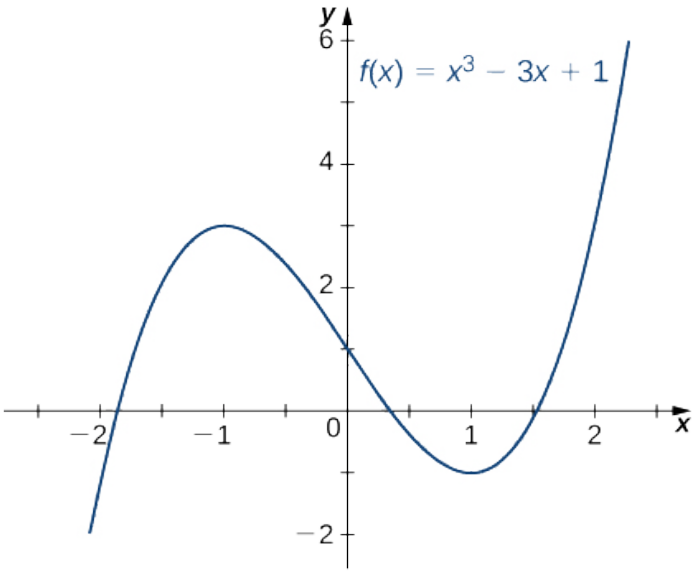

Figure 3.7.2**:** *The function* $f(x) = x^3 - 3x + 1$ *has one root over the interval* $[1, 2]$.

Checkpoint 3.7.1

$x_1 \approx 0.33333333$
$x_2 \approx 0.347222222$

Example 3.7.2

For $f(x) = x^2 - 2$, $f'(x) = 2x$. From Equation 3.7.1, we know that

$$\begin{aligned} x_n &= x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \\ &= x_{n-1} - \frac{x_{n-1}^2 - 2}{2x_{n-1}} \\ &= \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}} \\ &= \frac{1}{2}\left(x_{n-1} + \frac{2}{x_{n-1}}\right). \end{aligned}$$

Therefore,

$$x_1 = \frac{1}{2}\left(x_0 + \frac{2}{x_0}\right) = \frac{1}{2}\left(2 + \frac{2}{2}\right) = 1.5$$
$$x_2 = \frac{1}{2}\left(x_1 + \frac{2}{x_1}\right) = \frac{1}{2}\left(1.5 + \frac{2}{1.5}\right) \approx 1.416666667.$$

Continuing in this way, we find that

$$\begin{aligned} x_1 &= 1.5 \\ x_2 &\approx 1.416666667 \\ x_3 &\approx 1.414215686 \\ x_4 &\approx 1.414213562 \\ x_5 &\approx 1.414213562 \end{aligned}$$

Since we obtained the same value for x_4 and x_5, it is unlikely that the value x_n will change on any subsequent application of Newton's Method. We conclude that $\sqrt{2} \approx 1.414213562$.

Checkpoint 3.7.2

$x_1 = 2$
$x_2 = 1.75$

Example 3.7.3

For $f(x) = x^3 - 2x + 2$, the derivative is $f'(x) = 3x^2 - 2$.Therefore,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{f(0)}{f'(0)} = -\frac{2}{-2} = 1.$$

In the next step,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{1} = 0.$$

Consequently, the numbers $x_0,\ x_1,\ x_2,\ \ldots$ continue to bounce back and forth between 0 and 1 and never get closer to the root of f which is over the interval $[-2, -1]$ (Figure 3.7.5). Fortunately, if we choose an initial approximation x_0 closer to the actual root, we can avoid this situation.

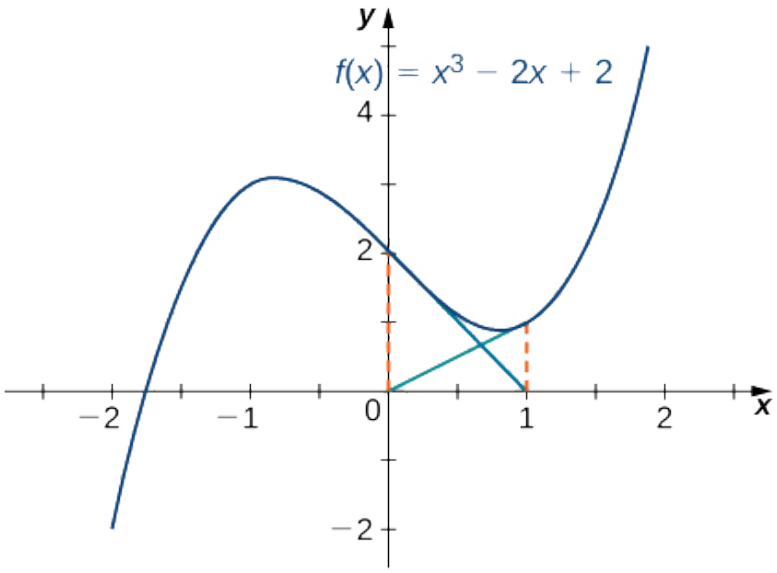

Figure 3.7.5**:** *The approximations continue to alternate between* 0 *and* 1 *and never approach the root of* f*.*

Checkpoint 3.7.3

$x_1 \approx -1.842105263$
$x_2 \approx -1.772826920$

Example 3.7.4

If $x_0 = 0$, then

- $x_1 = \frac{1}{2}(0) + 4 = 4$
- $x_2 = \frac{1}{2}(4) + 4 = 6$
- $x_3 = \frac{1}{2}(6) + 4 = 7$
- $x_4 = \frac{1}{2}(7) + 4 = 7.5$
- $x_5 = \frac{1}{2}(7.5) + 4 = 7.75$
- $x_6 = \frac{1}{2}(7.75) + 4 = 7.875$
- $x_7 = \frac{1}{2}(7.875) + 4 = 7.9375$
- $x_8 = \frac{1}{2}(7.9375) + 4 = 7.96875$
- $x_9 = \frac{1}{2}(7.96875) + 4 = 7.984375$.

From this list, we conjecture that the values x_n approach 8.

Figure 3.7.6 provides a graphical argument that the values approach 8 as $n \to \infty$. Starting at the point (x_0, x_0), we draw a vertical line to the point $(x_0, F(x_0))$. The next number in our list is $x_1 = F(x_0)$. We use x_1 to calculate x_2. Therefore, we draw a horizontal line connecting (x_0, x_1) to the point (x_1, x_1) on the line $y = x$, and then draw a vertical line connecting (x_1, x_1) to the point $(x_1, F(x_1))$. The output $F(x_1)$ becomes x_2. Continuing in this way, we could create infinite line segments. These line segments are trapped between the lines $F(x) = \frac{x}{2} + 4$ and $y = x$. The line segments get closer to the intersection point of these two lines, which occurs when $x = F(x)$. Solving the equation $x = \frac{x}{2} + 4$, we conclude they intersect at $x = 8$. Therefore, our graphical evidence agrees with our numerical evidence that the list of numbers $x_0,\ x_1,\ x_2,\ \ldots$ approaches $x^* = 8$ as $n \to \infty$.

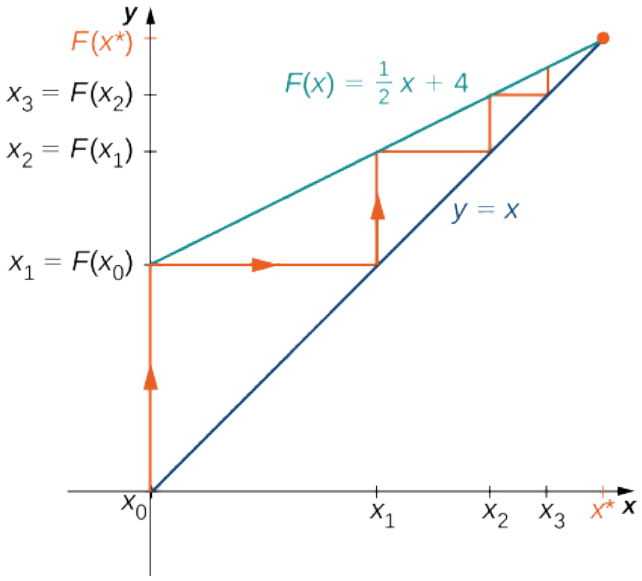

Figure 3.7.6**:** *This iterative process approaches the value* $x^* = 8$*.*

Checkpoint 3.7.4

$x_1 = 6,\ \ x_2 = 8,\ \ x_3 = \frac{26}{3},\ \ x_4 = \frac{80}{9},\ \ x_5 = \frac{242}{27};\ \ x^* = 9$

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3.7.1: Resources and Key Concepts

Resources

Videos

- Newton's Method
  - Newton's Method - Concept and Process
- Root finding and approximating solutions using Newton's Method
  - Using Newton's Method

Key Concepts

Definitions

- **Newton's Method Formula**: An iterative process to approximate a root of $f(x) = 0$, given by $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, where x_0 is an initial guess.
- **Iterative Process**: A process that creates a list of numbers $x_0, x_1, x_2, \ldots, x_n, \ldots$ where each term x_n (for $n \geq 1$) is defined in terms of the previous term x_{n-1} by repeating the same function or formula (e.g., $x_n = F(x_{n-1})$).
- **Fixed Point (of an iterative process)**: If an iterative process $x_n = F(x_{n-1})$ generates a list of numbers $x_1, x_2, \ldots$ that approaches a finite number x^*, then x^* satisfies $x^* = F(x^*)$, and x^* is called a fixed point of F.
- **2-cycle (in an iterative process)**: An iterative process where the sequence of values bounces back and forth between two distinct values.
- **Sensitive Dependence on Initial Conditions**: A property of chaotic iterative processes where small changes in initial conditions can generate drastically different behavior in the iterative process.
- **Mandelbrot Set**: A well-known example of a set of points generated by the iterative chaotic behavior of a relatively simple function.

Common Mistakes

- **Choosing a Poor Initial Guess (x_0)**:
  - If x_0 is too far from the desired root, Newton's method might converge to a different root or not converge at all.
  - If $f'(x_n) = 0$ (or is very close to 0) for some approximation x_n, the method fails because the tangent line is horizontal (or nearly so) and doesn't intersect the x-axis (or intersects it very far away).
- **Function Behavior Leading to Non-Convergence**:
  - The approximations may fail to approach a root entirely, possibly alternating between values.
- **Using Degree Mode for Trigonometric Functions**: If $f(x)$ involves trigonometric functions, calculators must be in radian mode.

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3.7.2: Homework

Reading Questions

1. What is the primary purpose of Newton's Method?
2. What is the formula for Newton's Method, $x_{n+1} = ...$?
3. Geometrically, how is the next approximation x_{n+1} found from the current approximation x_n in Newton's Method?
4. What is an "iterative process"? How does Newton's Method fit this description?
5. List three reasons why Newton's Method might fail to find a root or converge to the desired root.
6. What happens in Newton's Method if $f'(x_n) = 0$ for some approximation x_n?
7. Can Newton's Method be used to approximate square roots? If so, how would you set up the function $f(x)$ to approximate $\sqrt{k}$?
8. What is a "fixed point" of an iterative process $x_n = F(x_{n-1})$?
9. What is a "2-cycle" in an iterative process?
10. What does "sensitive dependence on initial conditions" mean in the context of chaotic iterative processes?

Homework

In exercises 1 - 5, write Newton's formula as $x_{n+1} = F(x_n)$ for solving $f(x) = 0$.

1) $f(x) = x^2 + 1$

2) $f(x) = x^3 + 2x + 1$

Answer

$$F(x_n) = x_n - \frac{x_n^3 + 2x_n + 1}{3x_n^2 + 2}$$

3) $f(x) = \sin x$

4) $f(x) = e^x$

Answer

$$F(x_n) = x_n - \frac{e^{x_n}}{e^{x_n}}$$

5) $f(x) = x^3 + 3xe^x$

In exercises 6 - 8, solve $f(x) = 0$ using the iteration $x_{n+1} = x_{n-c}f(x_n)$, which differs slightly from Newton's method. Find a c that works and a c that fails to converge, with the exception of $c = 0$.

6) $f(x) = x^2 - 4$, with $x_0 = 0$

Answer

$|c| > 0.5$ fails, $|c| \leq 0.5$ works

7) $f(x) = x^2 - 4x + 3$, with $x_0 = 2$

8) What is the value of "c" for Newton's method?

Answer

$$c = \frac{1}{f'(x_n)}$$

In exercises 9 - 16, compute x_1 and x_2 using the specified iterative method.

Start at

a. $x_0 = 0.6$ **and**

b. $x_0 = 2$.

9) $x_{n+1} = x_n^2 - \frac{1}{2}$

10) $x_{n+1} = 2x_n\,(1 - x_n)$

Answer

a. $x_1 = \frac{12}{25}, \;\; x_2 = \frac{312}{625}$;
b. $x_1 = -4, \;\; x_2 = -40$

11) $x_{n+1} = \sqrt{x_n}$

12) $x_{n+1} = \frac{1}{\sqrt{x_n}}$

Answer

a. $x_1 = 1.291, \;\; x_2 = 0.8801$;
b. $x_1 = 0.7071, \;\; x_2 = 1.189$

13) $x_{n+1} = 3x_n(1 - x_n)$

14) $x_{n+1} = x_n^2 + x_{n-2}$

Answer

a. $x_1 = -\frac{26}{25}, \;\; x_2 = -\frac{1224}{625}$;
b. $x_1 = 4, \;\; x_2 = 18$

15) $x_{n+1} = \frac{1}{2}x_n - 1$

16) $x_{n+1} = |x_n|$

Answer

a. $x_1 = \frac{6}{10}, \;\; x_2 = \frac{6}{10}$;
b. $x_1 = 2, \;\; x_2 = 2$

In exercises 17 - 26, solve to four decimal places using Newton's method and a computer or calculator. Choose any initial guess x_0 that is not the exact root.

17) $x^2 - 10 = 0$

18) $x^4 - 100 = 0$

Answer

3.1623 or -3.1623

19) $x^2 - x = 0$

20) $x^3 - x = 0$

Answer

0, -1 or 1

21) $x + 5\cos x = 0$

22) $x + \tan x = 0$, choose $x_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$

Answer

0

23) $\dfrac{1}{1-x} = 2$

24) $1 + x + x^2 + x^3 + x^4 = 2$

Answer

0.5188 or -1.2906

25) $x^3 + (x+1)^3 = 10^3$

26) $x = \sin^2(x)$

Answer

0

In exercises 27 - 30, use Newton's method to find the fixed points of the function where $f(x) = x$; round to three decimals.

27) $\sin x$

28) $\tan x$ on $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Answer

4.493

29) $e^x - 2$

30) $\ln(x) + 2$

Answer

$0.159, \;\; 3.146$

Newton's method can be used to find maxima and minima of functions in addition to the roots. In this case apply Newton's method to the derivative function $f'(x)$ to find its roots, instead of the original function. In exercises 31 - 32, consider the formulation of the method.

31) To find candidates for maxima and minima, we need to find the critical points $f'(x) = 0$. Show that to solve for the critical points of a function $f(x)$, Newton's method is given by $x_{n+1} = x_n - \dfrac{f'(x_n)}{f''(x_n)}$.

32) What additional restrictions are necessary on the function f?

Answer

We need f to be twice continuously differentiable.

In exercises 33 - 40, use Newton's method to find the location of the local minima and/or maxima of the following functions; round to three decimals.

33) Minimum of $f(x) = x^2 + 2x + 4$

34) Minimum of $f(x) = 3x^3 + 2x^2 - 16$

Answer

$x = 0$

35) Minimum of $f(x) = x^2 e^x$

36) Maximum of $f(x) = x + \dfrac{1}{x}$

Answer

$x = -1$

37) Maximum of $f(x) = x^3 + 10x^2 + 15x - 2$

38) Maximum of $f(x) = \dfrac{\sqrt{x} - \sqrt[3]{x}}{x}$

Answer

$x = 5.619$

39) Minimum of $f(x) = x^2 \sin x$, closest non-zero minimum to $x = 0$

40) Minimum of $f(x) = x^4 + x^3 + 3x^2 + 12x + 6$

Answer

$x = -1.326$

In exercises 41 - 44, use the specified method to solve the equation. If it does not work, explain why it does not work.

41) Newton's method, $x^2 + 2 = 0$

42) Newton's method, $0 = e^x$

Answer

There is no solution to the equation.

43) Newton's method, $0 = 1 + x^2$ starting at $x_0 = 0$

44) Solving $x_{n+1} = -x_n^3$ starting at $x_0 = -1$

Answer

It enters a cycle.

In exercises 45 - 48, use the *secant method*, an alternative iterative method to Newton's method. The formula is given by

$$x_n = x_{n-1} - f(x_{n-1}) \frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})}.$$

45) a root to $0 = x^2 - x - 3$ accurate to three decimal places.

46) Find a root to $0 = \sin x + 3x$ accurate to four decimal places.

Answer

0

47) Find a root to $0 = e^x - 2$ accurate to four decimal places.

48) Find a root to $\ln(x + 2) = \dfrac{1}{2}$ accurate to four decimal places.

Answer

-0.3513

49) Why would you use the secant method over Newton's method? What are the necessary restrictions on f?

In exercises 50 - 54, use both Newton's method and the secant method to calculate a root for the following equations. Use a calculator or computer to calculate how many iterations of each are needed to reach within three decimal places

of the exact answer. For the secant method, use the first guess from Newton's method.

50) $f(x) = x^2 + 2x + 1, \quad x_0 = 1$

Answer

Newton: 11 iterations, secant: 16 iterations

51) $f(x) = x^2, \quad x_0 = 1$

52) $f(x) = \sin x, \quad x_0 = 1$

Answer

Newton: three iterations, secant: six iterations

53) $f(x) = e^x - 1, \quad x_0 = 2$

54) $f(x) = x^3 + 2x + 4, \quad x_0 = 0$

Answer

Newton: five iterations, secant: eight iterations

In exercises 55 - 56, consider Kepler's equation regarding planetary orbits, $M = E - \epsilon \sin(E)$, where M is the mean anomaly, E is eccentric anomaly, and ϵ measures eccentricity.

55) Use Newton's method to solve for the eccentric anomaly E when the mean anomaly $M = \frac{\pi}{3}$ and the eccentricity of the orbit $\epsilon = 0.25$; round to three decimals.

56) Use Newton's method to solve for the eccentric anomaly E when the mean anomaly $M = \frac{3\pi}{2}$ and the eccentricity of the orbit $\epsilon = 0.8$; round to three decimals.

Answer

$E = 4.071$

In exercises 57 - 58, consider a bank investment. The initial investment is $\$10,000$. After 25 years, the investment has tripled to $\$30,000$.

57) Use Newton's method to determine the interest rate if the interest was compounded annually.

58) Use Newton's method to determine the interest rate if the interest was compounded continuously.

Answer

4.394

59) The cost for printing a book can be given by the equation $C(x) = 1000 + 12x + \frac{1}{2}x^{2/3}$. Use Newton's method to find the break-even point if the printer sells each book for $\$20$.

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4: INVESTIGATING INTEGRALS

4.1: Antiderivatives and Indefinite Integrals

4.1.1: Resources and Key Concepts

4.1.2: Homework

4.2: Approximating Areas

4.2.1: Resources and Key Concepts

4.2.2: Homework

4.3: The Definite Integral

4.3.1: Resources and Key Concepts

4.3.2: Homework

4.4: The Fundamental Theorem of Calculus

4.4.1: Resources and Key Concepts

4.4.2: Homework

4.5: The Net Change Theorem, Distances, and Symmetry

4.5.1: Resources and Key Concepts

4.5.2: Homework

4.6: Integration by Substitution

4.6.1: Resources and Key Concepts

4.6.2: Homework

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4.1: Antiderivatives and Indefinite Integrals

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

At this point, we have seen how to calculate derivatives of many functions and have been introduced to a variety of their applications. We now ask a question that turns this process around: Given a function f, how do we find a function with the derivative f, and why would we be interested in such a function?

We answer the first part of this question by defining antiderivatives. An **antiderivative** of a function f is a function whose derivative is f. The need for antiderivatives arises in many situations, and we look at various examples throughout the remainder of the text. Here, we examine one specific example that involves rectilinear motion.

In our examination of derivatives, we showed that given a position function, $s(t)$, of an object, its velocity function, $v(t)$, is the derivative of $s(t)$. That is, $v(t) = s'(t)$. Furthermore, the acceleration, $a(t)$, is the derivative of the velocity $v(t)$. That is, $a(t) = v'(t) = s''(t)$. Now, suppose we are given an acceleration function a but not the velocity function v or the position function s. Since $a(t) = v'(t)$, determining the velocity function requires us to find an antiderivative of the acceleration function. Since $v(t) = s'(t)$, determining the position function requires us to find an antiderivative of the velocity function.

Rectilinear motion is just one case where the need for antiderivatives arises. We will see many more examples throughout the remainder of the text. Let's look at the terminology and notation for antiderivatives and determine the antiderivatives for several functions. We examine techniques for finding antiderivatives of more complicated functions later in the text.

The Reverse of Differentiation

At this point, we know how to find derivatives of various functions. We now ask the opposite question. Given a function f, how can we find a function with derivative f? If we can find a function F with derivative f, we call F an **antiderivative** of f.

Definition: Antiderivative

A function F is an **antiderivative** of the function f if

$$F'(x) = f(x)$$

for all x in the domain of f.

Consider the function $f(x) = 2x$. Knowing the Power Rule of differentiation, we conclude that $F(x) = x^2$ is an antiderivative of f since $F'(x) = 2x$.

Are there any other antiderivatives of f?

Yes; since the derivative of any constant C is zero, $x^2 + C$ is also an antiderivative of $2x$. Therefore, $x^2 + 5$ and $x^2 - \sqrt{2}$ are also antiderivatives of $f(x) = 2x$.

Are there any others that are not of the form $x^2 + C$ for some constant C?

The answer is no. From the corollary following the Mean Value Theorem, we know that if F and G are differentiable functions such that $F'(x) = G'(x)$, then $F(x) - G(x) = C$ for some constant C. This fact leads to the following important theorem.

Theorem: General Form of an Antiderivative

Let F be an antiderivative of f over an interval I. Then,

I. for each constant C, the function $F(x)+C$ is also an antiderivative of f over I;
II. if G is an antiderivative of f over I, there is a constant C for which $G(x)=F(x)+C$ over I.

In other words, the most general form of the antiderivative of f over I is $F(x)+C$.

We use this fact and our knowledge of derivatives to find all the antiderivatives for several functions.

Example 4.1.1: Finding Antiderivatives

For each of the following functions, find all antiderivatives.

a. $f(x)=3x^2$
b. $f(x)=\frac{1}{x}$
c. $f(x)=\cos x$
d. $f(x)=e^x$

Solutions available at the end of this section.

Note: Breaks in the Domain and Our Choice of C

Consider the function $f(x)=\frac{1}{x}$. In Example 4.1.1*b*, we claimed that the antiderivative of $f(x)$ was $F(x)=\ln(|x|)+C$. In truth, however, because $F(x)$ has breaks in its domain, this antiderivative *could* technically be written as

$$F(x)=\begin{cases}\ln(x)+C_1, & \text{if } x>0\\ \ln(-x)+C_2, & \text{if } x<0\end{cases}$$

Likewise, if we consider the antiderivative of $g(x)=-\csc^2(x)$, rather than writing $G(x)=\cot(x)+C$ (I leave it to the reader to verify that $G(x)$ is the antiderivative of $g(x)$), we *could* write G (with some slightly abusive index notation) as

$$G(x)=\begin{cases}\cot(x)+C_1, & \text{if } 0\ltx<\pi\\ \cot(x)+C_{-1}, & \text{if } -\pi<x<0\\ \cot(x)+C_2, & \text{if } \pi\ltx<2\pi\\ \cot(x)+C_{-2}, & \text{if } -2\pi<x<-\pi\\ \vdots & \vdots\\ \cot(x)+C_n, & \text{if } (n-1)\pi\ltx<n\pi\\ \cot(x)+C_{-n}, & \text{if } n\pi<x<(n+1)\pi\\ \vdots & \vdots\end{cases}$$

While both of these piecewise versions of the antiderivatives of $f(x)$ and $g(x)$ are correct, I hope the reader would agree, they are incredibly bulky. The convention adopted in this text (and all others I have encountered on the subject) is to collapse these down to $F(x)=\ln(|x|)+C$ and $G(x)=\cot(x)+C$, respectively, with the understanding that these are valid on all disjoint intervals in their domains.

Checkpoint 4.1.1

Find all antiderivatives of $f(x)=\sin x$.

Answer available at the end of this section.

Indefinite Integrals

We now look at the formal notation used to represent antiderivatives and examine some of their properties. These properties allow us to find antiderivatives of more complicated functions. Given a function f, we use the notation $f'(x)$ or $\frac{df}{dx}$ to denote the derivative of f. Here, we introduce notation for antiderivatives. If F is an antiderivative of f, we say that $F(x)+C$ is the most general antiderivative of f and write

$$\int f(x)\,dx = F(x)+C.$$

The symbol $\int$ is called an **integral sign**, and $\int f(x)\ dx$ is called the **indefinite integral** of f.

Definition: Indefinite Integrals

Given a function f, the **indefinite integral** of f, denoted

$$\int f(x)\,dx,$$

is the most general antiderivative of f. If F is an antiderivative of f, then

$$\int f(x)\,dx = F(x)+C.$$

The expression $f(x)$ is called the **integrand** and the variable x is the **variable of integration**.

Given the terminology introduced in this definition, the act of finding the antiderivatives of a function f is usually referred to as **integrating** f.

For a function f and an antiderivative F, the set of functions $F(x)+C$, where C is any real number, is often referred to as the **family of antiderivatives** of f. For example, since x^2 is an antiderivative of $2x$ and any antiderivative of $2x$ is of the form x^2+C, we write

$$\int 2x\,dx = x^2+C.$$

The collection of all functions of the form x^2+C, where C is any real number, is known as the family of antiderivatives of $2x$. Figure $4.1.1$ shows a graph of this family of antiderivatives.

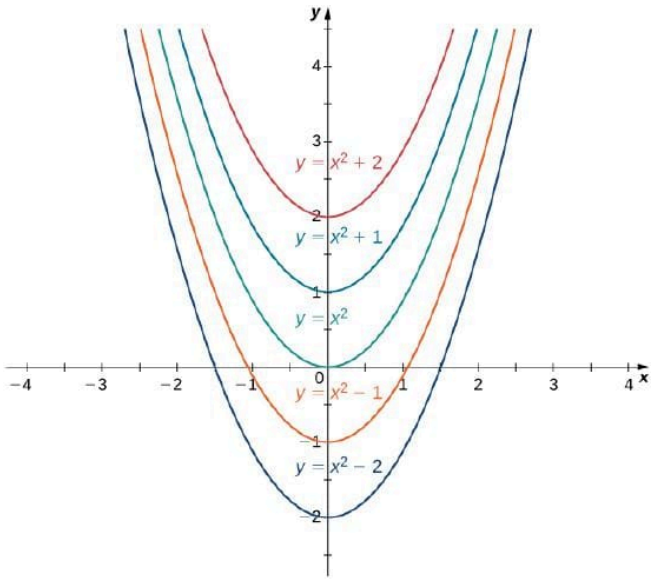

Figure 4.1.1**:** *The family of antiderivatives of $2x$ consists of all functions of the form $x^2 + C$, where C is any real number.*

Interactive Element: The Family of Antiderivatives

The following Interactive Element might help you visualize why every antiderivative of $f(x) = 2x$ differs only by a constant C.

Interact: Drag the red point up and down the y-axis to change the value of C.

Observation: The parabola slides vertically, but the tangent line at $x = 1$ always has slope $2 = f(1)$. Every member of the family $x^2 + C$ has the same derivative, which is exactly why all of them are antiderivatives of $f(x) = 2x$.

For some functions, evaluating indefinite integrals follows directly from properties of derivatives. For example, for $n \neq -1$,

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C,$$

which comes directly from

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = (n+1)\frac{x^n}{n+1} = x^n.$$

This fact is known as the Power Rule for integrals.

Theorem: Power Rule for Integrals

For $n \neq -1$,

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C.$$

Evaluating indefinite integrals for some other functions is also a straightforward calculation. The following table lists the indefinite integrals for several common functions.

Table 4.1.1***:*** *Integration Formulas*

Differentiation Formula	Indefinite Integral
$\frac{d}{dx}(k) = 0$	$\int k \, dx = \int kx^0 \, dx = kx + C$

Differentiation Formula	Indefinite Integral
$\frac{d}{dx}\left(x^n\right)=nx^{n-1}$	$\int x^n\ dx=\frac{x^{n+1}}{n+1}+C$ for $n\neq -1$
$\frac{d}{dx}\left(\ln\left\|x\right\|\right)=\frac{1}{x}$	$\int \frac{1}{x}\ dx=\ln\left\|x\right\|+C$
$\frac{d}{dx}\left(e^x\right)=e^x$	$\int e^x\ dx=e^x+C$
$\frac{d}{dx}\left(b^x\right)=b^x\ln\left(b\right)$	$\int b^x\ dx=\frac{b^x}{\ln\left(b\right)}+C$
$\frac{d}{dx}\left(\sin x\right)=\cos x$	$\int \cos x\ dx=\sin x+C$
$\frac{d}{dx}\left(\cos x\right)=-\sin x$	$\int \sin x\ dx=-\cos x+C$
$\frac{d}{dx}\left(\tan x\right)=\sec^2 x$	$\int \sec^2 x\ dx=\tan x+C$
$\frac{d}{dx}\left(\csc x\right)=-\csc x\cot x$	$\int \csc x\cot x\ dx=-\csc x+C$
$\frac{d}{dx}\left(\sec x\right)=\sec x\tan x$	$\int \sec x\tan x\ dx=\sec x+C$
$\frac{d}{dx}\left(\cot x\right)=-\csc^2 x$	$\int \csc^2 x\ dx=-\cot x+C$
$\frac{d}{dx}\left(\sin^{-1} x\right)=\frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\ dx=\sin^{-1} x+C$
$\frac{d}{dx}\left(\tan^{-1} x\right)=\frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\ dx=\tan^{-1} x+C$
$\frac{d}{dx}\left(\sec^{-1} x\right)=\frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}\ dx=\sec^{-1} x+C$
$\frac{d}{dx}\left(\sinh\left(x\right)\right)=\cosh\left(x\right)$	$\int \cosh\left(x\right)\ dx=\sinh\left(x\right)+C$
$\frac{d}{dx}\left(\cosh\left(x\right)\right)=\sinh\left(x\right)$	$\int \sinh\left(x\right)\ dx=\cosh\left(x\right)+C$

Notice that not all derivatives we have learned are represented in this table. For example, we learned $\frac{d}{dx}\left(\log_b\left(x\right)\right)=\frac{1}{x\ln\left(b\right)}$, however, the related indefinite integral is not listed. Moreover, most of the inverse trigonometric and hyperbolic functions are missing. This is because indefinite integrals involving these forms are rare. In the case of the inverse hyperbolic functions, if you find the antiderivative, you must convert that to a logarithm. This is not a pleasant process, but you will find a more efficient way to integrate these styles of functions in Calculus II. Rather than making an already long table even longer, we leave it to the reader to reference the relevant sections in Chapter 3 to determine the antiderivatives of those functions, if needed.

From the definition of indefinite integral of f, we know

$$\int f(x)\,dx=F(x)+C$$

if and only if F is an antiderivative of f.

Therefore, when claiming that

$$\int f(x)\,dx=F(x)+C,$$

it is essential to check whether this statement is correct by verifying that $F'(x)=f(x)$.

Example 4.1.2: Verifying an Indefinite Integral

Each of the following statements is of the form $\int f(x)\ dx = F(x) + C$. Verify that each statement is correct by showing that $F'(x) = f(x)$.

a. $\int (x + e^x)\ dx = \frac{x^2}{2} + e^x + C$

b. $\int xe^x\ dx = xe^x - e^x + C$

Solution available at the end of this section.

Checkpoint 4.1.2

Verify that $\int x \cos x\ \ dx = x \sin x + \cos x + C$.

Answer available at the end of this section.

In Table 4.1.1, we listed the indefinite integrals for many elementary functions. Let's now consider evaluating indefinite integrals for more complicated functions. For example, consider finding an antiderivative of a sum $f + g$. In Example 4.1.2a, we showed that an antiderivative of the sum $x + e^x$ is given by the sum $\frac{x^2}{2} + e^x$—that is, an antiderivative of a sum is given by a sum of antiderivatives. This result was not specific to this example. In general, if F and G are antiderivatives of any functions f and g, respectively, then

$$\frac{d}{dx}\left(F(x) + G(x)\right) = F'(x) + G'(x) = f(x) + g(x).$$

Therefore, $F(x) + G(x)$ is an antiderivative of $f(x) + g(x)$ and we have

$$\int \left(f(x) + g(x)\right)\ dx = F(x) + G(x) + C.$$

Similarly,

$$\int \left(f(x) - g(x)\right)\ dx = F(x) - G(x) + C.$$

In addition, consider the task of finding an antiderivative of $kf(x)$, where k is any real number. Since

$$\frac{d}{dx}\left(kF(x)\right) = k\frac{d}{dx}\left(F(x)\right) = kF'(x)$$

for any real number k, we conclude that

$$\int kf(x)\,dx = kF(x) + C.$$

These properties are summarized next.

Theorem: Properties of Indefinite Integrals

Let F and G be antiderivatives of f and g, respectively, and let k be any real number.

Sums and Differences

$$\int \left(f(x) \pm g(x)\right)\ dx = F(x) \pm G(x) + C$$

Constant Multiples

$$\int kf(x)\,dx = kF(x) + C$$

From this theorem, we can evaluate any integral involving a sum, difference, or constant multiple of functions with known antiderivatives. Evaluating integrals involving products, quotients, or compositions is more complicated. (See Example $4.1.2b$ for an example involving an antiderivative of a product.) We address integrals involving these more complicated functions in Integral Calculus (Calculus II). In the next example, we examine how to use this theorem to calculate the indefinite integrals of several functions.

Example 4.1.3: Evaluating Indefinite Integrals

Evaluate each of the following indefinite integrals:

a. $\int \left(5x^3 - 7x^2 + 3x + 4\right)\ dx$

b. $\int \dfrac{x^2 + 4\sqrt[3]{x}}{x}\ dx$

c. $\int \dfrac{4}{1 + x^2}\ dx$

d. $\int \tan x \cos x\ dx$

Solutions available at the end of this section.

Checkpoint 4.1.3

Evaluate $\int \left(4x^3 - 5x^2 + x - 7\right)\ dx$.

Answer available at the end of this section.

Initial-Value Problems

We look at techniques for integrating functions involving products, quotients, and compositions later in the text. Here, we turn to one everyday use for antiderivatives that arises often in many applications: solving differential equations.

A **differential equation** is an equation that relates an unknown function and one or more of its derivatives. The equation

$$\frac{dy}{dx} = f(x) \tag{4.1.1}$$

is a simple example of a differential equation. Solving this equation means finding a function y with a derivative f. Therefore, the solutions of Equation $4.1.1$ are the antiderivatives of f. If F is one antiderivative of f, every function of the form $y = F(x) + C$ is a solution of that differential equation. For example, the solutions of

$$\frac{dy}{dx} = 6x^2$$

are given by

$$y = \int 6x^2\,dx = 2x^3 + C.$$

Sometimes we are interested in determining whether a **particular solution curve** passes through a certain point (x_0, y_0)—that is, $y(x_0) = y_0$. The problem of finding a function y that satisfies a differential equation

$$\frac{dy}{dx} = f(x)$$

with the additional condition $y(x_0) = y_0$ is an example of an **initial-value problem**. The condition $y(x_0) = y_0$ is known as an **initial condition**. For example, looking for a function y that satisfies the differential equation

$$\frac{dy}{dx} = 6x^2$$

and the initial condition $y(1) = 5$ is an example of an initial-value problem. Since the solutions of the differential equation are $y = 2x^3 + C$, to find a function y that also satisfies the initial condition, we need to find C such that $y(1) = 2(1)^3 + C = 5$. This equation shows that $C = 3$, and we conclude that $y = 2x^3 + 3$ solves this initial-value problem, as shown in the following graph.

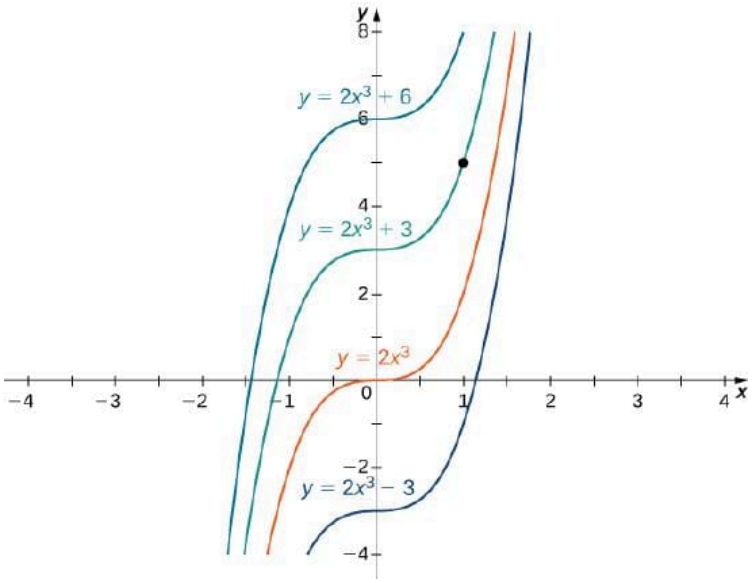

Figure 4.1.2**:** *Some of the solution curves of the differential equation* $\frac{dy}{dx} = 6x^2$ *are displayed. The function* $y = 2x^3 + 3$ *satisfies the differential equation and the initial condition* $y(1) = 5$.

Interactive Element: The Initial Condition Selects the Solution Curve

The following Interactive Element might help you visualize how a single initial condition picks out one curve from the family of solutions to $\frac{dy}{dx} = 6x^2$.

Interact: Drag the red point anywhere; the bold curve is the one member of the family $y = 2x^3 + C$ that passes through it.

Observation: Through every point there is exactly one curve of the family. Placing the point at $(1, 5)$ forces $C = 3$, giving $y = 2x^3 + 3$—the initial condition determines the constant.

Example 4.1.4: Solving an Initial-Value Problem

Solve the initial-value problem

$$\frac{dy}{dx} = \sin x, \quad y(0) = 5.$$

Solution available at the end of this section.

Checkpoint 4.1.4

Solve the initial-value problem $\frac{dy}{dx} = 3x^{-2}, \quad y(1) = 2.$

Answer available at the end of this section.

Initial-value problems arise in many applications. Next, we consider a situation where a driver applies brakes to a car. We are interested in how long it takes for the vehicle to stop. Recall that the velocity function $v(t)$ is the derivative of a position function

$s(t)$, and the acceleration $a(t)$ is the derivative of the velocity function. In earlier examples in the text, we could calculate the velocity from the position and then compute the acceleration from the velocity. In the following example, we work the other way around. Given an acceleration function, we calculate the velocity function. We then use the velocity function to determine the position function.

Example 4.1.5

A car is traveling at the rate of 88 ft/sec (60 mph) when the brakes are applied. The car begins decelerating at a constant rate of 15 ft/sec^2.

a. How many seconds elapse before the car stops?
b. How far does the car travel during that time?

Solutions available at the end of this section.

Checkpoint 4.1.5

Suppose the car is traveling at the rate of 44 ft/sec. How long does it take for the car to stop? How far will the car travel?

Answer available at the end of this section.

Success in Mathematics: Check Your Work

Checking your work is essential, but most students want to complete it and walk away from it. The fastest way to check your work in algebra is to plug your answers into the original problem. In trigonometry and calculus, this method may be a bit difficult. However, you can always check your solutions with graphs on the graphing calculator! It's a great idea.

1. Do you usually check your work?
2. Have you used the graphing calculator to check your work, or do you usually check "by hand?"

Miscellaneous Topics: Teachers and Role Models

Parents, teachers, and other role models influence our attitudes about math. What they say and do can change our view of the importance of math, how hard math is, and whether we are good at math. Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. Describe the attitudes towards math of your parent, teacher (not me, of course), or another personal role model.
2. Describe how the opinion of this role model has influenced your attitudes about math.

Answers

Section Preview

If you have taught from the Stewart Calculus textbook, the approach I take here is slightly different. Antidifferentiation is treated as part of this chapter and the indefinite integral is immediately introduced as a shorthand for the general antiderivative of a given function. This, to me, feels like a much better flow than the Stewart approach.

Section Preview

- Find the general antiderivative of a given function.
- Explain the terms and notation used for an indefinite integral.
- State the Power Rule for integrals.
- Use antidifferentiation to solve simple initial-value problems.

Example 4.1.1: Finding Antiderivatives

a. Because

$$\frac{d}{dx}\left(x^3\right) = 3x^2,$$

$F(x) = x^3$ is an antiderivative of $3x^2$. Therefore, every antiderivative of $3x^2$ is of the form $x^3 + C$ for some constant C, and every function of the form $x^3 + C$ is an antiderivative of $3x^2$.

b. Let $f(x) = \ln|x|$. For $x > 0$, $f(x) = \ln|x| = \ln(x)$ and

$$\frac{d}{dx}\left(\ln x\right) = \frac{1}{x}.$$

For $x < 0$, $f(x) = \ln|x| = \ln(-x)$ and

$$\frac{d}{dx}\left(\ln(-x)\right) = -\frac{1}{-x} = \frac{1}{x}.$$

Therefore,

$$\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x}.$$

Thus, $F(x) = \ln|x|$ is an antiderivative of $\frac{1}{x}$. Therefore, every antiderivative of $\frac{1}{x}$ is of the form $\ln|x| + C$ for some constant C and every function of the form $\ln|x| + C$ is an antiderivative of $\frac{1}{x}$.

c. We have

$$\frac{d}{dx}\left(\sin x\right) = \cos x,$$

so $F(x) = \sin x$ is an antiderivative of $\cos x$. Therefore, every antiderivative of $\cos x$ is of the form $\sin x + C$ for some constant C and every function of the form $\sin x + C$ is an antiderivative of $\cos x$.

d. Since

$$\frac{d}{dx}\left(e^x\right) = e^x,$$

then $F(x) = e^x$ is an antiderivative of e^x. Therefore, every antiderivative of e^x is of the form $e^x + C$ for some constant C and every function of the form $e^x + C$ is an antiderivative of e^x.

Checkpoint 4.1.1

$F(x) = -\cos x + C$

Example 4.1.2**: Verifying an Indefinite Integral**

a. Since

$$\frac{d}{dx}\left(\frac{x^2}{2} + e^x + C\right) = x + e^x,$$

the statement

$$\int \left(x + e^x\right)\,dx = \frac{x^2}{2} + e^x + C$$

is correct.

Note that we are verifying an indefinite integral for a sum. Furthermore, $\frac{x^2}{2}$ and e^x are antiderivatives of x and e^x, respectively, and the sum of the antiderivatives is an antiderivative of the sum. We discuss this fact again later in this section.

b. Using the Product Rule, we see that

$$\frac{d}{dx}\left(xe^x - e^x + C\right) = e^x + xe^x - e^x = xe^x.$$

Therefore, the statement

$$\int xe^x\,dx = xe^x - e^x + C$$

is correct.

Note that we are verifying an indefinite integral for a product. The antiderivative $xe^x - e^x$ is not a product of the antiderivatives. Furthermore, the product of antiderivatives, $x^2e^x/2$ is not an antiderivative of xe^x since

$$\frac{d}{dx}\left(\frac{x^2e^x}{2}\right) = xe^x + \frac{x^2e^x}{2} \neq xe^x.$$

Generally, the product of antiderivatives is not an antiderivative of a product.

Checkpoint 4.1.2

$\frac{d}{dx}\left(x\sin x + \cos x + C\right) = \sin x + x\cos x - \sin x = x\cos x$

Example 4.1.3**: Evaluating Indefinite Integrals**

a. Using Properties of Indefinite Integrals, we can integrate the four terms in the integrand separately. We obtain

$$\int \left(5x^3 - 7x^2 + 3x + 4\right)\,dx = \int 5x^3\,dx - \int 7x^2\,dx + \int 3x\,dx + \int 4\,dx.$$

From the second part of Properties of Indefinite Integrals, each coefficient can be written in front of the integral sign, which gives

$$\int 5x^3\,dx - \int 7x^2\,dx + \int 3x\,dx + \int 4\,dx = 5\int x^3\,dx - 7\int x^2\,dx + 3\int x\,dx + 4\int 1\,dx.$$

Using the Power Rule for integrals, we conclude that

$$\int \left(5x^3 - 7x^2 + 3x + 4\right)\,dx = \frac{5}{4}x^4 - \frac{7}{3}x^3 + \frac{3}{2}x^2 + 4x + C.$$

b. Rewrite the integrand as

$$\frac{x^2+4\sqrt[3]{x}}{x}=\frac{x^2}{x}+\frac{4\sqrt[3]{x}}{x}.$$

Then, to evaluate the integral, integrate each of these terms separately. Using the Power Rule, we have

$$\begin{aligned}\int\left(x+\frac{4}{x^{2/3}}\right)\,dx &= \int x\,dx+4\int x^{-2/3}\,dx\\ &= \frac{1}{2}x^2+4\frac{1}{\left(\frac{-2}{3}\right)+1}x^{(-2/3)+1}+C\\ &= \frac{1}{2}x^2+12x^{1/3}+C.\end{aligned}$$

c. Using Properties of Indefinite Integrals, write the integral as

$$4\int\frac{1}{1+x^2}\,dx.$$

Then, use the fact that $\tan^{-1}(x)$ is an antiderivative of $\frac{1}{1+x^2}$ to conclude that

$$\int\frac{4}{1+x^2}\,dx=4\tan^{-1}(x)+C.$$

d. Rewrite the integrand as

$$\tan x\cos x=\frac{\sin x}{\cos x}\cdot\cos x=\sin x.$$

Therefore,

$$\int\tan x\cos x\,dx=\int\sin x\,dx=-\cos x+C.$$

Checkpoint 4.1.3

$\int\left(4x^3-5x^2+x-7\right)\,dx=\quad x^4-\frac{5}{3}x^3+\frac{1}{2}x^2-7x+C$

Example 4.1.4**: Solving an Initial-Value Problem**

First, we need to solve the differential equation. If $\frac{dy}{dx}=\sin x$, then

$$y=\int\sin(x)\,dx=-\cos x+C.$$

Next we need to look for a solution y that satisfies the initial condition. The initial condition $y(0)=5$ means we need a constant C such that $-\cos x+C=5$. Therefore,

$$C=5+\cos(0)=6.$$

The solution of the initial-value problem is $y=-\cos x+6$.

Checkpoint 4.1.4

$y=-\frac{3}{x}+5$

Example 4.1.5

a. First, we introduce variables for this problem. Let t be the time (in seconds) after the brakes are applied. Let $a(t)$ be the acceleration of the car (in feet per second squared) at time t. Let $v(t)$ be the velocity of the car (in feet per second) at time t. Let $s(t)$ be the car's position (in feet) beyond the point where the brakes are applied at time t.

The car travels at a rate of 88 ft/sec. Therefore, the initial velocity is $v(0)=88$ ft/sec. Since the car is decelerating, the acceleration is

$$a(t)=-15\,\text{ft/sec}^2.$$

The acceleration is the derivative of the velocity,

$$v'(t)=-15.$$

Therefore, we have an initial-value problem to solve:

$$v'(t)=-15,\quad v(0)=88.$$

Integrating, we find that

$$v(t)=-15t+C.$$

Since $v(0)=88, C=88$. Thus, the velocity function is

$$v(t)=-15t+88.$$

To find how long it takes for the car to stop, we need to find the time t such that the velocity is zero. Solving $-15t+88=0$, we obtain $t=\frac{88}{15}$ sec.

b. To find how far the car travels during this time, we need to find the position of the car after $\frac{88}{15}$ sec. We know the velocity $v(t)$ is the derivative of the position $s(t)$. Consider the initial position as $s(0)=0$. Therefore, we need to solve the initial-value problem

$$s'(t)=-15t+88,\quad s(0)=0.$$

Integrating, we have

$$s(t)=-\frac{15}{2}t^2+88t+C.$$

Since $s(0)=0$, the constant is $C=0$. Therefore, the position function is

$$s(t)=-\frac{15}{2}t^2+88t.$$

After $t = \frac{88}{15}$ sec, the position is $s\left(\frac{88}{15}\right) \approx 258.133$ ft.

Checkpoint $4.1.5$

2.93 sec, 64.5 ft

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4.1.1: Resources and Key Concepts

Resources

Videos

- Antiderivative
  - Concept and Basic Theory
- Indefinite integral
  - Indefinite Integrals and Their Properties
- Properties of indefinite integrals
  - Finding Antiderivatives of Sums and Differences of Functions
- Evaluating indefinite integrals
  - Evaluating Indefinite Integrals
- Differential equation
  - Solving Basic Differential Equations
- Synthesizing previous topics with this new topic
  - Synthesis Questions

Key Concepts

Definitions

- **Antiderivative**: A function $F(x)$ is an antiderivative of the function $f(x)$ if $F'(x) = f(x)$ for all x in the domain of f.
- **Indefinite Integral**: Given a function f, the indefinite integral of f, denoted $\int f(x)\ dx$, is the most general antiderivative of f. If F is an antiderivative of f, then $\int f(x)\ dx = F(x) + C$, where C is an arbitrary constant called the constant of integration.
- **Integrand**: In the expression $\int f(x)\ dx$, the function $f(x)$ is called the integrand.
- **Variable of Integration**: In the expression $\int f(x)\ dx$, x is the variable of integration.
- **Integrating a function** f: The act of finding the antiderivatives of a function f.
- **Family of Antiderivatives**: For a function f and an antiderivative F, the set of functions $F(x) + C$, where C is any real number.
- **Differential Equation**: An equation that relates an unknown function and one or more of its derivatives (e.g., $\frac{dy}{dx} = f(x)$).
- **Initial-Value Problem**: The problem of finding a function y that satisfies a differential equation $\frac{dy}{dx} = f(x)$ along with an additional condition $y(x_0) = y_0$.
- **Initial Condition**: The condition $y(x_0) = y_0$ in an initial-value problem.

Theorems

- **Theorem: General Form of an Antiderivative**: Let F be an antiderivative of f over an interval I. Then,
  I. for each constant C, the function $F(x) + C$ is also an antiderivative of f over I;
  II. if G is an antiderivative of f over I, there is a constant C for which $G(x) = F(x) + C$ over I.
  In other words, the most general form of the antiderivative of f over I is $F(x) + C$.
- **Theorem: Power Rule for Integrals**: For $n \neq -1$, $\int x^n\ dx = \frac{x^{n+1}}{n+1} + C$.
- **Theorem: Properties of Indefinite Integrals**: Let F and G be antiderivatives of f and g, respectively, and let k be any real number.
  - **Sums and Differences**: $\int (f(x) \pm g(x))\ dx = F(x) \pm G(x) + C$ (or $\int (f(x) \pm g(x))\ dx = \int f(x)\ dx \pm \int g(x)\ dx$)
  - **Constant Multiples**: $\int kf(x)\ dx = kF(x) + C$ (or $\int kf(x)\ dx = k \int f(x)\ dx$)
- *(Table 1 lists several basic integration formulas derived from known differentiation formulas for constant, power, exponential, logarithmic, trigonometric, inverse trigonometric, and hyperbolic functions.)*

Common Mistakes

- **Forgetting the Constant of Integration** $+C$: When finding an indefinite integral, it's crucial to add the constant of integration C to represent the entire family of antiderivatives.
- **Incorrectly Applying the Power Rule for** $n = -1$: The power rule $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ is not valid for $n = -1$. $\int x^{-1}\ dx = \int \frac{1}{x} dx = \ln|x| + C$.
- **Errors in Antidifferentiating Basic Functions**: Mistakes in recalling or applying the basic antiderivative formulas (e.g., for trigonometric or exponential functions).
- **Product/Quotient of Integrals vs. Integral of Product/Quotient**: The integral of a product is NOT the product of the integrals ($\int f(x)g(x)\ dx \neq \int f(x)\ dx \cdot \int g(x)\ dx$). Similarly for quotients.

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4.1.2: Homework

Reading Questions

1. What is an antiderivative of a function $f(x)$?
2. If $F(x)$ is an antiderivative of $f(x)$, what is the most general form of an antiderivative of $f(x)$? What is the term for the arbitrary constant added?
3. What does the notation $\int f(x)\ dx$ represent? Identify the integrand and the variable of integration.
4. State the Power Rule for Integrals. For which value of n does this rule not apply? What is the antiderivative in that specific case?
5. If you know the antiderivatives of $f(x)$ and $g(x)$, how do you find the antiderivative of $f(x) + g(x)$?
6. If you know the antiderivative of $f(x)$, how do you find the antiderivative of $k \cdot f(x)$, where k is a constant?
7. What is a differential equation?
8. What is an initial-value problem? What two components does it typically consist of?
9. How is the constant of integration C determined when solving an initial-value problem?
10. If the acceleration of an object $a(t)$ is given, how can you find its velocity function $v(t)$? How can you then find its position function $s(t)$?

Homework

In exercises 1 - 20, find the antiderivative $F(x)$ of each function $f(x)$.

1) $f(x) = \dfrac{1}{x^2} + x$

2) $f(x) = e^x - 3x^2 + \sin x$

Answer

$$F(x) = e^x - x^3 - \cos x + C$$

3) $f(x) = e^x + 3x - x^2$

4) $f(x) = x - 1 + 4\sin(2x)$

Answer

$$F(x) = \frac{x^2}{2} - x - 2\cos(2x) + C$$

5) $f(x) = 5x^4 + 4x^5$

6) $f(x) = x + 12x^2$

Answer

$$F(x) = \tfrac{1}{2}x^2 + 4x^3 + C$$

7) $f(x) = \dfrac{1}{\sqrt{x}}$

8) $f(x) = \left(\sqrt{x}\right)^3$

Answer

$$F(x) = \tfrac{2}{5}\left(\sqrt{x}\right)^5 + C$$

9) $f(x) = x^{1/3} + \left(2x\right)^{1/3}$

10) $f(x) = \dfrac{x^{1/3}}{x^{2/3}}$

Answer

$$F(x) = \tfrac{3}{2}x^{2/3} + C$$

11) $f(x) = 2\sin(x) + \sin(2x)$

12) $f(x) = \sec^2 x + 1$

Answer

$$F(x) = x + \tan x + C$$

13) $f(x) = \sin x \cos x$

14) $f(x) = \sin^2(x)\cos(x)$

Answer

$$F(x) = \tfrac{1}{3}\sin^3(x) + C$$

15) $f(x) = 0$

16) $f(x) = \tfrac{1}{2}\csc^2 x + \dfrac{1}{x^2}$

Answer

$$F(x) = -\tfrac{1}{2}\cot x - \frac{1}{x} + C$$

17) $f(x) = \csc x \cot x + 3x$

18) $f(x) = 4\csc x \cot x - \sec x \tan x$

Answer

$$F(x) = -\sec x - 4\csc x + C$$

19) $f(x) = 8(\sec x)\Big(\sec x - 4\tan x\Big)$

20) $f(x) = \cosh(x) + \sinh(x)$

Answer

$$F(x) = \cosh(x) + \sinh(x) + C$$

For exercises 21 - 31, evaluate the integral.

21) $\displaystyle\int (-1)\ dx$

22) $\displaystyle\int \sin x\ dx$

Answer

$$\int \sin x\ dx = -\cos x + C$$

23) $\displaystyle\int \left(4x + \sqrt{x}\right)\ dx$

24) $\displaystyle\int \frac{3x^2 + 2}{x^2}\ dx$

Answer

$$\int \frac{3x^2+2}{x^2}\,dx = 3x - \frac{2}{x} + C$$

25) $\int \left(\sec x \tan x + 4x\right)\,dx$

26) $\int \left(4\sqrt{x} + \sqrt[4]{x}\right)\,dx$

Answer

$$\int \left(4\sqrt{x} + \sqrt[4]{x}\right)\,dx = \frac{8}{3}x^{3/2} + \frac{4}{5}x^{5/4} + C$$

27) $\int \left(x^{-1/3} - x^{2/3}\right)\,dx$

28) $\int \frac{14x^3+2x+1}{x^3}\,dx$

Answer

$$\int \frac{14x^3+2x+1}{x^3}\,dx = 14x - \frac{2}{x} - \frac{1}{2x^2} + C$$

29) $\int \left(e^x + e^{-x}\right)\,dx$

30) $\int \frac{dx}{\sqrt{x^2+1}}$

31) $\int -\frac{dx}{x\sqrt{1-x^2}}$

In exercises 32 - 36, solve the initial value problem.

32) $f'(x) = x^{-3}, \quad f(1) = 1$

Answer

$$f(x) = -\frac{1}{2x^2} + \frac{3}{2}$$

33) $f'(x) = \sqrt{x} + x^2, \quad f(0) = 2$

34) $f'(x) = \cos x + \sec^2(x), \quad f(\frac{\pi}{4}) = 2 + \frac{\sqrt{2}}{2}$

Answer

$$f(x) = \sin x + \tan x + 1$$

35) $f'(x) = x^3 - 8x^2 + 16x + 1, \quad f(0) = 0$

36) $f'(x) = \frac{2}{x^2} - \frac{x^2}{2}, \quad f(1) = 0$

Answer

$$f(x) = -\tfrac{1}{6}x^3 - \frac{2}{x} + \frac{13}{6}$$

In exercises 37 - 40, find two possible functions f given the second- or third-order derivatives

37) $f''(x) = x^2 + 2$

38) $f''(x) = e^{-x}$

Answer

Answers may vary; one possible answer is $f(x) = e^{-x}$

39) $f''(x) = 1 + x$

40) $f'''(x) = \cos x$

Answer

Answers may vary; one possible answer is $f(x) = -\sin x$

41) A car is being driven at a rate of 40 mph when the brakes are applied. The car decelerates at a constant rate of $10\ \text{ft/sec}^2$. How long before the car stops?

Answer

5.867 sec

42) In the preceding problem, calculate how far the car travels in the time it takes to stop.

43) You are merging onto the freeway, accelerating at a constant rate of $12\ \text{ft/sec}^2$. How long does it take you to reach merging speed at 60 mph?

Answer

7.333 sec

44) Based on the previous problem, how far does the car travel to reach merging speed?

45) A car company wants to ensure its newest model can stop in 8 sec when traveling at 75 mph. If we assume constant deceleration, find the value of deceleration that accomplishes this.

Answer

$13.75\ \text{ft/sec}^2$

46) A car company wants to ensure its newest model can stop in less than 450 ft when traveling at 60 mph. If we assume constant deceleration, find the value of deceleration that accomplishes this.

In exercises 47 - 51, find the antiderivative of the function, assuming $F(0) = 0$.

47) [Technology Required] $f(x) = x^2 + 2$

Answer

$F(x) = \frac{1}{3}x^3 + 2x$

48) [Technology Required] $f(x) = 4x - \sqrt{x}$

49) [Technology Required] $f(x) = \sin x + 2x$

Answer

$F(x) = x^2 - \cos x + 1$

50) [Technology Required] $f(x) = e^x$

51) [Technology Required] $f(x) = \dfrac{1}{(x+1)^2}$

Answer

$F(x) = -\dfrac{1}{x+1} + 1$

In exercises 52 - 55, determine whether the statement is true or false. Either prove it is true or find a counterexample if it is false.

52) If $f(x)$ is the antiderivative of $v(x)$, then $2f(x)$ is the antiderivative of $2v(x)$.

Answer

True

53) If $f(x)$ is the antiderivative of $v(x)$, then $f(2x)$ is the antiderivative of $v(2x)$.

54) If $f(x)$ is the antiderivative of $v(x)$, then $f(x)+1$ is the antiderivative of $v(x)+1$.

Answer

False

55) If $f(x)$ is the antiderivative of $v(x)$, then $(f(x))^2$ is the antiderivative of $(v(x))^2$.

In exercises 43 - 45, use the fact that a falling body with friction equal to velocity squared obeys the equation $\frac{dv}{dt} = g - v^2$.

43) Show that $v(t) = \sqrt{g}\tanh(\sqrt{g}t)$ satisfies this equation.

Answer

Answers may vary

44) Derive the previous expression for $v(t)$ by integrating $\frac{dv}{g-v^2} = dt$.

45) [Technology Required] Estimate how far a body has fallen in 12seconds by finding the area underneath the curve of $v(t)$.

Answer

37.30

In exercises 46 - 48, use this scenario: A cable hanging under its own weight has a slope $S = \frac{dy}{dx}$ that satisfies $\frac{dS}{dx} = c\sqrt{1+S^2}$. The constant c is the ratio of cable density to tension.

46) Show that $S = \sinh(cx)$ satisfies this equation.

47) Integrate $\frac{dy}{dx} = \sinh(cx)$ to find the cable height $y(x)$ if $y(0) = 1/c$.

Answer

$y = \frac{1}{c}\cosh(cx)$

48) Sketch the cable and determine how far down it sags at $x = 0$.

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4.2: Approximating Areas

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Archimedes was fascinated with calculating the areas of various shapes—in other words, the amount of space enclosed by the shape. He used a process that has come to be known as the **method of exhaustion**, which used smaller and smaller shapes, the areas of which could be calculated exactly, to fill an irregular region and thereby obtain closer and closer approximations to the total area. In this process, an area bounded by curves is filled with rectangles, triangles, and shapes with exact area formulas. These areas are then summed to approximate the area of the curved region.

In this section, we develop techniques to approximate the area between a curve, defined by a function $f(x)$, and the x-axis on a closed interval $[a, b]$. Like Archimedes, we first approximate the area under the curve using shapes of known area (namely, rectangles). By using smaller and smaller rectangles, we get closer and closer approximations to the area. Taking a limit allows us to calculate the exact area under the curve.

Let's start by introducing some notation to simplify calculations. We consider the case when $f(x)$ is continuous and nonnegative. Later in the chapter, we relax some of these restrictions and develop techniques that apply in more general cases.

Sigma (Summation) Notation

As mentioned, we will use shapes of known area to approximate the area of an irregular region bounded by curves. This process often requires adding up long strings of numbers. To make it easier to write down these lengthy sums, we look at some new notation here, called **sigma notation** (also known as **summation notation**). The Greek capital letter Σ, Sigma, is used to express long sums of values in a compact form. For example, if we want to add all the integers from 1 to 20 without sigma notation, we have to write

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20.$$

We could probably skip writing a couple of terms and write

$$1 + 2 + 3 + 4 + \cdots + 19 + 20,$$

which is better but still cumbersome. With sigma notation, we write this sum as

$$\sum_{i=1}^{20} i,$$

which is much more compact. Typically, sigma notation is presented in the form

$$\sum_{i=1}^{n} a_i,$$

where a_i describes the terms to be added, and the i is called the **index**. Each term is evaluated, then we sum all the values, beginning with the value when $i = 1$ and ending with the value when $i = n$. For example, an expression like $\sum_{i=2}^{7} s_i$ is interpreted as $s_2 + s_3 + s_4 + s_5 + s_6 + s_7$. Note that the index is used only to keep track of the terms to be added; it does not factor into calculating the sum itself. The index is, therefore, called a **dummy variable**. We can use any letter we like for the index. Typically,

mathematicians use i, j, k, m, and n for indices.

Let's try a couple of examples of using sigma notation.

Example 4.2.1: Using Sigma Notation

a. Write in sigma notation and evaluate the sum of terms 3^i for $i = 1, 2, 3, 4, 5$.
b. Write the sum in sigma notation:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}.$$

Solutions available at the end of this section.

Checkpoint 4.2.1

Write in sigma notation and evaluate the sum of terms 2^i for $i = 3, 4, 5, 6$.

Answer available at the end of this section.

The properties associated with the summation process are given in the following rule.

Theorem: Properties of Sigma Notation

Let $a_1,\ a_2,\ \ldots,\ a_n$ and $b_1,\ b_2,\ \ldots,\ b_n$ represent two sequences of terms and let c be a constant. The following properties hold for all positive integers n and for integers m, with $1 \le m \le n$.

i. $\displaystyle\sum_{i=1}^{n} c = nc$

ii. $\displaystyle\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$

iii. $\displaystyle\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

iv. $\displaystyle\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

v. $\displaystyle\sum_{i=1}^{n} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i$

Proof available at the end of this section.

A few more formulas for frequently found functions simplify the summation process further. These were (hopefully) covered in your Precalculus course and are shown in the next rule for **sums and powers of integers**.

Theorem: Sums and Powers of Integers

1. The sum of n integers is given by

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}. \tag{4.2.1}$$

2. The sum of consecutive integers squared is given by

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}. \tag{4.2.2}$$

3. The sum of consecutive integers cubed is given by

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}. \tag{4.2.3}$$

Example 4.2.2: Evaluation Using Sigma Notation

Write using sigma notation and evaluate:

a. The sum of the terms $(i-3)^2$ for $i = 1, 2, \ldots, 200$.
b. The sum of the terms $(i^3 - i^2)$ for $i = 1, 2, 3, 4, 5, 6$

Solutions available at the end of this section.

Checkpoint 4.2.2

Find the sum of the values of $4 + 3i$ for $i = 1, 2, \ldots, 100$.

Answer available at the end of this section.

Example 4.2.3: Finding the Sum of the Function Values

Find the sum of the values of $f(x) = x^3$ over the integers $1, 2, 3, \ldots, 10$.

Solution available at the end of this section.

Checkpoint 4.2.3

Evaluate the sum indicated by the notation $\sum_{k=1}^{20}(2k+1)$.

Answer available at the end of this section.

Approximating Area

Now that we have the necessary notation, we return to the problem: approximating the area under a curve. Let $f(x)$ be a continuous, nonnegative function defined on the closed interval $[a, b]$. We want to approximate the area A bounded by $f(x)$ above, the x-axis below, the line $x = a$ on the left, and the line $x = b$ on the right (Figure 4.2.1).

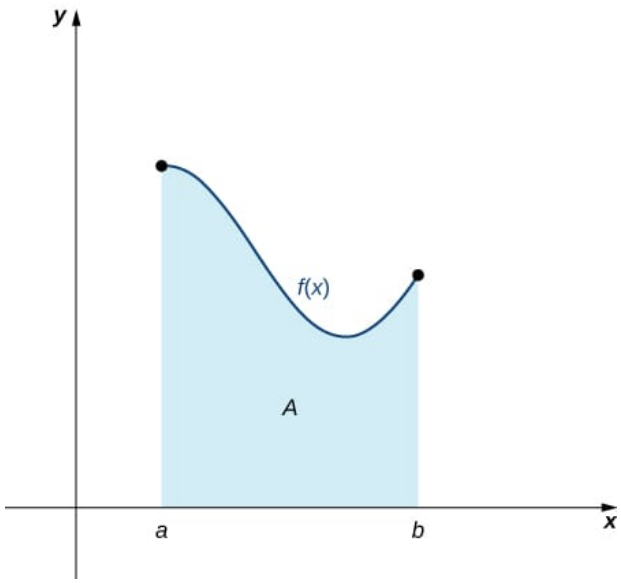

Figure 4.2.1**:** *An area (shaded region) bounded by the curve $f(x)$ at the top, the x-axis at the bottom, the line $x = a$ to the left, and the line $x = b$ at right.*

How do we approximate the area under this curve? The approach is a geometric one. By dividing a region into many small shapes with known area formulas, we can sum these areas and obtain a reasonable estimate of the true area. We begin by dividing the interval $[a, b]$ into n subintervals of equal width, $\frac{b-a}{n}$. We do this by selecting equally spaced points $x_0,\ x_1,\ x_2,\ \ldots,\ x_n$ with $x_0 = a$, $x_n = b$, and

$$x_i - x_{i-1} = \frac{b-a}{n}$$

for $i = 1,\ 2,\ 3,\ \ldots,\ n$.

We denote the width of each subinterval with the notation Δx, so $\Delta x = \frac{b-a}{n}$ and

$$x_i = x_0 + i\Delta x$$

for $i = 1,\ 2,\ 3,\ \ldots,\ n$. This notion of dividing an interval $[a, b]$ into subintervals by selecting points from within the interval approximates the area under a curve, so let's define some relevant terminology.

Definition: Partitions

A set of points $P = \{x_i\}$ for $i = 0,\ 1,\ 2,\ \ldots,\ n$ with $a = x_0 < x_1 < x_2 < \ldots < x_n = b$, which divides the interval $[a, b]$ into subintervals of the form $[x_0, x_1],\ [x_1, x_2],\ \ldots,\ [x_{n-1}, x_n]$ is called a **partition** of $[a, b]$. If the subintervals all have the same width, the set of points forms a **regular partition** (or uniform partition) of the interval $[a, b]$.

We can use this regular partition as the basis of a method for estimating the area under the curve. We next examine two methods: the **left-endpoint approximation** and the **right-endpoint approximation**.

Theorem: Left-Endpoint Approximation

On each subinterval $[x_{i-1}, x_i]$ (for $i = 1,\ 2,\ 3,\ \ldots,\ n$), construct a rectangle with width Δx and height equal to $f(x_{i-1})$, which is the function value at the left endpoint of the subinterval. Then the area of this rectangle is $f(x_{i-1})\Delta x$. Adding the areas of all these rectangles, we get an approximate value for A (Figure 4.2.2). We use the notation L_n to denote that this is a **left-endpoint approximation** of A using n subintervals.

$$A \approx L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

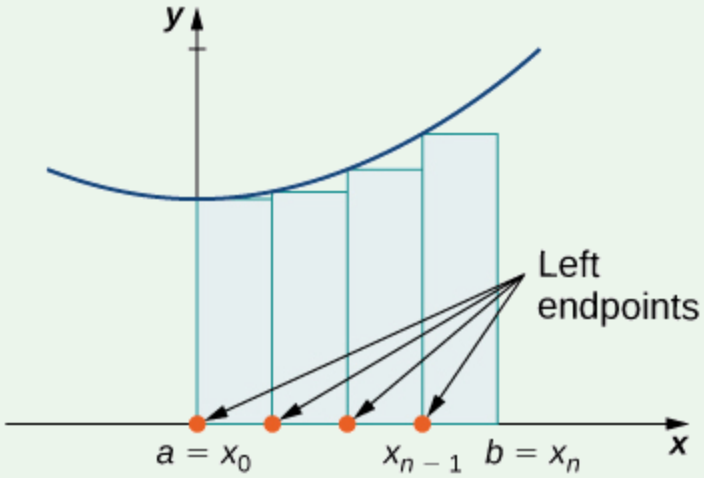

Figure 4.2.2**:** *In the left-endpoint approximation of the area under a curve, the height of each rectangle is determined by the function value at the left of each subinterval.*

The second method for approximating the area under a curve is the right-endpoint approximation. It is almost the same as the left-endpoint approximation, but now the heights of the rectangles are determined by the function values at the right of each subinterval.

Theorem: Right-Endpoint Approximation

Construct a rectangle on each subinterval $[x_{i-1}, x_i]$, only this time the height of the rectangle is determined by the function value $f(x_i)$ at the right endpoint of the subinterval. Then, the area of each rectangle is $f(x_i)\ \Delta x$ and the approximation for A is given by

$$A \approx R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x.$$

The notation R_n indicates this is a **right-endpoint approximation** for A (Figure 4.2.3).

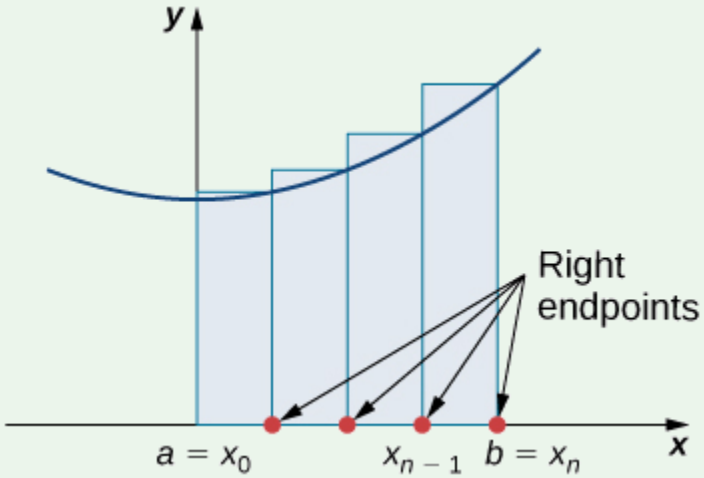

Figure 4.2.3**:** *In the right-endpoint approximation of the area under a curve, the height of each rectangle is determined by the function value at the right of each subinterval. Note that the right-endpoint approximation differs from the left-endpoint approximation in Figure* 4.2.2.

The graphs in Figure 4.2.4 represent the curve $f(x) = \frac{x^2}{2}$. In Figure 4.2.4*a*, we divide the region represented by the interval $[0, 3]$ into six subintervals, each of width 0.5. Thus, $\Delta x = 0.5$. We then form six rectangles by drawing vertical lines perpendicular to x_{i-1}, the left endpoint of each subinterval. We determine the height of each rectangle by calculating $f(x_{i-1})$ for $i = 1,\ 2,\ 3,\ 4,\ 5,\ 6$. The intervals are $[0, 0.5],\ [0.5, 1],\ [1, 1.5],\ [1.5, 2],\ [2, 2.5],\ [2.5, 3]$. We find the area of each rectangle by multiplying the height by the width. Then, the sum of the rectangular areas approximates the area between $f(x)$ and the x-axis. When the left endpoints

are used to calculate height, we have a left-endpoint approximation. Thus,

$$\begin{aligned} A &\approx L_6 \\ &= \sum_{i=1}^{6} f(x_{i-1})\Delta x \\ &= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x \\ &= f(0)0.5 + f(0.5)0.5 + f(1)0.5 + f(1.5)0.5 + f(2)0.5 + f(2.5)0.5 \\ &= (0)0.5 + (0.125)0.5 + (0.5)0.5 + (1.125)0.5 + (2)0.5 + (3.125)0.5 \\ &= 0 + 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 \\ &= 3.4375 \text{ units}^2 \end{aligned}$$

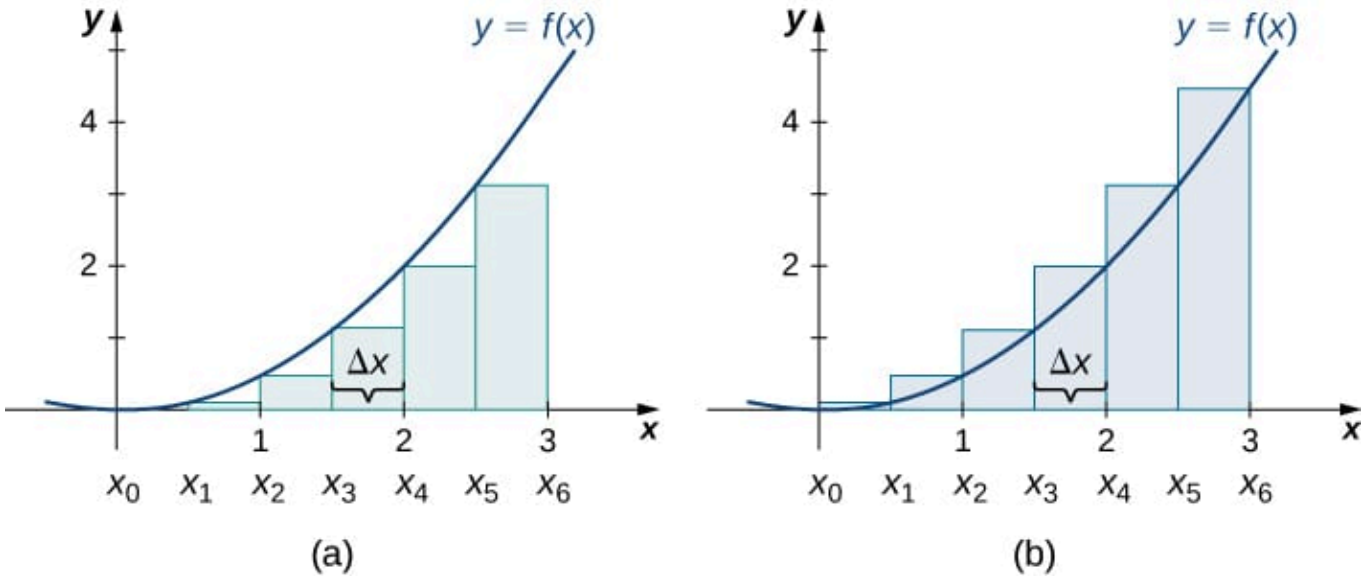

Figure 4.2.4**:** *Methods of approximating the area under a curve by using (a) the left endpoints and (b) the right endpoints.*

In Figure 4.2.4*b*, we draw vertical lines perpendicular to x_i such that x_i is the right endpoint of each subinterval, and calculate $f(x_i)$ for $i = 1,\ 2,\ 3,\ 4,\ 5,\ 6$. We multiply each $f(x_i)$ by Δx to find the rectangular areas and then add them. This is a right-endpoint approximation of the area under $f(x)$. Thus,

$$\begin{aligned} A &\approx R_6 \\ &= \sum_{i=1}^{6} f(x_i)\Delta x \\ &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x \\ &= f(0.5)0.5 + f(1)0.5 + f(1.5)0.5 + f(2)0.5 + f(2.5)0.5 + f(3)0.5 \\ &= (0.125)0.5 + (0.5)0.5 + (1.125)0.5 + (2)0.5 + (3.125)0.5 + (4.5)0.5 \\ &= 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 + 2.25 \\ &= 5.6875 \text{ units}^2. \end{aligned}$$

Example 4.2.4: Approximating the Area Under a Curve

Use both left-endpoint and right-endpoint approximations to approximate the area under the curve of $f(x) = x^2$ on the interval $[0, 2]$; use $n = 4$.

Solution available at the end of this section.

Checkpoint 4.2.4

Sketch left-endpoint and right-endpoint approximations for $f(x) = \frac{1}{x}$ on $[1, 2]$; use $n = 4$. Approximate the area using both methods.

Answer available at the end of this section.

Looking at Figure 4.2.4 and the graphs in Example 4.2.4, we can see that when we use a small number of intervals, neither the left-endpoint approximation nor the right-endpoint approximation is a particularly accurate estimate of the area under the curve. However, increasing the number of points in our partition will improve our estimate of A. We will have more rectangles, but each rectangle will be thinner so we can fit the rectangles to the curve more precisely.

We can demonstrate the improved approximation obtained through smaller intervals with an example. Let's explore the idea of increasing n, first in a left-endpoint approximation with four rectangles, then eight rectangles, and finally 32 rectangles. Then, let's do the same thing in a right-endpoint approximation, using the same sets of intervals of the same curved region. Figure 4.2.8 shows the area of the region under the curve $f(x) = (x-1)^3 + 4$ on the interval $[0, 2]$ using a left-endpoint approximation where $n = 4$. The width of each rectangle is

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}.$$

The summed areas of the rectangles approximate the area, or

$$L_4 = f(0)(0.5) + f(0.5)(0.5) + f(1)(0.5) + f(1.5)0.5 = 7.5 \text{ units}^2$$

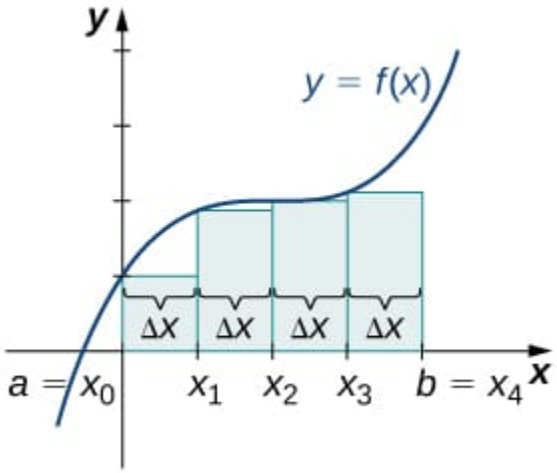

Figure 4.2.8: *With a left-endpoint approximation and dividing the region from a to b into four equal intervals, the area under the curve is approximately equal to the sum of the areas of the rectangles.*

Figure 4.2.9 shows the same curve divided into eight subintervals. Comparing the graph with four rectangles in Figure 4.2.8 with this graph with eight rectangles, we can see there appears to be less white space under the curve when $n = 8$. This white space is the area under the curve we cannot include using our approximation. The area of the rectangles is

$$L_8 = f(0)(0.25) + f(0.25)(0.25) + f(0.5)(0.25) + f(0.75)(0.25) + f(1)(0.25) + f(1.25)(0.25) + f(1.5)(0.25) + f(1.75)(0.25) = 7.75 \text{ unit}$$

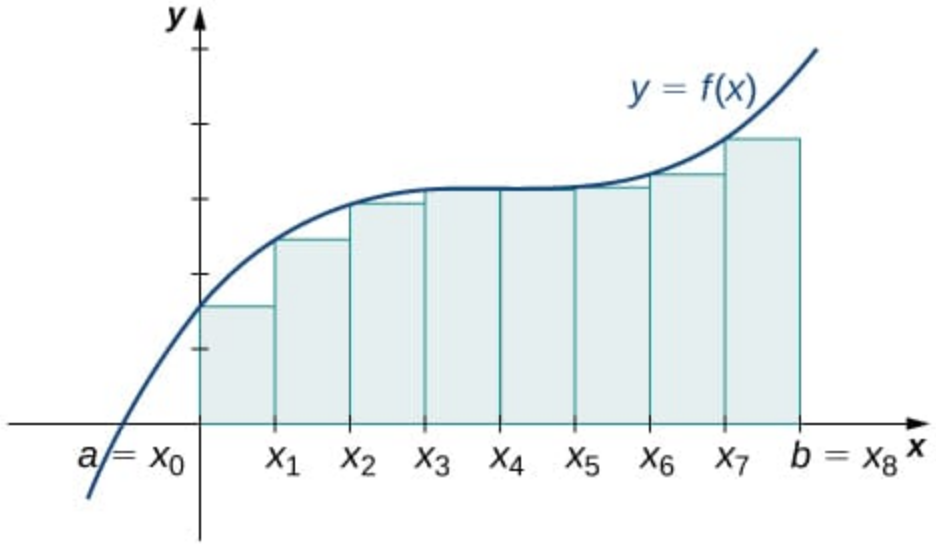

Figure $4.2.9$**:** *The region under the curve is divided into* $n = 8$ *rectangular areas of equal width for a left-endpoint approximation.*

The graph in Figure $4.2.10$ shows the same function with 32 rectangles inscribed under the curve. There appears to be little white space left. The area occupied by the rectangles is

$$L_{32} = f(0)(0.0625) + f(0.0625)(0.0625) + f(0.125)(0.0625) + \cdots + f(1.9375)(0.0625) = 7.9375 \text{ units}^2.$$

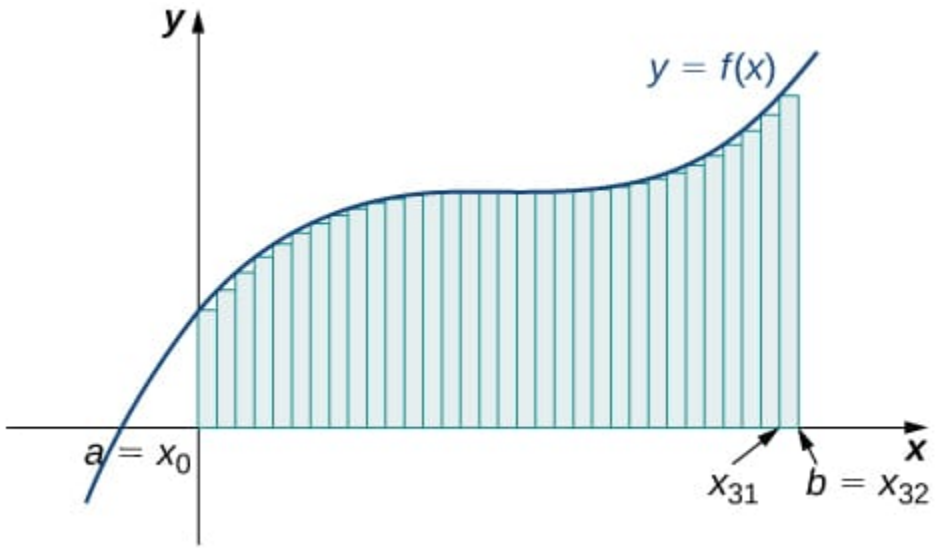

Figure $4.2.10$**:** *Here,* 32 *rectangles are inscribed under the curve for a left-endpoint approximation.*

We can perform a similar process for the right-endpoint approximation method. A right-endpoint approximation of the same curve, using four rectangles (Figure $4.2.11$), yields an area

$$R_4 = f(0.5)(0.5) + f(1)(0.5) + f(1.5)(0.5) + f(2)(0.5) = 8.5 \text{ units}^2.$$

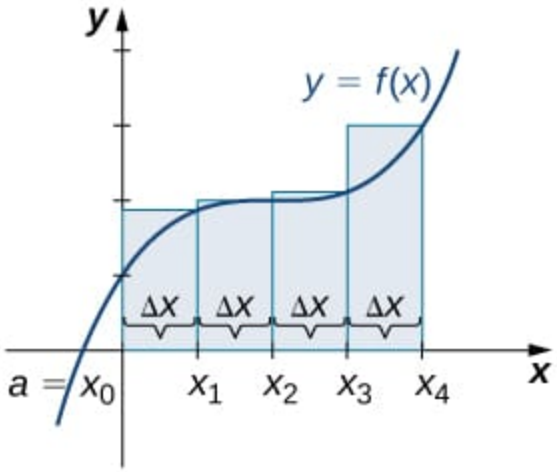

Figure 4.2.11**:** *Now, we divide the area under the curve into four equal subintervals for a right-endpoint approximation.*

Dividing the region over the interval $[0, 2]$ into eight rectangles results in $\Delta x = \frac{2-0}{8} = 0.25$. The graph is shown in Figure 4.2.12. The area is

$$R_8 = f(0.25)(0.25) + f(0.5)(0.25) + f(0.75)(0.25) + f(1)(0.25) + f(1.25)(0.25) + f(1.5)(0.25) + f(1.75)(0.25) + f(2)(0.25) = 8.25 \text{ unit}$$

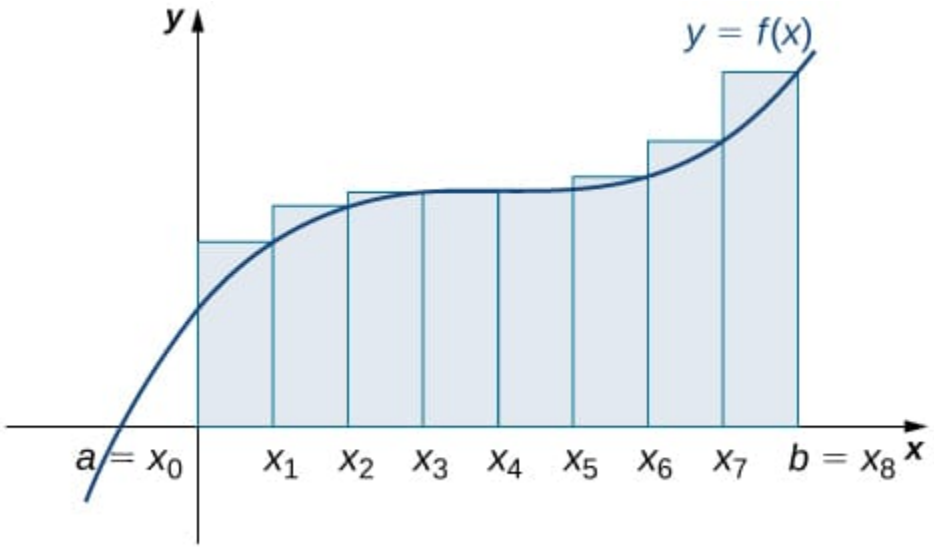

Figure 4.2.12**:** *Here, we use a right-endpoint approximation for a region divided into eight equal subintervals.*

Last, the right-endpoint approximation with $n = 32$ is close to the actual area (Figure 4.2.13). The area is approximately

$$R_{32} = f(0.0625)(0.0625) + f(0.125)(0.0625) + f(0.1875)(0.0625) + \cdots + f(2)(0.0625) = 8.0625 \text{ units}^2$$

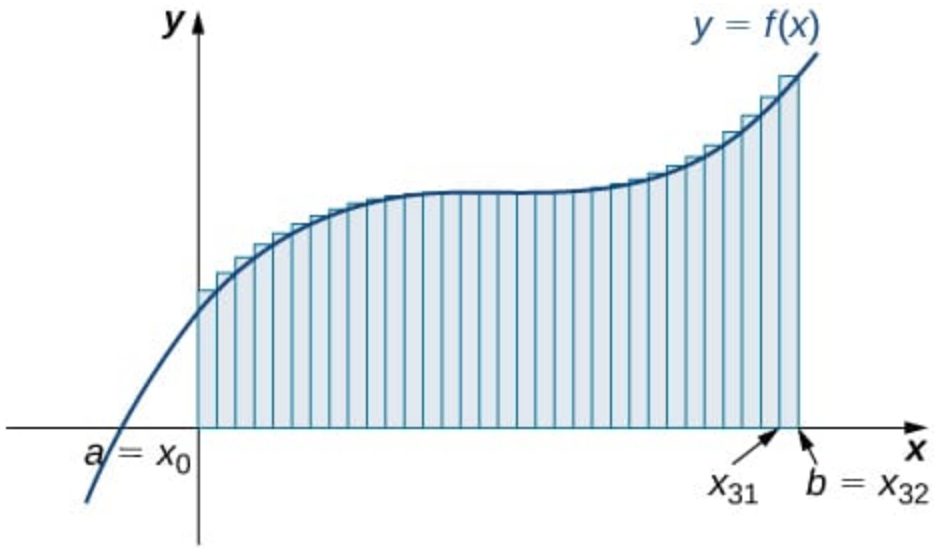

Figure 4.2.13**:** *The region is divided into* 32 *equal subintervals for a right-endpoint approximation.*

Based on these figures and calculations, we are on the right track; the rectangles better approximate the area under the curve as n gets larger. Furthermore, as n increases, both the left-end and right-endpoint approximations approach an area of 8 square units. Table $4.2.1$ shows a numerical comparison of the left- and right-endpoint methods. The idea that the approximations of the area under the curve get better and better as n gets larger and larger is very important, and we now explore this idea in more detail.

Table 4.2.1***:*** *Converging Values of Left- and Right-Endpoint Approximations as* n *Increases*

Value of n	Approximate Area L_n	Approximate Area R_n
$n = 4$	7.5	8.5
$n = 8$	7.75	8.25
$n = 32$	7.94	8.06

Interactive Element: Refining the Approximation

The following Interactive Element might help you visualize how the left- and right-endpoint approximations both close in on the exact area as the number of rectangles grows.

Interact: Drag the green handle along the lower track to change the number of rectangles n.

Observation: For this increasing curve the left-endpoint sum stays below and the right-endpoint sum stays above the true area, but as n grows both staircases squeeze onto the same value — 8 square units — matching Table $4.2.1$.

Forming Riemann Sums

So far, we have been using rectangles to approximate the area under a curve. The heights of these rectangles have been determined by evaluating the function at either the right or left endpoints of the subinterval $[x_{i-1}, x_i]$. In reality, there is no reason to restrict the evaluation of the function to only one of these two points. We could evaluate the function at any point x_i^* in the subinterval $[x_{i-1}, x_i]$, and use $f(x_i^*)$ as the height of our rectangle. This gives us an estimate for the area of the form

$$A \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$

A sum of this form is called a **Riemann sum**, named for the 19th-century mathematician Bernhard Riemann, who developed the idea.

Definition: Riemann sum

Let $f(x)$ be defined on a closed interval $[a, b]$ and let P be any partition of $[a, b]$. Let Δx_i be the width of each subinterval $[x_{i-1}, x_i]$ and for each i, let x_i^* be any point in $[x_{i-1},\ x_i]$. A **Riemann sum** is defined for $f(x)$ as

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i.$$

At this point, we'll choose a regular partition P, as we have in our examples above. This forces all Δx_i to be equal to $\Delta x = \frac{b-a}{n}$ for any natural number of intervals n.

Recall that with the left- and right-endpoint approximations, the estimates get better and better as n get larger and larger. The same thing happens with Riemann sums. Riemann sums give better approximations for larger values of n. We are now ready to define the area under a curve in terms of Riemann sums.

Definition: Area Under the Curve

Let $f(x)$ be a continuous, nonnegative function on an interval $[a, b]$, and let $\sum_{i=1}^{n} f(x_i^*)\ \Delta x$ be a Riemann sum for $f(x)$ with a regular partition P. Then, the **area under the curve** $y = f(x)$ on $[a, b]$ is given by

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$

Some subtleties here are worth discussing. First, taking the limit of a sum is different from taking the limit of a function $f(x)$ as x goes to infinity. Limits of sums are discussed in Calculus II; however, we can assume that the computational techniques we used to compute the limits of functions can also be used to calculate the limits of sums.

Second, we must consider what to do if the expression converges to different limits for different choices of x_i^*. Fortunately, this does not happen. Although the proof is beyond the scope of this text, it can be shown that if $f(x)$ is continuous on the closed interval $[a, b]$, then $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$ exists and is unique (in other words, it does not depend on the choice of x_i^*).

Interactive Element: Riemann Sums & the Sample Point

The following Interactive Element might help you visualize why the exact area doesn't depend on where inside each strip you measure the rectangle's height.

Interact: Drag the orange handle to move the sample point within every strip (left edge to right edge), and drag the green handle to change the number of strips n.

Observation: With only a few strips, moving the sample point changes the estimate a lot; with many strips, every choice gives nearly the same total — the limit is the same for any x_i^*.

We look at some examples shortly. But, before we do, let's take a moment and talk about some specific choices for x_i^*. Although any choice for x_i^* gives us an estimate of the area under the curve, we don't necessarily know whether that estimate is too high (overestimate) or too low (underestimate). If it is important to know whether our estimate is high or low, we can select our value for x_i^* to guarantee one result or the other.

If we want an overestimate, for example, we can choose x_i^* such that for $i = 1,\ 2,\ 3,\ \ldots,\ n$, $f(x_i^*) \geq f(x)$ for all $x \in [x_i - 1, x_i]$. In other words, we choose x_i^* so that for $i = 1,\ 2,\ 3,\ \ldots,\ n$, $f(x_i^*)$ is the maximum function value on the interval $[x_{i-1}, x_i]$. If we select x_i^* in this way, then the Riemann sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called an **upper sum**. Similarly, if we want an underestimate, we can choose x_i^* so that for $i = 1, 2, 3, \ldots, n$, $f(x_i^*)$ is the minimum function value on the interval $[x_{i-1}, x_i]$. The associated Riemann sum is called

a **lower sum**. Note that if $f(x)$ is either increasing or decreasing throughout the interval $[a, b]$, then the maximum and minimum values of the function occur at the endpoints of the subintervals, so the upper and lower sums are just the same as the left- and right-endpoint approximations.

