APPLIED CALCULUS



Shana Calaway, Dale Hoffman, & David Lippman

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Applied Calculus

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Licensing

*A detailed breakdown of this resource's licensing can be found in **Back Matter/Detailed Licensing**.*

CHAPTER OVERVIEW

1: The Derivative

Note: The videos for sections 2.1-2.5 were recorded based on an older edition of the book. This means that some of the section numbers I mention will no longer correspond to the same material, and screen-shots may look different. However, the content is essentially the same, and I've tried to put the videos in the correct location based on where the material was moved.

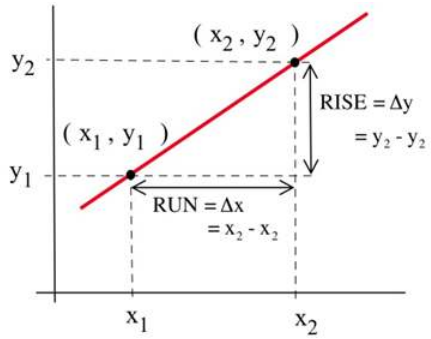
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1.1: Prelude to the Derivative

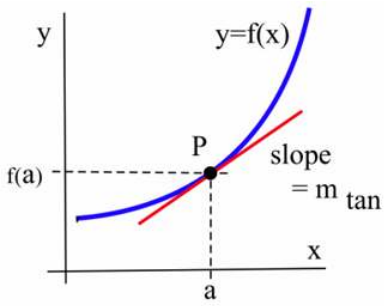
Precalculus Idea: Slope and Rate of Change

The slope of a line measures how fast a line rises or falls as we move from left to right along the line. It measures the rate of change of the y-coordinate with respect to changes in the x-coordinate. If the line represents the distance traveled over time, for example, then its slope represents the velocity. In the figure, you can remind yourself of how we calculate slope using two points on the line:



$$m = \text{Slope from } P \text{ to } Q = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

We would like to be able to get that same sort of information (how fast the curve rises or falls, velocity from distance) even if the graph is not a straight line. But what happens if we try to find the slope of a curve, as in the figure below?

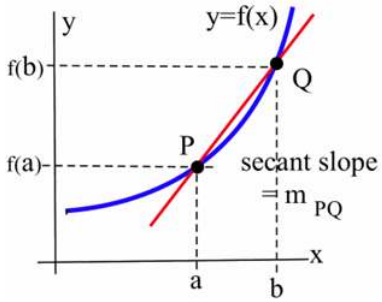


We need two points in order to determine the slope of a line. How can we find a slope of a curve, at just one point? The answer, as suggested in the figure, is to find the slope of the tangent line to the curve at that point. Most of us have an intuitive idea of what a tangent line is. Unfortunately, “tangent line” is hard to define precisely.

Definition: Secant Line

A **secant line** is a line between two points on a curve.

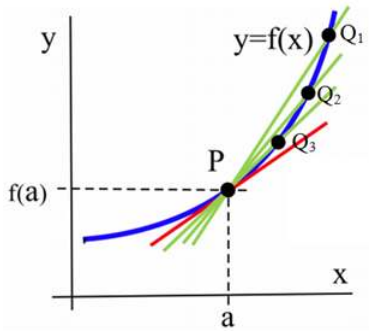
See the image below:



Can't-quite-do-it-yet Definition: Tangent Line

A **tangent line** is a line at one point on a curve… that does its best to be the curve at that point?

As you may be able to see in the image below, the closer the point Q is to the point P, the closer the secant slope gets to the tangent slope. This will be key to finding the tangent slope, but first we need to more carefully define the idea of "getting closer to."



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1.2: Limits and Continuity

Limit

In the last section, we saw that as the interval over which we calculated got smaller, the secant slopes approached the tangent slope. The **limit** gives us better language with which to discuss the idea of “approaches.”

The limit of a function describes the behavior of the function when the variable is near, **but does not equal**, a specified number (see the figure below).

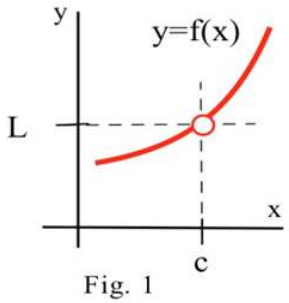


Fig. 1

Limit

If the values of $f(x)$ get closer and closer, as close as we want, to one number L as we take values of x very close to (but not equal to) a number c, then we say "**the limit of $f(x)$ as x approaches c is L**" and we write

$$\lim_{x \to c} f(x) = \mathbf{L}.$$

The symbol "$\to$" means "approaches" or, less formally, "gets very close to".

(This definition of the limit isn't stated as formally as it could be, but it is sufficient for our purposes in this course.)

Note:

- $\mathbf{f(c)}$ is a single number that describes the behavior (value) of $f(x)$ **AT** the point $x = c$.
- $\lim_{x \to c} f(x)$ is a single number that describes the behavior of $f(x)$ **NEAR, *BUT NOT AT***, the point $x = c$.

If we have a graph of the function near $x = c$, then it is usually easy to determine $\lim_{x \to c} f(x)$.

(Here is a link to the pictures used in the following video as well as elsewhere in this chapter: Graphs for Limits and Continuity Examples.)



Example 1.2.1

Use the graph of $y = f(x)$ in the figure below to determine the following limits:

1. $\lim_{x \to 1} f(x)$
2. $\lim_{x \to 2} f(x)$

3. $\lim_{x\to 3} f(x)$
4. $\lim_{x\to 4} f(x)$

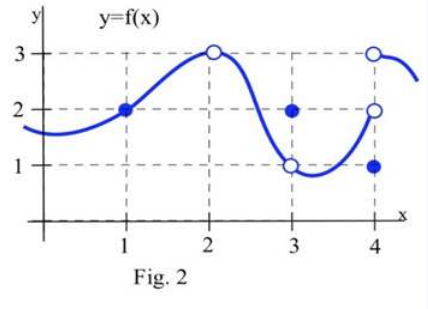


Fig. 2

Solution

1. $$\lim_{x\to 1} f(x) = 2$$

   When x is very close to 1, the values of $f(x)$ are very close to $y = 2$. In this example, it happens that $f(1) = 2$, but that is **irrelevant** for the limit. The only thing that matters is what happens for x *close to* 1 but $x \neq 1$.
2. $f(2)$ is undefined, but we only care about the behavior of $f(x)$ for x *close to* 2 but *not equal to* 2. When x is close to 2, the values of $f(x)$ are close to 3. If we restrict x close enough to 2, the values of y will be as close to 3 as we want, so $\lim_{x\to 2} f(x) = 3$.
3. When x is close to 3 (or "as x approaches the value 3"), the values of $f(x)$ are close to 1 (or "approach the value 1"), so $\lim_{x\to 3} f(x) = 1$. For this limit it is completely irrelevant that $f(3) = 2$, We only care about what happens to $f(x)$ for x close to and not equal to 3.
4. This one is harder and we need to be careful. When x is close to 4 and slightly **less than** 4 (x is just to the left of 4 on the x-axis), then the values of $f(x)$ are close to 2. But if x is close to 4 and slightly **larger than** 4 then the values of $f(x)$ are close to 3. If we only know that x is very close to 4, then we cannot say whether $y = f(x)$ will be close to 2 or close to 3 – it depends on whether x is on the right or the left side of 4. In this situation, the $f(x)$ values are not close to a *single* number so we say $\lim_{x\to 4} f(x)$ **does not exist**. It is irrelevant that $f(4) = 1$. The limit, as x approaches 4, would still be undefined if $f(4)$ was 3 or 2 or anything else.

We can also explore limits using tables and using algebra.

✔ Example 1.2.2

Find $\lim_{x\to 1} \dfrac{2x^2 - x - 1}{x - 1}$.

Solution

You might try to evaluate $f(x) = \frac{2x^2-x-1}{x-1}$ at $x = 1$, but $f(x)$ is not defined at $x = 1$. It is tempting, **but wrong**, to conclude that this function does not have a limit as x approaches 1.

Using tables: Trying some "test" values for x which get closer and closer to 1 from both the left and the right, we get

x	$f(x)$
0.9	2.82
0.9998	2.9996
0.999994	2.999988
0.9999999	2.9999998
$\to 1$	$\to 3$

x	$f(x)$
1.1	3.2
1.003	3.006
1.0001	3.0002
1.000007	3.000014
$\to 1$	$\to 3$

The function f is not defined at $x = 1$, but when x is **close to** 1, the values of $f(x)$ are getting very close to 3. We can get $f(x)$ as close to 3 as we want by taking x very close to 1 so

$$\lim_{x \to 1} \frac{2x^2 - x - 1}{x - 1} = 3.$$

Using algebra: We could have found the same result by noting that

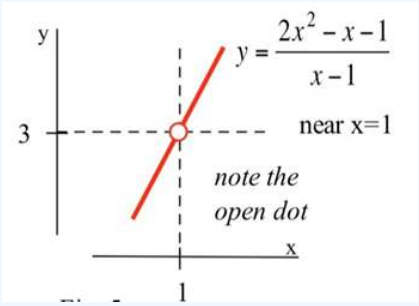
$$f(x) = \frac{2x^2 - x - 1}{x - 1} = \frac{(2x + 1)(x - 1)}{(x - 1)} = 2x + 1$$

as long as $x \neq 1$. (If $x \neq 1$, then $x - 1 \neq 0$ so it is valid to divide the numerator and denominator by the factor $x - 1$.) The "$x \to 1$" part of the limit means that x is close to 1 but **not equal to 1**, so our division step is valid and

$$\lim_{x \to 1} \frac{2x^2 - x - 1}{x - 1} = \lim_{x \to 1} 2x + 1 = 3,$$

which is our answer.

Using a graph: We can graph $y = f(x) = \dfrac{2x^2 - x - 1}{x - 1}$ for x close to 1:



Notice that whenever x is close to 1, the values of $y = f(x)$ are close to 3. Since f is not defined at $x = 1$, the graph has a hole above $x = 1$, but we only care about what $f(x)$ is doing for x close to but **not equal to** 1.

Example 1.2.3

Find $\lim\limits_{x\to 3}\dfrac{\frac{1}{x}-\frac{1}{3}}{x-3}$.

Solution

Notice that this function is not defined at $x=3$. We can find the limit using algebra. Giving the two terms in the numerator a common denominator, we can simplify:

$$\frac{\frac{1}{x}-\frac{1}{3}}{x-3}=\frac{\frac{1}{x}\cdot\frac{3}{3}-\frac{1}{3}\cdot\frac{x}{x}}{x-3}=\frac{\frac{3}{3x}-\frac{x}{3x}}{x-3}=\frac{\frac{3-x}{3x}}{x-3}$$

Remember that dividing a fraction is the same as multiplying by the reciprocal, so

$$\frac{\frac{3-x}{3x}}{x-3}=\frac{\frac{3-x}{3x}}{\frac{x-3}{1}}$$

is equivalent to

$$\frac{3-x}{3x}\cdot\frac{1}{x-3}$$

To simplify further, we need to factor a negative 1 out of the numerator. Then we can cancel the term $(x-3)$

$\dfrac{-1(x-3)}{3x}\cdot\dfrac{1}{x-3}=\dfrac{-1}{3x}$ as long as $x\neq 3$

Now we can evaluate the limit using this simplified form.

$$\lim_{x\to 3}\frac{\frac{1}{x}-\frac{1}{3}}{x-3}=\lim_{x\to 3}\frac{-1}{3x}=-\frac{1}{9}$$

One Sided Limits

Sometimes, what happens to us at a place depends on the direction we use to approach that place. If we approach Niagara Falls from the upstream side, then we will be 182 feet higher and have different worries than if we approach from the downstream side. Similarly, the values of a function near a point may depend on the direction we use to approach that point.

Left and Right Limits

The **left limit** of $f(x)$ as x approaches c is L if the values of $f(x)$ get as close to L as we want when x is very close to and **left of c** (i.e., $x<\mathbf{c}$). We write

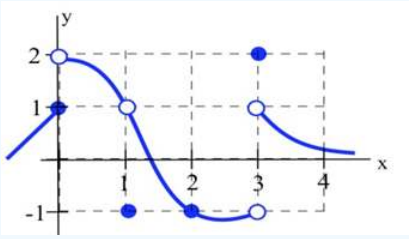
$$\lim_{x\to c^-}f(x)=L.$$

The **right limit** of $f(x)$ as x approaches c, written with $\mathbf{x}\to\mathbf{c}^+$, is L if the values of $f(x)$ get as close to L as we want when x is very close to and **right of c** (i.e., $x>\mathbf{c}$). We write

$$\lim_{x\to c^+}f(x)=L.$$

Example 1.2.4

Evaluate the one sided limits of the function $f(x)$ graphed below at $x = 0$ and $x = 1$.



Solution

As x approach 0 from the *left*, the value of the function is getting closer to 1, so $\lim_{x \to 0^-} f(x) = 1$.

As x approaches 0 from the *right*, the value of the function is getting closer to 2, so $\lim_{x \to 0^+} f(x) = 2$.

Notice that since the limit from the left and limit from the right are different, the general limit, $\lim_{x \to 0} f(x)$, does not exit.

At x approaches 1 from either direction, the value of the function is approaching 1, so

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x) = 1.$$

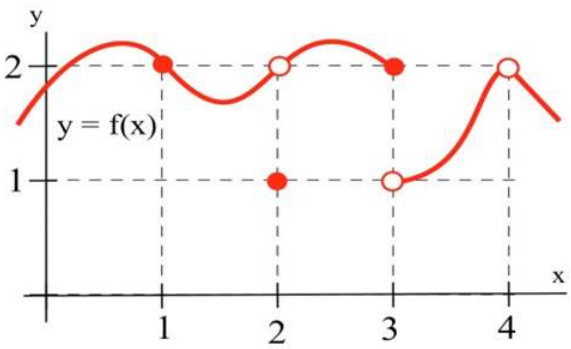
Continuity

A function that is "friendly" and doesn't have any breaks or jumps in it is called **continuous.** More formally,

Continuity at a Point

A function $\mathbf{f}$ **is continuous at** $\mathbf{x = a}$ if and only if $\lim_{x \to a} \mathbf{f(x) = f(a)}$.

The graph below illustrates some of the different ways a function can behave at and near a point, and the table contains some numerical information about the function and its behavior.



a	$f(a)$	$\lim_{x\to a} f(x)$
1	2	2
2	1	2
3	2	Does not exist (DNE)
4	Undefined	2

Based on the information in the table, we can conclude that f is continuous at 1 since $\lim_{x\to 1} f(x) = 2 = f(1)$.

We can also conclude from the information in the table that f is not continuous at 2 or 3 or 4, because $\lim_{x\to 2} f(x) \neq f(2)$, $\lim_{x\to 3} f(x) \neq f(3)$, and $\lim_{x\to 4} f(x) \neq f(4)$.

The behaviors at $x = 2$ and $x = 4$ exhibit a **hole** in the graph, sometimes called a **removable discontinuity**, since the graph could be made continuous by changing the value of a single point. The behavior at $x = 3$ is called a **jump discontinuity**, since the graph jumps between two values.

So which functions are continuous? It turns out pretty much every function you've studied is continuous where it is defined: **polynomial, radical, rational, exponential, and logarithmic functions are all continuous where they are defined**. Moreover, **any combination of continuous functions is also continuous**.

This is helpful, because the definition of continuity says that for a continuous function, $\lim_{x\to a} f(x) = f(a)$. That means for a continuous function, we can find the limit by **direct substitution** (evaluating the function) if the function is continuous at a.

Example 1.2.5

Evaluate using continuity, if possible:

1. $\lim_{x\to 2} x^3 - 4x$
2. $\lim_{x\to 2} \dfrac{x-4}{x+3}$
3. $\lim_{x\to 2} \dfrac{x-4}{x-2}$

Solution

1. The given function is polynomial, and is defined for all values of x, so we can find the limit by direct substitution:
$$\lim_{x\to 2} x^3 - 4x = 2^3 - 4(2) = 0.$$
2. The given function is rational. It is not defined at $x = -3$, but we are taking the limit as x approaches 2, and the function is defined at that point, so we can use direct substitution:
$$\lim_{x\to 2} \frac{x-4}{x+3} = \frac{2-4}{2+3} = -\frac{2}{5}.$$
3. This function is not defined at $x = 2$, and so is not continuous at $x = 2$. We cannot use direct substitution.

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1.3: Derivative using Limits of Difference Quotients/Average Rate of Change

Definition of the Derivative

When working with linear functions, we could find the slope of a line to determine the rate at which the function is changing. For an arbitrary function, we can determine the **average rate of change** of the function. This is the slope of the secant line through those two points on the graph.

Average Rate of Change

Average Rate of Change = $\dfrac{\Delta \text{output}}{\Delta \text{input}} = \dfrac{f(b)-f(a)}{b-a}$

= **the slope of the secant line through the two points** $(a, f(a))$ **and** $(b, f(b))$

Example 1.3.1

Suppose the total cost of producing x items is given by $TC(x) = 200 + 30x - 0.1x^2$. Determine the average rate of change of cost when increasing production from 25 units to 100 units.

Solution

The cost when producing 25 units is $TC(25) = 200 + 30(25) - 0.1(25)^2 = \$887.50$

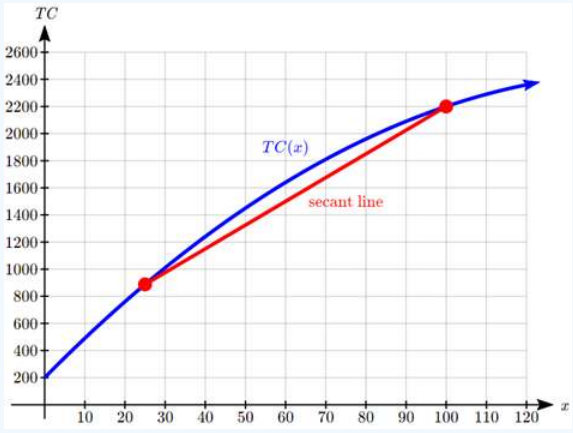
The cost when producing 100 units is $TC(100) = 200 + 30(100) - 0.1(100)^2 = \$2200$

The cost increased by \$2200-\$887.50 = \$1312.50 while production increased by 100 – 25 = 75 items. The average rate of change is:

$$\frac{TC(100)-TC(25)}{100-25} = \frac{2200-887.50}{100-25} = \frac{1312.50}{75} = 17.50 \text{ dollars per unit}$$

This tells us that on average the cost increases by \$17.50 for each unit produced.

The average rate of change is the slope of the secant line between (25, 887.50) and (100, 2200)



Thinking about the last example, suppose instead we asked the question "How fast are costs increasing when production is 25 units?" Notice this is a different kind of question. The question in the example asked for the rate of change over an interval, as production increased from one value to another. This question is again asking for a rate of change, but an **instantaneous rate of change**, at a particular moment.

We don't yet have a way to calculated rate of change except over an interval. In the next example we will explore a couple ways to estimate the instantaneous rate of change.

Example 1.3.2

Suppose the total cost of producing x items is given by $TC(x) = 200 + 30x - 0.1x^2$. Estimate the instantaneous rate of change when 25 items are being produced.

Solution

We can see in the previous graph that the secant slope on the interval $25 \le x \le 100$ is not a particularly good estimate of the instantaneous rate of change since the cost function seems steeper at $x = 25$ then over the secant slope. We could improve the estimate by choosing a smaller interval.

Approach 1: Using smaller intervals

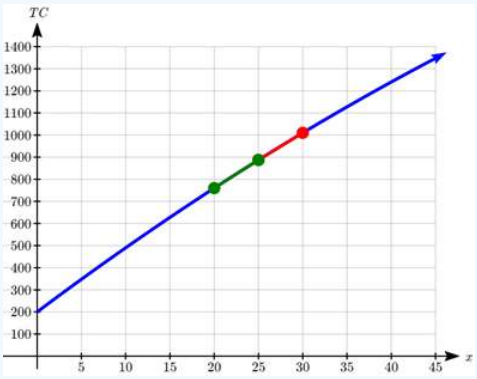
Using the interval $25 \le x \le 30$, the average rate of change would be:

$$\frac{TC(30) - TC(25)}{30 - 25} = \frac{1010 - 887.50}{30 - 25} = \frac{122.50}{5} = 24.50 \text{ dollars per unit}$$

Using an interval on the other side, $20 \le x \le 25$, the average rate of change would be:

$$\frac{TC(25) - TC(20)}{25 - 20} = \frac{887.50 - 760}{25 - 20} = \frac{127.5}{5} = 25.50 \text{ dollars per unit}$$

Visually, we can see both these secant lines seem to approximate the function pretty well.

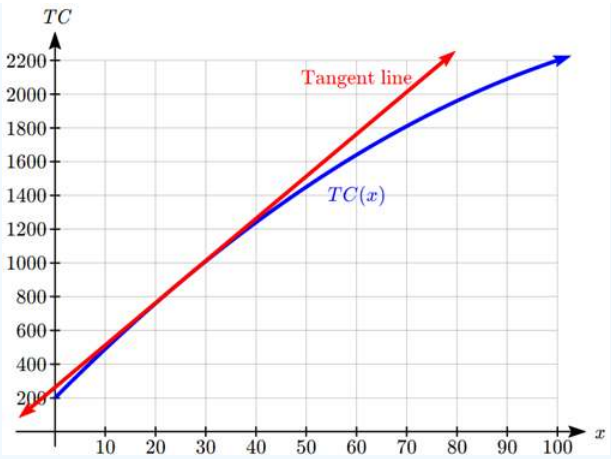


We would expect the instantaneous rate of change to be somewhere between these two values. Averaging them, we get an estimate of $25 per unit for the instantaneous rate of change.

Approach 2: Estimating from the graph.

This approach is more commonly used when we only have the graph of a function, and don't have a formula to evaluate, but we will illustrate it here using the same function.

The instantaneous rate of change is the slope of the tangent line, which is the line that just touches the graph at the point of interest, and has the same rate of change (slope) as the function does at the point. Given a graph, we can sketch in a **tangent line** by holding a ruler up to the graph, and drawing in a line that touches the graph when $x = 25$ and has the same slope.

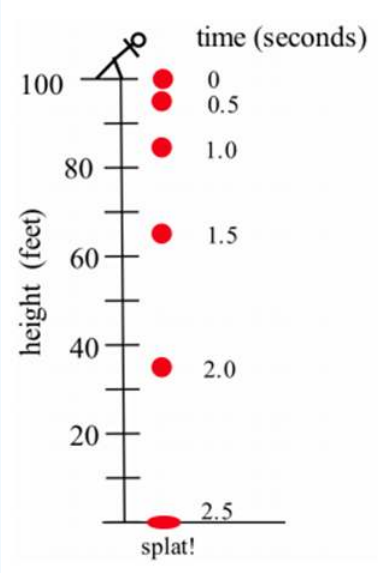


To estimate the instantaneous rate of change, we can now calculate the slope of this drawn-in line. Looking at the graph, the tangent line appears to pass approximately through (30, 1000) and (70, 2000) which would give a slope of $\frac{2000-1000}{70-30} = \frac{1000}{40} = 25$ dollars per unit.

Next we will explore the same ideas using a function defined in a table, and in another context.

Example 1.3.3

Suppose we drop a tomato from the top of a 100 foot building and time its fall.



Time (sec)	Height (ft)
0.0	100
0.5	96
1.0	84
1.5	64
2.0	36
2.5	0

1. How long did it take for the tomato to drop 100 feet?

2. How far did the tomato fall during the first second?
3. How far did the tomato fall during the last second?
4. How far did the tomato fall between $t = 0.5$ and $t = 1$?
5. What was the average velocity of the tomato during its fall?
6. What was the average velocity between $t = 1$ and $t = 2$ seconds?
7. How fast was the tomato falling 1 second after it was dropped?

Solution

Some of these questions are pretty easy to answer, while some are more complex.

1. From the table, it took 2.5 seconds for the tomato to drop 100 feet
2. During the first second, the tomato fell 100 – 84 = 16 feet
3. During the last second, the tomato fell 64 – 0 = 64 feet
4. Between $t = 0.5$ and $t = 1$ the tomato fell 96 – 84 = 12 feet
5. Velocity is similar to speed, and is a rate of change. We can calculate **average velocity** the same way we did average rate of change earlier.

$$\text{Average velocity} = \frac{\text{distance fallen}}{\text{total time}} = \frac{\Delta\text{position}}{\Delta\text{time}} = \frac{-100 \text{ ft}}{2.5 \text{ s}} = -40 \text{ ft/s}$$

6. Between $t = 1$ and $t = 2$ seconds,

$$\text{Average velocity} = \frac{\Delta\text{position}}{\Delta\text{time}} = \frac{36 \text{ ft} - 84 \text{ ft}}{2 \text{ s} - 1 \text{ s}} = \frac{-48 \text{ ft}}{1 \text{ s}} = -48 \text{ ft/s}$$

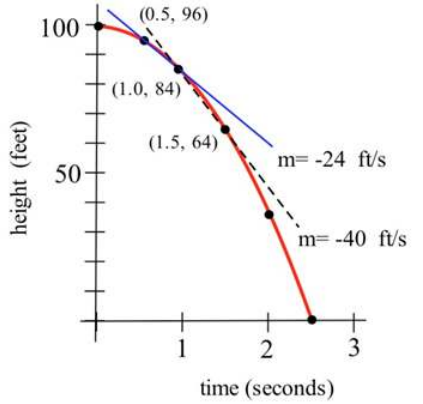
7. This question is significantly different from the previous two questions about average velocity. Here we want the **instantaneous velocity,** the velocity at an instant in time. Unfortunately the tomato is not equipped with a speedometer so we will have to give an approximate answer. Like in our Approach 1 in the previous example, we will estimate it using secant slopes.

One crude approximation of the instantaneous velocity after 1 second is simply the average velocity during the entire fall, -40 ft/s. But the tomato fell slowly at the beginning and rapidly near the end so the "-40 ft/s" estimate may or may not be a good answer.

We can get a better approximation of the instantaneous velocity at $t = 1$ by calculating the average velocities over a short time interval near $t = 1$. The average velocity between $t = 0.5$ and $t = 1$ is $\frac{-12 \text{ feet}}{0.5 \text{ s}} = -24 \text{ ft/s}$, and the average velocity between $t = 1$ and $t = 1.5$ is $\frac{-20 \text{ feet}}{0.5 \text{ s}} = -40 \text{ ft/s}$ so we can be reasonably sure that the instantaneous velocity is between -24 ft/s and -40 ft/s. The average, –32 ft/s, would be a good estimate for the instantaneous velocity.

In general, the shorter the time interval over which we calculate the average velocity, the better the average velocity will approximate the instantaneous velocity.

The average velocity over a time interval is $\frac{\Delta\text{position}}{\Delta\text{time}}$, which is the slope of the **secant line** through two points on the graph of height versus time. The instantaneous velocity at a particular time and height is the slope of the **tangent line** to the graph at the point given by that time and height.



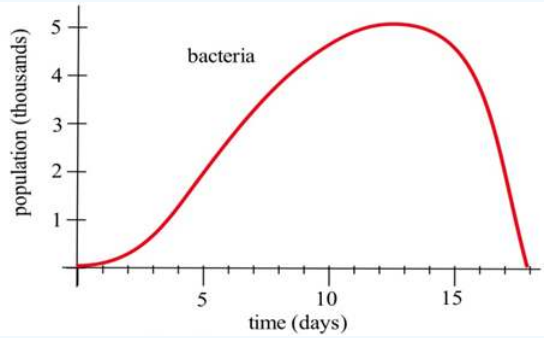
Average vs Instantaneous Velocity

Average velocity = $\dfrac{\Delta\text{position}}{\Delta\text{time}}$ = slope of the secant line through 2 points.

Instantaneous velocity = slope of the line tangent to the graph.

Example 1.3.4

Suppose we set up a machine to count the number of bacteria growing on a Petri plate. At first there are few bacteria so the population grows slowly. Then there are more bacteria to divide so the population grows more quickly. Later, there are more bacteria and less room and nutrients available for the expanding population, so the population grows slowly again. Finally, the bacteria have used up most of the nutrients, and the population declines as bacteria die.



Use the population graph to estimate the answer to the questions below.

1. What is the bacteria population at time $t = 3$ days?
2. What is the population increment from $t = 3$ to $t = 10$ days?
3. What is the **rate** of population growth from $t = 3$ to $t = 10$ days?
4. What is the **rate** of population growth on the third day, at $t = 3$?

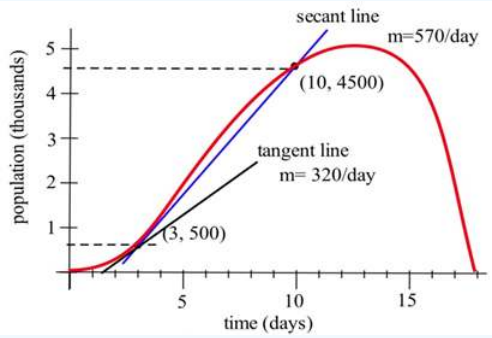
Solution

1. From the graph, at $t = 3$, the population is about 0.5 thousand, or 500 bacteria.
2. At $t = 10$, the population is about 4.5 thousand, so the increment is about 4000 bacteria.
3. The rate of growth from $t = 3$ to $t = 10$ is the average rate of change in population during that time:

$$\begin{aligned}\text{average change in population} &= \frac{\text{change in population}}{\text{change in time}} \\ &= \frac{\Delta\text{population}}{\Delta\text{time}} \\ &= \frac{4000 \text{ bacteria}}{7 \text{ days}} \\ &\approx 570 \text{ bacteria/day}.\end{aligned}$$

This is the slope of the secant line through the two points (3, 500) and (10, 4500).

4. This question is asking for the **instantaneous** rate of population change, the slope of the line which is **tangent** to the population curve at (3, 500). If we sketch a line approximately tangent to the curve at (3, 500) and pick two points near the ends of the tangent line segment, we can estimate that instantaneous rate of population growth is approximately 320 bacteria/day.



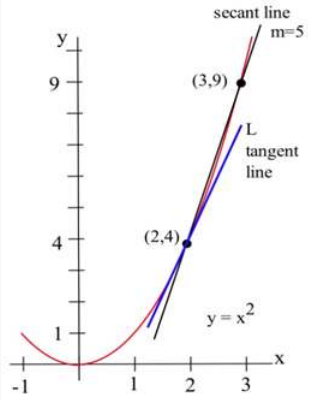
In the previous examples, we noticed that as the interval got smaller and smaller, the secant line got closer to the tangent line and its slope got closer to the slope of the tangent line. That's good news – we know how to find the slope of a secant line.

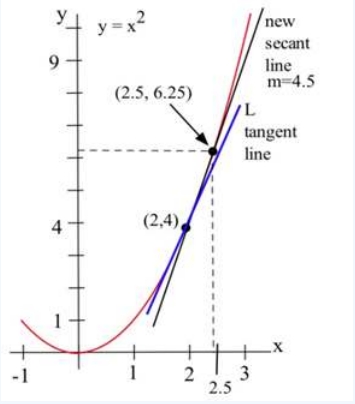
Let's explore further this idea of finding the tangent slope based on the secant slope.

Example 1.3.5

Find the slope of the line L in the graph below which is tangent to $f(x) = x^2$ at the point (2,4).

We could estimate the slope of L from the graph, but we won't. Instead, we will use the idea that secant lines over tiny intervals approximate the tangent line.





We can see that the line through (2,4) and (3,9) on the graph of f is an approximation of the slope of the tangent line, and we can calculate that slope exactly: $m = \frac{\Delta y}{\Delta x} = \frac{9-4}{3-2} = 5$. But $m = 5$ is only an estimate of the slope of the tangent line and not a very good estimate. It's too big. We can get a better estimate by picking a second point on the graph of f which is closer to (2,4). The point (2,4) is fixed and it must be one of the points we use.

From the second figure, we can see that the slope of the line through the points (2,4) and (2.5,6.25) is a better approximation of the slope of the tangent line at (2,4): $m = \frac{\Delta y}{\Delta x} = \frac{6.25-4}{2.5-2} = \frac{2.25}{0.5} = 4.5$, a better estimate, but still an approximation. We can continue picking points closer and closer to (2,4) on the graph of f, and then calculating the slopes of the lines through each of these points and the point (2,4):

Points to the left of (2,4)

x	$y = x^2$	Slope of line through (x, y) and (2,4).
1.5	2.25	3.5
1.9	3.61	3.9
1.99	3.9601	3.99

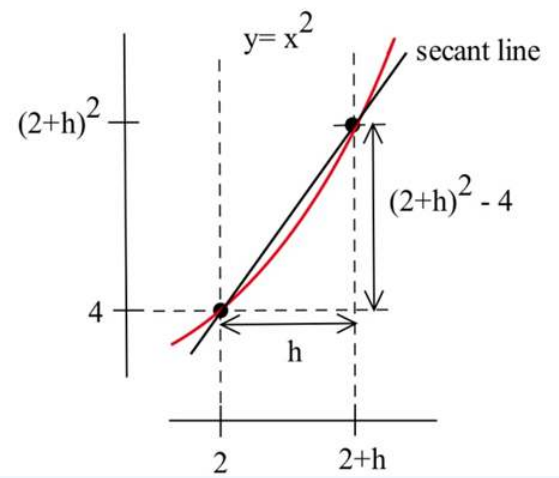
Points to the right of (2,4)

x	$y = x^2$	Slope of line through (x, y) and (2,4).
3	9	5
2.5	6.25	4.5
2.01	4.0401	4.01

The only thing special about the x–values we picked is that they are numbers which are close, and very close, to $x = 2$. Someone else might have picked other nearby values for x. As the points we pick get closer and closer to the point (2,4) on the

graph of $y = x^2$, the slopes of the lines through the points and (2,4) are better approximations of the slope of the tangent line, and these slopes are getting closer and closer to 4.

We can bypass much of the calculating by not picking the points one at a time: let's look at a general point near (2,4). Define $x = 2 + h$ so h is the increment from 2 to x. If h is small, then $x = 2 + h$ is close to 2 and the point $(2 + h, f(2 + h)) = \left(2 + h, (2 + h)^2\right)$ is close to (2,4). The slope m of the line through the points (2,4) and $\left(2 + h, (2 + h)^2\right)$ is a good approximation of the slope of the tangent line at the point (2,4):



We can calculate the secant slope for any value of h:

$$m = \frac{\Delta y}{\Delta x} = \frac{(2 + h)^2 - 4}{(2 + h) - 2} = \frac{\left(4 + 4h + h^2\right) - 4}{h} = \frac{4h + h^2}{h} = \frac{h(4 + h)}{h} = 4 + h$$

The value $m = 4 + h$ is the slope of the secant line through the two points (2,4) and $\left(2 + h, (2 + h)^2\right)$. As h gets smaller and smaller, this slope approaches the slope of the tangent line to the graph of f at (2,4).

More formally, we could write:

$$\text{Slope of the tangent line} = \lim_{h \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} (4 + h).$$

We can easily evaluate this limit using direct substitution, finding that as the interval h shrinks towards 0, the secant slope approaches the tangent slope, 4.

Use the applet below to explore this. You can drag the base point on the graph to explore the behavior at different locations on the graph. Once setting the base point, use the slider to see how the secant lines approach the tangent line as h approaches zero.

Finding tangent slopes and finding the instantaneous rate of change are the same problem. In each problem we wanted to know how rapidly something was **changing at an instant in time**, and the answer turned out to be finding the **slope of a tangent line**, which we *approximated* with the **slope of a secant line**. This idea is the key to defining the slope of a curve.

The Derivative

We can view the derivative in different ways. Here are a three of them:

- The **derivative** of a function f at a point $(x, f(x))$ is the instantaneous rate of change.
- The **derivative** is the slope of the tangent line to the graph of f at the point $(x, f(x))$.
- The **derivative** is the slope of the curve $f(x)$ at the point $(x, f(x))$.

A function is called **differentiable** at $(x, f(x))$ if its derivative exists at $(x, f(x))$.

Notation for the Derivative

The **derivative of $y = f(x)$ with respect to x** is written as

$$f'(x)$$

(read aloud as "f prime of x"), or

$$y'$$

(read aloud as "why prime") or

$$\frac{dy}{dx}$$

(read aloud as "dee why dee ex"), or

$$\frac{df}{dx}.$$

The notation that resembles a fraction is called **Leibniz notation**. It displays not only the name of the function (f or y), but also the name of the variable (in this case, x). It looks like a fraction because the derivative is a slope. In fact, this is simply $\frac{\Delta y}{\Delta x}$ written in Roman letters instead of Greek letters.

Verb Forms

We **find the derivative** of a function, or **take the derivative** of a function, or **differentiate** a function.

We use an adaptation of the $\frac{df}{dx}$ notation to mean "find the derivative of $f(x)$:"

$$\frac{d}{dx}[f(x)] = \frac{df}{dx}.$$

[The book uses parentheses instead of brackets–both are acceptable forms of the notation.]

Formal Algebraic Definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Practical Definition

The derivative can be approximated by looking at an average rate of change, or the slope of a secant line, over a very tiny interval. The tinier the interval, the closer this is to the true instantaneous rate of change, slope of the tangent line, or slope of the curve.

Looking Ahead

We will have methods for computing exact values of derivatives from formulas soon. If the function is given to you as a table or graph, you will still need to approximate this way.

This is the foundation for the rest of this chapter. It's remarkable that such a simple idea (the slope of a tangent line) and such a simple definition (for the derivative $f'(x)$) will lead to so many important ideas and applications.

Example 1.3.6

Find the slope of the tangent line to $f(x) = \frac{1}{x}$ at $x = 3$.

Solution

The slope of the tangent line is the value of the derivative $f'(3)$. $f(3) = \frac{1}{3}$ and $f(3+h) = \frac{1}{3+h}$, so, using the formal limit definition of the derivative,

$$f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h}.$$

We can simplify by giving the fractions a common denominator:

$$\begin{aligned}
\lim_{h\to 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h} &= \lim_{h\to 0} \frac{\frac{1}{3+h}\cdot\frac{3}{3} - \frac{1}{3}\cdot\frac{3+h}{3+h}}{h} \\
&= \lim_{h\to 0} \frac{\frac{3}{9+3h} - \frac{3+h}{9+3h}}{h} \\
&= \lim_{h\to 0} \frac{\frac{3-(3+h)}{9+3h}}{h} \\
&= \lim_{h\to 0} \frac{\frac{3-3-h}{9+3h}}{h} \\
&= \lim_{h\to 0} \frac{\frac{-h}{9+3h}}{h} \\
&= \lim_{h\to 0} \frac{-h}{9+3h}\cdot\frac{1}{h} \\
&= \lim_{h\to 0} \frac{-1}{9+3h}
\end{aligned}$$

and the evaluate using direct substitution:

$$\lim_{h\to 0} \frac{-1}{9+3h} = \frac{-1}{9+3(0)} = -\frac{1}{9}.$$

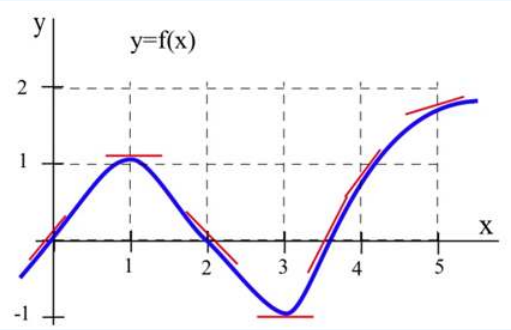
Thus, the slope of the tangent line to $f(x) = \frac{1}{x}$ at $x = 3$ is $-\frac{1}{9}$.

The Derivative as a Function

We now know how to find (or at least approximate) the derivative of a function for any x-value; this means we can think of the derivative as a function, too. The inputs are the same x's; the output is the value of the derivative at that x value.

Example 1.3.7

Below is the graph of a function $y = f(x)$. We can use the information in the graph to fill in a table showing values of $f'(x)$:



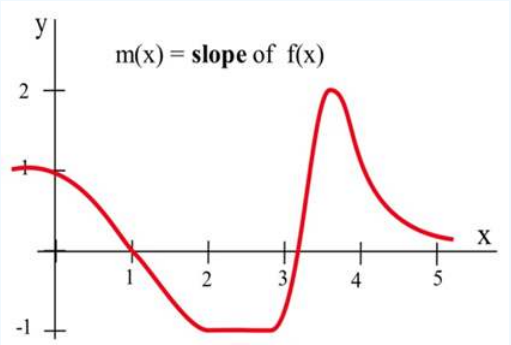
At various values of x, draw your best guess at the tangent line and measure its slope. You might have to extend your lines so you can read some points. In general, your estimate of the slope will be better if you choose points that are easy to read and far away from each other. Here are estimates for a few values of x (parts of the tangent lines used are shown above in the graph):

x	$y = f(x)$	$f'(x) =$ the estimated *slope* of the tangent line to the curve at the point (x, y).
0	0	1
1	1	0
2	0	-1
3	-1	0
3.5	0	2

We can estimate the values of $f'(x)$ at some non-integer values of x, too: $f'(0.5) \approx 0.5$ and $f'(1.3) \approx -0.3$.

We can even think about entire intervals. For example, if $0 < x < 1$, then $f(x)$ is increasing, all the slopes are positive, and so $f'(x)$ is positive.

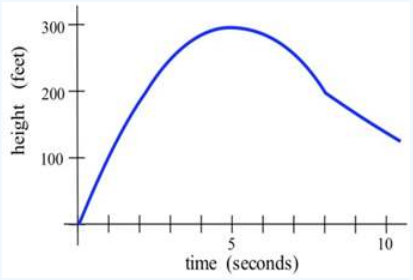
The values of $f'(x)$ definitely depend on the values of x, and $f'(x)$ is a function of x. We can use the results in the table to help sketch the graph of $f'(x)$.



To get a better feel for this, explore the applet below. The top graph is the graph of the original function $g(x)$. The bottom graph shows the slopes of $g(x)$, so is a graph of the derivative, $g'(x)$. Drag the point a and notice how the slope of the tangent line corresponds to the value of the derivative $g'(x)$.

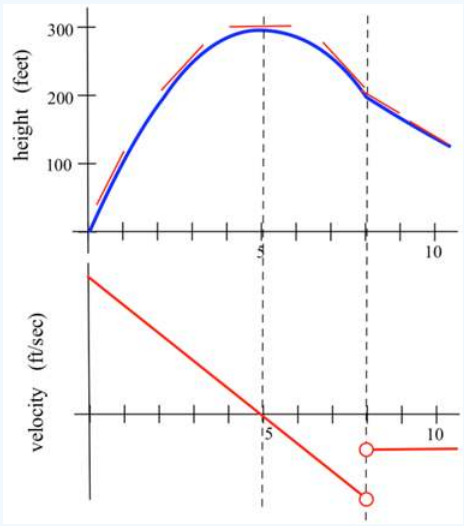
Example 1.3.8

Shown is the graph of the height $h(t)$ of a rocket at time t.



Sketch the graph of the **velocity** of the rocket at time t. (Velocity is the **derivative** of the height function, so it is the **slope of the tangent** to the graph of position or height.)

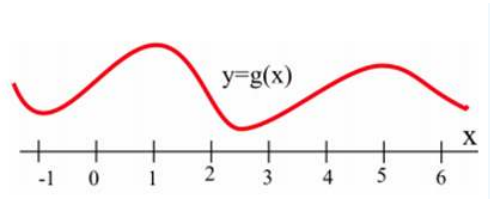
We can estimate the slope of the function at several points. The lower graph below shows the velocity of the rocket. This is $v(t) = h'(t)$.



In some applications, we need to know where the graph of a function $f(x)$ has horizontal tangent lines (slopes = 0).

Example 1.3.9

Below is the graph of $y = g(x)$. At what values of x does the graph of $g(x)$ have horizontal tangent lines?



The tangent lines to the graph of $g(x)$ are horizontal (slope = 0) when $x \approx -1, 1, 2.5,$ and 5.

We can also find derivative functions algebraically using limits.

Example 1.3.10

Find $\frac{d}{dx}\left(2x^2-4x-1\right)$.

Setting up the derivative using a limit,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

We will start by simplifying $f(x+h)$ by expanding:

$$\begin{aligned} f(x+h) &= 2(x+h)^2-4(x+h)-1 \\ &= 2(x^2+2xh+h^2)-4(x+h)-1 \\ &= 2x^2+4xh+2h^2-4x-4h-1 \end{aligned}$$

Now finding the limit:

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{(2x^2+4xh+2h^2-4x-4h-1)-(2x^2-4x-1)}{h} \\ &= \lim_{h\to 0}\frac{2x^2+4xh+2h^2-4x-4h-1-2x^2+4x+1}{h} && \text{(Substitute in the formulas.)} \\ &= \lim_{h\to 0}\frac{4xh+2h^2-4h}{h} && \text{(Now simplify.)} \\ &= \lim_{h\to 0}\frac{h(4x+2h-4)}{h} && \text{(Factor out the } h\text{, then cancel.)} \\ &= \lim_{h\to 0}(4x+2h-4) \end{aligned}$$

We can find the limit of this expression by direct substitution:

$$f'(x)=\lim_{h\to 0}(4x+2h-4)=4x-4$$

Notice that the derivative depends on x, and that this formula will tell us the slope of the tangent line to f at any value x. For example, if we wanted to know the tangent slope of f at $x=3$, we would simply evaluate: $f'(3)=4(3)-4=8$.

A formula for the derivative function is very powerful, but as you can see, calculating the derivative using the limit definition is **very** time consuming. In the next section, we will identify some patterns that will allow us to start building a set of rules for finding derivatives without needing the limit definition.

Interpreting the Derivative

So far we have emphasized the derivative as the slope of the line tangent to a graph. That interpretation is very visual and useful when examining the graph of a function, and we will continue to use it. Derivatives, however, are used in a wide variety of fields and applications, and some of these fields use other interpretations. The following are a few interpretations of the derivative that are commonly used.

General

Rate of Change: $f'(x)$ is the **rate of change** of the function at x. If the units for x are years and the units for $f(x)$ are people, then the units for $\frac{df}{dx}$ are $\frac{\text{people}}{\text{year}}$, a rate of change in population.

Graphical

Slope: $f'(x)$ is the **slope of the line tangent to the graph of f at the point $(x, f(x))$**.

Physical

Velocity: If $f(x)$ is the position of an object at time x, then $f'(x)$ is the **velocity** of the object at time x. If the units for x are hours and $f(x)$ is distance measured in miles, then the units for $\mathbf{f'(x) = \frac{df}{dx}}$ are $\frac{\text{miles}}{\text{hour}}$, miles per hour, which is a measure of velocity.

Acceleration: If $f(x)$ is the velocity of an object at time x, then $f'(x)$ is the **acceleration** of the object at time x. If the units are for x are hours and $f(x)$ has the units $\frac{\text{miles}}{\text{hour}}$, then the units for the acceleration $\mathbf{f'(x) = \frac{df}{dx}}$ are $\frac{\text{miles/hour}}{\text{hour}} = \frac{\text{miles}}{\text{hour}^2}$, miles per hour per hour.

Business

Marginal Cost, Marginal Revenue, and Marginal Profit: We'll explore these terms in more depth later in the section. Basically, the marginal cost is approximately the *additional* cost of making one more object once we have already made x objects. If the units for x are bicycles and the units for $f(x)$ are dollars, then the units for $f'(x) = \frac{df}{dx}$ are $\frac{\text{dollars}}{\text{bicycle}}$, the cost per bicycle.

In business contexts, the word "**marginal**" usually means the derivative or rate of change of some quantity.

One of the strengths of calculus is that it provides a unity and economy of ideas among diverse applications. The vocabulary and problems may be different, but the ideas and even the notations of calculus are still useful.

Example 1.3.11

Suppose the demand curve for widgets was given by $D(p) = \frac{1}{p}$, where D is the quantity of widgets, in thousands, at a price of p dollars. Interpret the derivative of D at $p =$\$3.

Note that we calculated $D'(3)$ earlier to be $D'(3) = -\frac{1}{9} \approx -0.111$.

Since D has units “thousands of widgets” and the units for p is dollars of price, the units for D' will be $\frac{\text{thousands of widgets}}{\text{dollar of price}}$. In other words, it shows how the demand will change as the price increases.

Specifically, $D'(3) \approx -0.111$ tells us that when the price is \$3, the demand will *decrease* by about 0.111 thousand items for every dollar the price increases.

(Note: The screen shots in the following video are from an earlier version of the book, so some of the section numbers or titles may not look the same. However, much of the content is the same, and the comments still apply.)

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1.4: Power and Sum-Difference Rules

In the next few sections, we'll get the derivative rules that will let us find formulas for derivatives when our function comes to us as a formula. This is a very algebraic section, and you should get lots of practice. When you tell someone you have studied calculus, this is the one skill they will expect you to have.

Building Blocks

These are the simplest rules – rules for the basic functions. We won't prove these rules; we'll just use them. But first, let's look at a few so that we can see they make sense.

Example 1.4.1

Find the derivative of $y = f(x) = mx + b$.

Solution

This is a linear function, so its graph is its own tangent line! The slope of the tangent line, the derivative, is the slope of the line:

$$f'(x) = m$$

Rule

The derivative of a linear function is its slope.

Example 1.4.2

Find the derivative of $f(x) = 135$.

Solution

Think about this one graphically, too. The graph of $f(x)$ is a horizontal line. So its slope is zero:

$$f'(x) = 0$$

Rule

The derivative of a constant is zero.

Example 1.4.3

Find the derivative of $f(x) = x^2$.

Solution

Recall the formal definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Using our function $f(x) = x^2$, $f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$.

Then

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \to 0} \frac{h(2x+h)}{h} \\ &= \lim_{h \to 0} (2x + h) \\ &= 2x \end{aligned}$$

From all that, we find that $f'(x) = 2x$.

Luckily, there is a handy rule we use to skip using the limit:

Power Rule

The derivative of $f(x) = x^n$ is

$$f'(x) = nx^{n-1}.$$

Example 1.4.4

Find the derivative of $g(x) = 4x^3$.

Solution

Using the power rule, we know that if $f(x) = x^3$, then $f'(x) = 3x^2$. Notice that g is 4 times the function f. Think about what this change means to the graph of g – it's now 4 times as tall as the graph of f. If we find the slope of a secant line, it will be $\frac{\Delta g}{\Delta x} = \frac{4\Delta f}{\Delta x} = 4\frac{\Delta f}{\Delta x}$; each slope will be 4 times the slope of the secant line on the f graph. This property will hold for the slopes of tangent lines, too:

$$\frac{d}{dx}\left(4x^3\right) = 4\frac{d}{dx}\left(x^3\right) = 4 \cdot 3x^2 = 12x^2.$$

Rule

Constants come along for the ride, i.e., $\frac{d}{dx}(kf) = kf'$.

Here are all the basic rules in one place.

Derivative Rules: Building Blocks

In what follows, f and g are differentiable functions of x.

Constant Multiple Rule

$$\frac{d}{dx}(kf) = kf'$$

Sum and Difference Rule

$$\frac{d}{dx}(f \pm g) = f' \pm g'$$

Power Rule

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Special cases:

$$\frac{d}{dx}(k) = 0 \quad (\text{Because } k = kx^0.)$$

$$\frac{d}{dx}(x) = 1 \quad (\text{Because } x = x^1.)$$

Exponential Functions

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(a^x) = \ln(a)\, a^x$$

Natural Logarithm

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

The sum, difference, and constant multiple rule combined with the power rule allow us to easily find the derivative of any polynomial.

Example 1.4.5

Find the derivative of $p(x) = 17x^{10} + 13x^8 - 1.8x + 1003$.

Solution

$$\begin{aligned}\frac{d}{dx}\left(17x^{10} + 13x^8 - 1.8x + 1003\right) &= \frac{d}{dx}\left(17x^{10}\right) + \frac{d}{dx}\left(13x^8\right) - \frac{d}{dx}(1.8x) + \frac{d}{dx}(1003) \\ &= 17\frac{d}{dx}\left(x^{10}\right) + 13\frac{d}{dx}\left(x^8\right) - 1.8\frac{d}{dx}(x) + \frac{d}{dx}(1003) \\ &= 17\left(10x^9\right) + 13\left(8x^7\right) - 1.8\,(1) + 0 \\ &= 170x^9 + 104x^7 - 1.8\end{aligned}$$

You don't have to show every single step. Do be careful when you're first working with the rules, but pretty soon you'll be able to just write down the derivative directly:

Example 1.4.6

Find $\frac{d}{dx}\left(17x^2 - 33x + 12\right)$.

Solution

Writing out the rules, we'd write

$$\frac{d}{dx}\left(17x^2 - 33x + 12\right) = 17(2x) - 33(1) + 0 = 34x - 33.$$

Once you're familiar with the rules, you can, in your head, multiply the 2 times the 17 and the 33 times 1, and just write

$$\frac{d}{dx}\left(17x^2 - 33x + 12\right) = 34x - 33.$$

The power rule works even if the power is negative or a fraction. In order to apply it, first translate all roots and basic rational expressions into exponents:

Example 1.4.7

Find the derivative of $y = 3\sqrt{t} - \frac{4}{t^4} + 5e^t$.

Solution

The first step is translate into exponents:

$$y = 3\sqrt{t} - \frac{4}{t^4} + 5e^t = 3t^{1/2} - 4t^{-4} + 5e^t$$

Now you can take the derivative:

$$\begin{aligned}\frac{d}{dt}\left(3t^{1/2} - 4t^{-4} + 5e^t\right) &= 3\left(\frac{1}{2}t^{-1/2}\right) - 4\left(-4t^{-5}\right) + 5\left(e^t\right) \\ &= \frac{3}{2}t^{-1/2} + 16t^{-5} + 5e^t\end{aligned}$$

If there is a reason to, you can rewrite the answer with radicals and positive exponents:

$$y' = \frac{3}{2}t^{-1/2} + 16t^{-5} + 5e^t = \frac{3}{2\sqrt{t}} + \frac{16}{t^5} + 5e^t$$

Be careful when finding the derivatives with negative exponents.

We can immediately apply these rules to solve the problem we started the chapter with - finding a tangent line.

Example 1.4.8

Find the equation of the line tangent to $g(t) = 10 - t^2$ when $t = 2$.

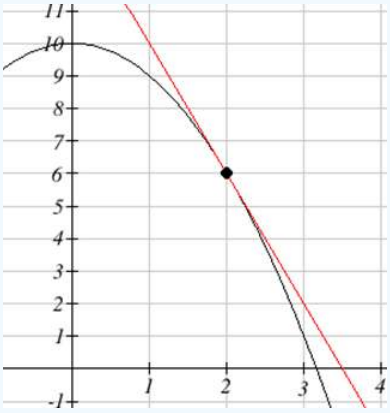
Solution

The slope of the tangent line is the value of the derivative. We can compute $g'(t) = -2t$. To find the slope of the tangent line when $t = 2$, evaluate the derivative at that point. The slope of the tangent line is -4.

To find the equation of the tangent line, we also need a point on the tangent line. Since the tangent line touches the original function at $t = 2$, we can find the point by evaluating the original function: $g(2) = 10 - 2^2 = 6$. The tangent line must pass through the point (2, 6).

Using the point-slope equation of a line, the tangent line will have equation $y - 6 = -4(t - 2)$. Simplifying to slope-intercept form, the equation is $y = -4t + 14$.

Graphing, we can verify this line is indeed tangent to the curve:



We can also use these rules to help us find the derivatives we need to interpret the behavior of a function.

Example 1.4.9

In a memory experiment, a researcher asks the subject to memorize as many words from a list as possible in 10 seconds. Recall is tested, then the subject is given 10 more seconds to study, and so on. Suppose the number of words remembered after t seconds of studying could be modeled by $W(t) = 4t^{2/5}$. Find and interpret $W'(20)$.

Solution

$W'(t) = 4 \cdot \frac{2}{5} t^{-3/5} = \frac{8}{5} t^{-3/5}$, so $W'(20) = \frac{8}{5}(20)^{-3/5} \approx 0.2652$.

Since W is measured in words, and t is in seconds, W' has units words per second. $W'(20) \approx 0.2652$ means that after 20 seconds of studying, the subject is learning about 0.27 more words for each additional second of studying.

Business and Economics Terms

Next we will delve more deeply into some business applications. To do that, we first need to review some terminology.

Suppose you are producing and selling some item. The profit you make is the amount of money you take in minus what you have to pay to produce the items. Both of these quantities depend on how many you make and sell. (So we have functions here.) Here is a list of definitions for some of the terminology, together with their meaning in algebraic terms and in graphical terms.

Cost

Your **cost** is the money you have to spend to produce your items.

Fixed Cost

The **Fixed Cost** (**FC**) is the amount of money you have to spend regardless of how many items you produce. FC can include things like rent, purchase costs of machinery, and salaries for office staff. You have to pay the fixed costs even if you don't produce anything.

Total Variable Cost

The **Total Variable Cost** (**TVC**) for q items is the amount of money you spend to actually produce them. TVC includes things like the materials you use, the electricity to run the machinery, gasoline for your delivery vans, maybe the wages of your production workers. These costs will vary according to how many items you produce.

Total Cost

The **Total Cost** (**TC**, or sometimes just **C**) for q items is the total cost of producing them. It's the sum of the fixed cost and the total variable cost for producing q items.

Average Cost

The **Average Cost** (**AC**) for q items is the total cost divided by q, or

$$AC(q) = \frac{TC}{q}$$

You can also talk about the average fixed cost, $\frac{FC}{q}$, or the average variable cost, $\frac{TVC}{q}$.

Marginal Cost

The **Marginal Cost** (**MC**) at q items is the cost of producing the next item. Really, it's

$$MC(q) = TC(q+1) - TC(q).$$

In many cases, though, it's easier to approximate this difference using calculus (see Example 1 below). And some sources define the marginal cost directly as the derivative,

$$MC(q) = TC'(q).$$

In this course, we will use both of these definitions as if they were interchangeable.

The units on marginal cost is cost per item.

For the purposes of this course, if a question asks for marginal cost, revenue, profit, etc., compute it using the derivative if possible, unless specifically told otherwise.

Why is it okay that there are two definitions for Marginal Cost (and Marginal Revenue, and Marginal Profit)?

We have been using slopes of secant lines over tiny intervals to approximate derivatives. In this example, we'll turn that around – we'll use the derivative to approximate the slope of the secant line.

Notice that the "cost of the next item" definition is actually the slope of a secant line, over an interval of 1 unit:

$$MC(q) = C(q+1) - 1 = \frac{C(q+1) - 1}{1}.$$

So this is approximately the same as the derivative of the cost function at q:

$$MC(q) = C'(q).$$

In practice, these two numbers are so close that there's no practical reason to make a distinction. For our purposes, the marginal cost **is** the derivative **is** the cost of the next item.

Example 1.4.10

The table shows the total cost (TC) of producing q items.

Items, q	TC
0	$20,000
100	$35,000
200	$45,000
300	$53,000

1. What is the fixed cost?
2. When 200 items are made, what is the total variable cost? The average variable cost?
3. When 200 items are made, estimate the marginal cost.

Solution

1. The fixed cost is $20,000, the cost even when no items are made.
2. When 200 items are made, the total cost is $45,000. Subtracting the fixed cost, the total variable cost is $45,000 - $20,000 = $25,000.

   The average variable cost is the total variable cost divided by the number of items, so we would divide the $25,000 total variable cost by the 200 items made. $25,000/200 = $125. On average, each item had a variable cost of $125.
3. We need to estimate the value of the derivative, or the slope of the tangent line at $q = 200$. Finding the secant line from $q = 100$ to $q = 200$ gives a slope of

$$\frac{45,000 - 35,000}{200 - 100} = 100.$$

   Finding the secant line from $q = 200$ to $q = 300$ gives a slope of

$$\frac{53,000 - 45,000}{300 - 200} = 80.$$

   We could estimate the tangent slope by averaging these secant slopes, giving us an estimate of $90/item.

   This tells us that after 200 items have been made, it will cost about $90 to make one more item.

Example 1.4.11

The cost to produce x items is $C(x) = \sqrt{x}$ hundred dollars.

1. What is the cost for producing 100 items? 101 items? What is cost of the 101st item?
2. Calculate $C'(x)$ and evaluate C' at $x = 100$. How does $C'(100)$ compare with the last answer in Part a?

Solution

1. $C(100) = 10$ hundred dollars = \$1000 and $C(101) =$10.0499 hundred dollars = \$1004.99, so it costs \$4.99 for that 101st item. Using this definition, the marginal cost is \$4.99.
2. $C'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$, so $C'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20}$ hundred dollars = \$5.00.

Note how close these answers are! This shows (again) why it's OK that we use both definitions for marginal cost.

Demand

Demand is the functional relationship between the price p and the quantity q that can be sold (that is demanded). Depending on your situation, you might think of p as a function of q, or of q as a function of p

Revenue

Your **revenue** is the amount of money you actually take in from selling your products.

Total Revenue

The **Total Revenue** (**TR**, or just **R**) for q items is the total amount of money you take in for selling q items. Total Revenue is price multiplied by quantity,

$$TR = p \cdot q.$$

Average Revenue

The **Average Revenue** (**AR**) for q items is the total revenue divided by q, or

$$\frac{TR}{q}.$$

Marginal Revenue

The Marginal Revenue (MR) at q items is the revenue from producing the next item,

$$MR(q) = TR(q+1) - TR(q).$$

Just as with marginal cost, we will use both this definition and the derivative definition:

$$MR(q) = TR'(q).$$

Profit

Your **profit** is what's left over from total revenue after costs have been subtracted.

The **Profit** (**P**) for q items is

$$TR(q) - TC(q),$$

the difference between total revenue and total costs.

The **average profit** for q items is

$$\frac{P}{q}.$$

The **marginal profit** at q items is

$$P(q+1) - P(q),$$

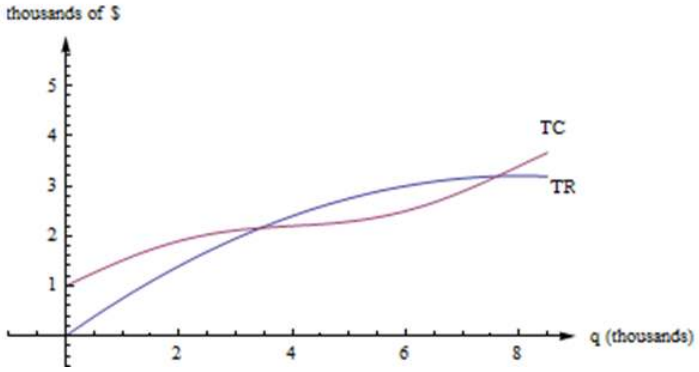
or

$$P'(q)$$

Graphical Interpretations of the Basic Business Math Terms

Illustration

Here are the graphs of TR and TC for producing and selling a certain item. The horizontal axis is the number of items, in thousands. The vertical axis is the number of dollars, also in thousands.



First, notice how to find the fixed cost and variable cost from the graph here. **FC is the y-intercept of the TC graph.** ($FC = TC(0)$.) The graph of TVC would have the same shape as the graph of TC, shifted down. ($TVC = TC - FC$.)

$MC(q) = TC(q+1) - TC(q)$, but that's impossible to read on this graph. How could you distinguish between TC(4022) and TC(4023)? On this graph, that interval is too small to see, and our best guess at the secant line is actually the tangent line to the TC curve at that point. (This is the reason we want to have the derivative definition handy.)

$MC(q)$ is the slope of the tangent line to the TC curve at $(q, TC(q))$.

$MR(q)$ is the slope of the tangent line to the TR curve at $(q, TR(q))$.

Profit is the distance between the TR and TC curve. If you experiment with a clear ruler, you'll see that the biggest profit occurs exactly when the tangent lines to the TR and TC curves are parallel. This is the rule **"profit is maximized when $MR = MC$"** which we'll explore later in the chapter.

Example 1.4.12

The demand, D, for a product at a price of p dollars is given by $D(p) = 200 - 0.2p^2$. Find the marginal revenue when the price is \$10.

Solution

First we need to form a revenue equation. Since Revenue = Price$\times$Quantity, and the demand equation shows the quantity of product that can be sold, we have

$$R(p) = D(p) \cdot p = \left(200 - 0.2p^2\right) p = 200p - 0.2p^3.$$

Now we can find marginal revenue by finding the derivative:

$$R'(p) = 200(1) - 0.2(3p^2) = 200 - 0.6p^2$$

At a price of \$10, $R'(10) = 200 - 0.6(10)^2 = 140$.

Notice the units for R' are $\frac{\text{dollars of Revenue}}{\text{dollar of price}}$, so $R'(10) = 140$ means that when the price is \$10, the revenue will increase by \$140 for each dollar that the price was increased.

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1.5: Product and Quotient Rules

The basic rules will let us tackle simple functions. But what happens if we need the derivative of a combination of these functions?

Example 1.5.1

Find the derivative of $h(x) = \left(4x^3 - 11\right)(x+3)$

This function is not a simple sum or difference of polynomials. It's a product of polynomials. We can simply multiply it out to find its derivative:

$$\begin{aligned} h(x) &= \left(4x^3 - 11\right)(x+3) \\ &= 4x^4 - 11x + 12x^3 - 33 \\ h'(x) &= 16x^3 - 11 + 36x^2 \end{aligned}$$

Now suppose we wanted to find the derivative of

$$f(x) = \left(4x^5 + x^3 - 1.5x^2 - 11\right)\left(x^7 - 7.25x^5 + 120x + 3\right)$$

This function is not a simple sum or difference of polynomials. It's a product of polynomials. We could 'simply' multiply it out to find its derivative as before – who wants to volunteer? Nobody?

We'll need a rule for finding the derivative of a product so we don't have to multiply everything out.

It would be great if we can just take the derivatives of the factors and multiply them, but unfortunately that won't give the right answer. To see that, consider finding derivative of $h(x) = \left(4x^3 - 11\right)(x+3)$. We already worked out the derivative, it is $h'(x) = 16x^3 - 11 + 36x^2$. What if we try differentiating the factors and multiplying them? We'd get $h'(x) = \left(12x^2\right)(1) = 12x^2$, which is radically different from the correct answer.

The rules for finding derivatives of products and quotients are a little complicated, but they save us the much more complicated algebra we might face if we were to try to multiply things out. They also let us deal with products where the factors are not polynomials. We can use these rules, together with the basic rules, to find derivatives of many complicated looking functions.

Derivative Rules: Product and Quotient Rules

In what follows, f and g are differentiable functions of x.

Product Rule

$\frac{d}{dx}(f \cdot g) = f' \cdot g + f \cdot g'$

The derivative of the first factor times the second left alone, plus the first left alone times the derivative of the second.

The product rule can extend to a product of several functions; the pattern continues – take the derivative of each factor in turn, multiplied by all the other factors left alone, and add them up:

$$\frac{d}{dx}(f \cdot g \cdot h) = f' \cdot g \cdot h + f \cdot g' \cdot h + f \cdot g \cdot h'$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f' \cdot g - f \cdot g'}{g^2}$$

The numerator of the result resembles the product rule, but there is a minus instead of a plus; the minus sign goes with the g'. The denominator is simply the square of the original denominator – no derivatives there.

Example 1.5.2

Find the derivative of $h(x) = \left(4x^3 - 11\right)(x+3)$

This is the same function we found the derivative of in Example 1, but let's use the product rule and check to see if we get the same answer. For this first example, we will provide a lot more detail and steps than one usually actually shows when working a problem like this.

Notice we can think of $h(x)$ as the product of two functions $f(x) = 4x^3 - 11$ and $g(x) = x + 3$. Finding the derivative of each of these,

$$f'(x) = 12x^2 \text{ and } g'(x) = 1.$$

Using the product rule,

$$\begin{aligned} h'(x) &= (f')(g) + (f)(g') \\ &= \left(12x^2\right)(x+3) + \left(4x^3 - 11\right)(1) \end{aligned}$$

To check if this is equivalent to the answer we found in Example 1 we could simplify:

$$\begin{aligned} h'(x) &= \left(12x^2\right)(x+3) + \left(4x^3 - 11\right)(1) \\ &= 12x^3 + 36x^2 + 4x^3 - 11 \\ &= 16x^3 + 36x^2 - 11 \end{aligned}$$

From this, we can see the answers are equivalent.

Example 1.5.3

Find the derivative of $F(t) = e^t \ln(t)$

This is a product, so we need to use the product rule. I like to put down empty parentheses to remind myself of the pattern; that way I don't forget anything:

$$F'(t) = (\,)(\,) + (\,)(\,)$$

Then I fill in the parentheses – the first set gets the derivative of e^t, the second gets $\ln t$ left alone, the third gets e^t left alone, and the fourth gets the derivative of $\ln t$.

$$F'(t) = \left(e^t\right)(\ln(t)) + \left(e^t\right)\left(\frac{1}{t}\right) = e^t \ln(t) + \frac{e^t}{t}.$$

Notice that this was one we could *not* have done by "multiplying out."

Example 1.5.4

Find the derivative of $y = \frac{x^4+4^x}{3+16x^3}$.

This is a quotient, so we need to use the quotient rule. Again, you find it helpful to put down the empty parentheses as a template:

$$y' = \frac{(\,)(\,) - (\,)(\,)}{(\,)^2}$$

We can find the derivative of the numerator and denominator separately:

For the numerator: $f(x) = x^4 + 4^x$ so $f'(x) = 4x^3 + \ln(4) \cdot 4^x$

For the denominator: $g(x) = 3 + 16x^3$ so $g'(x) = 48x^2$

Then fill in all the pieces:

$$y' = \frac{f' \cdot g - f \cdot g'}{g^2}$$

$$y' = \frac{\left(4x^3 + \ln(4) \cdot 4^x\right)\left(3 + 16x^3\right) - \left(x^4 + 4^x\right)\left(48x^2\right)}{\left(3 + 16x^3\right)^2}$$

Now for goodness' sake don't try to simplify that! Remember that "simple" depends on what you will do next; in this case, we were asked to find the derivative, and we've done that. I expect you to do any "basic" simplifications, such as multiplying constants together or doing obvious cancellations or combining of terms, but otherwise please STOP unless there is a reason to simplify further.

Example 1.5.5

Suppose a large tank contains 8 kg of a chemical dissolved in 50 liters of water. If a tap is opened and water is added to the tank at a rate of 5 liters per minute, at what rate is the concentration of chemical in the tank changing after 4 minutes?

First we need to set up a model for the concentration of chemical. The concentration would be measured as kg of chemical per liter of water, kg/L. The number of kg of chemical stays constant at 8 kg, but the quantity of water in the tank is increasing by 5 L/min. The total volume of water in the tank after t minutes is $50 + 5t$, so the concentration after t minutes is

$$c(t) = \frac{8}{50 + 5t}.$$

To find the rate at which the concentration is changing, we need the derivative:

$$\begin{aligned} c'(t) &= \frac{\frac{d}{dt}(8) \cdot (50 + 5t) - (8) \cdot \frac{d}{dt}(50 + 5t)}{(50 + 5t)^2} \\ &= \frac{(0) \cdot (50 + 5t) - (8)(5)}{(50 + 5t)^2} \\ &= -\frac{40}{(50 + 5t)^2} \end{aligned}$$

At $t = 4$,

$$c'(4) = -\frac{40}{(50+5(4))^2} \approx -0.00816.$$

Note that the units here are kg per liter, per minute, or $\frac{\text{kg/L}}{\text{min}}$. In other words, this tells us that after 4 minutes, the concentration of chemical is *decreasing* by 0.00816 kg/L each minute.

More Graphical Interpretations of the Basic Business Math Terms

Returning to our discussion of business and economics topics, in addition to total cost and marginal cost, we often also want to talk about average cost or average revenue.

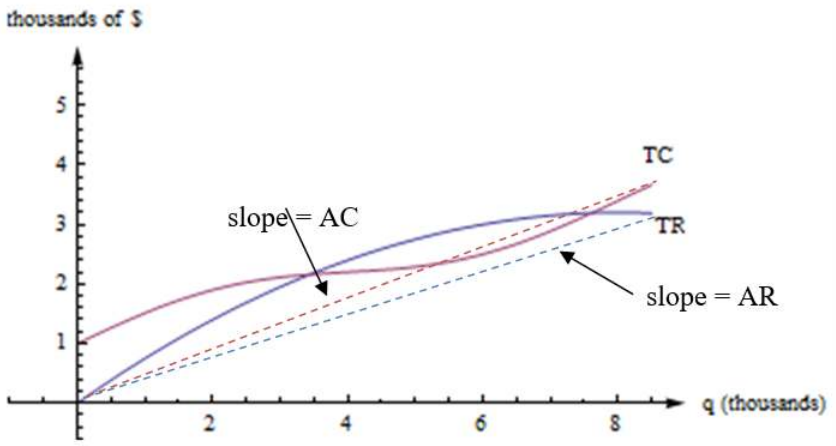
Recall from the previous section that the **Average Cost** (**AC**) for q items is the total cost divided by q, or $AC(q) = \frac{TC}{q}$. You can also talk about the average fixed cost, $\frac{FC}{q}$, or the average variable cost, $\frac{TVC}{q}$.

Also recall that the **Average Revenue** (**AR**) for q items is the total revenue divided by q, or $AR(q) = \frac{TR}{q}$.

We already know that we can find average rates of change by finding slopes of secant lines. AC, AR, MC, and MR are all rates of change, and we can find them with slopes, too.

$AC(q)$ is the slope of a diagonal line, from (0, 0) to $(q, TC(q))$.

$AR(q)$ is the slope of the line from (0, 0) to $(q, TR(q))$.



And just as we found marginal Total Cost, we can also find marginal Average Cost.

Example 1.5.6

The cost, in thousands of dollars, for producing x thousand cellphone cases is given by $C(x) = 22 + x - 0.004x^2$. Find

1. the Fixed Costs,
2. the Average Cost when 5 thousand, 10 thousand, or 20 thousand cases are produced,
3. the Marginal Average Cost when 5 thousand cases are produced.

1. The fixed costs are the costs when no items are produced: $C(0) = 22$ thousand dollars.
2. The average cost function is total cost divided by number of items, so

$$AC(x) = \frac{C(x)}{x} = \frac{22 + x - 0.004x^2}{x}.$$

Note the units are thousands of dollars per thousands of items, which simplifies to just dollars per item.

At a production of 5 thousand items: $AC(5) = \frac{22+5-0.004(5)^2}{5} = 5.38$ dollars per item.

At a production of 10 thousand items: $AC(10) = \frac{22+5-0.004(10)^2}{10} = 3.16$ dollars per item.

At a production of 20 thousand items: $AC(20) = \frac{22+5-0.004(20)^2}{20} = 2.02$ dollars per item.

Notice that while the total cost increases with production, the average cost per item decreases, because the initial fixed costs are being distributed across more items.

3. For the marginal average cost, we need to find the derivative of the average cost function. We can either calculate this using the quotient rule, or we could use algebra to simplify the equation first (this is the easier option – remember, simplifying before differentiating is almost always easier):

$$\begin{aligned} AC(x) &= \frac{22+x-0.004x^2}{x} \\ &= \frac{22}{x} + \frac{x}{x} - \frac{0.004x^2}{x} \\ &= \frac{22}{x} + 1 - 0.004x \\ &= 22x^{-1} + 1 - 0.004x \end{aligned}$$

(Note: we haven't differentiated yet, only simplified.)

Taking the derivative,

$$AC'(x) = -22x^{-2} - 0.004 = -\frac{22}{x^2} - 0.004.$$

When 5 thousand items are produced,

$$AC'(5) = -\frac{22}{5^2} - 0.004 = -0.884.$$

Since the units on AC are dollars per item, and the units on x are thousands of items, the units on AC' are dollars per item per thousands of items. This tells us that when 5 thousand items are produced, the average cost per item is *decreasing* by \$0.884 for each additional thousand items produced.

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1.6: Chain Rule

There is one more type of complicated function that we will want to know how to differentiate: composition. The Chain Rule will let us find the derivative of a composition. (This is the last derivative rule we will learn!)

Example 1.6.1

Find the derivative of $y = \left(4x^3 + 15x\right)^2$.

Solution

This is not a simple polynomial, so we can't use the basic building block rules yet. It is a product, so we could write it as $y = \left(4x^3 + 15x\right)^2 = \left(4x^3 + 15x\right)\left(4x^3 + 15x\right)$ and use the product rule. Or we could multiply it out and simply differentiate the resulting polynomial. I'll do it the second way:

$$\begin{aligned} y &= \left(4x^3 + 15x\right)^2 \\ &= 16x^6 + 120x^4 + 225x^2 \\ y' &= 96x^5 + 480x^3 + 450x \end{aligned}$$

Now suppose we want to find the derivative of $y = \left(4x^3 + 15x\right)^{20}$. We **could** write it as a product with 20 factors and use the product rule, or we **could** multiply it out. But I don't want to do that, do you?

We need an easier way, a rule that will handle a composition like this. The Chain Rule is a little complicated, but it saves us the much more complicated algebra of multiplying something like this out. It will also handle compositions where it wouldn't be possible to "multiply it out."

The Chain Rule is a common place for students to make mistakes. Part of the reason is that the notation takes a little getting used to. And part of the reason is that students often forget to use it when they should. When should you use the Chain Rule? Almost every time you take a derivative.

Derivative Rules: Chain Rule

In what follows, f and g are differentiable functions where $y = f(g(x))$. We could alternatively write $y = f(u)$ and $u = g(x)$.

Chain Rule (Leibniz notation)

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Notice that the du's seem to cancel. This is one advantage of the Leibniz notation – it can remind you of how the chain rule chains together.

Chain Rule (using prime notation)

$$\frac{d}{dx} f\left(g(x)\right) = f'\left(g(x)\right) \cdot g'(x)$$

Chain Rule (in words)

The derivative of a composition is the derivative of the outside (with the inside staying the same) TIMES the derivative of the inside.

I recite the version in words each time I take a derivative, especially if the function is complicated.

Example 1.6.2

Find the derivative of $y = \left(4x^3 + 15x\right)^2$.

Solution

This is the same one we did before by multiplying out. This time, let's use the Chain Rule: The inside function is what appears inside the parentheses: $4x^3 + 15x$. The outside function is the first thing we find as we come in from the outside – it's the square function, $(\text{inside})^2$.

The derivative of this outside function is $(2 \cdot \text{inside})$. Now using the chain rule, the derivative of our original function is $(2 \cdot \text{inside})$ TIMES the derivative of the inside (which is $12x^2 + 15$):

$$y' = 2\left(4x^3 + 15x\right)\left(12x^2 + 15\right)$$

If you multiply this out, you get the same answer we got before. Hurray! Algebra works!

Example 1.6.3

Find the derivative of $y = \left(4x^3 + 15x\right)^{20}$.

Solution

Now we have a way to handle this one. It's the derivative of the outside TIMES the derivative of the inside.

The outside function is $(\text{inside})^{20}$, which has derivative $20(\text{inside})^{19}$, so

$$y' = 20\left(4x^3 + 15x\right)^{19}\left(12x^2 + 15\right).$$

Example 1.6.4

Differentiate $y = e^{x^2+5}$.

Solution

This isn't a simple exponential function; it's a composition. Typical calculator or computer syntax can help you see what the "inside" function is here. On a TI calculator, for example, when you push the e^x key, it opens up parentheses: $\boxed{e\text{^}(}$. This tells you that the "inside" of the exponential function is the exponent. Here, the inside is the exponent $x^2 + 5$. Now we can use the Chain Rule: We want the derivative of the outside TIMES the derivative of the inside. The outside is the "e to the something" function, so its derivative is the same thing. The derivative of what's inside is $2x$. So

$$\frac{d}{dx}\left(e^{x^2+5}\right) = \left(e^{x^2+5}\right) \cdot (2x).$$

Example 1.6.5

The table gives values for f, f', g, and g' at a number of points. Use these values to determine $(f \circ g)(x)$ and $(f \circ g)'(x)$ at $x = -1$ and 0.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$	$(f\circ g)(x)$	$(g\circ f)(x)$
-1	2	3	1	0		
0	-1	1	3	2		
1	1	0	-1	3		
2	3	-1	0	1		
3	0	2	2	-1		

Solution

$$\begin{aligned}(f\circ g)(-1) &= f\left(g(-1)\right) = f(3) = 0\\ (f\circ g)(0) &= f\left(g(0)\right) = f(1) = 1\\ (f\circ g)'(-1) &= f'\left(g(-1)\right)\cdot g'(-1) = f'(3)\cdot(0) = (2)(0) = 0 \text{ and}\\ (f\circ g)'(0) &= f'\left(g(0)\right)\cdot g'(0) = f'(1)\cdot(2) = (-1)(2) = -2\end{aligned}$$

Example 1.6.6

If 2400 people now have a disease, and the number of people with the disease appears to double every 3 years, then the number of people expected to have the disease in t years is $y = 2400\cdot 2^{t/3}$.

1. How many people are expected to have the disease in 2 years?
2. When are 50,000 people expected to have the disease?
3. How fast is the number of people with the disease expected to grow now and 2 years from now?

1. In 2 years, $y = 2400\cdot 2^{2/3} \approx 3,810$ people.
2. We know $y = 50,000$, and we need to solve $50,000 = 2400\cdot 2^{t/3}$ for t. We could start by isolating the exponential by dividing both sides by 2400,

$$\begin{aligned}\frac{50000}{2400} &= 2^{t/3}\\ \ln\left(\frac{50000}{2400}\right) &= \ln\left(2^{t/3}\right) \qquad \text{(Taking the natural log of both sides.)}\\ \ln\left(\frac{50000}{2400}\right) &= \frac{t}{3}\ln(2) \qquad \text{(Using the exponent property for logs.)}\\ t &= \frac{3\ln\left(\frac{50000}{2400}\right)}{\ln(2)} \approx 13.14 \text{ years} \qquad \text{(Solving for } t.\text{)}\end{aligned}$$

   We expect 50,000 people to have the disease about 13.14 years from now.
3. This is asking for $\frac{dy}{dt}$ when $t = 0$ and 2 years. Using the chain rule,

$$\begin{aligned}\frac{dy}{dt} &= \frac{d}{dt}\left(2400 \cdot 2^{t/3}\right) \\ &= 2400 \cdot 2^{t/3} \cdot \ln(2) \cdot \frac{1}{3} \\ &\approx 554.5 \cdot 2^{t/3}\end{aligned}$$

So, at $t = 0$ the rate of growth of the disease is approximately $554.5 \cdot 2^0 \approx 554.5$ people/year. In 2 years the rate of growth will be approximately $554.5 \cdot 2^{2/3} \approx 880$ people/year.

Derivatives of Complicated Functions

You're now ready to take the derivative of some mighty complicated functions. But how do you tell what rule applies first? Work your way in from the outside – what do you encounter first? That's the first rule you need. Use the Product, Quotient, and Chain Rules to peel off the layers, one at a time, until you're all the way inside.

Example 1.6.7

Find $\frac{d}{dx}\left(e^{3x} \cdot \ln(5x+7)\right)$.

Solution

Coming in from the outside, we see that this is a product of two (complicated) functions. So we'll need the Product Rule first. we'll fill in the pieces we know, and then we can figure the rest as separate steps and substitute in at the end:

$$\frac{d}{dx}\left(e^{3x} \cdot \ln(5x+7)\right) = \left(\frac{d}{dx}\left(e^{3x}\right)\right) \cdot \ln(5x+7) + e^{3x} \cdot \left(\frac{d}{dx}(\ln(5x+7))\right)$$

Now as separate steps, we'll find

$$\frac{d}{dx}\left(e^{3x}\right) = 3e^{3x} \quad \text{(using the Chain Rule)}$$

and

$$\frac{d}{dx}(\ln(5x+7)) = \frac{1}{5x+7} \cdot 5 \quad \text{(also using the Chain Rule).}$$

Finally, to substitute these in their places:

$$\frac{d}{dx}\left(e^{3x} \cdot \ln(5x+7)\right) = \left(3e^{3x}\right) \cdot \ln(5x+7) + e^{3x} \cdot \left(\frac{1}{5x+7} \cdot 5\right)$$

(We can stop here – we don't need to try to simplify any further.)

Example 1.6.8

Differentiate $z = \left(\frac{3t^3}{e^t(t-1)}\right)^4$.

Solution

Don't panic! As we come in from the outside, what's the first thing we encounter? It's that fourth power. That tells us that this is a composition, a (complicated) function raised to the fourth power.

Step One: Use the Chain Rule. The derivative of the outside TIMES the derivative of the inside:

$$\frac{dz}{dt} = \frac{d}{dt}\left(\frac{3t^3}{e^t(t-1)}\right)^4 = 4\left(\frac{3t^3}{e^t(t-1)}\right)^3 \cdot \frac{d}{dt}\left(\frac{3t^3}{e^t(t-1)}\right)$$

Now we're one step inside, and we can concentrate on just the $\frac{d}{dt}\left(\frac{3t^3}{e^t(t-1)}\right)$ part. Now, as you come in from the outside, the first thing you encounter is a quotient – this is the quotient of two (complicated) functions.

Step Two: Use the Quotient Rule. The derivative of the numerator is straightforward, so we can just calculate it. The derivative of the denominator is a bit trickier, so we'll leave it for now:

$$\frac{d}{dt}\left(\frac{3t^3}{e^t(t-1)}\right)=\frac{\left(9t^2\right)\left(e^t(t-1)\right)-\left(3t^3\right)\left(\frac{d}{dt}\left(e^t(t-1)\right)\right)}{\left(e^t(t-1)\right)^2}$$

Now we've gone one more step inside, and we can concentrate on just the $\frac{d}{dt}\left(e^t(t-1)\right)$ part, which involves a product.

Step Three: Use the Product Rule:

$$\frac{d}{dt}\left(e^t(t-1)\right)=\left(e^t\right)(t-1)+\left(e^t\right)(1)$$

And now we're all the way in – no more derivatives to take!

Step Four: Now it's just a question of substituting back – be careful now!

$$\frac{d}{dt}\left(e^t(t-1)\right)=\left(e^t\right)(t-1)+\left(e^t\right)(1)$$

so

$$\frac{d}{dt}\left(\frac{3t^3}{e^t(t-1)}\right)=\frac{\left(9t^2\right)\left(e^t(t-1)\right)-\left(3t^3\right)\left(\left(e^t\right)(t-1)+\left(e^t\right)(1)\right)}{\left(e^t(t-1)\right)^2}$$

so

$$\frac{dz}{dt}=\frac{d}{dt}\left(\frac{3t^3}{e^t(t-1)}\right)^4=4\left(\frac{3t^3}{e^t(t-1)}\right)^3\cdot\left(\frac{\left(9t^2\right)\left(e^t(t-1)\right)-\left(3t^3\right)\left(\left(e^t\right)(t-1)+\left(e^t\right)(1)\right)}{\left(e^t(t-1)\right)^2}\right)$$

Phew!

What if the Derivative Doesn't Exist?

Differentiable

A function is called **differentiable** at a point if its derivative exists at that point.

We've been acting as if derivatives exist everywhere for every function. This is true for most of the functions that you will run into in this class. But there are some common places where the derivative doesn't exist.

Remember that the derivative is the slope of the tangent line to the curve. That's what to think about.

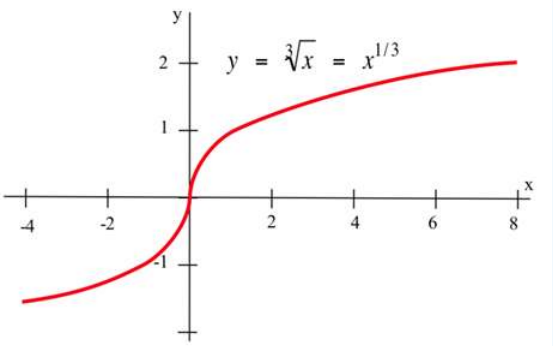
Where can a slope not exist? **If the tangent line is vertical, the derivative will not exist.**

Example 1.6.9

Show that $f(x)=\sqrt[3]{x}=x^{1/3}$ is not differentiable at $x=0$.

Solution

Finding the derivative, $f(x)=\frac{1}{3}x^{-2/3}=\frac{1}{3x^{2/3}}$. At $x=0$, this function is undefined. From the graph, we can see that the tangent line to this curve at $x=0$ is vertical with undefined slope, which is why the derivative does not exist at $x=0$.



Where can a tangent line not exist?

If there is a sharp corner (cusp) in the graph, the derivative will not exist at that point because there is no well-defined tangent line (a teetering tangent, if you will).

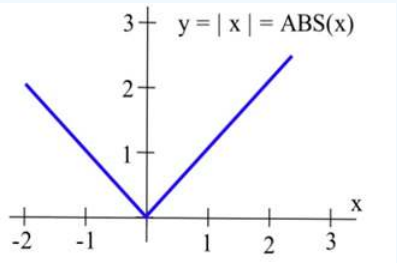
If there is a discontinuity in the graph (a jump, a break, a hole in the graph, or a vertical asymptote), the tangent line will be different on either side and **the derivative will not exist at that point**.

Example 1.6.10

Show that $f(x) = |x|$ is not differentiable at $x = 0$.

Solution

On the left side of the graph, the slope of the line is -1. On the right side of the graph, the slope is +1. There is no well-defined tangent line at the sharp corner at $x = 0$, so the function is not differentiable at that point.



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1.7: Higher Order Derivatives

The operation of differentiation takes as input one function, $f(x)$, and produces as output another function, $f'(x)$. Now $f'(x)$ is once again a function. So we can differentiate it again, assuming that it is differentiable, to create a third function, called the second derivative of f. And we can differentiate the second derivative again to create a fourth function, called the third derivative of f. And so on.

Definition 2.14.1.

- $f''(x)$ and $f^{(2)}(x)$ and $\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x)$ all mean $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right)$
- $f'''(x)$ and $f^{(3)}(x)$ and $\frac{\mathrm{d}^3}{\mathrm{d}x^3}(x)$ all mean $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right)\right)$
- $f^{(4)}(x)$ and $\frac{\mathrm{d}^4}{\mathrm{d}x^4}(x)$ both mean $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right)\right)\right)$
- and so on.

Here is a simple example. Then we'll think a little about the significance of second order derivatives. Then we'll do a more a computationally complex example.

Example 2.14.2 Derivatives of x^n.

Let n be a natural number and let $f(x) = x^n$. Then

$$\begin{aligned}\frac{d}{dx}x^n &= nx^{n-1}\\ \frac{\mathrm{d}^2}{\mathrm{d}x^2}x^n &= \frac{d}{dx}\left(nx^{n-1}\right) = n(n-1)x^{n-2}\\ \frac{\mathrm{d}^3}{\mathrm{d}x^3}x^n &= \frac{d}{dx}\left(n(n-1)x^{n-2}\right) = n(n-1)(n-2)x^{n-3}\end{aligned}$$

Each time we differentiate, we bring down the exponent, which is exactly one smaller than the previous exponent brought down, and we reduce the exponent by one. By the time we have differentiated $n-1$ times, the exponent has decreased to $n-(n-1)=1$ and we have brought down the factors $n(n-1)(n-2)\cdots 2$. So

$$\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}x^n = n(n-1)(n-2)\cdots 2x$$

and

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}x^n = n(n-1)(n-2)\cdots 1$$

The product of the first n natural numbers, $1\cdot 2\cdot 3\cdot\cdots\cdot n$, is called "$n$ factorial" and is denoted $n!$. So we can also write

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}x^n = n!$$

If $m > n$, then

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}x^n = 0$$

Example 2.14.3 Position, velocity and acceleration.

Recall that the derivative $v'(a)$ is the (instantaneous) rate of change of the function $v(t)$ at $t=a$. Suppose that you are walking on the x–axis and that $x(t)$ is your x–coordinate at time t. Also suppose, for simplicity, that you are moving from left to right. Then $v(t)=x'(t)$ is your velocity at time t and $v'(a)=x''(a)$ is the rate at which your velocity is changing at time $t=a$. It is called your acceleration. In particular, if $x''(a)>0$, then your velocity is increasing, i.e. you are speeding up, at time a. If $x''(a)<0$, then your velocity is decreasing, i.e. you are slowing down, at time a. That's one interpretation of the second derivative.

Example 2.14.4 2.11.1 continued.

Find y'' if $y = y^3 + xy + x^3$.

Solution This problem concerns some function $y(x)$ that is not given to us explicitly. All that we are told is that $y(x)$ satisfies

$$y(x) = y(x)^3 + xy(x) + x^3$$

for all x. We are asked to find $y''(x)$. We cannot solve this equation to get an explicit formula for $y(x)$. So we use implicit differentiation, as we did in Example 2.11.1. That is, we apply $\frac{d}{dx}$ to both sides of (E1). This gives

$$y'(x) = 3y(x)^2\, y'(x) + y(x) + x\, y'(x) + 3x^2$$

which we can solve for $y'(x)$, by moving all $y'(x)$'s to the left hand side, giving

$$\left[1 - x - 3y(x)^2\right] y'(x) = y(x) + 3x^2$$

and then dividing across.

$$y'(x) = \frac{y(x) + 3x^2}{1 - x - 3y(x)^2}$$

To get $y''(x)$, we have two options.

Method 1. Apply $\frac{d}{dx}$ to both sides of (E2). This gives

$$y''(x) = 3y(x)^2\, y''(x) + 6y(x)\, y'(x)^2 + 2y'(x) + x\, y''(x) + 6x$$

We can now solve for $y''(x)$, giving

$$y''(x) = \frac{6x + 2y'(x) + 6y(x)y'(x)^2}{1 - x - 3y(x)^2}$$

Then we can substitute in (E3), giving

$$\begin{aligned} y''(x) &= 2\frac{3x + \frac{y(x)+3x^2}{1-x-3y(x)^2} + 3y(x)\left(\frac{y(x)+3x^2}{1-x-3y(x)^2}\right)^2}{1 - x - 3y(x)^2} \\ &= 2\frac{3x\left[1-x-3y(x)^2\right]^2 + [y(x)+3x^2][1-x-3y(x)^2] + 3y(x)\left[y(x)+3x^2\right]^2}{\left[1 - x - 3y(x)^2\right]^3} \end{aligned}$$

Method 2. Alternatively, we can also differentiate (E3).

$$\begin{aligned} y''(x) &= \frac{[y'(x)+6x][1-x-3y(x)^2] - [y(x)+3x^2][-1-6y(x)y'(x)]}{\left[1 - x - 3y(x)^2\right]^2} \\ &= \frac{\left[\frac{y(x)+3x^2}{1-x-3y(x)^2} + 6x\right]\left[1 - x - 3y(x)^2\right] - [y(x)+3x^2][-1-6y(x)\frac{y(x)+3x^2}{1-x-3y(x)^2}]}{\left[1 - x - 3y(x)^2\right]^2} \\ &= \frac{2[y(x)+3x^2][1-x-3y(x)^2] + 6x\left[1-x-3y(x)^2\right]^2 + 6y(x)\left[y(x)+3x^2\right]^2}{\left[1 - x - 3y(x)^2\right]^3} \end{aligned}$$

Remark 1. We have now computed $y''(x)$ — sort of. The answer is in terms of $y(x)$, which we don't know. Since we cannot get an explicit formula for $y(x)$, there's not a great deal that we can do, in general.

Remark 2. Even though we cannot solve $y = y^3 + xy + x^3$ explicitly for $y(x)$, for general x, it is sometimes possible to solve equations like this for some special values of x. In fact, we saw in Example 2.11.1 that when $x = 1$, the given equation reduces to $y(1) = y(1)^3 + 1 \cdot y(1) + 1^3$, or $y(1)^3 = -1$, which we can solve to get $y(1) = -1$. Substituting into (E2), as we did in Example 2.11.1 gives

$$y'(1) = \frac{-1+3}{1-1-3(-1)^2} = -\frac{2}{3}$$

and substituting into (E4) gives

$$y''(1) = \frac{6+2\left(-\frac{2}{3}\right)+6(-1)\left(-\frac{2}{3}\right)^2}{1-1-3(-1)^2} = \frac{6-\frac{4}{3}-\frac{8}{3}}{-3} = -\frac{2}{3}$$

(It's a fluke that, in this example, $y'(1)$ and $y''(1)$ happen to be equal.) So we now know that, even though we can't solve $y = y^3+xy+x^3$ explicitly for $y(x)$, the graph of the solution passes through $(1,-1)$ and has slope $-\frac{2}{3}$ (i.e. is sloping downwards by between 30° and 45°) there and, furthermore, the slope of the graph decreases as x increases through $x=1$.



Here is a sketch of the part of the graph very near $(1,-1)$. The tangent line to the graph at $(1,-1)$ is also shown. Note that the tangent line is sloping down to the right, as we expect, and that the graph lies below the tangent line near $(1,-1)$. That's because the slope $f'(x)$ is decreasing (becoming more negative) as x passes through 1.

Warning 2.14.5.

Many people will suppress the (x) in $y(x)$ when doing computations like those in Example 2.14.4. This gives shorter, easier to read formulae, like $y' = \frac{y+3x^2}{1-x-3y^2}$. If you do this, you must never forget that y is a function of x and is *not* a constant. If you do forget, you'll make the very serious error of saying that $\dfrac{dy}{dx} = 0$, which is false.

Exercises

Stage 1

Exercise 1.7.1

What is the 180th derivative of the function $f(x) = e^x$?

Exercise 1.7.2

Suppose $f(x)$ is a differentiable function, with $f'(x) > 0$ and $f''(x) > 0$ for every $x \in (a,b)$. Which of the following must be true?

1. $f(x)$ is positive over (a,b)
2. $f(x)$ is increasing over (a,b)
3. $f(x)$ is increasing at a constant rate over (a,b)
4. $f(x)$ is increasing faster and faster over (a,b)
5. $f''''(x) > 0$ for some $x \in (a,b)$

Exercise 1.7.3

Let $f(x) = ax^{15}$ for some constant a. Which value of a results in $f^{(15)}(x) = 3$?

Exercise 1.7.4

Find the mistake(s) in the following work, and provide a corrected answer.

> *Suppose* $-14x^2+2xy+y^2 = 1$. *We find* $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ *at the point* $(1,3)$. *Differentiating implicitly:*
>
> $$-28x+2y+2xy'+2yy' = 0$$

Plugging in $x = 1$, $y = 3$:

$$-28 + 6 + 2y' + 6y' = 0$$
$$y' = \frac{11}{4}$$

Differentiating:

$$y'' = 0$$

Stage 2

? Exercise 1.7.5

Let $f(x) = (\log x - 1)x$. Evaluate $f''(x)$.

? Exercise 1.7.6

Evaluate $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}\{\arctan x\}$.

? Exercise 1.7.7

The unit circle consists of all point $x^2 + y^2 = 1$. Give an expression for $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ in terms of y.

? Exercise 1.7.8

Suppose the position of a particle at time t is given by $s(t) = \dfrac{e^t}{t^2 + 1}$. Find the acceleration of the particle at time $t = 1$.

? Exercise 1.7.9

Evaluate $\dfrac{\mathrm{d}^3}{\mathrm{d}x^3}\{\log(5x^2 - 12)\}$.

? Exercise 1.7.10

The height of a particle at time t seconds is given by $h(t) = -\cos t$. Is the particle speeding up or slowing down at $t = 1$?

? Exercise 1.7.11

The height of a particle at time t seconds is given by $h(t) = t^3 - t^2 - 5t + 10$. Is the particle's motion getting faster or slower at $t = 1$?

? Exercise 1.7.12

Suppose a curve is defined implicitly by

$$x^2 + x + y = \sin(xy)$$

What is $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ at the point $(0, 0)$?

? Exercise 1.7.13

Which statements below are true, and which false?

1. $\dfrac{\mathrm{d}^4}{\mathrm{d}x^4}\sin x = \sin x$
2. $\dfrac{\mathrm{d}^4}{\mathrm{d}x^4}\cos x = \cos x$
3. $\dfrac{\mathrm{d}^4}{\mathrm{d}x^4}\tan x = \tan x$

Stage 3

? Exercise 1.7.14

A function $f(x)$ satisfies $f'(x) < 0$ and $f''(x) > 0$ over (a, b). Which of the following curves below might represent $y = f(x)$?



? Exercise 1.7.15

Let $f(x) = 2^x$. What is $f^{(n)}(x)$, if n is a whole number?

? Exercise 1.7.16

Let $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c, and d are nonzero constants. What is the smallest integer n so that $\dfrac{\mathrm{d}^n f}{\mathrm{d}x^n} = 0$ for all x?

? Exercise 1.7.17 (✳)

$$f(x) = e^{x+x^2} \qquad\qquad h(x) = 1 + x + \frac{3}{2}x^2$$

1. Find the first and second derivatives of both functions
2. Evaluate both functions and their first and second derivatives at 0.
3. Show that for all $x > 0$, $f(x) > h(x)$.

Remark: for some applications, we only need to know that a function is "big enough." Since $f(x)$ is a difficult function to evaluate, it may be useful to know that it is bigger than $h(x)$ when x is positive.

? Exercise 1.7.18 (✳)

The equation $x^3y + y^3 = 10x$ defines y implicitly as a function of x near the point $(1, 2)$.

1. Compute y' at this point.
2. It can be shown that y'' is negative when $x = 1$. Use this fact and your answer to 2.14.2.18.a to make a sketch showing the relationship of the curve to its tangent line at $(1, 2)$.

? Exercise 1.7.19

Let $g(x) = f(x)e^x$. In Question 2.7.3.12, Section 2.7, we learned that $g'(x) = [f(x) + f'(x)]e^x$.

1. What is $g''(x)$?
2. What is $g'''(x)$?
3. Based on your answers above, guess a formula for $g^{(4)}(x)$. Check it by differentiating.

Exercise 1.7.20

Suppose $f(x)$ is a function whose first n derivatives exist over all real numbers, and $f^{(n)}(x)$ has precisely m roots. What is the maximum number of roots that $f(x)$ may have?

Exercise 1.7.21

How many roots does the function $f(x) = (x+1)\log(x+1) + \sin x - x^2$ have?

Exercise 1.7.22 (✳)

Let $f(x) = x|x|$.

1. Show that $f(x)$ is differentiable at $x = 0$, and find $f'(0)$.
2. Find the second derivative of $f(x)$. Explicitly state, with justification, the point(s) at which $f''(x)$ does not exist, if any.

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1.E: Instantaneous Rate of Change- The Derivative (Exercises)

These are homework exercises to accompany David Guichard's "General Calculus" Textmap. Complementary General calculus exercises can be found for other Textmaps and can be accessed here.

2.1: The Slope of a Function

Ex 2.1.1 Draw the graph of the function $y=f(x)=\sqrt{169-x^2}$ between $x=0$ and $x=13$. Find the slope $\Delta y/\Delta x$ of the chord between the points of the circle lying over (a) $x=12$ and $x=13$, (b) $x=12$ and $x=12.1$, (c) $x=12$ and $x=12.01$, (d) $x=12$ and $x=12.001$. Now use the geometry of tangent lines on a circle to find (e) the exact value of the derivative $f'(12)$. Your answers to (a)--(d) should be getting closer and closer to your answer to (e). (answer)

Ex 2.1.2 Use geometry to find the derivative $f'(x)$ of the function $f(x)=\sqrt{625-x^2}$ in the text for each of the following x: (a) 20, (b) 24, (c) -7, (d) -15. Draw a graph of the upper semicircle, and draw the tangent line at each of these four points. (answer)

Ex 2.1.3 Draw the graph of the function $y=f(x)=1/x$ between $x=1/2$ and $x=4$. Find the slope of the chord between (a) $x=3$ and $x=3.1$, (b) $x=3$ and $x=3.01$, (c) $x=3$ and $x=3.001$. Now use algebra to find a simple formula for the slope of the chord between $(3,f(3))$ and $(3+\Delta x,f(3+\Delta x))$. Determine what happens when Δx approaches 0. In your graph of $y=1/x$, draw the straight line through the point $(3,1/3)$ whose slope is this limiting value of the difference quotient as Δx approaches 0. (answer)

Ex 2.1.4 Find an algebraic expression for the difference quotient $\big(f(1+\Delta x)-f(1)\big)/\Delta x$ when $f(x)=x^2-(1/x)$. Simplify the expression as much as possible. Then determine what happens as Δx approaches 0. That value is $f'(1)$. (answer)

Ex 2.1.5 Draw the graph of $y=f(x)=x^3$ between $x=0$ and $x=1.5$. Find the slope of the chord between (a) $x=1$ and $x=1.1$, (b) $x=1$ and $x=1.001$, (c) $x=1$ and $x=1.00001$. Then use algebra to find a simple formula for the slope of the chord between 1 and $1+\Delta x$. (Use the expansion $(A+B)^3=A^3+3A^2B+3AB^2+B^3$.) Determine what happens as Δx approaches 0, and in your graph of $y=x^3$ draw the straight line through the point $(1,1)$ whose slope is equal to the value you just found. (answer)

Ex 2.1.6 Find an algebraic expression for the difference quotient $(f(x+\Delta x)-f(x))/\Delta x$ when $f(x)=mx+b$. Simplify the expression as much as possible. Then determine what happens as Δx approaches 0. That value is $f'(x)$. (answer)

Ex 2.1.7 Sketch the unit circle. Discuss the behavior of the slope of the tangent line at various angles around the circle. Which trigonometric function gives the slope of the tangent line at an angle \(\theta$? Why? Hint: think in terms of ratios of sides of triangles.

Ex 2.1.8 Sketch the parabola $y=x^2$. For what values of x on the parabola is the slope of the tangent line positive? Negative? What do you notice about the graph at the point(s) where the sign of the slope changes from positive to negative and vice versa?

2.2: An Example

Ex 2.2.1 An object is traveling in a straight line so that its position (that is, distance from some fixed point) is given by this table:

time (seconds)	0	1	2	3
distance (meters)	0	10	25	60

Find the average speed of the object during the following time intervals: $[0,1]$, $[0,2]$, $[0,3]$, $[1,2]$, $[1,3]$, $[2,3]$. If you had to guess the speed at $t=2$ just on the basis of these, what would you guess? (answer)

Ex 2.2.2 Let $y=f(t)=t^2$, where t is the time in seconds and y is the distance in meters that an object falls on a certain airless planet. Draw a graph of this function between $t=0$ and $t=3$. Make a table of the average speed of the falling object between (a) 2 sec and 3 sec, (b) 2 sec and 2.1 sec, (c) 2 sec and 2.01 sec, (d) 2 sec and 2.001 sec. Then use algebra to find a simple formula for the average speed between time 2 and time $2+\Delta t$. (If you substitute $\Delta t=1,\ 0.1,\ 0.01,\ 0.001$in this formula you should again get the answers to parts (a)--(d).) Next, in your formula for average speed (which should be in simplified form) determine what happens as Δt approaches zero. This is the instantaneous speed. Finally, in your graph of $y=t^2$ draw the

straight line through the point $(2,4)$ whose slope is the instantaneous velocity you just computed; it should of course be the tangent line. (answer)

Ex 2.2.3 If an object is dropped from an 80-meter high window, its height y above the ground at time t seconds is given by the formula $y = f(t) = 80 - 4.9t^2$. (Here we are neglecting air resistance; the graph of this function was shown in figure 1.0.1.) Find the average velocity of the falling object between (a) 1 sec and 1.1 sec, (b) 1 sec and 1.01 sec, (c) 1 sec and 1.001 sec. Now use algebra to find a simple formula for the average velocity of the falling object between 1 sec and $1+\Delta t$ sec. Determine what happens to this average velocity as Δt approaches 0. That is the instantaneous velocity at time $t = 1$ second (it will be negative, because the object is falling). (answer)

2.3: Limits

Compute the limits. If a limit does not exist, explain why.

Ex 2.3.1 $\lim_{x\to 3} \frac{x^2+x-12}{x-3}$ (answer)

Ex 2.3.2 $\lim_{x\to 1} \frac{x^2+x-12}{x-3}$ (answer)

Ex 2.3.3 $\lim_{x\to -4} \frac{x^2+x-12}{x-3}$ (answer)

Ex 2.3.4 $\lim_{x\to 2} \frac{x^2+x-12}{x-2}$ (answer)

Ex 2.3.5 $\lim_{x\to 1} \frac{\sqrt{x+8}-3}{x-1}$ (answer)

Ex 2.3.6 $\lim_{x\to 0^+} \sqrt{\frac{1}{x}+2} - \sqrt{\frac{1}{x}}$. (answer)

Ex 2.3.7 $\lim_{x\to 2} 3$ (answer)

Ex 2.3.8 $\lim_{x\to 4} 3x^3 - 5x$ (answer)

Ex 2.3.9 $\lim_{x\to 0} \frac{4x-5x^2}{x-1}$ (answer)

Ex 2.3.10 $\lim_{x\to 1} \frac{x^2-1}{x-1}$ (answer)

Ex 2.3.11 $\lim_{x\to 0^+} \frac{\sqrt{2-x^2}}{x}$ (answer)

Ex 2.3.12 $\lim_{x\to 0^+} \frac{\sqrt{2-x^2}}{x+1}$ (answer)

Ex 2.3.13 $\lim_{x\to a} \frac{x^3-a^3}{x-a}$ (answer)

Ex 2.3.14 $\lim_{x\to 2} (x^2+4)^3$ (answer)

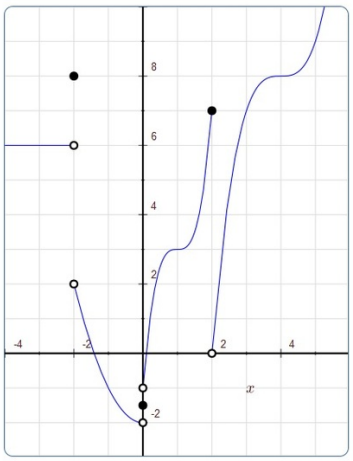
Ex 2.3.15 $\lim_{x\to 1} \begin{cases} x-5 & x \neq 1, \\ 7 & x = 1. \end{cases}$ (answer)

Ex 2.3.16 $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right)$ (Hint: Use the fact that $|\sin a| < 1$ for any real number a. You should probably use the definition of a limit here.) (answer)

Ex 2.3.17 Give an $\epsilon\$ - -\$\delta$ proof, similar to example 2.3.3, of the fact that $\lim_{x\to 4}(2x-5) = 3$.

Ex 2.3.18 Evaluate the expressions by reference to this graph:

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(a) $\lim_{x\to 4} f(x)$	(b) $\lim_{x\to -3} f(x)$	(c) $\lim_{x\to 0} f(x)$
(d) $\lim_{x\to 0^-} f(x)$	(e) $\lim_{x\to 0^+} f(x)$	(f) $f(-2)$
(g) $\lim_{x\to 2^-} f(x)$	(h) $\lim_{x\to -2^-} f(x)$	(i) $\lim_{x\to 0} f(x+1)$
(j) $f(0)$	(k) $\lim_{x\to 1^-} f(x-4)$	(l) $\lim_{x\to 0^+} f(x-2)$

(answer)

Ex 2.3.19 Use a calculator to estimate $\lim_{x\to 0} \frac{\sin x}{x}$.

Ex 2.3.20 Use a calculator to estimate $\lim_{x\to 0} \frac{\tan(3x)}{\tan(5x)}$.

2.4: The Derivative Function

Ex 2.4.1 Find the derivative of $y = f(x) = \sqrt{169 - x^2}$. (answer)

Ex 2.4.2 Find the derivative of $y = f(t) = 80 - 4.9t^2$. (answer)

Ex 2.4.3 Find the derivative of $y = f(x) = x^2 - (1/x)$. (answer)

Ex 2.4.4 Find the derivative of $y = f(x) = ax^2 + bx + c$ (where a, b, and c are constants). (answer)

Ex 2.4.5 Find the derivative of $y = f(x) = x^3$. (answer)

Ex 2.4.6 Shown is the graph of a function $f(x)$. Sketch the graph of $f'(x)$ by estimating the derivative at a number of points in the interval: estimate the derivative at regular intervals from one end of the interval to the other, and also at "special" points, as when the derivative is zero. Make sure you indicate any places where the derivative does not exist.

Ex 2.4.7 Shown is the graph of a function $f(x)$. Sketch the graph of $f'(x)$ by estimating the derivative at a number of points in the interval: estimate the derivative at regular intervals from one end of the interval to the other, and also at "special" points, as when the derivative is zero. Make sure you indicate any places where the derivative does not exist.

Ex 2.4.8 Find the derivative of $y = f(x) = 2/\sqrt{2x+1}$ (answer)

Ex 2.4.9 Find the derivative of $y = g(t) = (2t-1)/(t+2)$ (answer)

Ex 2.4.10 Find an equation for the tangent line to the graph of $f(x) = 5 - x - 3x^2$ at the point $x = 2$ (answer)

Ex 2.4.11 Find a value for a so that the graph of $f(x) = x^2 + ax - 3$ has a horizontal tangent line at $x = 4$. (answer)

2.5: Adjectives for Functions

Ex 2.5.1 Along the lines of Figure 2.5.1, for each part below sketch the graph of a function that is:

a. bounded, but not continuous.

b. differentiable and unbounded.
c. continuous at $x = 0$, not continuous at $x = 1$, and bounded.
d. differentiable everywhere except at $x = -1$, continuous, and unbounded.

Ex 2.5.2 Is $f(x) = \sin(x)$ a bounded function? If so, find the smallest M.

Ex 2.5.3 Is $s(t) = 1/(1+t^2)$ a bounded function? If so, find the smallest M.

Ex 2.5.4 Is $v(u) = 2\ln|u|$ a bounded function? If so, find the smallest M.

Ex 2.5.5 Consider the function h(x) = \cases{ 2x - 3, & if \(x < 1$\cr 0, & if \(x\geq 1\).} Show that it is continuous at the point $x = 0$. Is h a continuous function?

Ex 2.5.6 Approximate a root of $f = x^3 - 4x^2 + 2x + 2$ to one decimal place.

Ex 2.5.7 Approximate a root of $f = x^4 + x^3 - 5x + 1$ to one decimal place.

Contributors

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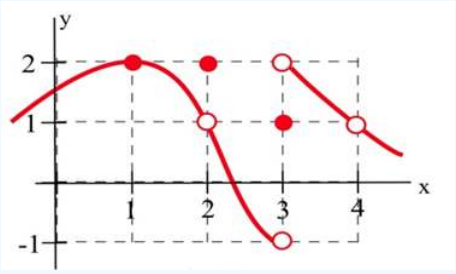
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1.E: The Derivative (Exercises)

2.1 Exercises

Exercise 1.E.1

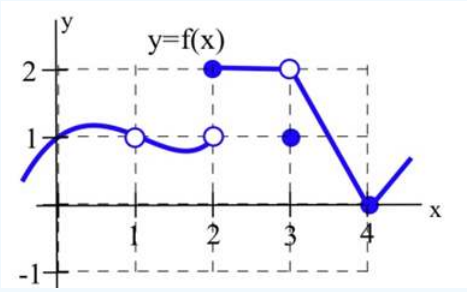
Use the graph to determine the following limits.



a. $\lim_{x\to 1} f(x)$	b. $\lim_{x\to 2} f(x)$
c. $\lim_{x\to 3} f(x)$	d. $\lim_{x\to 4} f(x)$

Exercise 1.E.2

Use the graph to determine the following limits.



a. $\lim_{x\to 1} f(x)$	b. $\lim_{x\to 2} f(x)$
c. $\lim_{x\to 3} f(x)$	d. $\lim_{x\to 4} f(x)$

Exercise 1.E.3

Evaluate

a. $\lim_{x\to 1} \frac{x^2+3x+3}{x-2}$	b. $\lim_{x\to 2} \frac{x^2+3x+3}{x-2}$

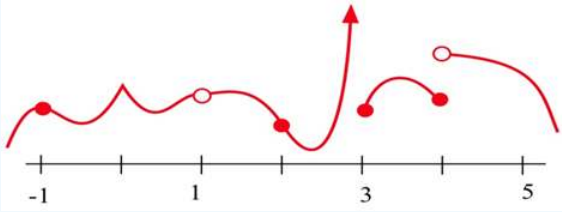
Exercise 1.E.4

Evaluate

a. $\lim_{x\to 0} \frac{x+7}{x^2+9x+14}$	b. $\lim_{x\to 3} \frac{x+7}{x^2+9x+14}$
c. $\lim_{x\to 4} \frac{x+7}{x^2+9x+14}$	d. $\lim_{x\to 7} \frac{x+7}{x^2+9x+14}$

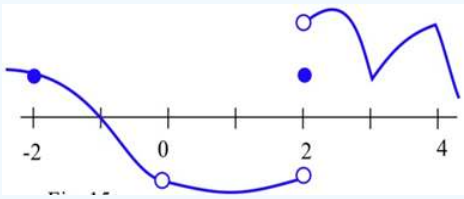
Exercise 1.E.5

At which points is the function shown discontinuous?



Exercise 1.E.6

At which points is the function shown discontinuous?



Exercise 1.E.7

Find at least one point at which each function is not continuous and state which of the 3 conditions in the definition of continuity is violated at that point

a. $\frac{x+5}{x-3}$	b. $\frac{x^2+x-6}{x-2}$	c. $\frac{x}{x}$
d. $\frac{\pi}{x^2-6x+9}$	e. $\ln(x^2)$	

2.2 Exercises

Exercise 1.E.1

What is the slope of the line through (3,9) and (x, y) for $y = x^2$ and $x = 2.97$? $x = 3.001$? $x = 3 + h$? What happens to this last slope when h is very small (close to 0)? Sketch the graph of $y = x^2$ for x near 3.

Exercise 1.E.2

What is the slope of the line through (–2,4) and (x, y) for $y = x^2$ and $x = -1.98$? $x = -2.03$? $x = -2 + h$? What happens to this last slope when h is very small (close to 0)? Sketch the graph of $y = x^2$ for x near –2.

Exercise 1.E.3

What is the slope of the line through (2,4) and (x, y) for $y = x^2 + x - 2$ and $x = 1.99$?

$x = 2.004$? $x = 2 + h$? What happens to this last slope when h is very small? Sketch the graph of $y = x^2 + x - 2$ for x near 2.

Exercise 1.E.4

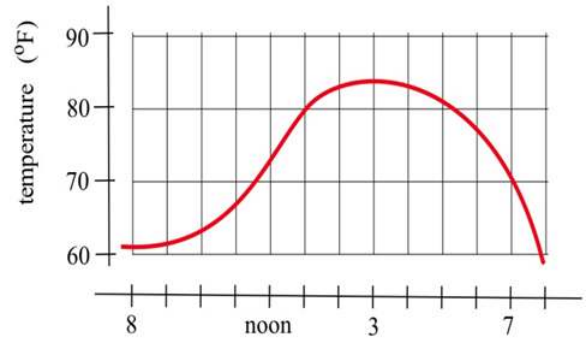
What is the slope of the line through (–1,–2) and (x, y) for $y = x^2 + x - 2$ and $x = -.98$?

$x = -1.03$? $x = -1 + h$? What happens to this last slope when h is very small? Sketch the graph of $y = x^2 + x - 2$ for x near –1.

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Exercise 1.E.5

The graph to the right shows the temperature during a day in Ames.

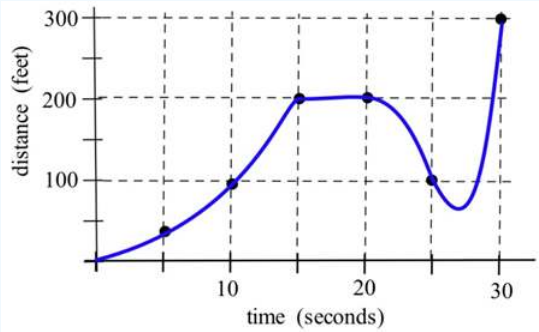


(a) What was the average change in temperature from 9 am to 1 pm?

(b) Estimate how fast the temperature was rising **at** 10 am and **at** 7 pm?

Exercise 1.E.6

The graph shows the distance of a car from a measuring position located on the edge of a straight road.



(a) What was the average velocity of the car from $t = 0$ to $t = 30$ seconds?

(b) What was the average velocity of the car from $t = 10$ to $t = 30$ seconds?

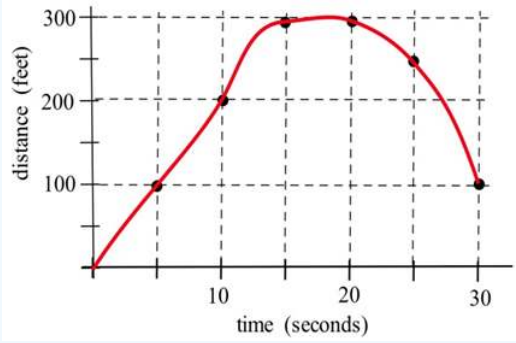
(c) About how fast was the car traveling **at** $t = 10$ seconds? **at** $t = 20$ s? **at** $t = 30$ s?

(d) What does the horizontal part of the graph between $t = 15$ and $t = 20$ seconds mean?

(e) What does the negative velocity at $t = 25$ represent?

Exercise 1.E.7

The graph shows the distance of a car from a measuring position located on the edge of a straight road.



(a) What was the average velocity of the car from $t = 0$ to $t = 20$ seconds?

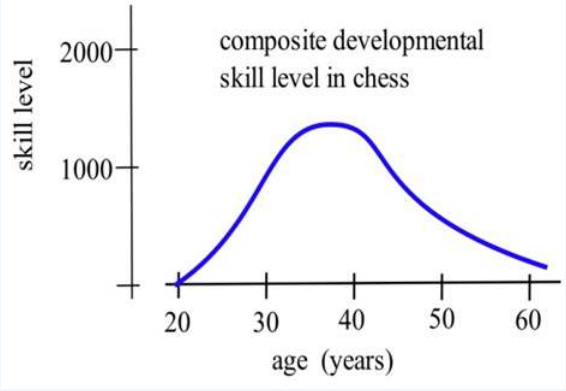
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(b) What was the average velocity from $t = 10$ to $t = 30$ seconds?

(c) About how fast was the car traveling **at** $t = 10$ seconds? **at** $t = 20$ s? **at** $t = 30$ s?

Exercise 1.E.8

The graph shows the composite developmental skill level of chessmasters at different ages as determined by their performance against other chessmasters. (From "Rating Systems for Human Abilities", by W.H. Batchelder and R.S. Simpson, 1988. UMAP Module 698.)



(a) At what age is the "typical" chessmaster playing the best chess?

(b) At approximately what age is the chessmaster's skill level increasing most rapidly?

(c) Describe the development of the "typical" chessmaster's skill in words.

(d) Sketch graphs which you **think** would reasonably describe the performance levels versus age for an athlete, a classical pianist, a rock singer, a mathematician, and a professional in your major field.

Exercise 1.E.9

Use the function in the graph to fill in the table and then graph $m(x)$.

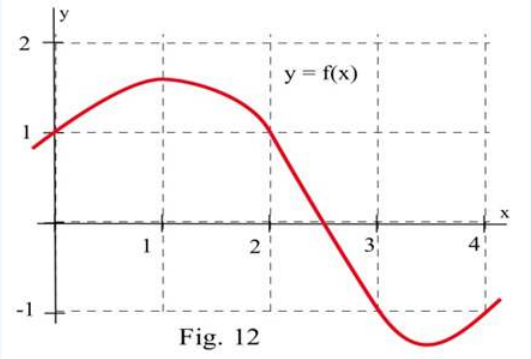


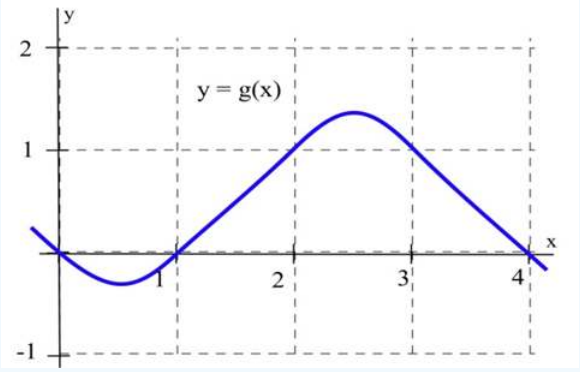
Fig. 12

x	$y = f(x)$	$m(x) =$ the estimated slope of the tangent line to $y = f(x)$ at the point (x, y)
0		
0.5		
1.0		
1.5		
2.0		

2.5		
3.0		
3.5		
4.0		

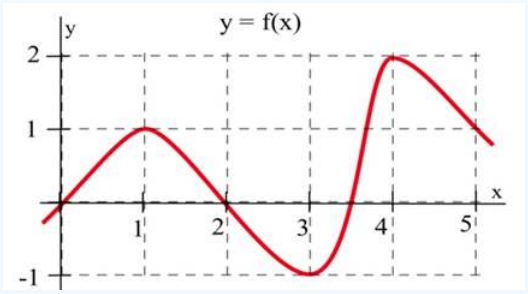
Exercise 1.E.10

Use the function in the graph to fill in the table and then graph $m(x)$.



x	$y = g(x)$	$m(x) =$ the estimated slope of the tangent line to $y = g(x)$ at the point (x, y)
0		
0.5		
1.0		
1.5		
2.0		
2.5		
3.0		
3.5		
4.0		

Exercise 1.E.11



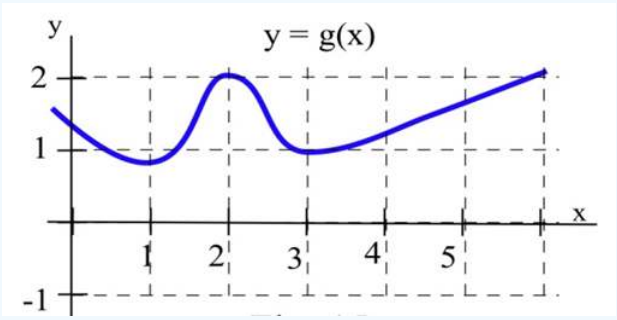
(a) At what values of x does the graph of f in the graph have a horizontal tangent line?

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(b) At what value(s) of x is the value of f the largest? smallest?

(c) Sketch the graph of $m(x)$ = the slope of the line tangent to the graph of f at the point (x, y)

Exercise $1.E.12$



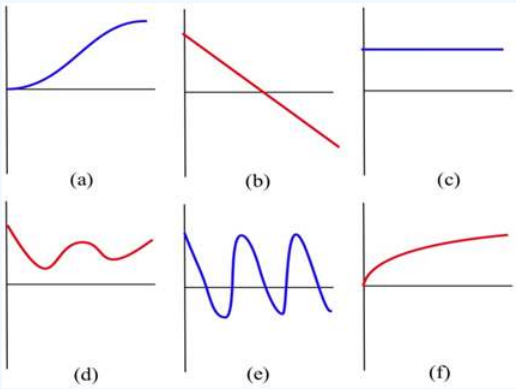
(a) At what values of x does the graph of g have a horizontal tangent line?

(b) At what value(s) of x is the value of g the largest? smallest?

(c) Sketch the graph of $m(x) =$ the slope of the line tangent to the graph of g at the point (x, y).

Exercise $1.E.13$

Match the situation descriptions with the corresponding **time–velocity** graph.



(a) A car quickly leaving from a stop sign.

(b) A car sedately leaving from a stop sign.

(c) A student bouncing on a trampoline.

(d) A ball thrown straight up.

(e) A student confidently striding across campus to take a calculus test.

(f) An unprepared student walking across campus to take a calculus test.

