MTH 165 & 175 COLLEGE ALGEBRA & PRECALCULUS



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MTH 165 & 175 College Algebra & Precalculus

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0.01: Review - Real Numbers: Notation and Operations

Real Numbers

Algebra is often described as the generalization of arithmetic. The systematic use of **variables**, letters used to represent numbers, allows us to communicate and solve a wide variety of real-world problems. For this reason, we begin by reviewing real numbers and their operations.

A **set** is a collection of objects, typically grouped within braces {}, where each object is called an **element**. When studying mathematics, we focus on special sets of numbers.

$$\underbrace{\mathbb{N}=\{1,2,3,4,5,\ldots\}}_{\textit{Natural Numbers}} \qquad \underbrace{W=\{0,1,2,3,4,5,\ldots\}}_{\textit{Whole Numbers}} \qquad \underbrace{\mathbb{Z}=\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}}_{\textit{Integers}}$$

The three periods (…) are called an ellipsis and indicate that the numbers continue without bound. A **subset**, denoted ⊆, is a set consisting of elements that belong to a given set. Notice that the sets of **natural** and **whole numbers** are both subsets of the set of integers and we can write $\mathbb{N} \subseteq \mathbb{Z}$ and $W \subseteq \mathbb{Z}$.

A set with no elements is called the **empty set** and has its own special notation:

$$\{\ \} = \varnothing \qquad \textit{Empty Set}$$

Rational numbers, denoted $\mathbb{Q}$, are defined as any number of the form $\frac{a}{b}$ where a and b are integers and b is nonzero. We can describe this set using **set notation**:

$$\mathbb{Q}=\left\{\frac{a}{b} \mid a,b \in \mathbb{Z}, b \neq 0\right\} \qquad \textit{Rational Numbers}$$

The vertical line | inside the braces reads, "*such that*" and the symbol ∈ indicates set membership and reads, "*is an element of*." The notation above in its entirety reads, "*the set of all numbers* $\frac{a}{b}$ *such that a and b are elements of the set of integers and b is not equal to zero.*" Decimals that terminate or repeat are rational. For example, $0.05=\frac{5}{100}$ and $0.\overline{6}=0.6666\ldots=\frac{2}{3}$.

The **set of integers** is a subset of the set of rational numbers, $\mathbb{Z} \subseteq \mathbb{Q}$, because every integer can be expressed as a ratio of the integer and 1. In other words, any integer can be written over 1 and can be considered a rational number. For example, $7=\frac{7}{1}$.

Irrational numbers are defined as any numbers that cannot be written as a ratio of two integers. Nonterminating decimals that do not repeat are irrational. For example, $\pi=3.14159\ldots$ and $\sqrt{2}=1.41421\ldots$.

Finally, the set of **real numbers**, denoted $\mathbb{R}$, is defined as the set of all rational numbers combined with the set of all irrational numbers. Therefore, all the numbers defined so far are subsets of the set of real numbers. In summary,

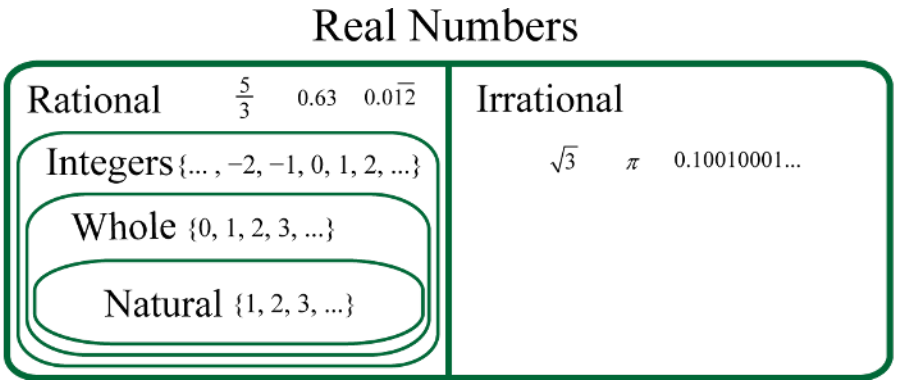


Figure 0.1.1: Real Numbers

Notation Used to Define Subsets of Real Numbers

Sets of numbers can be described in several ways, including Interval Notation, Set-Builder Notation and Inequality Notation.

Interval Notation

In **interval notation**, we use a square bracket [when the set includes the endpoint and a parenthesis (to indicate that the endpoint is either not included or the interval is unbounded.

- The smallest number in the interval, an endpoint, is written first.
- The largest number in the interval, another endpoint, is written second, and is written after a comma.
- Parentheses, (or), are used to signify that an endpoint is not included in the interval.
- Brackets, [or], are used to indicate that an endpoint is included in the interval.

Set Notation

Another way a set of numbers can be described is in **set-builder notation**. The braces {} are read as "the set of," and the vertical bar | is read as "such that," so we would read $\{x|10 \leq x < 30\}$ as "the set of x-values such that 10 is less than or equal to x, and x is less than 30." For a given value of x the statement is either true or false.

Figure 0.1.2 compares inequality notation, set-builder notation, and interval notation.

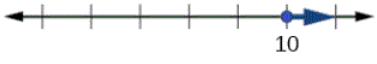
	Inequality Notation	Set-builder Notation	Interval Notation
	$5 < h \leq 10$	$\{h \mid 5 < h \leq 10\}$	$(5, 10]$
	$5 \leq h < 10$	$\{h \mid 5 \leq h < 10\}$	$[5, 10)$
	$5 < h < 10$	$\{h \mid 5 < h < 10\}$	$(5, 10)$
	$h < 10$	$\{h \mid h < 10\}$	$(-\infty, 10)$
	$h \geq 10$	$\{h \mid h \geq 10\}$	$[10, \infty)$
	All real numbers	$\mathbb{R}$	$(-\infty, \infty)$

Figure 0.1.2: Summary of notations for inequalities, sets, and intervals.

How to: Given a line graph, describe the set of values using interval notation.

1. Identify the intervals to be included in the set by determining where the heavy line overlays the real line.
2. At the left end of each interval, use [with each end value included in the set (solid dot) or (for each excluded end value (open dot).
3. At the right end of each interval, use] with each end value included in the set (filled dot) or) for each excluded end value (open dot).
4. Use the union symbol ∪ to combine all intervals into one set.
   NOTE: The word "**or**" (rather than ∪) is used to combine sets described by inequalities or Set-builder notation.

Compound Inequalities

For sets with a finite number of elements, like for example the sets $\{2,3,5\}$ and $\{5,6\}$, the **union** of these sets is the set $\{2,3,5,6\}$. The elements do not have to be listed in ascending order of numerical value, but if the original two sets have some elements in common, those elements should be listed only once in the union set. The **intersection** of these two set lists only those elements common to both sets, $\{5\}$

Union of intervals

There are several different ways to illustrate the union of two intervals, which combines all the elements of both sets.