Example 4.2.5: Finding Lower and Upper Sums

Find a lower sum for $f(x) = 10 - x^2$ on $[1, 2]$; let $n = 4$ subintervals.

Solution available at the end of this section.

Checkpoint 4.2.5

a. Find an upper sum for $f(x) = 10 - x^2$ on $[1, 2]$; let $n = 4$.
b. Sketch the approximation.

Answer available at the end of this section.

Example 4.2.6: Finding Lower and Upper Sums for $f(x) = \sin x$

Find a lower sum for $f(x) = \sin x$ over the interval $[a, b] = [0, \frac{\pi}{2}]$; let $n = 6$.

Solution available at the end of this section.

Checkpoint 4.2.6

Using the function $f(x) = \sin x$ over the interval $[0, \frac{\pi}{2}]$, find an upper sum; let $n = 6$.

Answer available at the end of this section.

Success in College: Dealing With Failure (Part 4)

1. A first-year student fails her math class. She hates math. Her advisor wants her to retake the course the next term. The student wants to wait a year or so before trying again. Discuss the advantages and disadvantages of waiting to retake this math course.
2. Some colleges restrict the number of times a student can repeat a course. Describe any restrictions on repeating a course at your school.

Success in College: Groupwork Learning Preferences

Most students are very intimidated by working in groups. It usually takes quite a while for students to realize that studying together can be very rewarding.

Think about the following questions regarding learning or practicing complicated math.

- Do you prefer to do homework by yourself, or do you like to do homework with others?
- Do you like to do homework at school, or do you like to do homework at home?
- If you need help, do you prefer to get it from friends or a spouse, or do you prefer to get it from your professor or a tutor?
- Do you concentrate on learning what you don't know by yourself, or do you prefer to help others because it helps you learn it better?

1. Do you work in conditions that match these preferences when you do your math homework? Why or why not?
2. Spending time with old and new friends is an integral part of college. However, studying with friends is not always practical. Do you ever feel peer pressure to study in ways that are not the best for you?

Success in Mathematics: Why Study Mathematics? (Part 5)

Mathematics reveals hidden patterns that help us understand the world around us. Now, much more than arithmetic and geometry, mathematics today is a diverse discipline that deals with data, measurements, and observations from science; with inference, deduction, and proof; and with mathematical models of natural phenomena, human behavior, and social systems.

The process of "doing" mathematics is far more than just calculation or deduction; it involves observing patterns, testing conjectures, and estimating results. As a practical matter, mathematics is a science of pattern and order. Its domain is not molecules or cells but numbers, chance, form, algorithms, and change. As a science of abstract objects, mathematics relies on logic rather than observation as its standard of truth yet employs observation, simulation, and even experimentation to discover the truth.

The unique role of mathematics in education is a consequence of its universal applicability. The mathematics results - theorems and theories - are significant and valuable; the best results are also elegant and profound. Through its theorems, mathematics offers science both a foundation of truth and a standard of certainty. In addition to theorems and theories, mathematics offers distinctive modes of thought that are versatile and powerful, including modeling, abstraction, optimization, logical analysis, inference from data, and the use of symbols.

As a major intellectual tradition, mathematics is a subject appreciated as much for its beauty as for its power. The enduring qualities of such abstract concepts as symmetry, proof, and change have been developed through more than 3,000 years of intellectual effort. Like language, religion, and music, mathematics is a universal part of human culture.

1. What was your favorite moment or lecture in this course? Explain.
2. Has this course helped your appreciation for mathematics? Explain.

Answers

Section Preview

This enter section should feel like a divergence from Calculus. Spend the time to develop the core concept of area approximations—this will pay you and your students back very soon.

Section Preview

- **Sequences and Series**
  - Summations and Their Properties: Students are expected to arrive with comfort reading summation notation and using the closed-form power-sum identities, which the text explicitly attributes to prior coursework ("hopefully covered in your Precalculus course").

Section Preview

- Use sigma (summation) notation to calculate sums and powers of integers.
- Use the sum of rectangular areas to approximate the area under a curve.
- Use Riemann sums to approximate area.
- Determine the precise area under polynomial curves of degree at most 3 using limits of Riemann sums.

Example $4.2.1$**: Using Sigma Notation**

a. $$\sum_{i=1}^{5} 3^i = 3 + 3^2 + 3^3 + 3^4 + 3^5 = 363.$$

b. The denominator of each term is a perfect square. Using sigma notation, this sum can be written as $\sum_{i=1}^{5} \frac{1}{i^2}$.

Checkpoint $4.2.1$

$$\sum_{i=3}^{6} 2^i = 2^3 + 2^4 + 2^5 + 2^6 = 120$$

Theorem: Properties of Sigma Notation

We prove properties (ii.) and (iii.) here and leave proof of the other properties to the Exercises.

(ii.) We have

$$\sum_{i=1}^{n} ca_i = ca_1 + ca_2 + ca_3 + \cdots + ca_n = c(a_1 + a_2 + a_3 + \cdots + a_n) = c\sum_{i=1}^{n} a_i.$$

(iii.) We have

$$\begin{aligned}\sum_{i=1}^{n}(a_i+b_i) &= (a_1+b_1)+(a_2+b_2)+(a_3+b_3)+\cdots+(a_n+b_n)\\ &= (a_1+a_2+a_3+\cdots+a_n)+(b_1+b_2+b_3+\cdots+b_n)\\ &= \sum_{i=1}^{n}a_i+\sum_{i=1}^{n}b_i.\end{aligned}$$

Q.E.D.

Example 4.2.2**: Evaluation Using Sigma Notation**

a. Multiplying out $(i-3)^2$, we can break the expression into three terms.

$$\begin{aligned}\sum_{i=1}^{200}(i-3)^2 &= \sum_{i=1}^{200}(i^2-6i+9)\\ &= \sum_{i=1}^{200}i^2-\sum_{i=1}^{200}6i+\sum_{i=1}^{200}9\\ &= \sum_{i=1}^{200}i^2-6\sum_{i=1}^{200}i+\sum_{i=1}^{200}9\\ &= \frac{200(200+1)(400+1)}{6}-6\left[\frac{200(200+1)}{2}\right]+9(200)\\ &= 2,686,700-120,600+1800\\ &= 2,567,900\end{aligned}$$

b. Use sigma notation property iv. and the rules for the sum of squared and cubed terms.

$$\begin{aligned}\sum_{i=1}^{6}(i^3-i^2) &= \sum_{i=1}^{6}i^3-\sum_{i=1}^{6}i^2\\ &= \frac{6^2(6+1)^2}{4}-\frac{6(6+1)(2(6)+1)}{6}\\ &= \frac{1764}{4}-\frac{546}{6}\\ &= 350\end{aligned}$$

Checkpoint 4.2.2

$15,550$

Example 4.2.3**: Finding the Sum of the Function Values**

Using Equation 4.2.3, we have

$$\sum_{i=0}^{10}i^3=\frac{(10)^2(10+1)^2}{4}=\frac{100(121)}{4}=3025.$$

Checkpoint 4.2.3

440

Example 4.2.4**: Approximating the Area Under a Curve**

First, divide the interval $[0,2]$ into n equal subintervals. Using $n=4,\quad \Delta x=\frac{(2-0)}{4}=0.5$. This is the width of each rectangle. The intervals $[0,0.5],\ [0.5,1],\ [1,1.5],$ $[1.5,2]$ are shown in Figure 4.2.5. Using a left-endpoint approximation, the heights are $f(0)=0,\ f(0.5)=0.25,\ f(1)=1$, and $f(1.5)=2.25$. Then,

$$\begin{aligned}L_4 &= f(x_0)\Delta x+f(x_1)\Delta x+f(x_2)\Delta x+f(x_3)\Delta x\\ &= 0(0.5)+0.25(0.5)+1(0.5)+2.25(0.5)\\ &= 1.75\,\text{units}^2\end{aligned}$$

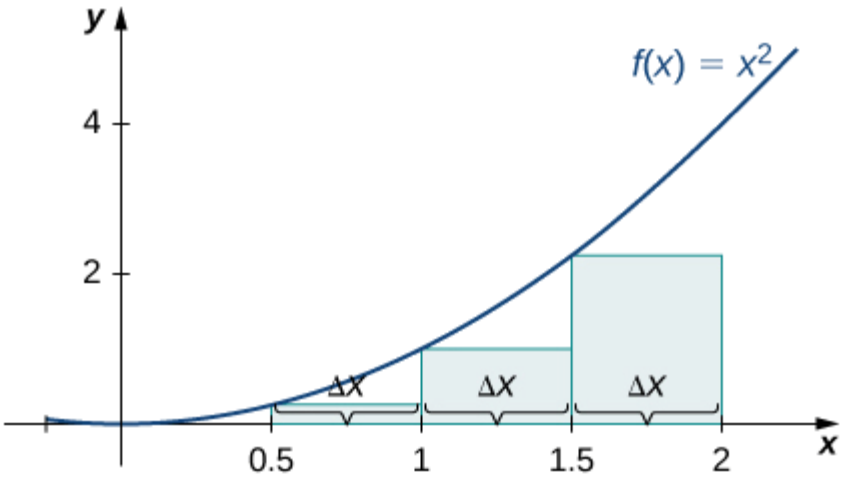

Figure 4.2.5**:** *The graph shows the left-endpoint approximation of the area under* $f(x) = x^2$ *from* 0 *to* 2.

The right-endpoint approximation is shown in Figure 4.2.6. The intervals are the same, $\Delta x = 0.5$, but now use the right endpoint to calculate the height of the rectangles. We have

$$\begin{aligned} R_4 &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\ &= 0.25(0.5) + 1(0.5) + 2.25(0.5) + 4(0.5) \\ &= 3.75 \text{ units}^2 \end{aligned}$$

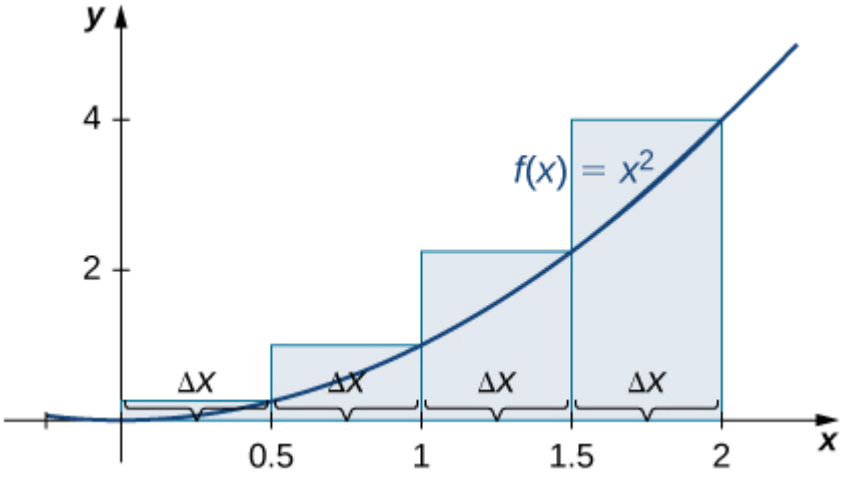

Figure 4.2.6**:** *The graph shows the right-endpoint approximation of the area under* $f(x) = x^2$ *from* 0 *to* 2.

The left-endpoint approximation is 1.75 units^2; the right-endpoint approximation is 3.75 units^2.

Checkpoint 4.2.4

The left-endpoint approximation is 0.7595 units^2. The right-endpoint approximation is 0.6345 units^2. See the figures below.

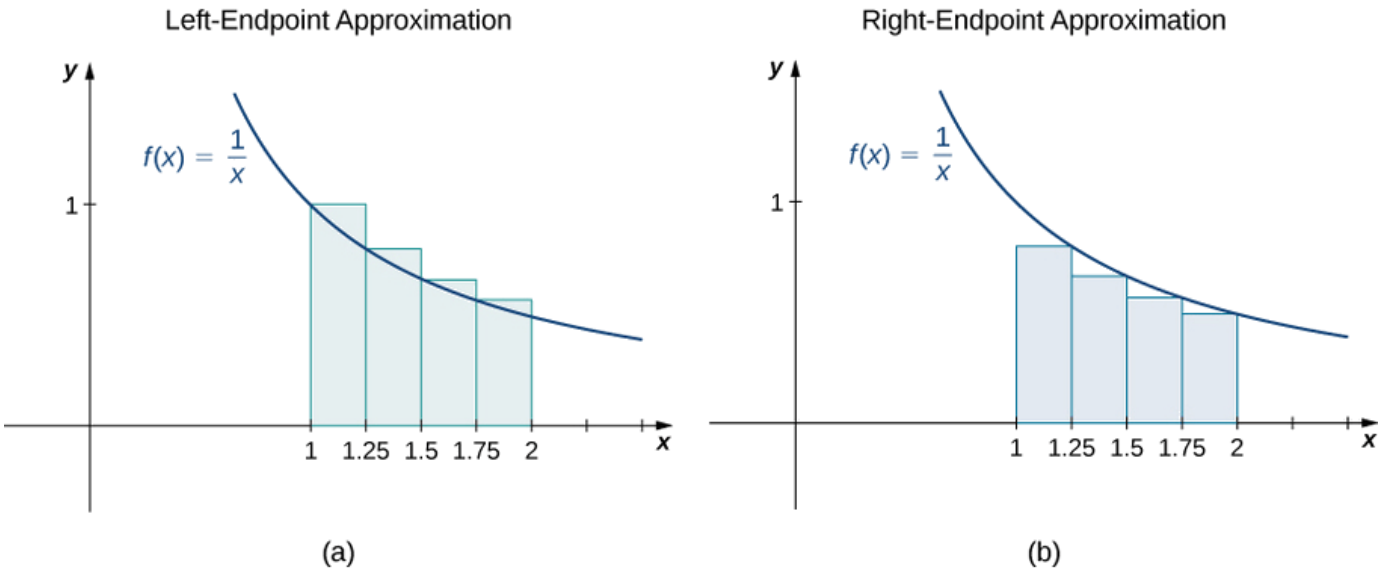

Figure 4.2.7

Example 4.2.5**: Finding Lower and Upper Sums**

With $n = 4$ over the interval $[1, 2]$, $\Delta x = \frac{1}{4}$. We can list the intervals as $[1, 1.25]$, $[1.25, 1.5]$, $[1.5, 1.75]$, and $[1.75, 2]$. Because the function is decreasing over the interval $[1, 2]$, Figure 4.2.14 shows that a lower sum is obtained by using the right endpoints.

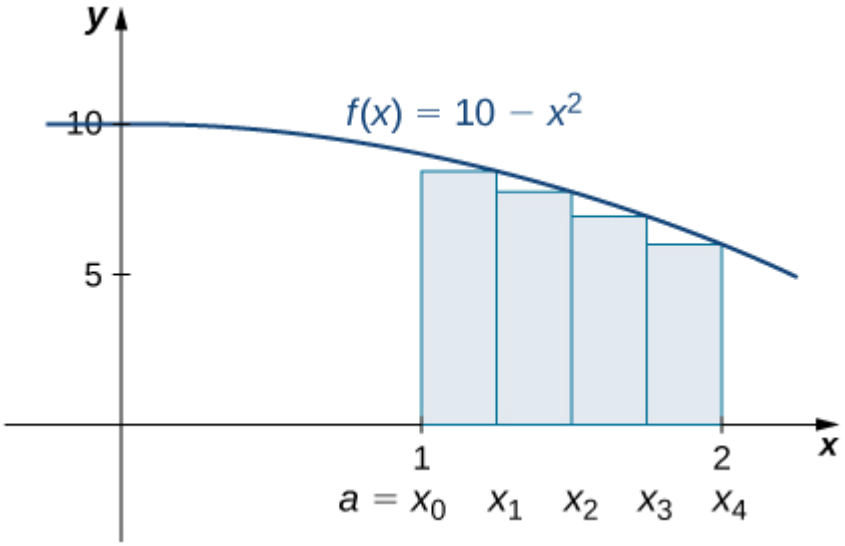

Figure 4.2.14**:** *The graph of $f(x) = 10 - x^2$ is set up for a right-endpoint approximation of the area bounded by the curve and the x-axis on $[1, 2]$, and it shows a lower sum.*

The Riemann sum is

$$\begin{aligned}\sum_{k=1}^{4}(10 - x^2)(0.25) &= 0.25[10 - (1.25)^2 + 10 - (1.5)^2 + 10 - (1.75)^2 + 10 - (2)^2] \\ &= 0.25[8.4375 + 7.75 + 6.9375 + 6] \\ &= 7.28\,\text{units}^2.\end{aligned}$$

The area of $7.28\ \text{units}^2$ is a lower sum and an underestimate.

Checkpoint 4.2.5

a. Upper sum=$8.0313\ \text{units}^2$.
b.

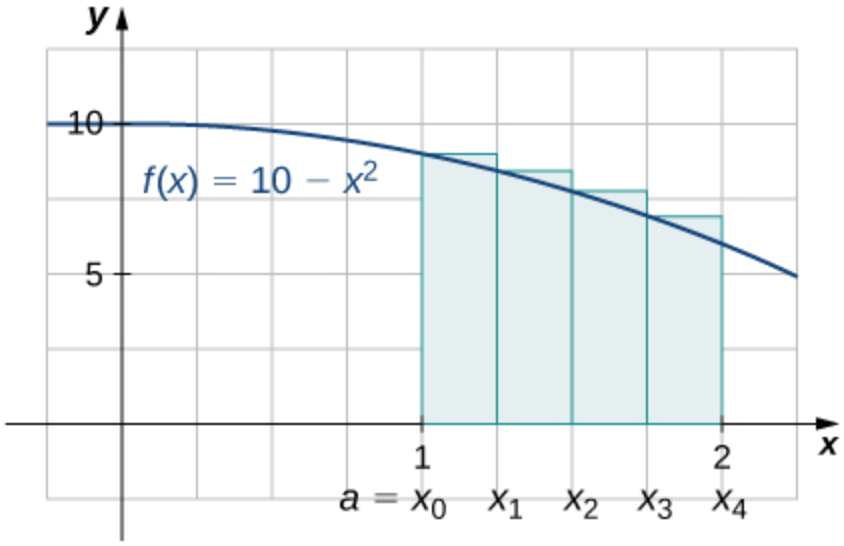

Figure 4.2.15

Example 4.2.6**: Finding Lower and Upper Sums for** $f(x) = \sin x$

Let's first look at the graph in Figure 4.2.16 to get a better idea of the area of interest.

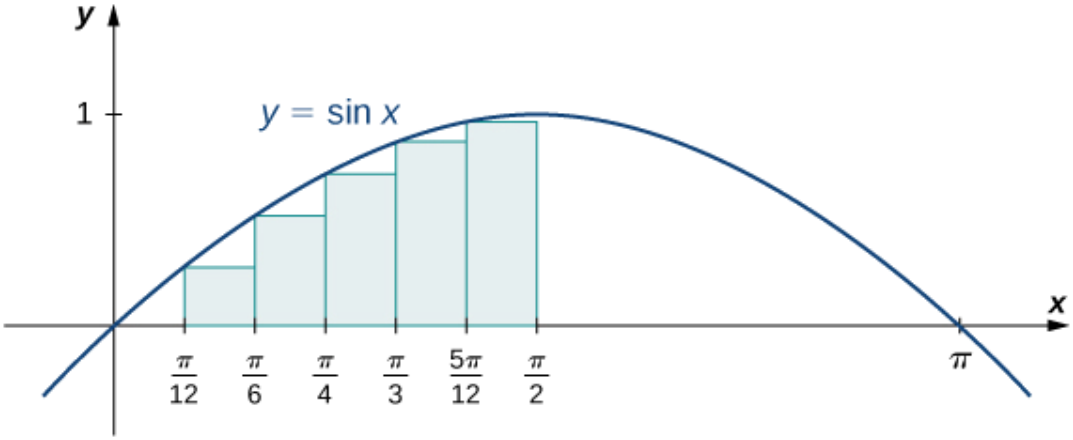

Figure 4.2.16**:** *The graph of* $y = \sin x$ *is divided into six regions:* $\Delta x = \frac{\pi/2}{6} = \frac{\pi}{12}$.

The intervals are $[0, \frac{\pi}{12}]$, $[\frac{\pi}{12}, \frac{\pi}{6}]$, $[\frac{\pi}{6}, \frac{\pi}{4}]$, $[\frac{\pi}{4}, \frac{\pi}{3}]$, $[\frac{\pi}{3}, \frac{5\pi}{12}]$, and $[\frac{5\pi}{12}, \frac{\pi}{2}]$. Note that $f(x) = \sin x$ is increasing on the interval $[0, \frac{\pi}{2}]$, so a left-endpoint approximation gives us the lower sum. A left-endpoint approximation is the Riemann sum $\sum_{i=0}^{5} \sin x_i \left(\frac{\pi}{12}\right)$.We have

$$A \approx \sin(0)\left(\tfrac{\pi}{12}\right) + \sin\left(\tfrac{\pi}{12}\right)\left(\tfrac{\pi}{12}\right) + \sin\left(\tfrac{\pi}{6}\right)\left(\tfrac{\pi}{12}\right) + \sin\left(\tfrac{\pi}{4}\right)\left(\tfrac{\pi}{12}\right) + \sin\left(\tfrac{\pi}{3}\right)\left(\tfrac{\pi}{12}\right) + \sin\left(\tfrac{5\pi}{12}\right)\left(\tfrac{\pi}{12}\right) \approx 0.863\,\text{units}^2.$$

Checkpoint 4.2.6

$A \approx 1.125\ \text{units}^2$

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4.2.1: Resources and Key Concepts

Resources

Prerequisite Topics

The following is a list of prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Sequences and Series**
  - Summations and Their Properties: Sigma notation (Σ) is introduced and used extensively. Properties of summations are listed and applied.

Videos

- Sigma notation
  - Sigma Notation and Its Properties
  - Using Sigma Notation
- Properties of summation
  - Review of Summation Properties Required for Integral Calculus
- Approximating areas
  - Approximating Areas Under Curves
- Riemann sum
  - Riemann Sums
  - Computing Areas “Under” Curves Using Riemann Sums
  - Computing Distances Using Riemann Sums

Key Concepts

Definitions

- **Sigma Notation (Summation Notation)**: A compact way to express long sums of values using the Greek capital letter Σ. In $\sum_{i=1}^{n} a_i$, a_i describes the terms to be added, and i is the index (a dummy variable).
- **Partition of an Interval**: A set of points $P = \{x_i\}$ for $i = 0, 1, 2, \ldots, n$ with $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, which divides the interval $[a, b]$ into n subintervals of the form $[x_{i-1}, x_i]$.
- **Regular Partition (Uniform Partition)**: A partition where all subintervals have the same width, $\Delta x = \frac{b-a}{n}$.
- **Left-Endpoint Approximation (L_n)**: An approximation of the area under a curve using rectangles where the height of each rectangle on the subinterval $[x_{i-1}, x_i]$ is determined by the function value at the left endpoint, $f(x_{i-1})$. $A \approx L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x$.
- **Right-Endpoint Approximation (R_n)**: An approximation of the area under a curve using rectangles where the height of each rectangle on the subinterval $[x_{i-1}, x_i]$ is determined by the function value at the right endpoint, $f(x_i)$. $A \approx R_n = \sum_{i=1}^{n} f(x_i)\Delta x$.
- **Riemann Sum**: Let $f(x)$ be defined on a closed interval $[a, b]$ and let P be any partition of $[a, b]$. Let Δx_i be the width of each subinterval $[x_{i-1}, x_i]$ and for each i, let x_i^* be any point in $[x_{i-1}, x_i]$. A Riemann sum is defined for $f(x)$ as $\sum_{i=1}^{n} f(x_i^*)\Delta x_i$.
- **Upper Sum**: A Riemann sum where $f(x_i^*)$ is the maximum function value on the interval $[x_{i-1}, x_i]$.
- **Lower Sum**: A Riemann sum where $f(x_i^*)$ is the minimum function value on the interval $[x_{i-1}, x_i]$.
- **Method of Exhaustion**: An ancient process of approximating areas of irregular regions by filling them with shapes of known

area, and then summing these areas.

Theorems

- **Theorem: Properties of Sigma Notation**:
  i. $\sum_{i=1}^{n} c = nc$
  ii. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$
  iii. $\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$
  iv. $\sum_{i=1}^{n}(a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$
  v. $\sum_{i=1}^{n} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i$ (for $1 \leq m < n$)
- **Theorem: Sums and Powers of Integers**:
  1. Sum of integers: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
  2. Sum of consecutive integers squared: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
  3. Sum of consecutive integers cubed: $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$
- **Theorem: Left-Endpoint Approximation** (Formula stated under Definitions).
- **Theorem: Right-Endpoint Approximation** (Formula stated under Definitions).
- **Area Under the Curve**: Let $f(x)$ be a continuous, nonnegative function on an interval $[a, b]$, and let $\sum_{i=1}^{n} f(x_i^*)\Delta x$ be a Riemann sum for $f(x)$ with a regular partition P. Then, the area under the curve $y = f(x)$ on $[a, b]$ is given by $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$.
- *(Uniqueness of Limit of Riemann Sums): Although not formally stated as a theorem with proof in this section, the text mentions that if $f(x)$ is continuous on the closed interval $[a, b]$, then $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$ exists and is unique (does not depend on the choice of x_i^*).*

Common Mistakes

- **Incorrectly Using Sigma Notation Properties**: For example, assuming $\sum(a_i b_i) = (\sum a_i)(\sum b_i)$.
- **Confusing Left and Right Endpoints**: Using $f(x_i)$ for a left-endpoint sum or $f(x_{i-1})$ for a right-endpoint sum.
- **Not Using Enough Rectangles for Good Approximation**: Realizing that approximations with a small number of rectangles (n) can be quite inaccurate.
- **Assuming Riemann Sum is Exact Area**: A Riemann sum with a finite n is an approximation, not the exact area (unless the function is constant or the rectangles perfectly fit). The exact area is the limit as $n \to \infty$.
- **Treating the Index of Summation as a Factor**: The index (e.g., i in $\sum a_i$) is a dummy variable and does not factor into the sum itself beyond indicating which terms are added.

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4.2.2: Homework

Reading Questions

1. What is sigma notation (or summation notation) used for? Explain the meaning of the components in $\sum_{i=1}^{n} a_i$.
2. What is a regular partition of an interval $[a, b]$? How is the width Δx of each subinterval calculated if there are n subintervals?
3. Describe the left-endpoint approximation method for estimating the area under a curve. How is the height of each rectangle determined?
4. Describe the right-endpoint approximation method. How does it differ from the left-endpoint method in determining rectangle height?
5. What is a Riemann sum? How does it generalize the left- and right-endpoint approximations?
6. What is an upper sum? What is a lower sum?
7. How is the exact area under a continuous, non-negative function $f(x)$ on an interval $[a, b]$ defined using Riemann sums?
8. What is the formula for the sum of the first n integers, $\sum_{i=1}^{n} i$?
9. What is the formula for the sum of the first n squares, $\sum_{i=1}^{n} i^2$?
10. Does the choice of sample points x_i^* in a Riemann sum affect the value of the limit $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$ if $f(x)$ is continuous?

Homework

1.) State whether the given sums are equal or unequal.

a. $\sum_{i=1}^{10} i$ and $\sum_{k=1}^{10} k$
b. $\sum_{i=1}^{10} i$ and $\sum_{i=6}^{15} (i-5)$
c. $\sum_{i=1}^{10} i(i-1)$ and $\sum_{j=0}^{9} (j+1)j$
d. $\sum_{i=1}^{10} i(i-1)$ and $\sum_{k=1}^{10} (k^2-k)$

Answer

a. They are equal; both represent the sum of the first 10 whole numbers.
b. They are equal; both represent the sum of the first 10 whole numbers.
c. They are equal by substituting $j = i - 1$.
d. They are equal; the first sum factors the terms of the second.

In exercises 2 - 3, use the rules for sums of powers of integers to compute the sums.

2) $\sum_{i=5}^{10} i$

3) $\sum_{i=5}^{10} i^2$

Answer

$$\sum_{i=5}^{10} i^2 = 385 - 30 = 355$$

Suppose that $\sum_{i=1}^{100} a_i = 15$ and $\sum_{i=1}^{100} b_i = -12$.

In exercises 4 - 7, compute the sums.

4) $\sum_{i=1}^{100}(a_i + b_i)$

5) $\sum_{i=1}^{100}(a_i - b_i)$

Answer

$$\sum_{i=1}^{100}(a_i - b_i) = 15 - (-12) = 27$$

6) $\sum_{i=1}^{100}(3a_i - 4b_i)$

7) $\sum_{i=1}^{100}(5a_i + 4b_i)$

Answer

$$\sum_{i=1}^{100}(5a_i + 4b_i) = 5(15) + 4(-12) = 27$$

In exercises 8 - 11, use summation properties and formulas to rewrite and evaluate the sums.

8) $\sum_{k=1}^{20} 100(k^2 - 5k + 1)$

9) $\sum_{j=1}^{50}(j^2 - 2j)$

Answer

$$\sum_{j=1}^{50} j^2 - 2\sum_{j=1}^{50} j = \frac{(50)(51)(101)}{6} - \frac{2(50)(51)}{2} = 40,375$$

10) $\sum_{j=11}^{20}(j^2 - 10j)$

11) $\sum_{k=1}^{25}[(2k)^2 - 100k]$

Answer

$$4\sum_{k=1}^{25} k^2 - 100\sum_{k=1}^{25} k = \frac{4(25)(26)(51)}{9} - 50(25)(26) = -10,400$$

Let L_n denote the left-endpoint sum using n subintervals and let R_n denote the corresponding right-endpoint sum.

In exercises 12 - 19, compute the indicated left and right sums for the given functions on the indicated interval.

12) L_4 for $f(x) = \dfrac{1}{x-1}$ on $[2,3]$

13) R_4 for $g(x) = \cos(\pi x)$ on $[0,1]$

Answer

$R_4 = 0.25$

14) L_6 for $f(x) = \dfrac{1}{x(x-1)}$ on $[2, 5]$

15) R_6 for $f(x) = \dfrac{1}{x(x-1)}$ on $[2, 5]$

Answer

$R_6 = 0.372$

16) R_4 for $\dfrac{1}{x^2+1}$ on $[-2, 2]$

17) L_4 for $\dfrac{1}{x^2+1}$ on $[-2, 2]$

Answer

$L_4 = 2.20$

18) R_4 for $x^2 - 2x + 1$ on $[0, 2]$

19) L_8 for $x^2 - 2x + 1$ on $[0, 2]$

Answer

$L_8 = 0.6875$

20) Compute the left and right Riemann sums— L_4 and R_4, respectively—for $f(x) = (2 - |x|)$ on $[-2, 2]$. Compute their average value and compare it with the area under the graph of f.

21) Compute the left and right Riemann sums— L_6 and R_6, respectively—for $f(x) = (3 - |3 - x|)$ on $[0, 6]$. Compute their average value and compare it with the area under the graph of f.

Answer

$L_6 = 9.000 = R_6$. The graph of f is a triangle with area 9 units^2.

22) Compute the left and right Riemann sums— L_4 and R_4, respectively—for $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$ and compare their values.

23) Compute the left and right Riemann sums— L_6 and R_6, respectively—for $f(x) = \sqrt{9 - (x-3)^2}$ on $[0, 6]$ and compare their values.

Answer

$L_6 = 13.12899 = R_6$. They are equal.

For exercises 24 - 27, express the following endpoint sums in sigma notation but do not evaluate them.

24) L_{30} for $f(x) = x^2$ on $[1, 2]$

25) L_{10} for $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$

Answer

$$L_{10} = \frac{4}{10} \sum_{i=1}^{10} \sqrt{4 - (-2 + 4\frac{(i-1)}{10})}$$

26) R_{20} for $f(x) = \sin x$ on $[0, \pi]$

27) R_{100} for $\ln x$ on $[1, e]$

Answer

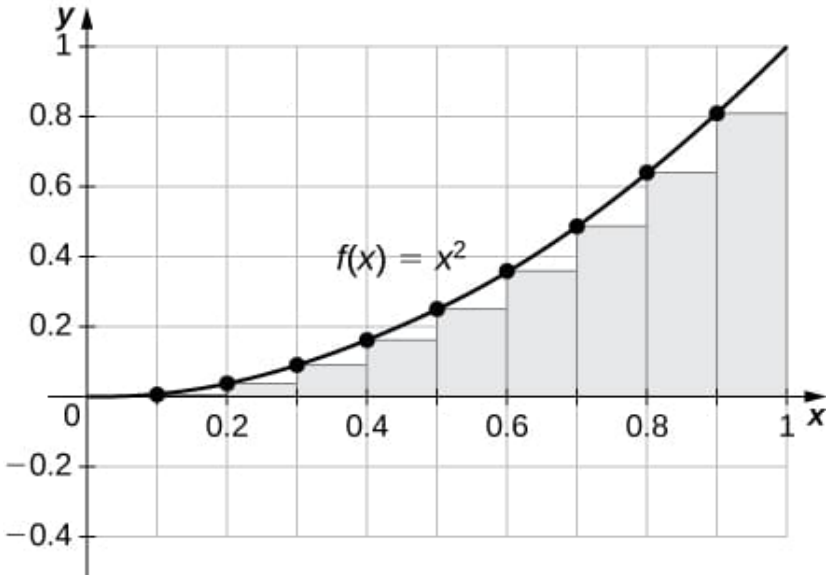
$$R_{100} = \frac{e-1}{100} \sum_{i=1}^{100} ln(1 + (e-1)\frac{i}{100})$$

In exercises 28 - 33, graph the function then use a calculator or a computer program to evaluate the following left and right endpoint sums. Is the area under the curve between the left and right endpoint sums?

28) [Technology Required] L_{100} and R_{100} for $y = x^2 - 3x + 1$ on the interval $[-1, 1]$

29) [Technology Required] L_{100} and R_{100} for $y = x^2$ on the interval $[0, 1]$

Answer

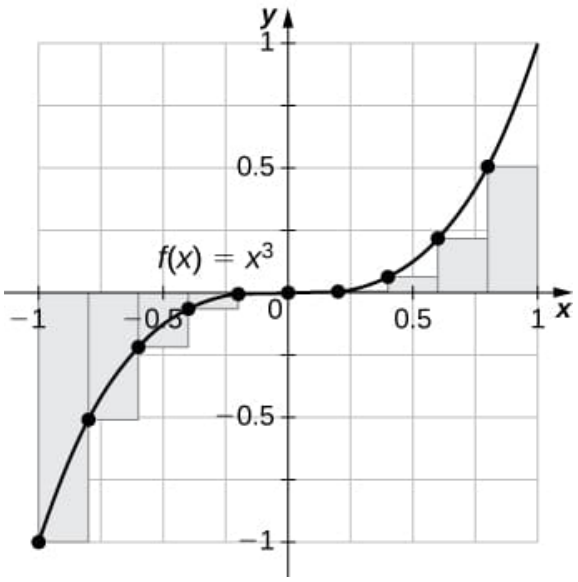


$R_{100} = 0.33835, \quad L_{100} = 0.32835.$
The plot shows that the left Riemann sum is an underestimate because the function is increasing. Similarly, the right Riemann sum is an overestimate. The area lies between the left and right Riemann sums. Ten rectangles are shown for visual clarity. This behavior persists for more rectangles.

30) [Technology Required] L_{50} and R_{50} for $y = \frac{x+1}{x^2-1}$ on the interval $[2, 4]$

31) [Technology Required] L_{100} and R_{100} for $y = x^3$ on the interval $[-1, 1]$

Answer

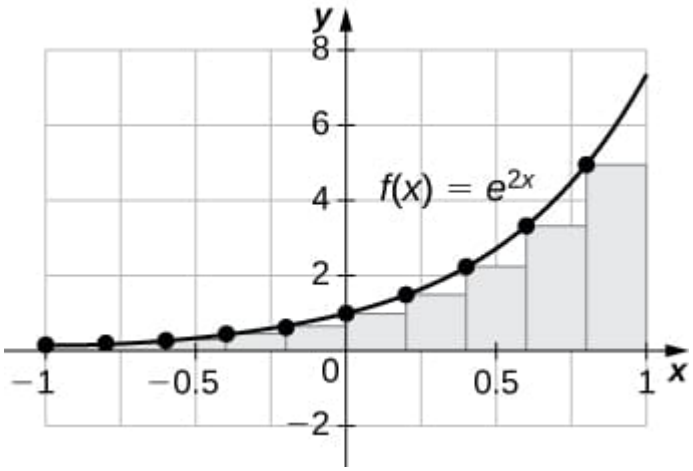


$L_{100} = -0.02, \quad R_{100} = 0.02.$
The left endpoint sum is an underestimate because the function is increasing. Similarly, a right endpoint approximation is an overestimate. The area lies between the left and right endpoint estimates.

32) [Technology Required] L_{50} and R_{50} for $y = \tan(x)$ on the interval $[0, \frac{\pi}{4}]$

33) [Technology Required] L_{100} and R_{100} for $y = e^{2x}$ on the interval $[-1, 1]$

Answer



$L_{100} = 3.555, \quad R_{100} = 3.670.$
The plot shows that the left Riemann sum is an underestimate because the function is increasing. Ten rectangles are shown for visual clarity. This behavior persists for more rectangles.

34) Let t_j denote the time that it took Tejay van Garteren to ride the jth stage of the Tour de France in 2014. If there were a total of 21 stages, interpret $\sum_{j=1}^{21} t_j$.

35) Let r_j denote the total rainfall in Portland on the jth day of the year in 2009. Interpret $\sum_{j=1}^{31} r_j$.

Answer

The sum represents the cumulative rainfall in January 2009.

36) Let d_j denote the hours of daylight and δ_j denote the increase in the hours of daylight from day $j-1$ to day j in Fargo, North Dakota, on the jth day of the year. Interpret $d1 + \sum_{j=2}^{365} \delta_j$.

37) To help get in shape, Joe gets a new pair of running shoes. If Joe runs 1 mi each day in week 1 and adds $\frac{1}{10}$ mi to his daily routine each week, what is the total mileage on Joe's shoes after 25 weeks?

Answer

The total mileage is $7 \times \sum_{i=1}^{25}(1 + \frac{(i-1)}{10}) = 7 \times 25 + \frac{7}{10} \times 12 \times 25 = 385$ mi.

38) The following table gives approximate values of the average annual atmospheric rate of increase in carbon dioxide (CO_2) each decade since 1960, in parts per million (ppm). Estimate the total increase in atmospheric CO_2 between 1964 and 2013.

Decade	Ppm/y
1964-1973	1.07
1976-1983	1.34
1984-1993	1.40
1994-2003	1.87
2004-2013	2.07

Average Annual Atmospheric CO2 Increase, 1964–2013 Source: http://www.esrl.noaa.gov/gmd/ccgg/trends/.

39) The following table gives the approximate increase in sea level in inches over 20 years starting in the given year. Estimate the net change in mean sea level from 1870 to 2010.

Starting Year	20- Year Change
1870	0.3
1890	1.5
1910	0.2
1930	2.8
1950	0.7
1970	1.1
1990	1.5

Approximate 20-Year Sea Level Increases, 1870–1990

Source: http://link.springer.com/article/10....712-011-9119-1

Answer

Add the numbers to get 8.1-in. net increase.

40) The following table gives the approximate increase in dollars in the average price of a gallon of gas per decade since 1950. If the average price of a gallon of gas in 2010 was $2.60, what was the average price of a gallon of gas in 1950?

Starting Year	10- Year Change
1950	0.03
1960	0.05
1970	0.86
1980	−0.03
1990	0.29
2000	1.12

Approximate 10-Year Gas Price Increases, 1950–2000

Source: epb.lbl.gov/homepages/Rick_Di...011-trends.pdf.

41) The following table gives the percent growth of the U.S. population beginning in July of the year indicated. If the U.S. population was 281,421,906 in July 2000, estimate the U.S. population in July 2010.

Year	% Change/Year
2000	1.12
2001	0.99
2002	0.93
2003	0.86
2004	0.93
2005	0.93
2006	0.97
2007	0.96
2008	0.95
2009	0.88

Annual Percentage Growth of U.S. Population, 2000–2009

Source: www.census.gov/popest/data.

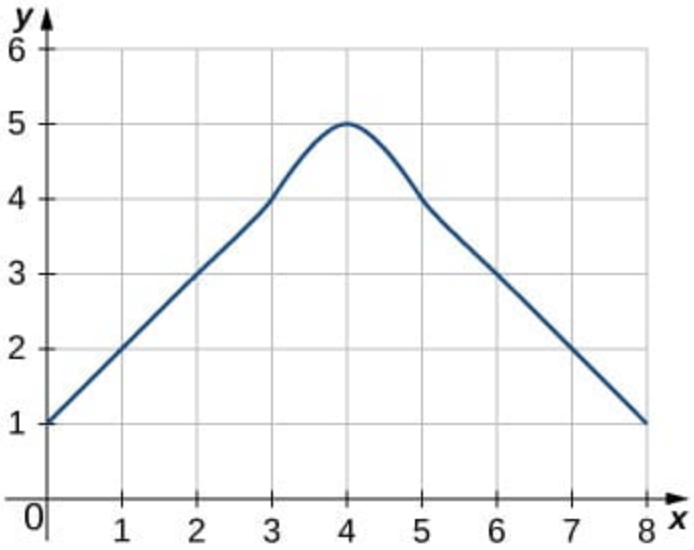
(Hint: To obtain the population in July 2001, multiply the population in July 2000 by 1.0112 to get 284,573,831.)

Answer

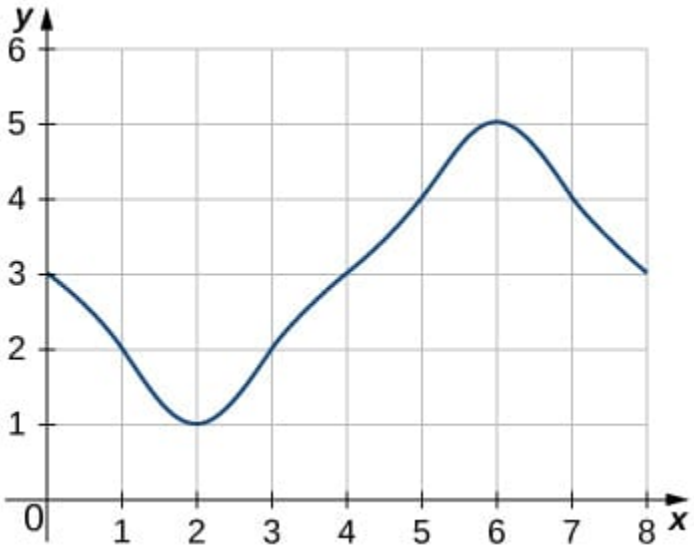
309,389,957

In exercises 42 - 45, estimate the areas under the curves by computing the left Riemann sums, L_8.

42)



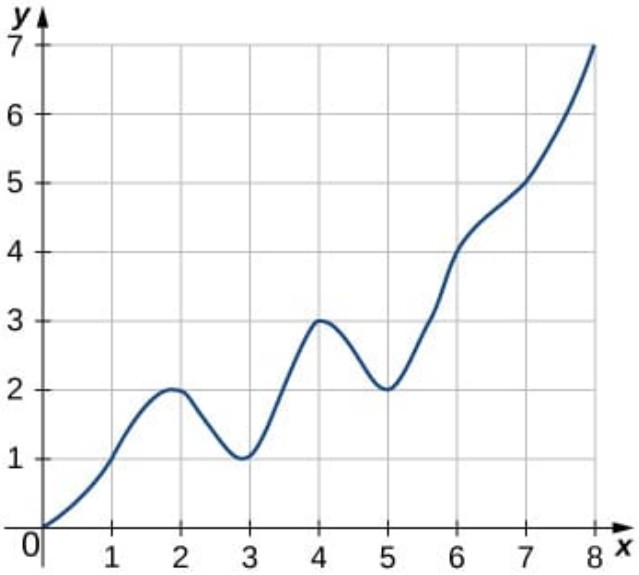
43)



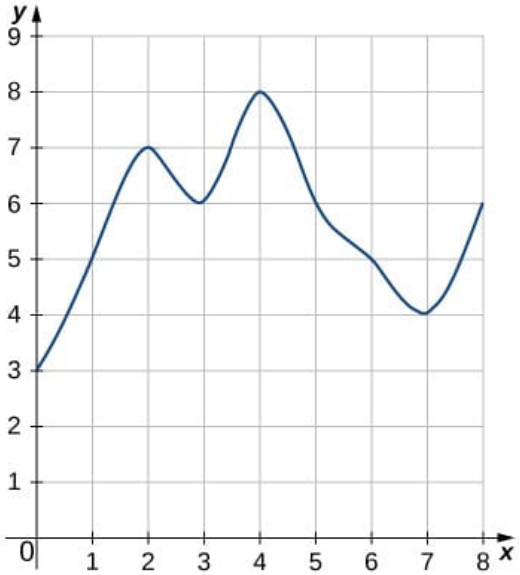
Answer

$L_8 = 3 + 2 + 1 + 2 + 3 + 4 + 5 + 4 = 24$

44)



45)



Answer

$$L_8 = 3 + 5 + 7 + 6 + 8 + 6 + 5 + 4 = 44$$

46) [Technology Required] Use a computer algebra system to compute the Riemann sum, L_N, for $N = 10, 30, 50$ for $f(x) = \sqrt{1 - x^2}$ on $[-1, 1]$.

47) [Technology Required] Use a computer algebra system to compute the Riemann sum, L_N, for $N = 10, 30, 50$ for $f(x) = \dfrac{1}{\sqrt{1 + x^2}}$ on $[-1, 1]$.

Answer

$$L_{10} \approx 1.7604, \quad L_{30} \approx 1.7625, \quad L_{50} \approx 1.76265$$

48) [Technology Required] Use a computer algebra system to compute the Riemann sum, L_N, for $N = 10, 30, 50$ for $f(x) = \sin^2 x$ on $[0, 2\pi]$. Compare these estimates with π.

In exercises 49-50, use a calculator or a computer program to evaluate the endpoint sums R_N and L_N for $N = 1, 10, 100$.

How do these estimates compare with the exact answers, which you can find via geometry?

49) [Technology Required] $y = \cos(\pi x)$ on the interval $[0, 1]$

Answer

$R_1 = -1, \quad L_1 = 1,$
$R_{10} = -0.1, \quad L_{10} = 0.1,$
$L_{100} = 0.01, \quad R_{100} = -0.01.$
By symmetry of the graph, the exact area is zero.

50) [Technology Required] $y = 3x + 2$ on the interval $[3, 5]$

In exercises 51 - 52, use a calculator or a computer program to evaluate the endpoint sums R_N and L_N for $N = 1, 10, 100$.

51) [Technology Required] $y = x^4 - 5x^2 + 4$ on the interval $[-2, 2]$, which has an exact area of $\frac{32}{15}$

Answer

$R_1 = 0, \quad L_1 = 0,$
$R_{10} = 2.4499, \quad L_{10} = 2.4499,$
$R_{100} = 2.1365, \quad L_{100} = 2.1365$

52) [Technology Required] $y = \ln x$ on the interval $[1, 2]$, which has an exact area of $2\ln(2) - 1$

53) Explain why, if $f(a) \geq 0$ and f is increasing on $[a, b]$, that the left endpoint estimate is a lower bound for the area below the graph of f on $[a, b]$.

Answer

If $[c, d]$is a subinterval of $[a, b]$ under one of the left-endpoint sum rectangles, then the area of the rectangle contributing to the left-endpoint estimate is $f(c)(d - c)$. But, $f(c) \leq f(x)$ for $c \leq x \leq d$, so the area under the graph of f between c and d is $f(c)(d - c)$ plus the area below the graph of f but above the horizontal line segment at height $f(c)$, which is positive. As this is true for each left-endpoint sum interval, it follows that the left Riemann sum is less than or equal to the area below the graph of f on $[a, b]$.

54) Explain why, if $f(b) \geq 0$ and f is decreasing on $[a, b]$, that the left endpoint estimate is an upper bound for the area below the graph of f on $[a, b]$.

55) Show that, in general, $R_N - L_N = (b - a) \times \dfrac{f(b) - f(a)}{N}$.

Answer

$$\begin{aligned} L_N &= \frac{b-a}{N} \sum_{i=1}^{N} f\left(a + (b-a)\frac{i-1}{N}\right) \\ &= \frac{b-a}{N} \sum_{i=0}^{N-1} f\left(a + (b-a)\frac{i}{N}\right) \end{aligned}$$

and

$$R_N = \frac{b-a}{N} \sum_{i=1}^{N} f\left(a + (b-a)\frac{i}{N}\right)$$

The left sum has a term corresponding to $i = 0$ and the right sum has a term corresponding to $i = N$. In $R_N - L_N$, any term corresponding to $i = 1, 2, \ldots, N - 1$ occurs once with a plus sign and once with a minus sign, so each such term cancels and one is left with

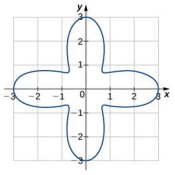
$$\begin{aligned} R_N - L_N &= \frac{b-a}{N} \cdot f\left(a + (b-a)\frac{N}{N}\right) - f\left(a + (b-a)\cdot\frac{0}{N}\right) \\ &= \frac{b-a}{N}(f(b) - f(a)) \end{aligned}$$

56) Explain why, if f is increasing on $[a, b]$, the error between either L_N or R_N and the area A below the graph of f is at most $(b - a)\dfrac{f(b) - f(a)}{N}$.

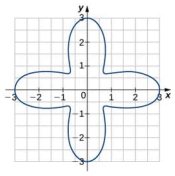
57) For each of the three graphs:

a. Obtain a lower bound $L(A)$ for the area enclosed by the curve by adding the areas of the squares enclosed completely by the curve.

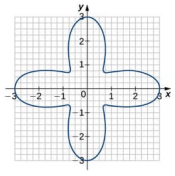
b. Obtain an upper bound $U(A)$ for the area by adding to $L(A)$ the areas $B(A)$ of the squares enclosed partially by the curve.



Graph 1



Graph 2



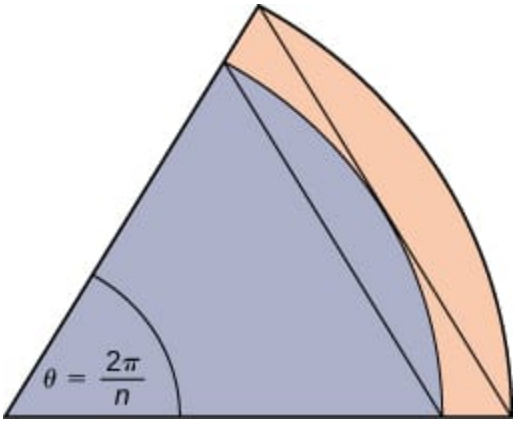
Graph 3

Answer

Graph 1: a. $L(A) = 0, \quad B(A) = 20; \quad b. U(A) = 20.$
Graph 2: $a. L(A) = 9; \quad b. B(A) = 11, \quad U(A) = 20.$
Graph 3: a. $L(A) = 11.0; \quad b. B(A) = 4.5, \quad U(A) = 15.5.$

58) In the previous exercise, explain why $L(A)$ gets no smaller while $U(A)$ gets no larger as the squares are subdivided into four boxes of equal area.

59) A unit circle is made up of n wedges equivalent to the inner wedge in the figure. The base of the inner triangle is 1 unit and its height is $\sin(\frac{\pi}{n})$. The base of the outer triangle is $B = \cos(\frac{\pi}{n}) + \sin(\frac{\pi}{n})\tan(\frac{\pi}{n})$ and the height is $H = B\sin(\frac{2\pi}{n})$. Use this information to argue that the area of a unit circle is equal to π.



Answer

Let A be the area of the unit circle. The circle encloses n congruent triangles each of area $\frac{\sin(\frac{2\pi}{n})}{2}$, so $\frac{n}{2}\sin(\frac{2\pi}{n}) \leq A$. Similarly, the circle is contained inside n congruent triangles each of area $\frac{BH}{2} = \frac{1}{2}(\cos(\frac{\pi}{n}) + \sin(\frac{\pi}{n})\tan(\frac{\pi}{n}))\sin(\frac{2\pi}{n})$, so $A \leq \frac{n}{2}\sin(\frac{2\pi}{n})(\cos(\frac{\pi}{n})) + \sin(\frac{\pi}{n})\tan(\frac{\pi}{n})$. As $n \to \infty, \quad \frac{n}{2}\sin(\frac{2\pi}{n}) = \frac{\pi\sin(\frac{2\pi}{n})}{(\frac{2\pi}{n})} \to \pi$, so we conclude $\pi \leq A$. Also, as $n \to \infty, \quad \cos(\frac{\pi}{n}) + \sin(\frac{\pi}{n})\tan(\frac{\pi}{n}) \to 1$, so we also have $A \leq \pi$. By the squeeze theorem for limits, we conclude that $A = \pi$.

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4.3: The Definite Integral

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

In the preceding section, we defined the area under a curve in terms of Riemann sums:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

However, this definition came with restrictions. We required $f(x)$ to be continuous and nonnegative. Unfortunately, real-world problems don't always meet these restrictions. In this section, we look at applying the concept of the area under the curve to a broader set of functions using the definite integral.

Definition and Notation

The **definite integral** generalizes the concept of the area under a curve. We lift the requirements that $f(x)$ be continuous and nonnegative, and define the definite integral as follows.

Definition: Definite Integral

If $f(x)$ is a function defined on an interval $[a, b]$, the **definite integral** of f from a to b is given by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

provided the limit exists. If this limit exists, the function $f(x)$ is said to be **integrable** on $[a, b]$, or is an **integrable function**.

The integral symbol in the previous definition should look familiar. We have seen a similar notation in a previous section, where we used the indefinite integral symbol (without the a and b above and below) to represent an antiderivative. Although the notation for indefinite integrals may look similar to that of a definite integral, they are not the same. A definite integral is a number. An indefinite integral is a family of functions. Later in this chapter, we examine how these concepts are related. However, close attention should always be paid to notation so we know whether we're working with a definite integral or an indefinite integral.

Integral notation goes back to the late seventeenth century. It is one of the contributions of Gottfried Wilhelm Leibniz, who is often considered the co-discoverer of Calculus, along with Isaac Newton. The integration symbol $\int$ is an elongated S, suggesting sigma or summation. On a definite integral, above and below the summation symbol are the boundaries of the interval, $[a, b]$. The numbers a and b are x-values and are called the **limits of integration**; specifically, a is the **lower limit** and b is the **upper limit**. To clarify, we use the word limit in two different ways in the context of the definite integral. First, we talk about the limit of a sum

as $n \to \infty$. Second, the boundaries of the region are called the limits of integration.

We call the function $f(x)$ the **integrand**, and the dx indicates that $f(x)$ is a function with respect to x, called the **variable of integration**. Note that, like the index in a sum, the variable of integration is a **dummy variable** and has no impact on the computation of the integral. We could use any variable we like as the variable of integration:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du$$

Previously, we discussed the fact that if $f(x)$ is continuous on $[a, b]$, then the limit $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$ exists and is unique. This leads to the following theorem, which we state without proof.

Theorem: Continuous Functions Are Integrable

If $f(x)$ is continuous on $[a, b]$, then f is integrable on $[a, b]$.

Functions that are not continuous on $[a, b]$ may still be integrable, depending on the nature of the discontinuities. For example, functions with a finite number of jump discontinuities or removable discontinuities on a closed interval are integrable.

It is also worth noting that we are using a regular partition in the Riemann sums. This is not strictly necessary. Any partition can be used to form a Riemann sum. However, if a non-regular partition defines the definite integral, it is not sufficient to take the limit as the number of subintervals goes to infinity. Instead, we must take the limit as the width of the largest subinterval goes to zero. This introduces a more complex notation in our limits and makes the calculations more difficult without really gaining much additional insight, so we stick with regular partitions for the Riemann sums.

Example 4.3.1: Evaluating an Integral Using the Definition

Use the definition of the definite integral to evaluate $\int_0^2 x^2\ dx$. Use a right-endpoint approximation to generate the Riemann sum.

Solution available at the end of this section.

Checkpoint 4.3.1

Use the definition of the definite integral to evaluate $\int_0^3 (2x - 1)\ dx$. Use a right-endpoint approximation to generate the Riemann sum.

Answer available at the end of this section.

Evaluating Definite Integrals Using Geometry

Evaluating definite integrals this way can be tedious because of the complexity of the calculations. Later in this chapter, we develop techniques for evaluating definite integrals *without* taking limits of Riemann sums. However, for now, we can rely on the fact that definite integrals represent the area under the curve, and we can evaluate definite integrals by using geometric formulas to calculate that area. We do this to confirm that definite integrals do, indeed, represent areas, so we can then discuss what to do in the case of a curve of a function dropping below the x-axis.

Example 4.3.2: Using Geometric Formulas to Calculate Definite Integrals

Use the formula for the area of a circle to evaluate $\int_3^6 \sqrt{9-(x-3)^2}\ dx$.

Solution available at the end of this section.

Checkpoint 4.3.2

Use the formula for the area of a trapezoid to evaluate $\int_2^4 (2x+3)\ dx$.

Answer available at the end of this section.

The ability to think geometrically can be *very* useful when it comes to evaluating definite integrals. It's always a good idea to see if the integrand represents a simple geometric curve for which you have an area formula. This will often be circles and lines.

Area and the Definite Integral

When we defined the definite integral, we lifted the requirement that $f(x)$ be nonnegative. But how do we interpret "the area under the curve" when $f(x)$ is negative?

Let us return to the Riemann sum. Consider, for example, the function $f(x) = 2 - 2x^2$ (shown in Figure 4.3.2) on the interval $[0, 2]$. Use $n = 8$ and choose x_i^* as the left-endpoint of each interval. Construct a rectangle on each subinterval of height $f(x_i^*)$ and width Δx. When $f(x_i^*)$ is positive, the product $f(x_i^*)\Delta x$ represents the area of the rectangle, as before. When $f(x_i^*)$ is negative, however, the product $f(x_i^*)\Delta x$ represents the negative of the area of the rectangle. The Riemann sum then becomes

$$\sum_{i=1}^{8} f(x_i^*)\Delta x = (\text{Area of rectangles above the } x\text{-axis}) - (\text{Area of rectangles below the } x\text{-axis})$$

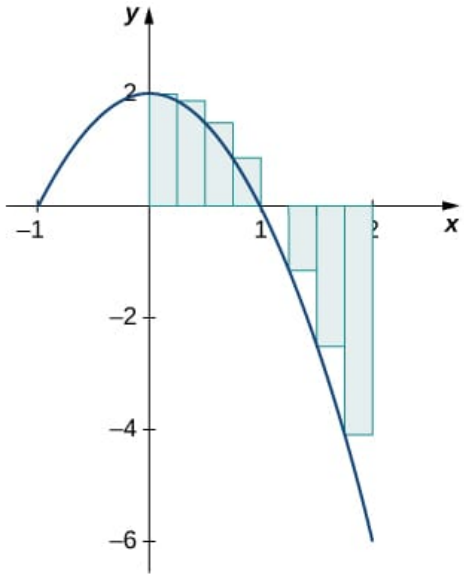

Figure 4.3.2: *For a function that is partly negative, the Riemann sum is the area of the rectangles above the x-axis less the area of the rectangles below the x-axis.*

Taking the limit as $n \to \infty$, the Riemann sum approaches the area between the curve above the x-axis and the x-axis, less the area between the curve below the x-axis and the x-axis, as shown in Figure 4.3.3. Then,

$$\int_0^2 f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x = A_1 - A_2.$$

The quantity $A_1 - A_2$ is called the **net signed area.**

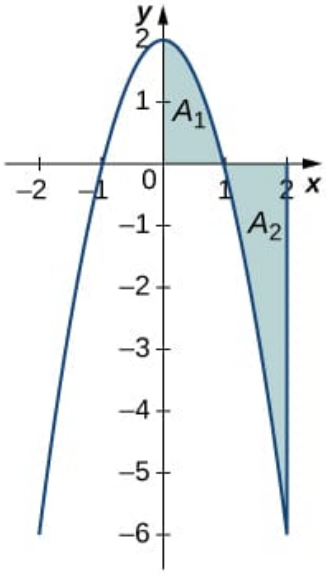

Figure 4.3.3**:** *In the limit, the definite integral equals area A_1 less area A_2, or the net signed area.*

Notice that the net signed area can be positive, negative, or zero. The net signed area is positive if the area above the x-axis is larger. The net signed area is negative if the area below the x-axis is larger. The net signed area is zero if the areas above and below the x-axis are equal.

Example 4.3.3: Finding the Net Signed Area

Find the net signed area between the curve of the function $f(x) = 2x$ and the x-axis over the interval $[-3, 3]$.

Solution available at the end of this section.

Checkpoint 4.3.3

Find the net signed area of $f(x) = x - 2$ over the interval $[0, 6]$, illustrated in the following image.

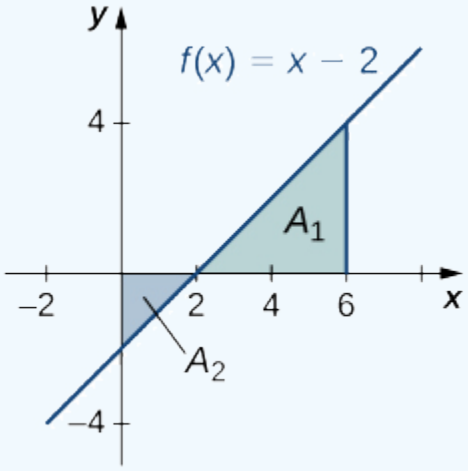

Figure 4.3.5

Answer available at the end of this section.

Interactive Element: Net Signed Area

The following Interactive Element might help you visualize how the definite integral adds the area above the x-axis and subtracts the area below it.

Interact: Drag the blue point along the x-axis to change the upper limit b, and watch the green (above-axis) and red (below-axis) regions grow.

Observation: The definite integral equals the net signed area—the green area minus the red area. It stays negative while the red region is larger, hits exactly zero when the two areas balance (at $b = 4$), and turns positive once the green region takes over.

One application of the definite integral is finding **displacement** when given a velocity function. If $v(t)$ represents the velocity of an object as a function of time, then the area under the curve tells us how far the object is from its original position. This is an essential application of the definite integral, and we examine it in more detail later in the chapter. We will look at some basic examples involving constant velocities to understand how this works.

When velocity is a constant, the area under the curve is just velocity times time. This idea is already very familiar. If a car travels away from its starting position in a straight line at a speed of 70 mph for 2 hours, then it is 140 miles away from its original position (Figure $4.3.6$). Using integral notation, we have

$$\int_0^2 70\,dt = 140\,\text{miles}.$$

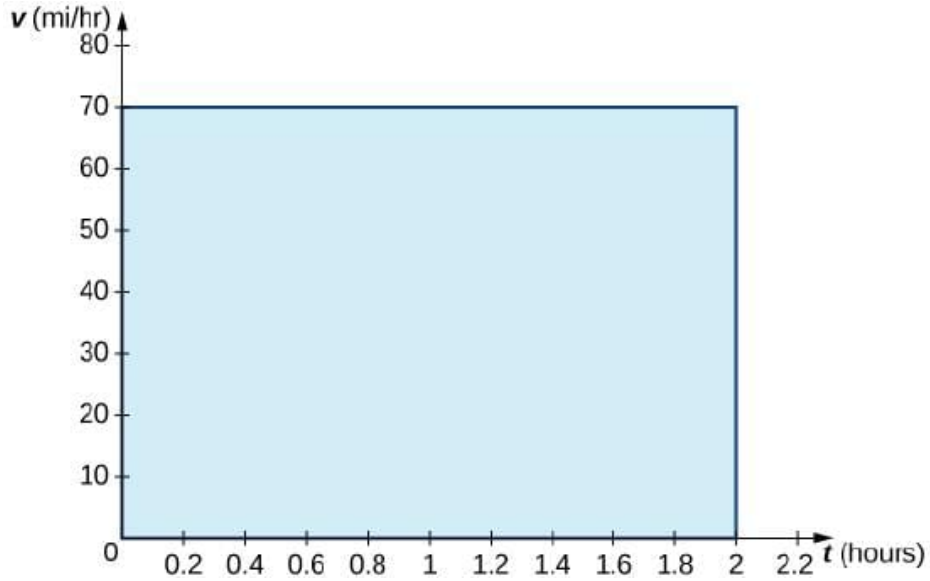

Figure $4.3.6$**:** *The area under the curve* $v(t) = 70$ *tells us how far the car is from its starting point at a given time.*

In the context of displacement, the net signed area allows us to consider direction. If a car travels straight north at 60 mph for 2 hours, it is 120 miles north of its starting position. If the car then turns around and travels south at a speed of 40 mph for 3 hours, it will be back at its starting position (Figure $4.3.7$). Again, using integral notation, we have

$$\int_0^2 60\,dt + \int_2^5 -40\,dt = 120 - 120 = 0.$$

In this case, the displacement is zero.

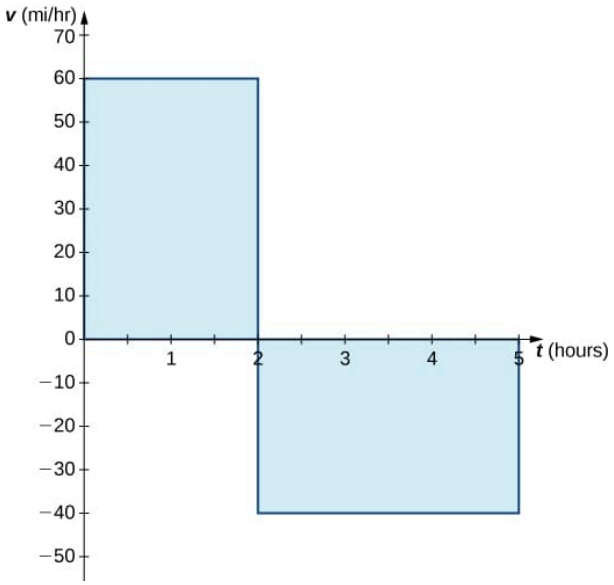

Figure 4.3.7**:** *The area above and below the axis is equal, so the net signed area is zero.*

Suppose we want to know how far the car travels, regardless of direction. In this case, we want to know the area between the curve and the t-axis, regardless of whether that area is above or below the axis. This is called the **total area.**

Graphically, it is easiest to calculate the total area by adding the areas above and below the axis (rather than subtracting the areas below the axis, as we did with the net signed area). To accomplish this mathematically, we use the absolute value function. Thus, the total distance traveled by the car is

$$\int_0^2 |60|\,dt + \int_2^5 |-40|\,dt = \int_0^2 60\,dt + \int_2^5 40\,dt = 120 + 120 = 240.$$

Bringing these ideas together formally, we state the following definitions.

Definition: Net Signed Area

Let $f(x)$ be an integrable function defined on an interval $[a, b]$. Let A_1 represent the area between $f(x)$ and the x-axis that lies above the axis, and let A_2 represent the area between $f(x)$ and the x-axis that lies below the axis. Then, the **net signed area** between $f(x)$ and the x-axis is given by

$$\int_a^b f(x)\,dx = A_1 - A_2.$$

The total area between $f(x)$ and the x-axis is given by

$$\int_a^b |f(x)|\,dx = A_1 + A_2.$$

Example 4.3.4: Finding the Total Area

Find the total area between $f(x) = x - 2$ and the x-axis over the interval $[0, 6]$.

Solution available at the end of this section.

Checkpoint 4.3.4

Find the total area between the function $f(x) = 2x$ and the x-axis over the interval $[-3, 3]$.

Answer available at the end of this section.

Properties of the Definite Integral

The properties of indefinite integrals also apply to definite integrals. Definite integrals also have properties that relate to the limits of integration. These properties, along with the rules of integration that we examine later in this chapter, help us manipulate expressions to evaluate definite integrals.

Theorem: Properties of the Definite Integral

1. If the limits of integration are the same, the integral is just a vertical line and contains no area. That is,
$$\int_a^a f(x)\,dx = 0.$$
2. If the limits are reversed, place a negative sign before the integral. That is,
$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$
3. The integral of a sum is the sum of the integrals. That is,
$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$
4. The integral of a difference is the difference of the integrals. That is,
$$\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$$
5. The integral of the product of a constant and a function is equal to the constant multiplied by the integral of the function. That is, for a constant c,
$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$$
6. An integral can be split into pieces. That is,
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$
Although this formula normally applies when c is between a and b, the formula holds for all values of a, b, and c, provided $f(x)$ is integrable on the largest interval.

Example 4.3.5: Using the Properties of the Definite Integral

Use the properties of the definite integral to express the definite integral of $f(x) = -3x^3 + 2x + 2$ over the interval $[-2, 1]$ as the sum of three definite integrals.

Solution available at the end of this section.

Checkpoint 4.3.5

Use the properties of the definite integral to express the definite integral of $f(x) = 6x^3 - 4x^2 + 2x - 3$ over the interval $[1, 3]$ as the sum of four definite integrals.

Answer available at the end of this section.

Example 4.3.6: Using the Properties of the Definite Integral

If it is known that $\int_0^8 f(x)\ dx = 10$ and $\int_0^5 f(x)\ dx = 5$, find the value of $\int_5^8 f(x)\ dx$.

Solution available at the end of this section.

Checkpoint 4.3.6

If it is known that $\int_1^5 f(x)\ dx = -3$ and $\int_2^5 f(x)\ dx = 4$, find the value of $\int_1^2 f(x)\ dx$.

Answer available at the end of this section.

Comparison Properties of Integrals

A picture can sometimes tell us more about a function than the results of computations. Comparing functions by their graphs and algebraic expressions can give new insight into the process of integration. Intuitively, we might say that if a function $f(x)$ is above another function $g(x)$, then the area between $f(x)$ and the x-axis is greater than the area between $g(x)$ and the x-axis. This is true depending on the interval over which the comparison is made. The properties of definite integrals are valid whether $a < b$, $a = b$, or="" $a = ″ ″ > b$. The following properties, however, concern only the case $a \leq b$ and are used when we want to compare the sizes of integrals.</b,\,a=b\),>

Theorem: Comparison Theorem

i. If $f(x) \geq 0$ for $a \leq x \leq b$, then

$$\int_a^b f(x)\,dx \geq 0.$$

ii. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

iii. If m and M are constants such that $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

Example 4.3.7: Comparing Two Functions over a Given Interval

Compare $f(x) = \sqrt{1+x^2}$ and $g(x) = \sqrt{1+x}$ over the interval $[0, 1]$.

Solution available at the end of this section.

Success in College: Learning From Tests (Part 5)

1. A math class includes one or two quizzes for each major test. A student is averaging about 85% on these quizzes. When he gets them back, he looks them over and drops them in the wastebasket as he leaves class. Explain why he should keep the quizzes.
2. A math professor tells her class to go over their tests so that they can learn from their mistakes. She does not require that students do this. Most do not. Do you think the professor should require the students to make corrections on the test and

turn them back in? (Assume that the corrections will not change their original test grade.) Explain.

Success in College: Preparing for the Final Exam (Part 1)

1. Describe what you will do to prepare for the final exam in this course.
2. Describe how well you think you are prepared right now to take the final exam.

Answers

Section Preview

We have arrived—the definite integral as a representation of the precise area under a curve! **Beware:** Save the reveal of the relationship between the definite integral and *ahem* something else for next section.

Section Preview

- **Conic Sections**
  - Circles — Equations and Graphs: Example 2 requires recognizing $\sqrt{9-(x-3)^2}$ as the upper half of the circle $(x-3)^2+y^2=9$ (a semicircle of radius 3); without the circle-equation form the integrand cannot be identified as a geometric shape.

Section Preview

- **Geometry**
  - Circles — Properties, Area, and Circumference: Example 2 evaluates the integral using the area of a semicircle drawn from $A=\pi r^2$ (here one quarter of the circle, $\frac{1}{4}\pi(3)^2$).

Section Preview

- State the definition of the definite integral.
- Explain the terms integrand, limits of integration, and variable of integration.
- Explain when a function is integrable.
- Describe the relationship between the definite integral and net area.
- Use Geometry and the properties of definite integrals to evaluate them.

Example $4.3.1$: Evaluating an Integral Using the Definition

We first want to set up a Riemann sum. Based on the limits of integration, we have $a=0$ and $b=2$. For $i=0,1,2,\ldots,n$, let $P=x_i$ be a regular partition of $[0,2]$. Then

$$\Delta x=\frac{b-a}{n}=\frac{2}{n}.$$

Since we are using a right-endpoint approximation to generate Riemann sums, for each i, we need to calculate the function value at the right endpoint of the interval $[x_{i-1},x_i]$. The right endpoint of the interval is x_i, and since P is a regular partition,

$$x_i=x_0+i\Delta x=0+i\left[\frac{2}{n}\right]=\frac{2i}{n}.$$

Thus, the function value at the right endpoint of the interval is

$$f(x_i)=x_i^2=\left(\frac{2i}{n}\right)^2=\frac{4i^2}{n^2}.$$

Then the Riemann sum takes the form

$$\sum_{i=1}^{n}f(x_i)\Delta x=\sum_{i=1}^{n}\left(\frac{4i^2}{n^2}\right)\frac{2}{n}=\sum_{i=1}^{n}\frac{8i^2}{n^3}=\frac{8}{n^3}\sum_{i=1}^{n}i^2.$$

Using the summation formula for $\sum_{i=1}^{n}i^2$, we have

$$\begin{aligned}\textstyle\sum_{i=1}^{n}f(x_i)\Delta x &= \frac{8}{n^3}\textstyle\sum_{i=1}^{n}i^2\\ &= \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]\\ &= \frac{8}{n^3}\left[\frac{2n^3+3n^2+n}{6}\right]\\ &= \frac{16n^3+24n^2+8n}{6n^3}\\ &= \frac{8}{3}+\frac{4}{n}+\frac{4}{3n^2}.\end{aligned}$$

Now, to calculate the definite integral, we need to take the limit as $n\to\infty$. We get

$$\begin{aligned}\int_0^2 x^2 dx &= \lim_{n\to\infty}\sum_{i=1}^n f(x_i)\Delta x \\ &= \lim_{n\to\infty}\left(\frac{8}{3}+\frac{4}{n}+\frac{4}{3n^2}\right) \\ &= \lim_{n\to\infty}\left(\frac{8}{3}\right)+\lim_{n\to\infty}\left(\frac{4}{n}\right)+\lim_{n\to\infty}\left(\frac{4}{3n^2}\right) \\ &= \frac{8}{3}+0+0 \\ &= \frac{8}{3}.\end{aligned}$$

Checkpoint 4.3.1

6

Example 4.3.2**: Using Geometric Formulas to Calculate Definite Integrals**

The function describes a semicircle with radius 3. To find

$$\int_3^6 \sqrt{9-(x-3)^2}\,dx$$

we want to find the area under the curve over the interval $[3,6]$. The formula for the area of a circle is $A=\pi r^2$. The area of a semicircle is just one-half the area of a circle, or $A=\frac{1}{2}\pi r^2$. The shaded area in Figure 4.3.1 covers one-half of the semicircle, or $A=\frac{1}{4}\pi r^2$. Thus,

$$\int_3^6 \sqrt{9-(x-3)^2}\,dx = \frac{1}{4}\pi(3)^2 = \frac{9}{4}\pi \approx 7.069.$$

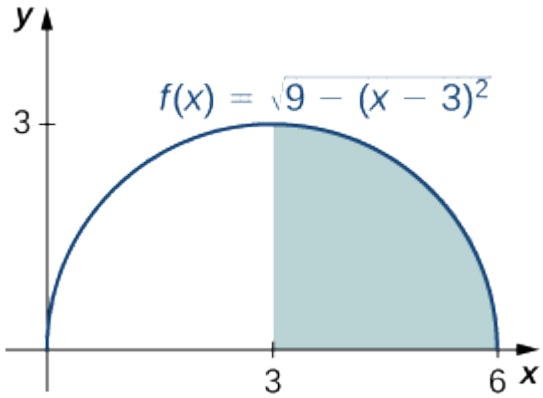

Figure 4.3.1**:** *The value of the integral of the function $f(x)$ over the interval $[3,6]$ is the area of the shaded region.*

Checkpoint 4.3.2

18 square units

Example 4.3.3**: Finding the Net Signed Area**

The function produces a straight line that forms two triangles: one from $x=-3$ to $x=0$ and the other from $x=0$ to $x=3$ (Figure 4.3.4). Using the geometric formula for the area of a triangle, $A=\frac{1}{2}bh$, the area of triangle A_1, above the axis, is

$$A_1 = \frac{1}{2}3(6) = 9,$$

where 3 is the base and $2(3)=6$ is the height. The area of triangle A_2, below the axis, is

$$A_2 = \frac{1}{2}(3)(6) = 9,$$

where 3 is the base and 6 is the height. Thus, the net area is

$$\int_{-3}^3 2x\,dx = A_1 - A_2 = 9-9 = 0.$$

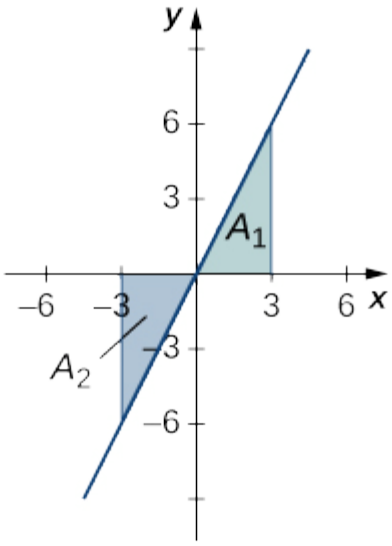

Figure 4.3.4**:** *The area above the curve and below the x-axis equals the area below the curve and above the x-axis.*

Analysis

If A_1 is the area above the x-axis and A_2 is the area below the x-axis, then the net area is $A_1 - A_2$. Since the areas of the two triangles are equal, the net area is zero.

Checkpoint 4.3.3

6

Example 4.3.4**: Finding the Total Area**

Calculate the x-intercept as $(2, 0)$ (set $y = 0$, solve for x). To find the total area, take the area below the x-axis over the subinterval $[0, 2]$ and add it to the area above the x-axis on the subinterval $[2, 6]$ (Figure 4.3.8).

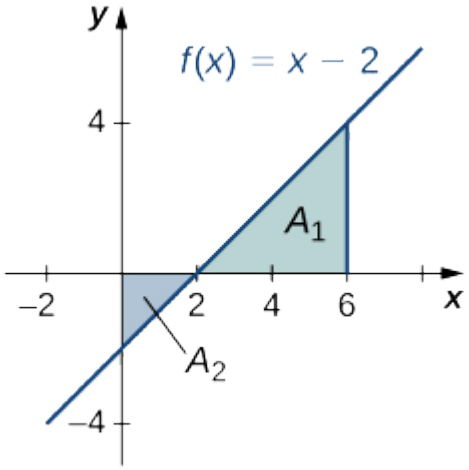

Figure 4.3.8**:** *The total area between the line and the x-axis over $[0, 6]$ is A_2 plus A_1.*

We have

$$\int_0^6 |x - 2|\, dx = A_2 + A_1.$$

Then, using the formula for the area of a triangle, we obtain

$$A_2 = \frac{1}{2}bh = \frac{1}{2} \cdot 2 \cdot 2 = 2$$
$$A_1 = \frac{1}{2}bh = \frac{1}{2} \cdot 4 \cdot 4 = 8.$$

The total area, then, is

$$A_1 + A_2 = 8 + 2 = 10 \text{ units}^2.$$

Checkpoint 4.3.4

18 units^2

Example 4.3.5**: Using the Properties of the Definite Integral**

Using integral notation, we have $\int_{-2}^{1} (-3x^3 + 2x + 2)\, dx$. We apply properties 3. and 5. to get

$$\begin{aligned}\int_{-2}^{1}(-3x^3+2x+2)\,dx &= \int_{-2}^{1}-3x^3\,dx+\int_{-2}^{1}2x\,dx+\int_{-2}^{1}2\,dx\\ &= -3\int_{-2}^{1}x^3\,dx+2\int_{-2}^{1}x\,dx+\int_{-2}^{1}2\,dx.\end{aligned}$$

Checkpoint 4.3.5

$$6\int_1^3 x^3\,dx-4\int_1^3 x^2\,dx+2\int_1^3 x\,dx-\int_1^3 3\,dx$$

Example 4.3.6**: Using the Properties of the Definite Integral**

By property 6,

$$\int_a^b f(x)\,dx=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx.$$

Thus,

$$\begin{aligned}\int_0^8 f(x)\,dx &= \int_0^5 f(x)\,dx+\int_5^8 f(x)\,dx\\ 10 &= 5+\int_5^8 f(x)\,dx\\ 5 &= \int_5^8 f(x)\,dx.\end{aligned}$$

Checkpoint 4.3.6

-7

Example 4.3.7**: Comparing Two Functions over a Given Interval**

Graphing these functions is necessary to understand how they compare over the interval $[0,1]$. Initially, when graphed on a graphing calculator, $f(x)$ appears to be above $g(x)$ everywhere. However, the graphs appear the same on the interval $[0,1]$. We need to zoom in to see that $g(x)$ is above $f(x)$ on the given interval. The two functions intersect at $x=0$ and $x=1$ (Figure 4.3.9).

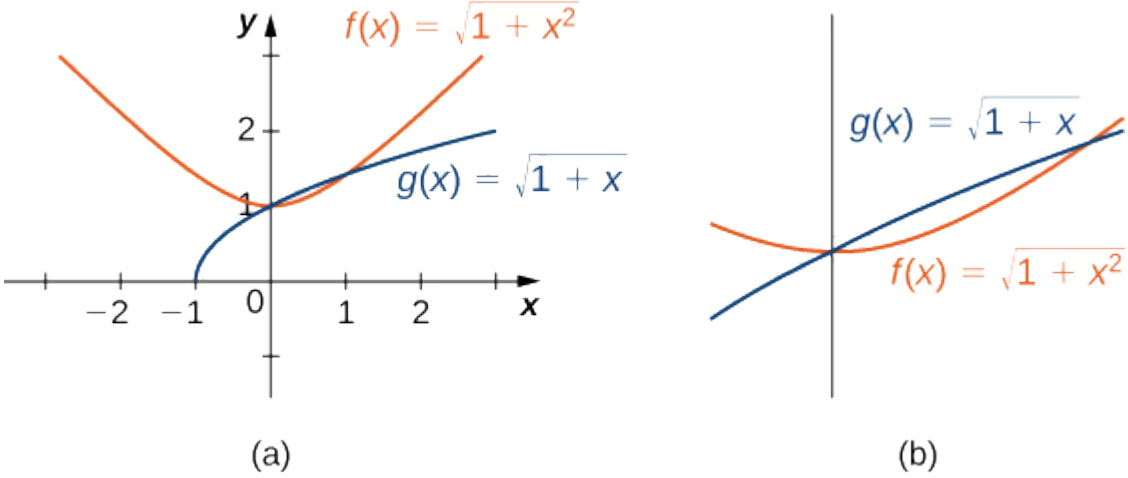

Figure 4.3.9*a***:** *The function $f(x)$ appears above the function $g(x)$ except over the interval $[0,1]$.*
Figure 4.3.9*b***:** *Viewing the same graph with a greater zoom shows this more clearly.*

We can see from the graph that over the interval $[0,1]$, $g(x)\geq f(x)$. Comparing the integrals over the specified interval $[0,1]$, we also see that $\int_0^1 g(x)\,dx\geq\int_0^1 f(x)\,dx$ (Figure 4.3.10). The thin, red-shaded area shows just how much difference there is between these two integrals over the interval $[0,1]$.

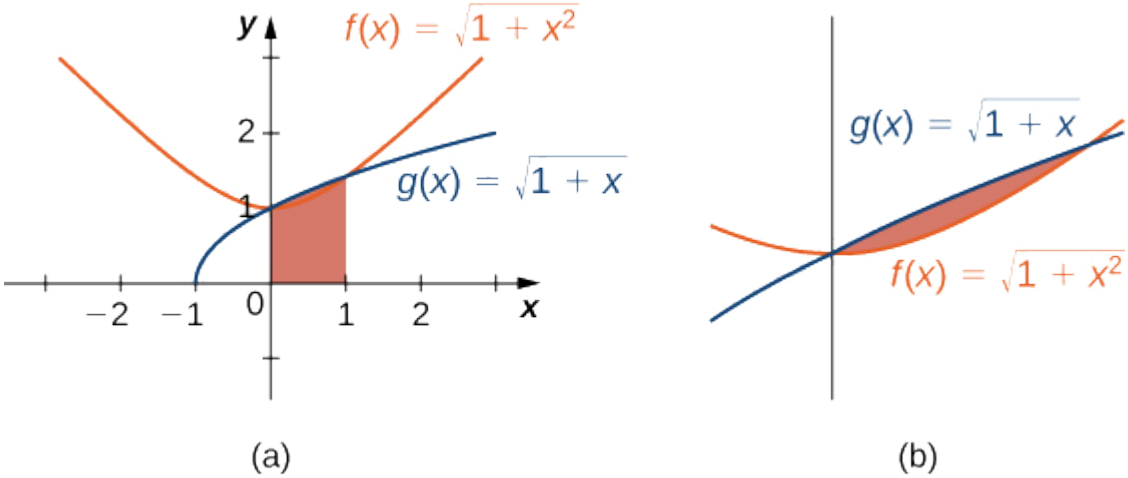

Figure 4.3.10*a***:** *The graph shows that over the interval $[0,1], g(x)\geq f(x)$, where equality holds only at the endpoints of the interval.*
Figure 4.3.10*b***:** *Viewing the same graph with a greater zoom shows this more clearly.*

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4.3.1: Resources and Key Concepts

Resources

Videos

- Definite integral
  - The Definition of a Definite Integral
- Riemann sum limit
  - Evaluating Definite Integrals Using Limits
- Properties of definite integrals
  - Properties of Definite Integrals
  - Evaluating Definite Integrals Using the Properties
- Evaluating integrals using geometry
  - Evaluating Definite Integrals: A Best Practice

Key Concepts

Definitions

- **Definite Integral**: If $f(x)$ is a function defined on an interval $[a, b]$, the definite integral of f from a to b is given by $\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$, provided the limit exists.
- **Integrable Function**: If the limit defining the definite integral exists, the function $f(x)$ is said to be integrable on $[a, b]$.
- **Limits of Integration**: In $\int_a^b f(x)\ dx$, a is the lower limit and b is the upper limit.
- **Integrand**: In $\int_a^b f(x)\ dx$, $f(x)$ is the integrand.
- **Variable of Integration**: In $\int_a^b f(x)\ dx$, x is the variable of integration (a dummy variable).
- **Net Signed Area**: The definite integral $\int_a^b f(x)\ dx$ represents the net signed area between the curve $y = f(x)$ and the x-axis over the interval $[a, b]$. It is the area above the x-axis minus the area below the x-axis. If A_1 is the area above and A_2 is the area below, then $\int_a^b f(x)\ dx = A_1 - A_2$.
- **Total Area**: The total area between $f(x)$ and the x-axis over $[a, b]$ is given by $\int_a^b |f(x)|\ dx = A_1 + A_2$, where A_1 is the area above the x-axis and A_2 is the area below the x-axis.
- **Displacement (from velocity)**: If $v(t)$ is a velocity function, then $\int_a^b v(t)\ dt$ represents the displacement of an object from time $t = a$ to $t = b$.

Theorems

- **Theorem: Continuous Functions Are Integrable**: If $f(x)$ is continuous on $[a, b]$, then f is integrable on $[a, b]$.
- **Theorem: Properties of the Definite Integral**:
  1. $\int_a^a f(x)\ dx = 0$
  2. $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$
  3. $\int_a^b [f(x) + g(x)]\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$

4. $\int_a^b [f(x) - g(x)]\ dx = \int_a^b f(x)\ dx - \int_a^b g(x)\ dx$
5. $\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$ (for a constant c)
6. $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$ (for any c such that f is integrable on the largest interval containing a, b, c)

- **Theorem: Comparison Theorem**:
  i. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\ dx \geq 0$.
  ii. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\ dx \geq \int_a^b g(x)\ dx$.
  iii. If m and M are constants such that $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)\ dx \leq M(b-a)$.

Common Mistakes

- **Confusing Definite and Indefinite Integrals**: A definite integral $\int_a^b f(x)\ dx$ is a number, while an indefinite integral $\int f(x)\ dx$ is a family of functions $(F(x) + C)$.
- **Ignoring the "Net" in Net Signed Area**: Forgetting that areas below the x-axis are subtracted when calculating a definite integral directly, unless the absolute value of the function is taken for total area.
- **Errors in Applying Properties of Definite Integrals**: For example, incorrectly assuming $\int_a^b f(x)g(x)\ \ dx = \left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)$.
- **Incorrectly Using Geometric Formulas**: Applying area formulas incorrectly or to shapes that don't match the integrand's graph.
- **Assuming All Functions are Integrable**: While continuous functions are integrable, not all functions are (e.g., some highly discontinuous functions). However, functions with a finite number of jump or removable discontinuities on a closed interval are still integrable.

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4.3.2: Homework

Reading Questions

1. How is the definite integral $\int_a^b f(x)\ dx$ defined in terms of a limit of Riemann sums?
2. What are the "limits of integration" in a definite integral? What do a and b represent?
3. What is the difference between a definite integral and an indefinite integral?
4. What does it mean for a function to be "integrable" on an interval $[a, b]$? Are all continuous functions on $[a, b]$ integrable?
5. How does the definite integral relate to the concept of "net signed area"? Explain what happens to areas below the x-axis.
6. How can you find the "total area" between a curve $y = f(x)$ and the x-axis over $[a, b]$ using a definite integral?
7. If $v(t)$ is a velocity function, what does $\int_a^b v(t)\ dt$ represent?
8. State the property of definite integrals: $\int_a^a f(x)\ dx =$?
9. State the property of definite integrals: $\int_a^b f(x)\ dx =? \cdot \int_b^a f(x)\ dx$
10. State the property of definite integrals regarding the integral of a sum: $\int_a^b [f(x) + g(x)]\ dx =$?
11. State the property of definite integrals: $\int_a^b cf(x)\ dx =$? (where c is a constant).
12. State the additivity property of definite integrals: $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx +$?
13. What does the Comparison Theorem state about $\int_a^b f(x)\ dx$ if $f(x) \geq 0$ on $[a, b]$?

Homework

In exercises 1 - 4, express the limits as integrals.

1) $\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^*)\Delta x$ over $[1, 3]$

2) $\lim_{n\to\infty} \sum_{i=1}^{n} (5(x_i^*)^2 - 3(x_i^*)^3)\Delta x$ over $[0, 2]$

Answer

$$\int_0^2 (5x^2 - 3x^3)\ dx$$

3) $\lim_{n\to\infty} \sum_{i=1}^{n} \sin^2(2\pi x_i^*)\Delta x$ over $[0, 1]$

4) $\lim_{n\to\infty} \sum_{i=1}^{n} \cos^2(2\pi x_i^*)\Delta x$ over $[0, 1]$

Answer

$$\int_0^1 \cos^2(2\pi x)\ dx$$

In exercises 5 - 10, given L_n or R_n as indicated, express their limits as $n \to \infty$ as definite integrals, identifying the correct intervals.

5) $L_n = \frac{1}{n} \sum_{i=1}^{n} \frac{i-1}{n}$

6) $R_n = \frac{1}{n}\sum_{i=1}^{n}\frac{i}{n}$

Answer

$$\int_0^1 x\ dx$$

7) $Ln = \frac{2}{n}\sum_{i=1}^{n}(1+2\frac{i-1}{n})$

8) $R_n = \frac{3}{n}\sum_{i=1}^{n}(3+3\frac{i}{n})$

Answer

$$\int_3^6 x\ dx$$

9) $L_n = \frac{2\pi}{n}\sum_{i=1}^{n}2\pi\frac{i-1}{n}\cos(2\pi\frac{i-1}{n})$

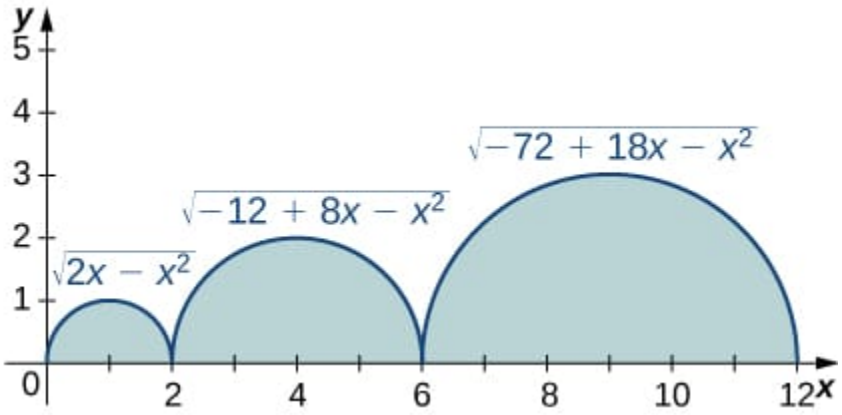
10) $R_n = \frac{1}{n}\sum_{i=1}^{n}(1+\frac{i}{n})\log((1+\frac{i}{n})^2)$

Answer

$$\int_1^2 x\log(x^2)\ dx$$

In exercises 11 - 16, evaluate the integrals of the functions graphed using the formulas for areas of triangles and circles, and subtracting the areas below the x-axis.