Exercise $1.E.14 - 1.E.19$

For each function $f(x)$ in problems 14 – 19, perform steps (a) – (d):

(a) calculate $m_{\text{sec}} = \frac{f(x+h)-f(x)}{h}$ and simplify

(b) determine $m_{\tan} = \lim_{h\to 0} m_{\text{sec}}$

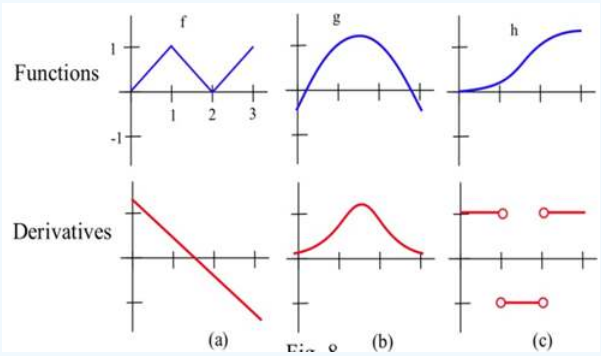
(c) evaluate $m_{\tan}$ at $x = 2$,

(d) find the equation of the line tangent to the graph of f at $(2, f(2))$

14. $f(x) = 3x - 7$	15. $f(x) = 2 - 7x$	16. $f(x) = ax + b$ where a and b are constants
17. $f(x) = x^2 + 3x$	18. $f(x) = 8 - 3x^2$	19. $f(x) = ax^2 + bx + c$ where a, b and c are constants

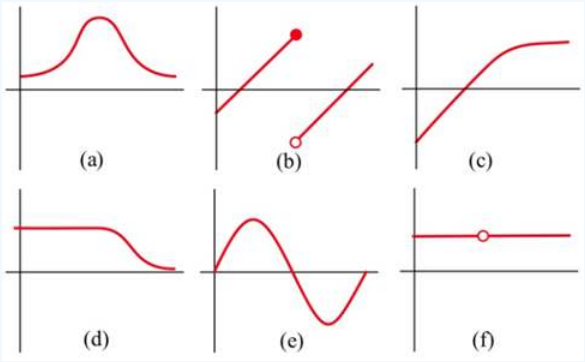
Exercise 1.E.20

Match the graphs of the three functions below with the graphs of their derivatives.



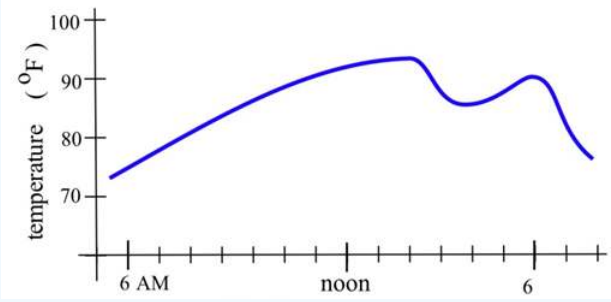
Exercise 1.E.21

Below are six graphs, three of which are derivatives of the other three. Match the functions with their derivatives.



Exercise 1.E.22

The graph below shows the temperature during a summer day in Chicago. Sketch the graph of the **rate** at which the temperature is changing. (This is just the graph of the **slopes** of the lines which are tangent to the temperature graph.)



Exercise 1.E. 23

Fill in the table with the appropriate units for $f'(x)$.

units for x	units for $f(x)$	units for $f'(x)$
hours	miles	
people	automobiles	
dollars	pancakes	
days	trout	
seconds	miles per second	
seconds	gallons	
study hours	test points	

Exercise 1.E. 24

If $C(x)$ is the total cost, in millions, of producing x thousand items, interpret $C'(4) = 2$.

Exercise 1.E. 25

Suppose $P(t)$ is the number of individuals infected by a disease t days after it was first detected. Interpret $P'(50) = -200$.

2.3 Exercises

Exercise 1.E. 1

Fill in the values in the table for $\frac{d}{dx}(3(f(x))$, $\frac{d}{dx}(2f(x)+g(x))$, and $\frac{d}{dx}(3(g(x)-f(x))$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$	$\frac{d}{dx}(3(f(x))$	$\frac{d}{dx}(2f(x)+g(x)$	$\frac{d}{dx}(3(g(x)-f(x))$
0	3	-2	-4	3			
1	2	-1	1	0			
2	4	2	3	1			

Exercise 1.E. 2

Find

(a) $D(x^{12})$

(b) $\frac{d}{dx}(\sqrt[7]{x})$

(c) $D(\frac{1}{x^3})$

(d) $\frac{dx^e}{dx}$

Exercise 1.E. 3

Find

(a) $D(x^9)$

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(b) $\frac{dx^{2/3}}{dx}$

(c) $D(\frac{1}{x^4})$

(d) $D(x^{\pi})$

Exercise $1.E.4 - 1.E.8$

In problems 4 – 8, (a) calculate $f'(1)$ and (b) determine when $f'(x) = 0$.

4. $f(x) = x^2 - 5x + 13$
5. $f(x) = 5x^2 - 40x + 73$
6. $f(x) = x^3 + 9x^2 + 6$
7. $f(x) = x^3 + 3x^2 + 3x - 1$
8. $f(x) = x^3 + 2x^2 + 2x - 1$

Exercise $1.E.9$

Where do $f(x) = x^2 - 10x + 3$ and $g(x) = x^3 - 12x$ have horizontal tangent lines?

Exercise $1.E.10$

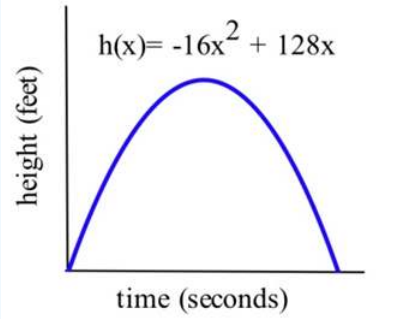
It takes $T(x) = x^2$ hours to weave x small rugs. What is the marginal production time to weave a rug? (Be sure to include the units with your answer.)

Exercise $1.E.11$

It costs $C(x) = \sqrt{x}$ dollars to produce x golf balls. What is the marginal production cost to make a golf ball? What is the marginal production cost when $x = 25$? when $x = 100$? (Include units.)

Exercise $1.E.12$

An arrow shot straight up from ground level with an initial velocity of 128 feet per second will be at height $h(x) = -16x^2 + 128x$ feet at x seconds.



(a) Determine the velocity of the arrow when $x = 0$, 1 and 2 seconds.

(b) What is the velocity of the arrow, $v(x)$, at any time x?

(c) At what time x will the velocity of the arrow be 0?

(d) What is the greatest height the arrow reaches?

(e) How long will the arrow be aloft?

(f) Use the answer for the velocity in part (b) to determine the acceleration, $a(x) = v'(x)$, at any time x.

Exercise 1.E.13

If an arrow is shot straight up from ground level on the moon with an initial velocity of 128 feet per second, its height will be $h(x) = -2.65x^2 + 128x$ feet at x seconds. Do parts (a) – (e) of problem 40 using this new equation for h.

Exercise 1.E.14

$f(x) = x^3 + Ax^2 + Bx + C$ with constants A, B and C. Can you find conditions on the constants A, B and C which will guarantee that the graph of $y = f(x)$ has two distinct "vertices"? (Here a "vertex" means a place where the curve changes from increasing to decreasing or from decreasing to increasing.)

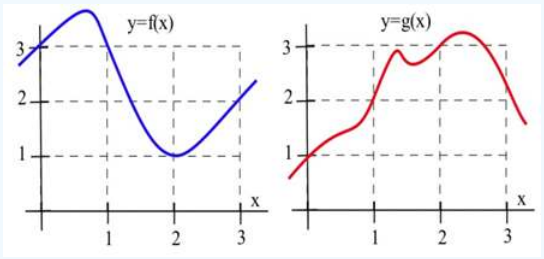
2.4 Exercises

Exercise 1.E.1

Use the values in the table to fill in the rest of the table.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$	$\frac{d}{dx}(f(x)\cdot g(x))$	$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)$	$\frac{d}{dx}\left(\frac{g(x)}{f(x)}\right)$
0	3	-2	-4	3			
1	2	-1	1	0			
2	4	2	3	1			

Exercise 1.E.2 – 1.E.3



2. Use the information in the graph to plot the values of the functions $f+g$, $f\cdot g$ and f/g and their derivatives at $x = 1$, 2 and 3.

3. Use the information in the graph to plot the values of the functions $2f$, $f-g$ and g/f and their derivatives at $x = 1$, 2 and 3.

Exercise 1.E.4

Calculate $\frac{d}{dx}((x-5)(3x+7))$ by (a) using the product rule and (b) expanding the product and then differentiating. Verify that both methods give the same result.

Exercise 1.E.5

If the product of f and g is a constant ($f(x)\cdot g(x) = k$ for all x), then how are $\frac{\frac{d}{dx}(f(x))}{f(x)}$ and $\frac{\frac{d}{dx}(g(x))}{g(x)}$ related?

Exercise 1.E.6

If the quotient of f and g is a constant ($\frac{f(x)}{g(x)} = k$ for all x), then how are $g\cdot f'$ and $f\cdot g'$ related?

Exercise $1.E.7 - 1.E.8$

In problems 7 – 8, (a) calculate $f'(1)$ and (b) determine when $f'(x) = 0$

7. $f(x) = \frac{7x}{x^2+4}$

8. $f(x) = \frac{3x^2}{2x-3}$

Exercise $1.E.9$

Determine $\frac{d}{dx}(x^2+1)(7x-3)$ and $\frac{d}{dt}(\frac{3t-2}{5t+1})$.

Exercise $1.E.10$

Find (a) $\frac{d}{dx}(x^3e^x)$ and (b) $\frac{d}{dx}(e^x)^3$.

Exercise $1.E.11$

Find (a) $\frac{d}{dt}(te^t)$, (b) $d(e^x)^5$

Exercise $1.E.12$

A manufacturer has determined that an employee with d days of production experience will be able to produce approximately $P(d) = 3 + 15(1 - e^{-0.2d})$ items per day. Graph $P(d)$.

(a) Approximately how many items will a beginning employee be able to produce each day?

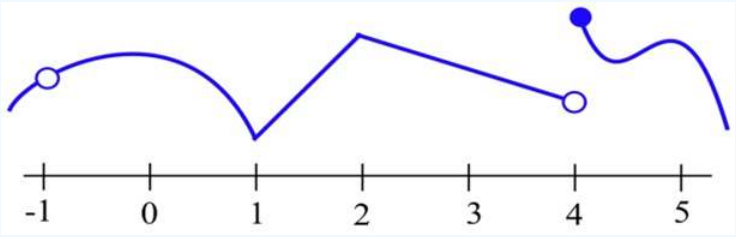
(b) How many items will an experienced employee be able to produce each day?

(c) What is the marginal production rate of an employee with 5 days of experience? (What are the units of your answer, and what does this answer mean?

2.5 Exercises

Exercise $1.E.1$

The graph of $y = f(x)$ is shown.



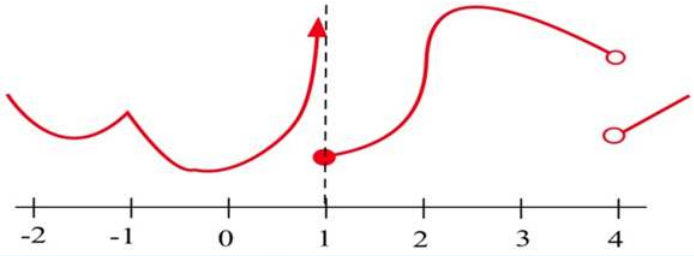
(a) At which integers is f continuous?

(b) At which integers is f differentiable?

Exercise $1.E.2$

The graph of $y = g(x)$ is shown.

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(a) At which integers is g continuous?

(b) At which integers is g differentiable?

Exercise $1.E.3 - 1.E.4$

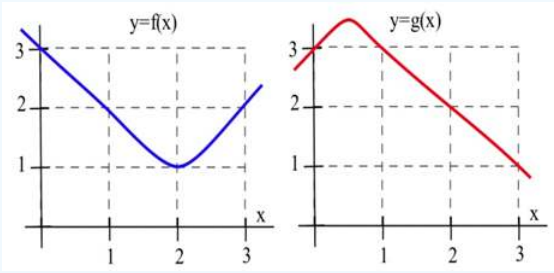
Problems 3 and 4 refer to the values given in this table:

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$	$(f \circ g)(x)$	$(f \circ g)'(x)$
-2	2	-1	1	1		
-1	1	2	0	2		
0	-2	1	2	-1		
1	0	-2	-1	2		
2	1	0	1	-1		

3. Use the table of values to determine $(f \circ g)(x)$ and $(f \circ g)'(x)$ at $x = 1$ and 2.

4. Use the table of values to determine $(f \circ g)(x)$ and $(f \circ g)'(x)$ at $x =$ –2, –1 and 0.

Exercise $1.E.5 - 1.E.6$



5. Use the graphs to estimate the values of $g(x)$, $\mathbf{g'(x)}$, $(f \circ g)(x)$, $\mathbf{f'(g(x))}$, and $\mathbf{(f \circ g)'(x)}$ at $x = 1$.

6. Use the graphs to estimate the values of $g(x)$, $\mathbf{g'(x)}$, $(f \circ g)(x)$, $\mathbf{f'(g(x))}$, and $\mathbf{(f \circ g)'(x)}$ for $x = 2$.

Exercise $1.E.7 - 1.E.12$

In problems 7 – 12, find the derivative of each function.

7. $f(x) = (2x-8)^5$	8. $f(x) = (6x-x^2)^{10}$	9. $f(x) = x \cdot (3x+7)^5$
10. $f(x) = (2x+3)^6 \cdot (x-2)^4$	11. $f(x) = \sqrt{x^2+6x-1}$	12. $f(x) = \frac{x-5}{(x+3)^4}$

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Exercise 1.E.13

If f is a differentiable function,

(a) how are the graphs of $y = f(x)$ and $y = f(x) + k$ related?

(b) how are the derivatives of $f(x)$ and $f(x) + k$ related?

2.6 Exercises

Exercise 1.E.1 – 1.E.2

In problems 1 and 2, each quotation is a statement about a quantity of something changing over time. Let $f(t)$ represent the quantity at time t. For each quotation, tell what f represents and whether the first and second derivatives of f are positive or negative.

1. (a) "Unemployment rose again, but the rate of increase is smaller than last month."

(b) "Our profits declined again, but at a slower rate than last month."

(c) "The population is still rising and at a faster rate than last year."

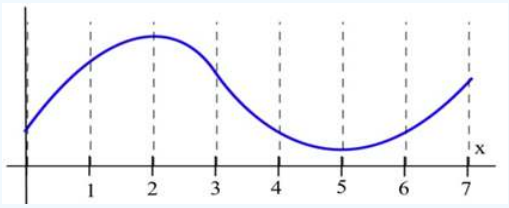
2. (a) "The child's temperature is still rising, but slower than it was a few hours ago."

(b) "The number of whales is decreasing, but at a slower rate than last year."

(c) "The number of people with the flu is rising and at a faster rate than last month."

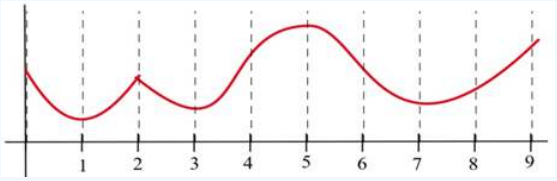
Exercise 1.E.3

On which intervals is the function in the graph (a) concave up? (b) concave down?



Exercise 1.E.4

On which intervals is the function in graph (a) concave up? (b) concave down?



Exercise 1.E.5

Sketch the graphs of functions which are defined and concave up everywhere and which have

(a) no roots.

(b) exactly 1 root.

(c) exactly 2 roots.

(d) exactly 3 roots.

Exercise 1.E.6 – 1.E.9

In problems 6 – 9, a function and values of x so that $f'(x) = 0$ are given. Use the Second Derivative Test to determine whether each point $(x, f(x))$ is a local maximum, a local minimum or neither

6. $f(x) = 2x^3 - 15x^2 + 6, x = 0, 5$.

7. $g(x) = x^3 - 3x^2 - 9x + 7, x = -1, 3$.

8. $h(x) = x^4 - 8x^2 - 2, x = -2, 0, 2$.

9. $f(x) = x \cdot \ln(x), x = 1/e$.

Exercise 1.E.10

Which of the labeled points in the graph are inflection points?

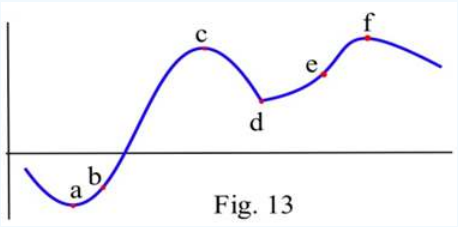
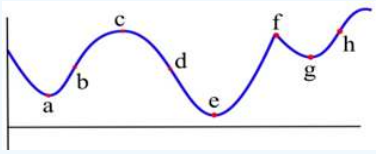


Fig. 13

Exercise 1.E.11

Which of the labeled points in the graph are inflection points?



Exercise 1.E.12

How many inflection points can a

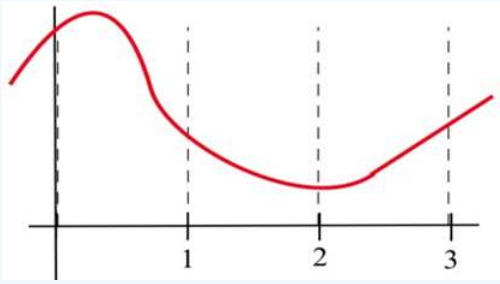
(a) quadratic polynomial have?

(b) cubic polynomial have?

(c) polynomial of degree n have?

Exercise 1.E.13

Fill in the table with "+", "–", or "0" for the function shown.



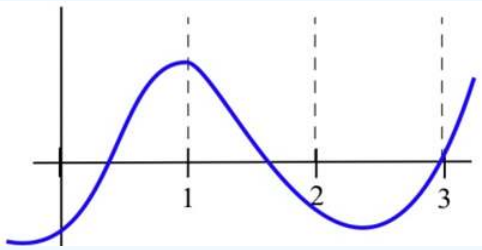
x	$f(x)$	$f'(x)$	$f''(x)$

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0			
1			
2			
3			

Exercise 1.E.14

Fill in the table with "+", "–", or "0" for the function shown.



x	$g(x)$	$g'(x)$	$g''(x)$
0			
1			
2			
3			

Exercise 1.E.15 – 1.E.21

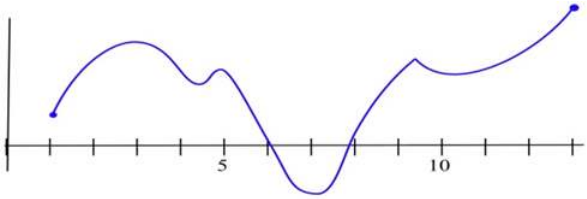
In problems 15 – 21, find the derivative and second derivative of each function.

15. $f(x) = 7x^2 + 5x - 3$
16. $f(x) = (2x - 8)^5$
17. $f(x) = (6x - x^2)^{10}$
18. $f(x) = x \cdot (3x + 7)^5$
19. $f(x) = (2x^3 + 3)^6$
20. $f(x) = \sqrt{x^2 + 6x - 1}$
21. $f(x) = \ln(x^2 + 4)$

2.7 Exercises

Exercise 1.E.1

Find all of the critical points of the function shown and identify them as local max, local min, or neither. Find the global max and min on the interval.



Exercise 1.E.2

Find all of the critical points of the function shown and identify them as local max, local min, or neither. Find the global max and min on the interval.

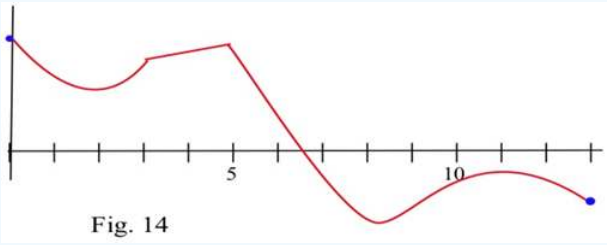


Fig. 14

Exercise 1.E.3 – 1.E.8

In problems 3 – 8, find all of the critical points and local maximums and minimums of each function.

3. $f(x) = x^2 + 8x + 7$	4. $f(x) = 2x^2 - 12x + 7$
5. $f(x) = x^3 - 6x^2 + 5$	6. $f(x) = (x-1)^2(x-3)$
7. $f(x) = \ln(x^2 - 6x + 11)$	8. $f(x) = 2x^3 - 96x + 42$

Exercise 1.E.9 – 1.E.16

In problems 9 – 16, find all critical points and global extremes of each function on the given intervals.

9. $f(x) = x^2 - 6x + 5$ on the entire real number line.
10. $f(x) = 2 - x^3$ on the entire real number line.
11. $f(x) = x^3 - 3x + 5$ on the entire real number line.
12. $f(x) = x - e^x$ on the entire real number line.
13. $f(x) = x^2 - 6x + 5$ on [–2, 5].
14. $f(x) = 2 - x^3$ on [–2, 1].
15. $f(x) = x^3 - 3x + 5$ on [–2, 1].
16. $f(x) = x - e^x$ on [1, 2].

Exercise 1.E.17

Suppose $f(1) = 5$ and $f'(1) = 0$. What can we conclude about the point (1,5) if

(a) $f'(x) < 0$ for $x < 1$, and $f'(x) > 0$ for $x > 1$?

(b) $f'(x) < 0$ for $x < 1$, and $f'(x) < 0$ for $x > 1$?

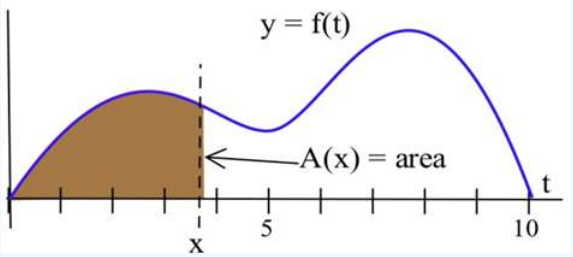
(c) $f'(x) > 0$ for $x < 1$, and $f'(x) < 0$ for $x > 1$?

(d) $f'(x) > 0$ for $x < 1$, and $f'(x) > 0$ for $x > 1$?

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Exercise 1.E.18

Define $A(x)$ to be the **area** bounded between the x–axis, the graph of f, and a vertical line at x.

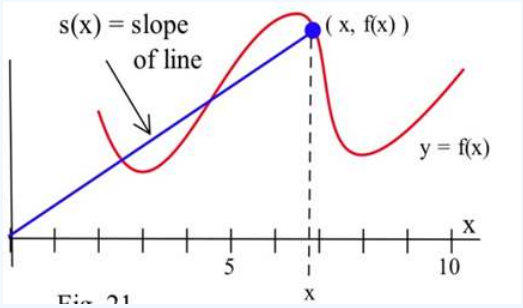


(a) At what value of x is $A(x)$ minimum?

(b) At what value of x is $A(x)$ maximum?

Exercise 1.E.19

Define $S(x)$ to be the **slope** of the line through the points $(0, 0)$ and $(x, f(x))$ based on the graph of f shown.

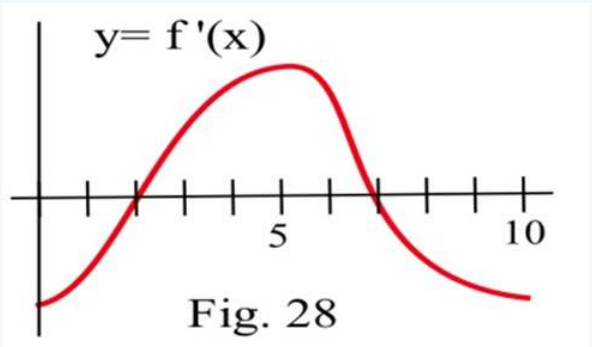


(a) At what value of x is $S(x)$ minimum?

(b) At what value of x is $S(x)$ maximum?

Exercise 1.E.20

The graph of the **derivative** of a continuous function f.



(a) List the critical numbers of f.

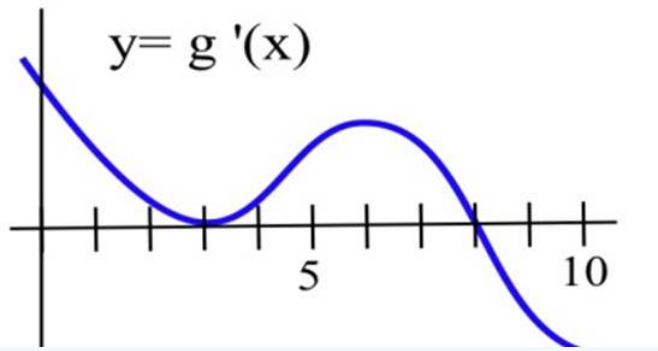
(b) For what values of x does f have a local maximum?

(c) For what values of x does f have a local minimum?

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Exercise 1.E.21

The graph of the **derivative** of a continuous function g.



(a) List the critical numbers of g.

(b) For what values of x does g have a local maximum?

(c) For what values of x does g have a local minimum?

Exercise 1.E.22 – 1.E.24

In problems 22 – 24, a function and values of x so that $f'(x) = 0$ are given. Use the Second Derivative Test to determine whether each point $(x, f(x))$ is a local maximum, a local minimum or neither

22. $f(x) = 2x^3 - 15x^2 + 6, x = 0, 5$.

23. $g(x) = x^3 - 3x^2 - 9x + 7, x = -1, 3$.

24. $h(x) = x^4 - 8x^2 - 2, x = -2, 0, 2$.

2.8 Exercises

Exercise 1.E.1

Sketch the graph of a continuous function f so that

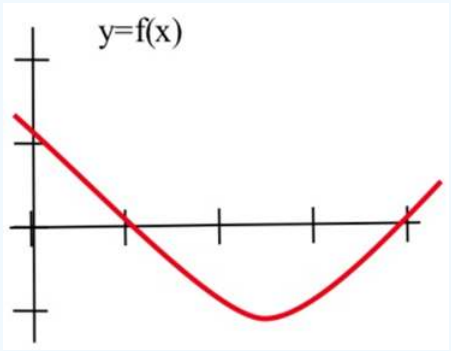
(a) $f(1) = 3$, $f'(1) = 0$, and the point (1,3) is a local maximum of f.

(b) $f(2) = 1$, $f'(2) = 0$, and the point (2,1) is a local minimum of f.

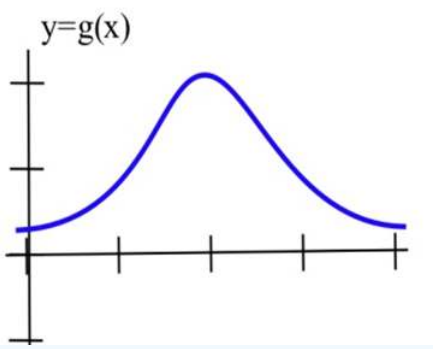
(c) $f(5) = 4$, $f'(5) = 0$, and the point (5,4) is not a local minimum or maximum of f.

Exercise 1.E.2 – 1.E.4

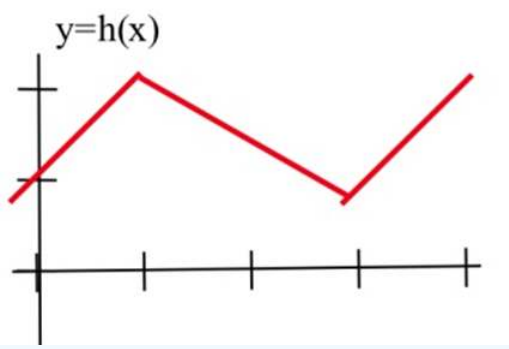
In problems 2–4, sketch the graph of the derivative of each function.



2.



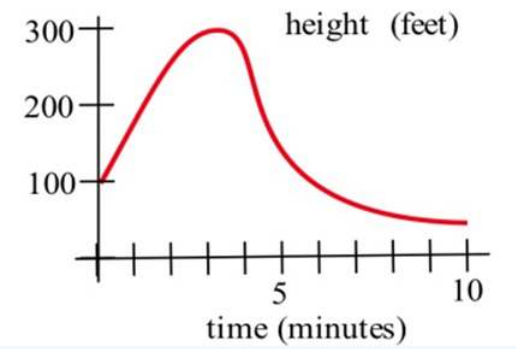
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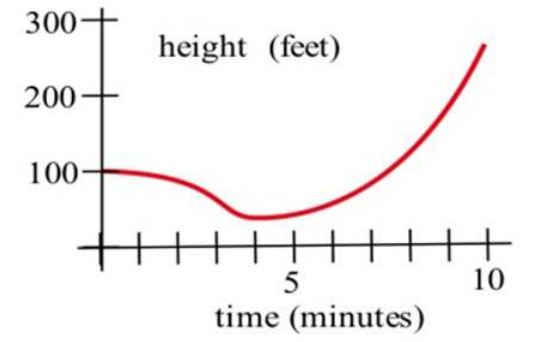
4.

Exercise $1.E.5 - 1.E.7$

In problems 5–7, the graph of the height of a helicopter is shown. Sketch the graph of the upward velocity of the helicopter.

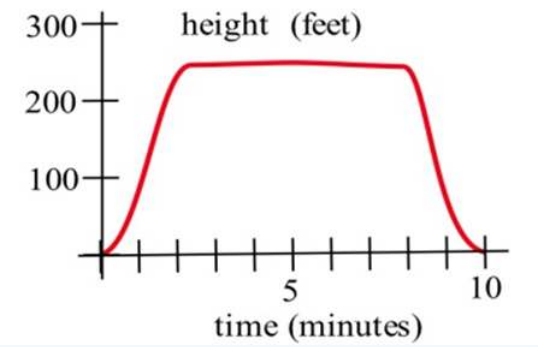


5.



6.

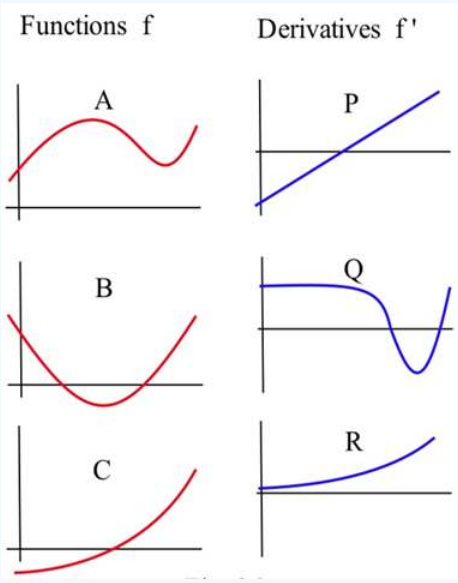
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7.

Exercise 1.E.8

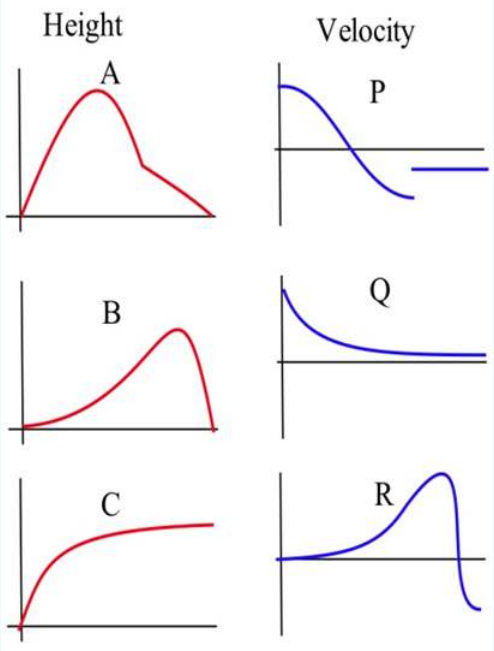
In the graphs to the right, match the graphs of the functions with those of their derivatives



Exercise 1.E.9

In the graphs below, match the graphs showing the heights of rockets with those showing their velocities.

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? Exercise $1.E.10 - 1.E.14$

In problems 10 – 14, use information from the derivatives of each function to help you graph the function. Find all local maximums and minimums of each function.

10. $f(x) = x^3 - 3x^2 - 9x - 5$	11. $g(x) = 2x^3 - 15x^2 + 6$	12. $h(x) = x^4 - 8x^2 + 3$
13. $r(t) = \frac{2}{t^2+1}$	14. $f(x) = \frac{x^2+3}{x}$	

2.9 Exercises

? Exercise $1.E.1$

(a) You have 200 feet of fencing available to construct a rectangular pen with a fence divider down the middle (see below). What dimensions of the pen enclose the largest total area?

(b) If you need 2 dividers, what dimensions of the pen enclose the largest area?

(c) What are the dimensions in parts (a) and (b) if one edge of the pen borders on a river and does not require any fencing?



? Exercise $1.E.2$

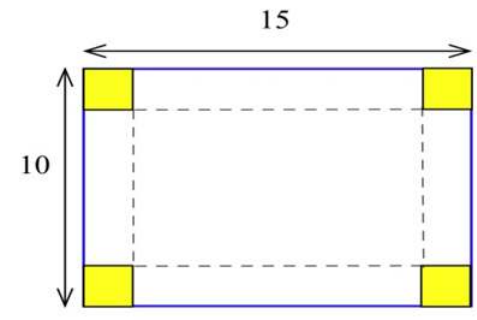
You have 120 feet of fencing to construct a pen with 4 equal sized stalls. If the pen is rectangular and shaped like the one below, what are the dimensions of the pen of largest area and what is that area?



Exercise 1.E.3

Suppose you decide to fence the rectangular garden in the corner of your yard. Then two sides of the garden are bounded by the yard fence which is already there, so you only need to use the 80 feet of fencing to enclose the other two sides. What are the dimensions of the new garden of largest area? What are the dimensions of the rectangular garden of largest area in the corner of the yard if you have F feet of new fencing available?

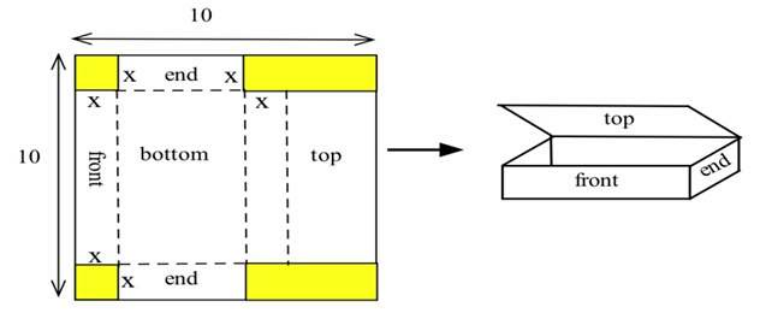
Exercise 1.E.4



(a) You have a 10 inch by 15 inch piece of tin which you plan to form into a box (without a top) by cutting a square from each corner and folding up the sides. How much should you cut from each corner so the resulting box has the greatest volume?

(b) If the piece of tin is A inches by B inches, how much should you cut from each corner so the resulting box has the greatest volume?

Exercise 1.E.5



You have a 10 inch by 10 inch piece of cardboard which you plan to cut and fold as shown to form a box with a top. Find the dimensions of the box which has the largest volume.

Exercise 1.E.6

(a) You have been asked to bid on the construction of a square-bottomed box with no top which will hold 100 cubic inches of water. If the bottom and sides are made from the same material, what are the dimensions of the box which uses the least material? (Assume that no material is wasted.)

(b) Suppose the box in part (a) uses different materials for the bottom and the sides. If the bottom material costs 5¢ per square inch and the side material costs 3¢ per square inch, what are the dimensions of the least expensive box which will hold 100 cubic inches of water?

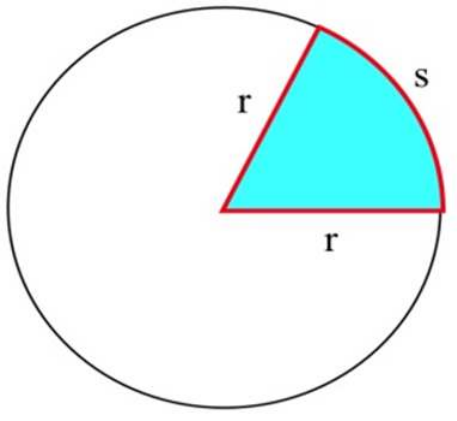
Exercise 1.E.7

(a) Determine the dimensions of the least expensive cylindrical can which will hold 100 cubic inches if the materials cost 2¢, 5¢ and 3¢ respectively for the top, bottom and sides.

(b) How do the dimensions of the least expensive can change if the bottom material costs more than 5¢ per square inch?

Exercise 1.E.8

You have 100 feet of fencing to build a pen in the shape of a circular sector, the "pie slice" shown. The area of such a sector is $(rs)/2$. What value of r maximizes the enclosed area?



Exercise 1.E.9

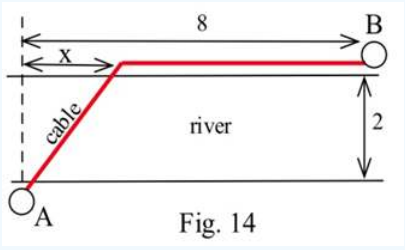


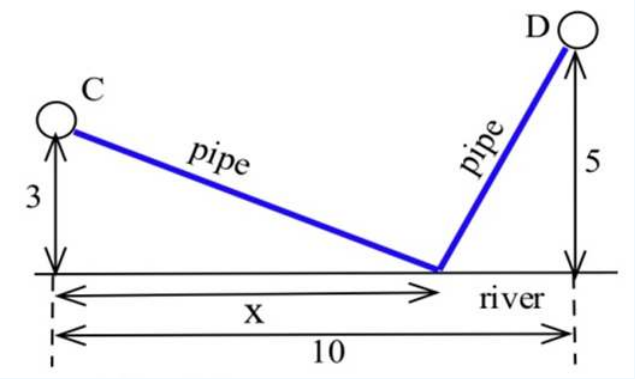
Fig. 14

(a) You have been asked to determine the least expensive route for a telephone cable which connects Andersonville with Beantown. If it costs \$5000 per mile to lay the cable on land and \$8000 per mile to lay the cable across the river and the cost of the cable is negligible, find the least expensive route.

(b) What is the least expensive route if the cable costs \$7000 per mile plus the cost to lay it.

Exercise 1.E.10

You have been asked to determine where a water works should be built along a river between Chesterville and Denton to minimize the total cost of the pipe to the towns.

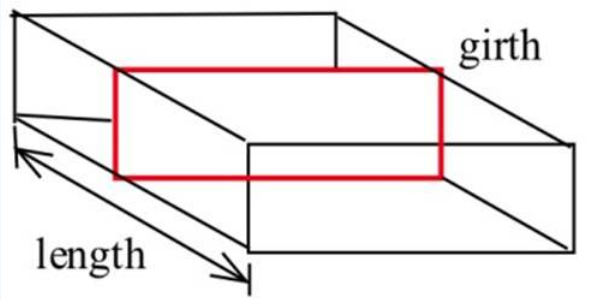


(a) Assume that the same size (and cost) pipe is used to each town. (This part can be done quickly without using calculus.)

(b) Assume that the pipe to Chesterville costs \$3000 per mile and to Denton it costs \$7000 per mile.

Exercise 1.E.11

U.S. postal regulations state that the sum of the length and girth (distance around) of a package must be no more than 108 inches.



(a) Find the dimensions of the acceptable box with a square end which has the largest volume.

(b) Find the dimensions of the acceptable box which has the largest volume if its end is a rectangle twice as long as it is width.

(c) Find the dimensions of the acceptable box with a circular end which has the largest volume.

Exercise 1.E.12

D. Simonton claims that the "productivity levels" of people in different fields can be described as a function of their "career age" t by $p(t) = e^{-at} - e^{-bt}$ where a and b are constants which depend on the field of work, and career age is approximately 20 less than the actual age of the individual.

(a) Based on this model, at what ages do mathematicians $(a = .03, b = .05)$, geologists $(a = .02, b = .04)$, and historians $(a = .02, b = .03)$ reach their maximum productivity?

(b) Simonton says "With a little calculus we can show that the curve $(p(t))$ maximizes at $t = \frac{1}{b-a}\ln(\frac{b}{a})$." Use calculus to show that Simonton is correct.

Note: Models of this type have uses for describing the behavior of groups, but it is dangerous and usually invalid to apply group descriptions or comparisons to **individuals** in the group.

(**Scientific Genius**, by Dean Simonton, Cambridge University Press, 1988, pp. 69 – 73)

Exercise 1.E.13

You own a small airplane which holds a maximum of 20 passengers. It costs you $100 per flight from St. Thomas to St. Croix for gas and wages plus an additional $6 per passenger for the extra gas required by the extra weight. The charge per passenger is $30 each if 10 people charter your plane (10 is the minimum number you will fly), and this charge is reduced by $1 per passenger for each passenger over 10 who goes (that is, if 11 go they each pay $29, if 12 go they each pay $28, etc.). What number of passengers on a flight will maximize your profits?

Exercise 1.E.14

In the planning of a coffee shop, we estimate that if there is seating for between 40 and 80 people, the daily profit will be $50 per seat. However, if the seating capacity is more than 80 places, the daily profit per seat will be decreased by $1 for each additional seat over 80. What should the seating capacity be in order to maximize the coffee shop's total profit?

Exercise 1.E.15

In the planning of a taco restaurant, we estimate that if there is seating for between 10 and 40 people, the daily profit will be $10 per seat. However, if the seating capacity is more than 40 places, the daily profit per seat will be decreased by $0.20 per seat. What should the seating capacity be in order to maximize the taco restaurant's total profit?

Exercise 1.E.16

The total cost in dollars for Alicia to make q oven mitts is given by $C(q) = 64 + 1.5q + .01q^2$.

(a) What is the fixed cost?

(b) Find a function that gives the marginal cost.

(c) Find a function that gives the average cost.

(d) Find the quantity that minimizes the average cost.

(e) Confirm that the average cost and marginal cost are equal at your answer to part (d).

Exercise 1.E.17

Shaki makes and sells backpack danglies. The total cost in dollars for Shaki to make q danglies is given by $C(q) = 75 + 2q + .015q^2$. Find the quantity that minimizes Shaki's average cost for making danglies.

2.10 Exercises

Exercise 1.E.1

If $g(20) = 35$ and $g'(20) = -2$, estimate the value of $g(22)$.

Exercise 1.E.2

If $g(1) = -17$ and $g'(1) = 5$, estimate the value of $g(1.2)$.

Exercise 1.E.3

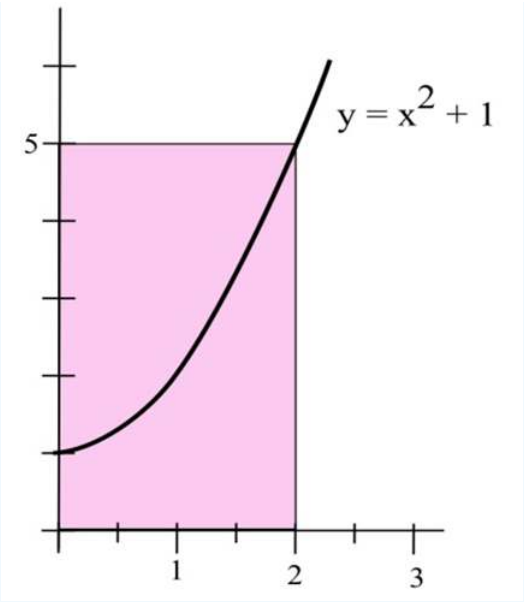
Use the Tangent Line Approximation to estimate the cube root of 9.

Exercise 1.E.4

Use the Tangent Line Approximation to estimate the fifth root of 30.

Exercise 1.E.5

A rectangle has one side on the x–axis, one side on the y–axis, and a corner on the graph of $y = x^2 + 1$.



(a) Use Linear Approximation of the area formula to estimate the increase in the area of the rectangle if the base grows from 2 to 2.3 inches.

(b) Calculate exactly the increase in the area of the rectangle as the base grows from 2 to 2.3 inches.

Exercise $1.E.6$

You can measure the diameter of a circle to within 0.3 cm.

(a) How large is the "error" in the calculated area of a circle with a measured diameter of 7.4 cm?

(b) How large is the "error" in the calculated area of a circle with a measured diameter of 13.6 cm?

(c) How large is the percentage error in the calculated area of a circle with a measured diameter of d?

Exercise $1.E.7$

The demand function for Alicia's oven mitts is given by $q = -8p + 80$ (q is the number of oven mitts, p is the price in dollars). Find the elasticity of demand when $p =$ \$7.50. Will revenue increase if Alicia raises her price from \$7.50?

Exercise $1.E.8$

The demand function for Shaki's danglies is given by $q = -35p + 205$ (q is the number of danglies, p is the price in dollars per dangly). Find the elasticity of demand when $p =$ \$5. Should Shaki raise or lower his price to increase revenue?

2.11 Exercises

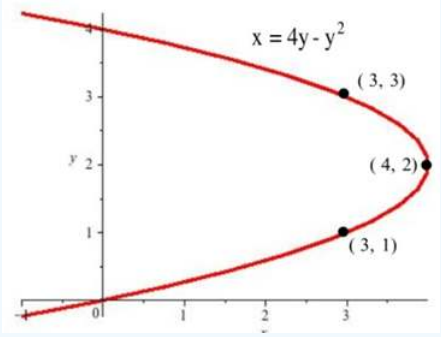
Exercise $1.E.1 - 1.E.10$

In problems 1 – 10 find dy/dx by differentiating implicitly then find the value of dy/dx at the given point.

1. $x^2 + y^2 = 100$, point (6, 8)	2. $x^2 + 5y^2 = 45$, point (5, 2)
3. $x^2 - 3xy + 7y = 5$, point (2,1)	4. $\sqrt{x} + \sqrt{y} = 5$, point (4,9)
5. $\frac{x^2}{9} + \frac{y^2}{16} = 1$, point (0,4)	6. $\frac{x^2}{9} + \frac{y^2}{16} = 1$, point (3,0)
7. $\ln(y) + 3x - 7 = 0$, point $(2, e)$	8. $x^2 - y^2 = 16$, point (5,3)
9. $x^2 - y^2 = 16$, point (5, –3)	10. $y^2 + 7x^3 - 3x = 8$, point (1,2)

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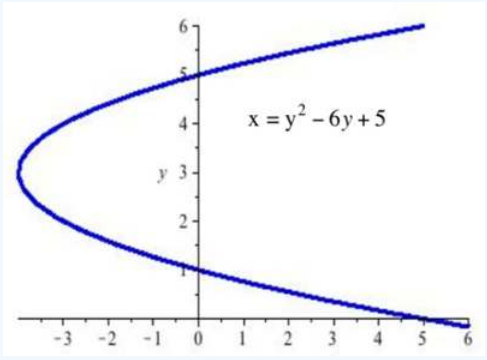
Exercise $1.E.11 - 1.E.12$



11. Find the slopes of the lines tangent to the graph in shown at the points (3,1), (3,3), and (4,2).

12. Find the slopes of the lines tangent to the graph in shown where the graph crosses the y–axis.

Exercise $1.E.13 - 1.E.14$



13. Find the slopes of the lines tangent to the graph in graph shown at the points ((5,0), (5,6), and (–4,3).

14. Find the slopes of the lines tangent to the graph in the graph shown where the graph crosses the y–axis.

Exercise $1.E.15 - 1.E.16$

In problems 15 – 16, find dy/dx using implicit differentiation and then find the slope of the line tangent to the graph of the equation at the given point.

15. $y^3 - 5y = 5x^2 + 7$, point (1,3)

16. $y^2 - 5xy + x^2 + 21 = 0$, point (2,5)

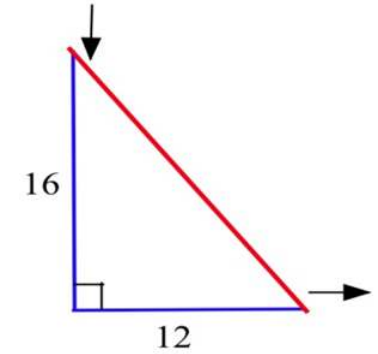
Exercise $1.E.17$

An expandable sphere is being filled with liquid at a constant rate from a tap (imagine a water balloon connected to a faucet). When the radius of the sphere is 3 inches, the radius is increasing at 2 inches per minute. How fast is the liquid coming out of the tap? $(V = \frac{4}{3}\pi r^3)$

Exercise $1.E.18$

The 12 inch base of a right triangle is growing at 3 inches per hour, and the 16 inch height is shrinking at 3 inches per hour.

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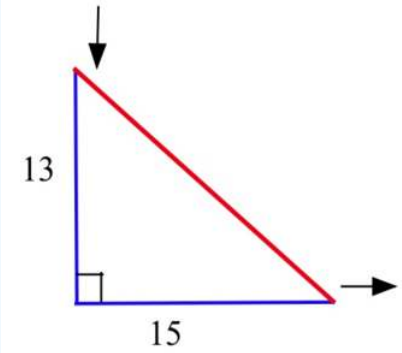
(a) Is the area increasing or decreasing?

(b) Is the perimeter increasing or decreasing?

(c) Is the hypotenuse increasing or decreasing?

Exercise 1.E.19

One hour later the right triangle in Problem 2 is 15 inches long and 13 inches high, and the base and height are changing at the same rate as in Problem 18.



(a) Is the area increasing or decreasing now?

(b) Is the hypotenuse increasing or decreasing now?

(c) Is the perimeter increasing or decreasing now?

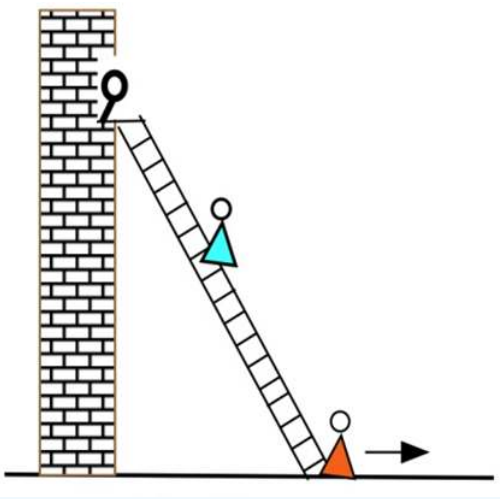
Exercise 1.E.20

A young woman and her boyfriend plan to elope, but she must rescue him from his mother who has locked him in his room. The young woman has placed a 20 foot long ladder against his house and is knocking on his window when his mother begins pulling the bottom of the ladder away from the house at a rate of 3 feet per second. How fast is the top of the ladder (and the young couple) falling when the bottom of the ladder is

(a) 12 feet from the bottom of the wall?

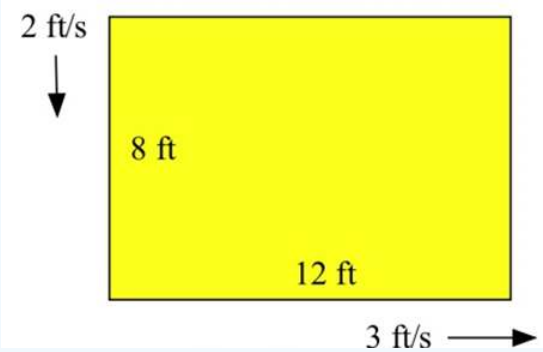
(b) 16 feet from the bottom of the wall?

(c) 19 feet from the bottom of the wall?



Exercise 1.E.21

The length of a 12 foot by 8 foot rectangle is increasing at a rate of 3 feet per second and the width is decreasing at 2 feet per second.



(a) How fast is the perimeter changing?

(b) How fast is the area changing?

Exercise 1.E.22

An oil tanker in Puget Sound has sprung a leak, and a circular oil slick is forming. The oil slick is 4 inches thick everywhere, is 100 feet in diameter, and the diameter is increasing at 12 feet per hour. Your job, as the Coast Guard commander or the tanker's captain, is to determine how fast the oil is leaking from the tanker.

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CHAPTER OVERVIEW

2: Exponential and Logarithmic functions

2.1: Exponents
2.2: Exponential Functions
2.3: Logarithmic Functions
2.4: Derivatives of the Exponential and Logarithmic Functions
2.5: Exponential Growth and Decay

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2.1: Exponents

The Laws of Exponents let you rewrite algebraic expressions that involve exponents. The last three listed here are really definitions rather than rules.

Laws of Exponents

All variables here represent real numbers and all variables in denominators are nonzero.

1. $x^a \cdot x^b = x^{a+b}$
2. $\dfrac{x^a}{x^b} = x^{a-b}$
3. $(x^a)^b = x^{ab}$
4. $(xy)^a = x^a y^a$
5. $\left(\dfrac{x}{y}\right)^b = \dfrac{x^b}{y^b}$
6. $x^0 = 1$, provided $x \neq 0$. [Although in some contexts 0^0 is still defined to be 1.]
7. $x^{-n} = \dfrac{1}{x^n}$, provided $x \neq 0$.
8. $x^{1/n} = \sqrt[n]{x}$, provided $x \neq 0$.

Example 2.1.1

Simplify $\left(2x^2\right)^3(4x)$.

Solution

We'll begin by simplifying the $\left(2x^2\right)^3$ portion. Using Property 4, we can write

$2^3\left(x^2\right)^3(4x)$	
$8x^6(4x)$	Evaluate 2^3, and use Property 3.
$32x^7$	Multiply the constants, and use Property 1, recalling $x = x^1$.

Being able to work with negative and fractional exponents will be very important later in this course.

Example 2.1.2

Rewrite $\dfrac{5}{x^3}$ using negative exponents.

Solution

Since $x^{-n} = \dfrac{1}{x^n}$, then $x^{-3} = \dfrac{1}{x^3}$ and thus

$$\frac{5}{x^3} = 5x^{-3}.$$

Example 2.1.3

Simplify $\left(\frac{x^{-2}}{y^{-3}}\right)^2$ as much as possible and write your answer using only positive exponents.

Solution

$$\begin{aligned}\left(\frac{x^{-2}}{y^{-3}}\right)^2 &= \frac{\left(x^{-2}\right)^2}{\left(y^{-3}\right)^2} \\ &= \frac{x^{-4}}{y^{-6}} \\ &= \frac{y^6}{x^4}\end{aligned}$$

Example 2.1.4

Rewrite $4\sqrt{x} - \frac{3}{\sqrt{x}}$ using exponents.

Solution

A square root is a radical with index of two. In other words, $\sqrt{x} = \sqrt[2]{x}$. Using the exponent rule above, $\sqrt{x} = \sqrt[2]{x} = x^{1/2}$. Rewriting the square roots using the fractional exponent,

$$4\sqrt{x} - \frac{3}{\sqrt{x}} = 4x^{1/2} - \frac{3}{x^{1/2}}.$$

Now we can use the negative exponent rule to rewrite the second term in the expression:

$$4x^{1/2} - \frac{3}{x^{1/2}} = 4x^{1/2} - 3x^{-1/2}.$$

Example 2.1.5

Rewrite $\left(\sqrt{p^5}\right)^{-1/3}$ using only positive exponents.

Solution

$$\begin{aligned}\left(\sqrt{p^5}\right)^{-1/3} &= \left(\left(p^5\right)^{1/2}\right)^{-1/3} \\ &= p^{-5/6} \\ &= \frac{1}{p^{5/6}}\end{aligned}$$

Example 2.1.6

Rewrite $x^{-4/3}$as a radical.

Solution

$$
\begin{aligned}
x^{-4/3} &= \frac{1}{x^{4/3}} \\
&= \frac{1}{\left(x^{1/3}\right)^4} \quad \left(\text{since } \frac{4}{3} = 4 \cdot \frac{1}{3}\right) \\
&= \frac{1}{\left(\sqrt[3]{x}\right)^4} \quad \text{(using the radical equivalence)}
\end{aligned}
$$

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2.2: Exponential Functions

Consider these two companies:

- Company A has 100 stores, and expands by opening 50 new stores a year
- Company B has 100 stores, and expands by increasing the number of stores by 50% of their total each year.

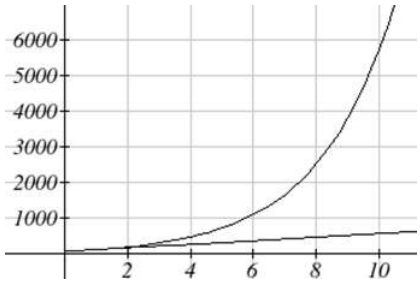
Company A is exhibiting linear growth. In linear growth, we have a constant rate of change – a constant *number* that the output increased for each increase in input. For company A, the number of new stores per year is the same each year.

Company B is different – we have a *percent* rate of change rather than a constant number of stores/year as our rate of change. To see the significance of this difference compare a 50% increase when there are 100 stores to a 50% increase when there are 1000 stores:

- 100 stores, a 50% increase is 50 stores in that year.
- 1000 stores, a 50% increase is 500 stores in that year.

Calculating the number of stores after several years, we can clearly see the difference in results.

Years	Company A	Company B
2	200	225
4	300	506
6	400	1139
8	500	2563
10	600	5767



Graphs of data from A and B, with B fit to a curve.

This percent growth can be modeled with an exponential function.

Exponential Function

An **exponential growth or decay function** is a function that grows or shrinks at a constant percent growth rate. The equation can be written in the form

$$f(x) = a(1+r)^x$$

or

$$f(x) = ab^x$$

where $b = 1 + r$.

Where

- a is the initial or starting value of the function,
- r is the percent growth or decay rate, written as a decimal,
- b is the growth factor or growth multiplier. Since powers of negative numbers behave strangely, we limit b to positive values.

Example 2.2.1

India's population was 1.14 billion in the year 2008 and is growing by about 1.34% each year. Write an exponential function for India's population, and use it to predict the population in 2020.

Solution

Using 2008 as our starting time ($t = 0$), our initial population will be 1.14 billion. Since the percent growth rate was 1.34%, our value for r is 0.0134.

Using the basic formula for exponential growth $f(x) = a(1+r)^x$ we can write the formula,

$$f(t) = 1.14(1+0.0134)^t$$

To estimate the population in 2020, we evaluate the function at $t = 12$, since 2020 is 12 years after 2008:

$$f(t) = 1.14(1+0.0134)^{12} \approx 1.337 \text{ billion people in } 2020.$$

Example 2.2.2

A certificate of deposit (CD) is a type of savings account offered by banks, typically offering a higher interest rate in return for a fixed length of time you will leave your money invested. If a bank offers a 24 month CD with an annual interest rate of 1.2% compounded monthly, how much will a $1000 investment grow to over those 24 months?

Solution

First, we must notice that the interest rate is an annual rate, but is compounded monthly, meaning interest is calculated and added to the account monthly. To find the monthly interest rate, we divide the annual rate of 1.2% by 12 since there are 12 months in a year: 1.2%/12 = 0.1%. Each month we will earn 0.1% interest. From this, we can set up an exponential function, with our initial amount of $1000 and a growth rate of $r = 0.001$, and our input m measured in months:

$$f(m) = 1000\left(1+\frac{0.012}{12}\right)^m = 1000(1.001)^m$$

After 24 months, the account will have grown to $f(24) = 1000(1.001)^{24} \approx \$1024.28$

Example 2.2.3

Bismuth-210 is an isotope that radioactively decays by about 13% each day, meaning 13% of the remaining Bismuth-210 transforms into another atom (polonium-210 in this case) each day. If you begin with 100 mg of Bismuth-210, how much remains after one week?

Solution

With radioactive decay, instead of the quantity increasing at a percent rate, the quantity is decreasing at a percent rate. Our initial quantity is $a = 100$ mg, and our growth rate will be negative 13%, since we are decreasing: $r = -0.13$. This gives the equation

$$Q(d) = 100(1-0.13)^d = 100(0.87)^d.$$

This can also be explained by recognizing that if 13% decays, then 87% remains.

After one week, 7 days, the quantity remaining would be $Q(7) = 100(0.87)^7 = 37.73$mg of Bismuth-210 remains.

Example 2.2.4

$T(q)$ represents the total number of Android smart phone contracts, in thousands, held by a certain Verizon store region measured quarterly since January 1, 2010. Interpret all of the parts of the equation $T(2) = 86(1.64)^2 = 231.3056$

Solution

Interpreting this from the basic exponential form, we know that 86 is our initial value. This means that on Jan. 1, 2010 this region had 86,000 Android smart phone contracts. Since $b = 1 + r = 1.64$, we know that every quarter the number of smart phone contracts grows by 64%. $T(2) = 231.3056$ means that in the second quarter (or at the end of the second quarter) there were approximately 231,305 Android smart phone contracts.

When working with exponentials, there is a special constant we must talk about. It arises when we talk about things growing continuously, such as continuous compounding, or natural phenomena like radioactive decay that happen continuously.

Euler's Number: e

$$e \approx 2.718282$$

Because e is often used as the base of an exponential, most scientific and graphing calculators have a button that can calculate powers of e, usually labeled e^x. Some computer software instead defines a function $exp(x)$, where exp(x) = e^x. Since calculus studies continuous change, we will almost always use the e-based form of exponential equations in this course.

Continuous Growth Formula

Continuous growth can be calculated using the formula

$$f(x) = ae^{rx}$$

where

- a is the starting amount,
- r is the continuous growth rate.

Example 2.2.5

Radon-222 decays at a continuous rate of 17.3% per day. How much will 100mg of Radon-222 decay to in 3 days?

Solution

Since we are given a continuous decay rate, we use the continuous growth formula. Since the substance is decaying, we know the growth rate will be negative: $r = -0.173$, $f(3) = 100e^{-0.173(3)} \approx 59.512$ mg of Radon-222 will remain.

Graphs of Exponential Functions

Graphical Features of Exponential Functions

Graphically, in the function $f(x) = ab^x$.

- a is the vertical intercept of the graph.
- b determines the rate at which the graph grows:
  - the function will increase if $b > 1$,
  - the function will decrease if $0 < b < 1$.
- The graph will have a horizontal asymptote at $y = 0$.

- The graph will be concave up if $a > 0$; concave down if $a < 0$.
- The domain of the function is all real numbers.
- The range of the function is $(0, \infty)$.

When sketching the graph of an exponential function, it can be helpful to remember that the graph will pass through the points $(0, a)$ and $(1, ab)$.

The value b will determine the function's long run behavior:

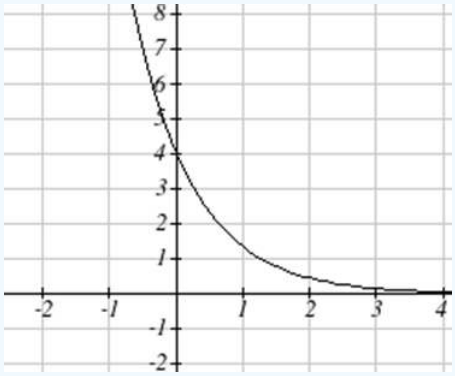
- If $b > 1$, as $x \to \infty$, $f(x) \to \infty$, and as $x \to -\infty$, $f(x) \to 0$.
- If $0 < b < 1$, as $x \to \infty$, $f(x) \to 0$, and as $x \to -\infty$, $f(x) \to \infty$.

Example 2.2.6

Sketch a graph of $f(x) = 4\left(\frac{1}{3}\right)^x$

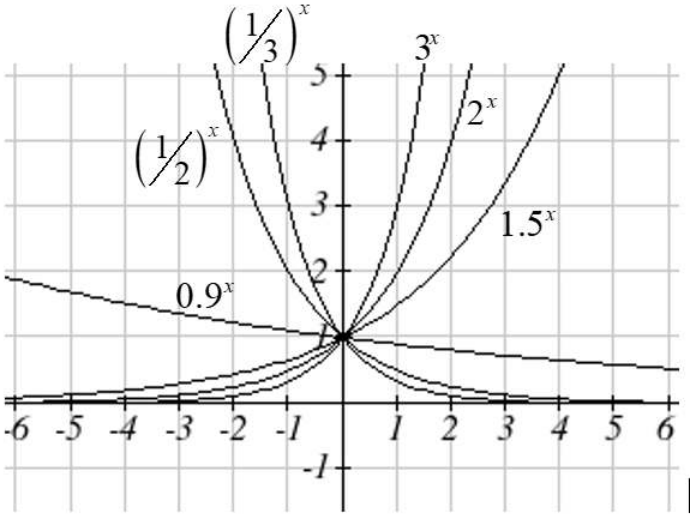
Solution

This graph will have a vertical intercept at (0,4), and pass through the point $\left(1, \frac{4}{3}\right)$. Since $b < 1$, the graph will be decreasing towards zero. Since $a > 0$, the graph will be concave up.



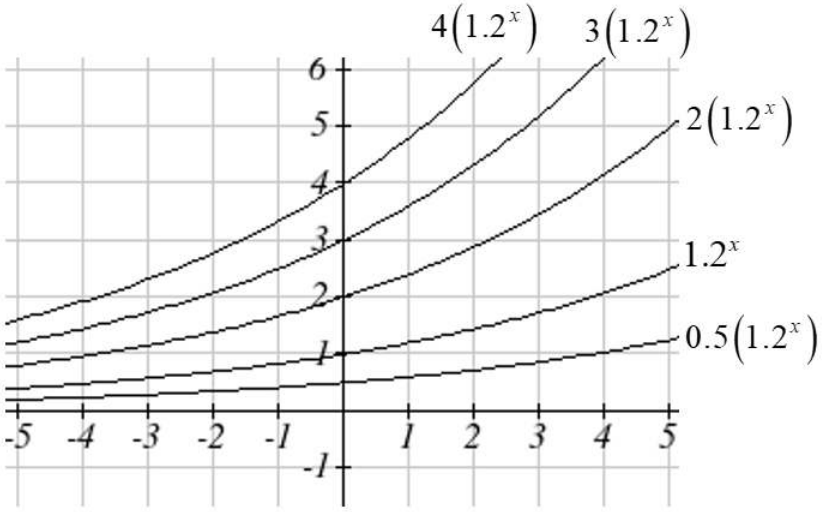
We can also see from the graph the long run behavior: as $x \to \infty$, $f(x) \to 0$, and as $x \to -\infty$, $f(x) \to \infty$.

To get a better feeling for the effect of a and b on the graph, examine the sets of graphs below. The first set shows various graphs, where a remains the same and we only change the value for b. Notice that the closer the value of b is to 1, the less steep the graph will be.



Changing the value of b.

In the next set of graphs, a is altered and our value for b remains the same.



Changing the value of a.

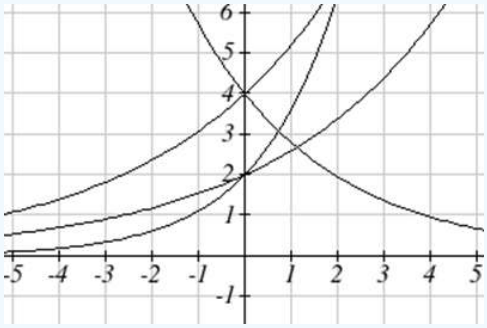
Notice that changing the value for a changes the vertical intercept. Since a is multiplying the b^x term, a acts as a vertical stretch factor, not as a shift. Notice also that the long run behavior for all of these functions is the same because the growth factor did not change and none of these a values introduced a vertical flip.

Try it for yourself using this applet:

Example 2.2.7

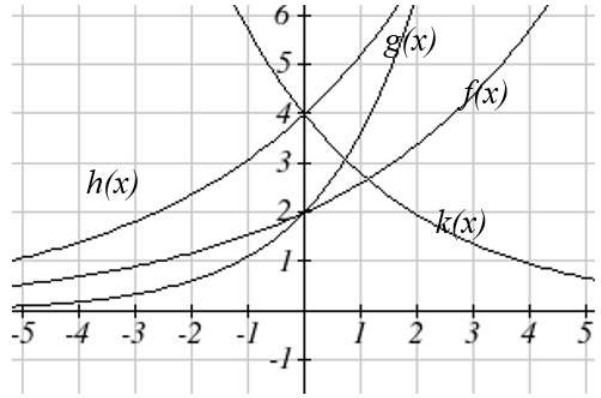
Match each equation with its graph.

- $f(x) = 2(1.3)^x$
- $g(x) = 2(1.8)^x$
- $h(x) = 4(1.3)^x$
- $k(x) = 4(0.7)^x$



Solution

The graph of $k(x)$ is the easiest to identify, since it is the only equation with a growth factor less than one, which will produce a decreasing graph. The graph of $h(x)$ can be identified as the only growing exponential function with a vertical intercept at (0,4). The graphs of $f(x)$ and $g(x)$ both have a vertical intercept at (0,2), but since $g(x)$ has a larger growth factor, we can identify it as the graph increasing faster.



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2.3: Logarithmic Functions

Logarithms are the inverse of exponential functions – they allow us to undo exponential functions and solve for the exponent. They are also commonly used to express quantities that vary widely in size.

Logarithm Equivalent to an Exponential

The logarithm (base b) function, written $\log_b(x)$, is the inverse of the exponential function (base b), b^x.

This means the statement $b^a = c$ is equivalent to the statement $\log_b(c) = a$.

Properties of Logs: Inverse Properties

- $\log_b(b^x) = x$
- $b^{\log_b(x)} = x$

Example 2.3.1

Write these exponential equations as logarithmic equations:

1. $2^3 = 8$
2. $5^2 = 25$
3. $10^{-4} = \frac{1}{10000}$

Solution

1. $2^3 = 8$ is equivalent to $\log_2(8) = 3$.
2. $5^2 = 25$ is equivalent to $\log_5(25) = 2$.
3. $10^{-4} = \frac{1}{10000}$ is equivalent to $\log_{10}\left(\frac{1}{10000}\right) = -4$.

Example 2.3.2

Solve $2^x = 10$ for x.

Solution

By rewriting this expression as a logarithm, we get $x = \log_2(10)$.

While this does define a solution, and an exact solution at that, you may find it somewhat unsatisfying since it is difficult to compare this expression to the decimal estimate we made earlier. Also, giving an exact expression for a solution is not always useful–often we really need a decimal approximation to the solution. Luckily, this is a task calculators and computers are quite adept at. Unluckily for us, most calculators and computers will only evaluate logarithms of two bases. Happily, this ends up not being a problem, as we'll see briefly.

Common and Natural Logarithms

The **common log** is the logarithm with base 10, and is typically written $\log(x)$.

The **natural log** is the logarithm with base e, and is typically written $\ln(x)$.

Example 2.3.3

Evaluate $\log(1000)$ using the definition of the common log.

Solution

To evaluate $\log(1000)$, we can say $x = \log(1000)$, then rewrite into exponential form using the common log base of 10:

$$10^x = 1000.$$

From this, we might recognize that 1000 is the cube of 10, so $x = 3$.

We also can use the inverse property of logs to write $log_{10}\left(10^3\right) = 3$.

Values of the common log

Number	Number as exponential	log(number)
1000	10^3	3
100	10^2	2
10	10^1	1
1	10^0	0
0.1	10^{-1}	-1
0.01	10^{-2}	-2
0.001	10^{-3}	-3

Example 2.3.4

Evaluate $\log(500)$ using you calculator or computer.

Solution

Using a computer or calculator, we can evaluate and find that $\log(500) \approx 2.69897$

Another property provides the basis for solving exponential equations.

Properties of Logs: Exponent Property

$\log_b(A^r) = r\ \log_b(A)$

Solving exponential equations:

1. Isolate the exponential expressions when possible.
2. Take the logarithm of both sides.
3. Utilize the exponent property for logarithms to pull the variable out of the exponent.
4. Use algebra to solve for the variable.

Example 2.3.5

In the last section, we predicted the population (in billions) of India t years after 2008 by using the function $f(t) = 1.14(1 + 0.0134)^t$. If the population continues following this trend, when will the population reach 2 billion?

Solution

We need to solve for the t so that $f(t) = 2$

$2 = 1.14(1.0134)^t$	Initial equation.
$\frac{2}{1.14} = 1.0134^t$	Divide by 1.14 to isolate the exponential expression.
$\ln\left(\frac{2}{1.14}\right) = \ln\left(1.0134^t\right)$	Take the logarithm of both sides of the equation.
$\ln\left(\frac{2}{1.14}\right) = t\ \ln(1.0134)$	Apply the exponent property on the right side.
$t = \frac{\ln\left(\frac{2}{1.14}\right)}{\ln(1.0134)}$	Divide both sides by $\ln(1.0134)$
$t \approx 42.23$ years	

If this growth rate continues, the model predicts the population of India will reach 2 billion about 42 years after 2008, or approximately in the year 2050.

Example 2.3.6

Solve $5e^{-0.3t} = 2$ for t.

Solution

First we divide by 5 to isolate the exponential:

$$e^{-0.3t} = \frac{2}{5}.$$

Since this equation involves e, it makes sense to use the natural log:

$\ln\left(e^{-0.3t}\right) = \ln\left(\frac{2}{5}\right)$	Take the natural log of both sides.
$-0.3t = \ln\left(\frac{2}{5}\right)$	Utilizing the inverse property for logs.
$t = \frac{\ln\left(\frac{2}{5}\right)}{-0.3}$	Now dividing by -0.3.
$t \approx 3.054$	

In addition to solving exponential equations, logarithmic expressions are common in many physical situations.

Example 2.3.7

In chemistry, pH is a measure of the acidity or basicity of a liquid. The pH is related to the concentration of hydrogen ions, $[H^+]$, measured in moles per liter, by the equation

$$\text{pH} = -\log\left(\left[H^+\right]\right)$$

If a liquid has concentration of 0.0001 moles per liter, determine the pH. Determine the hydrogen ion concentration of a liquid with pH of 7.

Solution

To answer the first question, we evaluate the expression $-\log(0.0001)$ While we could use our calculators for this, we do not really need them here, since we can use the inverse property of logs:

$$-\log(0.0001) = -\log\left(10^{-4}\right) = -(-4) = 4.$$

To answer the second question, we need to solve the equation $7 = -\log([H^+])$. Begin by isolating the logarithm on one side of the equation by multiplying both sides by -1: $-7 = \log([H^+])$. Rewriting into exponential form yields the answer:

$$\left[H^+\right] = 10^{-7} = 0.0000001 \text{ moles per liter.}$$

While we don't often need to sketch the graph of a logarithm, it is helpful to understand the basic shape.

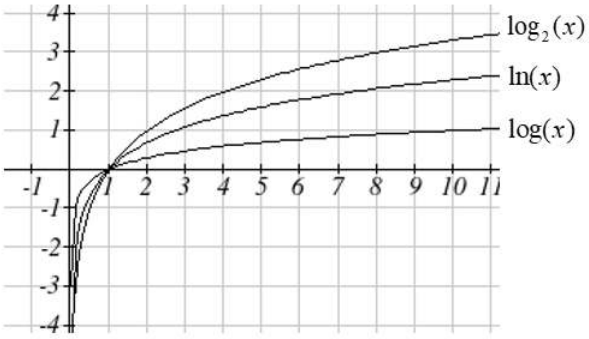
Graphical Features of the Logarithm

Graphically, given the function $g(x) = \log_b(x)$.

- The graph has a horizontal intercept at (1, 0).
- The graph has a vertical asymptote at $x = 0$.
- The graph is increasing and concave down.
- The domain of the function is $x > 0$, or $(0, \infty)$ in interval notation.
- The range of the function is all real numbers, or $(-\infty, \infty)$ in interval notation.

When sketching a general logarithm with base b, it can be helpful to remember that the graph will pass through the points $(1, 0)$ and $(b, 1)$.

To get a feeling for how the base affects the shape of the graph, examine the graphs below:



Another important observation made was the domain of the logarithm: $x > 0$. Like the reciprocal and square root functions, the logarithm has a restricted domain which must be considered when finding the domain of a composition involving a log.

Example 2.3.8

Find the domain of the function $f(x) = \log(5 - 2x)$.

Solution

The logarithm is only defined when the input is positive, so this function will only be defined when $5 - 2x > 0$. Solving this inequality, $-2x > -5$, so $x < \frac{5}{2}$.

The domain of this function is $x < \frac{5}{2}$, or, in interval notation, $\left(-\infty, \frac{5}{2}\right)$.

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2.4: Derivatives of the Exponential and Logarithmic Functions

As with the sine, we do not know anything about derivatives that allows us to compute the derivatives of the exponential and logarithmic functions without going back to basics. Let's do a little work with the definition again:

$$\begin{aligned}\frac{d}{dx}a^x &= \lim_{\Delta x\to 0}\frac{a^{x+\Delta x}-a^x}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{a^x a^{\Delta x}-a^x}{\Delta x}\\ &= \lim_{\Delta x\to 0} a^x\frac{a^{\Delta x}-1}{\Delta x}\\ &= a^x\lim_{\Delta x\to 0}\frac{a^{\Delta x}-1}{\Delta x}.\end{aligned}$$

There are two interesting things to note here: As in the case of the sine function we are left with a limit that involves Δx but not x, which means that whatever $\lim_{\Delta x\to 0}(a^{\Delta x}-1)/\Delta x$ is, we know that it is a number, that is, a constant. This means that a^x has a remarkable property: its derivative is a constant times itself.

We earlier remarked that the hardest limit we would compute is $\lim_{x\to 0}\sin x/x = 1$; we now have a limit that is just a bit too hard to include here. In fact the hard part is to see that $\lim_{\Delta x\to 0}(a^{\Delta x}-1)/\Delta x$ even exists---does this fraction really get closer and closer to some fixed value? Yes it does, but we will not prove this fact.

We can look at some examples. Consider $(2^x-1)/x$ for some small values of x: 1, 0.828427124, 0.756828460, 0.724061864, 0.70838051, $0.70070877$when x is 1, $1/2$, $1/4$, $1/8$, $1/16$, $1/32$, respectively. It looks like this is settling in around 0.7, which turns out to be true (but the limit is not exactly 0.7). Consider next $(3^x-1)/x$: 2, 1.464101616, 1.264296052, 1.177621520, 1.13720773, 1.11768854, at the same values of x. It turns out to be true that in the limit this is about 1.1.

Two examples don't establish a pattern, but if you do more examples you will find that the limit varies directly with the value of a: bigger a, bigger limit; smaller a, smaller limit. As we can already see, some of these limits will be less than 1 and some larger than 1. Somewhere between $a=2$ and $a=3$ the limit will be exactly 1; the value at which this happens is called e, so that

$$\lim_{\Delta x\to 0}\frac{e^{\Delta x}-1}{\Delta x} = 1.$$

As you might guess from our two examples, e is closer to 3 than to 2, and in fact $e\approx 2.718$.

Now we see that the function e^x has a truly remarkable property:

$$\begin{aligned}\frac{d}{dx}e^x &= \lim_{\Delta x\to 0}\frac{e^{x+\Delta x}-e^x}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{e^x e^{\Delta x}-e^x}{\Delta x}\\ &= \lim_{\Delta x\to 0} e^x\frac{e^{\Delta x}-1}{\Delta x}\\ &= e^x\lim_{\Delta x\to 0}\frac{e^{\Delta x}-1}{\Delta x}\\ &= e^x.\end{aligned}$$

That is, e^x is its own derivative, or in other words the slope of e^x is the same as its height, or the same as its second coordinate: The function $f(x)=e^x$ goes through the point (z,e^z) and has slope e^z there, no matter what z is. It is sometimes convenient to express the function e^x without an exponent, since complicated exponents can be hard to read. In such cases we use $\exp(x)$, e.g., $\exp(1+x^2)$ instead of e^{1+x^2}.

What about the logarithm function? This too is hard, but as the cosine function was easier to do once the sine was done, so the logarithm is easier to do now that we know the derivative of the exponential function. Let's start with $\log_e x$, which as you probably know is often abbreviated $\ln x$ and called the "natural logarithm" function.

Consider the relationship between the two functions, namely, that they are inverses, that one "undoes" the other. Graphically this means that they have the same graph except that one is "flipped" or "reflected" through the line $y = x$, as shown in Figure 2.4.1.

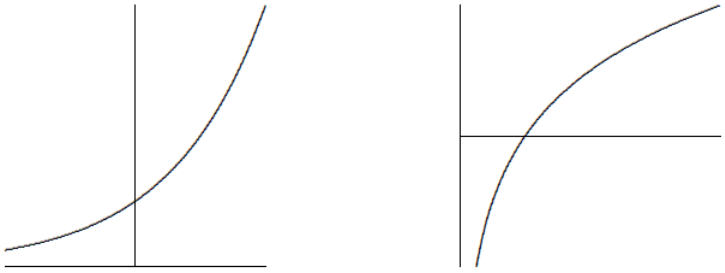


Figure 2.4.1: *The exponential (green) and logarithmic (blue) functions. As inverses of each other, their graphs are reflections of each other across the line $y = x$ (dashed).*

This means that the slopes of these two functions are closely related as well: For example, the slope of e^x is e at $x = 1$; at the corresponding point on the $\ln(x)$ curve, the slope must be $1/e$, because the "rise" and the "run" have been interchanged. Since the slope of e^x is e at the point $(1, e)$, the slope of $\ln(x)$ is $1/e$ at the point $(e, 1)$.

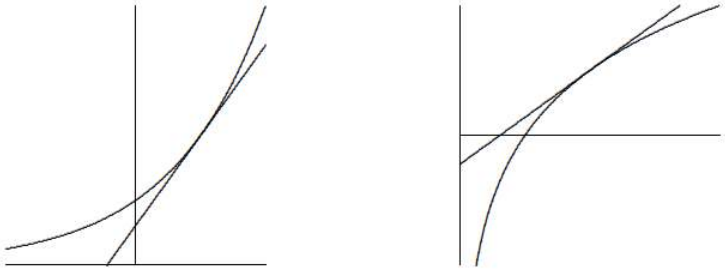


Figure 2.4.2: *The exponential (green) and logarithmic (blue) functions. The dashed lines indicate the slope of the respective functions at the points $(1, e)$ and $(e, 1)$. It is interesting to note that these lines intersect at the origin.*

More generally, we know that the slope of e^x is e^z at the point (z, e^z), so the slope of $\ln(x)$ is $1/e^z$ at (e^z, z), as indicated in Figure 2.4.2. In other words, the slope of $\ln x$ is the reciprocal of the first coordinate at any point; this means that the slope of $\ln x$ at $(x, \ln x)$ is $1/x$. The upshot is: $\frac{d}{dx} \ln x = \frac{1}{x}$. We have discussed this from the point of view of the graphs, which is easy to understand but is not normally considered a rigorous proof---it is too easy to be led astray by pictures that seem reasonable but that miss some hard point. It is possible to do this derivation without resorting to pictures, and indeed we will see an alternate approach soon.

Note that $\ln x$ is defined only for $x > 0$. It is sometimes useful to consider the function $\ln|x|$, a function defined for $x \neq 0$. When $x < 0$, $\ln|x| = \ln(-x)$ and

$$\frac{d}{dx} \ln|x| = \frac{d}{dx} \ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}.$$

Thus whether x is positive or negative, the derivative is the same.

What about the functions a^x and $\log_a x$? We know that the derivative of a^x is some constant times a^x itself, but what constant? Remember that "the logarithm is the exponent" and you will see that $a = e^{\ln a}$. Then $a^x = (e^{\ln a})^x = e^{x \ln a}$, and we can compute the derivative using the chain rule:

$$\frac{d}{dx} a^x = \frac{d}{dx} (e^{\ln a})^x = \frac{d}{dx} e^{x \ln a} = (\ln a) e^{x \ln a} = (\ln a) a^x.$$

The constant is simply $\ln a$. Likewise we can compute the derivative of the logarithm function $\log_a x$. Since $x = e^{\ln x}$ we can take the logarithm base a of both sides to get $\log_a(x) = \log_a(e^{\ln x}) = \ln x \log_a e$. Then

$$\frac{d}{dx} \log_a x = \frac{1}{x} \log_a e.$$

This is a perfectly good answer, but we can improve it slightly. Since

$$\begin{aligned} a &= e^{\ln a} \\ \log_a(a) &= \log_a(e^{\ln a}) = \ln a \log_a e \\ 1 &= \ln a \log_a e \\ \frac{1}{\ln a} &= \log_a e, \end{aligned}$$

we can replace $\log_a e$ to get $\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$.

You may if you wish memorize the formulas

$$\frac{d}{dx}a^x = (\ln a)a^x \quad \text{and} \quad \frac{d}{dx}\log_a x = \frac{1}{x \ln a}.$$

Because the "trick" $a = e^{\ln a}$ is often useful, and sometimes essential, it may be better to remember the trick, not the formula.

Example 2.4.1

Compute the derivative of $f(x) = 2^x$.

Solution

$$\begin{aligned} \frac{d}{dx}2^x &= \frac{d}{dx}(e^{\ln 2})^x \\ &= \frac{d}{dx}e^{x \ln 2} \\ &= \left(\frac{d}{dx}x \ln 2\right) e^{x \ln 2} \\ &= (\ln 2)e^{x \ln 2} = 2^x \ln 2 \end{aligned}$$

Example 2.4.2

Compute the derivative of $f(x) = 2^{x^2} = 2^{(x^2)}$.

$$\begin{aligned} \frac{d}{dx}2^{x^2} &= \frac{d}{dx}e^{x^2 \ln 2} \\ &= \left(\frac{d}{dx}x^2 \ln 2\right) e^{x^2 \ln 2} \\ &= (2 \ln 2)xe^{x^2 \ln 2} \\ &= (2 \ln 2)x2^{x^2} \end{aligned}$$

Example 2.4.3

Compute the derivative of $f(x) = x^x$. At first this appears to be a new kind of function: it is not a constant power of x, and it does not seem to be an exponential function, since the base is not constant. But in fact it is no harder than the previous example.

$$\begin{aligned} \frac{d}{dx}x^x &= \frac{d}{dx}e^{x \ln x} \\ &= \left(\frac{d}{dx}x \ln x\right) e^{x \ln x} \\ &= (x\frac{1}{x} + \ln x)x^x \\ &= (1 + \ln x)x^x \end{aligned}$$

Example 2.4.4

Recall that we have not justified the power rule except when the exponent is a positive or negative integer. We can use the exponential function to take care of other exponents.

$$\begin{aligned}\frac{d}{dx}x^r &= \frac{d}{dx}e^{r\ln x}\\ &= \left(\frac{d}{dx}r\ln x\right)e^{r\ln x}\\ &= (r\frac{1}{x})x^r\\ &= rx^{r-1}\end{aligned}$$

Contributors

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- Integrated by Justin Marshall.

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2.5: Exponential Growth and Decay

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CHAPTER OVERVIEW

3: Applications of Differentiation

3.1: Curve Sketching
3.2: Second Derivative and Concavity
3.3: Optimization
3.4: Applied Optimization
3.5: Implicit Differentiation and Related Rates

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3.1: Curve Sketching

This section examines some of the interplay between the shape of the graph of f and the behavior of f'. If we have a graph of f, we will see what we can conclude about the values of f'. If we know values of f', we will see what we can conclude about the graph of f. We will also utilize the information from f'' that we learning in the last section.

First Derivative Information

Definitions

The function $f(x)$ is **increasing** on (a,b) if $a < x_1 < x_2 < b$ implies $f(x_1) < f(x_2)$.

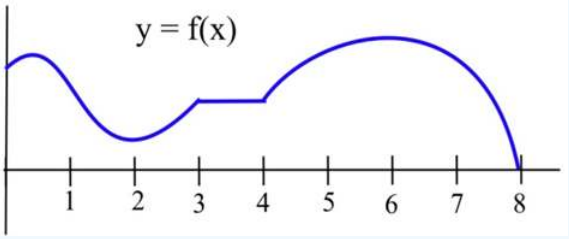
The function $f(x)$ is **decreasing** on (a,b) if $a < x_1 < x_2 < b$ implies $f(x_1) > f(x_2)$.

Graphically, f is **increasing** (decreasing) if, as we move from left to right along the graph of f, the height of the graph **increases** (decreases).

These same ideas make sense if we consider $h(t)$ to be the height (in feet) of a rocket at time t seconds. We naturally say that the rocket is rising or that its height is increasing if the height $h(t)$ increases over a period of time, as t increases.

Example 3.1.1

List the intervals on which the function shown is increasing or decreasing.



f is increasing on the intervals [0,0.5], [2,3] and [4,6].

f is decreasing on [0.5,2] and [6,8].

On the interval [3,4] the function is not increasing or decreasing – it is constant.

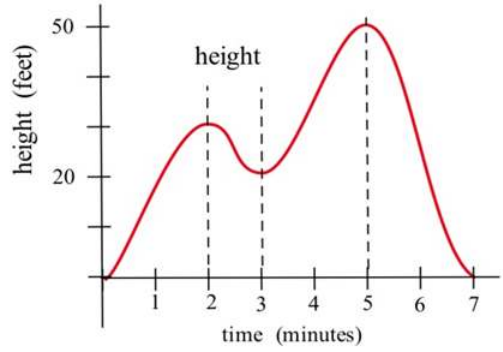
First Derivative Information about Shape

For a function f which is differentiable on an interval (a,b);

- if f is increasing on (a,b), then $f'(x) \geq 0$ for all x in (a,b).
- if f is decreasing on (a,b), then $f'(x) \leq 0$ for all x in (a,b).
- if f is constant on (a,b), then $f'(x) = 0$ for all x in (a,b).

Example 3.1.2

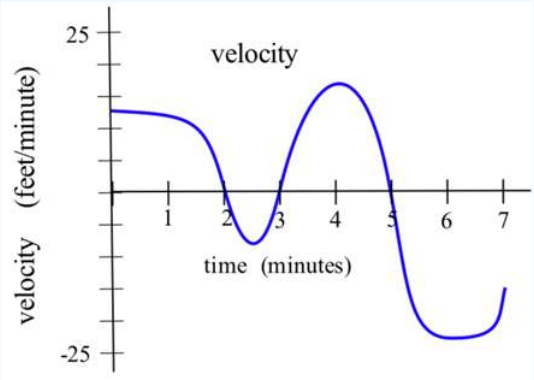
The graph shows the height of a helicopter during a period of time. Sketch the graph of the upward velocity of the helicopter, $\frac{dh}{dt}$.



Notice that the $h(t)$ has a local maximum when $t=2$ and $t=5$, and so $v(2)=0$ and $v(5)=0$. Similarly, $h(t)$ has a local minimum when $t=3$, so $v(3)=0$.

When h is increasing, v is positive. When h is decreasing, v is negative.

Using this information, we can sketch a graph of $v(t)=\frac{dh}{dt}$.



The next theorem is almost the converse of the First Shape Theorem and explains the relationship between the values of the derivative and the graph of a function from a different perspective. It says that if we know something about the values of f', then we can draw some conclusions about the shape of the graph of f.

First Derivative Information about Shape (part 2)

For a function f which is differentiable on an interval I;

- if $f'(x)>0$ for all x in the interval I, then f is increasing on I.
- if $f'(x)<0$ for all x in the interval I, then f is decreasing on I.
- if $f'(x)=0$ for all x in the interval I, then f is constant on I.

Example 3.1.3

Use information about the values of f' to help graph $f(x)=x^3-6x^2+9x+1$.

Solution

$f'(x)=3x^2-12x+9=3(x-1)(x-3)$ so $f'(x)=0$ only when $x=1$ or $x=3$. f' is a polynomial so it is always defined.

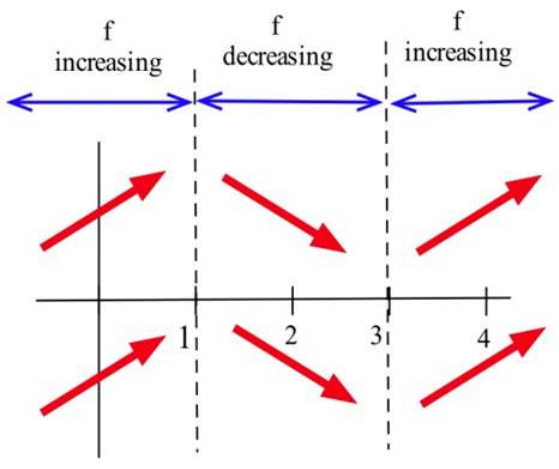
The only critical numbers for f are $x=1$ and $x=3$, and they divide the real number line into three intervals: $(-\infty,1)$, $(1,3)$, and $(3,\infty)$. On each of these intervals, the function is either always increasing or always decreasing.

If $x < 1$, then $f'(x) = 3$(negative number)(negative number) > 0 so f is increasing.

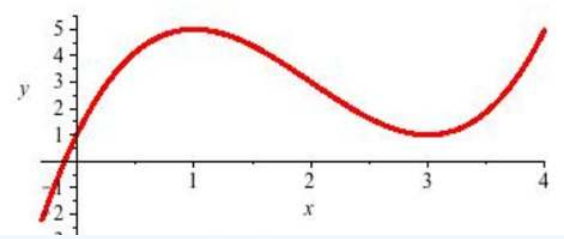
If $1 < x < 3$, then $f'(x) = 3$(positive number)(negative number) < 0 so f is decreasing.

If $x > 3$, then $f'(x) = 3$(positive number)(positive number) > 0 so f is increasing.

Even though we don't know the value of f anywhere yet, we do know a lot about the shape of the graph of f: as we move from left to right along the x-axis, the graph of f increases until $x = 1$, then the graph decreases until $x = 3$, and then the graph increases again. The graph of f makes "turns" when $x = 1$ and $x = 3$; it has a local maximum at $x = 1$, and a local minimum at $x = 3$.



To plot the graph of f, we still need to evaluate f at a few values of x, but only at a very few values. $f(1) = 5$, and (1,5) is a local maximum of f. $f(3) = 1$, and (3,1) is a local minimum of f. The resulting graph of f is shown here.



Second Derivative Information

Until now, we've only used first derivative information, but we could also use information from the second derivative to provide more information about the shape of the function.

Second Derivative Information about Shape

- If f is concave up on (a, b), then $f''(x) \geq 0$ for all x in (a, b).
- If f is concave down on (a, b), then $f''(x) \leq 0$ for all x in (a, b).

The converse of both of these are also true:

- If $f''(x) \geq 0$ for all x in (a, b), then f is concave up on (a, b).
- If $f''(x) \leq 0$ for all x in (a, b), then f is concave down on (a, b).

Example 3.1.4

Use information about the values of f'' to help determine the intervals on which the function $f(x) = x^3 - 6x^2 + 9x + 1$ is concave up and concave down.

Solution

For concavity, we need the second derivative: $f'(x) = 3x^2 - 12x + 9$, so $f''(x) = 6x - 12$.

To find possible inflection points, set the second derivative equal to zero. $6x - 12 = 0$, so $x = 2$. This divides the real number line into two intervals: $(-\infty, 2)$ and $(2, \infty)$.

For $x < 2$, the second derivative is negative (for example, $f''(0) = 6(0) - 12 = -12$), so f is concave down. For $x > 2$, the second derivative is positive, so f is concave up.

We could have incorporated this concavity information when sketching the graph for the previous example, and indeed we can see the concavity reflected in the graph shown.

Example 3.1.5

Use information about the values of f' and f'' to help graph $f(x) = x^{2/3}$.

Solution

$f'(x) = \frac{2}{3}x^{-1/3}$. This is undefined at $x = 0$.

$f''(x) = -\frac{2}{9}x^{-4/3}$. This is also undefined at $x = 0$.

This creates two intervals: $x < 0$, and $x > 0$.

On the interval $x < 0$, we could test out a value like $x = -1$:

$$f'(-1) = \frac{2}{3}(-1)^{-1/3} = -\frac{2}{3}$$

and

$$f''(-1) = -\frac{2}{9}(-1)^{-4/3} = -\frac{2}{9}.$$

$f'(x)$ is negative and $f''(x)$ is negative, so we can conclude that the function is decreasing and concave down on this interval.

On the interval $x > 0$, we could test out a value like $x = 1$:

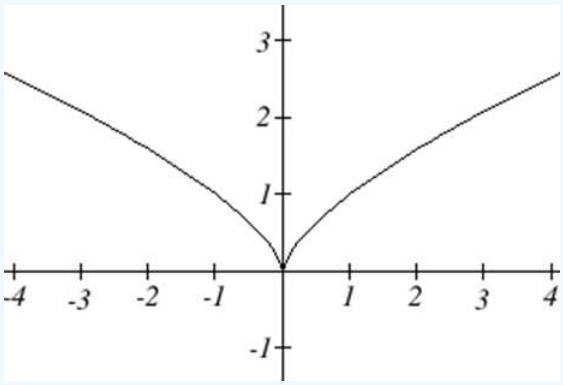
$$f'(1) = \frac{2}{3}(1)^{-1/3} = \frac{2}{3}$$

and

$$f''(1) = -\frac{2}{9}(1)^{-4/3} = -\frac{2}{9}.$$

$f'(x)$ is positive and $f''(x)$ is negative, so we can conclude that the function is increasing and concave down on this interval.

We can also calculate that $f(0) = 0$, giving us a base point for the graph. Using this information, we can conclude the graph must look like this:



Sketching without an Equation

Of course, graphing calculators and computers are great at graphing functions. Calculus provides a way to illuminate what may be hidden or out of view when we graph using technology. More importantly, calculus gives us a way to look at the derivatives of functions for which there is no equation given. We already saw the idea of this back in Section 2.3 where we sketched the derivative of two graphs by estimating slopes on the curves.

We can summarize all the derivative information about shape in a table.

Summary of Derivative Information about the Graph

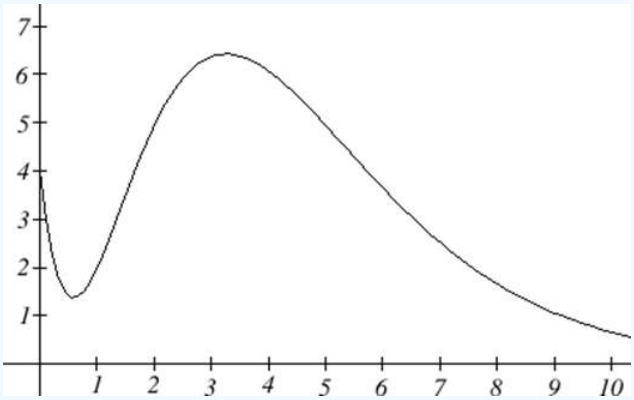
$f(x)$	Increasing	Decreasing	Concave Up	Concave Down
$f'(x)$	$+$	$-$	Increasing	Decreasing
$f''(x)$			$+$	$-$

When $f'(x) = 0$, the graph of f may have a local max or min.

When $f''(x) = 0$, the graph of f may have an inflection point.

Example 3.1.6

A company's bank balance, B, in millions of dollars, t weeks after releasing a new product is shown in the graph below. Sketch a graph of the marginal balance – the rate at which the bank balance was changing over time.



Solution

Notice that since the tangent line will be horizontal at about $t = 0.6$ and $t = 3.2$, the derivative will be 0 at those points.

We can then identifying intervals on which the original function is increasing or decreasing. This will tell us when the derivative function is positive or negative.

Interval	$B(t)$	$B'(t)$
$0 < t < 0.6$	Decreasing	Negative
$0.6 < t < 3.2$	Increasing	Positive
$t > 3.2$	Decreasing	Negative

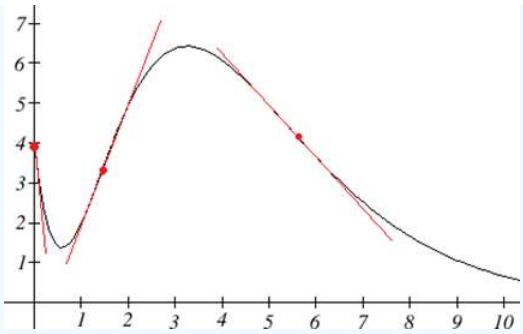
There appear to be inflection points at about $t = 1.5$ and $t = 5.5$. At these points, the derivative will be changing from increasing to decreasing or vice versa, so the derivative will have a local max or min at those points.

Looking at the intervals of concavity:

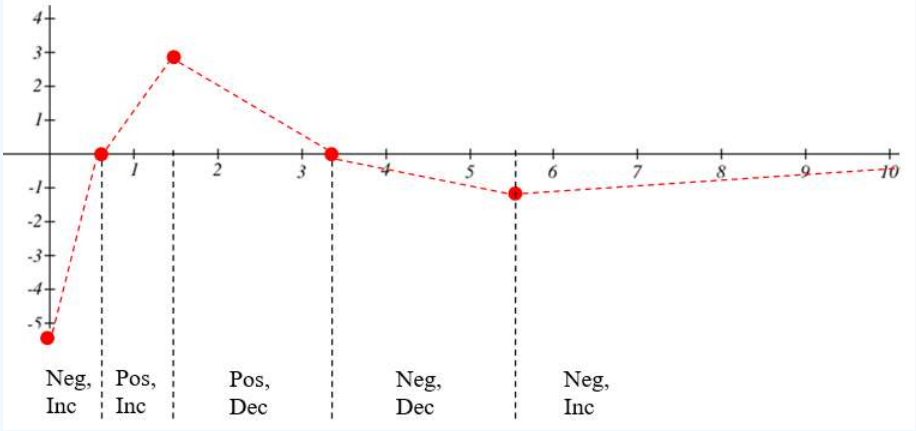
Interval	$B(t)$	$B'(t)$
$0 < t < 1.5$	Concave Up	Increasing
$1.5 < t < 5.5$	Concave Down	Decreasing
$t > 5.5$	Concave Up	Increasing

If we wanted a more accurate sketch of the derivative function, we could also estimate the derivative at a few points:

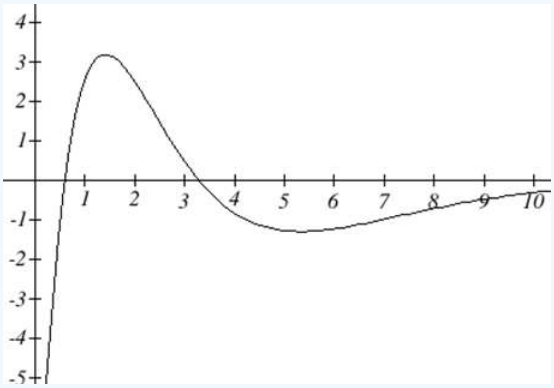
t	$B'(t)$
0	-10
1.5	3
6	-1



Now we can sketch the derivative. We know a few values for the derivative function, and on each interval we know the shape we need. We can use this to create a rough idea of what the graph should look like.



Smoothing this out gives us a good estimate for the graph of the derivative.



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3.2: Second Derivative and Concavity

Second Derivative and Concavity

Graphically, a function is **concave up** if its graph is curved with the opening upward (Figure $3.2.1a$). Similarly, a function is **concave down** if its graph opens downward (Figure $3.2.1b$).

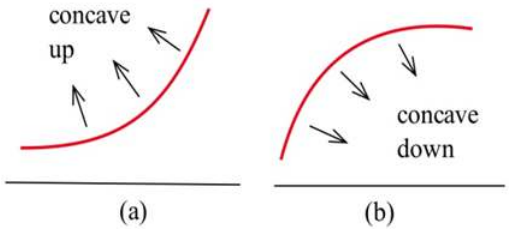
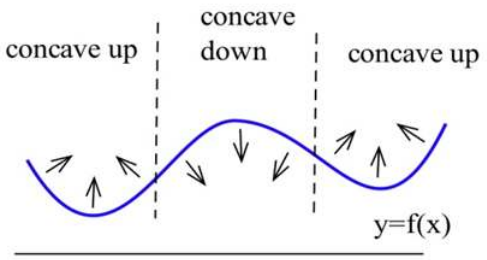


Figure 3.2.1

This figure shows the concavity of a function at several points. Notice that a function can be concave up regardless of whether it is increasing or decreasing.



For example, **an epidemic**: Suppose an epidemic has started, and you, as a member of congress, must decide whether the current methods are effectively fighting the spread of the disease or whether more drastic measures and more money are needed. In Figure $3.2.2$ below, $f(x)$ is the number of people who have the disease at time x, and two different situations are shown. In both Figure $3.2.2a$ and Figure $3.2.2b$, the number of people with the disease, $f(\text{now})$, and the rate at which new people are getting sick, $f'(\text{now})$, are the same. The difference in the two situations is the concavity of f, and that difference in concavity might have a big effect on your decision.

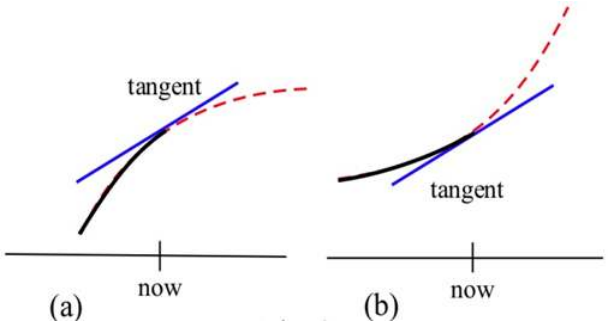


Figure 3.2.2

In Figure $3.2.2a$, f is concave down at "now", the slopes are decreasing, and it looks as if it is tailing off. We can say "f is increasing at a decreasing rate." It appears that the current methods are starting to bring the epidemic under control.

In Figure $3.2.2b$, f is concave up, the slopes are increasing, and it looks as if it will keep increasing faster and faster. It appears that the epidemic is still out of control.

The differences between the graphs come from whether the *derivative* is increasing or decreasing

The derivative of a function f is a function that gives information about the slope of f. **The derivative tells us if the original function is increasing or decreasing**.

Because f' is a function, we can take its derivative. This second derivative also gives us information about our original function f. The second derivative gives us a mathematical way to tell how the graph of a function is curved. **The second derivative tells us if the original function is concave up or down**.

Second Derivative

Let $y = f(x)$. The **second derivative of f** is the derivative of $y' = f'(x)$.

Using prime notation, this is $f''(x)$ or y''. You can read this aloud as "f double prime of x" or "y double prime."

Using Leibniz notation, the second derivative is written $\frac{d^2y}{dx^2}$ or $\frac{d^2f}{dx^2}$. This is read aloud as "the second derivative of y (or f)."

Derivatives and the Graph of a Function

The first derivative tells us if a function is increasing or decreasing

If $f'(x)$ is positive on an interval, the graph of $y = f(x)$ is **increasing** on that interval.

If $f'(x)$ is negative on an interval, the graph of $y = f(x)$ is **decreasing** on that interval.

The second derivative tells us if a function is concave up or concave down

If $f''(x)$ is positive on an interval, the graph of $y = f(x)$ is **concave up** on that interval. We can say that f is increasing (or decreasing) **at an increasing rate**.

If $f''(x)$ is negative on an interval, the graph of $y = f(x)$ is **concave down** on that interval. We can say that f is increasing (or decreasing) **at a decreasing rate**.

Example 3.2.1

Find $f''(x)$ for $f(x) = 3x^7$.

First, we need to find the first derivative:

$$f'(x) = 21x^6.$$

Then we take the derivative of that function:

$$f''(x) = \frac{d}{dx}(f'(x)) = \frac{d}{dx}\left(21x^6\right) = 126x^5.$$

If $f(x)$ represents the position of a particle at time x, then $v(x) = f'(x)$ will represent the velocity (rate of change of the position) of the particle and $a(x) = v'(x) = f''(x)$ will represent the acceleration (the rate of change of the velocity) of the particle.

You are probably familiar with acceleration from driving or riding in a car. The speedometer tells you your velocity (speed). When you leave from a stop and press down on the accelerator, you are accelerating – increasing your speed.

Example 3.2.2

The height (feet) of a particle at time t seconds is $f(t) = t^3 - 4t^2 + 8t$. Find the height, velocity and acceleration of the particle when $t = 0$, 1, and 2 seconds.

$f(t) = t^3 - 4t^2 + 8t$ so $f(0) = 0$ feet, $f(1) = 5$ feet, and $f(2) = 8$ feet.

The velocity is $v(t) = f'(t) = 3t^2 - 8t + 8$ so $v(0) = 8$ ft/s, $v(1) = 3$ ft/s, and $v(2) = 4$ ft/s. At each of these times the velocity is positive and the particle is moving upward, increasing in height.

The acceleration is $a(t) = f''(t) = 6t - 8$ so $a(0) = -8$ ft/s^2, $a(1) = -2$ ft/s^2 and $a(2) = 4$ ft/s^2.

At time 0 and 1, the acceleration is negative, so the particle's velocity would be decreasing at those points - the particle was slowing down. At time 2, the velocity is positive, so the particle was increasing in speed.

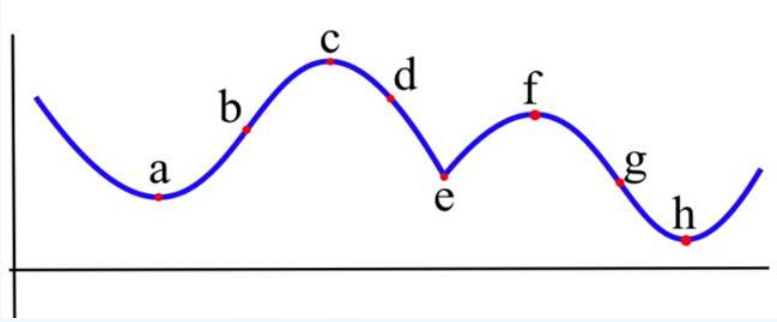
Inflection Points

Inflection Point

An **inflection point** is a point on the graph of a function where the concavity of the function changes, from concave up to down or from concave down to up.

Example 3.2.3

Which of the labeled points in the graph below are inflection points?



The concavity changes at points b and g. At points a and h, the graph is concave up on both sides, so the concavity does not change. At points c and f, the graph is concave down on both sides. At point e, even though the graph looks strange there, the graph is concave down on both sides – the concavity does not change.

Inflection points happen when the concavity changes. Because we know the connection between the concavity of a function and the sign of its second derivative, we can use this to find inflection points.

Working Definition: Inflection Point

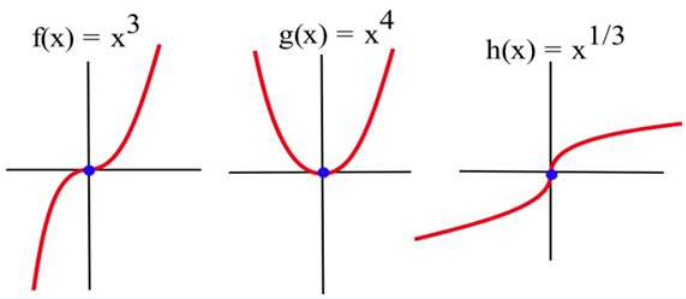
An **inflection point** is a point on the graph where the second derivative changes sign.

In order for the second derivative to change signs, it must either be zero or be undefined. So to find the inflection points of a function we only need to check the points where $f''(x)$ is 0 or undefined.

Note that it is not enough for the second derivative to be zero or undefined. We still need to check that the sign of f'' changes sign. The functions in the next example illustrate what can happen.

Example 3.2.4

Let $f(x) = x^3$, $g(x) = x^4$ and $h(x) = x^{1/3}$. For which of these functions is the point (0,0) an inflection point?



Graphically, it is clear that the concavity of $f(x) = x^3$ and $h(x) = x^{1/3}$ changes at (0,0), so (0,0) is an inflection point for f and h. The function $g(x) = x^4$ is concave up everywhere so (0,0) is not an inflection point of g.

We can also compute the second derivatives and check the sign change.

If $f(x) = x^3$, then $f'(x) = 3x^2$ and $f''(x) = 6x$. The only point at which $f''(x) = 0$ or is undefined (f' is not differentiable) is at $x = 0$. If $x < 0$, then $f''(x) < 0$ so f is concave down. If $x > 0$, then $f''(x) > 0$ so f is concave up. At $x = 0$ the concavity changes so the point $(0, f(0)) = (0, 0)$ is an inflection point of $f(x) = x^3$.

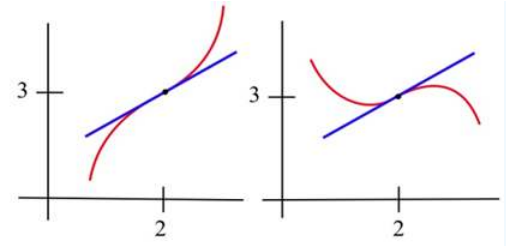
If $g(x) = x^4$, then $g'(x) = 4x^3$ and $g''(x) = 12x^2$. The only point at which $g''(x) = 0$ or is undefined is at $x = 0$. If $x < 0$, then $g''(x) > 0$ so g is concave up. If $x > 0$, then $g''(x) > 0$ so g is also concave up. At $x = 0$ the concavity **does not change** so the point $(0, g(0)) = (0, 0)$ is **not an inflection point** of $g(x) = x^4$. Keep this example in mind!

If $h(x) = x^{1/3}$, then $h'(x) = \frac{1}{3}x^{-2/3}$ and $h''(x) = -\frac{2}{9}x^{-5/3}$. h'' is not defined if $x = 0$, but $h''(\text{negative number}) > 0$ and $h''(\text{positive number}) < 0$ so h changes concavity at (0,0) and (0,0) is an inflection point of h.

Example 3.2.5

Sketch the graph of a function with $f(2) = 3$, $f'(2) = 1$, and an inflection point at (2,3).

Two possible solutions are shown here.



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3.3: Optimization

In theory and applications, we often want to maximize or minimize some quantity. An engineer may want to maximize the speed of a new computer or minimize the heat produced by an appliance. A manufacturer may want to maximize profits and market share or minimize waste. A student may want to maximize a grade in calculus or minimize the hours of study needed to earn a particular grade.

Without calculus, we only know how to find the optimum points in a few specific examples (for example, we know how to find the vertex of a parabola). But what if we need to optimize an unfamiliar function?

The best way we have without calculus is to examine the graph of the function, perhaps using technology. But our view depends on the viewing window we choose – we might miss something important. In addition, we'll probably only get an approximation this way. (In some cases, that will be good enough.)

Calculus provides ways of drastically narrowing the number of points we need to examine to find the exact locations of maximums and minimums, while at the same time ensuring that we haven't missed anything important.

Local Maxima and Minima

Before we examine how calculus can help us find maximums and minimums, we need to define the concepts we will develop and use.

Definition: Local Maxima and Minima

$f(x)$ has a **local maximum** at $x = a$ if $f(a) \geq f(x)$ for all x near a.

$f(x)$ has a **local minimum** at $x = a$ if $f(a) \leq f(x)$ for all x near a.

$f(x)$ has a **local extreme** at $x = a$ if $f(a)$ is a **local maximum or minimum**.

The plurals of these are maxima and minima. We often simply say "max" or "min;" it saves a lot of syllables.

Some books say "relative" instead of "local."

The process of finding maxima or minima is called **optimization**.

A point is a local max (or min) if it is higher (lower) than all the **nearby points**. These points come from the shape of the graph.

Definition: Global Maxima and Minima

$f(x)$ has a **global maximum** at $x = a$ if $f(a) \geq f(x)$ for all x in the domain of $f(x)$.

$f(x)$ has a **global minimum** at $x = a$ if $f(a) \leq f(x)$ for all x in the domain of $f(x)$.

$f(x)$ has a **global extreme** at $x = a$ if $f(a)$ is a **global maximum or minimum**.

Some books say "absolute" instead of "global."

A point is a global max (or min) if it is higher (lower) than every point on the graph. These points come from the shape of the graph **and** the window through which we view the graph.

The local and global extremes of the function in Figure 1 are labeled. You should notice that every global extreme is also a local extreme, but there are local extremes that are not global extremes.

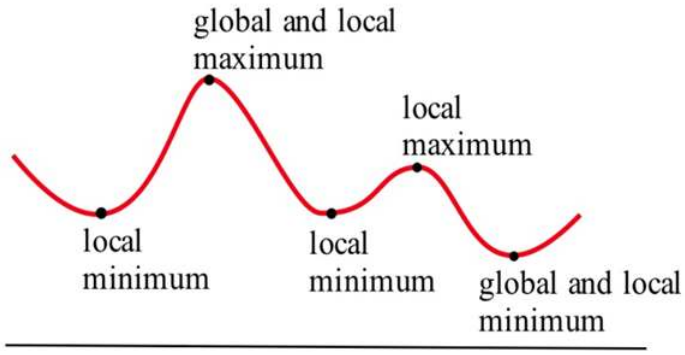


Figure 1

If $h(x)$ is the height of the earth above sea level at the location x, then the global maximum of h is h(summit of Mt. Everest) = 29,028 feet. The local maximum of h for the United States is h(summit of Mt. McKinley) = 20,320 feet. The local minimum of h for the United States is h(Death Valley) = -282 feet.

Example 3.3.1

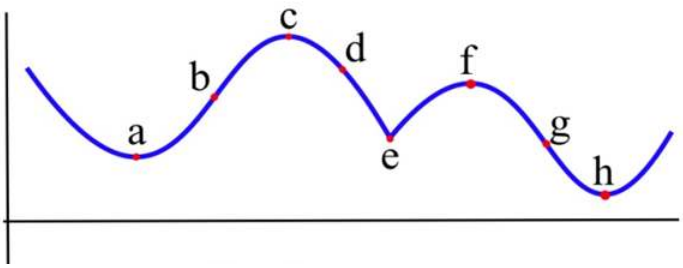
The table shows the annual calculus enrollments at a large university. Which years had local maximum or minimum calculus enrollments? What were the global maximum and minimum enrollments in calculus?

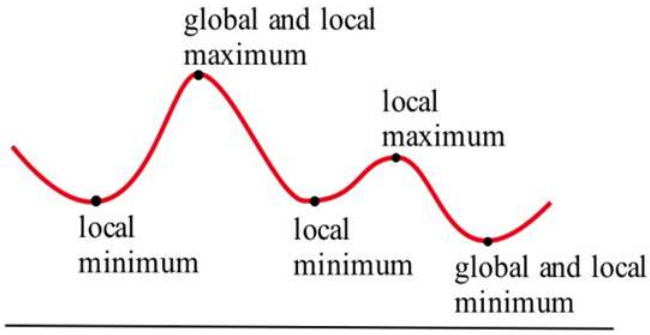
Year	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010
Enrollment	1257	1324	1378	1336	1389	1450	1523	1582	1567	1545	1571

There were local maxima in 2002 and 2007; the global maximum was 1582 students in 2007. There were local minima in 2003 and 2009; the global minimum was 1257 students in 2000. We choose not to think of 2000 as a local minimum or 2010 as a local maximum; however, some books would include the endpoints. We are allowed to have a global maximum or global minimum at an endpoint.

Finding Maxima and Minima of a Function

What must the tangent line look like at a local max or min? Look at these two graphs again – you'll see that at all the extreme points, the tangent line is horizontal (so $f' = 0$). There is one cusp in the blue graph – the tangent line is vertical there (so f' is undefined).





That gives us the clue how to find extreme values.

Critical number

A **critical number** for a function f is a value $x = a$ in the domain of f where either $f'(a) = 0$ **or** $f'(a)$ **is undefined.**

Critical point

A **critical point** for a function f is a point $(a, f(a))$ where a is a critical number of f.

A local max or min of f can only occur at a critical point.

Example 3.3.2

Find the critical points of $f(x) = x^3 - 6x^2 + 9x + 2$.

A critical number of f can occur only where $f'(x) = 0$ or where f' does not exist.

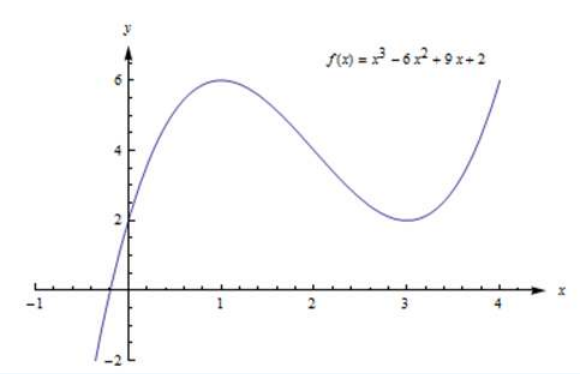
$$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x - 1)(x - 3)$$

So $f'(x) = 0$ at $x = 1$ and $x = 3$ (and no other values of x). There are no places where f' is undefined.

The critical numbers are $x = 1$ and $x = 3$. So the critical points are (1, 6) and (3, 2).

These are the only possible locations of local extremes of f. We haven't discussed yet how to tell whether either of these points is actually a local extreme of f, or which kind it might be. But we can be certain that no other point is a local extreme.

The graph of f below shows that $(1, f(1)) = (1, 6)$ is a local maximum and $(3, f(3)) = (3, 2)$ is a local minimum. This function does not have a global maximum or minimum.



Example 3.3.3

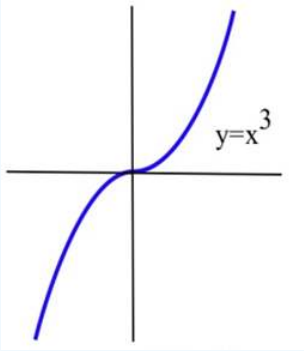
Find all local extremes of $f(x) = x^3$.

$f(x) = x^3$ is differentiable for all x, and $f'(x) = 3x^2$. The only place where $f'(x) = 0$ is at $x = 0$, so the only candidate is the critical point (0,0).

If $x > 0$ then $f(x) = x^3 > 0 = f(0)$, so $f(0)$ is not a local maximum.

Similarly, if $x < 0$ then $f(x) = x^3 < 0 = f(0)$ so $f(0)$ is not a local minimum.

The critical point (0,0) is the only candidate to be a local extreme of f, and, based on the graph, this candidate did not turn out to be a local extreme of f. The function $f(x) = x^3$ does not have any local extremes.



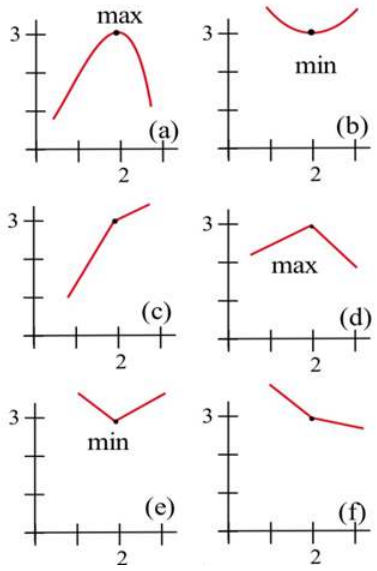
Remember this example! It is not enough to find the critical points – we can only say that f *might* have a local extreme at the critical points.

First and Second Derivative Tests

Is that critical point a Maximum or Minimum (or Neither)?

Once we have found the critical points of f, we still have the problem of determining whether these points are maxima, minima or neither.

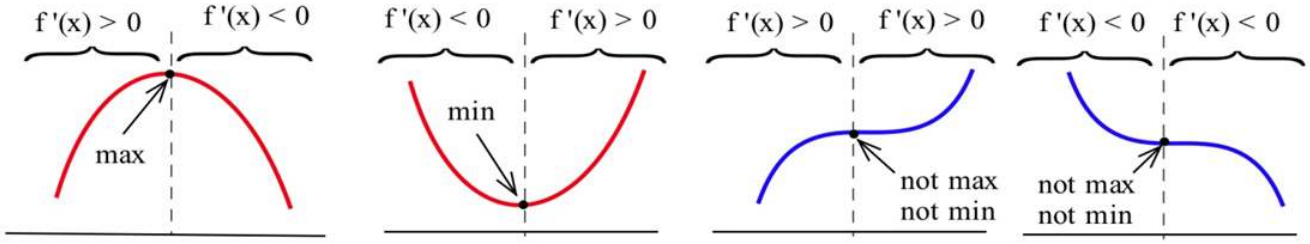
All of the graphs in the figure below have a critical point at (2, 3). It is clear from the graphs that the point (2,3) is a local maximum in (a) and (d), (2,3) is a local minimum in (b) and (e), and (2,3) is not a local extreme in (c) and (f).



The critical numbers only give the *possible* locations of extremes, and some critical numbers are not the locations of extremes. The critical numbers are the *candidates* for the locations of maxima and minima.

f' and Extreme Values of f

Four possible shapes of graphs are shown here – in each graph, the point marked by an arrow is a critical point, where $f'(x) = 0$. What happens to the derivative near the critical point?



At a local max, such as in the graph on the left, the function increases on the left of the local max, then decreases on the right. The derivative is first positive, then negative at a local max. At a local min, the function decreases to the left and increases to the right, so the derivative is first negative, then positive. When there isn't a local extreme, the function continues to increase (or decrease) right past the critical point – the derivative doesn't change sign.

The First Derivative Test for Extremes

Find the critical points of f.

For each critical number c, examine the sign of f' to the left and to the right of c. What happens to the sign as you move from left to right?

- If $f'(x)$ **changes from positive to negative** at $x = c$, then f has a **local maximum** at $(c, f(c))$.
- If $f'(x)$ **changes from negative to positive** at $x = c$, then f has a **local minimum** at $(c, f(c))$.
- If $f'(x)$ **does not change sign** at $x = c$, then $(c, f(c))$ is **neither** a local maximum nor a local minimum.

Example 3.3.4

Find the critical points of $f(x) = x^3 - 6x^2 + 9x + 2$ and classify them as local max, local min, or neither.

We already found the critical points; they are (1, 6) and (3, 2).

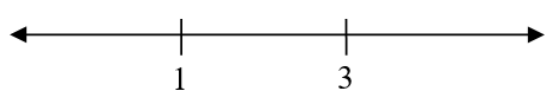
Now we can use the first derivative test to classify each. Recall that $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3)$. The factored form is easiest to work with here, so let's use that.

At (1, 6) we could choose a number slightly less than 1 to plug into the formula for f' – perhaps use $x = 0$, or $x = 0.9$. Then we could examine its sign. But we don't care about the numerical value, all we are interested in is its sign. And for that, we don't have to do any plugging in:

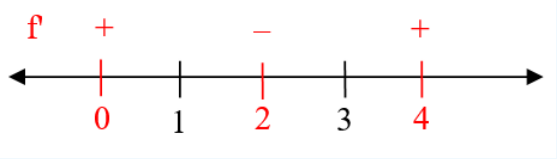
- If x is a little less than 1, then $x - 1$ is negative, and $x - 3$ is negative. So $f' = 3(x-1)(x-3)$ will be pos(neg)(neg) = positive.
- For x a little more than 1, we can evaluate f' at a number more than 1 (but less than 3, we don't want to go past the next critical point!) – perhaps $x = 2$. Or we can make a quick sign argument like what we did above: for x a little more than 1, $f' = 3(x-1)(x-3)$ will be pos(pos)(neg) = negative.

So f' changes from positive to negative, which means there is a local max at (1, 6).

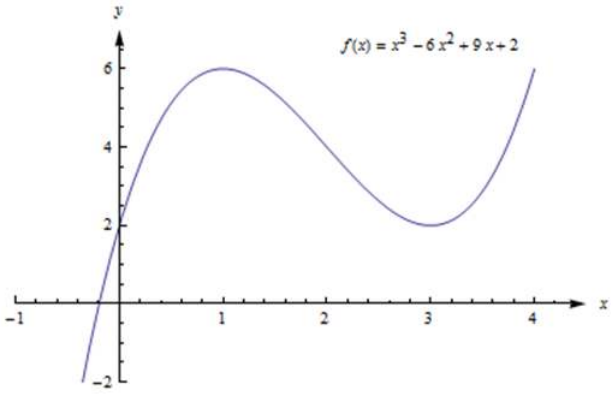
As another approach, we could draw a number line, and mark the critical numbers:



We already know the derivative is zero or undefined at the critical numbers. On each interval between these values, the derivative will stay the same sign. To determine the sign, we could pick a test value in each interval, and evaluate the derivative at those points (or use the sign approach used above).

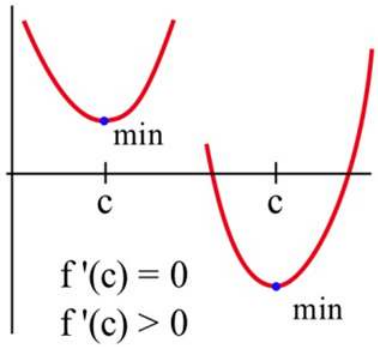


At (3, 2) f' changes from negative to positive, so there is a local min at (3, 2). This confirms what we saw before in the graph.



f'' and Extreme Values of f

The concavity of a function can also help us determine whether a critical point is a maximum or minimum or neither. For example, if a point is at the bottom of a concave up function, then the point is a minimum.