- For inequality notation or set-builder notation, the word "**or**" is used. (The logical "or" operation requires that either one or both conditions are true.)
- For interval notation, the union symbol $\cup$ is used to combine all the elements of both sets.

To find the union of two intervals, use the portion of the number line representing the total collection of numbers in the two number line graphs. For example,

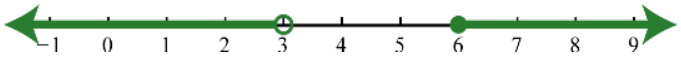


Figure **0.1.3** Number Line Graph of $x < 3$ or $x \geq 6$

Interval notation : $(-\infty, 3) \cup [6, \infty)$

Set notation : $\{x|\ x < 3 \text{ or } x \geq 6\}$

Example 0.1.1: Describing Sets on the Real-Number Line

Describe the intervals of values shown on the line graph below using inequality notation, set-builder notation, and interval notation.



Figure **0.1.4**: Line graph of $1 \leq x \leq 3$ or $x > 5$.

Solution

To describe the values, x, included in the intervals shown, we would say, "x is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5."

$1 \leq x \leq 3$ or $x > 5$	**Inequality**
$\{x \mid 1 \leq x \leq 3 \text{ or } x > 5\}$	**Set-Builder Notation**
$[1, 3] \cup (5, \infty)$	**Interval Notation**

Remember that when writing or reading interval notation, using a square bracket means the boundary is included in the set. Using a parenthesis means the boundary is not included in the set.

Try It 0.1.1

Given Figure **0.1.5**, specify the graphed set in

a. words
b. set-builder notation
c. interval notation

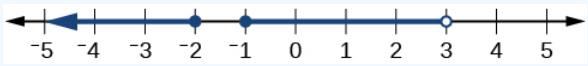


Figure **0.1.5**: Line graph

Answers

a. Values that are less than or equal to –2, or values that are greater than or equal to –1 and less than 3;
b. $\{x \mid x \leq -2 \text{ or } -1 \leq x < 3\}$

c. $(-\infty, -2] \cup [-1, 3)$

Intersection of intervals

There are several different ways to illustrate the intersection of two intervals.

- For inequality notation or set-builder notation, the word "**and**" is used to signify the intersection of two intervals. (The logical "and" operation requires that both conditions must be true.) The intersection can be more succinctly written as a compound inequality. For example, $-1 \leq x$ **and** $x < 3$ can be written more concisely as $-1 \leq x < 3$ which reads "*negative one is less than or equal to x and x is less than three*."
- When interval notation is used, the set composed of only the elements common to both sets is denoted by the intersection symbol, $\cap$.

To find the intersection of two intervals, take the portion of the number line that the two number line graphs have in common.

Example 0.1.2:

Graph and give the interval notation and set notation equivalent to $x < 3$ and $x \geq -1$.

Solution

Determine the intersection, or overlap, of the two solution sets to $x < 3$ and $x \geq -1$. The solutions to each inequality are sketched above the number line as a means to determine the intersection, which is graphed on the number line below.

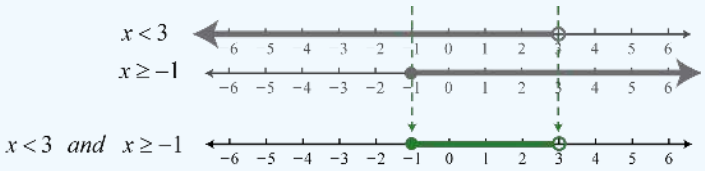


Figure 0.1.6

Here, 3 is not a solution because it solves only one of the inequalities. Alternatively, we may interpret $-1 \leq x < 3$ as all possible values for x between, or bounded by, -1 and 3 where -1 is included in the solution set.

Answer:

Interval notation: $[-1,\ 3)$; set notation: $\{x | -1 \leq x < 3\}$

Real Number Operations

Order of Operations

When we multiply a number by itself, we square it or raise it to a power of 2. For example, $4^2 = 4 \times 4 = 16$. We can raise any number to any power. In general, the exponential notation a^n means that the number or variable a is used as a factor n times.

$$a^n = a \cdot a \cdot a \cdots a \qquad \text{n factors}$$

In this notation, a^n is read as the nth power of a, where a is called the base and n is called the exponent. A term in exponential notation may be part of a mathematical expression, which is a combination of numbers and operations. For example, $24 + 6 \times \frac{2}{3} - 4^2$ is a mathematical expression.

To evaluate a mathematical expression, we perform the various operations. However, we do not perform them in any random order. We use the order of operations. This is a sequence of rules for evaluating such expressions.

Recall that in mathematics we use parentheses ( ), brackets [ ], and braces { } to group numbers and expressions so that anything appearing within the symbols is treated as a unit. Additionally, fraction bars, radicals, and absolute value bars are treated as grouping symbols. When evaluating a mathematical expression, begin by simplifying expressions within grouping symbols.

The next step is to address any exponents or radicals. Afterward, perform multiplication and division from left to right and finally addition and subtraction from left to right.

Let's take a look at the expression provided.

$$24+6\times\frac{2}{3}-4^2$$

There are no grouping symbols, so move on to exponents. The number 4 is raised to a power of 2, so simplify 4^2 as 16.

$$24+6\times\frac{2}{3}-4^2$$

$$24+6\times\frac{2}{3}-16$$

Next, perform multiplication or division, left to right.

$$24+6\times\frac{2}{3}-16$$

$$24+4-16$$

Lastly, perform addition or subtraction, left to right.

$$24+4-16$$

$$28-16$$

$$12$$

Therefore,

$$24+6\times\frac{2}{3}-4^2=12$$

For some complicated expressions, several passes through the order of operations will be needed. For instance, there may be a radical expression inside parentheses that must be simplified before the parentheses are evaluated. Following the order of operations ensures that anyone simplifying the same mathematical expression will get the same result.

ORDER OF OPERATIONS

Operations in mathematical expressions must be evaluated in a systematic order, which can be simplified using the acronym **PEMDAS**:

- **P**(arentheses)
- **E**(xponents)
- **M(**ultiplication) and **D**(ivision)
- **A**(ddition) and **S**(ubtraction)

How to: Given a mathematical expression, simplify it using the order of operations.

1. Simplify any expressions within grouping symbols. Grouping symbols include parentheses ( ), brackets [ ], braces { }, fraction bars, radicals, and absolute value bars
2. Simplify any expressions containing exponents or radicals.
3. Perform any multiplication and division in order, from left to right.
4. Perform any addition and subtraction in order, from left to right.