11)



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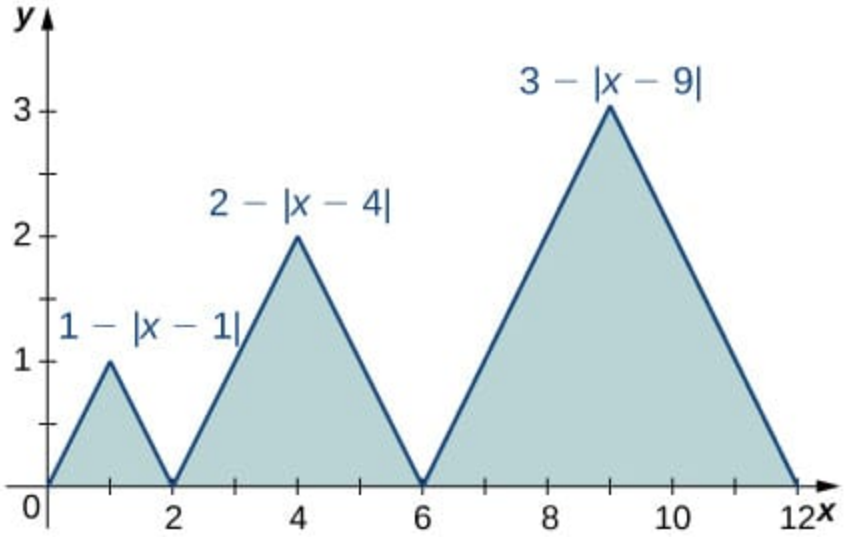
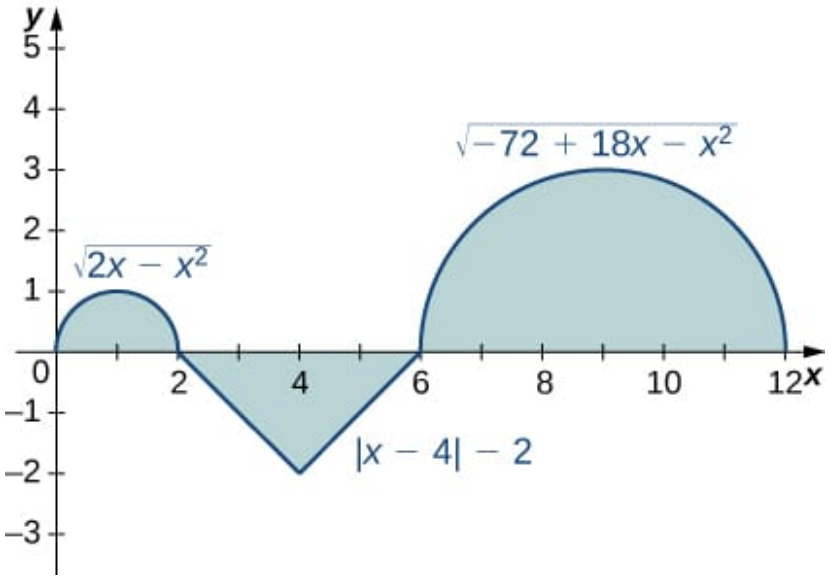


Answer

$1 + 2 \cdot 2 + 3 \cdot 3 = 14$

13)



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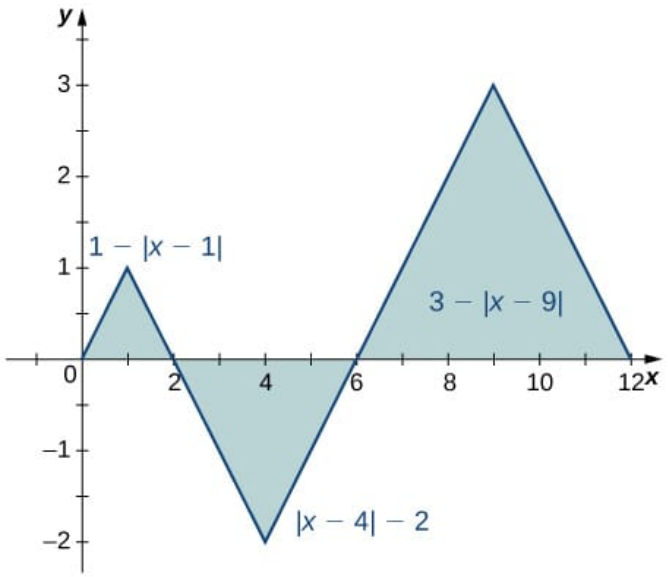
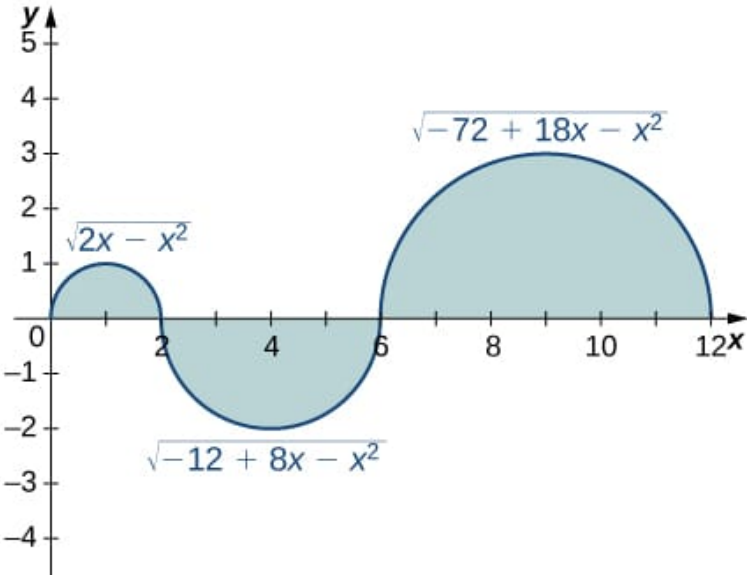


Answer

$1 - 4 + 9 = 6$

15)



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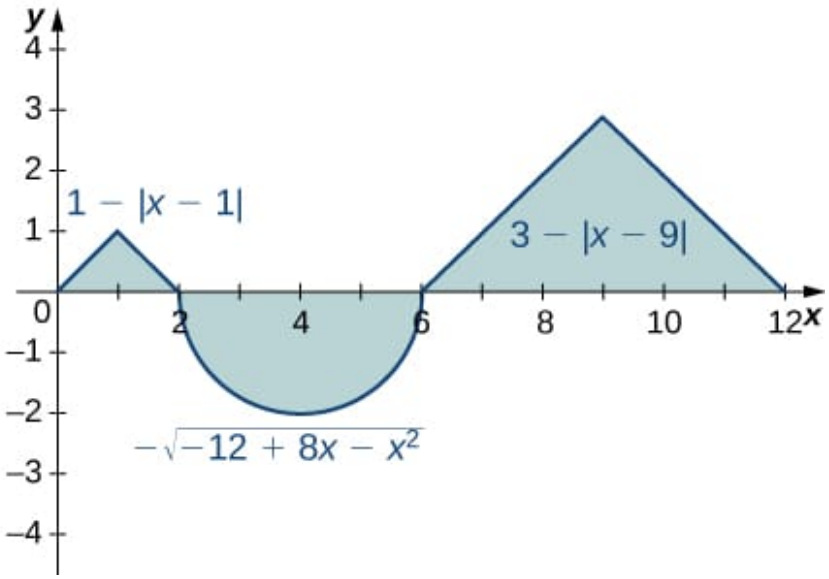


Answer

$1 - 2\pi + 9 = 10 - 2\pi$

In exercises 17 - 24, evaluate the integral using area formulas.

17) $\int_0^3 (3 - x)\ dx$

18) $\int_2^3 (3 - x)\ dx$

Answer

The integral is the area of the triangle, $\frac{1}{2}$.

19) $\int_{-3}^3 (3 - |x|)\ dx$

20) $\int_0^6 (3 - |x - 3|)\ dx$

Answer

The integral is the area of the triangle, 9.

21) $\int_{-2}^2 \sqrt{4 - x^2}\ dx$

22) $\int_1^5 \sqrt{4 - (x - 3)^2}\ dx$

Answer

The integral is the area $\frac{1}{2}\pi r^2 = 2\pi$.

23) $\int_0^{12} \sqrt{36 - (x - 6)^2}\ dx$

24) $\int_{-2}^3 (3 - |x|)\ dx$

Answer

The integral is the area of the "big" triangle less the "missing" triangle, $9 - \frac{1}{2}$.

Suppose that $\int_0^4 f(x)\ dx = 5$ and $\int_0^2 f(x)\ dx = -3$, and $\int_0^4 g(x)\ dx = -1$ and $\int_0^2 g(x)\ dx = 2$. In exercises 25 - 30, compute the integrals.

25) $\int_0^4 (f(x) + g(x))\ dx$

26) $\int_2^4 (f(x) + g(x))\ dx$

Answer

$$\int_2^4 f(x)\ dx + \int_2^4 g(x)\ dx = 8 - 3 = 5$$

27) $\int_0^2 (f(x) - g(x))\ dx$

28) $\int_2^4 (f(x) - g(x))\ dx$

Answer

$$\int_2^4 f(x)\ dx - \int_2^4 g(x)\ dx = 8 + 3 = 11$$

29) $\int_0^2 (3f(x) - 4g(x))\ dx$

30) $\int_2^4 (4f(x) - 3g(x))\ dx$

Answer

$$4\int_2^4 f(x)\ dx - 3\int_2^4 g(x)\ dx = 32 + 9 = 41$$

In exercises 31 - 34, use the identity $\int_{-A}^{A} f(x)\ dx = \int_{-A}^{0} f(x)\ dx + \int_0^A f(x)\ dx$ to compute the integrals.

31) $\int_{-\pi}^{\pi} \frac{\sin t}{1+t^2} dt$ (Hint: $\sin(-t) = -\sin(t)$)

32) $\int_{\sqrt{-\pi}}^{\sqrt{\pi}} \frac{t}{1+\cos t} dt$

Answer

The integrand is odd; the integral is zero.

33) $\int_1^3 (2 - x)\ dx$ (Hint: Look at the graph of f.)

34) $\int_2^4 (x-3)^3\ dx$ (Hint: Look at the graph of f.)

Answer

The integrand is antisymmetric with respect to $x = 3$. The integral is zero.

In exercises 35 - 40, given that $\int_0^1 x\ dx = \frac{1}{2}$, $\int_0^1 x^2\ dx = \frac{1}{3}$, and $\int_0^1 x^3\ dx = \frac{1}{4}$, compute the integrals.

35) $\int_0^1 (1 + x + x^2 + x^3)\ dx$

36) $\int_0^1 (1 - x + x^2 - x^3)\ dx$

Answer

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}$

37) $\int_0^1 (1 - x)^2\ dx$

38) $\int_0^1 (1 - 2x)^3\ dx$

Answer

$\int_0^1 (1 - 6x + 12x^2 - 8x^3)\ dx = 1 - 6\left(\frac{1}{2}\right) + 12\left(\frac{1}{3}\right) - 8\left(\frac{1}{4}\right) = 1 - 3 + 4 - 2\ = 0$

39) $\int_0^1 \left(6x - \frac{4}{3}x^2\right)\ dx$

40) $\int_0^1 (7 - 5x^3)\ dx$

Answer

$7 - \frac{5}{4} = \frac{23}{4}$

In exercises 41 - 46, use the comparison theorem.

41) Show that $\int_0^3 (x^2 - 6x + 9)\ dx \geq 0$.

42) Show that $\int_{-2}^3 (x - 3)(x + 2)\ dx \leq 0$.

Answer

The integrand is negative over $[-2, 3]$.

43) Show that $\int_0^1 \sqrt{1 + x^3}\ dx \leq \int_0^1 \sqrt{1 + x^2}\ dx$.

44) Show that $\int_1^2 \sqrt{1 + x}\ dx \leq \int_1^2 \sqrt{1 + x^2}\ dx$.

Answer

$x \leq x^2$ over $[1, 2]$, so $\sqrt{1 + x} \leq \sqrt{1 + x^2}$ over $[1, 2]$.

45) Show that $\int_0^{\pi/2} \sin t dt \geq \frac{\pi}{4}$ (Hint: $\sin t \geq \frac{2t}{\pi}$ over $[0, \frac{\pi}{2}]$)

46) Show that $\int_{-\pi/4}^{\pi/4} \cos t\ dt \geq \pi\sqrt{2}/4$.

Answer

$\cos(t) \geq \frac{\sqrt{2}}{2}$. Multiply by the length of the interval to get the inequality.

47) Suppose that $A = \int_0^{2\pi} \sin^2 t\ dt$ and $B = \int_0^{2\pi} \cos^2 t\ dt$. Show that $A + B = 2\pi$ and $A = B$.

48) Suppose that $A = \int_{-\pi/4}^{\pi/4} \sec^2 t\ dt = \pi$ and $B = \int_{-\pi/}^{\pi/4} 4\tan^2 t\ dt$. Show that $A - B = \frac{\pi}{2}$.

Answer

Use $\tan^2\theta + 1 = \sec^2\theta$. Then, $B - A = \int_{-\pi/4}^{\pi/4} 1\ dx = \frac{\pi}{2}$.

49) Explain why the graphs of a quadratic function (parabola) $p(x)$ and a linear function $\ell(x)$ can intersect in at most two points. Suppose that $p(a) = \ell(a)$ and $p(b) = \ell(b)$, and that $\int_a^b p(t)\ dt > \int_a^b \ell(t)dt$. Explain why $\int_c^d p(t) > \int_c^d \ell(t)\ dt$ whenever \(a \leq c

50) Suppose that parabola $p(x) = ax^2 + bx + c$ opens downward $(a < 0)$ and has a vertex of $y = \frac{-b}{2a} > 0$. For which interval $[A, B]$ is $\int_A^B (ax^2 + bx + c)\ dx$ as large as possible?

Answer

The integral is maximized when one uses the largest interval on which p is nonnegative. Thus, $A = \frac{-b-\sqrt{b^2-4ac}}{2a}$ and $B = \frac{-b+\sqrt{b^2-4ac}}{2a}$.

51) Suppose $[a, b]$ can be subdivided into subintervals \(a=a_0

a. Explain why $\int_a^b f(t)\ dt = A_1 + A_2 + \cdots + A_N$.

b. Then, explain why $\int_a^b f(t)\ dt \leq \int_a^b |f(t)|\ dt$.

52) Suppose f and g are continuous functions such that $\int_c^d f(t)\ dt \leq \int_c^d g(t)\ dt$ for every subinterval $[c, d]$ of $[a, b]$. Explain why $f(x) \leq g(x)$ for all values of x.

Answer

If $f(t_0) > g(t_0)$ for some $t_0 \in [a, b]$, then since $f - g$ is continuous, there is an interval containing t_0 such that $f(t) > g(t)$ over the interval $[c, d]$, and then $\int_d^d f(t)\ dt > \int_c^d g(t)\ dt$over this interval.

53) Suppose that for each i such that $1 \leq i \leq N$ one has $\int_{i-1}^{i} f(t)\ dt = i$. Show that $\int_0^N f(t)\ dt = \frac{N(N+1)}{2}$.

54) Suppose that for each i such that $1 \leq i \leq N$ one has $\int_{i-1}^{i} f(t)\ dt = i^2$. Show that $\int_0^N f(t)\ dt = \frac{N(N+1)(2N+1)}{6}$.

Answer

$$\begin{aligned}\int_0^N f(t)\,dt &= \sum_{i=1}^{N}\int_{i-1}^{i} f(t)\,dt \\ &= \sum_{i=1}^{N} i^2 \\ &= \frac{N(N+1)(2N+1)}{6}\end{aligned}$$

55) If $\int_1^5 \sqrt{1+t^4}\ dt = 41.7133\ldots$, what is $\int_1^5 \sqrt{1+u^4}\ du$?

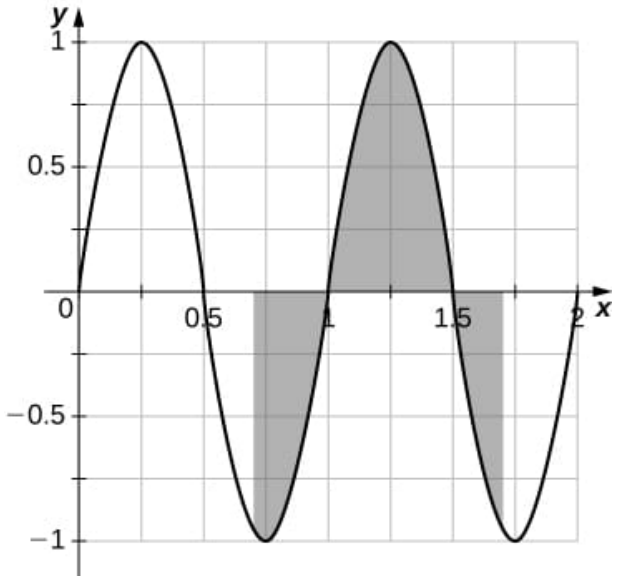
56) Estimate $\int_0^1 t\ dt$ by comparison with the area of a single rectangle with height equal to the value of t at the midpoint $t = \frac{1}{2}$. How does this midpoint estimate compare with the actual value $\int_0^1 t\ dt$?

57) From the graph of $\sin(2\pi x)$ shown:

a. Explain why $\int_0^1 \sin(2\pi t)\ dt = 0$.

b. Explain why, in general, $\int_a^{a+1} \sin(2\pi t)\ dt = 0$ for any value of a.

Answer



a. The graph is antisymmetric with respect to $t = \frac{1}{2}$ over $[0, 1]$, so the average value is zero.
b. For any value of a, the graph between $[a, a + 1]$ is a shift of the graph over $[0, 1]$, so the net areas above and below the axis do not change and the average remains zero.

58) If f is 1-periodic $(f(t + 1) = f(t))$, odd, and integrable over $[0, 1]$, is it always true that $\int_0^1 f(t)\ dt = 0$?

59) If f is 1-periodic and $\int 10f(t)\ dt = A$, is it necessarily true that $\int_a^{1+a} f(t)\ dt = A$ for all A?

Answer

Yes, the integral over any interval of length 1 is the same.

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4.4: The Fundamental Theorem of Calculus

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

In the previous section, we looked at the definite integral and its relationship to the area under the curve of a function. Unfortunately, the only tools available to calculate the value of a definite integral up to this point are geometric area formulas and limits of Riemann sums—both extremely cumbersome methods. In this section, we look at some more powerful and practical techniques for evaluating definite integrals.

These new techniques rely on the relationship between differentiation and integration. This relationship was discovered and explored by Sir Isaac Newton and Gottfried Wilhelm Leibniz (among others) during the late 1600s and early 1700s. It is codified in what we now call the Fundamental Theorem of Calculus, which has two parts that we examine in this section. Its very name indicates how central this theorem is to the entire development of Calculus.

Isaac Newton's contributions to Mathematics and Physics changed how we view the world. The relationships he discovered, codified as **Newton's Laws** and the **Law of Universal Gravitation**, are still taught as foundational material in Physics, and his Calculus has spawned entire fields of Mathematics.

Fundamental Theorem of Calculus Part 1: Integrals and Antiderivatives

As mentioned earlier, the Fundamental Theorem of Calculus is an extremely powerful theorem that establishes the relationship between differentiation and integration. It allows us to evaluate definite integrals without using Riemann sums or calculating areas. The theorem comprises two parts, the first of which, the **Fundamental Theorem of Calculus, Part 1** , is stated here. Part 1 establishes the relationship between differentiation and integration.

Theorem: The Fundamental Theorem of Calculus, Part 1 (FTC1)

If $f(x)$ is continuous over an interval $[a, b]$, and the function $F(x)$ is defined by

$$F(x) = \int_a^x f(t)\, dt,$$

then $F'(x) = f(x)$ over $[a, b]$.

Proof available at the end of this section.

$F(x)$ is called an **integral function** (also known as an **area function**) because it represents the definite integral of a function, $f(t)$, over an interval determined by the argument of F. That is, $F(x)$ is the area under the curve $f(t)$ from $t = a$ to some, yet to be determined value, $t = x$.

Interactive Element: Building the Area Function $F(x) = \int_a^x f(t)\ dt$

The following Interactive Element might help you visualize how the definite integral, viewed as a function of its upper limit, accumulates signed area — and why its rate of change is exactly f.

Interact: Drag the large red point along the x-axis to move the upper limit x.

Observation: The blue accumulated-area curve F rises where f (green) is above the axis and falls where f is below it, and levels off exactly where f crosses zero; its steepness at any x equals the height $f(x)$, which is the meaning of $F'(x) = f(x)$.

A couple of subtleties are worth mentioning here. First, a comment on the notation. Note that we have defined a function, $F(x)$, as the definite integral of another function, $f(t)$, from the number a to the number x. At first glance, this is confusing because we have said several times that a definite integral is a *number*, and here it looks like it's a function. The key here is to notice that for any particular value of x, the definite integral *is* a number. So, the function $F(x)$ returns a number (the value of the definite integral) for each value of x.

Second, it is worth commenting on some of the key implications of this theorem. There is a reason it is called the Fundamental Theorem of Calculus. Not only does it establish a relationship between integration and differentiation, but it also guarantees that any integrable function has an antiderivative. Specifically, it guarantees that any continuous function has an antiderivative.

Example 4.4.1: Finding a Derivative with the Fundamental Theorem of Calculus

Use the Fundamental Theorem of Calculus, Part 1 to find the derivative of

$$g(x) = \int_1^x \frac{1}{t^3 + 1}\, dt.$$

Solution available at the end of this section.

Checkpoint 4.4.1

Use the Fundamental Theorem of Calculus, Part 1 to find the derivative of $g(r) = \int_0^r \sqrt{x^2 + 4}\ dx$.

Answer available at the end of this section.

Example 4.4.2: Using the Fundamental Theorem and the Chain Rule to Calculate Derivatives

Let $F(x) = \int_1^{\sqrt{x}} \sin t\ dt$. Find $F'(x)$.

Solution available at the end of this section.

Checkpoint 4.4.2

Let $F(x) = \int_1^{x^3} \cos t\ dt$. Find $F'(x)$.

Answer available at the end of this section.

Example 4.4.3: Using the Fundamental Theorem of Calculus with Two Variable Limits of Integration

Let $F(x) = \int_x^{2x} t^3\ dt$. Find $F'(x)$.

Solution available at the end of this section.

Checkpoint 4.4.3

Let $F(x) = \int_x^{x^2} \cos t \; dt$. Find $F'(x)$.

Answer available at the end of this section.

If you study Example 4.4.3 in detail, it implies a more general form of the FTC1.

Corollary: General Form of the FTC1

If $f(x)$, $U(x)$, and $L(x)$ are continuous over an interval $[a, b]$, and the function $F(x)$ is defined by

$$F(x) = \int_{L(x)}^{U(x)} f(t)\, dt,$$

then

$$F'(x) = f(U(x))U'(x) - f(L(x))L'(x)$$

over $[a, b]$.

Proof available at the end of this section.

I encourage you to go back through Examples 4.4.1 through 4.4.3 applying this General Form of the FTC1 to verify it works in all cases. It is a wonderful preview of the structure of the Fundamental Theorem of Calculus, Part 2.

Fundamental Theorem of Calculus, Part 2: The Evaluation Theorem

The **Fundamental Theorem of Calculus, Part 2**, is perhaps the most important theorem in Calculus. After tireless efforts by mathematicians for approximately 500 years, new techniques emerged that provided scientists with the necessary tools to explain many phenomena. Using Calculus, astronomers could finally determine distances in space and map planetary orbits. Everyday financial problems such as calculating marginal costs or predicting total profit could now be handled with simplicity and accuracy. Engineers could calculate materials' bending strength or objects' three-dimensional motion. Our view of the world was forever changed with Calculus.

After finding approximate areas by adding the areas of n rectangles, applying this theorem is straightforward by comparison. It seems too simple that the area of an entire curved region can be calculated by just evaluating an antiderivative at the first and last endpoints of an interval.

Theorem: The Fundamental Theorem of Calculus, Part 2

If $f(x)$ is continuous over the interval $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)\, dx = F(b) - F(a). \tag{4.4.1}$$

Proof available at the end of this section.

We often see the notation $F(x)\,\big|_a^b$ to denote the expression $F(b) - F(a)$. We use this vertical bar and associated limits a and b to indicate that we should evaluate the function $F(x)$ at the upper limit (in this case, b), and subtract the value of the function $F(x)$ evaluated at the lower limit (in this case, a).

The Fundamental Theorem of Calculus, Part 2 (also known as the **Evaluation Theorem**) states that if we can find an antiderivative for the integrand, then we can evaluate the definite integral by evaluating the antiderivative at the endpoints of the interval and subtracting.

Example 4.4.4: Evaluating an Integral with the Fundamental Theorem of Calculus

Use Equation 4.4.1 to evaluate

$$\int_{-2}^{2} (t^2 - 4)\, dt.$$

Solution available at the end of this section.

Example 4.4.5: Evaluating a Definite Integral Using the Fundamental Theorem of Calculus, Part 2

Evaluate the following integral using the Fundamental Theorem of Calculus, Part 2 (Equation 4.4.1):

$$\int_{1}^{9} \frac{x-1}{\sqrt{x}} dx.$$

Solution available at the end of this section.

Checkpoint 4.4.5

Evaluate $\int_{1}^{2} x^{-4}\, dx$.

Answer available at the end of this section.

A Parachutist in Free Fall

Julie is an avid skydiver with more than 300 jumps under her belt and has mastered the art of adjusting her body position in the air to control how fast she falls. If she arches her back and points her belly toward the ground, she reaches a terminal velocity of approximately 120 mph (176 ft/sec). If, instead, she orients her body with her head straight down, she falls faster, reaching a terminal velocity of 150 mph (220 ft/sec).

Figure 4.4.3: *Skydivers can adjust the velocity of their dive by changing the position of their body during the free fall. (credit: Jeremy T. Lock)*

Since Julie will be moving (falling) downward, we assume the downward direction is positive to simplify our calculations. Julie executes her jumps from an altitude of 12,500 ft. After she exits the aircraft, she immediately starts falling at a velocity given by $v(t) = 32t$.

She continues accelerating according to this velocity function until she reaches terminal velocity. After she reaches terminal velocity, her speed remains constant until she pulls her ripcord and slows down to land.

On her first jump of the day, Julie orients herself in the slower "belly down" position (terminal velocity is 176 ft/sec). Using this information, answer the following questions.

1. How long after she exits the aircraft does Julie reach terminal velocity?
2. Based on your answer to question 1, set up an expression involving one or more integrals that represents the distance Julie falls after 30 sec.
3. If Julie pulls her ripcord at an altitude of 3000 ft, how long does she spend in a free fall?
4. Julie pulls her ripcord at 3000 ft. It takes 5 sec for her parachute to open completely and for her to slow down, during which time she falls another 400 ft. After her canopy is fully open, her speed is reduced to 16 ft/sec. Find the total time Julie spends in the air, from when she leaves the airplane to when her feet touch the ground. On Julie's second jump of the day, she decides she wants to fall a little faster and orients herself in the "head down" position. Her terminal velocity in this position is 220 ft/sec. Answer these questions based on this velocity:
5. How long does it take Julie to reach terminal velocity in this case?
6. Before pulling her ripcord, Julie reorients her body in the "belly down" position so she is not moving quite as fast when her parachute opens. If she begins this maneuver at an altitude of 4000 ft, how long does she spend in a free fall before beginning the reorientation?

Some jumpers wear "**wingsuits**" (Figure 4.4.4). These suits have fabric panels between the arms and legs, allowing the wearer to glide in a free fall, much like a flying squirrel. (Indeed, the suits are sometimes called "flying squirrel suits.") When wearing these suits, terminal velocity can be reduced to about 30 mph (44 ft/sec), allowing the wearers a much longer time in the air. Wingsuit flyers still use parachutes to land; although the vertical velocities are within the margin of safety, horizontal velocities can exceed 70 mph, much too fast to land safely.

Figure 4.4.4**:** *The fabric panels on the arms and legs of a wingsuit work to reduce the vertical velocity of a skydiver's fall. (credit: Richard Schneider)*

Answer the following question based on the velocity in a wingsuit.

7. If Julie dons a wingsuit before her third jump of the day, and she pulls her ripcord at an altitude of 3000 ft, how long does she get to spend gliding around in the air?

Success in College: Preparing for the Final Exam (Part 2)

1. Think about the concepts and skills you have learned this semester. Describe two concepts or skills that were most difficult for you.
2. Describe what you are doing to ensure you can answer the questions about these concepts or skills on the final exam.

Success in College: The Classroom and Learning Preferences

Read the strategies suggested for your learning preferences below.

Visual Learners

Strategies for visual learners include the following:

- take detailed notes during class
- reorganize and rewrite notes after class
- rework important examples
- watch the videos linked to your textbook
- use flashcards to study vocabulary

Auditory Learners

Strategies for auditory learners include the following:

- record your class, take brief notes of the big ideas and examples, and listen to the recording after class
- restate the main ideas from the lecture aloud to yourself
- talk to yourself as you do your homework, explaining each step to yourself
- study vocabulary by repeating definitions (listen to a recording of the words and definitions)

Kinesthetic Learners

Strategies for kinesthetic learners include the following:

- record the lectures, take brief notes of the big ideas and examples, and listen to the recording after class
- draw pictures, when possible
- with your finger, trace diagrams and graphs - don't just look at them
- imagine symbols as three-dimensional objects, or even cartoon characters
- study vocabulary as you exercise or walk to your car
- walk around your room as you repeat definitions

1. Do you use any of these strategies currently? If not, should you be?
2. Can you suggest any other ideas that I should add to this list?

Answers

Section Preview

This is what you've been waiting for (hopefully)—the big reveal! Here is where you relate the middle chunk of the course—derivatives—to the most recent material—integrals. If done right and developed properly, students should be genuinely excited about the connection (especially when you develop the Fundamental Theorem of Calculus, Part 1).

Section Preview

- State the meaning of the Fundamental Theorem of Calculus, Part 1.
- Use the Fundamental Theorem of Calculus, Part 1, to evaluate derivatives of integrals.
- State the meaning of the Fundamental Theorem of Calculus, Part 2.
- Use the Fundamental Theorem of Calculus, Part 2, to evaluate definite integrals.
- Explain the relationship between differentiation and integration.

Theorem: The Fundamental Theorem of Calculus, Part 1 (FTC1)

Applying the definition of the derivative, we have

$$\begin{aligned} F'(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} \\ &= \lim_{h\to 0}\frac{1}{h}\left[\int_a^{x+h} f(t)dt - \int_a^x f(t)\,dt\right] \\ &= \lim_{h\to 0}\frac{1}{h}\left[\int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt\right] \\ &= \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt. \end{aligned}$$

Before taking this limit, notice that we have shown, for $h \neq 0$,

$$\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)dt.$$

Suppose $h > 0$. Since $f(x)$ is continuous on $[a,b]$ and both x and $x+h$ are in $[a,b]$, $f(x)$ is continuous on $[x, x+h]$. By the Extreme Value Theorem, there must exist numbers $c_1, c_2 \in [x, x+h]$ such that $f(c_1) = m$ is the absolute minimum of f on $[x, x+h]$ and $f(c_2) = M$ is the absolute maximum of f on $[x, x+h]$. Therefore,

$$m \leq f(t) \leq M, \text{ for all } t \in [x, x+h].$$

By the Comparison Theorem,

$$\begin{aligned} & m\,[(x+h)-x] \leq \textstyle\int_x^{x+h} f(t)dt \leq M\,[(x+h)-x] \\ \Longrightarrow\quad & mh \leq \textstyle\int_x^{x+h} f(t)dt \leq Mh \\ \Longrightarrow\quad & f(c_1)h \leq \textstyle\int_x^{x+h} f(t)dt \leq f(c_2)h \\ \Longrightarrow\quad & f(c_1) \leq \textstyle\frac{1}{h}\int_x^{x+h} f(t)dt \leq f(c_2) \\ \Longrightarrow\quad & f(c_1) \leq \dfrac{F(x+h)-F(x)}{h} \leq f(c_2) \end{aligned}$$

However,

$$\begin{aligned} \lim_{h\to 0} f(c_1) &= \lim_{c_1\to x} f(c_1) \quad (\text{Since } c_1 \in [x, x+h], c_1 \to x \text{ as } h \to 0) \\ &= f(x) \end{aligned}$$

A similar argument shows $\lim_{h\to 0} f(c_2) = f(x)$. Since f is continuous, the Squeeze Theorem shows us that

$$f(x) \leq \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} \leq f(x).$$

That is,

$$\begin{aligned} F'(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} \\ &= f(x) \end{aligned}$$

The proof is done similarly if we had assumed $h < 0$.

Q.E.D.

Example 4.4.1**: Finding a Derivative with the Fundamental Theorem of Calculus**

According to the Fundamental Theorem of Calculus, Part 1, the derivative is given by

$$g'(x) = \frac{1}{x^3+1}.$$

Checkpoint 4.4.1

$g'(r) = \sqrt{r^2+4}$

Example 4.4.2**: Using the Fundamental Theorem and the Chain Rule to Calculate Derivatives**

Letting $u(x) = \sqrt{x}$, we have

$$F(x) = \int_1^{u(x)} \sin t\,dt.$$

Thus, by the Fundamental Theorem of Calculus and the Chain Rule,

$$F'(x) = \sin(u(x))\frac{du}{dx} = \sin(u(x)) \cdot \left(\frac{1}{2}x^{-1/2}\right) = \frac{\sin\sqrt{x}}{2\sqrt{x}}.$$

Checkpoint 4.4.2

$F'(x) = 3x^2\cos x^3$

Example 4.4.3**: Using the Fundamental Theorem of Calculus with Two Variable Limits of Integration**

We have $F(x) = \displaystyle\int_x^{2x} t^3\,dt$. Both limits of integration are variable, so we need to split this into two integrals. We get

$$\begin{aligned} F(x) &= \int_x^{2x} t^3\,dt \\ &= \int_x^0 t^3\,dt + \int_0^{2x} t^3\,dt \\ &= -\int_0^x t^3\,dt + \int_0^{2x} t^3\,dt. \end{aligned}$$

Differentiating the first term, we obtain

$$\frac{d}{dx}\left[-\int_0^x t^3\,dt\right] = -x^3.$$

Differentiating the second term, we first let $(x) = 2x$. Then,

$$\begin{aligned} \frac{d}{dx}\left[\int_0^{2x} t^3\,dt\right] &= \frac{d}{dx}\left[\int_0^{u(x)} t^3\,dt\right] \\ &= (u(x))^3\,du\;dx \\ &= (2x)^3 \cdot 2 = 16x^3. \end{aligned}$$

Thus,

$$\begin{aligned} F'(x) &= \frac{d}{dx}\left[-\int_0^x t^3\,dt\right] + \frac{d}{dx}\left[\int_0^{2x} t^3\,dt\right] \\ &= -x^3 + 16x^3 \\ &= 15x^3 \end{aligned}$$

Checkpoint 4.4.3

$F'(x) = 2x\cos x^2 - \cos x$

Corollary: General Form of the FTC1

$$\begin{aligned} \int_{L(x)}^{U(x)} f(t)\,dt &= \int_{L(x)}^{a} f(t)\,dt + \int_a^{U(x)} f(t)\,dt \qquad \text{(Properties of the Definite Integral)} \\ &= -\int_a^{L(x)} f(t)\,dt + \int_a^{U(x)} f(t)\,dt \quad \text{(Properties of the Definite Integral)} \\ &= \int_a^{U(x)} f(t)\,dt - \int_a^{L(x)} f(t)\,dt \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{d}{dx}\left(\int_{L(x)}^{U(x)} f(t)\,dt\right) &= \frac{d}{dx}\left(\int_a^{U(x)} f(t)\,dt - \int_a^{L(x)} f(t)\,dt\right) \\ &= \frac{d}{dx}\left(\int_a^{U(x)} f(t)\,dt\right) - \frac{d}{dx}\left(\int_a^{L(x)} f(t)\,dt\right) \quad \text{(Properties of the Definite Integral)} \\ &= f(U(x))U'(x) - f(L(x))L'(x) \qquad \text{(FTC1 and the Chain Rule)} \end{aligned}$$

Q.E.D.

Theorem: The Fundamental Theorem of Calculus, Part 2

Let $P = x_i, i = 0, 1, \ldots, n$ be a regular partition of $[a, b]$. Then, we can write

$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_0) \\ &= [F(x_n) - F(x_{n-1})] + [F(x_{n-1}) - F(x_{n-2})] + \ldots + [F(x_1) - F(x_0)] \\ &= \textstyle\sum_{i=1}^{n} [F(x_i) - F(x_{i-1})]. \end{aligned}$$

Now, we know F is an antiderivative of f over $[a, b]$, so, by the Mean Value Theorem, for $i = 0, 1, \ldots, n$ we can find c_i in $[x_{i-1}, x_i]$ such that

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) = f(c_i)\,\Delta x.$$

Then, substituting into the previous equation, we have

$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\,\Delta x.$$

Taking the limit of both sides as $n \to \infty$, we obtain

$$F(b) - F(a) = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)\,dx.$$

Q.E.D.

Example 4.4.4**: Evaluating an Integral with the Fundamental Theorem of Calculus**

Recall the Power Rule for Antiderivatives:

If $y = x^n$,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C.$$

Use this rule to find the antiderivative of the function and then apply the theorem. We have

$$\begin{aligned}\int_{-2}^{2}(t^2-4)dt &= \left(\frac{t^3}{3}-4t\right)\Big|_{-2}^{2}\\ &= \left[\frac{(2)^3}{3}-4(2)\right]-\left[\frac{(-2)^3}{3}-4(-2)\right]\\ &= \left[\frac{8}{3}-8\right]-\left[-\frac{8}{3}+8\right]\\ &= \frac{8}{3}-8+\frac{8}{3}-8\\ &= \frac{16}{3}-16\\ &= -\frac{32}{3}.\end{aligned}$$

Analysis

Notice that we did not include the "$+C$" term when we wrote the antiderivative. The reason is that, according to the Fundamental Theorem of Calculus, Part 2 (Equation 4.4.1), any antiderivative works. So, for convenience, we chose the antiderivative with $C = 0$. If we had chosen another antiderivative, the constant term would have been canceled. This always happens when evaluating a definite integral.

The region of the area we just calculated is depicted in Figure 4.4.1. Note that the region between the curve and the x-axis is all below the x-axis. Area is always positive, but a definite integral can still produce a negative number (a net signed area). For example, if this were a profit function, a negative number indicates the company is operating at a loss over the given interval.

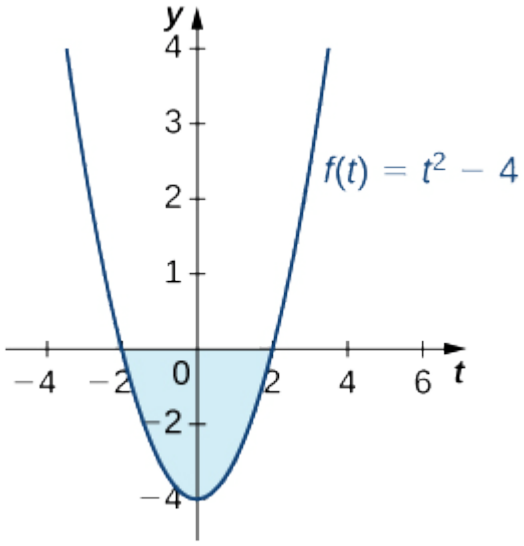

Figure 4.4.1**:** *The evaluation of a definite integral can produce a negative value, even though area is always positive.*

Example 4.4.5**: Evaluating a Definite Integral Using the Fundamental Theorem of Calculus, Part 2**

First, eliminate the radical by rewriting the integral using rational exponents. Then, separate the numerator terms by writing each one over the denominator:

$$\int_1^9 \frac{x-1}{x^{1/2}}\,dx = \int_1^9 \left(\frac{x}{x^{1/2}}-\frac{1}{x^{1/2}}\right)\,dx.$$

Use the properties of exponents to simplify:

$$\int_1^9 \left(\frac{x}{x^{1/2}}-\frac{1}{x^{1/2}}\right)\,dx = \int_1^9 (x^{1/2}-x^{-1/2})\,dx.$$

Now, integrate using the Power Rule:

$$\begin{aligned}\int_1^9 (x^{1/2}-x^{-1/2})\,dx &= \left(\frac{x^{3/2}}{\frac{3}{2}}-\frac{x^{1/2}}{\frac{1}{2}}\right)\Big|_1^9\\ &= \left[\frac{(9)^{3/2}}{\frac{3}{2}}-\frac{(9)^{1/2}}{\frac{1}{2}}\right]-\left[\frac{(1)^{3/2}}{\frac{3}{2}}-\frac{(1)^{1/2}}{\frac{1}{2}}\right]\\ &= \left[\frac{2}{3}(27)-2(3)\right]-\left[\frac{2}{3}(1)-2(1)\right]\\ &= 18-6-\frac{2}{3}+2\\ &= \frac{40}{3}.\end{aligned}$$

See Figure 4.4.2.

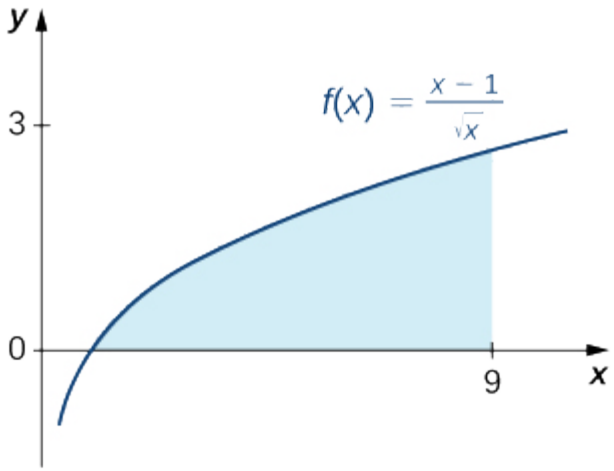

Figure $4.4.2$**:** *The area under the curve from* $x = 1$ *to* $x = 9$ *can be calculated by evaluating a definite integral.*

Checkpoint $4.4.5$

$\frac{7}{24}$

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4.4.1: Resources and Key Concepts

Resources

Videos

- The Integral Function
  - The Integral Function
- Fundamental Theorem of Calculus Part 1 (FTC1)
  - FTC1: Concept, Theorem, and Proof
  - Using the FTC1
- Fundamental Theorem of Calculus Part 2 (FTC2)
  - The FTC2: Concept, Theorem, and Proof
  - Using the FTC2
- Synthesis Questions

Key Concepts

Definitions

- **Integral Function (Area Function)**: A function $F(x)$ defined as a definite integral with a variable upper limit, such as $F(x) = \int_a^x f(t)\ dt$. It represents the definite integral of a function $f(t)$ over an interval determined by the argument of F, x.
- **Evaluation Theorem**: Another name for the Fundamental Theorem of Calculus, Part 2.
- **Notation** $F(x)|_a^b$: Represents the expression $F(b) - F(a)$.

Theorems

- **Theorem: The Fundamental Theorem of Calculus, Part 1 (FTC1)**: If $f(x)$ is continuous over an interval $[a, b]$, and the function $F(x)$ is defined by $F(x) = \int_a^x f(t)\ dt$, then $F'(x) = f(x)$ over $[a, b]$.
- **Corollary: General Form of the FTC1**: If $f(x)$, $U(x)$, and $L(x)$ are continuous over an interval $[a, b]$, and the function $F(x)$ is defined by $F(x) = \int_{L(x)}^{U(x)} f(t)\ dt$, then $F'(x) = f(U(x))U'(x) - f(L(x))L'(x)$.
- **Theorem: The Fundamental Theorem of Calculus, Part 2 (The Evaluation Theorem)**: If $f(x)$ is continuous over the interval $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then $\int_a^b f(x)\ dx = F(b) - F(a)$.

Common Mistakes

- **Forgetting the Chain Rule with FTC1**: When the upper limit of integration is a function $u(x)$ (not just x), forgetting to multiply by $u'(x)$ when finding the derivative of the integral. For example, $\frac{d}{dx}\int_a^{u(x)} f(t)\ dt = f(u(x))u'(x)$, not just $f(u(x))$.
- **Incorrectly Handling Two Variable Limits of Integration (FTC1)**: When both limits of integration are functions of x, $F(x) = \int_{L(x)}^{U(x)} f(t)\ dt$, applying the corollary correctly: $F'(x) = f(U(x))U'(x) - f(L(x))L'(x)$.
- **Errors in Finding Antiderivatives (FTC2)**: Making mistakes when determining an antiderivative $F(x)$ for the integrand $f(x)$.
- **Incorrectly Evaluating** $F(b) - F(a)$ **(FTC2)**:
  - Calculating $F(a) - F(b)$ instead of $F(b) - F(a)$.
  - Algebraic errors during the substitution and subtraction.
- **Forgetting the Constant of Integration** C **in Antiderivatives (FTC2 Context)**: While any antiderivative works for FTC2, and choosing $C = 0$ is simplest, understanding that the C would cancel out is important. The text notes that the "$+C$" term is

not included in the antiderivative for FTC2 because any antiderivative works, and the constant term would be canceled.
- **Confusing the Two Parts of FTC**: Misunderstanding which part applies to differentiating an integral (Part 1) versus evaluating a definite integral using an antiderivative (Part 2).

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4.4.2: Homework

Reading Questions

1. What does the Fundamental Theorem of Calculus, Part 1 (FTC1) state about the derivative of an integral function $F(x) = \int_a^x f(t)\ dt$?
2. If $g(x) = \int_1^x \frac{1}{t^3+1} dt$, what is $g'(x)$ according to FTC1?
3. How is the Chain Rule incorporated into FTC1 when the upper limit of integration is a function $u(x)$, i.e., for $F(x) = \int_a^{u(x)} f(t)\ dt$?
4. What is the general form of FTC1 when both limits of integration are functions, $F(x) = \int_{L(x)}^{U(x)} f(t)\ dt$?
5. What does the Fundamental Theorem of Calculus, Part 2 (FTC2), also known as the Evaluation Theorem, state?
6. What is an "antiderivative" of a function $f(x)$?
7. To evaluate $\int_a^b f(x)dx$ using FTC2, what are the two main steps?
8. When using FTC2, does it matter which antiderivative of $f(x)$ you choose? Explain why or why not regarding the constant of integration C.
9. How does FTC2 provide a more powerful and practical technique for evaluating definite integrals compared to using the limit of Riemann sums?
10. What is the relationship between differentiation and integration as established by the Fundamental Theorem of Calculus (both parts considered)?

Homework

In exercises 1 - 12, use the Fundamental Theorem of Calculus, Part 1, to find each derivative.

1) $\frac{d}{dx}\left[\int_1^x e^{-t^2}\,dt\right]$

2) $\frac{d}{dx}\left[\int_1^x e^{cost}\,dt\right]$

Answer

$$\frac{d}{dx}\left[\int_1^x e^{cost}\,dt\right] = e^{\cos t}$$

3) $\frac{d}{dx}\left[\int_3^x \sqrt{9-y^2}\,dy\right]$

4) $\frac{d}{dx}\left[\int_3^x \frac{ds}{\sqrt{16-s^2}}\right]$

Answer

$$\frac{d}{dx}\left[\int_3^x \frac{ds}{\sqrt{16-s^2}}\right] = \frac{1}{\sqrt{16-x^2}}$$

5) $\frac{d}{dx}\left[\int^{2x} x_t\,dt\right]$

6) $\frac{d}{dx}\left[\int_0^{\sqrt{x}} t\,dt\right]$

Answer

$$\frac{d}{dx}\left[\int_0^{\sqrt{x}} t\,dt\right] = \sqrt{x}\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2}$$

7) $\frac{d}{dx}\left[\int_0^{\sin x} \sqrt{1-t^2}\,dt\right]$

8) $\frac{d}{dx}\left[\int_{\cos x}^{1} \sqrt{1-t^2}\,dt\right]$

Answer

$$\frac{d}{dx}\left[\int_{\cos x}^{1} \sqrt{1-t^2}\,dt\right] = -\sqrt{1-\cos^2 x}\frac{d}{dx}\left(\cos x\right) = |\sin x|\sin x$$

9) $\frac{d}{dx}\left[\int_1^{\sqrt{x}} \frac{t^2}{1+t^4}\,dt\right]$

10) $\frac{d}{dx}\left[\int_1^{x^2} \frac{\sqrt{t}}{1+t}\,dt\right]$

Answer

$$\frac{d}{dx}\left[\int_1^{x^2} \frac{\sqrt{t}}{1+t}\,dt\right] = 2x\frac{|x|}{1+x^2}$$

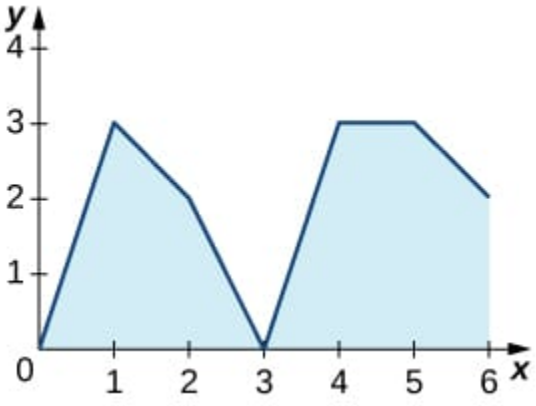
11) $\frac{d}{dx}\left[\int_0^{\ln x} e^t\,dt\right]$

12) $\frac{d}{dx}\left[\int_1^{e^x} \ln u^2\,du\right]$

Answer

$$\frac{d}{dx}\left[\int_1^{e^x} \ln u^2\,du\right] = \ln(e^{2x})\frac{d}{dx}\left(e^x\right) = 2xe^x$$

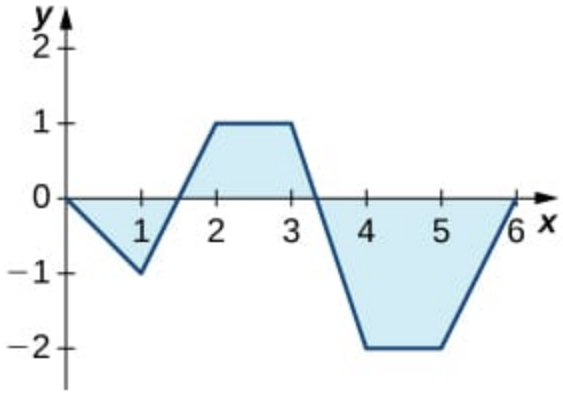
13) The graph of $y = \int_0^x f(t)\ dt$, where f is a piecewise constant function, is shown here.



a. Over which intervals is f positive? Over which intervals is it negative? Over which intervals, if any, is it equal to zero?

b. What are the maximum and minimum values of f?

14) The graph of $y = \int_0^x f(t)\ dt$, where f is a piecewise constant function, is shown here.



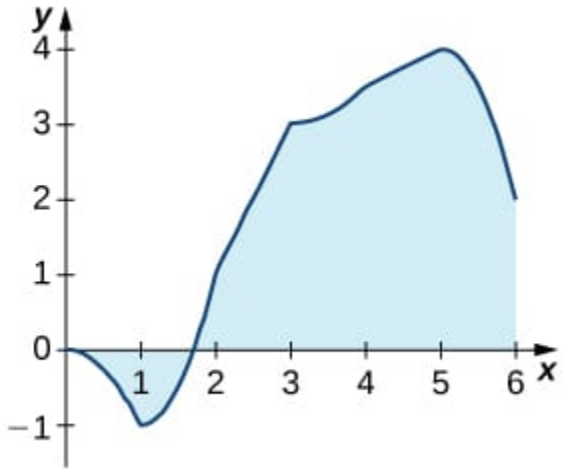
a. Over which intervals is f positive? Over which intervals is it negative? Over which intervals, if any, is it equal to zero?

b. What are the maximum and minimum values of f?

Answer

a. f is positive over $[1, 2]$ and $[5, 6]$, negative over $[0, 1]$ and $[3, 4]$, and zero over $[2, 3]$ and $[4, 5]$.
b. The maximum value is 2 and the minimum is -3.

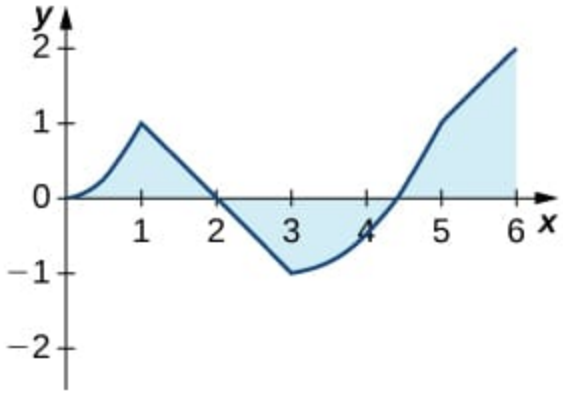
15) The graph of $y = \int_0^x \ell(t)\ dt$, where ℓ is a piecewise linear function, is shown here.



a. Over which intervals is ℓ positive? Over which intervals is it negative? Over which, if any, is it zero?

b. Over which intervals is ℓ increasing? Over which is it decreasing? Over which, if any, is it constant?

16) The graph of $y = \int_0^x \ell(t)\ dt$, where ℓ is a piecewise linear function, is shown here.



a. Over which intervals is ℓ positive? Over which intervals is it negative? Over which, if any, is it zero?

b. Over which intervals is ℓ increasing? Over which is it decreasing? Over which intervals, if any, is it constant?

Answer

a. ℓ is positive over $[0,1]$ and $[3,6]$, and negative over $[1,3]$.
b. It is increasing over $[0,1]$ and $[3,5]$, and it is constant over $[1,3]$ and $[5,6]$.

In exercises 17 - 36, evaluate each definite integral using the Fundamental Theorem of Calculus, Part 2.

17) $\int_{-1}^{2}(x^2-3x)\ dx$

18) $\int_{-2}^{3}(x^2+3x-5)\ dx$

Answer

$$F(x)=\frac{x^3}{3}+\frac{3x^2}{2}-5x,\quad F(3)-F(-2)=-\frac{35}{6}$$

19) $\int_{-2}^{3}(t+2)(t-3)dt$

20) $\int_{2}^{3}(t^2-9)(4-t^2)dt$

Answer

$$F(x)=-\frac{t^5}{5}+\frac{13t^3}{3}-36t,\quad F(3)-F(2)=\tfrac{62}{15}$$

21) $\int_{1}^{2}x^9\ dx$

22) $\int_{0}^{1}x^{99}\ dx$

Answer

$$F(x)=\frac{x^{100}}{100},\quad F(1)-F(0)=\frac{1}{100}$$

23) $\int_{4}^{8}(4t^{5/2}-3t^{3/2})dt$

24) $\int_{1/4}^{4}\left(x^2-\frac{1}{x^2}\right)\ dx$

Answer

$$F(x) = \frac{x^3}{3} + \frac{1}{x}, \quad F(4) - F\left(\tfrac{1}{4}\right) = \tfrac{1125}{64}$$

25) $\int_1^2 \frac{2}{x^3}\ dx$

26) $\int_1^4 \frac{1}{2\sqrt{x}}\ dx$

Answer

$$F(x) = \sqrt{x}, \quad F(4) - F(1) = 1$$

27) $\int_1^4 \frac{2-\sqrt{t}}{t^2}\ dt$

28) $\int_1^{16} \frac{dt}{t^{1/4}}$

Answer

$$F(t) = \tfrac{4}{3}t^{3/4}, \quad F(16) - F(1) = \tfrac{28}{3}$$

29) $\int_0^{2\pi} \cos\theta\ d\theta$

30) $\int_0^{\pi/2} \sin\theta\ d\theta$

Answer

$$F(\theta) = -\cos\theta, \quad F(\tfrac{\pi}{2}) - F(0) = 1$$

31) $\int_0^{\pi/4} \sec^2\theta\ d\theta$

32) $\int_0^{\pi/4} \sec\theta\tan\theta\ d\theta$

Answer

$$F(\theta) = \sec\theta, \quad F(\tfrac{\pi}{4}) - F(0) = \sqrt{2} - 1$$

33) $\int_{\pi/3}^{\pi/4} \csc\theta\cot\theta\ d\theta$

34) $\int_{\pi/4}^{\pi/2} \csc^2\theta\ d\theta$

Answer

$$F(\theta) = -\cot\theta, \quad F(\tfrac{\pi}{2}) - F(\tfrac{\pi}{4}) = 1$$

35) $\int_1^2 \left(\frac{1}{t^2} - \frac{1}{t^3}\right)\ dt$

36) $\int_{-2}^{-1} \left(\frac{1}{t^2} - \frac{1}{t^3}\right)\ dt$

Answer

$$F(t) = -\frac{1}{t} + \frac{1}{2t^2}, \quad F(-1) - F(-2) = \tfrac{7}{8}$$

In exercises 37 - 40, use the evaluation theorem to express the integral as a function $F(x)$.

37) $\int_a^x t^2 \ dt$

38) $\int_1^x e^t \ dt$

Answer

$$F(x) = e^x - e$$

39) $\int_0^x \cos t \ dt$

40) $\int_{-x}^x \sin t \ dt$

Answer

$$F(x) = 0$$

In exercises 41 - 44, identify the roots of the integrand to remove absolute values, then evaluate using the Fundamental Theorem of Calculus, Part 2.

41) $\int_{-2}^3 |x| \ dx$

42) $\int_{-2}^4 | \ t^2 - 2t - 3 \ | \ \ dt$

Answer

$$\int_{-2}^{-1} (t^2 - 2t - 3) \ dt - \int_{-1}^3 (t^2 - 2t - 3) \ dt + \int_3^4 (t^2 - 2t - 3) \ dt = \frac{46}{3}$$

43) $\int_0^\pi |\cos t| \ dt$

44) $\int_{-\pi/2}^{\pi/2} |\sin t| \ dt$

Answer

$$-\int_{-\pi/2}^0 \sin t \ dt + \int_0^{\pi/2} \sin t \ dt = 2$$

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4.5: The Net Change Theorem, Distances, and Symmetry

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

In this section, we use some basic integration formulas previously studied to solve key applied problems. It is important to note that these formulas are presented in terms of indefinite integrals. Although definite and indefinite integrals are closely related, there are some key differences to remember. A definite integral is either a number (when the limits of integration are constants) or a single function (when one or both of the limits of integration are variables). An indefinite integral represents a family of functions, all of which differ by a constant. As you become more familiar with integration, you will get a feel for when to use definite and indefinite integrals. You will naturally select the correct approach for a given problem without overthinking. However, until these concepts are cemented in your mind, think carefully about whether you need a definite or an indefinite integral and ensure you are using the proper notation based on your choice.

The Net Change Theorem

The **Net Change Theorem** considers the integral of a rate of change. It says that when a quantity changes, the new value equals the initial value plus the integral of the rate of change of that quantity. The formula can be expressed in two ways. The second is more familiar; it is simply the definite integral.

Theorem: Net Change Theorem

The new value of a changing quantity equals the initial value plus the integral of the rate of change:

$$F(b) = F(a) + \int_a^b F'(x)dx \tag{4.5.1}$$

or

$$\int_a^b F'(x)dx = F(b) - F(a). \tag{4.5.2}$$

Subtracting $F(a)$ from both sides of the Equation 4.5.1 yields Equation 4.5.2. Since they are equivalent formulas, which one we use depends on the application.

The significance of the Net Change Theorem lies in the results. Net change can be applied to area, distance, and volume, to name only a few applications. Net change accounts for negative quantities automatically without having to write more than one integral. To illustrate, let's apply the Net Change Theorem to a **velocity** function in which the result is **displacement**.

We looked at a simple example of this in a previous section. Suppose a car moves due north (the positive direction) at 40 mph between 2 p.m. and 4 p.m., then the car moves south at 30 mph between 4 p.m. and 5 p.m. We can graph this motion as shown in Figure 4.5.1.

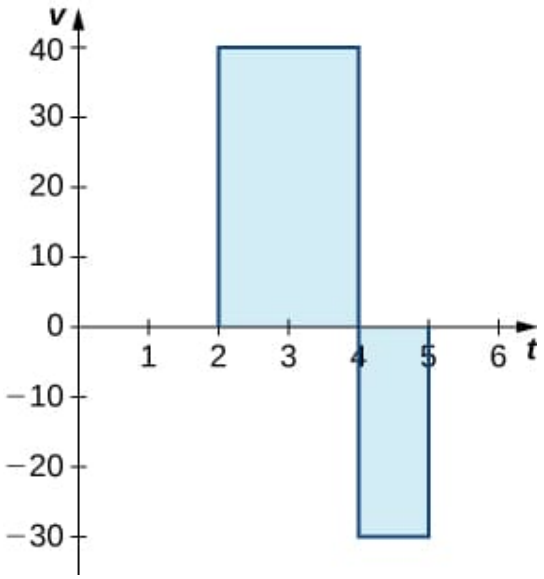

Figure 4.5.1**:** *The graph shows speed versus time for the given motion of a car.*

As before, we can use definite integrals to calculate the net displacement and the total distance traveled. The **net displacement** is given by

$$\int_2^5 v(t)\,dt = \int_2^4 40\,dt + \int_4^5 -30\,dt = 80 - 30 = 50.$$

Thus, at 5 p.m., the car is 50 mi north of its starting position. The **total distance** traveled is given by

$$\int_2^5 |v(t)|\,dt = \int_2^4 40\,dt + \int_4^5 30\,dt = 80 + 30 = 110.$$

Therefore, between 2 p.m. and 5 p.m., the car traveled a total of 110 mi.

To summarize, net displacement may include positive and negative values. In other words, the velocity function accounts for both forward and backward distances. To find net displacement, integrate the velocity function over the interval. Total distance traveled, on the other hand, is always positive. To find the total distance traveled by an object, regardless of direction, we need to integrate the absolute value of the velocity function.

Example 4.5.1: Finding Net Displacement

Given a velocity function $v(t) = 3t - 5$ (in meters per second) for a particle in motion from time $t = 0$ to time $t = 3$, find the net displacement of the particle.

Solution available at the end of this section.

Example 4.5.2: Finding the Total Distance Traveled

Use Example 4.5.1 to find the total distance traveled by a particle according to the velocity function $v(t) = 3t - 5$ m/sec over a time interval $[0, 3]$.

Solution available at the end of this section.

Interactive Element: Net Displacement vs. Total Distance

The following Interactive Element might help you visualize why net displacement uses signed area while total distance uses unsigned area, for the velocity function $v(t) = 3t - 5$ from Examples $4.5.1$ and $4.5.2$.

Interact: Drag the point along the horizontal (time) axis to change the upper time limit b.

Observation: The below-axis (negative) area subtracts from net displacement but adds to total distance, so once v changes sign the two quantities diverge — net displacement can fall while total distance keeps rising.

Checkpoint $4.5.2$

Find the net displacement and total distance traveled in meters given the velocity function $f(t) = \frac{1}{2}e^t - 2$ over the interval $[0, 2]$.

Answer available at the end of this section.

Applying the Net Change Theorem

The Net Change Theorem can be applied to the flow and consumption of fluids, as shown in Example $4.5.3$.

Example $4.5.3$: How Many Gallons of Gasoline Are Consumed?

If the motor on a motorboat is started at $t = 0$ and the boat consumes gasoline at the rate of $5 - t^3$ gal/hr, how much gasoline is used in the first hour?

Solution available at the end of this section.

Example $4.5.4$: Iceboats

Top **iceboat** racers can attain speeds of up to five times the wind speed. Andrew is an intermediate iceboater, though, so he attains speeds equal to only twice the wind speed.

Figure 4.5.3**:** *(credit: modification of work by Carter Brown, Flickr)*

Suppose Andrew takes his iceboat out one morning when a light 5-mph breeze has been blowing all morning. As Andrew sets up his iceboat, though, the wind picks up. During his first half hour of iceboating, the wind speed increases according to the function $v(t) = 20t + 5$. The wind remains steady at 15 mph for the second half hour of Andrew's outing. In other words, the wind speed is given by

$$v(t) = \begin{cases} 20t + 5, & \text{for } 0 \le t \le \frac{1}{2} \\ 15, & \text{for } \frac{1}{2} \le t \le 1 \end{cases}$$

Recalling that Andrew's iceboat travels at twice the wind speed, and assuming he moves in a straight line away from his starting point, how far is Andrew from his starting point after 1 hour?

Solution available at the end of this section.

Checkpoint 4.5.4

Suppose that, instead of remaining steady during the second half-hour of Andrew's outing, the wind starts to die down according to the function $v(t) = -10t + 15$. In other words, the wind speed is given by

$$v(t) = \begin{cases} 20t + 5, & \text{for } 0 \le t \le \frac{1}{2} \\ -10t + 15, & \text{for } \frac{1}{2} \le t \le 1 \end{cases}.$$

Under these conditions, how far from his starting point is Andrew after 1 hour?

Answer available at the end of this section.

Integrating Even and Odd Functions

We know that an **even function** is a function in which $f(-x) = f(x)$ for all x in the domain—that is, the graph of the curve is unchanged when x is replaced with $-x$. The graphs of even functions are symmetric about the y-axis. An **odd function** is one in which $f(-x) = -f(x)$ for all x in the domain, and the graph of the function is symmetric about the origin.

Integrals of even functions, when the limits of integration are from $-a$ to a, involve two equal areas because they are symmetric about the y-axis. Integrals of odd functions, when the limits of integration are similarly $[-a, a]$, evaluate to zero because the areas above and below the x-axis are equal.

Theorem: Integrals of Even and Odd Functions

For continuous even functions such that $f(-x) = f(x)$,

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

For continuous odd functions such that $f(-x) = -f(x)$,

$$\int_{-a}^{a} f(x)\,dx = 0.$$

Example 4.5.5: Integrating an Even Function

Integrate the even function $\int_{-2}^{2} (3x^8 - 2)\ dx$ and verify that the integration formula for even functions holds.

Solution available at the end of this section.

Interactive Element: Why an Even Function's Symmetric Integral Doubles the Half

The following Interactive Element might help you visualize why $\int_{-a}^{a} f(x)\ dx = 2\int_{0}^{a} f(x)\ dx$ for an even function.

Interact: Drag the point along the x-axis to change the symmetric limit a.

Observation: The left half mirrors the right half at every value of a, so the two areas are always equal and the whole integral is twice the half.

Example 4.5.6: Integrating an Odd Function

Evaluate the definite integral of the odd function $-5\sin x$ over the interval $[-\pi, \pi]$.

Solution available at the end of this section.

Interactive Element: Why an Odd Function's Symmetric Integral Is Zero

The following Interactive Element might help you visualize why $\int_{-a}^{a} f(x)\ dx = 0$ for an odd function, using $f(x) = -5\sin x$ from Example 4.5.6.

Interact: Drag the point along the x-axis to change the symmetric limit a.

Observation: No matter how wide you make the interval, the area on one side of the origin cancels the equal, opposite-signed area on the other, so the integral stays exactly 0.

Checkpoint 4.5.6

Integrate the function $\int_{-2}^{2} x^4\ dx$.

Answer available at the end of this section.

Success in College: Learning From Tests (Part 6)

1. How complex are mathematics tests for you compared to exams in other subjects?
2. How hard do you try on the mathematics test compared to your efforts in other subjects?

Success in College: Preparing for the Final Exam (Part 3)

One suggestion I like to make concerning preparing for a final exam is to generate your own tests. This is pretty simple; however, you wouldn't believe how many students fail to do this. When studying, most people focus on the wrong things. You get sidetracked and work on problems that may only make up 5% of the entire exam. Try the following:

Let's assume that your final exam covers nine chapters. Let's also suppose that chapter four has the most sections, with eight. Finally, suppose the greatest number of problems within any given section is 82.

- Open Excel (or Google Sheets)
- In the first cell, type "=RANDBETWEEN(1,9)". This will generate a random chapter number.
- In the second cell, type "=RANDBETWEEN(1,8)". This will generate a random section between 1 and 8.
- In the third cell, type "=RANDBETWEEN(1,82)". This will generate a random problem number between 1 and 82.

Of course, it is entirely possible to generate problem 5.7.77 (chapter 5, section 7, problem 77) when section 7 of chapter 5 only has, for example, 50 problems. Just hit F9, and Excel will generate a new combination for you!

All you have to do is repeat this a few times, and you have a randomly generated test!

1. What do you think are the advantages of generating an exam this way? Explain.
2. How do you usually prepare for final exams?

Answers

Section Preview

The material in this section didn't fit well with the previous material, and it should come before the Subsitution Rule. Hence, this section might feel like an afterthought, or a hodge-podge of topics. Despite that, they are each necessary.

Section Preview

- Apply the basic integration formulas.
- Explain the significance of the Net Change Theorem.
- Use the Net Change Theorem to solve applied problems.
- Apply the integrals of odd and even functions.

Example $4.5.1$: Finding Net Displacement

Applying the Net Change Theorem, we have

$$\int_0^3 (3t-5)\,dt = \left(\frac{3t^2}{2}-5t\right)\Big|_0^3 = \left[\frac{3(3)^2}{2}-5(3)\right]-0 = \frac{27}{2}-15 = \frac{27}{2}-\frac{30}{2} = -\frac{3}{2}.$$

The net displacement is $-\frac{3}{2}$ m (Figure $4.5.2$).

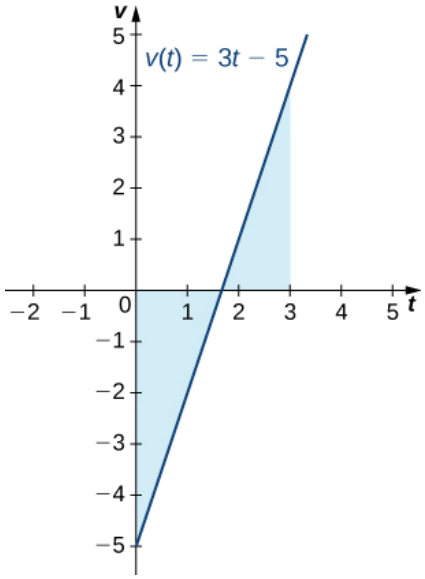

Figure 4.5.2**:** *The graph shows velocity versus time for a particle moving with a linear velocity function.*

Example 4.5.2**: Finding the Total Distance Traveled**

The total distance traveled includes both the positive and the negative values. Therefore, we must integrate the absolute value of the velocity function to find the total distance traveled.

To continue with the example, use two integrals to find the total distance. First, find the t-intercept of the function since that is where the division of the interval occurs. Set the equation equal to zero and solve for t. Thus,

$$\begin{aligned} 3t-5 &= 0 \\ 3t &= 5 \\ t &= \frac{5}{3}. \end{aligned}$$

The two subintervals are $\left[0,\frac{5}{3}\right]$ and $\left[\frac{5}{3},3\right]$. To find the total distance traveled, integrate the absolute value of the function. Since the function is negative over the interval $\left[0,\frac{5}{3}\right]$, we have $|v(t)|=-v(t)$ over that interval. Over $\left[\frac{5}{3},3\right]$, the function is positive, so $|v(t)|=v(t)$. Thus, we have

$$\begin{aligned} \int_0^3 |v(t)|\,dt &= \int_0^{5/3} -v(t)\,dt + \int_{5/3}^3 v(t)\,dt \\ &= \int_0^{5/3} 5-3t\,dt + \int_{5/3}^3 3t-5\,dt \\ &= \left(5t-\frac{3t^2}{2}\right)\Big|_0^{5/3} + \left(\frac{3t^2}{2}-5t\right)\Big|_{5/3}^3 \\ &= \left[5(\frac{5}{3})-\frac{3(5/3)^2}{2}\right]-0+\left[\frac{27}{2}-15\right]-\left[\frac{3(5/3)^2}{2}-\frac{25}{3}\right] \\ &= \frac{25}{3}-\frac{25}{6}+\frac{27}{2}-15-\frac{25}{6}+\frac{25}{3} \\ &= \frac{41}{6} \end{aligned}$$

So, the total distance traveled is $\frac{14}{6}$ m.

Checkpoint 4.5.2

Net displacement: $\frac{e^2-9}{2}\approx -0.8055$ m; total distance traveled: $4\ln 4-7.5+\frac{e^2}{2}\approx 1.740$ m.

Example 4.5.3**: How Many Gallons of Gasoline Are Consumed?**

Express the problem as a definite integral, integrate, and evaluate using the Fundamental Theorem of Calculus. The limits of integration are the endpoints of the interval $[0,1]$. We have

$$\begin{aligned} \int_0^1 (5-t^3)\,dt &= \left(5t-\frac{t^4}{4}\right)\Big|_0^1 \\ &= \left[5(1)-\frac{(1)^4}{4}\right]-0 \\ &= 5-\frac{1}{4} \\ &= 4.75. \end{aligned}$$

Thus, the motorboat uses 4.75 gallons of gas in the first hour.

Example 4.5.4**: Iceboats**

To determine how far Andrew has traveled, we need to integrate his velocity, which is twice the wind speed. Then

$$\text{Distance} = \int_0^1 2v(t)\,dt.$$

Substituting the expressions we were given for $v(t)$, we get

$$\begin{aligned}
\int_0^1 2v(t)\,dt &= \int_0^{1/2} 2v(t)\,dt + \int_{1/2}^1 2v(t)\,dt \\
&= \int_0^{1/2} 2(20t+5)\,dt + \int_{1/3}^1 2(15)\,dt \\
&= \int_0^{1/2} (40t+10)\,dt + \int_{1/2}^1 30\,dt \\
&= \left[20t^2+10t\right]\Big|_0^{1/2} + \left[30t\right]\Big|_{1/2}^1 \\
&= \left(\frac{20}{4}+5\right) - 0 + (30-15) \\
&= 25.
\end{aligned}$$

Andrew is 25 mi from his starting point after 1 hour.

Checkpoint 4.5.4

17.5 mi

Example 4.5.5: Integrating an Even Function

The symmetry appears in the graphs in Figure 4.5.4. Graph (a) shows the region below the curve and above the x-axis. We have to zoom in on this graph by a huge amount to see the region. Graph (b) shows the region above the curve and below the x-axis. The signed area of this region is negative. Both views illustrate the symmetry about the y-axis of an even function. We have

$$\begin{aligned}
\int_{-2}^2 (3x^8-2)\,dx &= \left(\frac{x^9}{3}-2x\right)\Big|_{-2}^2 \\
&= \left[\frac{(2)^9}{3}-2(2)\right] - \left[\frac{(-2)^9}{3}-2(-2)\right] \\
&= \left(\frac{512}{3}-4\right) - \left(-\frac{512}{3}+4\right) \\
&= \frac{1000}{3}.
\end{aligned}$$

To verify the integration formula for even functions, we can calculate the integral from 0 to 2 and double it, then check to make sure we get the same answer.

$$\int_0^2 (3x^8-2)\,dx = \left(\frac{x^9}{3}-2x\right)\Big|_0^2 = \frac{512}{3}-4 = \frac{500}{3}$$

Since $2\cdot\frac{500}{3} = \frac{1000}{3}$, we have verified the formula for even functions in this particular example.

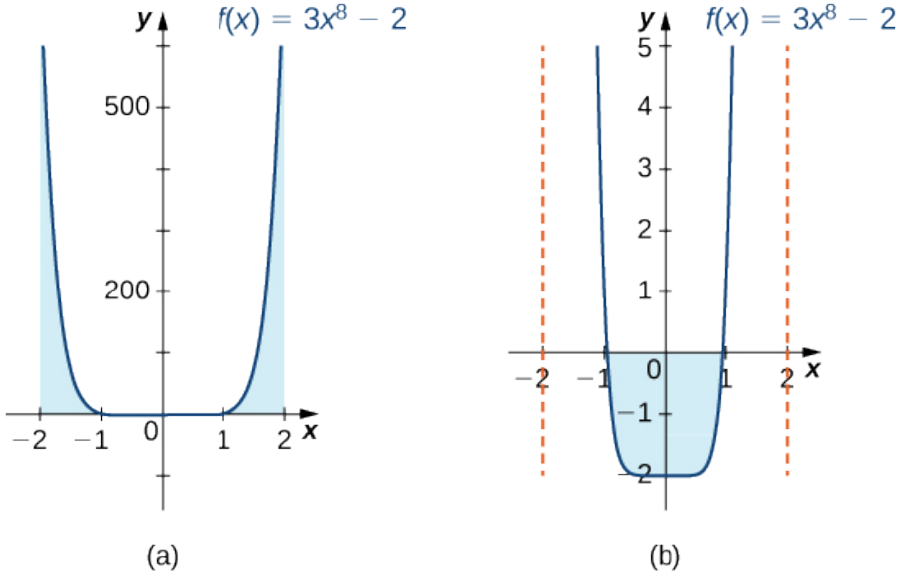

Figure 4.5.4: *Graph (a) shows the positive area between the curve and the x-axis, whereas graph (b) shows the negative area between the curve and the x-axis. Both views show the symmetry about the y-axis.*

Example 4.5.6: Integrating an Odd Function

The graph is shown in Figure 4.5.5. We can see the symmetry about the origin by the positive area above the x-axis over $[-\pi, 0]$, and the negative area below the x -axis over $[0, \pi]$. we have

$$\begin{aligned}\int_{-\pi}^{\pi} -5\sin x\,dx &= -5(-\cos x)\Big|_{-\pi}^{\pi} \\ &= 5\cos x\Big|_{-\pi}^{\pi} \\ &= [5\cos\pi] - [5\cos(-\pi)] \\ &= -5-(-5) \\ &= 0.\end{aligned}$$

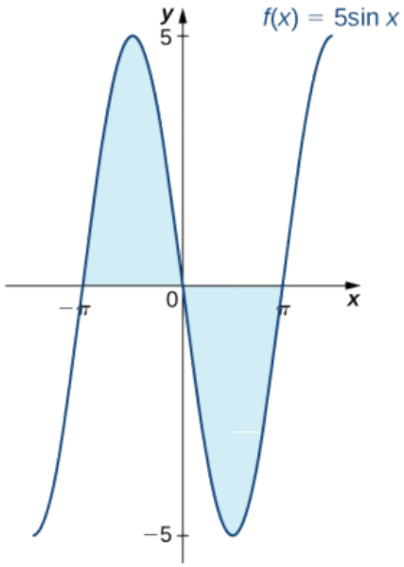

Figure 4.5.5**:** *The graph shows areas between a curve and the x-axis for an odd function.*

Checkpoint 4.5.6

$\frac{64}{5}$

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4.5.1: Resources and Key Concepts

Key Concepts

Definitions

- **Net Displacement**: The integral of the velocity function over an interval, $\int_a^b v(t)\ dt$. It accounts for direction and can be positive, negative, or zero.
- **Total Distance Traveled**: The integral of the absolute value of the velocity function (speed) over an interval, $\int_a^b |v(t)|\ dt$. It is always non-negative.

Theorems

- **Theorem: Net Change Theorem**: The new value of a changing quantity equals the initial value plus the integral of the rate of change of that quantity.
  - Form 1: $F(b) = F(a) + \int_a^b F'(x)\ dx$
  - Form 2 (Equivalent to FTC2): $\int_a^b F'(x)\ dx = F(b) - F(a)$
- **Theorem: Integrals of Even and Odd Functions**:
  - For continuous even functions such that $f(-x) = f(x)$: $\int_{-a}^{a} f(x)\ dx = 2\int_0^a f(x)\ dx$.
  - For continuous odd functions such that $f(-x) = -f(x)$: $\int_{-a}^{a} f(x)\ dx = 0$.

Common Mistakes

- **Confusing Net Displacement and Total Distance Traveled**: Using $\int_a^b v(t)\ dt$ when total distance is required, or $\int_a^b |v(t)|\ dt$ when net displacement is required. Total distance requires considering intervals where velocity is negative and taking the absolute value (or integrating $-v(t)$) over those intervals.
- **Incorrectly Identifying Even or Odd Functions**: Misclassifying a function's symmetry, leading to incorrect application of the integral properties for even/odd functions.
- **Applying Symmetry Rules to Incorrect Intervals**: The rules $\int_{-a}^{a} f(x)\ dx = 2\int_0^a f(x)\ dx$ (for even) and $\int_{-a}^{a} f(x)\ dx = 0$ (for odd) specifically require the interval of integration to be symmetric about the origin, i.e., of the form $[-a, a]$.
- **Errors in Evaluating Definite Integrals**: Basic mistakes in finding antiderivatives or applying the Fundamental Theorem of Calculus, Part 2.

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4.5.2: Homework

Reading Questions

1. State the Net Change Theorem in its two common forms. How does the second form relate to the Fundamental Theorem of Calculus, Part 2?
2. If $v(t)$ is the velocity of a particle, what does $\int_a^b v(t)\ dt$ represent? What does $\int_a^b |v(t)|\ dt$ represent?
3. When calculating total distance traveled, why might it be necessary to split the integral into multiple parts?
4. What is an even function? Describe its symmetry.
5. What is an odd function? Describe its symmetry.
6. What is the value of $\int_{-a}^{a} f(x)\ dx$ if $f(x)$ is a continuous odd function? Explain geometrically.
7. How can the integral $\int_{-a}^{a} f(x)\ dx$ be simplified if $f(x)$ is a continuous even function? Explain geometrically.
8. In Example 3 (gasoline consumption), how was the Net Change Theorem applied to find the total amount of gasoline used?
9. If $F'(x)$ represents the rate of change of a quantity $F(x)$, what does the definite integral $\int_a^b F'(x)\ dx$ tell you about the quantity $F(x)$?

Homework

1) Write an integral that expresses the increase in the perimeter $P(s)$ of a square when its side length s increases from 2 units to 4 units and evaluate the integral.

Answer

$P(s) = 4s$, so $\dfrac{dP}{ds} = 4$ and $\int_2^4 4\ ds = 8$.

2) Write an integral that quantifies the change in the area $A(s) = s^2$ of a square when the side length doubles from S units to $2S$ units and evaluate the integral.

3) A regular N-gon (an N-sided polygon with sides that have equal length s, such as a pentagon or hexagon) has perimeter Ns. Write an integral that expresses the increase in perimeter of a regular N-gon when the length of each side increases from 1 unit to 2 units and evaluate the integral.

Answer

$\int_1^2 N\ ds = N$

4) The area of a regular pentagon with side length $a > 0$ is pa^2 with $p = \frac{1}{4}\sqrt{5 + \sqrt{5 + 2\sqrt{5}}}$. The Pentagon in Washington, DC, has inner sides of length 360 ft and outer sides of length 920 ft. Write an integral to express the area of the roof of the Pentagon according to these dimensions and evaluate this area.

5) A dodecahedron is a Platonic solid with a surface that consists of 12 pentagons, each of equal area. By how much does the surface area of a dodecahedron increase as the side length of each pentagon doubles from 1 unit to 2 units?

Answer

With p as in the previous exercise, each of the 12 pentagons increases in area from $2p$ to $4p$ units so the net increase in the area of the dodecahedron is $36p$ units.

6) An icosahedron is a Platonic solid with a surface that consists of 20 equilateral triangles. By how much does the surface area of an icosahedron increase as the side length of each triangle doubles from a unit to $2a$ units?

7) Write an integral that quantifies the change in the area of the surface of a cube when its side length doubles from s unit to $2s$

units and evaluate the integral.

Answer

$$18s^2 = 6\int_s^{2s} 2x\ dx$$

8) Write an integral that quantifies the increase in the volume of a cube when the side length doubles from s unit to $2s$ units and evaluate the integral.

9) Write an integral that quantifies the increase in the surface area of a sphere as its radius doubles from R unit to $2R$ units and evaluate the integral.

Answer

$$12\pi R^2 = 8\pi\int_R^{2R} r\ dr$$

10) Write an integral that quantifies the increase in the volume of a sphere as its radius doubles from R unit to $2R$ units and evaluate the integral.

11) Suppose that a particle moves along a straight line with velocity $v(t) = 4 - 2t$, where $0 \le t \le 2$ (in meters per second). Find the displacement at time t and the total distance traveled up to $t = 2$.

Answer

$d(t) = \int_0^t v(s)\ ds = 4t - t^2$. The total distance is $d(2) = 4$ m.

12) Suppose that a particle moves along a straight line with velocity defined by $v(t) = t^2 - 3t - 18$, where $0 \le t \le 6$ (in meters per second). Find the displacement at time t and the total distance traveled up to $t = 6$.

13) Suppose that a particle moves along a straight line with velocity defined by $v(t) = |2t - 6|$, where $0 \le t \le 6$ (in meters per second). Find the displacement at time t and the total distance traveled up to $t = 6$.

Answer

$d(t) = \int_0^t v(s)\ ds$. For $t < 3,\ \ d(t) = \int_0^t (6 - 2t)\ dt = 6t - t^2$. For $t > 3,\ \ d(t) = d(3) + \int_3^t (2t - 6)\ dt = 9 + (t^2 - 6t)$. The total distance is $d(6) = 9$ m.

14) Suppose that a particle moves along a straight line with acceleration defined by $a(t) = t - 3$, where $0 \le t \le 6$ (in meters per second). Find the velocity and displacement at time t and the total distance traveled up to $t = 6$ if $v(0) = 3$ and $d(0) = 0$.

15) A ball is thrown upward from a height of 1.5 m at an initial speed of 40 m/sec. Acceleration resulting from gravity is $-9.8\ \mathrm{m/sec^2}$. Neglecting air resistance, solve for the velocity $v(t)$ and the height $h(t)$ of the ball t seconds after it is thrown and before it returns to the ground.

Answer

$v(t) = 40 - 9.8t;\ \ h(t) = 1.5 + 40t - 4.9t^2$ m/s

16) A ball is thrown upward from a height of 3 m at an initial speed of 60 m/sec. Acceleration resulting from gravity is $-9.8\ \mathrm{m/sec^2}$. Neglecting air resistance, solve for the velocity $v(t)$ and the height $h(t)$ of the ball t seconds after it is thrown and before it returns to the ground.

17) The area $A(t)$ of a circular shape is growing at a constant rate. If the area increases from 4π units to 9π units between times $t = 2$ and $t = 3$, find the net change in the radius during that time.

Answer

The net increase is 1 unit.

18) A spherical balloon is being inflated at a constant rate. If the volume of the balloon changes from $36\pi\ \mathrm{in^3}$ to $288\pi\ \mathrm{in^3}$ between time $t = 30$ and $t = 60$ seconds, find the net change in the radius of the balloon during that time.

19) Water flows into a conical tank with cross-sectional area πx^2 at height x and volume $\frac{\pi x^3}{3}$ up to height x. If water flows into the tank at a rate of 1 $\mathrm{m}^3/\mathrm{min}$, find the height of water in the tank after 5 min. Find the change in height between 5 min and 10 min.

Answer

At $t = 5$, the height of water is $x = \left(\frac{15}{\pi}\right)^{1/3}$ m. The net change in height from $t = 5$ to $t = 10$ is $\left(\left(\frac{30}{\pi}\right)^{1/3} - \left(\frac{15}{\pi}\right)^{1/3}\right)$ m.

20) A horizontal cylindrical tank has cross-sectional area $A(x) = 4(6x - x^2)$ m^2 at height x meters above the bottom when $x \leq 3$.

a. The volume V between heights a and b is $\int_a^b A(x)\ dx$. Find the volume at heights between 2 m and 3 m.

b. Suppose that oil is being pumped into the tank at a rate of 50 L/min. Using the chain rule, $\frac{dx}{dt} = \frac{dx}{dV}\frac{dV}{dt}$, at how many meters per minute is the height of oil in the tank changing, expressed in terms of x, when the height is at x meters?

c. How long does it take to fill the tank to 3 m starting from a fill level of 2 m?

21) The following table lists the electrical power in gigawatts—the rate at which energy is consumed—used in a certain city for different hours of the day, in a typical 24-hour period, with hour 1 corresponding to midnight to 1 a.m.

Hour	Power	Hour	Power
1	28	13	48
2	25	14	49
3	24	15	49
4	23	16	50
5	24	17	50
6	27	18	50
7	29	19	46
8	32	20	43
9	34	21	42
10	39	22	40
11	42	23	37
12	46	24	34

Find the total amount of power in gigawatt-hours (gW-h) consumed by the city in a typical 24-hour period.

Answer

The total daily power consumption is estimated as the sum of the hourly power rates, or 911 gW-h.

22) The average residential electrical power use (in hundreds of watts) per hour is given in the following table.

Hour	Power	Hour	Power
1	8	13	12
2	6	14	13
3	5	15	14
4	4	16	15
5	5	17	17

6	6	18	19
7	7	19	18
8	8	20	17
9	9	21	16
10	10	22	16
11	10	23	13
12	11	24	11

a. Compute the average total energy used in a day in kilowatt-hours (kWh).

b. If a ton of coal generates 1842 kWh, how long does it take for an average residence to burn a ton of coal?

c. Explain why the data might fit a plot of the form $p(t) = 11.5 - 7.5 \sin\left(\frac{\pi t}{12}\right)$.

23) The data in the following table are used to estimate the average power output produced by Peter Sagan for each of the last 18 sec of Stage 1 of the 2012 **Tour de France.**

Second	**Watts**	**Second**	**Watts**
1	600	10	1200
2	500	11	1170
3	575	12	1125
4	1050	13	1100
5	925	14	1075
6	950	15	1000
7	1050	16	950
8	950	17	900
9	1100	18	780

Average Power Output Source: sportsexercisengineering.com

Estimate the net energy used in kilojoules (kJ), noting that 1W = 1 J/s, and the average power output by Sagan during this time interval.

Answer

17 kJ

24) The data in the following table are used to estimate the average power output produced by Peter Sagan for each 15-min interval of Stage 1 of the 2012 Tour de France.

Minutes	**Watts**	**Minutes**	**Watts**
15	200	165	170
30	180	180	220
45	190	195	140
60	230	210	225
75	240	225	170
90	210	240	10

105	210	255	200
1120	220	270	220
135	210	285	250
150	150	300	400

Average Power Output *Source*: sportsexercisengineering.com

Estimate the net energy used in kilojoules, noting that 1W = 1 J/s.

25) The distribution of incomes as of 2012 in the United States in $5000 increments is given in the following table. The kth row denotes the percentage of households with incomes between $\$5000xk$ and $5000xk + 4999$. The row $k = 40$ contains all households with income between $200,000 and $250,000 and $k = 41$ accounts for all households with income exceeding $250,000.

0	3.5	21	1.5
1	4.1	22	1.4
2	5.9	23	1.3
3	5.7	24	1.3
4	5.9	25	1.1
5	5.4	26	1.0
6	5.5	27	0.75
7	5.1	28	0.8
8	4.8	29	1.0
9	4.1	30	0.6
10	4.3	31	0.6
11	3.5	32	.5
12	3.7	33	0.5
13	3.2	34	0.4
14	3.0	35	0.3
15	2.8	36	0.3
16	2.5	37	0.3
17	2.2	38	0.2
18	2.2	39	1.8
19	1.8	40	2.3
20	2.1	41	

Income Distributions Source: http://www.census.gov/prod/2013pubs/p60-245.pdf

a. Estimate the percentage of U.S. households in 2012 with incomes less than $55,000.

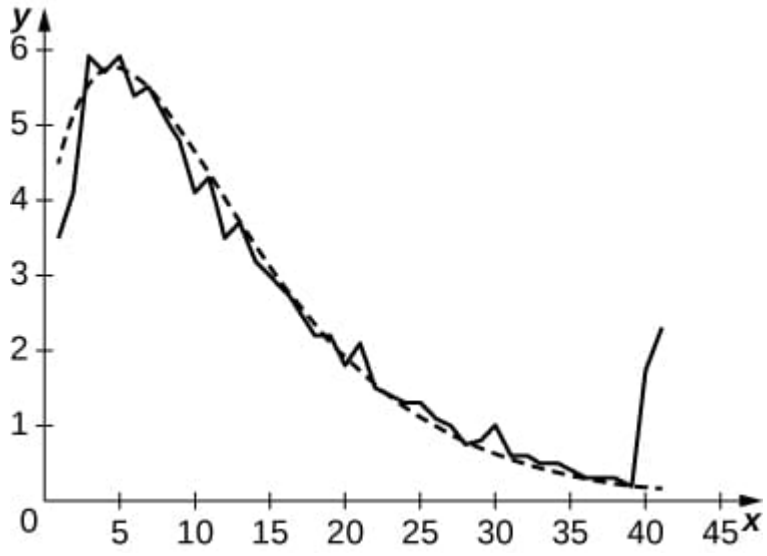
b. What percentage of households had incomes exceeding $85,000?

c. Plot the data and try to fit its shape to that of a graph of the form $a(x + c)e^{-b(x+e)}$ for suitable $a,\ b,\ c$.

Answer

a. 54.3%;

b. 27.00%;
c. The curve in the following plot is $2.35(t+3)e^{-0.15(t+3)}$.



26) Newton's law of gravity states that the gravitational force exerted by an object of mass M and one of mass m with centers that are separated by a distance r is $F = G\frac{mM}{r^2}$, with G an empirical constant $G = 6.67x10^{-11}\ \mathrm{m}^3/(\mathrm{kg}\cdot\mathrm{s}^2)$. The work done by a variable force over an interval $[a,b]$ is defined as $W = \int_a^b F(x)\ dx$. If Earth has mass 5.97219×10^{24} and radius 6371 km, compute the amount of work to elevate a polar weather satellite of mass 1400 kg to its orbiting altitude of 850 km above Earth.

27) For a given motor vehicle, the maximum achievable **deceleration** from braking is approximately $7\ \mathrm{m/sec}^2$ on dry concrete. On wet asphalt, it is approximately $2.5\ \mathrm{m/sec}^2$. Given that 1 mph corresponds to 0.447 m/sec, find the total distance that a car travels in meters on dry concrete after the brakes are applied until it comes to a complete stop if the initial velocity is 67 mph (30 m/sec) or if the initial braking velocity is 56 mph (25 m/sec). Find the corresponding distances if the surface is slippery wet asphalt.

Answer

In dry conditions, with initial velocity $v_0 = 30$ m/s, $D = 64.3$ and, if $v_0 = 25,\ \ D = 44.64$. In wet conditions, if $v_0 = 30$, and $D = 180$ and if $v_0 = 25,\ \ D = 125$.

28) John is a 25-year old man who weighs 160 lb. He burns $500 - 50t$ calories/hr while riding his bike for t hours. If an oatmeal cookie has 55 cal and John eats $4t$ cookies during the tth hour, how many net calories has he lost after 3 hours riding his bike?

29) Sandra is a 25-year old woman who weighs 120 lb. She burns $300 - 50t$ cal/hr while walking on her treadmill for t hours. Her caloric intake from drinking Gatorade is $100t$ calories during the tth hour. What is her net decrease in calories after walking for 3 hours?

Answer

225 cal

30) A motor vehicle has a maximum efficiency of 33 mpg at a cruising speed of 40 mph. The efficiency drops at a rate of 0.1 mpg/mph between 40 mph and 50 mph, and at a rate of 0.4 mpg/mph between 50 mph and 80 mph. What is the efficiency in miles per gallon if the car is cruising at 50 mph? What is the efficiency in miles per gallon if the car is cruising at 80 mph? If gasoline costs $3.50/gal, what is the cost of fuel to drive 50 mi at 40 mph, at 50 mph, and at 80 mph?

31) Although some engines are more efficient at given a horsepower than others, on average, fuel efficiency decreases with horsepower at a rate of $1/25$ mpg/horsepower. If a typical 50-horsepower engine has an average fuel efficiency of 32 mpg, what is the average fuel efficiency of an engine with the following horsepower: 150, 300, 450?

Answer

$E(150) = 28,\ \ E(300) = 22,\ \ E(450) = 16$

32) [Technology Required] The following table lists the 2013 schedule of **federal income tax** versus taxable income.