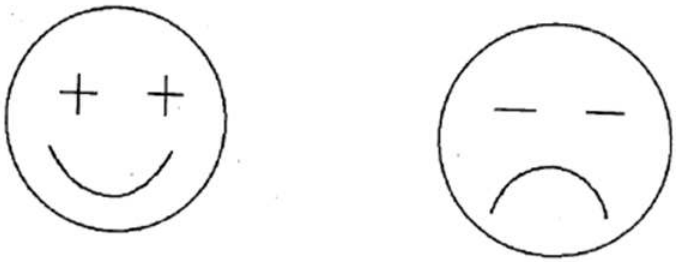
The Second Derivative Test for Extremes

Find all critical points of f. For those critical points where $f'(c) = 0$, find $f''(c)$.

- If $f''(c) < 0$ (negative) then f is concave down and has a local maximum at $x = c$.
- If $f''(c) > 0$ (positive) then f is concave up and has a local minimum at $x = c$.

- If $f''(c) = 0$ then f may have a local maximum, a minimum or neither at $x = c$.

The cartoon faces can help you remember the Second Derivative Test.



Example 3.3.5

$f(x) = 2x^3 - 15x^2 + 24x - 7$ has critical numbers $x = 1$ and 4. Use the Second Derivative Test for Extremes to determine whether $f(1)$ and $f(4)$ are maximums or minimums or neither.

We need to find the second derivative:

$$\begin{aligned} f(x) &= 2x^3 - 15x^2 + 24x - 7 \\ f'(x) &= 6x^2 - 30x + 24 \\ f''(x) &= 12x - 30 \end{aligned}$$

Then we just need to evaluate f'' at each critical number:

$x = 1$: $f''(1) = 12(1) - 30 < 0$, so there is a local maximum at $x = 1$.

$x = 4$: $f''(4) = 12(4) - 30 > 0$, so there is a local minimum at $x = 4$.

Many students like the Second Derivative Test. The Second Derivative Test is often easier to use than the First Derivative Test. You only have to find the sign of one number for each critical number rather than two. And if your function is a polynomial, its second derivative will probably be a simpler function than the derivative.

However, if you needed a product rule, quotient rule, or chain rule to find the first derivative, finding the second derivative can be a lot of work. Also, even if the second derivative is easy, the Second Derivative Test doesn't always give an answer. The First Derivative Test will always give you an answer.

Use whichever test you want to. But remember – you have to do some test to be sure that your critical point actually is a local max or min.

Global Maxima and Minima

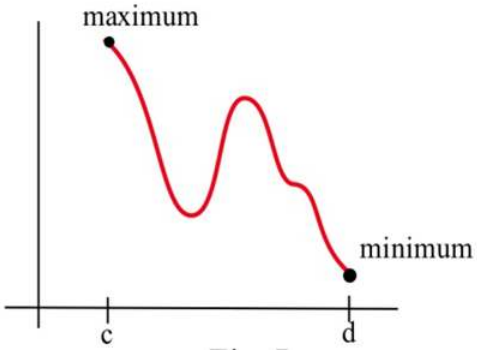
In applications, we often want to find the global extreme; knowing that a critical point is a local extreme is not enough.

For example, if we want to make the greatest profit, we want to make the absolutely greatest profit of all. How do we find global max and min?

There are just a few additional things to think about.

Endpoint Extremes

The local extremes of a function occur at critical points – these are points in the function that we can find by thinking about the shape (and using the derivative to help us). But if we're looking at a function on a closed interval, the endpoints could be extremes. These endpoint extremes are not related to the shape of the function; they have to do with the interval, the window through which we're viewing the function.



In the graph above, it appears that there are three critical points – one local min, one local max, and one that is neither one. But the global max, the highest point of all, is at the left endpoint. The global min, the lowest point of all, is at the right endpoint.

How do we decide if endpoints are global max or min? It's easier than you expected – simply plug in the endpoints, along with all the critical numbers, and compare y-values.

To find Global Extremes

This technique can be used when you are trying to find global extrema over a limited interval.

The only places where a function can have a global extreme on a limited interval are at critical points or endpoints.

- If the function has only one critical point, and it's a local extreme, then it is also the global extreme.
- If there are endpoints, find the global extremes by comparing y-values at all the critical points and at the endpoints.
- When in doubt, graph the function to be sure. (However, unless the problem explicitly tells you otherwise, it is **not** enough to just use the graph to get your answer.)

Example 3.3.6

Find the global max and min of $f(x) = x^3 - 3x^2 - 9x + 5$ for $-2 \le x \le 6$.

$f'(x) = 3x^2 - 6x - 9 = 3(x+1)(x-3)$. We need to find critical points, and we need to check the endpoints.

$f'(x) = 3(x+1)(x-3) = 0$ when $x = -1$ and $x = 3$. The endpoints of the interval are $x = -2$ and $x = 6$.

Now we simply compare the values of f at these four values of x:

x	$f(x)$
-2	3
-1	10
3	-22
6	59

The global minimum of f on $[-2, 6]$ is -22, when $x = 3$, and the global maximum of f on $[-2, 6]$ is 59, when $x = 6$.

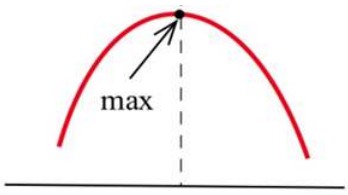
Caution

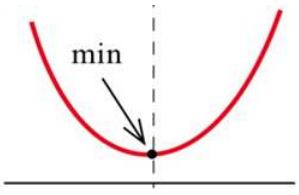
Be careful! You can **only** use this technique (evaluating the function at endpoints and critical values) to determine **global** extrema over a limited interval.

If a question asks for **local** extrema, you **must** use the first or second derivative test described earlier in the section.

If there's only one critical point

If the function has only one critical point and it's a local max (or min), then it must be the global max (or min). To see this, think about the geometry. Look at the graph on the left – there is a local max, and the graph goes down on either side of the critical point. Suppose there was some other point that was higher – then the graph would have to turn around. But that turning point would have shown up as another critical point. If there's only one critical point, then the graph can never turn back around.





When in doubt, graph it and look.

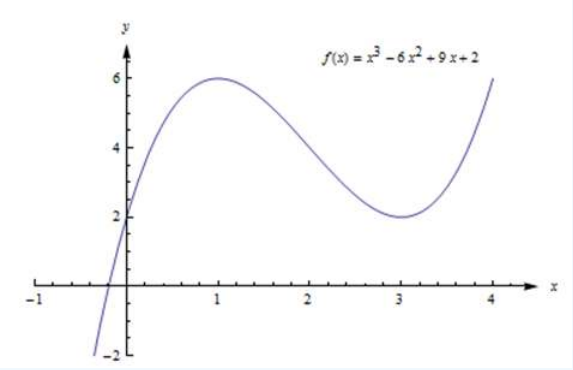
If you are trying to find a global max or min on an open interval (or the whole real line), and there is more than one critical point, then you need to look at the graph to decide whether there is a global max or min. Be sure that all your critical points show in your graph, and that you graph beyond them – that will tell you what you want to know.

Example 3.3.7

Find the global max and min of $f(x) = x^3 - 6x^2 + 9x + 2$.

We have previously found that (1, 6) is a local max and (3, 2) is a local min. This is not a closed interval, and there are two critical points, so we must turn to the graph of the function to find global max and min.

The graph of f shows that points to the left of $x = 4$ have y-values greater than 6, so (1, 6) is not a global max. Likewise, if x is negative, y is less than 2, so (3, 2) is not a global min. There are no endpoints, so we've exhausted all the possibilities. This function does not have a global maximum or minimum.



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3.4: Applied Optimization

We have used derivatives to help find the maximums and minimums of some functions given by equations, but it is very unlikely that someone will simply hand you a function and ask you to find its extreme values. More typically, someone will describe a problem and ask your help in maximizing or minimizing something: “What is the largest volume package which the post office will take?”; “What is the quickest way to get from here to there?”; or “What is the least expensive way to accomplish some task?” In this section, we’ll discuss how to find these extreme values using calculus.

Max/Min Applications

Example

The manager of a garden store wants to build a 600 square foot rectangular enclosure on the store’s parking lot in order to display some equipment. Three sides of the enclosure will be built of redwood fencing, at a cost of $7 per running foot. The fourth side will be built of cement blocks, at a cost of $14 per running foot. Find the dimensions of the least costly such enclosure.

The process of finding maxima or minima is called optimization. The function we're optimizing is called the **objective function** (or **objective equation**). The objective function can be recognized by its proximity to “est” words (greatest, least, highest, farthest, most, …). Look at the garden store example; the cost function is the objective function.

In many cases, there are two (or more) variables in the problem. In the garden store example again, the length and width of the enclosure are both unknown. If there is an equation that relates the variables we can solve for one of them in terms of the others, and write the objective function as a function of just one variable. Equations that relate the variables in this way are called **constraint equations**. The constraint equations are always equations, so they will have equals signs. For the garden store, the fixed area relates the length and width of the enclosure. This will give us our constraint equation.

Max-Min Story Problem Technique

1. Translate the English statement of the problem line by line into a picture (if that applies) and into math. This is often the hardest step!
2. Identify the objective function. Look for words indicating a largest or smallest value.
    1. If you seem to have two or more variables, find the constraint equation. Think about the English meaning of the word “constraint”, and remember that the constraint equation will have an equals sign.
    2. Solve the constraint equation for one variable and substitute into the objective function. Now you have an equation of one variable.
3. Use calculus to find the optimum values. (Take derivative, find critical points, test. Don't forget to check the endpoints!)
4. Look back at the question to make sure you answered what was asked. Translate your number answer back into English.

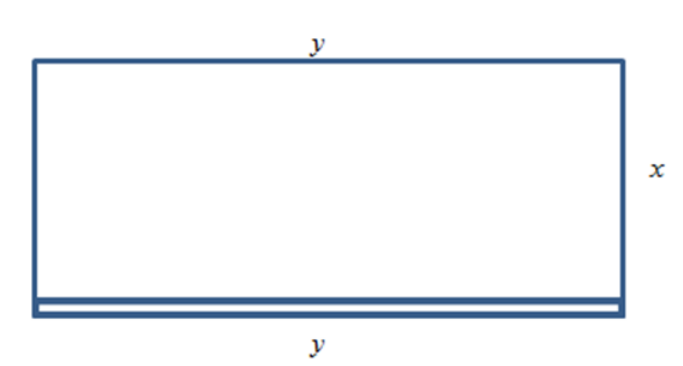
Here is a link to the examples used in the videos in this section: Applied Optimization Examples.

Example 3.4.1

The manager of a garden store wants to build a 600 square foot rectangular enclosure on the store's parking lot in order to display some equipment. Three sides of the enclosure will be built of redwood fencing, at a cost of $7 per running foot. The fourth side will be built of cement blocks, at a cost of $14 per running foot. Find the dimensions of the least costly such enclosure.

First, translate line by line into math and a picture:

Text	Translation
The manager of a garden store wants to build a *600 square foot rectangular* enclosure on the store's parking lot in order to display some equipment. *Three sides of the enclosure* will be built of redwood fencing, at a *cost of $7 per running foot*. The *fourth side* will be built of cement blocks, at a *cost of $14 per running foot*. Find the dimensions of the least costly such enclosure.	Let x and y be the dimensions of the enclosure, with y being the length of the side made of blocks. Then: $\text{Area} = A = xy = 600.$ $2x + y$ costs $7 per foot, y costs $14 per foot, so $\text{Cost} = C = 7(2x + y) + 14y = 14x + 21y.$ Find x and y so that C is minimized.



The objective function is the cost function, and we want to minimize it. As it stands, though, it has two variables, so we need to use the constraint equation. The constraint equation is the fixed area $A = xy = 600$. Solve A for x to get $x = \frac{600}{y}$, and then substitute into C:

$$C = 14\left(\frac{600}{y}\right) + 21y = \frac{8400}{y} + 21y.$$

Now we have a function of just one variable, so we can find the minimum using calculus.

$$C' = -\frac{8400}{y^2} + 21$$

C' is undefined for $y = 0$, and $C' = 0$ when $y = 20$ or $y = -20$.

Of these three critical numbers, only $y = 20$ makes sense (is in the domain of the actual function) – remember that y is a length, so it can't be negative, and $y = 0$ would mean there was no enclosure at all, so it couldn't have an area of 600 square feet.

Test $y = 20$ (here we chose the second derivative test):

$$C'' = \frac{16800}{y^3} > 0,$$

so this is a local minimum.

Since this is the only critical point in the domain, this must be the global minimum. Going back to our constraint function, we can find that when $y = 20$, $x = 30$. The dimensions of the enclosure that minimize the cost are 20 feet by 30 feet.

When trying to maximize their revenue, businesses also face the constraint of consumer demand. While a business would love to see lots of products at a very high price, typically demand decreases as the price of goods increases. In simple cases, we can construct that demand curve to allow us to maximize revenue.

Example 3.4.2

A concert promoter has found that if she sells tickets for \$50 each, she can sell 1200 tickets, but for each \$5 she raises the price, 50 less people attend. What price should she sell the tickets at to maximize her revenue?

We are trying to maximize revenue, and we know that $R = pq$, where p is the price per ticket, and q is the quantity of tickets sold.

The problem provides information about the demand relationship between price and quantity – as price increases, demand decreases. We need to find a formula for this relationship. To investigate, let's calculate what will happen to attendance if we raise the price:

Price, p	50	55	60	65
Quantity, q	1200	1150	1100	1050

You might recognize this as a linear relationship. We can find the slope for the relationship by using two points:

$$m = \frac{1150 - 1200}{55 - 50} = \frac{-50}{5} = -10.$$

You may notice that the second step in that calculation corresponds directly to the statement of the problem: the attendance drops 50 people for every \$5 the price increases.

Using the point-slope form of the line, we can write the equation relating price and quantity:

$$q - 1200 = -10(p - 50).$$

Simplifying to slope-intercept form gives the demand equation

$$q = 1700 - 10p.$$

Substituting this into our revenue equation, we get an equation for revenue involving only one variable:

$$R = pq = p(1700 - 10p) = 1700p - 10p^2.$$

Now, we can find the maximum of this function by finding critical numbers. $R' = 1700 - 20p$, so $R' = 0$ when $p = 85$.

Using the second derivative test, $R'' = -20 < 0$ (for any value of p), so the critical number is a local maximum. Since it is the only critical number, we can also conclude that it is the global maximum.

The promoter will be able to maximize revenue by charging \$85 per ticket. At this price, she will sell $q = 1700 - 10(85) = 850$ tickets, generating \$72,250 in revenue.

“Marginal Revenue = Marginal Cost”

You may have heard before that “profit is maximized when marginal cost and marginal revenue are equal.” Now you can see why people say that! (Even though it’s not completely true.)

Example 3.4.3

Suppose we want to maximize profit.

Now we know what to do – find the profit function, find its critical points, test them, etc.

But remember that Profit = Revenue - Cost. So Profit' = Revenue' - Cost'. That is, the derivative of the profit function is $MR - MC$.

Now let’s find the critical points – those will be where Profit' = 0 or is undefined. Profit' = 0 when $MR - MC = 0$, or where $MR = MC$.

Profit has critical points when Marginal Revenue and Marginal Cost are equal.

In all the cases we'll see in this class, Profit will be very well behaved, and we won't have to worry about looking for critical points where Profit' is undefined. But remember that not all critical points are local max! The places where $MR = MC$ could represent local max, local min, or neither one.

Example 3.4.4

A company sells q ribbon winders per year at $\$p$ per ribbon winder. The demand function for ribbon winders is given by: $p = 300 - 0.02q$. The ribbon winders cost \$30 apiece to manufacture, plus there are fixed costs of \$9000 per year. Find the quantity where profit is maximized.

We want to maximize profit, but there isn’t a formula for profit given. So let's make one. We can find a function for Revenue = pq using the demand function for p.

$$R(q) = (300 - 0.02q)q = 300q - 0.02q^2.$$

We can also find a function for Cost, using the variable cost of \$30 per ribbon winder, plus the fixed cost:

$$C(q) = 9000 + 30q.$$

Putting them together, we get a function for Profit:

$$P(q) = R(q) - C(q) = \left(300q - 0.02q^2\right) - (9000 + 30q) = -0.02q^2 + 270q - 9000$$

Now we have two choices. We can find the critical points of Profit by taking the derivative of $P(q)$ directly, or we can find MR and MC and set them equal. (Naturally, we'll get the same answer either way.)

Let's use $MR = MC$ this time.

$$\begin{aligned} MR &= 300 - 0.04q \\ MC &= 30 \\ 300 - 0.04q &= 30 \\ 270 &= 0.04q \\ q &= 6750 \end{aligned}$$

The only critical point is at $q = 6750$. Now we need to be sure this is a local max and not a local min. In this case, we'll look to the graph of $P(q)$ – it's a downward opening parabola, so this must be a local max. And since it's the only critical point, it must also be the global max.

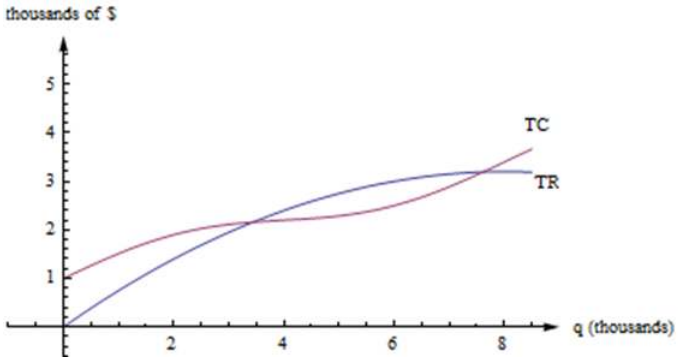
Profit is maximized when they sell 6750 ribbon winders.

"Average Cost = Marginal Cost"

"Average cost is minimized when average cost = marginal cost" is another saying that isn't quite true; in this case, the correct statement is:

Average Cost has critical points when Average Cost and Marginal Cost are equal.

Let's look at a geometric argument here. Remember that the average cost is the slope of the diagonal line, the line from the origin to the point on the total cost curve. If you move your clear plastic ruler around, you'll see (and feel) that the slope of the diagonal line is smallest when the diagonal line just touches the cost curve – when the diagonal line is actually a tangent line – when the average cost is equal to the marginal cost.



Example 3.4.5

The cost in dollars to produce q gift baskets is given by $C(q) = 160 + 2q + .1q^2$. Find the quantity where the average cost is minimum.

Solution

$A(q) = \frac{C(q)}{q} = \frac{160}{q} + 2 + .1q$. We could find the critical points by finding A', or by setting average cost to marginal cost; we'll do the latter this time.

$$MC(q) = 2 + .2q.$$

So we want to solve:

$$\begin{aligned} \frac{160}{q} + 2 + .1q &= 2 + .2q \\ \frac{160}{q} &= .1q \\ 1600 &= q^2 \\ q &= 40 \end{aligned}$$

The critical point of average cost is when $q = 40$.

Notice that we still have to confirm that the critical point is a minimum. For this, we can use the first or second derivative test on $A(q)$.

$$A'(q) = \frac{-160}{q^2} + .1$$

$$A''(q) = \frac{320}{q^3} > 0$$

The second derivative is positive for all positive q, so that means this is a local min. Average cost is minimized when they produce 40 gift baskets; at that quantity, the average cost is \$10 per basket.

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3.5: Implicit Differentiation and Related Rates

Implicit Differentiation

In our work up until now, the functions we needed to differentiate were either given **explicitly**, such as $y = x^2 + e^x$, or it was possible to get an explicit formula for them, such as solving $y^3 - 3x^2 = 5$ to get $y = \sqrt[3]{5 + 3x^2}$. Sometimes, however, we will have an equation relating x and y which is either difficult or impossible to solve explicitly for y, such as $y + e^y = x^2$. In any case, we can still find $y' = f'(x)$ by using implicit differentiation.

The key idea behind implicit differentiation is to **assume that y is a function of x** even if we cannot explicitly solve for y. This assumption does not require any work, but we need to be very careful to treat y as a function when we differentiate and to use the Chain Rule.

Example 3.5.1

Assume that y is a function of x. Calculate

1. $\frac{d}{dx}\left(y^3\right)$
2. $\frac{d}{dx}\left(x^3y^2\right)$
3. $\frac{d}{dx}(\ln(y))$

Solution

1. We need the chain rule since y is a function of x:

$$\frac{d}{dx}\left(y^3\right) = 3y^2\frac{dy}{dx} \overset{\text{or}}{=} 3y^2y'$$

2. We need to use the product rule and the Chain Rule:

$$\begin{aligned}\frac{d}{dx}\left(x^3y^2\right) &= x^3\frac{d}{dx}\left(y^2\right) + y^2\frac{d}{dx}\left(x^3\right)\\ &= x^3 2y\frac{dy}{dx} + y^2 3x^2\\ &\overset{\text{or}}{=} 2x^3yy' + 3y^2x^2\end{aligned}$$

3. We know $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$, so we use that and the Chain Rule:

$$\frac{d}{dx}(\ln(y)) = \frac{1}{y}\cdot y'$$

Implicit Differentiation

To determine y', differentiate each side of the defining equation, **treating y as a function of x**, and then algebraically solve for y'.

(The last example in the following video gets rather messy – don't worry too much if you can't follow all of the simplifications at the end.)

Example 3.5.2

Find the slope of the tangent line to the circle $x^2+y^2=25$ at the point (3,4) using implicit differentiation.

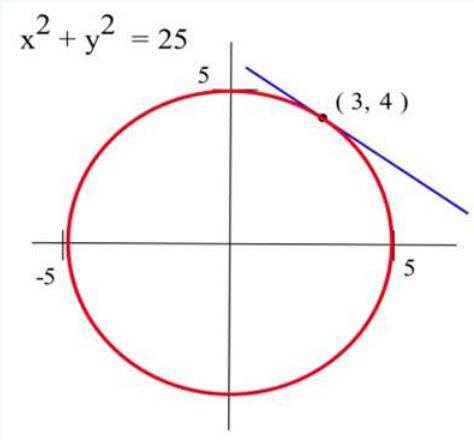
Solution

We differentiate each side of the equation $x^2+y^2=25$ and then solve for y':

$$\begin{aligned}\frac{d}{dx}\left(x^2+y^2\right) &= \frac{d}{dx}(25)\\ 2x+2yy' &= 0\end{aligned}$$

Solving for y', we have $y'=-\frac{2x}{2y}=-\frac{x}{y}$, and, at the point (3,4),

$$y'=-\frac{3}{4}.$$



In the previous example, it would have been easy to explicitly solve for y, and then we could differentiate y to get y'. Because we could explicitly solve for y, we had a choice of methods for calculating y'. Sometimes, however, we cannot explicitly solve for y, and the only way of determining y' is to use implicit differentiation.

Related Rates

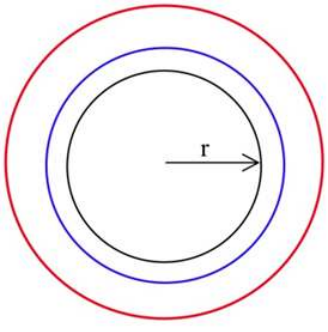
If several variables or quantities are related to each other and some of the variables are changing at a known rate, then we can use derivatives to determine how rapidly the other variables must be changing.

Here is a link to the examples used in the videos in this section: Related Rates.

Example 3.5.3

Suppose the border of a town is roughly circular, and the radius of that circle has been increasing at a rate of 0.1 miles each year. Find how fast the area of the town has been increasing when the radius is 5 miles.

Solution



We could get an *approximate* answer by calculating the area of the circle when the radius is 5 miles ($A = \pi r^2 = \pi(5 \text{ miles})^2 \approx 78.6 \text{ miles}^2$) and 1 year later when the radius is 0.1 feet larger than before ($A = \pi r^2 = \pi(5.1 \text{ miles})^2 \approx 81.7 \text{ miles}^2$) and then finding

$$\frac{\Delta \text{Area}}{\Delta \text{time}} = \frac{81.7 \text{ mi}^2 - 78.6 \text{ mi}^2}{1 \text{ year}} = 3.1 \text{ mi}^2/\text{yr}.$$

This approximate answer represents the average change in area during the 1 year period when the radius increased from 5 miles to 5.1 miles, and would correspond to the secant slope on the area graph.

To find the exact answer, though, we need derivatives. In this case both radius and area are functions of time:

$$r(t) = \text{ radius at time } t \qquad A(t) = \text{ area at time } t$$

We know how fast the radius is changing, which is a statement about the derivative: $\frac{dr}{dt} = 0.1 \frac{\text{mile}}{\text{year}}$. We also know that $r = 5$ at our moment of interest.

We are looking for how fast the area is increasing, which is $\frac{dA}{dt}$.

Now we need an equation relating our variables, which is the area equation:

$$A = \pi r^2.$$

Taking the derivative of both sides of that equation with respect to t, we can use implicit differentiation:

$$\frac{d}{dt}(A) = \frac{d}{dt}\left(\pi r^2\right)$$
$$\frac{dA}{dt} = \pi 2r \frac{dr}{dt}$$

Plugging in the values we know for r and $\frac{dr}{dt}$,

$$\frac{dA}{dt} = \pi 2(5 \text{ miles}) \left(0.1 \frac{\text{miles}}{\text{year}}\right) = \pi \frac{\text{miles}^2}{\text{year}}$$

So the area of the town is increasing by approximately 3.14 square miles per year when the radius is 5 miles.

Related Rates

When working with a related rates problem,

1. Draw a picture (if possible).
2. Identify the quantities that are changing, and assign them variables.
3. Find an equation that relates those quantities.
4. Differentiate both sides of that equation with respect to time.
5. Plug in any known values for the variables or rates of change.
6. Solve for the desired rate.

Example 3.5.4

A company has determined the demand curve for their product is $q = \sqrt{5000 - p^2}$, where p is the price in dollars, and q is the quantity in millions. If weather conditions are driving the price up \$2 a week, find the rate at which demand is changing when the price is \$40.

Solution

The quantities changing are p and q, and we assume they are both functions of time, t, in weeks. We already have an equation relating the quantities, so we can implicitly differentiate it.

$$\begin{aligned}\frac{d}{dt}(q) &= \frac{d}{dt}\left(5000-p^2\right)^{1/2}\\ \frac{dq}{dt} &= \frac{1}{2}\left(5000-p^2\right)^{-1/2}\frac{d}{dt}\left(5000-p^2\right)\\ \frac{dq}{dt} &= \frac{1}{2}\left(5000-p^2\right)^{-1/2}\left(-2p\frac{dp}{dt}\right)\end{aligned}$$

Using the given information, we know the price is increasing by \$2 per week when the price is \$40, giving $\frac{dp}{dt}=2$ when $p=40$. Plugging in these values,

$$\frac{dq}{dt}=\frac{1}{2}\left(5000-40^2\right)^{-1/2}\left(-2(40)(2)\right)\approx -1.37$$

Demand is falling by 1.37 million items per week.

Example 3.5.5

The total daily cost for producing x items in a day is $TC(x)=300,000+4x+\frac{200,000}{x}$. If production has been ramping up by 20 items a day, find the rate at which total daily cost is increasing, if they are currently producing 2,000 items.

Solution

The quantities changing are x and TC, and we assume they are both functions of time, t, in days. We already have an equation relating the quantities, so we can implicitly differentiate it.

$$\begin{aligned}\frac{d}{dt}(TC) &= \frac{d}{dt}\left(300,000+4x+200,000x^{-1}\right)\\ \frac{dTC}{dt} &= 4\frac{dx}{dt}-200,000x^{-2}\frac{dx}{dt}\\ \frac{dTC}{dt} &= \left(4-\frac{200,000}{x^2}\right)\frac{dx}{dt}\end{aligned}$$

We know the quantity produced is increasing by 20 items per week when the production is 2,000 items, giving $\frac{dx}{dt}=20$ when $x=2000$. Plugging in these values,

$$\frac{dTC}{dt}=\left(4-\frac{200,000}{2000^2}\right)(20)=79$$

Total daily cost is increasing by \$79 each day.

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CHAPTER OVERVIEW

4: The Integral

4.1: Prelude to the Integral
4.2: Antidifferentiation
4.3: Antiderivatives as Areas
4.4: The Fundamental Theorem and Antidifferentiation
4.5: Area, Average Value, and Volume
4.6: Substitution
4.7: Applications to Business
4.E: The Integral (Exercises)

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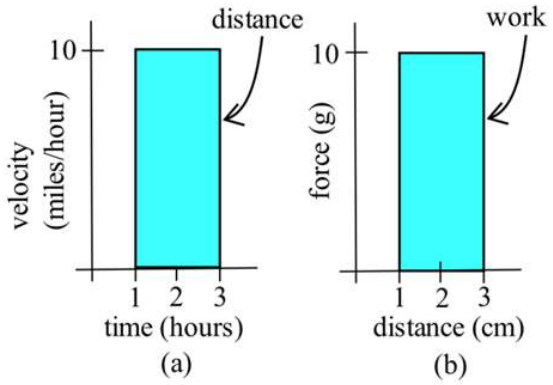
4.1: Prelude to the Integral

The previous chapters dealt with **Differential Calculus**. We started with the "simple" geometrical idea of the **slope of a tangent line** to a curve, developed it into a combination of theory about derivatives and their properties, techniques for calculating derivatives, and applications of derivatives. This chapter deals with **Integral Calculus** and starts with the "simple" geometric idea of **area**. This idea will be developed into another combination of theory, techniques, and applications.

PreCalculus Idea – The Area of a Rectangle

If you look on the inside cover of nearly any traditional math book, you will find a bunch of area and volume formulas – the area of a square, the area of a trapezoid, the volume of a right circular cone, and so on. Some of these formulas are pretty complicated. But you still won't find a formula for the area of a jigsaw puzzle piece or the volume of an egg. There are lots of things for which there is no formula. Yet we might still want to find their areas.

One reason areas are so useful is that they can represent quantities other than simple geometric shapes. If the units for each side of the rectangle are *meters*, then the area will have the units $meters \cdot meters = square\ meters = \mathrm{m}^2$. But if the units of the base of a rectangle are *hours* and the units of the height are *miles/hour*, then the units of the area of the rectangle are $hours \cdot (miles/hour) = miles$, a measure of distance. Similarly, if the base units are *centimeters* and the height units are *grams*, then the area units are *gram-centimeters*, a measure of work.



The basic shape we will use is the rectangle; the area of a rectangle is base·height. You should also know the area formulas for triangles, $A = \frac{1}{2}bh$, and for circles, $A = \pi r^2$.

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4.2: Antidifferentiation

Now we can put the ideas of areas and antiderivatives together to get a way of evaluating definite integrals that is exact and often easy. To evaluate a definite integral $\int_a^b f(t)\,dt$, we can find any antiderivative $F(t)$ of $f(t)$ and evaluate $F(b)-F(a)$. The problem of finding the exact value of a definite integral reduces to finding some (any) antiderivative F of the integrand and then evaluating $F(b)-F(a)$. Even finding one antiderivative can be difficult, and we will stick to functions that have easy antiderivatives.

Building Blocks

Antidifferentiation is going backwards through the derivative process. So the easiest antiderivative rules are simply backwards versions of the easiest derivative rules. Recall from Chapter 2:

Derivative Rules: Building Blocks

In what follows, f and g are differentiable functions of x.

Constant Multiple Rule

$$\frac{d}{dx}(kf) = kf'$$

Sum and Difference Rule

$$\frac{d}{dx}(f \pm g) = f' \pm g'$$

Power Rule

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Special cases:

$$\frac{d}{dx}(k) = 0 \quad (\text{Because } k = kx^0.)$$

$$\frac{d}{dx}(x) = 1 \quad (\text{Because } x = x^1.)$$

Exponential Functions

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(a^x) = \ln(a)\,a^x$$

Natural Logarithm

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

Thinking about these basic rules was how we came up with the antiderivatives of $2x$ and e^x before.

The corresponding rules for antiderivatives are next – each of the antiderivative rules is simply rewriting the derivative rule. All of these antiderivatives can be verified by differentiating.

There is one surprise – the antiderivative of $\frac{1}{x}$ is actually not simply $\ln(x)$, it's $\ln|x|$. This is a good thing – the antiderivative has a domain that matches the domain of $\frac{1}{x}$, which is bigger than the domain of $\ln(x)$, so we don't have to worry about whether our x's are positive or negative. But we *must* be careful to include those absolute values – otherwise, we could end up with domain problems.

Antiderivative Rules: Building Blocks

In what follows, f and g are differentiable functions of x, and k, n, and C are constants.

Constant Multiple Rule

$$\int k \cdot f(x)\,dx = k \cdot \int f(x)\,dx$$

Sum and Difference Rule

$$\int \left(f(x) \pm g(x)\right)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Power Rule

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \text{ provided that } n \neq -1$$

Special case:

$$\int k\,dx = kx + C \quad (\text{Because } k = kx^0.)$$

(The other special case ($n = -1$) is covered next.)

Natural Logarithm

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C$$

Exponential Functions

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\ln(a)} + C$$

Example 4.2.1

Find the antiderivative of $y = 3x^7 - 15\sqrt{x} + \frac{14}{x^2}$.

Solution

$$\begin{aligned}\int \left(3x^7 - 15\sqrt{x} + \frac{14}{x^2}\right)\,dx &= \int \left(3x^7 - 15x^{1/2} + 14x^{-2}\right)\,dx \\ &= 3\frac{x^8}{8} - 15\frac{x^{3/2}}{3/2} + 14\frac{x^{-1}}{-1} + C \\ &= \frac{3}{8}x^8 - 10x^{3/2} - 14x^{-1} + C\end{aligned}$$

Example 4.2.2

Find $\int \left(e^x + 12 - \frac{16}{x}\right)\, dx$.

Solution

$$\int \left(e^x + 12 - \frac{16}{x}\right)\, dx = e^x + 12x - 16\ln|x| + C$$

Example 4.2.3

Find $F(x)$ so that $F'(x) = e^x$ and $F(0) = 10$.

Solution

This time we are looking for a particular antiderivative; we need to find exactly the right constant. Let's start by finding the antiderivative:

$$\int e^x\, dx = e^x + C$$

So we know that $F(x) = e^x + (\text{some constant})$, now we just need to find which one. To do that, we'll use the other piece of information (the initial condition):

$$\begin{aligned} F(x) &= e^x + C \\ F(0) &= e^0 + C = 1 + C = 10 \\ C &= 9 \end{aligned}$$

The particular constant we need is 9; thus, $F(x) = e^x + 9$.

The reason we are looking at antiderivatives right now is so we can evaluate definite integrals exactly. Recall the Fundamental Theorem of Calculus:

Fundamental Theorem of Calculus

If $F(x)$ is a function where $F'(x) = f(x)$, then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

If we can find an antiderivative for the integrand, we can use that to evaluate the definite integral. The evaluation $F(b) - F(a)$ is represented as $F(x)\big|_a^b$ or $F(x)\Big|_a^b$.

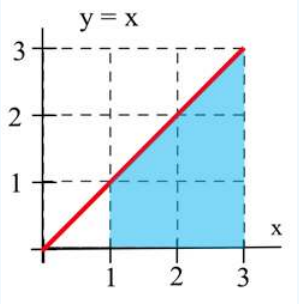
Example 4.2.4

Evaluate $\int_1^3 x\, dx$ in two ways:

1. By sketching the graph of $y = x$ and geometrically finding the area.
2. By finding an antiderivative of $F(x)$ of the integrand and evaluating $F(3) - F(1)$.

Solution

1. The graph of $y = x$ is shown below, and the shaded region corresponding to the integral has area 4.



2. One antiderivative of x is $F(x) = \frac{1}{2}x^2$, and

$$\begin{aligned}\int_1^3 x\,dx &= \left[\frac{1}{2}x^2\right]_1^3 \\ &= \left(\frac{1}{2}(3)^2\right) - \left(\frac{1}{2}(1)^2\right) \\ &= \frac{9}{2} - \frac{1}{2} \\ &= 4.\end{aligned}$$

Note that this answer agrees with the answer we got geometrically.

If we had used another antiderivative of x, say $F(x) = \frac{1}{2}x^2 + 7$, then

$$\begin{aligned}\int_1^3 x\,dx &= \left[\frac{1}{2}x^2 + 7\right]_1^3 \\ &= \left(\frac{1}{2}(3)^2 + 7\right) - \left(\frac{1}{2}(1)^2 + 7\right) \\ &= \frac{9}{2} + 7 - \frac{1}{2} - 7 \\ &= 4.\end{aligned}$$

In general, **whatever constant we choose gets subtracted away during the evaluation**, so we might as well always choose the easiest one, where the constant is 0.

Example 4.2.5

Find the area between the graph of $y = 3x^2$ and the horizontal axis for x between 1 and 2.

Solution

This is

$$\int_1^2 3x^2\,dx = x^3\Big|_1^2 = 2^3 - 1^3 = 7.$$

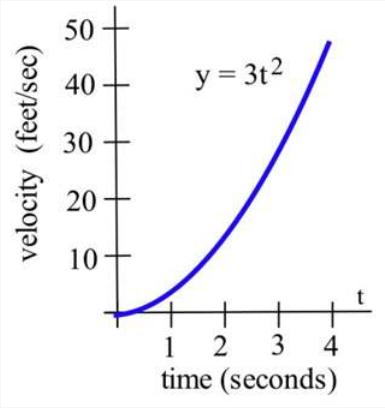
Example 4.2.6

A robot has been programmed so that when it starts to move, its velocity after t seconds will be $3t^2$ feet/second.

1. How far will the robot travel during its first 4 seconds of movement?
2. How far will the robot travel during its next 4 seconds of movement?

Solution

1. The distance during the first 4 seconds will be the area under the graph of velocity, from $t = 0$ to $t = 4$.



That area is the definite integral $\int_0^4 3t^2\,dt$. An antiderivative of $3t^2$ is t^3, so $\int_0^4 3t^2\,dt = t^3\big]_0^4 = 4^3 - 0^3 = 64$ feet.

2. $\int_4^8 3t^2\,dt = t^3\big]_4^8 = 8^3 - 4^3 = 512 - 64 = 448$ feet.

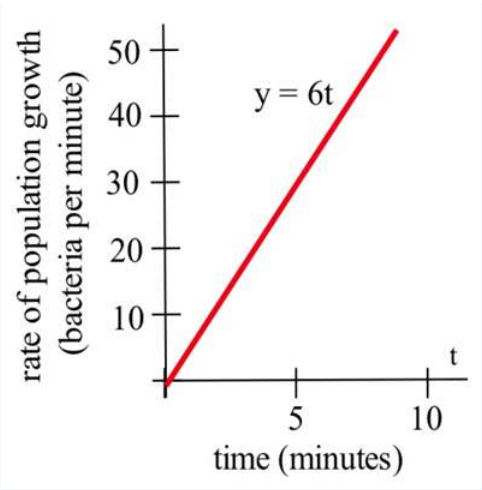
Example 4.2.7

Suppose that t minutes after putting 1000 bacteria on a Petri plate the rate of growth of the population is $6t$ bacteria per minute.

1. How many new bacteria are added to the population during the first 7 minutes?
2. What is the total population after 7 minutes?

Solution

1. The number of new bacteria is the area under the rate of growth graph, and one antiderivative of $6t$ is $3t^2$.



So

$$\text{new bacteria} = \int_0^7 6t\,dt = 3t^2\big|_0^7 = 3(7)^2 - 3(0)^2 = 147$$

2. The new population = (old population) + (new bacteria) = 1000 + 147 = 1147 bacteria.

Example 4.2.8

A company determines their marginal cost for production, in dollars per item, is $MC(x) = \frac{4}{\sqrt{x}} + 2$ when producing x thousand items. Find the cost of increasing production from 4 thousand items to 5 thousand items.

Solution

Remember that marginal cost is the rate of change of cost, and so the fundamental theorem tells us that $\int_a^b MC(x)\,dx = \int_a^b C'(x)\,dx = C(b) - C(a)$. In other words, the integral of marginal cost will give us a net change in cost. To find the cost of increasing production from 4 thousand items to 5 thousand items, we need to integrate $\int_4^5 MC(x)\,dx$.

We can write the marginal cost as $MC(x) = 4x^{-1/2} + 2$. We can then use the basic rules to find an antiderivative:

$$C(x) = 4\frac{x^{1/2}}{1/2} + 2x = 8\sqrt{x} + 2x.$$

Using this,

$$\begin{aligned}\text{Net change in cost} &= \int_4^5 \left(4x^{-1/2}+2\right)\,dx \\ &= \left[8\sqrt{x}+2x\right]_4^5 \\ &= \left(8\sqrt{5}+2(5)\right)-\left(8\sqrt{4}+2(4)\right) \\ &\approx 3.889\end{aligned}$$

It will cost 3.889 thousand dollars to increase production from 4 thousand items to 5 thousand items. (The final answer would be better written as $3889.)

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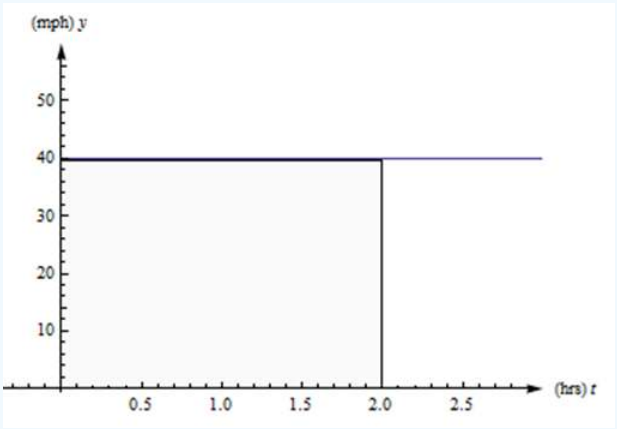
- **3.4: Antiderivatives of Formulas** by Shana Calaway, Dale Hoffman, & David Lippman is licensed CC BY 3.0. Original source: https://github.com/drlippman/buscalc.

4.3: Antiderivatives as Areas

Distance from Velocity

Example 4.3.1

Suppose a car travels on a straight road at a constant speed of 40 miles per hour for two hours. See the graph of its velocity below. How far has it gone?

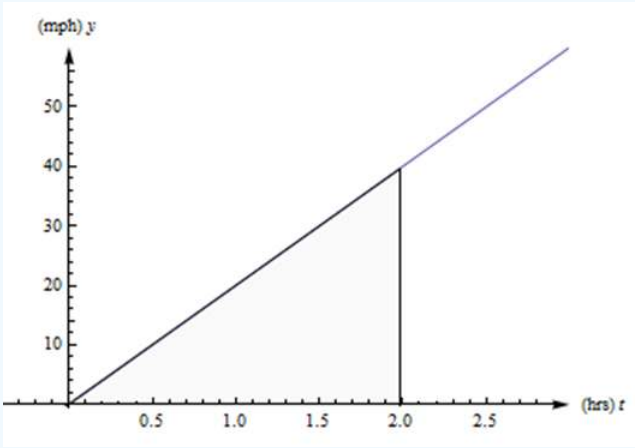


Solution

We all remember **distance = rate·time**, so this one is easy. The car has gone (40 miles/hour)·(2 hours) = 80 miles.

Example 4.3.2

Now suppose that a car travels so that its speed increases steadily from 0 to 40 miles per hour, for two hours. (Just be grateful you weren't stuck behind this car on the highway.) See the graph of its velocity in below. How far has this car gone?



Solution

The trouble with our old reliable **distance = rate·time** relationship is that it only works if the rate is constant. If the rate is changing, there isn't a good way to use this formula.

But look at the graph from the last example again. Notice that **distance = rate·time** also describes the area between the velocity graph and the t-axis, between $t = 0$ and $t = 2$ hours. The **rate** is the height of the rectangle, the **time** is the length of the rectangle, and the **distance** is the **area** of the rectangle. This is the way we can extend our simple formula to handle more complicated velocities: And this is the way we can answer the second example.

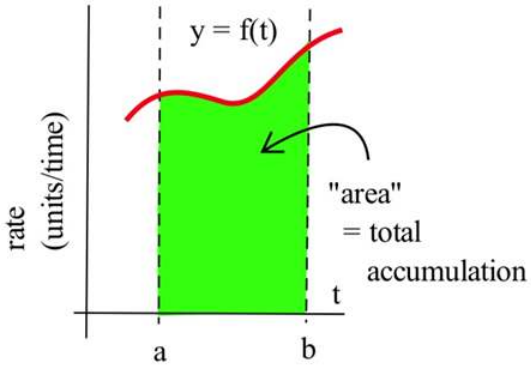
The distance the car travels is the area between its velocity graph, the t-axis, $t = 0$ and $t = 2$. This region is a triangle, so its area is

$$\frac{1}{2}bh = \frac{1}{2}(2 \text{ hours})(40 \text{ miles per hour}) = 40 \text{ miles}.$$

So the car travels 40 miles during its annoying trip.

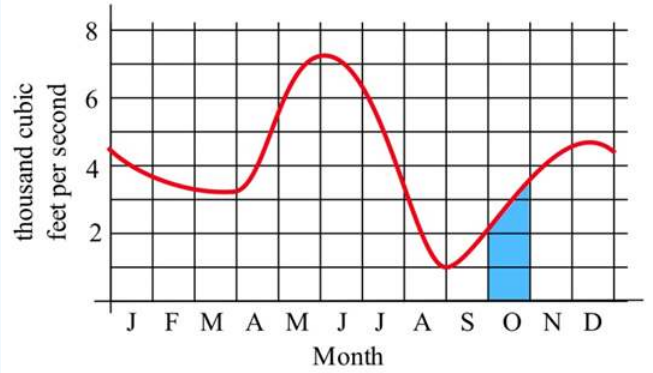
In our distance/velocity examples, the function represented a **rate** of travel (miles per hour), and the area represented the **total** distance traveled. This principle works more generally:

For functions representing other **rates** such as the production of a factory (bicycles per day), or the flow of water in a river (gallons per minute) or traffic over a bridge (cars per minute), or the spread of a disease (newly sick people per week), the area will still represent the **total** amount of something.



Example 4.3.3

The graph below shows the flow rate (cubic feet per second) of water in the Skykomish river at the town of Goldbar in Washington state.



Solution

The area of the shaded region represents the total volume (cubic feet or ft^3) of water flowing past the town during the month of October. We can approximate this area to approximate the total water by thinking of the shaded region as a rectangle with a triangle on top.

$$\begin{aligned}
\text{Total water} &= \text{total area} \\
&\approx \text{area of rectangle} + \text{area of the "triangle"} \\
&\approx \left(2000\frac{\text{ft}^3}{\text{s}}\right)(30 \text{ days}) + \frac{1}{2}\left(1500\frac{\text{ft}^3}{\text{s}}\right)(30 \text{ days}) \\
&= \left(2570\frac{\text{ft}^3}{\text{s}}\right)(30 \text{ days})
\end{aligned}$$

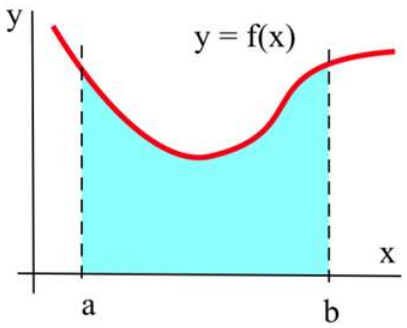
Note that we need to convert the units to make sense of our result:

$$\begin{aligned}\text{Total water} &\approx \left(2570\frac{\text{ft}^3}{\text{s}}\right)(30 \text{ days}) \\ &= \left(2570\frac{\text{ft}^3}{\text{s}}\right)(2{,}592{,}000 \text{ sec}) \\ &\approx 7.128 \cdot 10^9 \text{ ft}^3\end{aligned}$$

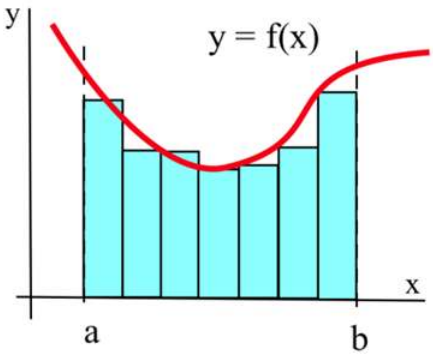
About 7 billion cubic feet of water flowed past Goldbar in October.

Approximating with Rectangles

How do we approximate the area if the rate curve is, well, "curvy"? We could use rectangles and triangles, like we did in the last example. But it turns out to be more useful (and easier) to simply use rectangles. The more rectangles we use, the better our approximation is.

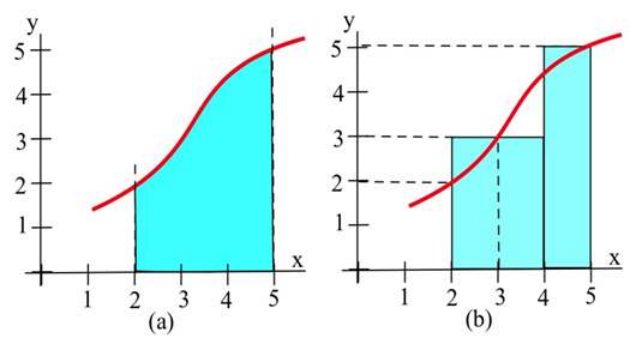


Suppose we want to calculate the area between the graph of a positive function f and the x-axis on the interval $[a, b]$ (graphed above). The **Riemann Sum method** is to build several rectangles with bases on the interval $[a, b]$ and sides that reach up to the graph of f (see below). Then the areas of the rectangles can be calculated and added together to get a number called a Riemann Sum of f on $[a, b]$. The area of the region formed by the rectangles is an **approximation** of the area we want.



Example 4.3.4

Approximate the area in the graph on the left between the graph of f and the x-axis on the interval $[2, 5]$ by summing the areas of the rectangles in the graph on the right.



Solution

The total area of rectangles is

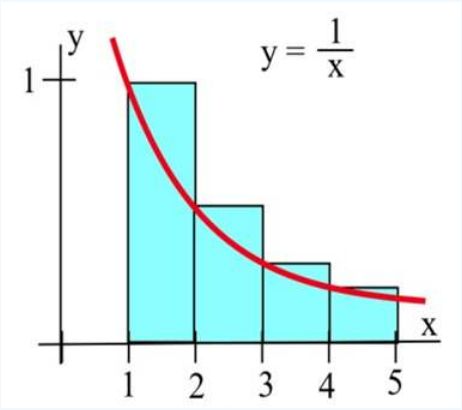
$$(2)(3)+(1)(5)=11 \text{ units}^2.$$

Example 4.3.5

Let A be the region bounded by the graph of $f(x)=\frac{1}{x}$, the x-axis, and vertical lines at $x=1$ and $x=5$. We can't find the area exactly (with what we know now), but we can approximate it using rectangles.

When we make our rectangles, we have a lot of choices. We could pick any (non-overlapping) rectangles whose bottoms lie within the interval on the x-axis, and whose tops intersect with the curve somewhere. But it is easiest to choose rectangles that (a) have all the same width and (b) take their heights from the function at one edge. Below are graphs showing two ways to use four rectangles to approximate this area. In the first graph, we used left-endpoints; the height of each rectangle comes from the function value at its left edge. In the second graph on the next page, we used right-hand endpoints.

Left-hand endpoints: The area is approximately the sum of the areas of the rectangles. Each rectangle gets its height from the function $f(x)=\frac{1}{x}$ and each rectangle has a width of 1.



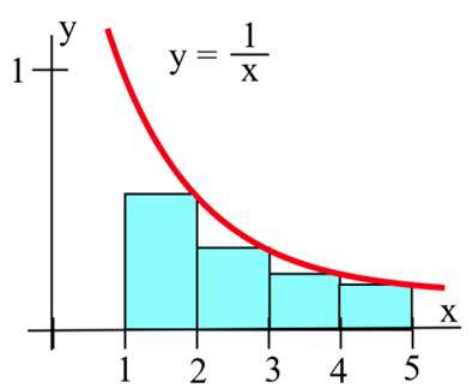
You can find the area of each rectangle using area = height·width. So the total area of the rectangles, the left-hand estimate of the area under the curve, is

$$f(1)\cdot 1+f(2)\cdot 1+f(3)\cdot 1+f(4)\cdot 1=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{25}{12}\approx 2.08.$$

Notice that because this function is decreasing, all the left endpoint rectangles stick out above the region we want – using left-hand endpoints will overestimate the area.

Right-hand endpoints: The right-hand estimate of the area is

$$f(2)\cdot 1 + f(3)\cdot 1 + f(4)\cdot 1 + f(5)\cdot 1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{77}{60} \approx 1.28.$$



All the right-hand rectangles lie completely under the curve, so this estimate will be an underestimate.

We can see that the true area is actually in between these two estimates. So we could take their average:

$$\text{Average} = \frac{\frac{25}{12} + \frac{77}{60}}{2} = \frac{101}{60} \approx 1.68$$

In general, the average of the left-hand and right-hand estimates will be closer to the real area than either individual estimate.

Our estimate of the area under the curve is about 1.68. (The actual area is about 1.61.)

This applet will allow you to see how the approximation changes if you use more rectangles; change the “position” slider to switch between using the left endpoints, right endpoints, and midpoints:

If we wanted a better answer, we could use even more and even narrower rectangles. But there is a limit to how much work we want to do by hand. In practice, choose a manageable number of rectangles. We will have better methods to get more accurate answers before long.

These sums of areas of rectangles are called **Riemann sums**. You may see a shorthand notation used when people talk about sums. We won't use it much in this book, but you should know what it means.

Riemann sum and Sigma Notation

Riemann Sum

A Riemann sum for a function $f(x)$ over an interval $[a, b]$ is a sum of areas of rectangles that approximates the area under the curve. Start by dividing the interval $[a, b]$ into n subintervals; each subinterval will be the base of one rectangle. We usually make all the rectangles the same width Δx. The height of each rectangle comes from the function evaluated at some point in its sub interval. Then the Riemann sum is

$$f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x$$

or, factoring out the Δx,

$$\Delta x \left(f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n)\right).$$

Sigma Notation

The upper-case Greek letter Sigma $\sum$ is used to stand for "sum". Sigma notation is a way to compactly represent a sum of many similar terms, such as a Riemann sum.

Using the Sigma notation, the Riemann sum can be written

$$\sum_{i=1}^{n} f\left(x_i\right) \Delta x.$$

This is read aloud as "the sum as i goes from 1 to n of f of x sub i times Delta x." The "i" is a counter or index, like you might have seen in a programming class.

Definition of the Definite Integral

Because the area under the curve is so important, it has a special vocabulary and notation.

The Definite Integral

The **definite integral** of a positive function $f(x)$ over an interval $[a, b]$ is the area between f, the x-axis, $x = a$ and $x = b$.

The **definite integral** of a positive function $f(x)$ from a to b is the area under the curve between a and b.

If $f(t)$ represents a positive rate (in y-units per t-units), then the definite integral of $f(t)$ from a to b is the total y-units that accumulate between $t = a$ and $t = b$.

Notation for the Definite Integral

The definite integral of $f(x)$ from a to b is written

$$\int_a^b f(x)\, dx.$$

The $\int$ symbol is called the **integral sign**; it is an elongated letter S, standing for "sum". (The $\int$ corresponds to the $\sum$ from the Riemann sum)

The dx on the end *must* be included! The dx tells what the variable is – in this example, the variable is x. (The dx corresponds to the Δx from the Riemann sum)

The function f is called the **integrand**.

The a and b are called the **limits of integration**.

Verb forms

We **integrate**, or **find the definite integral** of a function. This process is called **integration**.

Formal Algebraic Definition

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x.$$

Practical Definition

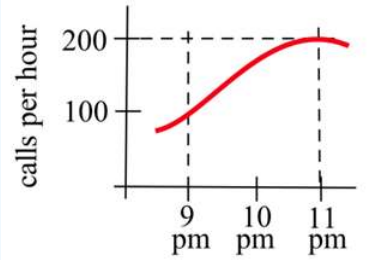
The definite integral can be approximated with a Riemann sum (dividing the area into rectangles where the height of each rectangle comes from the function, computing the area of each rectangle, and adding them up). The more rectangles we use, the narrower the rectangles are, the better our approximation will be.

Looking Ahead

We will have methods for computing exact values of some definite integrals from formulas soon. In many cases, including when the function is given to you as a table or graph, you will still need to approximate the definite integral with rectangles.

Example 4.3.6

The graph below shows $y = r(t)$, the number of telephone calls made per hours on a Tuesday. Approximately how many calls were made between 9 pm and 11 pm? Express this as a definite integral and approximate with a Riemann sum.

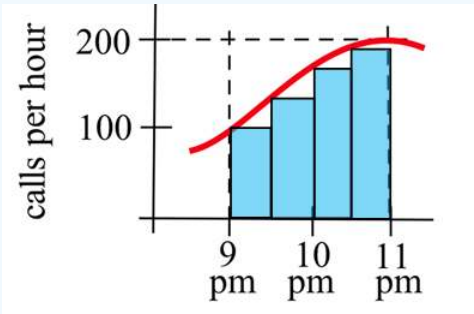


Solution

We know that the accumulated calls will be the area under this rate graph over that two-hour period, the definite integral of this rate from $t = 9$ to $t = 11$.

The total number of calls will be $\int_9^{11} r(t)\,dt$.

The top here is a curve, so we can't get an exact answer. But we can approximate the area using rectangles. Let's choose to use four rectangles and left-endpoints:



$$\int_9^{11} r(t)\,dt \approx 0.5(100+150+180+195) = 312.5$$

The units are *calls per hour·hours = calls*. Our estimate is that about 312 calls were made between 9 pm and 11 pm. Is this an under-estimate or an over-estimate?

Example 4.3.7

Describe the area between the graph of $f(x) = \frac{1}{x}$, the x-axis, and the vertical lines at $x = 1$ and $x = 5$ as a definite integral.

Solution

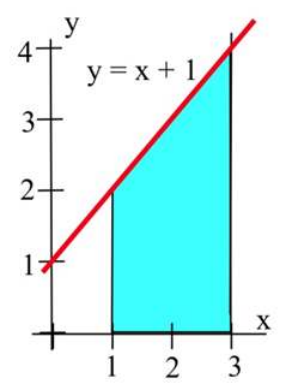
This is the same area we estimated to be about 1.68 before. Now we can use the notation of the definite integral to describe it. Our estimate of $\int_1^5 \frac{1}{x}\,dx$ was 1.68. The true value of $\int_1^5 \frac{1}{x}\,dx$ is about 1.61.

Example 4.3.8

Using the idea of area, determine the value of $\int_1^3 1 + x\,dx$.

Solution

$\int_1^3 1 + x\,dx$ represents the area between the graph of $f(x) = 1 + x$, the x-axis, and the vertical lines at 1 and 3.



Since this area can be broken into a rectangle and a triangle, we can find the area exactly. The area equals

$$4 + \frac{1}{2}(2)(2) = 6 \text{ units}^2.$$

Example 4.3.9

The table shows rates of population growth for Berrytown for several years. Use this table to estimate the total population growth from 1970 to 2000:

Year, t	1970	1980	1990	2000
Rate of population growth $R(t)$ in thousands of people per year	1.5	1.9	2.2	2.4

Solution

The definite integral of this rate will give the total change in population over the thirty-year period. We only have a few pieces of information, so we can only estimate. Even though we haven't made a graph, we're still approximating the area under the rate curve, using rectangles. How wide are the rectangles? We have information every 10 years, so the rectangles have a width of 10 years. How many rectangles? Be careful here – this is a thirty-year span, so there are three rectangles.

- Using left-hand endpoints: (1.5)(10) + (1.9)(10) + (2.2)(10) = 56
- Using right-hand endpoints: (1.9)(10) + (2.2)(10) + (2.4)(10) = 65

Taking the average of these two:

$$\frac{56+65}{2} = 60.5$$

Our best estimate of the total population growth from 1970 to 2000 is **60.5 thousand people**.

Signed Area

You may have noticed that until this point, we've insisted that the integrand (the function we're integrating) be positive. That's because we've been talking about area, which is always positive.

If the "height" (from the function) is a negative number, then multiplying it by the width doesn't give us actual area, it gives us the area with a negative sign.

But it turns out to be useful to think about the possibility of negative area. We'll expand our idea of a definite integral now to include integrands that might not always be positive. The "heights" of the rectangles, the values from the function, now might not always be positive.

The Definite Integral and Signed Area

The **definite integral** of a function $f(x)$ over an interval $[a, b]$ is the **signed area** between f, the x-axis, $x = a$ and $x = b$.

The **definite integral** of a function $f(x)$ from a to b is the **signed area** under the curve between a and b.

If the function is positive, the signed area is positive, as before (and we can call it area.)

If the function dips below the x-axis, the areas of the regions below the x-axis come in with a negative sign. In this case, we cannot call it simply "area." These negative areas take away from the definite integral.

$\int_a^b f(x)\,dx =$ (Area above x-axis) $-$ (Area below x-axis)

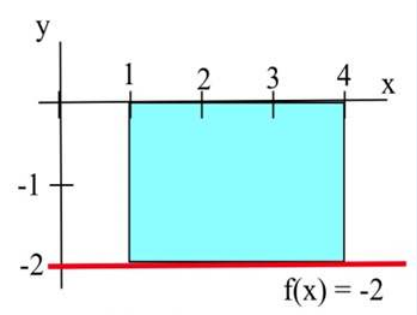
If $f(t)$ represents a positive rate (in y-units per t-units), then the **definite integral** of f from a to b is the **total** y-units that accumulate between $t = a$ and $t = b$.

If $f(t)$ represents any rate (in y-units per t-units), then the **definite integral** of f from a to b is the **net** y-units that accumulate between $t = a$ and $t = b$.

Example 4.3.10

Find the definite integral of $f(x) = -2$ on the interval [1,4].

Solution



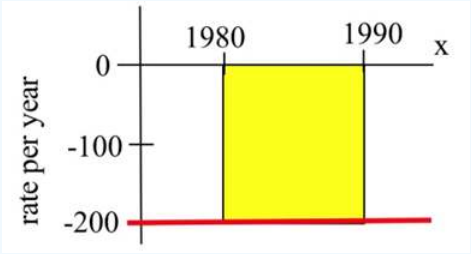
$\int_1^4 -2\,dx$ is the signed area of the region shown to the right. The region lies below the x-axis, so the area, 6, comes in with a negative sign. So the definite integral is $\int_1^4 -2\,dx = -6$.

Negative rates indicate that the amount is decreasing. For example, if $f(t)$ is the velocity of a car in the positive direction along a straight line at time t (miles/hour), then negative values of f indicate that the car is traveling in the negative direction, backwards. The definite integral of f is the change in position of the car during the time interval. If the velocity is positive, positive distance accumulates. If the velocity is negative, distance in the negative direction accumulates.

This is true of any rate. For example, if $f(t)$ is the rate of population change (people/year) for a town, then negative values of f would indicate that the population of the town was getting smaller, and the definite integral (now a negative number) would be the **change** in the population, a decrease, during the time interval.

Example 4.3.11

In 1980 there were 12,000 ducks nesting around a lake, and the **rate** of population change (in ducks per year) is shown in the figure below. Write a definite integral to represent the total change in the duck population from 1980 to 1990, and estimate the population in 1990.



Solution

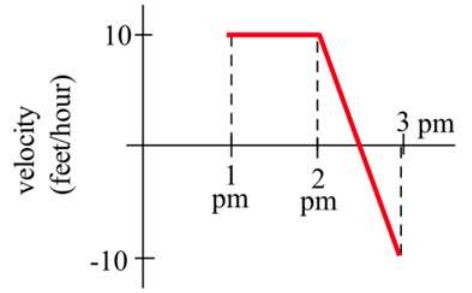
The change in population is:

$$\begin{aligned}\int_{1980}^{1990} f(t)\,dt &= -(\text{area between } f \text{ and axis}) \\ &\approx -(200 \text{ ducks/year}) \cdot (10 \text{ years}) \\ &= -2000 \text{ ducks}\end{aligned}$$

Then (1990 duck population) = (1980 population) + (change from 1980 to 1990) = (12,000) + (-2000) = 10,000 ducks.

Example 4.3.12

A bug starts at the location $x = 12$ on the x-axis at 1 pm walks along the axis with the velocity $v(x)$ shown in the graph. How far does the bug travel between 1 pm and 3 pm, and where is the bug at 3 pm?



Solution

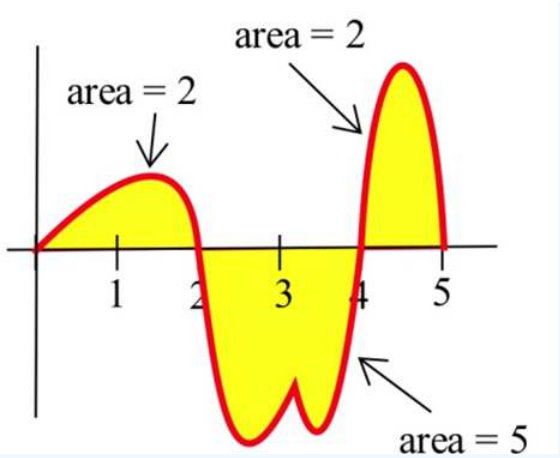
Note that the velocity is positive from 1 until 2:30, then becomes negative. So the bug moves in the positive direction from 1 until 2:30, then turns around and moves back toward where it started. The area under the velocity curve from 1 to 2:30 shows the total distance traveled by the bug in the positive direction; the bug moved 12.5 feet in the positive direction. The **area** between the velocity curve and the x-axis, between 2:30 and 3, shows the total distance traveled by the bug in the negative direction, back toward home; the bug traveled 2.5 feet in the negative direction. The definite integral of the velocity curve, $\int_1^3 v(t)\,dt$, shows the net change in distance:

$$\int_1^3 v(t)\,dt = 12.5 - 2.5 = 10.$$

The bug ended up 10 feet farther in the positive direction than he started. At 3 pm, the bug is at $x = 22$.

Example 4.3.13

Use the graph below to calculate $\int_0^2 f(x)\,dx$, $\int_2^4 f(x)\,dx$, $\int_4^5 f(x)\,dx$, and $\int_0^5 f(x)\,dx$.



Solution

Using the given areas, $\int_0^2 f(x)\,dx = 2$, $\int_2^4 f(x)\,dx = -5$, $\int_4^5 f(x)\,dx = 2$, and $\int_0^5 f(x)\,dx =$(area above)$-$(area below) $= (2+2)-(5) = -1$.

Approximating with Technology

If your function is given as a graph or table, you will still have to approximate definite integrals using areas, usually of rectangles. But if your function is given as a formula, you can turn to technology to get a better approximate answer. For example, most graphing calculators have some kind of numerical integration tool built in. You can also find many online tools that can do this; type numerical integration into any search engine to see a selection of these.

Most numerical integration tools use rectangles to estimate the signed area, just as you would do by hand. But they use many more rectangles than you would have the patience for, so they get a better answer. Some of them use computer algebra systems to find exact answers; we will learn how to do this ourselves later in this chapter.

When you turn to technology to find the value of a definite integral, be careful. Not every tool will be able to give you a correct answer for every integral. You should make an estimate of the answer yourself first so you can judge whether the answer you get makes sense.

Example 4.3.14

Use technology to approximate the definite integral $\int_1^5 \frac{1}{x}\,dx$. (This is the same definite integral we approximated with rectangles before.)

Solution

We could use the following command in GeoGebra to approximate this integral:

```
Integral[1 / x, 1, 5]
```

Or click this link to see how to evaluate this integral using Wolfram|Alpha.

Example 4.3.15

Use technology to approximate the definite integral $\int_1^2 e^{x^2+x}\,dx$.

Solution

The function here is positive, so there must be some area under the curve here. There isn't an algebraic way to find the exact answer, so some programs may just return the original integral rather than try to approximate it.

We could use the following command in GeoGebra to approximate this integral:

```
Integral[e^(x^2 + x), 1, 2]
```

Or click this link to see how to approximate this integral using Wolfram|Alpha. Although it looks like Wolfram|Alpha can evaluate the integral, the "Erfi" function that it shows in its answer is actually defined in terms of another integral, so it still hasn't found an algebraic answer.

Accumulation in Real Life

We have already seen that the "area" under a graph can represent quantities whose units are not the usual geometric units of square meters or square feet. For example, if t is a measure of time in seconds and $f(t)$ is a velocity with units feet/second, then the definite integral has units (feet/second)·(seconds) = feet.

In general, the units for the definite integral $\int_a^b f(x)\,dx$ are (y-units)·(x-units). A quick check of the units can help avoid errors in setting up an applied problem.

In previous examples, we looked at a function that represented a **rate** of travel (miles per hour); in that case, the area represented the **total** distance traveled. For functions representing other **rates** such as the production of a factory (bicycles per day), or the flow of water in a river (gallons per minute) or traffic over a bridge (cars per minute), or the spread of a disease (newly sick people per week), the area will still represent the **total** amount of something.

Example 4.3.16

Suppose $MR(q)$ is the marginal revenue in dollars/item for selling q items. What does $\int_0^{150} MR(q)\,dq$ represent?

Solution

$\int_0^{150} MR(q)\,dq$ has units (dollars/item)·(items) = dollars, and represents the accumulated dollars for selling from 0 to 150 items.

That is, $\int_0^{150} MR(q)\,dq = TR(150)$, the total revenue from selling 150 items.

Example 4.3.17

Suppose $r(t)$, in centimeters per year, represents how the diameter of a tree changes with time. What does $\int_{T_1}^{T_2} r(t)\,dt$ represent?

Solution

$\int_{T_1}^{T_2} r(t)\,dt$ has units of (centimeters per year)·(years) = centimeters, and represents the accumulated growth of the tree's diameter from T_1 to T_2. That is, $\int_{T_1}^{T_2} r(t)\,dt$ is the change in the diameter of the tree over this period of time.

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4.4: The Fundamental Theorem and Antidifferentiation

The Fundamental Theorem of Calculus

This section contains the most important and most used theorem of calculus, the Fundamental Theorem of Calculus. Discovered independently by Newton and Leibniz in the late 1600s, it establishes the connection between derivatives and integrals, provides a way of easily calculating many integrals, and was a key step in the development of modern mathematics to support the rise of science and technology. Calculus is one of the most significant intellectual structures in the history of human thought, and the Fundamental Theorem of Calculus is a most important brick in that beautiful structure.

The Fundamental Theorem of Calculus

If $F(x)$ is a function where $F'(x) = f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

This is actually not new for us; we've been using this relationship for some time; we just haven't written it this way. This says what we've said before: the definite integral of a rate from a to b is the net y-units, the change in y, that accumulate between $t = a$ and $t = b$. Here we've just made it plain that that the rate is a derivative.

Thinking about the relationship this way gives us the key to finding exact answers for some definite integrals. If the integrand is the derivative of some F, then maybe we could simply find F and subtract – that would be easier than approximating with rectangles. Going backwards through the differentiation process will help us evaluate definite integrals.

Example 4.4.1

Find $f(x)$ if $f'(x) = 2x$.

Solution

Oooh, I know this one. It's $f(x) = x^2 + 3$. Oh, wait, you were thinking something else? Yes, I guess you're right, $f(x) = x^2$ works too. So does $f(x) = x^2 - \pi$, and $f(x) = x^2 + 104{,}589.2$. In fact, there are lots of answers.

In fact, there are infinitely many functions that all have the same derivative. And that makes sense – the derivative tells us about the shape of the function, but it doesn't tell about the location. We could shift the graph up or down and the shape wouldn't be affected, so the derivative would be the same.

This leads to one of the trickier definitions – pay careful attention to the articles ("the" versus "an"), because they're important.

Antiderivatives

An **antiderivative** of a function $f(x)$ is any function $F(x)$ where $F'(x) = f(x)$.

The antiderivative of a function $f(x)$ is a whole family of functions, written $F(x) + C$, where $F'(x) = f(x)$ and C represents any constant.

The antiderivative is also called the **indefinite integral**.

Notation for the Antiderivative

The antiderivative of f is written

$$\int f(x)\,dx$$

This notation resembles the definite integral, because the Fundamental Theorem of Calculus says antiderivatives and definite integrals are intimately related. But in this notation, there are no limits of integration.

The $\int$ symbol is still called an **integral sign**; the dx on the end still must be included; you can still think of $\int$ and dx as left and right parentheses. The function f is still called the **integrand**.

Verb Forms

We **antidifferentiate**, or **integrate**, or **find the indefinite integral** of a function. This process is called **antidifferentiation** or **integration**.

There are no small families in the world of antiderivatives: if f has one antiderivative F, then f has an infinite number of antiderivatives and every one of them has the form $F(x)+C$.

Example 4.4.2

Find **an** antiderivative of $2x$.

Solution

We can choose any function we like as long as its derivative is $2x$, so we can pick, say, $F(x)=x^2-5.2$.

Example 4.4.3

Find **the** antiderivative of $2x$.

Solution

Now we need to write the entire family of functions whose derivatives are $2x$. We can use the $\int$ notation:

$$\int 2x\,dx = x^2 + C.$$

(Don't forget the "$+C$"!)

Example 4.4.4

Find $\int e^x\,dx$.

Solution

This is likely one you remember: e^x is its own derivative, so it is also its own antiderivative. The integral sign tells us that we need to include the entire family of functions, so we need that "$+C$" on the end:

$$\int e^x\,dx = e^x + C.$$

Antiderivatives Graphically or Numerically

Another way to think about the Fundamental Theorem of Calculus is to solve the expression for $F(b)$:

The Fundamental Theorem of Calculus (restated): Net Change

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

The definite integral of a derivative from a to b gives the net change in the original function.

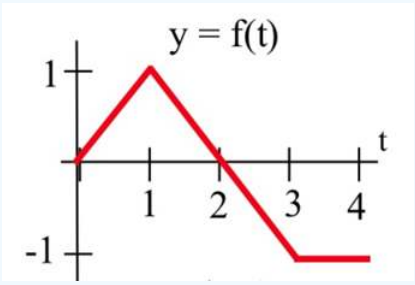
$$F(b) = F(a) + \int_a^b F'(x)\,dx$$

The amount we end up is the amount we start with plus the net change in the function.

This lets us get values for the antiderivative – as long as we have a starting point, and we know something about the area.

Example 4.4.5

Suppose $F(t)$ has the derivative $f(t)$ shown below, and suppose that we know $F(0) = 5$. Find values for $F(1)$, $F(2)$, $F(3)$, and $F(4)$.



Solution

Using the second way to think about the Fundamental Theorem of Calculus, $F(b) = F(a) + \int_a^b F'(x)\,dx$, we can see that $F(1) = F(0) + \int_0^1 f(x)\,dx$. We know the value of $F(0)$, and we can easily find $\int_0^1 f(x)\,dx$ from the graph – it's just the area of a triangle. So

$$
\begin{aligned}
F(1) &= \\
&= 5 + 0.5 = 5.5 \\
F(2) &= F(0) + \int_0^2 f(x)\,dx \\
&= 5 + 1 = 6
\end{aligned}
$$

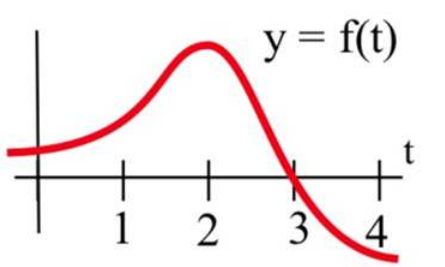
Note that we can start from any place of which we know the value; for example, now that we know $F(2)$, we can use that to find

$$
\begin{aligned}
F(3) &= F(2) + \int_2^3 f(x)\,dx \\
&= 6 - 0.5 = 5.5 \\
F(4) &= F(3) + \int_3^4 f(x)\,dx \\
&= 5.5 - 1 = 4.5
\end{aligned}
$$

Here is a copy of the applet used in this video:

Example 4.4.6

$F'(t) = f(t)$ is shown below. Where does $F(t)$ have maximum and minimum values on the interval [0, 4]?



Solution

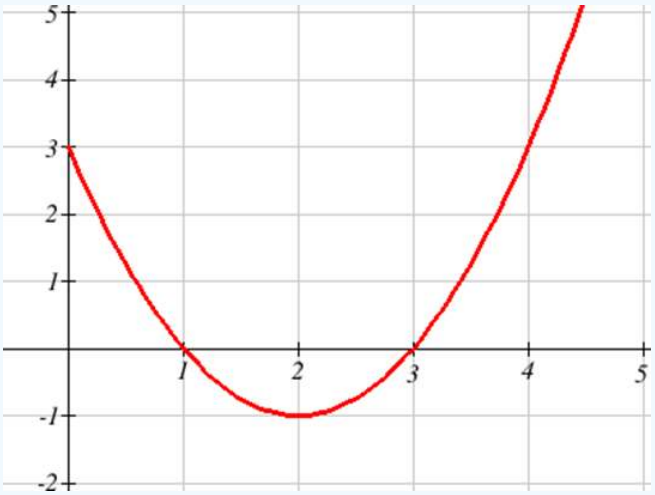
Since $F(b) = F(a) + \int_a^b F'(x)\,dx$, we know that F is increasing as long as the area accumulating under $F' = f$ is positive (until $t = 3$), and then decreases when the curve dips below the x-axis so that negative area starts accumulating. The area between $t = 3$ and $t = 4$ is much smaller than the positive area that accumulates between 0 and 3, so we know that $F(4)$ must be larger than $F(0)$. The maximum value is when $t = 3$; the minimum value is when $t = 0$.

Note that this is a different way to look at a problem we already knew how to solve – in Chapter 2, we would have found critical points of F, where $f = 0$: there's only one, when $t = 3$. $f = F'$ goes from positive to negative there, so F has a local max at that point. It's the only critical point, so it must be a global max. Then we would look at the values of F at the endpoints to find which was the global min.

We can also attempt to sketch a function based on the graph of the derivative.

Example 4.4.7

The graph below shows $f'(x)$ – the rate of change of $f(x)$. Use it to sketch a graph of $f(x)$ that satisfies $f(0) = 0$.



Solution

Recall from the last chapter the relationships between the function graph and the derivative graph:

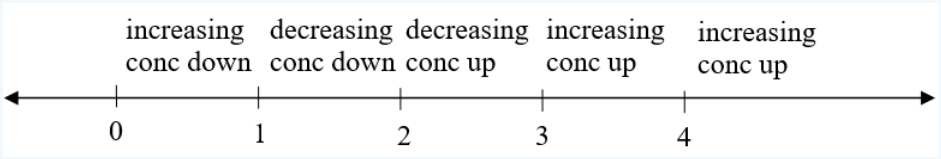
$f(x)$	Increasing	Decreasing	Concave Up	Concave Down
$f'(x)$	$+$	$-$	Increasing	Decreasing

In the graph shown, we can see the derivative is positive on the interval (0, 1) and $(3, \infty)$, so the graph of f should be increasing on those intervals. Likewise, f should be decreasing on the interval (1,3).

In the graph, f' is decreasing on the interval (0, 2), so f should be concave down on that interval. Likewise, f should be concave up on the interval $(2, \infty)$.

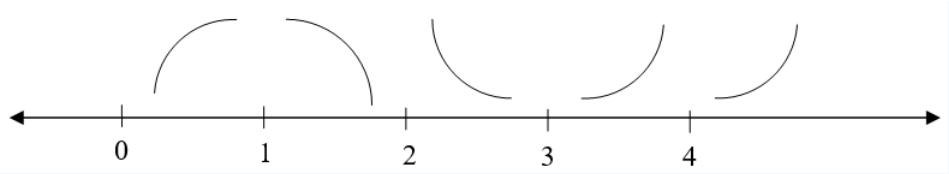
The derivative itself is not enough information to know where the function f starts, since there are a family of antiderivatives, but in this case we are given a specific point at which to start.

To start the sketch, we might note first the shapes we need:

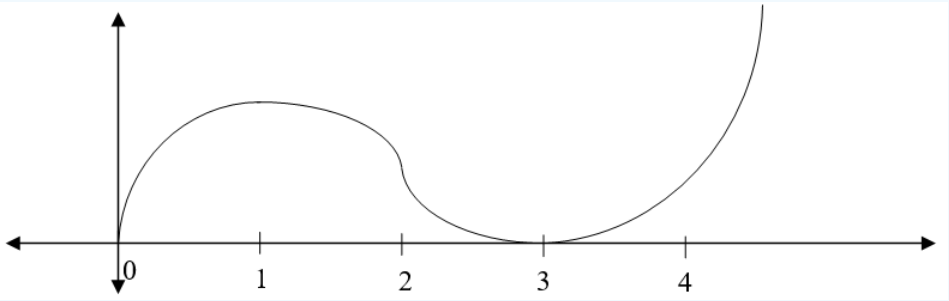


then sketch the basic shapes:

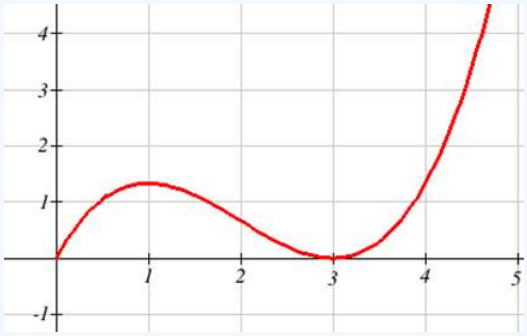
https://math.libretexts.org/@go/page/108798



Now we can attempt to sketch the graph, starting at the point (0, 0). Notice we are very roughly sketching this, as we don't have much information to work with. We can tell, though, from the graph that the area from $x = 0$ to $x = 1$ is about the same as the area from $x = 1$ to $x = 3$, so we would expect the net area from $x = 0$to $x = 3$ to be close to 0.



It turns out this graph isn't horribly bad. Smoothing it out would give a graph closer to the actual antiderivative graph, shown below.



Derivative of the Integral

There is another important connection between the integral and derivative.

The Fundamental Theorem of Calculus (part 2)

If

$$A(x) = \int_a^x f(t)\,dt$$

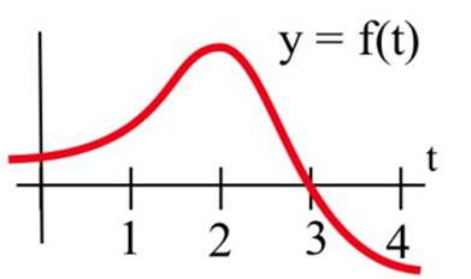
then

$$A'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

I.e., the derivative of the *accumulation function* is the original function.

Example 4.4.8

Let $F(x) = \int_0^x f(t)\,dt$, where f is graphed below. Estimate $F'(3)$.



Solution

The function $F(x)$ measures the area from $t = 0$ to some $t = x$. To estimate $F'(3)$, we want to estimate how much the area is increasing when $t = 3$. Since the value of the function $f(t)$ is 0 at $t = 3$, the area will not be increasing or decreasing, so we can estimate $F'(3) = 0$.

Directly using the fundamental theorem of calculus part 2,

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

so

$$F'(x) = f(3) = 0.$$

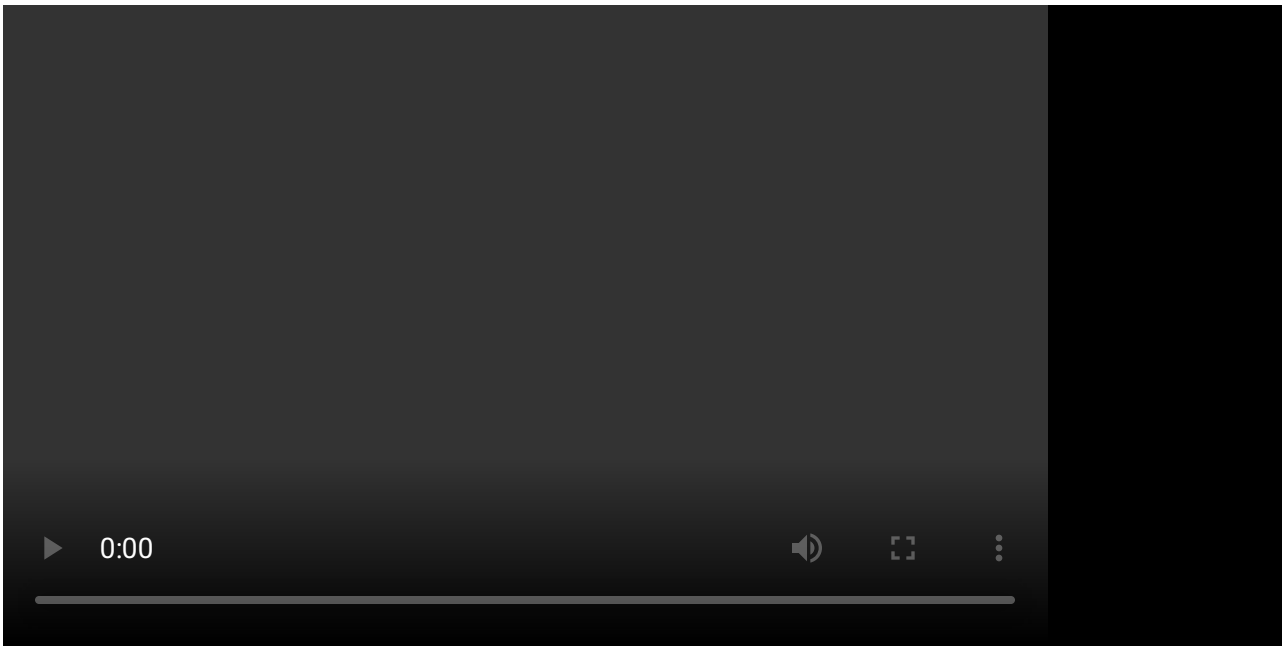
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4.5: Area, Average Value, and Volume

Average Value

We know the average of n numbers $a_1, a_2, \ldots, a_n$ is their sum divided by n. But what if we need to find the average temperature over a day's time – there are too many possible temperatures to add them up! This is a job for the definite integral.

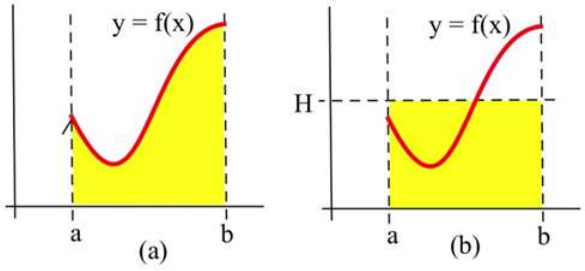


Average Value

The average value of a function $f(x)$ on the interval $[a, b]$ is given by

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

The average value of a positive f has a nice geometric interpretation. Imagine that the area under f (graph (a) below) is a liquid that can "leak" through the graph to form a rectangle with the same area (graph (b) below).



If the height of the rectangle is H, then the area of the rectangle is $H \cdot (b-a)$. We know the area of the rectangle is the same as the area under f so $H \cdot (b-a) = \int_a^b f(x)\,dx$. Then

$$H = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

the average value of f on $[a, b]$.

The average value of a positive function f is the height H of the rectangle whose area is the same as the area under f.

Example 4.5.1

During a 9 hour work day, the production rate at time t hours after the start of the shift was given by the function $r(t) = 5 + \sqrt{t}$ cars per hour. Find the average hourly production rate.

Solution

The average hourly production is $\frac{1}{9-0}\int_0^9 \left(5+\sqrt{t}\right)\,dt = 7$ cars per hour.

A note about the units – remember that the definite integral has units (cars per hour)·(hours) = cars. But the $\frac{1}{b-a}$ in front has units $\frac{1}{\text{hours}}$ so the units of the average value are cars per hour, just what we expect an average rate to be.

In general...

…the average value of a function will have the same units as the integrand.

Function averages, involving means and more complicated averages, are used to "smooth" data so that underlying patterns are more obvious and to remove high frequency "noise" from signals. In these situations, the original function f is replaced by some "average of f." If f is rather jagged time data, then the ten year average of f is the integral $g(x) = \frac{1}{10}\int_{x-5}^{x+5} f(t)\,dt$, an average of f over 5 units on each side of x.

For example, the figure below shows the graphs of a Monthly Average (rather "noisy" data) of surface temperature data, an Annual Average (still rather "jagged"), and a Five Year Average (a much smoother function).

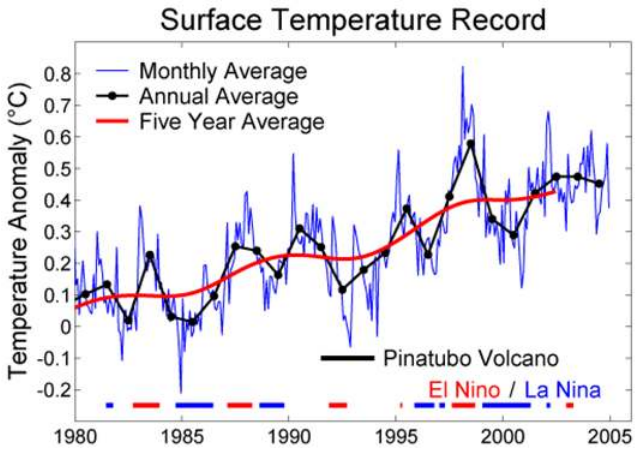
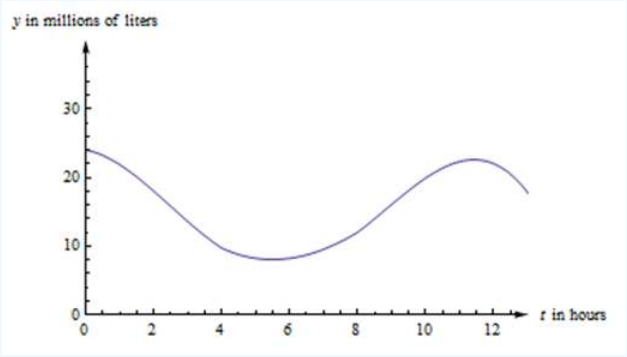


Image prepared by Robert A. Rohde, http://commons.wikimedia.org/wiki/File:Short_Instrumental_Temperature_Record.png.

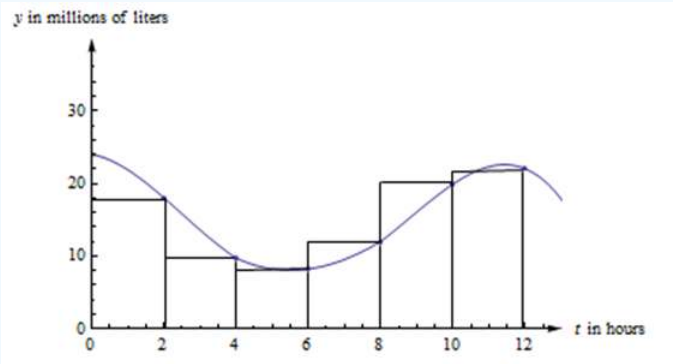
Typically the average function reveals the pattern much more clearly than the original data. This use of a "moving average" value of "noisy" data (weather information, stock prices) is a very common.

Example 4.5.2

The graph below shows the amount of water in a reservoir over a 12 hour period. Estimate the average amount of water in the reservoir over this period.



If $V(t)$ is the volume of the water (in millions of liters) after t hours, then the average amount is $\frac{1}{12}\int_0^{12} V(t)\,dt$. In order to find the definite integral, we'll have to estimate. Let's use 6 rectangles and take the heights from their right edges (there's nothing special about using 6 rectangles or right edges – other choices would still give you a valid estimate).



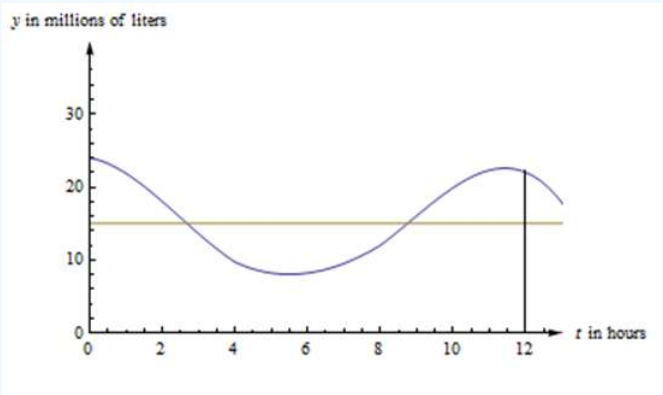
The estimate of the integral is

$$\int_0^{12} V(t)\,dt \approx (18)(2)+(9.7)(2)+(8.2)(2)+(12)(2)+(19.9)(2)+(22)(2) = 179.6.$$

Solution

The units of this integral are (millions of liters)·(feet). So our estimate of the average volume is $\frac{1}{12}\cdot 179.6 \approx 15$ millions of liters. (The estimate might change a little depending on how we estimate the function values from the graph.)

In the figure below, you can see the same graph with the line $y = 15$ drawn in. The area under the curve and the area under the rectangle are (approximately) the same.

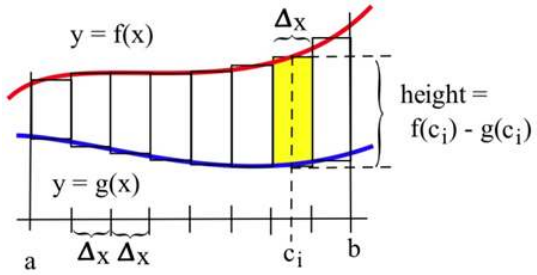


In fact, that would be a different way to estimate the average value. We could have estimated the placement of the horizontal line so that the area under the curve and under the line were equal.

Area

We have already used integrals to find the area between the graph of a function and the horizontal axis. Integrals can also be used to find the area between two graphs.

If $f(x) \geq g(x)$ for all x in $[a, b]$, then we can approximate the area between f and g by partitioning the interval $[a, b]$ and forming a Riemann sum, as shown in the picture. The height of each rectangle is top - bottom, $f(c_i) - g(c_i)$ so the area of the i-th rectangle is (height)·(base) = $(f(c_i) - g(c_i)) \cdot \Delta x$. Adding up these rectangles gives an approximation of the total area as $\sum_{i=1}^{n} (f(c_i) - g(c_i)) \cdot \Delta x$, a Riemann sum.



The limit of this Riemann sum, as the number of rectangles gets larger and their width gets smaller, is the definite integral $\int_a^b (f(x) - g(x))\, dx$.

Area Between Two Curves

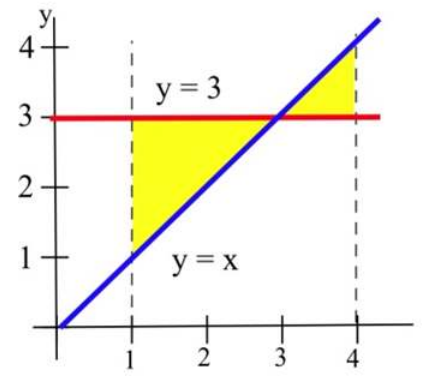
The area between two curves $f(x)$ and $g(x)$, where $f(x) \geq g(x)$, between $x = a$ and $x = b$ is

$$\int_a^b (f(x) - g(x))\, dx.$$

The integrand is "top - bottom." Make a graph or use test values to be sure which curve is which.

Example 4.5.3

Find the area bounded between the graphs of $f(x) = x$ and $g(x) = 3$ for $1 \leq x \leq 4$.



Solution

Always start with a graph so you can see which graph is the top and which is the bottom. In this example, the two curves cross, and they change positions; we'll need to split the area into two pieces. Geometrically, we can see that the area is 2 + 0.5 = 2.5.

Writing the area as a sum of definite integrals, we get

$$\text{Area} = \int_1^3 (3-x)\,dx + \int_3^4 (x-3)\,dx.$$

These integrals are easy to evaluate using antiderivatives:

$$\begin{aligned}\int_1^3 (3-x)\,dx &= \left[3x - \frac{x^2}{2}\right]_1^3 \\ &= \left(9 - \frac{9}{2}\right) - \left(3 - \frac{1}{2}\right) \\ &= 2\end{aligned}$$

$$\begin{aligned}\int_3^4 (x-3)\,dx &= \left[\frac{x^2}{2} - 3x\right]_3^4 \\ &= \left(\frac{16}{2} - 12\right) - \left(\frac{9}{2} - 9\right) \\ &= \frac{1}{2}\end{aligned}$$

The sum of these two integrals tells us that the total area between f and g is 2.5 square units, which we already knew from the picture.

Note that the single integral $\int_1^4 (3-x)\,dx = 1.5$ is not the **area** we want in the last example. The value of **the integral is 1.5**, and the value of **the area is 2.5**. That's because for the triangle on the right, the graph of $y = x$ is above the graph of $y = 3$, so the integrand $3 - x$ is negative; in the definite integral, the area of that triangle comes in with a negative sign.

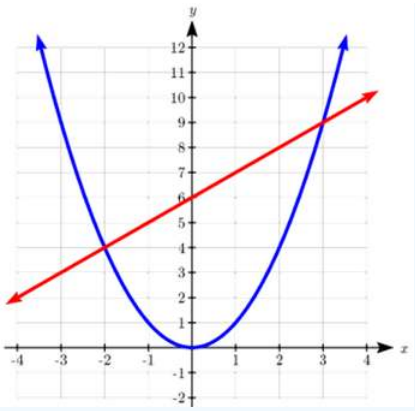
In this example, it was easy to see exactly where the two curves crossed so we could break the region into the two pieces to figure separately. In other examples, you might need to solve an equation to find where the curves cross.

Example 4.5.4

Find the area bounded by the graphs of $f(x) = x^2$ and $g(x) = x + 6$.

Solution

To begin, it will help to sketch a graph of the functions to see the area enclosed.



To set up the integral for the area, we need to determine where the line intersects the parabola. While the points look fairly clear in the graph, it will be good to verify the points algebraically:

$$\begin{aligned} x^2 &= x+6 \\ x^2 - x - 6 &= 0 \\ (x-3)(x+2) &= 1 \\ x &= -2, 3 \end{aligned}$$

Over this interval $-2 \leq x \leq 3$ notice the linear function is on top, so if we were to draw a rectangle between the curves, the height would be (top function) − (bottom function) = $(x+6) - x^2$.

Setting up a definite integral for this area:

$$\text{Area } = \int_{-2}^{3} (x+6-x^2)\,dx$$

We can evaluate this integral using antiderivatives:

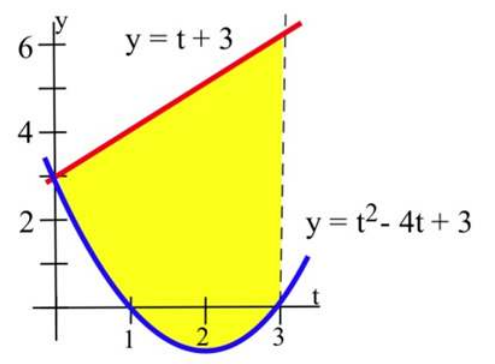
$$\begin{aligned} \int_{-2}^{3} (x+6-x^2)\,dx &= \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_{-2}^{3} \\ &= \left(\frac{3^2}{2} + 6\cdot 3 - \frac{3^3}{3}\right) - \left(\frac{(-2)^2}{2} + 6\cdot(-2) - \frac{(-2)^3}{3}\right) \\ &= \left(\frac{9}{2} + 18 - \frac{27}{3}\right) - \left(\frac{4}{2} - 12 + \frac{8}{3}\right) \\ &= \left(\frac{9}{2} + 18 - 9\right) - \left(2 - 12 + \frac{8}{3}\right) \\ &= \left(\frac{9}{2} + 9\right) - \left(-10 + \frac{8}{3}\right) \\ &= \frac{9}{2} + 9 + 10 - \frac{8}{3} \\ &= \frac{125}{6} \end{aligned}$$

The area between the curves is $\frac{125}{6} \approx 20.833$ square units.

Example 4.5.5

Two objects start from the same location and travel along the same path with velocities $v_A(t) = t + 3$ and $v_B(t) = t^2 - 4t + 3$ meters per second. How far ahead is A after 3 seconds?

Solution



Since $v_A(t) \geq v_B(t)$, the "area" between the graphs of v_A and v_B represents the distance between the objects.

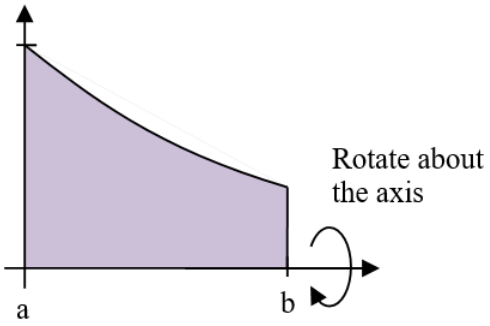
After 3 seconds, the distance apart is

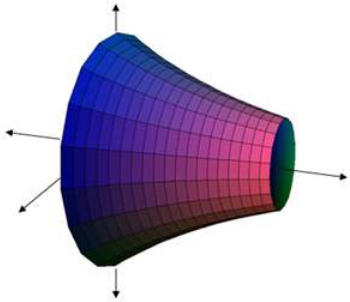
$$\begin{aligned}\int_0^3 (v_A(t) - v_B(t))\, dt &= \int_0^3 \left((t+3) - \left(t^2 - 4t + 3\right)\right)\, dt \\ &= \int_0^3 \left(5t - t^2\right)\, dt \\ &= \left[5\frac{t^2}{2} - \frac{t^3}{3}\right]_0^3 \\ &= \left(5\frac{9}{2} - \frac{27}{3}\right) - (0) \\ &= 13.5 \text{ meters.}\end{aligned}$$

Volume

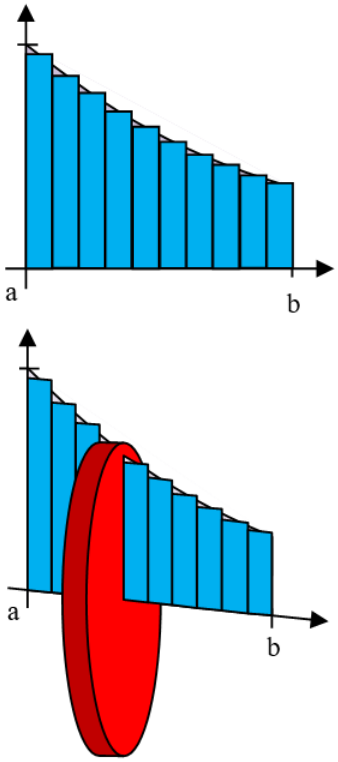
Just as we can partition an interval and imagine approximating an area with rectangles to find a formula for the area between curves, we can partition an interval and imagine approximating a volume with simple shapes to find a formula for the volume of a solid. While this approach works for a variety of shapes, our focus will be on shapes formed by revolving a curve around the horizontal axis.

We start with an area, the region below a function on the interval $a \leq x \leq b$. We are going to take that region, and rotate it around the x-axis, creating the solid shape shown.





To find the volume of this solid, we can start by partitioning the interval [0,1] and approximating the area with rectangles. As before, the width of each rectangle would be Δx and the height $f(c_i)$.



If we took just one of these rectangles and rotated it about the horizontal axis, it would form a cylindrical shape. The radius of that cylinder would be $f(c_i)$, so the volume would be

$$V = \pi r^2 h = \pi (f(c_i))^2 \Delta x.$$

The volume of the whole solid could be approximated by rotating each of the rectangles about the x axis. Adding up the volume of each of the little cylindrical discs gives an approximation of the total volume as $\sum_{i=1}^{n} \pi (f(c_i))^2 \Delta x$, a Riemann sum.

The limit of this sum as the width of the rectanges becomes small is the definite integral $\int_a^b \pi (f(c_i))^2 \, dx$

Volume

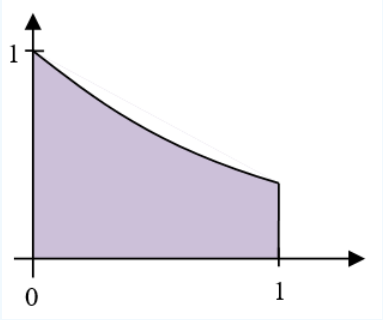
The volume of the solid obtained by rotating about the x-axis the area bounded by the curve $f(x)$, the x-axis, $x = a$, and $x = b$ is

$$\int_a^b \pi (f(c_i))^2 \, dx$$

Example 4.5.6

Find the volume of the solid formed by rotating the area under $f(x) = e^{-x}$ on the interval [0,1] about the x-axis.

Solution



This is the region pictured in the earlier example. We substitute in the function and bounds into the formula we derived to set up the definite integral:

$$\text{Volume} = \int_0^1 \pi \left(e^{-x}\right)^2 dx.$$

Using exponent rules, the integrand can be simplified. The constant π can be pulled out of the integral:

$$\text{Volume} = \pi \int_0^1 e^{-2x}\, dx.$$

Using the substitution $u = -2x$, we can integrate this function.

$$\begin{aligned} \pi \int_0^1 e^{-2x}\, dx &= (u\text{-substitution}) \\ &= -\frac{1}{2}\pi e^{-2x}\Big]_0^1 \\ &= \left(-\frac{1}{2}\pi e^{-2(1)}\right) - \left(-\frac{1}{2}\pi e^{-2(0)}\right) \\ &\approx 1.358 \text{ cubic units.} \end{aligned}$$

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4.6: Substitution

We don't have many integration rules. For quite a few of the problems we see, the rules won't directly apply; we'll have to do some algebraic manipulation first. In practice, it is much harder to write down the antiderivative of a function than it is to find a derivative. (In fact, it's very easy to write a function that doesn't have any antiderivative you can find with algebra, although *proving* that it doesn't have an antiderivative is much more difficult.)

The Substitution Method (also called u-Substitution) is one way of algebraically manipulating an integrand so that the rules apply. This is a way to unwind or undo the Chain Rule for derivatives. When you find the derivative of a function using the Chain Rule, you end up with a product of something like the original function *times* a derivative. We can reverse this to write an integral:

$$\frac{d}{dx} f\left(g(x)\right) = f'\left(g(x)\right) g'(x)$$

so

$$f\left(g(x)\right) = \int f'\left(g(x)\right) g'(x)\, dx$$

With substitution, we will substitute $u = g(x)$ (hence the name "u-substitution"). This means $\frac{du}{dx} = g'(x)$, so $du = g'(x)dx$. Making these substitutions, $\int f'\left(g(x)\right) g'(x)\, dx$ becomes $\int f'(u)\, du$, which will probably be easier to integrate.

Try u-Substitution when you see a product in your integral, especially if you recognize one factor as the derivative of some part of the other factor.

The u-Substitution Method for Antiderivatives

The goal is to turn $\int f\left(g(x)\right)\, dx$ into $\int f(u)\, du$, where $f(u)$ is much less messy than $f\left(g(x)\right)$.

1. Let u be some part of the integrand. A good first choice is "one step inside the messiest bit."
2. Compute $du = \frac{du}{dx}\, dx$.
3. Translate all your x's into u's everywhere in the integral, including the dx. When you're done, you should have a new integral that is entirely in u. If you have any x's left, then that's an indication that the substitution didn't work or isn't complete; you may need to go back to step 1 and try a different choice for u.
4. Integrate the new u-integral, if possible. If you still can't integrate it, go back to step 1 and try a different choice for u.
5. Finally, substitute back x's for u's everywhere in your answer.

Example 4.6.1

Evaluate $\displaystyle\int \frac{x}{\sqrt{4-x^2}}\, dx$.

Solution

This integrand is more complicated than anything in our list of basic integral formulas, so we'll have to try something else. The only tool we have is substitution, so let's try that!

1. Let u be some part of the integrand. A good first choice is "one step inside the messiest bit":

   Let $u = 4 - x^2$.

2. Compute $du = \frac{du}{dx}\,dx$:

   $du = -2x\,dx$. There is $x\,dx$ in the integrand, so that's a good sign; that will be $-\frac{1}{2}\,du$.

3. Translate all your x's into u's everywhere in the integral, including the dx:

$$\begin{aligned}\int \frac{x}{\sqrt{4-x^2}}\,dx &= \int \frac{1}{\sqrt{4-x^2}}(x\,dx)\\ &= \int \frac{1}{\sqrt{u}}\left(-\frac{1}{2}du\right)\\ &= -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du\\ &= -\frac{1}{2}\int u^{-1/2}\,du\end{aligned}$$

   Alternatively, we could have solved for dx and substituted that and simplified: $dx = \frac{du}{-2x}$, so

$$\begin{aligned}\int \frac{x}{\sqrt{4-x^2}}\,dx &= \int \frac{x}{\sqrt{u}}\left(\frac{du}{-2x}\right)\\ &= \int \frac{1}{\sqrt{u}}\left(-\frac{1}{2}du\right)\\ &= -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du\\ &= -\frac{1}{2}\int u^{-1/2}\,du\end{aligned}$$

4. Integrate the new u-integral, if possible:

$$-\frac{1}{2}\int u^{-1/2}\,du = -\frac{1}{2}\frac{u^{1/2}}{1/2} + C = -u^{1/2} + C$$

5. Finally, substitute back x's for u's everywhere in the answer:

   Undoing our $u = 4 - x^2$ substitution yields

$$-u^{1/2} + C = -\sqrt{4-x^2} + C.$$

Thus we have found

$$\int \frac{x}{\sqrt{4-x^2}}\,dx = -\sqrt{4-x^2} + C$$

How would we check this? By differentiating:

$$\begin{aligned}\frac{d}{dx}\left(-\sqrt{4-x^2}+C\right) &= \frac{d}{dx}\left(-\left(4-x^2\right)^{1/2}+C\right)\\ &= -\frac{1}{2}\left(4-x^2\right)^{-1/2}(-2x)\\ &= x\left(4-x^2\right)^{-1/2}\\ &= \frac{x}{\sqrt{4-x^2}}\end{aligned}$$

Example 4.6.2

Evaluate $\int \frac{e^x\,dx}{(e^x+15)^3}$.

Solution

This integral is not in our list of building blocks. But notice that the derivative of e^x+15 (which we see in the denominator) is just e^x (which we see in the numerator), so substitution will be a good choice for this.

Let $u=e^x+15$. Then $du=e^x\,dx$, and this integral becomes $\int \frac{du}{u^3}=\int u^{-3}\,du$.

Luckily, that is on our list of building block formulas: $\int \frac{du}{u^3}=\frac{u^{-2}}{-2}+C=-\frac{1}{2u^2}+C$.

Finally, translating back:

$$\int \frac{e^x\,dx}{(e^x+15)^3}=-\frac{1}{2(e^x+15)^2}+C.$$

Example 4.6.3

Evaluate

a. $\int \frac{x^2}{x^3+5}\,dx$

b. $\int \frac{x^3+5}{x^2}\,dx$

Solution

1. This is not a basic integral, but the composition is less obvious. Here, we can treat the denominator as the inside of the $\frac{1}{x}$ function.

   Let $u=x^3+5$. Then $du=3x^2\,dx$. Solving for dx, $dx=\frac{du}{3x^2}$. Substituting,

   $$\int \frac{x^2}{x^3+5}\,dx=\int \frac{x^2}{u}\frac{du}{3x^2}=\int \frac{1}{u}\frac{du}{3}=\frac{1}{3}\int \frac{1}{u}\,du$$

   Using our basic formulas,

   $$\frac{1}{3}\int \frac{1}{u}\,du=\frac{1}{3}\ln|u|+C.$$

   Undoing the substitution,

   $$\int \frac{x^2}{x^3+5}\,dx=\frac{1}{3}\ln|x^3+5|+C.$$

2. It is tempting to start this problem the same way we did the last, but if we try it will not work, since the numerator of this fraction is not the derivative of the denominator. Instead, we need to try a different approach. For this problem, we can use some basic algebra:

$$\int \frac{x^3+5}{x^2}\,dx = \int \left(\frac{x^3}{x^2}+\frac{5}{x^2}\right)\,dx$$
$$= \int \left(x+5x^{-2}\right)\,dx.$$

We can integrate this using our basic rules, without needing substitution:

$$\int \left(x+5x^{-2}\right)\,dx = \frac{x^2}{2}+5\frac{x^{-1}}{-1}+C$$
$$= \frac{1}{2}x^2-\frac{5}{x}+C.$$

Substitution and Definite Integrals

When you use substitution to help evaluate a definite integral, you have a choice for how to handle the limits of integration. You can do either of these, whichever seems better to you. The important thing to remember is that the original limits of integration were values of the original variable (say, x), not values of the new variable (say, u).

1. You can solve the antiderivative as a side problem, translating back to x's, and then use the antiderivative with the original limits of integration.

   Or…

2. You can substitute for the limits of integration at the same time as you're substituting for everything inside the integral, and then skip the "translate back into x" step.

   If the original integral had endpoints $x=a$ and $x=b$, and we make the substitution $u=g(x)$ and $du=g'(x)\,dx$, then the new integral will have endpoints $u=g(a)$ and $u=g(b)$ and

   $$\int_{x=a}^{x=b} (\text{original integrand})\,dx$$

   becomes

   $$\int_{u=g(a)}^{u=g(b)} (\text{new integrand})\,du.$$

Method 1 seems more straightforward for most students, but it can involve some messy algebra. Method 2 is often neater and usually involves fewer steps.

Example 4.6.4

Evaluate $\int_0^1 (3x-1)^4\,dx$.

Solution

We'll need substitution to find an antiderivative, so we'll need to handle the limits of integration carefully. Let's solve this example both ways.

1. Step One – find the antiderivative, using substitution:

   Let $u=3x-1$. Then $du=3\,dx$ and

   $$\int (3x-1)^4\,dx = \int u^4\left(\frac{1}{3}\,du\right) = \frac{1}{3}\frac{u^5}{5}+C.$$

   Translating back to x:

   $$\frac{1}{3}\frac{u^5}{5}+C = \frac{(3x-1)^5}{15}+C.$$

   Step Two – evaluate the definite integral:

$$\int_0^1 (3x-1)^4\,dx = \frac{(3x-1)^5}{15}\Big]_0^1 = \frac{(3(1)-1)^5}{15} - \frac{(3(0)-1)^5}{15} = \frac{32}{15} - \frac{-1}{15} = \frac{33}{15}.$$

2. Let $u = 3x-1$. Then $du = 3\,dx$ and, substituting for the limits of integration, when $x=0$, $u=-1$, and when $x=1$, $u=2$.

   So,

$$\begin{aligned}\int_{x=0}^{x=1} (3x-1)^4\,dx &= \int_{u=-1}^{u=2} u^4\left(\frac{1}{3}\,du\right)\\ &= \frac{u^5}{15}\Big]_{u=-1}^{u=2}\\ &= \frac{(2)^5}{15} - \frac{(-1)^5}{15}\\ &= \frac{32}{15} - \frac{-1}{15}\\ &= \frac{33}{15}\end{aligned}$$

Example 4.6.5

Evaluate $\int_2^{10} \frac{(\ln(x))^6}{x}\,dx$.

Solution

I can see the derivative of $\ln(x)$ in the integrand, so I can tell that substitution is a good choice.

Let $u = \ln(x)$. Then $du = \frac{1}{x}\,dx$. When $x=2$, $u=\ln(2)$. When $x=10$, $u=\ln(10)$. So the new definite integral is

$$\begin{aligned}\int_{x=2}^{x=10} \frac{(\ln(x))^6}{x}\,dx &= \int_{u=\ln(2)}^{u=\ln(10)} u^6\,du\\ &= \frac{u^7}{7}\Big]_{u=\ln(2)}^{u=\ln(10)}\\ &= \frac{1}{7}\left((\ln(10))^7 - (\ln(2))^7\right)\\ &\approx 49.01.\end{aligned}$$

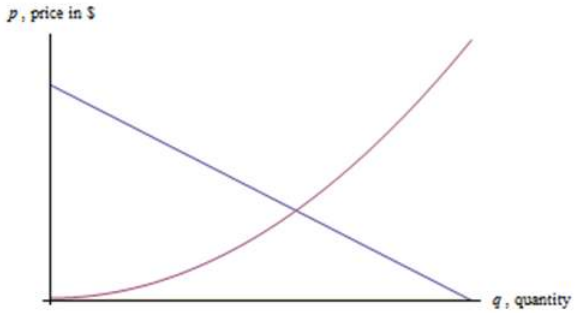
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4.7: Applications to Business

Consumer and Producer Surplus

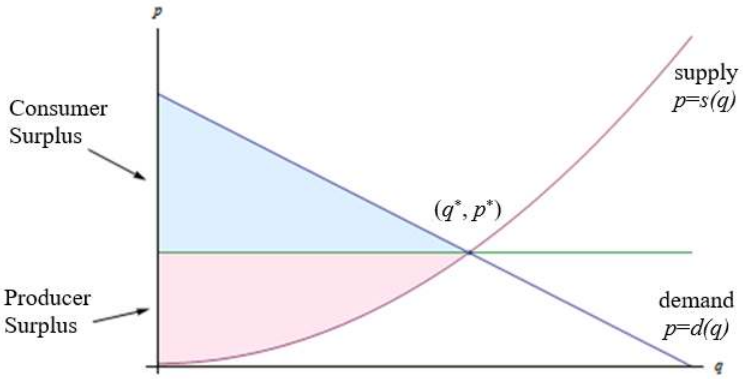
Here are a demand and a supply curve for a product. Which is which?



The demand curve is decreasing – lower prices are associated with higher quantities demanded, higher prices are associated with lower quantities demanded. Demand curves are often shown as if they were linear, but there's no reason they have to be.

The supply curve is increasing – lower prices are associated with lower supply, and higher prices are associated with higher quantities supplied.

The point where the demand and supply curve cross is called the equilibrium point (q^*, p^*).



Suppose that the price is set at the equilibrium price, so that the quantity demanded equals the quantity supplied. Now think about the folks who are represented on the left of the equilibrium point. The consumers on the left would have been willing to pay a higher price than they ended up having to pay, so the equilibrium price saved them money. On the other hand, the producers represented on the left would have been willing to supply these goods for a lower price – they made more money than they expected to. Both of these groups ended up with extra cash in their pockets!

Graphically, the amount of extra money that ended up in consumers' pockets is the area between the demand curve and the horizontal line at p^*. This is the difference in price, summed up over all the consumers who spent less than they expected to – a definite integral. Notice that since the area under the horizontal line is a rectangle, we can simplify the area integral:

$$\int_0^{q^*} (d(q) - p^*)\, dq = \int_0^{q^*} d(q)\, dq - \int_0^{q^*} p^*\, dq = \int_0^{q^*} d(q)\, dq - p^* q^*.$$

The amount of extra money that ended up in producers' pockets is the area between the supply curve and the horizontal line at p^*. This is the difference in price, summed up over all the producers who received more than they expected to. Similar to consumer surplus, this integral can be simplified:

$$\int_0^{q^*} (p^* - s(q))\, dq = \int_0^{q^*} p^*\, dq - \int_0^{q^*} s(q)\, dq = p^* q^* - \int_0^{q^*} s(q)\, dq.$$

Consumer and Producer Surplus

Given a demand function $p = d(q)$ and a supply function $p = s(q)$, and the equilibrium point (q^*, p^*)

- The **consumer surplus** is

$$\int_0^{q^*} d(q)\,dq - p^*q^*.$$

- The **producer surplus** is

$$p^*q^* - \int_0^{q^*} s(q)\,dq.$$

- The sum of the consumer surplus and producer surplus is the **total gains from trade**.

What are the units of consumer and producer surplus? The units are (price per item)(quantity of items) = money!

Example 4.7.1

Suppose the demand for a product is given by $p = d(q) = -0.8q + 150$ and the supply for the same product is given by $p = s(q) = 5.2q$. For both functions, q is the quantity and p is the price, in dollars.

1. Find the equilibrium point.
2. Find the consumer surplus at the equilibrium price.
3. Find the producer surplus at the equilibrium price.

Solution

1. The equilibrium point is where the supply and demand functions are equal.

$$\begin{aligned} -0.8q + 150 &= 5.2q \\ 150 &= 6.0q \\ q &= \frac{150}{6} \\ q &= 25 \end{aligned}$$

   The equilibrium quantity is 25 items. To find the corresponding equilibrium price we can use either the demand or supply equation.

   $s(25) = 5.2(25) = 130$. The equilibrium price is \$130.

2. The consumer surplus is

$$\begin{aligned}\int_0^{25}(-0.8q+150)\,dq-(130)(25) &= \left[-0.8\frac{q^2}{2}+150q\right]_0^{25}-3250\\ &=\left(-0.4\left(25^2\right)+150(25)\right)-\left(-0.4\left(0^2\right)+150(0)\right)-3250\\ &=3500-0-3250\\ &=\$250\end{aligned}$$

3. The producer surplus is

$$\begin{aligned}(130)(25)-\int_0^{25}5.2q\,dq &= 3250-\left[5.2\frac{q^2}{2}\right]_0^{25}\\ &=3250-\left(\left(2.6\left(25^2\right)\right)-\left(2.6\left(0^2\right)\right)\right)\\ &=3250-(1625-0)\\ &=\$1625\end{aligned}$$

Example 4.7.2

Suppose the demand for a product is given by $p=d(q)=300-5q$ and the supply for the same product is given by $p=s(q)=q^2$. For both functions, q is the quantity and p is the price, in dollars.

1. Find the equilibrium point.
2. Find the consumer surplus at the equilibrium price.
3. Find the producer surplus at the equilibrium price.

Solution

1. This is quadratic, so we can rearrange the equation so one side is zero: $q^2+5q-300=0$.

   We could solve this using the quadratic equation: $q=\frac{-5\pm\sqrt{5^2-4(1)(-300)}}{2(1)}$

   or we could factor: $(q+20)(q-15)=0$, so $q=-20$ or $q=15$.

   We want the positive solution. The equilibrium quantity is 15 items. The equilibrium price is $s(15)=15^2=\$225$.

2. The consumer surplus is

$$\begin{aligned}\int_0^{15}(300-5q)\,dq-(225)(15) &= \left[300q-5\frac{q^2}{2}\right]_0^{15}-3375\\ &=\left(300(15)-2.5\left(15^2\right)\right)-\left(300(0)-2.5\left(0^2\right)\right)-3375\\ &=3937.5-0-3375\\ &=\$562.50\end{aligned}$$

3. The producer surplus is

$$\begin{aligned}(225)(15)-\int_0^{15}q^2\,dq &= 3375-\left[\frac{q^3}{3}\right]_0^{15}s\\ &=3375-\left(\left(\frac{15^3}{3}\right)-\left(\frac{0^3}{3}\right)\right)\\ &=3375-(1125-0)\\ &=\$2250\end{aligned}$$

Example 4.7.3

The tables below show information about the demand and supply functions for a product. For both functions, q is the quantity and p is the price, in dollars.

q	0	100	200	300	400	500	600	700

p	70	61	53	46	40	35	31	28

q	0	100	200	300	400	500	600	700
p	14	21	28	33	40	47	54	61

1. Which is which? That is, which table represents demand and which represents supply?
2. What is the equilibrium price and quantity?
3. Find the consumer and producer surplus at the equilibrium price.

Solution

1. The first table shows decreasing price associated with increasing quantity; that is the demand function.
2. For both functions, $q = 400$ is associated with $p = 40$; the equilibrium price is \$40 and the equilibrium quantity is 400 units. Notice that we were lucky here, because the equilibrium point is actually one of the points shown. In many cases with a table, we would have to estimate.
3. The consumer surplus uses the demand function, which comes from the first table. We'll have to approximate the value of the integral using rectangles. There are 4 rectangles, and let's choose to use left endpoints.

   The consumer surplus is

$$\int_0^{400} (\text{demand})\, dq - (40)(400) \approx$$
$$(100)(70+61+53+46) - (40)(400) \;= \$7000$$

   So the consumer surplus is about \$7000.

   The producer surplus uses the supply function, which comes from the second table. Let's choose to use left endpoints for this integral also.

   The producer surplus is

$$(40)(400) - \int_0^{400} (\text{supply})\, dq \approx$$
$$(40)(400) - (100)(14+21+28+33) \;= \$6400$$

   So the producer surplus is about \$6400.

Continuous Income Stream

In precalculus, you learned about compound interest in that really simple situation where you made a single deposit into an interest-bearing account and let it sit undisturbed, earning interest, for some period of time. Recall:

Compound Interest Formulas

Let P be the principal (initial investment), r be the annual interest rate expressed as a decimal, and $A(t)$ be the amount in the account at the end of t years.

Compounding n times per year

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

Compounded continuously

$$A(t) = Pe^{rt}$$

If you're using this formula to find what an account will be worth in the future, $t > 0$ and $A(t)$ is called the **future value**.

If you're using the formula to find what you need to deposit today to have a certain value P sometime in the future, $t < 0$ and $A(t)$ is called the **present value**.

You may also have learned somewhat more complicated annuity formulas to deal with slightly more complicated situations – where you make equal deposits equally spaced in time.

But real life is not usually so neat.

Calculus allows us to handle situations where "deposits" are flowing continuously into an account that earns interest. As long as we can model the flow of income with a function, we can use a definite integral to calculate the present and future value of a continuous income stream. The idea here is that each little bit of income in the future needs to be multiplied by the exponential function to bring it back to the present, and then we'll add them all up (a definite integral).

Continuous Income Stream

Suppose money can earn interest at an annual interest rate of r, compounded continuously. Let $F(t)$ be a continuous income function (in dollars per year) that applies between year 0 and year T.

Then the present value of that income stream is given by

$$PV = \int_0^T F(t)e^{-rt}\,dt.$$

The future value can be computed by the ordinary compound interest formula

$$FV = PVe^{rt}.$$

This is a useful way to compare two investments – find the present value of each to see which is worth more today.

Example 4.7.4

You have an opportunity to buy a business that will earn \$75,000 per year continuously over the next eight years. Money can earn 2.8% per year, compounded continuously. Is this business worth its purchase price of \$630,000?

Solution

First, please note that we still have to make some simplifying assumptions. We have to assume that the interest rates are going to remain constant for that entire eight years. We also have to assume that the \$75,000 per year is coming in continuously, like a faucet dripping dollars into the business. Neither of these assumptions might be accurate.

But moving on: The present value of the \$630,000 is, well, \$630,000. This is one investment, where we put our \$630,000 in the bank and let it sit there.

To find the present value of the business, we think of it as an income stream. The function $F(t)$ in this case is a constant \$75,000 dollars per year, so $F(t) = 75{,}000$. The interest rate is 2.8% and the term we're interested in is 8 years, so $r = .028$, and $T = 8$:

$$PV = \int_0^8 75000e^{-0.028t}\,dt \approx 537{,}548.75$$

The present value of the business is about $537,500, which is less than the $630,000 asking price, so this is not a good deal.

While this integral could have been done using substitution, for many of the integrals in this section we don't have the techniques to use antiderivatives or, in some cases, no antiderivative exists. Technology will work quickly, and it will give you an answer that is good enough.

Example 4.7.5

A company is considering purchasing a new machine for its production floor. The machine costs $65,000. The company estimates that the additional income from the machine will be a constant $7000 for the first year, then will increase by $800 each year after that. In order to buy the machine, the company needs to be convinced that it will pay for itself by the end of 8 years with this additional income. Money can earn 1.7% per year, compounded continuously. Should the company buy the machine?

Solution

We'll assume that the income will come in continuously over the 8 years. We'll also assume that interest rates will remain constant over that 8-year time period.

We're interested in the present value of the machine, which we will compare to its $65,000 price tag. Let t be the time, in years, since the purchase of the machine. The income from the machine is different depending on the time.

From $t = 0$ to $t = 1$ (the first year), the income is constant $7000 per year. From $t = 1$ to $t = 8$, the income is increasing by $800 each year; the income flow function $F(t)$ will be $F(t) = 7000 + 800(t-1) = 6200 + 800t$. To find the present value, we'll have to divide the integral into the two pieces, one for each of the functions:

$$PV = \int_0^1 7000e^{-0.017t}\,dt + \int_1^8 (6200 + 800t)e^{-0.017t}\,dt \approx 70166.$$

The present value is greater than the cost of the machine, so the company should buy the machine.