Example 0.1.3: Using the Order of Operations

Use the order of operations to evaluate each of the following expressions.

1. $(3\times2)^2-4\times(6+2)$
2. $\frac{5^2-4}{7}-\sqrt{11-2}$
3. $6-|5-8|+3\times(4-1)$

4. $\dfrac{14-3\times 2}{2\times 5-3^2}$
5. $7\times(5\times 3)-2\times[(6-3)-4^2]+1$

Solution

a.

$$\begin{aligned}(3\times 2)^2-4\times(6+2) &= (6)^2-4\times(8) && \text{Simplify parentheses}\\ &= 36-4\times 8 && \text{Simplify exponent}\\ &= 36-32 && \text{Simplify multiplication}\\ &= 4 && \text{Simplify subtraction}\end{aligned}$$

b.

$$\begin{aligned}\frac{5^2-4}{7}-\sqrt{11-2} &= \frac{5^2-4}{7}-\sqrt{9} && \text{Simplify grouping symbols}\\ &= \frac{5^2-4}{7}-3 && \cdot\ \text{radical: Subtract}\\ &= \frac{25-4}{7}-3 && \cdot\ \text{numerator: Simplify exponent}\\ &= \frac{21}{7}-3 && \cdot\ \text{numerator: Subtract}\\ &= 3-3 && \text{Simplify division}\\ &= 0 && \text{Simplify subtraction}\end{aligned}$$

Note that in the first step, the radical is treated as a grouping symbol, like parentheses. Also, in the third step, the fraction bar is considered a grouping symbol so the numerator is considered to be grouped.

c.

$$\begin{aligned}6-|5-8|+3\times(4-1) &= 6-|-3|+3\times 3 && \text{Simplify inside grouping symbols}\\ &= 6-3+3\times 3 && \text{Simplify absolute value}\\ &= 6-3+9 && \text{Simplify multiplication}\\ &= 3+9 && \text{Simplify subtraction}\\ &= 12 && \text{Simplify addition}\end{aligned}$$

d.

$$\begin{aligned}\frac{14-3\times 2}{2\times 5-3^2} &= \frac{14-3\times 2}{2\times 5-9} && \text{Simplify exponent}\\ &= \frac{14-6}{10-9} && \text{Simplify products}\\ &= \frac{8}{1} && \text{Simplify differences}\\ &= 8 && \text{Simplify quotient}\end{aligned}$$

In this example, the fraction bar separates the numerator and denominator, which we simplify separately until the last step.

e.

$$\begin{aligned}7\times(5\times 3)-2\times[(6-3)-4^2]+1 &= 7\times(15)-2\times[(3)-4^2]+1 && \text{Simplify inside parentheses}\\ &= 7\times(15)-2\times[(3)-16]+1 && \text{Simplify exponent}\\ &= 7\times(15)-2\times(-13)+1 && \text{Subtract}\\ &= 105+26+1 && \text{Multiply}\\ &= 132 && \text{Add}\end{aligned}$$

Try It 0.1.3

Use the order of operations to evaluate each of the following expressions.

a. $\sqrt{5^2-4^2}+7\times(5-4)^2$
b. $1+\dfrac{7\times 5-8\times 4}{9-6}$
c. $|1.8-4.3|+0.4\times\sqrt{15+10}$
d. $\dfrac{1}{2}\times[5\times 3^2-7^2]+\dfrac{1}{3}\times 9^2$
e. $[(3-8^2)-4]-(3-8)$

Answer

a. 10 b. 2 c. 4.5 d. 25 e. −60

Real Number Properties

PROPERTIES OF REAL NUMBERS

The following properties hold for real numbers a, b, and c.

	Addition	Multiplication
Commutative Property	$a+b=b+a$	$a\times b=b\times a$
Associative Property	$a+(b+c)=(a+b)+c$	$a(bc)=(ab)c$
Distributive Property	$a\times(b+c)=a\times b+a\times c$	
Identity Property	There exists a unique real number called the additive identity, 0, such that, for any real number a $a+0=a$	There exists a unique real number called the multiplicative identity, 1, such that, for any real number a $a\times 1=a$
Inverse Property	Every real number a has an additive inverse, or opposite, denoted $-a$, such that $a+(-a)=0$	Every nonzero real number a has a multiplicative inverse, or reciprocal, denoted $\frac{1}{a}$, such that $a\times\left(\frac{1}{a}\right)=1$

Constants and Variables

So far, the mathematical expressions we have seen have involved real numbers only. In mathematics, we may see expressions such as $x+5$, $\frac{4}{3}\pi r^3$, or $\sqrt{2m^3n^2}$. In the expression $x+5$, 5 is called a **constant** because it does not vary and x is called a **variable** because it does. (In naming the variable, ignore any exponents or radicals included with the variable.) An **algebraic expression** is a collection of constants and variables joined together by the algebraic operations of addition, subtraction, multiplication, and division.

Example 0.1.4: Describing Algebraic Expressions

List the constants and variables for each algebraic expression.

	a. $x+5$	b. $\frac{4}{3}\pi r^3$	c. $\sqrt{2m^3n^2}$

Solution

	a. $x+5$	b. $\frac{4}{3}\pi r^3$	c. $\sqrt{2m^3n^2}$
Constants	5	$\frac{4}{3}$, π	2
Variables	x	r	m, n

Try It 0.1.4

List the constants and variables for each algebraic expression.

1. $2\pi r(r+h)$
2. $2(L+W)$
3. $4y^3+y$

Answer

	a. $2\pi r(r+h)$	b. $2(L+W)$	c. $4y^3+y$
Constants	$2,\pi$	2	4
Variables	r,h	L, W	y

Evaluating Algebraic Expressions

We have already seen some real number examples of exponential notation, a shorthand method of writing products of the same factor. When variables are used, the constants and variables are treated the same way.

$$(-3)^5 = (-3)\times(-3)\times(-3)\times(-3)\times(-3) \Rightarrow x^5 = x\times x\times x\times x\times x$$
$$(2\times 7)^3 = (2\times 7)\times(2\times 7)\times(2\times 7) \quad \Rightarrow (yz)^3 = (yz)\times(yz)\times(yz)$$

In each case, the exponent tells us how many factors of the base to use, whether the base consists of constants or variables.

Any variable in an algebraic expression may take on or be assigned different values. When that happens, the value of the algebraic expression changes. To evaluate an algebraic expression means to determine the value of the expression for a given value of each variable in the expression. Replace each variable in the expression with the given value, then simplify the resulting expression using the order of operations. Parentheses should be used to avoid ambiguity. If the algebraic expression contains more than one variable, replace each variable with its assigned value and simplify the expression as before.