Taxable Income Range	The Tax Is ...	... Of the Amount Over
$0–$8925	10%	$0
$8925–$36,250	$892.50 + 15%	$8925
$36,250–$87,850	$4,991.25 + 25%	$36,250
$87,850–$183,250	$17,891.25 + 28%	$87,850
$183,250–$398,350	$44,603.25 + 33%	$183,250
$398,350–$400,000	$115,586.25 + 35%	$398,350
> $400,000	$116,163.75 + 39.6%	$400,000

Federal Income Tax Versus Taxable Income Source: http://www.irs.gov/pub/irs-prior/i1040tt--2013.pdf.

Suppose that Steve just received a $10,000 raise. How much of this raise is left after federal taxes if Steve's salary before receiving the raise was $40,000? If it was $90,000? If it was $385,000?

33) [Technology Required] The following table provides hypothetical data regarding the level of service for a certain highway.

Highway Speed Range (mph)	Vehicles per Hour per Lane	Density Range (vehicles/mi)
>60	<600	<10
60-57	300-1000	10-20
57-54	1000-1500	20-30
57-54	1500-1900	30-45
46-30	1900-2100	48-70
<30	Unstable	70-200

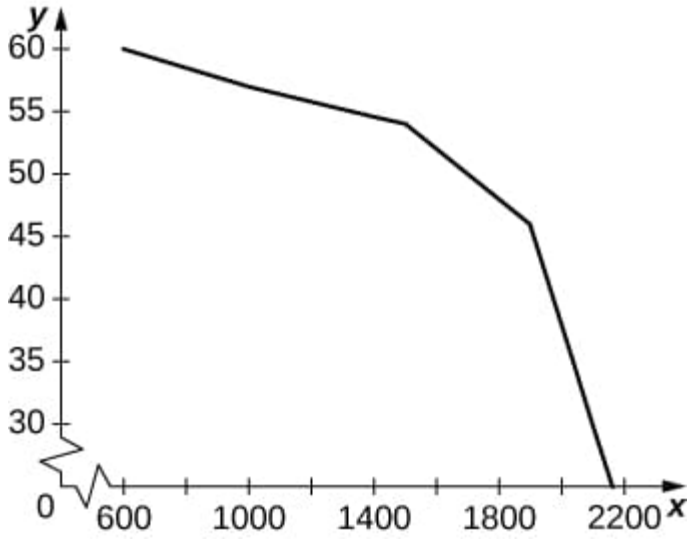
a. Plot vehicles per hour per lane on the x-axis and highway speed on the y-axis.

b. Compute the average decrease in speed (in miles per hour) per unit increase in congestion (vehicles per hour per lane) as the latter increases from 600 to 1000, from 1000 to 1500, and from 1500 to 2100. Does the decrease in miles per hour depend linearly on the increase in vehicles per hour per lane?

c. Plot minutes per mile (60 times the reciprocal of miles per hour) as a function of vehicles per hour per lane. Is this function linear?

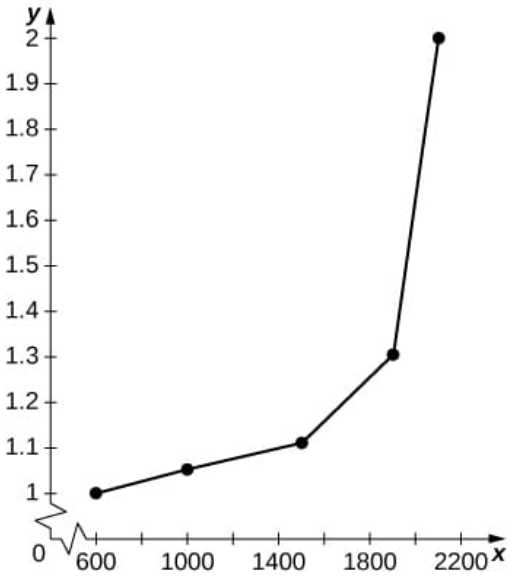
Answer

a.



b. Between 600 and 1000 the average decrease in vehicles per hour per lane is –0.0075. Between 1000 and 1500 it is –0.006 per vehicles per hour per lane, and between 1500 and 2100 it is –0.04 vehicles per hour per lane.

c.



The graph is nonlinear, with minutes per mile increasing dramatically as vehicles per hour per lane reach 2000.

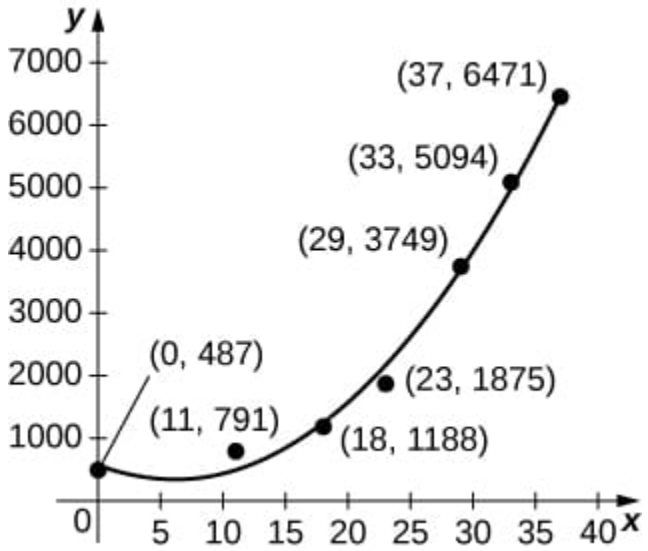
For the next two exercises use the data in the following table, which displays bald eagle populations from 1963 to 2000 in the continental United States.

Year	Population of Breeding Pairs of Bald Eagles
1963	487
1974	791
1981	1188
1986	1875

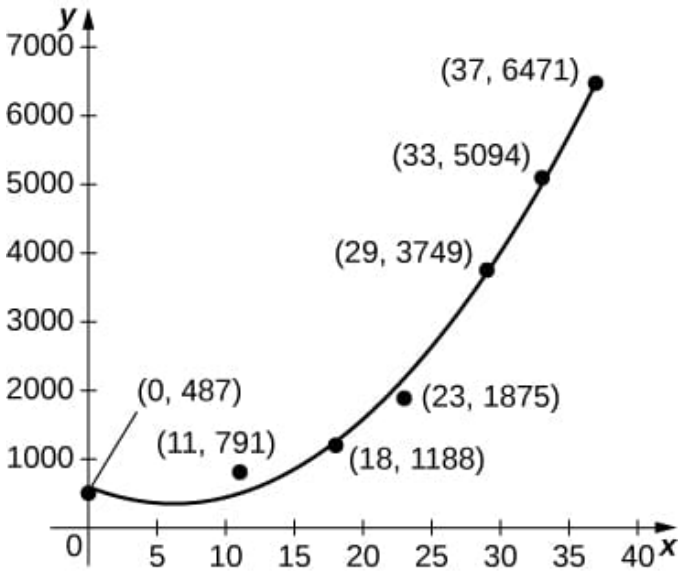
1992	3749
1996	5094
2000	6471

Population of Breeding Bald Eagle Pairs *Source*: http://www.fws.gov/Midwest/eagle/pop.../chtofprs.html.

34) [Technology Required] The graph below plots the quadratic $p(t) = 6.48t^2 - 80.31t + 585.69$ against the data in preceding table, normalized so that $t = 0$ corresponds to 1963. Estimate the average number of bald eagles per year present for the 37 years by computing the average value of p over $[0, 37]$.



35) [Technology Required] The graph below plots the cubic $p(t) = 0.07t^3 + 2.42t^2 - 25.63t + 521.23$ against the data in the preceding table, normalized so that $t = 0$ corresponds to 1963. Estimate the average number of bald eagles per year present for the 37 years by computing the average value of p over $[0, 37]$.



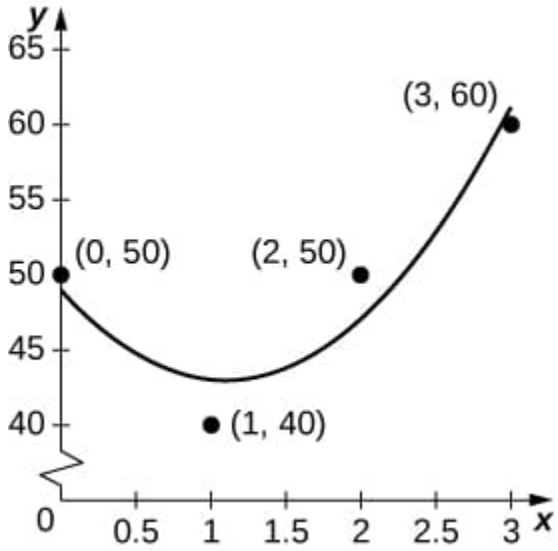
Answer

$$\frac{1}{37}\int_0^{37} p(t)\ dt = \frac{0.07(37)^3}{4} + \frac{2.42(37)^2}{3} - \frac{25.63(37)}{2} + 521.23 \approx 2037$$

Solution:

36) [Technology Required] Suppose you go on a road trip and record your speed at every half hour, as compiled in the following table. The best quadratic fit to the data is $q(t) = 5x^2 - 11x + 49$, shown in the accompanying graph. Integrate q to estimate the total distance driven over the 3 hours.

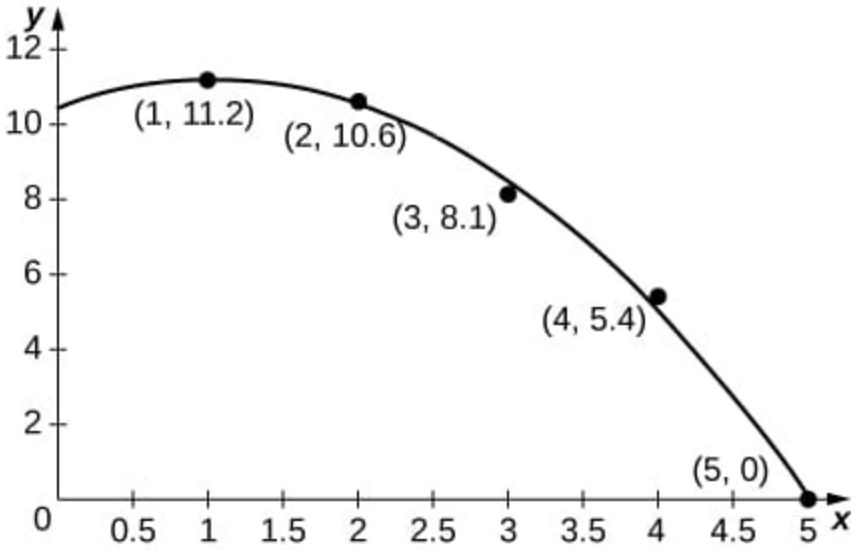
Time (hr)	**Speed (m[h)**
0 (start)	50
1	40
2	50
3	60



As a car accelerates, it does not accelerate at a constant rate; rather, the acceleration is variable. For the following exercises, use the following table, which contains the acceleration measured at every second as a driver merges onto a freeway.

Time (sec)	Acceleration (mph/sec)
1	11.2
2	10.6
3	8.1
4	5.4
5	0

37) [Technology Required] The accompanying graph plots the best quadratic fit, $a(t) = -0.70t^2 + 1.44t + 10.44$, to the data from the preceding table. Compute the average value of $a(t)$ to estimate the average acceleration between $t = 0$ and $t = 5$.



Answer

Average acceleration is $A = \frac{1}{5}\int_0^5 a(t)\ dt = -\frac{0.7(5^2)}{3} + \frac{1.44(5)}{2} + 10.44 \approx 8.2$ mph/s

38) [Technology Required] Using your acceleration equation from the previous exercise, find the corresponding velocity equation. Assuming the final velocity is 0 mph, find the velocity at time $t = 0$.

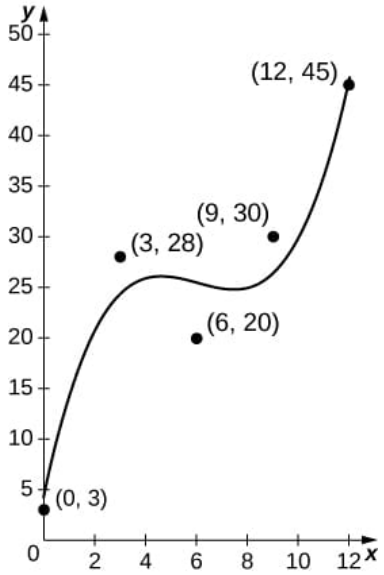
39) [Technology Required] Using your velocity equation from the previous exercise, find the corresponding distance equation, assuming your initial distance is 0 mi. How far did you travel while you accelerated your car? (Hint: You will need to convert time units.)

Answer

$d(t) = \int_0^1 |v(t)|\ dt = \int_0^t \left(\frac{7}{30}t^3 - 0.72t^2 - 10.44t + 41.033\right)\ dt = \frac{7}{120}t^4 - 0.24t^3 - 5.22t^3 + 41.033t$. Then, $d(5) \approx 81.12\ \text{mph} \times \text{sec} \approx 119$ feet.

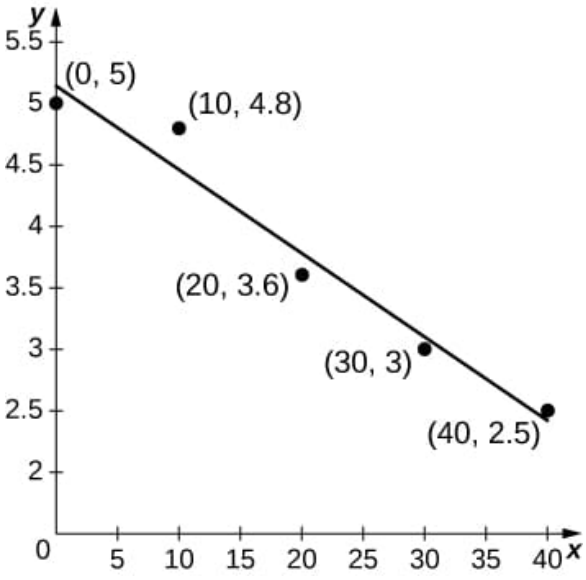
40) [Technology Required] The number of hamburgers sold at a restaurant throughout the day is given in the following table, with the accompanying graph plotting the best cubic fit to the data, $b(t) = 0.12t^3 - 2.13t^3 + 12.13t + 3.91$, with $t = 0$ corresponding to 9 a.m. and $t = 12$ corresponding to 9 p.m. Compute the average value of $b(t)$ to estimate the average number of hamburgers sold per hour.

Hours Past Midnight	No. of Burgers Sold
9	3
12	28
15	20
18	30
21	45



41) [Technology Required] An athlete runs by a motion detector, which records her speed, as displayed in the following table. The best linear fit to this data, $\ell(t) = -0.068t + 5.14$, is shown in the accompanying graph. Use the average value of $\ell(t)$ between $t = 0$ and $t = 40$ to estimate the runner's average speed.

Minutes	**Speed (m/sec)**
0	5
10	4.8
20	3.6
30	3.0
40	2.5



Answer

$$\frac{1}{40}\int_0^{40}(-0.068t + 5.14)\ dt = -\frac{0.068(40)}{2} + 5.14 = 3.78 \text{ m/sec}$$

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4.6: Integration by Substitution

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

The Fundamental Theorem of Calculus allows us to evaluate integrals without using Riemann sums. The drawback of this method is that we must be able to find an antiderivative, which can be challenging. This section examines **integration by substitution**—a technique to help us find antiderivatives. Specifically, this method allows us to find antiderivatives when the integrand is the result of a Chain Rule derivative.

At first, the approach to the substitution procedure may appear obscure. However, it is primarily a visual task—that is, the integrand shows you what to do; it is a matter of recognizing the form of the function. So, what are we supposed to see?

We are looking for an integrand of the form $f\left(g(x)\right)g'(x)\ dx$. For example, in the integral

$$\int (x^2-3)^3\,2x\,dx. \tag{4.6.1}$$

we have

$$f(x)=x^3$$

and

$$g(x)=x^2-3.$$

Then

$$g'(x)=2x.$$

Therefore,

$$f(g(x))g'(x)=(x^2-3)^3(2x),$$

and we see that our integrand is in the correct form. The method is called substitution, or the **Substitution Method**, because we substitute part of the integrand with the variable u and part of the integrand with du. It is also referred to as **change of variables** because we are changing variables to obtain an easier expression to work with for applying the integration rules.

Theorem: Substitution Method with Indefinite Integrals

Let $u=g(x)$, where $g'(x)$ is continuous over an interval, let $f(x)$ be continuous over the corresponding range of g, and let $F(x)$ be an antiderivative of $f(x)$. Then,

$$\begin{aligned}\int f\left(g(x)\right)g'(x)\,dx &= \int f(u)\,du\\ &= F(u)+C\\ &= F(g(x))+C\end{aligned}$$

Proof available at the end of this section.

Returning to the problem we looked at originally, we let $u = x^2 - 3$ and then $du = 2x\ dx$.

Rewrite the integral (Equation 4.6.1) in terms of u:

$$\int (x^2 - 3)^3 (2x\, dx) = \int u^3\, du.$$

Using the Power Rule for integrals, we have

$$\int u^3\, du = \frac{u^4}{4} + C.$$

Substitute the original expression for x back into the solution:

$$\frac{u^4}{4} + C = \frac{(x^2 - 3)^4}{4} + C.$$

At this point, it is important to note that integration is mostly a heuristic method. That is, integration is approached more often by using a "calculated guess" derived from previous experiences. As you move through the middle part of Calculus (also known as Calculus II, or Integral Calculus), you will learn many different integration techniques. You must practice as many problems as possible to get a "natural feel" for what method to use and when to use it.

In the case of Integration by Substitution, you are often (but not always) looking for an integrand involving a function, $g(x)$, and its *approximate* derivative, $g'(x)$. You let $u(x) = g(x)$ so that $du = g'(x)dx$. At that point, you will have rewritten your original integral into a form that (hopefully) looks like

$$\int u\, du.$$

Example 4.6.1: Using Substitution to Find an Antiderivative

Use substitution to find the antiderivative of $\int 6x(3x^2 + 4)^4\ dx$.

Solution available at the end of this section.

Checkpoint 4.6.1

Use substitution to find the antiderivative of $\int 3x^2(x^3 - 3)^2\ dx$.

Answer available at the end of this section.

Sometimes, we need to adjust the constants in our integral if they don't match up exactly with the expressions we are substituting.

Example 4.6.2: Using Substitution with Alteration

Use substitution to find the antiderivative of

$$\int z\sqrt{z^2 - 5}\, dz.$$

Solution available at the end of this section.

Checkpoint 4.6.2

Use substitution to find the antiderivative of $\int x^2(x^3 + 5)^9\ dx$.

Answer available at the end of this section.

Example 4.6.3: Using Substitution with Integrals of Trigonometric Functions

Use substitution to evaluate the integral $\int \frac{\sin t}{\cos^3 t}\, dt$.

Solution available at the end of this section.

Example 4.6.3 needs a little more explanation. How did we know to let $u = \cos(t)$ rather than the sine function? To be honest, this is something that becomes more of an instinct after a while; however, there is a science to it—it's the science of *considering ramifications*.

If we let $u = \sin(t)$, then $du = \cos(t)dt$. This might not jump off the page to you and scream, "This is bad," so let's see the ramifications of this choice.

The original integrand was $\frac{\sin(t)}{\cos^3(t)}$. Since this does not have a $\cos(t)$ to "steal" in the numerator, we need to solve $du = \cos(t)dt$ for dt. Doing so yields $dt = \frac{1}{\cos(t)}du$. If we make this substitution, the integrand would become

$$\frac{\sin(t)}{\cos^3(t)}\, dt = \frac{u}{\cos^3(t)} \cdot \frac{1}{\cos(t)}\, du = \frac{u}{\cos^4(t)}\, du.$$

There are two major issues here:

1. the integrand would have a mixture of variables, so we cannot leave it this way, and
2. the integrand would become *more* complex—not less.

The first issue can be overcome by using our Trigonometry knowledge.

$$\begin{aligned} \frac{u}{\cos^4(t)}\, du &= \frac{u}{(\cos^2(t))^2}\, du \\ &= \frac{u}{(1-\sin^2(t))^2}\, du \\ &= \frac{u}{(1-u^2)^2}\, du \qquad \text{(Substitution: Recall } u = \sin(t)\text{)} \end{aligned}$$

However, the second issue (complexity) still exists. Instead of

$$\int \frac{\sin(t)}{\cos^3(t)}\, dt,$$

we are now faced with

$$\int \frac{u}{(1-u^2)^2}\, du.$$

In either case, we do not immediately know of an appropriate antiderivative.

Checkpoint 4.6.3

Use substitution to evaluate each integral

a. $\int \frac{\cos t}{\sin^2 t}\, dt$
b. $\int \cos^3 t \sin t\ dt$

Answers available at the end of this section.

Sometimes, we need to manipulate an integral in more complicated ways than just multiplying or dividing by a constant. We need to eliminate all the expressions within the integrand that are in terms of the original variable. When done, u should be the only variable in the integrand. This sometimes means solving for the original variable in terms of u. This technique should become clear in the next example.

Example 4.6.4: Finding an Antiderivative Using u-Substitution

Use substitution to find the antiderivative of

$$\int \frac{x}{\sqrt{x-1}}\,dx.$$

Solution available at the end of this section.

The final example in this subsection illustrates a common tripping point for students—the dreaded substitution with resubstitution.

Example 4.6.5: Using Substitution with a Resubstitution

Evaluate.

$$\int 3x^{17}\sqrt[4]{4x^9+8}\,dx$$

Solution available at the end of this section.

Before moving away from Example 4.6.5, it's worthwhile to mention that we should simplify expressions when possible. Unfortunately, factoring out common expressions from the answer in Example 4.6.5 does not improve the look and feel of the solution.

$$\begin{aligned}
\frac{1}{12}\left(\frac{1}{9}(4x^9+8)^{9/4}-\frac{8}{5}(4x^9+8)^{5/4}\right)+C &= \frac{1}{12\cdot 9\cdot 5}(4x^9+8)^{5/4}\left(5(4x^9+8)-9\cdot 8\right)+C\\
&= \frac{1}{540}(4x^9+8)^{5/4}\left(20x^9+40-72\right)+C\\
&= \frac{1}{540}(4x^9+8)^{5/4}\left(20x^9-32\right)+C\\
&= \frac{4}{540}(4x^9+8)^{5/4}\left(5x^9-8\right)+C\\
&= \frac{1}{135}(4x^9+8)^{5/4}\left(5x^9-8\right)+C
\end{aligned}$$

Substitution for Definite Integrals

Substitution can be used with definite integrals, too. However, using substitution to evaluate a definite integral requires changing the limits of integration. If we change variables in the integrand, the limits of integration change as well.

Theorem: Substitution with Definite Integrals

Let $u=g(x)$ and let g' be continuous over an interval $[a,b]$, and let f be continuous over the range of $u=g(x)$. Then,

$$\int_a^b f(g(x))g'(x)\,dx=\int_{g(a)}^{g(b)} f(u)\,du.$$

Although we will not formally prove this theorem, we justify it with some calculations. From the Substitution Method for indefinite

integrals, if $F(x)$ is an antiderivative of $f(x)$, we have

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

Then

$$\begin{aligned}\int_a^b f[g(x)]g'(x)\,dx &= F(g(x))\Big|_{x=a}^{x=b} \\ &= F(g(b)) - F(g(a)) \\ &= F(u)\Big|_{u=g(a)}^{u=g(b)} \\ &= \int_{g(a)}^{g(b)} f(u)\,du\end{aligned}$$

and we have the desired result.

Example 4.6.6: Using Substitution to Evaluate a Definite Integral

Use substitution to evaluate

$$\int_0^1 x^2(1+2x^3)^5\,dx.$$

Solution available at the end of this section.

Checkpoint 4.6.6

Use substitution to evaluate the definite integral.

a. $\int_{-1}^{0} y(2y^2-3)^5\ dy$

b. $\int_0^1 x^2\cos\left(\frac{\pi}{2}x^3\right)\ dx$

Answers available at the end of this section.

Example 4.6.7: Using Substitution with an Exponential Function

Use substitution to evaluate

$$\int_0^1 xe^{4x^2+3}\,dx.$$

Solution available at the end of this section.

Caution: Change Limits of Integration!

When performing a substitution on a definite integral, it is critical that you *immediately* change the limits of integration as well.

Substitution may be only one of the techniques needed to evaluate a definite integral. All of the properties and rules of integration apply independently, and trigonometric functions may need to be rewritten using a trigonometric identity before we can apply substitution.

Example 4.6.8: Using Substitution to Evaluate a Trigonometric Integral

Use substitution to evaluate

$$\int_0^{\pi/2} \cos^2\theta\, d\theta.$$

Solution available at the end of this section.

Time-Sensitive Advice: Pass/No Pass Application Deadline

Within the Los Rios Community College District, the last day you can apply to switch the grading scheme on your class to Pass/No Pass is the last day of finals week.[1]

Why should you not apply for this for your math classes?

Generally, applying for a Pass/No Pass grade in mathematics looks sloppy on a transcript. When you apply for transfer to a four-year university, you are competing with thousands of other applicants, most of whom have letter grades. Individuals who exercise the Pass/No Pass grade scheme notoriously do so for fear of failing a class, and the application reviewers at four-year institutions know this! Do yourself a favor and stick with letter grades for math courses.

[1] Strangely, this was recently changed from Friday of the fifth week of class without any notice to faculty.

Success in College: Reflecting on the Course

1. Henry Ford said, "Failure is only the opportunity to begin again more intelligently." If you could start this course over, would you do anything differently? Explain.
2. Barack Obama said, "Making your mark on the world is hard. If it were easy, everybody would do it. But it's not. It takes patience, it takes commitment, and it comes with plenty of failure along the way. The real test is not whether you avoid this failure, because you won't. It's whether you let it harden or shame you into inaction, or whether you learn from it; whether you choose to persevere." Do you find it challenging to persevere, keep going, and get your college education? Explain.

Answers

Section Preview

You have arrived! This material **must** be given a **lot** of attention. It is the culmination of Differential Calculus, and students entering into Integral Calculus are assumed to have *mastered* this material. I know it might be too late, but you should reserve a week for this topic (just for students to practice in front of you and to challenge themselves).

Section Preview

- **Trigonometry**
  - Trigonometric Identities — Half-Angle and Power Reduction Identities: Example 8 rewrites $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ before substitution is even possible.

Section Preview

- Use substitution to evaluate indefinite integrals.
- Use substitution to evaluate definite integrals.

Theorem: Substitution Method with Indefinite Integrals

Let f, g, u, and F be as specified in the theorem. Then

$$\frac{d}{dx}\left[F(g(x))\right] = F'(g(x))g'(x) = f[g(x)]g'(x).$$

Integrating both sides with respect to x, we see that

$$\int f\left(g(x)\right) g'(x)\, dx = F(g(x)) + C.$$

If we now substitute $u = g(x)$, and $du = g'(x)\ dx$, we get

$$\int f\left(g(x)\right) g'(x)\, dx = \int f(u)\, du = F(u) + C = F(g(x)) + C.$$

Q.E.D.

Example 4.6.1**: Using Substitution to Find an Antiderivative**

The first step is to choose an expression for u. We choose $u = 3x^2 + 4$ because then $du = 6x\ dx$ and we already have du in the integrand. Write the integral in terms of u:

$$\int 6x(3x^2+4)^4\,dx = \int u^4\,du.$$

Remember that du is the derivative of the expression chosen for u, regardless of what is inside the integrand. Now we can evaluate the integral with respect to u:

$$\int u^4\,du = \frac{u^5}{5} + C = \frac{(3x^2+4)^5}{5} + C.$$

Analysis

We can check our answer by taking the derivative of the result of integration. We should obtain the integrand. Picking a value for C of 1, we let $y = \frac{1}{5}(3x^2+4)^5 + 1$. We have

$$y = \frac{1}{5}(3x^2+4)^5 + 1,$$

so

$$\begin{aligned} y' &= \left(\frac{1}{5}\right)5(3x^2+4)^4 6x \\ &= 6x(3x^2+4)^4. \end{aligned}$$

This is exactly the expression we started with inside the integrand.

Checkpoint 4.6.1

$\int 3x^2(x^3-3)^2\ dx = \frac{1}{3}(x^3-3)^3 + C$

Example 4.6.2**: Using Substitution with Alteration**

Rewrite the integral as $\int z(z^2-5)^{1/2}\ dz$. Let $u = z^2 - 5$ and $du = 2z\ dz$. We have a problem because $du = 2z\ dz$ and the original expression has only $z\ dz$. We must alter our expression for du, or the integral in u will be twice as large as it should be. If we multiply both sides of the du equation by $\frac{1}{2}$. we can solve this problem. Thus,

$$\begin{aligned} u &= z^2 - 5 \\ du &= 2z\,dz \\ \frac{1}{2}du &= \frac{1}{2}(2z)\,dz = z\,dz. \end{aligned}$$

Write the integral in terms of u, but pull the $\frac{1}{2}$ outside the integration symbol:

$$\int z(z^2-5)^{1/2}\,dz = \frac{1}{2}\int u^{1/2}\,du.$$

Integrate the expression in u:

$$\begin{aligned} \frac{1}{2}\int u^{1/2}\,du &= \left(\frac{1}{2}\right)\frac{u^{3/2}}{\frac{3}{2}} + C \\ &= \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)u^{3/2} + C \\ &= \frac{1}{3}u^{3/2} + C \\ &= \frac{1}{3}(z^2-5)^{3/2} + C \end{aligned}$$

Checkpoint 4.6.2

$\int x^2(x^3+5)^9\ dx = \frac{(x^3+5)^{10}}{30} + C$

Example 4.6.3**: Using Substitution with Integrals of Trigonometric Functions**

We know the derivative of $\cos t$ is $-\sin t$, so we set $u = \cos t$. Then $du = -\sin t\ dt$.

Substituting into the integral, we have

$$\int \frac{\sin t}{\cos^3 t}\,dt = -\int \frac{du}{u^3}.$$

Evaluating the integral, we get

$$-\int \frac{du}{u^3} = -\int u^{-3}\,du = -\left(-\frac{1}{2}\right)u^{-2} + C.$$

Putting the answer back in terms of t, we get

$$\int \frac{\sin t}{\cos^3 t}\,dt = \frac{1}{2u^2} + C = \frac{1}{2\cos^2 t} + C.$$

Checkpoint 4.6.3

a. $\int \frac{\cos t}{\sin^2 t}\ dt = -\frac{1}{\sin t} + C$

b. $\int \cos^3 t \sin t\ dt = -\frac{\cos^4 t}{4} + C$

Example 4.6.4**: Finding an Antiderivative Using u-Substitution**

If we let $u = x - 1$, then $du = dx$. But this does not account for the x in the numerator of the integrand. We must express x in terms of u. If $u = x - 1$, then $x = u + 1$. Now we can rewrite the integral in terms of u:

$$\int \frac{x}{\sqrt{x-1}}\,dx = \int \frac{u+1}{\sqrt{u}}\,du = \int \left(\sqrt{u} + \frac{1}{\sqrt{u}}\right)\,du = \int \left(u^{1/2} + u^{-1/2}\right)\,du.$$

Then we integrate in the usual way, replace u with the original expression, and factor and simplify the result. Thus,

$$\begin{aligned}
\int (u^{1/2} + u^{-1/2})\,du &= \frac{2}{3}u^{3/2} + 2u^{1/2} + C \\
&= \frac{2}{3}(x-1)^{3/2} + 2(x-1)^{1/2} + C \\
&= (x-1)^{1/2}\left[\frac{2}{3}(x-1) + 2\right] + C \\
&= (x-1)^{1/2}\left(\frac{2}{3}x - \frac{2}{3} + \frac{6}{3}\right) \\
&= (x-1)^{1/2}\left(\frac{2}{3}x + \frac{4}{3}\right) \\
&= \frac{2}{3}(x-1)^{1/2}(x+2) + C.
\end{aligned}$$

Example 4.6.5**: Using Substitution with a Resubstitution**

On first inspection, there doesn't seem to be an expression whose exact derivative (or constant multiple of the derivative) is contained within the integrand. However, practicing some faith, let's try the u-substitution, $u = 4x^9 + 8$. This would imply that $du = 36x^8\ dx$. Looking at the integrand, we don't have a $36x^8\ dx$ to "steal," but we *do* have $3x^8\ dx$. Thus, we let $\frac{1}{12}\ du = 3x^8\ dx$. Therefore,

$$\begin{aligned}
\int 3x^{17}\sqrt[4]{4x^9+8}\,dx &= \int x^9\sqrt[4]{4x^9+8}\cdot 3x^8\,dx \\
&= \frac{1}{12}\int x^9\sqrt[4]{u}\,du
\end{aligned}$$

At this point, I want to be clear that the notation in that final line is not traditionally acceptable. Our original integral was in terms of the variable x (and only that variable). The integral we arrived at in the last line *should* be in terms of only u; however, we have a pesky x^9 in there. Luckily, we know that $u = 4x^9 + 8$. Hence, $\frac{u-8}{4} = x^9$. Substituting (also known as resubstituting) this into the last line of the computation above, we get the following.

$$\begin{aligned}
\int 3x^{17}\sqrt[4]{4x^9+8}\,dx &= \frac{1}{12}\int x^9\sqrt[4]{u}\,du \\
&= \frac{1}{12}\int \frac{u-8}{4}u^{1/4}\,du \\
&= \frac{1}{48}\int (u-8)u^{1/4}\,du \\
&= \frac{1}{48}\int u^{5/4} - 8u^{1/4}\,du \\
&= \frac{1}{48}\left(\frac{4}{9}u^{9/4} - \frac{32}{5}u^{5/4}\right) + C \\
&= \frac{1}{12}\left(\frac{1}{9}u^{9/4} - \frac{8}{5}u^{5/4}\right) + C \\
&= \frac{1}{12}\left(\frac{1}{9}(4x^9+8)^{9/4} - \frac{8}{5}(4x^9+8)^{5/4}\right) + C
\end{aligned}$$

Example 4.6.6**: Using Substitution to Evaluate a Definite Integral**

Let $u = 1 + 2x^3$, so $du = 6x^2\ dx$. Since the original function includes one factor of x^2 and $du = 6x^2\ dx$, multiply both sides of the du equation by $1/6$. Then,

$$\begin{aligned}
du &= 6x^2\,dx \\
\implies \frac{1}{6}du &= x^2\,dx.
\end{aligned}$$

To adjust the limits of integration, note that when $x = 0, u = 1 + 2(0) = 1$, and when $x = 1,\ \ u = 1 + 2(1) = 3$.

Then

$$\int_0^1 x^2(1+2x^3)^5 dx = \frac{1}{6}\int_1^3 u^5\,du.$$

Evaluating this expression, we get

$$\begin{aligned}
\frac{1}{6}\int_1^3 u^5\,du &= \left(\frac{1}{6}\right)\left(\frac{u^6}{6}\right)\Big|_{u=1}^{u=3} \\
&= \frac{1}{36}\left[(3)^6 - (1)^6\right] \\
&= \frac{182}{9}.
\end{aligned}$$

Checkpoint 4.6.6

a. $\int_{-1}^{0} y(2y^2-3)^5\ dy = \dfrac{91}{3}$

b. $\int_{0}^{1} x^2 \cos\left(\dfrac{\pi}{2}x^3\right)\ dx = \dfrac{2}{3\pi} \approx 0.2122$

Example 4.6.7: Using Substitution with an Exponential Function

Let $u = 4x^3 + 3$. Then, $du = 8x\ dx$. To adjust the limits of integration, we note that when $x = 0,\ u = 3$, and when $x = 1,\ u = 7$. So our substitution gives

$$\begin{aligned} \int_0^1 xe^{4x^2+3}\,dx &= \frac{1}{8}\int_3^7 e^u\,du \\ &= \frac{1}{8}e^u\Big|_{u=3}^{u=7} \\ &= \frac{e^7-e^3}{8} \\ &\approx 134.568 \end{aligned}$$

Example 4.6.8: Using Substitution to Evaluate a Trigonometric Integral

First, use a trigonometric identity to rewrite the integral. The trigonometric identity $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ allows us to rewrite the integral as

$$\int_0^{\pi/2} \cos^2\theta\,d\theta = \int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\,d\theta.$$

Then,

$$\begin{aligned} \int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)\,d\theta &= \int_0^{\pi/2}\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)\,d\theta \\ &= \frac{1}{2}\int_0^{\pi/2}\,d\theta + \int_0^{\pi/2}\cos 2\theta\,d\theta. \end{aligned}$$

We can evaluate the first integral as it is. Still, we need to make a substitution to evaluate the second integral. Let $u = 2\theta$. Then, $du = 2\ d\theta$, or $\frac{1}{2}\ du = d\theta$. Also, when $\theta = 0,\ u = 0$, and when $\theta = \pi/2,\ u = \pi$. Expressing the second integral in terms of u, we have

$$\begin{aligned} \frac{1}{2}\int_0^{\pi/2}\,d\theta + \frac{1}{2}\int_0^{\pi/2}\cos 2\theta\,d\theta &= \frac{1}{2}\int_0^{\pi/2}\,d\theta + \frac{1}{2}\left(\frac{1}{2}\right)\int_0^{\pi}\cos u\,du \\ &= \frac{\theta}{2}\Big|_{\theta=0}^{\theta=\pi/2} + \frac{1}{4}\sin u\Big|_{u=0}^{u=\theta} \\ &= \left(\frac{\pi}{4}-0\right)+(0-0) \\ &= \frac{\pi}{4} \end{aligned}$$

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4.6.1: Resources and Key Concepts

Resources

Prerequisite Topics

The following is a list of prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Trigonometry**
  - Trigonometric Identities - Double-Angle Identities: May be needed to rewrite the integrand before or after substitution to make it integrable.
  - Trigonometric Identities - Half-Angle and Power Reduction Identities: May be needed to rewrite the integrand before or after substitution to make it integrable.

Videos

- Integration by substitution
  - The Substitution Rule - Concept, Theorem, and Proof
  - Evaluating Integrals Using the Substitution Rule
  - Quick Integration of Functions with Linear Arguments: Examples and Theory

Key Concepts

Definitions

- **Integration by Substitution (Substitution Method / u-Substitution)**: A method for finding antiderivatives (and evaluating definite integrals) by changing variables. If $u = g(x)$, then $du = g'(x)\ dx$, and the integral $\int f(g(x))g'(x)\ dx$ becomes $\int f(u)\ du$.
- **Change of Variables**: Another term for the substitution method, emphasizing that the variable of integration is changed (e.g., from x to u).
- **Resubstitution**: The process of substituting the original variable (e.g., x) back into the result after integrating with respect to the new variable (e.g., u) for indefinite integrals. For definite integrals, this step is avoided by changing the limits of integration.
- **Substitution with Alteration**: Adjusting constants in the integral if du doesn't perfectly match the remaining part of the integrand after choosing u. For example, if $u = g(x)$ and $du = c \cdot h(x)\ dx$, but the integral has $h(x)\ dx$, one substitutes $\frac{1}{c}du = h(x)\ dx$.

Theorems

- **Theorem: Substitution Method with Indefinite Integrals**: Let $u = g(x)$, where $g'(x)$ is continuous over an interval, let $f(x)$ be continuous over the corresponding range of g, and let $F(x)$ be an antiderivative of $f(x)$. Then, $\int f(g(x))g'(x)\ dx = \int f(u)\ du = F(u) + C = F(g(x)) + C$.
- **Theorem: Substitution with Definite Integrals**: Let $u = g(x)$ and let $g'(x)$ be continuous over an interval $[a, b]$, and let $f(x)$ be continuous over the range of $u = g(x)$. Then, $\int_a^b f(g(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du$.

Common Mistakes

- **Forgetting to Substitute dx with du (or an expression involving du)**: After choosing $u = g(x)$ and finding $du = g'(x)\ dx$, failing to replace all occurrences of x and dx with expressions involving u and du. The entire integral must be in terms of u and du before integrating.
- **Incorrectly Calculating du**: Errors in differentiating $g(x)$ to find du.
- **Not Changing Limits of Integration for Definite Integrals**: When performing u-substitution for a definite integral $\int_a^b f(g(x))g'(x)\ dx$, if the variable is changed to u, the limits of integration must also be changed from $x = a, x = b$ to $u = g(a), u = g(b)$. Failing to do this and substituting back x at the end with original limits is a common error, though it can sometimes yield the correct numerical answer if done carefully (but is procedurally incorrect and inefficient).
- **Mixing Variables**: Having both x and u (or dx and du) in the integral after substitution. The integral should be entirely in terms of u and du.
- **Forgetting the Constant of Integration $+C$ for Indefinite Integrals**.
- **Errors in Resubstitution**: After integrating with respect to u for an indefinite integral, making a mistake when substituting $g(x)$ back in for u.
- **Choosing an Inappropriate u**: Selecting a u such that du is not present (even with constant factor adjustments) or that doesn't simplify the integral.

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4.6.2: Homework

Reading Questions

1. What is the main goal of the integration by substitution method?
2. If we choose $u = g(x)$ for a substitution, what is du equal to?
3. When is integration by substitution particularly useful? What form of integrand are we often looking for?
4. After performing a u-substitution for an indefinite integral and finding the antiderivative in terms of u, what is the final step?
5. When performing a u-substitution for a definite integral $\int_a^b f(g(x))g'(x)\ dx$, what must be done to the limits of integration a and b?
6. In Example 2, where $u = z^2 - 5$ and $du = 2z\ dz$, but the integral had $z\ dz$, what "alteration" was made?
7. Is it permissible to have both x and u (or dx and du) in the integrand after making a substitution?
8. If, after choosing $u = g(x)$, there are still terms involving x in the integrand that are not part of du, what algebraic step might be necessary?
9. What is meant by "substitution with resubstitution" as illustrated in Example 5?
10. Why is it critical to change the limits of integration immediately when performing a substitution on a definite integral?

Homework

In exercises 1 - 10, find the antiderivatives for the given functions.

1) $\cosh(2x + 1)$

Answer

$\frac{1}{2}\sinh(2x + 1) + C$

2) $\tanh(3x + 2)$

3) $x\cosh(x^2)$

Answer

$\frac{1}{2}\sinh^2(x^2) + C$

4) $3x^3\tanh(x^4)$

5) $\cosh^2(x)\sinh(x)$

Answer

$\frac{1}{3}\cosh^3(x) + C$

6) $\tanh^2(x)\text{sech}^2(x)$

7) $\dfrac{\sinh(x)}{1 + \cosh(x)}$

Answer

$\ln(1 + \cosh(x)) + C$

8) $\coth(x)$

9) $\cosh(x) + \sinh(x)$

Answer

$\cosh(x) + \sinh(x) + C$

10) $(\cosh(x)+\sinh(x))^n$

In exercises 11 - 17, find the antiderivatives for the functions.

11) $\int \frac{dx}{4-x^2}$

12) $\int \frac{dx}{a^2-x^2}$

Answer

$$\frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)+C$$

13) $\int \frac{dx}{\sqrt{x^2+1}}$

14) $\int \frac{x\,dx}{\sqrt{x^2+1}}$

Answer

$$\sqrt{x^2+1}+C$$

15) $\int -\frac{dx}{x\sqrt{1-x^2}}$

16) $\int \frac{e^x\,dx}{\sqrt{e^{2x}-1}}$

Answer

$$\cosh^{-1}(e^x)+C$$

17) $\int -\frac{2x}{x^4-1}\,dx$

18) Why is u-substitution referred to as a *change of variable*?

19) If $f=g\circ h$, when reversing the chain rule, $\frac{d}{dx}(g\circ h)(x)=g'(h(x))h'(x)$, should you take $u=g(x)$ or $u=h(x)$?

Answer

$$u=h(x)$$

In exercises 20 - 24, verify each identity using differentiation. Then, using the indicated u-substitution, identify f such that the integral takes the form $\int f(u)\,du$.

20) $\int x\sqrt{x+1}\,dx=\frac{2}{15}(x+1)^{3/2}(3x-2)+C;\quad u=x+1$

21) $\int \frac{x^2}{\sqrt{x-1}}\,dx=\frac{2}{15}\sqrt{x-1}(3x^2+4x+8)+C,\quad (x>1);\quad u=x-1$

Answer

$$f(u)=\frac{(u+1)^2}{\sqrt{u}}$$

22) $\int x\sqrt{4x^2+9}\,dx=\frac{1}{12}(4x^2+9)^{3/2}+C;\quad u=4x^2+9$

23) $\int \frac{x}{\sqrt{4x^2+9}}\,dx=\frac{1}{4}\sqrt{4x^2+9}+C;\quad u=4x^2+9$

Answer

$$du = 8x\ dx; \quad f(u) = \frac{1}{8\sqrt{u}}$$

24) $\int \frac{x}{(4x^2+9)^2}\ dx = -\frac{1}{8(4x^2+9)} + C; \quad u = 4x^2 + 9$

In exercises 25 - 34, find the antiderivative using the indicated substitution.

25) $\int (x+1)^4\ dx; \quad u = x + 1$

Answer

$$\int (x+1)^4\ dx = \frac{1}{5}(x+1)^5 + C$$

26) $\int (x-1)^5\ dx; \quad u = x - 1$

27) $\int (2x-3)^{-7}\ dx; \quad u = 2x - 3$

Answer

$$\int (2x-3)^{-7}\ dx = -\frac{1}{12(2x-3)^6} + C$$

28) $\int (3x-2)^{-11}\ dx; \quad u = 3x - 2$

29) $\int \frac{x}{\sqrt{x^2+1}}\ dx; \quad u = x^2 + 1$

Answer

$$\int \frac{x}{\sqrt{x^2+1}}\ dx = \sqrt{x^2+1} + C$$

30) $\int \frac{x}{\sqrt{1-x^2}}\ dx; \quad u = 1 - x^2$

31) $\int (x-1)(x^2-2x)^3\ dx; \quad u = x^2 - 2x$

Answer

$$\int (x-1)(x^2-2x)^3\ dx = \frac{1}{8}(x^2-2x)^4 + C$$

32) $\int (x^2-2x)(x^3-3x^2)^2\ dx; \quad u = x^3 = 3x^2$

33) $\int \cos^3\theta\ d\theta; \quad u = \sin\theta$ (Hint: $\cos^2\theta = 1 - \sin^2\theta$)

Answer

$$\int \cos^3\theta\ d\theta = \sin\theta - \frac{\sin^3\theta}{3} + C$$

34) $\int \sin^3\theta\ d\theta; \quad u = \cos\theta$ (Hint: $\sin^2\theta = 1 - \cos^2\theta$)

In exercises 35 - 51, use a suitable change of variables to determine the indefinite integral.

35) $\int x(1-x)^{99}\ dx$

Answer

$$\int x(1-x)^{99}\,dx = \frac{(1-x)^{101}}{101} - \frac{(1-x)^{100}}{100} + C$$
$$= -\frac{(1-x)^{100}}{10100}\left[100x+1\right] + C$$

36) $\int t(1-t^2)^{10}\ dt$

37) $\int (11x-7)^{-3}\ dx$

Answer

$$\int (11x-7)^{-3}\ dx = -\frac{1}{22(11x-7)^2} + C$$

38) $\int (7x-11)^4\ dx$

39) $\int \cos^3\theta \sin\theta\ d\theta$

Answer

$$\int \cos^3\theta \sin\theta\ d\theta = -\frac{\cos^4\theta}{4} + C$$

40) $\int \sin^7\theta \cos\theta\ d\theta$

41) $\int \cos^2(\pi t)\sin(\pi t)\ dt$

Answer

$$\int \cos^2(\pi t)\sin(\pi t)\ dt = -\frac{cos^3(\pi t)}{3\pi} + C$$

42) $\int \sin^2 x \cos^3 x\ dx$ (Hint: $\sin^2 x + \cos^2 x = 1$)

43) $\int t\sin(t^2)\cos(t^2)\ dt$

Answer

$$\int t\sin(t^2)\cos(t^2)\ dt = -\frac{1}{4}\cos^2(t^2) + C$$

44) $\int t^2\cos^2(t^3)\sin(t^3)\ dt$

45) $\int \frac{x^2}{(x^3-3)^2}\ dx$

Answer

$$\int \frac{x^2}{(x^3-3)^2}\ dx = -\frac{1}{3(x^3-3)} + C$$

46) $\int \frac{x^3}{\sqrt{1-x^2}}\ dx$

47) $\int \frac{y^5}{(1-y^3)^{3/2}}\ dy$

Answer

$$\int \frac{y^5}{(1-y^3)^{3/2}}\ dy = -\frac{2(y^3-2)}{3\sqrt{1-y^3}} + C$$

48) $\int \cos\theta(1-\cos\theta)^{99}\sin\theta\ d\theta$

49) $\int (1-\cos^3\theta)^{10}\cos^2\theta\sin\theta\ d\theta$

Answer

$$\int (1-\cos^3\theta)^{10}\cos^2\theta\sin\theta\ d\theta = \frac{1}{33}(1-\cos^3\theta)^{11} + C$$

50) $\int (\cos\theta - 1)(\cos^2\theta - 2\cos\theta)^3\sin\theta\ d\theta$

51) $\int (\sin^2\theta - 2\sin\theta)(\sin^3\theta - 3\sin^2\theta)^3\cos\theta\ d\theta$

Answer

$$\int (\sin^2\theta - 2\sin\theta)(\sin^3\theta - 3\sin^2\theta)^3\cos\theta\ d\theta = \frac{1}{12}(\sin^3\theta - 3\sin^2\theta)^4 + C$$

In exercises 52 - 55, use a calculator to estimate the area under the curve using left Riemann sums with 50 terms, then use substitution to solve for the exact answer.

52) [Technology Required] $y = 3(1-x)^2$ over $[0, 2]$

53) [Technology Required] $y = x(1-x^2)^3$ over $[-1, 2]$

Answer

$L_{50} = -8.5779$. The exact area is $\frac{-81}{8}$ units^2.

54) [Technology Required] $y = \sin x(1-\cos x)^2$ over $[0, \pi]$

55) [Technology Required] $y = \dfrac{x}{(x^2+1)^2}$ over $[-1, 1]$

Answer

$L_{50} = -0.006399$. The exact area is 0.

In exercises 56 - 61, use a change of variables to evaluate the definite integral.

56) $\int_0^1 x\sqrt{1-x^2}\ dx$

57) $\int_0^1 \dfrac{x}{\sqrt{1+x^2}}\ dx$

Answer

$$u = 1+x^2, \quad du = 2x\ dx, \quad \int_0^1 \frac{x}{\sqrt{1+x^2}}\ dx = \frac{1}{2}\int_1^2 u^{-1/2}du = \sqrt{2}-1$$

58) $\int_0^2 \dfrac{t}{\sqrt{5+t^2}}\ dt$

59) $\int_0^1 \dfrac{t^2}{\sqrt{1+t^3}}\ dt$

Answer

$$u = 1 + t^3, \quad du = 3t^2, \quad \int_0^1 \frac{t^2}{\sqrt{1+t^3}}\ dt = \frac{1}{3}\int_1^2 u^{-1/2} du = \frac{2}{3}(\sqrt{2} - 1)$$

60) $\int_0^{\pi/4} \sec^2\theta \tan\theta\ d\theta$

61) $\int_0^{\pi/4} \frac{\sin\theta}{\cos^4\theta}\ d\theta$

Answer

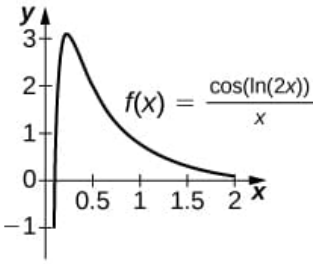
$$u = \cos\theta, \quad du = -\sin\theta\ d\theta, \quad \int_0^{\pi/4} \frac{\sin\theta}{\cos^4\theta}\ d\theta = -\int_1^{\sqrt{2}/2} u^{-4}\ du = \int_{\sqrt{2}/2}^1 u^{-4}\ du = \frac{1}{3}(2\sqrt{2} - 1)$$

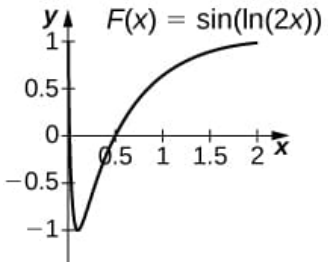
In exercises 62 - 67, evaluate the indefinite integral $\int f(x)\ dx$ with constant $C = 0$ using u-substitution. Then, graph the function and the antiderivative over the indicated interval. If possible, estimate a value of C that would need to be added to the antiderivative to make it equal to the definite integral $F(x) = \int_a^x f(t)\ dt$, with a the left endpoint of the given interval.

62) [Technology Required] $\int (2x+1)e^{x^2+x-6}\ dx$ over $[-3, 2]$

63) [Technology Required] $\int \frac{\cos(\ln(2x))}{x}\ dx$ on $[0, 2]$

Answer



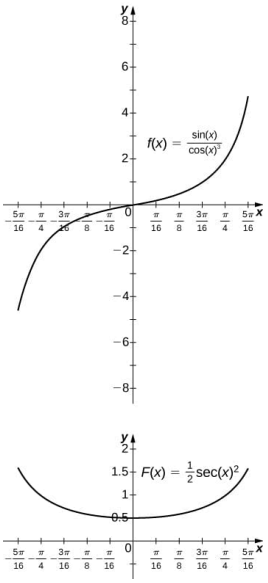


The antiderivative is $y = \sin(\ln(2x))$. Since the antiderivative is not continuous at $x = 0$, one cannot find a value of C that would make $y = \sin(\ln(2x)) - C$ work as a definite integral.

64) [Technology Required] $\int \frac{3x^2 + 2x + 1}{\sqrt{x^3 + x^2 + x + 4}}\ dx$ over $[-1, 2]$

65) [Technology Required] $\int \frac{\sin x}{\cos^3 x}\ dx$ over $[-\frac{\pi}{3}, \frac{\pi}{3}]$

Answer

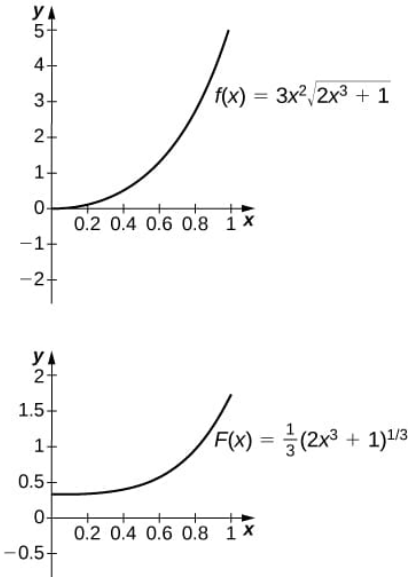


The antiderivative is $y = \frac{1}{2}\sec^2 x$. You should take $C = -2$ so that $F(-\frac{\pi}{3}) = 0$.

66) [Technology Required] $\int (x+2)e^{-x^2-4x+3}\ dx$ over $[-5, 1]$

67) [Technology Required] $\int 3x^2\sqrt{2x^3+1}\ dx$ over $[0, 1]$

Answer



The antiderivative is $y = \frac{1}{3}(2x^3+1)^{3/2}$. One should take $C = -\frac{1}{3}$.

68) If $h(a) = h(b)$ in $\int_a^b g'(h(x))h(x)\ dx$, what can you say about the value of the integral?

69) Is the substitution $u = 1 - x^2$ in the definite integral $\int_0^2 \frac{x}{1-x^2}\ dx$ okay? If not, why not?

Answer

No, because the integrand is discontinuous at $x = 1$.

In exercises 70 - 76, use a change of variables to show that each definite integral is equal to zero.

70) $\int_0^{\pi} \cos^2(2\theta)\sin(2\theta)\ d\theta$

71) $\int_0^{\sqrt{\pi}} t\cos(t^2)\sin(t^2)\ dt$

Answer

$u = \sin(t^2)$; the integral becomes $\frac{1}{2}\int_0^0 u\ du$.

72) $\int_0^1 (1-2t)\ dt$

73) $\int_0^1 \frac{1-2t}{1+(t-\frac{1}{2})^2}\ dt$

Answer

$u = 1 + (t-\frac{1}{2})^2$; the integral becomes $-\int_{5/4}^{5/4} \frac{1}{u}\ du$.

74) $\int_0^{\pi} \sin\left((t-\frac{\pi}{2})^3\right)\cos\left(t-\frac{\pi}{2}\right)\ dt$

75) $\int_0^2 (1-t)\cos(\pi t)\ dt$

Answer

$u = 1 - t$; Since the integrand is odd, the integral becomes

$$\int_1^{-1} u\cos\left(\pi(1-u)\right)du = \int_1^{-1} u[\cos\pi\cos u - \sin\pi\sin u]\,du = -\int_1^{-1} u\cos u\,du = \int_{-1}^{1} u\cos u\,du = 0$$

76) $\int_{\pi/4}^{3\pi/4} \sin^2 t\cos t\ dt$

77) Show that the average value of $f(x)$ over an interval $[a,b]$ is the same as the average value of $f(cx)$ over the interval $[\frac{a}{c},\frac{b}{c}]$ for $c > 0$.

Answer

Setting $u = cx$ and $du = c\ dx$ gets you $\frac{1}{\frac{b}{c}-\frac{a}{c}}\int_{a/c}^{b/c} f(cx)\ dx = \frac{c}{b-a}\int_{u=a}^{u=b} f(u)\frac{du}{c} = \frac{1}{b-a}\int_a^b f(u)\ du$.

78) Find the area under the graph of $f(t) = \frac{t}{(1+t^2)^a}$ between $t = 0$ and $t = x$ where $a > 0$ and $a \neq 1$ is fixed, and evaluate the limit as $x \to \infty$.

79) Find the area under the graph of $g(t) = \frac{t}{(1-t^2)^a}$ between $t = 0$ and $t = x$, where \(00\) is fixed. Evaluate the limit as $x \to 1$.

Answer

$\int_0^x g(t)\ dt = \frac{1}{2}\int_{u=1-x^2}^{1} \frac{du}{u^a} = \frac{1}{2(1-a)}u^{1-a} \mid 1u = \frac{1}{2(1-a)}(1-(1-x^2)^{1-a})$ As $x \to 1$ the limit is $\frac{1}{2(1-a)}$ if $a < 1$, and the limit diverges to $+\infty$ if $a > 1$.

80) The area of a semicircle of radius 1 can be expressed as $\int_{-1}^{1}\sqrt{1-x^2}\ dx$. Use the substitution $x = \cos t$ to express the area

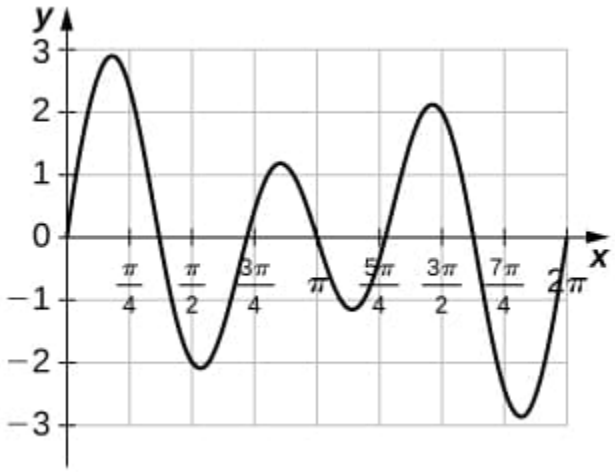
of a semicircle as the integral of a trigonometric function. You do not need to compute the integral.

81) The area of the top half of an ellipse with a major axis that is the x-axis from $x = -1$ to a and with a minor axis that is the y-axis from $y = -b$ to $y = b$ can be written as $\int_{-a}^{a} b\sqrt{1 - \frac{x^2}{a^2}}\ dx$. Use the substitution $x = a\cos t$ to express this area in terms of an integral of a trigonometric function. You do not need to compute the integral.

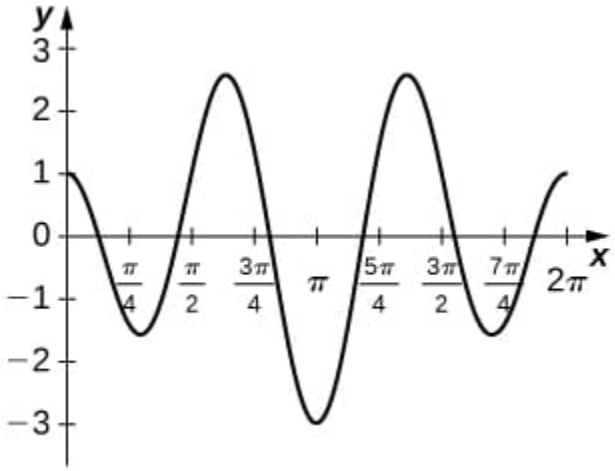
Answer

$$\int_{t=\pi}^{t=0} b\sqrt{1 - \cos^2 t} \times (-a\sin t)\ dt = \int_{t=0}^{t=\pi} ab\sin^2 t\ dt$$

82) [Technology Required] The following graph is of a function of the form $f(t) = a\sin(nt) + b\sin(mt)$. Estimate the coefficients a and b and the frequency parameters n and m. Use these estimates to approximate $\int_0^{\pi} f(t)\ dt$.



83) [Technology Required] The following graph is of a function of the form $f(x) = a\cos(nt) + b\cos(mt)$. Estimate the coefficients a and b and the frequency parameters n and m. Use these estimates to approximate $\int_0^{\pi} f(t)\ dt$.



Answer

$$f(t) = 2\cos(3t) - \cos(2t); \qquad \int_0^{\pi/2} (2\cos(3t) - \cos(2t))\ dt = -\frac{2}{3}$$

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5: APPENDIX

Calculus is the mathematics that describes changes in functions. In this chapter, we review all the functions necessary to study calculus. We define polynomial, rational, trigonometric, exponential, and logarithmic functions. We review how to evaluate these functions, and we show the properties of their graphs. We provide examples of equations with terms involving these functions and illustrate the algebraic techniques necessary to solve them. In short, this chapter provides the foundation for the material to come. It is essential to be familiar and comfortable with these ideas before proceeding to the formal introduction of calculus in the next chapter.

Caution

Due to time constraints, many professors skip most of the material in this chapter and leave it to you to review on your own. This is not done because your professor is lazy. The reality is that you are expected to have mastery of all topics in elementary algebra (Algebra 1), intermediate algebra (Algebra 2), college algebra, and trigonometry prior to enrolling in a calculus course.

If you have not had a full course in trigonometry, it is highly recommended that you enroll in one at your college and save calculus for next semester. I cannot emphasize this enough.

5.1: A.1- Algebraic Manipulations Critical for Calculus
5.1.1: A.1.1- Homework

5.2: A.2- Functions
5.2.1: A.2.1- Homework

5.3: A.3- Basic Classes of Functions and Their Graphs
5.3.1: A.3.1- Homework

5.4: A.4- Inverse Functions
5.4.1: A.4.1- Homework

5.5: A.5- Exponential and Logarithmic Functions
5.5.1: A.5.1- Homework

5.6: A.6- Trigonometry
5.6.1: A.6.1- Homework

5.7: A.7- Tangent Lines and Velocity
5.7.1: A.7.1- Homework

5.8: A.8- Answers to Selected Exercises
5.8.1: A.8.1- Chapter 1 Answers

5.8.2: Chapter 2 Answers

5.8.3: Chapter 3 Answers

5.8.4: Chapter 4 Answers

5.9: Diagnostic Examples
5.9.1: Chapter 1 Diagnostics

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5.1: A.1- Algebraic Manipulations Critical for Calculus

Learning Objectives

- State the Mathematical Mantra.
- Briefly review the following topics from Algebra:
  - nontrivial factoring techniques
  - understand when to distribute and, just as important, when not to distribute
  - simplifying compound fractions
  - rationalizing numerators and denominators
  - using the piecewise definition of the absolute value

It is crucial to note that the material reviewed in this section is not instruction. If you have never seen some of the topics being covered here, you should speak with your instructor about course alternatives that will better set you up for success in Calculus.[1]

The Mathematical Mantra

The following "mantra" is very helpful throughout all levels of mathematics, and significantly more so in Calculus.

Mathematical Mantra

Perform mathematics in the order you learned mathematics. That is, perform

Arithmetic before Algebra,

Algebra before Trigonometry,

Trigonometry before Precalculus, and

Precalculus before Differential Calculus.

Always be willing to ask yourself, "Is there something I learned previously that can simplify my work here?"

We will reference this mantra[2] continuously throughout this course.

Nontrivial Factoring Techniques

Each of the following factoring techniques will be required at many different points throughout Calculus. They will often be required as peripheral skills while working on homework, quizzes, and exams. As with all material in this section, it is assumed you have mastered these skills in your prerequisite courses; however, it is presented here as a gentle reminder for those who might have been away from their studies in mathematics for a while.

Sums and Differences of Cubes

Theorem: Sums and Differences of Cubes

$$F^3 \pm L^3 = (F \pm L)(F^2 \mp FL + L^2) \tag{5.1.1}$$

Example 5.1.1: Factor a Difference of Cubes

You are working on a complex Calculus problem during an exam and arrive at the following expression.

$$\frac{e^{6x}-8}{e^{2x}-2}$$

You are told that $x \neq \ln\left(2^{1/2}\right)$.

You cannot move forward on the problem without somehow removing that denominator (this is something that will be necessary, for reasons to be revealed later in the course). Simplify the expression to an equivalent expression not involving the denominator $e^{2x}-2$.

Solution available at the end of this section.

Factoring the GCF Out of Expressions Involving Negative Rational Exponents

One of the more common factoring techniques required in Calculus comes up when the result of some Calculus manipulations leads to an expression of the form

$$3(1+x)^{1/3}-x(1+x)^{-2/3}.$$

It will be necessary to factor this expression. But how?

When all terms of an algebraic expression share a common factor, but the common factor has different powers in each term, factor out the *lowest-powered* version of the shared factor.

Example 5.1.2: Factor Out a Common Expression Having a Negative Rational Exponent

It's the third exam in your Differential Calculus[4] course and you are trying to use Calculus to graph the function

$$f(x)=\frac{3}{2}(1+x)^{4/3}+3\,(1+x)^{1/3}+2.$$

You have done some great work so far, but at some point you arrive at the expression

$$3(1+x)^{1/3}-x(1+x)^{-2/3},$$

and you *need* to factor this to make the rest of your mathematics easy.

Solution available at the end of this section.

Knowing When to Distribute

Factoring is the ultimate end game for a lot of mathematics.[5] Therefore, if you have an expression that is already factored, then distributing things out is *almost* always a bad idea. However, in Calculus, there are going to be many times when you are going to multiply the numerator and denominator of a given expression by a fraction equivalent to 1 and, in doing so, you will need to "unfactor," or distribute. For example, you might be multiplying both the numerator and denominator by the LCD of all fractions within a compound fraction,[6] or you might be multiplying the numerator and denominator by the conjugate of the numerator.

In these cases, you are faced with a simple decision - do you distribute *everything* out? Do you only distribute out the numerator? Do you only distribute out the denominator? Do you leave things factored?

The following is a good rule of thumb to follow:

Rule of Thumb

When you have no choice but to multiply the numerator and denominator of a rational expression by either

a. a conjugate (to clear radicals), or
b. an LCD (to clear a compound fraction),

only perform distribution on the part (numerator or denominator) that caused the need for such an operation. That is, do not use distribution on the non-offending piece.

That "Rule of Thumb" probably doesn't make sense, so we will clarify it in Example 5.1.3.

Simplifying Compound Fractions

The first half of this course will be littered with "fractions containing fractions" - more commonly known as **compound fractions** (or **complex fractions**, in some texts). For example, it is inevitable that you will have to deal with a compound fraction of the form

$$\frac{(1+x+h)^{-1}-(1+x)^{-1}}{h}.$$

When faced with such an expression, you need to find the LCD of all fractions in the numerator and denominator. If you multiply both numerator and denominator of the compound fraction by this LCD, the entire compound fraction will "unravel."

Example 5.1.3: Simplify a Compound Rational Expression

On your first Calculus exam, you end up having to work with the following rational expression.

$$\frac{(1+x+h)^{-1}-(1+x)^{-1}}{h}$$

However, you cannot move past the first step of the given exam problem without having to simplify the compound rational expression (a fact that will *not* be explicitly stated).

Solution available at the end of this section.

Rationalizing Numerators and Denominators

As with simplifying compound fractions, you will need to instinctively know when to rationalize a numerator or denominator at many points throughout this course (without explicit instruction to do so from your instructor). The "Rule of Thumb" mentioned previously on knowing when to distribute and when not to distribute will be helpful here.

Example 5.1.4: Rationalize to Avoid Division By Zero

Early on in this course you will be working with expressions like

$$-\frac{\dfrac{1}{\sqrt{x}}-\dfrac{1}{\sqrt{x+h}}}{h}.$$

You must immediately identify the need to simplify the compound fraction. Moreover, the h in the denominator is going to be a problem (the reason for which will be discussed in Chapter 2). Simplify this expression and try to remove h from the denominator.

Solution available at the end of this section.

Using the Piecewise Definition of the Absolute Value

This topic is so important that it is covered *twice* in this chapter.

When we first learn of the absolute value, it is often from a flawed perspective. We are told, "The absolute value makes things positive," and, in essence, it does. It forces -3 to become 3 and π to stay π, but what about algebraic expressions like $-3x+\pi$? Since this expression has an unknown, there is no way of knowing whether or not this expression is negative. This is where knowledge of the **piecewise definition of the absolute value** (also known as the true definition of the absolute value) comes into play!

$$|\square| = \begin{cases} \square & \text{if } \square \geq 0 \\ -\square & \text{if } \square < 0 \end{cases}$$

In language, this is saying that the absolute value of $\square$ has two possibilities:

1. If $\square$ is already non-negative, then there is no need for the absolute values.
2. If $\square$ is negative, then the absolute value will force it to become positive by returning the *opposite* of $\square$.

Caution: Pay Attention to the Definition

The part people trip up on constantly are the inequalities. They assume the absolute value returns $\square$ if $x \geq 0$ and $-\square$ if $x < 0$. **This is not true!**

Example 5.1.5: Using the Piecewise Definition of the Absolute Value

You are diligently working through your first exam and are given a problem where you know that $x < 2$ and you need to simplify

$$\frac{x^2+x-6}{|x-2|}.$$

Solution available at the end of this section.

Caution

$$|\square| \text{ is not necessarily equal to } \square$$

Therefore,

$$\frac{|\square|}{\square} \text{ is not necessarily } 1.$$

Footnotes

[1] Trust me when I say it is better to step back to a previous course to fill in knowledge gaps than start a cycle of failure in Calculus.

[2] The Mathematical Mantra is a "rule of thumb" and, as such, there are times when it is not applicable. In such cases, I will endeavor to point out why it does not apply.

[3] All examples in this section will have step-by-step justifications; however, as we move forward in the text, the justifications occur less frequently. At some point, several elementary steps will be skipped between steps. As you can see by the justifications listed here, your success in Calculus is built on a solid foundation of Arithmetic, Algebra, and Trigonometry.

[4] "Differential Calculus" is another name for "Calculus I."

[5] As is often the case with absolute statements like this, there are exceptions.

[6] Also known as a complex fraction.

[7] In the distribution step (third line), notice that we did *not* distribute the denominator. This was because the entire reason we were forced to multiply by the LCD was the fractions in the *numerator*. Remember the Rule of Thumb: do not use distribution on the non-offending piece.

[8] In this distribution step, notice that we did *not* (and, technically, could not) distribute the denominator. The entire reason we were forced to multiply by the LCD was the fractions in the *numerator*. Remember the Rule of Thumb: do not use distribution on the non-offending piece.

[9] In this distribution step, notice that we did *not* distribute the denominator. The entire reason we were forced to multiply by the conjugate was the difference of radicals in the *numerator*. Remember the Rule of Thumb: do not use distribution on the non-offending piece.

Answers

Example 5.1.1**: Factor a Difference of Cubes**

Note that[3]

$$\begin{aligned} x \neq \ln\left(2^{1/2}\right) &\implies x \neq \frac{1}{2}\ln(2) && \text{(Algebra: Laws of Logarithms)} \\ &\implies 2x \neq \ln(2) && \text{(Algebra: Multiply both sides of an equation by a nonzero constant)} \\ &\implies e^{2x} \neq 2 && \text{(Algebra: Logarithms and exponentials are inverses)} \\ &\implies e^{2x}-2 \neq 0 && \text{(Algebra: Subtract a constant from both sides of an equation)} \end{aligned}$$

Therefore, we know the denominator of

$$\frac{e^{6x}-8}{e^{2x}-2}$$

is nonzero. Moreover, we see that the numerator is a difference of cubes. Thus, using the difference of cubes formula (Equation 5.1.1) we get

$$\begin{aligned} \frac{e^{6x}-8}{e^{2x}-2} &= \frac{\left(e^{2x}\right)^3-(2)^3}{e^{2x}-2} && \text{(Algebra: Laws of Exponents)} \\ &= \frac{\left(e^{2x}-2\right)\left(\left(e^{2x}\right)^2+2e^{2x}+(2)^2\right)}{e^{2x}-2} && \text{(Algebra: Difference of Cubes)} \\ &= \frac{\left(e^{2x}-2\right)\left(e^{4x}+2e^{2x}+4\right)}{e^{2x}-2} && \text{(Algebra: Laws of Exponents)} \\ &= \frac{\cancel{\left(e^{2x}-2\right)}\left(e^{4x}+2e^{2x}+4\right)}{\cancel{\left(e^{2x}-2\right)}} && \text{(Algebra: Cancel like factors)} \\ &= e^{4x}+2e^{2x}+4 \end{aligned}$$

Example 5.1.2**: Factor Out a Common Expression Having a Negative Rational Exponent**

Both terms in the expression share the factor $(1+x)$; however, the second term has the smallest-powered version of this factor. Therefore, we factor out $(1+x)^{-2/3}$.

$$\begin{aligned} 3(1+x)^{1/3}-x(1+x)^{-2/3} &= (1+x)^{-2/3}\left(\frac{3(1+x)^{1/3}}{(1+x)^{-2/3}}-\frac{x(1+x)^{-2/3}}{(1+x)^{-2/3}}\right) && \text{(Algebra: Factoring out the GCF is division)} \\ &= (1+x)^{-2/3}\left(3(1+x)^{1/3+2/3}-x(1+x)^{-2/3+2/3}\right) && \text{(Algebra: Laws of Exponents)} \\ &= (1+x)^{-2/3}\left(3(1+x)-x(1+x)^0\right) && \text{(Arithmetic)} \\ &= (1+x)^{-2/3}\left(3(1+x)-x\right) && \text{(Algebra: Laws of Exponents)} \\ &= (1+x)^{-2/3}(3+2x) && \text{(Algebra: Distribution and combining like terms)} \end{aligned}$$

Example 5.1.3**: Simplify a Compound Rational Expression**

$$\begin{aligned}
\frac{(1+x+h)^{-1}-(1+x)^{-1}}{h} &= \frac{\dfrac{1}{1+x+h}-\dfrac{1}{1+x}}{h} && \text{(Algebra: Laws of Exponents)}\\
&= \frac{\left(\dfrac{1}{1+x+h}-\dfrac{1}{1+x}\right)}{h}\cdot\frac{(1+x+h)(1+x)}{(1+x+h)(1+x)} && \text{(Algebra: Multiply numerator and denominator by the LCD)}\\
&= \frac{\dfrac{(1+x+h)(1+x)}{1+x+h}-\dfrac{(1+x+h)(1+x)}{1+x}}{h(1+x+h)(1+x)} && \left(\text{Algebra: Distribute}^{7}\right)\\
&= \frac{\dfrac{\cancel{(1+x+h)}\,(1+x)}{\cancel{(1+x+h)}}-\dfrac{(1+x+h)\,\cancel{(1+x)}}{\cancel{(1+x)}}}{h(1+x+h)(1+x)} && \text{(Algebra: Cancel like factors)}\\
&= \frac{(1+x)-(1+x+h)}{h(1+x+h)(1+x)}\\
&= \frac{1+x-1-x-h}{h(1+x+h)(1+x)} && \text{(Algebra: Distribution)}\\
&= \frac{-h}{h(1+x+h)(1+x)} && \text{(Algebra: Combine like terms)}\\
&= \frac{-\cancel{h}}{\cancel{h}(1+x+h)(1+x)} && \text{(Algebra: Cancel like factors)}\\
&= \frac{-1}{(1+x+h)(1+x)} && \text{(Algebra: Cancel like factors)}
\end{aligned}$$

Example 5.1.4**: Rationalize to Avoid Division By Zero**

The first step to simplifying this expression is to simplify the compound fraction.

$$\begin{aligned}
-\frac{\dfrac{1}{\sqrt{x}}-\dfrac{1}{\sqrt{x+h}}}{h} &= -\frac{\left(\dfrac{1}{\sqrt{x}}-\dfrac{1}{\sqrt{x+h}}\right)}{h}\cdot\frac{\sqrt{x}\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}} && \text{(Algebra: Multiply numerator and denominator by the LCD)}\\
&= -\frac{\sqrt{x+h}-\sqrt{x}}{h\sqrt{x}\sqrt{x+h}} && \left(\text{Algebra: Distribute}^{8}\right)\\
&= \frac{-\sqrt{x+h}+\sqrt{x}}{h\sqrt{x}\sqrt{x+h}} && \text{(Algebra: Distribute)}\\
&= \frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} && \text{(Arithmetic: Commutative property of addition)}
\end{aligned}$$

Now that the compound fraction has been simplified, we are faced with an expression that *still* has the h in the denominator. Remember, there is a hidden, underlying reason that will compel us to somehow remove that through creative uses of Algebra. One such use is conjugate multiplication!

$$\begin{aligned}
\frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} &= \frac{\left(\sqrt{x}-\sqrt{x+h}\right)}{h\sqrt{x}\sqrt{x+h}}\cdot\frac{\left(\sqrt{x}+\sqrt{x+h}\right)}{\left(\sqrt{x}+\sqrt{x+h}\right)} && \text{(Algebra: Multiply numerator and denominator by the conjugate of the numerator)}\\
&= \frac{x-(x+h)}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)} && \left(\text{Algebra: Distribute}^{9}\right)\\
&= \frac{-h}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)} && \text{(Algebra: Distribute)}\\
&= \frac{-\cancel{h}}{\cancel{h}\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)} && \text{(Algebra: Cancel like factors)}\\
&= \frac{-1}{\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}
\end{aligned}$$

Our final expression is no longer a compound fraction nor does it have that pesky h in the denominator!

Example 5.1.5**: Using the Piecewise Definition of the Absolute Value**

$$\frac{x^2+x-6}{|x-2|} = \frac{(x+3)(x-2)}{|x-2|} \qquad \text{(Algebra: Factoring)}$$

At this point, canceling $(x-2)$ and $|x-2|$ is impossible - you can only cancel like factors and these expressions are not exactly the same. Using the piecewise definition of the absolute value, we see

$$|x-2| = \begin{cases} x-2 & \text{if } x-2 \geq 0 \implies x \geq 2 \\ -(x-2) & \text{if } x-2 < 0 \implies x < 2 \end{cases}$$

Luckily, we know that $x < 2$, so we can replace $|x-2|$ with $-(x-2)$. Therefore,

$$
\begin{aligned}
\frac{x^2+x-6}{|x-2|} &= \frac{(x+3)(x-2)}{|x-2|} && \text{(Algebra: Factoring)} \\
&= \frac{(x+3)(x-2)}{-(x-2)} && \text{(Algebra: Piecewise definition of the absolute value)} \\
&= \frac{(x+3)\,\cancel{(x-2)}}{-\,\cancel{(x-2)}} && \text{(Algebra: Cancel like factors)} \\
&= \frac{(x+3)}{-1} \\
&= -(x+3)
\end{aligned}
$$

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5.1.1: A.1.1- Homework

1. Factor completely.
   a. $216x^3 + 125y^3$
   b. $64m^4 - 27m$
   c. $125h^2 - 216h^5$
   d. $64x^3 - 125$
   e. $108\cos^3(x) - 4e^{3x}$
   f. $9x(x+1)^{-3/4} + (x+1)^{1/4}$
   g. $-x(5x-3)^{1/4} - 9(5x-3)^{-3/4}$
   h. $6x(24x+1)^{5/6} - 5(24x+1)^{-1/6}$
2. Simplify the expression. When applicable, completely simplify your answer into a single rational expression without negative exponents.
   a. $\dfrac{(24-x^2)^{1/2} + x^2(24-x^2)^{-1/2}}{24-x^2}$
   b. $\dfrac{9(7+x)^{1/3} - x(7+x)^{-2/3}}{(7+x)^{2/3}}$
   c. $\dfrac{14(8+x)^{1/2} - x(8+x)^{-1/2}}{x+8}$
   d. $\dfrac{(11-5x)^{1/2} + \frac{5}{7}x(11-5x)^{-1/2}}{11-5x}$
3. Simplify the compound rational expression. When applicable, completely simplify your answer into a single rational expression without negative exponents.
   a. $15x - \dfrac{y}{\frac{x}{15y} + \frac{y}{15x}}$
   b. $11 - \dfrac{9}{1 - \frac{1}{x}}$
   c. $1 + \dfrac{1}{1 + \frac{1}{13+x}}$
   d. $\dfrac{\frac{15}{5+x+h} - \frac{15}{5+x}}{h}$
   e. $\dfrac{\frac{14}{(x+h)^2} - \frac{14}{x^2}}{h}$
   f. $\dfrac{25x^{-2} - 4y^{-2}}{5x^{-1} + 2y^{-1}}$
   g. $\dfrac{41x^{-1} + 97y^{-1}}{(97x+41y)^{-1}}$
   h. $\dfrac{-\frac{7}{\sqrt{x+h}} + \frac{7}{\sqrt{x}}}{h}$
4. Rationalize the numerator. When applicable, completely simplify your answer into a single rational expression without negative exponents.
   a. $\sqrt{x^2+36} - x$
   b. $\dfrac{\sqrt{100+h} - 10}{h}$
   c. $\dfrac{\sqrt{x} - \sqrt{x+9h}}{h\sqrt{x}\sqrt{x+h}}$
5. Let $x < \frac{5}{3}$. Simplify the expression.

$$\frac{|3x-5|}{3x^2 - 23x + 30}$$

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5.2: A.2- Functions

Learning Objectives

- Use functional notation to evaluate a function.
- Determine the domain and range of a function.
- Draw the graph of a function.
- Find the zeros of a function.
- Recognize a function from a table of values.
- Make new functions from two or more given functions.
- Describe the symmetry properties of a function.

In this section, we provide a formal definition of a function and examine several ways in which functions are represented - namely, through tables, formulas, and graphs. We study formal notation and terms related to functions. We also define composition of functions and symmetry properties. Most of this material will be a review for you, but it serves as a handy reference to remind you of some of the algebraic techniques useful for working with functions.

Functions

Given two sets A and B a set with elements that are ordered pairs (x, y) where x is an element of A and y is an element of B, is a relation from A to B. A relation from A to B defines a relationship between those two sets. A function is a special type of relation in which each element of the first set is related to exactly one element of the second set. The element of the first set is called the **input**; the element of the second set is called the **output**. Functions are used all the time in mathematics to describe relationships between two sets. For any function, when we know the input, the output is determined, so we say that the output is a function of the input. For example, the area of a square is determined by its side length, so we say that the area (the output) is a function of its side length (the input). The velocity of a ball thrown in the air can be described as a function of the amount of time the ball is in the air. The cost of mailing a package is a function of the weight of the package. Since functions have so many uses, it is important to have precise definitions and terminology to study them.

Figure 5.2.1: A function can be visualized as an input/output device

Definition: Functions

A **function** f consists of a set of inputs, a set of outputs, and a rule for assigning each input to exactly one output. The set of inputs is called the **domain** of the function. The set of outputs is called the **range** of the function.

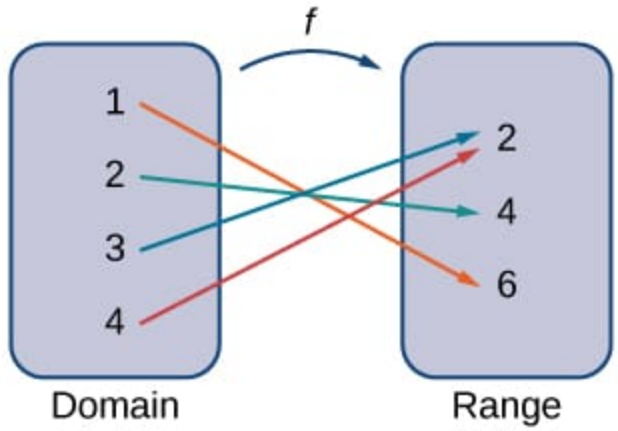


Figure 5.2.2: A function maps every element in the domain to exactly one element in the range. Although each input can be sent to only one output, two different inputs can be sent to the same output.

For example, consider the function f, where the domain is the set of all real numbers and the rule is to square the input. Then, the input $x = 3$ is assigned to the output $3^2 = 9$.

Since every nonnegative real number has a real-value square root, every nonnegative number is an element of the range of this function. Since there is no real number with a square that is negative, the negative real numbers are not elements of the range. We conclude that the range is the set of nonnegative real numbers.

For a general function f with domain D, we often use x to denote the input and y to denote the output associated with x. When doing so, we refer to x as the **independent variable** and y as the **dependent variable**, because it depends on x. Using function notation, we write $y = f(x)$, and we read this equation as "y equals f of x." For the squaring function described earlier, we write $f(x) = x^2$.

The concept of a function can be visualized using Figures 5.2.1 - 5.2.3.

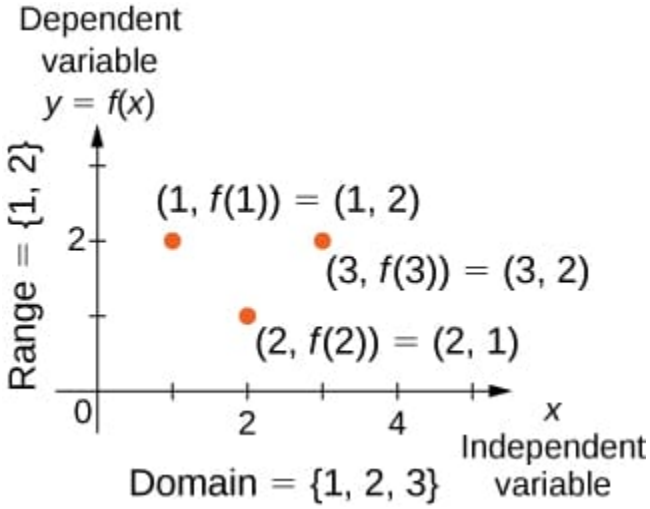


Figure 5.2.3: In this case, a graph of a function f has a domain of $\{1, 2, 3\}$ and a range of $\{1, 2\}$. The independent variable is x and the dependent variable is y.

We can also visualize a function by plotting points (x, y) in the coordinate plane where $y = f(x)$. The graph of a function is the set of all these points. For example, consider the function f, where the domain is the set $D = \{1, 2, 3\}$ and the rule is $f(x) = 3 - x$. In Figure 5.2.4, we plot a graph of this function.

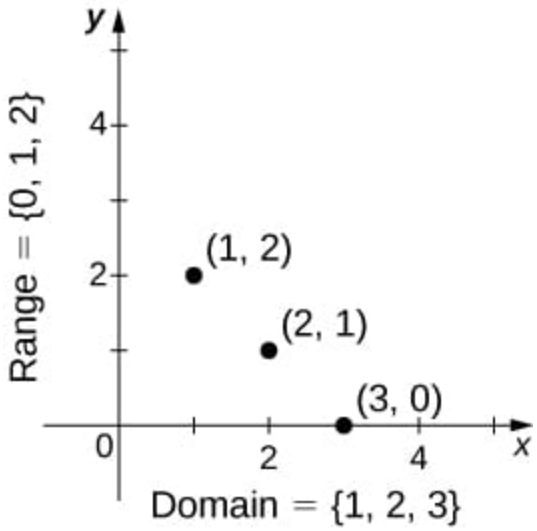


Figure 5.2.4: Here we see a graph of the function f with domain $\{1, 2, 3\}$ and rule $f(x) = 3 - x$. The graph consists of the points $(x, f(x))$ for all x in the domain.

Every function has a domain. However, sometimes a function is described by an equation, as in $f(x) = x^2$, with no specific domain given. In this case, the domain is taken to be the set of all real numbers x for which $f(x)$ is a real number. For example, since any real number can be squared, if no other domain is specified, we consider the domain of $f(x) = x^2$ to be the set of all real numbers. On the other hand, the square root function $f(x) = \sqrt{x}$ only gives a real output if x is nonnegative. Therefore, the domain of the function $f(x) = \sqrt{x}$ is the set of nonnegative real numbers, sometimes called the natural domain.

For the functions $f(x) = x^2$ and $f(x) = \sqrt{x}$, the domains are sets with an infinite number of elements. Clearly we cannot list all these elements. When describing a set with an infinite number of elements, it is often helpful to use **set-builder** or **interval notation**. When using set-builder notation to describe a subset of all real numbers, denoted R, we write

$$\{x \,|\, x \text{ has some property}\}.$$

We read this as the set of real numbers x such that x has some property. For example, if we were interested in the set of real numbers that are greater than one but less than five, we could denote this set using set-builder notation by writing\[\{x\,|\,1endpoints of this set. If we want to consider the set that includes the endpoints, we would denote this set by writing

$$[1, 5] = \{x \,|\, 1 \leq x \leq 5\}.$$

We can use similar notation if we want to include one of the endpoints, but not the other. To denote the set of nonnegative real numbers, we would use the set-builder notation

$$\{x \,|\, x \geq 0\}.$$

The smallest number in this set is zero, but this set does not have a largest number. Using interval notation, we would use the symbol ∞, which refers to positive infinity, and we would write the set as

$$[0, \infty) = \{x \,|\, x \geq 0\}.$$

Note

It is important to note that ∞ is not a real number.

∞ is used symbolically here to indicate that this set includes all real numbers greater than or equal to zero. Similarly, if we wanted to describe the set of all nonpositive numbers, we could write

$$(-\infty, 0] = \{x \,|\, x \leq 0\}.$$

Here, the notation $-\infty$ refers to negative infinity, and it indicates that we are including all numbers less than or equal to zero, no matter how small. The set

$$(-\infty, \infty) = \{x \,|\, x \text{ is any real number}\}$$

refers to the set of all real numbers. Some functions are defined using different equations for different parts of their domain.

These types of functions are known as **piecewise-defined functions**. For example, suppose we want to define a function f with a domain that is the set of all real numbers such that $f(x) = 3x + 1$ for $x \geq 2$ and $f(x) = x^2$ for $x < 2$. We denote this function by writing

$$f(x) = \begin{cases} 3x + 1, & \text{if } x \geq 2 \\ x^2, & \text{if } x < 2 \end{cases}$$

When evaluating this function for an input x, the equation to use depends on whether $x \geq 2$ or $x < 2$. For example, since $5 > 2$, we use the fact that $f(x) = 3x + 1$ for $x \geq 2$ and see that $f(5) = 3(5) + 1 = 16$. On the other hand, for $x = -1$, we use the fact that $f(x) = x^2$ for $x < 2$ and see that $f(-1) = 1$.

Example 5.2.1: Evaluating Functions

For the function $f(x) = 3x^2 + 2x - 1$, evaluate:

a. $f(-2)$
b. $f(\sqrt{2})$
c. $f(a + h)$

Solutions available at the end of this section.

Exercise 5.2.1

For $f(x) = x^2 - 3x + 5$, evaluate $f(1)$ and $f(a + h)$.

Hint available at the end of this section.

Example 5.2.2: Finding Domain and Range

For each of the following functions, determine the i. domain and ii. range.

a. $f(x) = (x - 4)^2 + 5$
b. $f(x) = \sqrt{3x + 2} - 1$
c. $f(x) = \dfrac{3}{x - 2}$

Solutions available at the end of this section.

Exercise 5.2.2

Find the domain and range for $f(x) = \sqrt{4 - 2x} + 5$.

Hint available at the end of this section.

Representing Functions

Typically, a function is represented using one or more of the following tools:

- A table
- A graph
- A formula

We can identify a function in each form, but we can also use them together. For instance, we can plot on a graph the values from a table or create a table from a formula.

Tables

Functions described using a **table of values** arise frequently in real-world applications. Consider the following simple example. We can describe temperature on a given day as a function of time of day. Suppose we record the temperature every hour for a 24-hour period starting at midnight. We let our input variable x be the time after midnight, measured in hours, and the output variable y be the temperature x hours after midnight, measured in degrees Fahrenheit. We record our data in Table $5.2.1$.

Table $5.2.1$: Temperature as a Function of Time of Day

Hour After Midnight	Temperature(°F)	Hour After Midnight	Temperature(°F)
0	58	12	84
1	54	13	85
2	53	14	85
3	52	15	83
4	52	16	82
5	55	17	80
6	60	18	77
7	64	19	74
8	72	20	69
9	75	21	65
10	78	22	60
11	80	23	58

We can see from the table that temperature is a function of time, and the temperature decreases, then increases, and then decreases again. However, we cannot get a clear picture of the behavior of the function without graphing it.

Graphs

Given a function f described by a table, we can provide a visual picture of the function in the form of a graph. Graphing the temperatures listed in Table $5.2.1$ can give us a better idea of their fluctuation throughout the day. Figure $5.2.5$ shows the plot of the temperature function.

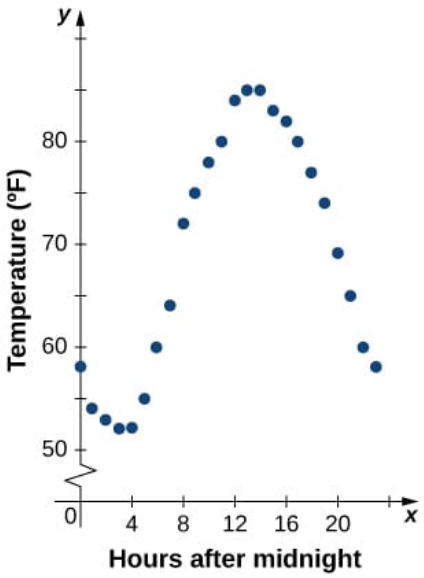


Figure 5.2.5: The graph of the data from Table 5.2.1 shows temperature as a function of time.

From the points plotted on the graph in Figure 5.2.5, we can visualize the general shape of the graph. It is often useful to connect the dots in the graph, which represent the data from the table. In this example, although we cannot make any definitive conclusion regarding what the temperature was at any time for which the temperature was not recorded, given the number of data points collected and the pattern in these points, it is reasonable to suspect that the temperatures at other times followed a similar pattern, as we can see in Figure 5.2.6.