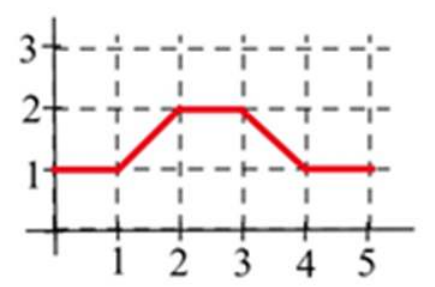
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4.E: The Integral (Exercises)

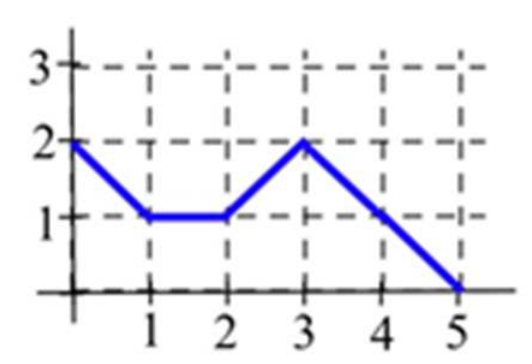
3.1 Exercises

Exercise $4.E.1$



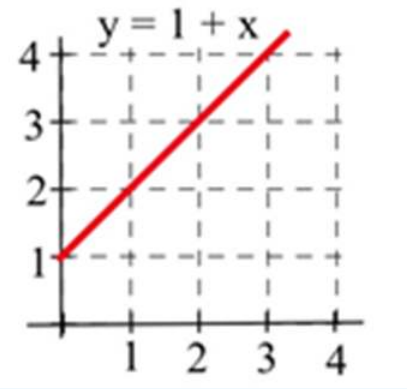
Let $A(x)$ represent the area bounded by the graph and the horizontal axis and vertical lines at $t = 0$ and $t = x$ for the graph shown. Evaluate $A(x)$ for $x = 1, 2, 3, 4$, and 5.

Exercise $4.E.2$



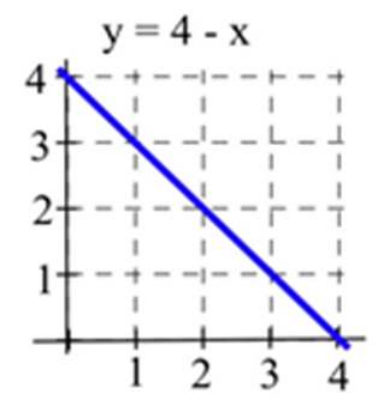
Let $B(x)$ represent the area bounded by the graph and the horizontal axis and vertical lines at $t = 0$ and $t = x$ for the graph shown. Evaluate $B(x)$ for $x = 1, 2, 3, 4$, and 5.

Exercise $4.E.3$



Let $C(x)$ represent the area bounded by the graph and the horizontal axis and vertical lines at $t = 0$ and $t = x$ for the graph shown. Evaluate $C(x)$ for $x = 1$, 2, and 3 and find a formula for $C(x)$.

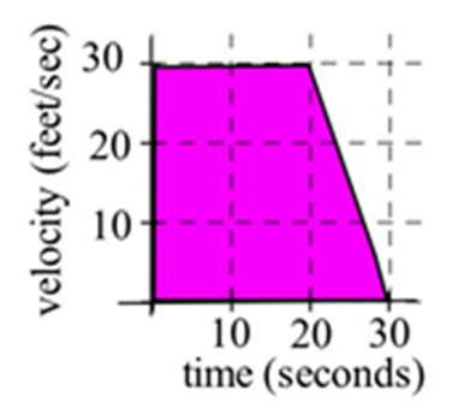
Exercise 4.E.4



Let $A(x)$ represent the area bounded by the graph and the horizontal axis and vertical lines at $t = 0$ and $t = x$ for the graph shown. Evaluate $A(x)$ for $x = 1$, 2, and 3 and find a formula for $A(x)$.

Exercise 4.E.5

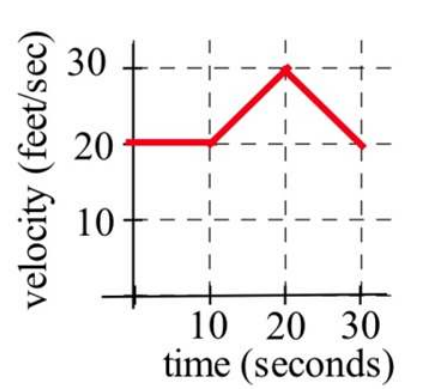
A car had the velocity shown in the graph to the right.



A car had the velocity shown in the graph to the right. How far did the car travel from $t = 0$ to $t = 30$ seconds?

Exercise 4.E.6

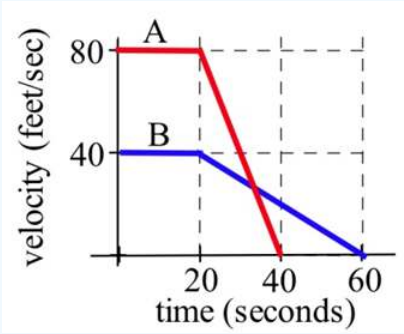
A car had the velocity shown below.



How afar did the car travel from $t = 0$ to $t = 30$ seconds?

Exercise 4.E.7

The velocities of two cars are shown in the graph.

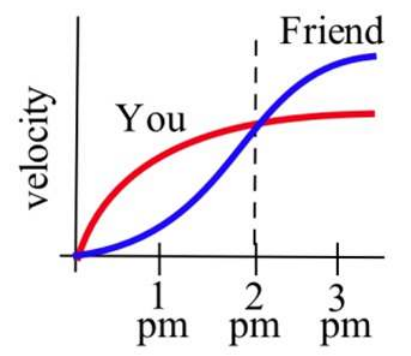


(a) From the time the brakes were applied, how many seconds did it take each car to stop?

(b) From the time the brakes were applied, which car traveled farther until it came to a complete stop?

Exercise 4.E.8

You and a friend start off at noon and walk in the same direction along the same path at the rates shown.

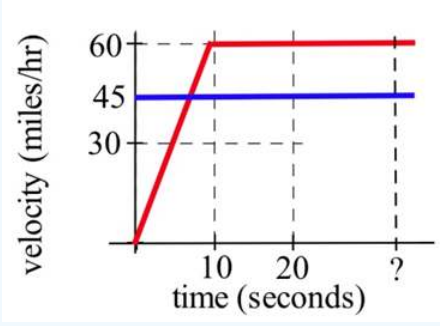


(a) Who is walking faster at 2 pm? Who is ahead at 2 pm?

(b) Who is walking faster at 3 pm? Who is ahead at 3 pm?

(c) When will you and your friend be together? (Answer in words.)

Exercise 4.E.9



Police chase: A speeder traveling 45 miles per hour (in a 25 mph zone) passes a stopped police car which immediately takes off after the speeder. If the police car speeds up steadily to 60 miles/hour in 20 seconds and then travels at a steady 60 miles/hour, **how long** and **how far** before the police car catches the speeder who continued traveling at 45 miles/hour?

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Exercise 4.E.10

Water is flowing into a tub. The table shows the rate at which the water flows, in gallons per minute. The tub is initially empty.

t, in minutes	0	1	2	3	4	5	6	7	8	9	10
Flow rate, in gal/min	0.5	1.0	1.2	1.4	1.7	2.0	2.3	1.8	0.7	0.5	0.2

Use the table to estimate how much water is in the tub after

a. five minutes

b. ten minutes

Exercise 4.E.11

The table shows the speedometer readings for a short car trip.

t, in minutes	0	5	10	15	20
Speed, in mph	0	30	40	65	40

a. Use the table to estimate how far the car traveled over the twenty minutes shown.

b. How accurate would you expect your estimate to be?

Exercise 4.E.12

The table shows values of $f(t)$. Use the table to estimate $\int_0^{40} f(t)dt$.

t	0	10	20	30	40
$f(t)$	17	22	18	11	35

Exercise 4.E.13

The table shows values of $g(x)$.

x	0	1	2	3	4	5	6
$g(x)$	140	142	144	152	154	165	200

Use the table to estimate

a. $\int_0^3 g(x)dx$	b $\int_3^6 g(x)dx$	c. $\int_0^6 g(x)dx$

Exercise 4.E.14

What are the units for the "area" of a rectangle with the given base and height units?

Base units	Height units	"Area" units
miles per second	seconds	

hours	dollars per hour	
square feet	feet	
kilowatts	hours	
houses	people per house	
meals	meals	

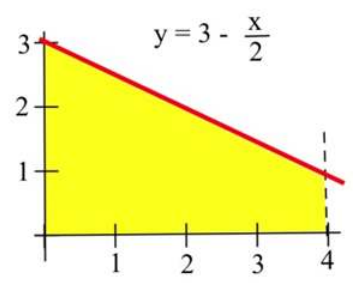
? Exercise $4.E.15 - 4.E.17$

In problems 15 – 17, represent the area of each bounded region as a definite integral, and use geometry to determine the value of the definite integral.

15. The region bounded by $y = 2x$, the x–axis, the line $x = 1$, and $x = 3$.

16. The region bounded by $y = 4 - 2x$, the x–axis, and the y–axis.

17. The shaded region in the graph to the right.



? Exercise $4.E.18$

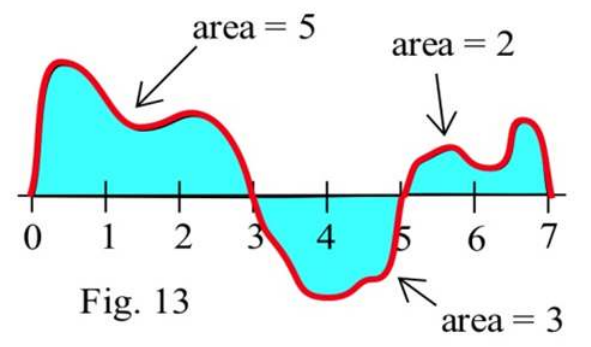


Fig. 13

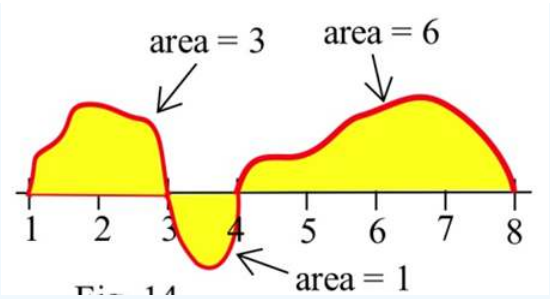
Using the graph of f shown and the given areas of several regions, evaluate:

(a) $\int_0^3 f(x)dx$

(b) $\int_3^5 f(x)dx$

(c) $\int_5^7 f(x)dx$

? Exercise $4.E.19$



Using the graph of f shown and the given areas of several regions, evaluate:

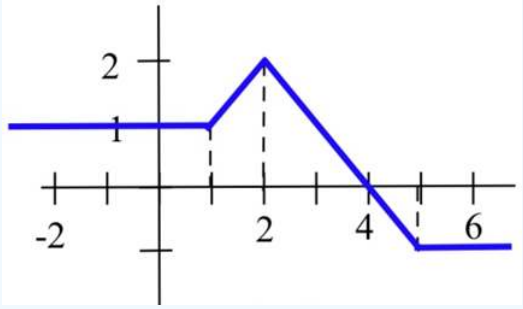
(a) $\int_1^3 g(x)dx$

(b) $\int_3^4 g(x)dx$

(c) $\int_4^8 g(x)dx$

(d) $\int_1^8 g(x)dx$

Exercise $4.E.20$



Use the graph to evaluate:

(a) $\int_{-2}^1 h(x)dx$

(b) $\int_4^6 h(x)dx$

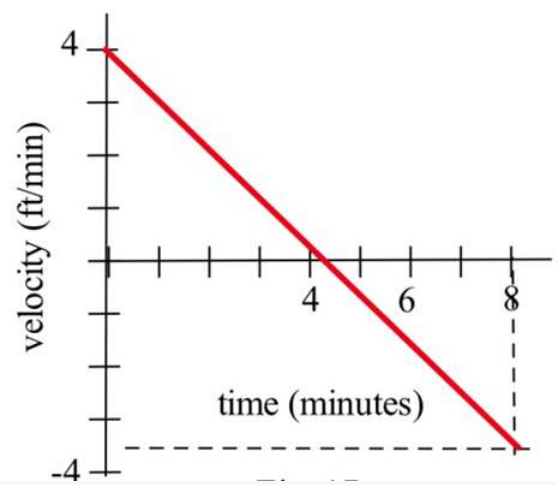
(c) $\int_{-2}^6 h(x)dx$

(d) $\int_{-2}^4 h(x)dx$

Exercise $4.E.21$

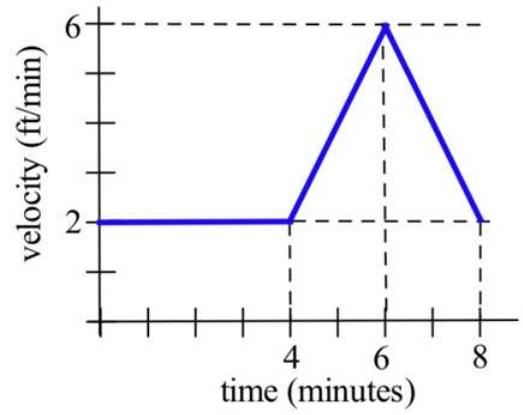
Your velocity along a straight road is shown to the right. How far did you travel in 8 minutes?

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Exercise $4.E.22$

Your velocity along a straight road is shown below. How many feet did you walk in 8 minutes?



Exercise $4.E.23 - 4.E.26$

In problems 23 - 26, the units are given for x and for $f(x)$. Give the units of $\int_a^b f(x)dx$.

23. x is time in "seconds", and $f(x)$ is velocity in "meters per second."

24. x is time in "hours", and $f(x)$ is a flow rate in "gallons per hour."

25. x is a position in "feet", and $f(x)$ is an area in "square feet."

26. x is a position in "inches", and $f(x)$ is a density in "pounds per inch."

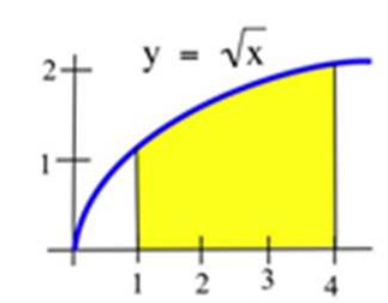
Exercise $4.E.27 - 4.E.31$

In problems 27 – 31, represent the area with a definite integral and use technology to find the approximate answer.

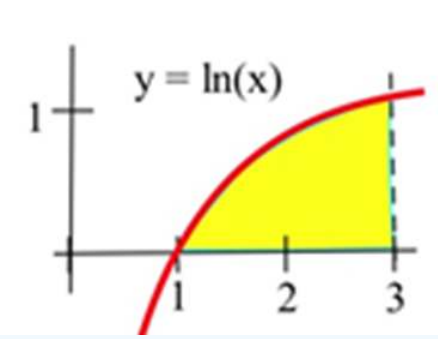
27. The region bounded by $y = x^3$, the x–axis, the line $x = 1$, and $x = 5$.

28. The region bounded by $y = \sqrt{x}$, the x–axis, and the line $x = 9$.

29. The shaded region shown to the right.



30. The shaded region below.



31. Consider the definite integral $\int_0^3 (3+x)dx$.

(a) Using six rectangles, find the left-hand Riemann sum for this definite integral.

(b) Using six rectangles, find the right-hand Riemann sum for this definite integral.

(c) Using geometry, find the exact value of this definite integral.

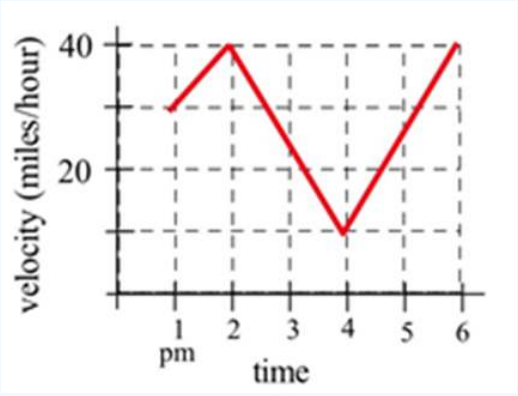
? Exercise 4.E. 32

Consider the definite integral $\int_0^2 x^3 dx$.

(a) Using four rectangles, find the left-hand Riemann sum for this definite integral.

(b) Using four rectangles, find the right-hand Riemann sum for this definite integral.

? Exercise 4.E. 33



Write the total distance traveled by the car in the graph between 1 pm and 4 pm as a definite integral and estimate the value of the integral.

Exercise $4.E.34 - 4.E.41$

Problems 34 – 41 refer to the graph of f shown.

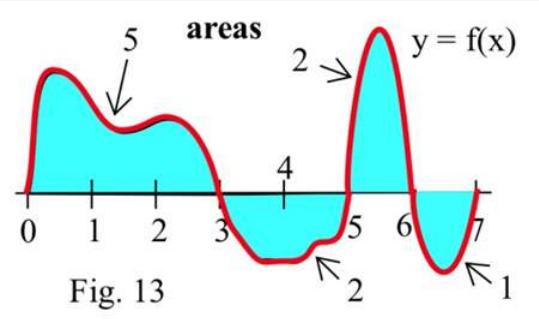


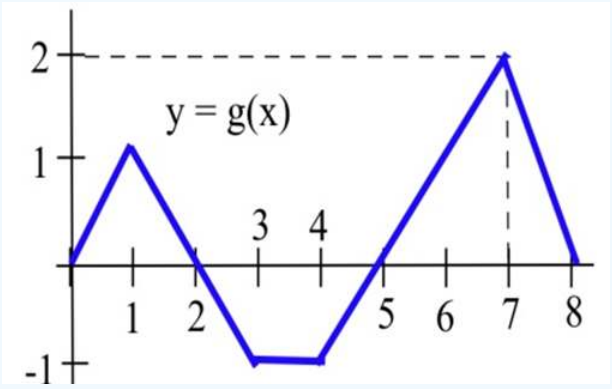
Fig. 13

Use the graph to determine the values of the definite integrals. (The bold numbers represent the **area** of each region.)

34. $\int_0^3 f(x)dx$	35. $\int_3^5 f(x)dx$	36. $\int_2^2 f(x)dx$	37. $\int_6^7 f(w)dw$
38. $\int_0^5 f(x)dx$	39. $\int_0^7 f(x)dx$	40. $\int_3^6 f(t)dt$	41. $\int_5^7 f(x)dx$

Exercise $4.E.42 - 4.E.47$

Problems 42 – 47 refer to the graph of g shown.



Use the graph to evaluate the integrals.

42. $\int_0^2 g(x)dx$	43. $\int_1^3 g(t)dt$	44. $\int_0^5 g(x)dx$
45. $\int_0^8 g(s)ds$	46. $\int_0^3 2g(t)dt$	47. $\int_5^8 1+g(x)dx$

3.2 Exercises

Exercise $4.E.1 - 4.E.5$

In problems 1 – 5, verify that $F(x)$ is an antiderivative of the integrand $f(x)$ and use Part 2 of the Fundamental Theorem to evaluate the definite integrals.

1. $\int_0^1 2xdx, F(x) = x^2 + 5$	2. $\int_1^4 3x^2dx, F(x) = x^3 + 2$	3. $\int_1^3 x^2dx, F(x) = \frac{1}{3}x^3$
4. $\int_0^3 (x^2 + 4x - 3)dx, F(x) = \frac{1}{3}x^3 + 2x^2 - 3x$	5. $\int_1^5 \frac{1}{x}dx, F(x) = \ln(x)$	

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Exercise $4.E.6$

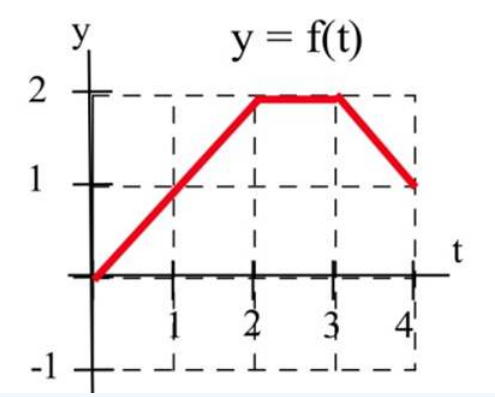
Given $A(x) = \int_0^x 2t dt$, find $A'(x)$

Exercise $4.E.7$

Given $A(x) = \int_0^x (3 - t^2) dt$, find $A'(x)$

Exercise $4.E.8$

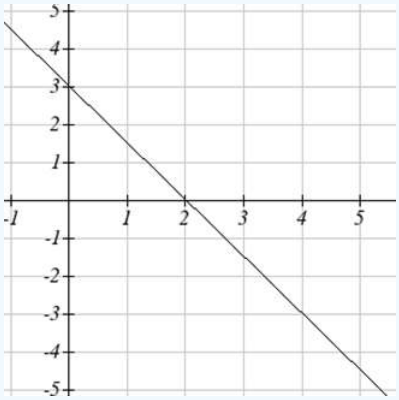
Let $A(x) = \int_0^x f(t) dt$ for the function graphed here.



Evaluate $A'(1)$, $A'(2)$, $A'(3)$.

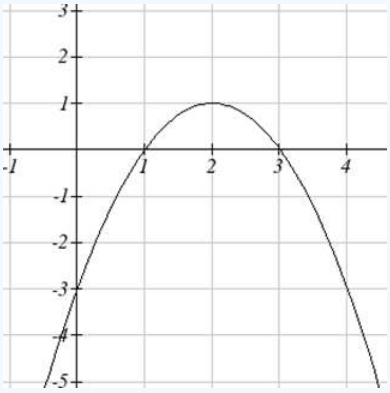
Exercise $4.E.9 - 4.E.10$

For problems 9-10, the graph provided shows $g'(x)$. Use it sketch a graph of $g(x)$ that satisfies $g(0) = 0$.



9.

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10.

3.3 Exercises

Exercise $4.E.1 - 4.E.10$

For problems 1-10, find the indicated antiderivative.

1. $\int(x^3-14x+5)dx$	2. $\int(2.5x^5-x-1.25)dx$
3. $\int 12.3dy$	4. $\int \pi^2 dw$
5. $\int e^P dP$	6. $\int\left(\sqrt{x}+e^x-\frac{1}{4x^3}\right)dx$
7. $\int \frac{1}{x}dx$	8. $\int \frac{1}{x^2}dx$
9. $\int(x-2)(x+2)dx$	10. $\int \frac{t^5-t^2}{t}dt$

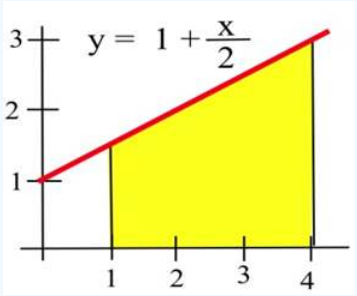
Exercise $4.E.11 - 4.E.18$

For problems 11-18, find an antiderivative of the integrand and use the Fundamental Theorem to evaluate the definite integral.

11. $\int_2^5 3x^2dx$	12. $\int_{-1}^2 x^2dx$	13. $\int_1^3(x^2+4x-3)dx$	14. $\int_1^e \frac{1}{x}dx$
15. $\int_{25}^{100}\sqrt{x}dx$	16. $\int_3^5\sqrt{x}dx$	17. $\int_1^{10}\frac{1}{x^2}dx$	18. $\int_1^{1000}\frac{1}{x^2}dx$

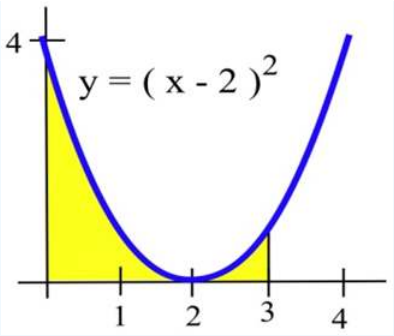
Exercise $4.E.19 - 4.E.21$

For problems 19 - 21 find the area shown in the figure.

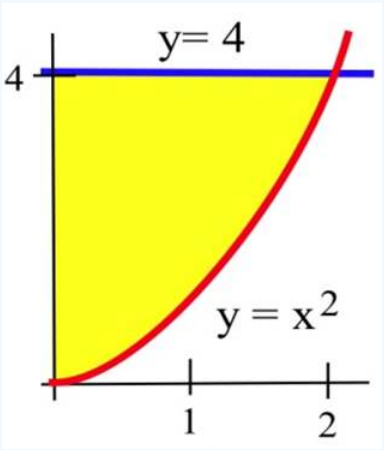


19.

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20.



21.

3.4 Exercises

Exercise $4.E.1 - 4.E.8$

For problems 1-8, find the indicated antiderivative.

1. $\int \frac{1}{(4x+1)^3}\,dx$	2. $\int e^{100x}\,dx$
3. $\int (1.0003)^{12t}\,dt$	4. $\int \frac{e^{10/x}}{x^2}\,dx$
5. $\int \sqrt{w+5}\,dw$	6. $\int 6x^2\sqrt{3x^3-1}\,dx$
7. $\int \frac{dx}{x\ln x}$	8. $\int \frac{x-3}{x^2-6x+5}dx$

Exercise $4.E.9 - 4.E.12$

For problems 9-12, find an antiderivative of the integrand and use the Fundamental Theorem to evaluate the definite integral.

9. $\int_{-2}^{2} \frac{2x}{1+x^2}\,dx$	10. $\int_0^1 e^{2x}\,dx$	11. $\int_2^4 (x-2)^3 dx$	12. $\int_0^1 x\sqrt{1-x^2}\,dx$

3.5 Exercises

Exercise $4.E.1 - 4.E.4$

In problems 1–4, a function u or dv is given. Find the piece u or dv which is not given, calculate du and v, and apply the Integration by Parts Formula.

1. $\int 12x \cdot \ln(x)dx$	$u = \ln(x)$	2. $\int x \cdot e^{-x}dx$	$u = x$
3. $\int x^4 \ln(x)dx$	$dv = x^4dx$	4. $\int x \cdot (5x+1)^{19}dx$	$u = x$

Exercise $4.E.5 - 4.E.10$

In problems 5 - 10 evaluate the integrals

5. $\int_0^1 \frac{x}{e^{3x}}dx$	6. $\int_0^1 10x \cdot e^{3x}dx$	7. $\int_1^3 \ln(2x+5)dx$
8. $\int x^3 \ln(5x)dx$	9. $\int x\ln(x+1)dx$	10. $\int_1^2 \frac{\ln(x)}{x^2}dx$

Exercise $4.E.11 - 4.E.14$

For problems 11 - 14 integrate each function.

11. $\int \frac{1}{4-x^2}$	12. $\int \frac{2}{9-x^2}$	13. $\int \sqrt{4+x^2}$	14. $\int \sqrt{9+x^2}$

3.6 Exercises

Exercise $4.E.1 - 4.E.4$

In problems 1 – 4, use the values in the table to estimate the areas.

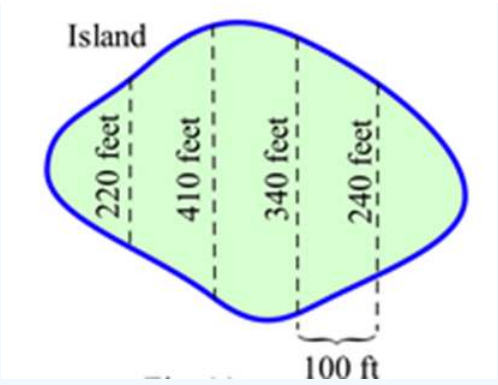
x	$f(x)$	$g(x)$	$h(x)$
0	5	2	5
1	6	1	6
2	6	2	8
3	4	2	6
4	3	3	5
5	2	4	4
6	2	0	2

1. Estimate the area between f and g, between $x = 0$ and $x = 4$.
2. Estimate the area between g and h, between $x = 0$ and $x = 6$.
3. Estimate the area between f and h, between $x = 0$ and $x = 4$.
4. Estimate the area between f and g, between $x = 0$ and $x = 6$.

Exercise $4.E.1$

Estimate the area of the island shown

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Exercise $4.E.6 - 4.E.15$

In problems 6 – 15, find the **area** between the graphs of f and g for x in the given interval. Remember to draw the graph!

6. $f(x) = x^2 + 3$, $g(x) = 1$ and $-1 \le x \le 2$.
7. $f(x) = x^2 + 3$, $g(x) = 1 + x$ and $0 \le x \le 3$.
8. $f(x) = x^2$, $g(x) = x$ and $0 \le x \le 2$.
9. $f(x) = (x-1)^2$, $g(x) = x + 1$ and $0 \le x \le 3$.
10. $f(x) = \frac{1}{x}$, $g(x) = x$ and $1 \le x \le e$.
11. $f(x) = \sqrt{x}$, $g(x) = x$ and $0 \le x \le 4$.
12. $f(x) = 4 - x^2$, $g(x) = x + 2$ and $0 \le x \le 2$.
13. $f(x) = e^x$, $g(x) = x$ and $0 \le x \le 2$.
14. $f(x) = 3$, $g(x) = \sqrt{1 - x^2}$ and $0 \le x \le 1$.
15. $f(x) = 2$, $g(x) = \sqrt{4 - x^2}$ and $-2 \le x \le 2$.

Exercise $4.E.16 - 4.E.18$

For problems 16-18, find the volume of the solid obtained by rotating the specified region about the x axis.

16. Region under $f(x) = x^2 + 3$ for $-1 \le x \le 2$.
17. Region under $f(x) = 4 - x^2$ for $0 \le x \le 2$.
18. Region under $f(x) = \frac{1}{x}$ for $1 \le x \le 2$.

Exercise $4.E.19 - 4.E.20$

In problems 19 and 20 use the values in the table to estimate the average values.

x	$f(x)$	$g(x)$
0	5	2
1	6	1
2	6	2
3	4	2
4	3	3
5	2	4

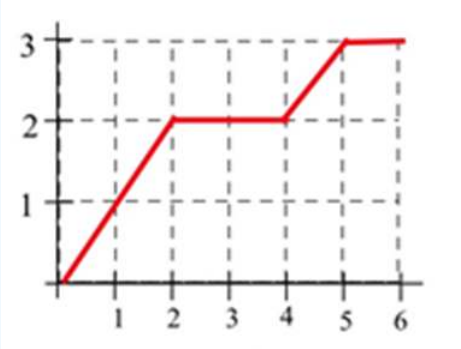
6	2	0

19. Estimate the average value of f on the interval [0, 6].

20. Estimate the average value of g on the interval [0, 6].

Exercise $4.E.21 - 4.E.26$

In problems 21 – 26, find the **average value** of f on the given interval.



21. $f(x)$ from the graph for $0 \leq x \leq 2$.

22. $f(x)$ from the graph for $0 \leq x \leq 4$.

23. $f(x)$ from the graph for $1 \leq x \leq 6$.

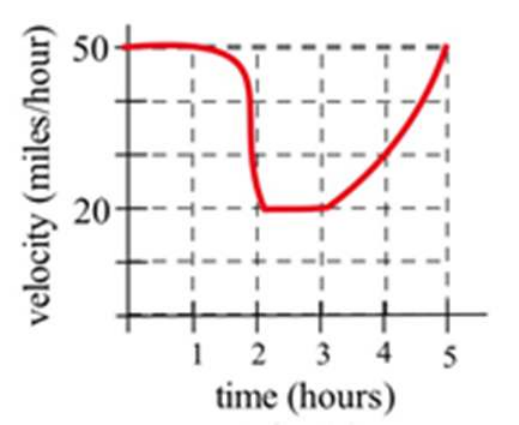
24. $f(x)$ from the graph for $4 \leq x \leq 6$.

25. $f(x) = 2x + 1$ for $0 \leq x \leq 4$.

26. $f(x) = x^2$ for $0 \leq x \leq 2$.

Exercise $4.E.27$

The graph shows the velocity of a car during a 5 hour trip.



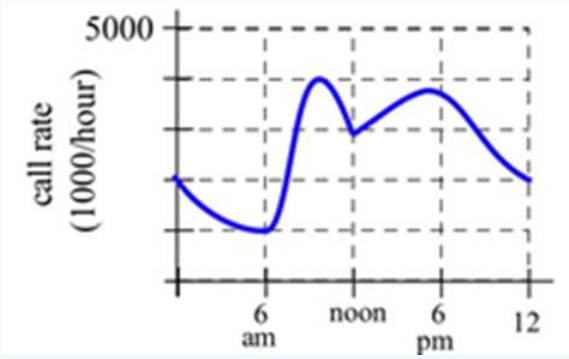
(a) Estimate how far the car traveled during the 5 hours.

(b) At what **constant** velocity should you drive in order to travel the same distance in 5 hours?

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Exercise 4.E.28

The graph shows the number of telephone calls per minute at a large company.



Estimate the average number of calls per minute

(a) From 8 am to 5 pm.

(b) From 9 am to 1 pm.

3.7 Exercises

Exercise 4.E.1

The demand and supply functions for a certain product are given by $p = 150 - .5q$ and $p = .002q^2 + 1.5$, where p is in dollars and q is the number of items.

(a) Which is the demand function?

(b) Find the equilibrium price and quantity

(c) Find the total gains from trade at the equilibrium price.

Exercise 4.E.2

Still thinking about the product from Exercise 1, with its demand and supply functions, suppose the price is set artificially at \$70 (which is above the equilibrium price).

(a) Find the quantity supplied and the quantity demanded at this price.

(b) Compute the consumer surplus at this price, using the quantity demanded.

(c) Compute the producer surplus at this price, using the quantity demanded (why?).

(d) Find the total gains from trade at this price.

(e) What do you observe?

Exercise 4.E.3

When the price of a certain product is \$40, 25 items can be sold. When the price of the same product costs \$20, 185 items can be sold. On the other hand, when the price of this product is \$40, 200 items will be produced. But when the price of this product is \$20, only 100 items will be produced. Use this information to find supply and demand functions (assume for simplicity that the functions are linear), and compute the consumer and producer surplus at the equilibrium price.

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Exercise 4.E.4

Find the present and future values of a continuous income stream of $5000 per year for 12 years if money can earn 1.3% annual interest compounded continuously.

Exercise 4.E.5

Find the present value of a continuous income stream of $40,000 per year for 35 years if money can earn

(a) 0.8% annual interest, compounded continuously,

(b) 2.5% annual interest, compounded continuously,

(c) 4.5% annual interest, compounded continuously.

Exercise 4.E.6

Find the present value of a continuous income stream $F(t) = 20 + t$, where t is in years and F is in tens of thousands of dollars per year, for 10 years, if money can earn 2% annual interest, compounded continuously.

Exercise 4.E.7

Find the present value of a continuous income stream $F(t) = 12 + 0.3t^t$, where t is in years and F is in thousands of dollars per year, for 8 years, if money can earn 3.7% annual interest, compounded continuously.

Exercise 4.E.8

Find the future value of a continuous income stream $F(t) = 8500 + \sqrt{640t + 100}$, where t is in years and F is in dollars per year, for 15 years, if money can earn 6% annual interest, compounded continuously.

Exercise 4.E.9

A business is expected to generate income at a continuous rate of $25,000 per year for the next eight years. Money can earn 3.4% annual interest, compounded continuously. The business is for sale for $153,000. Is this a good deal?

3.8 Exercises

Exercise 4.E.1 – 4.E.4

In problems 1 – 4, check that the function y is a solution of the given differential equation.

1. $y' + 3y = 6\,.\, y = e^{-3x} + 2\,.$	2. $y' - 2y = 8\,.\, y = e^{2x} - 4.$
3. $y' = -x/y\,.\, y = \sqrt{7 - x^2}\,.$	4. $y' = x - y\,.\, y = x - 1 + 2e^{-x}\,.$

Exercise 4.E.5 – 4.E.8

In problems 5 – 8 check that the function y is a solution of the given initial value problem.

5. $y' = 6x^2 - 3$ and $y(1) = 2\,.\, y = 2x^3 - 3x + 3\,.$

6. $y' = 6x + 4$ and $y(2) = 3\,.\, y = 3x^2 + 4x - 17\,.$

7. $y' = 5y$ and $y(0) = 7\,.\, y = 7e^{5x}\,.$

8. $y' = -2y$ and $y(0) = 3\,.\, y = 3e^{-2x}\,.$

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Exercise $4.E.9 - 4.E.12$

In problems 9 – 12, a family of solutions of a differential equation is given. Find the value of the constant C so the solution satisfies the initial value condition.

9. $y' = 2x$ and $y(3) = 7$. $y = x^2 + C$.	10. $y' = 3x^2 - 5$ and $y(1) = 2$. $y = x^3 - 5x + C$.
11. $y' = 3y$ and $y(0) = 5$. $y = Ce^{3x}$.	12. $y' = -2y$ and $y(0) = 3$. $y = Ce^{-2x}$.

Exercise $4.E.13 - 4.E.18$

In problems 13 – 18, solve the differential equation. (Assume that x and y are restricted so that division by zero does not occur.)

13. $y' = 2xy$	14. $y' = x/y$	15. $xy' = y + 3$
16. $y' = x^2y + 3y$	17. $y' = 4y$	18. $y' = 5(2 - y)$

Exercise $4.E.19 - 4.E.22$

In problems 19 – 22, solve the initial value separable differential equations.

19. $y' = 2xy$ for $y(0) = 3$, $y(0) = 5$, and $y(1) = 2$.

20. $y' = x/y$ for $y(0) = 3$, $y(0) = 5$, and $y(1) = 2$.

21. $y' = 3y$ for $y(0) = 4$, $y(0) = 7$, and $y(1) = 3$.

22. $y' = -2y$ for $y(0) = 4$, $y(0) = 7$, and $y(1) = 3$.

Exercise $4.E.23$

The rate of growth of a population $P(t)$ which starts with 3,000 people and increases by 4% per year is $P'(t) = 0.0392 \cdot P(t)$. Solve the differential equation and use the solution to estimate the population in 20 years.

Exercise $4.E.24$

The rate of growth of a population $P(t)$ which starts with 5,000 people and increases by 3% per year is $P'(t) = 0.0296 \cdot P(t)$. Solve the differential equation and use the solution to estimate the population in 20 years.

Exercise $4.E.25$

A manufacturer estimates that she can sell a maximum of 130 thousand cell phones in a city. By advertising heavily, her total sales grow at a rate proportional to the distance below this upper limit. If she enters a new market, and after 6 months her total sales are 59 thousand phones, find a formula for the total sales (in thousands) t months after entering the market, and use this to estimate the total sales at the end of the first year.

Exercise $4.E.26$

The temperature of a turkey in the oven will grow like limited growth. The turkey starts out at 40 degrees Fahrenheit, and is placed into a 350 degree oven. After 30 minutes, the turkey's temperature has risen to 55 degrees. How long will it take until the turkey's temperature reaches 165 degrees?

Exercise 4.E.27

A new cell phone is introduced into the market. It is predicted that sales will grow logistically. The manufacturer estimates that they can sell a maximum of 100 thousand cell phones. After 44 thousand cell phones have been sold, sales are increasing by 4 thousand phones per month. Use this to estimate the total sales at the end of the first year.

Exercise 4.E.28

Biologists stocked a lake with 400 fish and estimated the carrying capacity of the lake to be 8000 fish. The number of fish tripled in the first year. How long will it take the population to increase to 4000?

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CHAPTER OVERVIEW

5: Functions of Several Variables

From a formal point of view, the first 5 chapters of this text have been concerned with functions of one variable. More realistically, we have been looking at functions of several variables all along. If we consider the formula for finding how much is in a bank account in the future, we have the formula:

$$FutureAmount = InitialDeposit * \left(1 + \frac{rate}{ppy}\right)^{years*ppy}$$

where ppy is the number of periods per year, indicating how often we compound the interest. From a simple point of view, the future amount is a function of 4 variables, the initial deposit, annual rate, periods per year, and number of years. To consider this as a function of a single variable, we fixed 3 of the 4 variables as constants for a particular problem. In this chapter we want to address the more realistic situation where we treat more than one quantity as a variable at a time. This approach has the added advantage that most real world functions of interest have more than one variable. Before we look at functions of several variables, we want to create a list of tasks we have learned to accomplish with functions of one variable:

1. Evaluate the function at a particular point with Excel.
2. Make a table of values at a series of points with Excel.
3. Make a reasonable graph from a table of values.
4. Zoom in on a graph until it looks like a straight line.
5. Find the slope of the tangent line.
6. Give a formula for the tangent line at a point.
7. Identify the places where the tangent line is flat.
8. Find local extrema for the function.
9. Find global extrema for the function.
10. Learn applications of the derivative.

We would like to look at how to extend these tasks to functions of several variables. Most of this chapter simply notes how to modify the rules we learned for functions of a single variable to the multivariable case.

5.1: Functions of Two Variables
5.2: Calculus of Functions of Two Variables
5.3: Partial Derivatives
5.4: Maximum and Minimum Values
5.5: Optimization
5.6: Critical Points and Extrema
5.E: Functions of Two Variables (Exercises)

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5.1: Functions of Two Variables

Introduction

Real life is rarely as simple as one input / one output. Many relationships depend on lots of variables. Examples:

- If I put a deposit into an interest-bearing account and let it sit, the amount I have at the end of 3 years depends on P (how much my initial deposit is), r (the annual interest rate), and n (the number of compoundings per year).
- The air resistance on a wing in a wind tunnel depends on the shape of the wing, the speed of the wind, the wing's orientation (pitch, yaw, and roll), plus a myriad of other things that I can't begin to describe.
- The amount of your television cable bill depends on which basic rate structure you have chosen and how many pay-per-view movies you ordered.

Since the real world is so complicated, we want to extend our calculus ideas to functions of several variables.

Functions of Two Variables

If $x_1, x_2, x_3, \ldots, x_n$ are real numbers, then $(x_1, x_2, x_3, \ldots, x_n)$ is called an n-tuple. This is an extension of ordered pairs and triples. A function of n variables is a function whose domain is some set of n-tuples and whose range is some set of real numbers.

For much of what we do here, everything will work the same whether we were working with 2, 3, or 47 variables. Because we're trying to keep things a little bit simpler, we'll concentrate on functions of two variables.

A Function of Two Variables

A function of two variables is a function, that is, to each input is associated exactly one output.

The *inputs* are ordered pairs, (x, y). The outputs are real numbers (each output is a single real number). The domain of a function is the set of all possible inputs (ordered pairs); the range is the set of all possible outputs (real numbers).

The function can be written $z = f(x, y)$.

Functions of two variables can be described numerically (a table), graphically, algebraically (a formula), or in English.

We will often now call the familiar $y = f(x)$ a function of one variable.

Example 5.1.1

The cost of renting a car depends on how many days you keep it and how far you drive. Represent this using a function.

Solution

Let d be the number of days you rent the car, and m be the number of miles you drive. Then the cost of the car rental $C(d, m)$ is a function of two variables.

Example 5.1.2

The demand for hot dog buns depends on the price for the hot dog buns and also on the price for hot dogs. Represent this as a function.

Solution

The demand $q_B = f(p_B, p_D)$ is a function of two variables. (The demand for hot dogs also depends on the price of both dogs and buns).

Formulas and Tables

Just as in the case of functions of one variable, we can display a function of two variables in a table. The two inputs are shown in the margin (top row, left column), and the outputs are shown in the interior cells.

Example 5.1.3

Here is a table that shows the cost $C(d, m)$ in dollars for renting a car for d days and driving it m miles:

d (labels in left column), m (labels in top row)	100	200	300	400
1	55	70	85	100
2	95	110	125	140
3	135	*150*	165	180

1. What is the cost of renting a car for 3 days and driving it 200 miles?
2. What is $C(100, 4)$? What is $C(4, 100)$?
3. Suppose we rent the car for three days. Is C an increasing function of miles?

Solution

1. According to the table, renting the car for three days (row with $d = 3$) and driving it 200 miles (column with $m = 200$) will cost \$150 (italicized in the table).
2. Careful now – the input is an *ordered* pair, so in $C(100, 4)$, the 100 has to be a value of d and the 4 has to be a value of m. $C(100, 4)$ would be the cost of renting a car for 100 days and driving it 4 miles. That cost is not in the table. (And that would be a pretty silly way to rent a car.) On the other hand, $C(4, 100)$ is the cost of renting for 4 days and driving 100 miles – the table says that would cost \$175.
3. If we know that d is fixed at 3, we're looking at $C(3, m)$. This is now a function of one variable: just m. We can see the table that displays values of this function by focusing our attention on just the row where $d = 3$:

d (label in left column), m (labels in top row)	100	200	300	400
3	135	*150*	165	180

Now we can see that if we rent for 3 days, the cost appears to be an increasing function of the number of miles we drive, which shouldn't be surprising.

The idea of fixing one variable and watching what happens to the function as the other varies will come up again and again.

It's hard to display a function of more than two variables in a table. But it's convenient to work with formulas for functions of two variables, or as many variables as you like.

Example 5.1.4

The cost $C(d, m)$ in dollars for renting a car for d days and driving it m miles is given by the formula

$$C(d, m) = 40d + 0.15m.$$

1. What is the cost of renting a car for 3 days and driving it 200 miles?
2. What is $C(100, 4)$? What is $C(4, 100)$?
3. Suppose we rent the car for 3 days. Is C an increasing function of miles?

Solution

1. $C(3, 200) = 40(3) + 0.15(200) = \$150$. This is the same value we got from the table. The formula will give us the same answers for any of the table values.
2. $C(100, 4)$ makes perfect sense to the formula (even if it doesn't make sense for actually renting a car). So now we can get an answer. To rent the car for 100 days and drive it for 4 miles should cost \$4000.60. $C(4, 100) = \$175$, as before.
3. If we fix $d = 3$, then $C(d, m)$ becomes $C(3, m) = 40(3) + 0.15m = 120 + 0.15m$. Yes, this is an increasing function of m – we can tell because it's linear and its slope is $0.15 > 0$.

Reality check – the formula that gives the cost for the rental car makes sense for all values of d and m. But that's not how the real cost works – you can't rent the car for a negative number of days or drive a negative number of miles. (That is, there are domain restrictions.) In addition, most car rental agreements don't compute a charge for fractions of days; they round up to the next whole number of days.

Example 5.1.5

Let $f(x, y, z, w) = 35x^2w - \frac{1}{z} + yz^2$. Evaluate $f(0, 1, 2, 3)$.

Solution

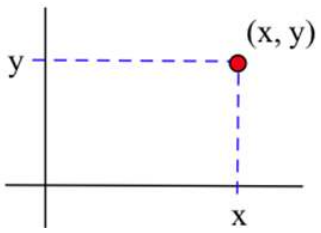
Remember that this is an ordered 4-tuple; make sure the numbers get substituted into the correct places:

$$f(0, 1, 2, 3) = 35(0)^2(3) - \frac{1}{2} + (1)(2)^2 = 3.5$$

Graphs

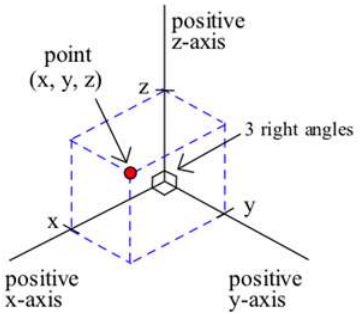
The graph of a function of two variables is a surface in three-dimensional space. Let's start by looking at the 3-dimensional rectangular coordinate system, how to locate points in three dimensions, and distance between points in three dimensions.

In the 2-dimensional rectangular coordinate system we have two coordinate axes that meet at right angles at the origin, and it takes two numbers, an ordered pair (x, y), to specify the rectangular coordinate location of a point in the plane (two dimensions).

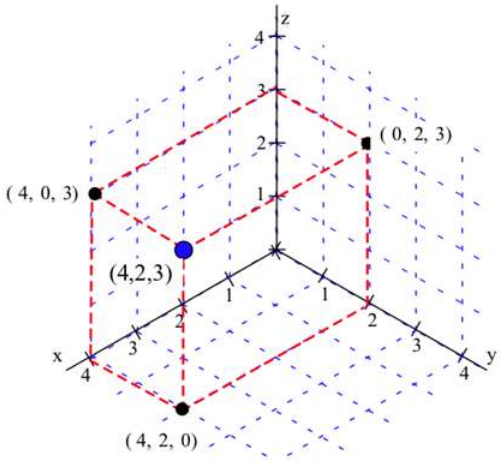


Each ordered pair (x, y) specifies the location of exactly one point, and the location of each point is given by exactly one ordered pair (x, y). The x and y values are the coordinates of the point (x, y).

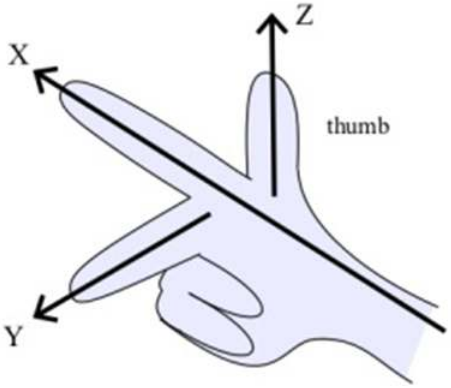
The situation in three dimensions is very similar. In the 3-dimensional rectangular coordinate system we have three coordinate axes that meet at right angles, and three numbers, an ordered triple (x, y, z), are needed to specify the location of a point.



Each ordered triple (x, y, z) specifies the location of exactly one point, and the location of each point is given by exactly one ordered triple (x, y, z). The x, y, and z values are the coordinates of the point (x, y, z). The figure below shows the location of the point (4, 2, 3).

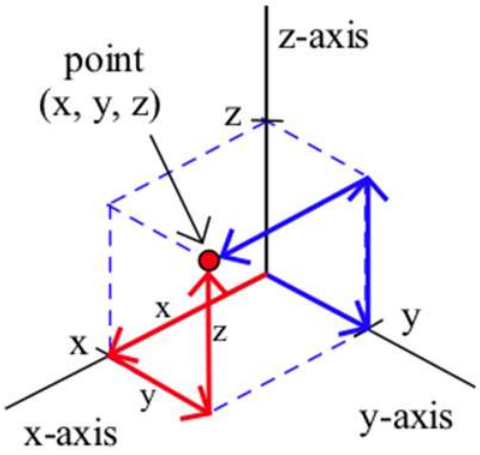


Typically we use a right-hand orientation. To see what this means, imagine your right hand in front hand in front of you with the palm toward your face, your thumb pointing up, you index finger straight out, and your next finger toward your face (and the two bottom fingers bent into the palm). Then, in the right hand coordinate system, your thumb points along the positive z-axis, your index finger along the positive x-axis, and the other finger along the positive y-axis.



Other orientations of the axes are possible and valid (with appropriate labeling), but the right-hand system is the most common orientation and is the one we will generally use. If another orientation is used, then the axes will be explicitly labeled.

Each ordered triple (x, y, z) specifies the location of a single point, and this location point can be plotted by locating the point $(x, y, 0)$ on the xy-plane and then going up z units (the red path in the figure below).



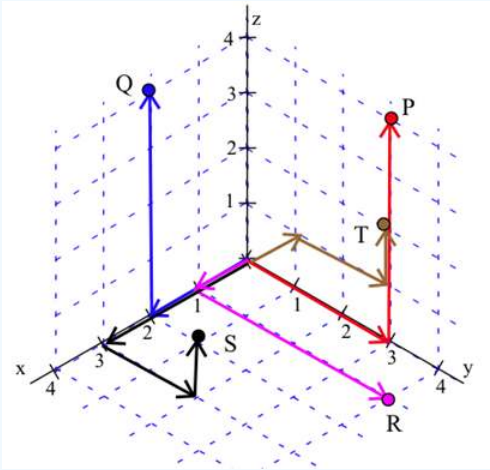
We could also get to the same (x, y, z) point in other ways. For instance, we could start by finding the point $(x, 0, z)$ on the xz-plane and then going y units parallel to the y-axis, or by finding $(0, y, z)$ on the yz-plane and then going x units parallel to the x-axis (the blue path in the figure above).

Example 5.1.6

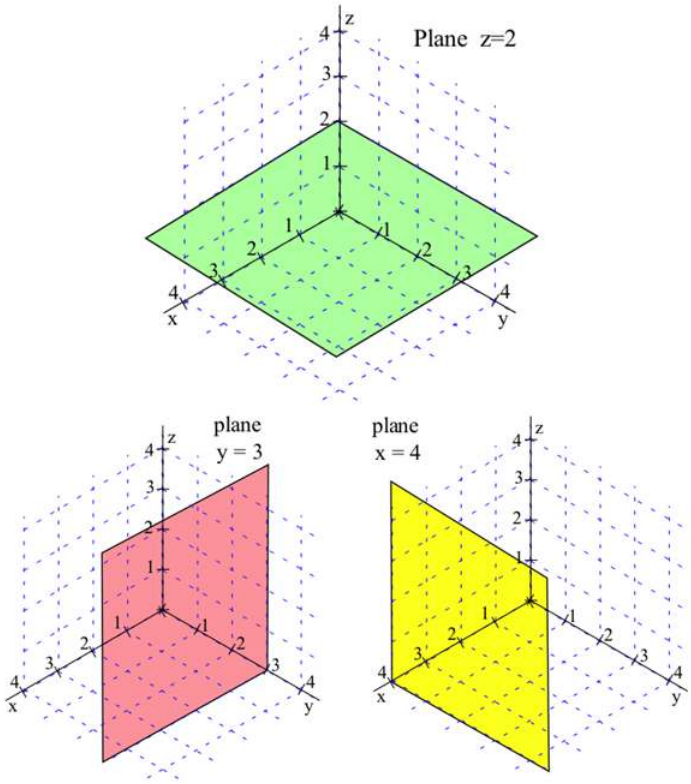
Plot the locations of the points

- $P = (0, 3, 4)$,
- $Q = (2, 0, 4)$,
- $R = (1, 4, 0)$,
- $S = (3, 2, 1)$, and
- $T = (-1, 2, 1)$.

The points are shown below.



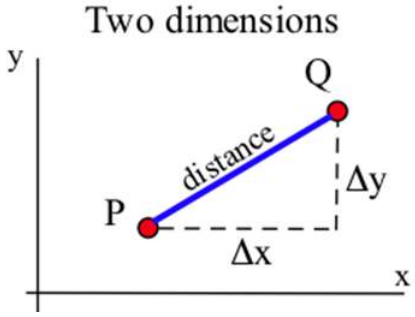
Once we can locate points, we can begin to consider the graphs of various collections of points. By the graph of "$z = 2$" we mean the collection of all points (x, y, z) which have the form "$(x, y, 2)$". Since no condition is imposed on the x and y variables, they take all possible values. The graph of $z = 2$ is a plane parallel to the xy-plane and 2 units above the xy-plane. Similarly, the graph of $y = 3$ is a plane parallel to the xz-plane, and $x = 4$ is a plane parallel to the yz-plane. (Note: The planes have been drawn as rectangles, but they actually extend infinitely far.)



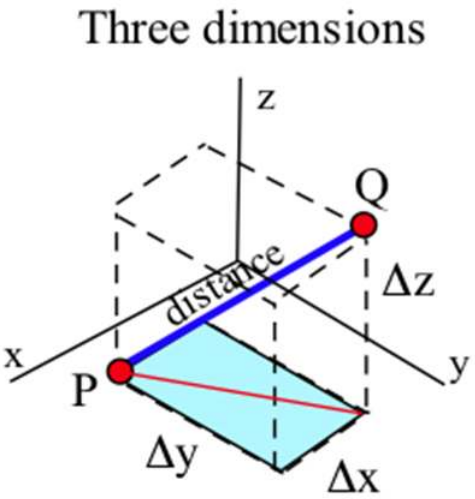
Distance Between Points

In two dimensions we can think of the distance between points as the length of the hypotenuse of a right triangle, and that leads to the Pythagorean formula:

$$\text{distance} = \sqrt{\Delta x^2 + \Delta y^2}\,.$$



In three dimensions we can also think of the distance between points as the length of the hypotenuse of a right triangle.



In this situation the calculations may appear more complicated, but they are straightforward and the final formula is what we hope it would be given the 2-dimensional formula:

$$\begin{aligned}\text{distance}^2 &= \text{base}^2 + \text{height}^2 \\ &= \left(\sqrt{\Delta x^2 + \Delta y^2}\right)^2 + \Delta z^2 \\ &= \Delta x^2 + \Delta y^2 + \Delta z^2\end{aligned}$$

so

$$\text{distance} = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}\,.$$

Distance in 3-Dimensions

If $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ are points in space, then the distance between P and Q is

$$\begin{aligned}\text{distance} &= \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\end{aligned}$$

The 3-dimensional pattern is very similar to the 2-dimensional pattern with the additional piece Δz^2.

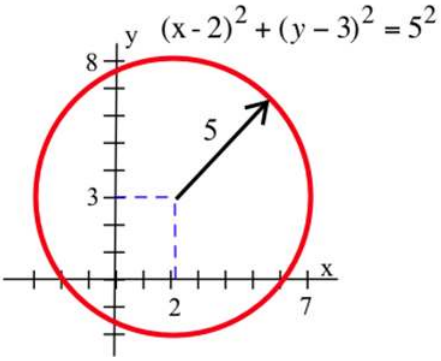
Example 5.1.7

Find the distances between points $A = (1, 2, 3)$ and $B = (7, 5, -3)$.

Solution

$$\text{Dist}(A, B) = \sqrt{6^2 + 3^2 + (-6)^2} = \sqrt{36 + 9 + 36} = \sqrt{81} = 9.$$

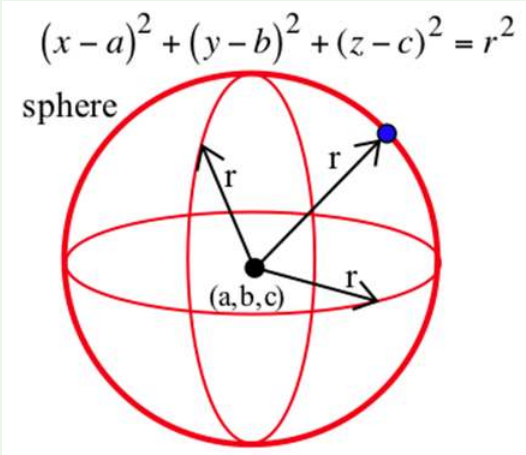
In two dimensions, the set of points at a fixed distance from a given point is a circle, and we used the distance formula to determine equations describing circles: the circle with center (2, 3) and radius 5 is given by $(x-2)^2 + (y-3)^2 = 5^2$ or $x^2 + y^2 - 4x - 6y = 12$.



The same ideas work for spheres in three dimensions.

Spheres

The set of points (x, y, z) at a fixed distance r from a point (a, b, c) is a sphere with center (a, b, c) and radius r.



The sphere is given by the equation

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2.$$

Example 5.1.8

Write the equations of a sphere with center (2, -3, 4) and radius 3.

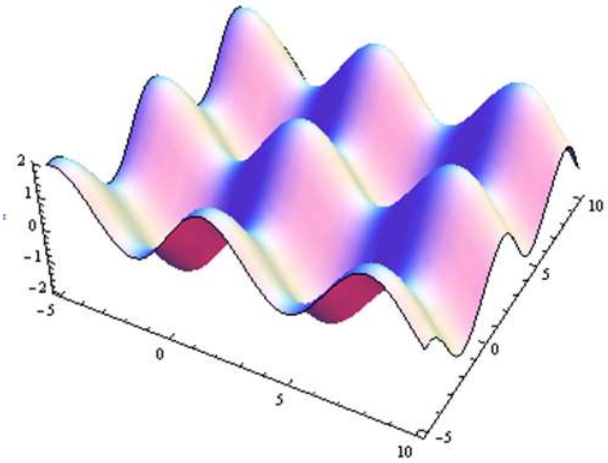
Solution

The equation is

$$(x-2)^2+(y+3)^2+(z-4)^2=3^2.$$

Now suppose that we want to graph a surface. We can think of each input (x, y) as a location on the plane, and plot the point $f(x, y)$ units above that point. Graphing that can be challenging. We have a few options:

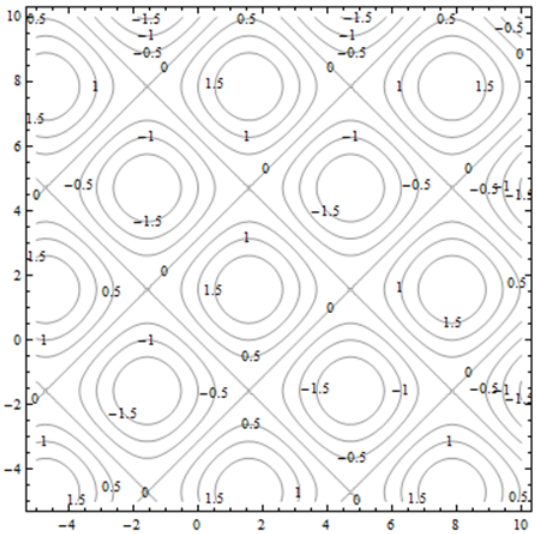
1. Use a computer program (such as GeoGebra or Mathematica) to draw beautiful perspective drawings.



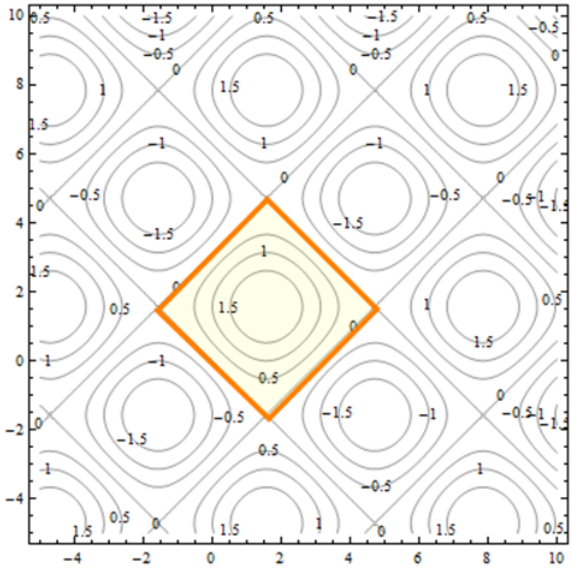
   If such a program is available, then this is usually the most accurate option.

2. Try to draw a perspective drawing by hand. This is very challenging, and usually not worth the effort.
3. Use level curves to draw **contour diagrams** (or **contour maps**), which is the approach we'll focus on here. A contour diagram is like a topographical map – points with the same elevation (outputs) are connected with curves. Each particular output is called a *level*, and these curves are called *level curves* or *contours*. The closer the curves are to each other, the steeper that section of the surface. Topographical maps give hikers information about elevation, steep and shallow grades, peaks and valleys. Contour diagrams give us the same kind of information about a function.

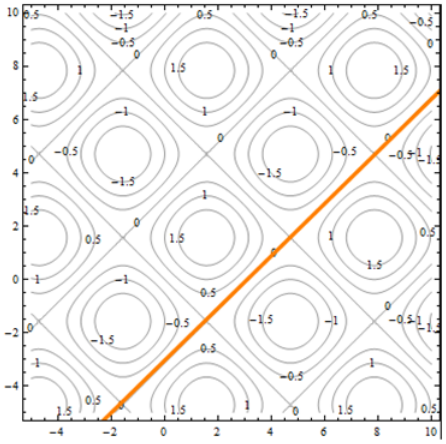
Below is a contour diagram of the same surface shown in above. The level curves are graphs in the xy-plane of curves $f(x, y) = c$ for various constants c.



Each of the squares corresponds to one of the bumps on the surface. If the contours are positive, as highlighted below, the bump is above the xy-plane. If the contours are negative, the bump extends below the xy-plane.



Everywhere on the crisscrossed pattern of diagonal lines, the height of the surface is 0, so the surface is on the xy-plane. This is a feature that we wouldn't necessarily have seen when we looked at the perspective drawing. Contour maps can help us see features of the surface that the 3-dimensional graph doesn't show.

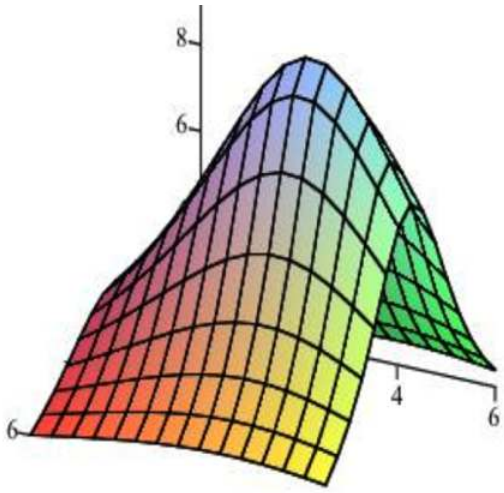


To better understand contour diagrams, suppose we had a table of elevation data. We could graph this by plotting the height at each point and connecting the dots with smooth curves, which would result in the something like the graph shown.

Table: Elevations arranged in a table

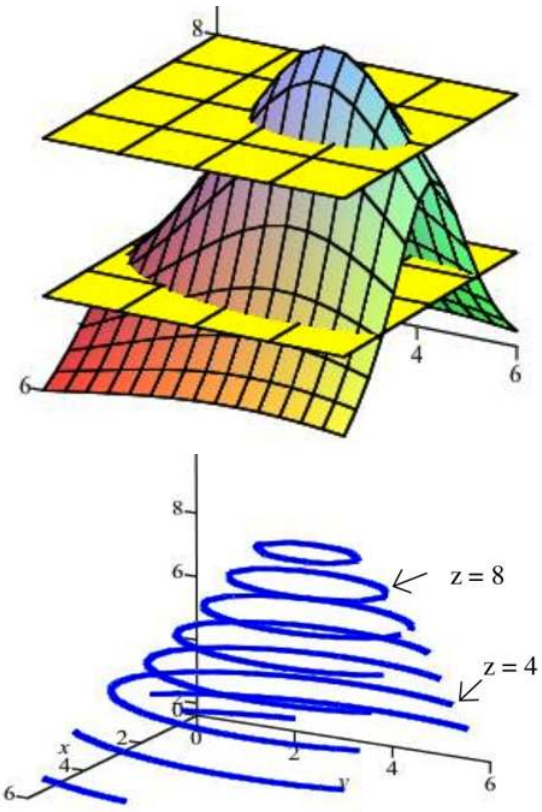
x (labels in left column), y (labels in top row)	0	1	2	3	4	5	6
0	1.6	1.8	2.0	2.1	2.2	2.1	2.0
1	2.4	2.9	3.3	3.7	5.9	3.7	3.3
2	3.3	4.3	5.6	6.7	7.1	6.7	5.6
3	3.8	5.3	7.1	9.1	9.9	9.1	7.1
4	3.3	4.3	5.6	6.7	7.1	6.7	5.6
5	2.4	2.9	3.3	3.7	5.9	3.7	3.3

6	1.6	1.8	2.0	2.1	2.2	2.1	2.0

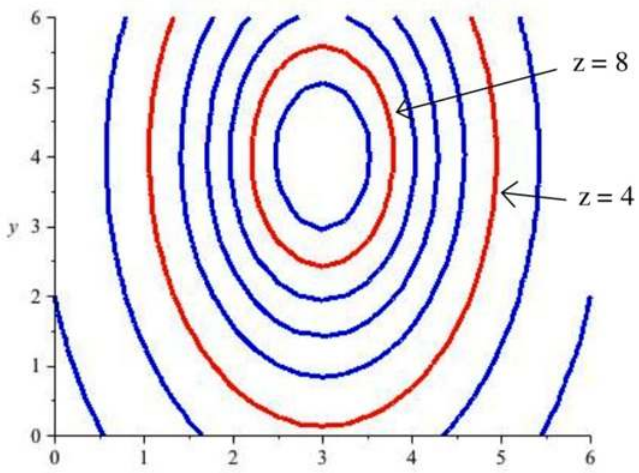


A surface from the data

If we "slice" the surface above with the plane $z = 8$, the points where the plane cuts the surface are those points where the elevation of the surface is 8 units above the xy-plane. The figure below shows the surface being sliced by the planes $z = 8$ and $z = 4$. Slicing the surface at different elevations and sketching the curves where the plane intersects the surface results in the second graph below.



If we move all of those curves to the xy–plane (or, equivalently, view them from directly overhead), the result is a 2-dimensional graph of the level curves of the original surface. This is the contour diagram.



Example 5.1.9

Create a contour diagram for our car rental example with cost function $C(d,m) = 40d + 0.15m$. Draw curves for when the cost is 0, 100, 200, 300, and 400.

Solution

We'll set $C(d,m) = 40d + 0.15m = c$ for $c = 0$, 100, 200, 300, and 400 and draw the curves in the dm-plane.

The first coordinate of the ordered pair is d, so the d-axis will be horizontal; the m-axis will be vertical. Remember that the domain for this function is really just where $d \geq 0$ and $m \geq 0$, so we will only draw the curves in the first quadrant.

When $c = 0$:

$$\begin{aligned} C(d,m) &= 40d + 0.15m = 0 \\ 0.15m &= -40d \\ m &= -\frac{40}{0.15}d \approx -267d \end{aligned}$$

This is the equation of a line, with slope about -267, passing through the origin. Because of the domain restrictions, the "curve" we will draw for this level is simply the origin. Putting this back into the car rental context, the only point where we pay \$0 for renting the car is when we rent the car for 0 days and drive it 0 miles – that is, if we don't rent it at all.

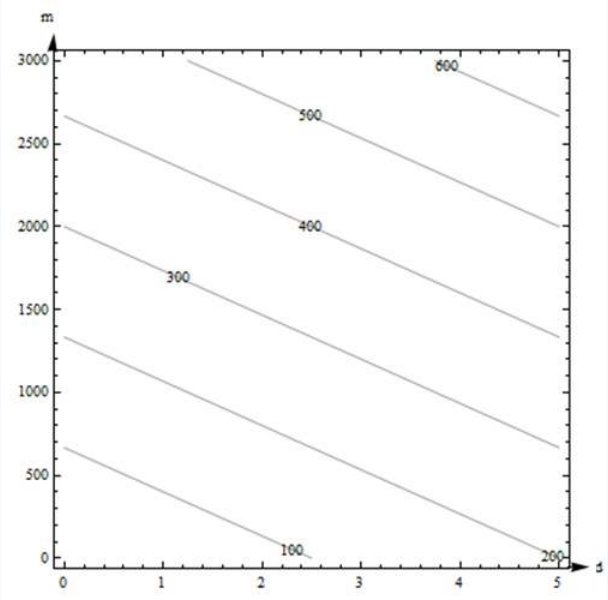
When $c = 100$:

$$\begin{aligned} C(d,m) &= 40d + 0.15m = 100 \\ 0.15m &= -40d + 100 \\ m &= -\frac{40}{0.15}d + \frac{100}{0.15} \approx -267d + 667 \end{aligned}$$

This is the equation of a line, with slope about -267, and d-intercept of about 667. This section of this line that lies in the first quadrant is shown with 100 labeling it.

Putting this into context, any point on that line represents a (d, m) combination of days and miles that will make the cost exactly \$100. So, for example – if we rent the car for 0 days and drive it 667 miles, it will cost us \$100. If we rent the car for 2.5 days and don't drive any miles, it will cost us \$100.

We continue for $c = 200$, 300, and 400 and sketch the curves in the plane, resulting in the contour diagram shown to the right.



Example 5.1.10

The contour diagram for the cost $C(d, m)$ in dollars for renting a car for d days and driving it m miles is shown in the previous example. Use the diagram to answer the following questions.

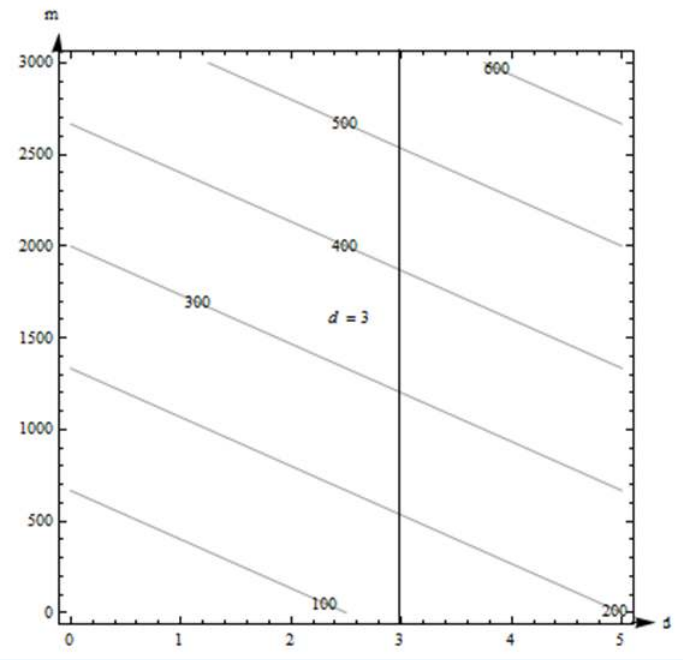
1. What is the cost of renting a car for 3 days and driving it 200 miles?
2. What is $C(100, 4)$? What is $C(4, 100)$?
3. Suppose we rent the car for 3 days. Is C an increasing function of miles?

Solution

1. The point (3, 200) is between contours on this graph, so we can't get an exact answer for $C(3, 200)$. (But it's typical for a graph that we would have to estimate). It looks to me as if (3, 200) is halfway between the 100 and the 200 contours, so let's estimate that $C(3, 200)$ is about \$150.

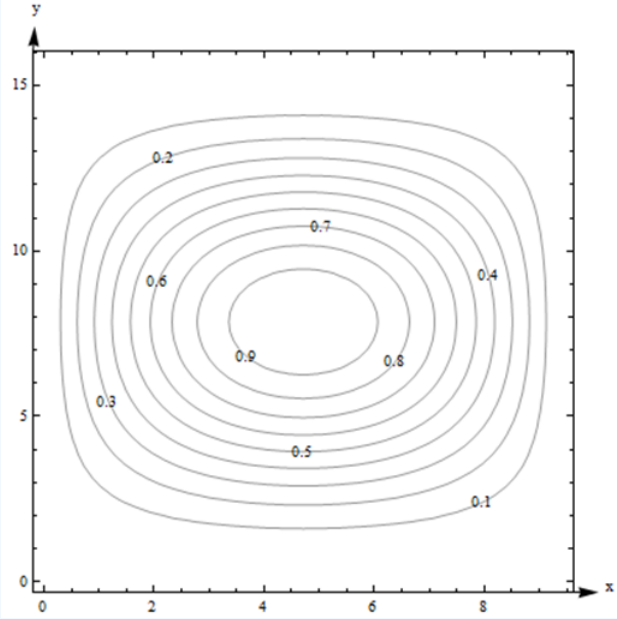
   Estimates from the graph are necessarily very rough. The graph only shows a little information (in this way, a contour diagram is like a table), so we have to extrapolate in between. But for most graphs, we don't actually know what happens between the contours. All we know for sure is that the output at (3, 200) is between the two levels we see. For this car rental example, we also know a formula, and my table showed this particular input, so we have other ways to get a better answer.

2. We can't find (100, 4) on this diagram, so we can't make an estimate of $C(100, 4)$ from this graph. (4, 100) lies between the contours for 100 and 200. It looks closer to 200, so let's estimate that $C(4, 100)$ is about \$180.
3. If we fix $d = 3$, we get a vertical line. What happens as m increases on this vertical line? As m increases, the function values shown on the contours increase, so C appears to be an increasing function of miles.



Example 5.1.11

Here is a contour diagram for a function $g(x, y)$.



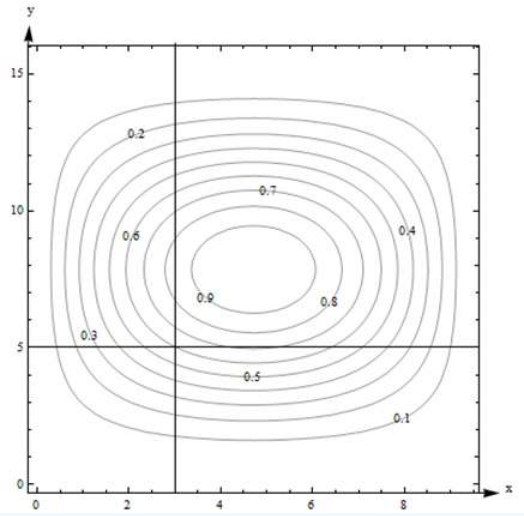
Use the diagram to answer the following questions:

1. What is $g(3, 5)$?
2. What is the highest point shown on the diagram? What is the lowest point shown?
3. If you start at (3, 5) and head in the positive x direction, do you go uphill or downhill first?

Solution

1. $g(3, 5)$ is 0.6. We can tell because the point is right on one of the contours, as illustrated in the image below.

https://math.libretexts.org/@go/page/108807

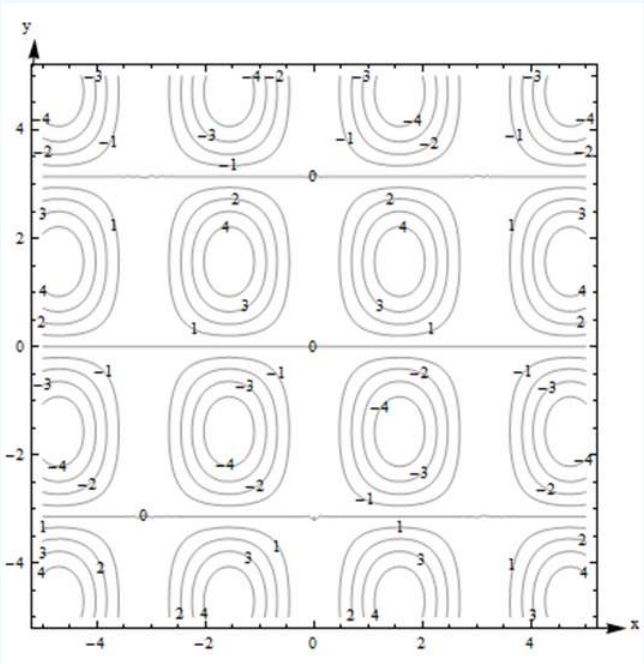


2. The highest contour shown is 0.9, and there would be a contour for 1.0 if the surface had ever got that high. However, the height seems to be increasing as we move in toward the center, so it appears that g gets to nearly 1 in the center. The lowest contour is 0.1. But again, we will guess that the height continues to decrease, so it appears that g is nearly 0 around the outside.
3. Starting at the point (3, 5, 0.6) on the surface and traveling to the right along the horizontal line shown in the previous part, we would cross the contour for 0.7 next. So the function increases first (we go uphill), and then decreases again.

Again, remember that we don't really know what happens between the contours. All we can do is estimate from the information in the graph.

Example 5.1.12

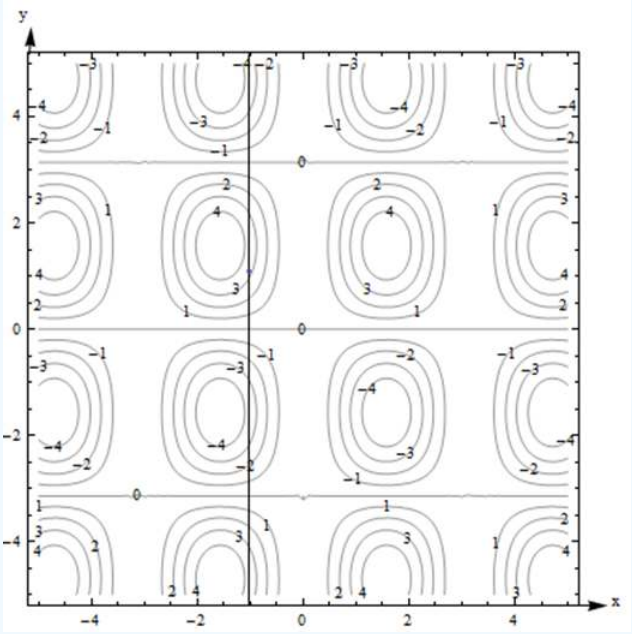
Here is a contour diagram for a function $F(x, y)$.



1. Describe the shape of the surface.
2. Suppose you travel along the surface in the positive y-direction, starting on the surface at the point above (or below) the point $(x, y) = (-1, 1)$. Describe your journey.

Solution

1. The surface is bumpy, with regularly spaced oval bumps. Notice that some of the bumps go up (positive contours), but others go down. Between the bumps, there are horizontal lines that are completely level, with an elevation of 0.
2. It looks as if $F(-1,1)$ is about 3. As we head in the positive y-direction along the line shown below, we first go uphill, nearly to 4, then we start going downhill. As we keep going north, we keep descending, going into the dip, until nearly -4. We're starting to go uphill again just as we leave the graph.



What happens if you have a function of more than two variables? Its graph will be a *hyper-surface*. For example, the graph of a function of four variables will be a hyper-surface in 5-dimensional space. This is very difficult (impossible for most of us) to visualize. Even the contours are hard to visualize – instead of curves in the plane, they're hyper-surfaces in 4-dimensional space. So if you have more than two variables, the graph isn't usually very useful.

Functions of Two Real-Life Variables

Complementary Goods and Substitute Goods

The demand for some pairs of goods have a relationship, where the quantity demanded for one product depends somehow on the prices for both

Complementary Goods

Two goods are **complementary** if an increase in the price of either decreases the demand for both.

Examples:

- The demand for cars depends on both the price for cars and the price of gasoline.
- The demand for hot dog buns depends on both the price for the buns and the price for the hot dogs.

Substitute Goods

Two goods are **substitutes** if an increase in the price of one increases the demand for the other.

Example:

- The demand for Brand A depends on its price and also on the price of its main competitor Brand B. If the Brand B raises its price, consumers will switch brands (substitute) and demand for Brand A will increase.

Think brands of soft drinks, detergent, or paper towels. A traditional example is coffee and tea: the idea is that consumers are simply looking for a hot drink and they'll buy whatever is cheaper. But this has always seemed fishy to me – I've never met any coffee- or tea-drinkers who would happily switch.

These demand functions are functions of two variables.

Example 5.1.13

The demand functions for two products are given below. p_1, p_2, q_1, and q_2 are the prices (in dollars) and quantities for Products 1 and 2:

$$\begin{aligned} q_1 &= 200 - 3p_1 - p_2 \\ q_2 &= 150 - p_1 - 2p_2 \end{aligned}$$

Solution

Are these two products complementary goods or substitute goods? What is the quantity demanded for each when the price for Product 1 is \$20 per item and the price for Product 2 is \$30 per item?

These products are complementary: an increase in either price decreases both demands. You can see that because the coefficients are both negative in each demand function.

When $p_1 = 20$ and $p_2 = 30$, we have

$$\begin{aligned} q_1 &= 200 - 3(20) - (30) = 110 \\ q_2 &= 150 - (20) - 2(30) = 70 \end{aligned}$$

So 110 units are demanded for Product 1 and 70 units are demanded for Product 2 when the price for Product 1 is \$20 per item and the price for Product 2 is \$30 per item.

Cobb-Douglas Production Function

Production functions are used to model the total output of a firm for a variety of inputs (doesn't this sound like a function of several variables?). One example is a Cobb-Douglas Production function:

$$P = AL^{\alpha}K^{\beta}$$

In this function P is the total production, A is a constant, α and β are constants between 0 and 1, L is the labor force, and K is the capital expenditure. (The units must be massaged well.)

You can read more about Cobb-Douglas Production functions at http://en.Wikipedia.org/wiki/Cobb-Douglas. You can read about other kinds of production functions at http://en.Wikipedia.org/wiki/Production_function.

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5.2: Calculus of Functions of Two Variables

Now that you have some familiarity with functions of two variables, it's time to start applying calculus to help us solve problems with them. In Chapter 2, we learned about the derivative for functions of two variables. Derivatives told us about the shape of the function, and let us find local max and min – we want to be able to do the same thing with a function of two variables.

First let's think. Imagine a surface, the graph of a function of two variables. Imagine that the surface is smooth and has some hills and some valleys. Concentrate on one point on your surface. What do we want the derivative to tell us? It ought to tell us how quickly the height of the surface changes as we move… Wait, which direction do we want to move? This is the reason that derivatives are more complicated for functions of several variables – there are so many (in fact, infinitely many) directions we could move from any point.

It turns out that our idea of fixing one variable and watching what happens to the function as the other changes is the key to extending the idea of derivatives to more than one variable.

Partial Derivatives

Partial Derivatives

Suppose that $z = f(x, y)$ is a function of two variables.

The **partial derivative of f with respect to x** is the derivative of the function $f(x, y)$ where we think of x as the only variable and act as if y is a constant.

The **partial derivative of f with respect to y** is the derivative of the function $f(x, y)$ where we think of y as the only variable and act as if x is a constant.

The "with respect to x" or "with respect to y" part is really important – you have to know and tell which variable you are thinking of as *THE* variable.

Geometrically

Geometrically the partial derivative with respect to x gives the slope of the curve as you travel along a cross-section, a curve on the surface parallel to the x-axis. The partial derivative with respect to y gives the slope of the cross-section parallel to the y-axis.

Notation for the Partial Derivative

The **partial derivative of $z = f(x, y)$ with respect to x** is written as

$$f_x(x, y)$$

or simply

$$f_x \quad \text{or} \quad z_x.$$

The Leibniz notation is

$$\frac{\partial f}{\partial x}$$

or

$$\frac{\partial z}{\partial x}.$$

We use an adaptation of the $\frac{\partial z}{\partial x}$ notation to mean "find the partial derivative of $f(x,y)$ with respect to x":

$$\frac{\partial}{\partial x}(f(x,y)) = \frac{\partial f}{\partial x}$$

To estimate a partial derivative from a table or contour diagram

The partial derivative with respect to x can be approximated by looking at an average rate of change, or the slope of a secant line, over a very tiny interval in the x-direction (holding y constant). The tinier the interval, the closer this is to the true partial derivative.

To compute a partial derivative from a formula

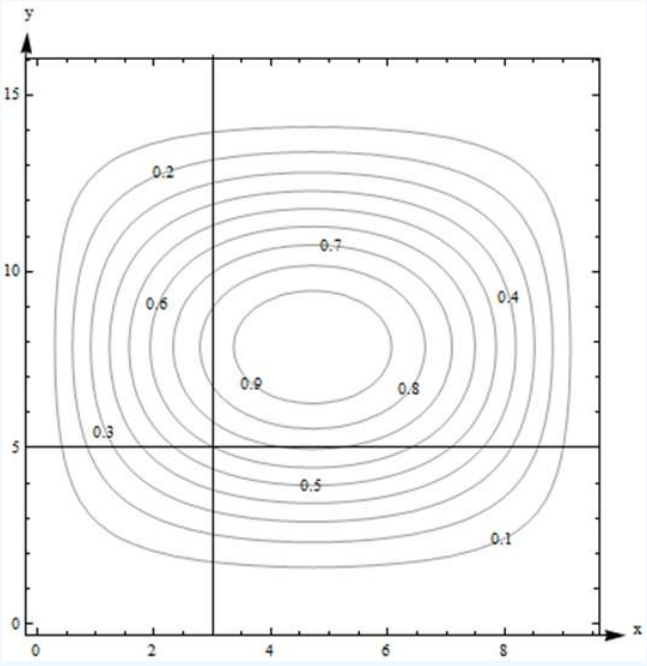
If $f(x,y)$ is given as a formula, you can find the partial derivative with respect to x algebraically by taking the ordinary derivative thinking of x as the only variable (holding y fixed).

Of course, **everything here works the same way if we're trying to find the partial derivative with respect to y** – just think of y as your only variable and act as if x is constant.

The idea of a partial derivative works perfectly well for a function of several variables: you focus on one variable to be *THE* variable and act as if all the other variables are constants.

Example 5.2.1

Here is a contour diagram for a function $g(x,y)$.



Use the diagram to answer the following questions:

1. Estimate $g_x(3,5)$ and $g_y(3,5)$.
2. Where on this diagram is g_x greatest? Where is g_y greatest?

Solution

1. $g_x(3,5)$ means we're thinking of x as the only variable, so we'll hold y fixed at $y=5$. That means we'll be looking along the horizontal line $y=5$. To estimate g_x, we need two function values. (3, 5) lies on the contour line, so we know that $g(3,5)=0.6$. The next point as we move to the right is $g(4.2,5)=0.7$.

   Now we can find the average rate of change:

$$\begin{aligned}\text{Average rate of change} &= \frac{(\text{change in output})}{(\text{change in input})}\\ &= \frac{\Delta g}{\Delta x}\\ &= \frac{0.7-0.6}{4.2-3}\\ &= \frac{1}{12} \approx 0.083\end{aligned}$$

   We can do the same thing by going to the next point we can read to the left, which is $g(2.4,5)=0.5$. Then the average rate of change is

$$\frac{\Delta g}{\Delta x} = \frac{0.5-0.6}{2.4-3} = \frac{1}{6} \approx 0.167.$$

   Either of these would be a fine estimate of $g_x(3,5)$ given the information we have, or we could take their average. We can estimate that $g_x(3,5) \approx 0.125$.

   Estimate $g_y(3,5)$ the same way, but moving on the vertical line. Using the next point up, we get the average rate of change is

$$\frac{\Delta g}{\Delta y} = \frac{0.7-0.6}{5.8-5} = \frac{1}{8} = 0.125.$$

   Using the next point down, we get

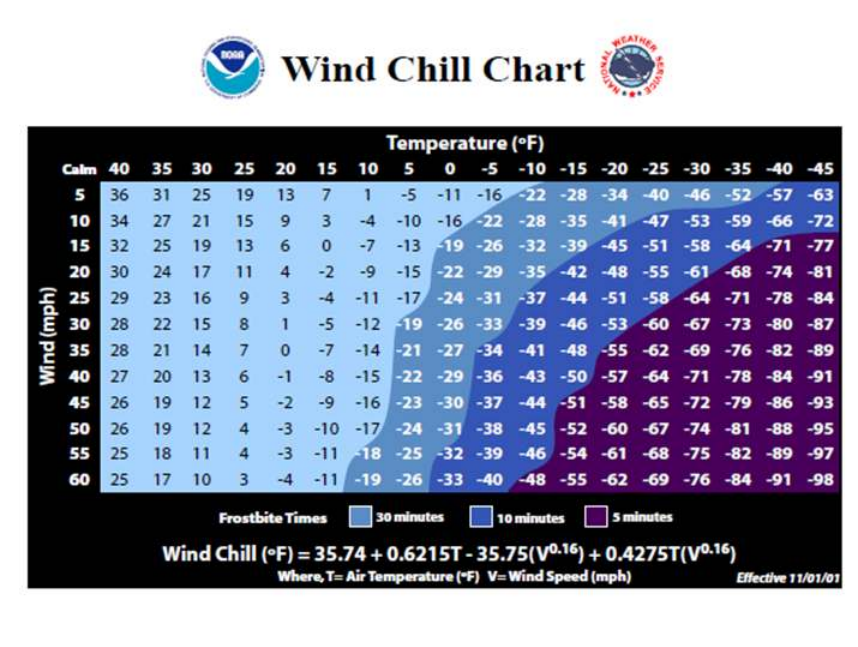
$$\frac{\Delta g}{\Delta y} = \frac{0.5-0.6}{4.5-5} = \frac{1}{5} = 0.2.$$

   Taking their average, we estimate $g_y(3,5) \approx 0.1625$.

2. g_x means x is our only variable, and we're thinking of y as a constant. So we're thinking about moving across the diagram on horizontal lines. g_x will be greatest when the contour lines are closest together, i.e., when the surface is steepest – then the denominator in $\frac{\Delta g}{\Delta x}$ will be small, so $\frac{\Delta g}{\Delta x}$ will be big. Scanning the graph, we can see that the contour lines are closest together when we head to the left or to the right from about (0.5, 8) and (9, 8). So g_x is greatest at about (0.5, 8) and (9, 8). For g_y, we want to look at vertical lines. g_y is greatest at about (5, 3.8) and (5, 12).

Example 5.2.2

Cold temperatures feel colder when the wind is blowing. Windchill is the perceived temperature, and it depends on both the actual temperature and the wind speed – a function of two variables! You can read more about windchill at www.nws.noaa.gov/om/windchill/. Below is a table that shows the perceived temperature for various temperatures and windspeeds.



Wind Chill Chart

Wind (mph) \ Temperature (°F)	Calm	40	35	30	25	20	15	10	5	0	-5	-10	-15	-20	-25	-30	-35	-40	-45
5		36	31	25	19	13	7	1	-5	-11	-16	-22	-28	-34	-40	-46	-52	-57	-63
10		34	27	21	15	9	3	-4	-10	-16	-22	-28	-35	-41	-47	-53	-59	-66	-72
15		32	25	19	13	6	0	-7	-13	-19	-26	-32	-39	-45	-51	-58	-64	-71	-77
20		30	24	17	11	4	-2	-9	-15	-22	-29	-35	-42	-48	-55	-61	-68	-74	-81
25		29	23	16	9	3	-4	-11	-17	-24	-31	-37	-44	-51	-58	-64	-71	-78	-84
30		28	22	15	8	1	-5	-12	-19	-26	-33	-39	-46	-53	-60	-67	-73	-80	-87
35		28	21	14	7	0	-7	-14	-21	-27	-34	-41	-48	-55	-62	-69	-76	-82	-89
40		27	20	13	6	-1	-8	-15	-22	-29	-36	-43	-50	-57	-64	-71	-78	-84	-91
45		26	19	12	5	-2	-9	-16	-23	-30	-37	-44	-51	-58	-65	-72	-79	-86	-93
50		26	19	12	4	-3	-10	-17	-24	-31	-38	-45	-52	-60	-67	-74	-81	-88	-95
55		25	18	11	4	-3	-11	-18	-25	-32	-39	-46	-54	-61	-68	-75	-82	-89	-97
60		25	17	10	3	-4	-11	-19	-26	-33	-40	-48	-55	-62	-69	-76	-84	-91	-98

Frostbite Times: 30 minutes, 10 minutes, 5 minutes

$$\text{Wind Chill (°F)} = 35.74 + 0.6215T - 35.75(V^{0.16}) + 0.4275T(V^{0.16})$$

Where, T= Air Temperature (°F) V= Wind Speed (mph) Effective 11/01/01

(Table courtesy of the National Weather Service, www.nws.noaa.gov/om/windchill/images/windchill.gif.)

Note that they also include the formula, but for this example we'll use the information in the table.

1. What is the perceived temperature when the actual temperature is 25°F and the wind is blowing at 15 miles per hour?
2. Suppose the actual temperature is 25°F. Use information from the table to describe how the perceived temperature would change if the wind speed increased from 15 miles per hour?

Solution

1. Reading the table, we see that the perceived temperature is 13°F
2. This is a question about a partial derivative. We're holding the temperature (T) fixed at 25°F, and asking what happens as wind speed (V) increases from 15 miles per hour. We're thinking of V as the only variable, so we want $\text{WindChill}_V = W_V$ when $T = 25$ and $V = 15$. We'll find the average rate of change by looking in the column where $T = 25$ and letting V increase, and use that to approximate the partial derivative.

$$W_V \approx \frac{\Delta W}{\Delta V} = \frac{11 - 13}{20 - 15} = -0.4$$

What are the units? W is measured in °F and V is measured in mph, so the units here are °F/mph. And that lets us describe what happens: The perceived temperature would decrease by about 0.4°F for each mph increase in wind speed.

Example 5.2.3

Find f_x and f_y at the points (0, 0) and (1, 1) if $f(x, y) = x^2 - 4xy + 4y^2$.

Solution

To find f_x, take the ordinary derivative of f with respect to x, acting as if y is constant:

$$f_x(x, y) = 2x - 4y.$$

Note that the derivative of the $4y^2$ term with respect to x is zero because it's a constant (as far as x is concerned).

Similarly,

$$f_y(x,y) = -4x + 8y.$$

Now we can evaluate these at the points:

$f_x(0,0) = 0$ and $f_y(0,0) = 0$; this tells us that the cross sections parallel to the x- and y- axes are both flat at (0,0).

$f_x(1,1) = -2$ and $f_y(1,1) = 4$; this tells us that above the point (1, 1), the surface decreases if we move to more positive x values and increases if we move to more positive y values.

Example 5.2.4

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ if $f(x,y) = \frac{e^{x+y}}{y^3+y} + y\ln(y)$.

Solution

$\frac{\partial f}{\partial x}$ means x is our only variable, we're thinking of y as a constant. Then we'll just find the ordinary derivative. From x's point of view, this is an exponential function, divided by a constant, with a constant added. The constant pulls out in front, the derivative of the exponential function is the same thing, and we need to use the chain rule, so we multiply by the derivative of that exponent (which is just 1):

$$\frac{\partial f}{\partial x} = \frac{1}{y^3+y} e^{x+y}.$$

$\frac{\partial f}{\partial y}$ means that we're thinking of y as the variable, acting as if x is constant. From y's point of view, f is a quotient plus a product, so we'll need the quotient rule and the product rule:

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{(\,)(\,) - (\,)(\,)}{(\,)^2} + (\,)(\,) + (\,)(\,) \\ &= \frac{\left(e^{x+y}(1)\right)\left(y^3+y\right) - \left(e^{x+y}\right)\left(3y^2+1\right)}{\left(y^3+y\right)^2} + (1)\left(\ln(y)\right) + (y)\left(\frac{1}{y}\right)\end{aligned}$$

Example 5.2.5

Find f_z if $f(x,y,z,w) = 35x^2w - \frac{1}{z} + yz^2$.

Solution

f_z means we act as if z is our only variable, so we'll act as if all the other variables (x, y, and w) are constants and take the ordinary derivative:

$$f_z(x,y,z,w) = \frac{1}{z^2} + 2yz.$$

Using Partial Derivatives to Estimate Function Values

We can use the partial derivatives to estimate values of a function. The geometry is similar to the tangent line approximation in one variable. Recall the one-variable case: if x is close enough to a known point a, then

$$f(x) \approx f(a) + f'(a)(x-a).$$

In two variables, we do the same thing in both directions at once:

Approximating Function Values with Partial Derivatives

To approximate the value of $f(x,y)$, find some point (a,b) where

1. (x,y) and (a,b) are close, that is, x and a are close and y and b are close.
2. You know the exact values of $f(a,b)$ and both partial derivatives there.

Then

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Notice that the total change in f is being approximated by adding the approximate changes coming from the x and y directions. Another way to look at the same formula:

$$\Delta f \approx f_x \Delta x + f_y \Delta y.$$

How close is close? It depends on the shape of the graph of f. In general, the closer the better.

Example 5.2.6

Use partial derivatives to estimate the value of $f(x,y) = x^2 - 4xy + 4y^2$ at (0.9, 1.1).

Solution

Note that the point (0.9, 1.1) is close to an "easy" point, (1, 1). In fact, we already worked out the partial derivatives at (1, 1): $f_x(x,y) = 2x - 4y$ so $f_x(1,1) = -2$, and $f_y(x,y) = -4x + 8y$ so $f_y(1,1) = 4$. We also know that $f(1,1) = 1$.

So,

$$f(0.9, 1.1) \approx 1 - 2(-0.1) + 4(0.1) = 1.6.$$

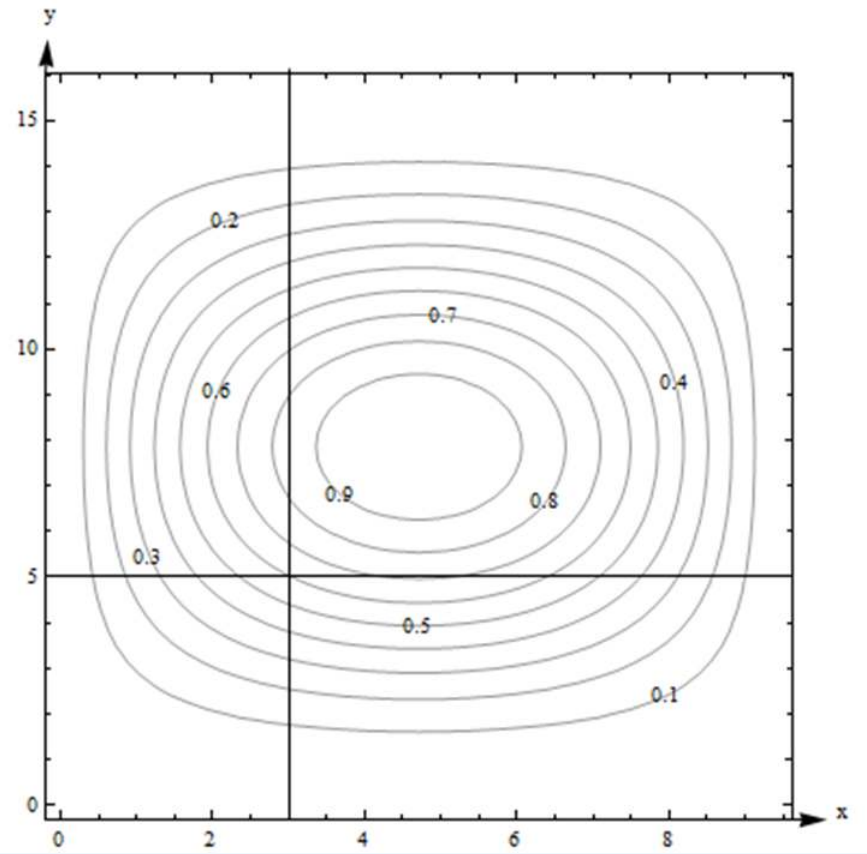
Note that in this example it would have been possible to simply compute the exact answer:

$$f(0.9, 1.1) = (0.9)^2 - 4(0.9)(1.1) + 4(1.1)^2 = 1.69.$$

Our estimate is not perfect, but it's pretty close.

Example 5.2.7

Here is a contour diagram for a function $g(x,y)$. Use partial derivatives to estimate the value of $g(3.2, 4.7)$.



Solution

This is the same diagram from before, so we already estimated the value of the function and the partial derivatives at the nearby point (3,5). $g(3,5)$ is 0.6, our estimate of $g_x(3,5) \approx 0.125$, and our estimate of $g_y(3,5) \approx 0.1625$. So

$$g(3.2, 4.7) \approx 0.6 + (0.125)(0.2) + (0.1625)(-0.3) = 0.57625.$$

Note that in this example we have no way to know how close our estimate is to the actual value.

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5.3: Partial Derivatives

We are now ready to define derivatives of functions of more than one variable. First, recall how we defined the derivative, $f'(a)$, of a function of one variable, $f(x)$. We imagined that we were walking along the x-axis, in the positive direction, measuring, for example, the temperature along the way. We denoted by $f(x)$ the temperature at x. The instantaneous rate of change of temperature that we observed as we passed through $x=a$ was

$$\frac{\mathrm{d}f}{\mathrm{d}x}(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

Next suppose that we are walking in the xy-plane and that the temperature at (x,y) is $f(x,y)$. We can pass through the point $(x,y)=(a,b)$ moving in many different directions, and we cannot expect the measured rate of change of temperature if we walk parallel to the x-axis, in the direction of increasing x, to be the same as the measured rate of change of temperature if we walk parallel to the y-axis in the direction of increasing y. We'll start by considering just those two directions. We'll consider other directions (like walking parallel to the line $y=x$) later.

Suppose that we are passing through the point $(x,y)=(a,b)$ and that we are walking parallel to the x-axis (in the positive direction). Then our y-coordinate will be constant, always taking the value $y=b$. So we can think of the measured temperature as the function of one variable $B(x)=f(x,b)$ and we will observe the rate of change of temperature

$$\frac{\mathrm{d}B}{\mathrm{d}x}(a) = \lim_{h\to 0} \frac{B(a+h)-B(a)}{h} = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

This is called the "partial derivative f with respect to x at (a,b)" and is denoted $\frac{\partial f}{\partial x}(a,b)$. Here

- the symbol ∂, which is read "partial", indicates that we are dealing with a function of more than one variable, and
- the x in $\frac{\partial f}{\partial x}$ indicates that we are differentiating with respect to x, while y is being held fixed, i.e. being treated as a constant.
- $\frac{\partial f}{\partial x}$ is read "partial dee f dee x".

Do not write $\frac{\mathrm{d}}{\mathrm{d}x}$ when $\frac{\partial}{\partial x}$ is appropriate. We shall later encounter situations when $\frac{\mathrm{d}}{\mathrm{d}x}f$ and $\frac{\partial}{\partial x}f$ are both defined and have different meanings.

If, instead, we are passing through the point $(x,y)=(a,b)$ and are walking parallel to the y-axis (in the positive direction), then our x-coordinate will be constant, always taking the value $x=a$. So we can think of the measured temperature as the function of one variable $A(y)=f(a,y)$ and we will observe the rate of change of temperature

$$\frac{\mathrm{d}A}{\mathrm{d}y}(b) = \lim_{h\to 0} \frac{A(b+h)-A(b)}{h} = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}$$

This is called the "partial derivative f with respect to y at (a,b)" and is denoted $\frac{\partial f}{\partial y}(a,b)$.

Just as was the case for the ordinary derivative $\frac{\mathrm{d}f}{\mathrm{d}x}(x)$ (see Definition 2.2.6 in the CLP-1 text), it is common to treat the partial derivatives of $f(x,y)$ as functions of (x,y) simply by evaluating the partial derivatives at (x,y) rather than at (a,b).

Definition 2.2.1. Partial Derivatives

The x- and y-partial derivatives of the function $f(x,y)$ are

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$
$$\frac{\partial f}{\partial y}(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

respectively. The partial derivatives of functions of more than two variables are defined analogously.

Partial derivatives are used a lot. And there many notations for them.

Definition 2.2.2

The partial derivative $\frac{\partial f}{\partial x}(x,y)$ of a function $f(x,y)$ is also denoted

$$\frac{\partial f}{\partial x} \qquad f_x(x,y) \qquad f_x \qquad D_x f(x,y) \qquad D_x f \qquad D_1 f(x,y) \qquad D_1 f$$

The subscript 1 on $D_1 f$ indicates that f is being differentiated with respect to its first variable. The partial derivative $\frac{\partial f}{\partial x}(a,b)$ is also denoted

$$\left.\frac{\partial f}{\partial x}\right|_{(a,b)}$$

with the subscript (a,b) indicating that $\frac{\partial f}{\partial x}$ is being evaluated at $(x,y)=(a,b)$.

The notation $\left(\frac{\partial f}{\partial x}\right)_y$ is used to make explicit that the variable y is being held fixed [1].

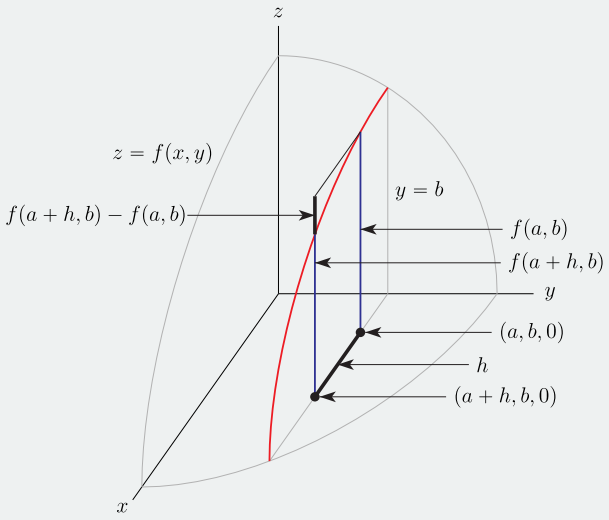
Remark 2.2.3. The Geometric Interpretation of Partial Derivatives

We'll now develop a geometric interpretation of the partial derivative

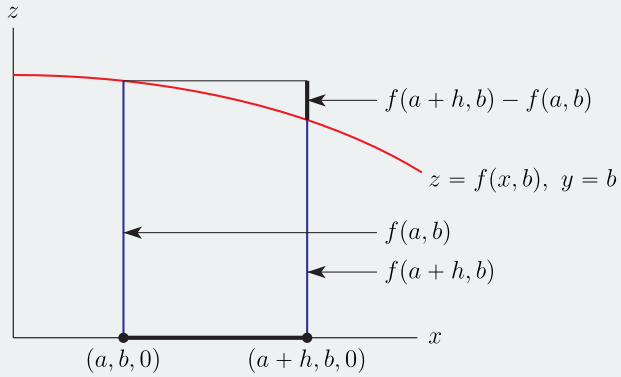
$$\frac{\partial f}{\partial x}(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

in terms of the shape of the graph $z=f(x,y)$ of the function $f(x,y)$. That graph appears in the figure below. It looks like the part of a deformed sphere that is in the first octant.

The definition of $\frac{\partial f}{\partial x}(a,b)$ concerns only points on the graph that have $y=b$. In other words, the curve of intersection of the surface $z=f(x,y)$ with the plane $y=b$. That is the red curve in the figure. The two blue vertical line segments in the figure have heights $f(a,b)$ and $f(a+h,b)$, which are the two numbers in the numerator of $\frac{f(a+h,b)-f(a,b)}{h}$.



A side view of the curve (looking from the left side of the y-axis) is sketched in the figure below.



Again, the two blue vertical line segments in the figure have heights $f(a,b)$ and $f(a+h,b)$, which are the two numbers in the numerator of $\frac{f(a+h,b)-f(a,b)}{h}$. So the numerator $f(a+h,b)-f(a,b)$ and denominator h are the rise and run, respectively, of the curve $z=f(x,b)$ from $x=a$ to $x=a+h$. Thus $\frac{\partial f}{\partial x}(a,b)$ is exactly *the slope of (the tangent to) the curve of intersection of the surface $z=f(x,y)$ and the plane $y=b$ at the point $\big(a,b,f(a,b)\big)$*. In the same way $\frac{\partial f}{\partial y}(a,b)$ is exactly *the slope of (the tangent to) the curve of intersection of the surface $z=f(x,y)$ and the plane $x=a$ at the point $\big(a,b,f(a,b)\big)$*.

Evaluation of Partial Derivatives

From the above discussion, we see that we can readily compute partial derivatives $\frac{\partial}{\partial x}$ by using what we already know about ordinary derivatives $\frac{\mathrm{d}}{\mathrm{d}x}$. More precisely,

- to evaluate $\frac{\partial f}{\partial x}(x,y)$, treat the y in $f(x,y)$ as a constant and differentiate the resulting function of x with respect to x.
- To evaluate $\frac{\partial f}{\partial y}(x,y)$, treat the x in $f(x,y)$ as a constant and differentiate the resulting function of y with respect to y.
- To evaluate $\frac{\partial f}{\partial x}(a,b)$, treat the y in $f(x,y)$ as a constant and differentiate the resulting function of x with respect to x. Then evaluate the result at $x=a$, $y=b$.
- To evaluate $\frac{\partial f}{\partial y}(a,b)$, treat the x in $f(x,y)$ as a constant and differentiate the resulting function of y with respect to y. Then evaluate the result at $x=a$, $y=b$.

Now for some examples.

Example 2.2.4

Let

$$f(x,y)=x^3+y^2+4xy^2$$

Then, since $\frac{\partial}{\partial x}$ treats y as a constant,

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(x^3)+\frac{\partial}{\partial x}(y^2)+\frac{\partial}{\partial x}(4xy^2)\\ &= 3x^2+0+4y^2\frac{\partial}{\partial x}(x)\\ &= 3x^2+4y^2\end{aligned}$$

and, since $\frac{\partial}{\partial y}$ treats x as a constant,

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(x^3)+\frac{\partial}{\partial y}(y^2)+\frac{\partial}{\partial y}(4xy^2)\\ &= 0+2y+4x\frac{\partial}{\partial y}(y^2)\\ &= 2y+8xy\end{aligned}$$

In particular, at $(x,y)=(1,0)$ these partial derivatives take the values

$$\begin{aligned}\frac{\partial f}{\partial x}(1,0) &= 3(1)^2+4(0)^2 &&= 3\\ \frac{\partial f}{\partial y}(1,0) &= 2(0)+8(1)(0) &&= 0\end{aligned}$$

Example 2.2.5

Let

$$f(x,y) = y\cos x + xe^{xy}$$

Then, since $\frac{\partial}{\partial x}$ treats y as a constant, $\frac{\partial}{\partial x}e^{yx} = ye^{yx}$ and

$$\begin{aligned}\frac{\partial}{\partial x}(x,y) &= y\frac{\partial}{\partial x}(\cos x)+e^{xy}\frac{\partial}{\partial x}(x)+x\frac{\partial}{\partial x}\left(e^{xy}\right) \qquad \text{(by the product rule)}\\ &= -y\sin x + e^{xy} + xye^{xy}\\ \frac{\partial}{\partial x}(x,y) &= \cos x\frac{\partial}{\partial y}(y)+x\frac{\partial}{\partial y}\left(e^{xy}\right)\\ &= \cos x + x^2e^{xy}\end{aligned}$$

Let's move up to a function of four variables. Things generalize in a quite straight forward way.

Example 2.2.6

Let

$$f(x,y,z,t) = x\sin(y+2z)+t^2e^{3y}\ln z$$

Then

$$\begin{aligned}\frac{\partial f}{\partial x}(x,y,z,t) &= \sin(y+2z)\\ \frac{\partial f}{\partial y}(x,y,z,t) &= x\cos(y+2z)+3t^2e^{3y}\ln z\\ \frac{\partial f}{\partial z}(x,y,z,t) &= 2x\cos(y+2z)+t^2e^{3y}/z\\ \frac{\partial f}{\partial t}(x,y,z,t) &= 2te^{3y}\ln z\end{aligned}$$

Now here is a more complicated example — our function takes a special value at $(0,0)$. To compute derivatives there we revert to the definition.

Example 2.2.7

Set

$$f(x,y) = \begin{cases}\frac{\cos x-\cos y}{x-y} & \text{if } x\neq y\\ 0 & \text{if } x=y\end{cases}$$

If $b\neq a$, then for all (x,y) sufficiently close to (a,b), $f(x,y) = \frac{\cos x-\cos y}{x-y}$ and we can compute the partial derivatives of f at (a,b) using the familiar rules of differentiation. However that is not the case for $(a,b)=(0,0)$. To evaluate $f_x(0,0)$, we need to set $y=0$ and find the derivative of

$$f(x,0) = \begin{cases}\frac{\cos x-1}{x} & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$$

with respect to x at $x=0$. As we cannot use the usual differentiation rules, we evaluate the derivative[2] by applying the definition

$$\begin{aligned}
f_x(0,0) &= \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} \\
&= \lim_{h\to 0} \frac{\frac{\cos h - 1}{h} - 0}{h} && \text{(Recall that } h \neq 0 \text{ in the limit.)} \\
&= \lim_{h\to 0} \frac{\cos h - 1}{h^2} \\
&= \lim_{h\to 0} \frac{-\sin h}{2h} && \text{(By l'Hôpital's rule.)} \\
&= \lim_{h\to 0} \frac{-\cos h}{2} && \text{(By l'Hôpital again.)} \\
&= -\frac{1}{2}
\end{aligned}$$

We could also evaluate the limit of $\frac{\cos h - 1}{h^2}$ by substituting in the Taylor expansion

$$\cos h = 1 - \frac{h^2}{2} + \frac{h^4}{4!} - \cdots$$

We can also use Taylor expansions to understand the behaviour of $f(x,y)$ for (x,y) near $(0,0)$. For $x \neq y$,

$$\begin{aligned}
\frac{\cos x - \cos y}{x-y} &= \frac{\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right] - \left[1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\right]}{x-y} \\
&= \frac{-\frac{x^2-y^2}{2!} + \frac{x^4-y^4}{4!} - \cdots}{x-y} \\
&= -\frac{1}{2!}\frac{x^2-y^2}{x-y} + \frac{1}{4!}\frac{x^4-y^4}{x-y} - \cdots \\
&= -\frac{x+y}{2!} + \frac{x^3+x^2y+xy^2+y^3}{4!} - \cdots
\end{aligned}$$

So for (x,y) near $(0,0)$,

$$f(x,y) \approx \begin{cases} -\frac{x+y}{2} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

So it sure looks like (and in fact it is true that)

- $f(x,y)$ is continuous at $(0,0)$ and
- $f(x,y)$ is not continuous at (a,a) for small $a \neq 0$ and
- $f_x(0,0) = f_y(0,0) = -\frac{1}{2}$

Example 2.2.8

Again set

$$f(x,y) = \begin{cases} \frac{\cos x - \cos y}{x-y} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

We'll now compute $f_y(x,y)$ for all (x,y).

The case $y \neq x$: When $y \neq x$,

$$\begin{aligned} f_y(x,y) &= \frac{\partial}{\partial y}\frac{\cos x-\cos y}{x-y} \\ &= \frac{(x-y)\frac{\partial}{\partial y}(\cos x-\cos y)-(\cos x-\cos y)\frac{\partial}{\partial y}(x-y)}{(x-y)^2} \\ &\qquad \text{(by the quotient rule)} \\ &= \frac{(x-y)\sin y+\cos x-\cos y}{(x-y)^2} \end{aligned}$$

The case $y=x$: When $y=x$,

$$\begin{aligned} f_y(x,y) &= \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h} \\ &= \lim_{h\to 0}\frac{f(x,x+h)-f(x,x)}{h} \\ &= \lim_{h\to 0}\frac{\frac{\cos x-\cos(x+h)}{x-(x+h)}-0}{h} \qquad \text{(Recall that } h\neq 0 \text{ in the limit.)} \\ &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h^2} \end{aligned}$$

Now we apply L'Hôpital's rule, remembering that, in this limit, x is a constant and h is the variable — so we differentiate with respect to h.

$$f_y(x,y) = \lim_{h\to 0}\frac{-\sin(x+h)}{2h}$$

Note that if x is not an integer multiple of π, then the numerator $-\sin(x+h)$ does *not* tend to zero as h tends to zero, and the limit giving $f_y(x,y)$ does not exist. On the other hand, if x is an integer multiple of π, both the numerator and denominator tend to zero as h tends to zero, and we can apply L'Hôpital's rule a second time. Then

$$\begin{aligned} f_y(x,y) &= \lim_{h\to 0}\frac{-\cos(x+h)}{2} \\ &= -\frac{\cos x}{2} \end{aligned}$$

The conclusion:

$$f_y(x,y) = \begin{cases} \frac{(x-y)\sin y+\cos x-\cos y}{(x-y)^2} & \text{if } x\neq y \\ -\frac{\cos x}{2} & \text{if } x=y \text{ with } x \text{ an integer multiple of } \pi \\ DNE & \text{if } x=y \text{ with } x \text{ not an integer multiple of } \pi \end{cases}$$

Example 2.2.9. Optional — A Little Weirdness

In this example, we will see that the function

$$f(x,y) = \begin{cases} \frac{x^2}{x-y} & \text{if } x\neq y \\ 0 & \text{if } x=y \end{cases}$$

is *not* continuous at $(0,0)$ and yet has both partial derivatives $f_x(0,0)$ and $f_y(0,0)$ perfectly well defined. We'll also see how that is possible. First let's compute the partial derivatives. By definition,

$$\begin{aligned} f_x(0,0) &= \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\overbrace{\frac{h^2}{h-0}}^{h} - 0}{h} = \lim_{h\to 0} 1 \\ &= 1 \\ f_y(0,0) &= \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{0^2}{0-h} - 0}{h} = \lim_{h\to 0} 0 \\ &= 0 \end{aligned}$$

So the first order partial derivatives $f_x(0,0)$ and $f_y(0,0)$ are perfectly well defined.

To see that, nonetheless, $f(x,y)$ is not continuous at $(0,0)$, we take the limit of $f(x,y)$ as (x,y) approaches $(0,0)$ along the curve $y = x - x^3$. The limit is

$$\lim_{x\to 0} f\big(x, x-x^3\big) = \lim_{x\to 0} \frac{x^2}{x-(x-x^3)} = \lim_{x\to 0} \frac{1}{x}$$

which does not exist. Indeed as x approoaches 0 through positive numbers, $\frac{1}{x}$ approaches $+\infty$, and as x approoaches 0 through negative numbers, $\frac{1}{x}$ approaches $-\infty$.

So how is this possible? The answer is that $f_x(0,0)$ only involves values of $f(x,y)$ with $y=0$. As $f(x,0)=x$, for all values of x, we have that $f(x,0)$ is a continuous, and indeed a differentiable, function. Similarly, $f_y(0,0)$ only involves values of $f(x,y)$ with $x=0$. As $f(0,y)=0$, for all values of y, we have that $f(0,y)$ is a continuous, and indeed a differentiable, function. On the other hand, the bad behaviour of $f(x,y)$ for (x,y) near $(0,0)$ only happens for x and y both nonzero.

Our next example uses implicit differentiation.

Example 2.2.10

The equation

$$z^5 + y^2 e^z + e^{2x} = 0$$

implicitly determines z as a function of x and y. That is, the function $z(x,y)$ obeys

$$z(x,y)^5 + y^2 e^{z(x,y)} + e^{2x} = 0$$

For example, when $x = y = 0,$ the equation reduces to

$$z(0,0)^5 = -1$$

which forces[3] $z(0,0) = -1$. Let's find the partial derivative $\frac{\partial z}{\partial x}(0,0)$.

We are not going to be able to explicitly solve the equation for $z(x,y)$. All we know is that

$$z(x,y)^5 + y^2 e^{z(x,y)} + e^{2x} = 0$$

for all x and y. We can turn this into an equation for $\frac{\partial z}{\partial x}(0,0)$ by differentiating[4] the whole equation with respect to x, giving

$$5z(x,y)^4 \frac{\partial z}{\partial x}(x,y) + y^2 e^{z(x,y)} \frac{\partial z}{\partial x}(x,y) + 2e^{2x} = 0$$

and then setting $x = y = 0,$ giving

$$5z(0,0)^4 \frac{\partial z}{\partial x}(0,0) + 2 = 0$$

As we already know that $z(0,0) = -1,$

$$\frac{\partial z}{\partial x}(0,0) = -\frac{2}{5z(0,0)^4} = -\frac{2}{5}$$

Next we have a partial derivative disguised as a limit.

Example 2.2.11

In this example we are going to evaluate the limit

$$\lim_{z\to 0}\frac{(x+y+z)^3-(x+y)^3}{(x+y)z}$$

The critical observation is that, in taking the limit $z\to 0,$ x and y are fixed. They do not change as z is getting smaller and smaller. Furthermore this limit is exactly of the form of the limits in the Definition 2.2.1 of partial derivative, disguised by some obfuscating changes of notation.

Set

$$f(x,y,z)=\frac{(x+y+z)^3}{(x+y)}$$

Then

$$\begin{aligned}\lim_{z\to 0}\frac{(x+y+z)^3-(x+y)^3}{(x+y)z} &= \lim_{z\to 0}\frac{f(x,y,z)-f(x,y,0)}{z}\\ &= \lim_{h\to 0}\frac{f(x,y,0+h)-f(x,y,0)}{h}\\ &= \frac{\partial f}{\partial z}(x,y,0)\\ &= \left[\frac{\partial}{\partial z}\frac{(x+y+z)^3}{x+y}\right]_{z=0}\end{aligned}$$

Recalling that $\frac{\partial}{\partial z}$ treats x and y as constants, we are evaluating the derivative of a function of the form $\frac{(\text{const}+z)^3}{\text{const}}$. So

$$\begin{aligned}\lim_{z\to 0}\frac{(x+y+z)^3-(x+y)^3}{(x+y)z} &= 3\frac{(x+y+z)^2}{x+y}\Big|_{z=0}\\ &= 3(x+y)\end{aligned}$$

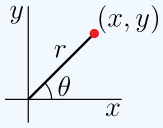
The next example highlights a potentially dangerous difference between ordinary and partial derivatives.

Example 2.2.12

In this example we are going to see that, in contrast to the ordinary derivative case, $\frac{\partial r}{\partial x}$ is not, in general, the same as $\left(\frac{\partial x}{\partial r}\right)^{-1}$.

Recall that Cartesian and polar coordinates [5] (for $(x,y)\neq(0,0)$ and $r>0$) are related by

$$\begin{aligned}x &= r\cos\theta\\ y &= r\sin\theta\\ r &= \sqrt{x^2+y^2}\\ \tan\theta &= \frac{y}{x}\end{aligned}$$



We will use the functions

$$x(r,\theta)=r\cos\theta \qquad \text{and} \qquad r(x,y)=\sqrt{x^2+y^2}$$

Fix any point $(x_0,y_0)\neq(0,0)$ and let (r_0,θ_0), $0\le\theta_0<2\pi,$ be the corresponding polar coordinates. Then

$$\frac{\partial x}{\partial r}(r,\theta)=\cos\theta \qquad \frac{\partial r}{\partial x}(x,y)=\frac{x}{\sqrt{x^2+y^2}}$$

so that

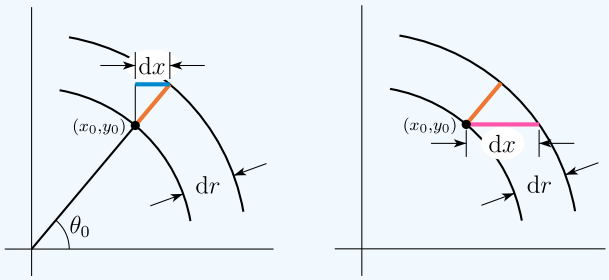
$$\begin{aligned}\frac{\partial x}{\partial r}(r_0,\theta_0)=\left(\frac{\partial r}{\partial x}(x_0,y_0)\right)^{-1} &\iff \cos\theta_0=\left(\frac{x_0}{\sqrt{x_0^2+y_0^2}}\right)^{-1}=(\cos\theta_0)^{-1}\\ &\iff \cos^2\theta_0=1\\ &\iff \theta_0=0,\pi\end{aligned}$$

We can also see pictorially why this happens. By definition, the partial derivatives

$$\begin{aligned}\frac{\partial x}{\partial r}(r_0,\theta_0)&=\lim_{\mathrm{d}r\to 0}\frac{x(r_0+\mathrm{d}r,\theta_0)-x(r_0,\theta_0)}{\mathrm{d}r}\\ \frac{\partial r}{\partial x}(x_0,y_0)&=\lim_{\mathrm{d}x\to 0}\frac{r(x_0+\mathrm{d}x,y_0)-r(x_0,y_0)}{\mathrm{d}x}\end{aligned}$$

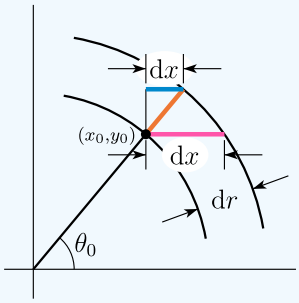
Here we have just renamed the h of Definition 2.2.1 to $\mathrm{d}r$ and to $\mathrm{d}x$ in the two definitions.

In computing $\frac{\partial x}{\partial r}(r_0,\theta_0)$, θ_0 is held fixed, r is changed by a small amount $\mathrm{d}r$ and the resulting $\mathrm{d}x=x(r_0+\mathrm{d}r,\theta_0)-x(r_0,\theta_0)$ is computed. In the figure on the left below, $\mathrm{d}r$ is the length of the orange line segment and $\mathrm{d}x$ is the length of the blue line segment.



On the other hand, in computing $\frac{\partial r}{\partial x}$, y is held fixed, x is changed by a small amount $\mathrm{d}x$ and the resulting $\mathrm{d}r=r(x_0+\mathrm{d}x,y_0)-r(x_0,y_0)$ is computed. In the figure on the right above, $\mathrm{d}x$ is the length of the pink line segment and $\mathrm{d}r$ is the length of the orange line segment.

Here are the two figures combined together. We have arranged that the same $\mathrm{d}r$ is used in both computations. In order for the $\mathrm{d}r$'s to be the same in both computations, the two $\mathrm{d}x$'s have to be different (unless $\theta_0=0,\pi$). So, in general, $\frac{\partial x}{\partial r}(r_0,\theta_0)\neq\left(\frac{\partial r}{\partial x}(x_0,y_0)\right)^{-1}$.