Example 0.1.5: Evaluating an Algebraic Expression at Different Values

Evaluate the expression $2x-7$ for each value for x.

a. $x=0$
b. $x=1$
c. $x=12$
d. $x=-4$

Solution

a. Substitute 0 for x. $\quad 2x-7=2(0)-7=0-7=-7$
Notice if parentheses were not used, $2(0)$ would have been 20, which is incorrect!!!
b. Substitute 1 for x. $\quad 2x-7=2(1)-7=2-7=-5$
c. Substitute $\frac{1}{2}$ for x. $\quad 2x-7=2\left(\frac{1}{2}\right)-7=1-7=-6$
d. Substitute -4 for x. $\quad 2x-7=2(-4)-7=-8-7=-15$
Notice if parentheses were not used, $2(-4)$ which is -6, would have been $2-4$ or -2, which is incorrect!!!

ALWAYS use parentheses in situations where not using them would create an incorrect expression!

Try It 0.1.5

Evaluate the expression $11-3y$ for each value for y.

a. $y=2$
b. $y=0$
c. $y=\frac{2}{3}$
d. $y=-5$

Answer

a. 5 b. 11 c. 9 d. 26

Example 0.1.6: Evaluating Algebraic Expressions

Evaluate each expression for the given values.

a. $x+5$ for $x=-5$
b. $\dfrac{t}{2t-1}$ for $t=10$
c. $\dfrac{4}{3}\pi r^3$ for $r=5$
d. $a+ab+b$ for $a=11$, $b=-8$
e. $\sqrt{2m^3n^2}$ for $m=2$, $n=3$

Solution

a. -5 for x.

$$x+5=(-5)+5=0$$

b. Substitute 10 for t.

$$\begin{aligned}\frac{t}{2t-1} &= \frac{(10)}{2(10)-1}\\ &= \frac{10}{20-1}=\frac{10}{19}\end{aligned}$$

c. Substitute 5 for r.

$$\begin{aligned}\frac{4}{3}\pi r^3 &= \frac{4}{3}\pi(5)^3\\ &= \frac{4}{3}\pi(125)=\frac{500}{3}\pi\end{aligned}$$

d. Substitute 11 for a and -8 for b.

$$\begin{aligned}a+ab+b &= (11)+(11)(-8)+(-8)\\ &= 11-88-8=-85\end{aligned}$$

e. Substitute 2 for m and 3 for n.

$$\begin{aligned}\sqrt{2m^3n^2} &= \sqrt{2(2)^3(3)^2}\\ &= \sqrt{2(8)(9)}\\ &= \sqrt{144}=12\end{aligned}$$

Try It 0.1.6

Evaluate each expression for the given values.

a. $\dfrac{y+3}{y-3}$ for $y=5$
b. $7-2t$ for $t=-2$
c. $\dfrac{1}{3}\pi r^2$ for $r=11$
d. $(p^2q)^3$ for $p=-2$, $q=3$
e. $4(m-n)-5(n-m)$ for $m=\dfrac{2}{3}$ $n=\dfrac{1}{3}$

Answer

a. 4 b. 11 c. $\dfrac{121}{3}\pi$ d. 1728 e. 3

Simplifying Algebraic Expressions

Sometimes we can simplify an algebraic expression to make it easier to evaluate or to use in some other way. To do so, we use the properties of real numbers. We can use the same properties in formulas because they contain algebraic expressions.

Example 0.1.7: Simplifying Algebraic Expressions

Simplify each algebraic expression.

a. $3x-2y+x-3y-7$
b. $2r-5(3-r)+4$
c. $\left(4t-\frac{5}{4}s\right)-\left(\frac{2}{3}t+2s\right)$
d. $2mn-5m+3mn+n$

Solution

a.
$$\begin{aligned} 3x-2y+x-3y-7 &= 3x+x-2y-3y-7 && \text{Commutative property of addition} \\ &= 4x-5y-7 && \text{Simplify} \end{aligned}$$

b.
$$\begin{aligned} 2r-5(3-r)+4 &= 2r-15+5r+4 && \text{Distributive property} \\ &= 2r+5r-15+4 && \text{Commutative property of addition} \\ &= 7r-11 && \text{Simplify} \end{aligned}$$

c.
$$\begin{aligned} \left(4t-\frac{5}{4}s\right)-\left(\frac{2}{3}t+2s\right) &= 4t-\frac{5}{4}s-\frac{2}{3}t-2s && \text{Distributive property} \\ &= 4t-\frac{2}{3}t-\frac{5}{4}s-2s && \text{Commutative property of addition} \\ &= \frac{10}{3}t-\frac{13}{4}s && \text{Simplify} \end{aligned}$$

d.
$$\begin{aligned} 2mn-5m+3mn+n &= 2mn+3mn-5m+n && \text{Commutative property of addition} \\ &= 5mn-5m+n && \text{Simplify} \end{aligned}$$

Try It 0.1.7

Simplify each algebraic expression.

a. $\frac{2}{3}y-2\left(\frac{4}{3}y+z\right)$
b. $\frac{5}{t}-2-\frac{3}{t}+1$
c. $4p(q-1)+q(1-p)$
d. $9r-(s+2r)+(6-s)$

Answer

a. $-2y-2z$ or $-2(y+z)$ b. $\frac{2}{t}-1$ c. $3pq-4p+q$ d. $7r-2s+6$

Example 0.1.8: Simplifying a Formula

A rectangle with length L and width W has a perimeter P given by $P=L+W+L+W$. Simplify this expression.

Solution

$$\begin{aligned} P &= L+W+L+W \\ P &= L+L+W+W && \text{Commutative property of addition} \\ P &= 2L+2W && \text{Simplify} \\ P &= 2(L+W) && \text{Distributive property} \end{aligned}$$

Try It 0.1.8

If the amount P is deposited into an account paying simple interest r for time t, the total value of the deposit A is given by $A = P + Prt$. Simplify the expression. (This formula will be explored in more detail later in the course.)