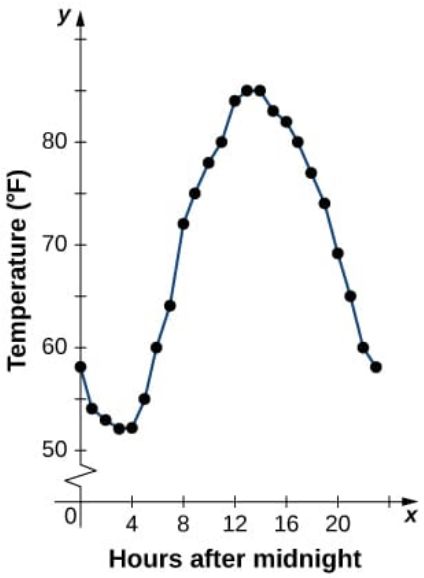


Figure 5.2.6: Connecting the dots in Figure 5.2.5 shows the general pattern of the data.

Algebraic Formulas

Sometimes we are not given the values of a function in table form, rather we are given the values in an explicit formula. Formulas arise in many applications. For example, the area of a circle of radius r is given by the formula $A(r) = \pi r^2$. When an object is thrown upward from the ground with an initial velocity v_0 ft/s, its height above the ground from the time it is thrown until it hits the ground is given by the formula $s(t) = -16t^2 + v_0 t$. When P dollars are invested in an account at an annual interest rate r compounded continuously, the amount of money after t years is given by the formula $A(t) = Pe^{rt}$. Algebraic formulas are important tools to calculate function values. Often we also represent these functions visually in graph form.

Given an algebraic formula for a function f, the graph of f is the set of points $(x, f(x))$, where x is in the domain of f and $f(x)$

is in the range. To graph a function given by a formula, it is helpful to begin by using the formula to create a table of inputs and outputs. If the domain of f consists of an infinite number of values, we cannot list all of them, but because listing some of the inputs and outputs can be very useful, it is often a good way to begin.

When creating a table of inputs and outputs, we typically check to determine whether zero is an output. Those values of x where $f(x) = 0$ are called the **zeros** of the function. For example, the zeros of $f(x) = x^2 - 4$ are $x = \pm 2$. The zeros determine where the graph of f intersects the x-axis, which gives us more information about the shape of the graph of the function. The graph of a function may never intersect the x-axis, or it may intersect multiple (or even infinitely many) times.

Another point of interest is the y -intercept, if it exists. The y-intercept is given by $(0, f(0))$.

Since a function has exactly one output for each input, the graph of a function can have, at most, one y-intercept. If $x = 0$ is in the domain of a function f, then f has exactly one y-intercept. If $x = 0$ is not in the domain of f, then f has no y-intercept. Similarly, for any real number c, if c is in the domain of f, there is exactly one output $f(c)$, and the line $x = c$ intersects the graph of f exactly once. On the other hand, if c is not in the domain of f, $f(c)$ is not defined and the line $x = c$ does not intersect the graph of f. This property is summarized in the Vertical Line Test.

Theorem 5.2.1: Vertical Line Test

Given a function f, every vertical line that may be drawn intersects the graph of f no more than once. If any vertical line intersects a set of points more than once, the set of points does not represent a function.

We can use this test to determine whether a set of plotted points represents the graph of a function (Figure 5.2.7).

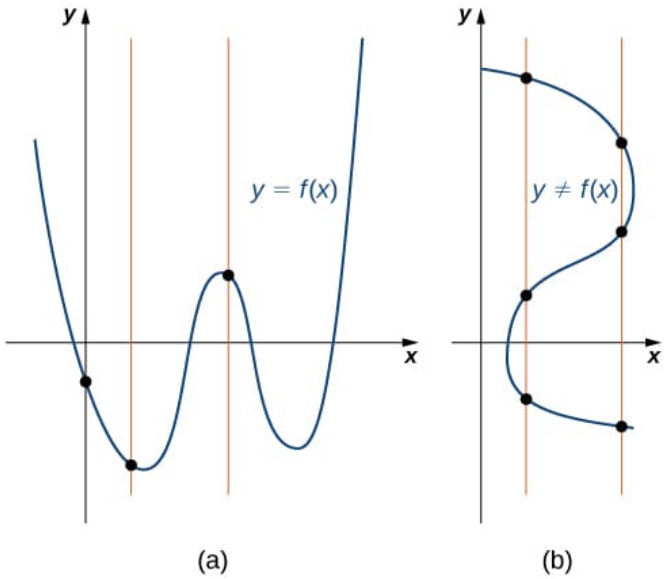


Figure 5.2.7: (a) The set of plotted points represents the graph of a function because every vertical line intersects the set of points, at most, once. (b) The set of plotted points does not represent the graph of a function because some vertical lines intersect the set of points more than once.

Example 5.2.3: Finding Zeros and y-Intercepts of a Function

Consider the function $f(x) = -4x + 2$.

a. Find all zeros of f.
b. Find the y-intercept (if any).
c. Sketch a graph of f.

Solutions available at the end of this section.

Example 5.2.4: Using Zeros and y-Intercepts to Sketch a Graph

Consider the function $f(x) = \sqrt{x+3} + 1$.

a. Find all zeros of f.
b. Find the y-intercept (if any).
c. Sketch a graph of f.

Solutions available at the end of this section.

Exercise 5.2.4

Find the zeros of $f(x) = x^3 - 5x^2 + 6x$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 5.2.5: Finding the Height of a Free-Falling Object

If a ball is dropped from a height of 100 ft, its height s at time t is given by the function $s(t) = -16t^2 + 100$, where s is measured in feet and t is measured in seconds. The domain is restricted to the interval $[0, c]$, where $t = 0$ is the time when the ball is dropped and $t = c$ is the time when the ball hits the ground.

a. Create a table showing the height s(t) when $t = 0,\ 0.5,\ 1,\ 1.5,\ 2$, and 2.5. Using the data from the table, determine the domain for this function. That is, find the time c when the ball hits the ground.
b. Sketch a graph of s.

Solutions available at the end of this section.

Definition: Increasing and Decreasing on an Interval

We say that a function f is **increasing** on the interval I if for all $x_1,\ x_2 \in I$,

$$f(x_1) \leq f(x_2) \text{ when } \backslash(x\_\{1\}$$

We say f is **strictly increasing** on the interval I if for all $x_1, x_2 \in I$,

$$\backslash(f(x\_\{1\})$$

We say that a function f is **decreasing** on the interval I if for all $x_1, x_2 \in I$,

$$f(x_1) \geq f(x_2) \text{ if } \backslash(x\_\{1\}$$

We say that a function f is **strictly decreasing** on the interval I if for all $x_1, x_2 \in I$,

$$f(x_1) > f(x_2) \text{ if } \backslash(x\_\{1\}$$

For example, the function $f(x) = 3x$ is increasing on the interval $(-\infty, \infty)$ because $3x_1 < 3x_2$ whenever \(x_{1}–x^3_{2}\) whenever \(x_{1}

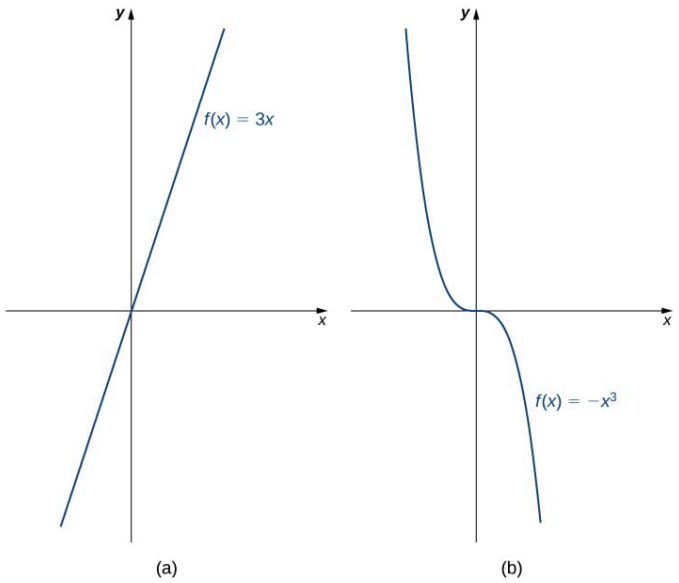


Figure 5.2.10: (a) The function $f(x) = 3x$ is increasing on the interval $(-\infty, \infty)$. (b) The function $f(x) = -x^3$ is decreasing on the interval $(-\infty, \infty)$.

Combining Functions

Now that we have reviewed the basic characteristics of functions, we can see what happens to these properties when we combine functions in different ways, using basic mathematical operations to create new functions. For example, if the cost for a company to manufacture x items is described by the function $C(x)$ and the revenue created by the sale of x items is described by the function $R(x)$, then the profit on the manufacture and sale of x items is defined as $P(x) = R(x) - C(x)$. Using the difference between two functions, we created a new function.

Alternatively, we can create a new function by **composing** two functions. For example, given the functions $f(x) = x^2$ and $g(x) = 3x + 1$, the **composite function** $f \circ g$ is defined such that

$$(f \circ g)(x) = f(g(x)) = (g(x))^2 = (3x + 1)^2.$$

The composite function $g \circ f$ is defined such that

$$(g \circ f)(x) = g(f(x)) = 3f(x) + 1 = 3x^2 + 1.$$

Note that these two new functions are different from each other.

Combining Functions with Mathematical Operators

To combine functions using mathematical operators, we simply write the functions with the operator and simplify. Given two functions f and g, we can define four new functions:

$(f + g)(x) = f(x) + g(x)$	Sum
$(f - g)(x) = f(x) - g(x)$	Difference
$(f \cdot g)(x) = f(x)g(x)$	Product
$(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$ for $g(x) \neq 0$	Quotient

Example 5.2.6: Combining Functions Using Mathematical Operations

Given the functions $f(x) = 2x - 3$ and $g(x) = x^2 - 1$, find each of the following functions and state its domain.

a. $(f+g)(x)$
b. $(f-g)(x)$
c. $(f \cdot g)(x)$
d. $\left(\frac{f}{g}\right)(x)$

Solutions available at the end of this section.

Exercise 5.2.6

For $f(x) = x^2 + 3$ and $g(x) = 2x - 5$, find $(f/g)(x)$ and state its domain.

Hint available at the end of this section.

Answer available at the end of this section.

Function Composition

When we compose functions, we take a function of a function. For example, suppose the temperature T on a given day is described as a function of time t (measured in hours after midnight) as in Table 5.2.1. Suppose the cost C, to heat or cool a building for 1 hour, can be described as a function of the temperature T. Combining these two functions, we can describe the cost of heating or cooling a building as a function of time by evaluating $C(T(t))$. We have defined a new function, denoted $C \circ T$, which is defined such that $(C \circ T)(t) = C(T(t))$ for all t in the domain of T. This new function is called a composite function. We note that since cost is a function of temperature and temperature is a function of time, it makes sense to define this new function $(C \circ T)(t)$. It does not make sense to consider $(T \circ C)(t)$, because temperature is not a function of cost.

Definition: Composite Functions

Consider the function f with domain A and range B, and the function g with domain D and range E. If B is a subset of D, then the **composite function** $(g \circ f)(x)$ is the function with domain A such that

$$(g \circ f)(x) = g(f(x))$$

A composite function $g \circ f$ can be viewed in two steps. First, the function f maps each input x in the domain of f to its output $f(x)$ in the range of f. Second, since the range of f is a subset of the domain of g, the output $f(x)$ is an element in the domain of g, and therefore it is mapped to an output $g(f(x))$ in the range of g. In Figure 5.2.11, we see a visual image of a composite function.

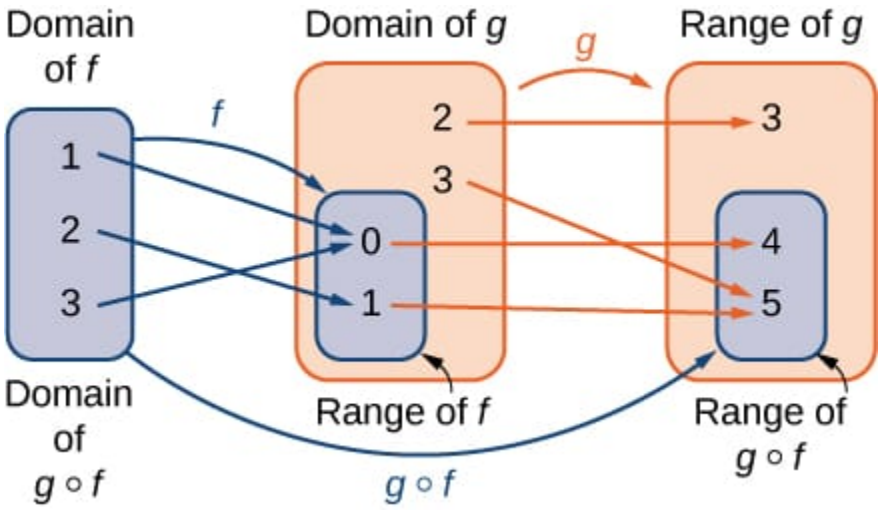


Figure 5.2.11: For the composite function $g \circ f$, we have $(g \circ f)(1) = 4$, $(g \circ f)(2) = 5$, and $(g \circ f)(3) = 4$.

Example 5.2.7: Compositions of Functions Defined by Formulas

Consider the functions $f(x) = x^2 + 1$ and $g(x) = 1/x$.

a. Find $(g \circ f)(x)$ and state its domain and range.
b. Evaluate $(g \circ f)(4)$, $(g \circ f)(-1/2)$.
c. Find $(f \circ g)(x)$ and state its domain and range.
d. Evaluate $(f \circ g)(4)$, $(f \circ g)(-1/2)$.

Solutions available at the end of this section.

In Example 5.2.7, we can see that $(f \circ g)(x) \neq (g \circ f)(x)$. This tells us, in general terms, that the order in which we compose functions matters.

Exercise 5.2.7

Let $f(x) = 2 - 5x$. Let $g(x) = \sqrt{x}$. Find $(f \circ g)(x)$.

Answer available at the end of this section.

Example 5.2.8: Composition of Functions Defined by Tables

Consider the functions f and g described by

x	-3	-2	-1	0	1	2	3	4
$f(x)$	0	4	2	4	-2	0	-2	4

x	-4	-2	0	2	4
$g(x)$	1	0	3	0	5

a. Evaluate $(g \circ f)(3)$,$(g \circ f)(0)$.
b. State the domain and range of $(g \circ f)(x)$.
c. Evaluate $(f \circ f)(3)$,$(f \circ f)(1)$.
d. State the domain and range of $(f \circ f)(x)$.

Solutions available at the end of this section.

Example 5.2.9: Application Involving a Composite Function

A store is advertising a sale of 20% off all merchandise. Caroline has a coupon that entitles her to an additional 15% off any item, including sale merchandise. If Caroline decides to purchase an item with an original price of x dollars, how much will she end up paying if she applies her coupon to the sale price? Solve this problem by using a composite function.

Solution available at the end of this section.

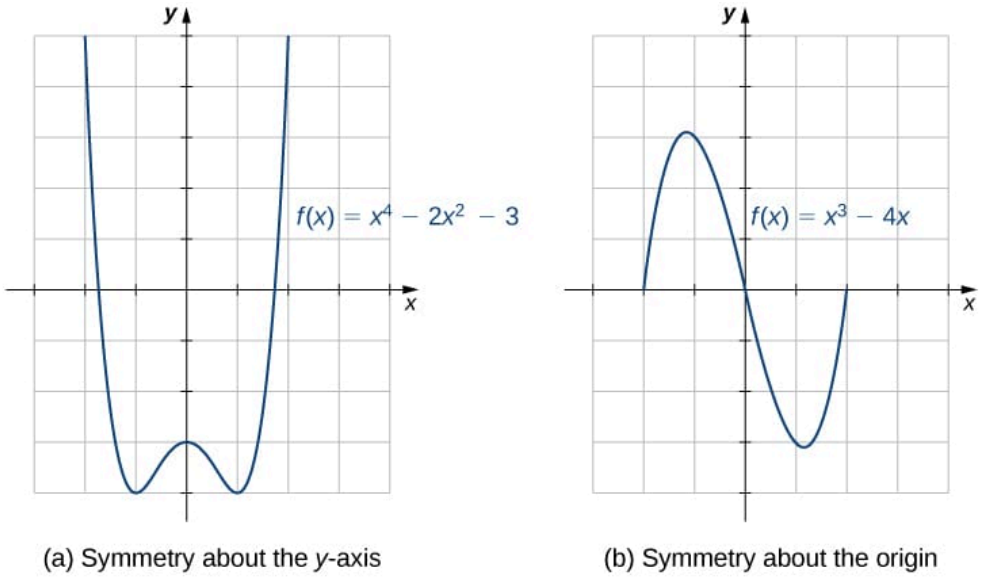
Exercise 5.2.9

If items are on sale for 10% off their original price, and a customer has a coupon for an additional 30% off, what will be the final price for an item that is originally x dollars, after applying the coupon to the sale price?

Hint available at the end of this section.

Symmetry of Functions

The graphs of certain functions have symmetry properties that help us understand the function and the shape of its graph. For example, consider the function $f(x) = x^4 - 2x^2 - 3$ shown in Figure $5.2.12a$. If we take the part of the curve that lies to the right of the y-axis and flip it over the y-axis, it lays exactly on top of the curve to the left of the y-axis. In this case, we say the function has **symmetry about the y-axis**. On the other hand, consider the function $f(x) = x^3 - 4x$ shown in Figure $5.2.12b$. If we take the graph and rotate it 180 ° about the origin, the new graph will look exactly the same. In this case, we say the function has **symmetry about the origin**.



(a) Symmetry about the y-axis (b) Symmetry about the origin

Figure $5.2.12$: (a) A graph that is symmetric about the y-axis. (b) A graph that is symmetric about the origin.

If we are given the graph of a function, it is easy to see whether the graph has one of these symmetry properties. But without a graph, how can we determine algebraically whether a function f has symmetry? Looking at Figure $5.2.12a$ again, we see that since f is symmetric about the y-axis, if the point (x, y) is on the graph, the point $(-x, y)$ is on the graph. In other words, $f(-x) = f(x)$. If a function f has this property, we say f is an **even function**, which has symmetry about the y-axis. For example, $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = x^2 = f(x).$$

In contrast, looking at Figure $5.2.12b$ again, if a function f is symmetric about the origin, then whenever the point (x, y) is on the graph, the point $(-x, -y)$ is also on the graph. In other words, $f(-x) = -f(x)$. If f has this property, we say f is an **odd function**, which has symmetry about the origin. For example, $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = -x^3 = -f(x).$$

Definition: Even and Odd Functions

- If $f(x) = f(-x)$ for all x in the domain of f, then f is an **even function**. An even function is symmetric about the y-axis.
- If $f(-x) = -f(x)$ for all x in the domain of f, then f is an **odd function**. An odd function is symmetric about the origin.

Example $5.2.10$: Even and Odd Functions

Determine whether each of the following functions is even, odd, or neither.

a. $f(x) = -5x^4 + 7x^2 - 2$

b. $f(x) = 2x^5 - 4x + 5$
c. $f(x) = \frac{3x}{x^2+1}$

Solutions available at the end of this section.

Exercise 5.2.10

Determine whether $f(x) = 4x^3 - 5x$ is even, odd, or neither.

Hint available at the end of this section.

Answer available at the end of this section.

One symmetric function that arises frequently is the **absolute value function**, written as $|x|$. The absolute value function is defined as

$$|x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Some students describe this function by stating that it "makes everything positive." By the definition of the absolute value function, we see that if $x < 0$, then $|x| = -x > 0$, and if $x > 0$, then $|x| = x > 0$. However, for $x = 0$, $|x| = 0$. Therefore, it is more accurate to say that for all nonzero inputs, the output is positive, but if $x = 0$, the output $|x| = 0$. We conclude that the range of the absolute value function is $\{y \mid y \geq 0\}$. In Figure 5.2.13, we see that the absolute value function is symmetric about the y-axis and is therefore an even function.

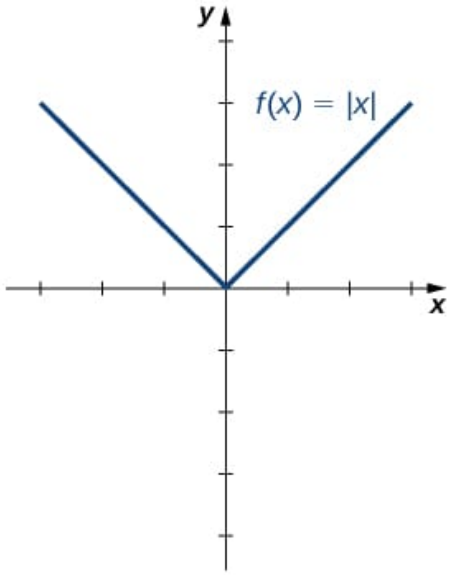


Figure 5.2.13: The graph of $f(x) = |x|$ is symmetric about the y-axis.

Example 5.2.11: Working with the Absolute Value Function

Find the domain and range of the function $f(x) = 2|x - 3| + 4$.

Solution available at the end of this section.

Exercise 5.2.11: Domain and Range

For the function $f(x) = |x + 2| - 4$, find the domain and range.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- A function is a mapping from a set of inputs to a set of outputs with exactly one output for each input.
- If no domain is stated for a function $y = f(x)$, the domain is considered to be the set of all real numbers x for which the function is defined.
- When sketching the graph of a function f, each vertical line may intersect the graph, at most, once.
- A function may have any number of zeros, but it has, at most, one y-intercept.
- To define the composition $g \circ f$, the range of f must be contained in the domain of g.
- Even functions are symmetric about the y-axis whereas odd functions are symmetric about the origin.

Key Equations

- **Composition of two functions**

$$(g \circ f)(x) = g\big(f(x)\big)$$

- **Absolute value function**

$$|x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Glossary

absolute value function
$$|x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

composite function
given two functions f and g, a new function, denoted $g \circ f$, such that $(g \circ f)(x) = g(f(x))$

decreasing on the interval I
a function decreasing on the interval I if, for all $x_1,\ x_2 \in I,\ f(x_1) \geq f(x_2)$ if \(x_1

dependent variable
the output variable for a function

domain
the set of inputs for a function

even function
a function is even if $f(-x) = f(x)$ for all x in the domain of f

function
a set of inputs, a set of outputs, and a rule for mapping each input to exactly one output

graph of a function
the set of points (x, y) such that x is in the domain of f and $y = f(x)$

increasing on the interval I
a function increasing on the interval I if for all $x_1,\ x_2 \in I,\ f(x_1) \leq f(x_2)$ if \(x_1

independent variable
: the input variable for a function

odd function
: a function is odd if $f(-x) = -f(x)$ for all x in the domain of f

range
: the set of outputs for a function

symmetry about the origin
: the graph of a function f is symmetric about the origin if $(-x, -y)$ is on the graph of f whenever (x, y) is on the graph

symmetry about the y-axis
: the graph of a function f is symmetric about the y-axis if $(-x, y)$ is on the graph of f whenever (x, y) is on the graph

table of values
: a table containing a list of inputs and their corresponding outputs

Vertical Line Test
: given the graph of a function, every vertical line intersects the graph, at most, once

zeros of a function
: when a real number x is a zero of a function f, $f(x) = 0$

Answers

Example 5.2.1**: Evaluating Functions**

a. $f(-2) = 3(-2)^2 + 2(-2) - 1 = 12 - 4 - 1 = 7$
b. $f(\sqrt{2}) = 3(\sqrt{2})^2 + 2\sqrt{2} - 1 = 6 + 2\sqrt{2} - 1 = 5 + 2\sqrt{2}$
c. $f(a+h) = 3(a+h)^2 + 2(a+h) - 1 = 3(a^2 + 2ah + h^2) + 2a + 2h - 1 = 3a^2 + 6ah + 3h^2 + 2a + 2h - 1$

Exercise 5.2.1

Substitute 1 and $a + h$ for x in the formula for $f(x)$.

$f(1) = 3$ and $f(a+h) = a^2 + 2ah + h^2 - 3a - 3h + 5$

Example 5.2.2**: Finding Domain and Range**

a. Consider $f(x) = (x-4)^2 + 5$.
   i. Since $f(x) = (x-4)^2 + 5$ is a real number for any real number x, the domain of f is the interval $(-\infty, \infty)$.
   ii. Since $(x-4)^2 \geq 0$, we know $f(x) = (x-4)^2 + 5 \geq 5$. Therefore, the range must be a subset of $\{y \mid y \geq 5\}$. To show that every element in this set is in the range, we need to show that for a given y in that set, there is a real number x such that $f(x) = (x-4)^2 + 5 = y$. Solving this equation for x, we see that we need x such that

   $$(x-4)^2 = y - 5.$$

   This equation is satisfied as long as there exists a real number x such that

   $$x - 4 = \pm\sqrt{y-5}$$

   Since $y \geq 5$, the square root is well-defined. We conclude that for $x = 4 \pm \sqrt{y-5}$, $f(x) = y$, and therefore the range is $\{y \mid y \geq 5\}$.
b. Consider $f(x) = \sqrt{3x+2} - 1$.
   i. To find the domain of f, we need the expression $3x + 2 \geq 0$. Solving this inequality, we conclude that the domain is $\{x \mid x \geq -2/3\}$.
   ii. To find the range of f, we note that since $\sqrt{3x+2} \geq 0$, $f(x) = \sqrt{3x+2} - 1 \geq -1$. Therefore, the range of f must be a subset of the set $\{y \mid y \geq -1\}$. To show that every element in this set is in the range of f, we need to show that for all y in this set, there exists a real number x in the domain such that $f(x) = y$. Let $y \geq -1$. Then, $f(x) = y$ if and only if

   $$\sqrt{3x+2} - 1 = y.$$

   Solving this equation for x, we see that x must solve the equation

   $$\sqrt{3x+2} = y + 1.$$

   Since $y \geq -1$, such an x could exist. Squaring both sides of this equation, we have $3x + 2 = (y+1)^2$. Therefore, we need

   $$3x = (y+1)^2 - 2,$$

   which implies

   $$x = \frac{1}{3}(y+1)^2 - \frac{2}{3}.$$

   We just need to verify that x is in the domain of f. Since the domain of f consists of all real numbers greater than or equal to $\frac{-2}{3}$, and

$$\frac{1}{3}(y+1)^2 - \frac{2}{3} \geq -\frac{2}{3},$$

there does exist an x in the domain of f. We conclude that the range of f is $\{y \mid y \geq -1\}$.

c. Consider $f(x) = \dfrac{3}{x-2}$.

i. Since $3/(x-2)$ is defined when the denominator is nonzero, the domain is $\{x \mid x \neq 2\}$.

ii. To find the range of f, we need to find the values of y such that there exists a real number x in the domain with the property that

$$\frac{3}{x-2} = y.$$

Solving this equation for x, we find that

$$x = \frac{3}{y} + 2.$$

Therefore, as long as $y \neq 0$, there exists a real number x in the domain such that $f(x) = y$. Thus, the range is $\{y \mid y \neq 0\}$.

Exercise 5.2.2

Use $4 - 2x \geq 0$.

Domain = $\{x \mid x \leq 2\}$ and range = $\{y \mid y \geq 5\}$

Example 5.2.3**: Finding Zeros and y-Intercepts of a Function**

a. To find the zeros, solve $f(x) = -4x + 2 = 0$. We discover that f has one zero at $x = 1/2$.

b. The y-intercept is given by $(0, f(0)) = (0, 2)$.

c. Given that f is a linear function of the form $f(x) = mx + b$ that passes through the points $(1/2, 0)$ and $(0, 2)$, we can sketch the graph of f (Figure 5.2.8).

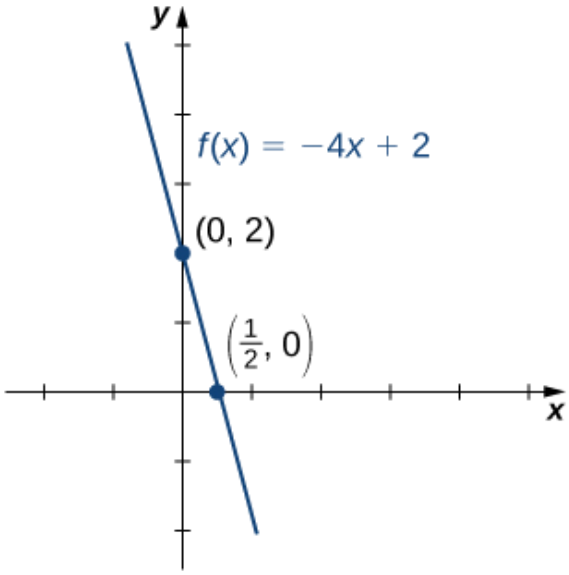


Figure 5.2.8: The function $f(x) = -4x + 2$ is a line with x-intercept $(1/2, 0)$ and y-intercept $(0, 2)$.

Example 5.2.4**: Using Zeros and y-Intercepts to Sketch a Graph**

a. To find the zeros, solve $\sqrt{x+3} + 1 = 0$. This equation implies $\sqrt{x+3} = -1$. Since $\sqrt{x+3} \geq 0$ for all x, this equation has no solutions, and therefore f has no zeros.

b. The y-intercept is given by $(0, f(0)) = (0, \sqrt{3} + 1)$.

c. To graph this function, we make a table of values. Since we need $x + 3 \geq 0$, we need to choose values of $x \geq -3$. We choose values that make the square-root function easy to evaluate.

x	-3	-2	1
$f(x)$	1	2	3

Making use of the table and knowing that, since the function is a square root, the graph of f should be similar to the graph of $y = \sqrt{x}$, we sketch the graph (Figure 5.2.9).

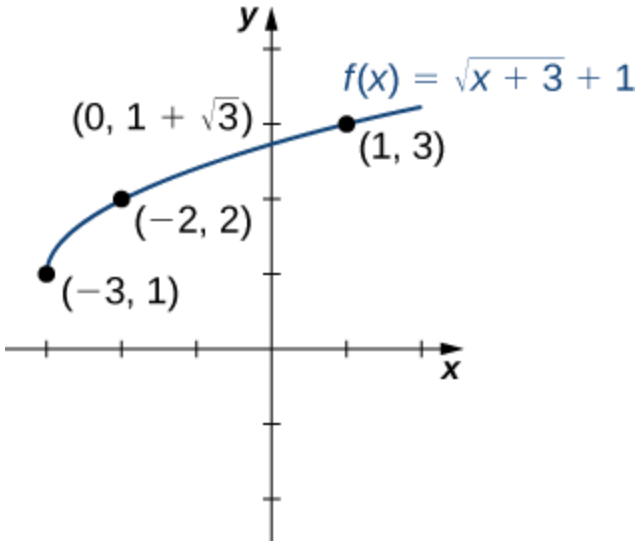


Figure 5.2.9: The graph of $f(x) = \sqrt{x+3} + 1$ has a y-intercept but no x-intercepts.

Exercise 5.2.4

Factor the polynomial.

Exercise 5.2.4

$x = 0, 2, 3$

Example 5.2.5**: Finding the Height of a Free-Falling Object**

a.

t	0	0.5	1	1.5	2	2.5
$s(t)$	100	96	84	64	36	0

Since the ball hits the ground when $t = 2.5$, the domain of this function is the interval $[0, 2.5]$.

b.

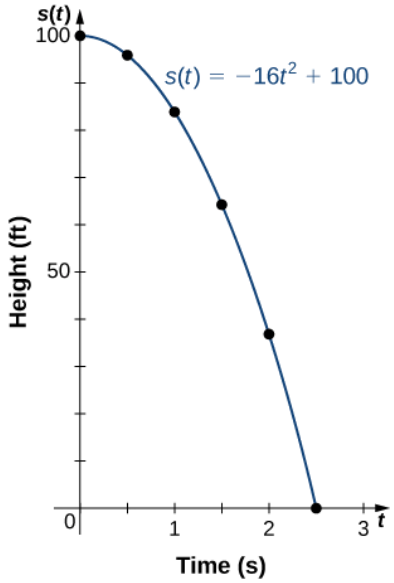


Figure 5.2.8, the values of $f(x)$ are getting smaller as x is getting larger. A function with this property is said to be decreasing. On the other hand, for the function $f(x) = \sqrt{x+3} + 1$ graphed in Figure 5.2.9, the values of $f(x)$ are getting larger as the values of x are getting larger. A function with this property is said to be increasing. It is important to note, however, that a function can be increasing on some interval or intervals and decreasing over a different interval or intervals. For example, using our temperature function plotted above, we can see that the function is decreasing on the interval $(0, 4)$, increasing on the interval $(4, 14)$, and then decreasing on the interval $(14, 23)$. We make the idea of a function increasing or decreasing over a particular interval more precise in the next definition.

Example 5.2.6**: Combining Functions Using Mathematical Operations**

a. $(f+g)(x) = (2x-3) + (x^2-1) = x^2 + 2x - 4$. The domain of this function is the interval $(-\infty, \infty)$.
b. $(f-g)(x) = (2x-3) - (x^2-1) = -x^2 + 2x - 2$. The domain of this function is the interval $(-\infty, \infty)$.

c. $(f \cdot g)(x) = (2x-3)(x^2-1) = 2x^3 - 3x^2 - 2x + 3$. The domain of this function is the interval $(-\infty, \infty)$.

d. $\left(\frac{f}{g}\right)(x) = \frac{2x-3}{x^2-1}$. The domain of this function is $\{x \mid x \neq \pm 1\}$.

Exercise 5.2.6

The new function $(f/g)(x)$ is a quotient of two functions. For what values of x is the denominator zero?

Exercise 5.2.6

$\left(\frac{f}{g}\right)(x) = \frac{x^2+3}{2x-5}$. The domain is $\{x \mid x \neq \frac{5}{2}\}$.

Example 5.2.7**: Compositions of Functions Defined by Formulas**

a. We can find the formula for $(g \circ f)(x)$ in two different ways. We could write

$$(g \circ f)(x) = g(f(x)) = g(x^2+1) = \frac{1}{x^2+1}.$$

Alternatively, we could write

$$(g \circ f)(x) = g(f(x)) = \frac{1}{f(x)} = \frac{1}{x^2+1}.$$

Since $x^2 + 1 \neq 0$ for all real numbers x, the domain of $(g \circ f)(x)$ is the set of all real numbers. Since $0 < 1/(x^2+1) \leq 1$, the range is, at most, the interval $(0, 1]$. To show that the range is this entire interval, we let $y = 1/(x^2+1)$ and solve this equation for x to show that for all y in the interval $(0, 1]$, there exists a real number x such that $y = 1/(x^2+1)$. Solving this equation for x, we see that $x^2 + 1 = 1/y$, which implies that

$$x = \pm\sqrt{\frac{1}{y} - 1}.$$

If y is in the interval $(0, 1]$, the expression under the radical is nonnegative, and therefore there exists a real number x such that $1/(x^2+1) = y$. We conclude that the range of $g \circ f$ is the interval $(0, 1]$.

b.

$$(g \circ f)(4) = g(f(4)) = g(4^2+1) = g(17) = \frac{1}{17}$$
$$(g \circ f)(-\frac{1}{2}) = g(f(-\frac{1}{2})) = g((-\frac{1}{2})^2+1) = g(\frac{5}{4}) = \frac{4}{5}$$

c. We can find a formula for $(f \circ g)(x)$ in two ways. First, we could write

$$(f \circ g)(x) = f(g(x)) = f(\frac{1}{x}) = (\frac{1}{x})^2 + 1.$$

Alternatively, we could write

$$(f \circ g)(x) = f(g(x)) = (g(x))^2 + 1 = (\frac{1}{x})^2 + 1.$$

The domain of $f \circ g$ is the set of all real numbers x such that $x \neq 0$. To find the range of f, we need to find all values y for which there exists a real number $x \neq 0$ such that

$$\left(\frac{1}{x}\right)^2 + 1 = y.$$

Solving this equation for x, we see that we need x to satisfy

$$\left(\frac{1}{x}\right)^2 = y - 1,$$

which simplifies to

$$\frac{1}{x} = \pm\sqrt{y-1}$$

Finally, we obtain

$$x = \pm\frac{1}{\sqrt{y-1}}.$$

Since $1/\sqrt{y-1}$ is a real number if and only if $y > 1$, the range of f is the set $\{y \mid y \geq 1\}$.

d.

$$(f \circ g)(4) = f(g(4)) = f(\frac{1}{4}) = (\frac{1}{4})^2 + 1 = \frac{17}{16}$$
$$(f \circ g)(-\frac{1}{2}) = f(g(-\frac{1}{2})) = f(-2) = (-2)^2 + 1 = 5$$

Exercise 5.2.7

$(f \circ g)(x) = 2 - 5\sqrt{x}$.

Example 5.2.8**: Composition of Functions Defined by Tables**

a.

$$(g \circ f)(3) = g(f(3)) = g(-2) = 0$$
$$(g \circ f)(0) = g(4) = 5$$

b. The domain of $g \circ f$ is the set $\{-3, -2, -1, 0, 1, 2, 3, 4\}$. Since the range of f is the set $\{-2, 0, 2, 4\}$, the range of $g \circ f$ is the set $\{0, 3, 5\}$.

c.

$$(f \circ f)(3) = f(f(3)) = f(-2) = 4$$
$$(f \circ f)(1) = f(f(1)) = f(-2) = 4$$

d. The domain of $f \circ f$ is the set $\{-3, -2, -1, 0, 1, 2, 3, 4\}$. Since the range of f is the set $\{-2, 0, 2, 4\}$, the range of $f \circ f$ is the set $\{0, 4\}$.

Example 5.2.9**: Application Involving a Composite Function**

Since the sale price is 20% off the original price, if an item is x dollars, its sale price is given by $f(x) = 0.80x$. Since the coupon entitles an individual to 15% off

the price of any item, if an item is y dollars, the price, after applying the coupon, is given by g(y)=0.85y. Therefore, if the price is originally x dollars, its sale price will be $f(x) = 0.80x$ and then its final price after the coupon will be $g(f(x)) = 0.85(0.80x) = 0.68x$.

Exercise 5.2.9

The sale price of an item with an original price of x dollars is $f(x) = 0.90x$. The coupon price for an item that is y dollars is $g(y) = 0.70y$.

$(g \circ f)(x) = 0.63x$

Example 5.2.10**: Even and Odd Functions**

a. $f(-x) = -5(-x)^4 + 7(-x)^2 - 2 = -5x^4 + 7x^2 - 2 = f(x)$. Therefore, f is even.
b. $f(-x) = 2(-x)^5 - 4(-x) + 5 = -2x^5 + 4x + 5$. Now, $f(-x) \neq f(x)$. Furthermore, noting that $-f(x) = -2x^5 + 4x - 5$, we see that $f(-x) \neq -f(x)$. Therefore, f is neither even nor odd.
c. $f(-x) = 3(-x)/((-x)2 + 1) = -3x/(x^2 + 1) = -[3x/(x^2 + 1)] = -f(x)$. Therefore, f is odd.

Exercise 5.2.10

Compare $f(-x)$ with $f(x)$ and $-f(x)$.

Exercise 5.2.10

$f(x)$ is odd.

Example 5.2.11**: Working with the Absolute Value Function**

Since the absolute value function is defined for all real numbers, the domain of this function is $(-\infty, \infty)$. Since $|x - 3| \geq 0$ for all x, the function $f(x) = 2|x - 3| + 4 \geq 4$. Therefore, the range is, at most, the set $\{y \mid y \geq 4\}$. To see that the range is, in fact, this whole set, we need to show that for $y \geq 4$ there exists a real number x such that

$$2|x - 3| + 4 = y$$

A real number x satisfies this equation as long as

$$|x - 3| = \frac{1}{2}(y - 4)$$

Since $y \geq 4$, we know $y - 4 \geq 0$, and thus the right-hand side of the equation is nonnegative, so it is possible that there is a solution. Furthermore,

$$|x - 3| = \begin{cases} -(x - 3), & \text{if } x < 3 \\ x - 3, & \text{if } x \geq 3 \end{cases}$$

Therefore, we see there are two solutions:

$$x = \pm\frac{1}{2}(y - 4) + 3.$$

The range of this function is $\{y \mid y \geq 4\}$.

Exercise 5.2.11**: Domain and Range**

$|x + 2| \geq 0$ for all real numbers x.

Exercise 5.2.11**: Domain and Range**

Domain = $(-\infty, \infty)$, range = $\{y \mid y \geq -4\}$.

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5.2.1: A.2.1- Homework

For exercises 1 - 6, (a) determine the domain and the range of each relation, and (b) state whether the relation is a function.

1)

x	y	x	y
-3	9	1	1
-2	4	2	4
-1	1	3	9
0	0		

Answer

a. Domain = $\{-3, -2, -1, 0, 1, 2, 3\}$, Range = $\{0, 1, 4, 9\}$

b. Yes, a function

2)

x	y	x	y
-3	-2	1	1
-2	-8	2	8
-1	-1	3	-2
0	0		

3)

x	y	x	y
1	-3	1	1
2	-2	2	2
3	-1	3	3
0	0		

Answer

a. Domain = $\{0, 1, 2, 3\}$, Range = $\{-3, -2, -1, 0, 1, 2, 3\}$

b. No, not a function

4)

x	y	x	y
1	1	5	1
2	1	6	1
3	1	7	1

x	y	x	y
4	1		

5)

x	y	x	y
3	3	15	1
5	2	21	2
8	1	33	3
10	0		

Answer

a. Domain = $\{3, 5, 8, 10, 15, 21, 33\}$, Range = $\{0, 1, 2, 3\}$

b. Yes, a function

6)

x	y	x	y
-7	11	1	-2
-2	5	3	4
-2	1	6	11
0	-1		

For exercises 7 - 13, find the values for each function, if they exist, then simplify.

a. $f(0)$ b. $f(1)$ c. $f(3)$ d. $f(-x)$ e. $f(a)$ f. $f(a+h)$

7) $f(x) = 5x - 2$

Answer

a. -2 b. 3 c. 13 d. $-5x - 2$ e. $5a - 2$ f. $5a + 5h - 2$

8) $f(x) = 4x^2 - 3x + 1$

9) $f(x) = \dfrac{2}{x}$

Answer

a. Undefined b. 2 c. $\frac{2}{3}$ d. $-\dfrac{2}{x}$ e. $\dfrac{2}{a}$ f. $\dfrac{2}{a+h}$

10) $f(x) = |x - 7| + 8$

11) $f(x) = \sqrt{6x + 5}$

Answer

a. $\sqrt{5}$ b. $\sqrt{11}$ c. $\sqrt{23}$ d. $\sqrt{-6x + 5}$ e. $\sqrt{6a + 5}$ f. $\sqrt{6a + 6h + 5}$

12) $f(x) = \dfrac{x - 2}{3x + 7}$

13) $f(x) = 9$

Answer

a. 9 b. 9 c. 9 d. 9 e. 9 f. 9

For exercises 14 - 21, find the domain, range, and all zeros/intercepts, if any, of the functions.

14) $f(x) = \dfrac{x}{x^2 - 16}$

15) $g(x) = \sqrt{8x - 1}$

Answer

$x \geq \frac{1}{8}; \quad y \geq 0; \quad x = \frac{1}{8}$; no y-intercept

16) $h(x) = \dfrac{3}{x^2 + 4}$

17) $f(x) = -1 + \sqrt{x + 2}$

Answer

$x \geq -2; \quad y \geq -1; \quad x = -1; \quad y = -1 + \sqrt{2}$

18) $f(x) = 1x - \sqrt{9}$

19) $g(x) = \dfrac{3}{x - 4}$

Answer

$x \neq 4; \quad y \neq 0$; no x-intercept; $y = -\frac{3}{4}$

20) $f(x) = 4|x + 5|$

21) $g(x) = \sqrt{\dfrac{7}{x - 5}}$

Answer

$x > 5; \quad y > 0$; no intercepts

For exercises 22 - 27, set up a table to sketch the graph of each function using the following values: $x = -3, -2, -1, 0, 1, 2, 3.$

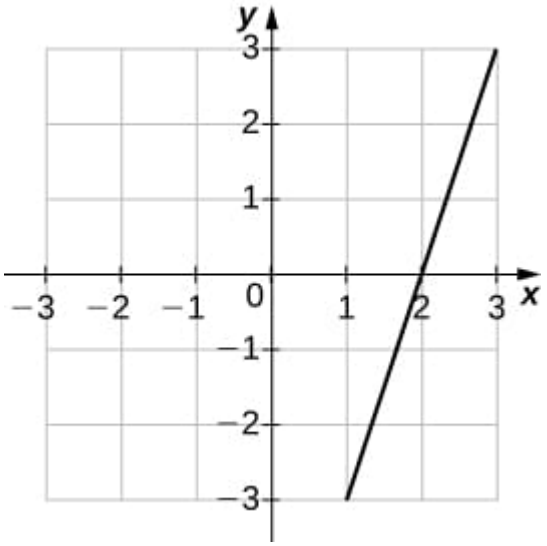
22) $f(x) = x^2 + 1$

x	y	x	y
-3	10	1	2
-2	5	2	5
-1	2	3	10
0	1		

23) $f(x) = 3x - 6$

x	y	x	y
-3	-15	1	-3
-2	-12	2	0
-1	-9	3	3
0	-6		

Answer



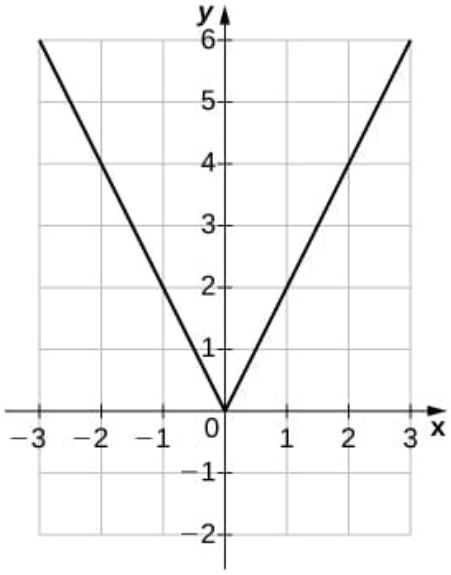
24) $f(x) = \frac{1}{2}x + 1$

x	y	x	y
-3	$-\frac{1}{2}$	1	$\frac{3}{2}$
-2	0	2	2
-1	$\frac{1}{2}$	3	$\frac{5}{2}$
0	1		

25) $f(x) = 2|x|$

x	y	x	y
-3	6	1	2
-2	4	2	4
-1	2	3	6
0	0		

Answer



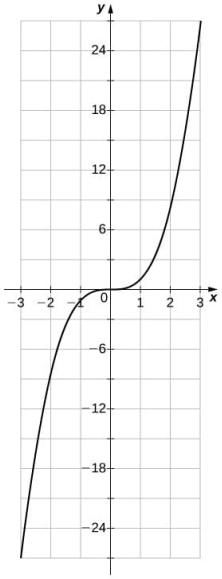
26) $f(x) = -x^2$

x	y	x	y
-3	-9	1	-1
-2	-4	2	-4
-1	-1	3	-9
0	0		

27) $f(x) = x^3$

x	y	x	y
-3	-27	1	1
-2	-8	2	8
-1	-1	3	27
0	0		

Answer



For exercises 28 - 35, use the vertical line test to determine whether each of the given graphs represents a function. Assume that a graph continues at both ends if it extends beyond the given grid. If the graph represents a function, then determine the following for each graph:

a. Domain and range

b. x -intercept, if any (estimate where necessary)

c. y-Intercept, if any (estimate where necessary)

d. The intervals for which the function is increasing

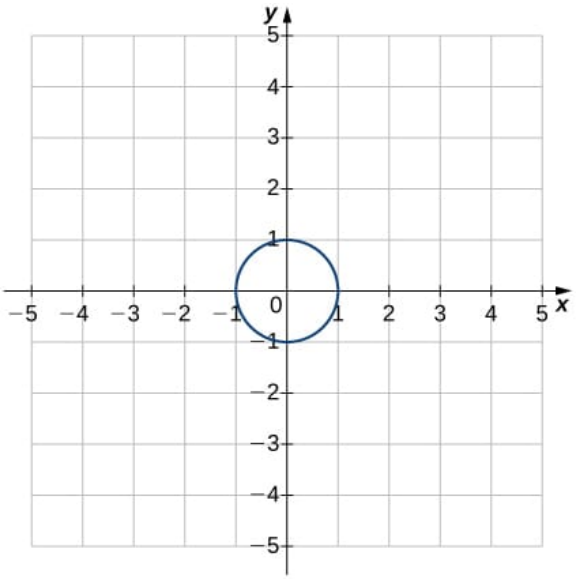
e. The intervals for which the function is decreasing

f. The intervals for which the function is constant

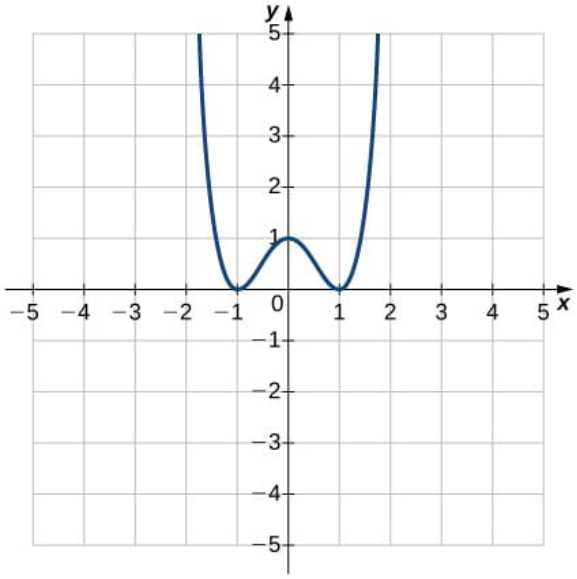
g. Symmetry about any axis and/or the origin

h. Whether the function is even, odd, or neither

28)



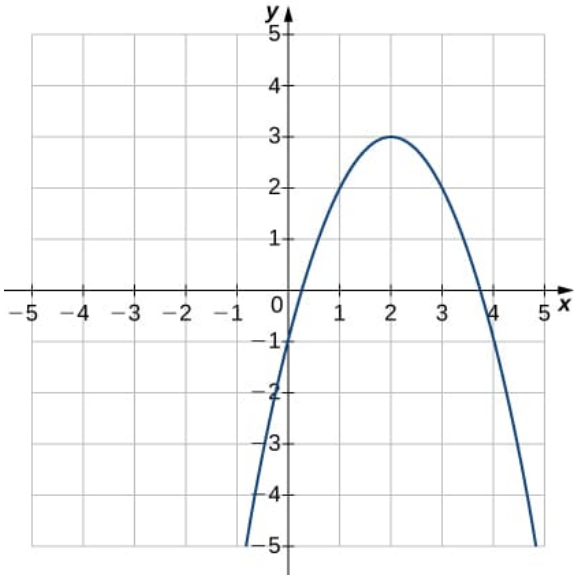
29)



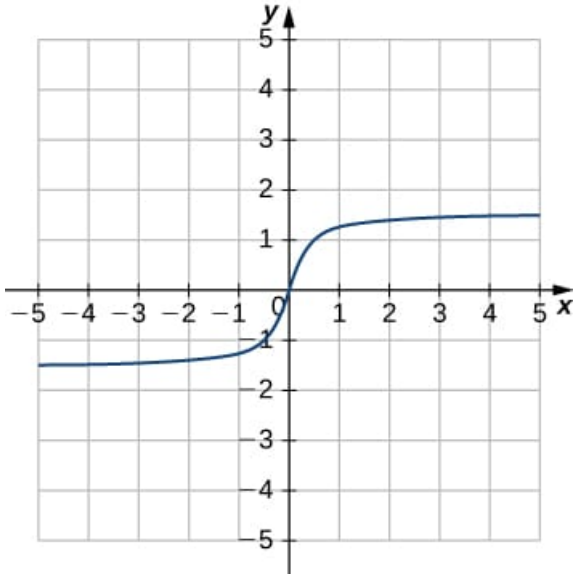
Answer

Function;
a. Domain: all real numbers, range: $y \geq 0$
b. $x = \pm 1$
c. $y = 1$
d. \(–1 e. \(–\infty f. Not constant
g. y-axis
h. Even

30)



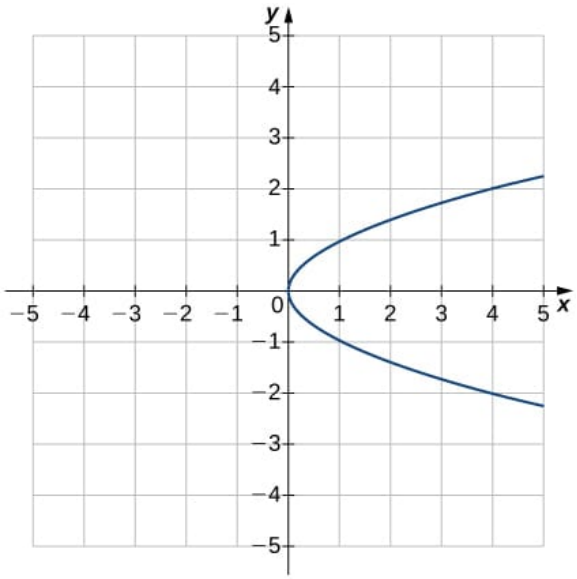
31)



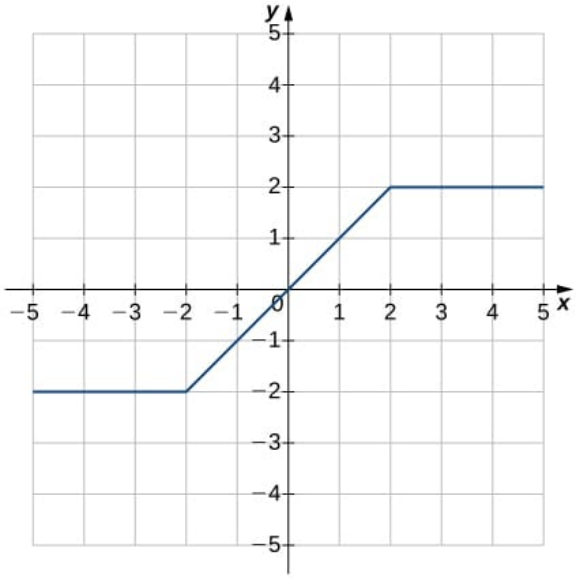
Answer

Function;
a. Domain: all real numbers, range: $-1.5 \leq y \leq 1.5$
b. $x = 0$
c. $y = 0$
d. all real numbers
e. None
f. Not constant
g. Origin
h. Odd

32)



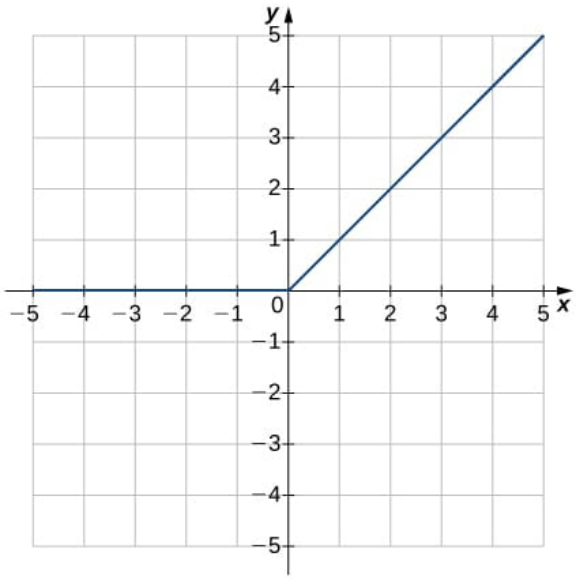
33)



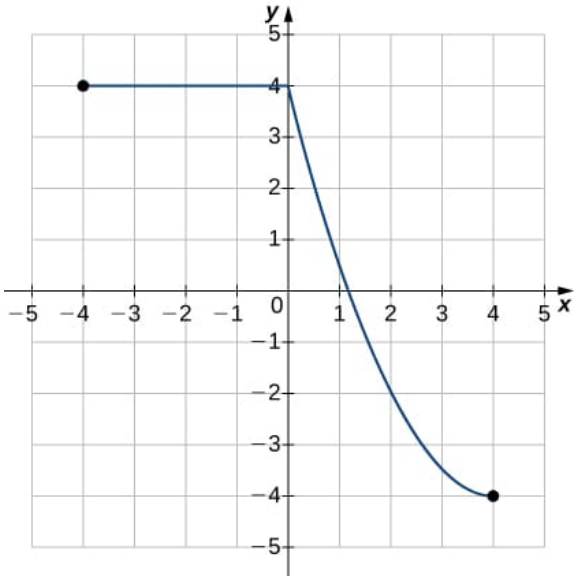
Answer

Function;
a. Domain: \(−\infty b. $x = 0$
c. $y = 0$
d. \(−2 e. Not decreasing
f. \(−\infty g. Origin
h. Odd

34)



35)



Answer

Function;
a. Domain: $-4 \leq x \leq 4$, range: $-4 \leq y \leq 4$
b. $x = 1.2$
c. $y = 4$
d. Not increasing
e. \(0 f. \(−4 g. No Symmetry
h. Neither

For exercises 36 - 41, for each pair of functions, find a. $f + g$ b. $f - g$ c. $f \cdot g$ d. f/g. Determine the domain of each of these new functions.

36) $f(x) = 3x + 4, \quad g(x) = x - 2$

37) $f(x) = x - 8, \quad g(x) = 5x^2$

Answer

a. $5x^2 + x - 8$; all real numbers
b. $-5x^2 + x - 8$; all real numbers
c. $5x^3 - 40x^2$; all real numbers
d. $\dfrac{x-8}{5x^2}; \quad x \neq 0$

38) $f(x) = 3x^2 + 4x + 1, \quad g(x) = x + 1$

39) $f(x) = 9 - x^2, \quad g(x) = x^2 - 2x - 3$

Answer

a. $-2x + 6$; all real numbers
b. $-2x^2 + 2x + 12$; all real numbers
c. $-x^4 + 2x^3 + 12x^2 - 18x - 27$; all real numbers
d. $-\dfrac{x+3}{x+1}; \quad x \neq -1, 3$

40) $f(x) = \sqrt{x}, \quad g(x) = x - 2$

41) $f(x) = 6 + \dfrac{1}{x}, \quad g(x) = \dfrac{1}{x}$

Answer

a. $6 + \dfrac{2}{x}; \quad x \neq 0$

b. $6; \quad x \neq 0$

c. $6x + \dfrac{1}{x^2}; \quad x \neq 0$

d. $6x + 1; \quad x \neq 0$

For exercises 42 - 48, for each pair of functions, find a. $(f \circ g)(x)$ and b. $(g \circ f)(x)$ Simplify the results. Find the domain of each of the results.

42) $f(x) = 3x, \quad g(x) = x + 5$

43) $f(x) = x + 4, \quad g(x) = 4x - 1$

Answer

a. $4x + 3$; all real numbers
b. $4x + 15$; all real numbers

44) $f(x) = 2x + 4, \quad g(x) = x^2 - 2$

45) $f(x) = x^2 + 7, \quad g(x) = x^2 - 3$

Answer

a. $x^4 - 6x^2 + 16$; all real numbers
b. $x^4 + 14x^2 + 46$; all real numbers

46) $f(x) = \sqrt{x}, \quad g(x) = x + 9$

47) $f(x) = \dfrac{3}{2x+1}, \quad g(x) = \dfrac{2}{x}$

Answer

a. $\dfrac{3x}{4+x}; \quad x \neq 0, -4$

b. $\frac{4x+2}{3}; \quad x \neq -\frac{1}{2}$

48) $f(x) = |x+1|, \quad g(x) = x^2 + x - 4$

49) The table below lists the NBA championship winners for the years 2001 to 2012.

Year	Winner
2001	LA Lakers
2002	LA Lakers
2003	Sam Antonio Spurs
2004	Detroit Pistons
2005	Sam Antonio Spurs
2006	Miami Heat
2007	Sam Antonio Spurs
2008	Boston Celtics
2009	LA Lakers
2010	LA Lakers
2011	Dallas Mavericks
2012	Miami Heat

a. Consider the relation in which the domain values are the years 2001 to 2012 and the range is the corresponding winner. Is this relation a function? Explain why or why not.

b. Consider the relation where the domain values are the winners and the range is the corresponding years. Is this relation a function? Explain why or why not.

Answer

a. Yes, because there is only one winner for each year.
b. No, because there are three teams that won more than once during the years 2001 to 2012.

50) [Technology Required] The area A of a square depends on the length of the side s.

a. Write a function $A(s)$ for the area of a square.

b. Find and interpret $A(6.5)$.

c. Find the exact and the two-significant-digit approximation to the length of the sides of a square with area 56 square units.

51) [Technology Required] The volume of a cube depends on the length of the sides s.

a. Write a function $V(s)$ for the area of a square.

b. Find and interpret $V(11.8)$.

Answer

a. $V(s) = s^3$
b. $V(11.8) \approx 1643$; a cube of side length 11.8 each has a volume of approximately 1643 cubic units.

52) [Technology Required] A rental car company rents cars for a flat fee of $20 and an hourly charge of $10.25. Therefore, the total cost C to rent a car is a function of the hours t the car is rented plus the flat fee.

a. Write the formula for the function that models this situation.

b. Find the total cost to rent a car for 2 days and 7 hours.

c. Determine how long the car was rented if the bill is $432.73.

53) [Technology Required] A vehicle has a 20-gal tank and gets 15 mpg. The number of miles N that can be driven depends on the amount of gas x in the tank.

a. Write a formula that models this situation.

b. Determine the number of miles the vehicle can travel on (i) a full tank of gas and (ii) 3/4 of a tank of gas.

c. Determine the domain and range of the function.

d. Determine how many times the driver had to stop for gas if she has driven a total of 578 mi.

Answer

a. $N(x) = 15x$
b. i. $N(20) = 15(20) = 300$; therefore, the vehicle can travel 300 mi on a full tank of gas.
ii. $N(15) = 225$; therefore, the vehicle can travel 225 mi on 3/4 of a tank of gas.
c. Domain: $0 \leq x \leq 20$; range: $[0, 300]$
d. The driver had to stop at least once, given that it takes approximately 39 gal of gas to drive a total of 578 mi.

54) [Technology Required] The volume V of a sphere depends on the length of its radius as $V = (4/3)\pi r^3$. Because Earth is not a perfect sphere, we can use the mean radius when measuring from the center to its surface. The mean radius is the average distance from the physical center to the surface, based on a large number of samples. Find the volume of Earth with mean radius 6.371×106 m.

55) [Technology Required] A certain bacterium grows in culture in a circular region. The radius of the circle, measured in centimeters, is given by $r(t) = 6 - \dfrac{5}{t^2 + 1}$, where t is time measured in hours since a circle of a 1-cm radius of the bacterium was put into the culture.

a. Express the area of the bacteria as a function of time.

b. Find the exact and approximate area of the bacterial culture in 3 hours.

c. Express the circumference of the bacteria as a function of time.

d. Find the exact and approximate circumference of the bacteria in 3 hours.

Answer

a. $A(t) = A(r(t)) = \pi \cdot (6 - \frac{5}{t^2+1})^2$
b. Exact: $\frac{121\pi}{4}$; approximately 95 cm^2
c. $C(t) = C(r(t)) = 2\pi(6 - \frac{5}{t^2+1})$
d. Exact: 11π; approximately 35 cm

56) [Technology Required] An American tourist visits Paris and must convert U.S. dollars to Euros, which can be done using the function $E(x) = 0.79x$, where x is the number of U.S. dollars and $E(x)$ is the equivalent number of Euros. Since conversion rates fluctuate, when the tourist returns to the United States 2 weeks later, the conversion from Euros to U.S. dollars is $D(x) = 1.245x$, where x is the number of Euros and $D(x)$ is the equivalent number of U.S. dollars.

a. Find the composite function that converts directly from U.S. dollars to U.S. dollars via Euros. Did this tourist lose value in the conversion process?

b. Use (a) to determine how many U.S. dollars the tourist would get back at the end of her trip if she converted an extra $200 when she arrived in Paris.

57) [Technology Required] The manager at a skateboard shop pays his workers a monthly salary S of $750 plus a commission of $8.50 for each skateboard they sell.

a. Write a function $y = S(x)$ that models a worker's monthly salary based on the number of skateboards x he or she sells.

b. Find the approximate monthly salary when a worker sells 25, 40, or 55 skateboards.

c. Determine the number of skateboards that must be sold for a worker to earn a monthly income of $1400.

Answer

a. $S(x) = 8.5x + 750$ b. $962.50, $1090, $1217.50 c. 77 skateboards

Contributors

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5.3: A.3- Basic Classes of Functions and Their Graphs

Learning Objectives

- Calculate the slope of a linear function and interpret its meaning.
- Recognize the degree of a polynomial.
- Find the roots of a quadratic polynomial.
- Describe the graphs of basic odd and even polynomial functions.
- Identify a rational function.
- Describe the graphs of power and root functions.
- Graph a piecewise-defined function.
- Sketch the graph of a function that has been shifted, stretched, or reflected from its initial graph position.

We have studied the general characteristics of functions, so now let's examine some specific classes of functions. We begin by reviewing the basic properties of linear and quadratic functions, and then generalize to include higher-degree polynomials. We finish the section with examples of piecewise-defined functions and take a look at how to sketch the graph of a function that has been shifted, stretched, or reflected from its initial form.

Linear Functions and Slope

The easiest type of function to consider is a **linear function**. Linear functions have the form $f(x) = ax + b$, where a and b are constants. In Figure $5.3.1$, we see examples of linear functions when a is positive, negative, and zero. Note that if $a > 0$, the graph of the line rises as x increases. In other words, $f(x) = ax + b$ is increasing on $(-\infty, \infty)$. If $a < 0$, the graph of the line falls as x increases. In this case, $f(x) = ax + b$ is decreasing on $(-\infty, \infty)$. If $a = 0$, the line is horizontal.

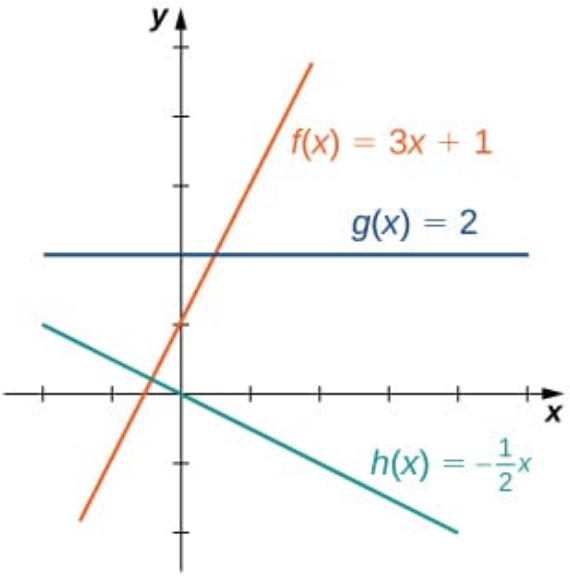


Figure $5.3.1$: These linear functions are increasing or decreasing on (∞, ∞) and one function is a horizontal line.

As suggested by Figure $5.3.1$, the graph of any linear function is a line. One of the distinguishing features of a line is its slope. The **slope** is the change in y for each unit change in x. The slope measures both the steepness and the direction of a line. If the slope is positive, the line points upward when moving from left to right. If the slope is negative, the line points downward when moving from left to right. If the slope is zero, the line is horizontal. To calculate the slope of a line, we need to determine the ratio of the change in y versus the change in x. To do so, we choose any two points (x_1, y_1) and (x_2, y_2) on the line and calculate $\frac{y_2 - y_1}{x_2 - x_1}$. In Figure $5.3.2$, we see this ratio is independent of the points chosen.

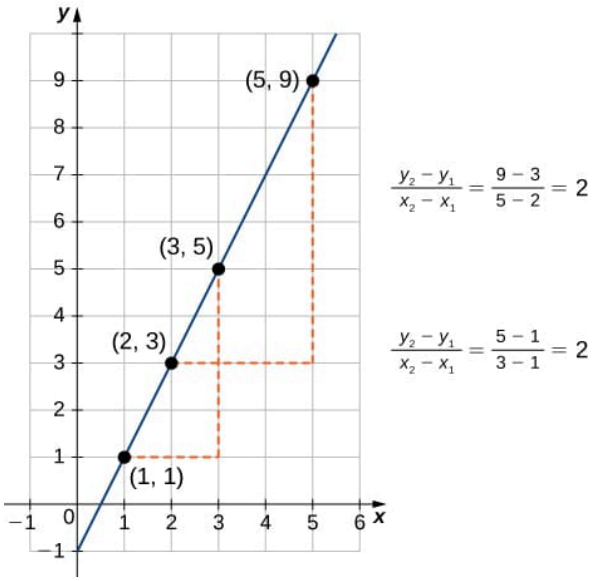


Figure 5.3.2: For any linear function, the slope $(y_2 - y_1)/(x_2 - x_1)$ is independent of the choice of points (x_1, y_1) and (x_2, y_2) on the line.

Definition: Slope of a Linear Function

Consider line L passing through points (x_1, y_1) and (x_2, y_2). Let $\Delta y = y_2 - y_1$ and $\Delta x = x_2 - x_1$ denote the changes in y and x,respectively. The **slope** of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

We now examine the relationship between slope and the formula for a linear function. Consider the linear function given by the formula $f(x) = ax + b$. As discussed earlier, we know the graph of a linear function is given by a line. We can use our definition of slope to calculate the slope of this line. As shown, we can determine the slope by calculating $(y_2 - y_1)/(x_2 - x_1)$ for any points (x_1, y_1) and (x_2, y_2) on the line. Evaluating the function f at $x = 0$, we see that $(0, b)$ is a point on this line. Evaluating this function at $x = 1$, we see that $(1, a + b)$ is also a point on this line. Therefore, the slope of this line is

$$\frac{(a + b) - b}{1 - 0} = a.$$

We have shown that the coefficient a is the slope of the line. We can conclude that the formula $f(x) = ax + b$ describes a line with slope a. Furthermore, because this line intersects the y-axis at the point $(0, b)$, we see that the y-intercept for this linear function is $(0, b)$. We conclude that the formula $f(x) = ax + b$ tells us the slope, a, and the y-intercept, $(0, b)$, for this line. Since we often use the symbol m to denote the slope of a line, we can write

$$\underbrace{f(x) = mx + b}_{\text{slope-intercept form}}$$

to denote the **slope-intercept form** of a linear function.

Sometimes it is convenient to express a linear function in different ways. For example, suppose the graph of a linear function passes through the point (x_1, y_1) and the slope of the line is m. Since any other point $(x, f(x))$ on the graph of f must satisfy the equation

$$m = \frac{f(x) - y_1}{x - x_1},$$

this linear function can be expressed by writing

$$\underbrace{f(x) - y_1 = m(x - x_1)}_{\text{point-slope equation}}.$$

We call this equation the **point-slope equation** for that linear function.

Since every nonvertical line is the graph of a linear function, the points on a nonvertical line can be described using the slope-intercept or point-slope equations. However, a vertical line does not represent the graph of a function and cannot be expressed in either of these forms. Instead, a vertical line is described by the equation $x = k$ for some constant k. Since neither the slope-intercept form nor the point-slope form allows for vertical lines, we use the notation

$$\underbrace{ax + by = c}_{\text{standard form}},$$

where a, b are both not zero, to denote the **standard form of a line.**

Definition: Point-Slope Equation, and the Slope-Intercept Form and Standard Form of the Equation of a Line

Consider a line passing through the point (x_1, y_1) with slope m. The equation

$$y - y_1 = m(x - x_1)$$

is the **point-slope form** for that line.

Consider a line with slope m and y-intercept $(0, b)$. The equation

$$y = mx + b$$

is an equation for that line in **slope-intercept form**.

The **standard form** of a line is given by the equation

$$ax + by = c,$$

where a and b are both not zero. This form is more general because it allows for a vertical line, $x = k$.

Example 5.3.1: Finding the Slope and Equations of Lines

Consider the line passing through the points $(11, -4)$ and $(-4, 5)$, as shown in Figure 5.3.3.

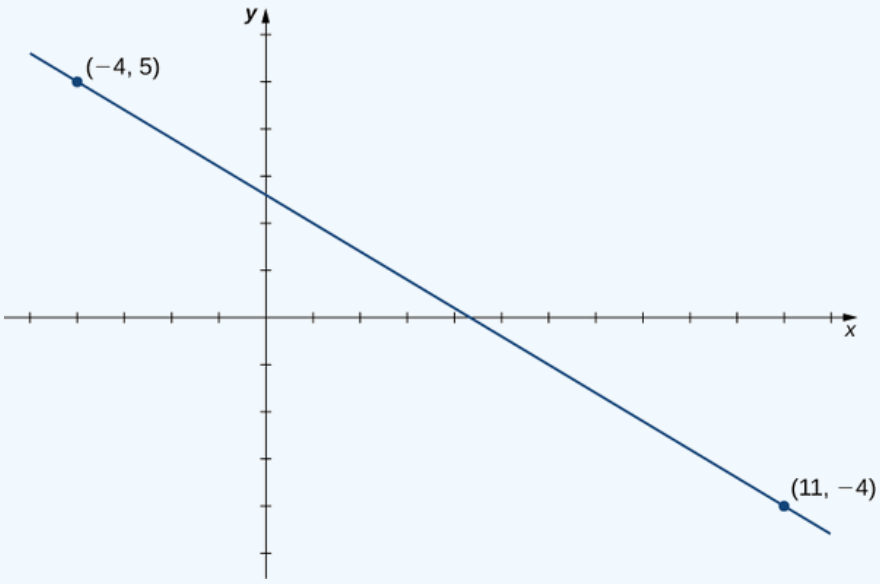


Figure 5.3.3: Finding the equation of a linear function with a graph that is a line between two given points.

1. Find the slope of the line.
2. Find an equation for this linear function in point-slope form.
3. Find an equation for this linear function in slope-intercept form.

Solution

1. The slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - (-4)}{-4 - 11} = -\frac{9}{15} = -\frac{3}{5}.$$

2. To find an equation for the linear function in point-slope form, use the slope $m = -3/5$ and choose any point on the line. If we choose the point $(11, -4)$, we get the equation

$$f(x) + 4 = -\frac{3}{5}(x - 11).$$

3. To find an equation for the linear function in slope-intercept form, solve the equation in part b. for $f(x)$. When we do this, we get the equation

$$f(x) = -\frac{3}{5}x + \frac{13}{5}.$$

Exercise 5.3.1

Consider the line passing through points $(-3, 2)$ and $(1, 4)$.

a. Find the slope of the line.
b. Find an equation of that line in point-slope form.
c. Find an equation of that line in slope-intercept form.

Hint available at the end of this section.

Answer a available at the end of this section.

Example 5.3.2:

Jessica leaves her house at 5:50 a.m. and goes for a 9-mile run. She returns to her house at 7:08 a.m. Answer the following questions, assuming Jessica runs at a constant pace.

a. Describe the distance D (in miles) Jessica runs as a linear function of her run time t (in minutes).
b. Sketch a graph of D.
c. Interpret the meaning of the slope.

Solution

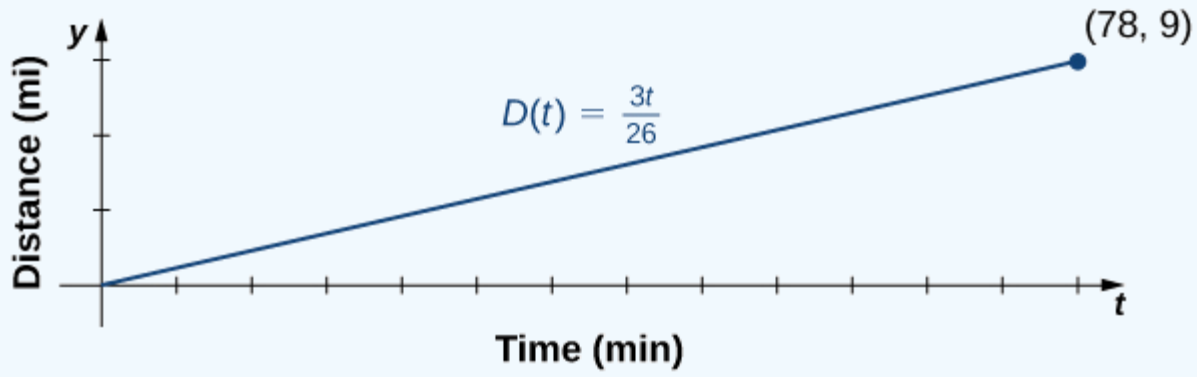
a. At time $t = 0$, Jessica is at her house, so $D(0) = 0$. At time $t = 78$ minutes, Jessica has finished running 9 mi, so $D(78) = 9$. The slope of the linear function is

$$m = \frac{9 - 0}{78 - 0} = \frac{3}{26}.$$

The y-intercept is $(0, 0)$, so the equation for this linear function is

$$D(t) = \frac{3}{26}t.$$

b. To graph D, use the fact that the graph passes through the origin and has slope $m = 3/26$.



c. The slope $m = 3/26 \approx 0.115$ describes the distance (in miles) Jessica runs per minute, or her average velocity.

Polynomials

A linear function is a special type of a more general class of functions: polynomials. A **polynomial function** is any function that can be written in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$$

for some integer $n \geq 0$ and constants $a_n, a_{n-1}, ..., a_0$, where $a_n \neq 0$. In the case when $n = 0$, we allow for $a_0 = 0$; if $a_0 = 0$, the function $f(x) = 0$ is called the **zero function**. The value n is called the **degree** of the polynomial; the constant a_n is called the **leading coefficient**. A linear function of the form $f(x) = mx + b$ is a polynomial of degree 1 if $m \neq 0$ and degree 0 if $m = 0$. A polynomial of degree 0 is also called a **constant function**. A polynomial function of degree 2 is called a **quadratic function**. In particular, a quadratic function has the form

$$f(x) = ax^2 + bx + c,$$

where $a \neq 0$. A polynomial function of degree 3 is called a **cubic function**.

Power Functions

Some polynomial functions are power functions. A **power function** is any function of the form $f(x) = ax^b$, where a and b are any real numbers. The exponent in a power function can be any real number, but here we consider the case when the exponent is a positive integer. (We consider other cases later.) If the exponent is a positive integer, then $f(x) = ax^n$ is a polynomial. If n is even, then $f(x) = ax^n$ is an even function because $f(-x) = a(-x)^n = ax^n$ if n is even. If n is odd, then $f(x) = ax^n$ is an odd function because $f(-x) = a(-x)^n = -ax^n$ if n is odd (Figure 5.3.4).

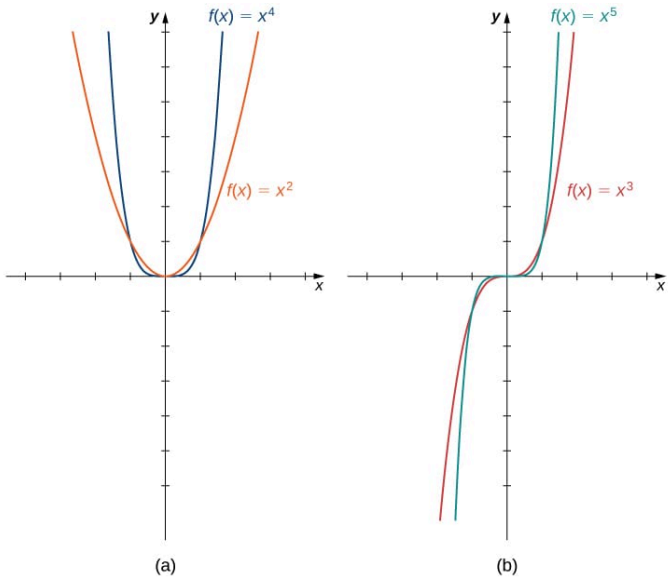


Figure 5.3.4: (a) For any even integer n,$f(x) = ax^n$ is an even function. (b) For any odd integer n,$f(x) = ax^n$ is an odd function.

Behavior at Infinity

To determine the behavior of a function f as the inputs approach infinity, we look at the values $f(x)$ as the inputs, x, become larger. For some functions, the values of $f(x)$ approach a finite number. For example, for the function $f(x) = 2 + 1/x$, the values $1/x$ become closer and closer to zero for all values of x as they get larger and larger. For this function, we say "$f(x)$ approaches two as x goes to infinity," and we write $f(x) \to 2$ as $x \to \infty$. The line $y = 2$ is a **horizontal asymptote** for the function $f(x) = 2 + 1/x$ because the graph of the function gets closer to the line as x gets larger.

For other functions, the values $f(x)$ may not approach a finite number but instead may become larger for all values of x as they get larger. In that case, we say "$f(x)$ approaches infinity as x approaches infinity," and we write $f(x) \to \infty$ as $x \to \infty$. For example, for the function $f(x) = 3x^2$, the outputs $f(x)$ become larger as the inputs x get larger. We can conclude that the function $f(x) = 3x^2$ approaches infinity as x approaches infinity, and we write $3x^2 \to \infty$ as $x \to \infty$. The behavior as $x \to -\infty$ and the meaning of $f(x) \to -\infty$ as $x \to \infty$ or $x \to -\infty$ can be defined similarly. We can describe what happens to the values of $f(x)$ as $x \to \infty$ and as $x \to -\infty$ as the **end behavior** of the function.

To understand the end behavior for polynomial functions, we can focus on quadratic and cubic functions. The behavior for higher-degree polynomials can be analyzed similarly. Consider a quadratic function $f(x) = ax^2 + bx + c$. If $a > 0$, the values $f(x) \to \infty$ as $x \to \pm\infty$. If $a < 0$, the values $f(x) \to -\infty$ as $x \to \pm\infty$. Since the graph of a quadratic function is a parabola, the parabola opens upward if $a > 0$; the parabola opens downward if $a < 0$ (Figure $5.3.5a$).

Now consider a cubic function $f(x) = ax^3 + bx^2 + cx + d$. If $a > 0$, then $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$. If $a < 0$, then $f(x) \to -\infty$ as $x \to \infty$ and $f(x) \to \infty$ as $x \to -\infty$. As we can see from both of these graphs, the leading term of the polynomial determines the end behavior (Figure $5.3.5b$).

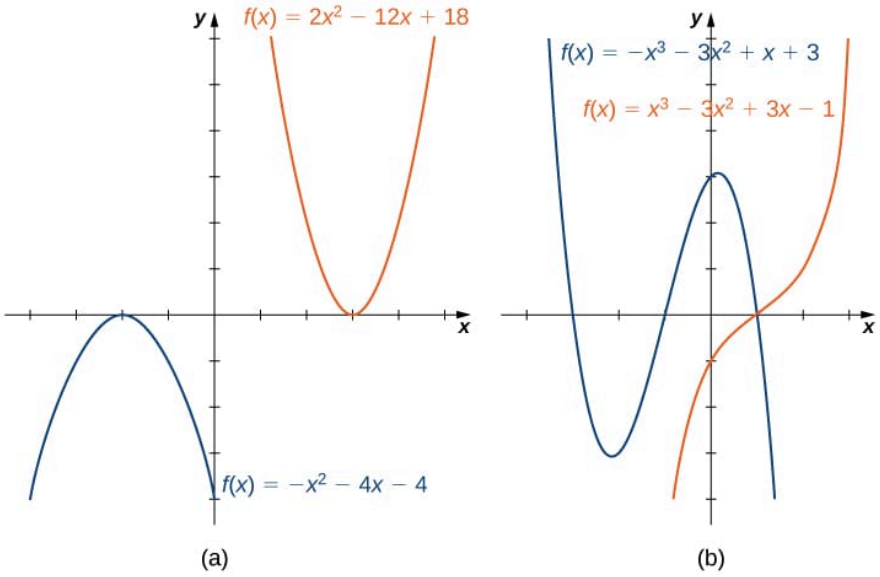


Figure 5.3.5: (a) For a quadratic function, if the leading coefficient $a > 0$,the parabola opens upward. If $a < 0$, the parabola opens downward. (b) For a cubic function f, if the leading coefficient $a > 0$, the values $f(x) \to \infty$ as $x \to \infty$ and the values $f(x) \to -\infty$ as $x \to -\infty$. If the leading coefficient $a < 0$, the opposite is true.

Zeros of Polynomial Functions

Another characteristic of the graph of a polynomial function is where it intersects the x-axis. To determine where a function f intersects the x-axis, we need to solve the equation $f(x) = 0$ for x. In the case of the linear function $f(x) = mx + b$, the x-intercept is given by solving the equation $mx + b = 0$. In this case, we see that the x-intercept is given by $(-b/m, 0)$. In the case of a quadratic function, finding the x-intercept(s) requires finding the zeros of a quadratic equation: $ax^2 + bx + c = 0$. In some cases, it is easy to factor the polynomial $ax^2 + bx + c$ to find the zeros. If not, we make use of the Quadratic Formula.

Theorem 5.3.1: Quadratic Formula

Consider the quadratic equation

$$ax^2 + bx + c = 0,$$

where $a \neq 0$. The solutions of this equation are given by the Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{5.3.1}$$

If the **discriminant** $b^2 - 4ac > 0$, Equation 5.3.1 tells us there are two real numbers that satisfy the quadratic equation. If $b^2 - 4ac = 0$, this formula tells us there is only one solution, and it is a real number. If $b^2 - 4ac < 0$, no real numbers satisfy the quadratic equation.

In the case of higher-degree polynomials, it may be more complicated to determine where the graph intersects the x-axis. In some instances, it is possible to find the x-intercepts by factoring the polynomial to find its zeros. In other cases, it is impossible to calculate the exact values of the x-intercepts. However, as we see later in the text, in cases such as this, we can use analytical tools to approximate (to a very high degree) where the x-intercepts are located. Here we focus on the graphs of polynomials for which we can calculate their zeros explicitly.

Example 5.3.3: Graphing Polynomial Functions

For the following functions,

a. $f(x) = -2x^2 + 4x - 1$
b. $f(x) = x^3 - 3x^2 - 4x$

i. describe the behavior of $f(x)$ as $x \to \pm\infty$,
ii. find all zeros of f, and
iii. sketch a graph of f.

Solution

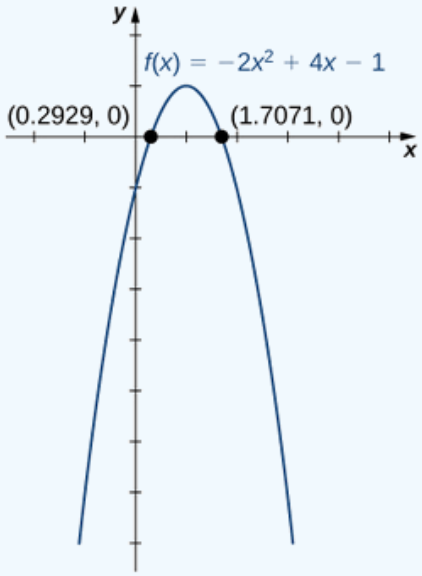
1. The function $f(x) = -2x^2 + 4x - 1$ is a quadratic function.

1. Because $a = -2 < 0$, as $x \to \pm\infty, f(x) \to -\infty$.

2. To find the zeros of f, use the Quadratic Formula. The zeros are

$$x = \frac{-4 \pm \sqrt{4^2 - 4(-2)(-1)}}{2(-2)} = \frac{-4 \pm \sqrt{8}}{-4} = \frac{-4 \pm 2\sqrt{2}}{-4} = \frac{2 \pm \sqrt{2}}{2}.$$

3. To sketch the graph of f,use the information from your previous answers and combine it with the fact that the graph is a parabola opening downward.



2. The function $f(x) = x^3 - 3x^2 - 4x$ is a cubic function.

1. Because $a = 1 > 0$, as $x \to \infty$, $f(x) \to \infty$. As $x \to -\infty$, $f(x) \to -\infty$.

2. To find the zeros of f, we need to factor the polynomial. First, when we factor x out of all the terms, we find

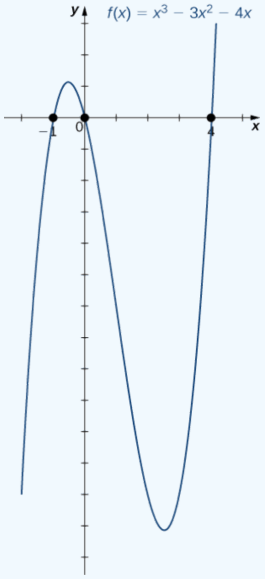
$$f(x) = x(x^2 - 3x - 4).$$

Then, when we factor the quadratic function $x^2 - 3x - 4$, we find

$$f(x) = x(x - 4)(x + 1).$$

Therefore, the zeros of f are $x = 0, 4, -1$.

3. Combining the results from parts i. and ii., draw a rough sketch of f.



Exercise 5.3.3

Consider the quadratic function $f(x) = 3x^2 - 6x + 2$. Find the zeros of f. Does the parabola open upward or downward?

Hint available at the end of this section.

Answer available at the end of this section.

Mathematical Models

A large variety of real-world situations can be described using **mathematical models**. A mathematical model is a method of simulating real-life situations with mathematical equations. Physicists, engineers, economists, and other researchers develop models by combining observation with quantitative data to develop equations, functions, graphs, and other mathematical tools to describe the behavior of various systems accurately. Models are useful because they help predict future outcomes. Examples of mathematical models include the study of population dynamics, investigations of weather patterns, and predictions of product sales.

As an example, let's consider a mathematical model that a company could use to describe its revenue for the sale of a particular item. The amount of revenue R a company receives for the sale of n items sold at a price of p dollars per item is described by the equation $R = p \cdot n$. The company is interested in how the sales change as the price of the item changes. Suppose the data in Table 5.3.1 show the number of units a company sells as a function of the price per item.

Table 5.3.1: Number of Units Sold n (in Thousands) as a Function of Price per Unit p (in Dollars)

p	6	8	10	12	14
n	19.4	18.5	16.2	13.8	12.2

In Figure 5.3.6, we see the graph the number of units sold (in thousands) as a function of price (in dollars). We note from the shape of the graph that the number of units sold is likely a linear function of price per item, and the data can be closely approximated by the linear function $n = -1.04p + 26$ for $0 \le p \le 25$, where n predicts the number of units sold in thousands. Using this linear function, the revenue (in thousands of dollars) can be estimated by the quadratic function

$$R(p) = p \cdot (-1.04p + 26) = -1.04p^2 + 26p \text{ for } 0 \le p \le 25.$$

In Example 5.3.4, we use this quadratic function to predict the amount of revenue the company receives depending on the price

the company charges per item. Note that we cannot conclude definitively the actual number of units sold for values of p, for which no data are collected. However, given the other data values and the graph shown, it seems reasonable that the number of units sold (in thousands) if the price charged is p dollars may be close to the values predicted by the linear function $n = -1.04p + 26$.

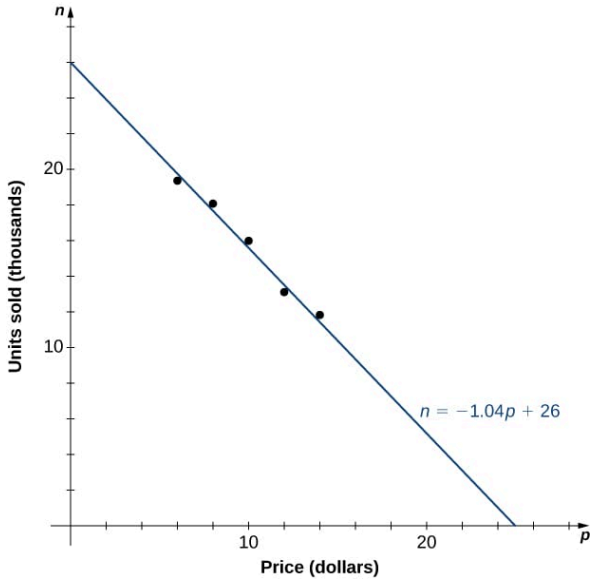


Figure 5.3.6: The data collected for the number of items sold as a function of price is roughly linear. We use the linear function $n = -1.04p + 26$ to estimate this function.

Example 5.3.4: Maximizing Revenue

A company is interested in predicting the amount of revenue it will receive depending on the price it charges for a particular item. Using the data from Table 5.3.1, the company arrives at the following quadratic function to model revenue R as a function of price per item p:

$$R(p) = p \cdot (-1.04p + 26) = -1.04p^2 + 26p$$

for $0 \le p \le 25$.

a. Predict the revenue if the company sells the item at a price of $p = \$5$ and $p = \$17$.
b. Find the zeros of this function and interpret the meaning of the zeros.
c. Sketch a graph of R.
d. Use the graph to determine the value of p that maximizes revenue. Find the maximum revenue.

Solution

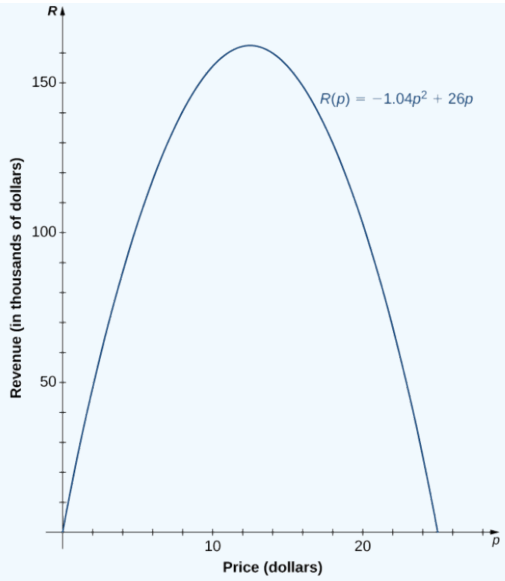
a. Evaluating the revenue function at $p = 5$ and $p = 17$, we can conclude that

$$R(5) = -1.04(5)^2 + 26(5) = 104, \text{ so revenue} = \$104,000;$$

$$R(17) = -1.04(17)^2 + 26(17) = 141.44, \text{ so revenue} = \$141,440.$$

b. The zeros of this function can be found by solving the equation $-1.04p^2 + 26p = 0$. When we factor the quadratic expression, we get $p(-1.04p + 26) = 0$. The solutions to this equation are given by $p = 0, 25$. For these values of p, the revenue is zero. When $p = \$0$, the revenue is zero because the company is giving away its merchandise for free. When $p = \$25$,the revenue is zero because the price is too high, and no one will buy any items.

c. Knowing the fact that the function is quadratic, we also know the graph is a parabola. Since the leading coefficient is negative, the parabola opens downward. One property of parabolas is that they are symmetric about the axis of symmetry, so since the zeros are at $p = 0$ and $p = 25$, the parabola must be symmetric about the line halfway between them, or $p = 12.5$.



d. The function is a parabola with zeros at $p = 0$ and $p = 25$, and it is symmetric about the line $p = 12.5$, so the maximum revenue occurs at a price of $p = \$12.50$ per item. At that price, the revenue is $R(p) = -1.04(12.5)^2 + 26(12.5) = \$162,500$.