Example 2.2.13. Optional — Example 2.2.12, continued

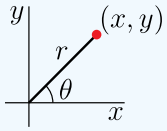
The inverse function theorem, for functions of one variable, says that, if $y(x)$ and $x(y)$ are inverse functions, meaning that $y\big(x(y)\big)=y$ and $x\big(y(x)\big)=x$, and are differentiable with $\frac{\mathrm{d}y}{\mathrm{d}x}\neq 0$, then

$$\frac{dx}{dy}(y) = \frac{1}{\frac{dy}{dx}(x(y))}$$

To see this, just apply $\frac{d}{dy}$ to both sides of $y\big(x(y)\big) = y$ to get $\frac{dy}{dx}\big(x(y)\big)\,\frac{dx}{dy}(y) = 1$, by the chain rule (see Theorem 2.9.3 in the CLP-1 text). In the CLP-1 text, we used this to compute the derivatives of the logarithm (see Theorem 2.10.1 in the CLP-1 text) and of the inverse trig functions (see Theorem 2.12.7 in the CLP-1 text).

We have just seen, in Example 2.2.12, that we can't be too naive in extending the single variable inverse function theorem to functions of two (or more) variables. On the other hand, there is such an extension, which we will now illustrate, using Cartesian and polar coordinates. For simplicity, we'll restrict our attention to $x > 0$, $y > 0$, or equivalently, $r > 0$, $0 < \theta < \frac{\pi}{2}$. The functions which convert between Cartesian and polar coordinates are

$$\begin{aligned} x(r,\theta) &= r\cos\theta \qquad & r(x,y) &= \sqrt{x^2+y^2} \\ y(r,\theta) &= r\sin\theta \qquad & \theta(x,y) &= \arctan\Big(\frac{y}{x}\Big) \end{aligned}$$



The two functions on the left convert from polar to Cartesian coordinates and the two functions on the right convert from Cartesian to polar coordinates. The inverse function theorem (for functions of two variables) says that,

- if you form the first order partial derivatives of the left hand functions into the matrix

$$\begin{bmatrix} \frac{\partial x}{\partial r}(r,\theta) & \frac{\partial r}{\partial \theta}(r,\theta) \\ \frac{\partial y}{\partial r}(r,\theta) & \frac{\partial y}{\partial \theta}(r,\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

- and you form the first order partial derivatives of the right hand functions into the matrix

$$\begin{bmatrix} \frac{\partial r}{\partial x}(x,y) & \frac{\partial r}{\partial y}(x,y) \\ \frac{\partial \theta}{\partial x}(x,y) & \frac{\partial \theta}{\partial y}(x,y) \end{bmatrix} = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-\frac{y}{x^2}}{1+(\frac{y}{x})^2} & \frac{\frac{1}{x}}{1+(\frac{y}{x})^2} \end{bmatrix} = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix}$$

- and if you evaluate the second matrix at $x = x(r,\theta)$, $y = y(r,\theta)$,

$$\begin{bmatrix} \frac{\partial r}{\partial x}\big(x(r,\theta),y(r,\theta)\big) & \frac{\partial r}{\partial y}\big(x(r,\theta),y(r,\theta)\big) \\ \frac{\partial \theta}{\partial x}\big(x(r,\theta),y(r,\theta)\big) & \frac{\partial \theta}{\partial y}\big(x(r,\theta),y(r,\theta)\big) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{bmatrix}$$

- and if you multiply [6] the two matrices together

$$\begin{aligned} &\begin{bmatrix} \frac{\partial r}{\partial x}\big(x(r,\theta),y(r,\theta)\big) & \frac{\partial r}{\partial y}\big(x(r,\theta),y(r,\theta)\big) \\ \frac{\partial \theta}{\partial x}\big(x(r,\theta),y(r,\theta)\big) & \frac{\partial \theta}{\partial y}\big(x(r,\theta),y(r,\theta)\big) \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial r}(r,\theta) & \frac{\partial x}{\partial \theta}(r,\theta) \\ \frac{\partial y}{\partial r}(r,\theta) & \frac{\partial y}{\partial \theta}(r,\theta) \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{bmatrix} \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \\ &= \begin{bmatrix} (\cos\theta)(\cos\theta)+(\sin\theta)(\sin\theta) & (\cos\theta)(-r\sin\theta)+(\sin\theta)(r\cos\theta) \\ (-\frac{\sin\theta}{r})(\cos\theta)+(\frac{\cos\theta}{r})(\sin\theta) & (-\frac{\sin\theta}{r})(-r\sin\theta)+(\frac{\cos\theta}{r})(r\cos\theta) \end{bmatrix} \end{aligned}$$

- then the result is the identity matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and indeed it is!

This two variable version of the inverse function theorem can be derived by applying the derivatives $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ to the equations

$$r\big(x(r,\theta),y(r,\theta)\big) = r$$
$$\theta\big(x(r,\theta),y(r,\theta)\big) = \theta$$

and using the two variable version of the chain rule, which we will see in §2.4.

Exercises

Stage 1

? 1

Let $f(x,y) = e^x \cos y$. The following table gives some values of $f(x,y)$.

	$x=0$	$x=0.01$	$x=0.1$
$y=-0.1$	0.99500	1.00500	1.09965
$y=-0.01$	0.99995	1.01000	1.10512
$y=0$	1.0	1.01005	1.10517

1. Find two different approximate values for $\frac{\partial f}{\partial x}(0,0)$ using the data in the above table.
2. Find two different approximate values for $\frac{\partial f}{\partial y}(0,0)$ using the data in the above table.
3. Evaluate $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$ exactly.

? 2

You are traversing an undulating landscape. Take the z-axis to be straight up towards the sky, the positive x-axis to be due south, and the positive y-axis to be due east. Then the landscape near you is described by the equation $z = f(x,y)$, with you at the point $(0,0,f(0,0))$.The function $f(x,y)$is differentiable.

Suppose $f_y(0,0) < 0$. Is it possible that you are at a summit? Explain.

? 3 ✳

Let

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Compute, directly from the definitions,

1. $\frac{\partial f}{\partial x}(0,0)$
2. $\frac{\partial f}{\partial y}(0,0)$
3. $\frac{d}{dt} f(t,t)\Big|_{t=0}$

Stage 2

? 4

Find all first partial derivatives of the following functions and evaluate them at the given point.

1. $f(x,y,z) = x^3y^4z^5 \qquad (0,-1,-1)$
2. $w(x,y,z) = \ln(1+e^{xyz}) \qquad (2,0,-1)$
3. $f(x,y) = \dfrac{1}{\sqrt{x^2+y^2}} \qquad (-3,4)$

? 5

Show that the function $z(x,y) = \frac{x+y}{x-y}$ obeys

$$x\frac{\partial z}{\partial x}(x,y) + y$$
$$frac\partial z\partial y(x,y) = 0$$

? 6 ✳

A surface $z(x,y)$ is defined by $zy - y + x = \ln(xyz)$.

1. Compute $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ in terms of x, y, z.
2. Evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(x,y,z) = (-1,-2,1/2)$.

? 7 ✳

Find $\frac{\partial U}{\partial T}$ and $\frac{\partial T}{\partial V}$ at $(1,1,2,4)$if (T,U,V,W) are related by

$$(TU-V)^2\ln(W-UV) = \ln 2$$

? 8 ✳

Suppose that $u = x^2 + yz,\ x = \rho r\cos(\theta),\ y = \rho r\sin(\theta)$ and $z = \rho r$. Find $\frac{\partial u}{\partial r}$ at the point $(\rho_0, r_0, \theta_0) = (2,3,\pi/2)$.

? 9

Use the definition of the derivative to evaluate $f_x(0,0)$ and $f_y(0,0)$ for

$$f(x,y) = \begin{cases} \frac{x^2-2y^2}{x-y} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Stage 3

? 10

Let f be any differentiable function of one variable. Define $z(x,y) = f(x^2+y^2)$. Is the equation

$$y\frac{\partial z}{\partial x}(x,y) - x\frac{\partial z}{\partial y}(x,y) = 0$$

necessarily satisfied?

? 11

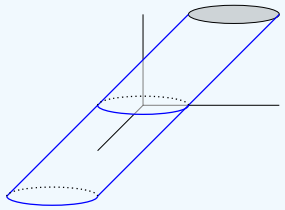
Define the function

$$f(x,y) = \begin{cases} \frac{(x+2y)^2}{x+y} & \text{if } x+y \neq 0 \\ 0 & \text{if } x+y = 0 \end{cases}$$

1. Evaluate, if possible, $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$.
2. Is $f(x,y)$ continuous at $(0,0)$?

? 12

Consider the cylinder whose base is the radius-1 circle in the xy-plane centred at $(0,0)$, and which slopes parallel to the line in the yz-plane given by $z = y$.



When you stand at the point $(0,-1,0)$, what is the slope of the surface if you look in the positive y direction? The positive x direction?

1. There are applications in which there are several variables that cannot be varied independently. For example, the pressure, volume and temperature of an ideal gas are related by the equation of state $PV = (\text{constant})T$. In those applications, it may not be clear from the context which variables are being held fixed.
2. It is also possible to evaluate the derivative by using the technique of the optional Section 2.15 in the CLP-1 text.
3. The only real number z which obeys $z^5 = -1$ is $z = -1$. However there are four other complex numbers which also obey $z^5 = -1$.
4. You should have already seen this technique, called implicit differentiation, in your first Calculus course. It is covered in Section 2.11 in the CLP-1 text.
5. If you are not familiar with polar coordinates, don't worry about it. There will be an introduction to them in §3.2.1.
6. Matrix multiplication is usually covered in courses on linear algebra, which you may or may not have taken. That's why this example is optional.

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5.4: Maximum and Minimum Values

One of the core topics in single variable calculus courses is finding the maxima and minima of functions of one variable. We'll now extend that discussion to functions of more than one variable [1]. Rather than leaping into the deep end, we'll not be too ambitious and concentrate on functions of two variables. That being said, many of the techniques work more generally. To start, we have the following natural extensions to some familiar definitions.

Definition 2.9.1

Let the function $f(x,y)$ be defined for all (x,y) in some subset R of $\mathbb{R}^2$. Let (a,b) be a point in R.

- (a,b) is a *local maximum* of $f(x,y)$ if $f(x,y) \le f(a,b)$ for all (x,y) close to (a,b). More precisely, (a,b) is a local maximum of $f(x,y)$ if there is an $r>0$ such that $f(x,y) \le f(a,b)$ for all points (x,y) within a distance r of (a,b).
- (a,b) is a *local minimum* of $f(x,y)$ if $f(x,y) \ge f(a,b)$ for all (x,y) close to (a,b).
- Local maximum and minimum values are also called extremal values.
- (a,b) is an *absolute maximum* or *global maximum* of $f(x,y)$ if $f(x,y) \le f(a,b)$ for all (x,y) in R.
- (a,b) is an *absolute minimum* or *global minimum* of $f(x,y)$ if $f(x,y) \ge f(a,b)$ for all (x,y) in R.

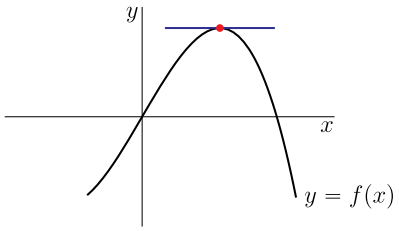
Local Maxima and Minima

One of the first things you did when you were developing the techniques used to find the maximum and minimum values of $f(x)$ was ask yourself [2]

- Suppose that the largest value of $f(x)$ is $f(a)$. What does that tell us about a?

After a little thought you answered

- If the largest value of $f(x)$ is $f(a)$ and f is differentiable at a, then $f'(a)=0$.



Let's recall why that's true. Suppose that the largest value of $f(x)$ is $f(a)$. Then for all $h>0$,

$$f(a+h) \le f(a) \implies f(a+h)-f(a) \le 0 \implies \frac{f(a+h)-f(a)}{h} \le 0 \quad \text{if } h>0$$

Taking the limit $h \to 0$ tells us that $f'(a) \le 0$. Similarly [3], for all $h<0$,

$$f(a+h) \le f(a) \implies f(a+h)-f(a) \le 0 \implies \frac{f(a+h)-f(a)}{h} \ge 0 \quad \text{if } h<0$$

Taking the limit $h \to 0$ now tells us that $f'(a) \ge 0$. So we have both $f'(a) \ge 0$ and $f'(a) \le 0$ which forces $f'(a)=0$.

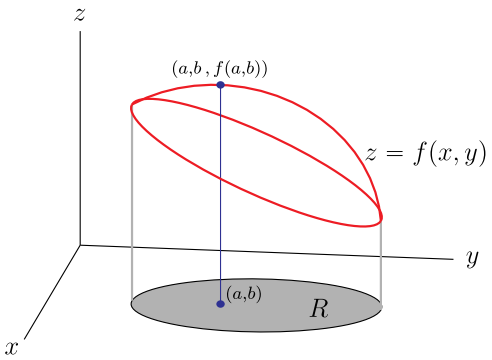
You also observed at the time that for this argument to work, you only need $f(x) \le f(a)$ for all x's close to a, not necessarily for all x's in the whole world. (In the above inequalities, we only used $f(a+h)$ with h small.) Since we care only about $f(x)$ for x near a, we can refine the above statement.

- If $f(a)$ is a local maximum for $f(x)$ and f is differentiable at a, then $f'(a)=0$.

Precisely the same reasoning applies to minima.

- If $f(a)$ is a local minimum for $f(x)$ and f is differentiable at a, then $f'(a)=0$.

Let's use the ideas of the above discourse to extend the study of local maxima and local minima to functions of more than one variable. Suppose that the function $f(x,y)$ is defined for all (x,y) in some subset R of $\mathbb{R}^2$, that (a,b) is point of R that is not on the boundary of R, and that f has a local maximum at (a,b). See the figure below.



Then the function $f(x,y)$ must decrease in value as (x,y) moves away from (a,b) in *any* direction. No matter which direction $\vec{d}$ we choose, the directional derivative of f at (a,b) in direction $\vec{d}$ must be zero or smaller. Writing this in mathematical symbols, we get

$$D_{\vec{d}}f(a,b) = \nabla f(a,b) \cdot \frac{\vec{d}}{|\vec{d}|} \leq 0$$

And the directional derivative of f at (a,b) in the direction $-\vec{d}$ also must be zero or negative.

$$D_{-\vec{d}}f(a,b) = \nabla f(a,b) \cdot \frac{-\vec{d}}{|\vec{d}|} \leq 0 \quad \text{which implies that} \quad \nabla f(a,b) \cdot \frac{\vec{d}}{|\vec{d}|} \geq 0$$

As $\vec{n}abla f(a,b) \cdot \frac{\vec{d}}{|\vec{d}|}$ must be both positive (or zero) and negative (or zero) at the same time, it must be zero. In particular, choosing $\vec{d} = \hat{\boldsymbol{\imath}}$ forces the x component of $\vec{n}abla f(a,b)$ to be zero, and choosing $\vec{d} = \hat{\boldsymbol{\jmath}}$ forces the y component of $\nabla f(a,b)$ to be zero. We have thus shown that $\nabla f(a,b) = \mathbf{0}$. The same argument shows that $\nabla f(a,b) = \mathbf{0}$ when (a,b) is a local minimum too. This is an important and useful result, so let's theoremise it.

Theorem 2.9.2

Let the function $f(x,y)$ be defined for all (x,y) in some subset R of $\mathbb{R}^2$. Assume that

- (a,b) is a point of R that is not on the boundary of R and
- (a,b) is a local maximum or local minimum of f and that
- the partial derivatives of f exist at (a,b).

Then

$$\nabla f(a,b) = \mathbf{0}.$$

Definition 2.9.3

Let $f(x,y)$ be a function and let (a,b) be a point in its domain. Then

- if $\nabla f(a,b)$ exists and is zero we call (a,b) a critical point (or a stationary point) of the function, and
- if $\nabla f(a,b)$ does not exist then we call (a,b) a singular point of the function.

Warning 2.9.4

Note that some people (and texts) combine both of these cases and call (a,b) a critical point when either the gradient is zero or does not exist.

Warning 2.9.5

Theorem 2.9.2 tells us that every local maximum or minimum (in the interior of the domain of a function whose partial derivatives exist) is a critical point. Beware that it does *not*[4] tell us that every critical point is either a local maximum or a local minimum.

In fact, we shall see later[5], in Examples 2.9.13 and 2.9.15, critical points that are neither local maxima nor a local minima. Nonetheless, Theorem 2.9.2 is very useful because often functions have only a small number of critical points. To find local maxima and minima of such functions, we only need to consider its critical and singular points. We'll return later to the question of how to tell if a critical point is a local maximum, local minimum or neither. For now, we'll just practice finding critical points.

Example 2.9.6. $f(x,y) = x^2 - 2xy + 2y^2 + 2x - 6y + 12$

Find all critical points of $f(x,y) = x^2 - 2xy + 2y^2 + 2x - 6y + 12$.

Solution

To find the critical points, we need to find the gradient. To find the gradient we need to find the first order partial derivatives. So, as a preliminary calculation, we find the two first order partial derivatives of $f(x,y)$.

$$\begin{aligned} f_x(x,y) &= 2x - 2y + 2 \\ f_y(x,y) &= -2x + 4y - 6 \end{aligned}$$

So the critical points are the solutions of the pair of equations

$$2x - 2y + 2 = 0 \qquad -2x + 4y - 6 = 0$$

or equivalently (dividing by two and moving the constants to the right hand side)

$$\begin{aligned} x - y &= -1 \qquad (E1) \\ -x + 2y &= 3 \qquad (E2) \end{aligned}$$

This is a system of two equations in two unknowns (x and y). One strategy for a solving system like this is to

- First use one of the equations to solve for one of the unknowns in terms of the other unknown. For example, (E1) tells us that $y = x + 1$. This expresses y in terms of x. We say that we have solved for y in terms of x.
- Then substitute the result, $y = x + 1$ in our case, into the other equation, (E2). In our case, this gives

$$-x + 2(x+1) = 3 \iff x + 2 = 3 \iff x = 1$$

- We have now found that $x = 1$, $y = x + 1 = 2$ is the only solution. So the only critical point is $(1,2)$. Of course it only takes a moment to verify that $\nabla f(1,2) = \langle 0, 0 \rangle$. It is a good idea to do this as a simple check of our work.

An alternative strategy for solving a system of two equations in two unknowns, like (E1) and (E2), is to

- add equations (E1) and (E2) together. This gives

$$(E1) + (E2): \quad (1-1)x + (-1+2)y = -1 + 3 \iff y = 2$$

  The point here is that adding equations (E1) and (E2) together eliminates the unknown x, leaving us with one equation in the unknown y, which is easily solved. For other systems of equations you might have to multiply the equations by some numbers before adding them together.
- We now know that $y = 2$. Substituting it into (E1) gives us

$$x - 2 = -1 \implies x = 1$$

- Once again (thankfully) we have found that the only critical point is $(1,2)$.

This was pretty easy because we only had to solve linear equations, which in turn was a consequence of the fact that $f(x,y)$ was a polynomial of degree two. Here is an example with some slightly more challenging algebra.

Example 2.9.7. $f(x,y) = 2x^3 - 6xy + y^2 + 4y$

Find all critical points of $f(x,y) = 2x^3 - 6xy + y^2 + 4y$.

Solution

As in the last example, we need to find where the gradient is zero, and to find the gradient we need the first order partial derivatives.

$$f_x = 6x^2 - 6y \qquad f_y = -6x + 2y + 4$$

So the critical points are the solutions of

$$6x^2 - 6y = 0 \qquad -6x + 2y + 4 = 0$$

We can rewrite the first equation as $y = x^2$, which expresses y as a function of x. We can then substitute $y = x^2$ into the second equation, giving

$$\begin{aligned} -6x + 2y + 4 = 0 &\iff -6x + 2x^2 + 4 = 0 \iff x^2 - 3x + 2 = 0 \\ &\iff (x-1)(x-2) = 0 \\ &\iff x = 1 \text{ or } 2 \end{aligned}$$

When $x = 1$, $y = 1^2 = 1$ and when $x = 2$, $y = 2^2 = 4$. So, there are two critical points: $(1,1)$, $(2,4)$.

Alternatively, we could have also used the second equation to write $y = 3x - 2$, and then substituted that into the first equation to get

$$6x^2 - 6(3x-2) = 0 \iff x^2 - 3x + 2 = 0$$

just as above.

And here is an example for which the algebra requires a bit more thought.

Example 2.9.8. $f(x,y) = xy(5x + y - 15)$

Find all critical points of $f(x,y) = xy(5x + y - 15)$.

Solution

The first order partial derivatives of $f(x,y) = xy(5x + y - 15)$ are

$$\begin{aligned} f_x(x,y) &= y(5x + y - 15) + xy(5) &&= y(5x + y - 15) + y(5x) \\ &= y(10x + y - 15) \\ f_y(x,y) &= x(5x + y - 15) + xy(1) &&= x(5x + y - 15) + x(y) \\ &= x(5x + 2y - 15) \end{aligned}$$

The critical points are the solutions of $f_x(x,y) = f_y(x,y) = 0$. That is, we need to find all x, y that satisfy the pair of equations

$$y(10x + y - 15) = 0 \tag{E1}$$
$$x(5x + 2y - 15) = 0 \tag{E2}$$

The first equation, $y(10x + y - 15) = 0$, is satisfied if at least one of the two factors y, $(10x + y - 15)$ is zero. So the first equation is satisfied if at least one of the two equations

$$y = 0 \tag{E1a}$$
$$10x + y = 15 \tag{E1b}$$

is satisfied. The second equation, $x(5x + 2y - 15) = 0$, is satisfied if at least one of the two factors x, $(5x + 2y - 15)$ is zero. So the second equation is satisfied if at least one of the two equations

$$x = 0 \tag{E2a}$$
$$5x + 2y = 15 \tag{E2b}$$

is satisfied.

So both critical point equations (E1) and (E2) are satisfied if and only if at least one of (E1a), (E1b) is satisfied and in addition at least one of (E2a), (E2b) is satisfied. So both critical point equations (E1) and (E2) are satisfied if and only if at least one of the following four possibilities hold.

- (E1a) and (E2a) are satisfied if and only if $x = y = 0$
- (E1a) and (E2b) are satisfied if and only if $y = 0,\ 5x + 2y = 15 \iff y = 0,\ 5x = 15$
- (E1b) and (E2a) are satisfied if and only if $10x + y = 15,\ x = 0 \iff y = 15,\ x = 0$
- (E1b) and (E2b) are satisfied if and only if $10x + y = 15,\ 5x + 2y = 15$. We can use, for example, the second of these equations to solve for x in terms of y: $x = \frac{1}{5}(15 - 2y)$. When we substitute this into the first equation we get $2(15 - 2y) + y = 15,$ which we can solve for y. This gives $-3y = 15 - 30$ or $y = 5$ and then $x = \frac{1}{5}(15 - 2 \times 5) = 1$.

In conclusion, the critical points are $(0, 0), (3, 0), (0, 15)$ and $(1, 5)$.

A more compact way to write what we have just done is

$$\begin{aligned}
& f_x(x,y) = 0 && \text{and} \quad f_y(x,y) = 0 \\
\iff \quad & y(10x + y - 15) = 0 && \text{and} \quad x(5x + 2y - 15) = 0 \\
\iff \quad & \{y = 0 \text{ or } 10x + y = 15\} && \text{and} \quad \{x = 0 \text{ or } 5x + 2y = 15\} \\
\iff \quad & \{y = 0,\ x = 0\} \text{ or } \{y = 0,\ 5x + 2y = 15\} \\
& \text{or } \{10x + y = 15,\ x = 0\} \text{ or } \{10x + y = 15,\ 5x + 2y = 15\} \\
\iff \quad & \{x = y = 0\} \text{ or } \{y = 0,\ x = 3\} \\
& \text{or } \{x = 0,\ y = 15\} \text{ or } \{x = 1,\ y = 5\}
\end{aligned}$$

Let's try a more practical example — something from the real world. Well, a mathematician's "real world". The interested reader should search-engine their way to a discussion of "idealisation", "game theory" "Cournot models" and "Bertrand models". But don't spend too long there. A discussion of breweries is about to take place.

Example 2.9.9

In a certain community, there are two breweries in competition [6], so that sales of each negatively affect the profits of the other. If brewery A produces x litres of beer per month and brewery B produces y litres per month, then the profits of the two breweries are given by

$$P = 2x - \frac{2x^2 + y^2}{10^6} \qquad Q = 2y - \frac{4y^2 + x^2}{2 \times 10^6}$$

respectively. Find the sum of the two profits if each brewery independently sets its own production level to maximize its own profit and assumes that its competitor does likewise. Then, assuming cartel behaviour, find the sum of the two profits if the two breweries cooperate so as to maximize that sum [7].

Solution

If A adjusts x to maximize P (for y held fixed) and B adjusts y to maximize Q (for x held fixed) then x and y are determined by the equations

$$P_x = 2 - \tfrac{4x}{10^6} = 0 \qquad \text{(E1)}$$
$$Q_y = 2 - \tfrac{8y}{2 \times 10^6} = 0 \qquad \text{(E2)}$$

Equation (E1) yields $x = \frac{1}{2}10^6$ and equation (E2) yields $y = \frac{1}{2}10^6$. Knowing x and y we can determine P, Q and the total profit

$$\begin{aligned}
P + Q &= 2(x + y) - \tfrac{1}{10^6}\left(\tfrac{5}{2}x^2 + 3y^2\right) \\
&= 10^6\left(1 + 1 - \tfrac{5}{8} - \tfrac{3}{4}\right) = \tfrac{5}{8}10^6
\end{aligned}$$

On the other hand if (A, B) adjust (x, y) to maximize $P + Q = 2(x + y) - \frac{1}{10^6}\left(\frac{5}{2}x^2 + 3y^2\right)$, then x and y are determined by

$$(P+Q)_x = 2 - \tfrac{5x}{10^6} = 0 \qquad \text{(E1)}$$
$$(P+Q)_y = 2 - \tfrac{6y}{10^6} = 0 \qquad \text{(E2)}$$

Equation (E1) yields $x = \frac{2}{5}10^6$ and equation (E2) yields $y = \frac{1}{3}10^6$. Again knowing x and y we can determine the total profit

$$\begin{aligned} P+Q &= 2(x+y) - \tfrac{1}{10^6}\left(\tfrac{5}{2}x^2 + 3y^2\right) \\ &= 10^6\left(\tfrac{4}{5} + \tfrac{2}{3} - \tfrac{2}{5} - \tfrac{1}{3}\right) = \tfrac{11}{15}10^6 \end{aligned}$$

So cooperating really does help their profits. Unfortunately, like a very small tea-pot, consumers will be a little poorer[8].

Moving swiftly away from the last pun, let's do something a little more geometric.

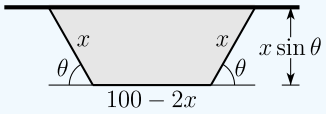
Example 2.9.10

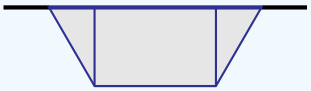
Equal angle bends are made at equal distances from the two ends of a 100 metre long fence so the resulting three segment fence can be placed along an existing wall to make an enclosure of trapezoidal shape. What is the largest possible area for such an enclosure?



Solution

This is a very geometric problem (fenced off from pun opportunities), and as such we should start by drawing a sketch and introducing some variable names.





The area enclosed by the fence is the area inside the blue rectangle (in the figure on the right above) plus the area inside the two blue triangles.

$$\begin{aligned} A(x,\theta) &= (100-2x)x\sin\theta + 2\cdot\frac{1}{2}\cdot x\sin\theta\cdot x\cos\theta \\ &= (100x - 2x^2)\sin\theta + x^2\sin\theta\,\cos\theta \end{aligned}$$

To maximize the area, we need to solve

$$\begin{aligned} 0 = \frac{\partial A}{\partial x} &= (100-4x)\sin\theta + 2x\sin\theta\cos\theta \\ 0 = \frac{\partial A}{\partial \theta} &= (100x - 2x^2)\cos\theta + x^2\left\{\cos^2\theta - \sin^2\theta\right\} \end{aligned}$$

Note that both terms in the first equation contain the factor $\sin\theta$ and all terms in the second equation contain the factor x. If either $\sin\theta$ or x are zero the area $A(x,\theta)$ will also be zero, and so will certainly not be maximal. So we may divide the first equation by $\sin\theta$ and the second equation by x, giving

$$(100-4x) + 2x\cos\theta = 0 \qquad \text{(E1)}$$
$$(100-2x)\cos\theta + x\left\{\cos^2\theta - \sin^2\theta\right\} = 0 \qquad \text{(E2)}$$

These equations might look a little scary. But there is no need to panic. They are not as bad as they look because θ enters only through $\cos\theta$ and $\sin^2\theta$, which we can easily write in terms of $\cos\theta$. Furthermore we can eliminate $\cos\theta$ by observing that the first equation forces $\cos\theta = -\frac{100-4x}{2x}$ and hence $\sin^2\theta = 1 - \cos^2\theta = 1 - \frac{(100-4x)^2}{4x^2}$. Substituting these into the second equation gives

$$
\begin{aligned}
-(100-2x)\frac{100-4x}{2x}+x\left[\frac{(100-4x)^2}{2x^2}-1\right] &= 0\\
\implies \quad -(100-2x)(100-4x)+(100-4x)^2-2x^2 &= 0\\
\implies \quad 6x^2-200x &= 0\\
\implies \quad x=\frac{100}{3} \quad \cos\theta=-\frac{-100/3}{200/3}=\frac{1}{2} \quad \theta &= 60^\circ
\end{aligned}
$$

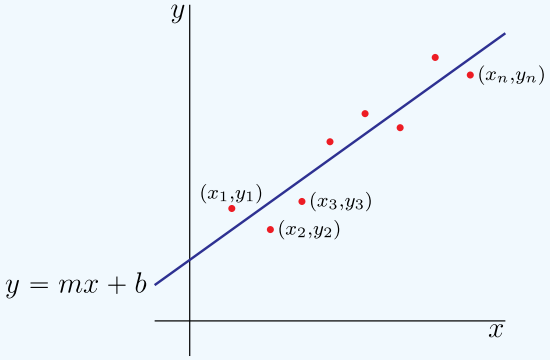
and the maximum area enclosed is

$$
A=\left(100\frac{100}{3}-2\frac{100^2}{3^2}\right)\frac{\sqrt{3}}{2}+\frac{1}{2}\frac{100^2}{3^2}\frac{\sqrt{3}}{2}=\frac{2500}{\sqrt{3}}
$$

Now here is a very useful (even practical!) statistical example — finding the line that best fits a given collection of points.

Example 2.9.11. Linear regression

An experiment yields n data points (x_i, y_i), $i=1,2,\cdots,n$. We wish to find the straight line $y=mx+b$ which "best" fits the data.



The definition of "best" is "minimizes the root mean square error", i.e. minimizes

$$
E(m,b)=\sum_{i=1}^{n}(mx_i+b-y_i)^2
$$

Note that

- term number i in $E(m,b)$ is the square of the difference between y_i, which is the i^{th} measured value of y, and $\Big[mx+b\Big]_{x=x_i}$, which is the approximation to y_i given by the line $y=mx+b$.
- All terms in the sum are positive, regardless of whether the points (x_i, y_i) are above or below the line.

Our problem is to find the m and b that minimizes $E(m,b)$. This technique for drawing a line through a bunch of data points is called "linear regression". It is used *a lot*[9] [10]. Even in the real world — and not just the real world that you find in mathematics problems. The actual real world that involves jobs.

Solution

We wish to choose m and b so as to minimize $E(m,b)$. So we need to determine where the partial derivatives of E are zero.

$$
\begin{aligned}
0=\frac{\partial E}{\partial m} &= \sum_{i=1}^{n}2(mx_i+b-y_i)x_i &&= m\Big[\sum_{i=1}^{n}2x_i^2\Big]+b\Big[\sum_{i=1}^{n}2x_i\Big]-\Big[\sum_{i=1}^{n}2x_iy_i\Big]\\
0=\frac{\partial E}{\partial b} &= \sum_{i=1}^{n}2(mx_i+b-y_i) &&= m\Big[\sum_{i=1}^{n}2x_i\Big]+b\Big[\sum_{i=1}^{n}2\Big]-\Big[\sum_{i=1}^{n}2y_i\Big]
\end{aligned}
$$

There are a lot of symbols here. But remember that all of the x_i's and y_i's are given constants. They come from, for example, experimental data. The only unknowns are m and b. To emphasize this, and to save some writing, define the constants

$$S_x = \sum_{i=1}^n x_i \qquad S_y = \sum_{i=1}^n y_i \qquad S_{x^2} = \sum_{i=1}^n x_i^2 \qquad S_{xy} = \sum_{i=1}^n x_i y_i$$

The equations which determine the critical points are (after dividing by two)

$$S_{x^2}\, m + S_x\, b = S_{xy} \tag{E1}$$
$$S_x\, m + n\, b = S_y \tag{E2}$$

These are two linear equations on the unknowns m and b. They may be solved in any of the usual ways. One is to use (E2) to solve for b in terms of m

$$b = \frac{1}{n}\left(S_y - S_x m\right)$$

and then substitute this into (E1) to get the equation

$$S_{x^2}\, m + \frac{1}{n} S_x \left(S_y - S_x m\right) = S_{xy} \qquad \Longrightarrow \qquad \left(nS_{x^2} - S_x^2\right) m = nS_{xy} - S_x S_y$$

for m. We can then solve this equation for m and substitute back into (E3) to get b. This gives

$$\begin{aligned} m &= \frac{nS_{xy} - S_x S_y}{nS_{x^2} - S_x^2} \\ b &= \frac{S_y}{n}\,\frac{nS_{x^2} - S_x^2}{nS_{x^2} - S_x^2} - \frac{S_x}{n}\,\frac{nS_{xy} - S_x S_y}{nS_{x^2} - S_x^2} = \frac{nS_y S_{x^2} - nS_x S_{xy}}{n(nS_{x^2} - S_x^2)} \\ &= -\frac{S_x S_{xy} - S_y S_{x^2}}{nS_{x^2} - S_x^2} \end{aligned}$$

Another way to solve the system of equations is

$$\begin{aligned} n(\text{E1}) - S_x(\text{E2}): \quad & \left[nS_{x^2} - S_x^2\right] m &&= nS_{xy} - S_x S_y \\ -S_x(\text{E1}) + S_{x^2}(\text{E2}): \quad & \left[nS_{x^2} - S_x^2\right] b &&= -S_x S_{xy} + S_y S_{x^2} \end{aligned}$$

which gives the same solution.

So given a bunch of data points, it only takes a quick bit of arithmetic — no calculus required — to apply the above formulae and so to find the best fitting line. Of course while you don't need any calculus to apply the formulae, you do need calculus to understand where they came from. The same technique can be extended to other types of curve fitting problems. For example, polynomial regression.

The Second Derivative Test

Now let's start thinking about how to tell if a critical point is a local minimum or maximum. Remember what happens for functions of one variable. Suppose that $x = a$ is a critical point of the function $f(x)$. Any (sufficiently smooth) function is well approximated, when x is close to a, by the first few terms of its Taylor expansion

$$f(x) = f(a) + f'(a)\,(x-a) + \tfrac{1}{2} f''(a)\,(x-a)^2 + \tfrac{1}{3!} f^{(3)}(a)\,(x-a)^3 + \cdots$$

As a is a critical point, we know that $f'(a) = 0$ and

$$f(x) = f(a) + \tfrac{1}{2} f''(a)\,(x-a)^2 + \tfrac{1}{3!} f^{(3)}(a)\,(x-a)^3 + \cdots$$

If $f''(a) \neq 0,\ f(x)$ is going to look a lot like $f(a) + \frac{1}{2} f''(a)\,(x-a)^2$ when x is really close to a. In particular

- if $f''(a) > 0,$ then we will have $f(x) > f(a)$ when x is close to (but not equal to) a, so that a will be a local minimum and
- if $f''(a) < 0,$ then we will have $f(x) < f(a)$ when x is close to (but not equal to) a, so that a will be a local maximum, but
- if $f''(a) = 0,$ then we cannot draw any conclusions without more work.

A similar, but messier, analysis is possible for functions of two variables. Here are some simple quadratic examples that provide a warmup for that messier analysis.

Example 2.9.12. $f(x,y) = x^2 + 3xy + 3y^2 - 6x - 3y - 6$

Consider $f(x,y) = x^2 + 3xy + 3y^2 - 6x - 3y - 6$. The gradient of f is

$$\nabla f(x,y) = \big(2x + 3y - 6\big)\,\hat{\boldsymbol{\imath}} + \big(3x + 6y - 3\big)\,\hat{\boldsymbol{\jmath}}$$

So (x,y) is a critical point of f if and only if

$$\begin{aligned} 2x + 3y &= 6 \qquad \text{(E1)} \\ 3x + 6y &= 3 \qquad \text{(E2)} \end{aligned}$$

Multiplying the first equation by 2 and subtracting the second equation gives

$$x = 9$$

Then substituting $x = 9$ back into the first equation gives

$$2 \times 9 + 3y = 6 \implies y = -4$$

So $f(x,y)$ has precisely one critical point, namely $(9\,,\,-4)$.

Now let's try to determine if $f(x,y)$ has a local minimum, or a local maximum, or neither, at $(9,-4)$. A good way to determine the behaviour of $f(x,y)$ for (x,y) near $(9,-4)$ is to make the change of variables [11]

$$x = 9 + \Delta x \qquad y = -4 + \Delta y$$

and study the behaviour of f for Δx and Δy near zero.

$$\begin{aligned} f\big(9+\Delta x\,,\,-4+\Delta y\big) &= (9+\Delta x)^2 + 3(9+\Delta x)(-4+\Delta y) + 3(-4+\Delta y)^2 \\ &\qquad - 6(9+\Delta x) - 3(-4+\Delta y) - 6 \\ &= (\Delta x)^2 + 3\Delta x\,\Delta y + 3(\Delta y)^2 - 27 \end{aligned}$$

And a good way to study the sign of quadratic expressions like $(\Delta x)^2 + 3\Delta x\,\Delta y + 3(\Delta y)^2$ is to complete the square. So far you have probably just completed the square for quadratic expressions that involve only a single variable. For example

$$x^2 + 3x + 3 = \left(x + \frac{3}{2}\right)^2 - \frac{9}{4} + 3$$

When there are two variables around, like Δx and Δy, you can just pretend that one of them is a constant and complete the square as before. For example, if you pretend that Δy is a constant,

$$\begin{aligned} (\Delta x)^2 + 3\Delta x\,\Delta y + 3(\Delta y)^2 &= \left(\Delta x + \frac{3}{2}\Delta y\right)^2 + \left(3 - \frac{9}{4}\right)(\Delta y)^2 \\ &= \left(\Delta x + \frac{3}{2}\Delta y\right)^2 + \frac{3}{4}(\Delta y)^2 \end{aligned}$$

To this point, we have expressed

$$f\big(9+\Delta x\,,\,-4+\Delta y\big) = \left(\Delta x + \frac{3}{2}\Delta y\right)^2 + \frac{3}{4}(\Delta y)^2 - 27$$

As the smallest values of $\left(\Delta x + \frac{3}{2}\Delta y\right)^2$ and $\frac{3}{4}(\Delta y)^2$ are both zero, we have that

$$f(x,y) = f\big(9+\Delta x\,,\,-4+\Delta y\big) \geq -27 = f(9,-4)$$

for all (x,y) so that $(9,-4)$ is both a local minimum and a global minimum for f.

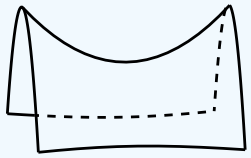
You have already encountered single variable functions that have a critical point which is neither a local max nor a local min. See Example 3.5.9 in the CLP-1 text. Here are a couple of examples which show that this can also happen for functions of two variables. We'll start with the simplest possible such example.

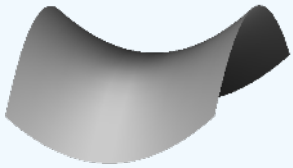
Example 2.9.13. $f(x,y) = x^2 - y^2$

The first partial derivatives of $f(x,y) = x^2 - y^2$ are $f_x(x,y) = 2x$ and $f_y(x,y) = -2y$. So the only critical point of this function is $(0,0)$. Is this a local minimum or maximum? Well let's start with (x,y) at $(0,0)$ and then move (x,y) away from $(0,0)$ and see if $f(x,y)$ gets bigger or smaller. At the origin $f(0,0) = 0$. Of course we can move (x,y) away from $(0,0)$ in many different directions.

- First consider moving (x,y) along the x-axis. Then $(x,y) = (x,0)$ and $f(x,y) = f(x,0) = x^2$. So when we start with $x = 0$ and then increase x, the value of the function f increases — which means that $(0,0)$ cannot be a local maximum for f.
- Next let's move (x,y) away from $(0,0)$ along the y-axis. Then $(x,y) = (0,y)$ and $f(x,y) = f(0,y) = -y^2$. So when we start with $y = 0$ and then increase y, the value of the function f decreases — which means that $(0,0)$ cannot be a local minimum for f.

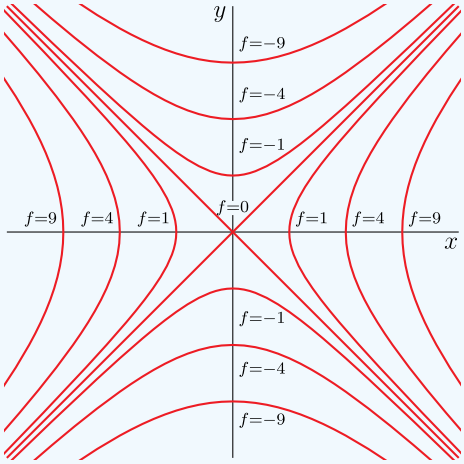
So moving away from $(0,0)$ in one direction causes the value of f to increase, while moving away from $(0,0)$ in a second direction causes the value of f to decrease. Consequently $(0,0)$ is neither a local minimum or maximum for f. It is called a saddle point, because the graph of f looks like a saddle. (The full definition of "saddle point" is given immediately after this example.) Here are some figures showing the graph of f.





The figure below show some level curves of f. Observe from the level curves that

- f increases as you leave $(0,0)$ walking along the x axis
- f decreases as you leave $(0,0)$ walking along the y axis



Approximately speaking, if a critical point (a,b) is neither a local minimum nor a local maximum, then it is a saddle point. For (a,b) to not be a local minimum, f has to take values bigger than $f(a,b)$ at some points nearby (a,b). For (a,b) to not be a local maximum, f has to take values smaller than $f(a,b)$ at some points nearby (a,b). Writing this more mathematically we get the following definition.

Definition 2.9.14

The critical point (a,b) is called a saddle point for the function $f(x,y)$ if, for each $r > 0$,

- there is at least one point (x,y), within a distance r of (a,b), for which $f(x,y) > f(a,b)$ and
- there is at least one point (x,y), within a distance r of (a,b), for which $f(x,y) < f(a,b)$.

Here is another example of a saddle point. This time we have to work a bit to see it.

Example 2.9.15. $f(x,y) = x^2 - 2xy - y^2 + 4y - 2$

Consider $f(x,y) = x^2 - 2xy - y^2 + 4y - 2$. The gradient of f is

$$\nabla f(x,y) = \big(2x - 2y\big)\,\hat{\boldsymbol{\imath}} + \big(-2x - 2y + 4\big)\,\hat{\boldsymbol{\jmath}}$$

So (x,y) is a critical point of f if and only if

$$\begin{aligned} 2x - 2y &= 0 \\ -2x - 2y &= -4 \end{aligned}$$

The first equation gives that $x = y$. Substituting $y = x$ into the second equation gives

$$-2y - 2y = -4 \implies x = y = 1$$

So $f(x,y)$ has precisely one critical point, namely $(1,1)$.

To determine if $f(x,y)$ has a local minimum, or a local maximum, or neither, at $(1,1)$, we proceed as in Example 2.9.12. We make the change of variables

$$x = 1 + \Delta x \qquad y = 1 + \Delta y$$

to give

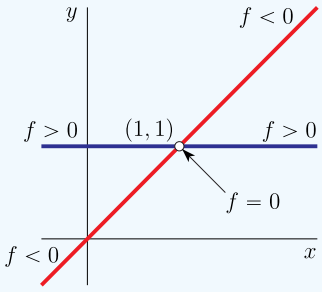
$$\begin{aligned} &f\big(1+\Delta x\,,\,1+\Delta y\big) \\ &\qquad = (1+\Delta x)^2 - 2(1+\Delta x)(1+\Delta y) - (1+\Delta y)^2 + 4(1+\Delta y) - 2 \\ &\qquad = (\Delta x)^2 - 2\Delta x\,\Delta y - (\Delta y)^2 \end{aligned}$$

Completing the square,

$$f\big(1+\Delta x\,,\,1+\Delta y\big) = (\Delta x)^2 - 2\Delta x\,\Delta y - (\Delta y)^2 = (\Delta x - \Delta y)^2 - 2(\Delta y)^2$$

Notice that f has now been written as the difference of two squares, much like the f in the saddle point Example 2.9.13.

- If Δx and Δy are such that the first square $(\Delta x - \Delta y)^2$ is nonzero, but the second square $(\Delta y)^2$ is zero, then $f\big(1+\Delta x\,,\,1+\Delta y\big) = (\Delta x - \Delta y)^2 > 0 = f(1,1)$. That is, whenever $\Delta y = 0$ and $\Delta x \neq \Delta y$, then $f\big(1+\Delta x\,,\,1+\Delta y\big) = (\Delta x - \Delta y)^2 > 0 = f(1,1)$.
- On the other hand, if Δx and Δy are such that the first square $(\Delta x - \Delta y)^2$ is zero but the second square $(\Delta y)^2$ is nonzero, then $f\big(1+\Delta x\,,\,1+\Delta y\big) = -2(\Delta y)^2 < 0 = f(1,1)$. That is, whenever $\Delta x = \Delta y \neq 0$, then $f\big(1+\Delta x\,,\,1+\Delta y\big) = -2(\Delta y)^2 < 0 = f(1,1)$.



So

- $f(x,y) > f(1,1)$ at all points on the blue line in the figure above, and
- $f(x,y) < f(1,1)$ at all point on the red line.

We conclude that $(1,1)$ is the only critical point for $f(x,y)$, and furthermore that it is a saddle point.

The above three examples show that we can find all critical points of quadratic functions of two variables. We can also classify each critical point as either a minimum, a maximum or a saddle point.

Of course not every function is quadratic. But by using the quadratic approximation 2.6.12 we can apply the same ideas much more generally. Suppose that (a,b) is a critical point of some function $f(x,y)$. For Δx and Δy small, the quadratic approximation 2.6.12 gives

$$\begin{aligned} &f\big(a+\Delta x\,,\,b+\Delta y\big)\\ &\quad\approx f\big(a\,,\,b\big)+f_x\big(a\,,\,b\big)\,\Delta x+f_y\big(a\,,\,b\big)\,\Delta y\\ &\qquad\quad+\frac{1}{2}\big\{f_{xx}\big(a,b\big)\,\Delta x^2+2f_{xy}\big(a,b\big)\,\Delta x\Delta y+f_{yy}\big(a,b\big)\,\Delta y^2\big\}\\ &\quad= f\big(a\,,\,b\big)+\frac{1}{2}\big\{f_{xx}\big(a,b\big)\,\Delta x^2+2f_{xy}\big(a,b\big)\,\Delta x\Delta y+f_{yy}\big(a,b\big)\,\Delta y^2\big\} \end{aligned}$$

since (a,b) is a critical point so that $f_x(a,b)=f_y(a,b)=0$. Then using the technique of Examples 2.9.12 and 2.9.15, we get [12] (details below).

Theorem 2.9.16. Second Derivative Test

Let $r>0$ and assume that all second order derivatives of the function $f(x,y)$ are continuous at all points (x,y) that are within a distance r of (a,b). Assume that $f_x(a,b)=f_y(a,b)=0$. Define

$$D(x,y)=f_{xx}(x,y)\,f_{yy}(x,y)-f_{xy}(x,y)^2$$

It is called the discriminant of f. Then

- if $D(a,b)>0$ and $f_{xx}(a,b)>0$, then $f(x,y)$ has a local minimum at (a,b),
- if $D(a,b)>0$ and $f_{xx}(a,b)<0$, then $f(x,y)$ has a local maximum at (a,b),
- if $D(a,b)<0$, then $f(x,y)$ has a saddle point at (a,b), but
- if $D(a,b)=0$, then we cannot draw any conclusions without more work.

Proof

We are putting quotation marks around the word "Proof", because we are not going to justify the fact that it suffices to analyse the quadratic approximation in equation $(*)$. Let's temporarily suppress the arguments (a,b). If $f_{xx}(a,b)\neq 0$, then by completing the square we can write

$$\begin{aligned} &f_{xx}\,\Delta x^2+2f_{xy}\,\Delta x\Delta y+f_{yy}\,\Delta y^2\\ &\qquad=f_{xx}\left(\Delta x+\frac{f_{xy}}{f_{xx}}\Delta y\right)^2+\left(f_{yy}-\frac{f_{xy}^2}{f_{xx}}\right)\,\Delta y^2\\ &\qquad=\frac{1}{f_{xx}}\big\{\big(f_{xx}\,\Delta x+f_{xy}\,\Delta y\big)^2+\big(f_{xx}f_{yy}-f_{xy}^2\big)\,\Delta y^2\big\} \end{aligned}$$

Similarly, if $f_{yy}(a,b)\neq 0$,

$$\begin{aligned} &f_{xx}\,\Delta x^2+2f_{xy}\,\Delta x\Delta y+f_{yy}\,\Delta y^2\\ &\qquad=\frac{1}{f_{yy}}\big\{\big(f_{xy}\,\Delta x+f_{yy}\,\Delta y\big)^2+\big(f_{xx}f_{yy}-f_{xy}^2\big)\,\Delta x^2\big\} \end{aligned}$$

Note that this algebra breaks down if $f_{xx}(a,b)=f_{yy}(a,b)=0$. We'll deal with that case shortly. More importantly, note that

- if $\big(f_{xx}f_{yy}-f_{xy}^2\big)>0$ then both f_{xx} and f_{yy} must be nonzero and of the same sign and furthermore, whenever Δx or Δy are nonzero,

$$\begin{aligned} \big\{\big(f_{xx}\,\Delta x+f_{xy}\,\Delta y\big)^2+\big(f_{xx}f_{yy}-f_{xy}^2\big)\,\Delta y^2\big\}&>0\quad\text{and}\\ \big\{\big(f_{xy}\,\Delta x+f_{yy}\,\Delta y\big)^2+\big(f_{xx}f_{yy}-f_{xy}^2\big)\,\Delta x^2\big\}&>0 \end{aligned}$$

so that, recalling $(*)$,

- if $f_{xx}(a,b) > 0$, then (a,b) is a local minimum and
- if $f_{xx}(a,b) < 0$, then (a,b) is a local maximum.

- If $\left(f_{xx}\,f_{yy} - f_{xy}^2\right) < 0$ and f_{xx} is nonzero then

$$\left\{\left(f_{xx}\,\Delta x + f_{xy}\,\Delta y\right)^2 + \left(f_{xx}\,f_{yy} - f_{xy}^2\right)\Delta y^2\right\}$$

is strictly positive whenever $\Delta x \neq 0$, $\Delta y = 0$ and is strictly negative whenever $f_{xx}\,\Delta x + f_{xy}\,\Delta y = 0$, $\Delta y \neq 0$, so that (a,b) is a saddle point. Similarly, (a,b) is also a saddle point if $\left(f_{xx}\,f_{yy} - f_{xy}^2\right) < 0$ and f_{yy} is nonzero.

- Finally, if $f_{xy} \neq 0$ and $f_{xx} = f_{yy} = 0$, then

$$f_{xx}\,\Delta x^2 + 2f_{xy}\,\Delta x\,\Delta y + f_{yy}\,\Delta y^2 = 2f_{xy}\,\Delta x\,\Delta y$$

is strictly positive for one sign of $\Delta x\,\Delta y$ and is strictly negative for the other sign of $\Delta x\,\Delta y$. So (a,b) is again a saddle point.

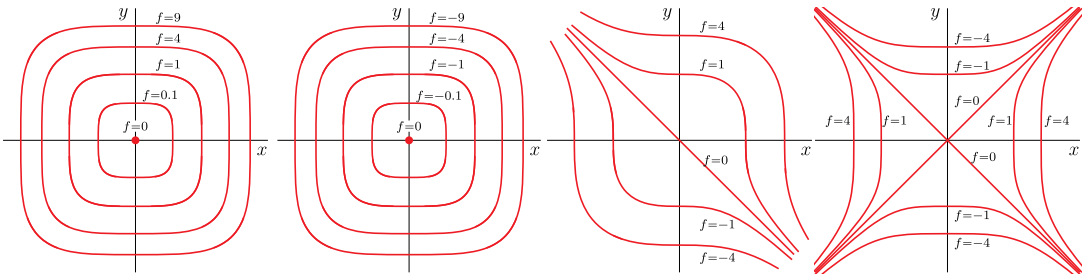
You might wonder why, in the local maximum/local minimum cases of Theorem 2.9.16, $f_{xx}(a,b)$ appears rather than $f_{yy}(a,b)$. The answer is only that x is before y in the alphabet[13]. You can use $f_{yy}(a,b)$ just as well as $f_{xx}(a,b)$. The reason is that if $D(a,b) > 0$ (as in the first two bullets of the theorem), then because $D(a,b) = f_{xx}(a,b)\,f_{yy}(a,b) - f_{xy}(a,b)^2 > 0$, we necessarily have $f_{xx}(a,b)\,f_{yy}(a,b) > 0$ so that $f_{xx}(a,b)$ and $f_{yy}(a,b)$ must have the same sign — either both are positive or both are negative.

You might also wonder why we cannot draw any conclusions when $D(a,b) = 0$ and what happens then. The second derivative test for functions of two variables was derived in precisely the same way as the second derivative test for functions of one variable is derived — you approximate the function by a polynomial that is of degree two in $(x-a)$, $(y-b)$ and then you analyze the behaviour of the quadratic polynomial near (a,b). For this to work, the contributions to $f(x,y)$ from terms that are of degree two in $(x-a)$, $(y-b)$ had better be bigger than the contributions to $f(x,y)$ from terms that are of degree three and higher in $(x-a)$, $(y-b)$ when $(x-a)$, $(y-b)$ are really small. If this is not the case, for example when the terms in $f(x,y)$ that are of degree two in $(x-a)$, $(y-b)$ all have coefficients that are exactly zero, the analysis will certainly break down. That's exactly what happens when $D(a,b) = 0$. Here are some examples. The functions

$$f_1(x,y) = x^4 + y^4 \qquad f_2(x,y) = -x^4 - y^4$$
$$f_3(x,y) = x^3 + y^3 \qquad f_4(x,y) = x^4 - y^4$$

all have $(0,0)$ as the only critical point and all have $D(0,0) = 0$. The first, f_1 has its minimum there. The second, f_2, has its maximum there. The third and fourth have a saddle point there.

Here are sketches of some level curves for each of these four functions (with all renamed to simply f).



level curves of $f(x,y) = x^4 + y^4$ level curves of $f(x,y) = -x^4 - y^4$ level curves of $f(x,y) = x^3 + y^3$ level curves of $f(x,y) = x^4 - y^4$

Example 2.9.17. $f(x,y) = 2x^3 - 6xy + y^2 + 4y$

Find and classify all critical points of $f(x,y) = 2x^3 - 6xy + y^2 + 4y$.

Solution

Thinking a little way ahead, to find the critical points we will need the gradient and to apply the second derivative test of Theorem 2.9.16 we will need all second order partial derivatives. So we need all partial derivatives of order up to two. Here they are.

$$
\begin{aligned}
f &= 2x^3 - 6xy + y^2 + 4y \\
f_x &= 6x^2 - 6y \qquad & f_{xx} &= 12x \qquad & f_{xy} &= -6 \\
f_y &= -6x + 2y + 4 \qquad & f_{yy} &= 2 \qquad & f_{yx} &= -6
\end{aligned}
$$

(Of course, f_{xy} and f_{yx} have to be the same. It is still useful to compute both, as a way to catch some mechanical errors.)

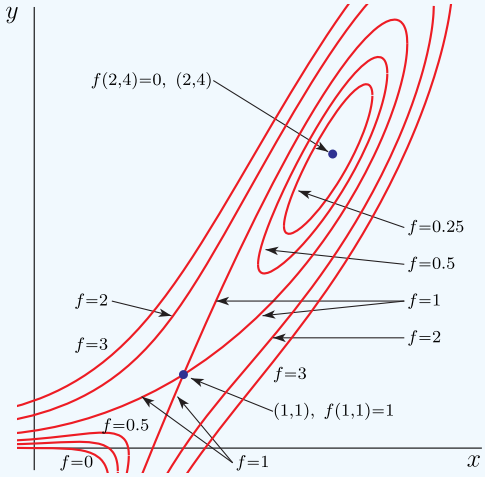
We have already found, in Example 2.9.7, that the critical points are $(1,1),\ (2,4)$. The classification is

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(1,1)$	$12 \times 2 - (-6)^2 < 0$		saddle point
$(2,4)$	$24 \times 2 - (-6)^2 > 0$	24	local min

We were able to leave the f_{xx} entry in the top row blank, because

- we knew that $f_{xx}(1,1)f_{yy}(1,1) - f_{xy}^2(1,1) < 0$, and
- we knew, from Theorem 2.9.16, that $f_{xx}(1,1)f_{yy}(1,1) - f_{xy}^2(1,1) < 0$, by itself, was enough to ensure that $(1,1)$ was a saddle point.

Here is a sketch of some level curves of our $f(x,y)$.



They are not needed to answer this question, but can give you some idea as to what the graph of f looks like.

Example 2.9.18. $f(x,y) = xy(5x + y - 15)$

Find and classify all critical points of $f(x,y) = xy(5x + y - 15)$.

Solution

We have already computed the first order partial derivatives

$$
f_x(x,y) = y(10x + y - 15) \qquad f_y(x,y) = x(5x + 2y - 15)
$$

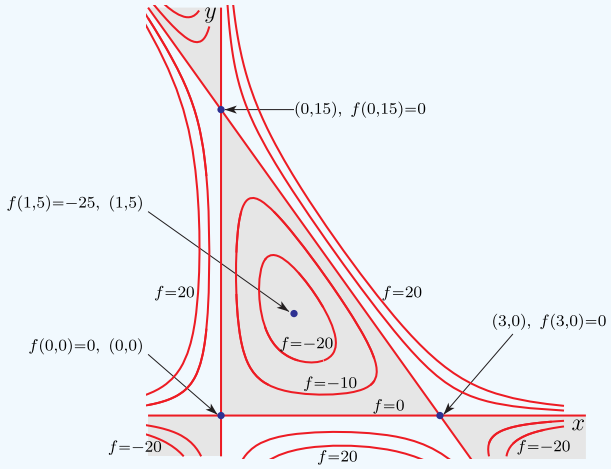
of $f(x,y)$ in Example 2.9.8. Again, to classify the critical points we need the second order partial derivatives. They are

$$
\begin{aligned}
f_{xx}(x,y) &= 10y \\
f_{yy}(x,y) &= 2x \\
f_{xy}(x,y) &= (1)(10x + y - 15) + y(1) = 10x + 2y - 15 \\
f_{yx}(x,y) &= (1)(5x + 2y - 15) + x(5) = 10x + 2y - 15
\end{aligned}
$$

(Once again, we have computed both f_{xy} and f_{yx} to guard against mechanical errors.) We have already found, in Example 2.9.8, that the critical points are $(0,0),\ (0,15),\ (3,0)$ and $(1,5)$. The classification is

critical point	$f_{xx}f_{yy}-f_{xy}^2$	f_{xx}	type
$(0,0)$	$0\times 0-(-15)^2<0$		saddle point
$(0,15)$	$150\times 0-15^2<0$		saddle point
$(3,0)$	$0\times 6-15^2<0$		saddle point
$(1,5)$	$50\times 2-5^2>0$	75	local min

Here is a sketch of some level curves of our $f(x,y)$. f is negative in the shaded regions and f is positive in the unshaded regions.



Again this is not needed to answer this question, but can give you some idea as to what the graph of f looks like.

Example 2.9.19

Find and classify all of the critical points of $f(x,y)=x^3+xy^2-3x^2-4y^2+4$.

Solution

We know the drill now. We start by computing all of the partial derivatives of f up to order 2.

$$\begin{aligned} f &= x^3+xy^2-3x^2-4y^2+4 \\ f_x &= 3x^2+y^2-6x \qquad & f_{xx} &= 6x-6 \qquad & f_{xy} &= 2y \\ f_y &= 2xy-8y & f_{yy} &= 2x-8 & f_{yx} &= 2y \end{aligned}$$

The critical points are then the solutions of $f_x=0,\ f_y=0$. That is

$$f_x = 3x^2+y^2-6x = 0 \tag{E1}$$
$$f_y = 2y(x-4) = 0 \tag{E2}$$

The second equation, $2y(x-4)=0$, is satisfied if and only if at least one of the two equations $y=0$ and $x=4$ is satisfied.

- When $y=0$, equation (E1) forces x to obey

$$0=3x^2+0^2-6x=3x(x-2)$$

so that $x=0$ or $x=2$.

- When $x=4$, equation (E1) forces y to obey

$$0=3\times 4^2+y^2-6\times 4=24+y^2$$

which is impossible.

So, there are two critical points: $(0,0),\ (2,0)$. Here is a table that classifies the critical points.

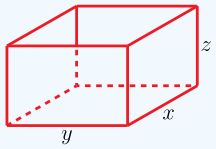
critical point	$f_{xx}f_{yy}-f_{xy}^2$	f_{xx}	type
$(0,0)$	$(-6)\times(-8)-0^2>0$	$-6<0$	local max
$(2,0)$	$6\times(-4)-0^2<0$		saddle point

Example 2.9.20

A manufacturer wishes to make an open rectangular box of given volume V using the least possible material. Find the design specifications.

Solution

Denote by x, y and z, the length, width and height, respectively, of the box.



The box has two sides of area xz, two sides of area yz and a bottom of area xy. So the total surface area of material used is

$$S=2xz+2yz+xy$$

However the three dimensions x, y and z are not independent. The requirement that the box have volume V imposes the constraint

$$xyz=V$$

We can use this constraint to eliminate one variable. Since z is at the end of the alphabet (poor z), we eliminate z by substituting $z=\frac{V}{xy}$. So we have find the values of x and y that minimize the function

$$S(x,y)=\frac{2V}{y}+\frac{2V}{x}+xy$$

Let's start by finding the critical points of S. Since

$$\begin{aligned} S_x(x,y) &= -\frac{2V}{x^2}+y \\ S_y(x,y) &= -\frac{2V}{y^2}+x \end{aligned}$$

(x,y) is a critical point if and only if

$$x^2y=2V \tag{E1}$$
$$xy^2=2V \tag{E2}$$

Solving (E1) for y gives $y=\frac{2V}{x^2}$. Substituting this into (E2) gives

$$x\frac{4V^2}{x^4}=2V \implies x^3=2V \implies x=\sqrt[3]{2V} \quad \text{and} \quad y=\frac{2V}{(2V)^{2/3}}=\sqrt[3]{2V}$$

As there is only one critical point, we would expect it to give the minimum [14]. But let's use the second derivative test to verify that at least the critical point is a local minimum. The various second partial derivatives are

$$S_{xx}(x,y)=\frac{4V}{x^3} \qquad S_{xx}\left(\sqrt[3]{2V}\,,\,\sqrt[3]{2V}\right)=2$$
$$S_{xy}(x,y)=1 \qquad S_{xy}\left(\sqrt[3]{2V}\,,\,\sqrt[3]{2V}\right)=1$$
$$S_{yy}(x,y)=\frac{4V}{y^3} \qquad S_{yy}\left(\sqrt[3]{2V}\,,\,\sqrt[3]{2V}\right)=2$$

So

$$S_{xx}\left(\sqrt[3]{2V}\,,\,\sqrt[3]{2V}\right)\,S_{yy}\left(\sqrt[3]{2V}\,,\,\sqrt[3]{2V}\right)-S_{xy}\left(\sqrt[3]{2V}\,,\,\sqrt[3]{2V}\right)^2=3>0$$
$$S_{xx}\left(\sqrt[3]{2V}\,,\,\sqrt[3]{2V}\right)=2>0$$

and, by Theorem 2.9.16.b, $\left(\sqrt[3]{2V}\,,\,\sqrt[3]{2V}\right)$ is a local minimum and the desired dimensions are

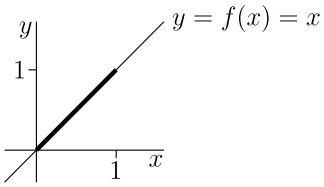
$$x=y=\sqrt[3]{2V} \qquad z=\sqrt[3]{\frac{V}{4}}$$

Note that our solution has $x=y$. That's a good thing — the function $S(x,y)$ is symmetric in x and y. Because the box has no top, the symmetry does not extend to z.

Absolute Minima and Maxima

Of course a local maximum or minimum of a function need not be the absolute maximum of minimum. We'll now consider how to find the absolute maximum and minimum. Let's start by reviewing how one finds the absolute maximum and minimum of a function of one variable on an interval.

For concreteness, let's suppose that we want to find the extremal[15] values of a function $f(x)$ on the interval $0\le x\le 1$. If an extremal value is attained at some $x=a$ which is in the interior of the interval, i.e. if $0<a<1$, then a is also a local maximum or minimum and so has to be a critical point of f. But if an extremal value is attained at a boundary point a of the interval, i.e. if $a=0$ or $a=1$, then a need not be a critical point of f. This happens, for example, when $f(x)=x$. The largest value of $f(x)$ on the interval $0\le x\le 1$ is 1 and is attained at $x=1$, but $f'(x)=1$ is never zero, so that f has no critical points.



So to find the maximum and minimum of the function $f(x)$ on the interval $[0,1]$, you

1. build up a list of all candidate points $0\le a\le 1$ at which the maximum or minimum could be attained, by finding all a's for which either
   1. $0<a<1$ and $f'(a)=0$ or
   2. $0<a<1$ and $f'(a)$ does not exist[16] or
   3. a is a boundary point, i.e. $a=0$ or $a=1$,
2. and then you evaluate $f(a)$ at each a on the list of candidates. The biggest of these candidate values of $f(a)$ is the absolute maximum and the smallest of these candidate values is the absolute minimum.

The procedure for finding the maximum and minimum of a function of two variables, $f(x,y)$ in a set like, for example, the unit disk $x^2+y^2\le 1$, is similar. You again

1. build up a list of all candidate points (a,b) in the set at which the maximum or minimum could be attained, by finding all (a,b)'s for which either[17]
   1. (a,b) is in the interior of the set (for our example, $a^2+b^2<1$) and $f_x(a,b)=f_y(a,b)=0$ or
   2. (a,b) is in the interior of the set and $f_x(a,b)$ or $f_y(a,b)$ does not exist or
   3. (a,b) is a boundary[18] point, (for our example, $a^2+b^2=1$), and could give the maximum or minimum on the boundary — more about this shortly —

2. and then you evaluate $f(a,b)$ at each (a,b) on the list of candidates. The biggest of these candidate values of $f(a,b)$ is the absolute maximum and the smallest of these candidate values is the absolute minimum.

The boundary of a set, like $x^2+y^2\le 1,$ in $\mathbb{R}^2$ is a curve, like $x^2+y^2=1.$ This curve is a one dimensional set, meaning that it is like a deformed x-axis. We can find the maximum and minimum of $f(x,y)$ on this curve by converting $f(x,y)$ into a function of one variable (on the curve) and using the standard function of one variable techniques. This is best explained by some examples.

Example 2.9.21

Find the maximum and minimum of $T(x,y)=(x+y)e^{-x^2-y^2}$ on the region defined by $x^2+y^2\le 1$ (i.e. on the unit disk).

Solution

Let's follow our checklist. First critical points, then points where the partial derivatives don't exist, and finally the boundary.

Interior Critical Points: If T takes its maximum or minimum value at a point in the interior, $x^2+y^2<1,$ then that point must be either a critical point of T or a singular point of T. To find the critical points we compute the first order derivatives.

$$T_x(x,y)=(1-2x^2-2xy)e^{-x^2-y^2}\qquad T_y(x,y)=(1-2xy-2y^2)e^{-x^2-y^2}$$

Because the exponential $e^{-x^2-y^2}$ is never zero, the critical points are the solutions of

$$\begin{aligned} T_x=0 &\iff 2x(x+y)=1\\ T_y=0 &\iff 2y(x+y)=1\end{aligned}$$

- As both $2x(x+y)$ and $2y(x+y)$ are nonzero, we may divide the two equations, which gives $\frac{x}{y}=1,$ forcing $x=y$.
- Substituting this into either equation gives $2x(2x)=1$ so that $x=y=\pm\frac12$.

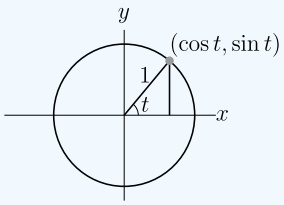
So the only critical points are $(\frac12,\frac12)$ and $(-\frac12,-\frac12)$. Both are in $x^2+y^2<1$.

Singular points: In this problem, there are no singular points.

Boundary: Points on the boundary satisfy $x^2+y^2=1.$ That is they lie on a circle. We may use the figure below to express $x=\cos t$ and $y=\sin t,$ in terms of the angle t. This will make the formula for T on the boundary quite a bit easier to deal with. On the boundary,

$$T=(\cos t+\sin t)e^{-\cos^2 t-\sin^2 t}=(\cos t+\sin t)e^{-1}$$

As all t's are allowed, this function takes its max and min at zeroes of



$$\frac{dT}{dt}=\big(-\sin t+\cos t\big)e^{-1}$$

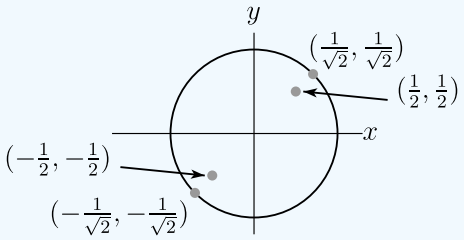
That is, $(\cos t+\sin t)e^{-1}$ takes its max and min

- when $\sin t=\cos t,$
- that is, when $x=y$ and $x^2+y^2=1,$
- which forces $x^2+x^2=1$ and hence $x=y=\pm\frac{1}{\sqrt2}$.

All together, we have the following candidates for max and min, with the max and min indicated.

point	$(\frac12,\frac12)$	$(-\frac12,-\frac12)$	$(\frac{1}{\sqrt2},\frac{1}{\sqrt2})$	$(-\frac{1}{\sqrt2},-\frac{1}{\sqrt2})$
value of T	$\frac{1}{\sqrt e}\approx 0.61$	$-\frac{1}{\sqrt e}$	$\frac{\sqrt2}{e}\approx 0.52$	$-\frac{\sqrt2}{e}$
	max	min		

The following sketch shows all of the critical points. It is a good idea to make such a sketch so that you don't accidentally include a critical point that is outside of the allowed region.



In the last example, we analyzed the behaviour of f on the boundary of the region of interest by using the parametrization $x = \cos t$, $y = \sin t$ of the circle $x^2 + y^2 = 1$. Sometimes using this parametrization is not so clean. And worse, some curves don't have such a simple parametrization. In the next problem we'll look at the boundary a little differently.

Example 2.9.22

Find the maximum and minimum values of $f(x, y) = x^3 + xy^2 - 3x^2 - 4y^2 + 4$ on the disk $x^2 + y^2 \le 1$.

Solution

Again, we first find all critical points, then find all singular points and, finally, analyze the boundary.

Interior Critical Points: If f takes its maximum or minimum value at a point in the interior, $x^2 + y^2 < 1$, then that point must be either a critical point of f or a singular point of f. To find the critical points [19] we compute the first order derivatives.

$$f_x = 3x^2 + y^2 - 6x \qquad f_y = 2xy - 8y$$

The critical points are the solutions of

$$f_x = 3x^2 + y^2 - 6x = 0 \tag{E1}$$
$$f_y = 2y(x - 4) = 0 \tag{E2}$$

The second equation, $2y(x-4) = 0$, is satisfied if and only if at least one of the two equations $y = 0$ and $x = 4$ is satisfied.

- When $y = 0$, equation (E1) forces x to obey

  $$0 = 3x^2 + 0^2 - 6x = 3x(x - 2)$$

  so that $x = 0$ or $x = 2$.
- When $x = 4$, equation (E1) forces y to obey

  $$0 = 3 \times 4^2 + y^2 - 6 \times 4 = 24 + y^2$$

  which is impossible.

So, there are only two critical points: $(0, 0)$, $(2, 0)$.

Singular points: In this problem, there are no singular points.

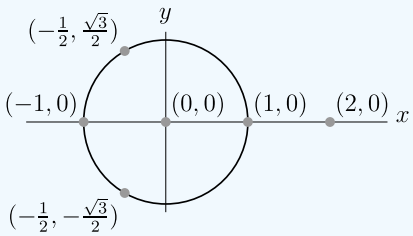
Boundary: On the boundary, $x^2 + y^2 = 1$, we could again take advantage of having a circle and write $x = \cos t$ and $y = \sin t$. But, for practice, we'll use another method [20]. We know that (x, y) satisfies $x^2 + y^2 = 1$, and hence $y^2 = 1 - x^2$. Examining the formula for $f(x, y)$, we see that it contains only even [21] powers of y, so we can eliminate y by substituting $y^2 = 1 - x^2$ into the formula.

$$f = x^3 + x(1 - x^2) - 3x^2 - 4(1 - x^2) + 4 = x + x^2$$

The max and min of $x + x^2$ for $-1 \le x \le 1$ must occur either

- when $x = -1$ ($\Rightarrow y = f = 0$) or
- when $x = +1$ ($\Rightarrow y = 0$, $f = 2$) or
- when $0 = \frac{d}{dx}(x + x^2) = 1 + 2x$ ($\Rightarrow x = -\frac{1}{2}$, $y = \pm\sqrt{\frac{3}{4}}$, $f = -\frac{1}{4}$).

Here is a sketch showing all of the points that we have identified.



Note that the point $(2,0)$ is outside the allowed region[22]. So all together, we have the following candidates for max and min, with the max and min indicated.

point	$(0,0)$	$(-1,0)$	$(1,0)$	$(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$
value of f	4	0	2	$-\frac{1}{4}$
	max			min

Example 2.9.23

Find the maximum and minimum values of $f(x,y) = xy - x^3y^2$ when (x,y) runs over the square $0 \le x \le 1,\ 0 \le y \le 1$.

Solution

As usual, let's examine the critical points, singular points and boundary in turn.

Interior Critical Points: If f takes its maximum or minimum value at a point in the interior, $0 < x < 1,\ 0 < y < 1$, then that point must be either a critical point of f or a singular point of f. To find the critical points we compute the first order derivatives.

$$f_x(x,y) = y - 3x^2y^2 \qquad f_y(x,y) = x - 2x^3y$$

The critical points are the solutions of

$$\begin{aligned} f_x = 0 &\iff y(1-3x^2y) = 0 \iff y = 0 \text{ or } 3x^2y = 1 \\ f_y = 0 &\iff x(1-2x^2y) = 0 \iff x = 0 \text{ or } 2x^2y = 1 \end{aligned}$$

- If $y = 0$, we cannot have $2x^2y = 1$, so we must have $x = 0$.
- If $3x^2y = 1$, we cannot have $x = 0$, so we must have $2x^2y = 1$. Dividing gives $1 = \frac{3x^2y}{2x^2y} = \frac{3}{2}$ which is impossible.

So the only critical point in the square is $(0,0)$. There $f = 0$.

Singular points: Yet again there are no singular points in this problem.

Boundary: The region is a square, so its boundary consists of its four sides.

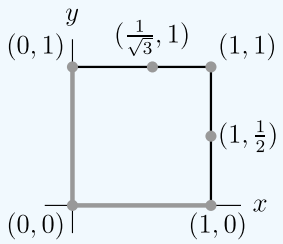
- First, we look at the part of the boundary with $x = 0$. On that entire side $f = 0$.
- Next, we look at the part of the boundary with $y = 0$. On that entire side $f = 0$.
- Next, we look at the part of the boundary with $y = 1$. There $f = f(x,1) = x - x^3$. To find the maximum and minimum of $f(x,y)$ on the part of the boundary with $y = 1$, we must find the maximum and minimum of $x - x^3$ when $0 \le x \le 1$.

  Recall that, in general, the maximum and minimum of a function $h(x)$ on the interval $a \le x \le b$, must occur either at $x = a$ or at $x = b$ or at an x for which either $h'(x) = 0$ or $h'(x)$ does not exist. In this case, $\frac{d}{dx}(x - x^3) = 1 - 3x^2$, so the max and min of $x - x^3$ for $0 \le x \le 1$ must occur

  - either at $x = 0$, where $f = 0$,
  - or at $x = \frac{1}{\sqrt{3}}$, where $f = \frac{2}{3\sqrt{3}}$,
  - or at $x = 1$, where $f = 0$.

- Finally, we look at the part of the boundary with $x = 1$. There $f = f(1, y) = y - y^2$. As $\frac{d}{dy}(y - y^2) = 1 - 2y$, the only critical point of $y - y^2$ is at $y = \frac{1}{2}$. So the the max and min of $y - y^2$ for $0 \le y \le 1$ must occur
  - either at $y = 0$, where $f = 0$,
  - or at $y = \frac{1}{2}$, where $f = \frac{1}{4}$,
  - or at $y = 1$, where $f = 0$.

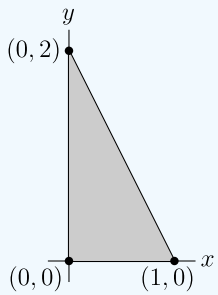
All together, we have the following candidates for max and min, with the max and min indicated.



point	$(0,0)$	$(0,0 \le y \le 1)$	$(0 \le x \le 1, 0)$	$(1,0)$	$(1, \frac{1}{2})$	$(1,1)$	$(0,1)$	$(\frac{1}{\sqrt{3}}, 1)$
value of f	0	0	0	0	$\frac{1}{4}$	0	0	$\frac{2}{3\sqrt{3}} \approx 0.385$
	min	min	min	min		min	min	max

Example 2.9.24

Find the maximum and minimum values of $f(x, y) = xy + 2x + y$ when (x, y) runs over the triangular region with vertices $(0, 0)$, $(1, 0)$ and $(0, 2)$. The triangular region is sketched in



Solution

As usual, let's examine the critical points, singular points and boundary in turn.

Interior Critical Points: If f takes its maximum or minimum value at a point in the interior, then that point must be either a critical point of f or a singular point of f. The critical points are the solutions of

$$f_x(x, y) = y + 2 = 0 \qquad f_y(x, y) = x + 1 = 0$$

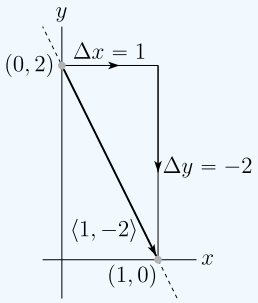
So there is exactly one critical point, namely $(-1, -2)$. This is well outside the triangle and so is not a candidate for the location of the max and min.

Singular points: Yet again there are no singular points for this f.

Boundary: The region is a triangle, so its boundary consists of its three sides.

- First, we look at the side that runs from $(0, 0)$ to $(0, 2)$. On that entire side $x = 0$, so that $f(0, y) = y$. The smallest value of f on that side is $f = 0$ at $(0, 0)$ and the largest value of f on that side is $f = 2$ at $(0, 2)$.
- Next, we look at the side that runs from $(0, 0)$ to $(1, 0)$. On that entire side $y = 0$, so that $f(x, 0) = 2x$. The smallest value of f on that side is $f = 0$ at $(0, 0)$ and the largest value of f on that side is $f = 2$ at $(1, 0)$.

- Finally, we look at the side that runs from $(0,2)$ to $(1,0)$. Or first job is to find the equation of the line that contains $(0,2)$ and $(1,0)$. By way of review, we'll find the equation using three different methods.
  - *Method 1:* You (probably) learned in high school that any line in the xy-plane[23] has equation $y = mx + b$ where b is the y intercept and m is the slope. In this case, the line crosses the y axis at $y = 2$ and so has y intercept $b = 2$. The line passes through $(0,2)$ and $(1,0)$ and so, as we see in the figure below, has slope $m = \frac{\Delta y}{\Delta x} = \frac{0-2}{1-0} = -2$. Thus the side of the triangle that runs from $(0,2)$ to $(1,0)$ is $y = 2 - 2x$ with $0 \le x \le 1$.



- *Method 2:* Every line in the xy-plane has an equation of the form $ax + by = c$. In this case $(0,0)$ is *not* on the line so that $c \neq 0$ and we can divide the equation by c, giving $\frac{a}{c}x + \frac{b}{c}y = 1$. Rename $\frac{a}{c} = A$ and $\frac{b}{c} = B$. Thus, because the line does not pass through the origin, it has an equation of the form $Ax + By = 1$, for some constants A and B. In order for $(0,2)$ to lie on the line, $x = 0$, $y = 2$ has to be a solution of $Ax + By = 1$. That is, $Ax\big|_{x=0} + By\big|_{y=2} = 1$, so that $B = \frac{1}{2}$. In order for $(1,0)$ to lie on the line, $x = 1$, $y = 0$ has to be a solution of $Ax + By = 1$. That is $Ax\big|_{x=1} + By\big|_{y=0} = 1$, so that $A = 1$. Thus the line has equation $x + \frac{1}{2}y = 1$, or equivalently, $y = 2 - 2x$.
- *Method 3:* The vector from $(0,2)$ to $(1,0)$ is $\langle 1-0\,,\ 0-2\rangle = \langle 1, -2\rangle$. As we see from the figure above, it is a direction vector for the line. One point on the line is $(0,2)$. So a parametric equation for the line (see Equation 1.3.1) is

$$\langle x-0\,,\ y-2\rangle = t\,\langle 1,-2\rangle \qquad \text{or} \qquad x = t,\ y = 2-2t$$

By any of these three methods[24], we have that the side of the triangle that runs from $(0,2)$ to $(1,0)$ is $y = 2 - 2x$ with $0 \le x \le 1$. On that side of the triangle

$$f(x, 2-2x) = x(2-2x) + 2x + (2-2x) = -2x^2 + 2x + 2$$

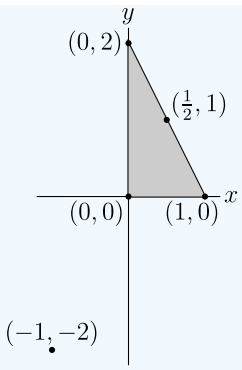
Write $g(x) = -2x^2 + 2x + 2$. The maximum and minimum of $g(x)$ for $0 \le x \le 1$, and hence the maximum and minimum values of f on the hypotenuse of the triangle, must be achieved either at

- $x = 0$, where $f(0,2) = g(0) = 2$, or at
- $x = 1$, where $f(1,0) = g(1) = 2$, or when
- $0 = g'(x) = -4x + 2$ so that $x = \frac{1}{2}$, $y = 2 - \frac{2}{2} = 1$ and

$$f(\frac{1}{2}, 1) = g(\frac{1}{2}) = -\frac{2}{4} + \frac{2}{2} + 2 = \frac{5}{2}$$

All together, we have the following candidates for max and min, with the max and min indicated.

point	$(0,0)$	$(0,2)$	$(1,0)$	$(\frac{1}{2},1)$
value of f	0	2	2	$\frac{5}{2}$
	min			max



Example 2.9.25

Find the high and low points of the surface $z = \sqrt{x^2+y^2}$ with (x, y) varying over the square $|x| \le 1$, $|y| \le 1$.

Solution

The function $f(x, y) = \sqrt{x^2+y^2}$ has a particularly simple geometric interpretation — it is the distance from the point (x, y) to the origin. So

- the minimum of $f(x, y)$ is achieved at the point in the square that is nearest the origin — namely the origin itself. So $(0, 0, 0)$ is the lowest point on the surface and is at height 0.
- The maximum of $f(x, y)$ is achieved at the points in the square that are farthest from the origin — namely the four corners of the square $(\pm 1, \pm 1)$. At those four points $z = \sqrt{2}$. So the highest points on the surface are $(\pm 1, \pm 1, \sqrt{2})$.

Even though we have already answered this question, it will be instructive to see what we would have found if we had followed our usual protocol. The partial derivatives of $f(x, y) = \sqrt{x^2+y^2}$ are defined for $(x, y) \ne (0, 0)$ and are

$$f_x(x, y) = \frac{x}{\sqrt{x^2+y^2}} \qquad f_y(x, y) = \frac{y}{\sqrt{x^2+y^2}}$$

- There are no critical points because
  - $f_x = 0$ only for $x = 0$, and
  - $f_y = 0$ only for $y = 0$, but
  - $(0, 0)$ is not a critical point because f_x and f_y are not defined there.
- There is one singular point — namely $(0, 0)$. The minimum value of f is achieved at the singular point.
- The boundary of the square consists of its four sides. One side is

  $$\{(x, y) | x = 1, \ -1 \le y \le 1\}$$

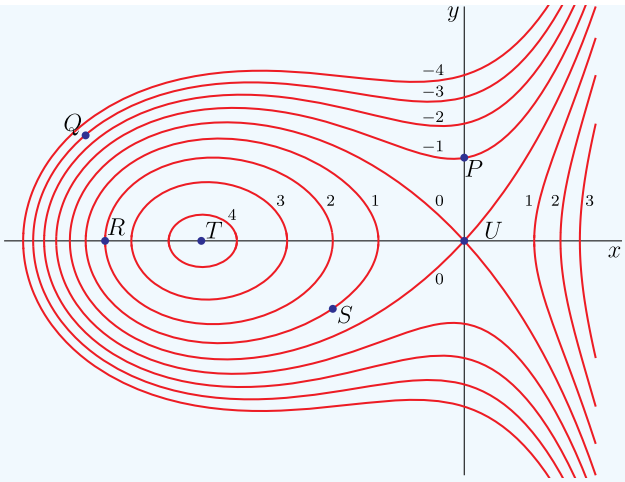
  On this side $f = \sqrt{1+y^2}$. As $\sqrt{1+y^2}$ increases with $|y|$, the smallest value of f on that side is 1 (when $y = 0$) and the largest value of f is $\sqrt{2}$ (when $y = \pm 1$). The same thing happens on the other three sides. The maximum value of f is achieved at the four corners. Note that f_x and f_y are both nonzero at all four corners.

Exercises

Stage 1

? 1 ✳

a. Some level curves of a function $f(x, y)$ are plotted in the xy--plane below.

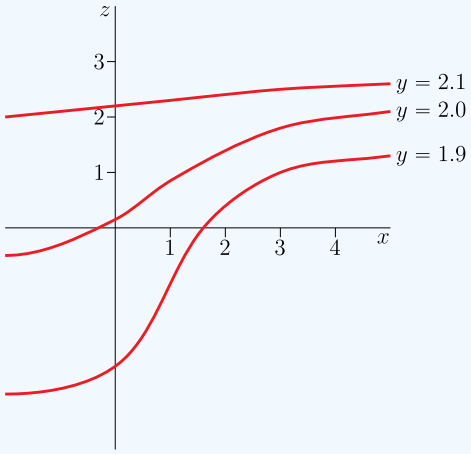


For each of the four statements below, circle the letters of *all* points in the diagram where the situation applies. For example, if the statement were "These points are on the y--axis", you would circle both P and U, but none of the other letters. You may assume that a local maximum occurs at point T.

(i) $\vec{n}abla f$ is zero	P R S T U
(ii) f has a saddle point	P R S T U
(iii) the partial derivative f_y is positive	P R S T U
(iv) the directional derivative of f in the direction $\langle 0, -1 \rangle$ is negative	P R S T U

b. The diagram below shows three "y traces" of a graph $z = F(x, y)$ plotted on xz--axes. (Namely the intersections of the surface $z = F(x, y)$ with the three planes ($y = 1.9$, $y = 2$, $y = 2.1$). For each statement below, circle the correct word.

(i) the first order partial derivative $F_x(1, 2)$ is	positive/negative/zero
(ii) F has a critical point at $(2, 2)$	true/false
(iii) the second order partial derivative $F_{xy}(1, 2)$ is	positive/negative/zero



? 2

Find the high and low points of the surface $z = \sqrt{x^2 + y^2}$ with (x, y) varying over the square $|x| \leq 1$, $|y| \leq 1$. Discuss the values of z_x, z_y there. Do not evaluate any derivatives in answering this question.

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? 3

If t_0 is a local minimum or maximum of the smooth function $f(t)$ of one variable (t runs over all real numbers) then $f'(t_0) = 0$. Derive an analogous necessary condition for $\vec{x}_0$ to be a local minimum or maximium of the smooth function $g(\vec{x})$ ***restricted*** to points on the line $\vec{x} = \vec{a} + t\vec{d}$. The test should involve the gradient of $g(\vec{x})$.

Stage 2

? 4 ✳

Let $z = f(x,y) = (y^2 - x^2)^2$.

1. Make a reasonably accurate sketch of the level curves in the xy--plane of $z = f(x,y)$ for $z = 0,\ 1$ and 16. Be sure to show the units on the coordinate axes.
2. Verify that $(0,0)$ is a critical point for $z = f(x,y)$, and determine from part (a) or directly from the formula for $f(x,y)$ whether $(0,0)$ is a local minimum, a local maximum or a saddle point.
3. Can you use the Second Derivative Test to determine whether the critical point $(0,0)$ is a local minimum, a local maximum or a saddle point? Give reasons for your answer.

? 5 ✳

Use the Second Derivative Test to find all values of the constant c for which the function $z = x^2 + cxy + y^2$ has a saddle point at $(0,0)$.

? 6 ✳

Find and classify all critical points of the function

$$f(x,y) = x^3 - y^3 - 2xy + 6.$$

? 7 ✳

Find all critical points for $f(x,y) = x(x^2 + xy + y^2 - 9)$. Also find out which of these points give local maximum values for $f(x,y)$, which give local minimum values, and which give saddle points.

? 8 ✳

Find the largest and smallest values of x^2y^2z in the part of the plane $2x + y + z = 5$ where $x \geq 0$, $y \geq 0$ and $z \geq 0$. Also find all points where those extreme values occur.

? 9

Find and classify all the critical points of $f(x,y) = x^2 + y^2 + x^2y + 4$.

? 10 ✳

Find all saddle points, local minima and local maxima of the function

$$f(x,y) = x^3 + x^2 - 2xy + y^2 - x.$$

? 11 ✳

For the surface

$$z = f(x,y) = x^3 + xy^2 - 3x^2 - 4y^2 + 4$$

Find and classify [as local maxima, local minima, or saddle points] all critical points of $f(x,y)$.

? 12

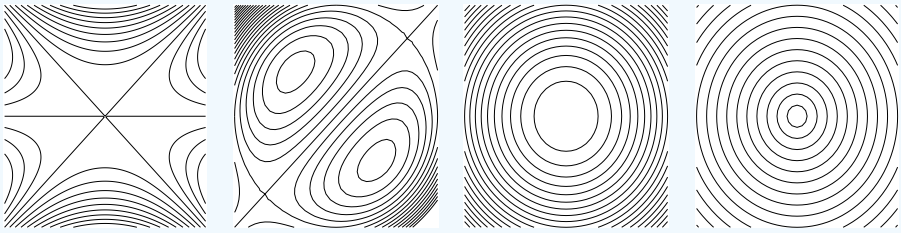
Find the maximum and minimum values of $f(x,y) = xy - x^3y^2$ when (x,y) runs over the square $0 \le x \le 1,\ 0 \le y \le 1$.

? 13

The temperature at all points in the disc $x^2 + y^2 \le 1$ is given by $T(x,y) = (x+y)e^{-x^2-y^2}$. Find the maximum and minimum temperatures at points of the disc.

? 14 ✳

1. For the function $z = f(x,y) = x^3 + 3xy + 3y^2 - 6x - 3y - 6$. Find and classify as [local maxima, local minima, or saddle points] all critical points of $f(x,y)$.
2. The images below depict level sets $f(x,y) = c$ of the functions in the list at heights $c = 0, 0.1, 0.2, \ldots, 1.9, 2$.Label the pictures with the corresponding function and mark the critical points in each picture. (Note that in some cases, the critical points might not be drawn on the images already. In those cases you should add them to the picture.)
    1. $f(x,y) = (x^2 + y^2 - 1)(x - y) + 1$
    2. $f(x,y) = \sqrt{x^2 + y^2}$
    3. $f(x,y) = y(x+y)(x-y) + 1$
    4. $f(x,y) = x^2 + y^2$



? 15 ✳

Let the function

$$f(x,y) = x^3 + 3xy + 3y^2 - 6x - 3y - 6$$

Classify as $\left[\text{local maxima, minima or saddle points}\right]$ all critical points of $f(x,y)$.

? 16 ✳

Let $h(x,y) = y(4 - x^2 - y^2)$.

1. Find and classify the critical points of $h(x,y)$ as local maxima, local minima or saddle points.
2. Find the maximum and minimum values of $h(x,y)$ on the disk $x^2 + y^2 \le 1$.

? 17 ✳

Find the absolute maximum and minimum values of the function $f(x,y) = 5 + 2x - x^2 - 4y^2$ on the rectangular region

$$R = \{(x,y) | -1 \le x \le 3,\ -1 \le y \le 1\}$$

? 18 ✳

Find the minimum of the function $h(x,y) = -4x - 2y + 6$ on the closed bounded domain defined by $x^2 + y^2 \leq 1$.

? 19 ✳

Let $f(x,y) = xy(x+y-3)$.

1. Find all critical points of f, and classify each one as a local maximum, a local minimum, or saddle point.
2. Find the location and value of the absolute maximum and minimum of f on the triangular region $x \geq 0$, $y \geq 0$, $x + y \leq 8$.

? 20 ✳

Find and classify the critical points of $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 4$.

? 21 ✳

Consider the function

$$f(x,y) = 2x^3 - 6xy + y^2 + 4y$$

1. Find and classify all of the critical points of $f(x,y)$.
2. Find the maximum and minimum values of $f(x,y)$ in the triangle with vertices $(1,0)$, $(0,1)$ and $(1,1)$.

? 22 ✳

Find all critical points of the function $f(x,y) = x^4 + y^4 - 4xy + 2$, and for each determine whether it is a local minimum, maximum or saddle point.

? 23 ✳

Let

$$f(x,y) = xy(x + 2y - 6)$$

1. Find every critical point of $f(x,y)$ and classify each one.
2. Let D be the region in the plane between the hyperbola $xy = 4$ and the line $x + 2y - 6 = 0$. Find the maximum and minimum values of $f(x,y)$ on D.

? 24 ✳

Find all the critical points of the function

$$f(x,y) = x^4 + y^4 - 4xy$$

defined in the xy-plane. Classify each critical point as a local minimum, maximum or saddle point.

? 25 ✳

A metal plate is in the form of a semi-circular disc bounded by the x-axis and the upper half of $x^2 + y^2 = 4$. The temperature at the point (x,y) is given by $T(x,y) = \ln\left(1 + x^2 + y^2\right) - y$. Find the coldest point on the plate, explaining your steps carefully. (Note: $\ln 2 \approx 0.693, \ln 5 \approx 1.609$)

? 26 ✳

Find all the critical points of the function

$$f(x,y) = x^3 + xy^2 - x$$

defined in the xy-plane. Classify each critical point as a local minimum, maximum or saddle point. Explain your reasoning.

? 27 ✳

Consider the function $g(x,y) = x^2 - 10y - y^2$.

1. Find and classify all critical points of g.
2. Find the absolute extrema of g on the bounded region given by

$$x^2 + 4y^2 \leq 16,\ y \leq 0$$

? 28 ✳

Find and classify all critical points of

$$f(x,y) = x^3 - 3xy^2 - 3x^2 - 3y^2$$

? 29 ✳

Find the maximum value of

$$f(x,y) = xye^{-(x^2+y^2)/2}$$

on the quarter-circle $D = \left\{(x,y) | x^2 + y^2 \leq 4,\ x \geq 0,\ y \geq 0\right\}$.

? 30

Equal angle bends are made at equal distances from the two ends of a 100 metre long fence, so that the resulting three segment fence can be placed along an existing wall to make an enclosure of trapezoidal shape. What is the largest possible area for such an enclosure?

? 31

Find the most economical shape of a rectangular box that has a fixed volume V and that has no top.

Stage 3

? 32 ✳

The temperature $T(x,y)$ at a point of the xy--plane is given by

$$T(x,y) = 20 - 4x^2 - y^2$$

1. Find the maximum and minimum values of $T(x,y)$ on the disk D defined by $x^2 + y^2 \leq 4$.
2. Suppose an ant lives on the disk D. If the ant is initially at point $(1,1)$, in which direction should it move so as to increase its temperature as quickly as possible?
3. Suppose that the ant moves at a velocity $\vec{v} = \langle -2, -1 \rangle$. What is its rate of increase of temperature as it passes through $(1,1)$?
4. Suppose the ant is constrained to stay on the curve $y = 2 - x^2$. Where should the ant go if it wants to be as warm as possible?

? 33 ✳

Consider the function

$$f(x,y) = 3kx^2y + y^3 - 3x^2 - 3y^2 + 4$$

where $k > 0$ is a constant. Find and classify all critical points of $f(x,y)$ as local minima, local maxima, saddle points or points of indeterminate type. Carefully distinguish the cases $k < \frac{1}{2}$, $k = \frac{1}{2}$ and $k > \frac{1}{2}$.

? 34 ✳

1. Show that the function $f(x,y) = 2x + 4y + \frac{1}{xy}$ has exactly one critical point in the first quadrant $x > 0$, $y > 0$, and find its value at that point.
2. Use the second derivative test to classify the critical point in part (a).
3. Hence explain why the inequality $2x + 4y + \frac{1}{xy} \geq 6$ is valid for all positive real numbers x and y.

? 35

An experiment yields data points (x_i, y_i), $i = 1, 2, \cdots, n$. We wish to find the straight line $y = mx + b$ which "best" fits the data. The definition of "best" is "minimizes the root mean square error", i.e. minimizes $\sum_{i=1}^{n} (mx_i + b - y_i)^2$. Find m and b.

1. Life is not (always) one-dimensional and sometimes we have to embrace it.
2. Or perhaps your instructor asked you.
3. Recall that if $h < 0$ and $A \leq B$, then $hA \geq hB$. This is because the product of any two negative numbers is positive, so that $h < 0,\ A \leq B \implies A - B \leq 0 \implies h(A-B) \geq 0 \implies hA \geq hB$.
4. A very common error of logic that people make is "Affirming the consequent". "If P then Q" is true, does not imply that "If Q then P" is true . The statement "If he is Shakespeare then he is dead" is true. But concluding from "That man is dead" that "He must be Shakespeare" is just silly.
5. And you also saw, for example in Example 3.6.4 of the CLP-1 text, that critical points that are also inflection points are neither local maxima nor local minima.
6. We have both types of music here — country and western.
7. This sort of thing is generally illegal.
8. Sorry about the pun.
9. Proof by search engine.
10. And has been used for a long time. It was introduced by the French mathematician Adrien-Marie Legendre, 1752--1833, in 1805, and by the German mathematician and physicist Carl Friedrich Gauss, 1777--1855, in 1809.
11. This is equivalent to translating the graph so that the critical point lies at $(0,0)$.
12. There are analogous results in higher dimensions that are accessible to people who have learned some linear algebra. They are derived by diagonalizing the matrix of second derivatives, which is called the Hessian matrix.
13. The shackles of convention are not limited to mathematics. Election ballots often have the candidates listed in alphabetic order.
14. Indeed one can use the facts that $0 < x < \infty$, that $0 < y < \infty$, and that $S \to \infty$ as $x \to 0$ and as $y \to 0$ and as $x \to \infty$ and as $y \to \infty$ to prove that the single critical point gives the global minimum.
15. Recall that "extremal value" means "either maximum value or minimum value".
16. Recall that if $f'(a)$ does not exist, then a is called a singular point of f.
17. This is probably a good time to review the statement of Theorem 2.9.2.
18. It should intuitively obvious from a sketch that the boundary of the disk $x^2 + y^2 \leq 1$ is the circle $x^2 + y^2 = 1$. But if you really need a formal definition, here it is. A point (a,b) is on the boundary of a set S if there is a sequence of points in S that converges to (a,b) and there is also a sequence of points in the complement of S that converges to (a,b).
19. We actually found the critical points in Example 2.9.19. But, for the convenience of the reader, we'll repeat that here.
20. Even if you don't believe that "you can't have too many tools", it is pretty dangerous to have to rely on just one tool.
21. If it contained odd powers too, we could consider the cases $y \geq 0$ and $y \leq 0$ separately and substitute $y = \sqrt{1 - x^2}$ in the former case and $y = -\sqrt{1 - x^2}$ in the latter case.

22. We found $(2, 0)$ as a solution to the critical point equations (E1), (E2). That's because, in the course of solving those equations, we ignored the constraint that $x^2 + y^2 \leq 1$.
23. To be picky, any line the xy-plane that is not parallel to the y axis.
24. In the third method, x has just be renamed to t.

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5.5: Optimization

The partial derivatives tell us something about where a surface has local maxima and minima. Remember that even in the one-variable cases, there were critical points which were neither maxima nor minima – this is also true for functions of many variables. In fact, as you might expect, the situation is even more complicated.

Second Derivatives

When you find a partial derivative of a function of two variables, you get another function of two variables – you can take its partial derivatives, too. We've done this before, in the one-variable setting. In the one-variable setting, the second derivative gave information about how the graph was curved. In the two-variable setting, the second partial derivatives give some information about how the surface is curved, as you travel on cross-sections – but that's not very complete information about the entire surface.

Imagine that you have a surface that's ruffled around a point, like what happens near a button on an overstuffed sofa, or a pinched piece of fabric, or the wrinkly skin near your thumb when you make a fist. Right at that point, every direction you move, something different will happen – it might increase, decrease, curve up, curve down… A simple phrase like “concave up” or “concave down” can't describe all the things that can happen on a surface.

Surprisingly enough, though, there is still a second derivative test that can help you decide if a point is a local max or min or neither, so we still do want to find second derivatives.

Second Partial Derivatives

Suppose $f(x,y)$ is a function of two variables. Then it has four **second partial derivatives**:

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = (f_x)_x \qquad f_{xy} = \frac{\partial}{\partial y}(f_x) = (f_x)_y$$
$$f_{yx} = \frac{\partial}{\partial x}(f_y) = (f_y)_x \qquad f_{yy} = \frac{\partial}{\partial y}(f_y) = (f_y)_y$$

f_{xy} and f_{yx} are called the **mixed (second) partial derivatives of f**.

Leibniz notation for the second partial derivatives is a bit confusing, and we won't use it as often:

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \qquad f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$
$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} \qquad f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Notice that the order of the variables for the mixed partials goes from right to left in the Leibniz notation instead of left to right.

Example 5.5.1

Find all four partial derivatives of $f(x,y) = x^2 - 4xy + 4y^2$.

Solution

We have to start by finding the (first) partial derivatives:

$$f_x(x,y) = 2x - 4y$$
$$f_y(x,y) = -4x + 8y$$

Now we’re ready to take the second partial derivatives:

$$
\begin{aligned}
f_{xx}(x,y) &= \frac{\partial}{\partial x}(2x-4y) = 2 \\
f_{xy}(x,y) &= \frac{\partial}{\partial y}(2x-4y) = -4 \\
f_{yx}(x,y) &= \frac{\partial}{\partial x}(-4x+8y) = -4 \\
f_{yy}(x,y) &= \frac{\partial}{\partial y}(-4x+8y) = 8
\end{aligned}
$$

You might have noticed that the two mixed partial derivatives were equal in this last example. It turns out that it's not a coincidence – it's a theorem!

Mixed Partial Derivative Theorem

If f, f_x, f_y, f_{xy}, and f_{yx} are all continuous (no breaks in their graphs), then

$$f_{xy} = f_{yx}.$$

In fact, as long as f and all its appropriate partial derivatives are continuous, the mixed partials are equal even if they are of higher order, and even if the function has more than two variables.

This theorem means that the confusing Leibniz notation for second derivatives is not a big problem – in almost every situation the mixed partials are equal, so the order in which we compute them doesn't matter.

Example 5.5.2

Find $\frac{\partial^2 f}{\partial x \partial y}$ for $f(x,y) = \frac{e^{x+y}}{y^3+y} + y\ln(y)$.

Solution

We already found the first partial derivatives in an earlier example:

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{1}{y^3+y}e^{x+y} \\
\frac{\partial f}{\partial y} &= \frac{\left(e^{x+y}(1)\right)\left(y^3+y\right) - \left(e^{x+y}\right)\left(3y^2+1\right)}{\left(y^3+y\right)^2} + (1)\left(\ln(y)\right) + (y)\left(\frac{1}{y}\right)
\end{aligned}
$$

Now we need to find the mixed partial derivative. The theorem says that $\frac{\partial f^2}{\partial x \partial y} = \frac{\partial f^2}{\partial y \partial x}$, so it doesn't matter whether we find the partial derivative of $\frac{\partial f}{\partial x}$ with respect to y or the partial derivative of $\frac{\partial f}{\partial y}$ with respect to x. Which would you rather do?

It looks like it will be easier to compute the mixed partial by finding the partial derivative of $\frac{\partial f}{\partial x} = \frac{1}{y^3+y}e^{x+y}$ with respect to y – it still looks messy, but it looks less messy:

$$\frac{\partial f^2}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{1}{y^3+y}e^{x+y}\right) = \frac{\left(e^{x+y}\right)\left(y^3+y\right) - \left(e^{x+y}\right)\left(3y^2+1\right)}{\left(y^3+y\right)^2}$$

If we had decided to do this the other way, we'd end up in the same place. Eventually…

Local Maxima, Local Minima, and Saddle Points

Let's briefly review max-min problems in one variable.

A local max is a point on a curve that is higher than all the nearby points. A local min is lower than all the nearby points. We know that local max or min can only occur at critical points, where the derivative is zero or undefined. But we also know that not all critical points are max or min, so we also need to test them, with the First Derivative or Second Derivative Test.

The situation with a function of two variables is much the same. Just as in the one-variable case, the first step is to find critical points, places where both the partial derivatives are either zero or undefined.

Definition: Local Maximum and Minimum

- f has a **local maximum** at (a, b) if $f(a, b) \geq f(x, y)$ for all points (x, y) near (a, b).
- f has a **local minimum** at (a, b) if $f(a, b) \leq f(x, y)$ for all points (x, y) near (a, b).

A **critical point** for a function $f(x, y)$ is a point (x, y) (or $(x, y, f(x, y))$) where both the following are true:

1. $f_x = 0$ or is undefined, **and**
2. $f_y = 0$ or is undefined.

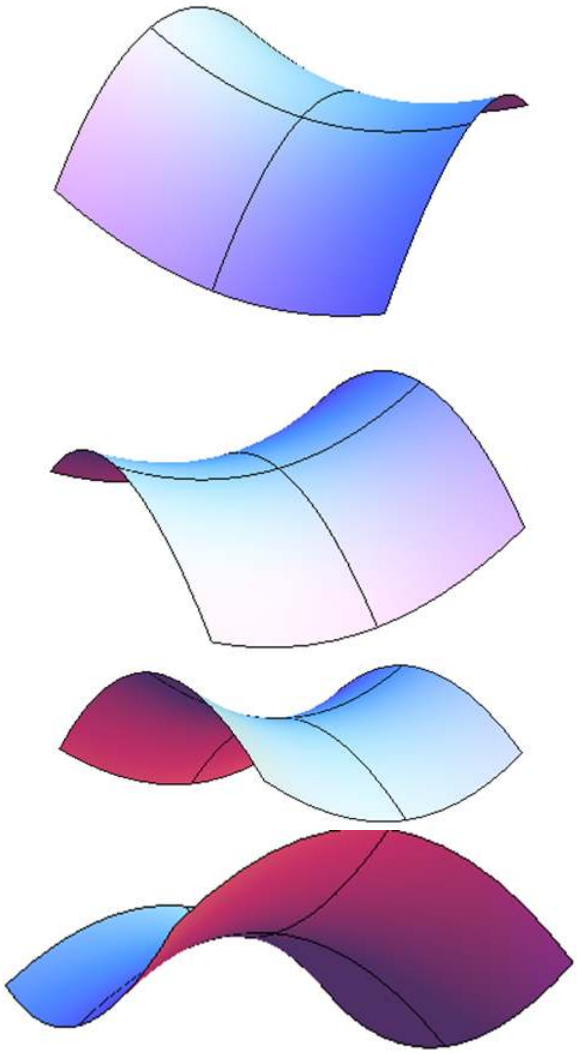
Just as in the one-variable case, a local max or min of f can only occur at a critical point.

Just as in the one-variable setting, not all critical points are local max or min. For a function of two variables, the critical point could be a local max, local min, or a saddle point.

A point on a surface is a local maximum if it's higher than all the points nearby; a point is a local minimum if it's lower than all the points nearby.

A saddle point is a point on a surface that is a minimum along some paths and a maximum along some others. It's called this because it's shaped a bit like a saddle you might use to ride a horse. You can see a saddle point by making a fist – between the knuckles of your index and middle fingers, you can see a place that is a minimum as you go across your knuckles, but a maximum as you go along your hand toward your fingers.

Here is a picture of a saddle point from a few different angles. This is the surface $f(x, y) = 5x^2 - 3y^2 + 10$, and there is a saddle point above the origin. The lines show what the surface looks like above the x- and y- axes. Notice how the point above the origin, where the lines cross, is a local minimum in one direction, but a local maximum in the other direction.



Second Derivative Test

Just as in the one-variable case, we'll need a way to test critical points to see whether they are local max or min. There is a second derivative test for functions of two variables that can help, but, just as in the one-variable case, it won't always give an answer.

The Second Derivative Test for Functions of Two Variables

1. Find all critical points of $f(x, y)$.
2. Compute

$$D(x, y) = (f_{xx})(f_{yy}) - (f_{xy})(f_{yx}),$$

and evaluate it at each critical point.

1. If $D > 0$, then f has a local max or min at the critical point. To see which, look at the sign of f_{xx}:
   - If $f_{xx} > 0$, then f has a local minimum at the critical point.
   - If $f_{xx} < 0$, then f has a local maximum at the critical point.
2. If $D < 0$ then f has a saddle point at the critical point.
3. If $D = 0$, there could be a local max, local min, or neither (i.e., the test in inconclusive).

Example 5.5.3

Find all local maxima, minima, and saddle points for the function

$$f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8.$$

Solution

First we need the partial derivatives: $f_x = 3x^2 + 6x$ and $f_y = 3y^2 - 6y$.

Critical points are the places where both of these are zero (neither is ever undefined): $f_x = 3x^2 + 6x = 3x(x + 2) = 0$ when $x = 0$ or when $x = -2$. $f_y = 3y^2 - 6y = 3y(y - 2) = 0$ when $y = 0$ or when $y = 2$.

Putting these together, we get four critical points: (0, 0), (-2, 0), (0, 2), and (-2, 2).

Now to classify them, we'll use the Second Derivative Test. We'll need all the second partial derivatives:

$$f_{xx} = 6x + 6, \; f_{yy} = 6y - 6, \; f_{xy} = f_{yx} = 0.$$

Then

$$D(x, y) = (6x + 6)(6y - 6) - (0)(0) = (6x + 6)(6y - 6).$$

Now look at each critical point in turn:

- At (0, 0): $D(0, 0) = (6(0) + 6)(6(0) - 6) = (6)(-6) = -36 < 0$, so there is a saddle point at (0, 0).
- At (-2, 0): $D(-2, 0) = (6(-2) + 6)(6(0) - 6) = (-6)(-6) = 36 > 0$ and $f_{xx}(-2, 0) = 6(-2) + 6 = -6 < 0$, so there is a local maximum at (-2, 0).
- At (0, 2): $D(0, 2) = (6(0) + 6)(6(2) - 6) = (6)(6) = 36 > 0$ and $f_{xx}(0, 2) = 6(0) + 6 = 6 > 0$, so there is a local minimum at (0, 2).
- At (-2, 2): $D(-2, 2) = (6(-2) + 6)(6(2) - 6) = (-6)(6) = -36 < 0$, so there is another saddle point at (-2, 2).

Example 5.5.4

Find all local maxima, minima, and saddle points for the function

$$z = 9x^3 + \frac{y^3}{3} - 4xy.$$

Solution

We'll need all the partial derivatives and second partial derivatives, so let's compute them all first:

$$\begin{aligned} z_x &= 27x^2 - 4y, \quad z_y = y^2 - 4x, \\ z_{xx} &= 54x, \quad z_{xy} = z_{yx} = -4, \quad z_y = 2y. \end{aligned}$$

Now to find the critical points: We need both z_x and z_y to be zero (neither is ever undefined), so we need to solve this set of equations simultaneously:

$$\begin{aligned} z_x &= 27x^2 - 4y = 0 \\ z_y &= y^2 - 4x = 0 \end{aligned}$$

Perhaps it's been a while since you solved systems of equations. One solution method is the substitution method – solve one equation for one variable and substitute into the other equation:

$$\left.\begin{aligned} 27x^2 - 4y &= 0 \\ y^2 - 4x &= 0 \end{aligned}\right\} \rightarrow \text{Solve } y^2 - 4x = 0 \text{ for } x = \frac{y^2}{4} \ldots$$

…then substitute into the other equation:

$$\begin{aligned} 27\left(\frac{y^2}{4}\right)^2 - 4y &= 0 \\ \frac{27}{16}y^4 - y &= 0 \end{aligned}$$

Now we have just one equation in one variable to solve. Factoring out a y gives

$$y\left(\frac{27}{16}y^3 - 1\right) = 0,$$

so $y = 0$ or $\frac{27}{16}y^3 - 1 = 0$, giving $y = \sqrt[3]{\frac{1}{27/16}} = \frac{\sqrt[3]{4}}{3}$.

Plugging back in to the equation $x = \frac{y^2}{4}$ to find x gives us the two critical points: (0,0) and $\left(\frac{4}{9}, \frac{4}{3}\right)$.

Now to test them. First compute

$$\begin{aligned} D(x,y) &= (f_{xx})(f_{yy}) - (f_{xy})(f_{yx}) \\ &= (54x)(2y) - (-4)(-4) \\ &= 108xy - 16 \end{aligned}$$

Then evaluate D at the two critical points:

- At (0,0): $D(0,0) = -16 < 0$, so there is a saddle point at (0, 0).
- At $\left(\frac{4}{9}, \frac{\sqrt[3]{4}}{3}\right)$: $D\left(\frac{4}{9}, \frac{\sqrt[3]{4}}{3}\right) = 16(\sqrt[3]{4} - 1) > 0$, and $f_{xx}\left(\frac{4}{9}, \frac{\sqrt[3]{4}}{3}\right) > 0$, so there is a local minimum at $\left(\frac{4}{9}, \frac{\sqrt[3]{4}}{3}\right)$.

Applied Optimization

Example 5.5.5

A company makes two products. The demand equations for the two products are given below. p_1, p_2, q_1,and q_2 are the prices and quantities for Products 1 and 2.

$$\begin{aligned} q_1 &= 200 - 3p_1 - p_2 \\ q_2 &= 150 - p_1 - 2p_2 \end{aligned}$$

Solution

Find the price the company should charge for each product in order to maximize total revenue. What is that maximum revenue?

Revenue is still price·quantity. If we're selling two products, the total revenue will be the sum of the revenues from the two products:

$$\begin{aligned} R(p_1, p_2) &= p_1 q_1 + p_2 q_2 \\ &= p_1(200 - 3p_1 - p_2) + p_2(150 - p_1 - 2p_2) \\ &= 200p_1 - 3p_1^2 - 2p_1 p_2 + 150p_2 - 2p_2^2 \end{aligned}$$

This is a function of two variables, the two prices, and we need to optimize it (just as in the previous examples). First we find critical points. The notation here gets a bit hard to look at, but hang in there – this is the same stuff we've done before.

$$R_{p_1} = 200 - 6p_1 - 2p_2 \text{ and } R_{p_2} = 150 - 2p_1 - 4p_2.$$

Solving these simultaneously gives the one critical point $(p_1, p_2) = (25, 25)$

To confirm that this gives maximum revenue, we need to use the Second Derivative Test. Find all the second derivatives:

$$R_{p_1 p_2} = -6,\ R_{p_2 p_2} = -4, \text{ and } R_{p_1 p_2} = R_{p_2 p_1} = -2.$$

So $D(25, 25) = (-6)(-4) - (-2)(-2) > 0$ and $R_{p_1 p_2}(25, 25) < 0$, so this really is a local maximum.

Thus, to maximize revenue the company should charge \$25 per unit for both products. This will yield a maximum revenue of \$4375.

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5.6: Critical Points and Extrema

With functions of one variable we were interested in places where the derivative is zero, since they made candidate points for the maximum or minimum of a function. If the derivative is not zero, we have a direction that is downhill and moving a little in that direction gives a lower value of the function. Similarly, with functions of two variables we can only find a minimum or maximum for a function if both partial derivatives are 0 at the same time. Such points are called critical points.

The point (a,b) is a ***critical point*** for the multivariable function $f(x,y)$, if both partial derivatives are 0 at the same time.

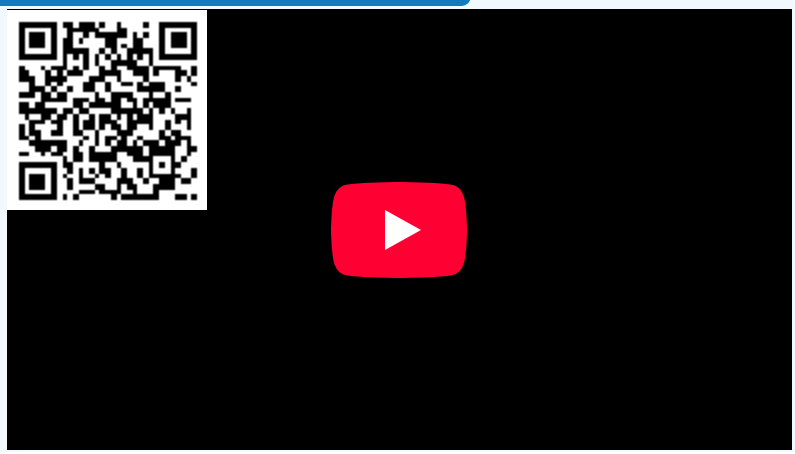
In other words

$$\frac{\partial}{\partial x} f(x,y)|_{x=a,y=b} = 0$$

and

$$\frac{\partial}{\partial y} f(x,y)|_{x=a,y=b} = 0$$

Example 6.3.1: Finding a Local Minimum of a Function.



Use the partial derivatives of $f(x,y) = x^2 + 2xy + 3y^2 - 4x - 3y$ to find the minimum of the graph.

Solution

Critical Point by Algebra: In the previous section, we already computed

$$\frac{\partial}{\partial x} f(x,y) = 2x + 2y - 4$$

$$\frac{\partial}{\partial y} f(x,y) = 2x + 6y - 3.$$

We need to find the places where both partial derivatives are 0. With this simple system, I can solve this system algebraically and find the only critical point is $(9/4, -1/4)$.

$$0 = 2x + 2y - 4$$

$$0 = 2x + 6y - 3$$

Subtract the equations to eliminate x.

$$0 = 0 - 4y - 1$$

Solve for y.

$$4y = -1$$

$$y = -1/4$$

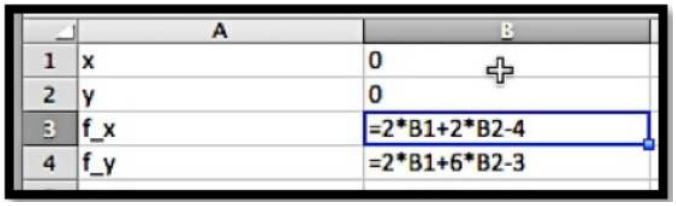
Substitute back and solve for x.

$$0 = 2x + 2(-1/4) - 4$$

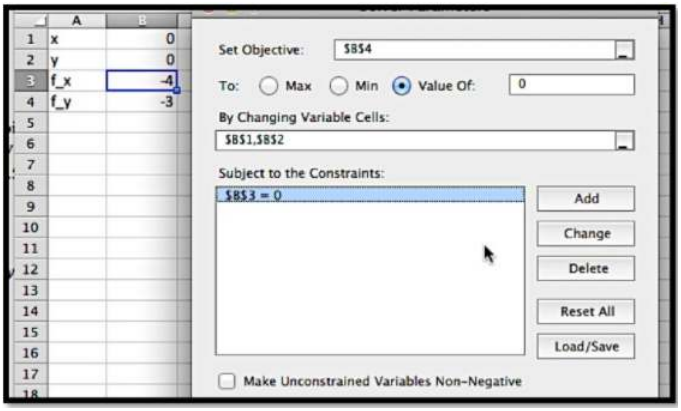
$$2x = 9/2$$

$$x = 9/4$$

Critical Point by Solver: However, if the partials are more complicated, I will want to find the critical points another way. I can find the point with Solver.



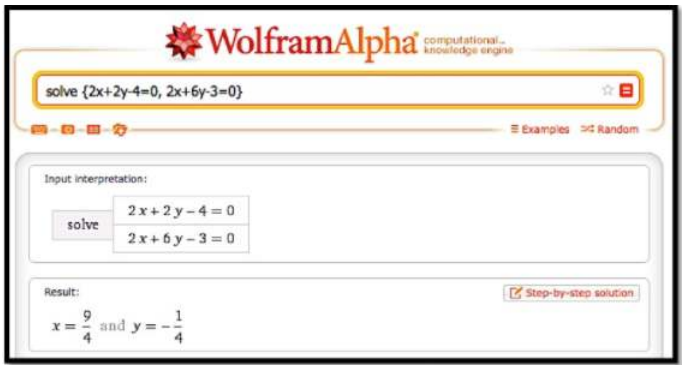
To get solver to set both partials to 0 at the same time, I ask it to solve for $f_y = 0$, while setting $f_x = 0$ as a constraint. Make sure to uncheck the box that makes unconstrained variables non-negative.



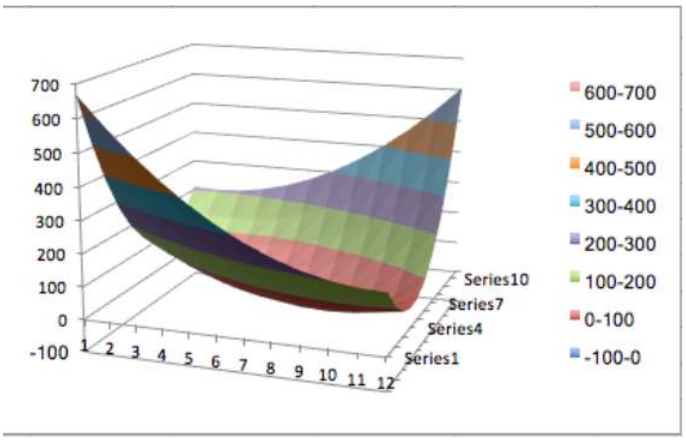
This finds our critical point within our error tolerance.

	A	B
1	x	2.2500092
2	y	-0.2500092
3	f_x	-3.576E-08
4	f_y	-3.696E-05
5	f_x+f_y	-3.7E-05
6		

Critical Point by CAS: We can also use Wolfram|Alpha to find the solution to our system of equations.



Determining the Critical Point is a Minimum We thus get a critical point at (9/4,-1/4) with any of the three methods of solving for both partial derivatives being zero at the same time. Once we have a critical point we want to determine if it is a maximum, minimum, or something else. The easiest way is to look at the graph near the critical point.



It is clear from the graph that this critical point is a local minimum.

It is easy to see that $f(x,y)=x^2+y^2$ has a critical point at $(0,0)$ and that that point is a minimum for the function. Similarly, $f(x,y)=-x^2-y^2$ has a critical point at $(0,0)$ and that that point is a maximum for the function. For some functions, like $f(x,y)=x^2-y^2$, which has a critical point at $(0,0)$, we can have a maximum in one direction and a minimum in another direction. Such a point is called a ***saddle point***. We note that we can have a saddle point even if the x and y slice curves both indicate a minimum.

Example 6.3.2: A Saddle Point at a Minimum on Both Axes.



Show that $f(x,y)=x^2-3xy+y^2$ has a critical point at $(0,0)$, which is a minimum of both slice curves, but is not a local minimum.

Solution

We look at the two partial derivatives, and notice they are both zero at the origin.

$$\frac{\partial}{\partial x} f(x,y) = 2x - 3y$$

$$\frac{\partial}{\partial x} f(x,y) = -3x + 2y.$$

We then see that both slice curves are parabolas that bend up, with a minimum at 0.

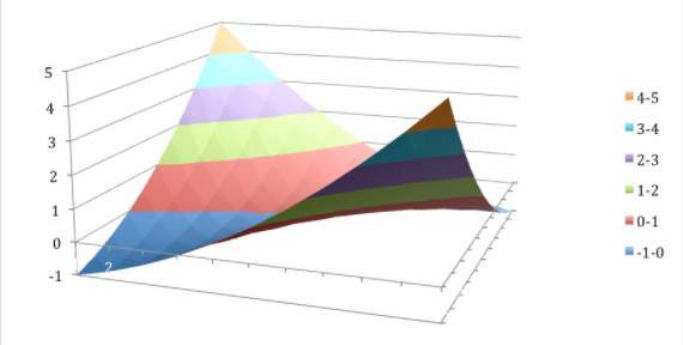
$$f(x,0) = x^2$$

$$f(0,y) = y^2.$$

However if we take the slice with $x = y$, we get a parabola bending down, so we don't have a minimum.

$$f(x,-x) = x^2 - 3xx + x^2 = -x^2.$$

Looking at the graph, we see that this graph does not have a minimum.



With only first derivatives, we can just find the critical points. To check if a critical point is maximum, a minimum, or a saddle point, using only the first derivative, the best method is to look at a graph to determine the kind of critical point. For some applications we want to categorize the critical points symbolically.

Second Partial Derivatives

With functions of one variable we used the second derivative to test if a critical point was a maximum or minimum. In the two variable case we need to define the second derivatives and use them to define the discriminant of a function to test if a critical point is a minimum, maximum, or saddle point. We first need to define second partial derivatives.

Second partials:

$$f_{ab} = (f_a)_b = \frac{\partial}{\partial b}(\frac{\partial}{\partial a} f).$$

Note that f_{xx} is simply the old second derivative of the curve $f(x, y_0)$ and f_{yy} is simply the old second derivative of the curve $f(x_0, y)$. For curves with continuous second partial derivatives, the mixed partials, f_{yx} and f_{xy} are the same.

Example 6.3.3: Finding Second Partial Derivatives.

Find the second partial derivatives of

$$f(x,y) = x^2 + 3xy + 5y^3 - 7x - 11y.$$

Solution

We start by computing the first partial derivatives.

$$f_x = \frac{\partial}{\partial x} f(x,y) = 2x + 3y - 7$$

$$f_y = \frac{\partial}{\partial y} f(x,y) = 3x + 15y^2 - 11.$$

Then we compute the second partial derivatives.

$$f_{xx} = \frac{\partial}{\partial x} f_x = 2$$

$$f_{xy} = \frac{\partial}{\partial y} f_x = 3$$

$$f_{yx} = \frac{\partial}{\partial x} f_y = 3$$

$$f_{yy} = \frac{\partial}{\partial y} f_y = 30y$$

As expected, the mixed partials are the same.

Using the Discriminant to Test Critical Points

To test if a critical point is a maximum, minimum, or saddle point we compute the discriminant of the function.

Discriminant:

$$D(f(x,y)) = f_{xx} f_{yy} - f_{xy}^2.$$

Example 6.3.4: Finidng the Discriminant of a Function.