Answer

$A = P(1 + rt)$

Glossary

algebraic expression
constants and variables combined using addition, subtraction, multiplication, and division

associative property of addition
the sum of three numbers may be grouped differently without affecting the result; in symbols, a+(b+c) = (a+b)+c

associative property of multiplication
the product of three numbers may be grouped differently without affecting the result; in symbols, a(bc) = (ab)c

base
in exponential notation, the expression that is being multiplied

commutative property of addition
two numbers may be added in either order without affecting the result; in symbols, a + b = b + a

commutative property of multiplication
two numbers may be multiplied in any order without affecting the result; in symbols, ab = ba

constant
a quantity that does not change value

distributive property
the product of a factor times a sum is the sum of the factor times each term in the sum; in symbols, a(b+c) = ab + ac

equation
a mathematical statement indicating that two expressions are equal

exponent
in exponential notation, the raised number or variable that indicates how many times the base is being multiplied

exponential notation
a shorthand method of writing products of the same factor

formula
an equation expressing a relationship between constant and variable quantities

identity property of addition
there is a unique number, called the additive identity, 0, which, when added to a number, results in the original number; in symbols, a + 0 = a

identity property of multiplication
there is a unique number, called the multiplicative identity, 1, which, when multiplied by a number, results in the original number; in symbols, $a \cdot 1 = a$

integers

the set consisting of the natural numbers, their opposites, and 0: { ... -3, -2, -1, 0, 1, 2, 3, ... }

inverse property of addition

for every real number a there is a unique number, called the additive inverse (or opposite), denoted $-a$; which, when added to the original number, results in the additive identity, 0; in symbols, $a + (-a) = 0$

inverse property of multiplication

for every non-zero real number a there is a unique number, called the multiplicative inverse (or reciprocal), denoted $\frac{1}{a}$ which, when multiplied by the original number, results in the multiplicative identity, 1; in symbols $a \cdot \frac{1}{a} = 1$.

irrational numbers

the set of all numbers that are not rational; they cannot be written as either a terminating or repeating decimal; they cannot be expressed as a fraction of two integers

natural numbers

the set of counting numbers: {1, 2, 3, ...}.

order of operations

a set of rules governing how mathematical expressions are to be evaluated, assigning priorities to operations

rational numbers

the set of all numbers of the form $\frac{m}{n}$ where m and n are integers and $n \neq 0$. Any rational number may be written as a fraction or a terminating or repeating decimal.

real number line

a horizontal line used to represent the real numbers. An arbitrary fixed point is chosen to represent 0; positive numbers lie to the right of 0 and negative numbers to the left.

real numbers

the sets of rational numbers and irrational numbers taken together

variable

a quantity that may change value

whole numbers

the set consisting of 0 plus the natural numbers: {0, 1, 2, 3, ...}.

8/16/2025

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0.01e: Exercises - Real Number Operations

A: The Number Line, Set and Interval Notation

Exercises 0.1.1

★ **Graph the solution set and give the interval notation and set-builder notation equivalents.**

1. $x < -1$	5. $-10 \le x < 4$	9. $x < 5$ and $x \ge 0$	13. $x < -2$ or $x \ge 5$
2. $x > -3$	6. $3 < x \le 7$	10. $x \le -10$ and $x \ge -40$	14. $x \le 0$ or $x \ge 4$
3. $x \ge -8$	7. $-40 < x < 0$	11. $x \le 7$ and $x < 10$	15. $x < 6$ or $x > 2$
4. $x \le 6$	8. $-12 \le x \le -4$	12. $x < 1$ and $x > 3$	16. $x < 0$ or $x \le 5$

Answers to odd exercises:

1. $(-\infty, -1)$; $\{x \mid x < -1\}$;

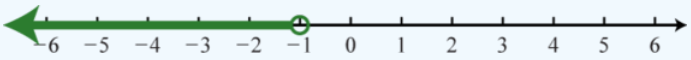

Figure 0.1e.1

3. $[-8, \infty)$; $\{x \mid x \ge -8\}$;

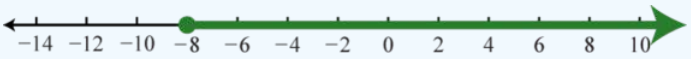

Figure 0.1e.3

5. $[-10, 4)$; $\{x \mid -10 \le x < 4\}$;

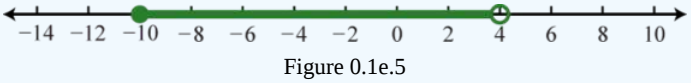

Figure 0.1e.5

7. $(-40, 0)$; $\{x \mid -40 < x < 0\}$;

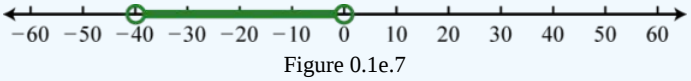

Figure 0.1e.7

9. $[0, 5)$; $\{x \mid 0 \le x < 5\}$;

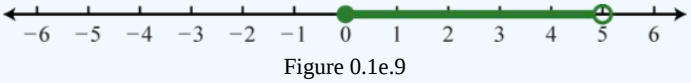

Figure 0.1e.9

11. $(-\infty, 7]$; $\{x \mid x \le 7\}$;



Figure 0.1e.11

13. $(-\infty, -2) \cup [5, \infty)$; $\{x \mid x < -2 \text{ or } x \ge 5\}$

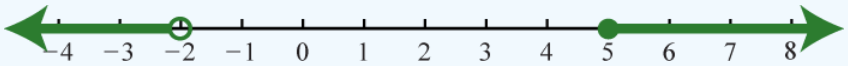

Figure 0.1e.13

15. $(-\infty, \infty) = \mathbb{R}$;

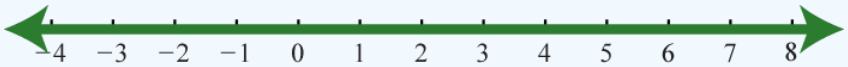

Figure 0.1e.15

B: Convert a Description to an Inequality

Exercises 0.1.2

★ **Write an equivalent inequality.**

17. All real numbers less than -15.
18. All real numbers greater than or equal to -7.
19. All real numbers less than 6 and greater than zero.
20. All real numbers less than zero and greater than -5.
21. All real numbers less than or equal to 5 or greater than 10.
22. All real numbers between -2 and 2.

Answers to odd exercises:

17. $x < -15$
19. $0 < x < 6$
21. $x \leq 5$ or $x > 10$

C: Convert Interval Notation to an Inequality

Exercises 0.1.3

★ **Determine the inequality given the answers expressed in interval notation.**

23. $(-\infty, 12)$
24. $[-8, \infty)$
25. $(-\infty, 0]$
26. $(0, \infty)$
27. $(-6, 14)$
28. $(0, 12]$
29. $[5, 25)$
30. $[-30, -10]$
31. $(-\infty, 2) \cup [3, \infty)$
32. $(-\infty, -19] \cup [-12, \infty)$
33. $(-\infty, -2) \cup (0, \infty)$
34. $(-\infty, -15] \cup (-5, \infty)$

Answers to odd exercises:

23. $x < 12$
25. $x \leq 0$
27. $-6 < x < 14$
29. $5 \leq x < 25$
31. $x < 2$ or $x \geq 3$
33. $x < -2$ or $x > 0$