Algebraic Functions

By allowing for quotients and fractional powers in polynomial functions, we create a larger class of functions. An **algebraic function** is one that involves addition, subtraction, multiplication, division, rational powers, and roots. Two types of algebraic functions are rational functions and root functions.

Just as rational numbers are quotients of integers, rational functions are quotients of polynomials. In particular, a **rational function** is any function of the form $f(x) = p(x)/q(x)$,where $p(x)$ and $q(x)$ are polynomials. For example,

$$f(x) = \frac{3x-1}{5x+2} \text{ and } g(x) = \frac{4}{x^2+1}$$

are rational functions. A **root function** is a power function of the form $f(x) = x^{1/n}$, where n is a positive integer greater than one. For example, $f(x) = x^{1/2} = \sqrt{x}$ is the square-root function and $g(x) = x^{1/3} = \sqrt[3]{x}$ is the cube-root function. By allowing for compositions of root functions and rational functions, we can create other algebraic functions. For example, $f(x) = \sqrt{4-x^2}$ is an algebraic function.

Example 5.3.5: Finding Domain and Range for Algebraic Functions

For each of the following functions, find the domain and range.

a. $f(x) = \dfrac{3x-1}{5x+2}$
b. $f(x) = \sqrt{4-x^2}$

Solution

1. It is not possible to divide by zero, so the domain is the set of real numbers x such that $x \neq -2/5$. To find the range, we need to find the values y for which there exists a real number x such that

$$y = \frac{3x-1}{5x+2}$$

When we multiply both sides of this equation by $5x+2$, we see that x must satisfy the equation

$$5xy+2y=3x-1.$$

From this equation, we can see that x must satisfy

$$2y+1=x(3-5y).$$

If y=$3/5$, this equation has no solution. On the other hand, as long as $y \neq 3/5$,

$$x=\frac{2y+1}{3-5y}$$

satisfies this equation. We can conclude that the range of f is $\{y \mid y \neq 3/5\}$.

2. To find the domain of f, we need $4-x^2 \geq 0$. When we factor, we write $4-x^2=(2-x)(2+x) \geq 0$. This inequality holds if and only if both terms are positive or both terms are negative. For both terms to be positive, we need to find x such that

$$2-x \geq 0 \text{ and } 2+x \geq 0.$$

These two inequalities reduce to $2 \geq x$ and $x \geq -2$. Therefore, the set $\{x \mid -2 \leq x \leq 2\}$ must be part of the domain. For both terms to be negative, we need

$$2-x \leq 0 \text{ and } 2+x \leq 0.$$

These two inequalities also reduce to $2 \leq x$ and $x \leq -2$. There are no values of x that satisfy both of these inequalities. Thus, we can conclude the domain of this function is $\{x \mid -2 \leq x \leq 2\}$.

If $-2 \leq x \leq 2$, then $0 \leq 4-x^2 \leq 4$. Therefore, $0 \leq \sqrt{4-x2} \leq 2$, and the range of f is $\{y \mid 0 \leq y \leq 2\}$.

Exercise 5.3.5

Find the domain and range for the function $f(x)=(5x+2)/(2x-1)$.

Hint available at the end of this section.

The root functions $f(x)=x^{1/n}$ have defining characteristics depending on whether n is odd or even. For all even integers $n \geq 2$, the domain of $f(x)=x^{1/n}$ is the interval $[0,\infty)$. For all odd integers $n \geq 1$, the domain of $f(x)=x^{1/n}$ is the set of all real numbers. Since $x^{1/n}=(-x)^{1/n}$ for odd integers n,$f(x)=x^{1/n}$ is an odd function if$n$ is odd. See the graphs of root functions for different values of n in Figure 5.3.7.

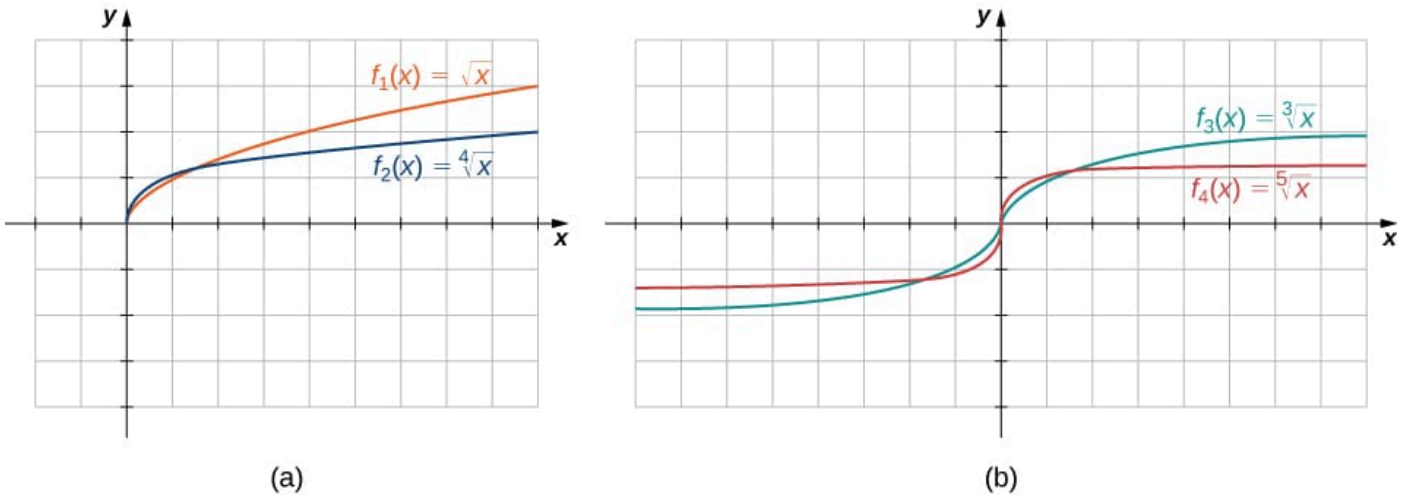


Figure 5.3.7: (a) If n is even, the domain of $f(x) = \sqrt[n]{x}$ is $[0, \infty)$. (b) If n is odd, the domain of $f(x) = \sqrt[n]{x}$ is $(-\infty, \infty)$ and the function $f(x) = \sqrt[n]{x}$ is an odd function.

Example 5.3.6: Finding Domains for Algebraic Functions

For each of the following functions, determine the domain of the function.

a. $f(x) = \dfrac{3}{x^2 - 1}$
b. $f(x) = \dfrac{2x + 5}{3x^2 + 4}$
c. $f(x) = \sqrt{4 - 3x}$
d. $f(x) = \sqrt[3]{2x - 1}$

Solution

a. You cannot divide by zero, so the domain is the set of values x such that $x^2 - 1 \neq 0$. Therefore, the domain is $\{x \mid x \neq \pm 1\}$.
b. You need to determine the values of x for which the denominator is zero. Since $3x^2 + 4 \geq 4$ for all real numbers x, the denominator is never zero. Therefore, the domain is $(-\infty, \infty)$.
c. Since the square root of a negative number is not a real number, the domain is the set of values x for which $4 - 3x \geq 0$. Therefore, the domain is $\{x \mid x \leq 4/3\}$.
d. The cube root is defined for all real numbers, so the domain is the interval $(-\infty, \infty)$.

Exercise 5.3.6

Find the domain for each of the following functions: $f(x) = (5 - 2x)/(x^2 + 2)$ and $g(x) = \sqrt{5x - 1}$.

Hint available at the end of this section.

Answer available at the end of this section.

Piecewise-Defined Functions

Sometimes a function is defined by different formulas on different parts of its domain. A function with this property is known as a **piecewise-defined function**. The absolute value function is an example of a piecewise-defined function because the formula changes with the sign of x:

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}.$$

Other piecewise-defined functions may be represented by completely different formulas, depending on the part of the domain in which a point falls. To graph a piecewise-defined function, we graph each part of the function in its respective domain, on the same coordinate system. If the formula for a function is different for \(xa\), we need to pay special attention to what happens at $x = a$ when we graph the function. Sometimes the graph needs to include an open or closed circle to indicate the value of the function at $x = a$. We examine this in the next example.

Example 5.3.7: Graphing a Piecewise-Defined Function

Sketch a graph of the following piecewise-defined function:

$$f(x) = \begin{cases} x + 3, & \text{if } x < 1 \\ (x-2)^2, & \text{if } x \geq 1 \end{cases}$$

Solution

Graph the linear function $y = x + 3$ on the interval $(-\infty, 1)$ and graph the quadratic function $y = (x-2)^2$ on the interval $[1, \infty)$. Since the value of the function at $x = 1$ is given by the formula $f(x) = (x-2)^2$, we see that $f(1) = 1$. To indicate this on the graph, we draw a closed circle at the point $(1, 1)$. The value of the function is given by $f(x) = x + 3$ for all $x < 1$, but not at $x = 1$. To indicate this on the graph, we draw an open circle at $(1, 4)$.

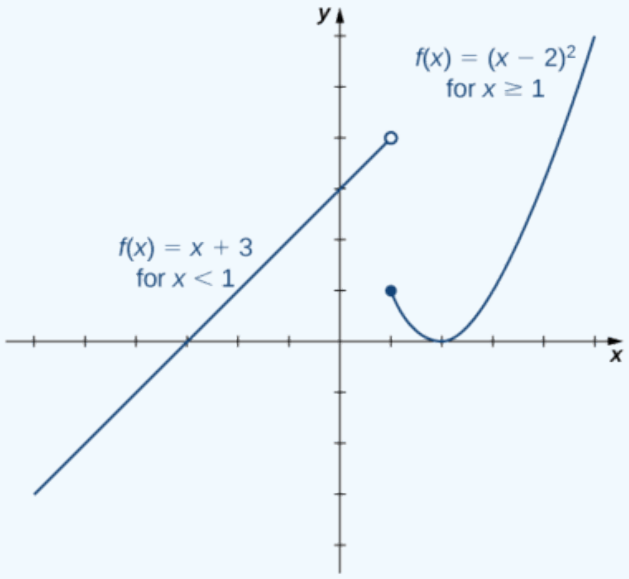
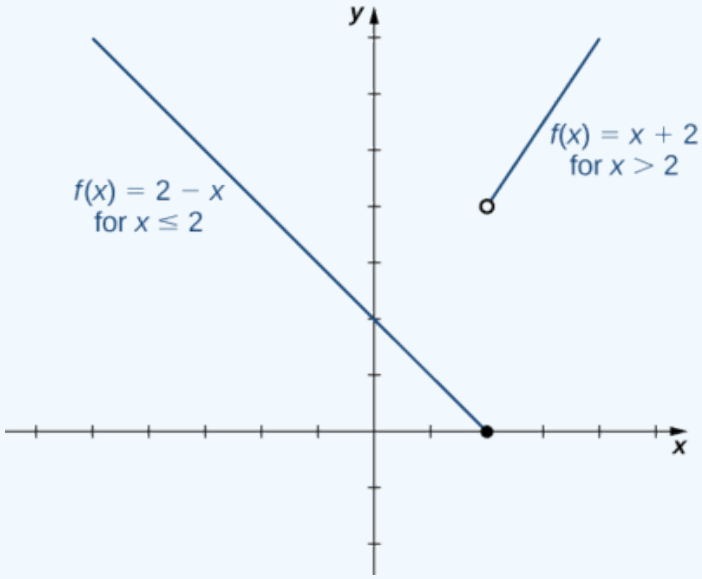


Figure 5.3.8: This piecewise-defined function is linear for $x < 1$ and quadratic for $x \geq 1$.

2) Sketch a graph of the function

$$f(x) = \begin{cases} 2 - x, & \text{if } x \leq 2 \\ x + 2, & \text{if } x > 2 \end{cases}.$$

Solution:



Example 5.3.8: Parking Fees Described by a Piecewise-Defined Function

In a big city, drivers are charged variable rates for parking in a parking garage. They are charged \$10 for the first hour or any part of the first hour and an additional \$2 for each hour or part thereof up to a maximum of \$30 for the day. The parking garage is open from 6 a.m. to 12 midnight.

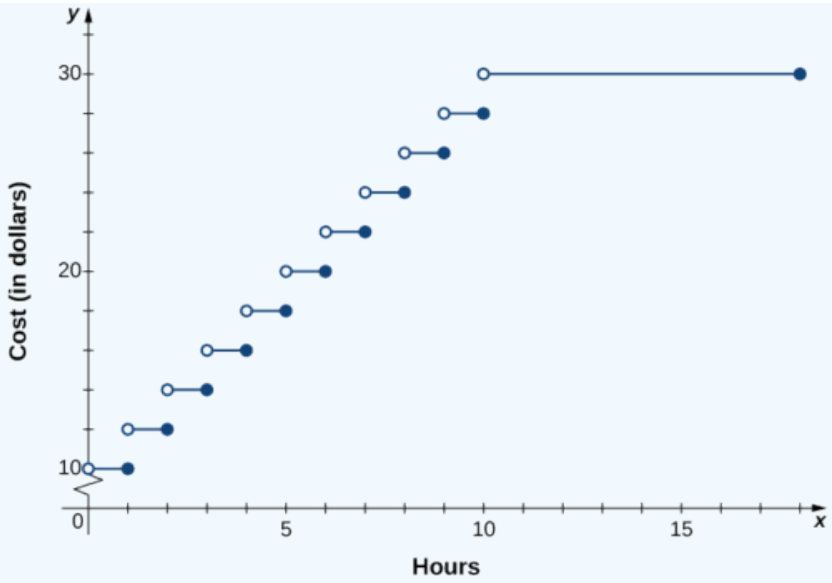
a. Write a piecewise-defined function that describes the cost C to park in the parking garage as a function of hours parked x.
b. Sketch a graph of this function $C(x)$.

Solution

1.Since the parking garage is open 18 hours each day, the domain for this function is \(\{x\,|\,0

\[C(x)=\begin{cases}10, & \text{for } 0

2.The graph of the function consists of several horizontal line segments.



Exercise 5.3.8

The cost of mailing a letter is a function of the weight of the letter. Suppose the cost of mailing a letter is $49¢$ for the first ounce and $21¢$ for each additional ounce. Write a piecewise-defined function describing the cost C as a function of the weight x for \(0

Hint available at the end of this section.

Answer available at the end of this section.

Transformations of Functions

We have seen several cases in which we have added, subtracted, or multiplied constants to form variations of simple functions. For instance, suppose we subtract 2 from the argument of the function $y = x^2$ to get the function $f(x) = (x - 2)^2$. This subtraction represents a shift of the function $y = x^2$ two units to the right. A shift, horizontally or vertically, is a type of **transformation of a function**. Other transformations include horizontal and vertical scalings, and reflections about the axes.

A **vertical shift** of a function occurs if we add or subtract the same constant to each output y. For $c > 0$, the graph of $f(x) + c$ is a shift of the graph of $f(x)$ up c units, whereas the graph of $f(x) - c$ is a shift of the graph of $f(x)$ down c units. For example, the graph of the function $f(x) = x^3 + 4$ is the graph of $y = x^3$ shifted up 4 units; the graph of the function $f(x) = x^3 - 4$ is the graph of $y = x^3$ shifted down 4 units (Figure $5.3.9$).

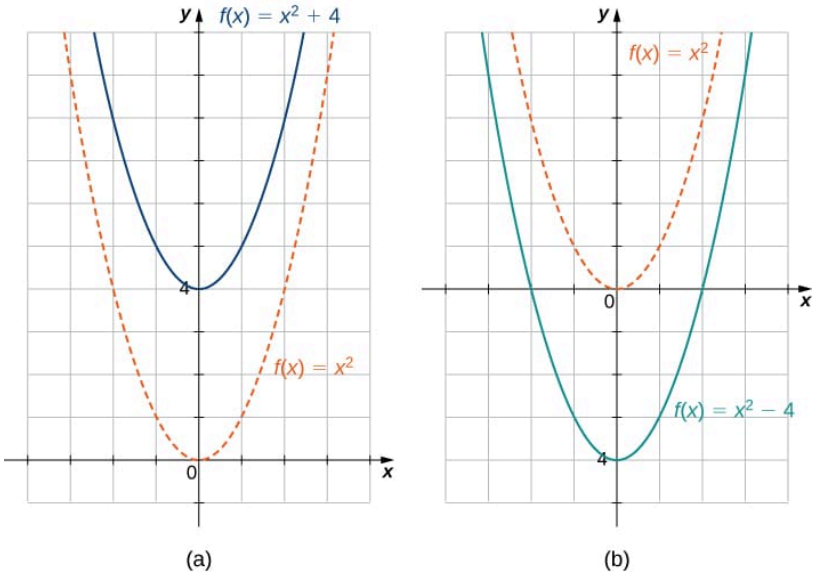


Figure 5.3.9: (a) For $c > 0$, the graph of $y = f(x) + c$ is a vertical shift up c units of the graph of $y = f(x)$. (b) For $c > 0$, the graph of $y = f(x) - c$ is a vertical shift down c units of the graph of $y = f(x)$.

A **horizontal shift** of a function occurs if we add or subtract the same constant to each input x. For $c > 0$, the graph of $f(x + c)$ is a shift of the graph of $f(x)$ to the left c units; the graph of $f(x - c)$ is a shift of the graph of $f(x)$ to the right c units. Why does the graph shift left when adding a constant and shift right when subtracting a constant? To answer this question, let's look at an example.

Consider the function $f(x) = |x + 3|$ and evaluate this function at $x - 3$. Since $f(x - 3) = |x|$ and \(x–3

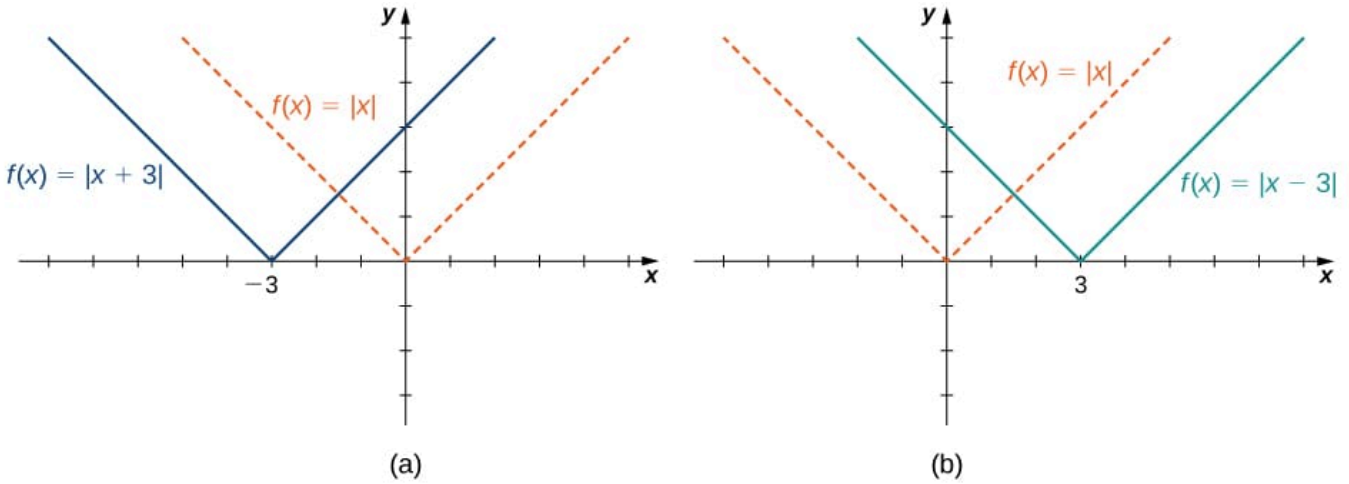


Figure 5.3.10: (a) For $c > 0$, the graph of $y = f(x + c)$ is a horizontal shift left c units of the graph of $y = f(x)$. (b) For $c > 0$, the graph of $y = f(x - c)$ is a horizontal shift right c units of the graph of $y = f(x)$.

A **vertical scaling** of a graph occurs if we multiply all outputs y of a function by the same positive constant. For $c > 0$, the graph of the function $cf(x)$ is the graph of $f(x)$ scaled vertically by a factor of c. If $c > 1$, the values of the outputs for the function $cf(x)$ are larger than the values of the outputs for the function $f(x)$; therefore, the graph has been stretched vertically. If \(0

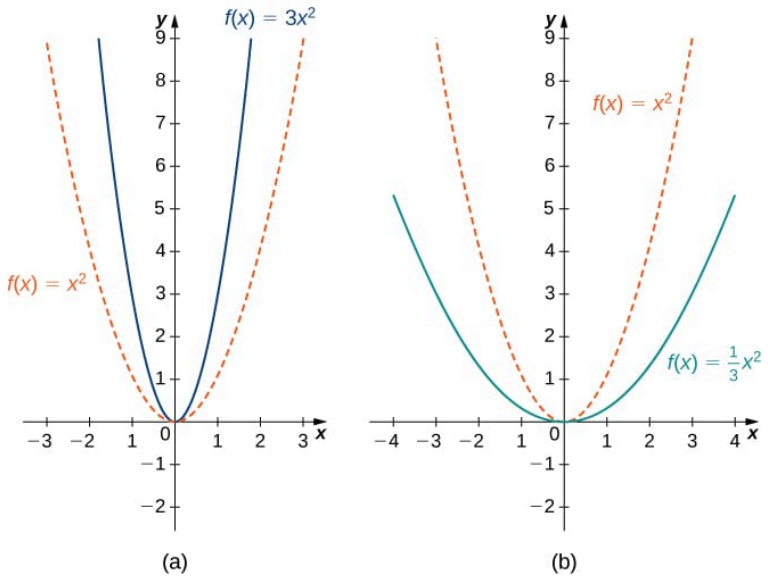


Figure 5.3.11: (a) If $c > 1$, the graph of $y = cf(x)$ is a vertical stretch of the graph of $y = f(x)$. (b) If \(0

The **horizontal scaling** of a function occurs if we multiply the inputs x by the same positive constant. For $c > 0$, the graph of the function $f(cx)$ is the graph of $f(x)$ scaled horizontally by a factor of c. If $c > 1$, the graph of $f(cx)$ is the graph of $f(x)$ compressed horizontally. If \(0

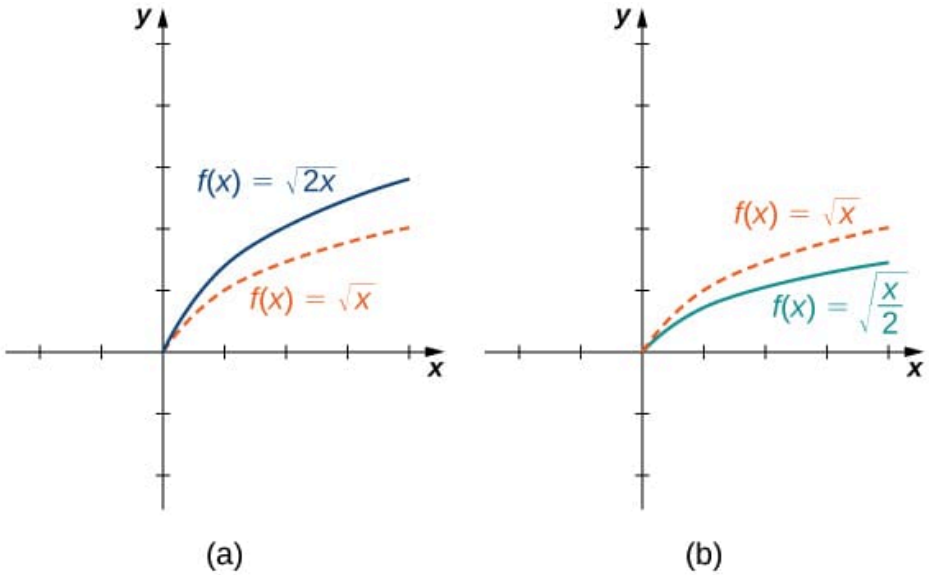


Figure 5.3.12: (a) If $c > 1$, the graph of $y = f(cx)$ is a horizontal compression of the graph of $y = f(x)$. (b) If \(0

We have explored what happens to the graph of a function f when we multiply f by a constant $c > 0$ to get a new function $cf(x)$. We have also discussed what happens to the graph of a function $f$when we multiply the independent variable x by $c > 0$ to get a new function $f(cx)$. However, we have not addressed what happens to the graph of the function if the constant c is negative. If we have a constant $c < 0$, we can write c as a positive number multiplied by -1; but, what kind of transformation do we get when we multiply the function or its argument by -1? When we multiply all the outputs by -1, we get a **reflection** about the x-axis. When we multiply all inputs by -1, we get a reflection about the y-axis. For example, the graph of $f(x) = -(x^3 + 1)$ is the graph of $y = (x^3 + 1)$ reflected about the x-axis. The graph of $f(x) = (-x)^3 + 1$ is the graph of $y = x^3 + 1$ reflected about the y-axis (Figure 5.3.13).

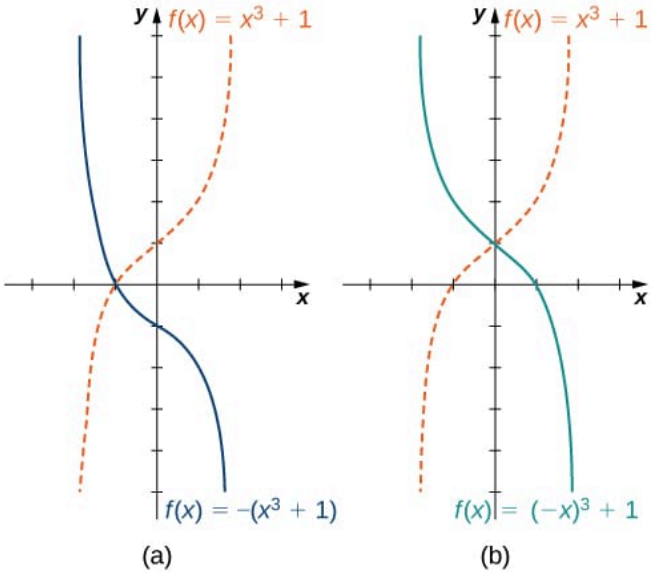


Figure 5.3.13: (a) The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about the x-axis. (b) The graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected about the y-axis.

If the graph of a function consists of more than one transformation of another graph, it is important to transform the graph in the correct order. Given a function $f(x)$, the graph of the related function $y = cf(a(x + b)) + d$ can be obtained from the graph of $y = f(x)$by performing the transformations in the following order.

- Horizontal shift of the graph of $y = f(x)$. If $b > 0$, shift left. If $b < 0$ shift right.
- Horizontal scaling of the graph of $y = f(x + b)$ by a factor of $|a|$. If $a < 0$, reflect the graph about the y-axis.
- Vertical scaling of the graph of $y = f(a(x + b))$ by a factor of $|c|$. If $c < 0$, reflect the graph about the x -axis.
- Vertical shift of the graph of $y = cf(a(x + b))$. If $d > 0$, shift up. If $d < 0$, shift down.

We can summarize the different transformations and their related effects on the graph of a function in the following table.

Transformation of $f(c > 0)$	Effect of the graph of f
$f(x) + c$	Vertical shift up c units
$f(x) - c$	Vertical shift down c units
$f(x + c)$	Shift left by c units
$f(x - c)$	Shift right by c units
$cf(x)$	Vertical stretch if $c > 1$; vertical compression if \(0
$f(cx)$	Horizontal stretch if \(0 horizontal compression if $c > 1$
$-f(x)$	Reflection about the x-axis
$f(-x)$	Reflection about the y-axis

Example 5.3.9: Transforming a Function

For each of the following functions, a. and b., sketch a graph by using a sequence of transformations of a well-known function.

a. $f(x) = -|x + 2| - 3$
b. $f(x) = \sqrt{x} + 1$

Solution

1.Starting with the graph of $y = |x|$, shift 2 units to the left, reflect about the x-axis, and then shift down 3 units.

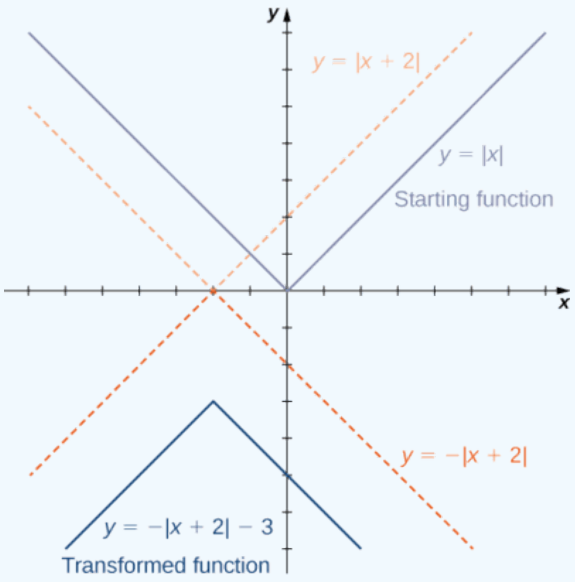


Figure 5.3.14: The function $f(x) = -|x + 2| - 3$ can be viewed as a sequence of three transformations of the function $y = |x|$.

2. Starting with the graph of $y = \sqrt{x}$, reflect about the y-axis, stretch the graph vertically by a factor of 3, and move up 1 unit.

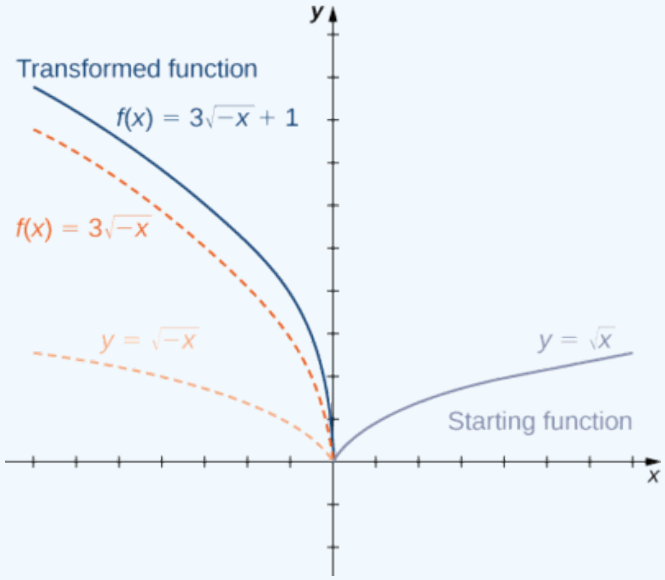


Figure 5.3.15: The function $f(x) = \sqrt{x} + 1$can be viewed as a sequence of three transformations of the function $y = \sqrt{x}$.

Exercise 5.3.9

Describe how the function $f(x) = -(x+1)^2 - 4$ can be graphed using the graph of $y = x^2$ and a sequence of transformations

Answer available at the end of this section.

Key Concepts

- The power function $f(x) = x^n$ is an even function if n is even and $n \neq 0$, and it is an odd function if n is odd.
- The root function $f(x) = x^{1/n}$ has the domain $[0, \infty)$ if n is even and the domain $(-\infty, \infty)$ if n is odd. If n is odd, then $f(x) = x^{1/n}$ is an odd function.
- The domain of the rational function $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomial functions, is the set of x such that $q(x) \neq 0$.
- Functions that involve the basic operations of addition, subtraction, multiplication, division, and powers are algebraic functions.
- A polynomial function f with degree $n \geq 1$ satisfies $f(x) \to \pm\infty$ as $x \to \pm\infty$. The sign of the output as $x \to \infty$ depends on the sign of the leading coefficient only and on whether n is even or odd.
- Vertical and horizontal shifts, vertical and horizontal scalings, and reflections about the x- and y-axes are examples of transformations of functions.

Key Equations

- **Point-slope equation of a line**

$$y - y_1 = m(x - x_1)$$

- **Slope-intercept form of a line**

$$y = mx + b$$

- **Standard form of a line**

$$ax + by = c$$

- **Polynomial function**

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Glossary

algebraic function
: a function involving any combination of only the basic operations of addition, subtraction, multiplication, division, powers, and roots applied to an input variable x

cubic function
: a polynomial of degree 3; that is, a function of the form $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$

degree
: for a polynomial function, the value of the largest exponent of any term

linear function
: a function that can be written in the form $f(x) = mx + b$

logarithmic function
: a function of the form $f(x) = \log_b(x)$ for some base $b > 0,\ b \neq 1$ such that $y = \log_b(x)$ if and only if $b^y = x$

mathematical model
A method of simulating real-life situations with mathematical equations

piecewise-defined function
a function that is defined differently on different parts of its domain

point-slope equation
equation of a linear function indicating its slope and a point on the graph of the function

polynomial function
a function of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$

power function
a function of the form $f(x) = x^n$ for any positive integer $n \geq 1$

quadratic function
a polynomial of degree 2; that is, a function of the form $f(x) = ax^2 + bx + c$ where $a \neq 0$

rational function
a function of the form $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials

root function
a function of the form $f(x) = x^{1/n}$ for any integer $n \geq 2$

slope
the change in y for each unit change in x

slope-intercept form
equation of a linear function indicating its slope and y-intercept

transformation of a function
a shift, scaling, or reflection of a function

Answers

Exercise 5.3.1
The slope $m = \Delta y/\Delta x$.

Exercise 5.3.1
$m = 1/2$.
The point-slope form is $y - 4 = \frac{1}{2}(x - 1)$.
The slope-intercept form is $y = \frac{1}{2}x + \frac{7}{2}$.

Exercise 5.3.3
Use the Quadratic Formula.

Exercise 5.3.3
The zeros are $x = 1 \pm \sqrt{3}/3$. The parabola opens upward.

Exercise 5.3.5
The denominator cannot be zero. Solve the equation $y = (5x + 2)/(2x - 1)$ for x to find the range.
The domain is the set of real numbers x such that $x \neq 1/2$. The range is the set $\{y \mid y \neq 5/2\}$.

Exercise 5.3.6
Determine the values of x when the expression in the denominator of f is nonzero, and find the values of x when the expression inside the radical of g is nonnegative.

Exercise 5.3.6
The domain of f is $(-\infty, \infty)$. The domain of g is $\{x \mid x \geq 1/5\}$.

Exercise 5.3.8
The piecewise-defined function is constant on the intervals $(0, 1]$, $(1, 2]$, ...

Exercise 5.3.8
\[C(x)=\begin{cases}49, 0

Exercise 5.3.9
Shift the graph $y = x^2$ to the left 1 unit, reflect about the x-axis, then shift down 4 units.

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5.3.1: A.3.1- Homework

In exercises 1 - 8, for each pair of points,

a. find the slope of the line passing through the points and

b. indicate whether the line is increasing, decreasing, horizontal, or vertical.

1) $(-2, 4)$ and $(1, 1)$

Answer

a. $m = -1$
b. Decreasing

2) $(-1, 4)$ and $(3, -1)$

3) $(3, 5)$ and $(-1, 2)$

Answer

a. $m = 3/4$
b. Increasing

4) $(6, 4)$ and $(4, -3)$

5) $(2, 3)$ and $(5, 7)$

Answer

a. $m = 4/3$
b. Increasing

6) $(1, 9)$ and $(-8, 5)$

7) $(2, 4)$ and $(1, 4)$

Answer

a. $m = 0$
b. Horizontal

8) $(1, 4)$ and $(1, 0)$

In exercises 9 - 16, write the equation of the line satisfying the given conditions in slope-intercept form.

9) Slope $=-6$, passes through $(1, 3)$

Answer

$y = -6x + 9$

10) Slope $=3$, passes through $(-3, 2)$

11) Slope $=\frac{1}{3}$, passes through $(0, 4)$

Answer

$y = \frac{1}{3}x + 4$

12) Slope $=\frac{2}{5}$, x-intercept $=8$

13) Passing through $(2, 1)$ and $(-2, -1)$

Answer

$y = \frac{1}{2}x$

14) Passing through $(-3, 7)$ and $(1, 2)$

15) x-intercept =5 and y-intercept =-3

Answer

$y = \frac{3}{5}x - 3$

16) x-Intercept =-6 and y-intercept =9

In exercises 17 - 24, for each linear equation,

a. give the slope m and y-intercept b, if any, and

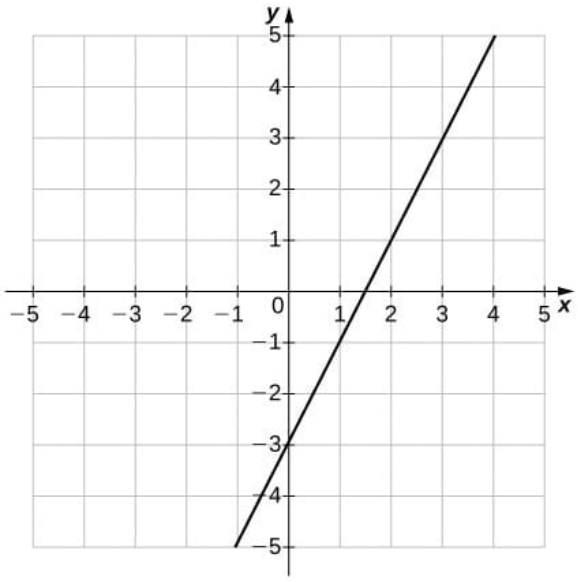
b. graph the line.

17) $y = 2x - 3$

Answer

a. $m = 2, \quad b = -3$

b.



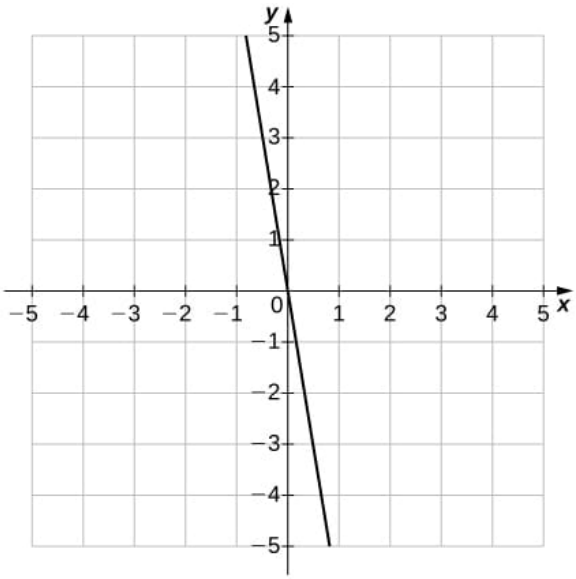
18) $y = -\frac{1}{7}x + 1$

19) $f(x) = -6x$

Answer

a. $m = -6, \quad b = 0$

b.



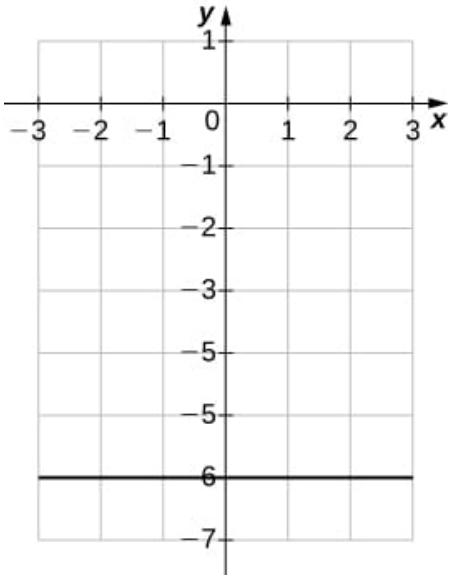
20) $f(x) = -5x + 4$

21) $4y + 24 = 0$

Answer

a. $m = 0, \quad b = -6$

b.



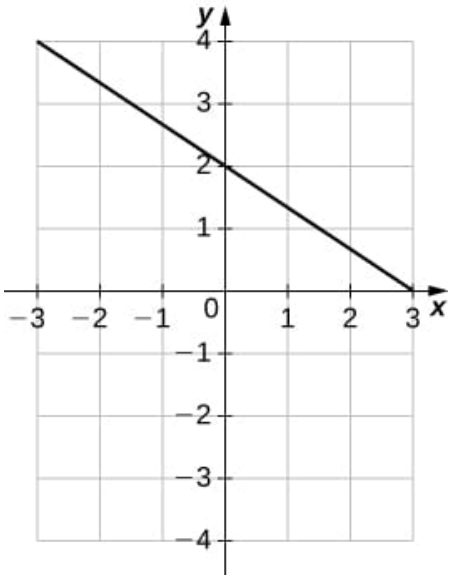
22) $8x - 4 = 0$

23) $2x + 3y = 6$

Answer

a. $m = -\frac{2}{3}, \quad b = 2$

b.



24) $6x - 5y + 15 = 0$

In exercises 25 - 29, for each polynomial,

a. find the degree;

b. find the zeros, if any;

c. find the y-intercept(s), if any;

d. use the leading coefficient to determine the graph's end behavior; and

e. determine algebraically whether the polynomial is even, odd, or neither.

25) $f(x) = 2x^2 - 3x - 5$

Answer

a. 2
b. $\frac{5}{2},\ -1$;
c. -5
d. Both ends rise
e. Neither

26) $f(x) = -3x^2 + 6x$

27) $f(x) = \frac{1}{2}x^2 - 1$

Answer

a. 2
b. \pm $\sqrt{2}$
c. -1
d. Both ends rise
e. Even

28) $f(x) = x^3 + 3x^2 - x - 3$

29) $f(x) = 3x - x^3$

Answer

a. 3

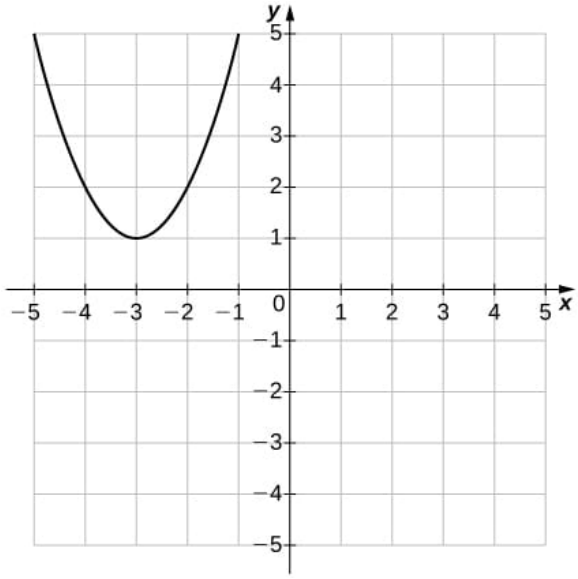
b. 0, \pm $\sqrt{3}$
c. 0
d. Left end rises, right end falls
e. Odd

For exercises 30 - 31, use the graph of $f(x) = x^2$ to graph each transformed function g.

30) $g(x) = x^2 - 1$

31) $g(x) = (x + 3)^2 + 1$

Answer

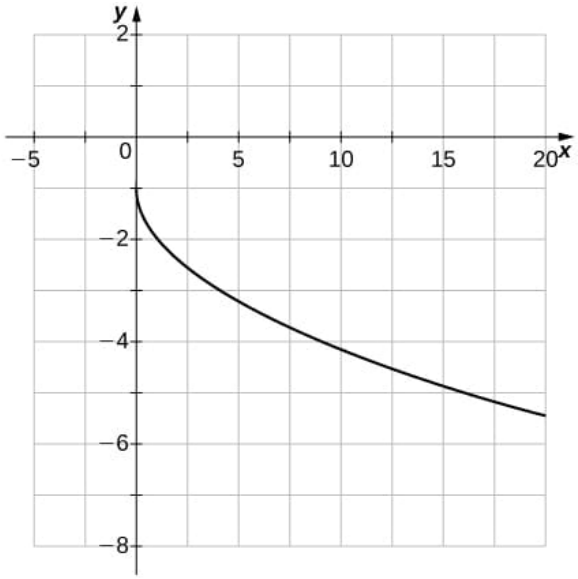


For exercises 32 - 33, use the graph of $f(x) = \sqrt{x}$ to graph each transformed function g.

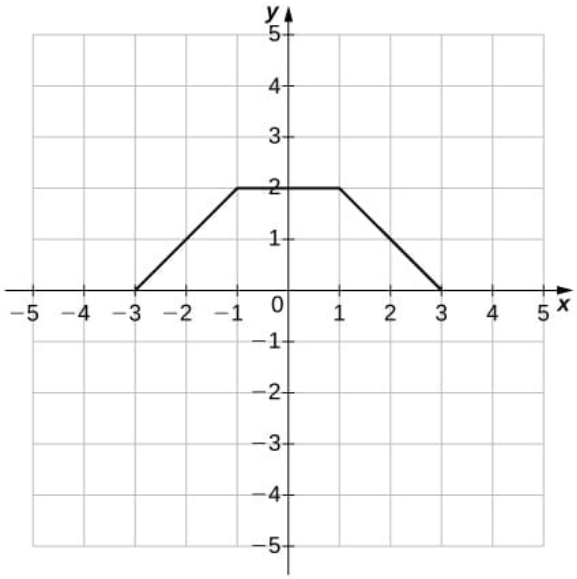
32) $g(x) = \sqrt{x + 2}$

33) $g(x) = -\sqrt{x} - 1$

Answer



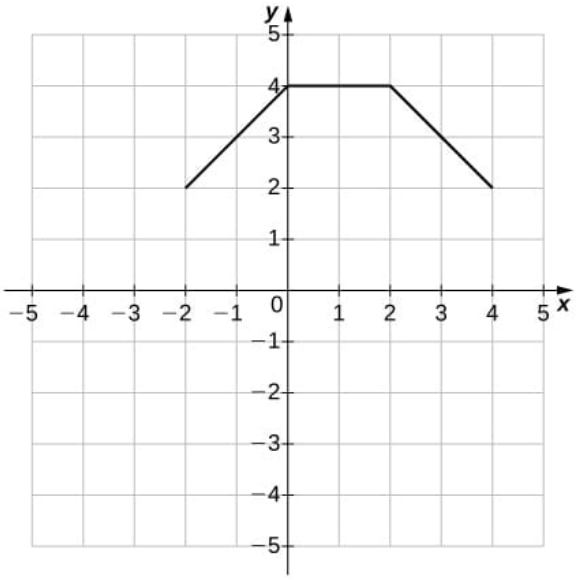
For exercises 34 - 35, use the graph of $y = f(x)$ to graph each transformed function g.



34) $g(x) = f(x) + 1$

35) $g(x) = f(x - 1) + 2$

Answer



In exercises 36 - 39, for each of the piecewise-defined functions,

a. evaluate at the given values of the independent variable, and

b. sketch the graph.

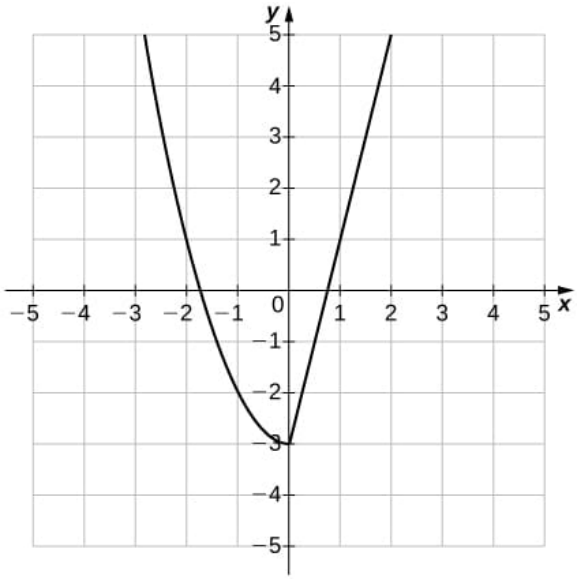
36) $f(x) = \begin{cases} 4x + 3, & \text{if } x \le 0 \\ -x + 1, & \text{if } x > 0 \end{cases}$; $f(-3)$; $f(0)$; $f(2)$

37) $f(x) = \begin{cases} x^2 - 3, & \text{if } x \le 0 \\ 4x - 3, & \text{if } x > 0 \end{cases}$; $f(-4)$; $f(0)$; $f(2)$

Answer

a. $f(-4) = 13, \quad f(0) = -3, \quad f(2) = 5$

b.



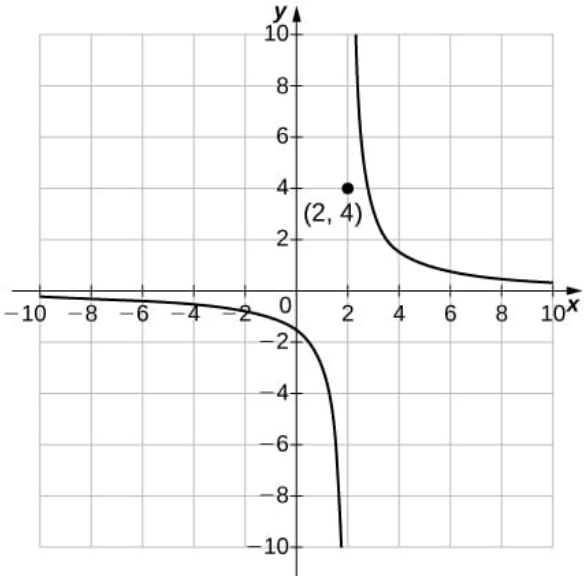
38) $h(x) = \begin{cases} x+1, & \text{if } x \le 5 \\ 4, & \text{if } x > 5 \end{cases}; \quad h(0); \; h(\pi); \; h(5)$

39) $g(x) = \begin{cases} \dfrac{3}{x-2}, & \text{if } x \neq 2 \\ 4, & \text{if } x = 2 \end{cases}; \quad g(0); \; g(-4); \; g(2)$

Answer

a. $g(0) = -\frac{3}{2}, \; g(-4) = -\frac{1}{2}, \; g(2) = 4$

b.



In exercises 40 - 43, determine whether the statement is *true or false*. Explain why.

40) $g(x) = \sqrt[3]{x}$ is an odd root function.

Answer

True; $n = 3$

41) A logarithmic function is an algebraic function.

42) A function of the form $f(x) = x^b$, where b is a real valued constant, is an exponential function.

Answer

False; $f(x) = x^b$, where b is a real-valued constant, is a power function

43) The domain of an even root function is all real numbers.

44) [Technology Required] A company purchases some computer equipment for $20,500. At the end of a 3-year period, the value of the equipment has decreased linearly to $12,301.

a. Find a function $y = V(t)$ that determines the value V of the equipment at the end of t years.

b. Find and interpret the meaning of the x- and y-intercepts for this situation.

c. What is the value of the equipment at the end of 5 years?

d. When will the value of the equipment be $3000?

Answer

a. $V(t) = -2733t + 20500$
b. $(0, 20, 500)$ means that the initial purchase price of the equipment is $20,500; $(7.5, 0)$ means that in 7.5 years the computer equipment has no value.
c. $6835
d. In approximately 6.4 years

45) [Technology Required] Total online shopping during the Christmas holidays has increased dramatically during the past 5 years. In 2012 $(t = 0)$,total online holiday sales were $42.3 billion, whereas in 2013 they were $48.1 billion.

a. Find a linear function S that estimates the total online holiday sales in the year t.

b. Interpret the slope of the graph of S.

c. Use part a. to predict the year when online shopping during Christmas will reach $60 billion.

46) [Technology Required] A family bakery makes cupcakes and sells them at local outdoor festivals. For a music festival, there is a fixed cost of $125 to set up a cupcake stand. The owner estimates that it costs $0.75 to make each cupcake. The owner is interested in determining the total cost C as a function of number of cupcakes made.

a. Find a linear function that relates cost C to x, the number of cupcakes made.

b. Find the cost to bake 160 cupcakes.

c. If the owner sells the cupcakes for $1.50 apiece, how many cupcakes does she need to sell to start making profit? (Hint: Use the INTERSECTION function on a calculator to find this number.)

Answer

a. $C = 0.75x + 125$
b. $245
c. 167 cupcakes

47) [Technology Required] A house purchased for $250,000 is expected to be worth twice its purchase price in 18 years.

a. Find a linear function that models the price P of the house versus the number of years t since the original purchase.

b. Interpret the slope of the graph of P.

c. Find the price of the house 15 years from when it was originally purchased.

48) [Technology Required] A car was purchased for $26,000. The value of the car depreciates by $1500 per year.

a. Find a linear function that models the value V of the car after t years.

b. Find and interpret $V(4)$.

Answer

a. $V(t) = -1500t + 26,000$
b. In 4 years, the value of the car is $20,000.

49) [Technology Required] A condominium in an upscale part of the city was purchased for $432,000. In 35 years it is worth $60,500. Find the rate of depreciation.

50) [Technology Required] The total cost C (in thousands of dollars) to produce a certain item is modeled by the function $C(x) = 10.50x + 28,500$, where x is the number of items produced. Determine the cost to produce 175 items.

Answer

$30,337,500

51) [Technology Required] A professor asks her class to report the amount of time t they spent writing two assignments. Most students report that it takes them about 45 minutes to type a four-page assignment and about 1.5 hours to type a nine-page assignment.

a. Find the linear function $y = N(t)$ that models this situation, where N is the number of pages typed and t is the time in minutes.

b. Use part a. to determine how many pages can be typed in 2 hours.

c. Use part a. to determine how long it takes to type a 20-page assignment.

52) [Technology Required] The output (as a percent of total capacity) of nuclear power plants in the United States can be modeled by the function $P(t) = 1.8576t + 68.052$, where t is time in years and $t = 0$ corresponds to the beginning of 2000. Use the model to predict the percentage output in 2015.

Answer

96% of the total capacity

53) [Technology Required] The admissions office at a public university estimates that 65% of the students offered admission to the class of 2019 will actually enroll.

a. Find the linear function $y = N(x)$, where N is the number of students that actually enroll and x is the number of all students offered admission to the class of 2019.

b. If the university wants the 2019 freshman class size to be 1350, determine how many students should be admitted.

Contributors

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5.4: A.4- Inverse Functions

Learning Objectives

- Determine the conditions for when a function has an inverse.
- Use the Horizontal Line Test to recognize when a function is one-to-one.
- Find the inverse of a given function.
- Draw the graph of an inverse function.

An inverse function reverses the operation done by a particular function. In other words, whatever a function does, the inverse function undoes it. In this section, we define an inverse function formally and state the necessary conditions for an inverse function to exist. We examine how to find an inverse function and study the relationship between the graph of a function and the graph of its inverse.

Existence of an Inverse Function

We begin with an example. Given a function f and an output $y = f(x)$, we are often interested in finding what value or values x were mapped to y by f. For example, consider the function $f(x) = x^3 + 4$. Since any output $y = x^3 + 4$, we can solve this equation for x to find that the input is $x = \sqrt[3]{y - 4}$. This equation defines x as a function of y. Denoting this function as f^{-1}, and writing $x = f^{-1}(y) = \sqrt[3]{y - 4}$, we see that for any x in the domain of f, $f^{-1}(f(x)) = f^{-1}(x^3 + 4) = x$. Thus, this new function, f^{-1}, "undid" what the original function f did. A function with this property is called the **inverse function** of the original function.

Definition: Inverse Functions

Given a function f with domain D and range R, its **inverse function** (if it exists) is the function f^{-1} with domain R and range D such that $f^{-1}(y) = x$ if and only if $f(x) = y$. In other words, for a function f and its inverse f^{-1},

$$f^{-1}(f(x)) = x$$

for all x in D and

$$f(f^{-1}(y)) = y$$

for all y in R.

Note that f^{-1} is read as "f inverse." Here, the -1 is not used as an exponent so

$$f^{-1}(x) \neq \frac{1}{f(x)}.$$

Figure $5.4.1$shows the relationship between the domain and range of f and the domain and range of f^{-1}.

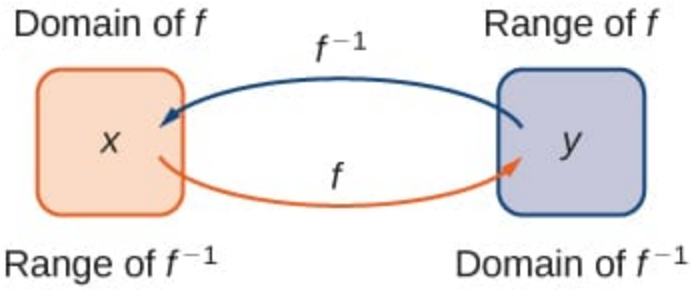


Figure 5.4.1: Given a function f and its inverse f^{-1}, $f^{-1}(y) = x$ if and only if $f(x) = y$. The range of f becomes the domain of f^{-1} and the domain of f becomes the range of f^{-1}.

Recall that a function has exactly one output for each input. Therefore, to define an inverse function, we need to map each input to exactly one output. For example, let's try to find the inverse function for $f(x) = x^2$. Solving the equation $y = x^2$ for x, we arrive at the equation $x = \pm\sqrt{y}$. This equation does not describe x as a function of y because there are two solutions to this equation for every $y > 0$. The problem with trying to find an inverse function for $f(x) = x^2$ is that two inputs are sent to the same output for each output $y > 0$. The function $f(x) = x^3 + 4$ discussed earlier did not have this problem. For that function, each input was sent to a different output. A function that sends each input to a different output is called a **one-to-one function**.

Definition: One-to-One functions

We say a function f is a **one-to-one function** if $f(x_1) \neq f(x_2)$ when $x_1 \neq x_2$.

One way to determine whether a function is one-to-one is by looking at its graph. If a function is one-to-one, then no two inputs can be sent to the same output. Therefore, if we draw a horizontal line anywhere in the xy-plane, according to the **Horizontal Line Test**, it cannot intersect the graph more than once. We note that the Horizontal Line Test is different from the Vertical Line Test. The Vertical Line Test determines whether a graph is the graph of a function. The Horizontal Line Test determines whether a function is one-to-one (Figure 5.4.2).

Theorem 5.4.1: Horizontal Line Test

A function f is one-to-one if and only if every horizontal line intersects the graph of f no more than once.

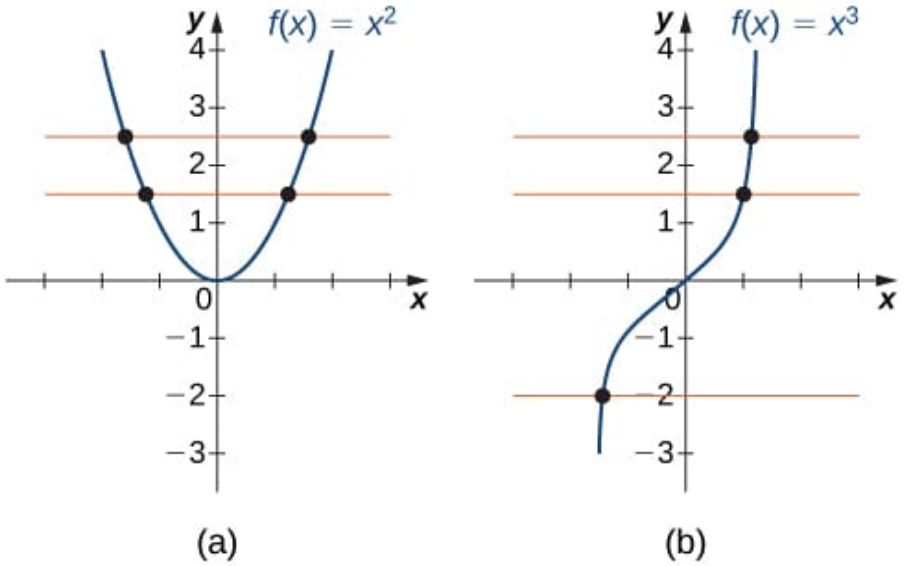
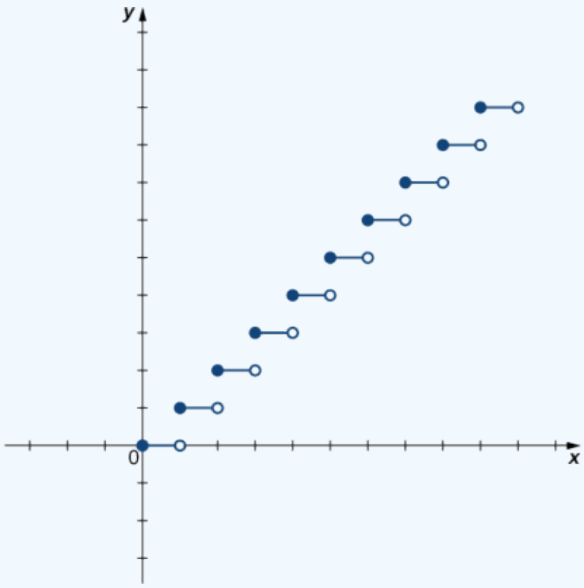


Figure 5.4.2: (a) The function $f(x) = x^2$ is not one-to-one because it fails the Horizontal Line Test. (b) The function $f(x) = x^3$ is one-to-one because it passes the Horizontal Line Test.

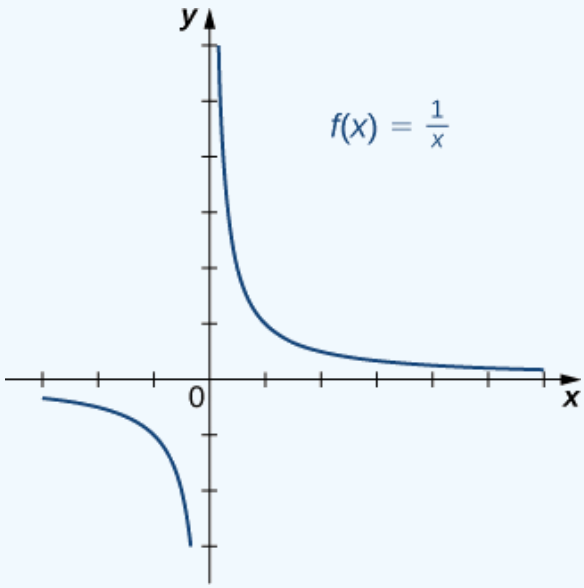
Example 5.4.1: Determining Whether a Function Is One-to-One

For each of the following functions, use the Horizontal Line Test to determine whether it is one-to-one.

a)

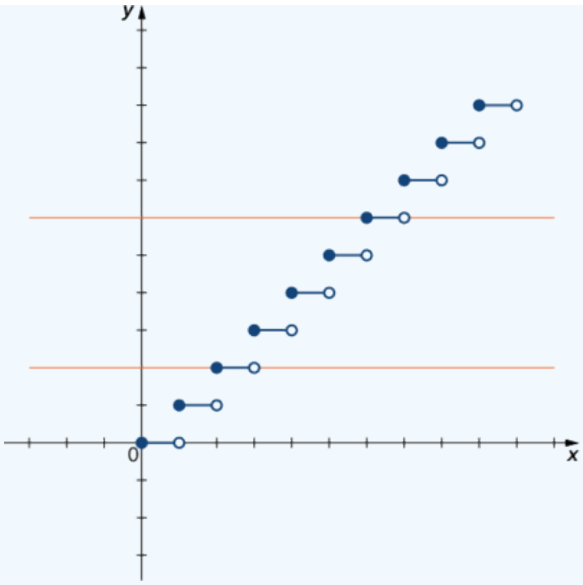


b)

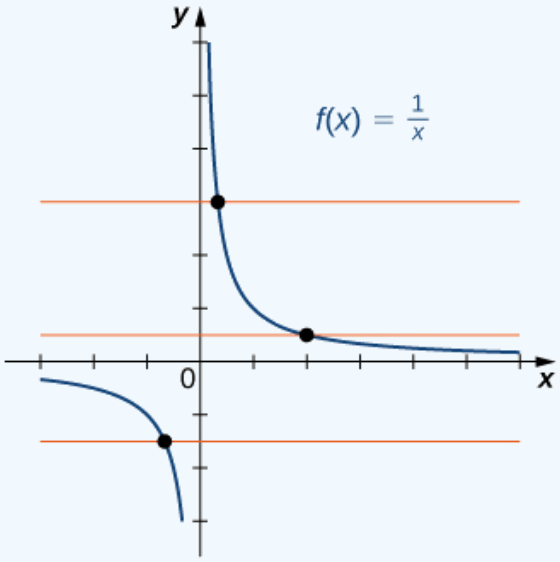


Solution

a) Since the horizontal line $y = n$ for any integer $n \geq 0$ intersects the graph more than once, this function is not one-to-one.

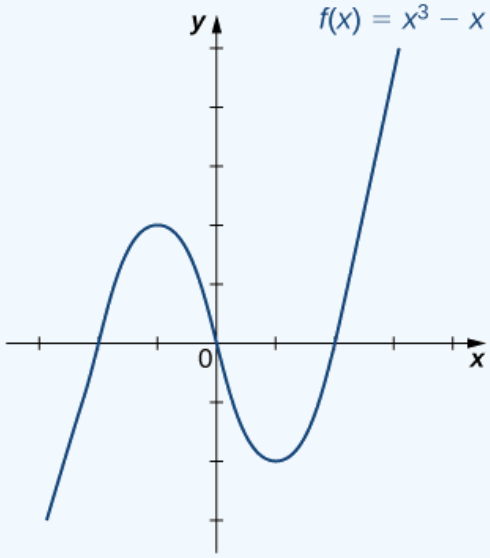


b) Since every horizontal line intersects the graph once (at most), this function is one-to-one.



Exercise 5.4.1

Is the function f graphed in the following image one-to-one?



Solution available at the end of this section.

Answer available at the end of this section.

Finding the Inverse of a Function

We can now consider one-to-one functions and show how to find their inverses. Recall that a function maps elements in the domain of f to elements in the range of f. The inverse function maps each element from the range of f back to its corresponding element from the domain of f. Therefore, to find the inverse function of a one-to-one function f, given any y in the range of f, we need to determine which x in the domain of f satisfies $f(x) = y$. Since f is one-to-one, there is exactly one such value x. We can find that value x by solving the equation $f(x) = y$ for x. Doing so, we are able to write x as a function of y where the domain of this function is the range of f and the range of this new function is the domain of f. Consequently, this function is the inverse of f , and we write $x = f^{-1}(y)$. Since we typically use the variable x to denote the independent variable and y to denote the dependent variable, we often interchange the roles of x and y, and write $y = f^{-1}(x)$. Representing the inverse function in this way is also helpful later when we graph a function f and its inverse f^{-1} on the same axes.

Example 5.4.2: Finding an Inverse Function

Find the inverse for the function $f(x) = 3x - 4$. State the domain and range of the inverse function. Verify that $f^{-1}(f(x)) = x$.

Solution

If $y = 3x - 4$, then $3x = y + 4$ and $x = \frac{1}{3}y + \frac{4}{3}$.

Rewrite as $y = \frac{1}{3}x + \frac{4}{3}$ and let $y = f^{-1}(x)$.Therefore, $f^{-1}(x) = \frac{1}{3}x + \frac{4}{3}$.

Since the domain of f is $(-\infty, \infty)$, the range of f^{-1} is $(-\infty, \infty)$. Since the range of f is $(-\infty, \infty)$, the domain of f^{-1} is $(-\infty, \infty)$.

You can verify that $f^{-1}(f(x)) = x$ by writing

$$f^{-1}(f(x)) = f^{-1}(3x - 4) = \tfrac{1}{3}(3x - 4) + \tfrac{4}{3} = x - \tfrac{4}{3} + \tfrac{4}{3} = x.$$

Note that for $f^{-1}(x)$ to be the inverse of $f(x)$, both $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$ for all x in the domain of the inside function.

Exercise 5.4.2

Find the inverse of the function $f(x) = 3x/(x-2)$. State the domain and range of the inverse function.

Answer available at the end of this section.

Graphing Inverse Functions

Let's consider the relationship between the graph of a function f and the graph of its inverse. Consider the graph of f shown in Figure 5.4.3 and a point (a, b) on the graph. Since $b = f(a)$, then $f^{-1}(b) = a$. Therefore, when we graph f^{-1}, the point (b, a) is on the graph. As a result, the graph of f^{-1} is a reflection of the graph of f about the line $y = x$.

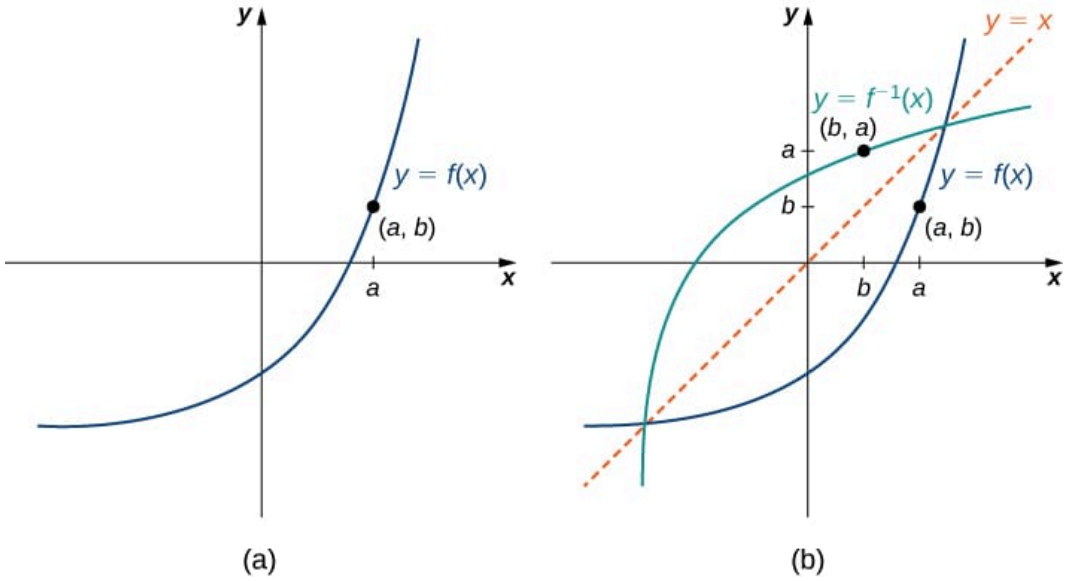
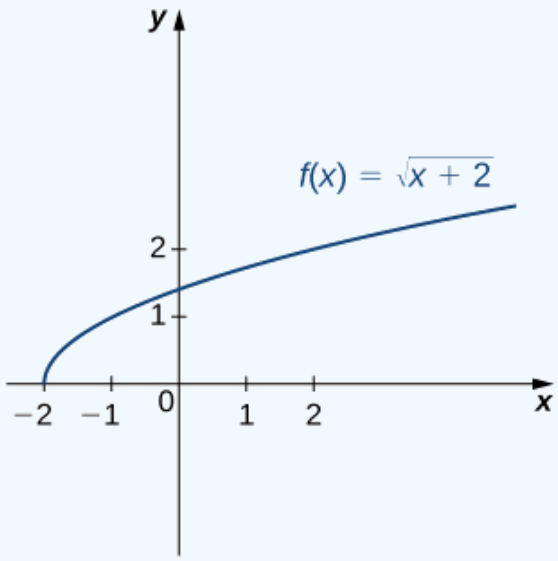


Figure 5.4.3: (a) The graph of this function f shows point (a, b) on the graph of f. (b) Since (a, b) is on the graph of f, the point (b, a) is on the graph of f^{-1}. The graph of f^{-1} is a reflection of the graph of f about the line $y = x$.

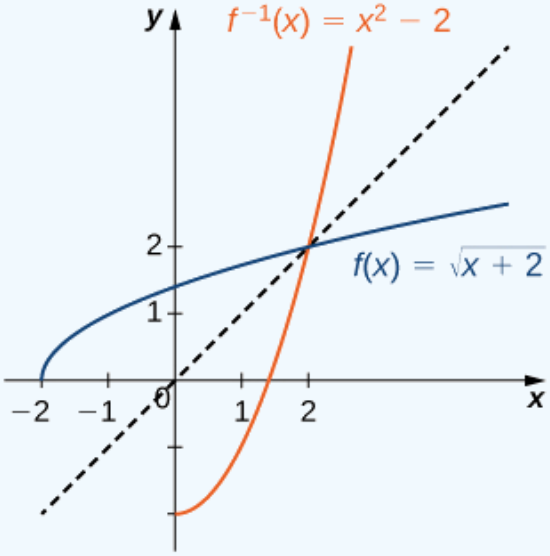
Example 5.4.3: Sketching Graphs of Inverse Functions

For the graph of f in the following image, sketch a graph of f^{-1} by sketching the line $y = x$ and using symmetry. Identify the domain and range of f^{-1}.



Solution

Reflect the graph about the line $y = x$. The domain of f^{-1} is $[0, \infty)$. The range of f^{-1} is $[-2, \infty)$. By using the preceding strategy for finding inverse functions, we can verify that the inverse function is $f^{-1}(x) = x^2 - 2$, as shown in the graph.



Exercise 5.4.3

Sketch the graph of $f(x) = 2x + 3$ and the graph of its inverse using the symmetry property of inverse functions.

Hint available at the end of this section.

Answer available at the end of this section.

Restricting Domains

As we have seen, $f(x) = x^2$ does not have an inverse function because it is not one-to-one. However, we can choose a subset of the domain of f such that the function is one-to-one. This subset is called a **restricted domain**. By restricting the domain of f, we can define a new function g such that the domain of g is the restricted domain of f and $g(x) = f(x)$ for all x in the domain of g. Then we can define an inverse function for g on that domain. For example, since $f(x) = x^2$ is one-to-one on the interval $[0, \infty)$, we can define a new function g such that the domain of g is $[0, \infty)$ and $g(x) = x^2$ for all x in its domain. Since g is a one-to-one function, it has an inverse function, given by the formula $g^{-1}(x) = \sqrt{x}$. On the other hand, the function $f(x) = x^2$ is also one-to-one on the domain $(-\infty, 0]$. Therefore, we could also define a new function h such that the domain of h is $(-\infty, 0]$ and $h(x) = x^2$ for all x in the domain of h. Then h is a one-to-one function and must also have an inverse. Its inverse is given by the formula $h^{-1}(x) = -\sqrt{x}$ (Figure 5.4.4).

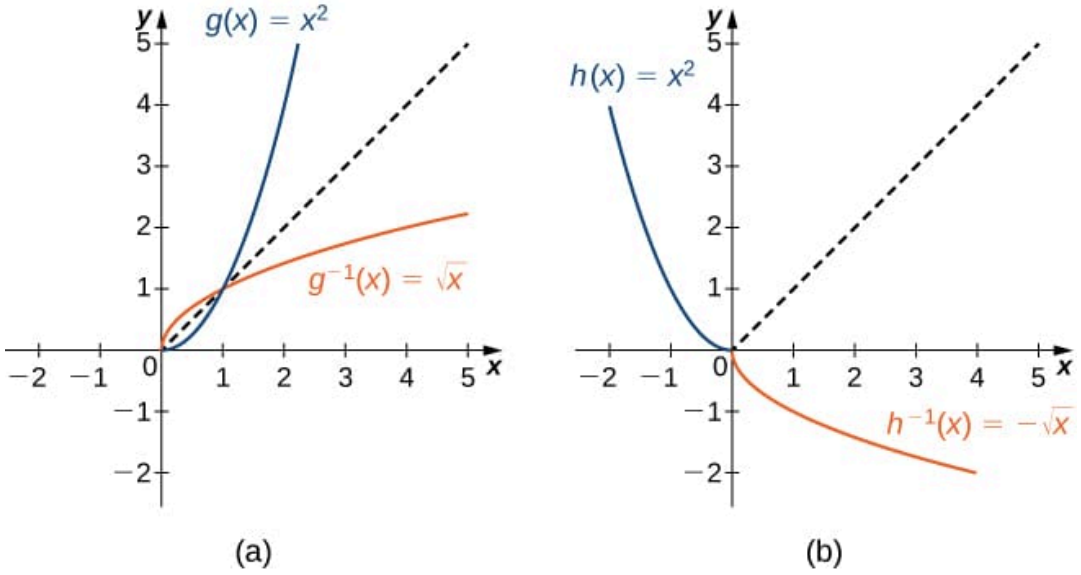


Figure 5.4.4: (a) For $g(x) = x^2$ restricted to $[0, \infty)$, $g^{-1}(x) = \sqrt{x}$. (b) For $h(x) = x^2$ restricted to $(-\infty, 0]$, $h^{-1}(x) = -\sqrt{x}$.

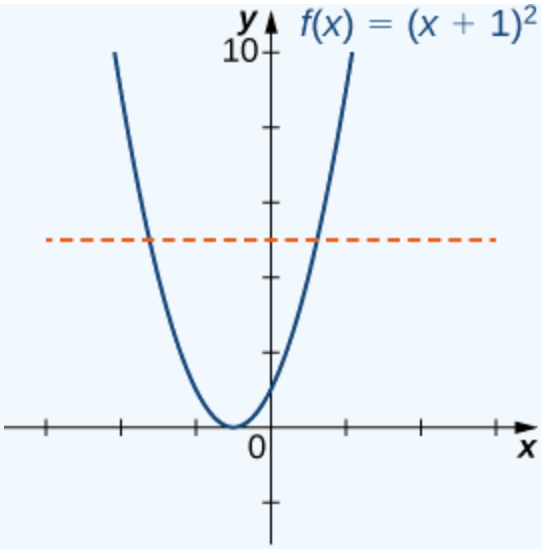
Example 5.4.4: Restricting the Domain

Consider the function $f(x) = (x + 1)^2$.

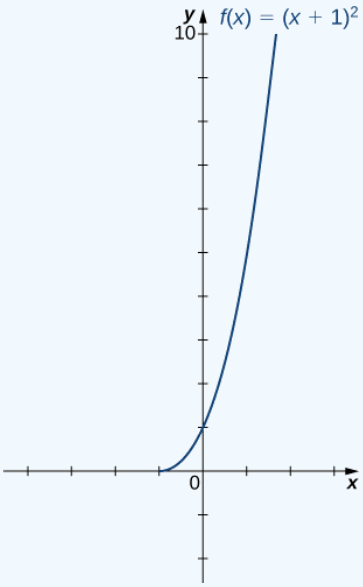
a. Sketch the graph of f and use the Horizontal Line Test to show that f is not one-to-one.
b. Show that f is one-to-one on the restricted domain $[-1, \infty)$. Determine the domain and range for the inverse of f on this restricted domain and find a formula for f^{-1}.

Solution

a) The graph of f is the graph of $y = x^2$ shifted left 1 unit. Since there exists a horizontal line intersecting the graph more than once, f is not one-to-one.



b) On the interval $[-1, \infty)$, f is one-to-one.



The domain and range of f^{-1} are given by the range and domain of f, respectively. Therefore, the domain of f^{-1} is $[0, \infty)$ and the range of f^{-1} is $[-1, \infty)$. To find a formula for f^{-1}, solve the equation $y = (x + 1)^2$ for x. If $y = (x + 1)^2$, then $x = -1 \pm \sqrt{y}$. Since we are restricting the domain to the interval where $x \geq -1$, we need $\pm\sqrt{y} \geq 0$. Therefore, $x = -1 + \sqrt{y}$. Interchanging x and y, we write $y = -1 + \sqrt{x}$ and conclude that $f^{-1}(x) = -1 + \sqrt{x}$.

Exercise 5.4.4

Consider $f(x) = 1/x^2$ restricted to the domain $(-\infty, 0)$. Verify that f is one-to-one on this domain. Determine the domain and range of the inverse of f and find a formula for f^{-1}.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- For a function to have an inverse, the function must be one-to-one. Given the graph of a function, we can determine whether the function is one-to-one by using the Horizontal Line Test.
- If a function is not one-to-one, we can restrict the domain to a smaller domain where the function is one-to-one and then define the inverse of the function on the smaller domain.
- For a function f and its inverse f^{-1}, $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1} and $f^{-1}(f(x)) = x$ for all x in the domain of f.
- The graph of a function f and its inverse f^{-1} are symmetric about the line $y = x$.

Key Equations

- **Inverse function**

  $f^{-1}(f(x)) = x$ for all x in D, and $f(f^{-1}(y)) = y$ for all y in R.

Glossary

Horizontal Line Test
: a function f is one-to-one if and only if every horizontal line intersects the graph of f, at most, once

inverse function
: for a function f, the inverse function f^{-1} satisfies $f^{-1}(y) = x$ if $f(x) = y$

one-to-one function
: a function f is one-to-one if $f(x_1) \neq f(x_2)$ if $x_1 \neq x_2$

restricted domain
: a subset of the domain of a function f

Answers

Exercise 5.4.1

Use the Horizontal Line Test.

Exercise 5.4.1

No

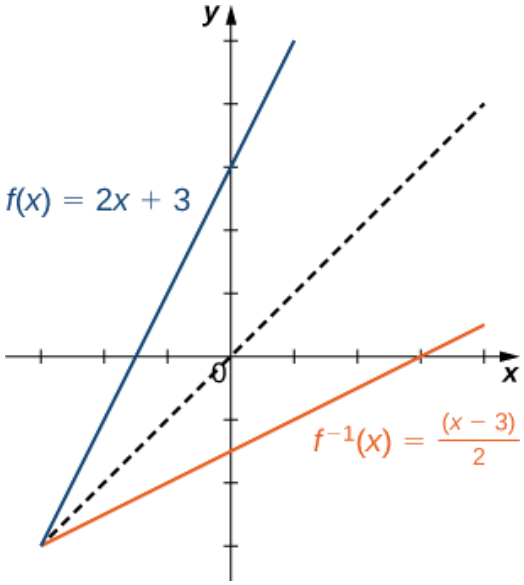
Exercise 5.4.2

$f^{-1}(x) = \dfrac{2x}{x-3}$. The domain of f^{-1} is $\{x \mid x \neq 3\}$. The range of f^{-1} is $\{y \mid y \neq 2\}$.

Exercise 5.4.3

The graphs are symmetric about the line $y = x$

Exercise 5.4.3



Exercise 5.4.4

The domain and range of f^{-1} is given by the range and domain of f, respectively. To find f^{-1}, solve $y = 1/x^2$ for x.

Exercise 5.4.4

The domain of f^{-1} is $(0, \infty)$. The range of f^{-1} is $(-\infty, 0)$. The inverse function is given by the formula $f^{-1}(x) = -1/\sqrt{x}$.

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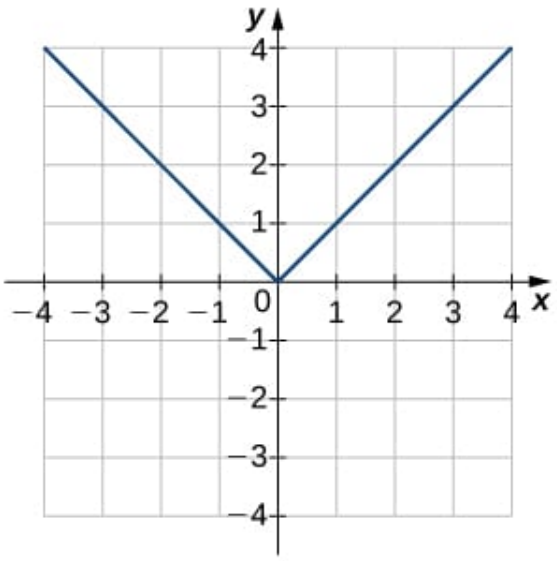
5.4.1: A.4.1- Homework

In exercises 1 - 6, use the horizontal line test to determine whether each of the given graphs is one-to-one.

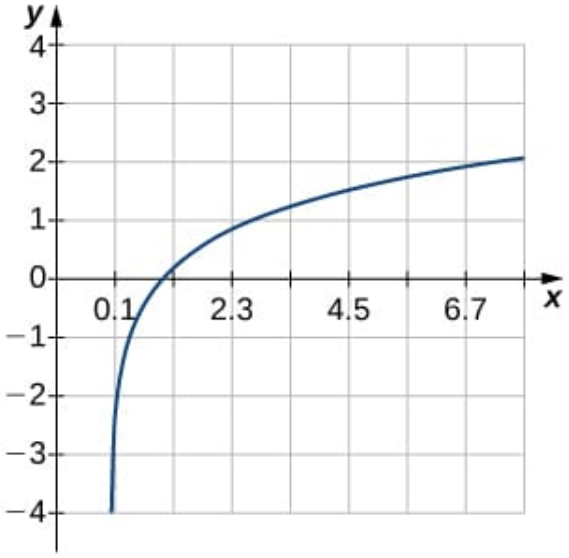
1)



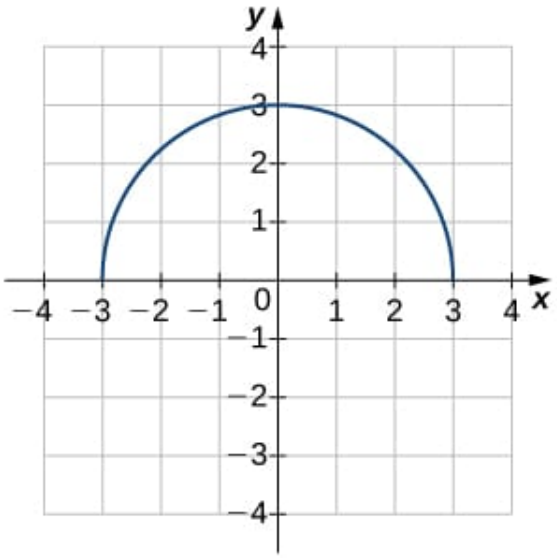
Answer

Not one-to-one

2)



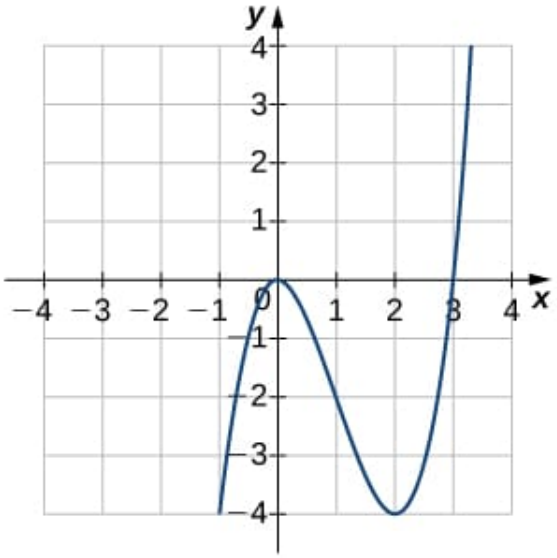
3)



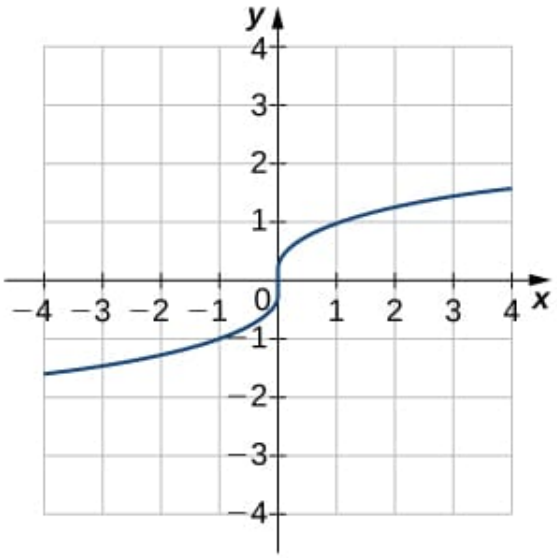
Answer

Not one-to-one

4)



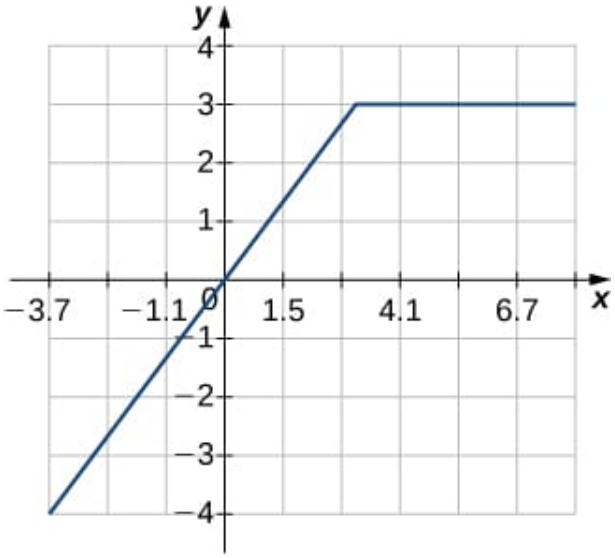
5)



Answer

One-to-one

6)



In exercises 7 - 12,

a. find the inverse function, and

b. find the domain and range of the inverse function.

7) $f(x) = x^2 - 4, \quad x \geq 0$

Answer

a. $f^{-1}(x) = \sqrt{x+4}$

b. Domain: $x \geq -4$, Range: $y \geq 0$

8) $f(x) = \sqrt[3]{x-4}$

9) $f(x) = x^3 + 1$

Answer

a. $f^{-1}(x) = \sqrt[3]{x-1}$
b. Domain: all real numbers, Range: all real numbers

10) $f(x) = (x-1)^2, \quad x \leq 1$

11) $f(x) = \sqrt{x-1}$

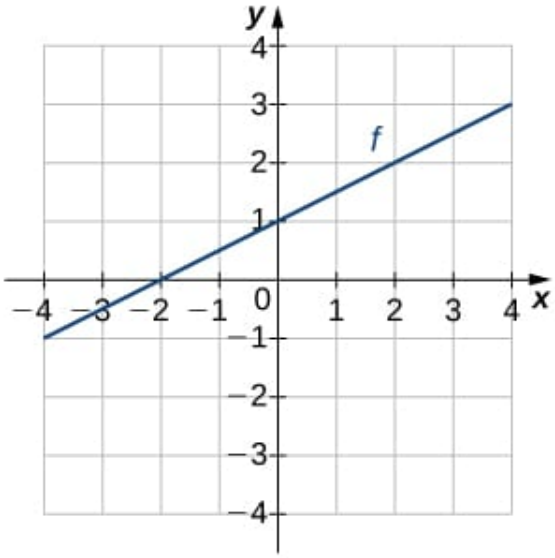
Answer

a. $f^{-1}(x) = x^2 + 1$,
b. Domain: $x \geq 0$, Range: $y \geq 1$

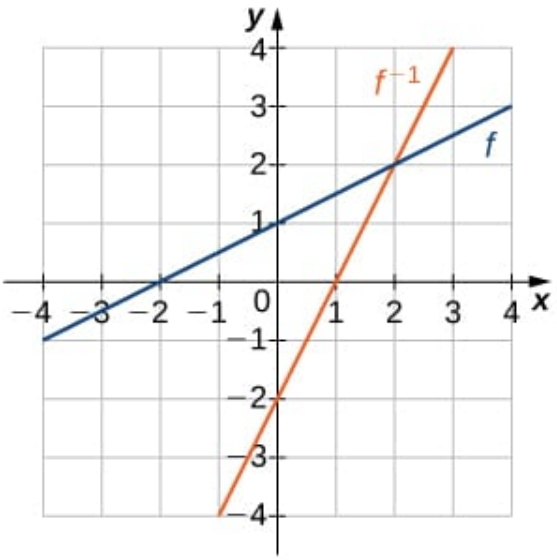
12) $f(x) = \dfrac{1}{x+2}$

In exercises 13 - 16, use the graph of f to sketch the graph of its inverse function.

13)



Answer



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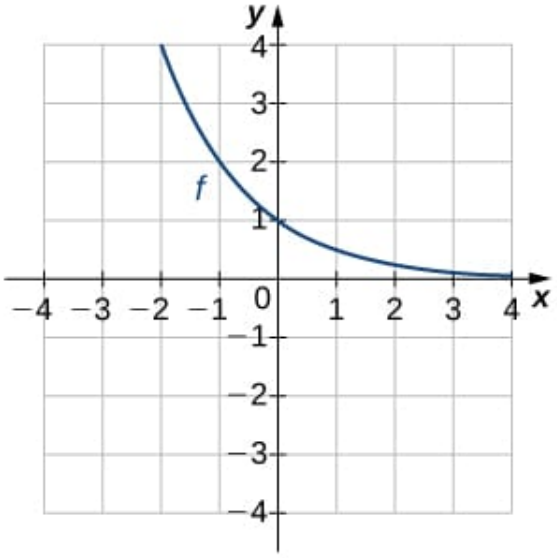

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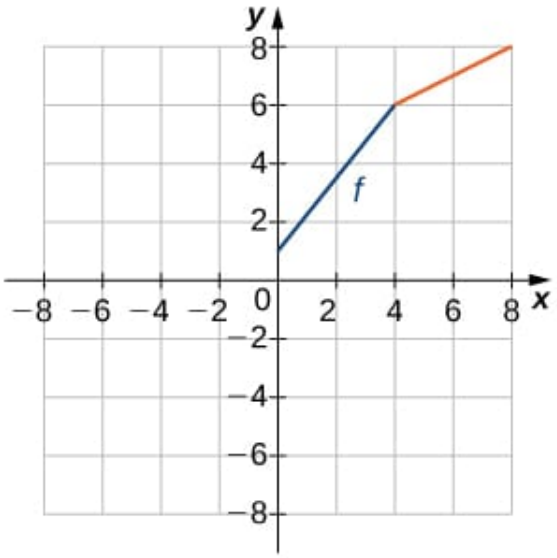
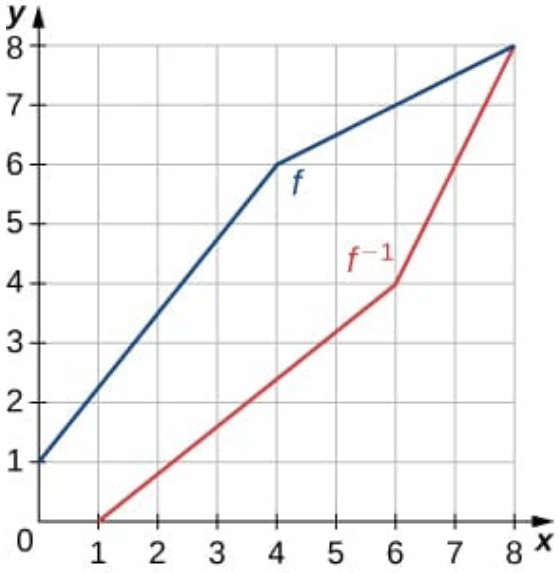
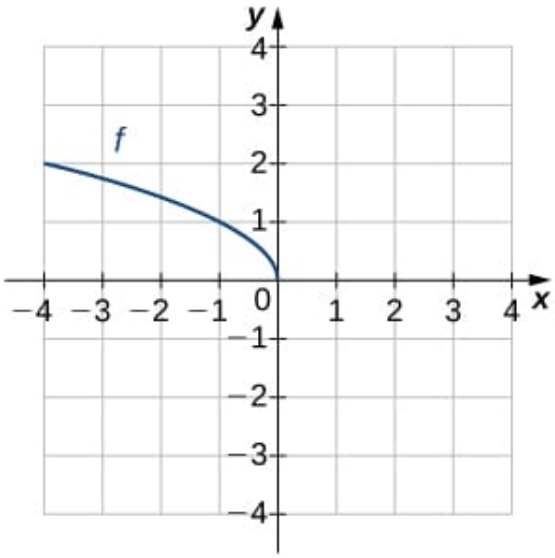


Answer



16)



In exercises 17 - 24, use composition to determine which pairs of functions are inverses.