Find the discriminant of

$$f(x,y) = x^2 + 3xy + 5y^3 - 7x - 11y.$$

Solution

We have already computed the second partial derivatives.

$$f_x x = 2, \quad f_x y = 3, \quad f_y y = 30y.$$

Substituting into the formula

$$D = (2)(30y) - 3^2 = 60y - 9.$$

Discriminant test: Let (a,b) be a critical point of $f(x,y)$.

If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$ then (a,b) is a local minimum of $f(x,y)$.

If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$ then (a,b) is a local maximum of $f(x,y)$.

If $D(a,b) < 0$ then (a,b) is a saddle point of $f(x,y)$.

If $D(a,b) = 0$ we do not have enough information to classify the point.

Example 6.3.5: Using the Discriminant to Classify Critical Points.

Based on the information given, classify each of the following points as a local maximum, local minimum, saddle point, not a critical point, or not enough information to classify.

p	f_x	f_y	f_{xx}	f_{xy}	f_{yy}
A	0	0	0	0	1
B	0	1	3	2	4
C	1	0	0	2	3
D	0	0	1	2	0

E	0	0	-1	2	3
F	0	0	-3	1	-2
G	0	0	3	3	3

Solution

We need to compute the discriminant and apply the test.

p	f_x	f_y	f_{xx}	f_{xy}	f_{yy}	Discriminant	Classification
A	0	0	0	0	1	0	Not enough information
B	0	1	3	2	4	8	Not a critical point
C	1	0	0	2	3	-4	Not a critical point
D	0	0	1	2	0	-4	Saddle point
E	0	0	-1	2	3	-7	Saddle point
F	0	0	-3	1	-2	5	Maximum
G	0	0	3	3	3	0	Not enough information

Example 6.3.6: Finding and Classifying Critical Points.

Let $f(x,y) = x^3 - 3x + y^3 - 3y^2$. Find the critical points and classify them using the discriminant.

Solution

We start by computing the first partial derivatives.

$$f_x = 3x^2 - 3 = 3(x-1)(x+1)$$

$$f_y = 3y^2 - 6y = 3(y-2)(y).$$

Then we compute the second partial derivatives and the discriminant.

$$f_{xx} = 6x, \quad f_{xy} = 0, \quad f_{yy} = 6y - 6, \quad D = (6x)(6y-6) - 0^2 = 36xy - 36x.$$

We have critical points when both first partials are 0, so at $(1,2)$, $(-1,2)$, $(1,0)$, and $(-1,0)$.

At $(1,2)$, both D and f_{xx} are positive, so we have a local minimum.

At $(-1,2)$ and $(1,0)$, D is negative, so we have a saddle point.

At $((-1,0)$, D is positive and f_{xx} is negative, so we have a local maximum.

Exercises: Critical Points and Extrema Problems

class="

For exercises 1-6, for the given functions and region:

- Find the partial derivatives of the original function.
- Find any critical points in the region.
- Produce a small graph around any critical point.
- Determine if the critical points are maxima, minima, or saddle points.

Exercise 1:

The function is $f(x,y)=x^2+2xy+4y^2+5x-6y,$ for the region $-10 \leq x \leq 10,$ and $-10 \leq y \leq 10.$

Answer

- \[f_x (x,y)=2x+2y+5 \nonumber \]

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5.E: Functions of Two Variables (Exercises)

4.1 Exercises

Exercise 5.E.1

$F(x,y) = x^2 - y^2$. Find

a) $F(0,4)$

b) $F(4,0)$

c) $F(x,4)$

d) $F(4,y)$

e) $F(800,800)$

f) $F(x,x)$

g) $F(x,-x)$

Exercise 5.E.2

$g(s,t) = \sqrt{st^2}$. Find

a) $g(1,9)$

b) $g(9,1)$

c) $g(1,t)$

d) $g(s,9)$

e) $g(w,z+1)$

Exercise 5.E.3

Let $f(x,y,z,w) = x^2 - \frac{1}{zw} + xyz^2$. Evaluate $f(1,2,3,4)$.

Exercise 5.E.4

Let $f(x,y,z,w) = \sqrt{xy} - w^2 + 102yz$. Evaluate $f(1,2,3,4)$.

Exercise 5.E.5

Here is a table showing the function $A(t,r)$

$\downarrow t$ $r \rightarrow$	.03	.04	.05	.06	.07
1	30.45	40.81	51.27	61.84	72.51
2	61.84	83.29	105.17	127.50	150.27
3	94.17	127.50	161.83	197.22	233.68

a) Find $A(2,.05)$

b) Find $A(.05,.2)$

c) Is $A(t,.06)$ an increasing or decreasing function of t?

d) Is $A(3,r)$ an increasing or decreasing function of r?

Exercise $5.E.6$

Here is a table showing values for the function $H(t, h)$.

$\overset{t}{\downarrow}\ h \rightarrow$	100	150	200
0	100	150	200
1	110.1	160.1	210.1
2	110.4	160.4	210.4
3	100.9	150.9	200.9
4	81.6	131.6	181.6
5	52.5	102.5	152.5

a) Is $H(t, 150)$ an increasing or decreasing function of t?

b) Is $H(4, h)$ an increasing or decreasing function of h?

c) Fill in the blanks: The maximum value shown on this table is H(___,___) =____.

d) Fill in the blanks: The minimum value shown on this table is H(___,___) =____.

Exercise $5.E.7 - 5.E.10$

In problems 7 – 10, plot the given points.

7. $A = (0, 3, 4), B = (1, 4, 0), C = (1, 3, 4), D = (1, 4, 2)$

8. $E = (4, 3, 0), F = (3, 0, 1), G = (0, 4, 1), H = (3, 3, 1)$

9. $P = (2, 3, -4), Q = (1, -2, 3), R = (4, -1, -2), S = (-2, 1, 3)$

10. $T = (-2, 3, -4), U = (2, 0, -3), V = (-2, 0, 0), W = (-3, -1, -2)$

Exercise $5.E.11 - 5.E.14$

In problems 11 – 14, calculate the distances between the given points

11. $A = (5, 3, 4), B = (3, 4, 4)$

12. $A = (6, 2, 1), B = (3, 2, 1)$

13. $A = (3, 4, 2), B = (-1, 6, -2)$

14. $A = (-1, 5, 0), B = (1, 3, 2)$

Exercise $5.E.15 - 5.E.18$

In problems 15 – 18, graph the given planes.

15. $y = 1$ and $z = 2$	16. $x = 4$ and $y = 2$
17. $x = 1$ and $y = 0$	18. $x = 2$ and $z = 0$

Exercise $5.E.19 - 5.E.22$

In problems 19 – 22, the center and radius of a sphere are given. Find an equation for the sphere.

19. Center = (4, 3, 5), radius = 3	20. Center = (0, 3, 6), radius = 2

21. Center = (5, 1, 0), radius = 5	22. Center = (1, 2, 3), radius = 4

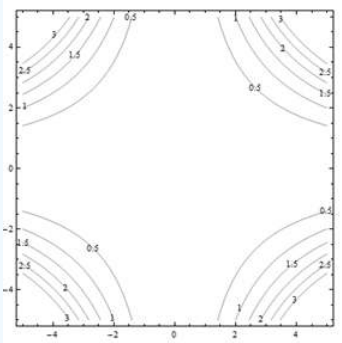
Exercise $5.E.23 - 5.E.24$

In problems 23 – 24, the equation of a sphere is given. Find the center and radius of the sphere.

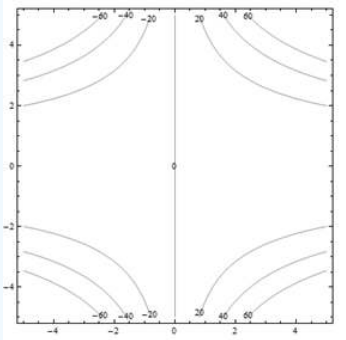
23. $(x-3)^2+(y+4)^2+(z-1)^2=16$	24. $(x+2)^2+y^2+(z-4)^2=25$

Exercise $5.E.25 - 5.E.30$

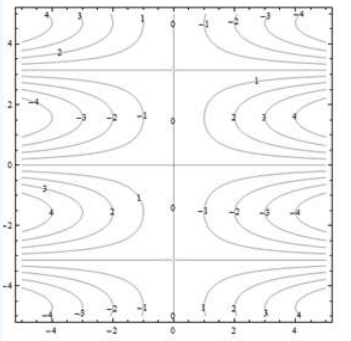
For problems 25 through 30. Match the contour diagram to the computer-generated, perspective drawing (a through f) it matches. Briefly explain your answer.



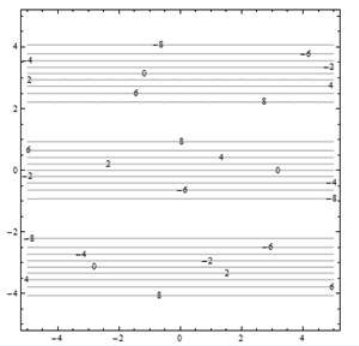
25.



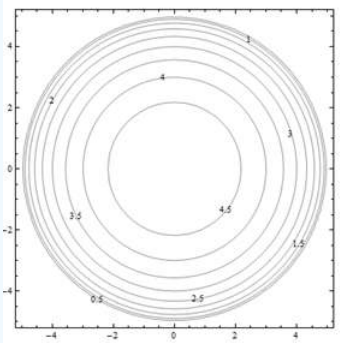
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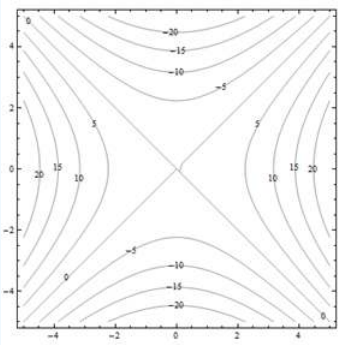
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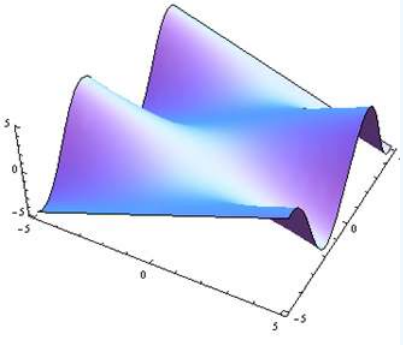
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29.

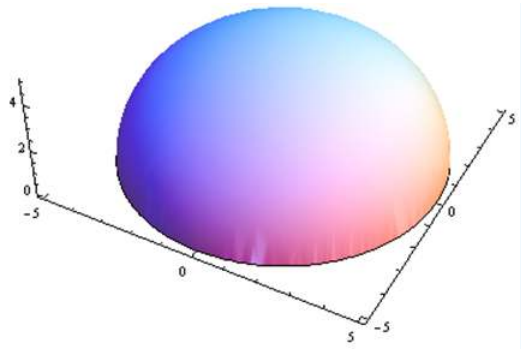


30.

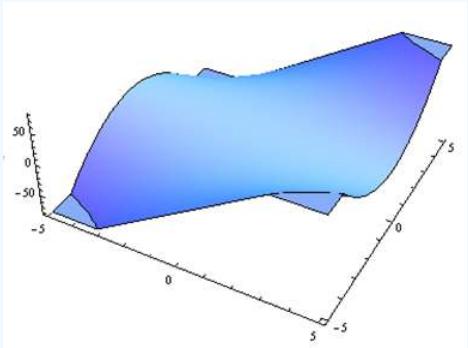


a.

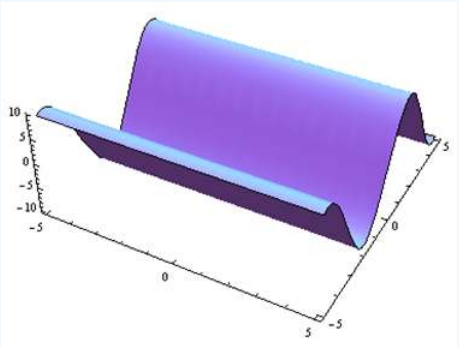
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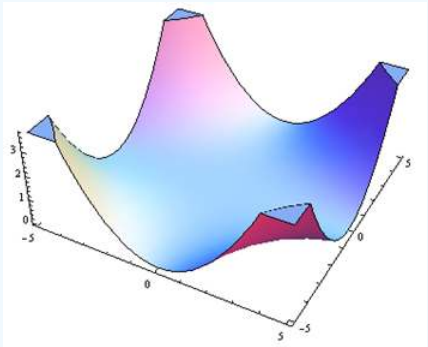
b.



c.

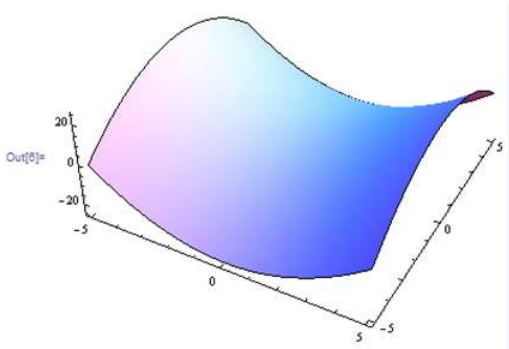


d.



e.

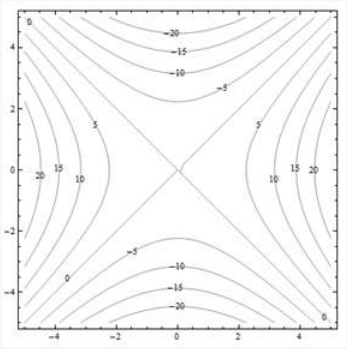
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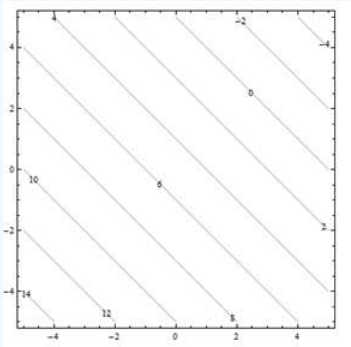
f.

Exercise $5.E.31 - 5.E.36$

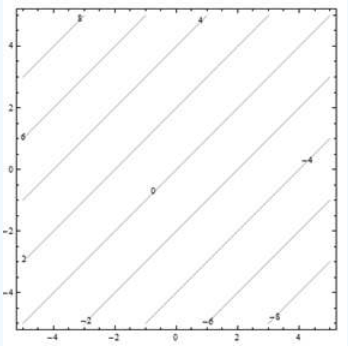
For problems 31 through 36. Match the contour diagram to the equation (a through f) it matches. Briefly explain your answer.



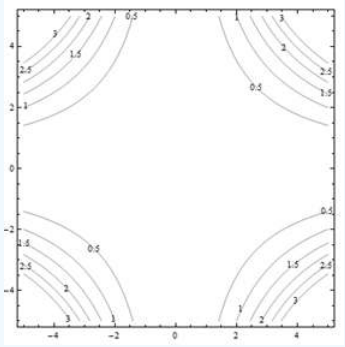
31.



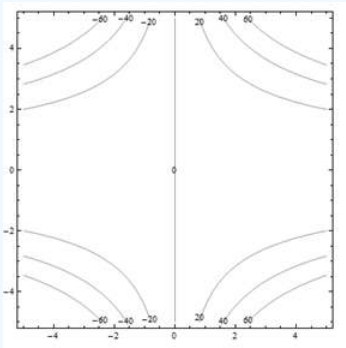
32.



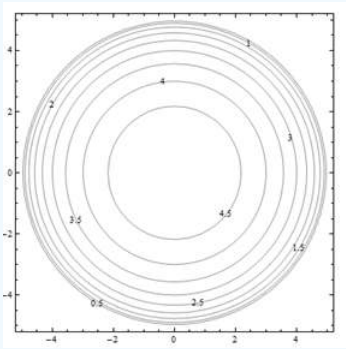
33.



34.



35.

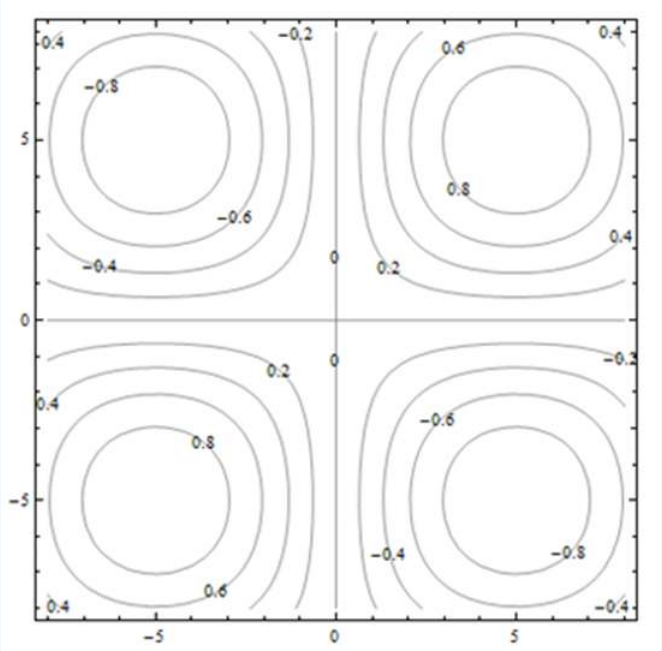


36.

a. $f(x,y) = y - x$	b. $f(x,y) = xy^2$
c. $f(x,y) = \sqrt{25 - x^2 - y^2}$	d. $f(x,y) = 5 - x - y$
e. $f(x,y) = 0.01x^2y^2$	f. $f(x,y) = x^2 - y^2$

Exercise $5.E.37$

The contour diagram shown is for a function $M(x,y)$.



Use the diagram to answer the following:

a) Estimate $M(1,3)$

b) Estimate $M(3,1)$

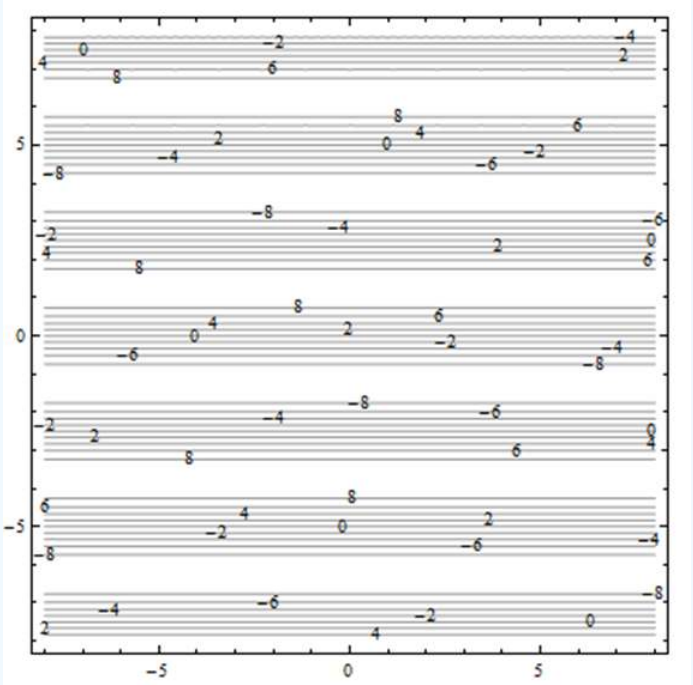
c) Is $M(x,3)$ an increasing or decreasing function of x?

d) Is $M(3,y)$ an increasing or decreasing function of y?

e) Find a value of c so that $M(c,y)$ is a constant function of y.

Exercise $5.E.38$

The contour diagram shown is for a function $G(x,y)$.



Use the diagram to answer the following:

a) Estimate $G(2,3)$

b) Suppose you travel north (in the direction of increasing y) along the surface, starting above (2, 3). Describe your journey.

c) Suppose you travel east (in the direction of increasing x) along the surface, starting above (2, 3). Describe your journey.

Exercise $5.E.39$

The demand functions for two products are given below. p_1, p_2, q_1, and q_2 are the prices (in dollars) and quantities for products 1 and 2. Are these two products complementary goods or substitute goods

$$q_1 = 200 - 3p_1 + p_2$$

$$q_2 = 150 + p_1 - 2p_2$$

Exercise $5.E.40$

The demand functions for two products are given below. p_1, p_2, q_1, and q_2 are the prices (in dollars) and quantities for products 1 and 2. Are these two products complementary goods or substitute goods

$$q_1 = 350 + p_1 + 2p_2$$

$$q_2 = 225 + p_1 + p_2$$

Exercise $5.E.41$

Consider the Cobb-Douglas Production function: $P(L,K) = 11L^{0.3}K^{0.7}$. Find the total units of production when 19 units of labor and 12 units of capital are invested.

Exercise $5.E.42$

Consider the Cobb-Douglas Production function: $P(L,K) = 6L^{0.6}K^{0.4}$. Find the total units of production when 24 units of labor and 8 units of capital are invested.

4.2 Exercises

Exercise $5.E.1 - 5.E.16$

For problems 1 through 16, find f_x and f_y for the function given

1. $f(x,y) = x^2 - 5y^2$
2. $f(x,y) = \frac{x^2-5y^2}{x+4}$
3. $f(x,y) = e^{x+6y}$
4. $f(x,y) = (x^2 - 5y^2)e^x$
5. $f(x,y) = (x^2 - 5y^2)\left(\frac{1}{3y} + 4\right)$
6. $f(x,y) = x$
7. $f(x,y) = 6$
8. $f(x,y) = \ln(xy + 2x - 6y)$
9. $f(x,y) = \frac{x^2-5y^2}{y^4-5x^4}$
10. $f(x,y) = e^{\sqrt{x-4y}}(x - 4y)$

11. $f(x,y) = y^5 e^x$
12. $f(x,y) = \frac{1}{16xy}$
13. $f(x,y) = (x+e^y)^7$
14. $f(x,y) = x^4 + 4x^3y - 6x^2y^2 - 4xy^3 + y^4$
15. $f(x,y) = \sqrt{x+\sqrt{y}}$
16. $f(x,y) = x^2y^3 - 4x^3$

Exercise $5.E.17$

Here is a table showing the function $A(t,r)$

$\overset{t}{\downarrow}\ r \rightarrow$	.03	.04	.05	.06	.07
1	30.45	40.81	51.27	61.84	72.51
2	61.84	83.29	105.17	127.50	150.27
3	94.17	127.50	161.83	197.22	233.68

a. Estimate $A_t(2,.05)$.

b. Estimate $A_r(2,.05)$

c. Use your answers to parts a and b to estimate the value of $A(2.5,.054)$

d. The values in the table came from $A(t,r) = 1000(e^{rt}-1)$, which shows the interest earned if 1000 dollars is deposited in an account earning r annual interest, compounded continuously, and left there for t years. How close are your estimates from parts a, b, and c?

Exercise $5.E.18$

18. Here is a table showing values for the function $H(t,h)$.

$\overset{t}{\downarrow}\ h \rightarrow$	100	150	200
0	100	150	200
1	110.1	160.1	210.1
2	110.4	160.4	210.4
3	100.9	150.9	200.9
4	81.6	131.6	181.6
5	52.5	102.5	152.5

a. Estimate the value of $\frac{\partial H}{dt}$ at (3, 150).

b. Estimate the value of $\frac{\partial H}{dh}$ at (3, 150).

c. Use your answers to parts a and b to estimate the value of $H(2.6,156)$.

d. The values in the table came from $H(t,h) = h + 15t - 4.9t^2$, which gives the height in meters above the ground after t seconds of an object that is thrown upward from an initial height of h meters with an initial velocity of 15 meters per second. How close are your estimates from parts a, b, and c?

Exercise $5.E.19$

Given the function $f(x,y) = x^2\sqrt{y}$

a. Calculate $f(2,4)$, $f_x(2,4)$, and $f_y(2,4)$

b. Use your answers from part a to estimate $f(1.9, 4.1)$

Exercise $5.E.20$

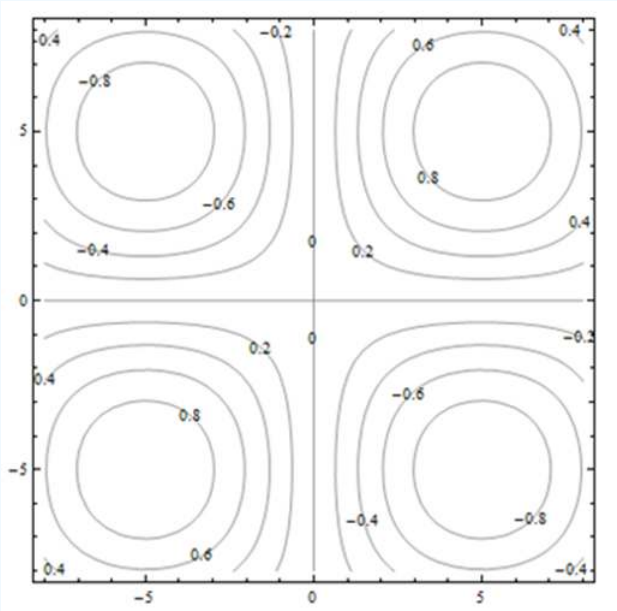
Given the function $f(x,y) = \ln(10 - x^2 - y)$

a. Calculate $f(2,5)$, $f_x(2,5)$, and $f_y(2,5)$

b. Use your answers from part a to estimate $f(1.8, 4.8)$

Exercise $5.E.21 - 5.E.26$

In problems 21 - 26, use the contour plot shown to estimate the desired value.



21. $f_x(1,-5)$
22. $f_x(-5,2)$
23. $f_x(5,5)$
24. $f_x(0,0)$
25. $f_y(1,-5)$
26. $f_y(-5,2)$

4.3 Exercises

Exercise $5.E.1 - 5.E.6$

For problems 1 through 6, find f_{xx}, f_{yy}, f_{xy} and f_{yx} for the function given. Confirm that $f_{xy} = f_{yx}$.

1. $f(x,y) = x^2 - 5y^2$
2. $f(x,y) = x^4 + 4x^3y - 6x^2y^y - 4xy^3 + y^4$

https://math.libretexts.org/@go/page/108810

3. $f(x,y)=5x^2y^2$

4. $f(x,y)=e^{x+6y}$

5. $f(x,y)=\ln(xy+2x-6y)$

6. $f(x,y)=\frac{x^2}{y^4-5}$

Exercise $5.E.7$

Find the critical points of $f(x,y)=y^3-x^3+15x^2-12y+12$ and use the Second Derivative Test to classify them. If the test fails, say "the test fails."

Exercise $5.E.8$

Find the critical points of $f(x,y)=2xy-x^2-2y^2+6x+4$ and use the Second Derivative Test to classify them. If the test fails, say "the test fails."

Exercise $5.E.9$

Find the critical points of $f(x,y)=y^2-4\ln(x)+4x$ and use the Second Derivative Test to classify them. If the test fails, say "the test fails."

Exercise $5.E.10$

Find the critical points of $f(x,y)=xy-6x^2+3x-y+2$ and use the Second Derivative Test to classify them. If the test fails, say "the test fails."

Exercise $5.E.11$

The origin is a critical point for the function $f(x,y)=x^3+y^3$, and $D=0$ there. That is, the Second Derivative Test fails. Use what you know about shapes of functions to decide if there is a local minimum, local maximum, or saddle point for this function at (0, 0).

Exercise $5.E.12$

The origin is a critical point for the function $f(x,y)=15-x^2y^2$, and $D=0$ there. That is, the Second Derivative Test fails. Use what you know about shapes of functions to decide if there is a local minimum, local maximum, or saddle point for this function at (0, 0).

Exercise $5.E.13-5.E.18$

For problems 13 through 18, find all local maxima, minima, and saddle points for the function.

13. $f(x,y)=xy-5x^2-5y^2+33y$

14. $f(x,y)=10xy-x^2-y^2+3x$

15. $f(x,y)=x^3+y^3-3xy$

16. $f(x,y)=5x^2-4xy+2y^2+4x-4y+10$

17. $f(x,y)=y^2e^x+x^2$

18. $f(x,y)=xy+2x-\ln(x^2y)$, for $x>0$ and $y>0$.

Exercise 5.E.19

The demand functions for two products are given below. p_1, p_2, q_1, and q_2 are the prices (in dollars) and quantities for products 1 and 2.

$$q_1 = 200 + 3p_1 + p_2$$

$$q_2 = 150 + p_1 + 2p_2$$

a. Are these two products complementary goods or substitute goods?

b. What is the quantity demanded for each when the price for product 1 is \$20 per item and the price for product 2 is \$30 per item?

c. Write a function $R(p_1, p_2)$ that expresses the total revenue from these two products.

d. Find the price and quantity for each product that maximizes the total revenue.

Exercise 5.E.20

The demand functions for two products are given below. p_1, p_2, q_1, and q_2 are the prices (in dollars) and quantities for products 1 and 2.

$$q_1 = 350 + p_1 + 2p_2$$

$$q_2 = 225 + p_1 + p_2$$

a. Are these two products complementary goods or substitute goods?

b. What is the quantity demanded for each when the price for product 1 is \$20 per item and the price for product 2 is \$30 per item?

c. Write a function $R(p_1, p_2)$ that expresses the total revenue from these two products.

d. Find the price and quantity for each product that maximizes the total revenue.

Exercise 5.E.21

Suppose the demand functions for two products are $q_1 = f(p_1, p_2)$ and $q_2 = g(p_1, p_2)$, where p_1, p_2, q_1, and q_2 are the prices (in dollars) and quantities for products 1 and 2. Consider the four partial derivatives $\frac{\partial q_1}{\partial p_1}$, $\frac{\partial q_1}{\partial p_2}$, $\frac{\partial q_2}{\partial p_1}$, and $\frac{\partial q_2}{\partial p_2}$. Tell the sign of each of these partial derivatives if

a. the products are complementary goods.

b. the products are substitute goods.

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CHAPTER OVERVIEW

6: Algebra Review

- 6.1: Prelude to Functions and Graphs
- 6.2: Functions
- 6.3: Relations, Graphs, and Functions
- 6.4: Set-Builder and Interval Notations
- 6.5: Operations on Functions
- 6.6: Linear Functions
  - 6.6.1: Slope
  - 6.6.2: Equations of Lines
    - 6.6.2.1: The Point-Slope Form of a Line
- 6.7: Quadratics
- 6.8: Polynomials and Rational Functions
- 6.E: Review (Exercises)

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6.1: Prelude to Functions and Graphs

In the past few years, major earthquakes have occurred in several countries around the world. In January 2010, an earthquake of magnitude 7.3 hit Haiti. A magnitude 9 earthquake shook northeastern Japan in March 2011. In April 2014, an 8.2-magnitude earthquake struck off the coast of northern Chile. What do these numbers mean? In particular, how does a magnitude 9 earthquake compare with an earthquake of magnitude 8.2? Or 7.3? Later in this chapter, we show how logarithmic functions are used to compare the relative intensity of two earthquakes based on the magnitude of each earthquake.



Figure 6.1.1: A portion of the San Andreas Fault in California. Major faults like this are the sites of most of the strongest earthquakes ever recorded. (credit: modification of work by Robb Hannawacker, NPS)

Calculus is the mathematics that describes changes in functions. In this chapter, we review all the functions necessary to study calculus. We define polynomial, rational, trigonometric, exponential, and logarithmic functions. We review how to evaluate these functions, and we show the properties of their graphs. We provide examples of equations with terms involving these functions and illustrate the algebraic techniques necessary to solve them. In short, this chapter provides the foundation for the material to come. It is essential to be familiar and comfortable with these ideas before proceeding to the formal introduction of calculus in the next chapter.

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6.2: Functions

What is a Function?

The natural world is full of relationships between quantities that change. When we see these relationships, it is natural for us to ask "If I know one quantity, can I then determine the other?" This establishes the idea of an input quantity, or independent variable, and a corresponding output quantity, or dependent variable. From this we get the notion of a functional relationship in which the output can be determined from the input.

For some quantities, like height and age, there are certainly relationships between these quantities. Given a specific person and any age, it is easy enough to determine their height, but if we tried to reverse that relationship and determine height from a given age that would be problematic, since most people maintain the same height for many years.

Function

A **function** is rule for a relationship between an input, or independent, quantity and an output, or dependent, quantity in which each input value uniquely determines one output value. We say "the output is a function of the input."

Example 6.2.1

In the height and age example above, is height a function of age? Is age a function of height?

In the height and age example above, it would be correct to say that height is a function of age, since each age uniquely determines a height. For example, on my 18th birthday, I had exactly one height of 69 inches.

However, age is not a function of height, since one height input might correspond with more than one output age. For example, for an input height of 70 inches, there is more than one output of age since I was 70 inches at the age of 20 and 21.

Function Notation

To simplify writing out expressions and equations involving functions, a simplified notation is often used. We also use descriptive variables to help us remember the meaning of the quantities in the problem.

Rather than write "height is a function of age," we could use the descriptive variable h to represent height and we could use the descriptive variable a to represent age.

"height is a function of age"	if we name the function f we write
"h is f of a"	or more simply
$h = f(a)$	we could instead name the function h and write
$h(a)$	which is read "h of a"

Remember we can use any variable to name the function; the notation $h(a)$ shows us that h depends on a. The value "a" must be put into the function "h" to get a result. Be careful – the parentheses indicate that age is input into the function (Note: do not confuse these parentheses with multiplication!).

Function Notation

The notation output = f(input) defines a function named f. This would be read "output is f of input."

Example 6.2.2

A function $N = f(y)$ gives the number of police officers, N, in a town in year y. What does $f(2005) = 300$ tell us?

Solution

When we read $f(2005) = 300$, we see the input quantity is 2005, which is a value for the input quantity of the function, the year (y). The output value is 300, the number of police officers (N), a value for the output quantity. Remember $N = f(y)$. So

this tells us that in the year 2005 there were 300 police officers in the town.

Tables as Functions

Functions can be represented in many ways: Words (as we did in the last few examples), tables of values, graphs, or formulas. Represented as a table, we are presented with a list of input and output values.

This table represents the age of children in years and their corresponding heights. While some tables show all the information we know about a function, this particular table represents just some of the data available for height and ages of children.

(input) a, age in years	4	5	6	7	8	9	10
(output) h, height in inches	40	42	44	47	50	52	54

Example 6.2.3

Which of these tables define a function (if any)?

Input	Output
2	1
5	3
8	6

Input	Output
-3	5
0	1
4	5

Input	Output
1	0
5	2
5	

Solution

The first and second tables define functions. In both, each input corresponds to exactly one output. The third table does not define a function since the input value of 5 corresponds with two different output values.

To view this video please enable JavaScript, and consider upgrading to a web browser that supports HTML5 video

Solving and Evaluating Functions

When we work with functions, there are two typical things we do: evaluate and solve. Evaluating a function is what we do when we know an input, and use the function to determine the corresponding output. Evaluating will always produce one result, since each input of a function corresponds to exactly one output.

Solving equations involving a function is what we do when we know an output, and use the function to determine the inputs that would produce that output. Solving a function could produce more than one solution, since different inputs can produce the same output.

Example 6.2.4

Using the table shown, where $Q = g(n)$

n	1	2	3	4	5
Q	8	6	7	6	8

a. Evaluate $g(3)$
b. Solve $g(n) = 6$

Solution

Evaluating g(3) (read: “g of 3”) means that we need to determine the output value, Q, of the function g given the input value of n=3. Looking at the table, we see the output corresponding to n=3 is Q=7, allowing us to conclude g(3) = 7.

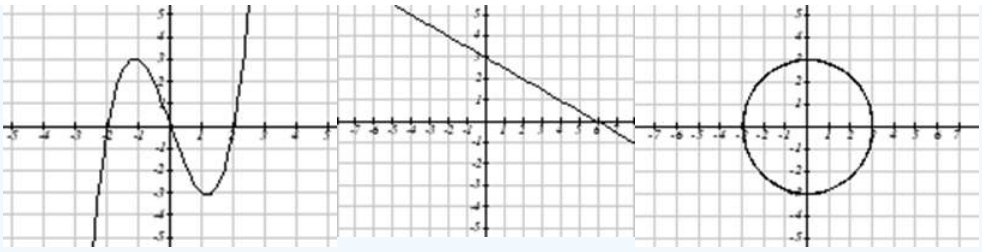
Solving g(n) = 6 means we need to determine what input values, n, produce an output value of 6. Looking at the table we see there are two solutions: n = 2 and n = 4. When we input 2 into the function g, our output is Q = 6 When we input 4 into the function g, our output is also Q = 6

Graphs as Functions

Oftentimes a graph of a relationship can be used to define a function. By convention, graphs are typically created with the input quantity along the horizontal axis and the output quantity along the vertical.

Example 6.2.5

Which of these graphs defines a function $y = f(x)$?



Solution

Looking at the three graphs above, the first two define a function y=f(x), since for each input value along the horizontal axis there is exactly one output value corresponding, determined by the y-value of the graph. The 3rd graph does not define a function y=f(x) since some input values, such as x=2, correspond with more than one output value.

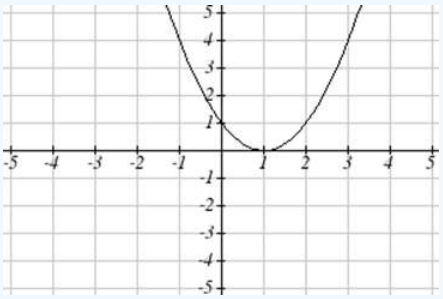
Vertical Line Test

The **vertical line test** is a handy way to think about whether a graph defines the vertical output as a function of the horizontal input. Imagine drawing vertical lines through the graph. If any vertical line would cross the graph more than once, then the graph does not define only one vertical output for each horizontal input.

Evaluating a function using a graph requires taking the given input and using the graph to look up the corresponding output. Solving a function equation using a graph requires taking the given output and looking on the graph to determine the corresponding input.

Example 6.2.6

Given the graph below,



a. Evaluate $f(2)$.
b. Solve $f(x) = 4$.

Solution

a) To evaluate f(2), we find the input of x=2 on the horizontal axis. Moving up to the graph gives the point (2, 1), giving an output of y=1. So f(2) = 1

b) To solve f(x) = 4, we find the value 4 on the vertical axis because if f(x) = 4 then 4 is the output. Moving horizontally across the graph gives two points with the output of 4: (-1,4) and (3,4). These give the two solutions to f(x) = 4: x = -1 or x = 3 This means f(-1)=4 and f(3)=4, or when the input is -1 or 3, the output is 4.

Notice that while the graph in the previous example is a function, getting two input values for the output value of 4 shows us that this function is not one-to-one.

Formulas as Functions

When possible, it is very convenient to define relationships using formulas. If it is possible to express the output as a formula involving the input quantity, then we can define a function.

Example 6.2.7

Express the relationship $2n+6p=12$ as a function $p=f(n)$ if possible.

Solution

To express the relationship in this form, we need to be able to write the relationship where p is a function of n, which means writing it as $p=$ [something involving n].

$$\begin{aligned} 2n+6p &= 12 \quad \text{subtract } 2n \text{ from both sides} \\ 6p &= 12-2n \quad \text{divide both sides by 6 and simplify} \\ p &= \tfrac{12-2n}{6} = \tfrac{12}{6} - \tfrac{2n}{6} = 2 - \tfrac{1}{3}n \end{aligned}$$

Having rewritten the formula as $p=$, we can now express p as a function:

$$p = f(n) = 2 - \frac{1}{3}n$$

Not every relationship can be expressed as a function with a formula. As with tables and graphs, it is common to evaluate and solve functions involving formulas. Evaluating will require replacing the input variable in the formula with the value provided and calculating. Solving will require replacing the output variable in the formula with the value provided, and solving for the input(s) that would produce that output.

Example 6.2.8

Given the function $k(t)=t^3+2$:

a. Evaluate $k(2)$.
b. Solve $k(t)=1$.

Solution

a) To evaluate $k(2)$, we plug in the input value 2 into the formula wherever we see the input variable t, then simplify

$$\begin{aligned} k(2) &= 2^3+2 \\ k(2) &= 8+2 \end{aligned}$$

So $k(2)=10$

b) To solve $k(t)=1$, we set the formula for $k(t)$ equal to 1 , and solve for the input value that will produce that output

$$k(t)=1 \quad \text{substitute the original formula } k(t)=t^3+2$$

$t^3+2=1$ subtract 2 from each side
$t^3=-1$ take the cube root of each side
$t=-1$

When solving an equation using formulas, you can check your answer by using your solution in the original equation to see if your calculated answer is correct.

We want to know if $k(t)=1$ is true when $t=-1$.

$$\begin{aligned} k(-1) &= (-1)^3+2 \\ &= -1+2 \\ &= 1 \text{ which was the desired result.} \end{aligned}$$

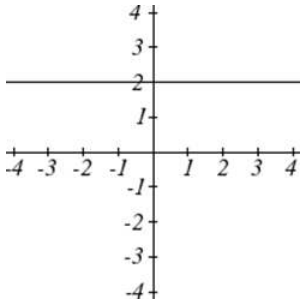
Basic Toolkit Functions

There are some basic functions that it is helpful to know the name and shape of. We call these the basic "toolkit of functions." For these definitions we will use x as the input variable and $f(x)$ as the output variable.

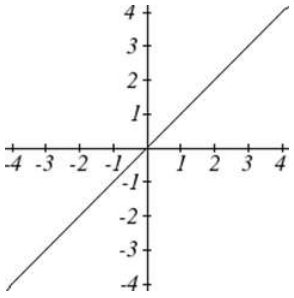
Toolkit Functions

Linear	
Constant:	$f(x) = c$, where c is a constant (number)
Identity:	$f(x) = x$
Absolute Value:	$f(x) = \|x\|$
Power	
Quadratic:	$f(x) = x^2$
Cubic:	$f(x) = x^3$
Reciprocal:	$f(x) = \frac{1}{x}$
Reciprocal squared:	$f(x) = \frac{1}{x^2}$
Square root:	$f(x) = \sqrt[2]{x} = \sqrt{x}$
Cube root:	$f(x) = \sqrt[3]{x}$

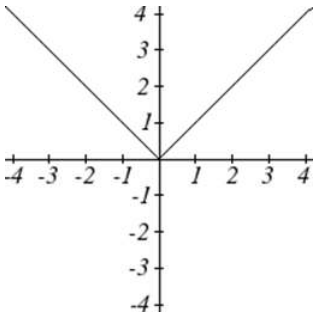
Graphs of the Toolkit Functions



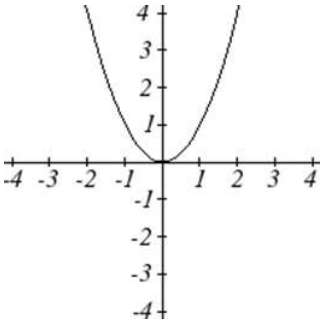
Constant: $f(x) = c$ (c is 2 in this picture).



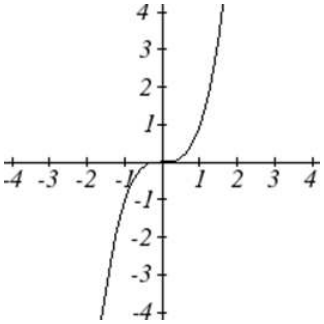
Identity: $f(x) = x$



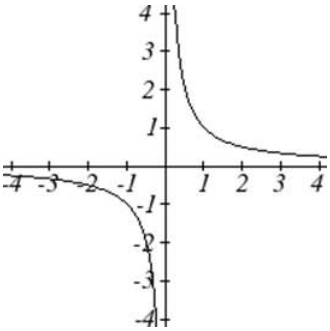
Absolute Value: $f(x) = |x|$



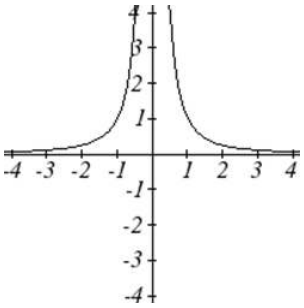
Quadratic: $f(x) = x^2$



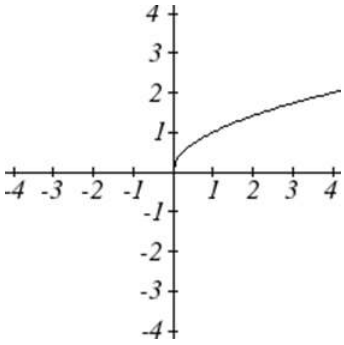
Cubic: $f(x) = x^3$



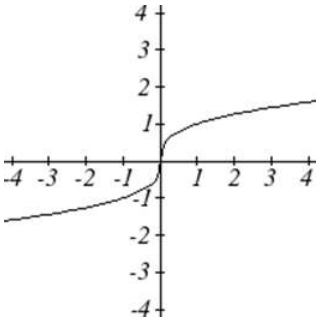
Reciprocal: $f(x) = \frac{1}{x}$



Reciprocal squared: $f(x) = \frac{1}{x^2}$



Square root: $f(x) = \sqrt[2]{x} = \sqrt{x}$



Cube root: $f(x) = \sqrt[3]{x}$

One of our main goals in mathematics is to model the real world with mathematical functions. In doing so, it is important to keep in mind the limitations of those models we create.

This table shows a relationship between circumference and height of a tree as it grows.

Circumference, c	1.7	2.5	5.5	8.2	13.7
Height, h	24.5	31	45.2	54.6	92.1

While there is a strong relationship between the two, it would certainly be ridiculous to talk about a tree with a circumference of -3 feet, or a height of 3000 feet. When we identify limitations on the inputs and outputs of a function, we are determining the domain and range of the function.

Domain and Range

- **Domain:** The set of possible input values to a function.
- **Range:** The set of possible output values of a function

Example 6.2.9

Using the tree table above, determine a reasonable domain and range.

Solution

We could combine the data provided with our own experiences and reason to approximate the domain and range of the function $h = f(c)$. For the domain, possible values for the input circumference c, it doesn't make sense to have negative values, so $c > 0$. We could make an educated guess at a maximum reasonable value, or look up that the maximum circumference measured is about 119 feet. With this information we would say a reasonable domain is feet.

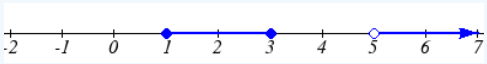
Similarly for the range, it doesn't make sense to have negative heights, and the maximum height of a tree could be looked up to be 379 feet, so a reasonable range is feet.

A more compact alternative to inequality notation is **interval notation**, in which intervals of values are referred to by the starting and ending values. Curved parentheses are used for "strictly less than," and square brackets are used for "less than or equal to." Since infinity is not a number, we can't include it in the interval, so we always use curved parentheses with ∞ and $-\infty$. The table below will help you see how inequalities correspond to interval notation:

Inequality	Interval notation
$5 < h \leq 10$	(5, 10]
$5 \leq h < 10$	[5, 10)
$5 < h < 10$	(5, 10)
$h < 10$	$(-\infty, 10)$
$h \geq 10$	$[10, \infty)$
All real numbers ($\mathbb{R}$)	$(-\infty, \infty)$

Example 6.2.10

Describe the intervals of values shown on the line graph below using set builder and interval notations:



Solution

To describe the values, x, that lie in the intervals shown above we would say, "x is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5." As an inequality it is "$1 \leq x \leq 3$ or $x > 5$". In interval notation it is "$[1, 3] \cup (5, \infty)$".

Example 6.2.11

Find the domain of each function:

a. $f(x) = 2\sqrt{x+4}$

b. $g(x) = \dfrac{3}{6-3x}$

Solution

a. Since we cannot take the square root of a negative number, we need the inside of the square root to be non-negative. $x + 4 \geq 0$ when $x \geq -4$, so the domain of $f(x)$ is $[-4, \infty)$.

b. We cannot divide by zero, so we need the denominator to be non-zero. $6 - 3x = 0$ when $x = 2$, so we must exclude 2 from the domain. The domain of $g(x)$ is $(-\infty, 2) \cup (2, \infty)$.

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6.3: Relations, Graphs, and Functions

Learning Objectives

- State the domain and range of a relation.
- Identify a function.
- Use function notation

Graphs, Relations, Domain, and Range

The **rectangular coordinate system**[1] consists of two real number lines that intersect at a right angle. The horizontal number line is called the **x-axis**[2], and the vertical number line is called the **y-axis**[3]. These two number lines define a flat surface called a **plane**[4], and each point on this plane is associated with an **ordered pair**[5] of real numbers (x, y). The first number is called the x-coordinate, and the second number is called the y-coordinate. The intersection of the two axes is known as the **origin**[6], which corresponds sto the point $(0, 0)$.

The x- and y-axes break the plane into four regions called **quadrants**[7], named using roman numerals I, II, III, and IV, as pictured. The ordered pair (x, y) represents the position of points relative to the origin. For example, the ordered pair $(-4, 3)$ represents the position 4 units to the left of the origin, and 3 units above in the second quadrant.

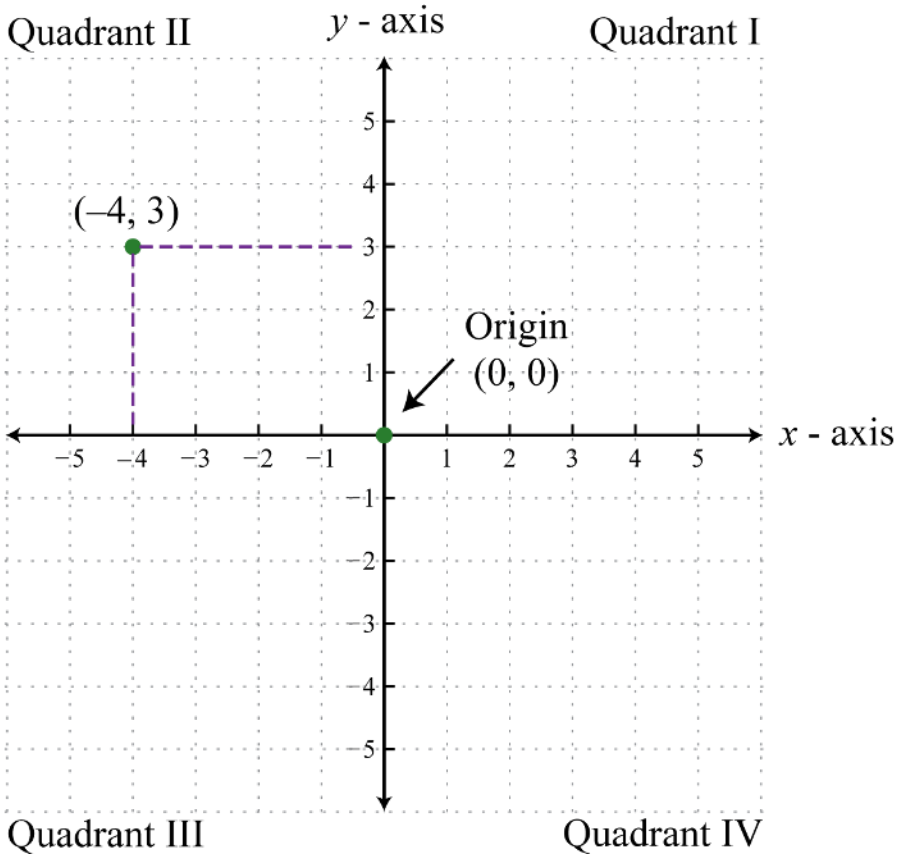


Figure 6.3.1

This system is often called the **Cartesian coordinate system**[8], named after the French mathematician René Descartes (1596–1650).



Figure 6.3.2

Next, we define a **relation**[9] as any set of ordered pairs. In the context of algebra, the relations of interest are sets of ordered pairs (x, y) in the rectangular coordinate plane. Typically, the coordinates are related by a rule expressed using an algebraic equation. For example, both the algebraic equations $y = |x| - 2$ and $x = |y| + 1$ define relationships between x and y. Following are some integers that satisfy both equations:

$y = |x| - 2$

x	y
-3	1
-2	0
-1	-1
0	-2
1	-1
2	0
3	1

$x = |y| + 1$

x	y
4	-3
3	-2
2	-1
1	0
2	1
3	2
4	3

Figure 6.3.3

Here two relations consisting of seven ordered pair solutions are obtained:

$$y = |x| - 2 \quad \textit{has solutions} \quad \{(-3,1),(-2,0),(-1,-1),(0,-2),(1,-1),(2,0),(3,1)\}$$

$$\textit{and}$$

$$x = |y| + 1 \quad \textit{has solutions} \quad \{(4,-3),(3,-2),(2,-1),(1,0),(2,1),(3,2),(4,3)\}$$

We can visually display any relation of this type on a coordinate plane by plotting the points.

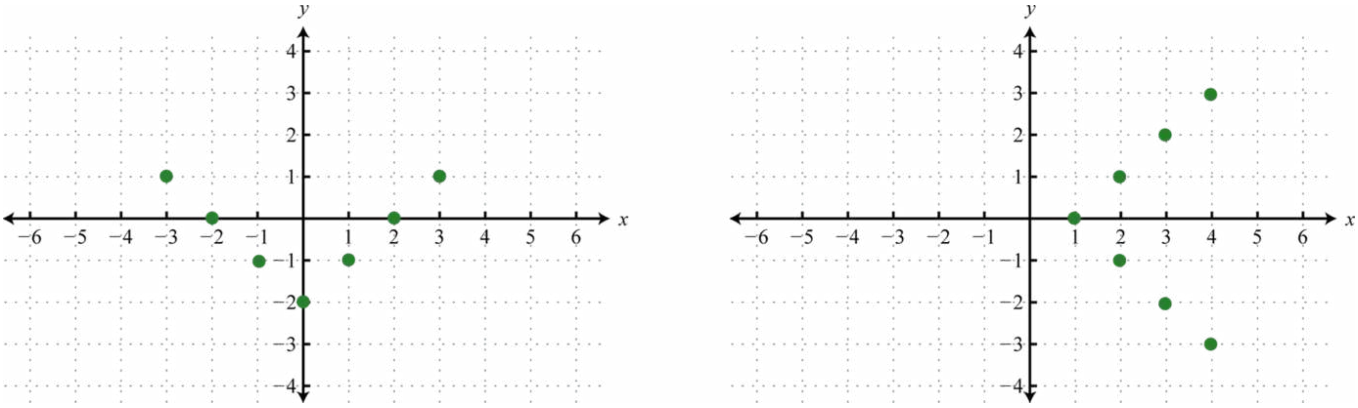


Figure 6.3.4

The solution sets of each equation will form a relation consisting of infinitely many ordered pairs. We can use the given ordered pair solutions to estimate all of the other ordered pairs by drawing a line through the given points. Here we put an arrow on the ends of our lines to indicate that this set of ordered pairs continues without bounds.

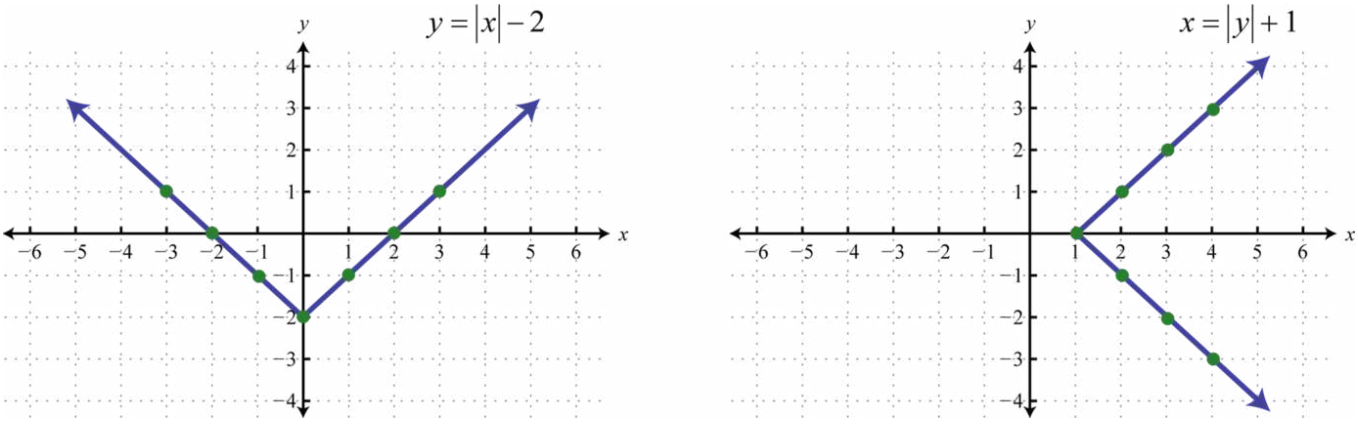


Figure 6.3.5

The representation of a relation on a rectangular coordinate plane, as illustrated above, is called a **graph**[10]. Any curve graphed on a rectangular coordinate plane represents a set of ordered pairs and thus defines a relation.

The set consisting of all of the first components of a relation, in this case the *x*-values, is called the **domain**[11]. And the set consisting of all second components of a relation, in this case the *y*-values, is called the **range**[12] (or **codomain**[13]). Often, we can determine the domain and range of a relation if we are given its graph.

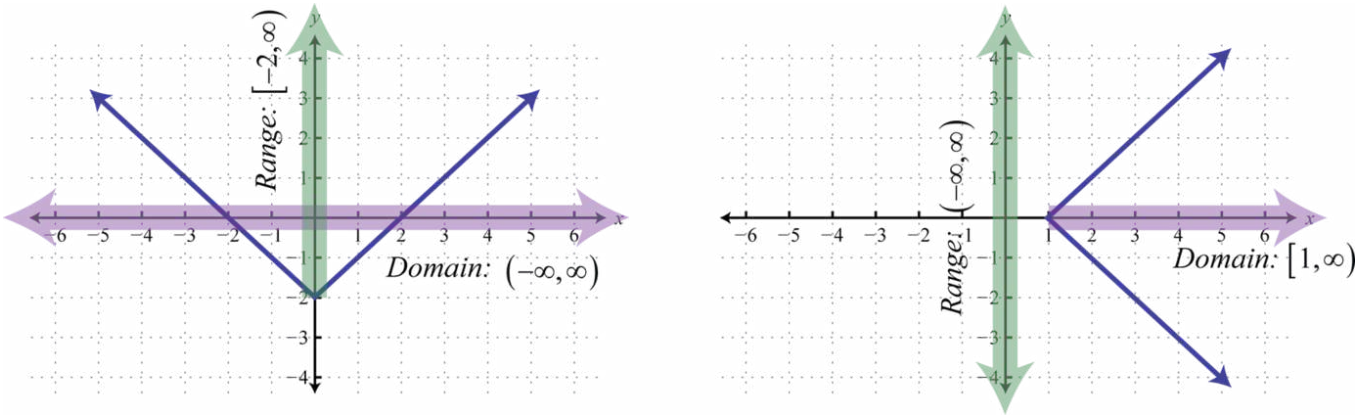


Figure 6.3.6

Here we can see that the graph of $y = |x| - 2$ has a domain consisting of all real numbers, $\mathbb{R} = (-\infty, \infty)$, and a range of all *y*-values greater than or equal to -2, $[-2, \infty)$. The domain of the graph of $x = |y| + 1$ consists of all *x*-values greater than or equal to 1, $[1, \infty)$, and the range consists of all real numbers, $\mathbb{R} = (-\infty, \infty)$.

Example 6.3.1:

Determine the domain and range of the following relation:

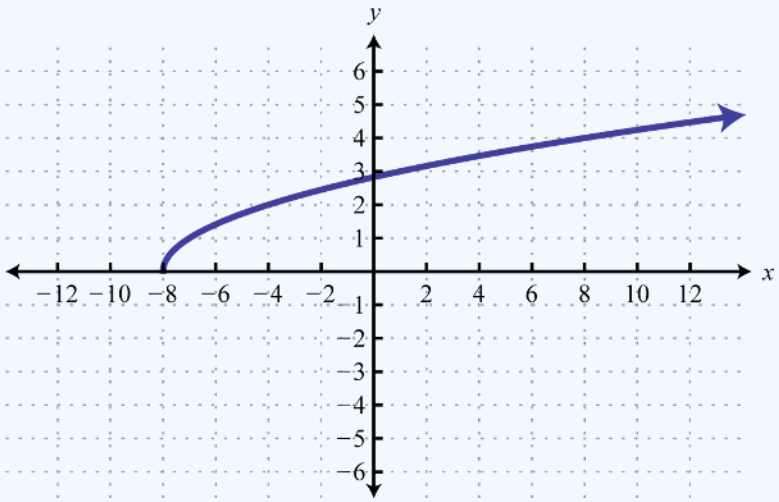


Figure 6.3.7

Solution

The minimum x-value represented on the graph is -8 all others are larger. Therefore, the domain consists of all x-values in the interval $[-8, \infty)$. The minimum y-value represented on the graph is 0; thus, the range is $[0, \infty)$.

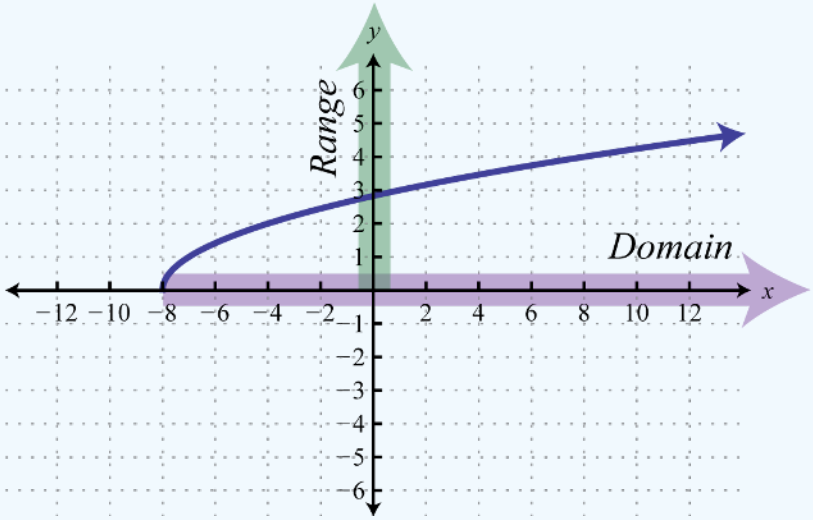


Figure 6.3.8

Answer:

Domain: $[-8, \infty)$; range: $[0, \infty)$

Functions

Of special interest are relations where every x-value corresponds to exactly one y-value. A relation with this property is called a **function**[14].

Example 6.3.2:

Determine the domain and range of the following relation and state whether it is a function or not: $\{(-1, 4), (0, 7), (2, 3), (3, 3), (4, -2)\}$

Solution

Here we separate the domain (*x-values*), and the range (*y-values*), and depict the correspondence between the values with arrows.

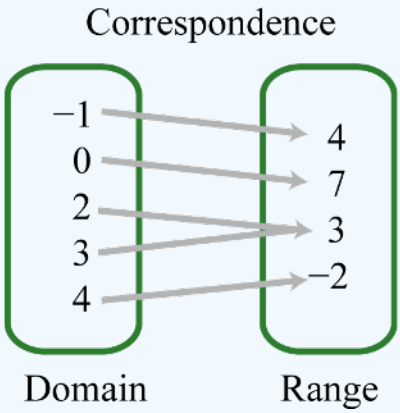


Figure 6.3.9

The relation is a function because each *x*-value corresponds to exactly one *y*-value.

Answer:

The domain is $\{-1, 0, 2, 3, 4\}$and the range is $\{-2, 3, 4, 7\}$. The relation is a function.

Example 6.3.3:

Determine the domain and range of the following relation and state whether it is a function or not: $\{(-4, -3), (-2, 6), (0, 3), (3, 5), (3, 7)\}$

Solution

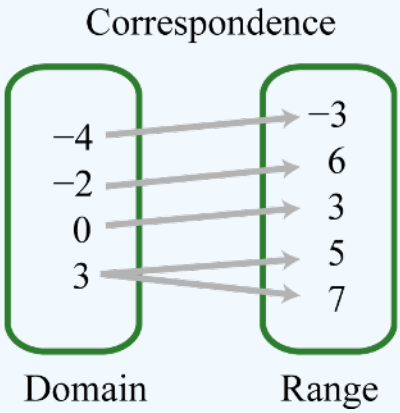


Figure 6.3.10

The given relation is not a function because the x-value 3 corresponds to two y-values. We can also recognize functions as relations where no x-values are repeated.

Answer:

The domain is $\{-4, -2, 0, 3\}$and the range is $\{-3, 3, 5, 6, 7\}$. This relation is not a function.

Consider the relations consisting of the seven ordered pair solutions to $y = |x| - 2$and $x = |y| + 1$. The correspondence between the domain and range of each can be pictured as follows:

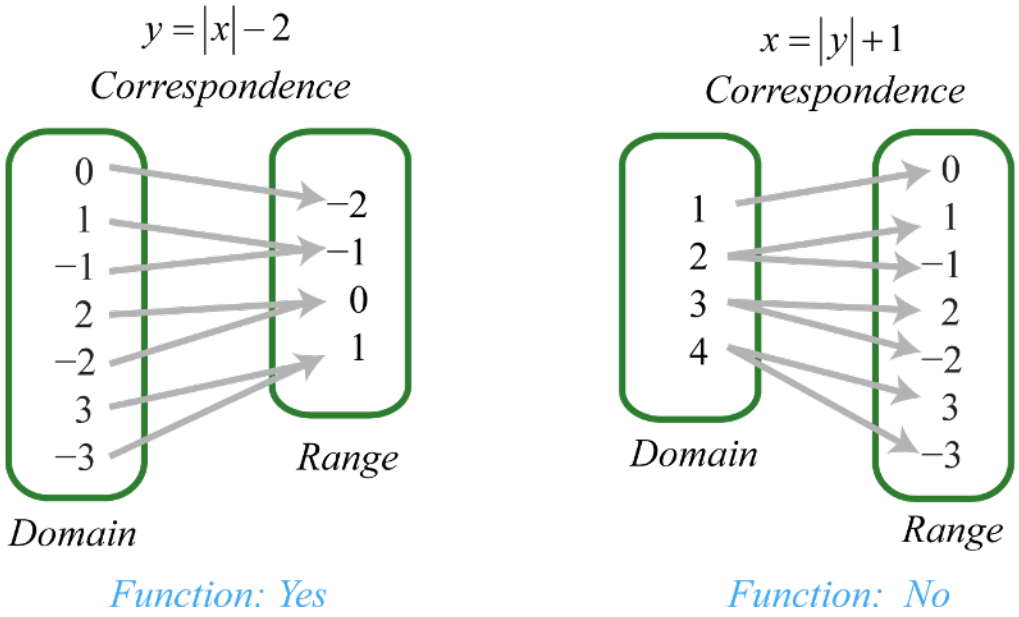


Figure 6.3.11

Notice that every element in the domain of the solution set of $y = |x| - 2$ corresponds to only one element in the range; it is a function. The solutions to $x = |y| + 1$, on the other hand, have values in the domain that correspond to two elements in the range. In particular, the x-value 4 corresponds to two y-values -3 and 3. Therefore, $x = |y| + 1$ does not define a function.

We can visually identify functions by their graphs using the **vertical line test**[15]. If any vertical line intersects the graph more than once, then the graph does not represent a function.

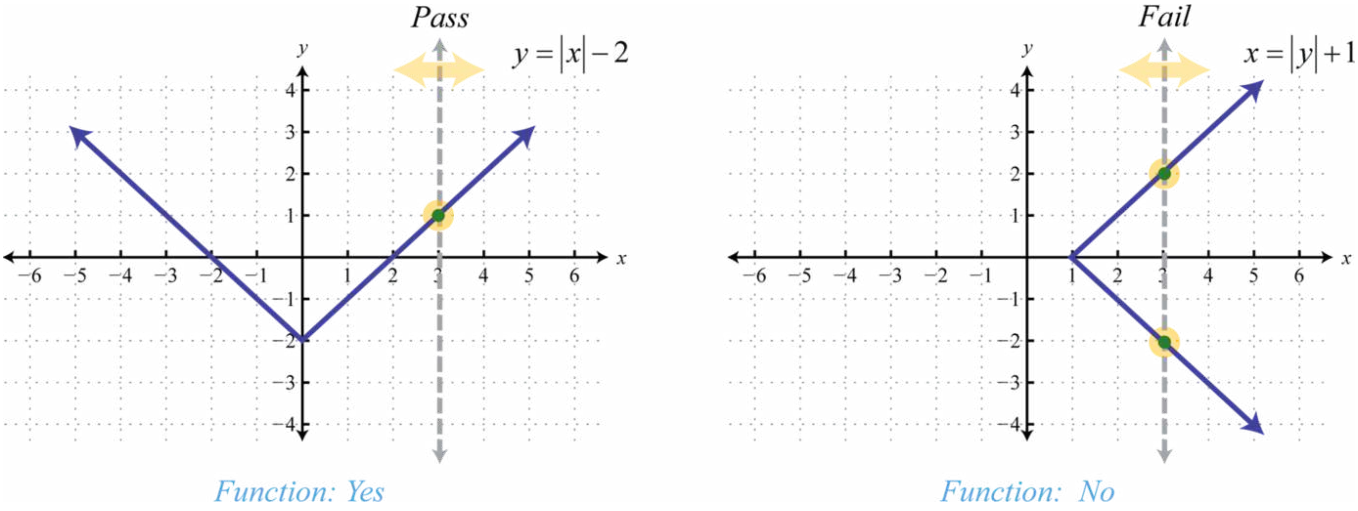


Figure 6.3.12

The vertical line represents a value in the domain, and the number of intersections with the graph represent the number of values to which it corresponds. As we can see, any vertical line will intersect the graph of $y = |x| - 2$ only once; therefore, it is a function. A vertical line can cross the graph of $x = |y| + 1$ more than once; therefore, it is not a function. As pictured, the x-value 3 corresponds to more than one y-value.

Example 6.3.1:

Given the graph, state the domain and range and determine whether or not it represents a function:

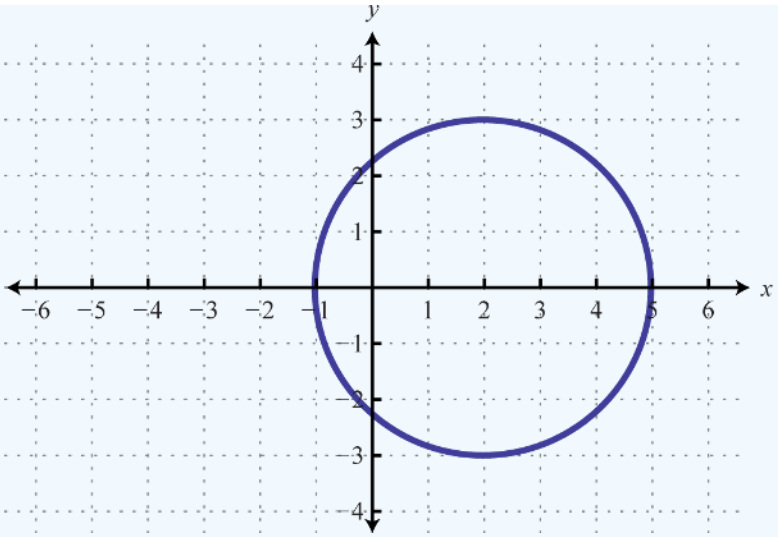


Figure 6.3.13

Solution

From the graph we can see that the minimum x-value is -1 and the maximum x-value is 5. Hence, the domain consists of all the real numbers in the set from $[-1, 5]$. The maximum y-value is 3 and the minimum is -3; hence, the range consists of y-values in the interval $[-3, 3]$.

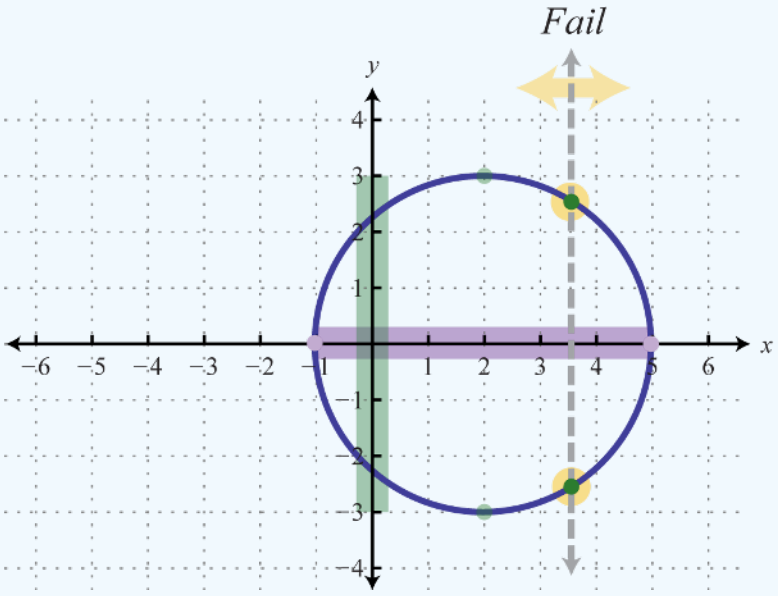


Figure 6.3.14

In addition, since we can find a vertical line that intersects the graph more than once, we conclude that the graph is not a function. There are many x-values in the domain that correspond to two y-values.

Answer:

Domain: $[-1, 5]$; range: $[-3, 3]$; function: no

Exercise 6.3.1

Given the graph, determine the domain and range and state whether or not it is a function:

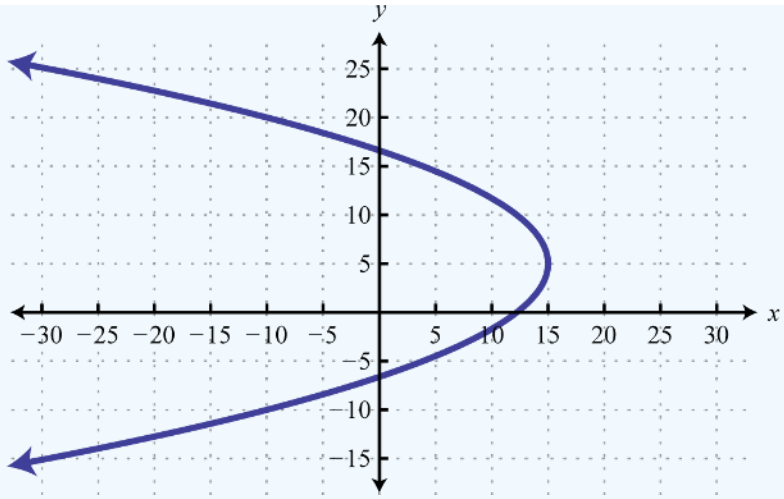


Figure 6.3.15

Answer

Domain: $(-\infty, 15]$; range:$\mathbb{R}$; function: no

www.youtube.com/v/uEtVDkjVSX8

Function Notation

With the definition of a function comes special notation. If we consider each x-value to be the input that produces exactly one output, then we can use **function notation**[16]:

$$f(x) = y$$

The notation $f(x)$ reads, "*f of x*" and should not be confused with multiplication. Algebra frequently involves functions, and so the notation becomes useful when performing common tasks. Here f is the function name, and $f(x)$ denotes the value in the range associated with the value x in the domain. Functions are often named with different letters; some common names for functions are f, g, h, C, and R. We have determined that the set of solutions to $y = |x| - 2$ is a function; therefore, using function notation we can write:

$$\begin{aligned} y &= |x| - 2 \\ &\downarrow \\ f(x) &= |x| - 2 \end{aligned}$$

It is important to note that y and $f(x)$ are used interchangeably. This notation is used as follows:

$$\begin{aligned} f(x) &= |x| - 2 \\ \downarrow &\quad \downarrow \\ f(-5) &= |-5| - 2 = 5 - 2 = 3 \end{aligned}$$

Here the compact notation $f(-5) = 3$ indicates that where $x = -5$ (*the input*), the function results in $y = 3$ (*the output*). In other words, replace the variable with the value given inside the parentheses.

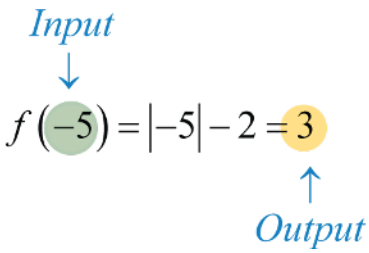


Figure 6.3.16

Functions are compactly defined by an algebraic equation, such as $f(x)=|x|-2$. Given values for x in the domain, we can quickly calculate the corresponding values in the range. As we have seen, functions are also expressed using graphs. In this case, we interpret $f(-5)=3$ as follows:

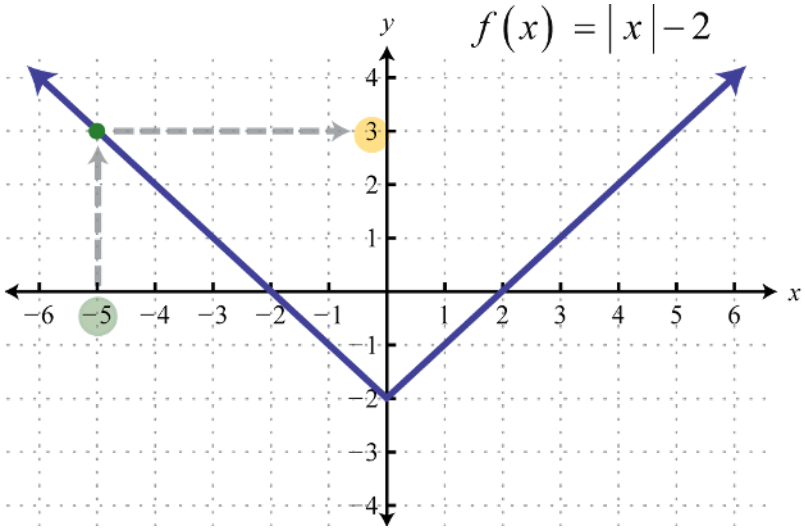


Figure 6.3.17

Function notation streamlines the task of evaluating. For example, use the function h defined by $h(x)=\frac{1}{2}x-3$ to evaluate for x-values in the set $\{-2,0,7\}$.

$$h(-2)=\tfrac{1}{2}(-2)-3=-1-3=-4$$
$$h(0)=\tfrac{1}{2}(0)-3=0-3=-3$$
$$h(7)=\tfrac{1}{2}(7)-3=\tfrac{7}{2}-3=\tfrac{1}{2}$$

Given any function defined by $h(x)=y$, the value x is called the **argument of the function**[17]. The argument can be any algebraic expression. For example:

$$h(4a^3)=\frac{1}{2}(4a^3)-3=2a^3-3$$
$$h(2x-1)=\frac{1}{2}(2x-1)-3=x-\frac{1}{2}-3=x-\frac{7}{2}$$

Example 6.3.5:

Given $g(x)=x^2$, find $g(-2)$, $g(\frac{1}{2})$, and $g(x+h)$.

Solution

Recall that when evaluating, it is a best practice to begin by replacing the variables with parentheses and then substitute the appropriate values. This helps with the order of operations when simplifying expressions.

$$g(-2)=(-2)^2=4$$
$$g(\frac{1}{2})=(\frac{1}{2})^2=\frac{1}{4}$$
$$g(x+h)=(x+h)^2=x^2+2xh+h^2$$

Answer:

$g(-2)=4,\ g(\frac{1}{2})=\frac{1}{4},\ g(x+h)=x^2+2xh+h^2$

At this point, it is important to note that, in general, $f(x+h)\neq f(x)+f(h)$. The previous example, where $g(x)=x^2$, illustrates this nicely

$$g(x+h)\neq g(x)+g(h)$$
$$(x+h)^2\neq x^2+h^2$$

Example 6.3.6:

Given $f(x)=\sqrt{2x+4}$, find $f(-2)$, $f(0)$, and $f\left(\frac{1}{2}a^2-2\right)$.

Solution

$$f(-2)=\sqrt{2(-2)+4}=\sqrt{-4+4}=\sqrt{0}=0$$
$$f(0)=\sqrt{2(0)+4}=\sqrt{0+4}=\sqrt{4}=2$$
$$f\left(\frac{1}{2}a^2-2\right)=\sqrt{2\left(\frac{1}{2}a^2-2\right)+4}=\sqrt{a^2-4+4}=\sqrt{a^2}=|a|$$

Answer:

$f(-2)=0,\ f(0)=2,\ f\left(\frac{1}{2}a^2-2\right)=|a|$

Example 6.3.7:

Given the graph of $g(x)$, find $g(-8)$, $g(0)$, and $g(8)$.

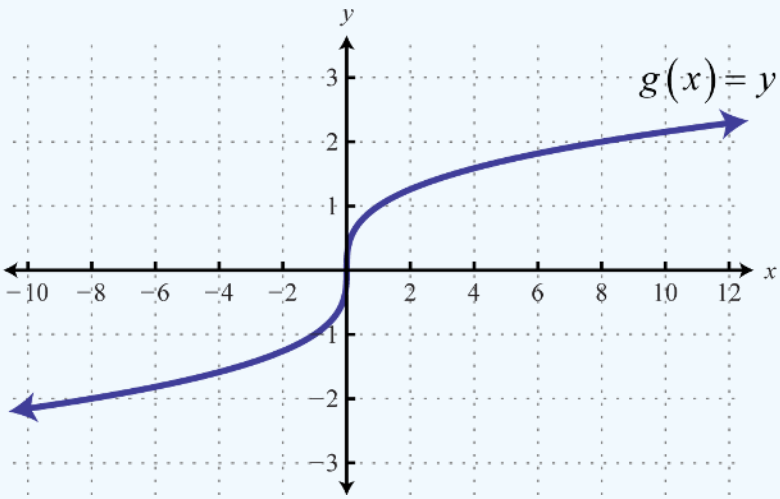


Figure 6.3.18

Solution

Use the graph to find the corresponding y-values where $x=-8, 0$, and 8.

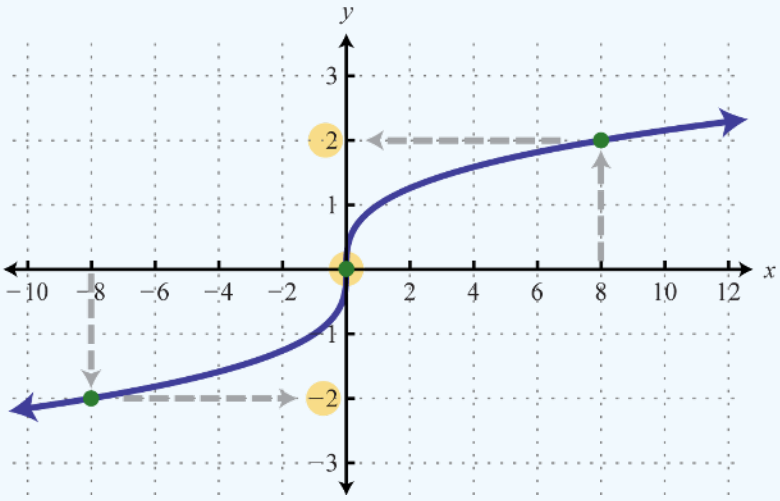


Figure 6.3.19

Answer:

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$g(-8) = -2,\ g(0) = 0,\ g(8) = 2$

Sometimes the output is given and we are asked to find the input.

Example 6.3.8:

Given $f(x) = 5x + 7$, find x where $f(x) = 27$.

Solution

In this example, the output is given and we are asked to find the input. Substitute $f(x)$ with 27 and solve.

$$\begin{aligned} f(x) &= 5x + 7 \\ &\downarrow \\ 27 &= 5x + 7 \\ 20 &= 5x \\ 4 &= x \end{aligned}$$

Therefore, $f(4) = 27$. As a check, we can evaluate $f(4) = 5(4) + 7 = 27$.

Answer:

$x = 4$

Example 6.3.9:

Given the graph of g, find x where $g(x) = 2$.

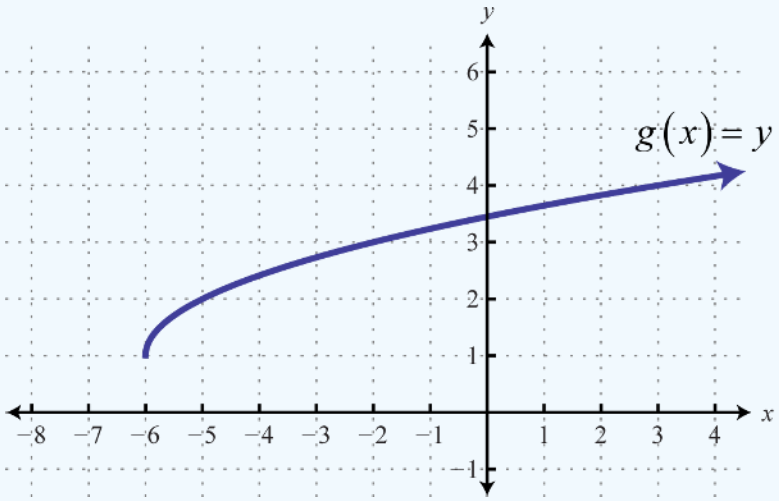


Figure 6.3.20

Solution

Here we are asked to find the *x*-value given a particular *y*-value. We begin with 2 on the *y*-axis and then read the corresponding *x*-value.

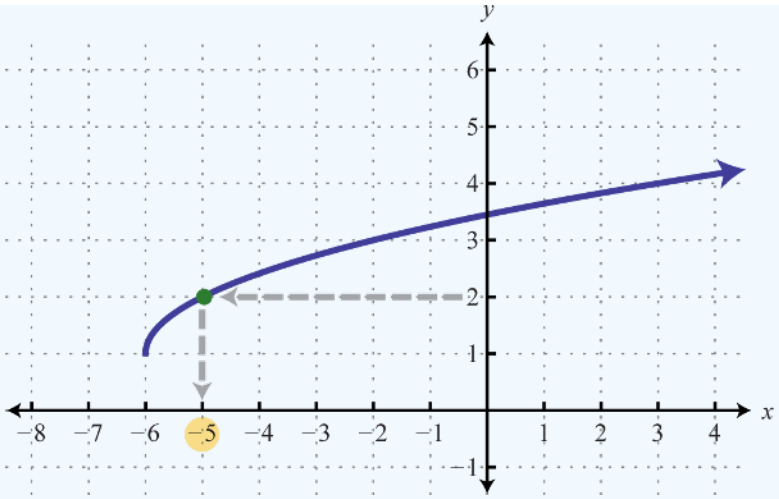


Figure 6.3.21

We can see that $g(x) = 2$ where $x = -5$; in other words, $g(-5) = 2$.

Answer:

$x = -5$

Exercise 6.3.2

Given the graph of h, find x where $h(x) = -4$

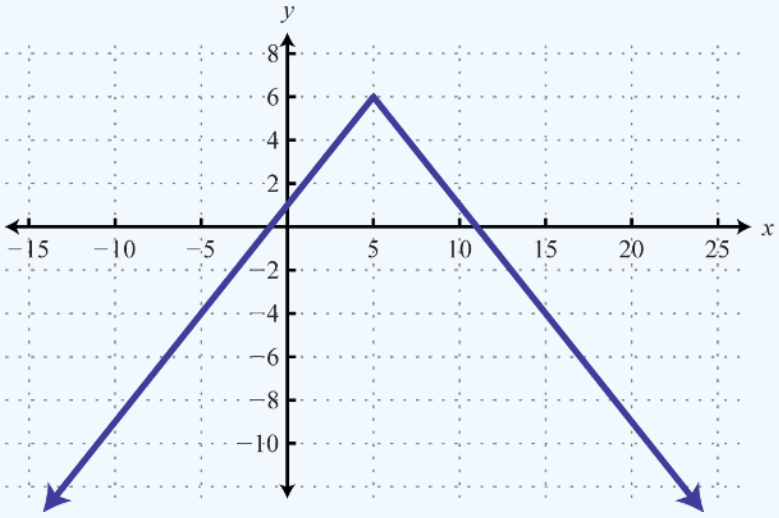


Figure 6.3.22

Answer

$x = -15$ and $x = 15$

www.youtube.com/v/NtvyMlfPqWs

Key Takeaways

- A relation is any set of ordered pairs. However, in this course, we will be working with sets of ordered pairs (x, y) in the rectangular coordinate system. The set of x-values defines the domain and the set of y-values defines the range.
- Special relations where every x-value (input) corresponds to exactly one y-value (output) are called functions.
- We can easily determine whether or not an equation represents a function by performing the vertical line test on its graph. If any vertical line intersects the graph more than once, then the graph does not represent a function.

- If an algebraic equation defines a function, then we can use the notation $f(x) = y$. The notation $f(x)$ is read "f *of* x" and should not be confused with multiplication. When working with functions, it is important to remember that y and $f(x)$ are used interchangeably.
- If asked to find $f(a)$, we substitute the argument a in for the variable and then simplify. The argument could be an algebraic expression.
- If asked to find x where $f(x) = a$, we set the function equal to a and then solve for x.

Exercise 6.3.3

Determine the domain and range and state whether the relation is a function or not.

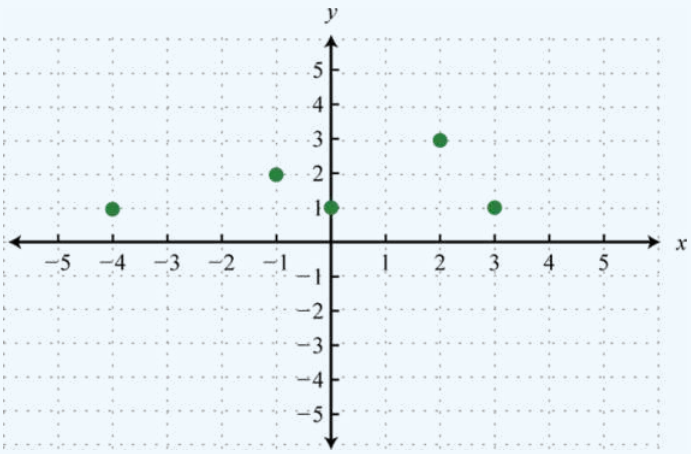
1. $\{(3,1),(5,2),(7,3),(9,4),(12,4)\}$
2. $\{(2,0),(4,3),(6,6),(8,6),(10,9)\}$
3. $\{(7,5),(8,6),(10,7),(10,8),(15,9)\}$
4. $\{(1,1),(2,1),(3,1),(4,1),(5,1)\}$
5. $\{(5,0),(5,2),(5,4),(5,6),(5,8)\}$
6. $\{(-3,1),(-2,2),(-1,3),(0,4),(0,5)\}$
7. 

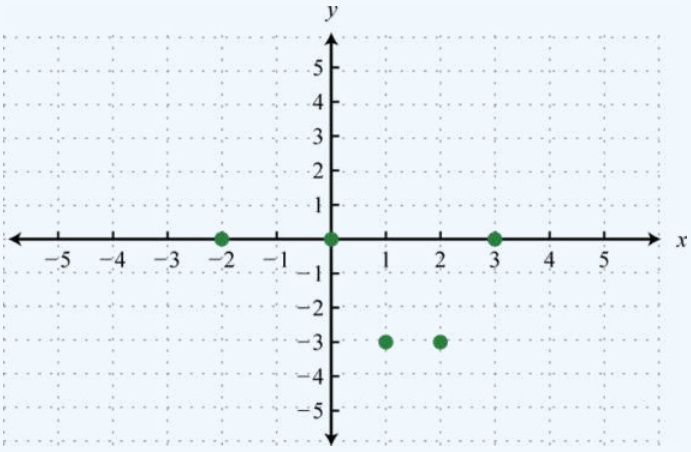
Figure 6.3.23
8. 

Figure 6.3.24
9.

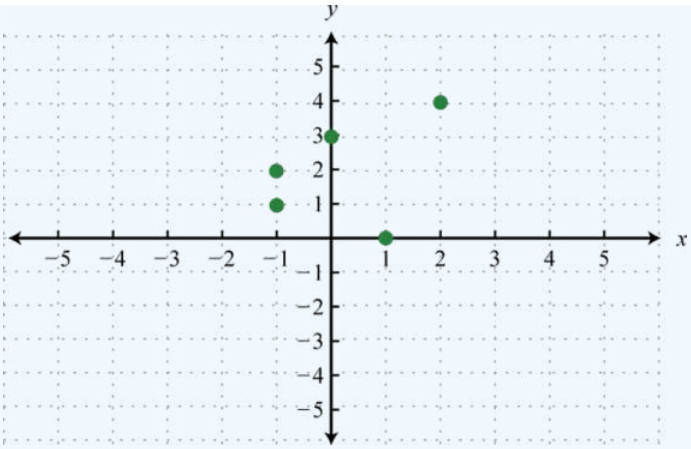


Figure 6.3.25

10.

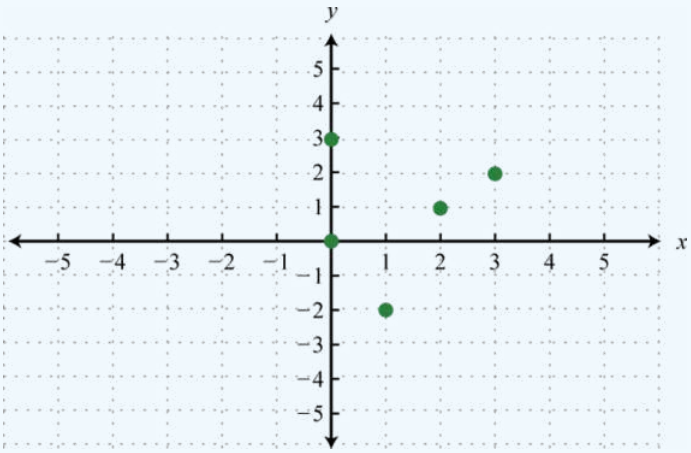


Figure 6.3.26

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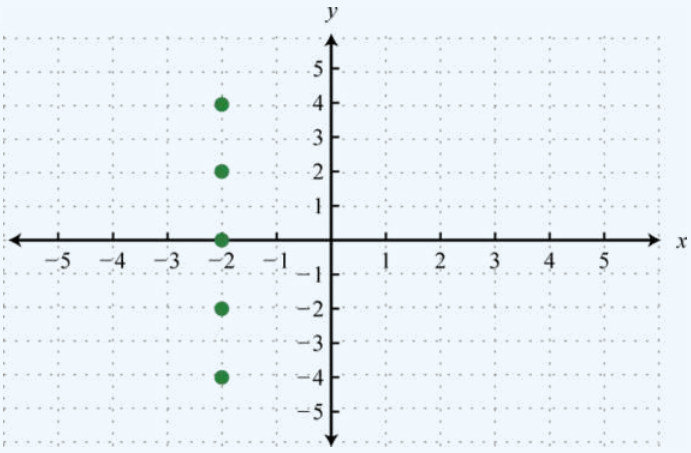


Figure 6.3.27

12.

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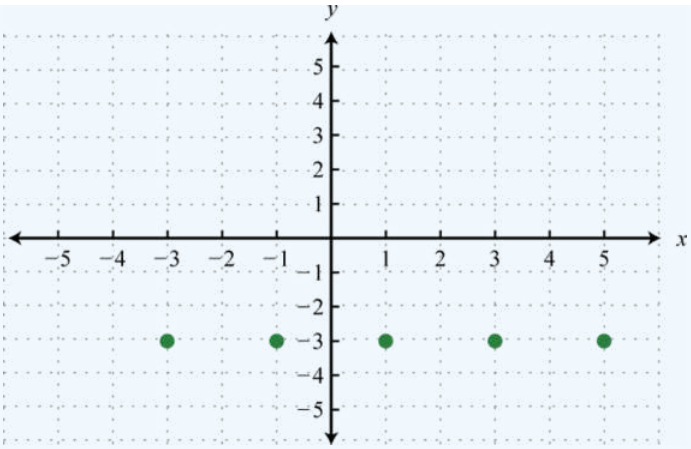


Figure 6.3.28

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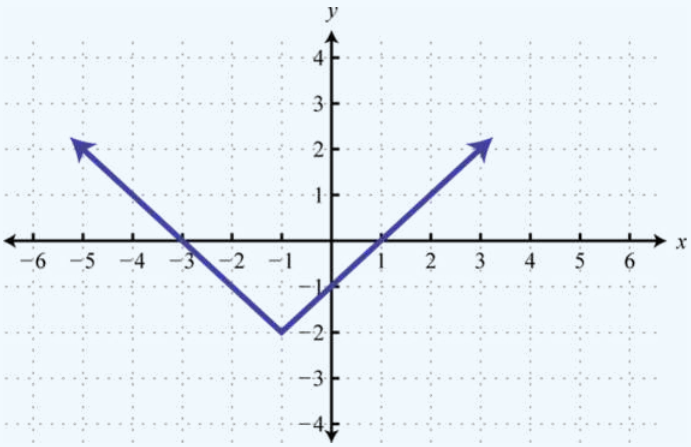


Figure 6.3.29

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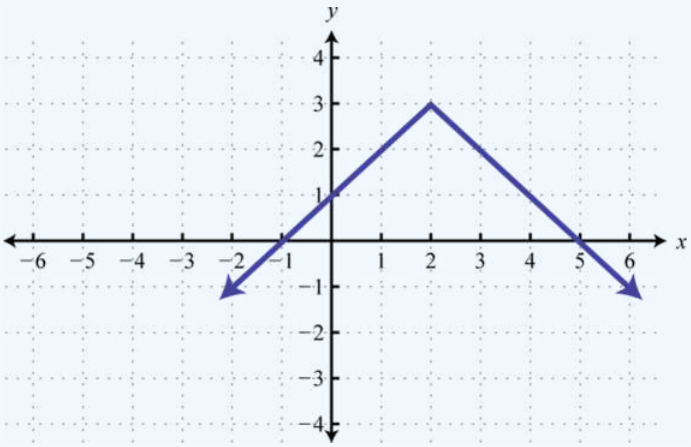


Figure 6.3.30

15.

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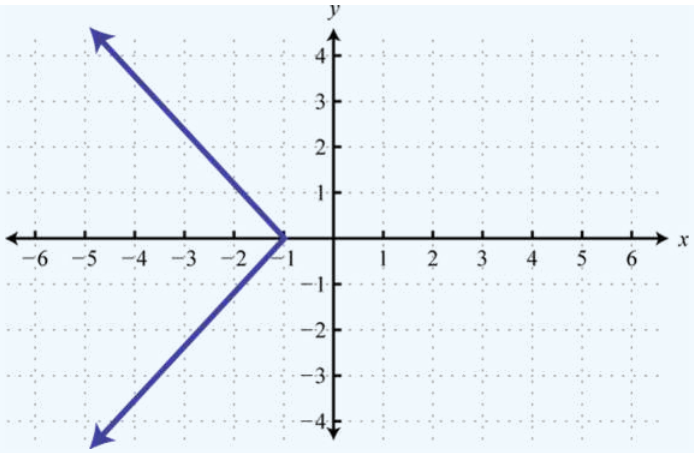


Figure 6.3.31

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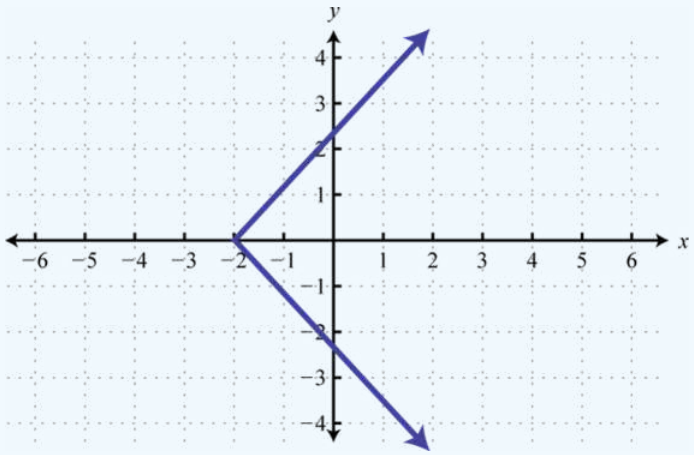


Figure 6.3.32

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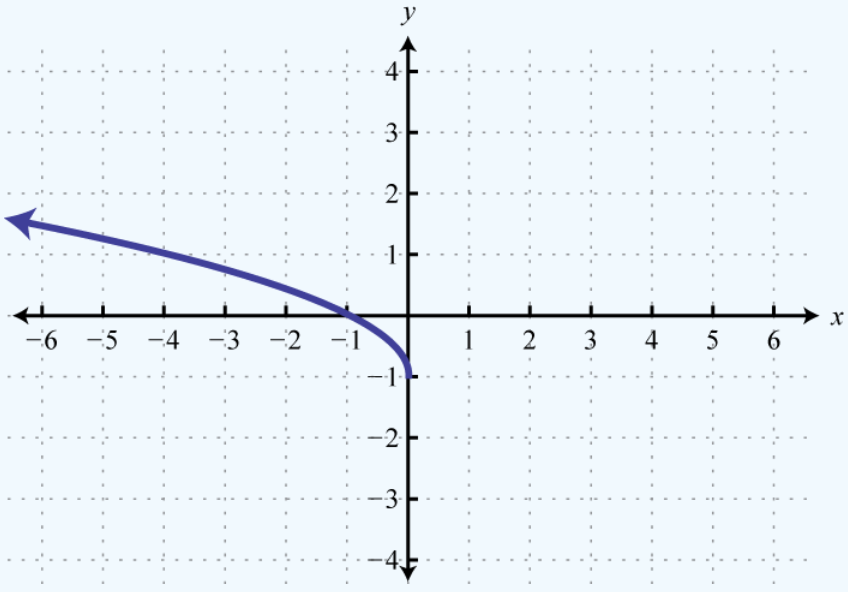


Figure 6.3.33

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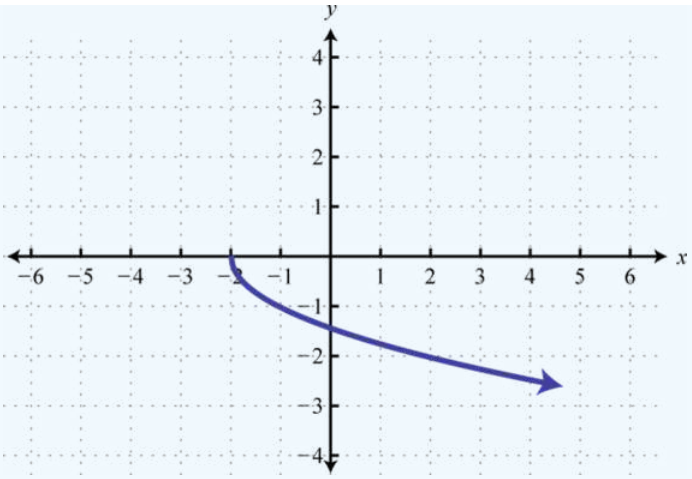


Figure 6.3.34

19.

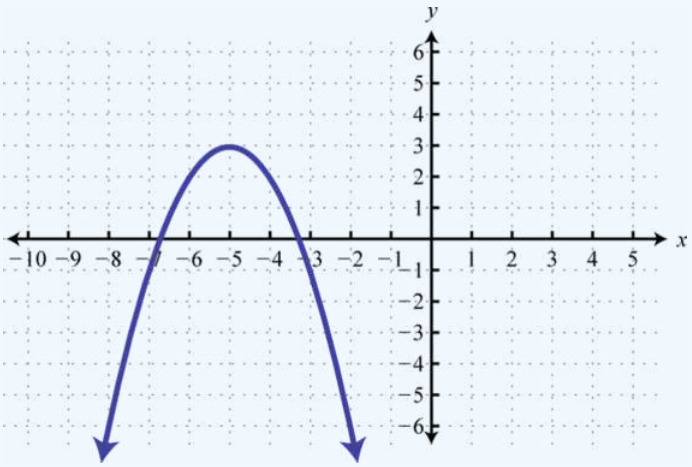


Figure 6.3.35

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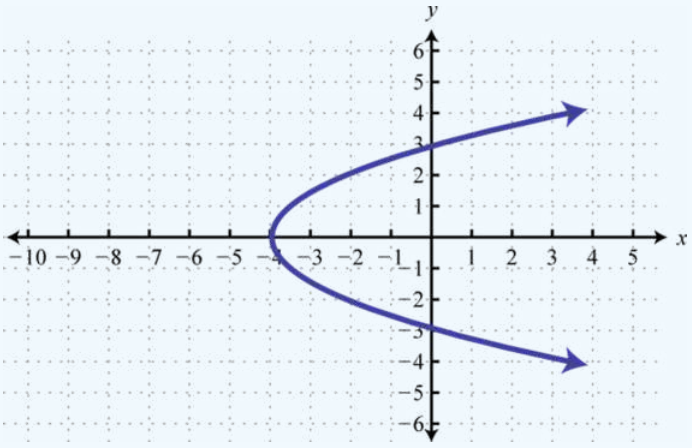


Figure 6.3.36

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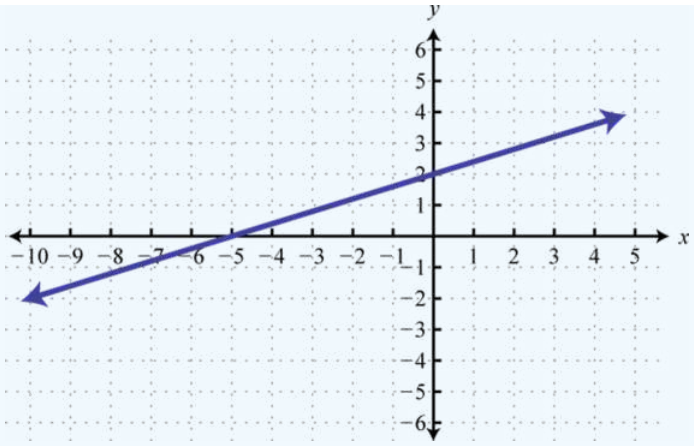


Figure 6.3.37

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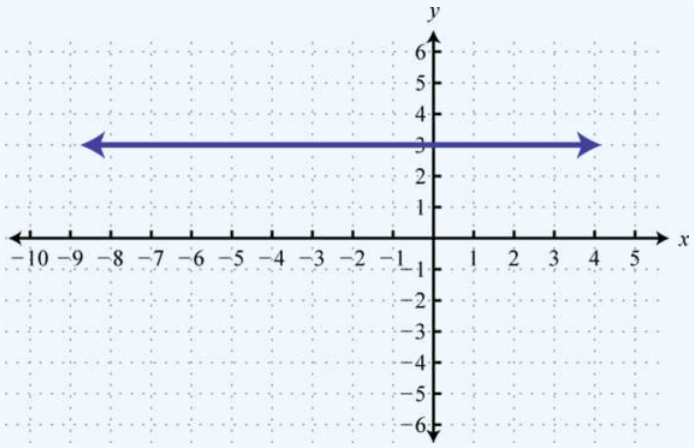


Figure 6.3.38

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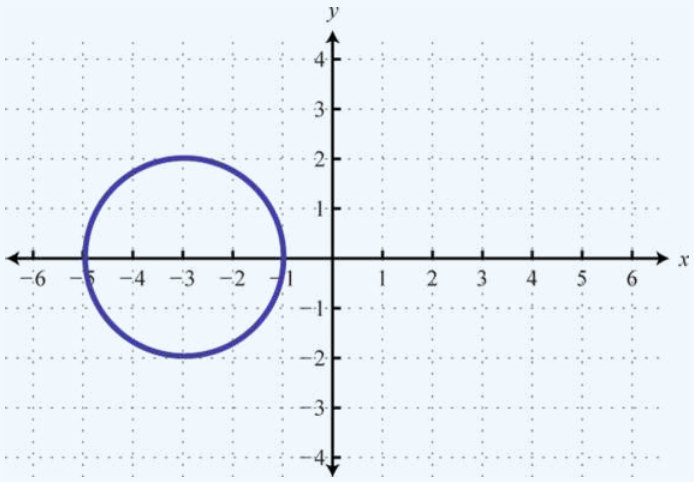


Figure 6.3.39

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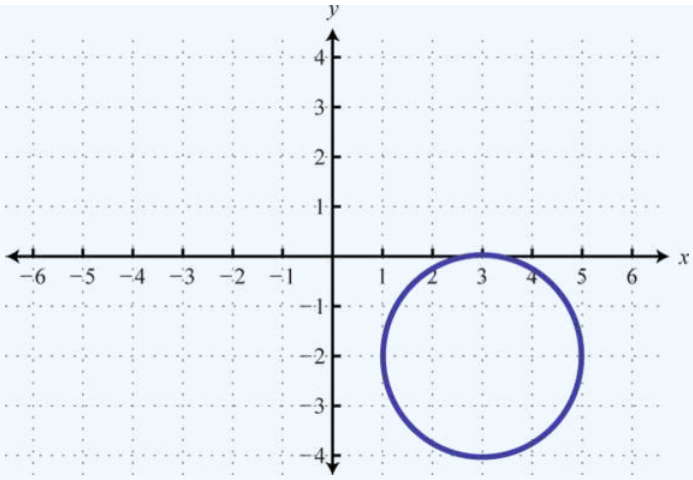


Figure 6.3.40

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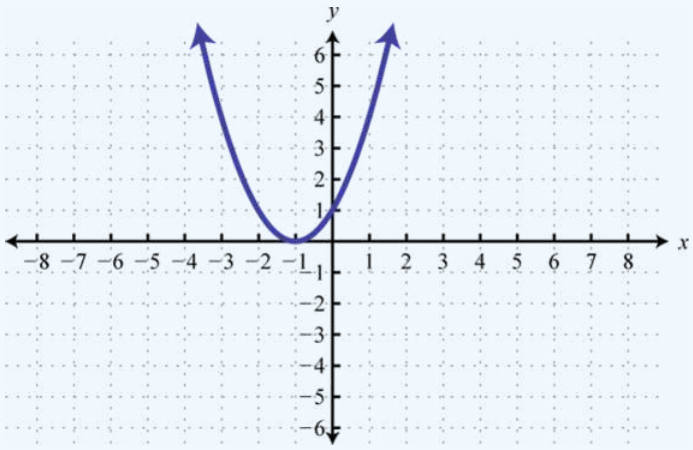


Figure 6.3.41

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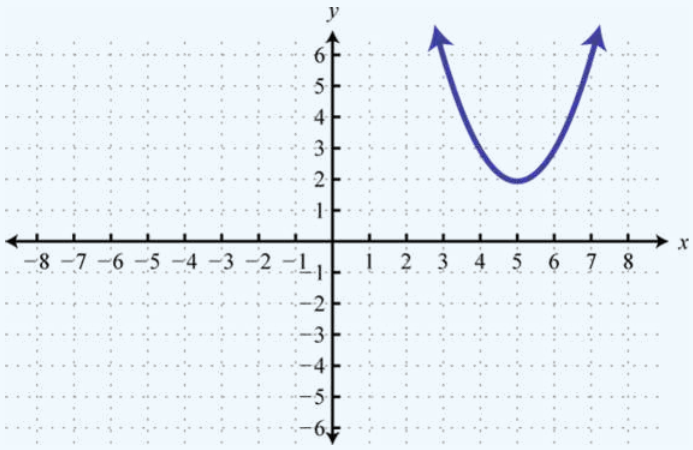


Figure 6.3.42

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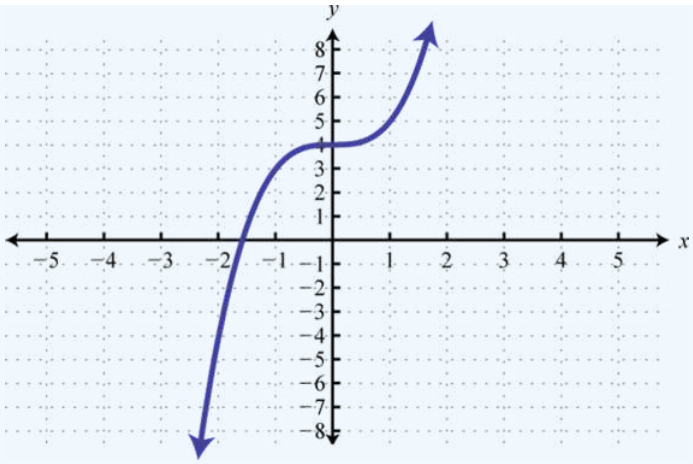


Figure 6.3.43

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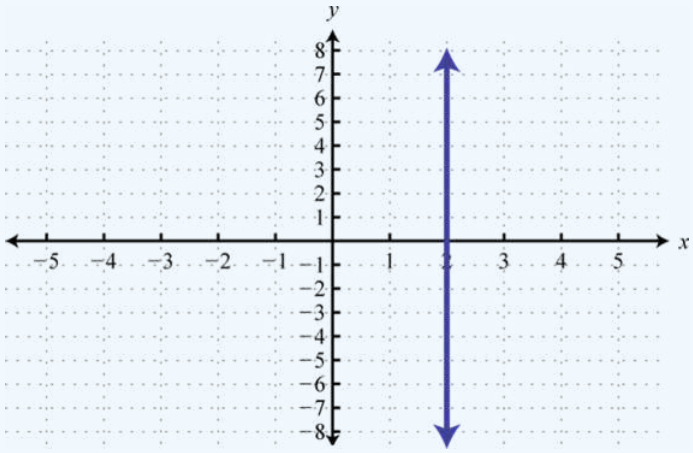


Figure 6.3.44

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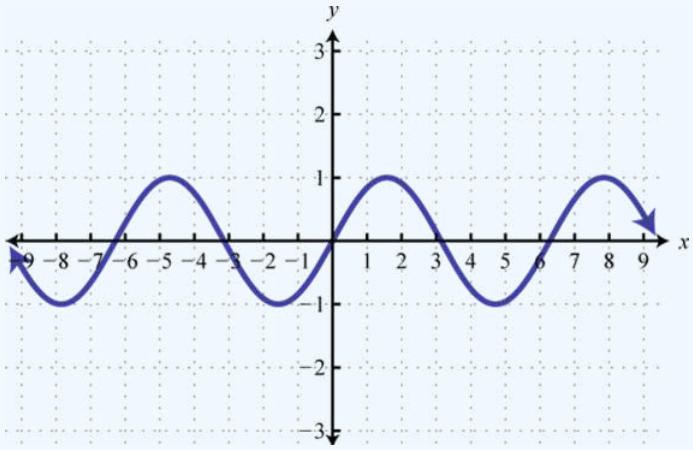


Figure 6.3.45

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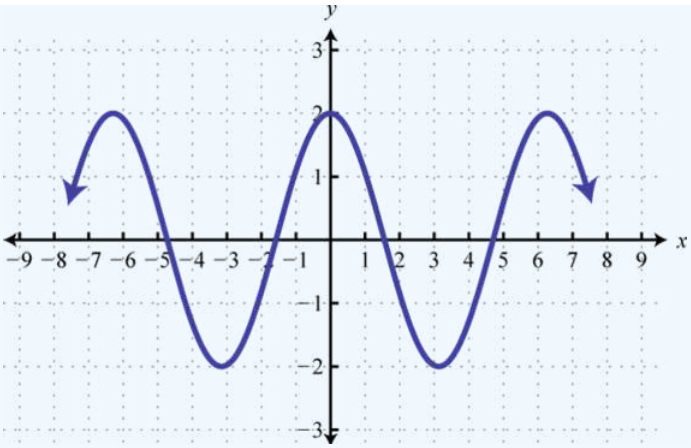


Figure 6.3.46

31.

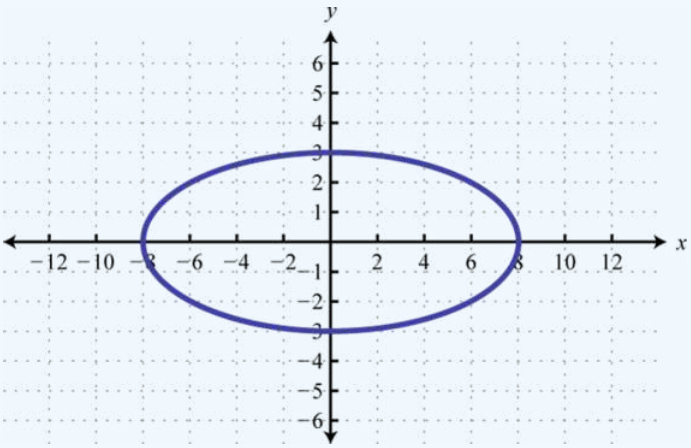


Figure 6.3.47

32.

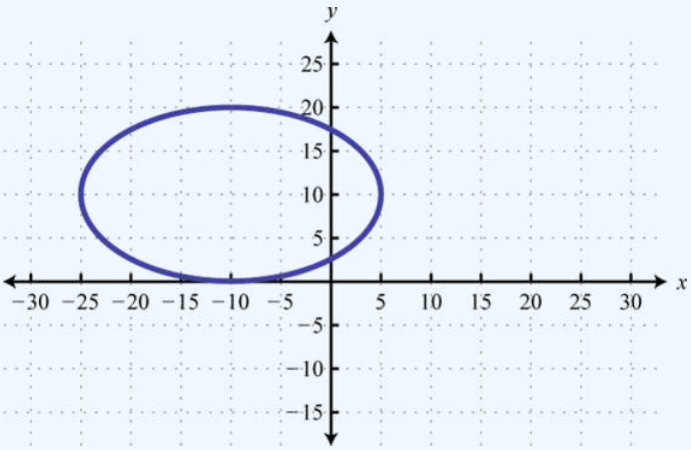


Figure 6.3.48

33.

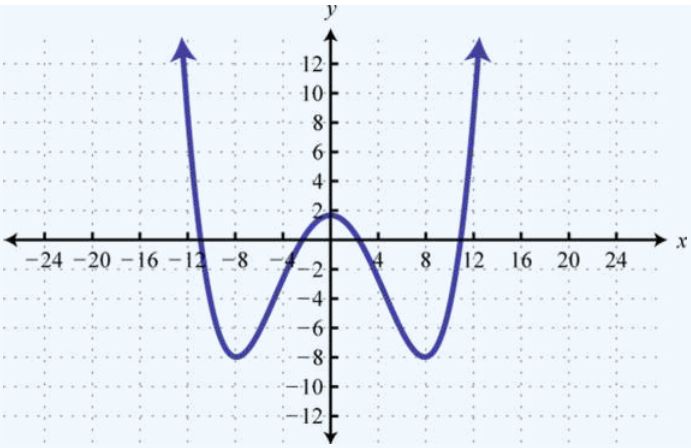


Figure 6.3.49

34.

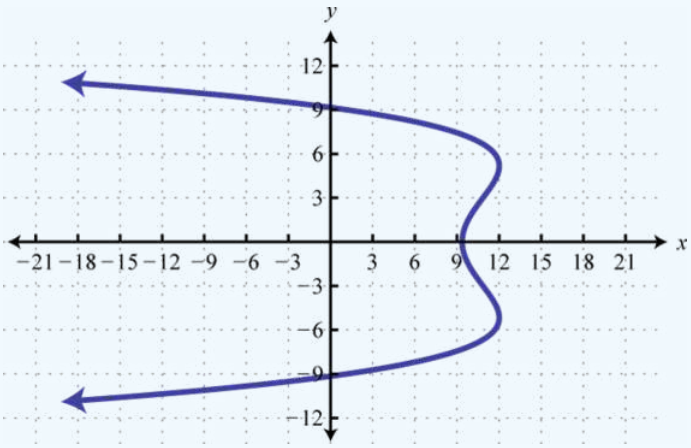


Figure 6.3.50

Answer

1. Domain: $\{3, 5, 7, 9, 12\}$ range: $\{1, 2, 3, 4\}$ function: yes
3. Domain: $\{7, 8, 10, 15\}$ range: $\{5, 6, 7, 8, 9\}$ function: no
5. Domain: $\{5\}$; range: $\{0, 2, 4, 6, 8\}$ function: no
7. Domain: $\{-4, -1, 0, 2, 3\}$ range: $\{1, 2, 3\}$ function: yes
9. Domain: $\{-1, 0, 1, 2\}$ range: $\{0, 1, 2, 3, 4\}$ function: no
11. Domain: $\{-2\}$; range: $\{-4, -2, 0, 2, 4\}$ function: no
13. Domain: $\mathbb{R}$; range: $[-2, \infty)$; function: yes
15. Domain: $(-\infty, -1]$; range: $\mathbb{R}$; function: no
17. Domain: $(-\infty, 0]$; range: $[-1, \infty)$; function: yes
19. Domain: $\mathbb{R}$; range: $(-\infty, 3]$; function: yes
21. Domain: $\mathbb{R}$; range: $\mathbb{R}$; function: yes
23. Domain: $[-5, -1]$; range: $[-2, 2]$; function: no
25. Domain: $\mathbb{R}$; range: $[0, \infty]$; function: yes
27. Domain: $\mathbb{R}$; range: $\mathbb{R}$; function: yes
29. Domain: $\mathbb{R}$; range: $[-1, 1]$; function: yes

31. Domain: $[-8,8]$; range: $[-3,3]$; function: no

33. Domain: $\mathbb{R}$; range: $[-8,\infty]$; function: yes

Exercise 6.3.4

Evaluate.

1. $g(x)=|x-5|$ find $g(-5),g(0),$ and $g(5)$
2. $g(x)=|x|-5;$ find $g(-5),g(0),$ and $g(5)$
3. $g(x)=|2x-3|;$ find $g(-1),g(0),$ and $g\left(\frac{3}{2}\right)$
4. $g(x)=3-|2x|;$ find $g(-3),g(0),$ and $g(3)$
5. $f(x)=2x-3;$ find $f(-2),f(0),$ and $f(x-3)$
6. $f(x)=5x-1;$ find $f(-2),f(0),$ and $f(x+1)$
7. $g(x)=\frac{2}{3}x+1;$ find $g(-3),g(0),$ and $f(9x+6)$
8. $g(x)=-\frac{3}{4}x-\frac{1}{2};$ find $g(-4),g(0),$ and $g(6x-2)$
9. $g(x)=x^2;$ find $g(-5),g(\sqrt{3}),$ and $g(x-5)$
10. $g(x)=x^2+1;$ find $g(-1),g(\sqrt{6}),$ and $g(2x-1)$
11. $f(x)=x^2-x-2;$ find $f(0),f(2),$ and $f(x+2)$
12. $f(x)=-2x^2+x-4;$ find $f(-2),f\left(\frac{1}{2}\right),$ and $f(x-3)$
13. $h(t)=-16t^2+32;$ find $h\left(\frac{1}{4}\right),h\left(\frac{1}{2}\right),$ and $h(2a-1)$
14. $h(t)=-16t^2+32;$ find $h(0),h(\sqrt{2}),h(2a+1)$
15. $f(x)=\sqrt{x+1}-2$ find $f(-1),f(0),f(x-1)$
16. $f(x)=\sqrt{x-3}+1;$ find $f(12),f(3),f(x+3)$
17. $g(x)=\sqrt{x+8};$ find $g(0),g(-8),$ and $g(x-8)$
18. $g(x)=\sqrt{3x-1};$ find $g\left(\frac{1}{3}\right),g\left(\frac{5}{3}\right),$ and $g\left(\frac{1}{3}a^2+\frac{1}{3}\right)$
19. $f(x)=x^3+1;$ find $f(-1),f(0),f\left(a^2\right)$
20. $f(x)=x^3-8;$ find $f(2),f(0),f\left(a^3\right)$

Answer

1. $g(-5)=10,g(0)=5,g(5)=0$
3. $g(-1)=5,g(0)=3,g\left(\frac{3}{2}\right)=0$
5. $f(-2)=-7,f(0)=-3,f(x-3)=2x-9$
7. $g(-3)=-1,g(0)=1,g(9x+6)=6x+5$
9. $g(-5)=25,g(\sqrt{3})=3,g(x-5)=x^2-10x+25$
11. $f(0)=-2,f(2)=0,f(x+2)=x^2+3x$
13. $h\left(\frac{1}{4}\right)=31,h\left(\frac{1}{2}\right)=28,h(2a-1)=-64a^2+64a+16$
15. $f(-1)=-2,f(0)=-1,f(x-1)=\sqrt{x}-2$
17. $g(0)=2\sqrt{2},g(-8)=0,g\left(a^2-8\right)=|a|$
19. $f(-1)=0,f(0)=1,f\left(a^2\right)=a^6+1$

Exercise 6.3.5

Given the function find $f(x+h)$.

1. $f(x)=3x-1$
2. $f(x)=-5x+2$
3. $f(x)=x^2+x+1$
4. $f(x)=2x^2-x-1$

5. $f(x) = x^3$
6. $f(x) = 2x^3 - 1$

Answer

1. $f(x+h) = 3x + 3h - 1$
3. $f(x+h) = x^2 + 2xh + h^2 + x + h + 1$
5. $f(x+h) = x^3 + 3hx^2 + 3h^2x + h^3$

Exercise 6.3.6

Find x given the function.

1. $f(x) = 2x - 3;$ find x where $f(x) = 25$
2. $f(x) = 7 - 3x;$ find x where $f(x) = -27$
3. $f(x) = 2x + 5;$ find x where $f(x) = 0$
4. $f(x) = -2x + 1;$ find x where $f(x) = 0$
5. $g(x) = 6x + 2;$ find x where $g(x) = 5$
6. $g(x) = 4x + 5;$ find x where $g(x) = 2$
7. $h(x) = \frac{2}{3}x - \frac{1}{2};$ find x where $h(x) = \frac{1}{6}$
8. $h(x) = \frac{5}{4}x + \frac{1}{3};$ find x where $h(x) = \frac{1}{2}$
9. The value of a new car in dollars is given by the function $V(t) = -1,800t + 22,000$ where t represents the age of the car in years. Use the function to determine the value of the car when it is 4 years old. What was the value of the car new?
10. The monthly income in dollars of a commissioned car salesperson is given by the function $I(n) = 350n + 1,450$ where n represents the number of cars sold in the month. Use the function to determine the salesperson's income if he sells 3 cars this month. What is his income if he does not sell any cars in one month?

Answer

1. $x = 14$
3. $x = -\frac{5}{2}$
5. $x = \frac{1}{2}$
7. $x = 1$
9. New: $\$22,000$; 4 years old: $\$14,800$

Exercise 6.3.7

Given the graph of the function f, find the function values.