D: Order of Operations

Exercises 0.1.4

★ **Use order of operations to evaluate the given expression.**

36. $10+2\times(5-3)$
37. $6\div2-(81\div3^2)$
38. $18+(6-8)^3$
39. $-2\times[16\div(8-4)^2]^2$
40. $4-6+2\times7$
41. $3(5-8)$
42. $4+6-10\div2$
43. $12\div(36\div9)+6$
44. $(4+5)^2\div3$
45. $3-12\times2+19$
46. $2+8\times7\div4$
47. $5+(6+4)-11$
48. $9-18\div3^2$
49. $14\times3\div7-6$
50. $9-(3+11)\times2$
51. $6+2\times2-1$
52. $64\div(8+4\times2)$
53. $9+4(2^2)$
54. $(12\div3\times3)^2$
55. $25\div5^2-7$
56. $(15-7)\times(3-7)$
57. $2\times4-9(-1)$
58. $4^2-25\cdot\frac{1}{5}$
59. $12(3-1)\div6$

Answers to odd exercises:

37. -6 39. -2 41. -9 43. 9 45. -2 47. 4
49. 0 51. 9 53. 25 55. -6 57. 17 59. 4

E: Substitute and Evaluate Expressions

Exercises 0.1.5

★ **Evaluate.**

61. $-2x+3$ where $x=-2$
62. $8x-5$ where $x=-1$
63. x^2-x+5 where $x=-5$
64. $2x^2-8x+1$ where $x=3$
65. $\dfrac{x^2-x+2}{2x-1}$ where $x=-\frac{1}{2}$
66. $\dfrac{9x^2+x-2}{3x-4}$ where $x=-\frac{2}{3}$
67. $(3y-2)(y+5)$ where $y=\frac{2}{3}$
68. $(3x+2)(5x+1)$ where $x=-\frac{1}{5}$
69. $(3x-1)(x-8)$ where $x=-1$
70. $(7y+5)(y+1)$ where $y=-2$
71. y^6-y^3+2 where $y=-1$
72. y^5+y^3-3 where $y=-2$
73. a^2-5b^2 where $a=-2$ and $b=-1$
74. a^3-2b^3 where $a=-3$ and $b=2$
75. $(x-2y)(x+2y)$ where $x=2$ and $y=-5$
76. $(4x-3y)(x-y)$ where $x=-4$ and $y=-3$
77. a^2-ab+b^2 where $a=-1$ and $b=-2$
78. x^2y^2-xy+2 where $x=-3$ and $y=-2$
79. a^4-b^4 where $a=-2$ and $b=-3$
80. $a^6-2a^3b^3-b^6$ where $a=2$ and $b=-1$

81. Evaluate $\sqrt{b^2-4ac}$ given the following values.

a. $a=6, b=1$ and $c=-1$
b. $a=15, b=4$ and $c=-4$
c. $a=\dfrac{3}{4}, b=-2$ and $c=-4$
d. $a=\dfrac{1}{2}, b=-2$ and $c=-30$
e. $a=1, b=2$ and $c=-1$
f. $a=1, b=-4$ and $c=-50$
g. $a=1, b=-1$ and $c=-\dfrac{1}{16}$
h. $a=-2, b=-\dfrac{1}{3}$ and $c=1$

Answers to odd exercises:

61. 7 63. 35 65. $-\frac{11}{8}$ 67. 0 69. 36 71. 4 73. -1
75. -96 77. 3 79. -65 81 a. 5 81 c. 4 81 e. $2\sqrt{2}$ 81 g. $\frac{\sqrt{5}}{2}$

F: Simplify Algebraic Expressions

Exercises 0.1.6

★ **Simplify.**

85. $5-2(4x+8)$
86. $8-6(2x-1)$
87. $2(x^2-7x+1)+3x-7$
88. $-5(x^2+4x-1)+8x^2-5$
89. $5ab-4(ab+5)$
90. $5(7-ab)+2ab$
91. $2-a^2+3(a^2+4)$
92. $7-3y+2(y^2-3y-2)$
93. $8x^2-3x-5(x^2+4x-1)$
94. $2-5y-6(y^2-y+2)$
95. $a^2b^2-5+3(a^2b^2-3ab+2)$
96. $a^2-3ab-2(a^2-ab+1)$
97. $10y^2+6-(3y^2+2y+4)$
98. $4m^2-3mn-(m^2-3mn+n^2)$
99. $x^{2n}-3x^n+5(x^{2n}-x^n+1)$
100. $-3(y^{2n}-2y^n+1)+4y^{2n}-5$
101. $7x-x\div 2\times 4+\dfrac{x}{3}$
102. $5v\div 3v\times(9-6+2)$
103. $7z-3+z\times 6^2$
104. $4\times 3+18x\div 9-12$
105. $8b-1-4b\times 3+2$
106. $a\div 64\times 2^3-12a\div 6$
107. $2(9-4^2)y-11$
108. $4x+x(13-7)$
109. $9(y+8)-27$
110. $\left(\dfrac{9}{6}t-4\right)2$
111. $6+12b-3(6b)$
112. $18y-2(1+7y)$
113. $\left(\dfrac{4}{9}\right)^2 27x$
114. $8(3-m)+1(-8)$
115. $9x+4x(2+3)-4(2x+3x)$
116. $5^2-4(3x)$

Answers to odd exercises:

85. $-8x-11$ 87. $2x^2-11x-5$ 89. $ab-20$ 91. $2a^2+14$ 93. $3x^2-23x+5$
95. $4a^2b^2-9ab+1$ 97. $7y^2-2y+2$ 99. $6x^{2n}-8x^n+5$ 101. $\frac{16x}{3}$ 103. $43z-3$
105. $-4b+1$ 107. $-14y-11$ 109. $9y+45$ 111. $-6b+6$ 113. $\frac{16x}{3}$ 115. $9x$

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0.02: Review - whole number exponents

Using the Product Rule of Exponents

Consider the product $x^3 \times x^4$. Both terms have the same base, x, but they are raised to different exponents. Expand each expression, and then rewrite the resulting expression.

$$\begin{aligned} x^3 \times x^4 &= \overbrace{x \times x \times x}^{3 \text{ factors}} \times \overbrace{x \times x \times x \times x}^{4 \text{ factors}} \\ &= \overbrace{x \times x \times x \times x \times x \times x \times x}^{7 \text{ factors}} \\ &= x^7 \end{aligned}$$

Notice that the exponent of a product is the sum of the exponents of the two factors. In other words, when multiplying exponential expressions with the same base, we write the result with the common base and add the exponents. This is the product rule of exponents.

$$x^3 \times x^4 = x^{3+4} = x^7$$

Now consider an example with real numbers.

$$2^3 \times 2^4 = 2^{3+4} = 2^7$$

We can always check that this is true by simplifying each exponential expression. We find that 2^3 is 8, 2^4 is 16, and 2^7 is 128. The product 8×16 equals 128, so the relationship is true. We can use the product rule of exponents to simplify expressions that are a product of two numbers or expressions with the same base but different exponents.