17) $f(x) = 8x, \quad g(x) = \dfrac{x}{8}$

Answer

These are inverses.

18) $f(x) = 8x + 3, \quad g(x) = \dfrac{x - 3}{8}$

19) $f(x) = 5x - 7, \quad g(x) = \dfrac{x + 5}{7}$

Answer

These are not inverses.

20) $f(x) = \frac{2}{3}x + 2, \quad g(x) = \frac{3}{2}x + 3$

21) $f(x) = \dfrac{1}{x - 1}, \ x \neq 1, \quad g(x) = \dfrac{1}{x} + 1, \ x \neq 0$

Answer

These are inverses.

22) $f(x) = x^3 + 1, \quad g(x) = (x - 1)^{1/3}$

23) $f(x) = x^2 + 2x + 1, \ x \geq -1, \quad g(x) = -1 + \sqrt{x}, \ x \geq 0$

Answer

These are inverses.

24) $f(x) = \sqrt{4 - x^2}, \ 0 \leq x \leq 2, \quad g(x) = \sqrt{4 - x^2}, \ 0 \leq x \leq 2$

25) The function $C = T(F) = (5/9)(F - 32)$ converts degrees Fahrenheit to degrees Celsius.

a) Find the inverse function $F = T^{-1}(C)$

b) What is the inverse function used for?

26) [Technology Required] The velocity V (in centimeters per second) of blood in an artery at a distance x cm from the center of

the artery can be modeled by the function $V = f(x) = 500(0.04 - x^2)$ for $0 \le x \le 0.2$.

a) Find $x = f^{-1}(V)$.

b) Interpret what the inverse function is used for.

c) Find the distance from the center of an artery with a velocity of 15 cm/sec, 10 cm/sec, and 5 cm/sec.

Answer

a. $x = f^{-1}(V) = \sqrt{0.04 - \frac{V}{500}}$
b. The inverse function determines the distance from the center of the artery at which blood is flowing with velocity V.
c. 0.1 cm; 0.14 cm; 0.17 cm

27) A function that converts dress sizes in the United States to those in Europe is given by $D(x) = 2x + 24$.

a) Find the European dress sizes that correspond to sizes 6, 8, 10, and 12 in the United States.

b) Find the function that converts European dress sizes to U.S. dress sizes.

c) Use part b. to find the dress sizes in the United States that correspond to 46, 52, 62, and 70.

28) [Technology Required] The cost to remove a toxin from a lake is modeled by the function $C(p) = \frac{75p}{85 - p}$, where C is the cost (in thousands of dollars) and p is the amount of toxin in a small lake (measured in parts per billion [ppb]). This model is valid only when the amount of toxin is less than 85 ppb.

a) Find the cost to remove 25 ppb, 40 ppb, and 50 ppb of the toxin from the lake.

b) Find the inverse function.

c) Use part b. to determine how much of the toxin is removed for $50,000.

Answer

a. $31,250, $66,667, $107,143
b. $p = \frac{85C}{C + 75}$
c. 34 ppb

29) [Technology Required] A race car is accelerating at a velocity given by $v(t) = \frac{25}{4}t + 54$,

where v is the velocity (in feet per second) at time t.

a) Find the velocity of the car at 10 sec.

b) Find the inverse function.

c) Use part b. to determine how long it takes for the car to reach a speed of 150 ft/sec.

Contributors

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5.5: A.5- Exponential and Logarithmic Functions

Learning Objectives

- Explain the difference between algebraic and transcendental functions.
- Identify the form of an exponential function.
- Explain the difference between the graphs of x^b and b^x.
- Recognize the significance of the number e.
- Identify the form of a logarithmic function.
- Explain the relationship between exponential and logarithmic functions.
- Describe how to calculate a logarithm to a different base.

In this section we examine exponential and logarithmic functions. We use the properties of these functions to solve equations involving exponential or logarithmic terms, and we study the meaning and importance of the number e.

Transcendental Functions

Thus far, we have discussed algebraic functions. Some functions, however, cannot be described by basic algebraic operations. These functions are known as **transcendental functions** because they are said to "transcend," or go beyond, algebra. The most common transcendental functions are trigonometric, exponential, and logarithmic functions. A trigonometric function relates the ratios of two sides of a right triangle. They are $\sin x, \cos x, \tan x, \cot x, \sec x, \text{ and } \csc x$. (We discuss trigonometric functions later in the chapter.) An exponential function is a function of the form $f(x) = b^x$, where the base $b > 0, \ b \neq 1$. A logarithmic function is a function of the form $f(x) = \log_b(x)$ for some constant $b > 0, \ b \neq 1$, where $\log_b(x) = y$ if and only if $b^y = x$.

Example 5.5.1: Classifying Algebraic and Transcendental Functions

Classify each of the following functions, a. through c., as algebraic or transcendental.

a. $f(x) = \dfrac{\sqrt{x^3+1}}{4x+2}$
b. $f(x) = 2^{x^2}$
c. $f(x) = \sin(2x)$

Solution

a. Since this function involves basic algebraic operations only, it is an algebraic function.
b. This function cannot be written as a formula that involves only basic algebraic operations, so it is transcendental. (Note that algebraic functions can only have powers that are rational numbers.)
c. As in part b, this function cannot be written using a formula involving basic algebraic operations only; therefore, this function is transcendental.

Exercise 5.5.1:

Is $f(x) = x/2$ an algebraic or a transcendental function?

Answer available at the end of this section.

Exponential Functions

Exponential functions arise in many applications. One common example is **population growth.** For example, if a population starts with P_0 individuals and then grows at an annual rate of 2%, its population after 1 year is

$$P(1) = P_0 + 0.02P_0 = P_0(1 + 0.02) = P_0(1.02).$$

Its population after 2 years is

$$P(2) = P(1) + 0.02P(1) = P(1)(1.02) = P_0(1.02)^2.$$

In general, its population after t years is

$$P(t) = P_0(1.02)^t,$$

which is an exponential function. More generally, any function of the form $f(x) = b^x$, where $b > 0$, $b \neq 1$, is an exponential function with **base** b and **exponent** x. Exponential functions have constant bases and variable exponents. Note that a function of the form $f(x) = x^b$ for some constant b is *not* an exponential function but a *power* function.

To see the difference between an exponential function and a power function, we compare the functions $y = x^2$ and $y = 2^x$. In Table $5.5.1$, we see that both 2^x and x^2 approach infinity as $x \to \infty$. Eventually, however, 2^x becomes larger than x^2 and grows more rapidly as $x \to \infty$. In the opposite direction, as $x \to -\infty$, $x^2 \to \infty$, whereas $2^x \to 0$. The line $y = 0$ is a horizontal asymptote for $y = 2^x$.

Table $5.5.1$

x	-3	-2	-1	0	1	2	3	4	5	6
x^2	9	4	1	0	1	4	9	16	25	36
2^x	1/8	1/4	1/2	1	2	4	8	16	32	64

In Figure $5.5.1$, we graph both $y = x^2$ and $y = 2^x$ to show how the graphs differ.

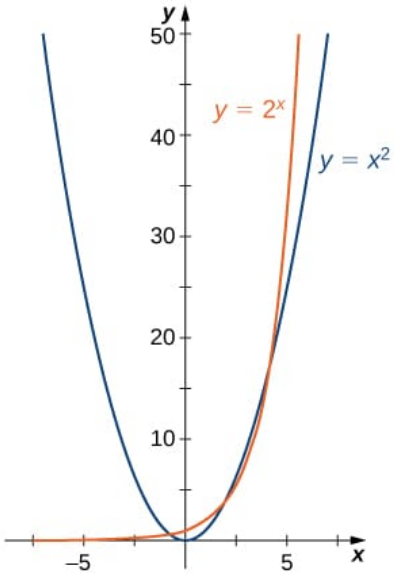

Figure $5.5.1$: Both 2^x and x^2 approach infinity as $x \to \infty$, but 2^x grows more rapidly than x^2. As $x \to -\infty$, $x^2 \to \infty$, whereas $2^x \to 0$.

Evaluating Exponential Functions

Recall the properties of exponents: If x is a positive integer, then we define $b^x = b \cdot b \cdots b$ (with x factors of b). If x is a negative integer, then $x = -y$ for some positive integer y, and we define $b^x = b^{-y} = 1/b^y$. Also, b^0 is defined to be 1. If x is a rational number,

then $x = p/q$, where p and q are integers and $b^x = b^{p/q} = \sqrt[q]{b^p}$. For example, $9^{3/2} = \sqrt{9^3} = \left(\sqrt{9}\right)^3 = 27$. However, how is b^x defined if x is an irrational number? For example, what do we mean by $2^{\sqrt{2}}$? This is too complex a question for us to answer fully right now; however, we can make an approximation.

Table 5.5.2: Values of 2^x for a List of Rational Numbers Approximating $\sqrt{2}$

x	1.4	1.41	1.414	1.4142	1.41421	1.414213
2^x	2.639	2.65737	2.66475	2.665119	2.665138	2.665143

In Table 5.5.2, we list some rational numbers approaching $\sqrt{2}$, and the values of 2^x for each rational number x are presented as well. We claim that if we choose rational numbers x getting closer and closer to $\sqrt{2}$, the values of 2^x get closer and closer to some number L. We define that number L to be $2^{\sqrt{2}}$.

It's often strange for students when they first encounter this conversation that b^x has never been defined for them for irrational values of x; however, there was no need for such discussion until calculus. For example, up to this point in your mathematical career, you needed to compute values like

$$2^{10}, 2^{-2}, \text{ and } 8^{4/3}.$$

However, do you recall the need to compute 2^{π}?

Example 5.5.2: Bacterial Growth

Suppose a particular population of bacteria is known to double in size every 4 hours. If a culture starts with 1000 bacteria, the number of bacteria after 4 hours is $n(4) = 1000 \cdot 2$. The number of bacteria after 8 hours is $n(8) = n(4) \cdot 2 = 1000 \cdot 2^2$. In general, the number of bacteria after $4m$ hours is $n(4m) = 1000 \cdot 2^m$. Letting $t = 4m$, we see that the number of bacteria after t hours is $n(t) = 1000 \cdot 2^{t/4}$. Find the number of bacteria after 6 hours, 10 hours, and 24 hours.

Solution

The number of bacteria after 6 hours is given by

$$n(6) = 1000 \cdot 2^{6/4} \approx 2828 \,\text{bacteria}.$$

The number of bacteria after 10 hours is given by

$$n(10) = 1000 \cdot 2^{10/4} \approx 5657 \,\text{bacteria}.$$

The number of bacteria after 24 hours is given by $n(24) = 1000 \cdot 2^6 = 64,000$ bacteria.

Exercise 5.5.2

Given the exponential function $f(x) = 100 \cdot 3^{x/2}$, evaluate $f(4)$ and $f(10)$.

Answer available at the end of this section.

Graphing Exponential Functions

For any base $b > 0$, $b \neq 1$, the exponential function $f(x) = b^x$ is defined for all real numbers x and $b^x > 0$. Therefore, the domain of $f(x) = b^x$ is $(-\infty, \infty)$ and the range is $(0, \infty)$. To graph b^x, we note that for $b > 1$, b^x is increasing on $(-\infty, \infty)$ and $b^x \to \infty$ as $x \to \infty$, whereas $b^x \to 0$ as $x \to -\infty$. On the other hand, if \(0

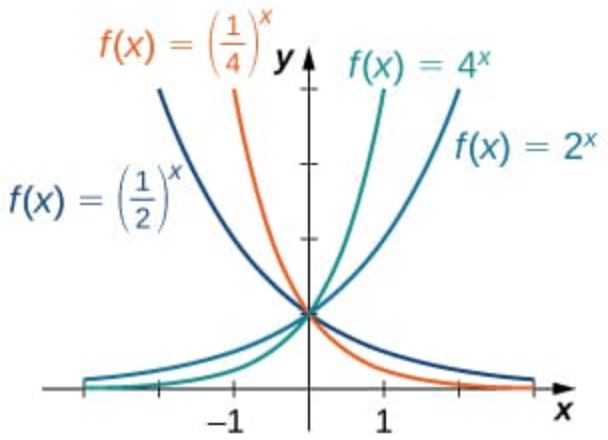

Figure 5.5.2: If $b > 1$, then b^x is increasing on $(-\infty, \infty)$. If \(0

Note that exponential functions satisfy the general Laws of Exponents. To remind you of these laws, we state them as rules.

Theorem 5.5.1: Laws of Exponents

For any constants $a > 0$, $b > 0$, and for all x and y,

1. $b^x \cdot b^y = b^{x+y}$
2. $\dfrac{b^x}{b^y} = b^{x-y}$
3. $(b^x)^y = b^{xy}$
4. $(ab)^x = a^x b^x$
5. $\dfrac{a^x}{b^x} = \left(\dfrac{a}{b}\right)^x$

Example 5.5.3: Using the Laws of Exponents

Use the Laws of Exponents to simplify each of the following expressions.

a. $\dfrac{(2x^{2/3})^3}{(4x^{-1/3})^2}$

b. $\dfrac{(x^3y^{-1})^2}{(xy^2)^{-2}}$

Soution

a. We can simplify as follows:

$$\frac{(2x^{2/3})^3}{(4x^{-1/3})^2} = \frac{2^3(x^{2/3})^3}{4^2(x^{-1/3})^2} = \frac{8x^2}{16x^{-2/3}} = \frac{x^2x^{2/3}}{2} = \frac{x^{8/3}}{2}.$$

b. We can simplify as follows:

$$\frac{(x^3y^{-1})^2}{(xy^2)^{-2}} = \frac{(x^3)^2(y^{-1})^2}{x^{-2}(y^2)^{-2}} = \frac{x^6y^{-2}}{x^{-2}y^{-4}} = x^6x^2y^{-2}y^4 = x^8y^2.$$

Exercise 5.5.3

Use the Laws of Exponents to simplify $\dfrac{6x^{-3}y^2}{12x^{-4}y^5}$.

Hint available at the end of this section.

Answer available at the end of this section.

The Number e

A special type of exponential function appears frequently in applications related to STEM fields. To describe it, consider the following example of exponential growth, which arises from **compounding interest** in a savings account. Suppose a person invests P dollars in a savings account with an annual interest rate r, compounded annually. The amount of money after 1 year is

$$A(1) = P + rP = P(1 + r).$$

The amount of money after 2 years is

$$A(2) = A(1) + rA(1) = P(1 + r) + rP(1 + r) = P(1 + r)^2.$$

More generally, the amount after t years is

$$A(t) = P(1 + r)^t.$$

If the money is compounded 2 times per year, the amount of money after half a year is

$$A\left(\frac{1}{2}\right) = P + \left(\frac{r}{2}\right)P = P\left(1 + \left(\frac{r}{2}\right)\right).$$

The amount of money after 1 year is

$$A(1) = A\left(\frac{1}{2}\right) + \left(\frac{r}{2}\right)A\left(\frac{1}{2}\right) = P\left(1 + \frac{r}{2}\right) + \frac{r}{2}\left(\left(P(1 + \frac{r}{2}\right)\right) = P\left(1 + \frac{r}{2}\right)^2.$$

After t years, the amount of money in the account is

$$A(t) = P\left(1 + \frac{r}{2}\right)^{2t}.$$

More generally, if the money is compounded n times per year, the amount of money in the account after t years is given by the function

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}.$$

What happens as $n \to \infty$? To answer this question, we let $m = n/r$ and write

$$\left(1 + \frac{r}{n}\right)^{nt} = \left(1 + \frac{1}{m}\right)^{mrt},$$

and examine the behavior of $(1 + 1/m)^m$ as $m \to \infty$, using a table of values (Table $5.5.3$).

Table $5.5.3$: Values of $\left(1 + \dfrac{1}{m}\right)^m$ as $m \to \infty$

m	10	100	1000	10,000	100,000	1,000,000
$\left(1 + \dfrac{1}{m}\right)^m$	2.5937	2.7048	2.71692	2.71815	2.718268	2.718280

Looking at this table, it appears that $(1+1/m)^m$ is approaching a number between 2.7 and 2.8 as $m \to \infty$. In fact, $(1+1/m)^m$ does approach some number as $m \to \infty$. We call this number e. To six decimal places of accuracy,

$$e \approx 2.718282.$$

Leonhard Euler

The letter e was first used to represent this number by the Swiss mathematician Leonhard Euler during the 1720s. Although Euler did not discover the number, he showed many important connections between e and logarithmic functions. We still use the notation e today to honor Euler's work because it appears in many areas of mathematics and because we can use it in many practical applications.

Returning to our savings account example, we can conclude that if a person puts P dollars in an account at an annual interest rate r, compounded continuously, then $A(t) = Pe^{rt}$. This function may be familiar. Since functions involving base e arise often in applications, we call the function $f(x) = e^x$ the **natural exponential function**. Not only is this function interesting because of the definition of the number e, but also, as discussed next, its graph has an important property.

Since $e > 1$, we know $f(x) = e^x$ is increasing on $(-\infty, \infty)$. In Figure $5.5.3$, we show a graph of $f(x) = e^x$ along with a **tangent line** to the graph of f at $x = 0$. We give a precise definition of tangent line in the next chapter; but, informally, we say a tangent line to a graph of f at $x = a$ is a line that passes through the point $(a, f(a))$ and has the same "slope" as f at that point (as we will learn, using the word "slope" for a nonlinear curve is generally frowned upon, but it will have to suffice for now). The function $f(x) = e^x$ is the only exponential function b^x with tangent line at $x = 0$ that has a slope of 1. As we see later in the text, having this property makes the natural exponential function the most simple exponential function to use in many instances.

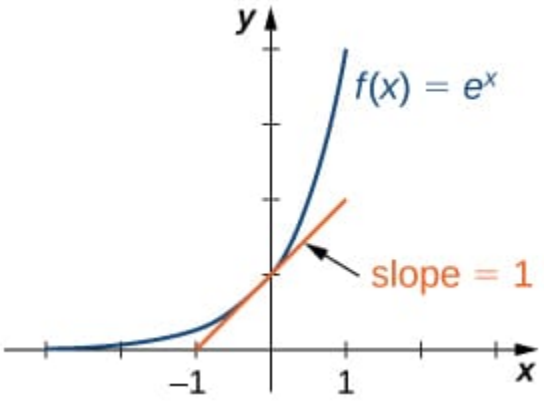

Figure $5.5.3$: The graph of $f(x) = e^x$ has a tangent line with slope 1 at $x = 0$.

Example $5.5.4$: Compounding Interest

Suppose $\$500$ is invested in an account at an annual interest rate of $r = 5.5\%$, compounded continuously.

a. Let t denote the number of years after the initial investment and $A(t)$ denote the amount of money in the account at time t. Find a formula for $A(t)$.
b. Find the amount of money in the account after 10 years and after 20 years.

Solution

a. If P dollars are invested in an account at an annual interest rate r, compounded continuously, then $A(t) = Pe^{rt}$. Here $P = \$500$ and $r = 0.055$. Therefore, $A(t) = 500e^{0.055t}$.

b. After 10 years, the amount of money in the account is

$$A(10) = 500e^{0.055 \cdot 10} = 500e^{0.55} \approx \$866.63.$$

After 20 years, the amount of money in the account is

$$A(20) = 500e^{0.055 \cdot 20} = 500e^{1.1} \approx \$1,502.08.$$

Exercise 5.5.4

If $\$750$ is invested in an account at an annual interest rate of 4%, compounded continuously, find a formula for the amount of money in the account after t years. Find the amount of money after 30 years.

Hint available at the end of this section.

Answer available at the end of this section.

Logarithmic Functions

Using our understanding of exponential functions, we can discuss their inverses, which are the logarithmic functions. These come in handy when we need to consider any phenomenon that varies over a wide range of values, such as the pH scale in chemistry or decibels in sound levels.

The exponential function $f(x) = b^x$ is one-to-one, with domain $(-\infty, \infty)$ and range $(0, \infty)$. Therefore, it has an inverse function, called the **logarithmic function** with base b. For any $b > 0,\ \ b \neq 1$, the logarithmic function with base b, denoted $\log_b$, has domain $(0, \infty)$ and range $(-\infty, \infty)$,and satisfies

$$\log_b(x) = y$$

if and only if $b^y = x$.

For example,

$$\log_2(8) = 3$$

since $2^3 = 8$,

$$\log_{10}\left(\frac{1}{100}\right) = -2$$

since $10^{-2} = \dfrac{1}{10^2} = \dfrac{1}{100}$,

$$\log_b(1) = 0$$

since $b^0 = 1$ for any base $b > 0$.

Furthermore, since $y = \log_b(x)$ and $y = b^x$ are inverse functions,

$$\log_b(b^x) = x$$

and

$$b^{\log_b(x)} = x.$$

The most commonly used logarithmic function is the function $\log_e$. Since this function uses natural e as its base, it is called the **natural logarithm**. Here we use the notation $\ln(x)$ or $\ln x$ to mean $\log_e(x)$. For example,

$$\ln(e) = \log_e(e) = 1$$
$$\ln(e^3) = \log_e(e^3) = 3$$
$$\ln(1) = \log_e(1) = 0.$$

Since the functions $f(x) = e^x$ and $g(x) = \ln(x)$ are inverses of each other,

$$\ln(e^x) = x \text{ and } e^{\ln x} = x,$$

and their graphs are symmetric about the line $y = x$ (Figure $5.5.4$).

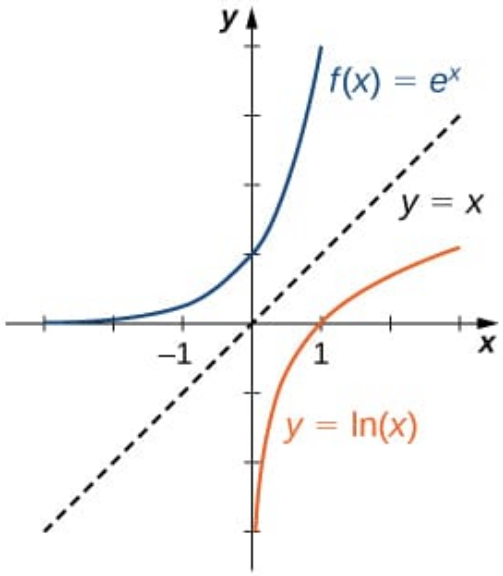

Figure $5.5.4$: The functions $y = e^x$ and $y = \ln(x)$ are inverses of each other, so their graphs are symmetric about the line $y = x$.

In general, for any base $b > 0$, $b \neq 1$, the function $g(x) = \log_b(x)$ is symmetric about the line $y = x$ with the function $f(x) = b^x$. Using this fact and the graphs of the exponential functions, we graph functions $\log_b$ for several values of $b > 1$ (Figure $5.5.5$).

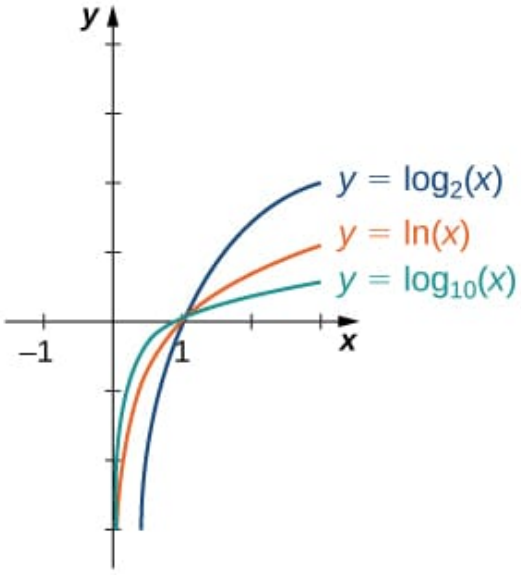

Figure $5.5.5$: Graphs of $y = \log_b(x)$ are depicted for $b = 2,\ e,\ 10$.

Before solving some equations involving exponential and logarithmic functions, let's review the basic properties of logarithms.

Theorem $5.5.2$: Laws of Logarithms

If $a,\ b,\ c > 0,\ b \neq 1$, and r is any real number, then

- **Product property**

$$\log_b(ac) = \log_b(a) + \log_b(c) \tag{5.5.1}$$

- **Quotient property**

$$\log_b\left(\frac{a}{c}\right) = \log_b(a) - \log_b(c) \tag{5.5.2}$$

- **Power property**

$$\log_b(a^r) = r\log_b(a) \tag{5.5.3}$$

Example 5.5.5: Solving Equations Involving Exponential Functions

Solve each of the following equations for x.

a. $5^x = 2$
b. $e^x + 6e^{-x} = 5$

Solution

a. Applying the natural logarithm function to both sides of the equation, we have

$$\ln 5^x = \ln 2.$$

Using the power property of logarithms,

$$x \ln 5 = \ln 2.$$

Therefore,

$$x = \frac{\ln 2}{\ln 5}.$$

b. Multiplying both sides of the equation by e^x,we arrive at the equation

$$e^{2x} + 6 = 5e^x.$$

Rewriting this equation as

$$e^{2x} - 5e^x + 6 = 0,$$

we can then rewrite it as a quadratic equation in e^x:

$$(e^x)^2 - 5(e^x) + 6 = 0.$$

Now we can solve the quadratic equation. Factoring this equation, we obtain

$$(e^x - 3)(e^x - 2) = 0.$$

Therefore, the solutions satisfy $e^x = 3$ and $e^x = 2$. Taking the natural logarithm of both sides gives us the solutions $x = \ln 3, \ln 2$.

Exercise 5.5.5

Solve

$$e^{2x}/(3+e^{2x}) = 1/2.$$

Hint available at the end of this section.

Answer available at the end of this section.

Example 5.5.6: Solving Equations Involving Logarithmic Functions

Solve each of the following equations for x.

a. $\ln\left(\frac{1}{x}\right) = 4$
b. $\log_{10}\sqrt{x} + \log_{10} x = 2$
c. $\ln(2x) - 3\ln(x^2) = 0$

Solution

a. By the definition of the natural logarithm function,

$$\ln\left(\frac{1}{x}\right) = 4$$

if and only if $e^4 = \frac{1}{x}$.

Therefore, the solution is $x = 1/e^4$.

b. Using the product (Equation 5.5.1) and power (Equation 5.5.3) properties of logarithmic functions, rewrite the left-hand side of the equation as

$$\begin{aligned}\log_{10}\sqrt{x} + \log_{10} x &= \log_{10} x\sqrt{x} \\ &= \log_{10} x^{3/2} \\ &= \frac{3}{2}\log_{10} x.\end{aligned}$$

Therefore, the equation can be rewritten as

$$\frac{3}{2}\log_{10} x = 2$$

or

$$\log_{10} x = \frac{4}{3}.$$

The solution is $x = 10^{4/3} = 10\sqrt[3]{10}$.

c. Using the power property (Equation 5.5.3) of logarithmic functions, we can rewrite the equation as $\ln(2x) - \ln(x^6) = 0$.

Using the quotient property (Equation 5.5.2), this becomes

$$\ln\left(\frac{2}{x^5}\right) = 0$$

Therefore, $2/x^5 = 1$, which implies $x = \sqrt[5]{2}$. We should then check for any extraneous solutions.

Exercise 5.5.6

Solve $\ln(x^3) - 4\ln(x) = 1$.

Hint available at the end of this section.

Answer available at the end of this section.

When evaluating a logarithmic function with a calculator, you may have noticed that the only options are $\log_{10}$ or $\log$, called the **common logarithm**, or $\ln$, which is the natural logarithm. However, exponential functions and logarithm functions can be expressed in terms of any desired base b. If you need to use a calculator to evaluate an expression with a different base, you can apply the **Change of Base Formulas** first. Using this change of base, we typically write a given exponential or logarithmic function in terms of the natural exponential and natural logarithmic functions.

Theorem 5.5.3: Change of Base Formula

Let $a > 0,\ b > 0$, and $a \neq 1,\ b \neq 1$.

1. $a^x = b^{x \log_b a}$ for any real number x.

   If $b = e$, this equation reduces to $a^x = e^{x \log_e a} = e^{x \ln a}$.

2. $\log_a x = \dfrac{\log_b x}{\log_b a}$ for any real number $x > 0$.

   If $b = e$, this equation reduces to $\log_a x = \dfrac{\ln x}{\ln a}$.

Proof available at the end of this section.

Example 5.5.7: Changing Bases

Use a calculating utility to evaluate $\log_3 7$ with the Change of Base Formula presented earlier.

Solution

Use the second equation with $a = 3$ and $b = e$: $\log_3 7 = \dfrac{\ln 7}{\ln 3} \approx 1.77124$.

Exercise 5.5.7

Use the Change of Base Formula and a calculating utility to evaluate $\log_4 6$.

Hint available at the end of this section.

Example 5.5.8: The Richter Scale for Earthquakes

In 1935, Charles Richter developed a scale (now known as the Richter scale) to measure the magnitude of an earthquake. The scale is a base-10 logarithmic scale, and it can be described as follows: Consider one earthquake with magnitude R_1 on the Richter scale and a second earthquake with magnitude R_2 on the Richter scale. Suppose $R_1 > R_2$, which means the earthquake of magnitude R_1 is stronger, but how much stronger is it than the other earthquake?

Figure 5.5.6: (credit: modification of work by Robb Hannawacker, NPS)

A way of measuring the intensity of an earthquake is by using a seismograph to measure the amplitude of the earthquake waves. If A_1 is the amplitude measured for the first earthquake and A_2 is the amplitude measured for the second earthquake, then the amplitudes and magnitudes of the two earthquakes satisfy the following equation:

$$R_1 - R_2 = \log_{10}\left(\frac{A1}{A2}\right).$$

Consider an earthquake that measures 8 on the Richter scale and an earthquake that measures 7 on the Richter scale. Then,

$$8 - 7 = \log_{10}\left(\frac{A1}{A2}\right).$$

Therefore,

$$\log_{10}\left(\frac{A1}{A2}\right) = 1,$$

which implies $A_1/A_2 = 10$ or $A_1 = 10A_2$. Since A_1 is 10 times the size of A_2, we say that the first earthquake is 10 times as intense as the second earthquake. On the other hand, if one earthquake measures 8 on the Richter scale and another measures 6, then the relative intensity of the two earthquakes satisfies the equation

$$\log_{10}\left(\frac{A1}{A2}\right) = 8 - 6 = 2.$$

Therefore, $A_1 = 100A_2$.That is, the first earthquake is 100 times more intense than the second earthquake.

How can we use logarithmic functions to compare the relative severity of the magnitude 9 earthquake in Japan in 2011 with the magnitude 7.3 earthquake in Haiti in 2010?

Solution

To compare the Japan and Haiti earthquakes, we can use an equation presented earlier:

$$9 - 7.3 = \log_{10}\left(\frac{A1}{A2}\right).$$

Therefore, $A_1/A_2 = 10^{1.7}$, and we conclude that the earthquake in Japan was approximately 50 times more intense than the earthquake in Haiti.

Exercise 5.5.8

Compare the relative severity of a magnitude 8.4 earthquake with a magnitude 7.4 earthquake.

Hint available at the end of this section.

Key Concepts

- Trigonometric, exponential, and logarithmic functions are examples of transcendental functions.

- The exponential function $y = b^x$ is increasing if $b > 1$ and decreasing if \(0
- The logarithmic function $y = \log_b(x)$ is the inverse of $y = b^x$. Its domain is $(0, \infty)$ and its range is $(-\infty, \infty)$.
- The natural exponential function is $y = e^x$ and the natural logarithmic function is $y = \ln x = \log_e x$.
- Given an exponential function or logarithmic function in base a, we can make a change of base to convert this function to any base $b > 0$, $b \neq 1$. We typically convert to base e.

Glossary

base
: the number b in the exponential function $f(x) = b^x$ and the logarithmic function $f(x) = \log_b x$

exponent
: the value x in the expression b^x

natural exponential function
: the function $f(x) = e^x$

natural logarithm
: the function $\ln x = \log_e x$

number e
: as m gets larger, the quantity $(1 + (1/m)^m$ gets closer to some real number; we define that real number to be e; the value of e is approximately 2.718282

transcendental function
: a function that cannot be expressed by a combination of basic arithmetic operations

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Answers

Exercise $5.5.1$**:**

Algebraic

Exercise $5.5.2$

$f(4) = 900$

$f(10) = 24,300.$

Exercise $5.5.3$

$x^a/x^b = x^{a-b}$

Exercise $5.5.3$

$x/(2y^3)$

Exercise $5.5.4$

$A(t) = Pe^{rt}$

Exercise $5.5.4$

$A(t) = 750e^{0.04t}$. After 30 years, there will be approximately $\$2,490.09$.

Exercise $5.5.5$

First solve the equation for e^{2x}

Exercise $5.5.5$

$x = \dfrac{\ln 3}{2}$.

Exercise $5.5.6$

First use the power property, then use the product property of logarithms.

Exercise $5.5.6$

$x = \dfrac{1}{e}$

Theorem $5.5.3$**: Change of Base Formula**

For the first Change of Base Formula, we begin by making use of the power property of logarithmic functions. We know that for any base $b > 0,\ b \neq 1$, $\log_b(a^x) = x \log_b a$. Therefore,

$$b^{\log_b(a^x)} = b^{x \log_b a}.$$

In addition, we know that b^x and $\log_b(x)$ are inverse functions. Therefore,

$$b^{\log_b(a^x)} = a^x.$$

Combining these last two equalities, we conclude that $a^x = b^{x \log_b a}$.

To prove the second property, we show that

$$(\log_b a) \cdot (\log_a x) = \log_b x.$$

Let $u = \log_b a, v = \log_a x$, and $w = \log_b x$. We will show that $u \cdot v = w$. By the definition of logarithmic functions, we know that $b^u = a,\ \ a^v = x$, and $b^w = x$. From the previous equations, we see that

$$b^{uv} = (b^u)^v = a^v = x = b^w.$$

Therefore, $b^{uv} = b^w$. Since exponential functions are one-to-one, we can conclude that $u \cdot v = w$.

Q.E.D.

Exercise $5.5.7$

Use the change of base to rewrite this expression in terms of expressions involving the natural logarithm function.

$\log_4 6 = \dfrac{\ln 6}{\ln 4} \approx 1.29248$

Exercise $5.5.8$

$R_1 - R_2 = \log_{10}(A1/A2)$.

The magnitude 8.4 earthquake is roughly 10 times as severe as the magnitude 7.4 earthquake.

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5.5.1: A.5.1- Homework

In exercise 1, determine whether the statement is *true or false*. Explain why.

1) $f(x) = \dfrac{4x+1}{7x-2}$ is a transcendental function.

In exercises 2 - 5, evaluate the given exponential functions as indicated, accurate to two significant digits after the decimal.

2) $f(x) = 5^x$

a. $x = 3$

b. $x = \frac{1}{2}$

c. $x = \sqrt{2}$

Answer

a. 125
b. 2.24
c. 9.74

3) $f(x) = (0.3)^x$

a. $x = -1$

b. $x = 4$

c. $x = -1.5$

4) $f(x) = 10^x$

a. $x = -2$

b. $x = 4$

c. $x = \frac{5}{3}$

Answer

a. 0.01
b. $10,000$
c. 46.42

5) $f(x) = e^x$

a. $x = 2$

b. $x = -3.2$

c. $x = \pi$

In exercises 6 - 11, match the exponential equation to the correct graph.

a. $y = 4^{-x}$

b. $y = 3^{x-1}$

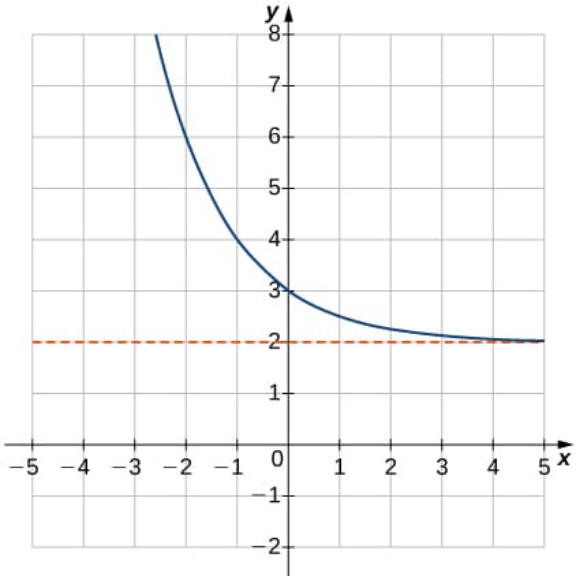
c. $y = 2^{x+1}$

d. $y = \left(\frac{1}{2}\right)^x + 2$

e. $y = -3^{-x}$

f. $y = 1 - 5^x$

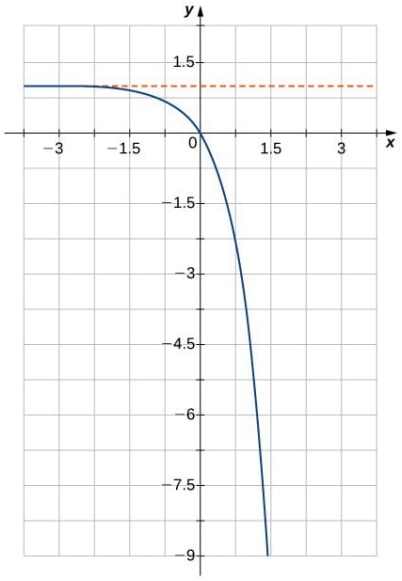
6)



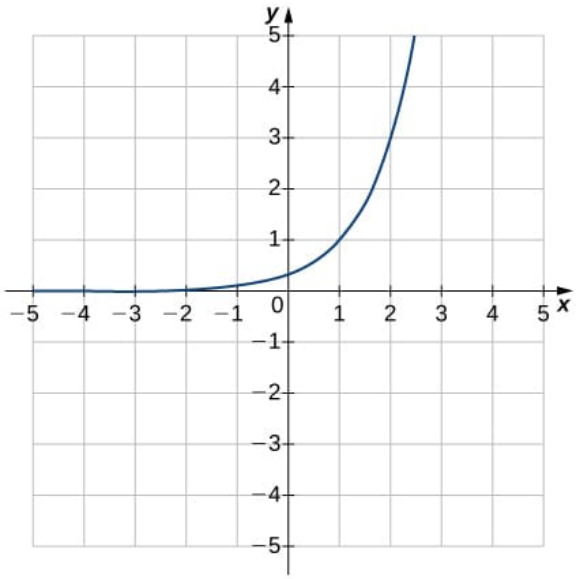
Answer

d

7)



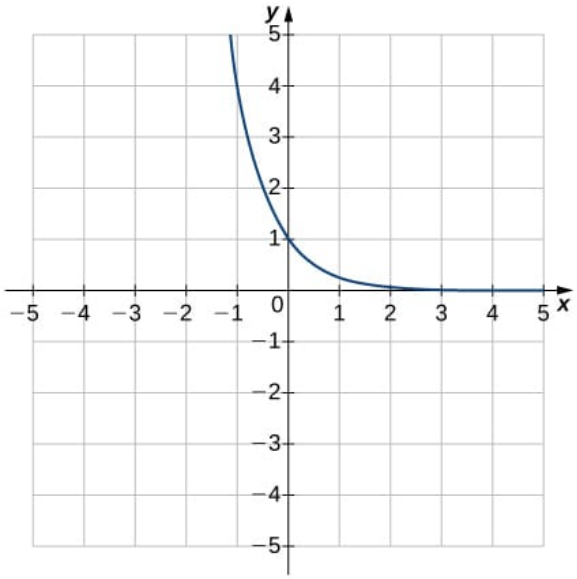
8)



Answer

b

9)



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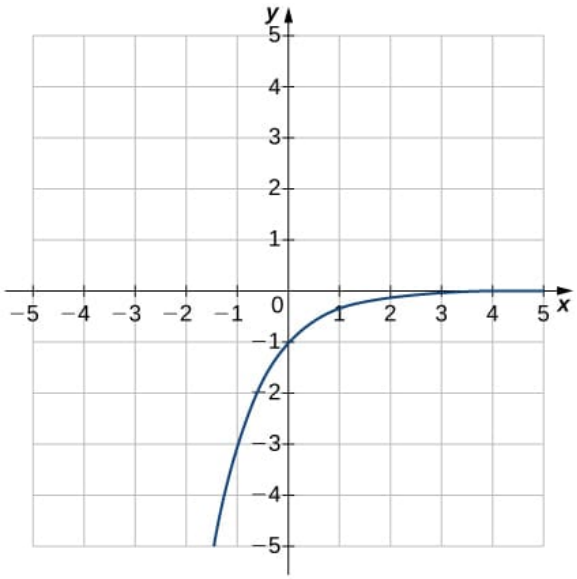
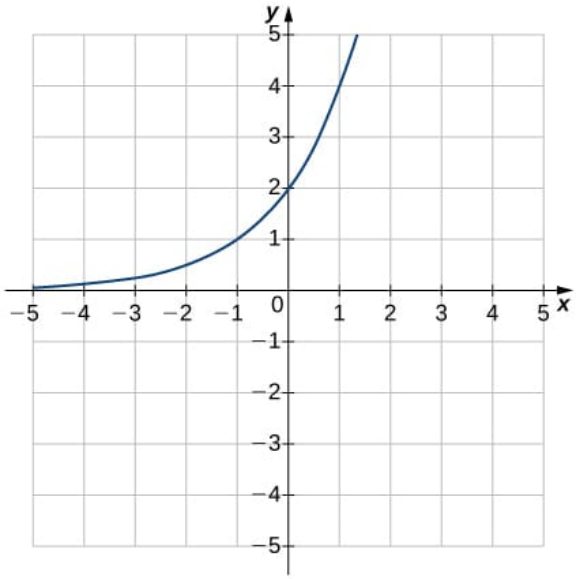


Answer

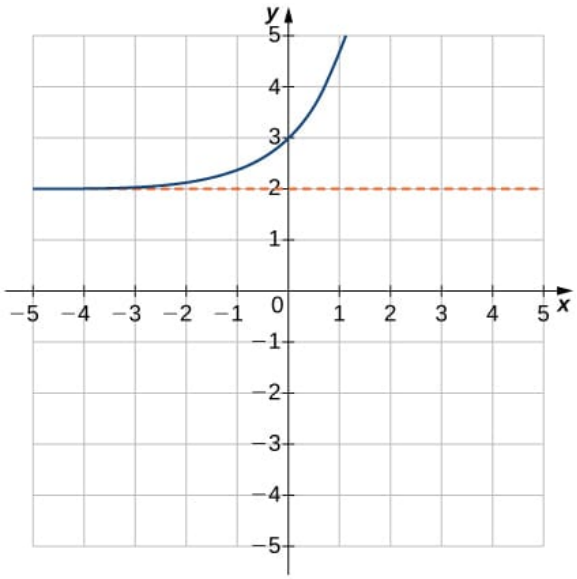
e

11)



In exercises 12 - 18, sketch the graph of the exponential function. Determine the domain, range, and horizontal asymptote.

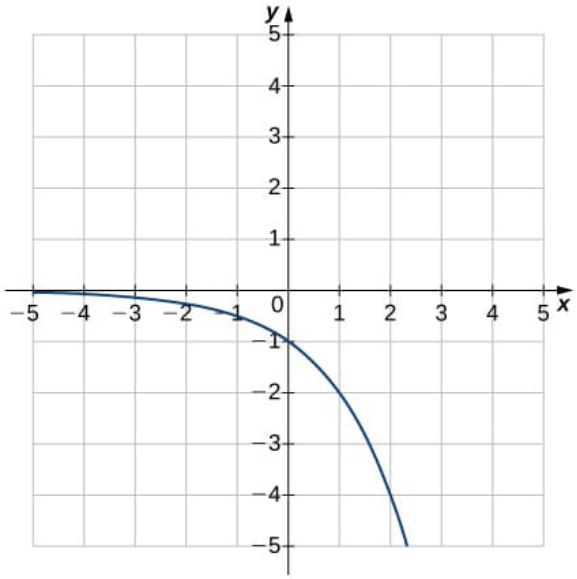
12) $f(x) = e^x + 2$



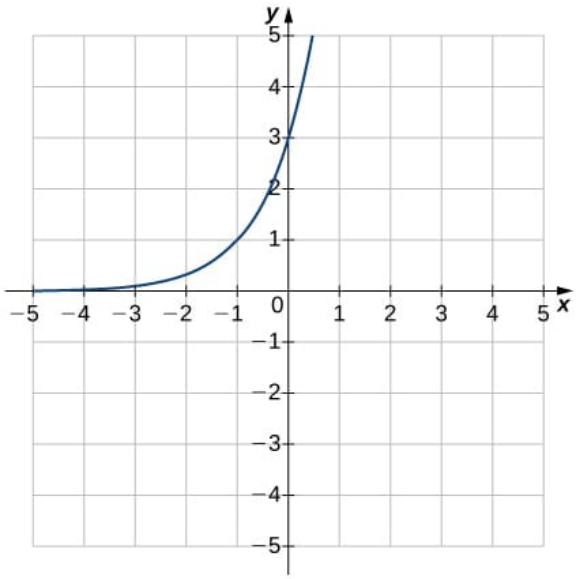
Answer

Domain: all real numbers, Range: $(2, \infty)$, $y = 2$

13) $f(x) = -2^x$



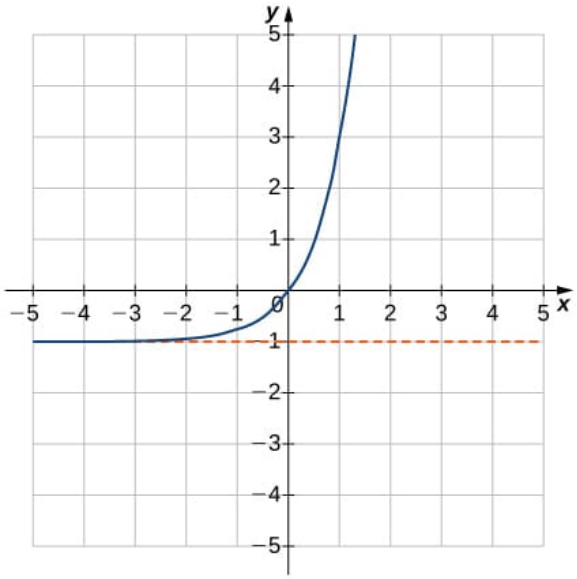
14) $f(x) = 3^{x+1}$



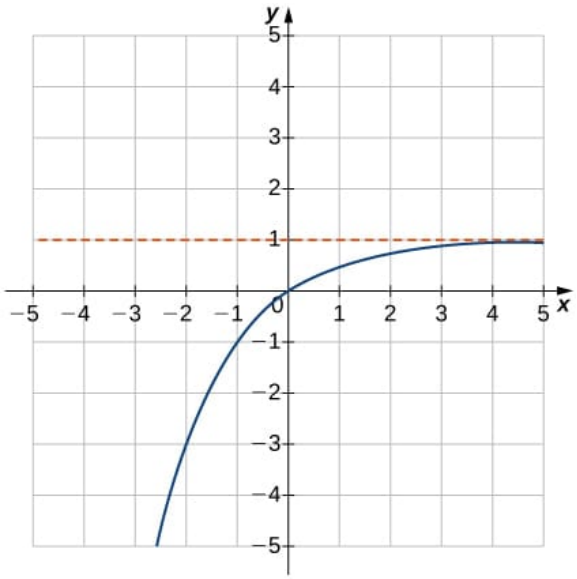
Answer

Domain: all real numbers, Range: $(0, \infty)$, $y = 0$

15) $f(x) = 4^x - 1$



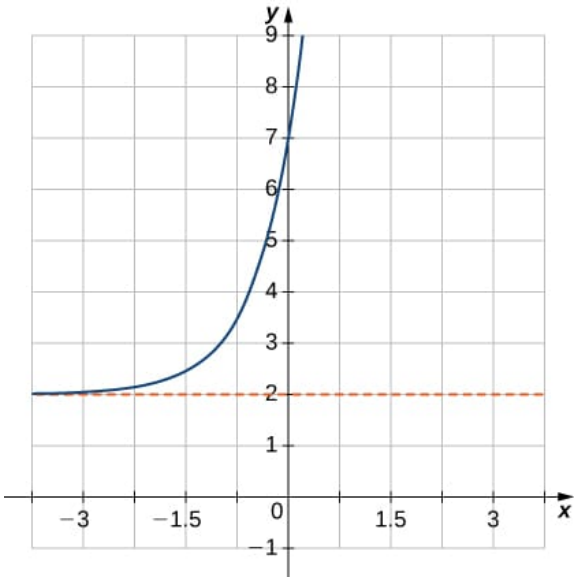
16) $f(x) = 1 - 2^{-x}$



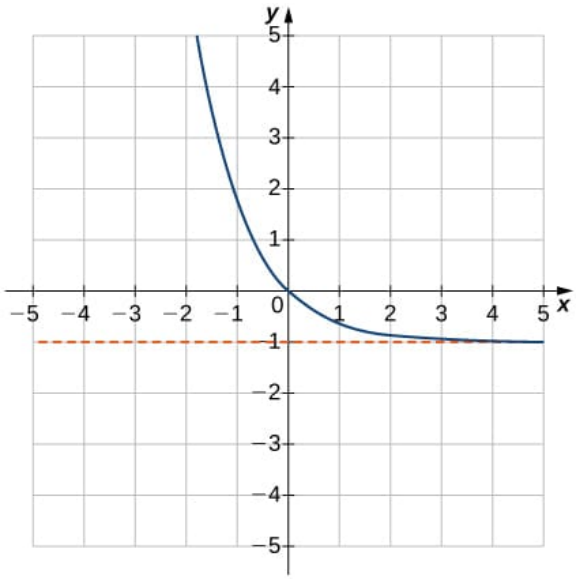
Answer

Domain: all real numbers, Range: $(-\infty, 1), \ \ y = 1$

17) $f(x) = 5^{x+1} + 2$



18) $f(x) = e^{-x} - 1$



Answer

Domain: all real numbers, Range: $(-1, \infty)$, $y = -1$

In exercises 19 - 26, write the equation in equivalent exponential form.

19) $\log_3 81 = 4$

20) $\log_8 2 = \frac{1}{3}$

Answer

$8^{1/3} = 2$

21) $\log_5 1 = 0$

22) $\log_5 25 = 2$

Answer

$5^2 = 25$

23) $\log 0.1 = -1$

24) $\ln\left(\frac{1}{e^3}\right) = -3$

Answer

$e^{-3} = \dfrac{1}{e^3}$

25) $\log_9 3 = 0.5$

26) $\ln 1 = 0$

Answer

$e^0 = 1$

In exercises 27 - 36, write the equation in equivalent logarithmic form.

27) $2^3 = 8$

28) $4^{-2} = \frac{1}{16}$

Answer

$\log_4\left(\frac{1}{16}\right) = -2$

29) $10^2 = 100$

30) $9^0 = 1$

Answer

$\log_9 1 = 0$

31) $\left(\frac{1}{3}\right)^3 = \frac{1}{27}$

32) $\sqrt[3]{64} = 4$

Answer

$\log_{64} 4 = \frac{1}{3}$

33) $e^x = y$

34) $9^y = 150$

Answer

$\log_9 150 = y$

35) $b^3 = 45$

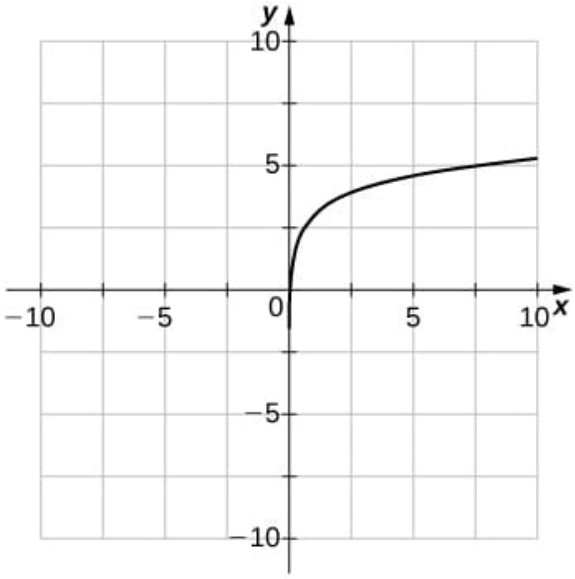
36) $4^{-3/2} = 0.125$

Answer

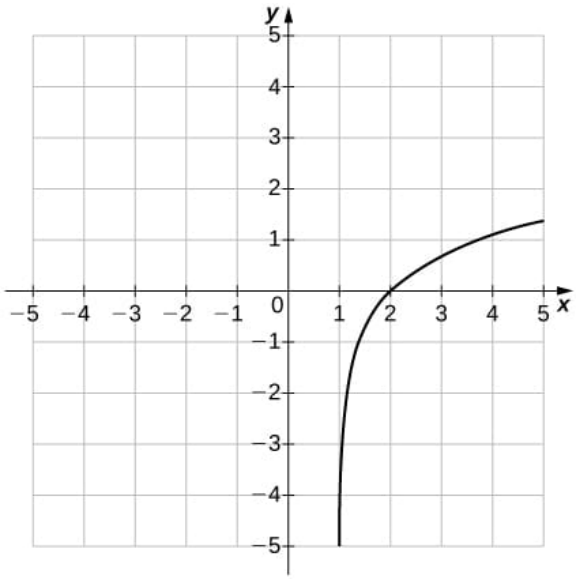
$\log_4 0.125 = -\frac{3}{2}$

In exercises 37 - 42, sketch the graph of the logarithmic function. Determine the domain, range, and vertical asymptote.

37) $f(x) = 3 + \ln x$



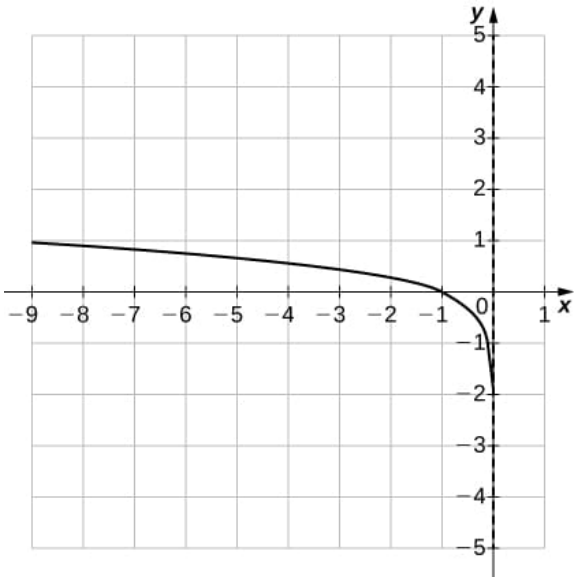
38) $f(x) = \ln(x - 1)$



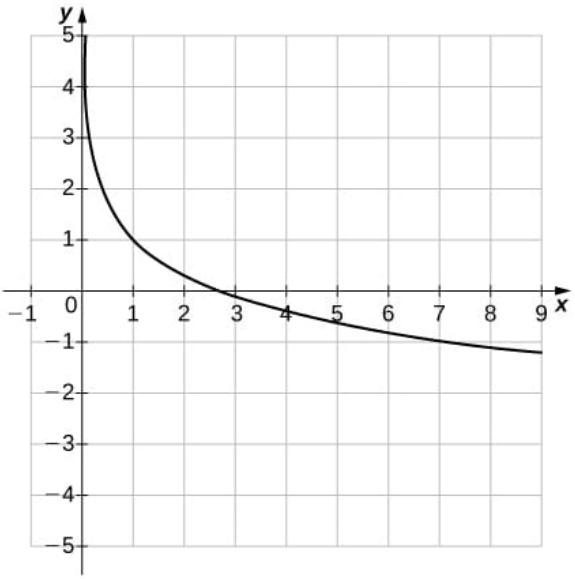
Answer

Domain: $(1, \infty)$, Range: $(-\infty, \infty)$, $x = 1$

39) $f(x) = \ln(-x)$



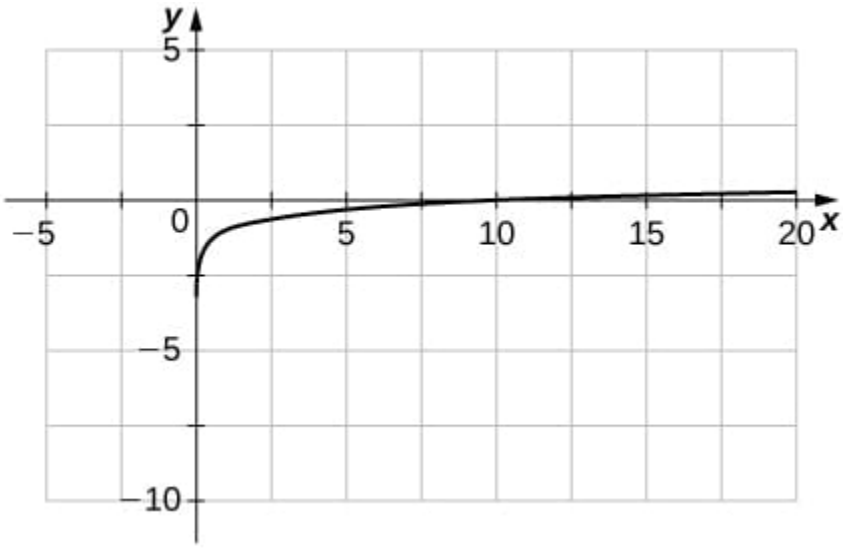
40) $f(x) = 1 - \ln x$



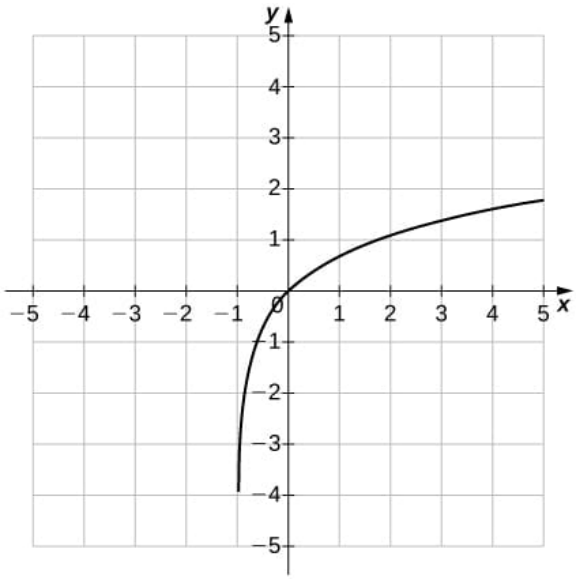
Answer

Domain: $(0, \infty)$, Range: $(-\infty, \infty)$, $x = 0$

41) $f(x) = \log x - 1$



42) $f(x) = \ln(x + 1)$



Answer

Domain: $(-1, \infty)$, Range: $(-\infty, \infty)$, $x = -1$

In exercises 43 - 48, use properties of logarithms to write the expressions as a sum, difference, and/or product of logarithms.

43) $\log x^4 y$

44) $\log_3 \frac{9a^3}{b}$

Answer

$2 + 3\log_3 a - \log_3 b$

45) $\ln a\sqrt[3]{b}$

46) $\log_5 \sqrt{125xy^3}$

Answer

$\frac{3}{2} + \frac{1}{2}\log_5 x + \frac{3}{2}\log_5 y$

47) $\log_4 \frac{\sqrt[3]{xy}}{64}$

48) $\ln\left(\frac{6}{\sqrt{e^3}}\right)$

Answer

$-\frac{3}{2} + \ln 6$

In exercises 49 - 56, solve the exponential equation exactly.

49) $5^x = 125$

50) $e^{3x} - 15 = 0$

Answer

$\frac{\ln 15}{3}$

51) $8^x = 4$

52) $4^{x+1} - 32 = 0$

Answer

$\frac{3}{2}$

53) $3^{x/14} = \frac{1}{10}$

54) $10^x = 7.21$

Answer

$\log 7.21$

55) $4 \cdot 2^{3x} - 20 = 0$

56) $7^{3x-2} = 11$

Answer

$\frac{2}{3} + \frac{\log 11}{3 \log 7}$

In exercises 57 - 64, solve the logarithmic equation exactly, if possible.

57) $\log_3 x = 0$

58) $\log_5 x = -2$

Answer

$x = \frac{1}{25}$

59) $\log_4(x + 5) = 0$

60) $\log(2x - 7) = 0$

Answer

$x = 4$

61) $\ln \sqrt{x + 3} = 2$

62) $\log_6(x + 9) + \log_6 x = 2$

Answer

$x = 3$

63) $\log_4(x + 2) - \log_4(x - 1) = 0$

64) $\ln x + \ln(x - 2) = \ln 4$

Answer

$1 + \sqrt{5}$

In exercises 65 - 70, use the Change of Base Formula and either base 10 or base e to evaluate the given expressions. Answer in exact form and in approximate form, rounding to four decimal places.

65) $\log_5 47$

66) $\log_7 82$

Answer

$\frac{\log 82}{\log 7} \approx 2.2646$

67) $\log_6 103$

68) $\log_{0.5} 211$

Answer

$$\frac{\log 211}{\log 0.5} \approx -7.7211$$

69) $\log_2 \pi$

70) $\log_{0.2} 0.452$

Answer

$$\frac{\log 0.452}{\log 0.2} \approx 0.4934$$

71) [Technology Required] The number of bacteria N in a culture after t days can be modeled by the function $N(t) = 1300 \cdot (2)^{t/4}$. Find the number of bacteria present after 15 days.

Answer

$\sim 17,491$

72) [Technology Required] The demand D (in millions of barrels) for oil in an oil-rich country is given by the function $D(p) = 150 \cdot (2.7)^{-0.25p}$, where p is the price (in dollars) of a barrel of oil. Find the amount of oil demanded (to the nearest million barrels) when the price is between $15 and $20.

73) [Technology Required] The amount A of a $100,000 investment paying continuously and compounded for t years is given by $A(t) = 100,000 \cdot e^{0.055t}$. Find the amount A accumulated in 5 years.

Answer

Approximately $131,653 is accumulated in 5 years.

74) [Technology Required] An investment is compounded monthly, quarterly, or yearly and is given by the function $A = P\left(1 + \frac{j}{n}\right)^{nt}$, where A is the value of the investment at time t, P is the initial principle that was invested, j is the annual interest rate, and n is the number of time the interest is compounded per year. Given a yearly interest rate of 3.5% and an initial principle of $100,000, find the amount A accumulated in 5 years for interest that is compounded a. daily, b., monthly, c. quarterly, and d. yearly.

75) [Technology Required] The concentration of hydrogen ions in a substance is denoted by $[H+]$, measured in moles per liter. The pH of a substance is defined by the logarithmic function $pH = -\log[H+]$. This function is used to measure the acidity of a substance. The pH of water is 7. A substance with a pH less than 7 is an acid, whereas one that has a pH of more than 7 is a base.

a. Find the pH of the following substances. Round answers to one digit.

b. Determine whether the substance is an acid or a base.

i. Eggs: $[H+] = 1.6 \times 10^{-8}$ mol/L

ii. Beer: $[H+] = 3.16 \times 10^{-3}$ mol/L

iii. Tomato Juice: $[H+] = 7.94 \times 10^{-5}$ mol/L

Answer

i. a. pH = 8 b. Base
ii. a. pH = 3 b. Acid
iii. a. pH = 4 b. Acid

76) [Technology Required] Iodine-131 is a radioactive substance that decays according to the function $Q(t) = Q_0 \cdot e^{-0.08664t}$, where Q_0 is the initial quantity of a sample of the substance and t is in days. Determine how long it takes (to the nearest day) for 95% of a quantity to decay.

77) [Technology Required] According to the World Bank, at the end of 2013 $(t = 0)$ the U.S. population was 316 million and was increasing according to the following model:

$$P(t) = 316e^{0.0074t},$$

where P is measured in millions of people and t is measured in years after 2013.

a. Based on this model, what will be the population of the United States in 2020?

b. Determine when the U.S. population will be twice what it is in 2013.

Answer

a. ~ 333 million
b. 94 years from 2013, or in 2107

78) [Technology Required] The amount A accumulated after 1000 dollars is invested for t years at an interest rate of 4% is modeled by the function $A(t) = 1000(1.04)^t$.

a. Find the amount accumulated after 5 years and 10 years.

b. Determine how long it takes for the original investment to triple.

79) [Technology Required] A bacterial colony grown in a lab is known to double in number in 12 hours. Suppose, initially, there are 1000 bacteria present.

a. Use the exponential function $Q = Q_0 e^{kt}$ to determine the value k, which is the growth rate of the bacteria. Round to four decimal places.

b. Determine approximately how long it takes for $200,000$ bacteria to grow.

Answer

a. $k \approx 0.0578$
b. \approx 92 hours

80) [Technology Required] The rabbit population on a game reserve doubles every 6 months. Suppose there were 120 rabbits initially.

a. Use the exponential function $P = P_0 a^t$ to determine the growth rate constant a. Round to four decimal places.

b. Use the function in part a. to determine approximately how long it takes for the rabbit population to reach 3500.

81) [Technology Required] The 1906 earthquake in San Francisco had a magnitude of 8.3 on the Richter scale. At the same time, in Japan, an earthquake with magnitude 4.9 caused only minor damage. Approximately how much more energy was released by the San Francisco earthquake than by the Japanese earthquake?

Answer

The San Francisco earthquake had $10^{3.4}$ or ~ 2512 times more energy than the Japan earthquake.

Contributors

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5.6: A.6- Trigonometry

Learning Objectives

- Convert angle measures between degrees and radians.
- Recognize the triangular and circular definitions of the basic trigonometric functions.
- Write the basic trigonometric identities.
- Identify the graphs and periods of the trigonometric functions.
- Describe the shift of a sine or cosine graph from the equation of the function.

Trigonometric functions are used to model many phenomena, including sound waves, vibrations of strings, alternating electrical current, and the motion of pendulums. In fact, almost any repetitive, or cyclical, motion can be modeled by some combination of trigonometric functions. In this section, we define the six basic trigonometric functions and look at some of the main identities involving these functions.

Radian Measure

To use trigonometric functions, we first must understand how to measure the angles. Although we can use both radians and degrees, **radians** are a more natural measurement because they are related directly to the **unit circle**, a circle with radius 1. The radian measure of an angle is defined as follows. Given an angle θ, let s be the length of the corresponding arc on the unit circle (Figure $5.6.1$). We say the angle corresponding to the arc of length 1 has radian measure 1.

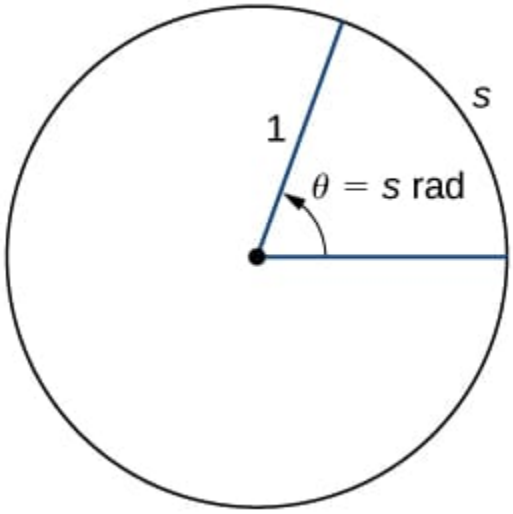


Figure $5.6.1$: The radian measure of an angle θ is the arc length s of the associated arc on the unit circle.

Since an angle of 360° corresponds to the circumference of a circle, or an arc of length 2π, we conclude that an angle with a degree measure of 360° has a radian measure of 2π. Similarly, we see that 180° is equivalent to π radians. Table $5.6.1$ shows the relationship between common degree and radian values.

Table $5.6.1$: Common Angles Expressed in Degrees and Radians

Degrees	Radians	Degrees	Radians
0	0	120	$2\pi/3$
30	$\pi/6$	135	$3\pi/4$
45	$\pi/4$	150	$5\pi/6$

Degrees	Radians	Degrees	Radians
60	$\pi/3$	180	π
90	$\pi/2$		

Example 5.6.1: Converting between Radians and Degrees

a. Express 225° using radians.
b. Express $5\pi/3$ rad using degrees.

Solution

Use the fact that 180° is equivalent to π radians as a conversion factor (Table $5.6.1$):

$$1 = \frac{\pi \text{ rad}}{180^\circ} = \frac{180^\circ}{\pi \text{ rad}}.$$

a. $225^\circ = 225^\circ \cdot \left(\frac{\pi}{180^\circ}\right) = \left(\frac{5\pi}{4}\right)$ rad
b. $\frac{5\pi}{3}$ rad = $\frac{5\pi}{3} \cdot \frac{180^\circ}{\pi} = 300^\circ$

Exercise 5.6.1

a. Express 210° using radians.
b. Express $11\pi/6$ rad using degrees.

Hint available at the end of this section.

Answer available at the end of this section.

The Six Basic Trigonometric Functions

Trigonometric functions allow us to use angle measures, in radians or degrees, to find the coordinates of a point on any circle - not only on a unit circle - or to find an angle given a point on a circle. They also define the relationship between the sides and angles of a triangle.

To define the trigonometric functions, first consider the unit circle centered at the origin and a point $P = (x, y)$ on the unit circle. Let θ be an angle with an initial side that lies along the positive x-axis and with a terminal side that is the line segment OP. An angle in this position is said to be in standard position (Figure $5.6.2$). We can then define the values of the six trigonometric functions for θ in terms of the coordinates x and y.

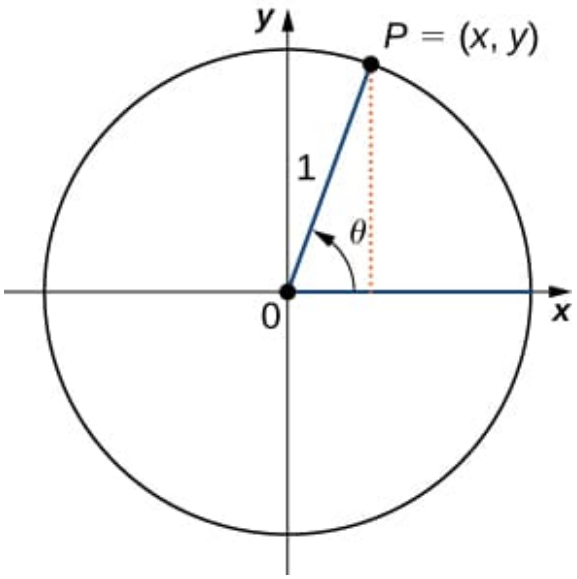


Figure 5.6.2: The angle θ is in standard position. The values of the trigonometric functions for θ are defined in terms of the coordinates x and y.

Definition: Trigonometric functions

Let $P = (x, y)$ be a point on the unit circle centered at the origin O. Let θ be an angle with an initial side along the positive x -axis and a terminal side given by the line segment OP. The trigonometric functions are then defined as

$\sin\theta = y$	$\csc\theta = \dfrac{1}{y}$
$\cos\theta = x$	$\sec\theta = \dfrac{1}{x}$
$\tan\theta = \dfrac{y}{x}$	$\cot\theta = \dfrac{x}{y}$

If $x = 0, \sec\theta$ and $\tan\theta$ are undefined. If $y = 0$, then $\cot\theta$ and $\csc\theta$ are undefined.

We can see that for a point $P = (x, y)$ on a circle of radius r with a corresponding angle θ, the coordinates x and y satisfy

$$\cos\theta = \frac{x}{r} \tag{5.6.1}$$

$$x = r\cos\theta \tag{5.6.2}$$

and

$$\sin\theta = \frac{y}{r} \tag{5.6.3}$$

$$y = r\sin\theta. \tag{5.6.4}$$

The values of the other trigonometric functions can be expressed in terms of x, y, and r (Figure 5.6.3).

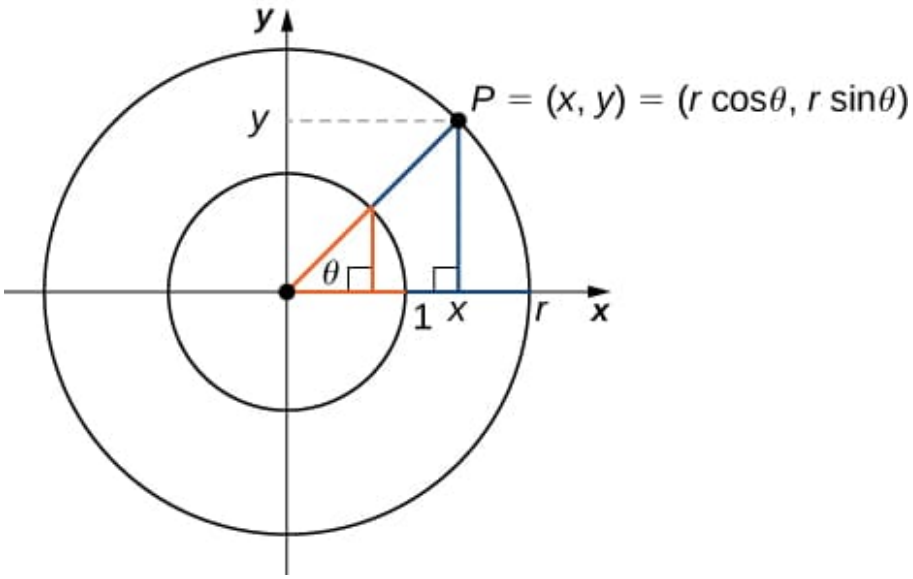


Figure 5.6.3: For a point $P = (x, y)$ on a circle of radius r, the coordinates x and y satisfy $x = r\cos\theta$ and $y = r\sin\theta$.

Table 5.6.2 shows the values of sine and cosine at the major angles in the first quadrant. From this table, we can determine the values of sine and cosine at the corresponding angles in the other quadrants. The values of the other trigonometric functions are calculated easily from the values of $\sin\theta$ and $\cos\theta$.

Table 5.6.2: Values of $\sin\theta$ and $\cos\theta$ at Major Angles θ in the First Quadrant

θ	$\sin\theta$	$\cos\theta$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0

Example 5.6.2: Evaluating Trigonometric Functions

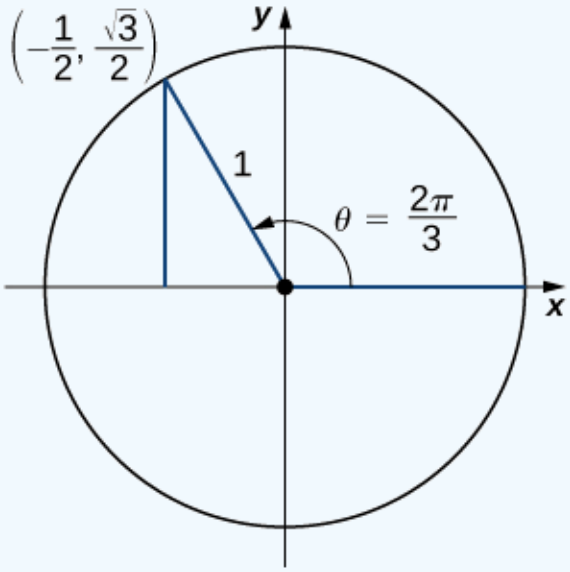
Evaluate each of the following expressions.

a. $\sin\left(\frac{2\pi}{3}\right)$
b. $\cos\left(-\frac{5\pi}{6}\right)$
c. $\tan\left(\frac{15\pi}{4}\right)$

Solution

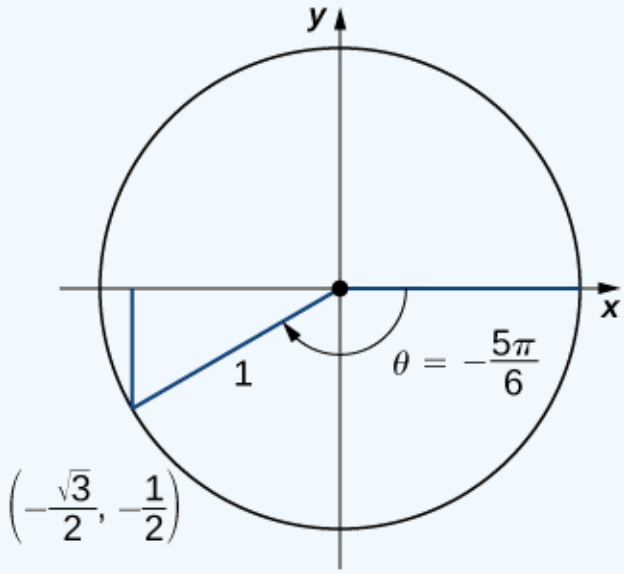
a) On the unit circle, the angle $\theta = \frac{2\pi}{3}$ corresponds to the point $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Therefore,

$$\sin\left(\frac{2\pi}{3}\right) = y = \left(\frac{\sqrt{3}}{2}\right).$$



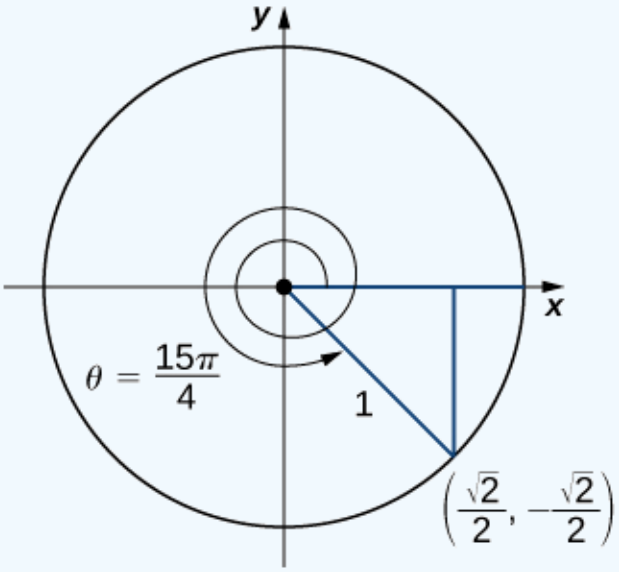
b) An angle $\theta = -\frac{5\pi}{6}$ corresponds to a revolution in the negative direction, as shown. Therefore,

$$\cos\left(-\frac{5\pi}{6}\right) = x = -\frac{\sqrt{3}}{2}.$$



c) An angle $\theta = \frac{15\pi}{4} = 2\pi + \frac{7\pi}{4}$. Therefore, this angle corresponds to more than one revolution, as shown. Knowing the fact that an angle of $\frac{7\pi}{4}$ corresponds to the point $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, we can conclude that

$$\tan\left(\frac{15\pi}{4}\right) = \frac{y}{x} = -1.$$



Exercise 5.6.2

Evaluate $\cos(3\pi/4)$ and $\sin(-\pi/6)$.

Hint available at the end of this section.

Answer available at the end of this section.

As mentioned earlier, the ratios of the side lengths of a right triangle can be expressed in terms of the trigonometric functions evaluated at either of the acute angles of the triangle. Let θ be one of the acute angles. Let A be the length of the adjacent leg, O be the length of the opposite leg, and H be the length of the hypotenuse. By inscribing the triangle into a circle of radius H, as shown in Figure 5.6.4, we see that A, H, and O satisfy the following relationships with θ:

$\sin\theta = \frac{O}{H}$	$\csc\theta = \frac{H}{O}$
$\cos\theta = \frac{A}{H}$	$\sec\theta = \frac{H}{A}$
$\tan\theta = \frac{O}{A}$	$\cot\theta = \frac{A}{O}$

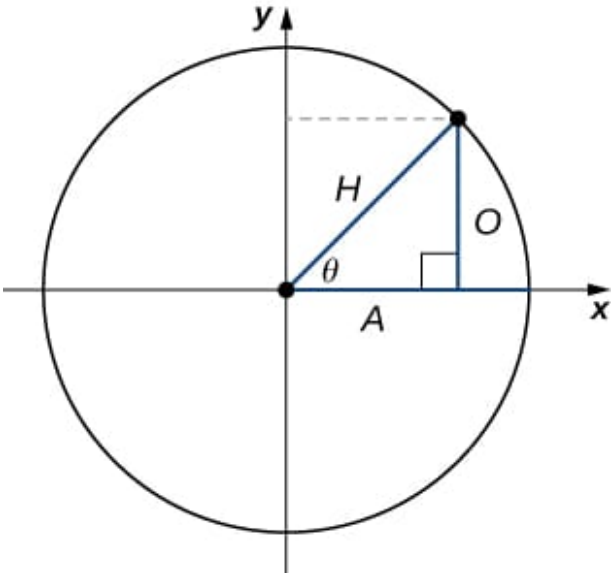


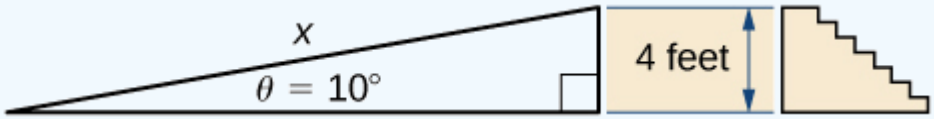
Figure 5.6.4: By inscribing a right triangle in a circle, we can express the ratios of the side lengths in terms of the trigonometric functions evaluated at θ.

Example 5.6.3: Constructing a Wooden Ramp

A wooden ramp is to be built with one end on the ground and the other end at the top of a short staircase. If the top of the staircase is 4 ft from the ground and the angle between the ground and the ramp is to be 10°, how long does the ramp need to be?

Solution

Let x denote the length of the ramp. In the following image, we see that x needs to satisfy the equation $\sin(10^\circ) = 4/x$. Solving this equation for x, we see that $x = 4\sin(10^\circ) \approx 23.035$ ft.



Exercise 5.6.3

A house painter wants to lean a 20-ft ladder against a house. If the angle between the base of the ladder and the ground is to be 60°, how far from the house should she place the base of the ladder?

Hint available at the end of this section.

Answer available at the end of this section.

Trigonometric Identities

A **trigonometric identity** is an equation involving trigonometric functions that is true for all angles θ for which the functions are defined. We can use the identities to help us solve or simplify equations. The main trigonometric identities are listed below.

Theorem 5.6.1: Trigonometric Identities

Reciprocal Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \tag{5.6.5}$$

$$1 + \tan^2\theta = \sec^2\theta \tag{5.6.6}$$

$$1 + \cot^2\theta = \csc^2\theta \tag{5.6.7}$$

Addition and Subtraction Identities

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Double-Angle Identities

$$\sin(2\theta) = 2\sin\theta\cos\theta \tag{5.6.8}$$

$$\cos(2\theta) = 2\cos^2\theta - 1 \tag{5.6.9}$$

$$= 1 - 2\sin^2\theta \tag{5.6.10}$$

$$= \cos^2\theta - \sin^2\theta \tag{5.6.11}$$

Example 5.6.4: Solving Trigonometric Equations

For each of the following equations, use a trigonometric identity to find all solutions.

a. $1 + \cos(2\theta) = \cos\theta$
b. $\sin(2\theta) = \tan\theta$

Solution

a) Using the Double-Angle Identity for $\cos(2\theta)$, we see that θ is a solution of

$$1 + \cos(2\theta) = \cos\theta$$

if and only if

$$1 + 2\cos^2\theta - 1 = \cos\theta,$$

which is true if and only if

$$2\cos^2\theta - \cos\theta = 0.$$

To solve this equation, it is important to note that we need to factor the left-hand side and not divide both sides of the equation by $\cos\theta$. The problem with dividing by $\cos\theta$ is that it is possible that $\cos\theta$ is zero. In fact, if we did divide both sides of the equation by $\cos\theta$, we would miss some of the solutions of the original equation. Factoring the left-hand side of the equation, we see that θ is a solution of this equation if and only if

$$\cos\theta(2\cos\theta - 1) = 0.$$

Since $\cos\theta = 0$ when

$$\theta = \frac{\pi}{2}, \frac{\pi}{2} \pm \pi, \frac{\pi}{2} \pm 2\pi, \ldots,$$

and $\cos\theta = 1/2$ when

$$\theta = \frac{\pi}{3}, \frac{\pi}{3} \pm 2\pi, \ldots \text{ or } \theta = -\frac{\pi}{3}, -\frac{\pi}{3} \pm 2\pi, \ldots,$$

we conclude that the set of solutions to this equation is

$$\theta = \frac{\pi}{2} + n\pi, \ \theta = \frac{\pi}{3} + 2n\pi$$

and

$$\theta = -\frac{\pi}{3} + 2n\pi, \ n = 0, \pm 1, \pm 2, \ldots.$$

b) Using the Double-Angle Identity for $\sin(2\theta)$ and the reciprocal identity for $\tan(\theta)$, the equation can be written as

$$2\sin\theta\cos\theta = \frac{\sin\theta}{\cos\theta}.$$

To solve this equation, we multiply both sides by $\cos\theta$ to eliminate the denominator, and say that if θ satisfies this equation, then θ satisfies the equation

$$2\sin\theta\cos^2\theta - \sin\theta = 0.$$

However, we need to be a little careful here. Even if θ satisfies this new equation, it may not satisfy the original equation because, to satisfy the original equation, we would need to be able to divide both sides of the equation by $\cos\theta$. However, if $\cos\theta = 0$, we cannot divide both sides of the equation by $\cos\theta$. Therefore, it is possible that we may arrive at extraneous solutions. So, at the end, it is important to check for extraneous solutions. Returning to the equation, it is important that we factor $\sin\theta$ out of both terms on the left-hand side instead of dividing both sides of the equation by $\sin\theta$. Factoring the left-hand side of the equation, we can rewrite this equation as

$$\sin\theta(2\cos^2\theta - 1) = 0.$$

Therefore, the solutions are given by the angles θ such that $\sin\theta = 0$ or $\cos^2\theta = 1/2$. The solutions of the first equation are $\theta = 0, \pm\pi, \pm 2\pi, \ldots$. The solutions of the second equation are $\theta = \pi/4, (\pi/4) \pm (\pi/2), (\pi/4) \pm \pi, \ldots$. After checking for extraneous solutions, the set of solutions to the equation is

$$\theta = n\pi$$

and

$$\theta = \frac{\pi}{4} + \frac{n\pi}{2}$$

with $n = 0, \pm 1, \pm 2, \ldots$.

Exercise 5.6.4

Find all solutions to the equation $\cos(2\theta) = \sin\theta$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 5.6.5: Proving a Trigonometric Identity

Prove the trigonometric identity $1 + \tan^2\theta = \sec^2\theta$.

Solution

We start with the Pythagorean Identity (Equation 5.6.5)

$$\sin^2\theta + \cos^2\theta = 1.$$

Dividing both sides of this equation by $\cos^2\theta$, we obtain

$$\frac{\sin^2\theta}{\cos^2\theta} + 1 = \frac{1}{\cos^2\theta}.$$

Since $\sin\theta/\cos\theta = \tan\theta$ and $1/\cos\theta = \sec\theta$, we conclude that

$$\tan^2\theta + 1 = \sec^2\theta.$$

Exercise 5.6.5

Prove the trigonometric identity $1 + \cot^2\theta = \csc^2\theta$.

Answer available at the end of this section.

Graphs and Periods of the Trigonometric Functions

We have seen that as we travel around the unit circle, the values of the trigonometric functions repeat. We can see this pattern in the graphs of the functions. Let $P = (x, y)$ be a point on the unit circle and let θ be the corresponding angle . Since the angle θ and $\theta + 2\pi$ correspond to the same point P, the values of the trigonometric functions at θ and at $\theta + 2\pi$ are the same. Consequently, the trigonometric functions are **periodic functions**. The period of a function f is defined to be the smallest positive value p such that $f(x + p) = f(x)$ for all values x in the domain of f. The sine, cosine, secant, and cosecant functions have a period of 2π. Since the tangent and cotangent functions repeat on an interval of length π, their period is π (Figure 5.6.5).

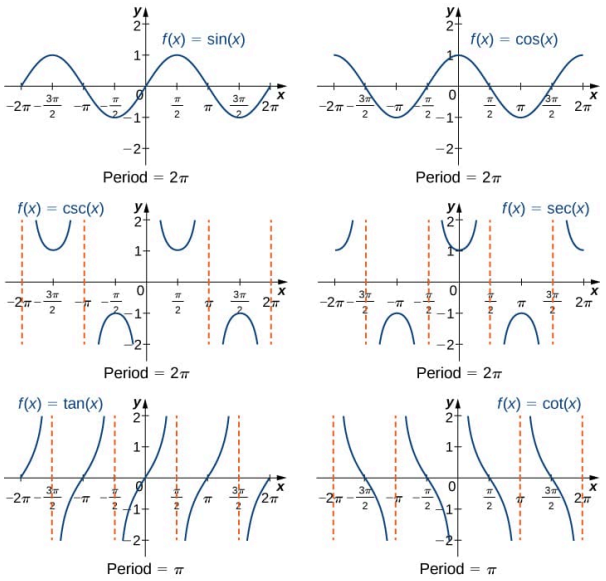


Figure 5.6.5: The six trigonometric functions are periodic.

Just as with algebraic functions, we can apply transformations to trigonometric functions. In particular, consider the following function:

$$f(x) = A\sin(B(x-\alpha)) + C.$$

In Figure 5.6.6, the constant α causes a horizontal or phase shift. The factor B changes the period. This transformed sine function will have a period $2\pi/|B|$. The factor A results in a vertical stretch by a factor of $|A|$. We say $|A|$ is the "amplitude of f." The constant C causes a vertical shift.

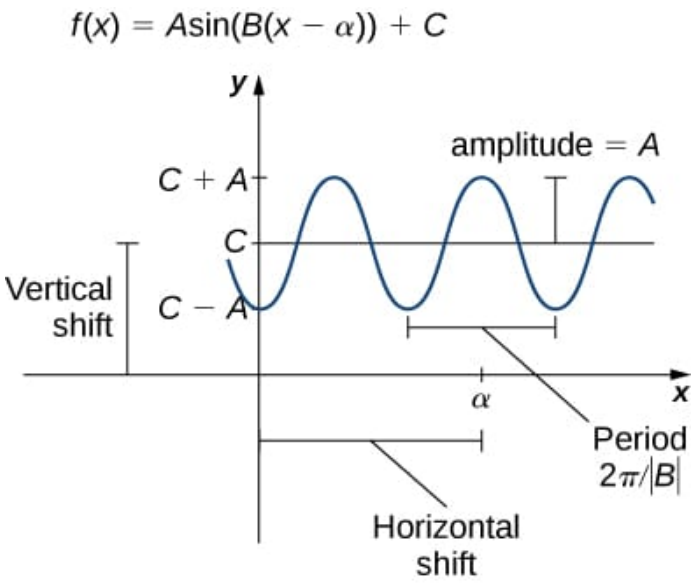


Figure 5.6.6: A graph of a general sine function.

Notice in Figure 5.6.6 that the graph of $y = \cos x$ is the graph of $y = \sin x$ shifted to the left $\pi/2$ units. Therefore, we can write

$$\cos x = \sin(x + \pi/2).$$

Similarly, we can view the graph of $y = \sin x$ as the graph of $y = \cos x$ shifted right $\pi/2$ units, and state that $\sin x = \cos(x - \pi/2)$.

A shifted sine curve arises naturally when graphing the number of hours of daylight in a given location as a function of the day of the year. For example, suppose a city reports that June 21 is the longest day of the year with 15.7 hours and December 21 is the

shortest day of the year with 8.3 hours. It can be shown that the function

$$h(t) = 3.7 \sin\left(\frac{2\pi}{365}(x - 80.5)\right) + 12$$

is a model for the number of hours of daylight h as a function of day of the year t (Figure $5.6.7$).

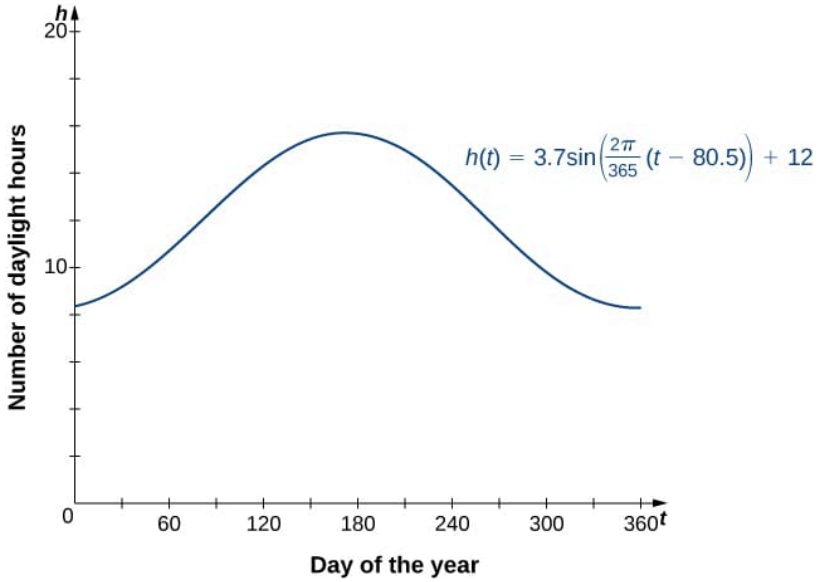


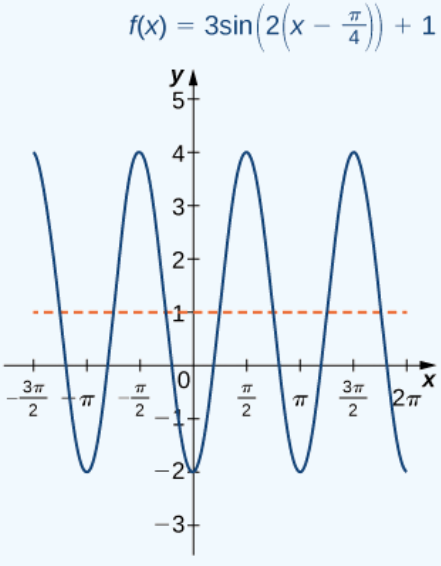
Figure $5.6.7$: The hours of daylight as a function of day of the year can be modeled by a shifted sine curve.

Example $5.6.6$: Sketching the Graph of a Transformed Sine Curve

Sketch a graph of $f(x) = 3\sin(2(x - \frac{\pi}{4})) + 1$.

Solution

This graph is a phase shift of $y = \sin(x)$ to the right by $\pi/4$ units, followed by a horizontal compression by a factor of 2, a vertical stretch by a factor of 3, and then a vertical shift by 1 unit. The period of f is π.