1. Find $f(0)$, $f(2)$, and $f(4)$.

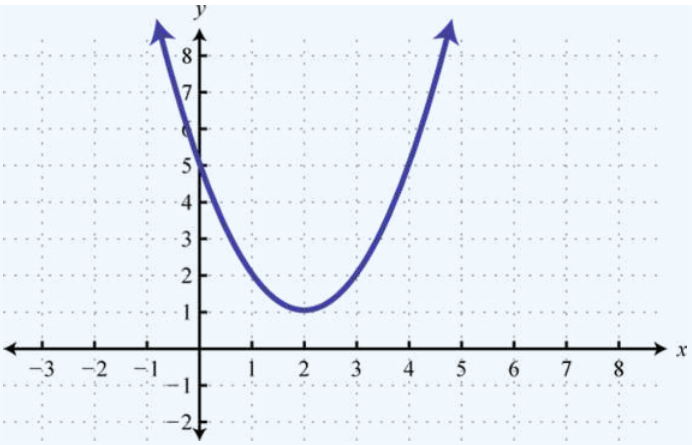


Figure 6.3.51

2. Find $f(-1)$, $f(0)$, and $f(1)$.

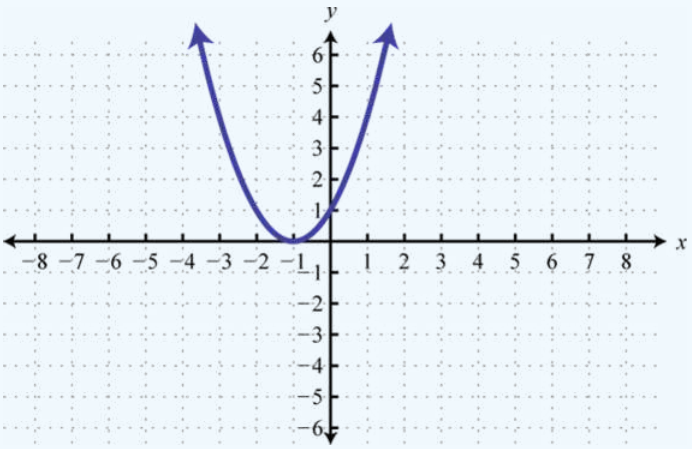


Figure 6.3.52

3. Find $f(0)$, $f(2)$, and $f(4)$.

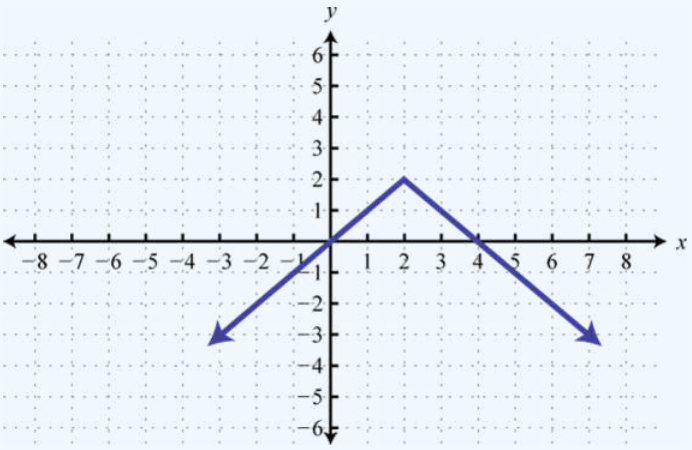


Figure 6.3.53

4. Find $f(-3)$, $f(0)$, and $f(3)$.

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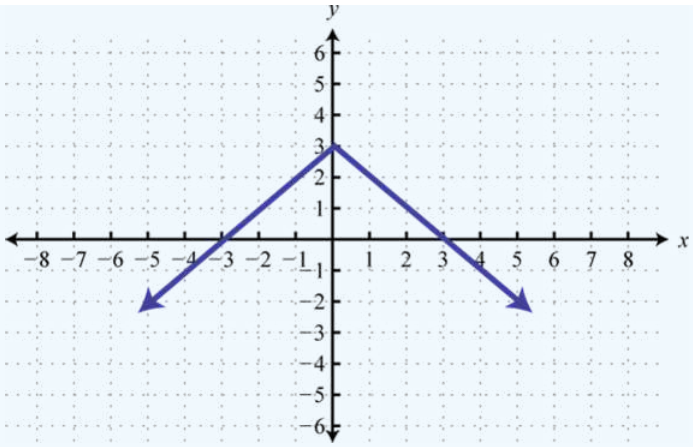


Figure 6.3.54

5. Find $f(-4)$, $f(0)$, and $f(2)$.\

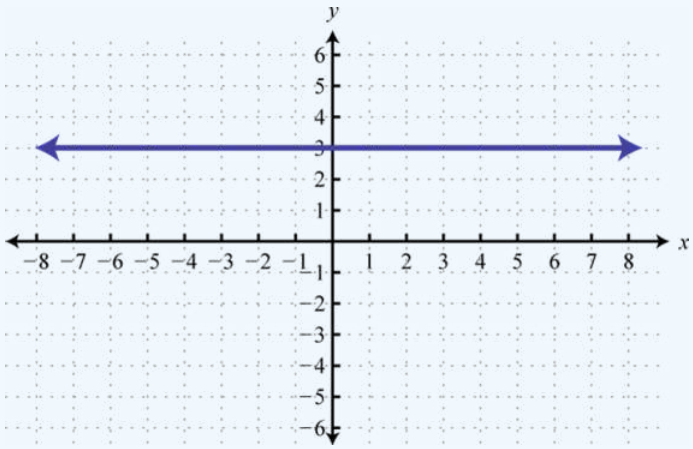


Figure 6.3.55

6. Find $f(-6)$, $f(0)$, and $f(6)$.

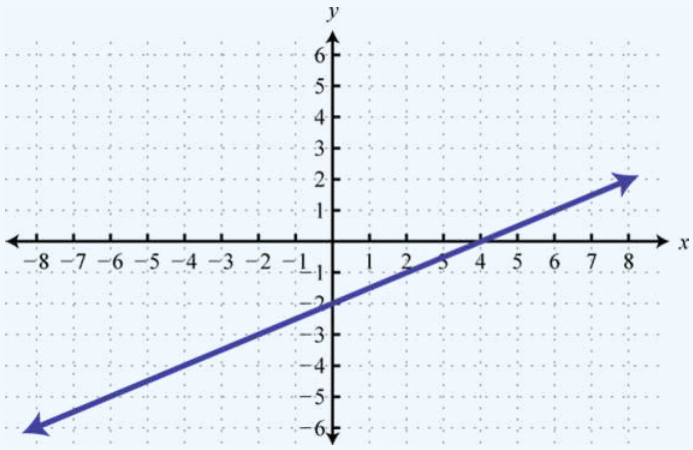


Figure 6.3.56

7. Find $f(-2)$, $f(2)$, and $f(7)$.

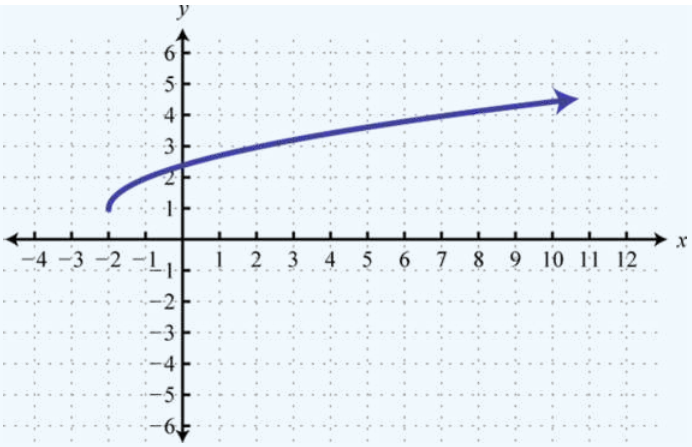


Figure 6.3.57

8. Find $f(0)$, $f(5)$, and $f(9)$.

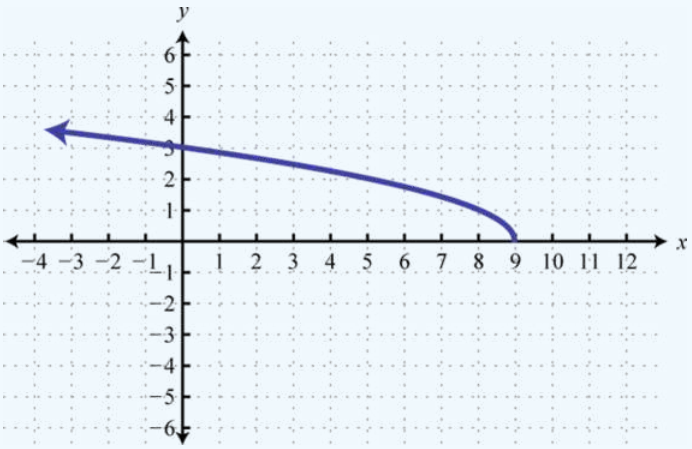


Figure 6.3.58

9. Find $f(-8)$, $f(0)$, and $f(8)$.

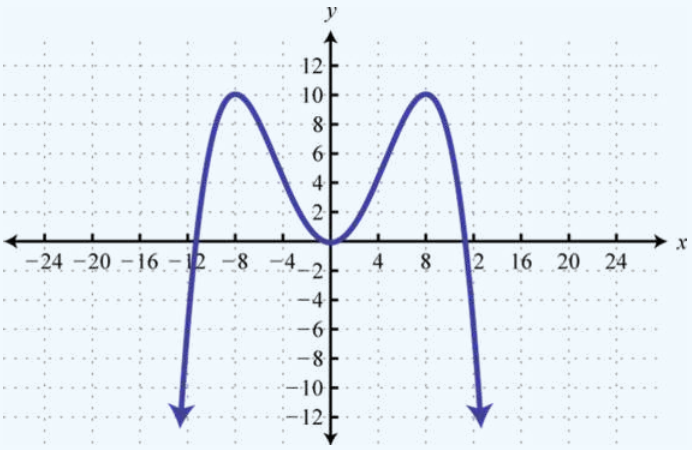


Figure 6.3.59

10. Find $f(-12)$, $f(0)$, and $f(12)$.

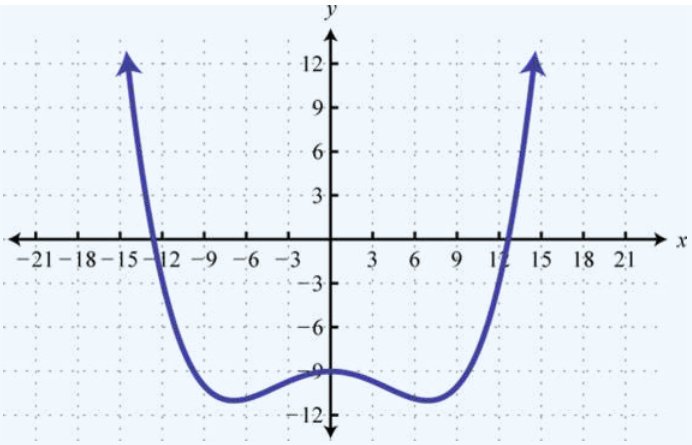


Figure 6.3.60

Answer

1. $f(0) = 5, f(2) = 1, f(4) = 5$
3. $f(0) = 0, f(2) = 2, f(4) = 0$
5. $f(-4) = 3, f(0) = 3, f(2) = 3$
7. $f(-2) = 1, f(2) = 3, f(7) = 4$
9. $f(-8) = 10, f(0) = 0, f(8) = 10$

Exercise 6.3.8

Given the graph of a function g, find the x-values:

1. Find x where $g(x) = 3, g(x) = 0$, and $g(x) = -2$.

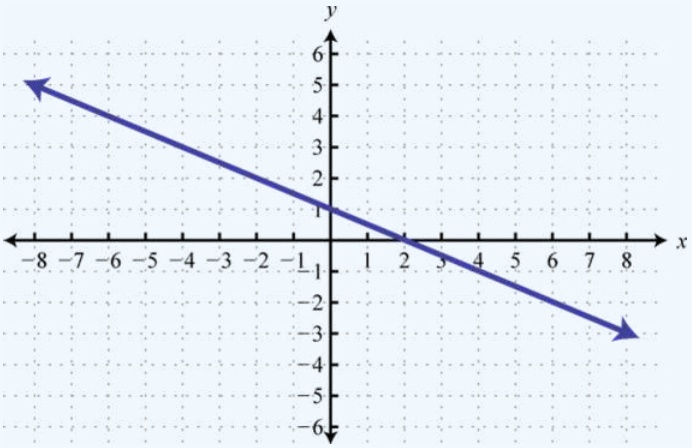


Figure 6.3.61

2. Find x where $g(x) = 0, g(x) = 1$, and $g(x) = 4$.

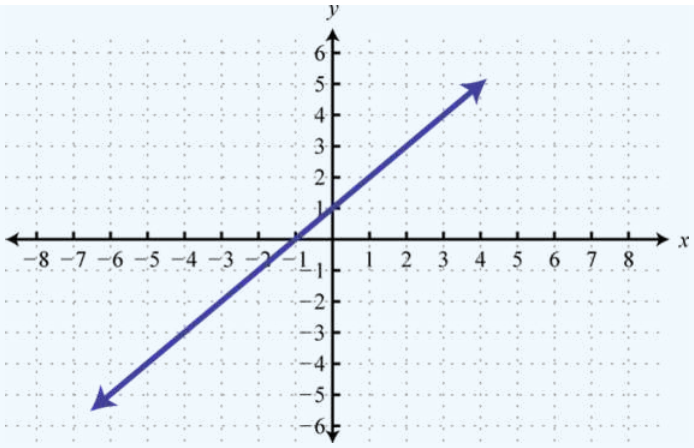


Figure 6.3.62

3. Find x where $g(x) = -5$, $g(x) = 0$, and $g(x) = 10$.

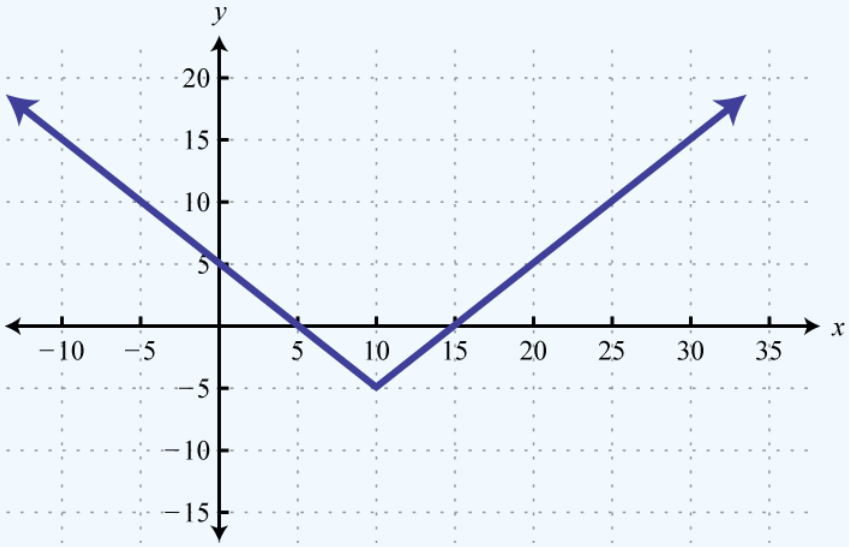


Figure 6.3.63

4. Find x where $g(x) = 0$, $g(x) = 10$, and $g(x) = 15$.

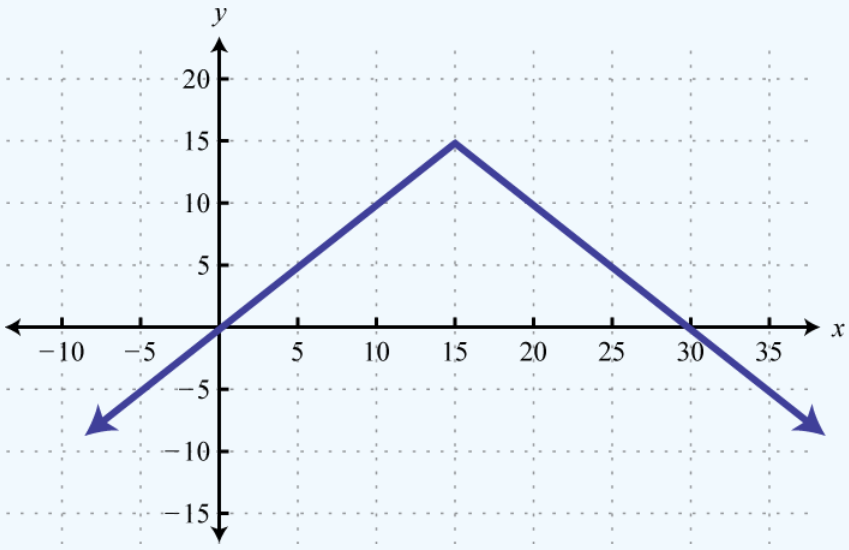


Figure 6.3.64

5. Find x where $g(x) = -5$, $g(x) = -4$, and $g(x) = 4$.\

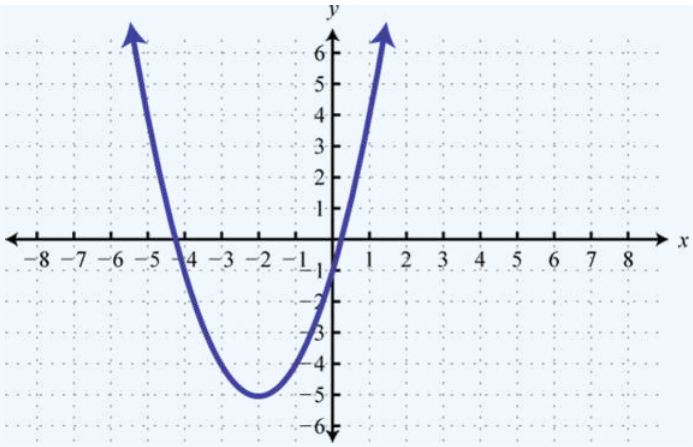


Figure 6.3.65

6. Find x where $g(x) = 1, g(x) = 0$, and $g(x) = -3$.

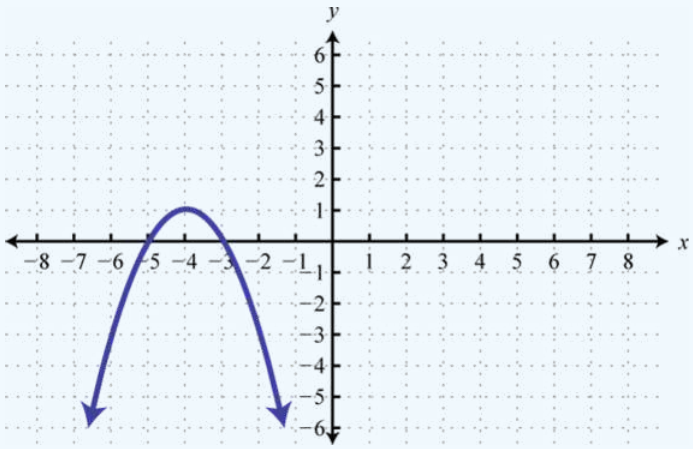


Figure 6.3.66

7. Find x where $g(x) = -4, g(x) = 3$, and $g(x) = 4$.

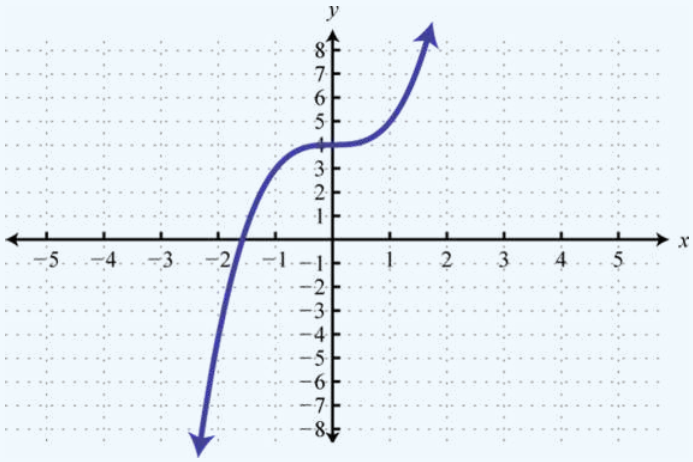


Figure 6.3.67

8. Find x where $g(x) = -5, g(x) = -4$, and $g(x) = 4$.

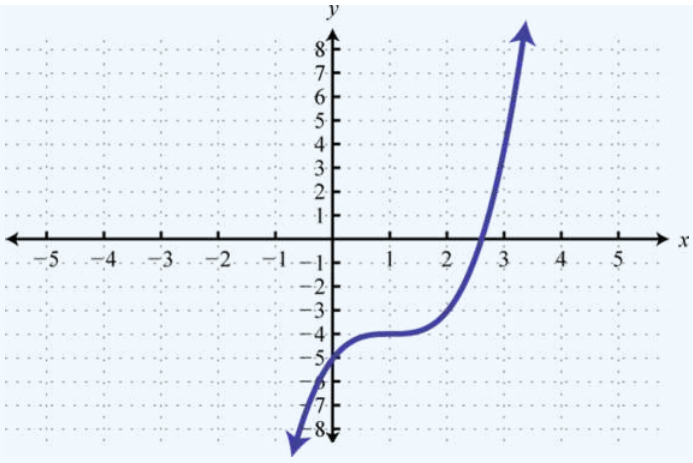


Figure 6.3.68

9. Find x where $g(x) = -10$ and $g(x) = 5$.

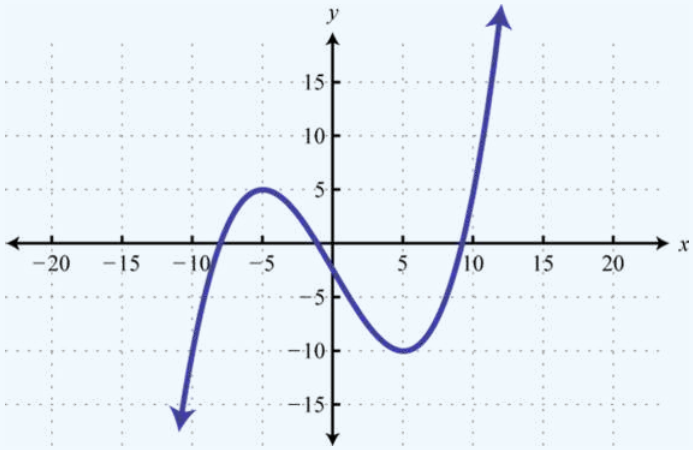


Figure 6.3.69

10. Find x where $g(x) = 2$.

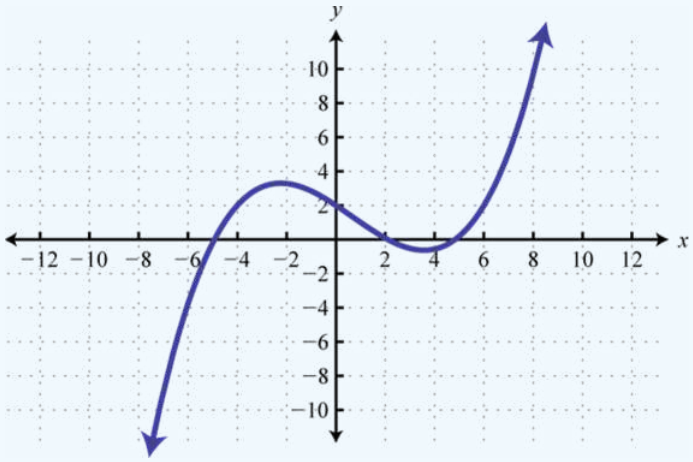


Figure 6.3.70

The value of a certain automobile in dollars depends on the number of years since it was purchased in 1970 according to the following function:

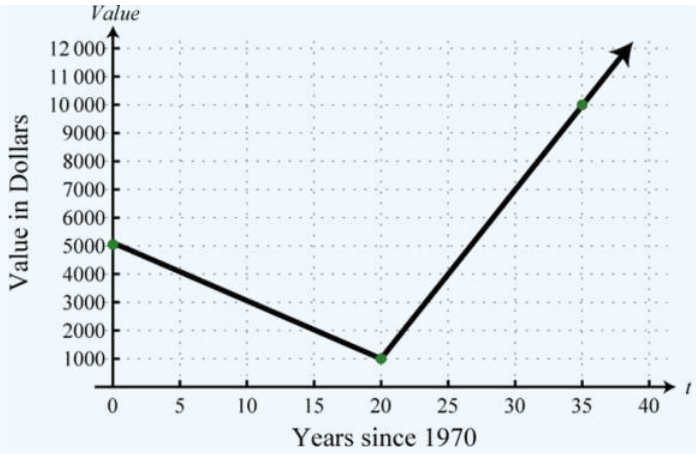


Figure 6.3.71

11. What was the value of the car when it was new in 1970?
12. In what year was the value of the car at a minimum?
13. What was the value of the car in 2005?
14. In what years was the car valued at $\$4,000$?

Answer

1. $g(-4)=3, g(2)=0$, and $g(6)=-2$.
3. $g(10)=-5, g(5)=0$ and $g(15)=0, g(-5)=10$ and $g(25)=10$
5. $g(-2)=-5, g(-3)=-4$ and $g(-1)=-4, g(-5)=4$ and $g(1)=4$
7. $g(-2)=-4, g(-1)=3, g(0)=4$
9. $g(-10)=-10$ and $g(5)=-10; g(-5)=5$ and $g(10)=5$
11. $\$5,000$
13. $\$10,000$

Exercise 6.3.9

Given the linear function defined by $f(x)=2x-5$, simplify the following.

1. $f(5)-f(3)$
2. $f(0)-f(7)$
3. $f(x+2)-f(2)$
4. $f(x+7)-f(7)$
5. $f(x+h)-f(x)$
6. $\frac{f(x+h)-f(x)}{h}$
7. Simplify $\frac{c(x+h)-c(x)}{h}$ given $c(x)=3x+1$.
8. Simplify $\frac{p(x+h)-p(x)}{h}$ given $p(x)=7x-3$.
9. Simplify $\frac{g(x+h)-g(x)}{h}$ given $g(x)=mx+b$.
10. Simplify $\frac{q(x+h)-q(x)}{h}$ given $q(x)=ax$.

Answer

1. 4
3. $2x$
5. $2h$

7. 3

9. m

Exercise 6.3.10

1. Who is credited with the introduction of the notation $y = f(x)$? Provide a brief summary of his life and accomplishments.
2. Explain to a beginning algebra student what the vertical line test is and why it works.
3. Research and discuss the life and contributions of René Descartes.
4. Conduct an Internet search for the vertical line test, functions, and evaluating functions. Share a link to a page that you think others may find useful.

Answer

1. Answer may vary

3. Answer may vary

Footnotes

[1]A system with two number lines at right angles specifying points in a plane using ordered pairs (x, y).

[2]The horizontal number line used as reference in a rectangular coordinate system.

[3]The vertical number line used as reference in a rectangular coordinate system.

[4]The flat surface defined by x- and y-axes.

[5]Pairs (x, y) that identify position relative to the origin on a rectangular coordinate plane.

[6]The point where the x- and y-axes cross, denoted by $(0, 0)$.

[7]The four regions of a rectangular coordinate plane partly bounded by the x- and y-axes and numbered using the Roman numerals I, II, III, and IV.

[8]Term used in honor of René Descartes when referring to the rectangular coordinate system.

[9]Any set of ordered pairs.

[10]A visual representation of a relation on a rectangular coordinate plane.

[11]The set consisting of all of the first components of a relation. For relations consisting of points in the plane, the domain is the set of all x-values.

[12]The set consisting of all of the second components of a relation. For relations consisting of points in the plane, the range is the set of all y-values.

[13]Used when referencing the range.

[14]A relation where each element in the domain corresponds to exactly one element in the range.

[15]If any vertical line intersects the graph more than once, then the graph does not represent a function.

[16]The notation $f(x) = y$, which reads "f of x is equal to y." Given a function, y and $f(x)$ can be used interchangeably.

[17]The value or algebraic expression used as input when using function notation.

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6.4: Set-Builder and Interval Notations

Learning Objectives

- Find the domain of a function defined by an equation.
- Graph piecewise-defined functions.

If you're in the mood for a scary movie, you may want to check out one of the five most popular horror movies of all time—I am Legend, Hannibal, The Ring, The Grudge, and The Conjuring. Figure 6.4.1 shows the amount, in dollars, each of those movies grossed when they were released as well as the ticket sales for horror movies in general by year. Notice that we can use the data to create a function of the amount each movie earned or the total ticket sales for all horror movies by year. In creating various functions using the data, we can identify different independent and dependent variables, and we can analyze the data and the functions to determine the domain and range. In this section, we will investigate methods for determining the domain and range of functions such as these.

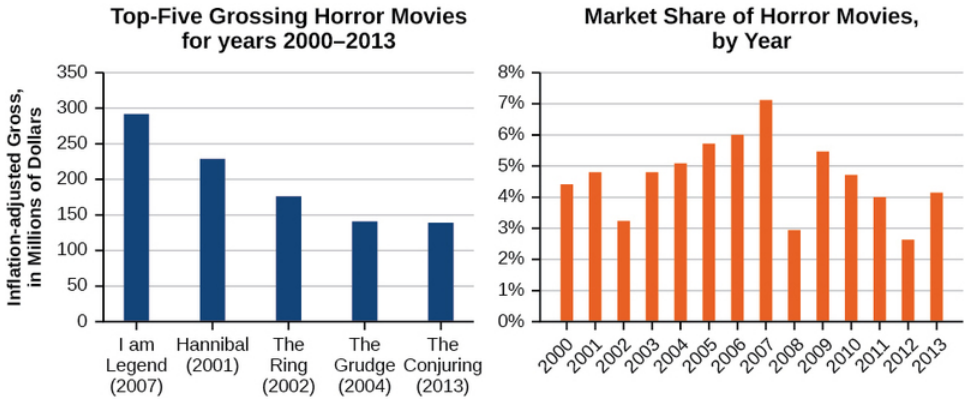


Figure 6.4.1: Graph of the Top-Five Grossing Horror Movies for years 2000-2003, and a Graph of the Market Share of Horror Movies by Year. Based on data compiled by www.the-numbers.com.

Finding the Domain of a Function Defined by an Equation

In Functions and Function Notation, we were introduced to the concepts of **domain and range**. In this section, we will practice determining domains and ranges for specific functions. Keep in mind that, in determining domains and ranges, we need to consider what is physically possible or meaningful in real-world examples, such as tickets sales and year in the horror movie example above. We also need to consider what is mathematically permitted. For example, we cannot include any input value that leads us to take an even root of a negative number if the domain and range consist of real numbers. Or in a function expressed as a formula, we cannot include any input value in the domain that would lead us to divide by 0.

We can visualize the domain as a "holding area" that contains "raw materials" for a "function machine" and the range as another "holding area" for the machine's products (Figure 6.4.2).

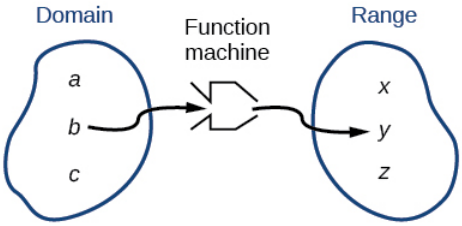


Figure 6.4.2: Diagram of how a function relates two relations

We can write the **domain and range** in **interval notation**, which uses values within brackets to describe a set of numbers. In interval notation, we use a square bracket [when the set includes the endpoint and a parenthesis (to indicate that the endpoint is either not included or the interval is unbounded. For example, if a person has $100 to spend, he or she would need to express the interval that is more than 0 and less than or equal to 100 and write $(0, 100]$. We will discuss interval notation in greater detail later.

Let's turn our attention to finding the domain of a function whose equation is provided. Oftentimes, finding the domain of such functions involves remembering three different forms. First, if the function has no denominator or an even root, consider whether the domain could be all real numbers. Second, if there is a denominator in the function's equation, exclude values in the domain that force the denominator to be zero. Third, if there is an even root, consider excluding values that would make the radicand negative.

Before we begin, let us review the conventions of interval notation:

- The smallest term from the interval is written first.
- The largest term in the interval is written second, following a comma.
- Parentheses, (or), are used to signify that an endpoint is not included, called exclusive.
- Brackets, [or], are used to indicate that an endpoint is included, called inclusive.

See Figure 6.4.3 for a summary of interval notation.

Inequality	Interval Notation	Graph on Number Line	Description
$x > a$	(a, ∞)		x is greater than a
$x < a$	$(-\infty, a)$		x is less than a
$x \geq a$	$[a, \infty)$		x is greater than or equal to a
$x \leq a$	$(-\infty, a]$		x is less than or equal to a
$a < x < b$	(a, b)		x is strictly between a and b
$a \leq x < b$	$[a, b)$		x is between a and b, to include a
$a < x \leq b$	$(a, b]$		x is between a and b, to include b
$a \leq x \leq b$	$[a, b]$		x is between a and b, to include a and b

Figure 6.4.3: Summary of interval notation.

Example 6.4.1: Finding the Domain of a Function as a Set of Ordered Pairs

Find the domain of the following function: $\{(2, 10), (3, 10), (4, 20), (5, 30), (6, 40)\}$

Solution

First identify the input values. The input value is the first coordinate in an ordered pair. There are no restrictions, as the ordered pairs are simply listed. The domain is the set of the first coordinates of the ordered pairs.

$$\{2, 3, 4, 5, 6\}$$

Exercse 6.4.1

Find the domain of the function:

$$\{(-5, 4), (0, 0), (5, -4), (10, -8), (15, -12)\}$$

Answer

$\{-5, 0, 5, 10, 15\}$

How To: Given a function written in equation form, find the domain.

1. Identify the input values.
2. Identify any restrictions on the input and exclude those values from the domain.
3. Write the domain in interval form, if possible.

Example 6.4.2: Finding the Domain of a Function

Find the domain of the function $f(x) = x^2 - 1$.

Solution

The input value, shown by the variable x in the equation, is squared and then the result is lowered by one. Any real number may be squared and then be lowered by one, so there are no restrictions on the domain of this function. The domain is the set of real numbers.

In interval form, the domain of f is $(-\infty, \infty)$.

Exercse 6.4.2

Find the domain of the function:

$$f(x) = 5 - x + x^3$$

Answer

$(-\infty, \infty)$

Howto: Given a function written in an equation form that includes a fraction, find the domain

1. Identify the input values.
2. Identify any restrictions on the input. If there is a denominator in the function's formula, set the denominator equal to zero and solve for x . If the function's formula contains an even root, set the radicand greater than or equal to 0, and then solve.
3. Write the domain in interval form, making sure to exclude any restricted values from the domain.

Example 6.4.3: Finding the Domain of a Function Involving a Denominator

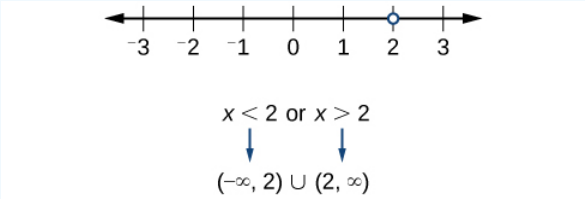
Find the domain of the function $f(x) = \dfrac{x+1}{2-x}$.

Solution

When there is a denominator, we want to include only values of the input that do not force the denominator to be zero. So, we will set the denominator equal to 0 and solve for x.

$$2-x=0$$
$$-x=-2$$
$$x=2$$

Now, we will exclude 2 from the domain. The answers are all real numbers where $x<2$ or $x>2$. We can use a symbol known as the union, $\cup$,to combine the two sets. In interval notation, we write the solution:$(-\infty,2)\cup(2,\infty)$.



In interval form, the domain of f is $(-\infty,2)\cup(2,\infty)$.

Exercse 6.4.3

Find the domain of the function:

$$f(x)=\frac{1+4x}{2x-1}$$

Answer

$$(-\infty,\frac{1}{2})\cup(\frac{1}{2},\infty)$$

How To: Given a function written in equation form including an even root, find the domain.

1. Identify the input values.
2. Since there is an even root, exclude any real numbers that result in a negative number in the radicand. Set the radicand greater than or equal to zero and solve for x.
3. The solution(s) are the domain of the function. If possible, write the answer in interval form.

Example 6.4.4: Finding the Domain of a Function with an Even Root

Find the domain of the function:

$$f(x)=\sqrt{7-x}.$$

Solution

When there is an even root in the formula, we exclude any real numbers that result in a negative number in the radicand.

Set the radicand greater than or equal to zero and solve for x.

$$7-x\geq 0$$
$$-x\geq -7$$
$$x\leq 7$$

Now, we will exclude any number greater than 7 from the domain. The answers are all real numbers less than or equal to 7, or $(-\infty,7]$.

Exercse 6.4.4

Find the domain of the function

$$f(x)=\sqrt{5+2x}.$$

Answer

$$[-2.5, \infty)$$

Q&A: Can there be functions in which the domain and range do not intersect at all?

Yes. For example, the function $f(x) = -\frac{1}{\sqrt{x}}$ *has the set of all positive real numbers as its domain but the set of all negative real numbers as its range. As a more extreme example, a function's inputs and outputs can be completely different categories (for example, names of weekdays as inputs and numbers as outputs, as on an attendance chart), in such cases the domain and range have no elements in common.*

Using Notations to Specify Domain and Range

In the previous examples, we used inequalities and lists to describe the domain of functions. We can also use inequalities, or other statements that might define sets of values or data, to describe the behavior of the variable in set-builder notation. For example, $\{x|10 \leq x < 30\}$ describes the behavior of x in set-builder notation. The braces $\{\}$ are read as "the set of," and the vertical bar $|$ is read as "such that," so we would read $\{x|10 \leq x < 30\}$ as "the set of x-values such that 10 is less than or equal to x, and x is less than 30."

Figure $6.4.4$ compares inequality notation, set-builder notation, and interval notation.

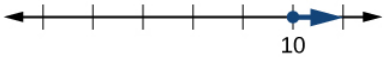
	Inequality Notation	Set-builder Notation	Interval Notation
	$5 < h \leq 10$	$\{h \mid 5 < h \leq 10\}$	$(5, 10]$
	$5 \leq h < 10$	$\{h \mid 5 \leq h < 10\}$	$[5, 10)$
	$5 < h < 10$	$\{h \mid 5 < h < 10\}$	$(5, 10)$
	$h < 10$	$\{h \mid h < 10\}$	$(-\infty, 10)$
	$h \geq 10$	$\{h \mid h \geq 10\}$	$[10, \infty)$
	All real numbers	$\mathbb{R}$	$(-\infty, \infty)$

Figure 6.4.4: Summary of notations for inequalities, set-builder, and interval.

To combine two intervals using inequality notation or set-builder notation, we use the word "or." As we saw in earlier examples, we use the union symbol, $\cup$,to combine two unconnected intervals. For example, the union of the sets $\{2, 3, 5\}$ and $\{4, 6\}$ is the set $\{2, 3, 4, 5, 6\}$. It is the set of all elements that belong to one or the other (or both) of the original two sets. For sets with a finite number of elements like these, the elements do not have to be listed in ascending order of numerical value. If the original two sets have some elements in common, those elements should be listed only once in the union set. For sets of real numbers on intervals, another example of a union is

$$\{x||x| \geq 3\} = (-\infty, -3] \cup [3, \infty) \tag{6.4.1}$$

Set-Builder Notation and Interval Notation

Set-builder notation is a method of specifying a set of elements that satisfy a certain condition. It takes the form $\{x|$ statement about x$\}$ which is read as, "the set of all x such that the statement about x is true." For example,

$$\{x|4 < x \leq 12\}$$

Interval notation is a way of describing sets that include all real numbers between a lower limit that may or may not be included and an upper limit that may or may not be included. The endpoint values are listed between brackets or parentheses. A square bracket indicates inclusion in the set, and a parenthesis indicates exclusion from the set. For example,

$$(4, 12]$$

Given a line graph, describe the set of values using interval notation.

1. Identify the intervals to be included in the set by determining where the heavy line overlays the real line.
2. At the left end of each interval, use [with each end value to be included in the set (solid dot) or (for each excluded end value (open dot).
3. At the right end of each interval, use] with each end value to be included in the set (filled dot) or) for each excluded end value (open dot).
4. Use the union symbol $\cup$ to combine all intervals into one set.

Example 6.4.5: Describing Sets on the Real-Number Line

Describe the intervals of values shown in Figure 6.4.5 using inequality notation, set-builder notation, and interval notation.



Figure 6.4.5: Line graph of $1 \leq x \leq 3$ or $x > 5$.

Solution

To describe the values, x, included in the intervals shown, we would say, "x is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5."

Inequality

$$1 \leq x \leq 3 \text{ or } x > 5$$

Set-builder Notation

$$\{x|1 \leq x \leq 3 \text{ or } x > 5\}$$

Interval notation

$$[1, 3] \cup (5, \infty)$$

Remember that, when writing or reading interval notation, using a square bracket means the boundary is included in the set. Using a parenthesis means the boundary is not included in the set.

Exercse 6.4.5

Given Figure 6.4.6, specify the graphed set in

a. words
b. set-builder notation
c. interval notation



Figure 6.4.6: Line graph of $-2 \leq x$, $-1 \leq x < 3$.leq}x\), $-1 \leq x < 3$.

https://math.libretexts.org/@go/page/108756

Answer a

Values that are less than or equal to –2, or values that are greater than or equal to –1 and less than 3;

Answer b

$\{x|x \leq -2 or -1 \leq x < 3\}$

Answer c

$(-\infty, -2] \cup [-1, 3)$

Finding Domain and Range from Graphs

Another way to identify the domain and range of functions is by using graphs. Because the domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the x-axis. The range is the set of possible output values, which are shown on the y-axis. Keep in mind that if the graph continues beyond the portion of the graph we can see, the domain and range may be greater than the visible values. See Figure 6.4.7.

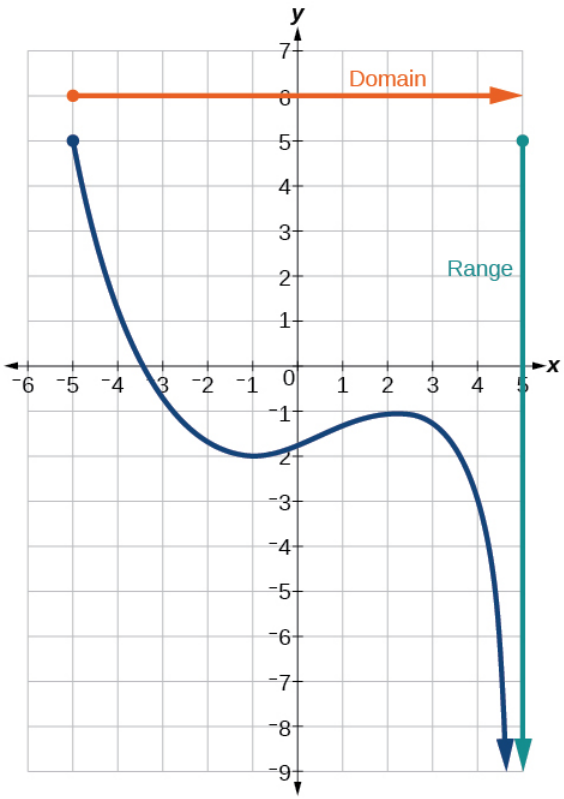


Figure 6.4.7: Graph of a polynomial that shows the x-axis is the domain and the y-axis is the range

We can observe that the graph extends horizontally from −5 to the right without bound, so the domain is $[-5, \infty)$. The vertical extent of the graph is all range values 5 and below, so the range is $(-\infty, 5]$. Note that the domain and range are always written from smaller to larger values, or from left to right for domain, and from the bottom of the graph to the top of the graph for range.

Example 6.4.6A: Finding Domain and Range from a Graph

Find the domain and range of the function f whose graph is shown in Figure 1.2.8.

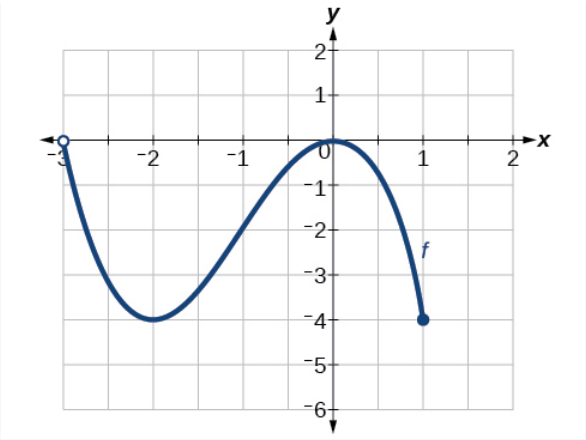


Figure 6.4.8: Graph of a function from (-3, 1].

Solution

We can observe that the horizontal extent of the graph is –3 to 1, so the domain of f is $(-3, 1]$.

The vertical extent of the graph is 0 to –4, so the range is $[-4, 0)$. See Figure 6.4.9.

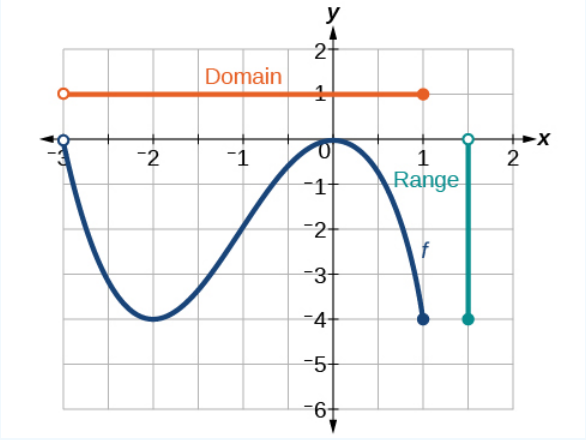


Figure 6.4.9: Graph of the previous function shows the domain and range

Example 6.4.6B: Finding Domain and Range from a Graph of Oil Production

Find the domain and range of the function f whose graph is shown in Figure 6.4.10.

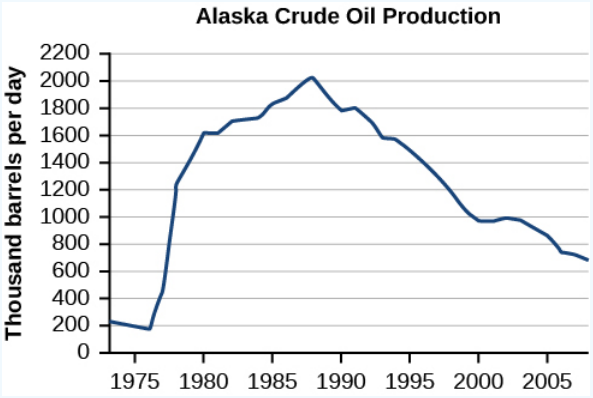


Figure 6.4.10: Graph of the Alaska Crude Oil Production where the y-axis is thousand barrels per day and the -axis is the years(credit: modification of work by the U.S. Energy Information Administration)

Solution

The input quantity along the horizontal axis is “years,” which we represent with the variable t for time. The output quantity is “thousands of barrels of oil per day,” which we represent with the variable b for barrels. The graph may continue to the left and right beyond what is viewed, but based on the portion of the graph that is visible, we can determine the domain as $1973 \leq t \leq 2008$ and the range as approximately $180 \leq b \leq 2010$.

In interval notation, the domain is $[1973, 2008]$ and the range is about $[180, 2010]$. For the domain and the range, we approximate the smallest and largest values since they do not fall exactly on the grid lines.

Exercse 6.4.6

Given Figure 6.4.11, identify the domain and range using interval notation.

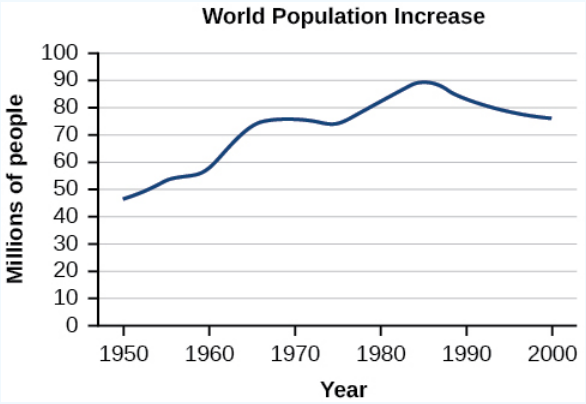


Figure 6.4.11: Graph of World Population Increase where the y-axis represents millions of people and the x-axis represents the year.

Answer

domain =$[1950, 2002]$

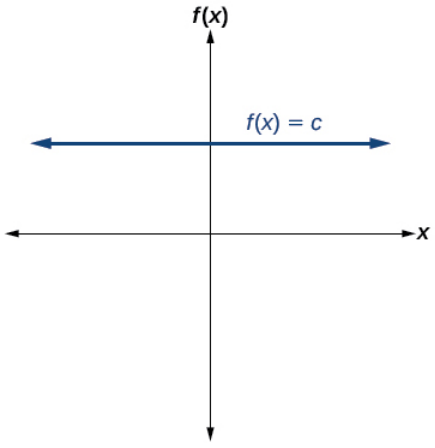
range = $[47,000,000, 89,000,000]$

Can a function’s domain and range be the same?

Yes. For example, the domain and range of the cube root function are both the set of all real numbers.

Finding Domains and Ranges of the Toolkit Functions

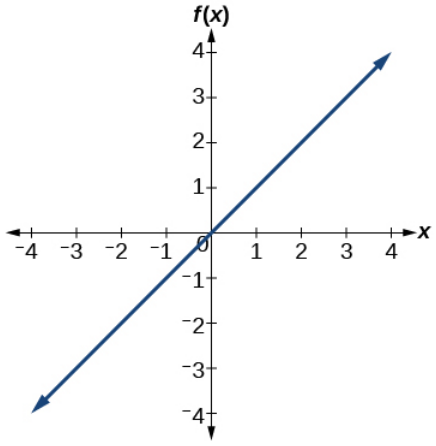
We will now return to our set of toolkit functions to determine the domain and range of each.



Domain: $(-\infty, \infty)$
Range: $[c,c]$

Figure 6.4.12: Constant function $f(x) = c$.

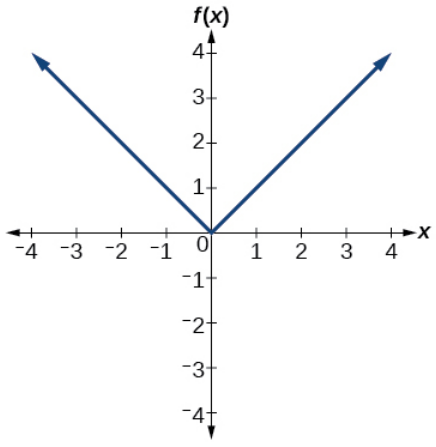
For the **constant function** $f(x) = c$, the domain consists of all real numbers; there are no restrictions on the input. The only output value is the constant c, so the range is the set $\{c\}$ that contains this single element. In interval notation, this is written as $[c, c]$, the interval that both begins and ends with c.



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

***Figure* 6.4.13:** *Identity function f(x)=x.*

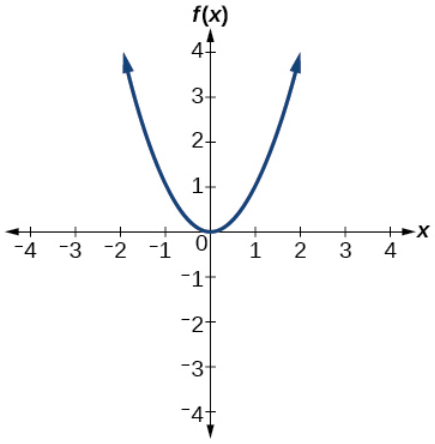
For the **identity function** $f(x) = x$, there is no restriction on x. Both the domain and range are the set of all real numbers.



Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

Figure 6.4.14: Absolute function $f(x) = |x|$.

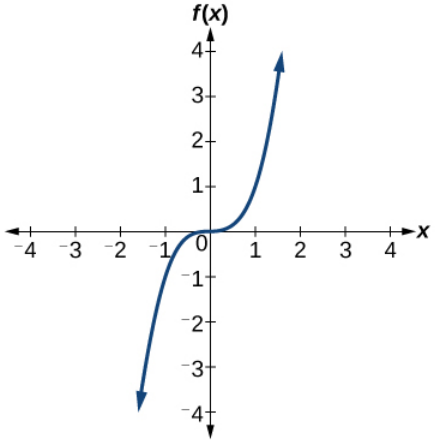
For the **absolute value function** $f(x) = |x|$, there is no restriction on x. However, because absolute value is defined as a distance from 0, the output can only be greater than or equal to 0.



Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

Figure 6.4.15: Quadratic function $f(x) = x^2$.

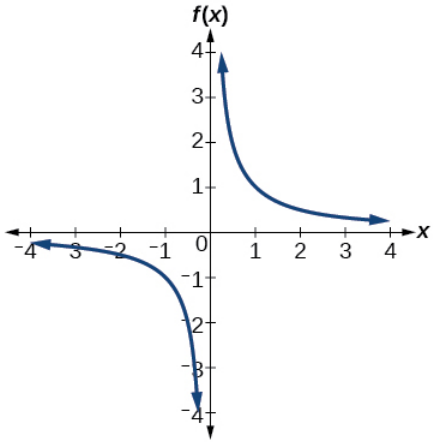
For the **quadratic function** $f(x) = x^2$, the domain is all real numbers since the horizontal extent of the graph is the whole real number line. Because the graph does not include any negative values for the range, the range is only nonnegative real numbers.



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Figure 6.4.16: Cubic function $f(x) - x^3$.

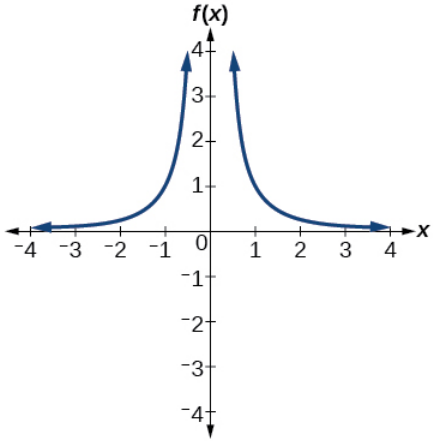
For the **cubic function** $f(x) = x^3$, the domain is all real numbers because the horizontal extent of the graph is the whole real number line. The same applies to the vertical extent of the graph, so the domain and range include all real numbers.



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$

Figure 6.4.17: Reciprocal function $f(x) = \frac{1}{x}$.

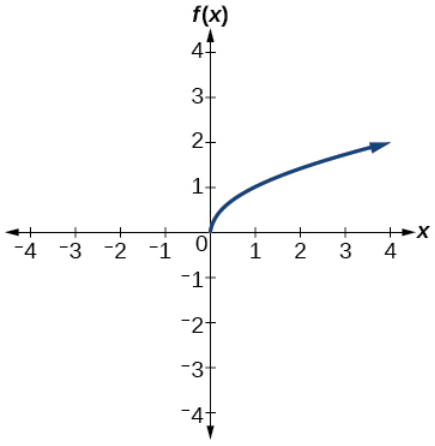
For the **reciprocal function** $f(x) = \frac{1}{x}$, we cannot divide by 0, so we must exclude 0 from the domain. Further, 1 divided by any value can never be 0, so the range also will not include 0. In set-builder notation, we could also write $\{x | x \neq 0\}$,the set of all real numbers that are not zero.



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(0, \infty)$

Figure 6.4.18: Reciprocal squared function $f(x) = \frac{1}{x^2}$

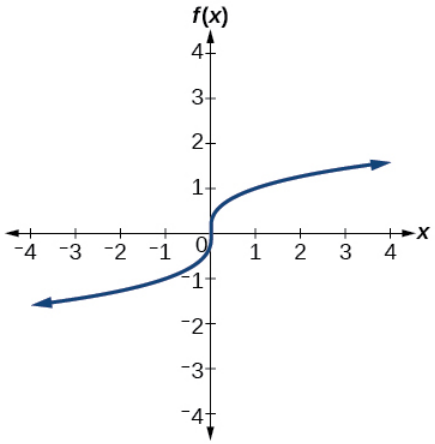
For the **reciprocal squared function** $f(x) = \frac{1}{x^2}$,we cannot divide by 0, so we must exclude 0 from the domain. There is also no x that can give an output of 0, so 0 is excluded from the range as well. Note that the output of this function is always positive due to the square in the denominator, so the range includes only positive numbers.



Domain: [0, ∞)
Range: [0, ∞)

Figure* 6.4.19*: *Square root function* $f(x)=\sqrt{(x)}$.

For the **square root function** $f(x)=\sqrt{x}$, we cannot take the square root of a negative real number, so the domain must be 0 or greater. The range also excludes negative numbers because the square root of a positive number x is defined to be positive, even though the square of the negative number $-\sqrt{x}$ also gives us x.



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Figure 6.4.20: Cube root function $f(x)=\sqrt[3]{x}$.

For the **cube root function** $f(x)=\sqrt[3]{x}$, the domain and range include all real numbers. Note that there is no problem taking a cube root, or any odd-integer root, of a negative number, and the resulting output is negative (it is an odd function).

Given the formula for a function, determine the domain and range.

1. Exclude from the domain any input values that result in division by zero.
2. Exclude from the domain any input values that have nonreal (or undefined) number outputs.
3. Use the valid input values to determine the range of the output values.
4. Look at the function graph and table values to confirm the actual function behavior.

Finding the Domain and Range Using Toolkit Functions

Find the domain and range of $f(x)=2x^3-x$.

Solution

There are no restrictions on the domain, as any real number may be cubed and then subtracted from the result.

The domain is $(-\infty, \infty)$ and the range is also $(-\infty, \infty)$.

Example 6.4.7B: Finding the Domain and Range

Find the domain and range of $f(x) = \frac{2}{x+1}$.

Solution

We cannot evaluate the function at -1 because division by zero is undefined. The domain is $(-\infty, -1) \cup (-1, \infty)$. Because the function is never zero, we exclude 0 from the range. The range is $(-\infty, 0) \cup (0, \infty)$.

Example 6.4.7C: Finding the Domain and Range

Find the domain and range of $f(x) = 2\sqrt{x+4}$.

Solution

We cannot take the square root of a negative number, so the value inside the radical must be nonnegative.

$x + 4 \geq 0$ when $x \geq -4$

The domain of $f(x)$ is $[-4, \infty)$.

We then find the range. We know that $f(-4) = 0$, and the function value increases as x increases without any upper limit. We conclude that the range of f is $[0, \infty)$.

Analysis

Figure 6.4.19represents the function f.

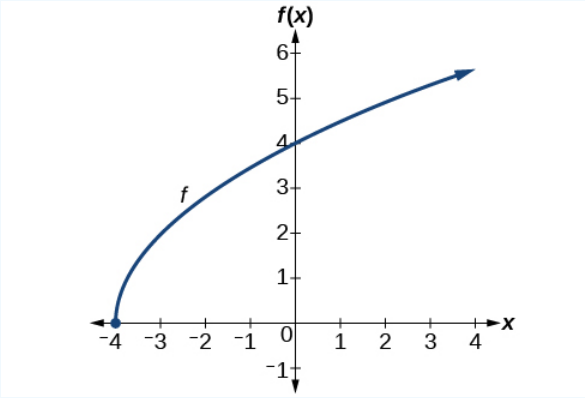


Figure 6.4.19: Graph of a square root function at $(-4, 0)$.

Exercise 6.4.7

Find the domain and range of

$f(x) = \sqrt{-2 - x}$.

Answer

domain: $(-\infty, -2]$

range: $[0, \infty)$

Graphing Piecewise-Defined Functions

Sometimes, we come across a function that requires more than one formula in order to obtain the given output. For example, in the toolkit functions, we introduced the absolute value function $f(x) = |x|$. With a domain of all real numbers and a range of values greater than or equal to 0, **absolute value** can be defined as the **magnitude**, or **modulus**, of a real number value regardless of sign. It is the distance from 0 on the number line. All of these definitions require the output to be greater than or equal to 0.

If we input 0, or a positive value, the output is the same as the input.

$$f(x) = x \text{ if } x \geq 0$$

If we input a negative value, the output is the opposite of the input.

$$f(x) = -x \text{ if } x < 0$$

Because this requires two different processes or pieces, the absolute value function is an example of a **piecewise function**. A piecewise function is a function in which more than one formula is used to define the output over different pieces of the domain.

We use piecewise functions to describe situations in which a rule or relationship changes as the input value crosses certain "boundaries." For example, we often encounter situations in business for which the cost per piece of a certain item is discounted once the number ordered exceeds a certain value. Tax brackets are another real-world example of piecewise functions. For example, consider a simple tax system in which incomes up to $10,000 are taxed at 10%, and any additional income is taxed at 20%. The tax on a total income S would be $0.1S$ if $S \leq \$10,000$ and $\$1000 + 0.2(S - \$10,000)$ if $S > \$10,000$.

Piecewise Function

A piecewise function is a function in which more than one formula is used to define the output. Each formula has its own domain, and the domain of the function is the union of all these smaller domains. We notate this idea like this:

$$f(x) = \begin{cases} \text{formula 1} & \text{if x is in domain 1} \\ \text{formula 2} & \text{if x is in domain 2} \\ \text{formula 3} & \text{if x is in domain 3} \end{cases}$$

In piecewise notation, the absolute value function is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

How To: Given a piecewise function, write the formula and identify the domain for each interval.

1. Identify the intervals for which different rules apply.
2. Determine formulas that describe how to calculate an output from an input in each interval.
3. Use braces and if-statements to write the function.

Example 6.4.8A: Writing a Piecewise Function

A museum charges $5 per person for a guided tour with a group of 1 to 9 people or a fixed $50 fee for a group of 10 or more people. Write a **function** relating the number of people, n, to the cost, C.

Solution

Two different formulas will be needed. For n-values under 10, $C = 5n$. For values of n that are 10 or greater, $C = 50$.

$$C(n) = \begin{cases} 5n & \text{if } n < 10 \\ 50 & \text{if } n \geq 10 \end{cases}$$

Analysis

The function is represented in Figure 6.4.20. The graph is a diagonal line from $n = 0$ to $n = 10$ and a constant after that. In this example, the two formulas agree at the meeting point where $n = 10$, but not all piecewise functions have this property.

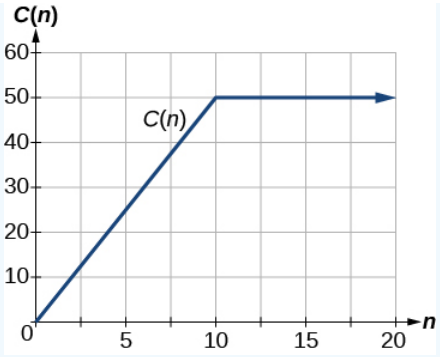


Figure 6.4.20

Example $6.4.8B$: Working with a Piecewise Function

A cell phone company uses the function below to determine the cost, C, in dollars for g gigabytes of data transfer.

$$C(g) = \begin{cases} 25 & \text{if } 0 < g < 2 \\ 25 + 10(g-2) & \text{if } g \geq 2 \end{cases}$$

Find the cost of using 1.5 gigabytes of data and the cost of using 4 gigabytes of data.

Soltuion

To find the cost of using 1.5 gigabytes of data, $C(1.5)$, we first look to see which part of the domain our input falls in. Because 1.5 is less than 2, we use the first formula.

$$C(1.5) = \$25$$

To find the cost of using 4 gigabytes of data, C(4), we see that our input of 4 is greater than 2, so we use the second formula.

$$C(4) = 25 + 10(4-2) = \$45$$

Analysis

The function is represented in Figure $6.4.21$. We can see where the function changes from a constant to a shifted and stretched identity at $g = 2$. We plot the graphs for the different formulas on a common set of axes, making sure each formula is applied on its proper domain.

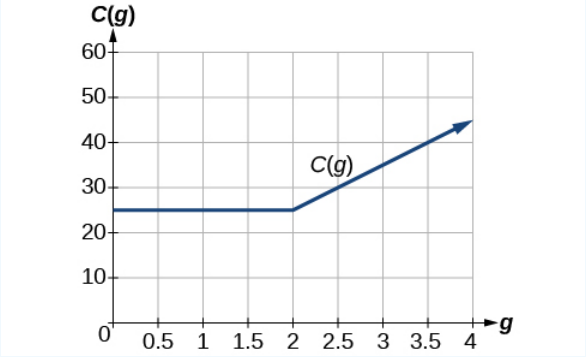


Figure 6.4.21

Given a piecewise function, sketch a graph.

1. Indicate on the x-axis the boundaries defined by the intervals on each piece of the domain.
2. For each piece of the domain, graph on that interval using the corresponding equation pertaining to that piece. Do not graph two functions over one interval because it would violate the criteria of a function.

Example 6.4.8C: Graphing a Piecewise Function

Sketch a graph of the function.

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ 3 & \text{if } 1 < x \leq 2 \\ x & \text{if } x > 2 \end{cases}$$

Solution

Each of the component functions is from our library of toolkit functions, so we know their shapes. We can imagine graphing each function and then limiting the graph to the indicated domain. At the endpoints of the domain, we draw open circles to indicate where the endpoint is not included because of a less-than or greater-than inequality; we draw a closed circle where the endpoint is included because of a less-than-or-equal-to or greater-than-or-equal-to inequality.

Figure 6.4.20 shows the three components of the piecewise function graphed on separate coordinate systems.

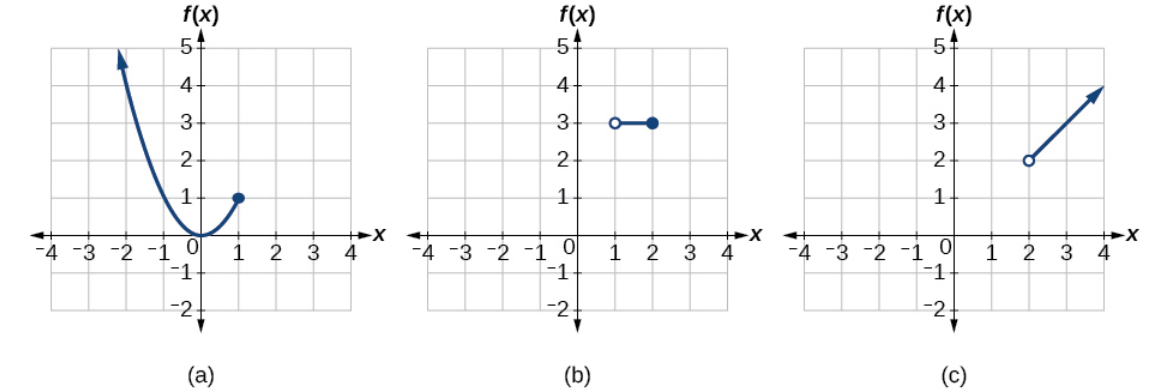


Figure 6.4.20: *Graph of each part of the piece-wise function f(x)*

(a) $f(x) = x^2$ if $x \leq 1$; (b) $f(x) = 3$ if $1 < x \leq 2$; (c) $f(x) = x$ if $x > 2$

Now that we have sketched each piece individually, we combine them in the same coordinate plane. See Figure 6.4.21.

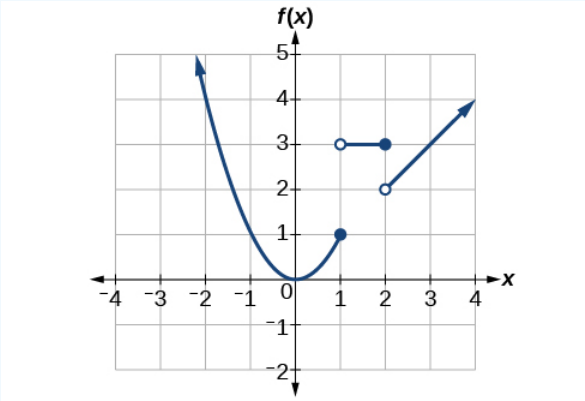


Figure 6.4.21: Graph of the entire function.

Analysis

Note that the graph does pass the vertical line test even at $x = 1$ and $x = 2$ because the points $(1, 3)$ and $(2, 2)$ are not part of the graph of the function, though $(1, 1)$ and $(2, 3)$ are.

Exercise 6.4.8

Graph the following piecewise function.

$$f(x) = \begin{cases} x^3 & \text{if } x < -1 \\ -2 & \text{if } -1 < x < 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$$

Answer

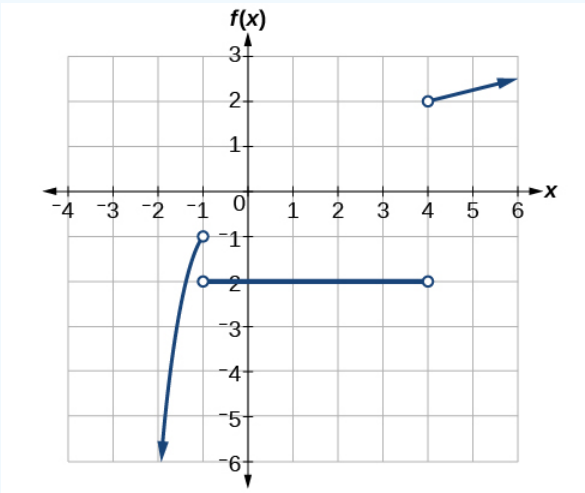


Figure 6.4.22

Can more than one formula from a piecewise function be applied to a value in the domain?

No. Each value corresponds to one equation in a piecewise formula.

Key Concepts

- The domain of a function includes all real input values that would not cause us to attempt an undefined mathematical operation, such as dividing by zero or taking the square root of a negative number.
- The domain of a function can be determined by listing the input values of a set of ordered pairs.
- The domain of a function can also be determined by identifying the input values of a function written as an equation.
- Interval values represented on a number line can be described using inequality notation, set-builder notation, and interval notation.
- For many functions, the domain and range can be determined from a graph.
- An understanding of toolkit functions can be used to find the domain and range of related functions.
- A piecewise function is described by more than one formula.
- A piecewise function can be graphed using each algebraic formula on its assigned subdomain.

Footnotes

1 The Numbers: Where Data and the Movie Business Meet. “Box Office History for Horror Movies.” http://www.the-numbers.com/market/genre/Horror. Accessed 3/24/2014

2 www.eia.gov/dnav/pet/hist/Lea...s=MCRFPAK2&f=A.

Glossary

interval notation

a method of describing a set that includes all numbers between a lower limit and an upper limit; the lower and upper values are listed between brackets or parentheses, a square bracket indicating inclusion in the set, and a parenthesis indicating exclusion

piecewise function

a function in which more than one formula is used to define the output

set-builder notation

a method of describing a set by a rule that all of its members obey; it takes the form {x| statement about x}

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6.5: Operations on Functions

Composition of Functions

Suppose we wanted to calculate how much it costs to heat a house on a particular day of the year. The cost to heat a house will depend on the average daily temperature, and the average daily temperature depends on the particular day of the year. Notice how we have just defined two relationships: The temperature depends on the day, and the cost depends on the temperature. Using descriptive variables, we can notate these two functions.

The first function, $C(T)$, gives the cost C of heating a house when the average daily temperature is T degrees Celsius, and the second, $T(d)$, gives the average daily temperature of a particular city on day d of the year. If we wanted to determine the cost of heating the house on the fifth day of the year, we could do this by linking our two functions together, an idea called composition of functions. Using the function $T(d)$, we could evaluate $T(5)$ to determine the average daily temperature on the fifth day of the year. We could then use that temperature as the input to the $C(T)$ function to find the cost to heat the house on the fifth day of the year: $C(T(5))$.

Composition of Functions

When the output of one function is used as the input of another, we call the entire operation a **composition of functions**. We write $f(g(x))$, and read this as "f of g of x" or "f composed with g at x".

An alternate notation for composition uses the composition operator: $\circ$. $(f \circ g)(x)$ is read "f of g of x" or "f composed with g at x", just like $f(g(x))$.

Example 6.5.1

Suppose $c(s)$ gives the number of calories burned doing s sit-ups, and $s(t)$ gives the number of sit-ups a person can do in t minutes. Interpret $c(s(3))$.

Solution

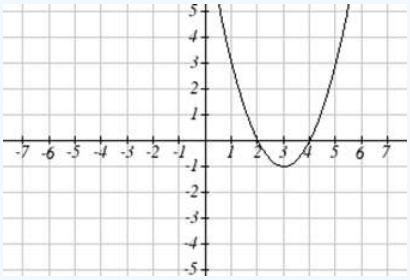
When we are asked to interpret, we are being asked to explain the meaning of the expression in words. The inside expression in the composition is $s(3)$. Since the input to the s function is time, the 3 is representing 3 minutes, and $s(3)$ is the number of sit-ups that can be done in 3 minutes. Taking this output and using it as the input to the $c(s)$ function will give us the calories that can be burned by the number of sit-ups that can be done in 3 minutes.

Composition of Functions using Tables and Graphs

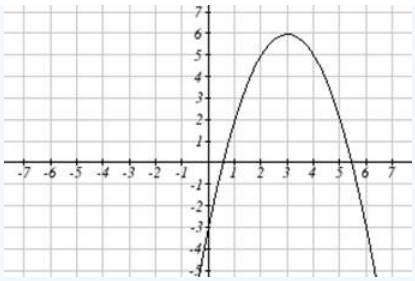
When working with functions given as tables and graphs, we can look up values for the functions using a provided table or graph. We start evaluation from the provided input, and first evaluate the inside function. We can then use the output of the inside function as the input to the outside function. To remember this, always work from the inside out.

Example 6.5.2

Using the graphs below, evaluate $f(g(1))$.



$g(x)$



$f(x)$

Solution

To evaluate $f(g(1))$, we again start with the inside evaluation. We evaluate $g(1)$ using the graph of the $g(x)$ function, finding the input of 1 on the horizontal axis and finding the output value of the graph at that input. Here, $g(1) = 3$. Using this value as the input to the f function, $f(g(1)) = f(3)$. We can then evaluate this by looking to the graph of the $f(x)$ function, finding the input of 3 on the horizontal axis, and reading the output value of the graph at this input. Here, $f(3) = 6$, so $f(g(1)) = 6$.

Compositions using Formulas

When evaluating a composition of functions where we have either created or been given formulas, the concept of working from the inside out remains the same. First we evaluate the inside function using the input value provided, then use the resulting output as the input to the outside function.

Example 6.5.3

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate $f(h(1))$.

Solution

Since the inside evaluation is $h(1)$ we start by evaluating the $h(x)$ function at 1:

$$h(1) = 3(1) + 2 = 5$$

Then $f(h(1)) = f(5)$, so we evaluate the $f(t)$ function at an input of 5:

$$f(h(1)) = f(5) = 5^2 - 5 = 20$$

We are not limited, however, to using a numerical value as the input to the function. We can put anything into the function: a value, a different variable, or even an algebraic expression, provided we use the input expression everywhere we see the input variable.

Example 6.5.4

Let $f(x) = x^2$ and $g(x) = \frac{1}{x} - 2x$. Find $f(g(x))$ and $g(f(x))$.

Solution

To find $f(g(x))$, we start by evaluating the inside, writing out the formula for $g(x)$:

$$g(x) = \frac{1}{x} - 2x$$

We then use the expression $\left(\frac{1}{x} - 2x\right)$ as the input for the function f:

$$f(g(x)) = f\left(\frac{1}{x} - 2x\right)$$

We then evaluate the function $f(x)$ using the formula for $g(x)$ as the input. Since $f(x) = x^2$ then

$$f\left(\frac{1}{x} - 2x\right) = \left(\frac{1}{x} - 2x\right)^2$$

This gives us the formula for the composition:

$$f(g(x)) = \left(\frac{1}{x} - 2x\right)^2$$

Likewise, to find $g(f(x))$, we evaluate the inside, writing out the formula for $f(x)$: $g(f(x)) = g(x^2)$.

Now we evaluate the function $g(x)$ using x^2 as the input:

$$g(f(x)) = \frac{1}{x^2} - 2x^2$$

Example 6.5.5

A city manager determines that the tax revenue, R, in millions of dollars collected on a population of p thousand people is given by the formula $R(p) = 0.03p + \sqrt{p}$, and that the city's population, in thousands, is predicted to follow the formula $p(t) = 60 + 2t + 0.3t^2$, where t is measured in years after 2010. Find a formula for the tax revenue as a function of the year.

Solution

Since we want tax revenue as a function of the year, we want year to be our initial input, and revenue to be our final output. To find revenue, we will first have to predict the city population, and then use that result as the input to the tax function. So we need to find $R(p(t))$. Evaluating this,

$$\begin{aligned} R(p(t)) &= R(60 + 2t + 0.3t^2) \\ &= 0.03(60 + 2t + 0.3t^2) + \sqrt{60 + 2t + 0.3t^2} \end{aligned}$$

This composition gives us a single formula which can be used to predict the tax revenue during a given year, without needing to find the intermediary population value.

For example, to predict the tax revenue in 2017, when $t = 7$ (because t is measured in years after 2010),

$$\begin{aligned} R(p(7)) &= 0.03\left(60 + 2(7) + 0.3\left(7^2\right)\right) + \sqrt{60 + 2(7) + 0.3\left(7^2\right)} \\ &\approx 12.079 \text{ million dollars} \end{aligned}$$

Later in this course, it will be desirable to decompose a function – to write it as a composition of two simpler functions.

Example 6.5.6

Write $f(x) = 3 + \sqrt{5 - x^2}$ as the composition of two functions.

Solution

We are looking for two functions, g and h, so $f(x) = g(h(x))$. To do this, we look for a function inside a function in the formula for $f(x)$. As one possibility, we might notice that $5 - x^2$ is the inside of the square root. We could then decompose the function as:

$$h(x) = 5 - x^2, \quad g(x) = 3 + \sqrt{x}$$

We can check our answer by recomposing the functions:

$$g(h(x)) = g(5 - x^2) = 3 + \sqrt{5 - x^2}$$

Note that this is not the only solution to the problem. Another non-trivial decomposition would be

$$h(x) = x^2, \quad g(x) = 3 + \sqrt{5 - x}.$$

Transformations of Functions

Transformations allow us to construct new equations from our basic toolkit functions. The most basic transformations are shifting the graph vertically or horizontally.

Vertical Shift

Given a function $f(x)$, if we define a new function $g(x)$ as $g(x) = f(x) + k$, where k is a constant, then $g(x)$ is a **vertical shift** of the function $f(x)$, where all the output values have been increased by k.

If k is positive, then the graph will shift up. If k is negative, then the graph will shift down.

Horizontal Shift

Given a function $f(x)$, if we define a new function $g(x)$ as $g(x) = f(x + k)$, where k is a constant, then $g(x)$ is a **horizontal shift** of the function $f(x)$, where all the output values have been increased by k.

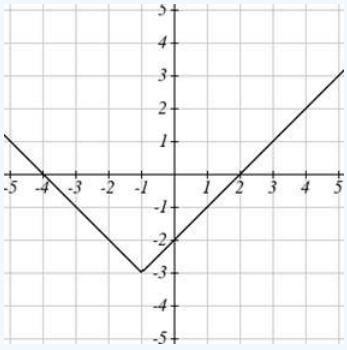
If k is positive, then the graph will shift left. If k is negative, then the graph will shift right.

Example 6.5.7

Given $f(x) = |x|$, sketch a graph of $h(x) = f(x+1) - 3 = |x+1| - 3$.

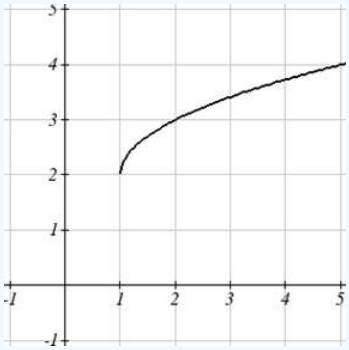
Solution

The function f is our toolkit absolute value function. We know that this graph has a V shape, with the point at the origin. The graph of h has transformed f in two ways: $f(x+1)$ is a change on the inside of the function, giving a horizontal shift left by 1, then the subtraction by 3 in $f(x+1)-3$ is a change to the outside of the function, giving a vertical shift down by 3. Transforming the graph gives:



Example 6.5.8

Write a formula for the graph shown, a transformation of the toolkit square root function.



Solution

The graph of the toolkit function starts at the origin, so this graph has been shifted 1 to the right, and up 2. In function notation, we could write that as $h(x) = f(x-1)+2$. Using the formula for the square root function we can write $h(x) = \sqrt{x-1}+2$.

Note that this transformation has changed the domain and range of the function. This new graph has domain $[1, \infty)$ and range $[2, \infty)$.

Another transformation that can be applied to a function is a reflection over the horizontal or vertical axis.

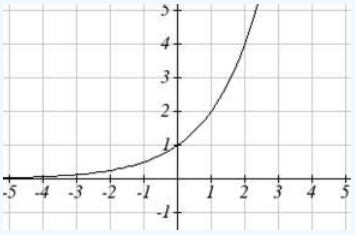
Reflections

Given a function $f(x)$, if we define a new function $g(x)$ as $-f(x)$, then $g(x)$ is a **vertical reflection** of the function $f(x)$, sometimes called a reflection about the x-axis

If we define a new function $g(x)$ as $f(-x)$, then $g(x)$ is a **horizontal reflection** of the function $f(x)$, sometimes called a reflection about the y-axis.

Example 6.5.9

A common model for learning has an equation similar to $k(t) = -2^t + 1$, where k is the percentage of mastery that can be achieved after t practice sessions. This is a transformation of the function $f(t) = 2^t$ shown here. Sketch a graph of $k(t)$.



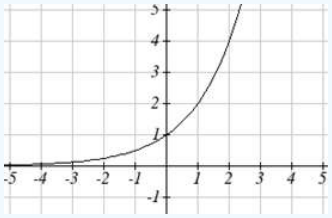
$f(t) = 2^t$

Solution

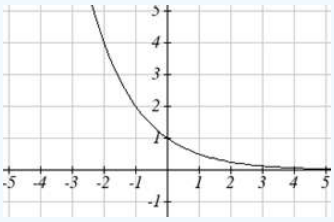
This equation combines three transformations into one equation.

A horizontal reflection:	$f(-t) = 2^{-t}$	combined with
a vertical reflection:	$-f(-t) = -2^{-t}$	combined with
a vertical shift up 1:	$-f(-t)+1 = -2^{-t}+1$.	

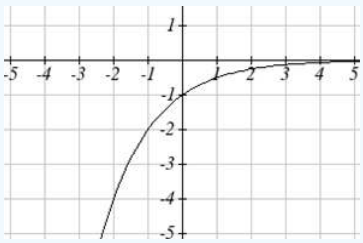
We can sketch a graph by applying these transformations one at a time to the original function:



The original graph



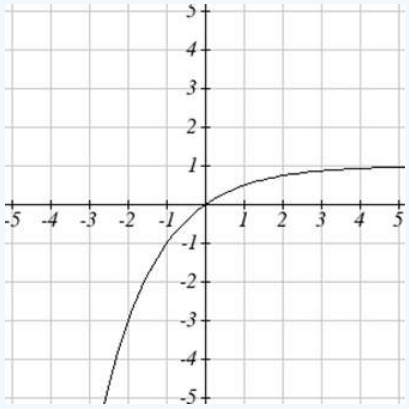
Horizontally reflected



Then vertically reflected

Then, after shifting up 1, we get the final graph:

$$k(t) = -f(-t) + 1 = -2^{-t} + 1$$



Final graph

Note: As a model for learning, this function would be limited to a domain of $t \geq 0$, with corresponding range $[0, 1)$.

With shifts, we saw the effect of adding or subtracting to the inputs or outputs of a function. We now explore the effects of multiplying the outputs.

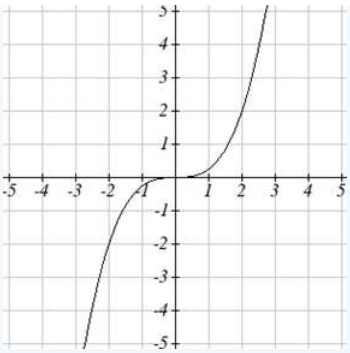
Vertical Stretch/Compression

Given a function $f(x)$, if we define a new function $g(x)$ as $g(x) = k \cdot f(x)$, where k is a constant, then $g(x)$ is a **vertical stretch or compression** of the function $f(x)$.

- If $k > 1$, then the graph will be stretched
- If $0 < k < 1$, then the graph will be compressed
- If $k < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection

Example 6.5.10

The graph below is a transformation of the toolkit function $f(x) = x^3$. Relate this new function $g(x)$ to $f(x)$, then find a formula for $g(x)$.



Solution

When trying to determine a vertical stretch or shift, it is helpful to look for a point on the graph that is relatively clear. In this graph, it appears that $g(2) = 2$. With the basic cubic function at the same input, $f(2) = 2^3 = 8$. Based on that, it appears that the outputs of g are $\frac{1}{4}$ the outputs of the function f, since $g(2) = \frac{1}{4}f(x)$. From this we can fairly safely conclude that

$$g(x) = \frac{1}{4}f(x).$$

We can write a formula for g by using the definition of the function f:

$$g(x) = \frac{1}{4}f(x) = \frac{1}{4}x^3.$$

Combining Transformations

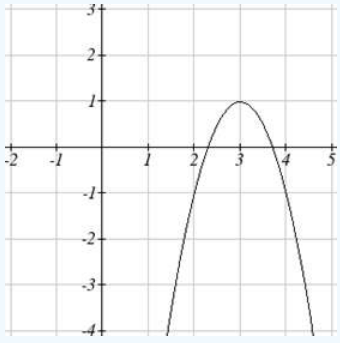
When combining vertical transformations, it is very important to consider the order of the transformations. For example, vertically shifting by 3 and then vertically stretching by 2 does not create the same graph as vertically stretching by 2 and then vertically shifting by 3. The order follows nicely from order of operations.

Combining Vertical Transformations

When combining vertical transformations written in the form $a \cdot f(x) + k$, first vertically stretch by a, then vertically shift by k.

Example 6.5.11

Write an equation for the transformed graph of the quadratic function shown.



Solution

Since this is a quadratic function, first consider what the basic quadratic tool kit function looks like and how this has changed. Observing the graph, we notice several transformations: The original tool kit function has been flipped over the x axis, some kind of stretch or compression has occurred, and we can see a shift to the right 3 units and a shift up 1 unit. In total there are four operations:

1. Vertical Reflection, requiring a negative sign outside the function.

2. Vertical Stretch
3. Horizontal Shift Right 3 units, which tells us to put x-3 on the inside of the function.
4. Vertical Shift up 1 unit, telling us to add 1 on the outside of the function.

By observation, the basic tool kit function has a vertex at (0, 0) and symmetrical points at (1, 1) and (-1, 1). These points are 1 unit up and 1 unit over from the vertex. The new points on the transformed graph are 1 unit away horizontally but 2 units away vertically. They have been stretched vertically by 2.

Not everyone can see this by simply looking at the graph. If you can, great, but if not we can solve for it. First, we will write the equation for this graph, with an unknown vertical stretch:

$f(x) = x^2$	The original function
$-f(x) = -x^2$	Vertically reflected
$-a \cdot f(x) = -ax^2$	Vertically stretched
$-a \cdot f(x-3) = -a(x-3)^2$	Shifted right 3
$-a \cdot f(x-3)+1 = -a(x-3)^2+1$	Shifted up 1

We now know our graph is going to have an equation of the form $g(x) = -a(x-3)^2+1$. To find the vertical stretch, we can identify any point on the graph (other than the highest point), such as the point (2,-1), which tells us $g(2) = -1$. Using our general formula, and substituting 2 for x, and -1 for $g(x)$.

This tells us that to produce the graph we need a vertical stretch by two.

Thus the function that produces this graph is

$$g(x) = -2(x-3)^2+1.$$

Example 6.5.12

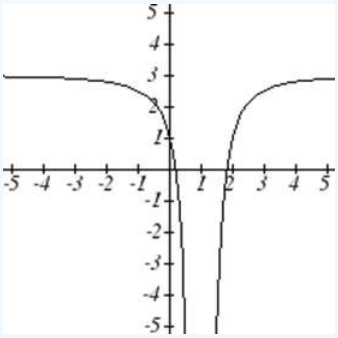
On what interval(s) is the function $g(x) = \dfrac{-2}{(x-1)^2}+3$ increasing and decreasing?

Solution

This is a transformation of the toolkit reciprocal squared function, $f(x) = \dfrac{1}{x^2}$

$-2f(x) = \dfrac{-2}{x^2}$	A vertical flip and vertical stretch by 2
$-2f(x-1) = \dfrac{-2}{(x-1)^2}$	A shift right by 1
$-2f(x-1)+3 = \dfrac{-2}{(x-1)^2}+3$	A shift up by 3

The basic reciprocal squared function is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. Because of the vertical flip, the $g(x)$ function will be decreasing on the left and increasing on the right. The horizontal shift right by 1 will also shift these intervals to the right one. From this, we can determine $g(x)$ will be increasing on $(1, \infty)$ and decreasing on $(-\infty, 1)$. We also could graph the transformation to help us determine these intervals.



Graph of the transformed function.

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6.6: Linear Functions

As you hop into a taxicab in Allentown, the meter will immediately read $3.30; this is the "drop" charge made when the taximeter is activated. After that initial fee, the taximeter will add $2.40 for each mile the taxi drives. In this scenario, the total taxi fare depends upon the number of miles ridden in the taxi, and we can ask whether it is possible to model this type of scenario with a function. Using descriptive variables, we choose m for miles and C for Cost in dollars as a function of miles: $C(m)$.

We know for certain that $C(0) = 3.30$, since the $3.30 drop charge is assessed regardless of how many miles are driven. Since $2.40 is added for each mile driven, we could write that if m miles are driven, $C(m) = 3.30 + 2.40m$ because we start with a $3.30 drop fee and then for each mile increase we add $2.40.

It is good to verify that the units make sense in this equation. The $3.30 drop charge is measured in dollars; the $2.40 charge is measured in dollars per mile. So

$$C(m) = 3.30 \text{ dollars} + \left(2.40 \frac{\text{dollars}}{\text{mile}}\right)(m \text{ miles})$$

When dollars per mile are multiplied by a number of miles, the result is a number of dollars, matching the units on the 3.30, and matching the desired units for the C function.

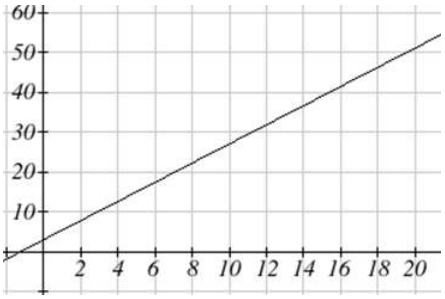
Notice this equation $C(m) = 3.30 + 2.40m$ consisted of two quantities. The first is the fixed $3.30 charge which does not change based on the value of the input. The second is the $2.40 dollars per mile value, which is a **rate of change**. In the equation this rate of change is multiplied by the input value.

Looking at this same problem in table format we can also see the cost changes by $2.40 for every 1 mile increase:

m	0	1	2	3
$C(m)$	3.30	5.70	8.10	10.50

It is important here to note that in this equation, the **rate of change is constant**; over any interval, the rate of change is the same.

Graphing this equation, $C(m) = 3.30 + 2.40m$ we see the shape is a line, which is how these functions get their name: **linear functions**.



When the number of miles is zero the cost is $3.30, giving the point (0, 3.30) on the graph. This is the vertical or $C(m)$ intercept. The graph is increasing in a straight line from left to right because for each mile the cost goes up by $2.40; this rate remains consistent.

Linear Function

A **linear function** is a function whose graph produces a line. Linear functions can always be written in the form

$$f(x) = b + mx$$

or

$$f(x) = mx + b$$

where b is the initial or starting value of the function (with input $x = 0$), and m is the constant rate of change of the function.

This form of a line is called **slope-intercept form** of a line.

Many people like to write linear functions in the form $f(x) = b + mx$ because it corresponds to the way we tend to speak: "The output starts at b and increases at a rate of m."

For this reason alone we will use the $f(x) = b + mx$ form for many of the examples, but remember they are equivalent and can be written correctly both ways. [While this is the book's convention, in class and in the videos I will likely use $f(x) = mx + b$.]

Slope and Increasing/Decreasing

m is the constant rate of change of the function (also called **slope**). The slope determines if the function is an increasing function or a decreasing function.

- $f(x) = b + mx$ is an **increasing** function if $m > 0$.
- $f(x) = b + mx$ is a **decreasing** function if $m < 0$.

If $m = 0$, the rate of change is 0, and the function $f(x) = b + 0x = b$ is just a horizontal line passing through the point (0, b), neither increasing nor decreasing.

Example 6.6.1

Marcus currently owns 200 songs in his iTunes collection. Every month, he adds 15 new songs. Write a formula for the number of songs, N, in his iTunes collection as a function of the number of months, m. How many songs will he own in a year?

Solution

The initial value for this function is 200, since he currently owns 200 songs, so $N(0) = 200$. The number of songs increases by 15 songs per month, so the rate of change is 15 songs per month. With this information, we can write the formula:

$$N(m) = 200 + 15m.$$

$N(m)$ is an increasing linear function.

With this formula we can predict how many songs he will have in 1 year (12 months):

$$N(12) = 200 + 15(12) = 200 + 180 = 380.$$

Marcus will have 380 songs in 12 months.

Calculating Rate of Change

Given two values for the input, x_1 and x_2, and two corresponding values for the output, y_1 and y_2, or a set of points, (x_1, y_1) and (x_2, y_2), if we wish to find a linear function that contains both points we can calculate the rate of change, m:

$$m = \frac{\text{change in output}}{\text{change in input}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Rate of change of a linear function is also called the **slope** of the line.

Note in function notation, $y_1 = f(x_1)$ and $y_2 = f(x_2)$, so we could equivalently write

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Example 6.6.2

The population of a city increased from 23,400 to 27,800 between 2002 and 2006. Find the rate of change of the population during this time span.

Solution

The rate of change will relate the change in population to the change in time. The population increased by 27800-23400=4400 people over the 4 year time interval. To find the rate of change, the number of people per year the population changed by:

$$\frac{4400 \text{ people}}{4 \text{ years}} = 1100 \frac{\text{people}}{\text{year}} = 1100 \text{ people per year.}$$

Notice that we knew the population was increasing, so we would expect our value for m to be positive. This is a quick way to check to see if your value is reasonable.

Example 6.6.3

The pressure, P, in pounds per square inch (PSI) on a diver depends upon their depth below the water surface, d, in feet, following the equation $P(d) = 14.696 + 0.43d$. Interpret the components of this function.

Solution

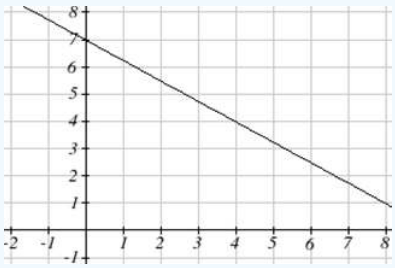
The rate of change, or slope, 0.434 would have units $\frac{\text{output}}{\text{input}} = \frac{\text{pressure}}{\text{depth}} = \frac{\text{PSI}}{\text{ft}}$. This tells us the pressure on the diver increases by 0.434 PSI for each foot their depth increases.

The initial value, 14.696, will have the same units as the output, so this tells us that at a depth of 0 feet, the pressure on the diver will be 14.696 PSI.

We can now find the rate of change given two input-output pairs, and could write an equation for a linear function if we had the rate of change and initial value. If we have two input-output pairs and they do not include the initial value of the function, then we will have to solve for it.

Example 6.6.4

Write an equation for the linear function graphed below.



Solution

Looking at the graph, we might notice that it passes through the points (0, 7) and (4, 4). From the first value, we know the initial value of the function is $b = 7$, so in this case we will only need to calculate the rate of change:

$$m = \frac{4-7}{4-0} = \frac{-3}{4}$$

This allows us to write the equation:

$$f(x) = 7 - \frac{3}{4}x$$

Example 6.6.5

If $f(x)$ is a linear function, $f(3) = -2$, and $f(8) = 1$, find an equation for the function.

Solution

In Example 6.6.3, we computed the rate of change to be $m = \frac{3}{5}$. In this case, we do not know the initial value $f(0)$, so we will have to solve for it. Using the rate of change, we know the equation will have the form $f(x) = b + \frac{3}{5}x$. Since we know the value of the function when $x = 3$, we can evaluate the function at 3: $f(x) = b + \frac{3}{5}(3)$.

Since we know that $f(3) = -2$, we can substitute on the left side: $-2 = b + \frac{3}{5}(3)$.

This leaves us with an equation we can solve for the initial value: $b = -2 - \frac{9}{5} = -\frac{19}{5}$.

Combining this with the value for the rate of change, we can now write a formula for this function:

$$f(x) = -\frac{19}{5} + \frac{3}{5}x.$$

As an alternative to the approach used above to find the initial value, b, we can use the **point-slope** form of a line instead.

Point-Slope Equation of a Line

An equation for the line passing through the point (x_1, y_1) with slope m can be written as

$$y - y_1 = m(x - x_1)$$

This is called the **point-slope form of a line**. It is a little easier to write if you know a point and the slope, but requires a bit of work to rewrite into slope-intercept form, and requires memorizing another formula.

Example 6.6.6

Working as an insurance salesperson, Ilya earns a base salary and a commission on each new policy, so Ilya's weekly income, I, depends on the number of new policies, n, he sells during the week. Last week he sold 3 new policies, and earned $760 for the week. The week before, he sold 5 new policies, and earned $920. Find an equation for $I(n)$, and interpret the meaning of the components of the equation.

Solution

The given information gives us two input-output pairs: (3,760) and (5,920). We start by finding the rate of change:

$$m = \frac{920-760}{5-3} = \frac{160}{2} = 80.$$

Keeping track of units can help us interpret this quantity. Income increased by \$160 when the number of policies increased by 2, so the rate of change is \$80 per policy; Ilya earns a commission of \$80 for each policy sold during the week.

We can now write the equation using the point-slope form of the line, using the slope we just found and the point (3,760):

$$I - 760 = 80(n-3)$$

If we wanted this in function form (slope intercept form), we could rewrite the equation into that form:

$$\begin{aligned} I - 760 &= 80(n-3) \\ I - 760 &= 80n - 240 \\ I(n) &= 520 + 80n \end{aligned}$$

This form allows us to see the starting value for the function: 520. This is Ilya's income when $n = 0$, which means no new policies are sold. We can interpret this as Ilya's base salary for the week, which does not depend upon the number of policies sold.

Our final interpretation is: Ilya's base salary is \$520 per week and he earns an additional \$80 commission for each policy sold each week.

Graphs of Linear Functions

Graphical Interpretation of a Linear Equation

Graphically, in the equation $f(x) = b + mx$,

- b is the **vertical intercept** of the graph and tells us we can start our graph at $(0, b)$
- and m is the **slope of the line** and tells us how far to rise and run to get to the next point.

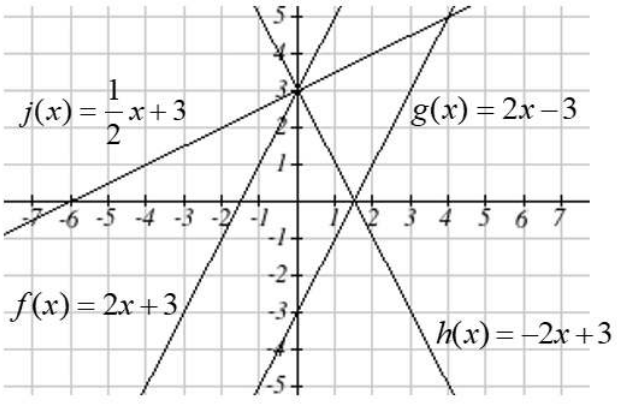
Once we have at least 2 points, we can extend the graph of the line to the left and right.

Example 6.6.7

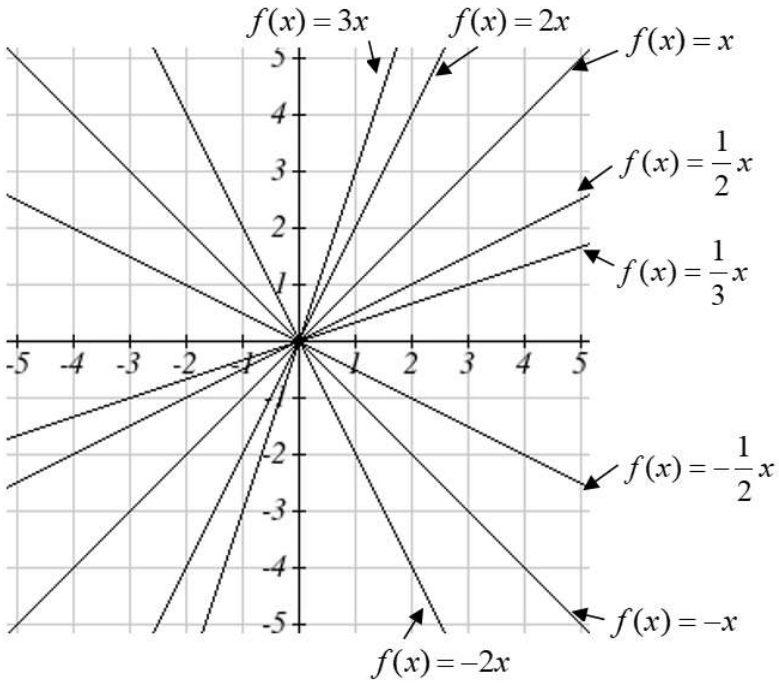
Graph $f(x) = 5 - \frac{2}{3}x$ using the vertical intercept and slope.

Solution

The vertical intercept of the function is (0, 5), giving us a point on the graph of the line. The slope is $-\frac{2}{3}$. This tells us that for every 3 units the graph "runs" in the horizontal, the vertical "rise" decreases by 2 units. In graphing, we can use this by first plotting our vertical intercept on the graph, then using the slope to find a second point. From the initial value (0, 5) the slope tells us that if we move to the right 3, we will move down 2, moving us to the point (3, 3). We can continue this again to find a third point at (6, 1). Finally, extend the line to the left and right, containing these points.

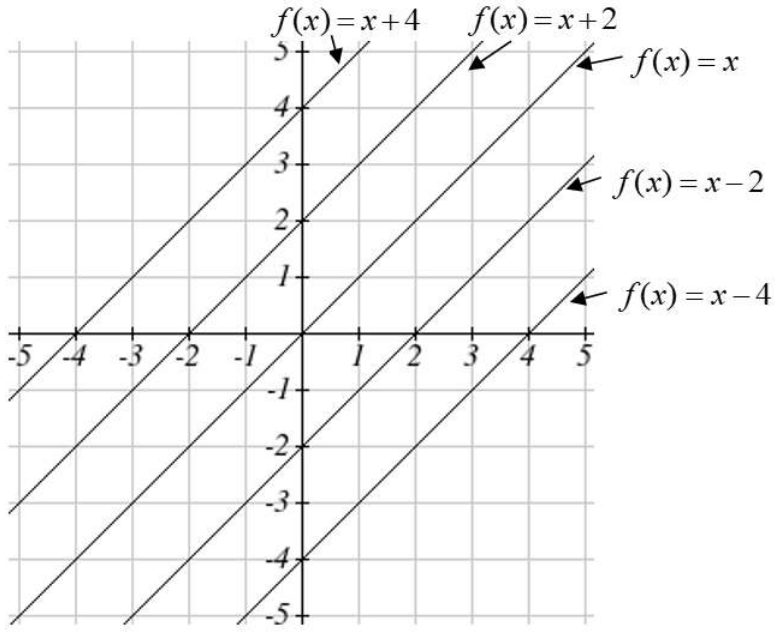


Another option for graphing is to use transformations of the identity function $f(x)=x$. In the equation $f(x)=mx$, the m is acting as the vertical stretch of the identity function. When m is negative, there is also a vertical reflection of the graph. Looking at some examples will also help show the effect of slope on the shape of the graph:



$f(x)=mx$ for several values of m.

In $f(x)=mx+b$, the b acts as the vertical shift, moving the graph up and down without affecting the slope of the line. Some examples:



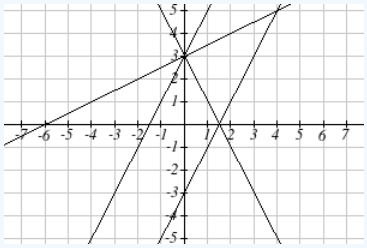
$f(x)=mx+b$ for several values of b.

Try it for yourself using this applet:

Example 6.6.8

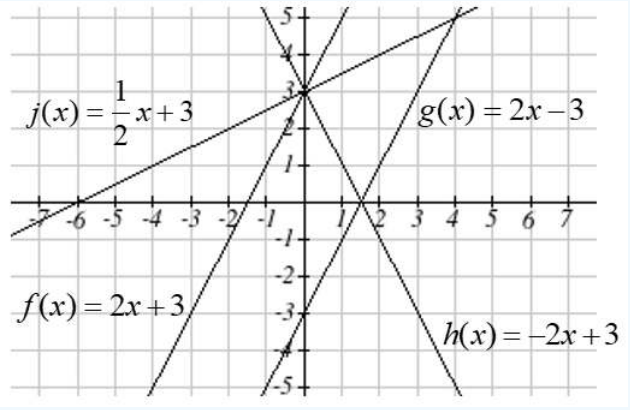
Match each equation with one of the lines in the graph below

$$f(x) = 2x + 3$$
$$g(x) = 2x - 3$$
$$h(x) = -2x + 3$$
$$j(x) = \frac{1}{2}x + 3$$



Solution

Only one graph has a vertical intercept of -3, so we can immediately match that graph with $g(x)$. For the three graphs with a vertical intercept at 3, only one has a negative slope, so we can match that line with $h(x)$. Of the other two, the steeper line would have a larger slope, so we can match that graph with equation $f(x)$, and the flatter line with the equation $j(x)$.



In addition to understanding the basic behavior of a linear function (increasing or decreasing, recognizing the slope and vertical intercept), it is often helpful to know the horizontal intercept of the function – where it crosses the horizontal axis.

Finding Horizontal Intercept

The **horizontal intercept** of the function is where the graph crosses the horizontal axis. If a function has a horizontal intercept, you can always find it by solving $f(x) = 0$.

Example 6.6.9

Find the horizontal intercept of $f(x) = -3 + \frac{1}{2}x$

Solution

Setting the function equal to zero to find what input will put us on the horizontal axis:

$$0 = -3 + \frac{1}{2}x$$
$$3 = \frac{1}{2}x$$
$$x = 6$$

Thus the graph crosses the horizontal axis at (6,0).

Intersections of Lines

The graphs of two lines will intersect if they are not parallel. They will intersect at the point that satisfies both equations. To find this point when the equations are given as functions, we can solve for an input value so that $f(x) = g(x)$. In other words, we can set the formulas for the lines equal, and solve for the input that satisfies the equation.

Economics tells us that in a free market, the price for an item is related to the quantity that producers will supply and the quantity that consumers will demand. Increases in prices will decrease demand, while supply tends to increase with prices. Sometimes supply and demand are modeled with linear functions.

Example 6.6.10

The supply, in thousands of items, for custom phone cases can be modeled by the equation $s(p) = 0.5 + 1.2p$ while the demand can be modeled by $d(p) = 8.7 - 0.7p$, where p is in dollars. Find the equilibrium price and quantity, the intersection of the supply and demand curves.

Solution

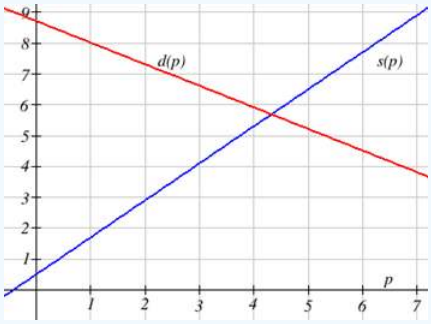
Setting $s(p) = d(p)$, we find

$$\begin{aligned} 0.5 + 1.2p &= 8.7 - 0.7p \\ 1.9p &= 8.2 \\ p &\approx \$4.32 \end{aligned}$$

We can find the output value of the intersection point by evaluating either function at this input:

$$s(4.32) = 0.5 + 1.2(4.32) \approx 5.68$$

These lines intersect at the point (4.32, 5.68). Looking at the graph, this result seems reasonable.



Business Applications

In business, a very common application of functions is to model cost, revenue, and profit.

Cost, Revenue, Profit

When a company produces q items, the **total cost** is the cost of total cost of producing those items. The total cost includes both **fixed costs**, which are startup costs, like equipment and buildings, and **variable costs**, which are costs that depend on the number of items produced, like materials and labor.

In the most simple case, **Total Cost = (Fixed Costs) + (Variable Costs) · q**

Revenue is the amount of money a company brings in from sales.

In the most simple case, **Revenue = (Price per item) · q**

Profit is the amount of money brings in, after expenses.

Profit = Revenue – Costs

We often talk about the **break-even** point. This is the level of production where Revenue equals Cost, or equivalently where Profit is zero. This is typically the minimum level of sales necessary for the company to make a profit.

Example 6.6.11

A tech startup is looking at developing and launching a new mobile app. Initial development of the app will cost $300,000, and they estimate marketing and support for each user will cost $0.50. While the app will be free, they estimate they will be able to bring in $2 per user on average from in-app purchases. How many users will the company need to break even?

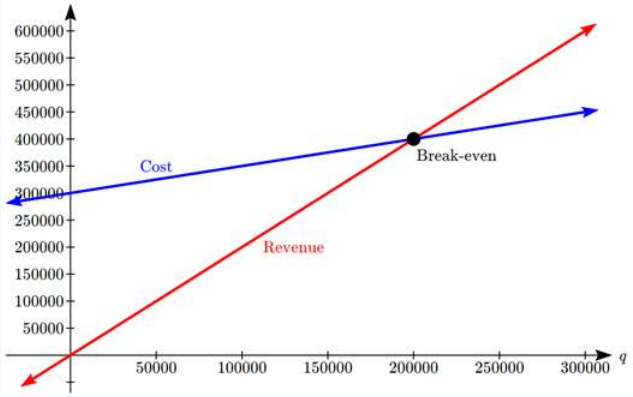
Solution

We start by modeling the cost, revenue, and profit. Let q = number of users.

The fixed (initial) costs are $300,000, and the variable (per-item) costs as $0.50 per user. We can write the total cost equation:
$TC(q) = 300,000 + 0.50q$

The revenue will be $2 per user, so the revenue equation will be:
$R(q) = 2q$

We could find the break-even point by setting the total cost equal to the revenue, which is equivalent to finding the intersection of the lines.



Alternatively, we could go ahead and find a profit equation first:
$P(q) = R(q) - TC(q) = 2q - (300,000 + 0.50q) = 1.5q - 300,000$

The break even point can be found by setting the profit equal to zero:

$$\begin{aligned} 0 &= 1.5q - 300,000 \\ q &= 200,000 \end{aligned}$$

The company will have to acquire 200,000 users to break even.

In economics, there is a model for how prices are determined in a free market which states that **supply and demand** for a product is related to the price. The demand relationship shows the quantity of a certain product that consumers are willing to buy at a certain price. Typically the quantity demanded will decrease for an item if the price increases. The supply relationship shows the quantity of a product that suppliers are willing to produce at a certain sales price. Typically the supply demanded will increase if the price increases. Economic theory says that supply and demand will interact, and the intersection will be the equilibrium price, or market price, where the quantity supplied and demanded will be equal.

Supply and Demand

If p is the price of a product, then

Q_d is the quantity demanded

Q_s is the quantity supplied

The demand curve is a decreasing function, while the supply curve is an increasing function.

The intersection of the curves is the **equilibrium price and quantity**, also called the **market price and quantity**. This point is often notated as p^*, Q^*.

Later in the course you will explore supply and demand curves that are non-linear, but in this section we will focus on linear supply and demand functions.

In most economic books, you will see the supply and demand curve written with price as the input and quantity as the output, like $Q_d = 140 - 2p$. However, supply and demand graphs are drawn with price on the vertical axis and quantity on the horizontal. We can write the equation in whichever way is more convenient for us.

Example 6.6.12

At a price of \$2.50 per gallon, there is a demand in a certain town for 42.5 thousand gallons of gas and a supply of 20 thousand gallons. At a price of \$3.50, there is demand for 25.5 thousand gallons and a supply of 28 thousand gallons. Assuming supply and demand are linear, find the equilibrium price and quantity.

Solution

We start by finding a linear equation for both supply and demand. We will use price, p in dollars, as the output and quantity, q in thousands of gallons, as the input.

For supply, we have the points (20, 2.50) and (28, 3.50).

Finding the slope: $m = \dfrac{3.50 - 2.50}{28 - 20} = \frac{1}{8}$

We know the equation will look like $p = \frac{1}{8}q + b$, so substituting in (20, 2.50)

$$\begin{aligned} 2.5 &= \frac{1}{8}(20) + b \\ 2.5 &= 2.5 + b \\ q &= 0 \end{aligned}$$

The supply equation is: $p = \frac{1}{8}q$

For demand, we have the points (42.5, 2.50) and (25.5, 3.50). Using a similar approach, we can find the demand equation is: $p = -\frac{1}{17}q + 5$

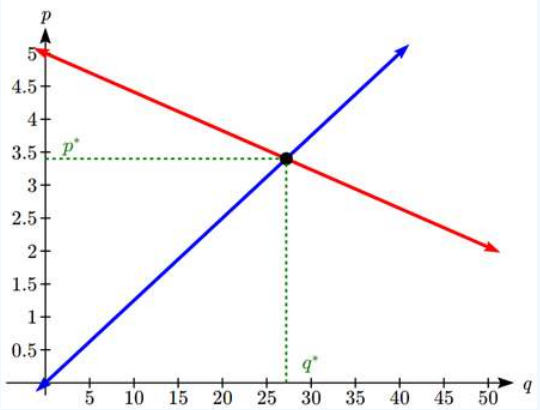
To find the equilibrium, we set the supply equal to the demand:
$\frac{1}{8}q = -\frac{1}{17}q + 5$

Multiplying through by 8(17) = 136 to clear the fractions,
$136\left(\frac{1}{8}q\right) = 136\left(-\frac{1}{17}q + 5\right)$
$17q = -8q + 680$

Now we solve for q
$25q = 680$
$q = 27.2$

To find the equilibrium price, we can substitute that value back into either equation:
$p = \frac{1}{8}(27.2) = 3.4$

The equilibrium quantity will be 27.2 thousand gallons of gas at a price of \$3.40.



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6.6.1: Slope

In the previous section on Linear Models, we saw that if the dependent variable was changing at a constant rate with respect to the independent variable, then the graph was a line. If the rate was positive, then as we swept our eyes from left to right, the line rose upward, the dependent variable increasing with increasing changes in the independent variable. If the rate was negative, then the graph fell downward, the dependent variable decreasing with increasing changes in the independent variable. You may have also learned that higher rates led to steeper lines (lines that rose more quickly) and lower rates led to lines that were less steep.

In this section, we will connect the intuitive concept of rate developed in the previous section with a formal definition of the slope of a line. To start, let's state up front what is meant by the slope of a line.

Definition

Slope is a number that tells us how quickly a line rises or falls.

If slope is a number that is directly connected to the "steepness" of a line, then we should have certain expectations.

Expectations

1. Lines with positive slope should slant uphill (as our eyes sweep from left to right).
2. Lines with negative slope should slant downhill (as our eyes sweep from left to right).
3. Because any horizontal line neither slants uphill nor downhill, we expect that it should have slope equal to zero.
4. Lines with a larger positive slope should rise more quickly than lines with a smaller positive slope.
5. If two lines have negative slope, then the line having the slope with larger absolute value should fall more quickly than the other line.

It remains to define how to compute the slope of a particular line. Whatever definition we choose, it should conform with the expectations outlined above. We also would like the definition of slope to conform with the concept of rate developed in the previous section. Thus, we make the following definition.

Definition: Slope

The **slope** of a line is the rate at which the dependent variable is changing with respect to the independent variable.

Note how the word "change" is used Definition. It is important to understand that the change in some quantity can be positive, negative, or zero. For example, if the temperature outside is $40^{\circ}\mathrm{F}$ when I leave my home at 6 AM, and at noon the temperature is $65\circ$ F, then the change in temperature is a positive $25^{\circ}\mathrm{F}$. On the other hand, if the temperature outside is $65^{\circ}\mathrm{F}$ at noon, and the temperature is $50^{\circ}\mathrm{F}$ when I return home in the evening, then the change in temperature is a negative 15 degrees Fahrenheit.

In calculating the change in a quantity, follow this rule.

Definition: Change in Quantity

Change in Quantity = Latter Measurement − Former Measurement.

Thus, if T represents the temperature and ΔT represents the change in the temperature, then in our first case (taking the temperature in the morning then later at noon), the change in temperature is

$$\Delta T = \text{ Latter } - \text{ Former } = 65^{\circ}\mathrm{F} - 40^{\circ}\mathrm{F} = 25^{\circ}\mathrm{F}$$

This positive result represents an increase in the temperature of $25^{\circ}\mathrm{F}$.

In the second case (taking the temperature at noon then later in the evening), the change in temperature is

$$\begin{aligned} \Delta T &= \text{Latter} - \text{Former} \\ &= 50^{\circ}\mathrm{F} - 65^{\circ}\mathrm{F} \\ &= -15^{\circ}\mathrm{F} \end{aligned}$$

This negative result represents a decrease in the temperature of $15^\circ \mathrm{F}$

Tip 3

Readers should note that the direction of subtraction is extremely important. To detect the change in a quantity, always subtract the former (earlier) measurement from the latter (later) measurement.

Example 6.6.1.1

A ball is perched at rest at the top of a long ramp. It's given a little tap and it begins to roll down the ramp. The speed v of the ball (in meters per second) is plotted versus the time t (in seconds) in Figure 6.6.1.1.

Determine the slope of the line.

Solution

We've defined the slope as the rate at which the dependent variable is changing with respect to the independent variable. In this case, the speed v of the ball "depends" upon the amount of time t that has elapsed. Consequently, v is the dependent variable and has been placed on the vertical axis.3 On the other hand, t is the independent variable and has been assigned the horizontal axis.

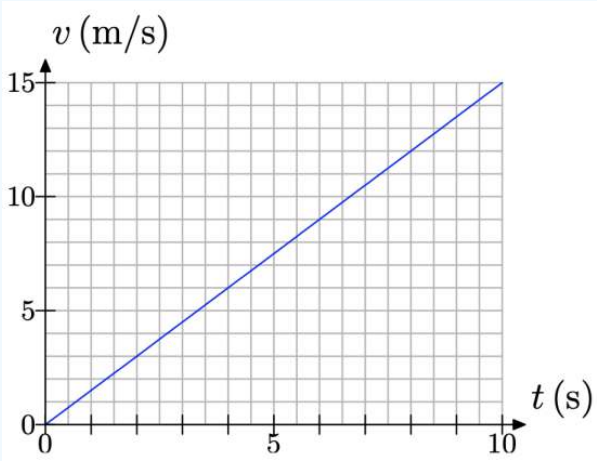


Figure 6.6.1.1. Speed versus time.

To determine the rate at which v is changing with respect to t (the slope of the line), we first select two points P(2, 3) and Q(8, 12) on the line, as shown in Figure 6.6.1.2 As we sweep our eyes from left to right (a convention we will always follow when dealing with slope), the point P occurs before the point Q. Hence, we consider P the "former" measurement and point Q the "latter" measurement.

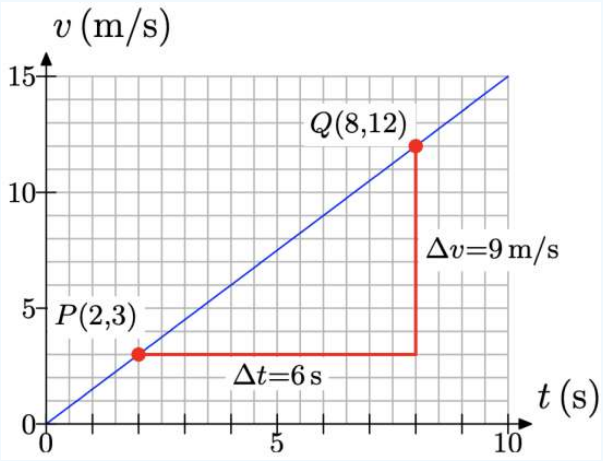


Figure 6.6.1.2. Determining the slope of the line.

At point P, the time is t = 2 seconds, then at point Q the time is t = 8 seconds. The change in t is found by subtracting the former measurement from the latter measurement.

$$\Delta t = 8s - 2s = 6s$$

At point P, the speed is v = 3 meters per second, then at point Q the speed is v = 12 meters per second. Hence, the change in v is

$$\Delta v = 12\text{m/s} - 3\text{m/s} = 9\text{m/s}$$

Finally, the slope of the line is defined as the rate at which the dependent variable v is changing with respect to the independent variable t. That is,

$$\text{Slope} = \frac{\Delta v}{\Delta t} = \frac{9\text{m/s}}{6\text{s}} = \frac{3\text{m/s}}{\text{s}}$$

Scientists prefer to write this as 1.5 m/s^2, but this might not be as intuitive as writing 1.5 (m/s)/s, which indicates that the speed is increasing at a rate of 1.5 m/s every second. This makes good sense as a ball rolling down a ramp will pick up speed with the passage of time. The slope provides an exact numerical description of how the speed increases with respect to time.

Note that our definition of the slope of the line satisfies one of our goals: the slope is precisely the same as the notion of rate described in the previous section. Indeed, note the right triangle we've drawn in Figure 6.6.1.2 The bottom edge of the triangle is 12 boxes long, but every 2 boxes represents one second, so this displacement in the time t direction is 6 seconds. The vertical side of the right triangle is 9 boxes in height where each box represents 1 meter per second. Consequently, this vertical edge of the right triangle represents a positive displacement of 9 meters per second. Thus, every 6 seconds, there is an increase in speed of 9 meters per second. Hence, the ball is picking up speed at the rate of 9 meters per second every 6 seconds, or equivalently, 1.5 meters per second every second.

Note

In Figure 6.6.1.2 the rate at which the speed is increasing with respect to time is equivalent to the slope of the line.

Suppose that we had labeled our points $P\left(t_{\text{initial}}, v_{\text{initial}}\right)$ and $Q\left(t_{\text{final}}, v_{\text{final}}\right)$ as shown in Figure 6.6.1.3

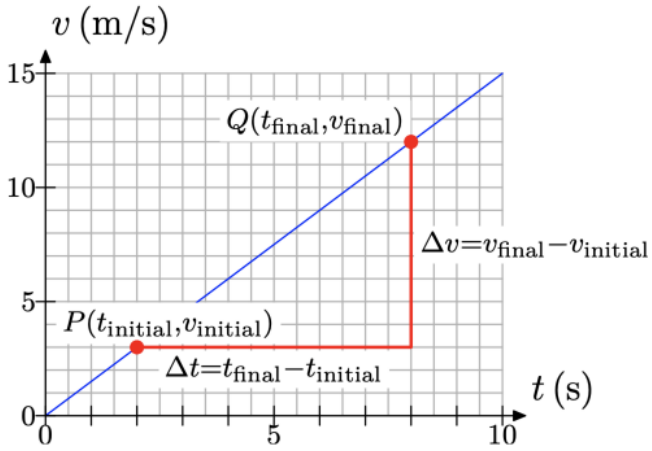


Figure 6.6.1.3. Initial and final measurements.

Now the change in speed v would be

$$\Delta v = v_{\text{final}} - v_{\text{initial}}$$

and the change in time t would be

$$\Delta t = t_{\text{final}} - t_{\text{initial}}$$

Therefore, the slope of the line would be computed with the following formula.

$$\text{Slope} = \frac{\Delta v}{\Delta t} = \frac{v_{\text{final}} - v_{\text{initial}}}{t_{\text{final}} - t_{\text{initial}}}$$

With $P\left(t_{\text{initial}}, v_{\text{initial}}\right) = (2\text{s}, 3\text{m/s})$ and $Q\left(t_{\text{final}}, v_{\text{final}}\right) = (8\text{s}, 12\text{m/s})$, this becomes

$$\text{Slope} = \frac{12\text{m/s} - 3\text{m/s}}{8\text{s} - 2\text{s}} = \frac{9\text{m/s}}{6\text{s}} = 1.5\text{m/s}^2$$

The Slope Formula

The last calculation in Example 6.6.1.1 allows us to discuss the slope of a line as a purely mathematical concept, one that is not rooted in a supporting application as in Example 6.6.1.1. Take, for example, the line shown in Figure 6.6.1.4 that passes through the points $P(-3, -3)$ and $Q(2, 1)$.

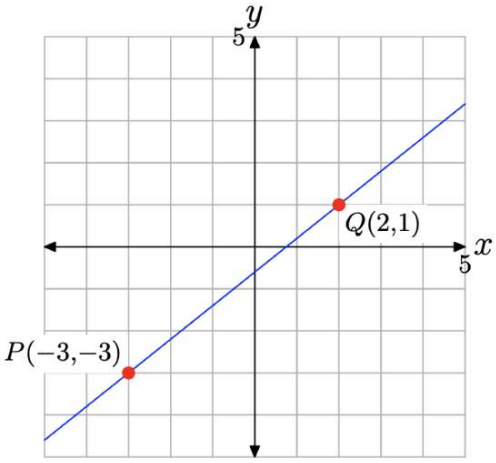


Figure 6.6.1.4. Computing the slope of a line in an xy-coordinate system.

In this example, the dependent variable is y and the independent variable is x, so the slope of the line is Δy (the change in y) divided by Δx (the change in x).

$$\text{Slope} = \frac{\Delta y}{\Delta x}$$

Sweeping our eyes from left to right, the point P comes first, followed by the point Q. Keeping "latter minus former" in mind, the change in y is computed by subtracting the y-value of point P from the y-value of point Q. That is,

$$\Delta y = 1 - (-3) = 4$$

Similarly, the change in x is computed by subtracting the x-value of point P from the x-value of point Q. That is,

$$\Delta x = 2 - (-3) = 5$$

Thus, the slope of the line is

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{4}{5}$$

Alternatively, we can use the points P and Q as vertices of a right triangle with sides parallel to the axes (shown in Figure 6.6.1.5(a)). The horizontal edge of the right triangle is 5 boxes (each representing 1 unit), so the displacement in x is 5 units. The vertical edge is 4 boxes (each representing 1 unit), so the displacement in y is 4 units. Hence, each time x is increased by 5 units, y experiences an increase of 4 units. Therefore, the slope of the line is again 4/5.

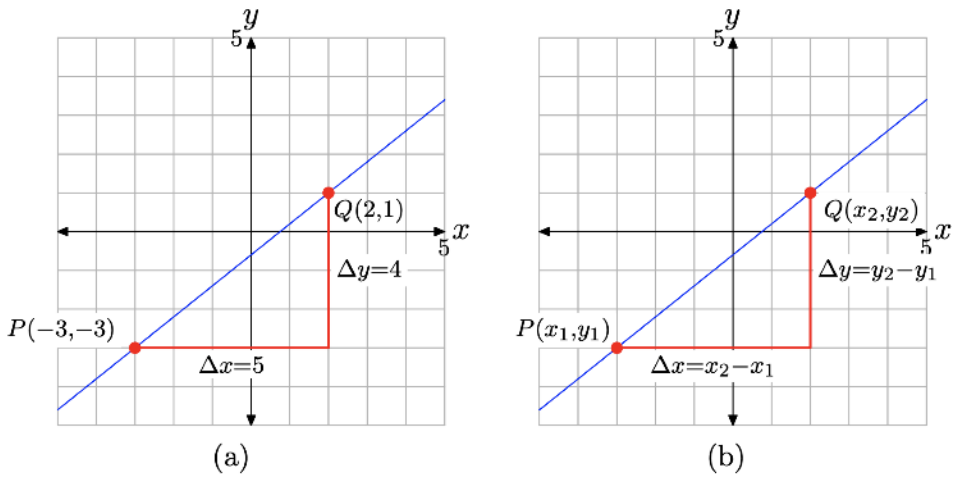


Figure 6.6.1.5. Using a right triangle to determine the slope.

Suppose that we had labeled our points $P\left(x_1, y_1\right)$ and $Q\left(x_2, y_2\right)$ as shown in Figure 6.6.1.5(b). Now the change in y would be

$$\Delta y = y_2 - y_1$$

and the change in x would be

$$\Delta x = x_2 - x_1$$

Therefore, the slope of the line would be computed with the following formula.

$$\text{Slope } = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

With $P\left(x_1, y_1\right) = (-3, -3)$ and $Q\left(x_2, y_2\right) = (2, 1)$, this becomes

$$\text{Slope } = \frac{1 - (-3)}{2 - (-3)} = \frac{4}{5}$$

The slope formula is worth summarizing in a definition.

Definition 5

The slope of the line that passes through the points $P\left(x_1, y_1\right)$ and $Q\left(x_2, y_2\right)$ is given by the formula

$$\text{Slope } = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Let's look at some more examples.

Example 6.6.1.2

Find the slope of the line passing through the points P(−3, −2) and Q(3, 1).