THE PRODUCT RULE OF EXPONENTS

For any real number a and natural numbers m and n, the product rule of exponents states that

$$a^m \times a^n = a^{m+n} \tag{0.2.1}$$

Example 0.2.1: Using the Product Rule

Write each of the following products with a single base. Do not simplify further.

a. $t^5 \times t^3$
b. $(-3)^5 \times (-3)$
c. $x^2 \times x^5 \times x^3$

Solution

Use the product rule (Equation 0.2.1) to simplify each expression.

a. $t^5 \times t^3 = t^{5+3} = t^8$
b. $(-3)^5 \times (-3) = (-3)^5 \times (-3)^1 = (-3)^{5+1} = (-3)^6$
c. $x^2 \times x^5 \times x^3$

At first, it may appear that we cannot simplify a product of three factors. However, using the associative property of multiplication, begin by simplifying the first two.

$$x^2 \times x^5 \times x^3 = (x^2 \times x^5) \times x^3 = (x^{2+5}) \times x^3 = x^7 \times x^3 = x^{7+3} = x^{10}$$

Notice we get the same result by adding the three exponents in one step.

$$x^2 \times x^5 \times x^3 = x^{2+5+3} = x^{10}$$

Try It 0.2.1

Write each of the following products with a single base. Do not simplify further.

a. $k^6 \times k^9$

b. $\left(\frac{2}{y}\right)^4 \times \left(\frac{2}{y}\right)$

c. $t^3 \times t^6 \times t^5$

Answers

a. k^{15} b. $\left(\frac{2}{y}\right)^5$ c. t^{14}

Using the Power Rule of Exponents

Suppose an exponential expression is raised to some power. Can we simplify the result? Yes. To do this, we use the power rule of exponents. Consider the expression $(x^2)^3$. The expression inside the parentheses is multiplied twice because it has an exponent of 2. Then the result is multiplied three times because the entire expression has an exponent of 3.

$$\begin{aligned}(x^2)^3 &= (x^2) \times (x^2) \times (x^2) \\ &= x \times x \times x \times x \times x \times x \\ &= x^6\end{aligned}$$

The exponent of the answer is the product of the exponents. In other words, when raising an exponential expression to a power, we write the result with the common base and multiply the exponents.

$$(x^2)^3 = x^{2 \cdot 3} = x^6$$

Be careful to distinguish between uses of the product rule and the power rule. When using the product rule, different factors with the same bases are raised to exponents. In this case, you add the exponents. When using the power rule, a factor in exponential notation is raised to a power. In this case, you multiply the exponents.

Product Rule	Power Rule
$5^3 \times 5^4 = 5^{3+4} = 5^7$	$(5^3)^4 = 5^{3 \times 4} = 5^{12}$
$x^5 \times x^2 = x^{5+2} = x^7$	$(x^5)^2 = x^{5 \times 2} = x^{10}$
$(3a)^7 \times (3a)^{10} = (3a)^{7+10} = (3a)^{17}$	$((3a)^7)^{10} = (3a)^{7 \times 10} = (3a)^{70}$

THE POWER RULE OF EXPONENTS

For any real number a and natural numbers m and n, the power rule of exponents states that

$$(a^m)^n = a^{m \cdot n} \tag{0.2.2}$$

Example 0.2.2: Using the Power Rule

Write each of the following products with a single base. Do not simplify further.

a. $(x^2)^7$

b. $((2t)^5)^3$

c. $((-3)^5)^{11}$

Solution

Use the power rule (Equation 0.2.2) to simplify each expression.

a. $(x^2)^7 = x^{2 \cdot 7} = x^{14}$

b. $((2t)^5)^3 = (2t)^{5\cdot 3} = (2t)^{15}$
c. $((-3)^5)^{11} = (-3)^{5\cdot 11} = (-3)^{55}$

Try It 0.2.2

Write each of the following products with a single base. Do not simplify further.

a. $((3y)^8)^3$
b. $(t^5)^7$
c. $((-g)^4)^4$

Answers

a. $(3y)^{24}$ b. t^{35} c. $(-g)^{16}$

Using the Quotient Rule of Exponents

The quotient rule of exponents allows us to simplify an expression that divides two numbers with the same base but different exponents. In a similar way to the product rule, we can simplify an expression such as $\frac{y^m}{y^n}$. Consider the example $\frac{y^9}{y^5}$. Perform the division by canceling common factors.

$$\begin{aligned}\frac{y^9}{y^5} &= \frac{y\cdot y\cdot y\cdot y\cdot y\cdot y\cdot y\cdot y\cdot y}{y\cdot y\cdot y\cdot y\cdot y} \\ &= \frac{y\cdot y\cdot y\cdot y}{1} \\ &= y^4\end{aligned}$$

Notice that the exponent of the quotient is the difference between the exponents of the divisor and dividend. In other words, when dividing exponential expressions with the same base, we write the result with the common base and subtract the exponents.

$$\frac{y^9}{y^5} = y^{9-5} = y^4$$

THE QUOTIENT RULE OF EXPONENTS

For any real number $a, a \neq 0$ and natural numbers m and n, the quotient rule of exponents states that

$$\frac{a^m}{a^n} = a^{m-n} \tag{0.2.3}$$

Instead of qualifying variables as nonzero each time, we will simplify matters and assume from here on that all variables represent nonzero real numbers.

Example 0.2.3: Using the Quotient Rule

Write each of the following products with a single base. Do not simplify further.

a. $\frac{(-2)^{14}}{(-2)^9}$
b. $\frac{t^{23}}{t^{15}}$
c. $\frac{(z\sqrt{2})^5}{z\sqrt{2}}$

Solution

Use the quotient rule (Equation 0.2.3) to simplify each expression.

a. $\frac{(-2)^{14}}{(-2)^9} = (-2)^{14-9} = (-2)^5$

b. $\frac{t^{23}}{t^{15}} = t^{23-15} = t^8$

c. $\frac{(z\sqrt{2})^5}{z\sqrt{2}} = (z\sqrt{2})^{5-1} = (z\sqrt{2})^4$

Try It 0.2.3

Write each of the following products with a single base. Do not simplify further.

a. $\frac{s^{75}}{s^{68}}$

b. $\frac{(-3)^6}{-3}$

c. $\frac{(ef^2)^5}{(ef^2)^3}$

Answers

a. s^7 b. $(-3)^5$ c. $(ef^2)^2$

Using the Zero Exponent Rule of Exponents

What would happen if the Quotient Rule were used and $m = n$? Consider the example.

$$\frac{t^8}{t^8} = 1 \qquad \textbf{because a number divided by itself is 1}$$

If we were to simplify the expression using the quotient rule instead, we would have

$$\frac{t^8}{t^8} = t^{8-8} = t^0$$

If we equate the two answers, the result is $t^0 = 1$. This is true for any nonzero real number, or any variable representing a real number. The sole exception is the expression 0^0, whose value is undefined.