Exercise 5.6.6

Describe the relationship between the graph of $f(x) = 3\sin(4x) - 5$ and the graph of $y = \sin(x)$.

Hint available at the end of this section.

Answer available at the end of this section.

Inverse Trigonometric Functions

The six basic trigonometric functions are periodic, and therefore they are not one-to-one. However, if we restrict the domain of a trigonometric function to an interval where it is one-to-one, we can define its inverse. Consider the sine function. The sine function is one-to-one on an infinite number of intervals, but the standard convention is to restrict the domain to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. By doing so, we define the inverse sine function on the domain $[-1, 1]$ such that for any x in the interval $[-1, 1]$, the inverse sine function tells us which angle θ in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ satisfies $\sin\theta = x$. Similarly, we can restrict the domains of the other trigonometric functions to define **inverse trigonometric functions**, which are functions that tell us which angle in a certain interval has a specified trigonometric value.

Definition: Inverse Trigonometric Functions

The inverse sine function, denoted $\sin^{-1}$ or $\arcsin$, and the inverse cosine function, denoted $\cos^{-1}$ or $\arccos$, are defined on the domain $D = \{y \mid -1 \le y \le 1\}$ as follows:

$$\begin{aligned} \sin^{-1}(y) = \theta \quad &\text{if and only if} \quad \sin(\theta) = y \quad \text{where} \quad \theta \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right] \\ \cos^{-1}(y) = \theta \quad &\text{if and only if} \quad \cos(\theta) = y \quad \text{where} \quad \theta \in [0, \pi]\,. \end{aligned}$$

The inverse tangent function, denoted $\tan^{-1}$ or $\arctan$, and inverse cotangent function, denoted $\cot^{-1}$ or arccot, are defined on the domain $D = \{y \mid -\infty < y < \infty\}$ as follows:

$$\begin{aligned} \tan^{-1}(y) = \theta \quad &\text{if and only if} \quad \tan(\theta) = y \quad \text{where} \quad \theta \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \\ \cot^{-1}(y) = \theta \quad &\text{if and only if} \quad \cot(\theta) = y \quad \text{where} \quad \theta \in (0, \pi)\,. \end{aligned}$$

The inverse cosecant function, denoted $\csc^{-1}$ or arccsc, and inverse secant function, denoted $\sec^{-1}$ or arcsec, are defined on the domain $D = \{y \mid |y| \ge 1\}$ as follows:

$$\begin{aligned} \csc^{-1}(y) = \theta \quad &\text{if and only if} \quad \csc(\theta) = y \quad \text{where} \quad \theta \in \left(0, \tfrac{\pi}{2}\right] \cup \left(\pi, \tfrac{3\pi}{2}\right] \\ \sec^{-1}(y) = \theta \quad &\text{if and only if} \quad \sec(\theta) = y \quad \text{where} \quad \theta \in \left[0, \tfrac{\pi}{2}\right) \cup \left[\pi, \tfrac{3\pi}{2}\right)\,. \end{aligned}$$

To graph the inverse trigonometric functions, we use the graphs of the trigonometric functions restricted to the domains defined earlier and reflect the graphs about the line $y = x$ (Figures $5.6.8 - 5.6.10$).

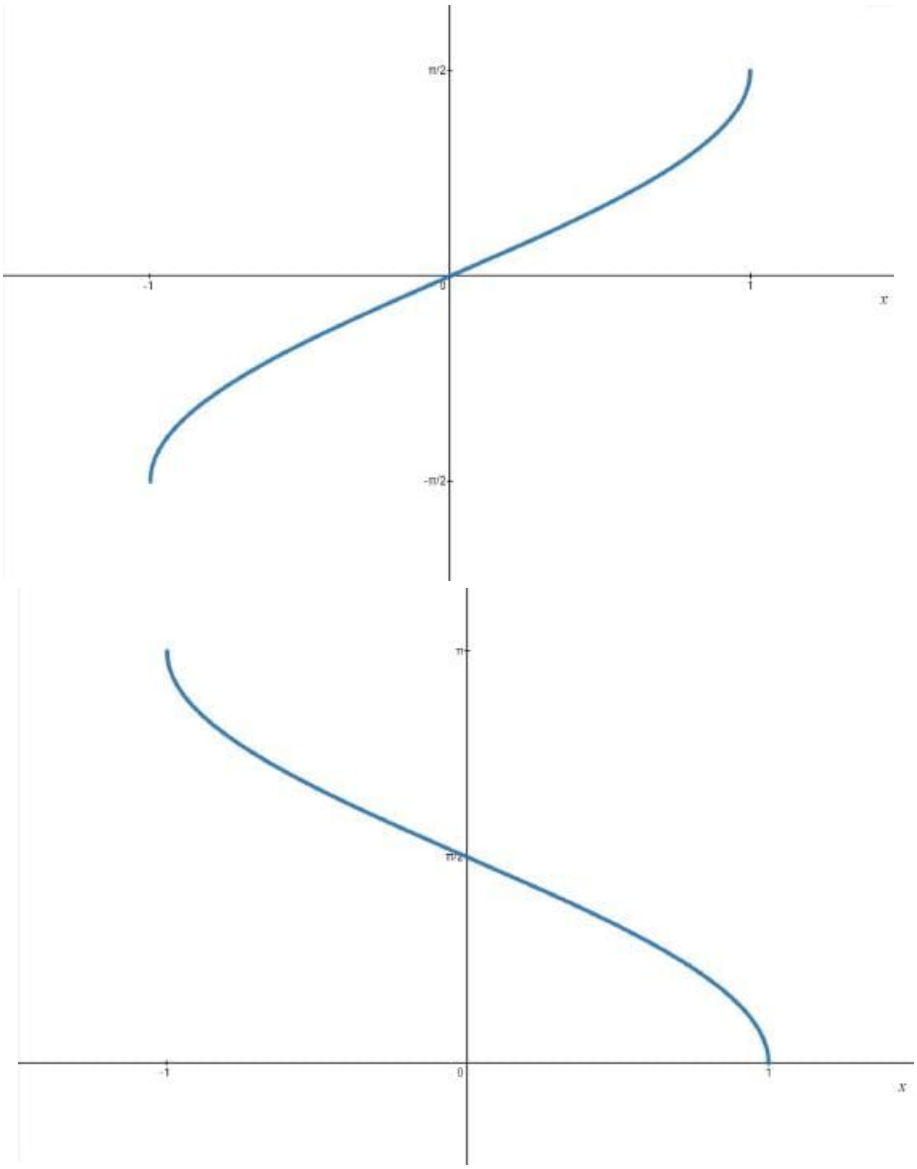


Figure 5.6.8: The graphs of the arcsine (left) and arccosine (right).

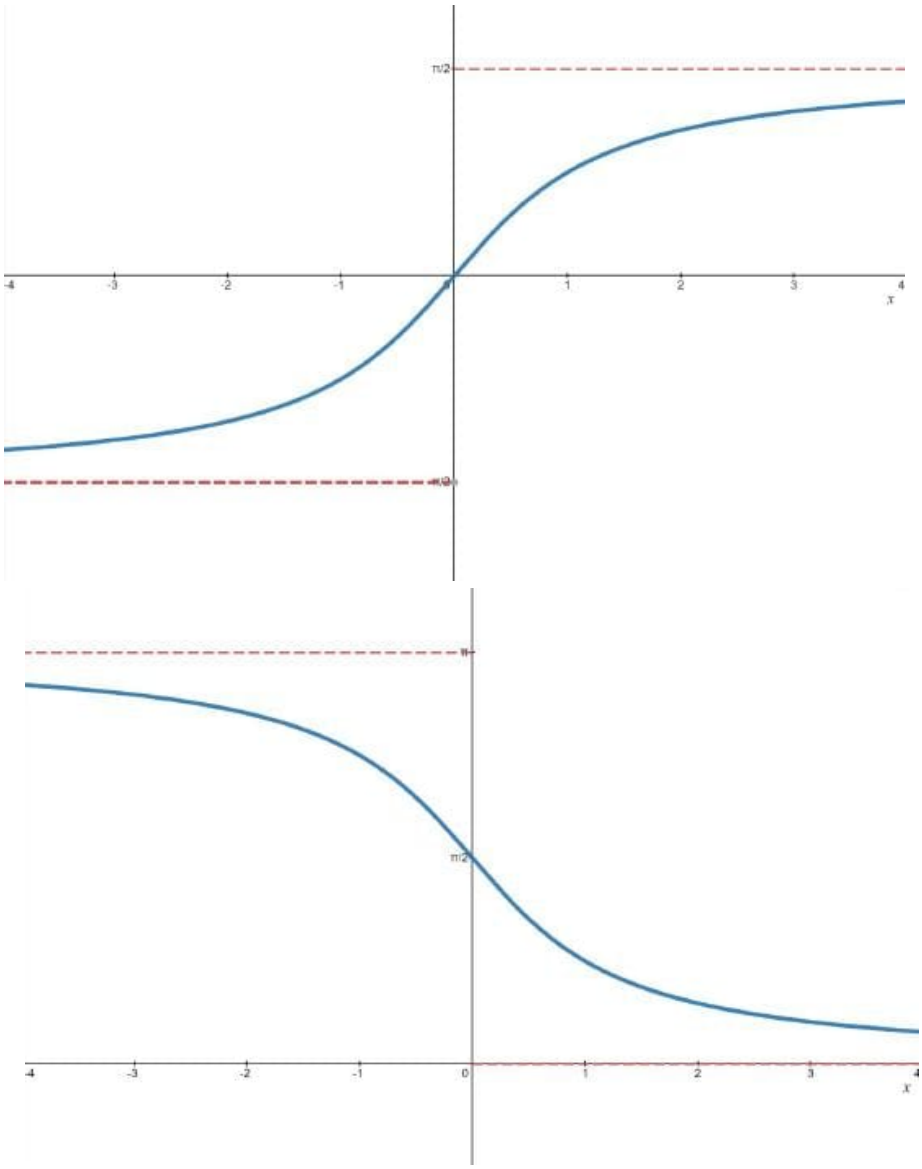


Figure 5.6.9: The graphs of the arctangent (left) and arccotangent (right).

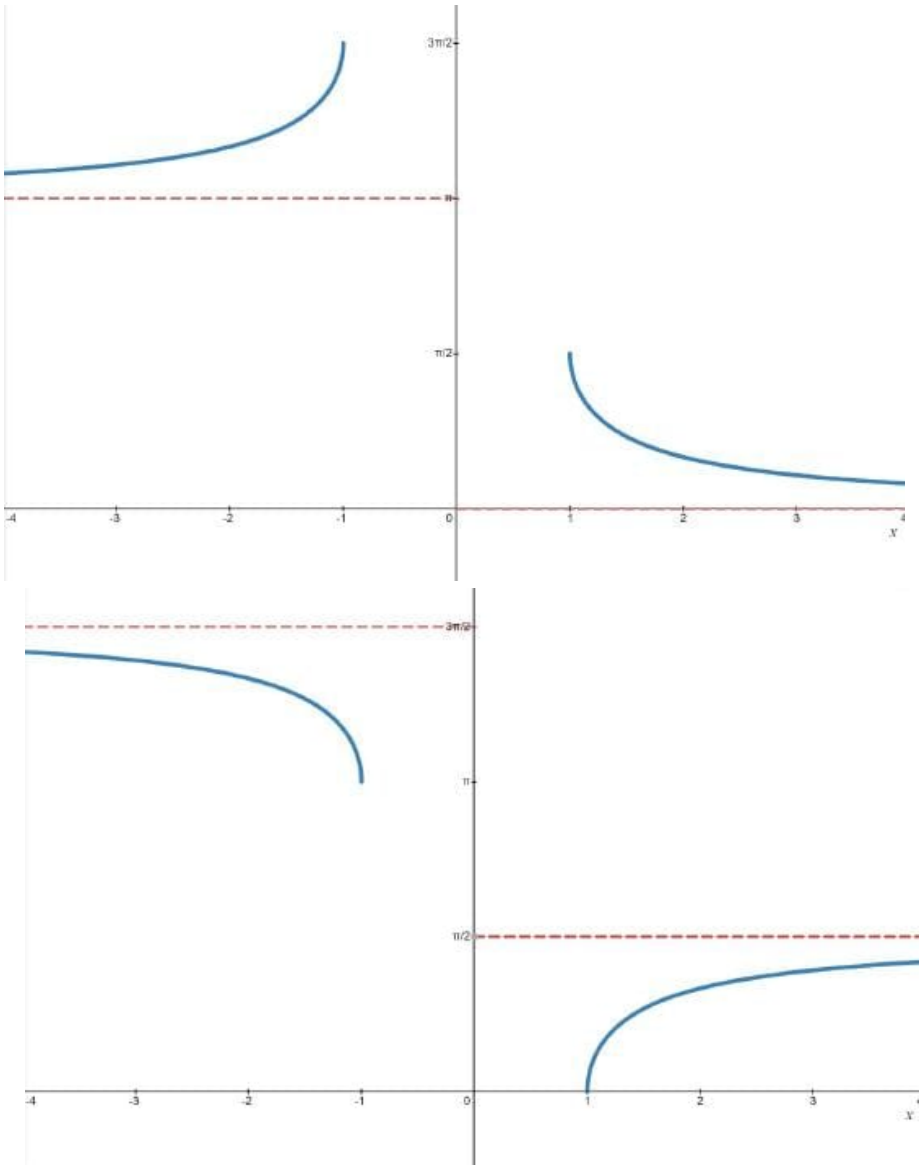


Figure 5.6.10: The graphs of the arccosecant (left) and arcsecant (right).

Note

All the inverse trigonometric functions return angles in quadrant I (where all trigonometric functions are positive) and one other quadrant where the corresponding trigonometric function is negative. The returned quadrants where the corresponding trigonometric functions are negative are fairly standard and agreed upon (quadrant IV for arctangent and arcsine, and quadrant II for arccosine and arccotangent); **however**, the quadrants chosen for the ranges of the arcsecant and arccosecant are *not* universally agreed upon.

In this text, we choose arcsecant and arccosecant to return angles in quadrants I and **III** for a special reason - it makes our work in calculus slightly easier. I mention this because you might have seen a slightly different choice for the ranges of these functions in another textbook. The difference is insignificant other than the ease our choice makes for our work in calculus.

When evaluating an inverse trigonometric function, the output is an angle. For example, to evaluate $\cos^{-1}\left(\frac{1}{2}\right)$, we need to find an angle θ such that $\cos\theta = \frac{1}{2}$. Clearly, many angles have this property. However, given the definition of $\cos^{-1}$, we need the angle θ that not only solves this equation, but also lies in the interval $[0, \pi]$. We conclude that $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

We now consider a composition of a trigonometric function and its inverse. For example, consider the two expressions $\sin\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$ and $\sin^{-1}(\sin(\pi))$.

For the first one, we simplify as follows:

$$\sin\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

For the second one, we have

$$\sin^{-1}(\sin(\pi)) = \sin^{-1}(0) = 0.$$

The inverse function is supposed to "undo" the original function, so why isn't $\sin^{-1}(\sin(\pi)) = \pi$? Recalling our definition of inverse functions, a function f and its inverse f^{-1} satisfy the conditions $f(f^{-1}(y)) = y$ for all y in the domain of f^{-1} and $f^{-1}(f(x)) = x$ for all x in the domain of f, so what happened here? The issue is that the inverse sine function, $\sin^{-1}$, is the inverse of the restricted sine function defined on the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Therefore, for x in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, it is true that $\sin^{-1}(\sin x) = x$. However, for values of x outside this interval, the equation does not hold, even though $\sin^{-1}(\sin x)$ is defined for all real numbers x.

What about $\sin(\sin^{-1} y)$? Does that have a similar issue? The answer is no. Since the domain of $\sin^{-1}$ is the interval $[-1, 1]$, we conclude that $\sin\left(\sin^{-1} y\right) = y$ if $-1 \leq y \leq 1$ and the expression is not defined for other values of y. To summarize,

$$\sin(\sin^{-1} y) = y \text{ if } -1 \leq y \leq 1$$

and

$$\sin^{-1}(\sin x) = x \text{ if } -\tfrac{\pi}{2} \leq x \leq \tfrac{\pi}{2}.$$

Similarly, for the cosine function,

$$\cos(\cos^{-1} y) = y \text{ if } -1 \leq y \leq 1$$

and

$$\cos^{-1}(\cos x) = x \text{ if } 0 \leq x \leq \pi.$$

Similar properties hold for the other trigonometric functions and their inverses.

Example 5.6.7: Evaluating Expressions Involving Inverse Trigonometric Functions

Evaluate each of the following expressions.

a. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
b. $\tan\left(\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right)$
c. $\cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right)$
d. $\sin^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right)$

Solution

a. Evaluating $\sin^{-1}(-\sqrt{3}/2)$ is equivalent to finding the angle θ such that $\sin\theta = -\sqrt{3}/2$ and $-\pi/2 \leq \theta \leq \pi/2$. The angle $\theta = -\pi/3$ satisfies these two conditions. Therefore, $\sin^{-1}(-\sqrt{3}/2) = -\pi/3$.
b. First we use the fact that $\tan^{-1}(-1/\sqrt{3}) = -\pi/6$. Then $\tan(-\pi/6) = -1/\sqrt{3}$. Therefore, $\tan(\tan^{-1}(-1/\sqrt{3})) = -1/\sqrt{3}$.
c. To evaluate $\cos^{-1}(\cos(5\pi/4))$,first use the fact that $\cos(5\pi/4) = -\sqrt{2}/2$. Then we need to find the angle θ such that $\cos(\theta) = -\sqrt{2}/2$ and $0 \leq \theta \leq \pi$. Since $3\pi/4$ satisfies both these conditions, we have $\cos^{-1}(\cos(5\pi/4)) = \cos^{-1}(-\sqrt{2}/2)) = 3\pi/4$.
d. Since $\cos(2\pi/3) = -1/2$, we need to evaluate $\sin^{-1}(-1/2)$. That is, we need to find the angle θ such that $\sin(\theta) = -1/2$ and $-\pi/2 \leq \theta \leq \pi/2$. Since $-\pi/6$ satisfies both these conditions, we can conclude that $\sin^{-1}(\cos(2\pi/3)) = \sin^{-1}(-1/2) = -\pi/6$.

The Maximum Value of a Function

In many areas of science, engineering, and mathematics, it is useful to know the maximum value a function can obtain, even if we don't know its exact value at a given instant. For instance, if we have a function describing the strength of a roof beam, we would want to know the maximum weight the beam can support without breaking. If we have a function that describes the speed of a train, we would want to know its maximum speed before it jumps off the rails. Safe design often depends on knowing maximum values.

This project describes a simple example of a function with a maximum value that depends on two equation coefficients. We will see that maximum values can depend on several factors other than the independent variable x.

1. Consider the graph in Figure $5.6.9$ of the function $y = \sin x + \cos x$. Describe its overall shape. Is it periodic? How do you know?

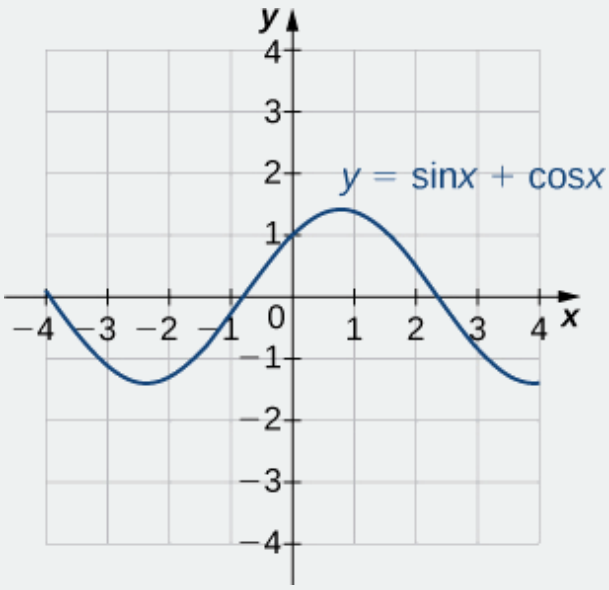


Figure $5.6.9$: The graph of $y = \sin x + \cos x$.

Using a graphing calculator or other graphing device, estimate the x- and y-values of the maximum point for the graph (the first such point where $x > 0$). It may be helpful to express the x-value as a multiple of π.

2. Now consider other graphs of the form $y = A \sin x + B \cos x$ for various values of A and B. Sketch the graph when $A = 2$ and $B = 1$, and find the x- and y-values for the maximum point. (Remember to express the x-value as a multiple of π, if possible.) Has it moved?

3. Repeat for $A = 1,\ \ B = 2$. Is there any relationship to what you found in part (2)?

4. Complete the following table, adding a few choices of your own for A and B :

A	B	x	y	A	B	x	y
0	1			3	4		
1	0			4	3		
1	1			$\sqrt{3}$	1		
1	2			1	$\sqrt{3}$		
2	1			12	5		
2	2			5	12		

5. Try to figure out the formula for the y-values.

6. The formula for the x-values is a little harder. The most helpful points from the table are $(1,1),\ (1,\sqrt{3}),\ (\sqrt{3},1)$. (Hint: Consider inverse trigonometric functions.)

7. If you found formulas for parts (5) and (6), show that they work together. That is, substitute the x-value formula you found into $y = A\sin x + B\cos x$ and simplify it to arrive at the y-value formula you found.

Key Concepts

- Radian measure is defined such that the angle associated with the arc of length 1 on the unit circle has radian measure 1. An angle with a degree measure of 180° has a radian measure of π rad.
- For acute angles θ,the values of the trigonometric functions are defined as ratios of two sides of a right triangle in which one of the acute angles is θ.
- For a general angle θ, let (x,y) be a point on a circle of radius r corresponding to this angle θ. The trigonometric functions can be written as ratios involving x, y, and r.
- The trigonometric functions are periodic. The sine, cosine, secant, and cosecant functions have period 2π. The tangent and cotangent functions have period π.

Key Equations

- **Generalized sine function**

$$f(x) = A\sin(B(x-\alpha)) + C$$

Glossary

periodic function
: a function is periodic if it has a repeating pattern as the values of x move from left to right

radians
: for a circular arc of length s on a circle of radius 1, the radian measure of the associated angle θ is s

trigonometric functions
: functions of an angle defined as ratios of the lengths of the sides of a right triangle

trigonometric identity
: an equation involving trigonometric functions that is true for all angles θ for which the functions in the equation are defined

Answers

Exercise $5.6.1$

π radians is equal to 180°

Exercise $5.6.1$

a. $7\pi/6$
b. 330°

Exercise $5.6.2$

Look at angles on the unit circle.

Exercise $5.6.2$

$$\cos(3\pi/4) = -\sqrt{2}/2$$
$$\sin(-\pi/6) = -1/2$$

Exercise $5.6.3$

Draw a right triangle with hypotenuse 20.

Exercise 5.6.3

10 ft

Exercise 5.6.4

Use the Double-Angle Identity for cosine (Equation 5.6.8).

Exercise 5.6.4

$\theta = \frac{3\pi}{2} + 2n\pi, \frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi$

for $n = 0, \pm 1, \pm 2, \ldots$.

Exercise 5.6.5

Divide both sides of the identity $\sin^2\theta + \cos^2\theta = 1$ by $\sin^2\theta$.

Exercise 5.6.6

The graph of f can be sketched using the graph of $y = \sin(x)$ and a sequence of three transformations.

Exercise 5.6.6

To graph $f(x) = 3\sin(4x) - 5$, the graph of $y = \sin(x)$ needs to be compressed horizontally by a factor of 4, then stretched vertically by a factor of 3, then shifted down 5 units. The function f will have a period of $\pi/2$ and an amplitude of 3.

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5.6.1: A.6.1- Homework

In exercises 1 - 5, convert each angle in degrees to radians. Write the answer as a multiple of π.

1) 240°

Answer

$\frac{4\pi}{3}$ rad

2) 15°

3) 60°

Answer

$\frac{\pi}{3}$ rad

4) -225°

5) 330°

Answer

$\frac{11\pi}{6}$ rad

In exercises 6 - 10, convert each angle in radians to degrees.

6) $\frac{\pi}{2}$ rad

7) $\frac{7\pi}{6}$ rad

Answer

210°

8) $\frac{11\pi}{2}$ rad

9) -3π rad

Answer

-540°

10) $\frac{5\pi}{12}$ rad

In exercises 11 - 16, evaluate the functional values.

11) $\cos\frac{4\pi}{3}$

Answer

$\cos\frac{4\pi}{3} = -0.5$

12) $\tan\frac{19\pi}{4}$

13) $\sin\left(-\frac{3\pi}{4}\right)$

Answer

$\sin\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

14) $\sec\left(-\frac{\pi}{6}\right)$

15) $\sin\left(-\frac{\pi}{12}\right)$

Answer

$$\sin\left(-\frac{\pi}{12}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

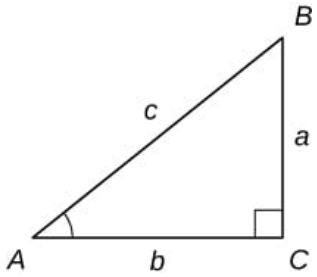
16) $\cos\left(-\frac{5\pi}{12}\right)$

In exercises 17 - 22, consider triangle ABC, a right triangle with a right angle at C.

a. Find the missing side of the triangle.

b. Find the six trigonometric function values for the angle at A.

Where necessary, round to one decimal place.



= 225^{\circ} \cdot

17) $a = 4,\ \ c = 7$

Answer

a. $b = 5.7$
b. $\sin A = \frac{4}{7},\ \cos A = \frac{5.7}{7},\ \tan A = \frac{4}{5.7},\ \csc A = \frac{7}{4},\ \sec A = \frac{7}{5.7},\ \cot A = \frac{5.7}{4}$

18) $a = 21,\ \ c = 29$

19) $a = 85.3,\ \ b = 125.5$

Answer

a. $c = 151.7$
b. $\sin A = 0.5623,\ \cos A = 0.8273,\ \tan A = 0.6797,\ \csc A = 1.778,\ \sec A = 1.209,\ \cot A = 1.471$

20) $b = 40,\ \ c = 41$

21) $a = 84,\ \ b = 13$

Answer

a. $c = 85$
b. $\sin A = \frac{84}{85},\ \cos A = \frac{13}{85},\ \tan A = \frac{84}{13},\ \csc A = \frac{85}{84},\ \sec A = \frac{85}{13},\ \cot A = \frac{13}{84}$

22) $b = 28,\ \ c = 35$

In exercises 23 - 26, P is a point on the unit circle.

a. Find the (exact) missing coordinate value of each point and

b. find the values of the six trigonometric functions for the angle θ with a terminal side that passes through point P.

Rationalize denominators.

23) $P\left(\frac{7}{25}, y\right), \quad y > 0$

Answer

a. $y = \frac{24}{25}$
b. $\sin\theta = \frac{24}{25}, \ \cos\theta = \frac{7}{25}, \ \tan\theta = \frac{24}{7}, \ \csc\theta = \frac{25}{24}, \ \sec\theta = \frac{25}{7}, \ \cot\theta = \frac{7}{24}$

24) $P\left(-\frac{15}{17}, y\right), \quad y > 0$

25) $P\left(x, \frac{\sqrt{7}}{3}\right), \quad x > 0$

Answer

a. $x = -\frac{\sqrt{2}}{3}$
b. $\sin\theta = \frac{\sqrt{7}}{3}, \ \cos\theta = -\frac{\sqrt{2}}{3}, \ \tan\theta = -\frac{\sqrt{14}}{2}, \ \csc\theta = \frac{3\sqrt{7}}{7}, \ \sec\theta = -\frac{3\sqrt{2}}{2}, \ \cot\theta = -\frac{\sqrt{14}}{7}$

26) $P\left(x, -\frac{\sqrt{15}}{4}\right), \quad y > 0$

In exercises 27 - 34, simplify each expression by writing it in terms of sines and cosines, then simplify. The final answer does not have to be in terms of sine and cosine only.

27) $\tan^2 x + \sin x \csc x$

Answer

$\sec^2 x$

28) $\sec x \sin x \cot x$

29) $\frac{\tan^2 x}{\sec^2 x}$

Answer

$\sin^2 x$

30) $\sec x - \cos x$

31) $(1 + \tan\theta)^2 - 2\tan\theta$

Answer

$\sec^2\theta$

32) $(\sin x)(\csc x - \sin x)$

33) $\frac{\cos t}{\sin t} + \frac{\sin t}{1 + \cos t}$

Answer

$\frac{1}{\sin t} = \csc t$

34) $\frac{1 + \tan^2\alpha}{1 + \cot^2\alpha}$

In exercises 35 - 42, verify that each equation is an identity.

35) $\frac{\tan\theta\cot\theta}{\csc\theta} = \sin\theta$

36) $\frac{\sec^2\theta}{\tan\theta} = \sec\theta\csc\theta$

37) $\frac{\sin t}{\csc t} + \frac{\cos t}{\sec t} = 1$

38) $\dfrac{\sin x}{\cos x + 1} + \dfrac{\cos x - 1}{\sin x} = 0$

39) $\cot \gamma + \tan \gamma = \sec \gamma \csc \gamma$

40) $\sin^2 \beta + \tan^2 \beta + \cos^2 \beta = \sec^2 \beta$

41) $\dfrac{1}{1 - \sin \alpha} + \dfrac{1}{1 + \sin \alpha} = 2 \sec^2 \alpha$

42) $\dfrac{\tan \theta - \cot \theta}{\sin \theta \cos \theta} = \sec^2 \theta - \csc^2 \theta$

In exercises 43 - 50, solve the trigonometric equations on the interval $0 \leq \theta < 2\pi$.

43) $2 \sin \theta - 1 = 0$

Answer

$\left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$

44) $1 + \cos \theta = \frac{1}{2}$

45) $2 \tan^2 \theta = 2$

Answer

$\left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$

46) $4 \sin^2 \theta - 2 = 0$

47) $\sqrt{3} \cot \theta + 1 = 0$

Answer

$\left\{\frac{2\pi}{3}, \frac{5\pi}{3}\right\}$

48) $3 \sec \theta - 2\sqrt{3} = 0$

49) $2 \cos \theta \sin \theta = \sin \theta$

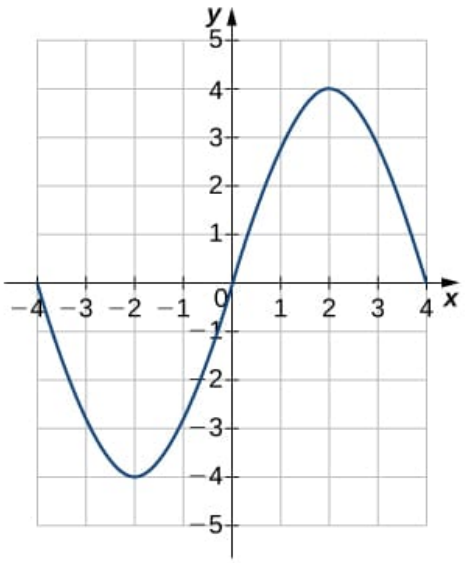
Answer

$\left\{0, \pi, \frac{\pi}{3}, \frac{5\pi}{3}\right\}$

50) $\csc^2 \theta + 2 \csc \theta + 1 = 0$

In exercises 51 - 54, each graph is of the form $y = A \sin Bx$ **or** $y = A \cos Bx$**, where** $B > 0$. **Write the equation of the graph.**

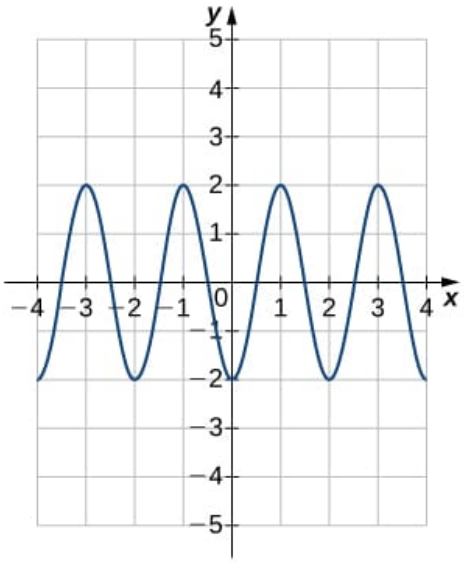
51)



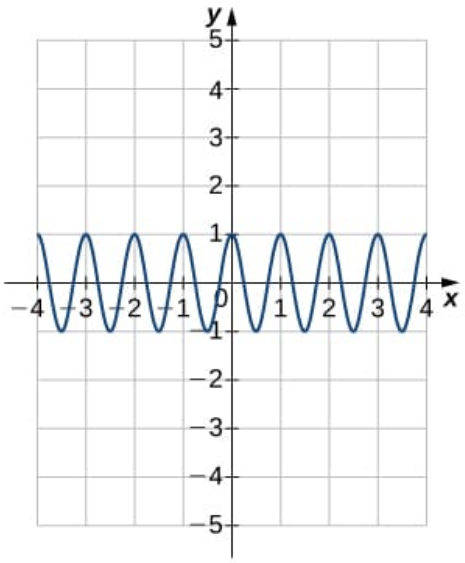
Answer

$y = 4\sin\left(\frac{\pi}{4}x\right)$

52)



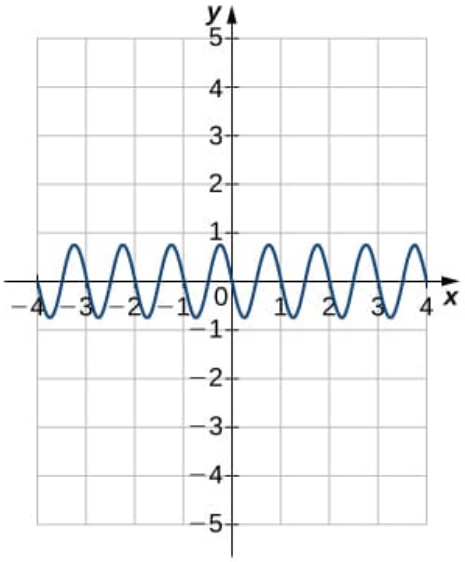
53)



Answer

$y = \cos 2\pi x$

54)



In exercises 55 - 60, find

a. the amplitude,

b. the period, and

c. the phase shift with direction for each function.

55) $y = \sin\left(x - \frac{\pi}{4}\right)$

Answer

a. 1
b. 2π
c. $\frac{\pi}{4}$ units to the right

56) $y = 3\cos(2x + 3)$

57) $y = -\frac{1}{2}\sin\left(\frac{1}{4}x\right)$

Answer

a. $\frac{1}{2}$
b. 8π
c. No phase shift

58) $y = 2\cos\left(x - \frac{\pi}{3}\right)$

59) $y = -3\sin(\pi x + 2)$

Answer

a. 3
b. 2
c. $\frac{2}{\pi}$ units to the left

60) $y = 4\cos\left(2x - \frac{\pi}{2}\right)$

61) [Technology Required] The diameter of a wheel rolling on the ground is 40 in. If the wheel rotates through an angle of 120°, how many inches does it move? Approximate to the nearest whole inch.

Answer

Approximately 42 in.

62) [Technology Required] Find the length of the arc intercepted by central angle θ in a circle of radius r. Round to the nearest hundredth.

a. $r = 12.8$ cm, $\theta = \frac{5\pi}{6}$ rad b. $r = 4.378$ cm, $\theta = \frac{7\pi}{6}$ rad c. $r = 0.964$ cm, $\theta = 50^\circ$ d. $r = 8.55$ cm, $\theta = 325^\circ$

63) [Technology Required] As a point P moves around a circle, the measure of the angle changes. The measure of how fast the angle is changing is called angular speed, ω, and is given by $\omega = \theta/t$, where θ is in radians and t is time. Find the angular speed for the given data. Round to the nearest thousandth.

a. $\theta = \frac{7\pi}{4}$ rad, $t = 10$ sec b. $\theta = \frac{3\pi}{5}$ rad, $t = 8$ sec c. $\theta = \frac{2\pi}{9}$ rad, $t = 1$ min d. $\theta = 23.76$ rad, $t = 14$ min

Answer

a. 0.550 rad/sec
b. 0.236 rad/sec
c. 0.698 rad/min
d. 1.697 rad/min

64) [Technology Required] A total of $250,000\ \text{m}^2$ of land is needed to build a nuclear power plant. Suppose it is decided that the area on which the power plant is to be built should be circular.

a)Find the radius of the circular land area.

b)If the land area is to form a 45° sector of a circle instead of a whole circle, find the length of the curved side.

65) [Technology Required] The area of an isosceles triangle with equal sides of length x is $\frac{1}{2}x^2\sin\theta$, where θ is the angle formed by the two sides. Find the area of an isosceles triangle with equal sides of length 8 in. and angle $\theta = \frac{5\pi}{12}$ rad.

Answer

$\approx 30.9\ \text{in}^2$

66) [Technology Required] A particle travels in a circular path at a constant angular speed ω. The angular speed is modeled by the function $\omega = 9\,|\cos(\pi t - \pi/12)|$. Determine the angular speed at $t = 9$ sec.

67) [Technology Required] An alternating current for outlets in a home has voltage given by the function $V(t) = 150\cos 368t$,

where V is the voltage in volts at time t in seconds.

a) Find the period of the function and interpret its meaning.

b) Determine the number of periods that occur when 1 sec has passed.

Answer

a. $\frac{\pi}{184}$; the voltage repeats every $\frac{\pi}{184}$ sec
b. Approximately 59 periods

68) [Technology Required] The number of hours of daylight in a northeast city is modeled by the function

$$N(t) = 12 + 3\sin\left[\frac{2\pi}{365}(t-79)\right],$$

where t is the number of days after January 1.

a) Find the amplitude and period.

b) Determine the number of hours of daylight on the longest day of the year.

c) Determine the number of hours of daylight on the shortest day of the year.

d) Determine the number of hours of daylight 90 days after January 1.

e) Sketch the graph of the function for one period starting on January 1.

69) [Technology Required] Suppose that $T = 50 + 10\sin\left[\frac{\pi}{12}(t-8)\right]$ is a mathematical model of the temperature (in degrees Fahrenheit) at t hours after midnight on a certain day of the week.

a) Determine the amplitude and period.

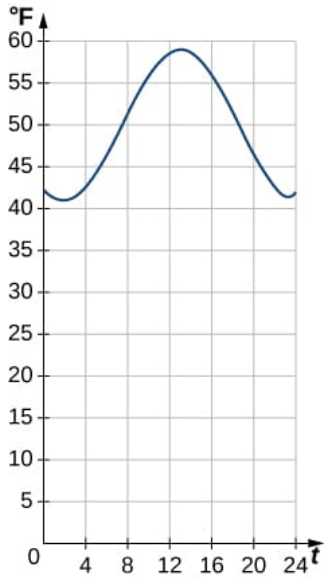
b) Find the temperature 7 hours after midnight.

c) At what time does $T = 60^\circ$?

d) Sketch the graph of T over $0 \le t \le 24$.

Answer

a. Amplitude = 10; Period=24
b. 47.4° F
c. 14 hours later, or 2 p.m.
d.



70) [Technology Required] The function $H(t) = 8\sin\left(\frac{\pi}{6}t\right)$ models the height H (in feet) of the tide t hours after midnight. Assume that $t = 0$ is midnight.

a) Find the amplitude and period.

b) Graph the function over one period.

c) What is the height of the tide at 4:30 a.m.?

In exercises 71 - 79, evaluate the functions. Give the exact value.

71) $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$

Answer

$\frac{\pi}{6}$

72) $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

73) $\cot^{-1}(1)$

Answer

$\frac{\pi}{4}$

74) $\sin^{-1}(-1)$

75) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Answer

$\frac{\pi}{6}$

76) $\cos\left(\tan^{-1}(\sqrt{3})\right)$

77) $\sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$

Answer

$\frac{\sqrt{2}}{2}$

78) $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$

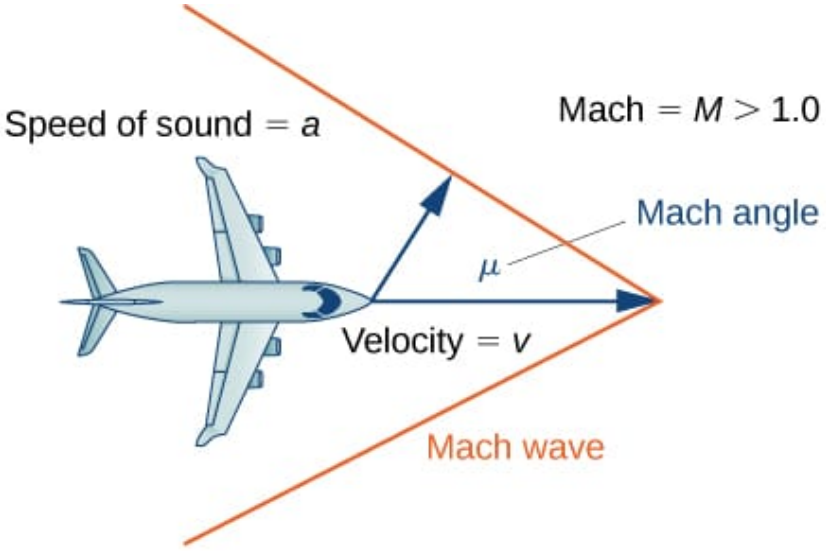
79) $\tan^{-1}\left(\tan\left(-\frac{\pi}{6}\right)\right)$

Answer

$-\frac{\pi}{6}$

80) [Technology Required] An airplane's Mach number M is the ratio of its speed to the speed of sound. When a plane is flying at a constant altitude, then its Mach angle is given by $\mu = 2\sin^{-1}\left(\frac{1}{M}\right)$.

Find the Mach angle (to the nearest degree) for the following Mach numbers.



a. $\mu = 1.4$

b. $\mu = 2.8$

c. $\mu = 4.3$

Answer

a. $\sim 92^\circ$ b. $\sim 42^\circ$ c. $\sim 27^\circ$

81) [Technology Required] Using $\mu = 2\sin^{-1}\left(\frac{1}{M}\right)$, find the Mach number M for the following angles.

a. $\mu = \frac{\pi}{6}$

b. $\mu = \frac{2\pi}{7}$

c. $\mu = \frac{3\pi}{8}$

82) [Technology Required] The temperature (in degrees Celsius) of a city in the northern United States can be modeled by the function

$$T(x) = 5 + 18\sin\left[\tfrac{\pi}{6}(x - 4.6)\right],$$

where x is time in months and $x = 1.00$ corresponds to January 1. Determine the month and day when the temperature is $21^\circ C$.

Answer

$x \approx 6.69,\ 8.51$; so, the temperature occurs on June 21 and August 15

83) [Technology Required] The depth (in feet) of water at a dock changes with the rise and fall of tides. It is modeled by the function $D(t) = 5\sin\left(\frac{\pi}{6}t - \frac{7\pi}{6}\right) + 8$, where t is the number of hours after midnight. Determine the first time after midnight when

the depth is 11.75 ft.

84) [Technology Required] An object moving in simple harmonic motion is modeled by the function $s(t) = -6\cos\left(\frac{\pi t}{2}\right)$, where s is measured in inches and t is measured in seconds. Determine the first time when the distance moved is 4.5 in.

Answer

~ 1.5 sec

85) [Technology Required] A local art gallery has a portrait 3 ft in height that is hung 2.5 ft above the eye level of an average person. The viewing angle θ can be modeled by the function $\theta = \tan^{-1}\frac{5.5}{x} - \tan^{-1}\frac{2.5}{x}$, where x is the distance (in feet) from the portrait. Find the viewing angle when a person is 4 ft from the portrait.

86) [Technology Required] Use a calculator to evaluate $\tan^{-1}(\tan(2.1))$ and $\cos^{-1}(\cos(2.1))$. Explain the results of each.

Answer

$\tan^{-1}(\tan(2.1)) \approx -1.0416$; the expression does not equal 2.1 since $2.1 > 1.57 = \frac{\pi}{2}$—in other words, it is not in the restricted domain of $\tan x$. $\cos^{-1}(\cos(2.1)) = 2.1$, since 2.1 is in the restricted domain of $\cos x$.

87) [Technology Required] Use a calculator to evaluate $\sin(\sin^{-1}(-2))$ and $\tan(\tan^{-1}(-2))$. Explain the results of each.

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5.7: A.7- Tangent Lines and Velocity

Absolute Prerequisites

Templates

- Algebra
  - Functions
    - Evaluating at a constant
    - Linear Functions
      - Equation of a line
      - Slope
    - Rate of Change

Templates

Learning Objectives

- Relate the rate of change of a function to the slope of a secant line.
- Describe the concept and process of approximating the tangent line to a function at a given point.
- Recognize a tangent to a curve at a point as the limit of secant lines.
- Identify instantaneous velocity as the limit of average velocity over a small time interval.
- Distinguish when technology is appropriate to use within a given problem.

As we embark on our study of Calculus, we shall see how its development arose from common solutions to practical problems in areas such as engineering physics. Two key problems led to the initial formulation of calculus: (1) the tangent problem, or how to determine the slope of a line tangent to a curve at a point; and (2) the area problem, or how to determine the area under a curve. Chapters two through four focus on the first of these problems. We will approach the area problem at the end of this course.

Rates of Change

Rate of change is one of the most critical concepts in Calculus. We begin our investigation of rates of change by looking at the graphs of the three lines $f(x) = -2x - 3$, $g(x) = \frac{1}{2}x + 1$, and $h(x) = 2$, shown in Figure $5.7.1$.

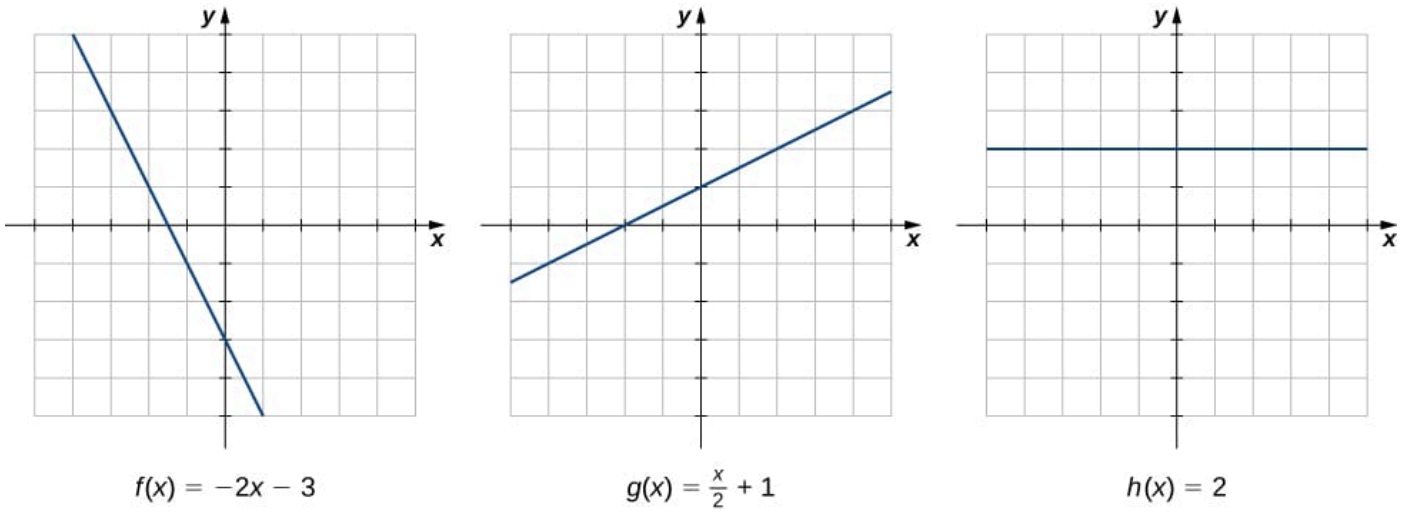


Figure $5.7.1$: The rate of change of a linear function is constant in each of these three graphs, with the constant determined by the slope.

As we move from left to right along the graph of $f(x) = -2x - 3$, we see that the graph decreases at a constant rate. For every 1 unit we move to the right along the x-axis, the y-coordinate decreases by 2 units. This rate of change is determined by the slope

(-2) of the line. Similarly, the slope of $1/2$ in the function $g(x)$ tells us that for every change in x of 1 unit there is a corresponding change in y of $1/2$ unit. The function $h(x) = 2$ has a slope of zero, indicating that the values of the function remain constant. We see that the slope of each linear function indicates the rate of change of the function.

Theorem: Rates of Change of Linear Functions

The rate of change at any given point, $(a, f(a))$, of a linear function is constant and equal to the slope of the function.

Compare the graphs of these three functions with the graph of $k(x) = x^2$ (Figure $5.7.2$). The graph of $k(x) = x^2$ starts from the left by decreasing rapidly, then begins to decrease more slowly and level off, and then finally begins to increase - slowly at first, followed by an increasing rate of increase as it moves toward the right. Unlike a linear function, no single number represents the rate of change for this function. We quite naturally ask: How do we measure the rate of change of a nonlinear function? Since the rate of change for a nonlinear function is ever-changing, is it possible to determine the rate of change at a specific point?

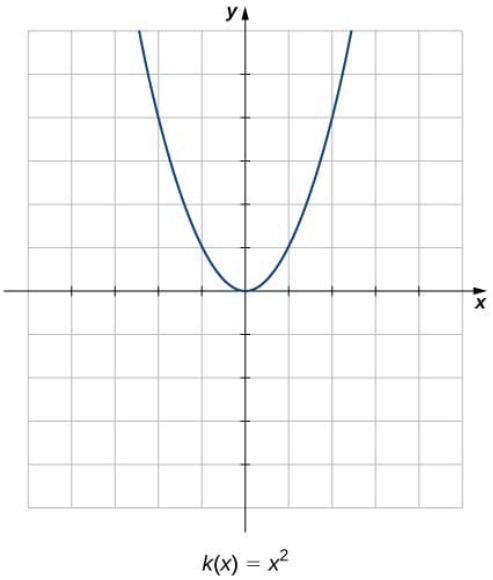


Figure $5.7.2$: The function $k(x) = x^2$ does not have a constant rate of change.

Before we go any further, a word of caution is needed.

Caution: Nonlinear functions do not have slopes

Lines have slopes; however, nonlinear curves do not. That is, it's incorrect to talk about the "slope" of the nonlinear function $k(x) = x^2$.

How do we overcome this restriction in language? When referencing the "steepness" of a nonlinear function at a point $(a, f(a))$, we instead refer to the slope of the **tangent line** at this point.

Definition: Tangent Line (Informal)

In geometry, the **tangent line** (or simply **tangent**) to a plane curve at a given point is the straight line that "just touches" the curve at that point.[1] As it passes through the point where the tangent line and the curve meet, called the **point of tangency**, the tangent line is "going in the same direction" as the curve, and is thus the best straight-line approximation to the curve at that point.

Therefore, if you wanted to tell someone how steep the function $k(x) = x^2$ is at the point $(a, f(a))$, you would reference the slope of the tangent line to the function at that point. Another phrase used for the slope of the tangent line is the **instantaneous rate**

of change of $f(x)$ at $(a, f(a))$.

Interactive Element: Visualizing the Tangent Line

The blue line that intersects $k(x) = x^2$ locally at the purple point is called the *tangent line* to the curve $k(x)$ at $(a, k(a))$.

The graph of $k(x) = x^2$ and the tangent line to $k(x)$ at $(a, k(a))$.

Interact: Move the point.

Observation: The slope of the tangent line changes. For example, if you move the point to $(1/2, 1/4)$, you will see that the slope of the tangent line is 1. Therefore, the instantaneous rate of change of $k(x) = x^2$ at $(1/2, 1/4)$ is 1.

Tangent and Secant Lines

We can approximate the instantaneous rate of change of a function $f(x)$ at a point $(a, f(a))$ on its graph by taking another point $(x, f(x))$ on the graph of $f(x)$, drawing a line through the two points, and calculating the slope of the resulting line. Such a line is called a **secant line**. Figure $5.7.3$ shows a secant line to a function $f(x)$ at a point $(a, f(a))$.

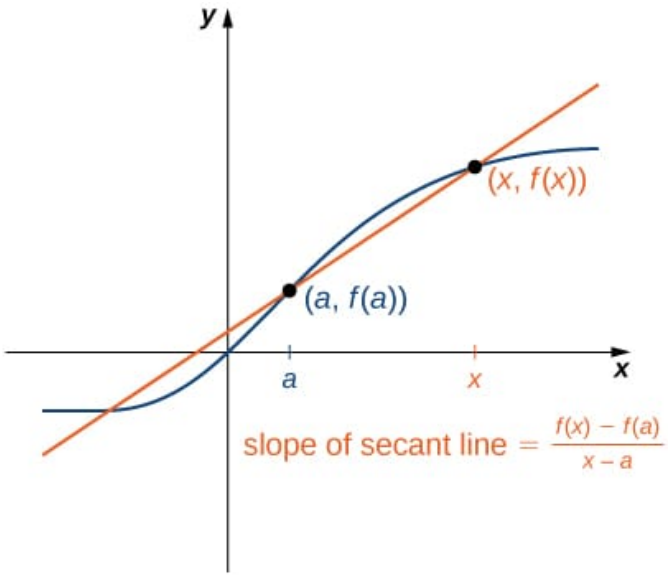


Figure $5.7.3$: The slope of a secant line through a point $(a, f(a))$ estimates the instantaneous rate of change of the function at the point $(a, f(a))$.

We formally define a secant line as follows:

Definition: Secant Line

The **secant line** (also known simply as the **secant**) to the function $f(x)$ through the points $(a, f(a))$ and $(x, f(x))$ is the line passing through these points. Its slope is given by

$$m_{sec} = \frac{f(x) - f(a)}{x - a}. \tag{5.7.1}$$

The accuracy of approximating the instantaneous rate of change of the function with a secant line depends on how close x is to a. As we see in Figure $5.7.4$, if x is closer to a, the slope of the secant line looks to be a better measure of the instantaneous rate of change of $f(x)$ at a. There are times when dragging $(x, f(x))$ closer to $(a, f(a))$ will not provide a better estimate of the tangent line, but we will avoid those instances for now.

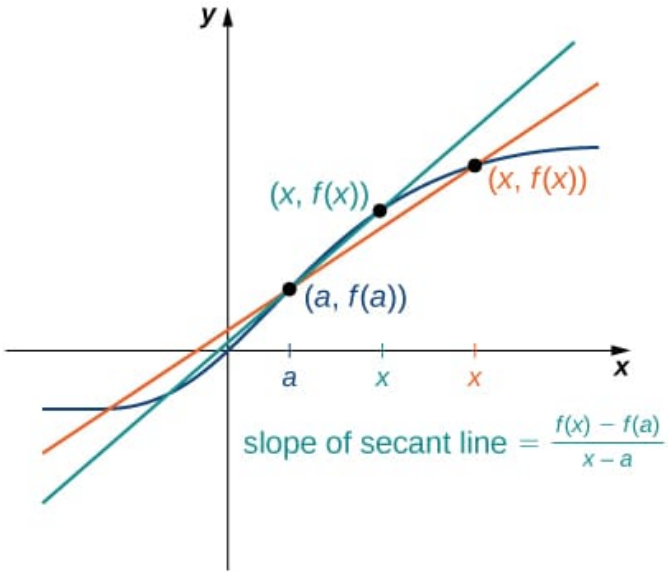


Figure $5.7.4$: As x gets closer to a, the slope of the secant line becomes a better approximation to the rate of change of the function $f(x)$ at a.

The secant lines themselves approach the tangent line to the function $f(x)$ at a (Figure $5.7.5$). As stated previously, the slope of the tangent line to the graph at a is a measure of the instantaneous rate of change of the function at a. This value is incredibly important in Calculus.

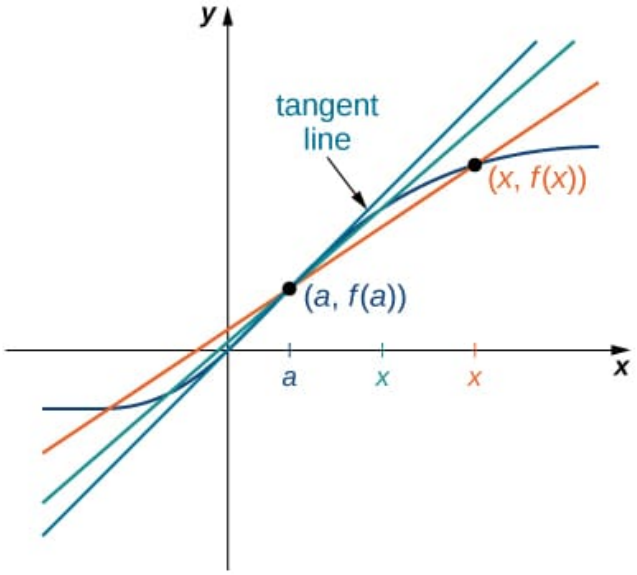


Figure $5.7.5$: Solving the Tangent Problem: As x approaches a, the secant lines approach the tangent line.

Note: Anchor Points

This entire section is focused on using the slopes of the secant lines to *approximate* the slope of the tangent line to $f(x)$ at $(a, f(a))$. The point $(a, f(a))$ is commonly referred to as the **anchor point** because it is "anchored" to a single spot while the secondary point, $(x, f(x))$, is moved toward it. In Figures $5.7.4$ and $5.7.5$, you can see the secant lines each include the anchor point $(a, f(a))$.

Interactive Element: Secant Lines and the Tangent Line

Use this Interactive Element to help you understand the difference between secant lines, the tangent line, rates of change, and the instantaneous rate of change of a function at a point.

Interact: Move the point P close to the anchor point. You can also move the anchor point elsewhere.

Observation: As P gets closer to the anchor point, the slope of the secant line begins to approximate the slope of the tangent line at the anchor point.

Question: What happens when the two points meet? Why do you think this happens?

Approximating the Slope of the Tangent Line

The next example illustrates how to find slopes of secant lines. These slopes estimate the slope of the tangent line or, equivalently, the instantaneous rate of change of the function at the point at which the slopes are calculated.

Example 5.7.1: Finding Slopes of Secant Lines

Estimate the instantaneous rate of change (slope of the tangent line) to $f(x) = x^2$ at $x = 1$ by finding slopes of secant lines through $(1, 1)$ and each of the following points on the graph of $f(x) = x^2$.

a. $(2, 4)$
b. $\left(\frac{3}{2}, \frac{9}{4}\right)$

Solutions available at the end of this section.

Checkpoint 5.7.1

Estimate the slope of the tangent line (instantaneous rate of change) to $f(x) = x^2$ at $x = 1$ by finding slopes of secant lines through $(1, 1)$ and the point $\left(\frac{5}{4}, \frac{25}{16}\right)$ on the graph of $f(x) = x^2$.

Answer available at the end of this section.

Example 5.7.2: Finding Slopes of Secant Lines

Estimate the slope of the tangent line to the function $g(x) = -2\sin(\pi x) + 2$ at $x = \frac{1}{4}$ to three decimal places.

Solution available at the end of this section.

Instantaneous Velocity

We continue our investigation by exploring a related question. Keeping in mind that **velocity** may be thought of as the rate of change of position, suppose that we have a function, $s(t)$, that gives the position of an object along a coordinate axis at any given time t. Can we use these same ideas to create a reasonable definition of the instantaneous velocity at a given time $t = a$? We start by approximating the instantaneous velocity with an average velocity. First, recall that the speed of an object traveling at a constant rate is the ratio of the distance traveled to the length of time it has traveled. We define the **average velocity** of an object over a time period to be the change in its position divided by the length of the time period.

Definition: Average Velocity

Let $s(t)$ be the position of an object moving along a coordinate axis at time t. The **average velocity** of the object over a time interval $[a, t]$ where \(a

As t is chosen closer to a, the average velocity becomes closer to the instantaneous velocity. Note that finding the average velocity of a position function over a time interval is essentially the same as finding the slope of a secant line to a function. Furthermore, to find the slope of a tangent line at a point a, we let the x-values approach a in the slope of the secant line. Similarly, to find the instantaneous velocity at time a, we let the t-values approach a in the average velocity. This process of letting x or t approach a in an expression is called taking a **limit**. Thus, we may define the **instantaneous velocity** as follows.

Definition: Instantaneous Velocity

For a position function $s(t)$, the **instantaneous velocity** at a time $t = a$ is the value that the average velocities approach on intervals of the form $[a, t]$ and $[t, a]$ as the values of t become closer to a, provided such a value exists.

Example $5.7.3$ illustrates this concept of limits and average velocity.

Example $5.7.3$: Finding Average Velocity

A rock is dropped from a height of 64 ft. It is determined that its height (in feet) above ground t seconds later (for $0 \leq t \leq 2$) is given by $s(t) = -16t^2 + 64$. Find the average velocity of the rock over each of the given time intervals. Use this information to guess the instantaneous velocity of the rock at time $t = 0.5$.

a. $[0.49, 0.5]$
b. $[0.5, 0.51]$

Solutions available at the end of this section.

Checkpoint $5.7.3$

An object moves along a coordinate axis so that its position at time t is given by $s(t) = t^3$. Estimate its instantaneous velocity at time $t = 2$ by computing its average velocity over the time interval $[2, 2.001]$.

Hint available at the end of this section.

Answer available at the end of this section.

Technology in Calculus

Before we leave this section, it is a good time to mention an important point about using technology in this course. Most of your Calculus course should be done *without* the use of technology. As such, you might find it difficult to identify when you are allowed to use technology to help you with a problem. Moreover, if you are allowed technology on a given problem, it is critical to know how to use that technology.

When to Use Technology

An exercise or homework problem typically requires the using of technology if any of the following are true:

- you are asked to round the answer to a certain number of decimal places or significant figures
- the problem specifically states that a calculator or graphing technology must be used

- you are asked to "build a table of values" or "obtain a numerical approximation"

How to Efficiently Use Technology to Build Tables

For the material in this section, understanding how to effectively and efficiently use technology to build tables of values is critical. The following example illustrates the importance of knowing how to employ these skills, along with the significance of having a mastery of our prerequisite algebra.

Example 5.7.2 Revisted: Efficiently Approximating the Instantaneous Rate of Change Using Algebra and Technology

Estimate the instantaneous rate of change of the function $g(x) = -2\sin(\pi x) + 2$ at $x = \frac{1}{4}$ to three decimal places.

Solution available at the end of this section.

Footnotes

[1] Leibniz defined the tangent line to a curve at a point as the line through a pair of infinitely close points on the curve.

Answers

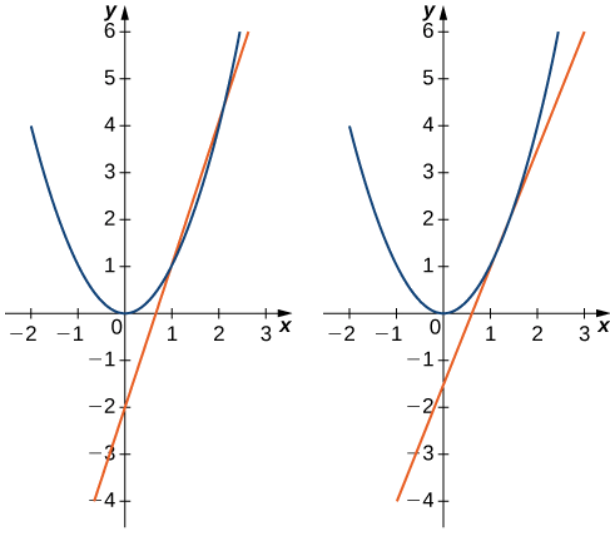
Example 5.7.1: Finding Slopes of Secant Lines

Use the formula for the slope of a secant line (Equation 5.7.1).

a. $m_{sec} = \frac{4-1}{2-1} = 3$

b. $m_{sec} = \frac{\frac{9}{4}-1}{\frac{3}{2}-1} = \frac{5}{2} = 2.5$

The point in part b. is closer to the point $(1,1)$, so the slope of 2.5 is closer to the slope of the tangent line. A good estimate for the slope of the tangent would be in the range of 2 to 2.5 (Figure 5.7.6).



The secant lines to $f(x) = x^2$ at $(1,1)$ through (a) $(2,4)$ and (b) $\left(\frac{3}{2}, \frac{9}{4}\right)$ provide successively closer approximations to the tangent line to $f(x) = x^2$ at $(1,1)$.

Checkpoint 5.7.1

2.25

Example 5.7.2: Finding Slopes of Secant Lines

In this example, we are not given an *explicit* list of points to try. Instead, there is an *implicit* assumption that we will create our own list of x-values approaching $\frac{1}{4}$. It is important to include x-values both greater than and less than $\frac{1}{4}$ in our trials. I will demonstrate two different approaches *both of which should be read for understanding*.

In either approach, we will need to use the formula for the slope of the secant line (Equation 5.7.1); however, the implementation of the formula will differ between

methods. Before going into the methods, we need to compute $g\left(\frac{1}{4}\right)$.

$$\begin{aligned} g\left(\frac{1}{4}\right) &= -2\sin\left(\frac{\pi}{4}\right)+2 \\ &= -\frac{2}{\sqrt{2}}+2 \\ &= -\sqrt{2}+2 \\ &= 2-\sqrt{2} \end{aligned}$$

Method #1: Explicitly Choosing Values Near $1/4 = 0.25$

The first approach is the one most students will start with because it mimics how they learned slope. We successively pick numbers closer and closer to 0.25 (from the right and left of 0.25 on the number line) to get an approximation to the slope of the tangent line.

Prior to "plugging in" values near 0.25, we write the secant slope formula, substitute any values we have been given (or have computed), and then simplify.

$$\begin{aligned} m_{sec} &= \frac{g(x)-g(a)}{x-a} \\ &= \frac{-2\sin(\pi x)+2-g\left(\frac{1}{4}\right)}{x-\frac{1}{4}} \\ &= \frac{-2\sin(\pi x)+2-\left(2-\sqrt{2}\right)}{x-\frac{1}{4}} \\ &= \frac{-2\sin(\pi x)+2-2+\sqrt{2}}{x-\frac{1}{4}} \\ &= \frac{-2\sin(\pi x)+\sqrt{2}}{x-\frac{1}{4}} \\ &= \frac{-8\sin(\pi x)+4\sqrt{2}}{4x-1} \end{aligned}$$

> Being able to do the algebraic "pre-work" for a problem is critical to success in Calculus. If you were lost on any of the manipulations above, it is best to review Chapter 1 of this text, or review your College Algebra textbook.

At this point, we are ready to substitute values for x that approach 0.25. The following table contains values approaching 0.25 from the right (on the number line).

x	$m_{sec} = \frac{-8\sin(\pi x)+4\sqrt{2}}{4x-1}$
0.35	≈ -3.6779949
0.26	≈ -4.3723692
0.251	≈ -4.4358968
0.2501	≈ -4.442185
0.25001	≈ -4.4428131
0.250001	≈ -4.442876

The next table shows similar computations for values of x approaching 0.25 from the left (on the number line).

x	$m_{sec} = \frac{-8\sin(\pi x)+4\sqrt{2}}{4x-1}$
0.15	≈ -5.0623256
0.24	≈ -4.5119351
0.249	≈ -4.4498545
0.2499	≈ -4.4435808
0.24999	≈ -4.4429527
0.249999	≈ -4.4428899

In both cases, it looks like the slopes of the secant lines are approaching roughly -4.443 (when rounded to the third decimal place).

Caution: Calculus Requires Radians

Are your values not matching those in the table?

Remember, Calculus requires angles in radians. From this point forward, whenever you are working on a problem involving trigonometric functions and you require a calculator, be sure to set the calculator to radian mode!

Method #2: Choosing "Wiggles"

The second approach is the preferred approach for the rest of Calculus. We successively pick smaller and smaller numbers to decrease the "wiggled distance" between a and x.

The "pre-work" is slightly different than what we did previously. Instead working with x, we let $x = a + h$ and allow h to get smaller and smaller. The lower the value of h, the closer $x = a + h$ is to a. Like before, prior to "plugging in" smaller and smaller values of h, we write the secant slope formula, substitute any values we have been given (or have computed), and then simplify.

$$\begin{aligned} m_{sec} &= \frac{g(x)-g(a)}{x-a} \\ &= \frac{g(a+h)-g(a)}{a+h-a} \\ &= \frac{g(a+h)-g(a)}{h} \\ &= \frac{g(0.25+h)-g(0.25)}{h} \\ &= \frac{-2\sin(\pi(0.25+h))+2-g(0.25)}{h} \\ &= \frac{-2\sin(\pi(0.25+h))+2-\left(2-\sqrt{2}\right)}{h} \\ &= \frac{-2\sin(\pi(0.25+h))+2-2+\sqrt{2}}{h} \\ &= \frac{-2\sin(\pi(0.25+h))+\sqrt{2}}{h} \end{aligned}$$

At this point, we are ready to substitute values for h that approach 0. The following table contains values approaching 0 from the right (on the number line).

h	$m_{sec} = \frac{-2\sin(\pi(0.25+h))+\sqrt{2}}{h}$
0.1	≈ -3.6779949
0.01	≈ -4.3723692
0.001	≈ -4.4358968
0.0001	≈ -4.442185
0.00001	≈ -4.4428131
0.000001	≈ -4.442876

The next table shows similar computations for values of x approaching 0 from the left (on the number line).

h	$m_{sec} = \frac{-2\sin(\pi(0.25+h))+\sqrt{2}}{h}$
-0.1	≈ -5.0623256
-0.01	≈ -4.5119351
-0.001	≈ -4.4498545
-0.0001	≈ -4.4435808
-0.00001	≈ -4.4429527
-0.000001	≈ -4.4428899

In both cases, it looks like the slopes of the secant lines are approaching roughly -4.443 (when rounded to the third decimal place).

Result

In the end, both methods produce the same results, however, the second method demonstrates a significant building block for the study of Calculus. While the "pre-work" was slightly more involved, the inputs (the values that were being substituted in for h) not only were easier to work with, but conceptually made sense. We are just decreasing the "space" between the anchor point and $(x, f(x))$ to something near zero.

Example $5.7.3$**: Finding Average Velocity**

Substitute the data into Equation ??? for the definition of average velocity.

a. $v_{avg} = \frac{s(0.49)-s(0.5)}{0.49-0.5} = -15.84$
b. $v_{avg} = \frac{s(0.51)-s(0.5)}{0.51-0.5} = -16.016$

The instantaneous velocity is somewhere between –15.84 and –16.16 ft/sec. A good guess might be –16 ft/sec.

Checkpoint $5.7.3$

Use Equation ??? with $v_{avg} = \frac{s(2.001)-s(2)}{2.001-2}$.

Checkpoint $5.7.3$

12.006001

Example $5.7.2$ **Revisted: Efficiently Approximating the Instantaneous Rate of Change Using Algebra and Technology**

The fact that we are being asked to approximate to three decimal places means that we can grab some type of technology to help us with the computations. Recall that, with the second method, we eventually arrived at

$$m_{sec} = \frac{-2\sin(\pi(0.25+h))+\sqrt{2}}{h}.$$

If you go to the Desmos graphing calculator, you can insert a table by clicking on the "plus" sign and selecting "table".

You will be presented with an unfilled table where the independent variable is x_1 and the dependent variable is y_1. Replace y_1 with $\frac{-2\sin(\pi(0.25+x_1))+\sqrt{2}}{x_1}$ (notice I have traded out h for x_1 in our secant slope formula). At this point, you can fill in the values for x_1 (which are our values of h).

x_1	$\frac{-2\sin\left(\pi\left(0.25+x_1\right)\right)+\sqrt{2}}{x_1}$
0.1	−3.6779949
0.01	−4.3723692
0.001	−4.4358968
0.0001	−4.442185
0.00001	−4.4428131
0.000001	−4.442876
−0.1	−5.0623256
−0.01	−4.5119351
−0.001	−4.4498545
−0.0001	−4.4435808
−0.00001	−4.4429527
−0.000001	−4.4428899

The graphing calculator in Desmos is in radian mode by default, so you do not have to worry about incorrect values in a Calculus course (however, the scientific calculator in Desmos is in degree mode by default).

Note

This table of values also can be built in Excel, Google Sheets, and many other applications very easily.

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5.7.1: A.7.1- Homework

For exercises 1 - 3 , points $P(1,2)$ and $Q(x,y)$ are on the graph of the function $f(x)=x^2+1$.

1. [Technology Required] Complete the following table with the appropriate values: y-coordinate of Q, the point $Q(x,y)$, and the slope of the secant line passing through points P and Q. Round your answer to eight significant digits.

x	y	$Q(x,y)$	m_{sec}
1.1			
1.01			
1.001			
1.0001			

2. Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the line tangent to f at $x=1$.
3. Use the value in the preceding exercise to find the equation of the tangent line at point P. Graph $f(x)$ and the tangent line.

Answer

$y=2x$

For the exercises 4-6, points $P(1,1)$ and $Q(x,y)$ are on the graph of the function $f(x)=x^3$.

4. [Technology Required] Complete the following table with the appropriate values: y-coordinate of Q, the point $Q(x,y)$, and the slope of the secant line passing through points P and Q. Round your answer to eight significant digits.

x	y	$Q(x,y)$	m_{sec}
1.1			
1.01			
1.001			
1.0001			

5. Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the tangent line to f at $x=1$.

Answer

3

6. Use the value in the preceding exercise to find the equation of the tangent line at point P. Graph $f(x)$ and the tangent line.

For the exercises 7 - 9, points $P(4,2)$ and $Q(x,y)$ are on the graph of the function $f(x)=\sqrt{x}$.

7) [Technology Required] Complete the following table with the appropriate values: y-coordinate of Q, the point $Q(x,y)$, and the slope of the secant line passing through points P and Q. Round your answer to eight significant digits.

x	y	$Q(x,y)$	m_{sec}
4.1	a.	e.	i.
4.01	b.	f.	j.
4.001	c.	g.	k.
4.0001	d.	h.	l.

Answer

a. 2.0248457
b. 2.0024984
c. 2.0002500
d. 2.0000250
e. (4.1000000,2.0248457)
f. (4.0100000,2.0024984)
g. (4.0010000,2.0002500)
h. (4.0001000,2.0000250)
i. 0.24845673
j. 0.24984395
k. 0.24998438
l. 0.24999844

8) Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the tangent line to f at $x = 4$.

9) Use the value in the preceding exercise to find the equation of the tangent line at point P.

Answer

$y = \frac{x}{4} + 1$

For exercises 10 - 12, points $P(1.5, 0)$ and $Q(\phi, y)$ are on the graph of the function $f(\phi) = \cos(\pi\phi)$.

10) [Technology Required] Complete the following table with the appropriate values: y-coordinate of Q, the point $Q(\phi, y)$, and the slope of the secant line passing through points P and Q. Round your answer to eight significant digits.

x	y	$Q(\phi, y)$	m_{sec}
1.4	a.	e.	i.
1.49	b.	f.	j.
1.499	c.	g.	k.
1.4999	d.	h.	l.

11) Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the tangent line to f at $\phi = 1.5$.

Answer

π

12) Use the value in the preceding exercise to find the equation of the tangent line at point P.

For exercises 13 - 15, points $P(-1, -1)$ and $Q(x, y)$ are on the graph of the function $f(x) = \frac{1}{x}$.

13) [Technology Required] Complete the following table with the appropriate values: y-coordinate of Q, the point $Q(x, y)$, and the slope of the secant line passing through points P and Q. Round your answer to eight significant digits.

x	y	$Q(x, y)$	m_{sec}
-1.05	a.	e.	i.
-1.01	b.	f.	j.
-1.005	c.	g.	k.
-1.001	d.	h.	l.

Answer

a. –0.95238095

b. −0.99009901
c. −0.99502488
d. −0.99900100
e. (−1;.0500000,−0;.95238095)
f. (−1;.0100000,−0;.9909901)
g. (−1;.0050000,−0;.99502488)
h. (1.0010000,−0;.99900100)
i. −0.95238095
j. −0.99009901
k. −0.99502488
l. −0.99900100

14) Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the line tangent to f at $x = -1$.

15) Use the value in the preceding exercise to find the equation of the tangent line at point P.

Answer

$y = -x - 2$

For exercises 16 - 17, the position function of a ball dropped from the top of a 200-meter tall building is given by $s(t) = 200 - 4.9t^2$, where position s is measured in meters and time t is measured in seconds. Round your answer to eight significant digits.

16) [Technology Required] Compute the average velocity of the ball over the given time intervals.

a. [4.99,5]

b. [5,5.01]

c. [4.999,5]

d. [5,5.001]

17) Use the preceding exercise to guess the instantaneous velocity of the ball at $t = 5$ sec.

Answer

-49 m/sec (velocity of the ball is 49 m/sec downward)

For exercises 18 - 19, consider a stone tossed into the air from ground level with an initial velocity of 15 m/sec. Its height in meters at time t seconds is $h(t) = 15t - 4.9t^2$.

18) [Technology Required] Compute the average velocity of the stone over the given time intervals.

a. [1,1.05]

b. [1,1.01]

c. [1,1.005]

d. [1,1.001]

19) Use the preceding exercise to guess the instantaneous velocity of the stone at $t = 1$ sec.

Answer

5.2 m/sec

For exercises 20 - 21, consider a rocket shot into the air that then returns to Earth. The height of the rocket in meters is given by $h(t) = 600 + 78.4t - 4.9t^2$, where t is measured in seconds.

20) [Technology Required] Compute the average velocity of the rocket over the given time intervals.

a. [9,9.01]

b. [8.99,9]

c. [9,9.001]

d. [8.999,9]

21) Use the preceding exercise to guess the instantaneous velocity of the rocket at $t = 9$ sec.

Answer

-9.8 m/sec

For exercises, consider an athlete running a 40-m dash. The position of the athlete is given by $d(t) = \frac{t^3}{6} + 4t$, where d is the position in meters and t is the time elapsed, measured in seconds.

22) [Technology Required] Compute the average velocity of the runner over the given time intervals.

a. [1.95,2.05]

b. [1.995,2.005]

c. [1.9995,2.0005]

d. [2,2.00001]

23) Use the preceding exercise to guess the instantaneous velocity of the runner at $t = 2$ sec.

Answer

6 m/sec

24) Estimate the slope of the tangent line to the function at $t = 1$ for the given function.

a. $\sinh(x)$

b. $\cosh(x)$

c. $\tanh(x)$

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5.8: A.8- Answers to Selected Exercises

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5.8.1: A.8.1- Chapter 1 Answers

Chapter 1

1.1

2) $\lim_{x\to 1} f(x)$ does not exist because $\lim_{x\to 1^-} f(x) = -2 \neq \lim_{x\to 1^+} f(x) = 2$.

4) $\lim_{x\to 0} (1+x)^{1/x} \approx 2.7183$.

6) a. 1.98669331; b. 1.99986667; c. 1.99999867; d. 1.99999999; e. 1.98669331; f. 1.99986667; g. 1.99999867; h. 1.99999999; $\lim_{x\to 0} \frac{\sin 2x}{x} = 2$

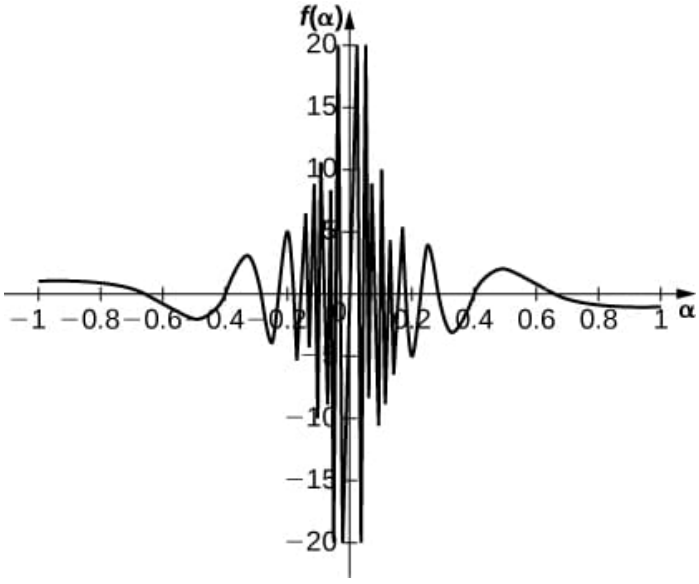
8) $\lim_{x\to 0} \frac{\sin ax}{x} = a$

10) a. –0.80000000; b. –0.98000000; c. –0.99800000; d. –0.99980000; e. –1.2000000; f. –1.0200000; g. –1.0020000; h. –1.0002000; $\lim_{x\to 1}(1-2x) = -1$

12) a. –37.931034; b. –3377.9264; c. –333,777.93; d. –33,337,778; e. –29.032258; f. –3289.0365; g. –332,889.04; h. –33,328,889; $\lim_{x\to 0} \frac{z-1}{z^2(z+3)} = -\infty$

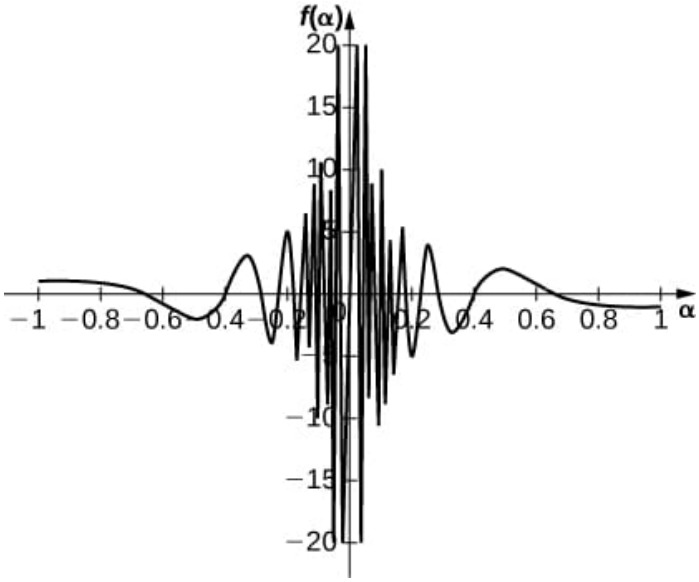
14) a. 0.13495277; b. 0.12594300; c. 0.12509381; d. 0.12500938; e. 0.11614402; f. 0.12406794; g. 0.12490631; h. 0.12499063; $\therefore \lim_{x\to 2} \frac{1-\frac{2}{x}}{x^2-4} = 0.1250 = \frac{1}{8}$

16) a. 10.00000; b. 100.00000; c. 1000.0000; d. 10,000.000;
Guess: $\lim_{\alpha\to 0^+} \frac{1}{\alpha}\cos\left(\frac{\pi}{\alpha}\right) = \infty$;
Actual: DNE , since the graph shows the function oscillates wildly between values approaching positive infinity and values approaching negative infinity, as the value of α approaches 0 from the positive side.



1.2

2) a. 10.00000; b. 100.00000; c. 1000.0000; d. 10,000.000;
Guess: $\lim_{\alpha \to 0^+} \frac{1}{\alpha} \cos\left(\frac{\pi}{\alpha}\right) = \infty$;
Actual: DNE , since the graph shows the function oscillates wildly between values approaching positive infinity and values approaching negative infinity, as the value of α approaches 0 from the positive side.



4) False; $\lim_{x \to -2^+} f(x) = +\infty$

6) False; $\lim_{x \to 6} f(x)$ DNE since $\lim_{x \to 6^-} f(x) = 2$ and $\lim_{x \to 6^+} f(x) = 5$.

8) 2

10) 1

12) 1

14) DNE

16) 0

18) DNE

20) 2

22) 3

24) DNE

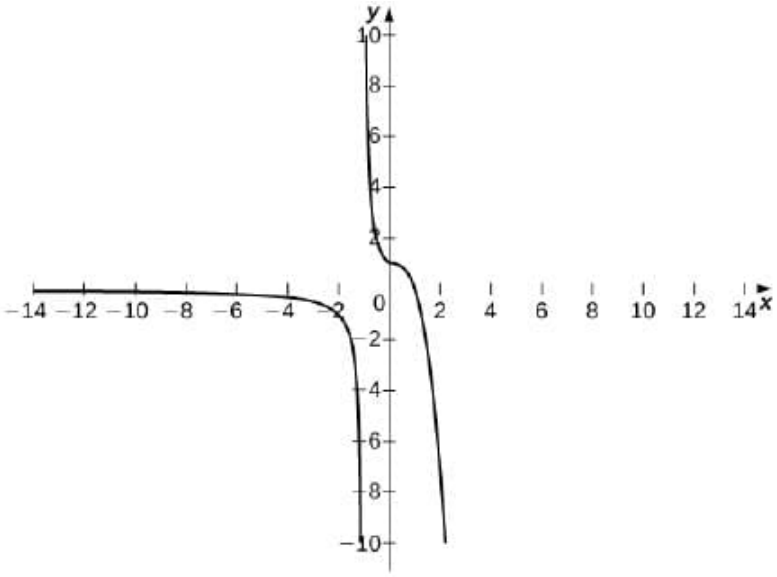
26) 0

28) -2

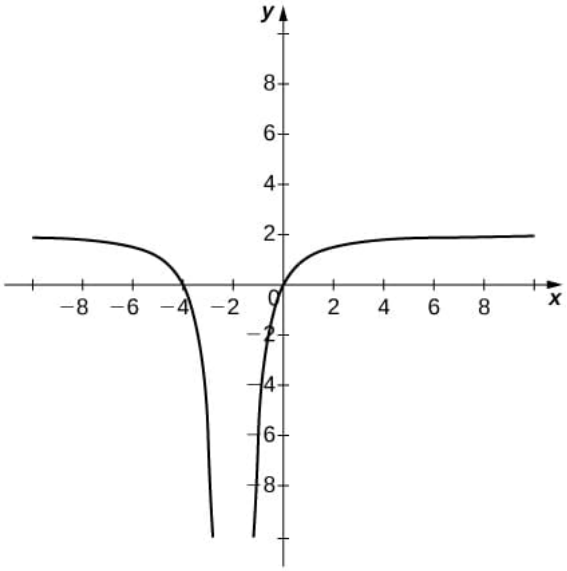
30) DNE

32) 0

34) Answers may vary



36) Answer may vary



38) a. ρ_2 b. ρ_1 c. DNE unless $\rho_1 = \rho_2$. As you approach X_{SF} from the right, you are in the high-density area of the shock. When you approach from the left, you have not experienced the "shock" yet and are at a lower density.

1.3

1) Use constant multiple law and difference law:

$$\begin{aligned} \lim_{x\to 0}(4x^2 - 2x + 3) &= 4\lim_{x\to 0} x^2 - 2\lim_{x\to 0} x + \lim_{x\to 0} 3 \\ &= 0 + 0 + 3 \\ &= 3 \end{aligned}$$

3) Use root law: $\lim_{x\to -2}\sqrt{x^2 - 6x + 3} = \sqrt{\lim_{x\to -2}(x^2 - 6x + 3)} = \sqrt{19}$

5) $\lim_{x\to 7} x^2 = 49$

7) $\lim_{x\to 0} \dfrac{1}{1+\sin x} = 1$

8) $\lim_{x\to 1} \dfrac{2-7x}{x+6} = -\dfrac{5}{7}$

9)

$$\begin{aligned} \lim_{x\to 6} 2f(x)g(x) &= 2\left(\lim_{x\to 6} f(x)\right)\left(\lim_{x\to 6} g(x)\right) \\ &= 2(4)(9) \\ &= 72 \end{aligned}$$

11)

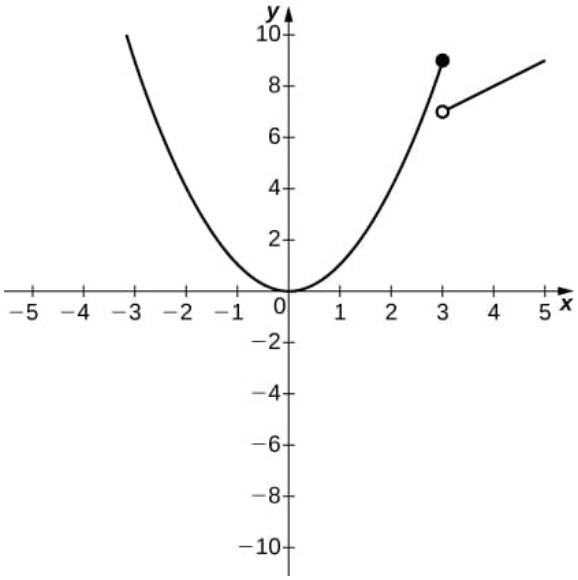
$$\begin{aligned} \lim_{x\to 6}\left(f(x)+\frac{1}{3}g(x)\right) &= \lim_{x\to 6} f(x) + \frac{1}{3}\lim_{x\to 6} g(x) \\ &= 4+\frac{1}{3}(9) \\ &= 7 \end{aligned}$$

13)

$$\begin{aligned} \lim_{x\to 6}\sqrt{g(x)-f(x)} &= \sqrt{\lim_{x\to 6} g(x) - \lim_{x\to 6} f(x)} \\ &= \sqrt{9-4} \\ &= \sqrt{5} \end{aligned}$$

15) $\lim_{x\to 6}[(x+1)f(x)] = \left(\lim_{x\to 6}(x+1)\right)\left(\lim_{x\to 6} f(x)\right) = 7(4) = 28$

17)



a. 9; b.7

21)

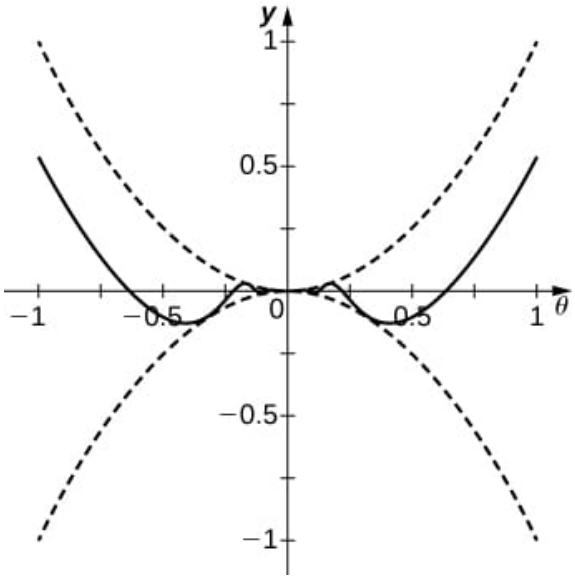
$$\begin{aligned}\lim_{x\to -3^-}(f(x)-3g(x)) &= \lim_{x\to -3^-} f(x) - 3\lim_{x\to -3^-} g(x)\\ &= 0+6\\ &= 6\end{aligned}$$

23) $\lim_{x\to -5}\dfrac{2+g(x)}{f(x)} = \dfrac{2+\left(\lim_{x\to -5} g(x)\right)}{\lim_{x\to -5} f(x)} = \dfrac{2+0}{2} = 1$

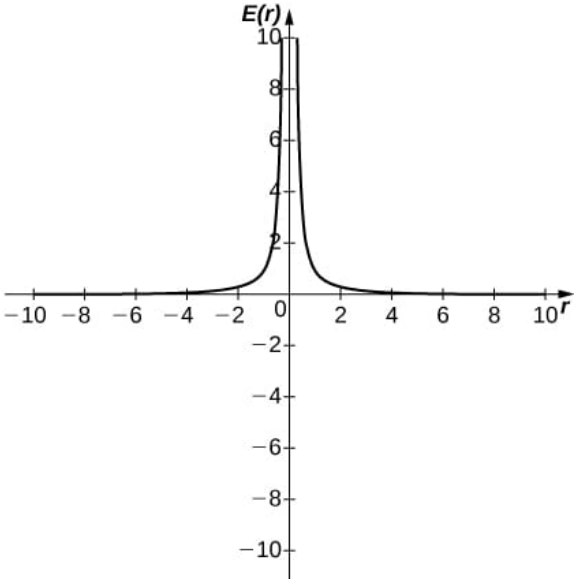
25) $\lim_{x\to 1}\sqrt[3]{f(x)-g(x)} = \sqrt[3]{\lim_{x\to 1} f(x) - \lim_{x\to 1} g(x)} = \sqrt[3]{2+5} = \sqrt[3]{7}$

27) $\lim_{x\to -9}(xf(x)+2g(x)) = \left(\lim_{x\to -9} x\right)\left(\lim_{x\to -9} f(x)\right) + 2\lim_{x\to -9} g(x) = (-9)(6)+2(4) = -46$

29) The limit is zero.



31) a.



b. ∞. The magnitude of the electric field as you approach the particle q becomes infinite. It does not make physical sense to evaluate negative distance.

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5.9: Diagnostic Examples

Ideally, the diagnostic assessments would be given a couple of lecture days *prior* to the start of a new chapter. This gives the students time to complete the diagnostics and the instructor time to prepare lecture and support activities based upon the strengths and weaknesses of their class.

Assignment Structure of the Diagnostic Assessments

- Questions are displayed in groups (e.g., all questions related to trigonometry would be on one page of the assessment and all questions pertaining to simplifying algebraic expressions on another).
  LOGIC: Breaking up the assessment into smaller chunks is easier to digest for the student and allows the instructor the ability to assign only one or two chunks a day, or all at once.
- Students are given 1 randomized version of each question, which cannot be rerandomized.
  LOGIC: This is an assessment. I don't want students gaming the system.
- All questions are multiple choice.
  LOGIC: Faster and less prone to typo mistakes from the student. Moreover, it makes auto-grading simple. Finally, consistency is key. Having the same format for all questions reduces mental fatigue for the student.
- Students will be given two tries per question. In any case, they will not know their score on that question (right or wrong) until the end of the assessment.
  LOGIC: If they suddenly have an epiphany during the assessment and decide to go back to change an answer, they can!
- Question order is *not* shuffled.
  LOGIC: It keeps logically similar questions (e.g., trig) together and reduces friction for the student.
- Question categories *are* displayed. The categories identically match the section of the Corequisite Codex to which the question pertains.
  LOGIC: This is to let the astute student know where they might be struggling.
- Latepasses are typically *not* allowed; however, this is ultimately the instructor's choice.
  LOGIC: Since the instructor needs this information as soon as possible, Latepasses just muddy the assessment process.
- There is a time limit.
  LOGIC: These are base skills (and multiple choice). Given too much time, the students might search out helpful resources. Moreover, the skills assessed in these diagnostics should be "at hand" for the student (i.e., they should be competent and capable with the skills without needing a lot of time). Less time exposes weaknesses.
- Help videos, hints, and examples are turned off.
  LOGIC: Time and it's an assessment!

A Work in Progress

Please understand that I am still writing the questions, so the diagnostic questions (on the next page) will get filled in as time goes on.

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5.9.1: Chapter 1 Diagnostics

The following is for demonstration purposes. The actual assessment is taken within Canvas, is timed, does not include any links to help resources, and does not offer "Try a Similar Question."

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Part 4: Solving Equations & Inequalities

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