Solution

We can use the slope formula in Definition 5 to determine the slope. With $(x_1, y_1) = P(-3, -2)$ and $(x_2, y_2) = Q(3, 1)$,

$$\text{Slope } = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - (-2)}{3 - (-3)} = \frac{3}{6} = \frac{1}{2}$$

Readers will sometimes ask, "Which point should be (x_1, y_1) and which should be (x_2, y_2)?" The short answer is, "It doesn't matter!" Suppose instead, that we let $(x_1, y_1) = Q(3, 1)$ and $(x_2, y_2) = P(-3, -2)$. Then,

$$\text{Slope } = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 1}{-3 - 3} = \frac{-3}{-6} = \frac{1}{2}$$

Because the change in any quantity is found by subtracting the earlier measurement from the later measurement, we will continue to stress the first order. However, if we reverse the points as we did in our second calculation, both numerator and denominator reverse sign with this interchange, so we get the same answer.

Of course, we can also determine the slope by plotting P(−3, −2) and Q(3, 1) and the line that passes through P and Q, as we've done in Figure 6.6.1.6

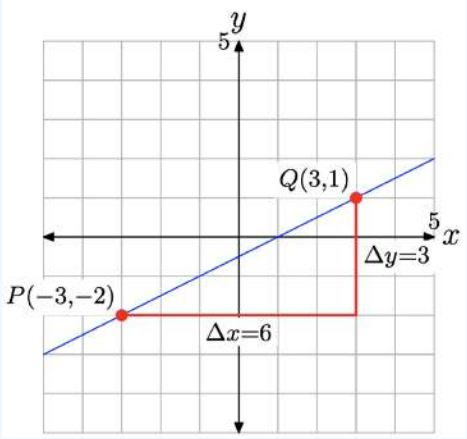


Figure 6.6.1.6. Determining the slope from the graph.

Starting at the point P, to get to the point Q, we move 6 boxes to the right, then 3 boxes up, as shown in Figure 6.6.1.6 Hence, the slope of the line is

$$\text{Slope } = \frac{\Delta y}{\Delta x} = \frac{3}{6} = \frac{1}{2}$$

Note that two of our expectations regarding the slope of a line are met with this example.

1. The line through P(−3, −2) and Q(3, 1) in Figure 6.6.1.6has slope 1/2. This is a positive number and the line slants uphill (as expected) as we sweep our eyes from left to right.
2. The slope in this example is 1/2, which is less than the slope of the line in Figure 6.6.1.5(a), which was 4/5. Note that the line in Figure 6.6.1.6is less steep than the line in Figure 6.6.1.5(a), which was another of our earlier expectations regarding the slope of a line.

Example 6.6.1.3

Find the slope of the line passing through the points P(−4, 4) and Q(4, −2).

Solution

We can use the slope formula in Definition 5 to determine the slope. With $(x_1, y_1) = P(-4, 4)$ and $(x_2, y_2) = Q(4, -2)$,

$$\text{Slope } = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-4}{4-(-4)} = \frac{-6}{8} = -\frac{3}{4}$$

We can also get the slope of the line from the graph in Figure 6.6.1.7. Starting at the point P(−4, 4), move 8 units to the right, then 6 units downward, as shown in Figure 6.6.1.7.

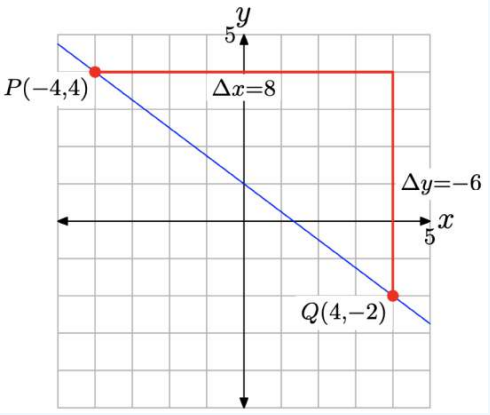


Figure 6.6.1.7. Determining the slope from the graph.

Thus, the slope of the line is

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{-6}{8} = -\frac{3}{4}$$

Again, one of our earlier expectations regarding the slope of a line is met in this example. The slope is −3/4, which is a negative number, and the line in Figure 6.6.1.7 slants downhill (as we sweep our eyes from left to right).

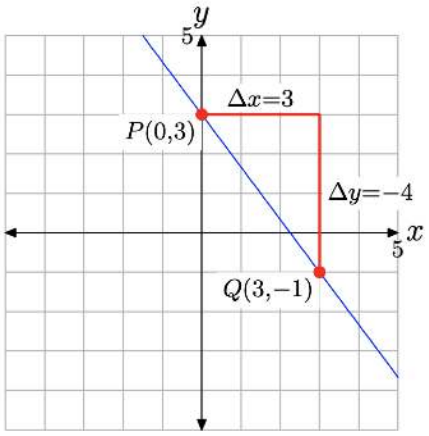
Example 6.6.1.4

Draw a line that intercepts the y-axis at (0, 3) so that the line has slope −4/3. Draw a second line that passes through the point P(−1, −1) with slope 3/5.

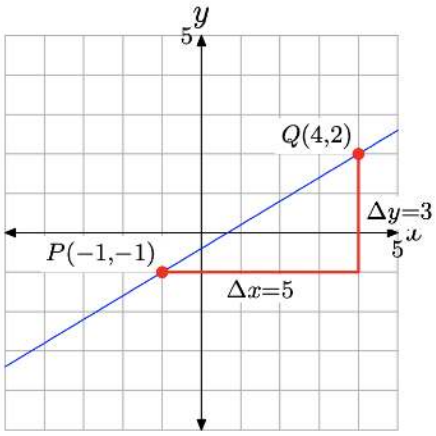
Solution

The slope of the first line is −4/3. This means that our line must slant downhill (as we sweep our eyes from left to right). The slope is the change in y over the change in x. Therefore, every time x increases by 3 units, y must decrease by 4 units. Plot the point P(0, 3), as shown in Figure 6.6.1.8(a). Then, starting at P, move 3 units to the right, followed by 4 units downward to the point Q(3, −1), as shown in Figure 6.6.1.8(a). Draw the required line, which must pass through the points P and Q.

To draw the second line, first plot the point P(−1, −1), as shown in Figure 6.6.1.8(b). Starting at the point P, move 5 units to the right, then upward 3 units to the point Q(4, 2), as shown in Figure 6.6.1.8(b). Draw the required line passing through the points P and Q.



(a) A line having y-intercept at $(0, 3)$ with slope $-4/3$.



(b) A line having slope $3/5$ that passes through the point $P(-1, -1)$.

Figure 6.6.1.8

Parallel Lines

Because slope controls the “steepness” of a line, it is a simple matter to see that parallel lines must have the same slope.

property

Let $\boldsymbol{L}_1$ be a line having slope m_1. Let $\boldsymbol{L}_2$ be a line having slope m_2. If $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ are parallel, then

$$m_1 = m_2$$

That is, any two parallel lines have the same slope.

Example 6.6.1.5

What is the slope of any horizontal line? What is the slope of any vertical line?

Solution

One would expect that our definition would verify that the slope of any horizontal line is zero. Select, for example, the horizontal line shown in Figure 9(a). Select the points (−3, 3) and (3, 3) on this line.

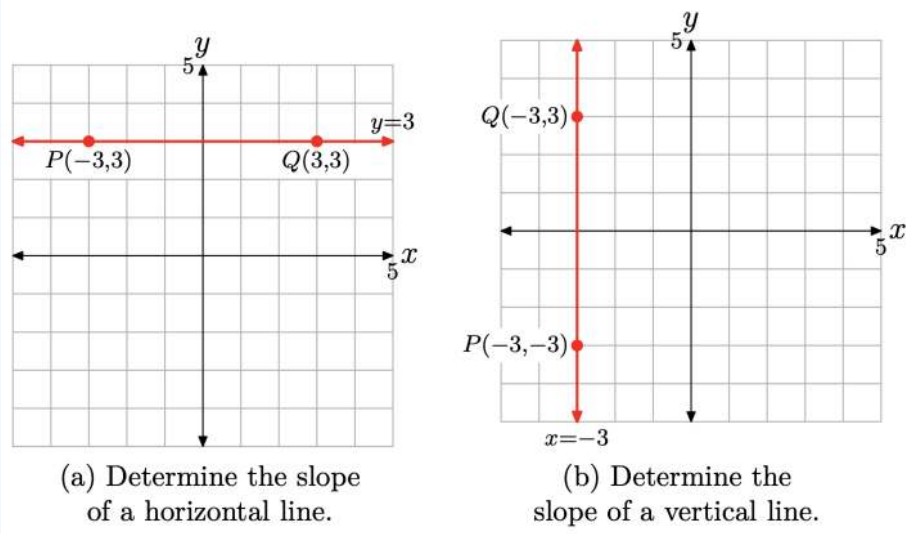


Figure 6.6.1.9

With $(x_1, y_1) = (-3, 3)$ and $(x_2, y_2) = (3, 3)$

$$\text{Slope } = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-3}{3-(-3)} = \frac{0}{6} = 0$$

Thus, the horizontal line in Figure 6.6.1.9(a) has slope equal to zero, exactly as expected. Further, all horizontal lines are parallel to this horizontal line and have the same slope. Therefore, all horizontal lines have slope zero.

We would surmise that the vertical line in Figure 6.6.1.9(b) has undefined slope (we’ll explore this more fully in the exercises). In Figure 6.6.1.9(b), we’ve selected the points P(−3, −3) and Q(−3, 3) on the vertical line. With $(x_1, y_1) = P(-3, -3)$ and $(x_2, y_2) = Q(-3, -3)$,

$$\text{Slope } = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-(-3)}{-3-(-3)} = \frac{6}{0}, \text{ which is undefined.}$$

The slope of the vertical line in Figure 6.6.1.9(b) is undefined because division by zero is meaningless. Further, all vertical lines are parallel to this vertical line and have undefined slope.

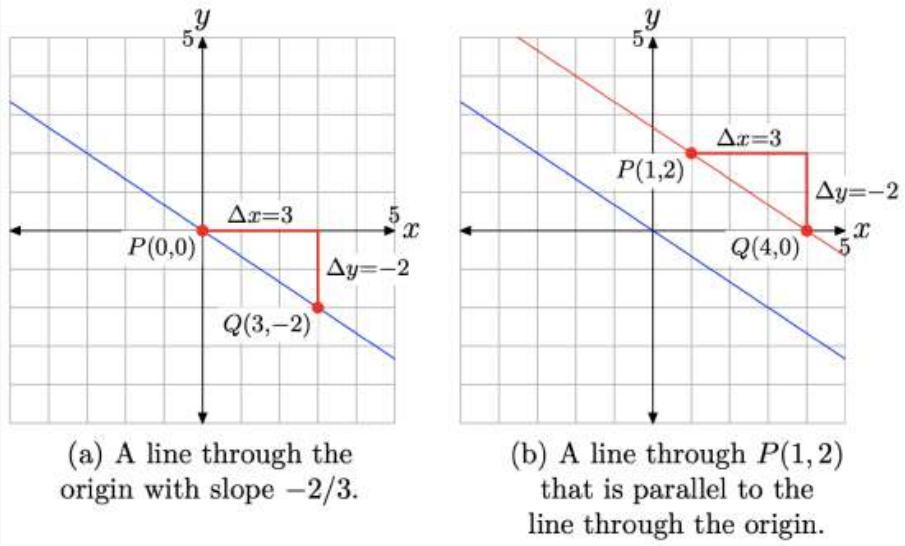
Example 6.6.1.6

Draw a line through the point P(1, 2) that is parallel to the line passing through the origin with slope −2/3.

Solution

We will first draw a line through the origin with slope −2/3. Plot the point P(0, 0), then move 3 units to the right and 2 units downward to the point Q(3, −2), as shown in Figure 6.6.1.10(a). Draw a line through the points P and Q as shown in Figure 6.6.1.10(a).

Next, plot the point P(1, 2) as shown in Figure 6.6.1.10(b). To draw a line through this point that is parallel to the line through the origin, this second line must have the same slope as the first line. Therefore, start at the point P(1, 2), as shown in Figure 6.6.1.10(b), then move 3 units to the right and 2 units downward to the point Q(4, 0). Draw a line through the points P and Q as shown in Figure 6.6.1.10(b). Note that this second line is parallel to the first.



(a) A line through the origin with slope −2/3.

(b) A line through $P(1,2)$ that is parallel to the line through the origin.

Figure 6.6.1.10

Perpendicular Lines

The relationship between the slopes of two perpendicular lines is not as straightforward as the relation between the slopes of two parallel lines. Let's begin by stating the pertinent property.

Property 13

Let L_1 be a line having slope m_1. Let L_2 be a line having slope m_2. If L_1 and L_2 are perpendicular, then

$$m_1 m_2 = -1$$

That is, the product of the slopes of two perpendicular lines is −1.

We can solve equation (14) for m_1 in terms of m_2.

$$m_1 = -\frac{1}{m_2}$$

Equation (15) tells us that the slope of the first line is the negative reciprocal of the slope of the second line.

For example, suppose that L_1 and L_2 are perpendicular lines with slopes m_1 and m_2, respectively

- If $m_2 = 2$, then $m_1 = -\frac{1}{2}$
- If $m_2 = \frac{3}{5}$, then $m_1 = -\frac{5}{3}$
- If $m_2 = -\frac{2}{3}$, then $m_1 = \frac{3}{2}$

Note that in each bulleted item, the product of the slopes is −1.

We won't provide a proof of equation (15), but we will provide some motivating evidence in the form of a graph.

Example 6.6.1.7

Sketch the graphs of the lines passing through the origin having slopes 2 and −1/2.

Solution

In Figure 6.6.1.11(a), we've plotted the point P(0, 0) at the origin, then moved 1 unit to the right and 2 units upward to the point Q(1, 2). The resulting line passes through the origin and has slope $m_1 = 2$ (alternatively, $m_1 = 2/1$).

In Figure 6.6.1.11(b), we've again plotted the point P(0, 0) at the origin, then moved 2 units to the right and 1 unit downward to the point Q(2, −1). The resulting line passes through the origin and has slope $m_2 = -1/2$.

There are two important points that need to be made about the lines in Figure 6.6.1.11(b).

The two lines in Figure 6.6.1.11(b) are perpendicular. They meet and form a right angle of 90°. If you have a protractor available, you might want to measure the angle between the two lines and note that the measure of the angle is 90°.

The product of the two slopes is

$$m_1 m_2 = 2 \cdot \left(-\frac{1}{2}\right) = -1$$

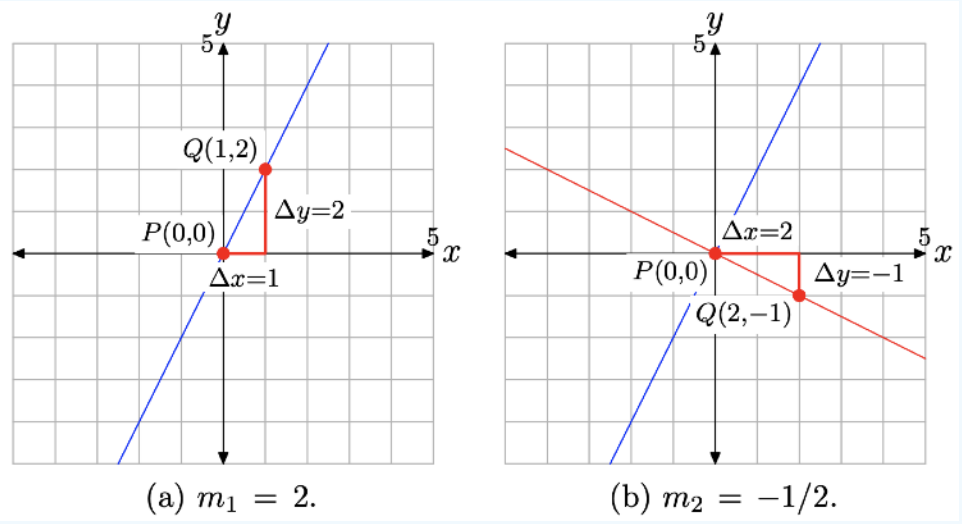


Figure 6.6.1.11. Sketching perpendicular lines.

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6.6.2: Equations of Lines

In this section we will develop the slope-intercept form of a line. When you have completed the work in this section, you should be able to look at the graph of a line and determine its equation in slope-intercept form.

The Slope-Intercept Form

In the previous section, we developed the formula for the slope of a line. Let's assume that the dependent variable is y and the independent variable is x and we have a line passing through the points $P\left(x_1, y_1\right)$ and $Q\left(x_2, y_2\right)$, as shown in Figure 6.6.2.1.

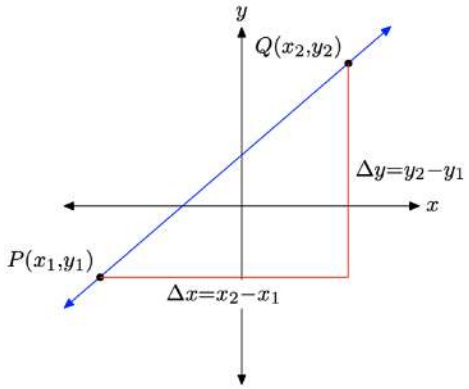


Figure 6.6.2.1. Determining the slope of a line through two points.

As we sweep our eyes from left to right, note that the change in x is $\Delta x = x_2 - x_1$ and the change in y is $\Delta y = y_2 - y_1$. Thus, the slope of the line is determined by the formula

$$\text{Slope } = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Now consider the line in Figure 6.6.2.2 Suppose that we are given two facts about this line:

1. The point where the line crosses the y-axis (the y-intercept) is (0, b).
2. The "slope" of the line is some number m.

To find the equation of the line pictured in Figure 6.6.2.2 select an arbitrary point Q(x, y) on the line, then compute the slope of the line using $(x_1, y_1) = P(0, b)$ and $(x_2, y_2) = Q(x, y)$ in the slope formula (1).

$$\text{Slope } = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y - b}{x - 0}$$

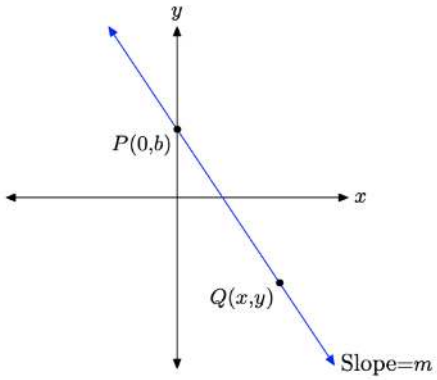


Figure 6.6.2.2. Find the equation of the line in slope-intercept form.

Simplify.

$$\text{Slope } = \frac{y - b}{x}$$

We're given that the slope is the number m, so substitute this number for the word "Slope" in the last result.

$$m = \frac{y-b}{x}$$

Multiply both sides of the last equation by x.

$$mx = y - b$$

Add b to both sides of the last equation to obtain

$$mx + b = y$$

or upon exchanging sides of the equation,

$$y = mx + b$$

The above discussion leads to the following result.

Definition: The Slope-Intercept Form of a Line

If the line L intercepts the y-axis at the point (0, b) and has slope m, then the equation of the line is

$$y = mx + b. \tag{6.6.2.1}$$

This form of the equation of a line is called the slope-intercept form. The function defined by the equation

$$f(x) = mx + b$$

is called a **linear function**.

It is important to note two key facts about the slope-intercept form y = mx + b.

- The coefficient of x (the m in y = mx + b) is the slope of the line.
- The constant term (the b in y = mx+b) is the y-coordinate of the y-intercept (0, b).

Procedure for Using the Slope-Intercept Form of a Line

When given the slope of a line and the y-intercept of the line, use the slope-intercept form as follows:

1. Substitute the given slope for m in the formula $y = mx + b$.
2. Substitute the y-coordinate of the y-intercept for b in the formula $y = mx + b$.

For example, if the line has slope −2 and the y-intercept (the point where the line crosses the y-axis) is (0, 3), then substitute m = −2 and b = 3 into Equation 6.6.2.1to obtain

$$y = -2x + 3$$

.

Let's look at some examples of use of this all-important formula.

Example 6.6.2.1

What is the equation of the line having slope −2/3 and y-intercept at (0, 3)? Sketch the line on graph paper.

Solution

The equation of the line is

$$y = mx + b$$

We're given that the slope is −2/3. Hence, m = −2/3. Secondly, we're given that the line intercepts the y-axis at the point (0, 3). In the slope-intercept form y = mx + b, recall that b represents the y-coordinate of the y-intercept. Hence, b = 3. Substitute m = −2/3 and b = 3 in equation (4), obtaining

$$y = -\frac{2}{3}x + 3$$

To sketch the graph of the line, first locate the y-intercept at P(0, 3), as shown in Figure 6.6.2.3. Starting from the y-intercept at P(0, 3), move 3 units to the right and 2 units downward to the point Q(3, 1). The required line passes through the points P and Q.

Note that the line "intercepts" the y-axis at 3 and slants downhill, in accordance with the fact that the slope is negative in this example.

Example 6.6.2.2

Given the graph of the line in Figure 6.6.2.4(a), determine the equation of the line.

Solution

First, locate the y-intercept of the line, which we've labeled P(0, −1) in Figure 6.6.2.4(b). In the formula y = mx + b, recall that b represents the y-coordinate of the y-intercept. Thus, b = −1.

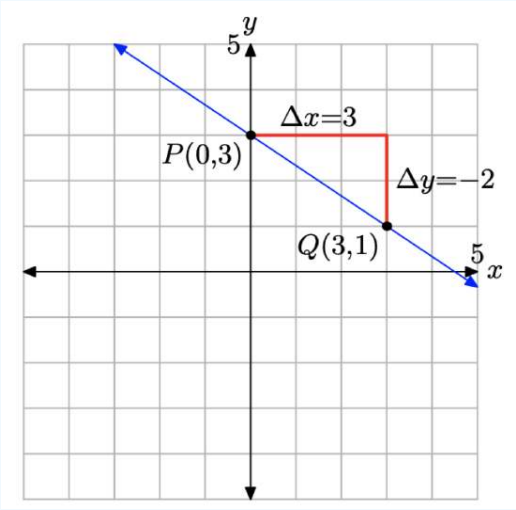


Figure 6.6.2.3. The line has y-intercept at (0, 3) and slope −2/3.

Secondly, we need to determine the slope of the line. In Figure 6.6.2.4(b), start at the point P, move 2 units to the right, then 3 units upward to the point Q(2, 2). This makes the slope

$$m = \frac{\Delta y}{\Delta x} = \frac{3}{2}$$

Substitute m = 3/2 and b = −1 into the slope-intercept form y = mx + b to obtain

$$y = \frac{3}{2}x - 1$$

which is the desired equation of the line.

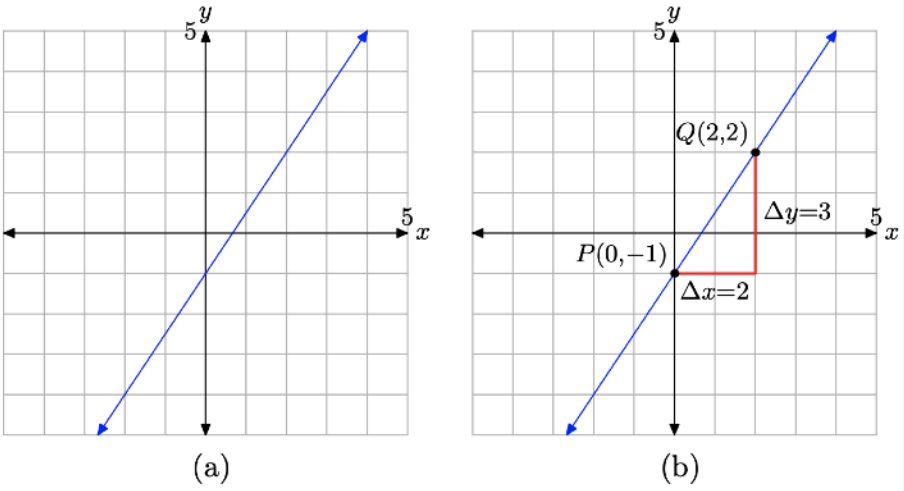


Figure 6.6.2.4. Determining the equation of a line from its graph.

Making Connections

If the connection between rate and slope is still not clear, let's recall an example we did earlier in the chapter.

Sebastian waves good-bye to his brother, who is talking to a group of his friends approximately 20 feet away. Sebastian then begins to walk away from his brother at a constant rate of 4 feet per second.

The distance between the brothers depends upon the amount of time that has passed, so we set distance d on the vertical axis and time t on the horizontal axis, as shown in Figure 6.6.2.5 Note that d and t are taking the "usual" place of y and x, respectively. The distance separating the brothers at time t = 0 is d = 20 feet. This is indicated with the "d-intercept" at P(0, 20) in Figure 6.6.2.5

Next, the distance between the brothers is increasing at a rate of 4 feet per second. Starting at the point P(0, 20), move 1 second to the right (2 boxes) and 4 units up (1 box) to the point Q(1, 24), as shown in Figure 6.6.2.5 The line through the points P and Q then models the distance between the brothers as a function of time.

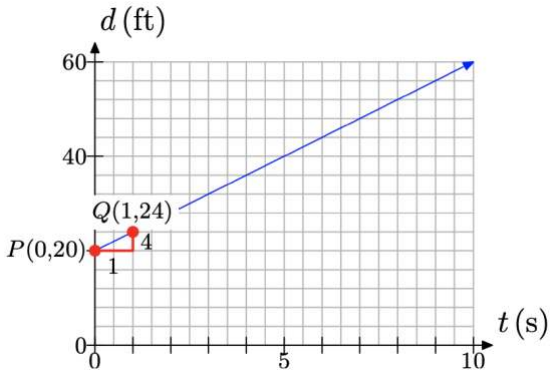


Figure 6.6.2.5. Distance between brothers as a function of time.

If you recall, we then determined a relation between distance d and time t by examining the distance between the brothers at times t = 0, 1, 2, and 3, and summarizing the results in Table 6.6.2.1

Table 6.6.2.1 Determining a model equation.

t	d
0	20
1	20+4(1)
2	20+4(2)
3	20+4(3)

Intuition then led to the following model, which provides the distance d between the brothers as a function of time t.

$$d = 20 + 4t$$

Again, readers should check that equation (6) produces the results in Table $6.6.2.1$at t = 0, 1, 2, and 3.

Alternatively, with the theory developed in this section, we would develop the equation of the line by using the slope-intercept form of the line; that is,

$$y = mx + b$$

However, in this case, the dependent variable is d, not y, and the independent variable is t, not x. So, replace y and x in equation (7) with d and t, respectively, obtaining

$$d = mt + b$$

Next, the line intercepts the d-axis at P(0, 20), so b = 20. Furthermore, the slope of the line is 4 feet per second, so m = 4. Substitute m = 4 and b = 20 in equation (8) to obtain

$$d = 4t + 20$$

or using function notation, $d(t) = 4t + 20$. Note that equation (9) is identical to the intuitively generated model of equation (6).

Hopefully, this development should cement for all time the idea that slope of the line is the rate at which the dependent variable is changing with respect to the independent variable.

The Standard Form of a Line

We now know that if our equation has the form $y = mx + b$ (or can be manipulated into this form), the graph will be a line. Let's take a moment to demonstrate that the graph of the equation $Ax + By = C$, where A, B, and C are constants, is a line.

If we can place the form $Ax + By = C$ into slope-intercept form $y = mx + b$, then that will demonstrate that the graph of $Ax + By = C$ is a line. So, start with $Ax + By = C$ and subtract Ax from both sides of the equation.

$$By = -Ax + C$$

Divide both sides of this last equation by B. Note that there is an assumption here that $B \neq 0$. We will handle the case when B = 0 separately, at the end of this section.

$$\frac{By}{B} = \frac{-Ax + C}{B}$$
$$y = -\frac{A}{B}x + \frac{C}{B}$$

When we compare $y = -(A/B)x + (C/B)$ with y = mx + b, we note that the slope is m = −A/B and the y-coordinate of the y-intercept is b = C/B. Because we were successful in placing the equation $Ax + By = C$ into slope-intercept form, we now know that the graph of $Ax + By = C$ is a line (we'll need this result in later work).

The Standard Form of a Line

The graph of the equation Ax + By = C is a line. The form

$$Ax + By = C$$

is called the standard form of a line.

Let's look at an example.

Example $6.6.2.3$

The equation 3x+4y = 12 is in standard form. Place this equation in slope-intercept form, determine the slope and y-intercept, then use these results to draw the graph of the line.

Solution

First, solve the equation 3x + 4y = 12 for y.

$$\begin{aligned} 3x+4y &= 12 \\ 4y &= -3x+12 \\ y &= -\frac{3}{4}x+3 \end{aligned}$$

Note in the last step how the distributive property came into play. When we divided −3x + 12 by 4, we divided each term by 4, getting (−3/4)x + 3.

When we compare y = (−3/4)x+3 with the general slope-intercept form y = mx+b, we determine that the slope is m = −3/4 and the y-coordinate of the y-intercept is b = 3. To sketch the graph of the line, as we've done in Figure 6.6.2.6, plot the y-intercept at P(0, 3), then move 4 units to the right and 3 units down to the point Q(4, 0). The line passing through the points P and Q is the required line.

Note again that the slope is m = −3/4 and the line slants "downhill." Also, b = 3 and the line "intercepts" the y-axis at P(0, 3).

Let's look at another example.

Example 6.6.2.4

In Example 6.6.2.1, we determined that the given line had the equation

$$y = \frac{3}{2}x - 1$$

Place this equation in standard from $Ax + By = C$, where A, B, and C are integers and $A > 0$.

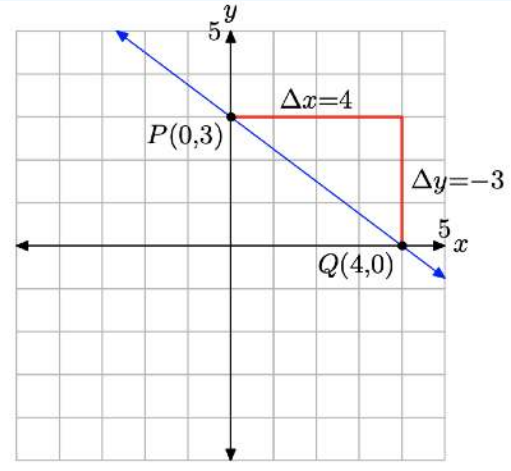


Figure 6.6.2.6. The graph of $3x + 4y = 12$.

Solution

We're requested to place the equation $y = (3/2)x - 1$ in the form $Ax + By = C$, where A, B, and C are integers, so let's begin by clearing fractions from the equation. Multiply both sides of the equation by the common denominator 2.

$$\begin{aligned} y &= \frac{3}{2}x - 1 \\ 2y &= 2\left(\frac{3}{2}x - 1\right) \\ 2y &= 3x - 2 \end{aligned}$$

Now, subtract 2y from both sides of the equation, then add 2 to both sides of the equation to obtain

$$2 = 3x - 2y$$

or equivalently,

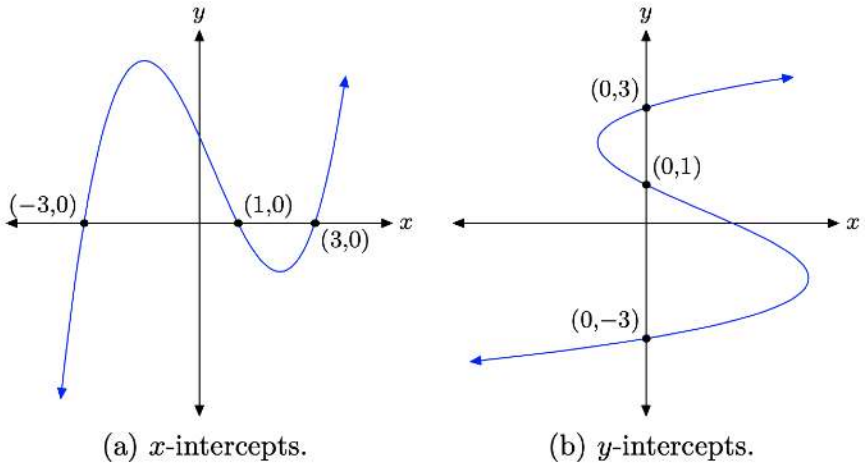
$$3x - 2y = 2$$

Note that this last result is in standard form $Ax + By = C$, where A, B, and C are integers and A > 0.

Intercepts

We now know that the graph of the equation $Ax + By = C$, where A, B, and C are constants, is a line. Because the graph of $Ax + By = C$ is a line, to draw the graph of the line, we need only find two points that satisfy the equation, plot them, then draw a line through them. Our two favorite points to work with are the x- and y-intercepts, because each involves the number zero, an easy number to work with.

Consider the graph in Figure 6.6.2.7(a). Note that the graph passes through the x-axis three times. The points where the graph intercepts the x-axis are called x-intercepts. Note that each of these points has a defining value in common: the y-value of each of these x-intercepts is equal to zero.



(a) x-intercepts. (b) y-intercepts.

Figure 6.6.2.7.

How to Find an x-intercept

To find an x intercept, let y = 0 in the equation and solve for x.

On the other hand, consider the graph in Figure 6.6.2.7(b). Note that this is not a function (fails the vertical line test) but the graph intercepts the y-axis three separate times. The points where the graph intercepts the y-axis are called y-intercepts. Note that each of the y-intercepts in Figure 6.6.2.7(b) has a defining value in common: the x-value of each of the y-intercepts is equal to zero.

How to Find a y-intercept

To find a y-intercept, let x = 0 in the equation and solve for y.

Let's put what we've learned into practice.

Example 6.6.2.5

Sketch the graph of 3x + 4y = 12.

Solution

We drew the graph of the equation 3x + 4y = 12 in Figure 6.6.2.6 There we solved the equation for y to determine the slope and the y-intercept. These, in turn, were used to draw the graph of 3x + 4y = 12 in Figure 6.6.2.6

Here, our approach will be different. We will first determine the x- and y-intercepts, plot them, then draw a line through these intercepts. Hopefully, we will get a result that matches that in Figure 6.6.2.6

To find the x-intercept, let $y = 0$ in $3x + 4y = 12$ and solve for x.

$$\begin{aligned} 3x+4y &= 12 \\ 3x+4(0) &= 12 \\ 3x &= 12 \\ x &= 4 \end{aligned}$$

Hence, the x-intercept is the point Q(4, 0). To find the y-intercept, let x = 0 in $3x+4y=12$ and solve for y.

$$\begin{aligned} 3x+4y &= 12 \\ 3(0)+4y &= 12 \\ 4y &= 12 \\ y &= 3 \end{aligned}$$

Hence, the y-intercept is the point P(0, 3). In Figure 6.6.2.8 we've plotted both x- and y-intercepts and drawn a line through them. Note that the resulting line in Figure 6.6.2.8 matches the same line drawn in Figure 6.6.2.6 (where we used a different method).

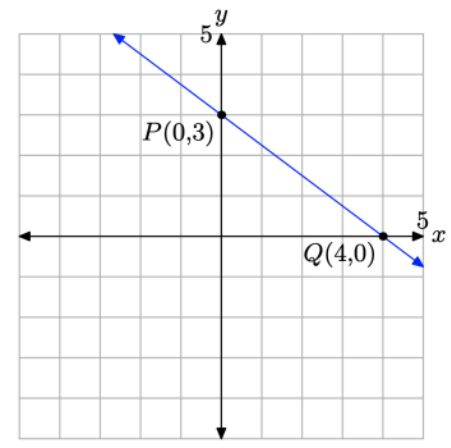


Figure 6.6.2.8 Plotting the x- and y-intercepts.

We recommend that whenever the line is given in Standard Form $Ax+By=C$, find the x- and y-intercepts, plot them, then draw a line through them. This technique is quite efficient because working with the number zero greatly simplifies the calculations.

Horizontal and Vertical Lines

We've introduced the standard form of the line $Ax+By=C$. The case where A and B are simultaneously equal to zero is not very interesting.4 However, the following two cases are of interest.

1. If we let A = 0 and B 6= 0 in the standard form $Ax+By=C$, then By = C, or equivalently y = C/B. Note that this has the form y = b, where b is some constant.
2. Similarly, if we let B = 0 and A 6= 0 in the standard form $Ax+By=C$, then Ax = C, or equivalently, x = C/A. Note that this has the form x = a, where a is some constant.

The lines having the form x = a and y = b are two of the easiest lines to plot. Let's look at an example of each.

Example 6.6.2.6

Sketch the graph of the equation x = 3.

Solution

The direction "sketch the graph of the equation x = 3" can be quite vexing unless one remembers that a graph of an equation is the set of all points that satisfy the equation. Thus, the direction is better posed if we say "sketch the set of all points (x, y) that satisfy x = 3," or equivalently, "sketch the set of all points (x, y) that have an x-value of 3." Then it is an easy matter to sketch the vertical line shown in Figure 6.6.2.9

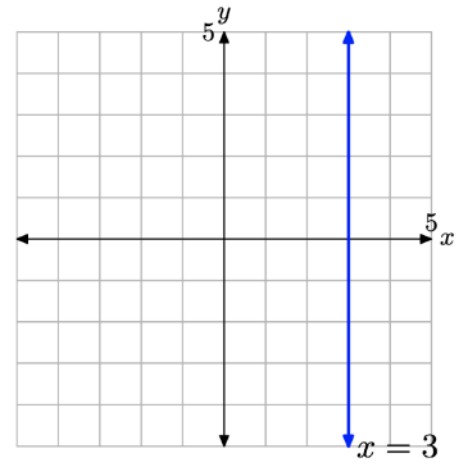


Figure 6.6.2.9. The graph of the equation x = 3.

Note that each point on the line has an x-value equal to 3. Also, note that the slope of this vertical line is undefined.

Example 6.6.2.7

Sketch the graph of the equation y = 3.

Solution

This direction is better posed if we say "sketch the set of all points (x, y) that satisfy y = 3," or equivalently, "sketch the set of all points (x, y) that have a y-value of 3." Then it is an easy matter to sketch the horizontal line shown in Figure 6.6.2.10

Note that each point on the line has a y-value equal to 3. Also, note that this horizontal line has slope zero.

Two final comments are in order. Because the line in Figure 10 has slope zero and y-intercept (0, 3), we can insert m = 0 and b = 3 into the slope intercept form y = mx + b and obtain

$$y = 0x + 3$$

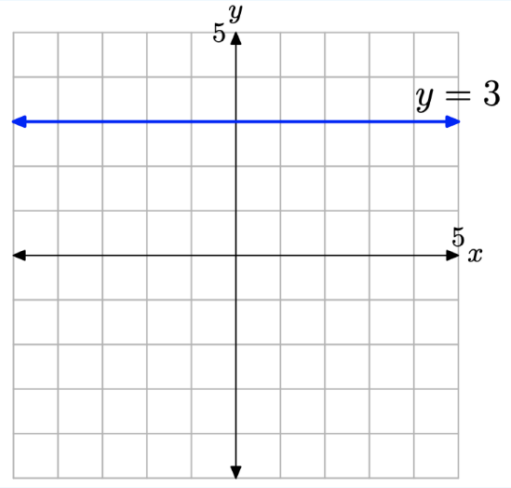


Figure 6.6.2.10. The graph of the equation y = 3.

which of course, is equivalent to y = 3. However, the vertical line shown in Figure 6.6.2.9 has "undefined" slope, so this approach is unavailable. We must simply recognize that the vertical line in Figure 6.6.2.9 consists of all points having an x-value equal to 3, and then intuit that its equation is x = 3.

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6.6.2.1: The Point-Slope Form of a Line

In the last section, we developed the slope-intercept form of a line (y = mx + b). The slope-intercept form of a line is applicable when you're given the slope and y-intercept of the line. However, there will be times when the y-intercept is unknown.

Suppose for example, that you are asked to find the equation of a line that passes through a particular point $P\left(x_0, y_0\right)$ with slope = m. This situation is pictured in Figure 6.6.2.1.1

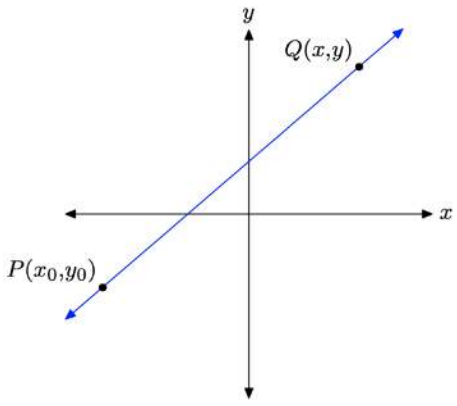


Figure 6.6.2.1.1 . A line through $\left(x_0, y_0\right)$ with slope m.

Let the point Q(x, y) be an arbitrary point on the line. We can determine the equation of the line by using the slope formula with points P and Q. Hence,

$$\text{Slope } = \frac{\Delta y}{\Delta x} = \frac{y - y_0}{x - x_0}$$

Because the slope equals m, we can set Slope = m in this last result to obtain

$$m = \frac{y - y_0}{x - x_0}$$

If we multiply both sides of this last equation by $x - x_0$, we get

$$m\left(x - x_0\right) = y - y_0$$

or exchanging sides of this last equation,

$$y - y_0 = m\left(x - x_0\right)$$

This last result is the equation of the line.

The Point-Slope Form of a Line

If line L passes through the point $\left(x_0, y_0\right)$ and has slope m, then the equation of the line is

$$y - y_0 = m\left(x - x_0\right)$$

This form of the equation of a line is called the point-slope form.

To use the point-slope form of a line, follow these steps.

Procedure for Using the Point-Slope Form of a Line

When given the slope of a line and a point on the line, use the point-slope form as follows:

1. Substitute the given slope for m in the formula $y - y_0 = m\left(x - x_0\right)$.
2. Substitute the coordinates of the given point for x0 and y0 in the formula $y - y_0 = m\left(x - x_0\right)$.

For example, if the line has slope −2 and passes through the point (3, 4), then substitute $m = -2, x_0 = 3,$ and $y_0 = 4$ in the formula $y - y_0 = m\left(x - x_0\right)$ to obtain

$$y-4=-2(x-3)$$

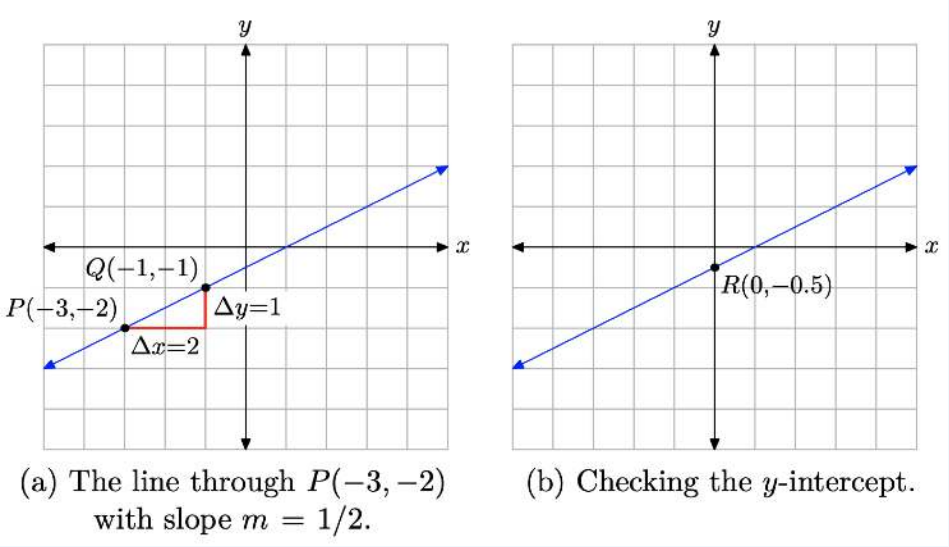
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Example 6.6.2.1.1

Draw the line that passes through the point P(−3, −2) and has slope m = 1/2. Use the point-slope form to determine the equation of the line.

Solution

First, plot the point P(−3, −2), as shown in Figure 6.6.2.1.2(a). Starting from the point P(−3, −2), move 2 units to the right and 1 unit up to the point Q(−1, −1). The line through the points P and Q in Figure 6.6.2.1.2(a) now has slope m = 1/2.



(a) The line through $P(-3,-2)$ with slope $m = 1/2$. (b) Checking the y-intercept.

Figure 6.6.2.1.2

To determine the equation of the line in Figure 6.6.2.1.2(a), we will use the point-slope form of the line

$$y-y_0=m\,(x-x_0)$$

The slope of the line is m = 1/2 and the given point is P(−3, −2), so $(x_0, y_0) = (-3, -2)$. In equation (3), set $m = 1/2$, $x_0 = -3$, and $y_0 = -2$, obtaining

$$y-(-2)=\frac{1}{2}(x-(-3))$$

or equivalently,

$$y+2=\frac{1}{2}(x+3)$$

This is the equation of the line in Figure 6.6.2.1.2(a).

As a check, we've estimated the y-intercept of the line in Figure 6.6.2.1.2(b) as R(0, −0.5). Let's place equation (4) in slope-intercept form to determine the exact value of the y-intercept. First, distribute 1/2 to get

$$y+2=\frac{1}{2}x+\frac{3}{2}$$

Subtract 2 from both sides of this last equation.

$$y=\frac{1}{2}x+\frac{3}{2}-2$$

Make equivalent fractions with a common denominator and simplify.

$$y = \frac{1}{2}x + \frac{3}{2} - \frac{4}{2}$$
$$y = \frac{1}{2}x - \frac{1}{2}$$

Comparing equation (5) with y = mx+b gives us b = −1/2. This is the exact y-value of the y-intercept. Note that this result compares exactly with the y-value of point R in Figure 6.6.2.1.2(b). This is a bit lucky. Don't expect to get an exact comparison every time. However, if the comparison is not close, look for an error in your work, either in your computations or in your graph.

Let's look at another example.

Example 6.6.2.1.2

Find the equation of the line passing through the points P(−3, 2) and Q(2, −1). Place your final answer in standard form.

Solution

Again, to help keep our focus, we draw the line passing through the points P(−3, 2) and Q(2, −1) in Figure 6.6.2.1.3

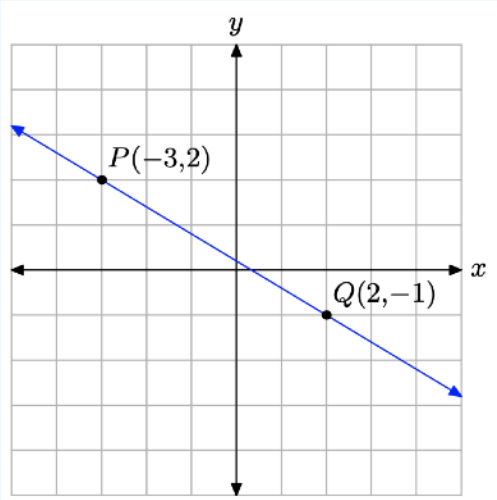


Figure 6.6.2.1.3. The line through points P(−3, 2) and Q(2, −1).

Use the slope formula to determine the slope of the line through the points P(−3, 2) and Q(2, −1).

$$m = \frac{\Delta y}{\Delta x} = \frac{-1-2}{2-(-3)} = -\frac{3}{5}$$

We'll use the point-slope form of the line

$$y - y_0 = m\,(x - x_0)$$

Let's use point P(−3, 2) as the given point (x_0, y_0). That is, $(x_0, y_0) = (-3, 2)$. Substitute $m = -3/5, x_0 = -3,$ and $y_0 = 2$ in equation (7), obtaining

$$y - 2 = -\frac{3}{5}(x - (-3))$$

This is the equation of the line passing through the points P and Q.

Alternatively, we could also use the point Q(2, −1) as the given point (x_0, y_0). That is, $(x_0, y_0) = (2, -1)$. Substitute $m = -3/5, x_0 = 2,$ and $y_0 = -1$ in the point-slope form (7), obtaining

$$y - (-1) = -\frac{3}{5}(x - 2)$$

This too, is the equation of the line passing through the points P and Q.

How can the equations (8) and (9) both be the equation of the line through P and Q, yet look so distinctly different? Let's place each equation in standard form $Ax + By = C$ and compare the results.

If we start with equation (8) and distribute the slope,

$$y-2=-\tfrac{3}{5}(x-(-3))$$
$$y-2=-\tfrac{3}{5}x-\tfrac{9}{5}$$

Multiply both sides by the common denominator 5 to clear the fractions.

$$\begin{aligned}5(y-2)&=5\left(-\frac{3}{5}x-\frac{9}{5}\right)\\5y-10&=-3x-9\end{aligned}$$

Add 3x to both sides of the equation, then add 10 to both sides of the equation to obtain

$$3x+5y=1$$

Place equation (9) in standard form in a similar manner. First, start with equation (9) and distribute the slope,

$$\begin{aligned}y-(-1)&=-\frac{3}{5}(x-2)\\y+1&=-\frac{3}{5}x+\frac{6}{5}\end{aligned}$$

Next, multiply both sides of this last result by 5 to clear the fractions from the equation.

$$\begin{aligned}5(y+1)&=5\left(-\frac{3}{5}x+\frac{6}{5}\right)\\5y+5&=-3x+6\end{aligned}$$

Finally, add 3x to both sides of the equation, then subtract 5 from both sides of the equation to obtain

$$3x+5y=1$$

Note that equation (11) is identical to equation (10). Thus, it doesn't matter which point you use in the point-slope form. Both lead to the same result.

Parallel Lines

Recall that slope controls the "steepness" of a line. Consequently, if two lines are parallel, they must have the same "steepness" or slope. Let's look at an example of parallel lines.

Example 6.6.2.1.3

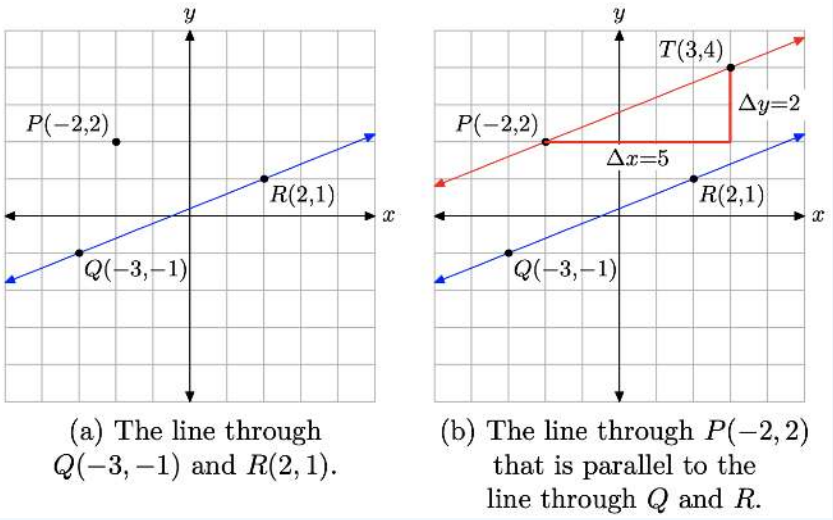
Find the equation of the line that passes through the point P(−2, 2) that is parallel to the line passing through the points Q(−3, −1) and R(2, 1).

Solution

First, to help us stay focused, we draw the line through the points Q(−3, −1) and R(2, 1), then plot the point P(−2, 2), as shown in Figure 6.6.2.1.4(a).

We can use the slope formula to calculate the slope of the line passing through the points Q(−3, −1) and R(2, 1).

$$m=\frac{\Delta y}{\Delta x}=\frac{1-(-1)}{2-(-3)}=\frac{2}{5}$$



(a) The line through $Q(-3,-1)$ and $R(2,1)$.

(b) The line through $P(-2,2)$ that is parallel to the line through Q and R.

Figure 6.6.2.1.4.

We now draw a line through the point P(−2, 2) that is parallel to the line through the points Q and R. Parallel lines must have the same slope, so we start at the point P(−2, 2), "run" 5 units to the right, then "rise" 2 units up to the point T(3, 4), as shown in Figure 6.6.2.1.4(b).

We seek the equation of the line through the points P and T. We'll use the pointslope form of the line

$$y - y_0 = m\,(x - x_0)$$

We'll use the point P(−2, 2) as the given point (x_0, y_0). That is, $(x_0, y_0) = (-2, 2)$. The line through P has slope 2/5. Substitute $m = 2/5, x_0 = -2,$ and $y_0 = 2$ in equation (13) to obtain

$$y - 2 = \frac{2}{5}(x - (-2))$$

Let's place the equation (14) in standard form. Distribute the slope, then clear fractions by multiplying both sides of the resulting equation by 5.

$$\begin{aligned} y - 2 &= \frac{2}{5}x + \frac{4}{5} \\ 5(y-2) &= 5\left(\frac{2}{5}x + \frac{4}{5}\right) \\ 5y - 10 &= 2x + 4 \end{aligned}$$

Finally, subtract 5y from both sides of the last equation, then subtract 4 from both sides of the equation, obtaining

$$-14 = 2x - 5y$$

or equivalently,

$$2x - 5y = -14$$

This is the standard form of the equation of the line passing through the point P and parallel to the line passing through the points Q and R.

Perpendicular Lines

Suppose that two lines L_1 and L_2 have slopes m_1 and m_2, respectively. Recall (see the section on Slope) that if L_1 and L_2 are perpendicular, then the product of their slopes is $m_1 m_2 = -1$. Alternatively, the slope of the first line is the negative reciprocal of the second line, and vice-versa; i.e., $m_1 = -1/m_2$ and $m_2 = -1/m_1$. Let's look at an example of perpendicular lines.

Example 6.6.2.1.4

Find the equation of the line passing through the point P(−4, −4) that is perpendicular to the line 4x + 3y = 12.

Solution

It will help our focus if we draw the given line 4x + 3y = 12. The easiest way to plot a line in standard form Ax + By = C is to find the x- and y-intercepts.

$$\begin{aligned} 4x+3y &= 12 \\ 4x+3(0) &= 12 \\ 4x &= 12 \\ x &= 3 \end{aligned} \qquad \begin{aligned} 4x+3y &= 12 \\ 4(0)+3y &= 12 \\ 3y &= 12 \\ y &= 4 \end{aligned}$$

Plot the x- and y-intercepts R(3, 0) and S(0, 4) as shown in Figure 6.6.2.1.5(a). The line through points R and S is the graph of the equation 4x + 3y = 12.

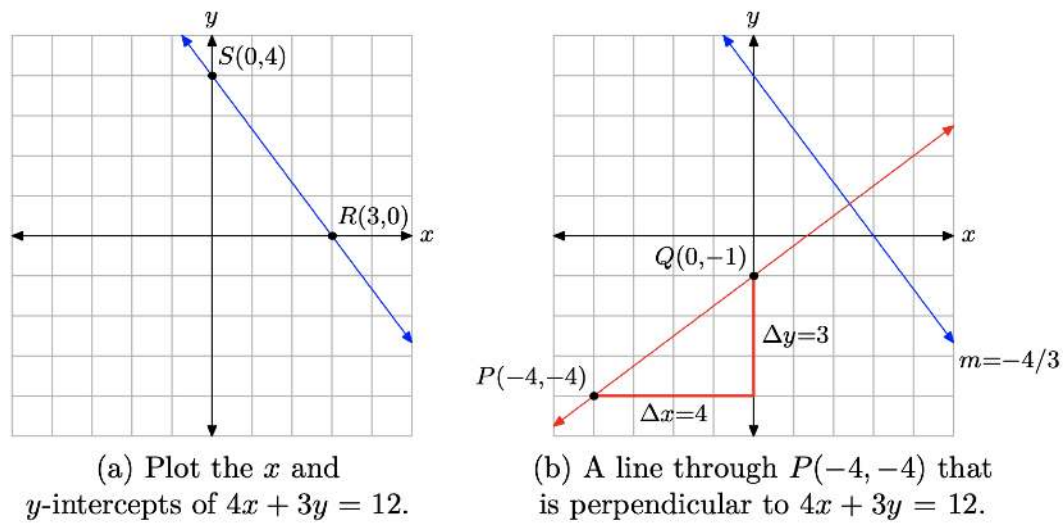


Figure 6.6.2.1.5

Next, determine the slope of the line 4x + 3y = 12 by placing this equation in slope-intercept form (i.e., solve the equation 4x + 3y = 12 for y)

$$\begin{aligned} 4x+3y &= 12 \\ 3y &= -4x+12 \\ y &= -\frac{4}{3}x+4 \end{aligned}$$

If two lines are perpendicular, then their slopes are negative reciprocals of one another. Therefore, the slope of the line that is perpendicular to the line 4x + 3y = 12 (which has slope −4/3) is m = 3/4. Our second line must pass through the point P(−4, −4). To draw this second line, first plot the point P(−4, −4), then move 4 units to the right and 3 units upward to the point Q(0, −1), as shown in Figure 6.6.2.1.5(b). The line through the points P and Q is perpendicular to the line 4x + 3y = 12.

To determine the equation of the line through the points P and Q, we will use the point-slope form of the line, namely

$$y-y_0 = m\,(x-x_0)$$

The slope of the line through points P and Q is m = 3/4. If we use the point P(−4, −4), then $(x_0, y_0) = (-4, -4)$. Set $m = 3/4$, $x_0 = -4$, and $y_0 = -4$ in equation (16), obtaining

$$y-(-4) = \frac{3}{4}(x-(-4))$$

or equivalently,

$$y+4=\frac{3}{4}(x+4)$$

Alternatively, we could use the slope-intercept form of the line. We know that the line through points P and Q in Figure 6.6.2.1.5(b) crosses the y-axis at Q(0, −1). So, with slope m = 3/4 and y-coordinate of the y-intercept b = −1, the slope-intercept form y = mx + b becomes

$$y=\frac{3}{4}x-1$$

On the other hand, if we solve equation (17) for y,

$$\begin{aligned} y+4 &= \frac{3}{4}(x+4) \\ y+4 &= \frac{3}{4}x+3 \\ y &= \frac{3}{4}x-1 \end{aligned}$$

Note that this is identical to the result found using the slope-intercept form above.

It is comforting to note that the two forms (point-slope and slope-intercept) give the same result, but how do we determine the most efficient form to use for a particular problem? Here's a good hint.

Determining the Form of the Line to Use

Here is some sound advice when you are trying to determine whether to use the slope-intercept form or the pointslope form of a line.

- If you are given the slope and the y-intercept, use the slope-intercept form y = mx + b.
- If you are given a point (other than the y-intercept) and the slope, use the point-slope form $y-y_0=m\,(x-x_0)$.

Applications of Linear Functions

In this section we will look at some applications of linear functions. We begin by developing a function relating Fahrenheit and Celsius temperature.

Example 6.6.2.1.5

Water freezes at $32^\circ F$ and 0°C. Water boils at $212^\circ F$ and 100°C. F and C are abbreviations for Fahrenheit and Celsius temperature scales, respectively. Assuming a linear relationship, develop a model relating Fahrenheit and Celsius temperature.

Solution

First, to help keep our focus, we set up a coordinate system on a sheet of graph paper. In Figure 6.6.2.1.6 we've decided to make the Celsius temperature the dependent variable and have assigned the Celsius temperature to the vertical axis. Similarly, we've declared the Fahrenheit temperature the independent variable and assigned it to the horizontal axis.

Interpret the given data:

- Water freezes at $32^\circ F$ and $0^\circ C$. This gives us the point (F, C) = (32, 0), which we plot in Figure 6.6.2.1.6
- Water boils at $212^\circ F$ and $100^\circ C$. This gives us the point (F, C) = (212, 100), which we plot in Figure 6.6.2.1.6

Now we are on familiar ground. We want to find the equation of the line through these two points, which is the same type of problem we tackled in Example 6. First, use the points (32, 0) and (212, 100) to determine the slope of the line.

$$m=\frac{\Delta C}{\Delta F}=\frac{100-0}{212-32}=\frac{100}{180}=\frac{5}{9}$$

We will now use the point-slope form of the line, $y-y_0=m\,(x-x_0)$ with m = 5/9 and $(x_0,y_0)=(32,0)$. Substitute $m=5/9, x_0=32$, and $y_0=0$ in $y-y_0=m\,(x-x_0)$ to obtain

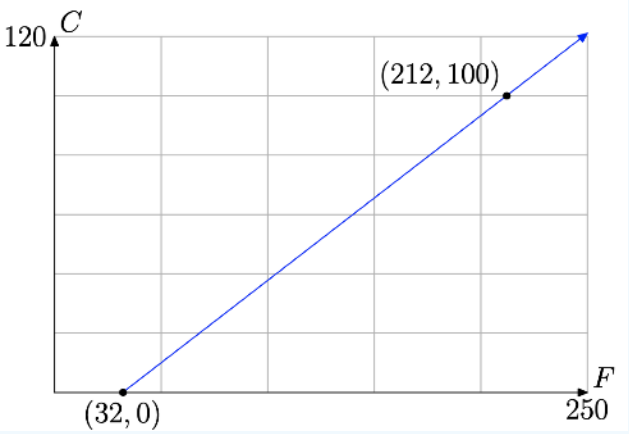


Figure 6.6.2.1.6 . Plotting Celsius temperature versus Fahrenheit temperature

$$y - 0 = \frac{5}{9}(x - 32)$$

However, our dependent axis is labeled C, not y, and our independent axis is labeled F, not x. So, we must replace y and x in equation (20) with C and F, respectively, obtaining

$$C = \frac{5}{9}(F - 32)$$

This result in equation (21) expresses the Celsius temperature as a function of the Fahrenheit temperature. Alternatively, we could also use function notation and write

$$C(F) = \frac{5}{9}(F - 32)$$

Suppose that we know that the Fahrenheit temperature outside is $80^\circ\mathrm{F}$ and we wish to express this using the Celsius scale. To do so, we simply evaluate C(80), as in

$$C(80) = \frac{5}{9}(80 - 32) \approx 26.6$$

Hence, the Celsius temperature is approximately $26.6^\circ\mathrm{C}$.

On the other hand, suppose that we know the Celsius temperature on a metal roof is $80^\circ\mathrm{C}$ and we wish to find the Fahrenheit temperature. To do so, we need to solve

$$C(F) = 80$$

for F, or equivalently,

$$\frac{5}{9}(F - 32) = 80$$

Multiply both sides by 9 to obtain

$$5(F - 32) = 720$$

then divide both sides of the result by 5 to obtain

$$F - 32 = 144$$

Adding 32 to both sides of this last result produces the Fahrenheit temperature $F = 176^\circ\mathrm{F}$. Wow, that's hot!

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6.7: Quadratics

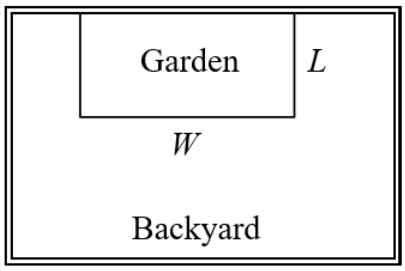
Quadratics are transformations of the function $f(x) = x^2$. Quadratics commonly arise from problems involving area and projectile motion, providing some interesting applications.

Example 6.7.1

A backyard farmer wants to enclose a rectangular space for a new garden. She has purchased 80 feet of wire fencing to enclose three sides, and will put the fourth side against the backyard fence. Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length L.

Solution

In a scenario like this involving geometry, it is often helpful to draw a picture. It might also be helpful to introduce a temporary variable, W, to represent the side of fencing parallel to the fourth side or backyard fence.



Since we know we only have 80 feet of fence available, we know that $L + W + L = 80$, or more simply, $2L + W = 80$. This allows us to represent the width, W, in terms of L: $W = 80 - 2L$

Now we are ready to write an equation for the area the fence encloses. We know the area of a rectangle is length multiplied by width, so $A = LW = L(80 - 2l)$, so

$$A(L) = 80L - 2L^2.$$

This formula represents the area of the fence in terms of the variable length L.

Example 6.7.2

A local newspaper currently has 84,000 subscribers, at a quarterly charge of $30. Market research has suggested that if they raised the price to $32, they would lose 5,000 subscribers. Assuming that subscriptions are linearly related to the price, create an equation to model their revenue as a function of the quarterly charge.

Solution

Revenue is the amount of money a company brings in. In this case, the revenue can be found by multiplying the charge per subscription times the number of subscribers. We can introduce variables, C for charge per subscription and S for the number subscribers, giving us the equation:

Revenue = CS

Since the number of subscribers changes with the price, we need to find a relationship between the variables.

We know that currently $S = 84,000$ and $C = 30$, and that if they raise the price to $32 they would lose 5,000 subscribers, giving a second pair of values, $S = 79,000$ and $C = 32$. From this we can find a linear equation relating the two quantities. Treating C as the input and S as the output, the equation will have form $S = mC + b$. The slope will be

$$m = \frac{79,000 - 84,000}{32 - 30} = \frac{-5,000}{2} = -2,500$$

This tells us the paper will lose 2,500 subscribers for each dollar they raise the price. We can then solve for the vertical intercept.

$$S = -2500C + b$$

Plug in the point $S = 84,000$ and $C = 30$

$$84000 = -2500(30) + b$$

Solve for b

$$b = 159,000$$

This gives us the linear equation $S = -2,500C + 159,000$ relating cost and subscribers. Note this is a demand equation where C is the price and S is the quantity demanded. We now return to our revenue equation.

Revenue = CS

Substituting the equation for S from above:
Revenue = $C\left(-2,500C + 159,000\right)$

Expanding
Revenue = $-2,500C^2 + 159,000C$

We now have a quadratic equation for revenue as a function of the subscription charge. Later in the course we will use equations like this to determine the price to charge to maximize revenue.

Forms of Quadratic Functions

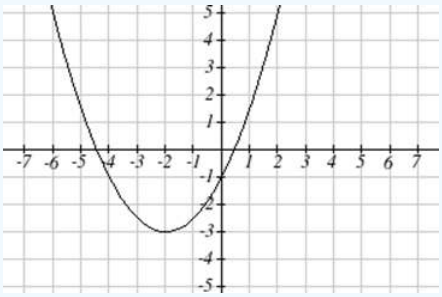
The **standard form** of a quadratic function is $f(x) = ax^2 + bx + c$.

The **transformation form** of a quadratic function is $f(x) = a(x - h)^2 + k$.

The **vertex** of the quadratic function is located at (h, k), where h and k are the numbers in the transformation form of the function. Because the vertex appears in the transformation form, it is often called the **vertex form**.

Example 6.7.3

Write an equation for the quadratic graphed below as a transformation of $f(x) = x^2$.



Solution

We can see the graph is the basic quadratic shifted to the left 2 and down 3, putting the vertex at $(-2, -3)$, giving a formula in the form $g(x) = a(x + 2)^2 - 3$. By plugging in a point that falls on the grid, such as $(0, -1)$, we can solve for the stretch factor:

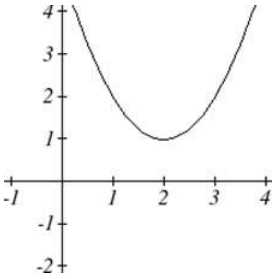
$$\begin{aligned} -1 &= a(0 + 2)^2 - 3 \\ 2 &= 4a \\ a &= \frac{1}{2} \end{aligned}$$

The equation for this formula is

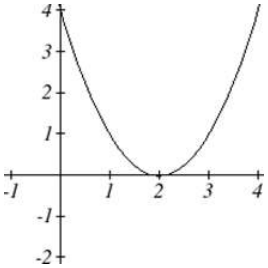
$$g(x) = \frac{1}{2}(x + 2)^2 - 3$$

Short run Behavior: Intercepts

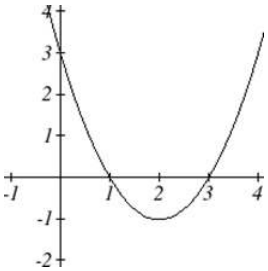
As with any function, we can find the vertical intercepts of a quadratic by evaluating the function at an input of zero, and we can find the horizontal intercepts by solving for when the output will be zero. Notice that depending upon the location of the graph, we might have zero, one, or two horizontal intercepts.



Zero horizontal intercepts



One horizontal intercept



Two horizontal intercepts

Notice that in the standard form of a quadratic, the constant term c reveals the vertical intercept of the graph, since $f(0) = a(0)^2 + b(0) + c = c$.

Example 6.7.4

Find the vertical and horizontal intercepts of the quadratic $f(x) = 3x^2 + 5x - 2$.

Solution

We can find the vertical intercept by evaluating the function at an input of zero:

$$f(0) = 3(0)^2 + 5(0) - 2 = -2$$

So the vertical intercept is at (0,-2)

For the horizontal intercepts, we solve for when the output will be zero:

$$0 = 3x^2 + 5x - 2.$$

In this case, the quadratic can be factored easily, providing the simplest method for solution.:

$$0 = (3x - 1)(x + 2),$$

so either

$$\begin{aligned} 0 &= 3x - 1 \\ x &= \frac{1}{3} \end{aligned}$$

or

$$\begin{aligned} 0 &= x + 2 \\ x &= -2 \end{aligned}$$

So the Horizontal intercepts are at $\left(\frac{1}{3}, 0\right)$ and $(-2, 0)$.

When a quadratic is not factorable or is hard to factor, we can turn to the quadratic formula.

Quadratic Formula

For a quadratic function given in standard form $f(x) = ax^2 + bx + c$, the **quadratic formula** gives the horizontal intercepts of the graph of this function:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 6.7.5

A ball is thrown upwards from the top of a 40 foot high building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation

$$H(t) = -16t^2 + 80t + 40.$$

When does the ball hit the ground?

Solution

To find when the ball hits the ground, we need to determine when the height is zero, i.e., when $H(t) = 0$. While we could do this using the transformation form of the quadratic, we can also use the quadratic formula:

$$t = \frac{-80 \pm \sqrt{80^2 - 4(-16)(40)}}{2(-16)} = \frac{-80 \pm \sqrt{8960}}{-32}$$

Since the square root does not simplify nicely, we can use a calculator to approximate the values of the solutions:

$$t = \frac{-80 - \sqrt{8960}}{-32} \approx 5.458 \quad \text{or} \quad t = \frac{-80 + \sqrt{8960}}{-32} \approx -0.458$$

The second answer is outside the reasonable domain of our model, so we conclude the ball will hit the ground after about 5.458 seconds.

Example 6.7.6

The supply for a certain product can be modeled by $p = 3q^2$ and the demand can be modeled by $p = 1620 - 2q^2$, where p is the price in dollars, and q is the quantity in thousands of items. Find the equilibrium price and quantity.

Solution

Recall that the equilibrium price and quantity is found by finding where the supply and demand curve intersect. We can find that by setting the equations equal:

$$3q^2 = 1620 - 2q^2$$

Add $2q^2$ to both sides: $5q^2 = 1620$

Divide by 5 on both sides: $q^2 = 324$

Take the square root of both sides:

$$q = \pm\sqrt{324} = \pm 18$$

Since it doesn't make sense to talk about negative quantities, the equilibrium quantity is $q = 18$. To find the equilibrium price, we evaluate either function at the equilibrium quantity.

$$p = 3(18)^2 = 972$$

The equilibrium is 18 thousand items, at a price of $972.

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6.8: Polynomials and Rational Functions

Polynomial Functions

Terminology of Polynomial Functions

A **polynomial** is function that can be written as $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$

Each of the a_i constants are called **coefficients** and can be positive, negative, or zero, and be whole numbers, decimals, or fractions.

A **term** of the polynomial is any one piece of the sum, that is any a_ix^i. Each individual term is a transformed power function.

The **degree** of the polynomial is the highest power of the variable that occurs in the polynomial.

The **leading term** is the term containing the highest power of the variable: the term with the highest degree.

The **leading coefficient** is the coefficient of the leading term.

Because of the definition of the "leading" term we often rearrange polynomials so that the powers are descending.

$$f(x) = a_nx^n + \ldots + a_2x^2 + a_1x + a_0$$

Example 6.8.1

Identify the degree, leading term, and leading coefficient of the polynomial

$$f(x) = 3 + 2x^2 - 4x^3$$

Solution

For the function $f(x)$, the degree is 3, the highest power on x. The leading term is the term containing that power, $-4x^3$. The leading coefficient is the coefficient of that term, -4.

Short run Behavior: Intercepts

As with any function, the vertical intercept can be found by evaluating the function at an input of zero. Since this is evaluation, it is relatively easy to do it for a polynomial of any degree. To find horizontal intercepts, we need to solve for when the output will be zero. For general polynomials, this can be a challenging prospect. Consequently, we will limit ourselves to three cases:

1. The polynomial can be factored using known methods: greatest common factor and trinomial factoring.
2. The polynomial is given in factored form.
3. Technology is used to determine the intercepts.

Example 6.8.2

Find the horizontal intercepts of $f(x) = x^6 - 3x^4 + 2x^2$.

Solution

We can attempt to factor this polynomial to find solutions for $f(x) = 0$.

$$x^6 - 3x^4 + 2x^2 = 0$$

$$\begin{aligned} x^2(x^4 - 3x^2 + 2) &= 0 && \text{Factoring out the greatest common factor} \\ x^2(x^2 - 1)(x^2 - 2) &= 0 && \text{Factoring the inside as a quadratic in } x^2 \end{aligned}$$

Then break apart to find solutions:

$$\begin{aligned} x^2 &= 0 \\ x &= 0 \end{aligned}$$

or

$$(x^2-1)=0$$
$$x^2=1$$
$$x=\pm 1$$

or

$$(x^2-2)=0$$
$$x^2=2$$
$$x=\pm\sqrt{2}$$

This gives us 5 horizontal intercepts.

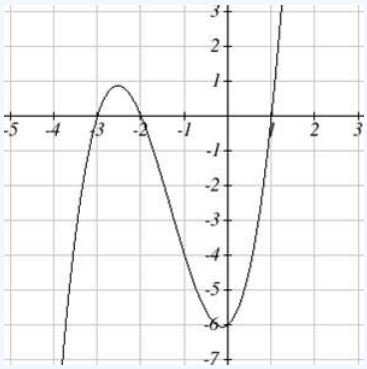
Example 6.8.3

Find the horizontal intercepts of $h(t)=t^3+4t^2+t-6$

Solution

Since this polynomial is not in factored form, has no common factors, and does not appear to be factorable using techniques we know, we can turn to technology to find the intercepts.

Graphing this function, it appears there are horizontal intercepts at t = -3, -2, and 1.



We could check these are correct by plugging in these values for t and verifying that $h(-3)=h(-2)=h(1)=0$.

Solving Polynomial Inequalities

One application of our ability to find intercepts and sketch a graph of polynomials is the ability to solve polynomial inequalities. It is a very common question to ask when a function will be positive and negative, and one we will use later in this course.

Example 6.8.4

Solve $(x+3)(x+1)^2(x-4)>0$

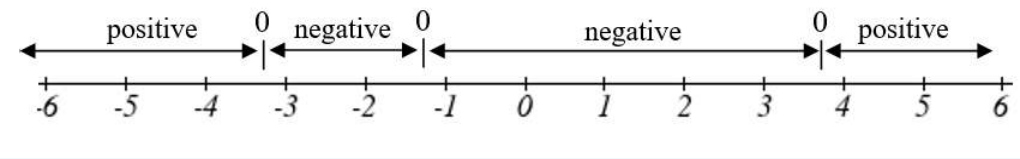
Solution

As with all inequalities, we start by solving the equality $(x+3)(x+1)^2(x-4)=0$, which has solutions at x = -3, -1, and 4. We know the function can only change from positive to negative at these values, so these divide the inputs into 4 intervals.

We could choose a test value in each interval and evaluate the function $f(x)=(x+3)(x+1)^2(x-4)$ at each test value to determine if the function is positive or negative in that interval

Interval	Test x in interval	f(test value)	>0 or <0?
$x<-3$	-4	72	> 0
$-3<x<-1$	-2	-6	< 0
$-1<x<4$	0	-12	< 0
$x>4$	5	288	> 0

On a number line this would look like:



From our test values, we can determine this function is positive when $x < -3$ or $x > 4$, or in interval notation, $(-\infty, -3) \cup (4, \infty)$

Rational Functions

Rational functions are the ratios, or fractions, of polynomials. They can arise from both simple and complex situations.

Example 6.8.5

You plan to drive 100 miles. Find a formula for the time the trip will take as a function of the speed you drive.

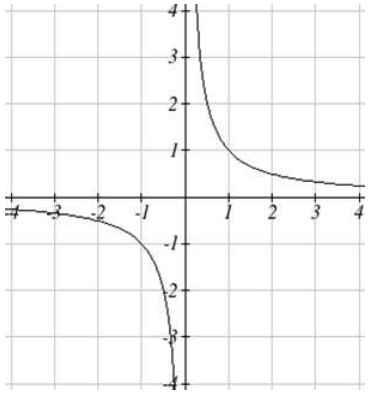
Solution

You may recall that multiplying speed by time will give you distance. If we let t represent the drive time in hours, and v represent the velocity (speed or rate) at which we drive, then $vt =$ distance. Since our distance is fixed at 100 miles, $vt = 100$. Solving this relationship for the time gives us the function we desired:

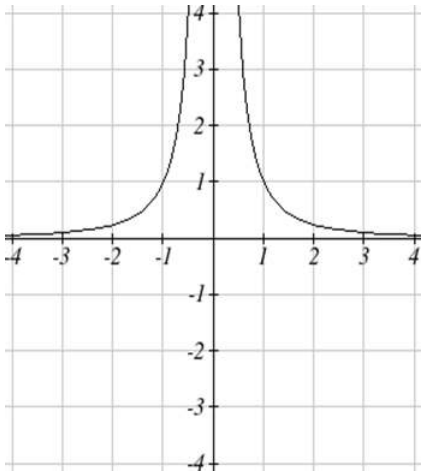
$$t(v) = \frac{100}{v}$$

Notice that this is a transformation of the reciprocal toolkit function, $f(x) = \frac{1}{x}$. Several natural phenomena, such as gravitational force and volume of sound, behave in a manner **inversely proportional to the square** of another quantity. For example, the volume, V, of a sound heard at a distance d from the source would be related by $V = \frac{k}{d^2}$ for some constant value k. These functions are transformations of the reciprocal squared toolkit function $f(x) = \frac{1}{x^2}$.

We have seen the graphs of the basic reciprocal function and the squared reciprocal function from our review of toolkit functions. These graphs have several important features.



$f(x) = \frac{1}{x}$



$f(x) = \frac{1}{x^2}$

Let's begin by looking at the reciprocal function, $f(x) = \frac{1}{x}$. As you well know, dividing by zero is not allowed and therefore zero is not in the domain, and so the function is undefined at an input of zero.

Short run behavior:

As the input values approach zero from the left side (taking on very small, negative values), the function values become very large in the negative direction (in other words, they approach negative infinity).

We write: as $x \to 0^-$, $f(x) \to -\infty$.

As we approach zero from the right side (small, positive input values), the function values become very large in the positive direction (approaching infinity).

We write: as $x \to 0^+$, $f(x) \to \infty$.

This behavior creates a **vertical asymptote**. An asymptote is a line that the graph approaches. In this case the graph is approaching the vertical line $x = 0$ as the input becomes close to zero.

Long run behavior:

As the values of x approach infinity, the function values approach 0.

As the values of x approach negative infinity, the function values approach 0.

Symbolically: as $x \to \pm\infty$, $f(x) \to 0$

Based on this long run behavior and the graph we can see that the function approaches 0 but never actually reaches 0, it just "levels off" as the inputs become large. This behavior creates a **horizontal asymptote**. In this case the graph is approaching the horizontal line $f(x) = 0$ as the input becomes very large in the negative and positive directions.

Vertical and Horizontal Asymptotes

A **vertical asymptote** of a graph is a vertical line $x = a$ where the graph tends towards positive or negative infinity as the inputs approach a. As $x \to a$, $f(x) \to \pm\infty$.

A **horizontal asymptote** of a graph is a horizontal line $y = b$ where the graph approaches the line as the inputs get large. As $x \to \pm\infty$, $f(x) \to b$.

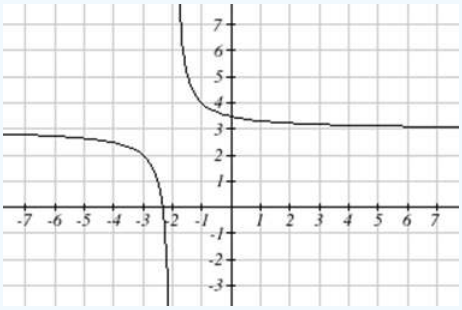
Example 6.8.6

Sketch a graph of the reciprocal function shifted two units to the left and up three units. Identify the horizontal and vertical asymptotes of the graph, if any.

Solution

Transforming the graph left 2 and up 3 would result in the function $f(x)=\frac{1}{x+2}+3$, or equivalently, by giving the terms a common denominator, $f(x)=\frac{3x+7}{x+2}$

Shifting the toolkit function would give us this graph.



Notice that this equation is undefined at $x=-2$, and the graph also is showing a vertical asymptote at $x=-2$.

As $x\to -2^-$, $f(x)\to -\infty$, and as $x\to -2^+$, $f(x)\to \infty$

As the inputs grow large, the graph appears to be leveling off at output values of 3, indicating a horizontal asymptote at $y=3$.

As $x\to \pm\infty$, $f(x)\to 3$.

Notice that horizontal and vertical asymptotes get shifted left 2 and up 3 along with the function.

A general rational function is the ratio of any two polynomials.

Rational Function

A **rational function** is a function that can be written as the ratio of two polynomials, $P(x)$ and $Q(x)$.

$$f(x)=\frac{P(x)}{Q(x)}=\frac{a_0+a_1x+a_2x^2+\ldots+a_px^p}{b_0+b_1x+b_2x^2+\ldots+b_qx^q}$$

Rational functions can arise from real situations.

Example 6.8.7

A large mixing tank currently contains 100 gallons of water, into which 5 pounds of sugar have been mixed. A tap will open pouring 10 gallons per minute of water into the tank at the same time sugar is poured into the tank at a rate of 1 pound per minute. Find the concentration (pounds per gallon) of sugar in the tank after t minutes.

Solution

Notice that the amount of water in the tank is changing linearly, as is the amount of sugar in the tank. We can write an equation independently for each:

$$\begin{aligned} water &= 100+10t \\ sugar &= 5+1t \end{aligned}$$

The concentration, C, will be the ratio of pounds of sugar to gallons of water

$$C(t)=\frac{5+t}{100+10t}$$

Vertical Asymptotes of Rational Functions

The **vertical asymptotes** of a rational function will occur where the denominator of the function is equal to zero and the numerator is not zero.

Horizontal Asymptote of Rational Functions

The **horizontal asymptote** of a rational function can be determined by looking at the degrees of the numerator and denominator.

Degree of denominator > degree of numerator: Horizontal asymptote at $y = 0$

Degree of denominator < degree of numerator: No horizontal asymptote

Degree of denominator = degree of numerator: Horizontal asymptote at ratio of leading coefficients.

Example 6.8.8

In the sugar concentration problem from earlier, we created the equation $C(t) = \frac{5+t}{100+10t}$.

Find the horizontal asymptote and interpret it in context of the scenario.

Solution

Both the numerator and denominator are linear (degree 1), so since the degrees are equal, there will be a horizontal asymptote at the ratio of the leading coefficients. In the numerator, the leading term is t, with coefficient 1. In the denominator, the leading term is $10t$, with coefficient 10. The horizontal asymptote will be at the ratio of these values: As $t \to \infty$, $C(t) \to \frac{1}{10}$. This function will have a horizontal asymptote at $y = \frac{1}{10}$.

This tells us that as the input gets large, the output values will approach 1/10. In context, this means that as more time goes by, the concentration of sugar in the tank will approach one tenth of a pound of sugar per gallon of water or 1/10 pounds per gallon.

Example 6.8.9

Find the horizontal and vertical asymptotes of the function

$$f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$$

Solution

First, note this function has no inputs that make both the numerator and denominator zero, so there are no potential holes. The function will have vertical asymptotes when the denominator is zero, causing the function to be undefined. The denominator will be zero at x = 1, -2, and 5, indicating vertical asymptotes at these values.

The numerator has degree 2, while the denominator has degree 3. Since the degree of the denominator is greater than the degree of the numerator, the denominator will grow faster than the numerator, causing the outputs to tend towards zero as the inputs get large, and so as $x \to \pm\infty$, $f(x) \to 0$. This function will have a horizontal asymptote at $y = 0$.

As with all functions, a rational function will have a vertical intercept when the input is zero, if the function is defined at zero. It is possible for a rational function to not have a vertical intercept if the function is undefined at zero.

Likewise, a rational function will have horizontal intercepts at the inputs that cause the output to be zero (unless that input corresponds to a hole). It is possible there are no horizontal intercepts. Since a fraction is only equal to zero when the numerator is zero, horizontal intercepts will occur when the numerator of the rational function is equal to zero.

Example 6.8.10

Find the intercepts of $f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$

Solution

We can find the vertical intercept by evaluating the function at zero

$$f(0)=\frac{(0-2)(0+3)}{(0-1)(0+2)(0-5)}=\frac{-6}{10}=-\frac{3}{5}$$

The horizontal intercepts will occur when the function is equal to zero:

$$0=\frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$$

This is zero when the numerator is zero.

$$\begin{aligned} 0 &= (x-2)(x+3) \\ x &= 2,-3 \end{aligned}$$

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6.E: Review (Exercises)

1.1 Exercises

Exercise $6.E.1$

The amount of garbage, G, produced by a city with population p is given by $G = f(p)$. G is measured in tons per week, and p is measured in thousands of people.

1. The town of Tola has a population of 40,000 and produces 13 tons of garbage each week. Express this information in terms of the function f.
2. Explain the meaning of the statement $f(5) = 2$.

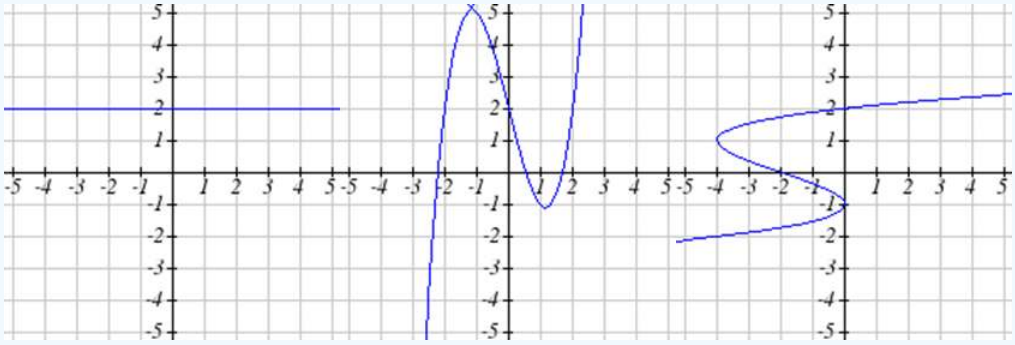
Exercise $6.E.2$

The number of cubic yards of dirt, D, needed to cover a garden with area a square feet is given by $D = g(a)$.

1. A garden with area 5000 ft^2 requires 50 cubic yards of dirt. Express this information in terms of the function g.
2. Explain the meaning of the statement $g(100) = 1$.

Exercise $6.E.3$

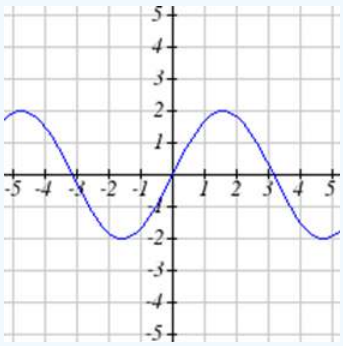
Select all of the following graphs which represent y as a function of x.



a.

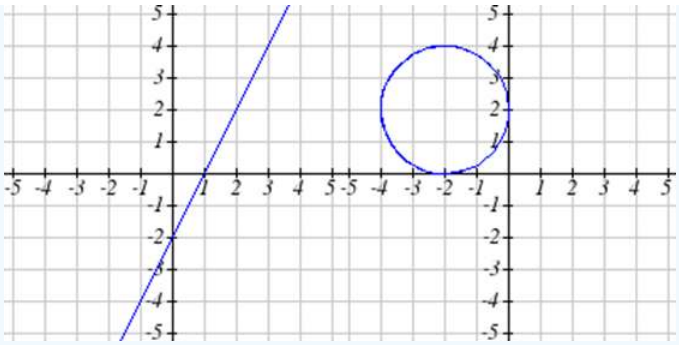
b.

c.



d.

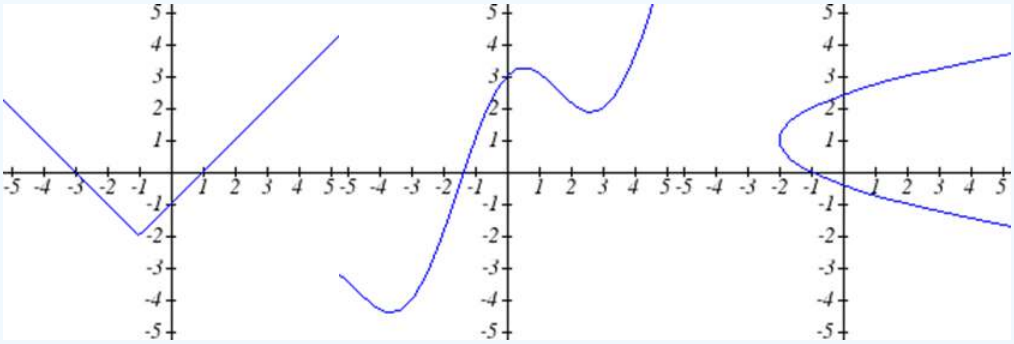
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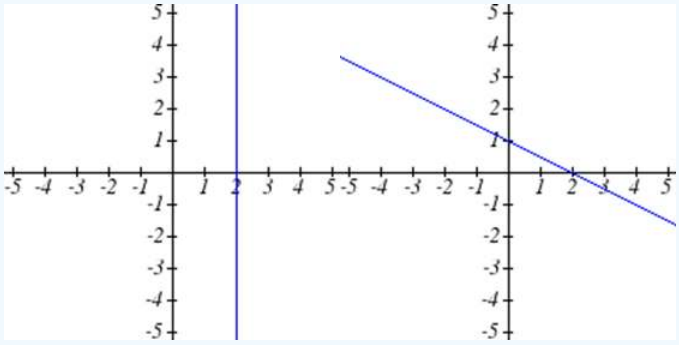
e. f.

Exercise $6.E.4$

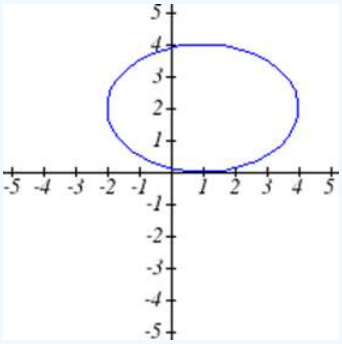
Select all of the following graphs which represent y as a function of x.



a. b. c.



d. e.



f.

Exercise 6.E.5

Select all of the following tables which represent y as a function of x.

a.

x	5	10	15
y	3	8	14

b.

x	5	10	15
y	3	8	8

c.

x	5	10	10
y	3	8	14

Exercise 6.E.6

Select all of the following tables which represent y as a function of x.

a.

x	2	6	13
y	3	10	10

b.

x	2	6	6
y	3	10	14

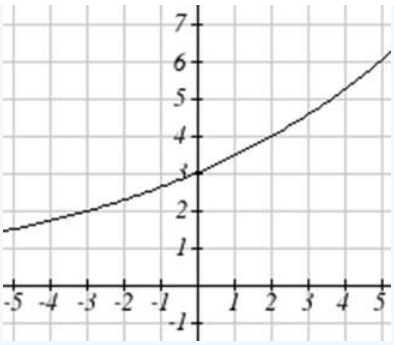
c.

x	2	6	13
y	3	10	14

Exercise 6.E.7

Given the function $g(x)$ graphed here,

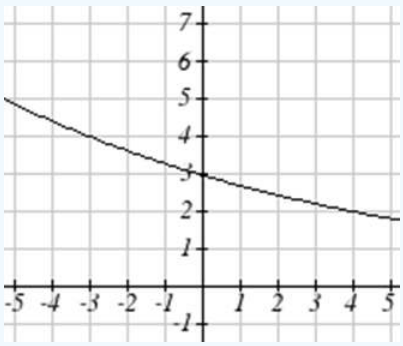
1. Evaluate $g(2)$
2. Solve $g(x) = 2$



Exercise 6.E.8

Given the function $f(x)$ graphed here.

1. Evaluate $f(4)$
2. Solve $f(x) = 4$



Exercise 6.E.9

Based on the table below,

1. Evaluate $f(3)$
2. Solve $f(x) = 1$

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	74	28	1	53	56	3	36	45	14	47

Exercise 6.E.10

Based on the table below,

1. Evaluate $f(8)$
2. Solve $f(x) = 7$

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	62	8	7	38	86	73	70	39	75	34

Exercises 6.E.11 – 6.E.18

For each of the following functions, evaluate: $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, and $f(2)$

11. $f(x) = 4 - 2x$	12. $f(x) = 8 - 3x$

13. $f(x) = 8x^2 - 7x + 3$	14. $f(x) = 6x^2 - 7x + 4$
15. $f(x) = 3 + \sqrt{x+3}$	16. $f(x) = 4 - \sqrt[3]{x-2}$
17. $f(x) = \frac{x-3}{x+1}$	18. $f(x) = \frac{x-2}{x+2}$

Exercise 6.E.19

Let $f(t) = 3t + 5$

1. Evaluate $f(0)$
2. Solve $f(t) = 0$

Exercise 6.E.20

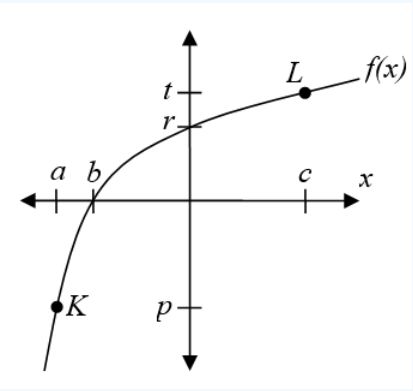
Let $g(p) = 6 - 2p$

1. Evaluate $g(0)$
2. Solve $g(p) = 0$

Exercise 6.E.21

Using the graph shown,

1. Evaluate $f(c)$
2. Solve $f(x) = p$
3. What are the coordinates of points L and K?

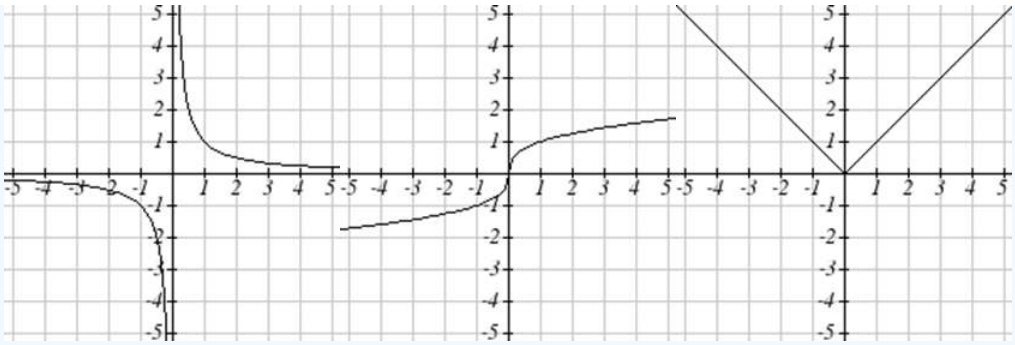


Exercise 6.E.22

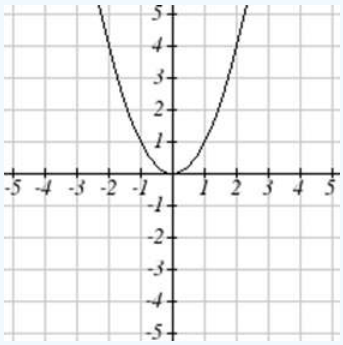
Match each graph with its equation.

a. $y = x$	b. $y = x^3$	$y = \sqrt[3]{x}$	d. $y = \frac{1}{x}$
e. $y = x^2$	f. $y = \sqrt{x}$	g. $y = \lvert x \rvert$	h. $y = \frac{1}{x^2}$

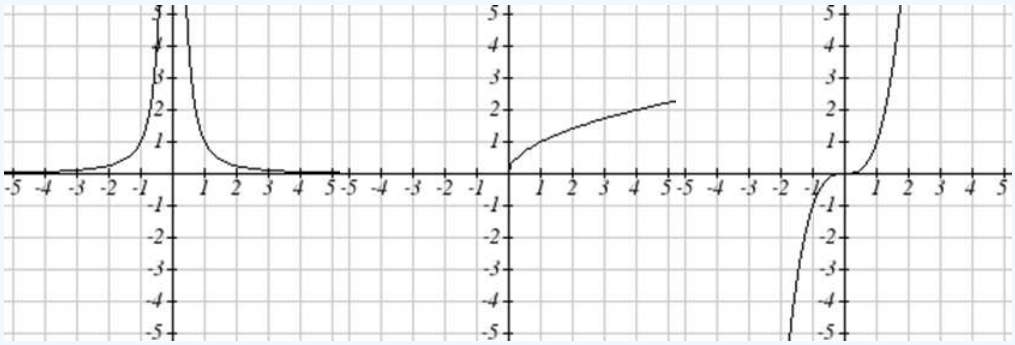
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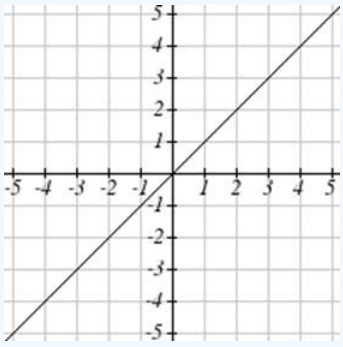
i. ii. iii.



iv.



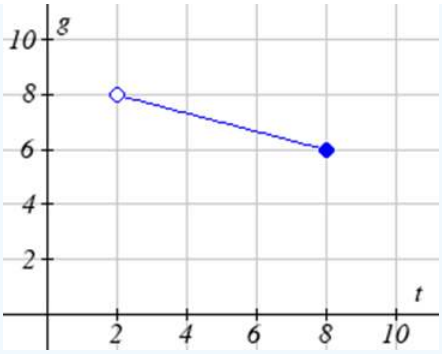
v. vi. vii.



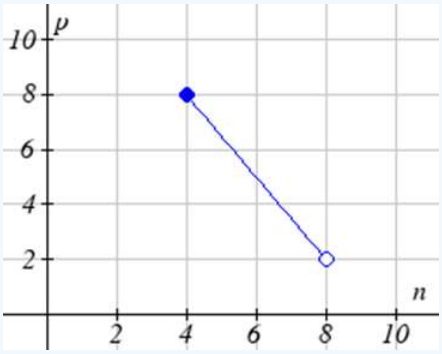
viii.

For Exercises $6.E.23 - 6.E.24$, write the domain and range of each graph as an inequality.

? Exercise $6.E.23$



Exercise 6.E.24



Exercise 6.E.25 – 6.E.30

Find the domain of each function.

25. $f(x) = 3\sqrt{x-2}$	26. $f(x) = 5\sqrt{x+3}$
27. $f(x) = \frac{9}{x-6}$	28. $f(x) = \frac{6}{x-8}$
29. $f(x) = \frac{3x+1}{4x+2}$	30. $f(x) = \frac{5x+3}{4x-1}$

1.2 Exercises

Exercise 6.E.1 – 6.E.4

1. $f(x) = 4x + 8, g(x) = 7 - x^2$	2. $f(x) = 5x + 7, g(x) = 4 - 2x^2$
3. $f(x) = \sqrt{x+4}, g(x) = 12 - x^3$	4. $f(x) = \frac{1}{x+2}, g(x) = 4x + 3$

Exercise 6.E.5 – 6.E.12

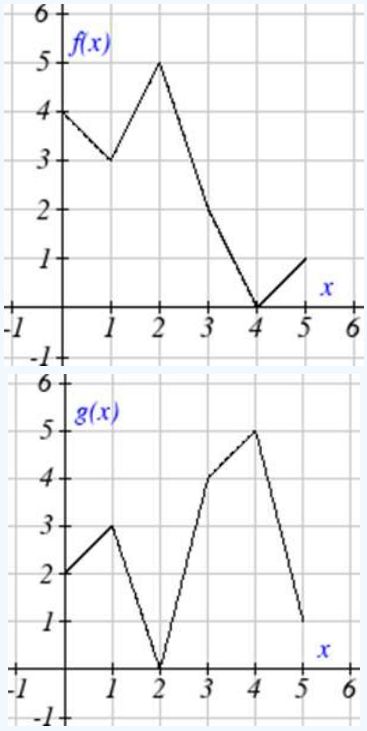
x	$f(x)$	$g(x)$
0	7	9
1	6	5
2	5	6
3	8	2
4	4	1

5	0	8
6	2	7
7	1	3
8	9	4
9	3	0

Use the table of values to evaluate each expression.

5. $f(g(8))$
6. $f(g(5))$
7. $g(f(5))$
8. $g(f(3))$
9. $f(f(4))$
10. $f(f(1))$
11. $g(g(2))$
12. $g(g(6))$

Exercise $6.E.13 - 6.E.20$



Use the graphs to evaluate the expressions below.

13. $f(g(3))$
14. $f(g(1))$
15. $g(f(1))$
16. $g(f(0))$

17. $f(f(5))$
18. $f(f(4))$
19. $g(g(2))$
20. $g(g(0))$

Exercise $6.E.21 - 6.E.26$

For each pair of functions, find $f(g(x))$ and $g(f(x))$. Simplify your answers.

21. $f(x)=\frac{1}{x-6}, g(x)=\frac{7}{x}+6$	22. $f(x)=\frac{1}{x-4}, g(x)=\frac{2}{x}+4$
23. $f(x)=x^2+1, g(x)=\sqrt{x+2}$	24. $f(x)=\sqrt{x}+2, g(x)=x^2+3$
25. $f(x)=\lvert x\rvert, g(x)=5x+1$	26. $f(x)=\sqrt[3]{x}, g(x)=\frac{x+1}{x^3}$

Exercise $6.E.27$

If $f(x)=x^4+6$, $g(x)=x-6$, and $h(x)=\sqrt{x}$, find $f(g(h(x)))$

Exercise $6.E.28$

If $f(x)=x^2+1$, $g(x)=\frac{1}{x}$, and $h(x)=x+3$, find $f(g(h(x)))$

Exercise $6.E.29$

The function $D(p)$ gives the number of items that will be demanded when the price is p. The production cost, $C(x)$ is the cost of producing x items. To determine the cost of production when the price is \$6, you would do which of the following:

1. Evaluate $D(C(6))$
2. Solve $C(D(6))$
3. Evaluate $D(C(x))=6$
4. Solve $C(D(p))=6$

Exercise $6.E.30$

The function $A(d)$ gives the pain level on a scale of 0-10 experienced by a patient with d milligrams of a pain reduction drug in their system. The milligrams of drug in the patient's system after t minutes is modeled by $m(t)$. To determine when the patient will be at a pain level of 4, you would need to:

1. Evaluate $A(m(4))$
2. Solve $m(A(4))$
3. Evaluate $A(m(t))=4$
4. Solve $m(A(d))=4$

Exercise $6.E.31 - 6.E.36$

Find functions $f(x)$ and $g(x)$ so the given function can be expressed as $h(x)=f(g(x))$.

31. $h(x)=(x+2)^2$	32. $h(x)=(x-5)^3$
33. $h(x)=\frac{3}{x-5}$	34. $h(x)=\frac{4}{(x+2)^2}$
35. $h(x)=3+\sqrt{x-2}$	36. $h(x)=4+\sqrt[3]{x}$

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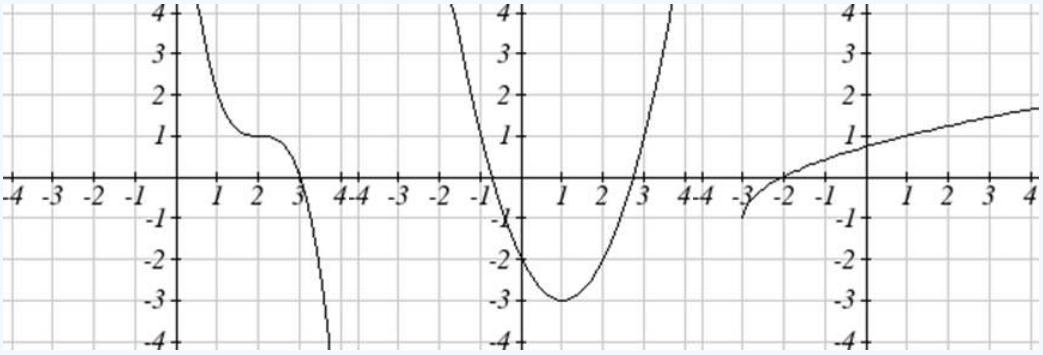
Exercise $6.E.37 - 6.E.44$

Sketch a graph of each function as a transformation of a toolkit function.

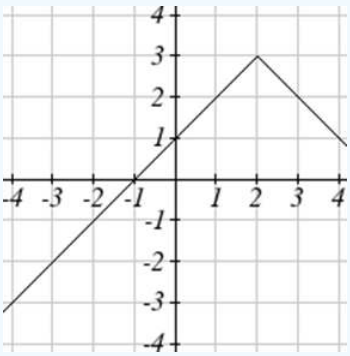
37. $f(t) = (t+1)^2 - 3$	38. $h(x) = \lvert x-1 \rvert + 4$
39. $k(x) = (x-2)^3 - 1$	40. $m(t) = 3 + \sqrt{t+2}$
41. $f(x) = 4(x+1)^2 - 5$	42. $g(x) = 5(x+3)^2 - 2$
43. $h(x) = -2\lvert x-4 \rvert + 3$	44. $k(x) = -3\sqrt{x} - 1$

Exercise $6.E.45 - 6.E.50$

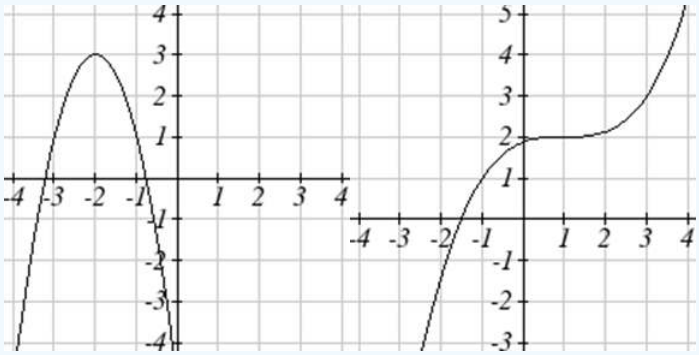
Write an equation for each function graphed below.



45. 46. 47.



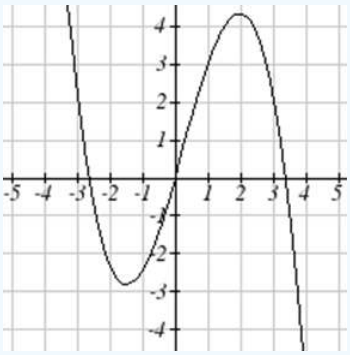
48.



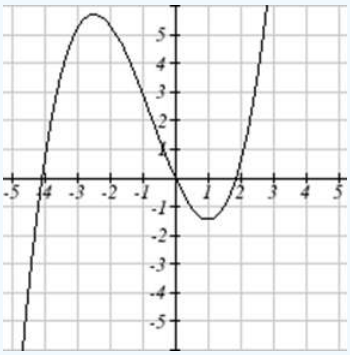
49. 50.

Exercise $6.E.51 - 6.E.52$

For each function graphed, estimate the intervals on which the function is increasing and decreasing.



51.



52.

1.3 Exercises

Exercise $6.E.1$

A town's population has been growing linearly. In 2003, the population was 45,000, and the population has been growing by 1700 people each year. Write an equation, $P(t)$, for the population t years after 2003.

Exercise $6.E.2$

A town's population has been growing linearly. In 2005, the population was 69,000, and the population has been growing by 2500 people each year. Write an equation, $P(t)$, for the population t years after 2005.

Exercise $6.E.3$

Timmy goes to the fair with $40. Each ride costs $2. How much money will he have left after riding n rides?

Exercise $6.E.4$

At noon, a barista notices she has $20 in her tip jar. If she makes an average of $0.50 from each customer, how much will she have in her tip jar if she serves n more customers during her shift?

Exercise $6.E.5$

A phone company charges for service according to the formula: $C(n) = 24 + 0.1n$, where n is the number of minutes talked, and $C(n)$ is the monthly charge, in dollars.
Find and interpret the rate of change and initial value.

Exercise $6.E.6$

A phone company charges for service according to the formula: $C(n) = 26 + 0.04n$, where n is the number of minutes talked, and $C(n)$ is the monthly charge, in dollars.
Find and interpret the rate of change and initial value.

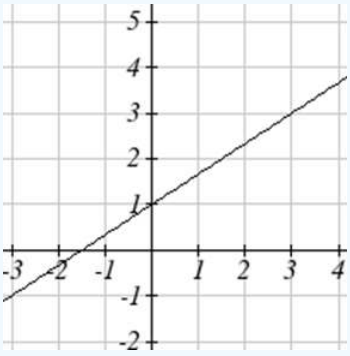
Exercise $6.E.7 - 6.E.14$

Given each set of information, find a linear equation satisfying the conditions, if possible.

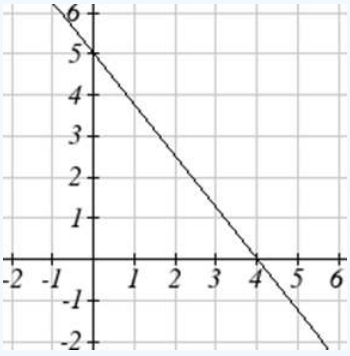
7. $f(-5) = -4$, and $f(5) = 2$	8. $f(-1) = 4$, and $f(5) = 1$
9. Passes through (2,4) and (4,10)	10. Passes through (1, 5) and (4, 11)
11. Passes through (-1,4) and (5, 2)	12. Passes through (-2, 8) and (4, 6)
13. x intercept at (-2, 0) and y intercept at (0, -3)	14. x intercept at (-5, 0) and y intercept at (0, 4)

Exercise $6.E.15 - 6.E.16$

Find an equation for the function graphed.



15.



16.

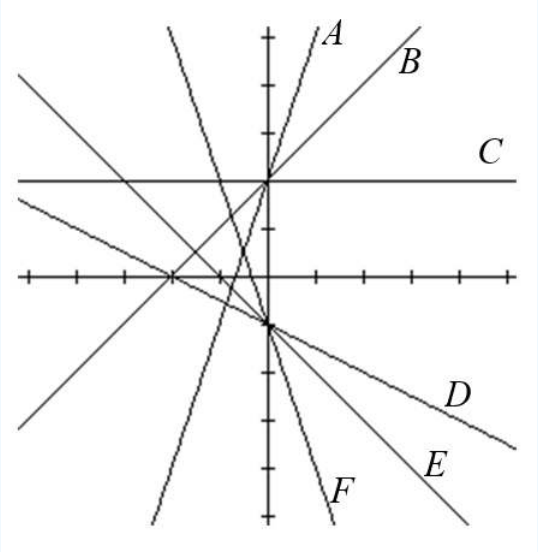
Exercise $6.E.17$

A clothing business finds there is a linear relationship between the number of shirts, n, it can sell and the price, p, it can charge per shirt. In particular, historical data shows that 1000 shirts can be sold at a price of \$30, while 3000 shirts can be sold at a price of \$22. Find a linear equation in the form $p = mn + b$ that gives the price p they can charge for n shirts.

Exercise $6.E.18$

A farmer finds there is a linear relationship between the number of bean stalks, n, she plants and the yield, y, each plant produces. When she plants 30 stalks, each plant yields 30 oz of beans. When she plants 34 stalks, each plant produces 28 oz of beans. Find a linear relationships in the form $y = mn + b$ that gives the yield when n stalks are planted.

Exercise $6.E.19 - 6.E.24$



Match each linear equation with its graph

19. $f(x) = -x - 1$
20. $f(x) = -2x - 1$
21. $f(x) = -\frac{1}{2}x - 1$
22. $f(x) = 2$
23. $f(x) = 2 + x$
24. $f(x) = 3x + 2$

Exercise $6.E.25 - 6.E.28$

Sketch the graph of each equation

25. $f(x) = -2x - 1$	26. $g(x) = -3x + 2$
27. $h(x) = \frac{1}{3}x + 2$	28. $k(x) = \frac{2}{3}x - 3$

Exercise $6.E.29$

Find the point at which the line $f(x) = -2x - 1$ intersects the line $g(x) = -x$

Exercise $6.E.30$

Find the point at which the line $f(x) = 2x + 5$ intersects the line $g(x) = -3x - 5$

Exercise 6.E.31

A car rental company offers two plans for renting a car.

Plan A: 30 dollars per day and 18 cents per mile

Plan B: 50 dollars per day with free unlimited mileage

How many miles would you need to drive for plan B to save you money?

Exercise 6.E.32

A cell phone company offers two data options for its prepaid phones

Pay per use: \$0.002 per Kilobyte (KB) used

Data Package: \$5 for 5 Megabytes (5120 Kilobytes) + \$0.002 per addition KB

Assuming you will use less than 5 Megabytes, under what circumstances will the data package save you money?

Exercise 6.E.33

The Federal Helium Reserve held about 16 billion cubic feet of helium in 2010, and is being depleted by about 2.1 billion cubic feet each year.

1. Give a linear equation for the remaining federal helium reserves, R, in terms of t, the number of years since 2010.
2. In 2015, what will the helium reserves be?
3. If the rate of depletion doesn't change, when will the Federal Helium Reserve be depleted?

Exercise 6.E.34

Suppose the world's current oil reserves are 1820 billion barrels. If, on average, the total reserves is decreasing by 25 billion barrels of oil each year:

1. Give a linear equation for the remaining oil reserves, R, in terms of t, the number of years since now.
2. Seven years from now, what will the oil reserves be?
3. If the rate of depletion isn't change, when will the world's oil reserves be depleted?

1.4 Exercises

Exercise 6.E.1 – 6.E.8

Simplify each expression

1. x^3x^5	2. x^4x^2
3. $\left(x^3\right)^4$	4. $\left(x^7\right)^2$
5. $\left(2x^2\right)^3x^4$	6. $\left(5x^4\right)^2x^5$
7. $\frac{\left(3x^2\right)^2}{6x^3}$	8. $\frac{5x(4x)^2}{2x^2}$

Exercise 6.E.9 – 6.E.14

Simplify, and rewrite without negative exponents

9. $4x^{-3}$	10. $2x^{-5}$
11. $x^{-4}x^2$	12. $x^{-2}x$

13. $\frac{5x^{-3}}{2x^{-6}}$	14. $\frac{2x^{-4}}{6x^{-2}}$

Exercise $6.E.15 - 6.E.20$

Rewrite using negative or fractional exponents

15. $\frac{4}{x^{-5}}$	16. $\frac{4}{x^{-3}}$
17. $3\sqrt{x}$	18. $\sqrt[4]{x}$
19. $\frac{4}{\sqrt[3]{x}}$	20. $\frac{1}{5\sqrt{x}}$

Exercise $6.E.21 - 6.E.24$

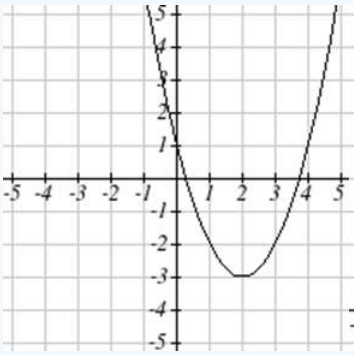
Rewrite as a radical

21. $4x^{-\frac{1}{2}}$	22. $5x^{-\frac{1}{3}}$
23. $2x^{\frac{1}{3}}$	24. $5x^{\frac{3}{2}}$

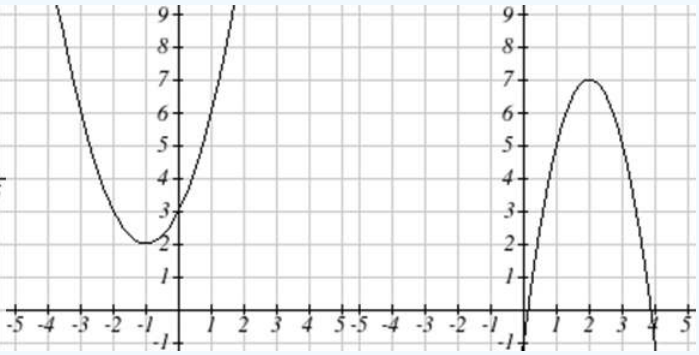
1.5 Exercises

Exercise $6.E.1 - 6.E.6$

Write an equation for the quadratic function graphed.

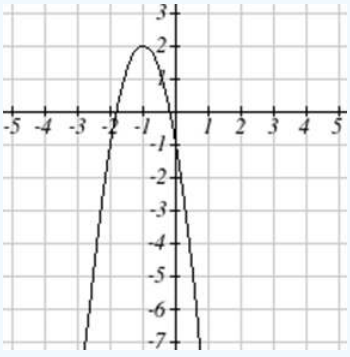


1.



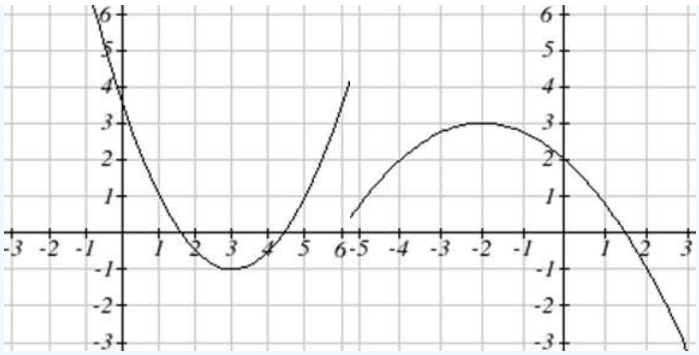
2.

3.



4.

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5. 6.

Exercise 6.E.7 – 6.E.12

7. $y(x) = 2x^2 + 10x + 12$	8. $z(p) = 3x^2 + 6x - 9$
9. $f(x) = 2x^2 - 10x + 4$	10. $g(x) = -2x^2 - 14x + 12$
11. $h(t) = -4t^2 + 6t - 1$	12. $k(t) = 2x^2 + 4x - 15$

Exercise 6.E.13

The height of a ball thrown in the air is given by $h(x) = -\frac{1}{12}x^2 + 6x + 3$, where x is the horizontal distance in feet from the point at which the ball is thrown.

1. How high is the ball when it was thrown?
2. What is the maximum height of the ball?
3. How far from the thrower does the ball strike the ground?

Exercise 6.E.14

A javelin is thrown in the air. Its height is given by $h(x) = -\frac{1}{20}x^2 + 8x + 6$, where x is the horizontal distance in feet from the point at which the javelin is thrown.

1. How high is the javelin when it was thrown?
2. What is the maximum height of the javelin?
3. How far from the thrower does the javelin strike the ground?

1.6 Exercises

Exercise 6.E.1 – 6.E.6

Find the degree and leading coefficient of each polynomial

1. $4x^7$	2. $5x^6$
3. $5 - x^2$	4. $6 + 3x - 4x^3$
5. $-2x^4 - 3x^2 + x - 1$	6. $6x^5 - 2x^4 + x^2 + 3$

Exercise 6.E.7 – 6.E.12

Find the vertical and horizontal intercepts of each function.

7. $f(t) = 2(t-1)(t+2)(t-3)$	8. $f(x) = 3(x+1)(x-4)(x+5)$

9. $g(n) = -2(3n-1)(2n+1)$	10. $k(u) = -3(4-n)(4n+3)$
11. $C(t) = 2t^4 - 8t^3 + 6t^2$	12. $C(t) = 4t^4 + 12t^3 - 40t^2$

Exercise $6.E.13 - 6.E.14$

Use your calculator or other graphing technology to solve graphically for the zeros of the function.

13. $f(x) = x^3 - 7x^2 + 4x + 30$	14. $g(x) = x^3 - 6x^2 + x + 28$

Exercise $6.E.15 - 6.E.18$

Solve each inequality.

15. $(x-3)(x-2)^2 > 0$	16. $(x-5)(x+1)^2 > 0$
17. $(x-1)(x+2)(x-3) < 0$	18. $(x-4)(x+3)(x+6) < 0$

Exercise $6.E.19 - 6.E.26$

For each function, find the horizontal intercepts, the vertical intercept, the vertical asymptotes, and the horizontal asymptote.

19. $p(x) = \frac{2x-3}{x+4}$	20. $q(x) = \frac{x-5}{3x-1}$
21. $s(x) = \frac{4}{(x-2)^2}$	22. $r(x) = \frac{5}{(x+1)^2}$
23. $f(x) = \frac{3x^2-14x-5}{3x^2+8x-16}$	24. $g(x) = \frac{2x^2+7x-15}{3x^2-14+15}$
25. $h(x) = \frac{2x^2+x-1}{x-4}$	26. $k(x) = \frac{2x^2-3x-20}{x-5}$

Exercise $6.E.27$

A scientist has a beaker containing 20 mL of a solution containing 20% acid. To dilute this, she adds pure water.

1. Write an equation for the concentration in the beaker after adding n mL of water.
2. Find the concentration if 10 mL of water has been added.
3. How many mL of water must be added to obtain a 4% solution?
4. What is the behavior as $n \to \infty$, and what is the physical significance of this?

Exercise $6.E.28$

A scientist has a beaker containing 30 mL of a solution containing 3 grams of potassium hydroxide. To this, she mixes a solution containing 8 milligrams per mL of potassium hydroxide.

1. Write an equation for the concentration in the tank after adding n mL of the second solution.
2. Find the concentration if 10 mL of the second solution has been added.
3. How many mL of water must be added to obtain a 50 mg/mL solution?
4. What is the behavior as $n \to \infty$, and what is the physical significance of this?

1.7 Exercises

Exercise 6.E.1

A population numbers 11,000 organisms initially and grows by 8.5% each year. Write an exponential model for the population.

Exercise 6.E.2

A population is currently 6,000 and has been increasing by 1.2% each day. Write an exponential model for the population.

Exercise 6.E.3

A vehicle purchased for $32,500 depreciates at a constant rate of 5% each year. Determine the approximate value of the vehicle 12 years after purchase.

Exercise 6.E.4

A business purchases $125,000 of office furniture which depreciates at a constant rate of 12% each year. Find the residual value of the furniture 6 years after purchase.

Exercise 6.E.5

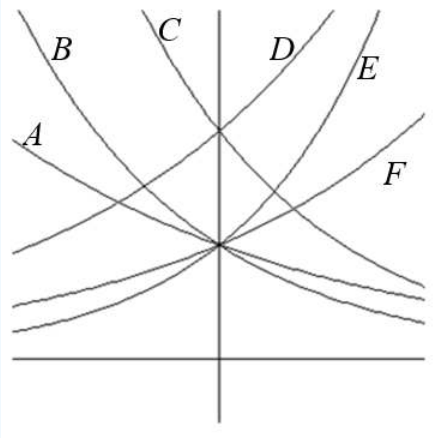
If $4,000 is invested in a bank account at an interest rate of 7 per cent per year, find the amount in the bank after 9 years if interest is compounded annually, quarterly, monthly, and continuously.

Exercise 6.E.6

If $6,000 is invested in a bank account at an interest rate of 9 per cent per year, find the amount in the bank after 5 years if interest is compounded annually, quarterly, monthly, and continuously.

Exercise 6.E.7 – 6.E.12

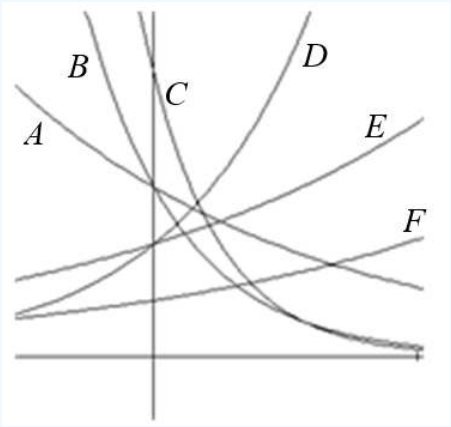
Match each function with one of the graphs below.



7. $f(x) = 2(0.69)^x$
8. $f(x) = 2(1.28)^x$
9. $f(x) = 2(0.81)^x$
10 $f(x) = 4(1.28)^x$
11. $f(x) = 2(1.59)^x$
12. $f(x) = 4(0.69)^x$

Exercise $6.E.13 - 6.E.16$

If all the graphs to the right have equations with form $f(t) = ae^{kt}$,



13. Which graph has the largest value for k?
14. Which graph has the smallest value for k?
15. Which graph has the largest value for a?
16. Which graph has the smallest value for a?

1.8 Exercises

Exercise $6.E.1 - 6.E.4$

Rewrite each equation in exponential form

1. $\log(v) = t$	2. $\log(r) = s$	3. $\ln(w) = n$	4. $\ln(x) = y$

Exercise $6.E.5 - 6.E.8$

Rewrite each equation in logarithmic form.

5. $10^a = b$	6. $10^p = v$	7. $e^k = h$	8. $e^y = x$

Exercise $6.E.9 - 6.E.24$

Solve each equation for the variable.

9. $5^x = 14$	10. $3^x = 23$	11. $7^x = \frac{1}{15}$	12. $3^x = \frac{1}{4}$
13. $e^{5x} = 17$	14. $e^{3x} = 12$	15. $3^{4x-5} = 38$	16. $4^{2x-3} = 44$

17. $1000(1.03)^t = 5000$	18. $200(1.06)^t = 550$
19. $3(1.04)^{3t} = 8$	20. $2(1.08)^{4t} = 7$
21. $50e^{-0.12t} = 10$	22. $10e^{-0.03t} = 4$
23. $10 - 8\left(\frac{1}{2}\right)^x = 5$	24. $100 - 100\left(\frac{1}{4}\right)^x = 70$

Exercise $6.E.25$

The population of Kenya was 39.8 million in 2009 and has been growing by about 2.6% each year. If this trend continues, when will the population exceed 45 million?

Exercise $6.E.26$

The population of Algeria was 34.9 million in 2009 and has been growing by about 1.5% each year. If this trend continues, when will the population exceed 45 million?

Exercise $6.E.27$

If $1000 is invested in an account earning 3% compounded monthly, how long will it take the account to grow in value to $1500?

Exercise $6.E.28$

If $1000 is invested in an account earning 2% compounded quarterly, how long will it take the account to grow in value to $1300?

Exercise $6.E.29$

Sketch a graph of: $f(x) = \log(x)$, $g(x) = \ln(x)$

Exercise $6.E.30 - 6.E.33$

Find the domain of each function.

30. $f(x) = \log(x-5)$	31. $f(x) = \ln(3-x)$
32. $f(x) = \ln(1-3x)$	33. $f(x) = \log(2x+5)$

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Table of Integrals

1. $\int \frac{1}{x(ax+b)}dx = \frac{1}{b}\ln\left|\frac{x}{ax+b}\right| + C$
2. $\int \frac{x}{ax+b}dx = \frac{x}{a} - \frac{b}{a^2}\ln|ax+b| + C$
3. $\int \frac{1}{(x+a)(x+b)}dx = \frac{1}{b-a}(\ln|x+a| - \ln|x+b|) + C = \frac{1}{b-a}\ln\left|\frac{x+a}{x+b}\right| + C, \quad a \neq b$
4. $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + C$
5. $\int \frac{1}{\sqrt{x^2 \pm a^2}}dx = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$
6. $\int \sqrt{x^2 \pm a^2}\,dx = \frac{x}{2}\sqrt{x^2 \pm a^2} + \frac{a^2}{2}\ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$
7. $\int x(ax+b)^n dx = \frac{(ax+b)^{n+1}}{a}\left(\frac{ax+b}{n+2} - \frac{b}{n+1}\right) + C, \quad n \neq -1, -2$
8. $\int x^n e^{ax}dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}dx + C$

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