THE ZERO EXPONENT RULE OF EXPONENTS

For any nonzero real number a, the zero exponent rule of exponents states that

$$a^0 = 1 \tag{0.2.4}$$

Example 0.2.4: Using the Zero Exponent Rule

Simplify each expression using the zero exponent rule of exponents.

a. $\frac{c^3}{c^3}$

b. $\frac{-3x^5}{x^5}$

c. $\frac{(j^2k)^4}{(j^2k) \times (j^2k)^3}$

d. $\frac{5(rs^2)^2}{(rs^2)^2}$

Solution

Use the zero exponent and other rules to simplify each expression.

a.

$$\begin{aligned}\frac{c^3}{c^3} &= c^{3-3} \\ &= c^0 \\ &= 1\end{aligned}$$

b.

$$\begin{aligned}\frac{-3x^5}{x^5} &= -3 \times \frac{x^5}{x^5} \\ &= -3 \times x^{5-5} \\ &= -3 \times x^0 \\ &= -3 \times 1 \\ &= -3\end{aligned}$$

c.

$$\begin{aligned}\frac{(j^2k)^4}{(j^2k) \times (j^2k)^3} &= \frac{(j^2k)^4}{(j^2k)^{1+3}} && \text{Use the product rule in the denominator} \\ &= \frac{(j^2k)^4}{(j^2k)^4} && \text{Simplify} \\ &= (j^2k)^{4-4} && \text{Use the quotient rule} \\ &= (j^2k)^0 && \text{Simplify} \\ &= 1\end{aligned}$$

d.

$$\begin{aligned}\frac{5(rs^2)^2}{(rs^2)^2} &= 5(rs^2)^{2-2} && \text{Use the quotient rule} \\ &= 5(rs^2)^0 && \text{Simplify} \\ &= 5 \times 1 && \text{Use the zero exponent rule} \\ &= 5 && \text{Simplify}\end{aligned}$$

Try It 0.2.4

Simplify each expression using the zero exponent rule of exponents.

a. $\frac{t^7}{t^7}$

b. $\frac{(de^2)^{11}}{2(de^2)^{11}}$

c. $\frac{w^4 \times w^2}{w^6}$

d. $\frac{t^3 \times t^4}{t^2 \times t^5}$

Answers

a. 1 b. $\frac{1}{2}$ c. 1 d. 1

Using the Negative Rule of Exponents

Consider the situation where one exponential expression is divided by another exponential expression with a larger exponent. For example, $\frac{t^3}{t^5}$.

$$\begin{aligned}\frac{t^3}{t^5} &= \frac{t \times t \times t}{t \times t \times t \times t \times t} \\ &= \frac{1}{t \times t} \\ &= \frac{1}{t^2}\end{aligned}$$

If we were to simplify the original expression using the quotient rule, we would have

$$\begin{aligned}\frac{t^3}{t^5} &= t^{3-5} \\ &= t^{-2}\end{aligned}$$

Putting the answers together, we have $t^{-2} = \frac{1}{t^2}$. This is true for any nonzero real number t.

In general, a factor with a negative exponent becomes the same factor with a positive exponent if it is moved across the fraction bar—from numerator to denominator or vice versa.

$$a^{-n} = \frac{1}{a^n} \text{ and } a^n = \frac{1}{a^{-n}}$$

We have shown that the exponential expression a^n is defined when n is a natural number, 0, or the negative of a natural number. That means that a^n is defined for any integer n. Also, the product and quotient rules and all of the rules we will look at soon hold for any integer n.

THE NEGATIVE RULE OF EXPONENTS

For any nonzero real number a and integer n, the negative rule of exponents states that

$$a^{-n} = \frac{1}{a^n} \qquad \text{and} \qquad a^n = \frac{1}{a^{-n}} \tag{0.2.5}$$

Example 0.2.5: Using the Negative Exponent Rule

Write each of the following quotients with a single base. Do not simplify further. Write answers with positive exponents.

a. $\frac{\theta^3}{\theta^{10}}$
b. $\frac{z^2 \times z}{z^4}$
c. $\frac{(-5t^3)^4}{(-5t^3)^8}$

Solution

a. $\frac{\theta^3}{\theta^{10}} = \theta^{3-10} = \theta^{-7} = \frac{1}{\theta^7}$
b. $\frac{z^2 \times z}{z^4} = \frac{z^{2+1}}{z^4} = \frac{z^3}{z^4} = z^{3-4} = z^{-1} = \frac{1}{z}$
c. $\frac{(-5t^3)^4}{(-5t^3)^8} = (-5t^3)^{4-8} = (-5t^3)^{-4} = \frac{1}{(-5t^3)^4}$

Try It 0.2.5

Write each of the following quotients with a single base. Do not simplify further. Write answers with positive exponents.

a. $\frac{(-3t)^2}{(-3t)^8}$
b. $\frac{f^{47}}{f^{49} \times f}$
c. $\frac{2k^4}{5k^7}$

Answers

a. $\frac{1}{(-3t)^6}$ b. $\frac{1}{f^3}$ c. $\frac{2}{5k^3}$

Example 0.2.6: Using the Product and Quotient Rules

Write each of the following products with a single base. Do not simplify further. Write answers with positive exponents.

a. $b^2 \times b^{-8}$
b. $(-x)^5 \times (-x)^{-5}$
c. $\frac{-7z}{(-7z)^5}$

Solution

a. $b^2 \times b^{-8} = b^{2+-8} = b^{-6} = \frac{1}{b^6}$

b. $(-x)^5 \times (-x)^{-5} = (-x)^{5+-5} = (-x)^0 = 1$

c. $\frac{-7z}{(-7z)^5} = \frac{(-7z)^1}{(-7z)^5} = (-7z)^{1-5} = (-7z)^{-4} = \frac{1}{(-7z)^4}$

Try It 0.2.6

Write each of the following products with a single base. Do not simplify further. Write answers with positive exponents.

a. $t^{-11} \times t^6$

b. $\frac{25^{12}}{25^{13}}$

Answers

a. $t^{-5} = \frac{1}{t^5}$ b. $\frac{1}{25}$

Finding the Power of a Product

To simplify the power of a product of two exponential expressions, we can use the power of a product rule of exponents, which breaks up the power of a product of factors into the product of the powers of the factors. For instance, consider $(pq)^3$. We begin by using the associative and commutative properties of multiplication to regroup the factors.

$$\begin{aligned}(pq)^3 &= (pq) \times (pq) \times (pq) \\ &= p \times q \times p \times q \times p \times q \\ &= p^3 \times q^3\end{aligned}$$

In other words, $(pq)^3 = p^3 \times q^3$.

THE POWER OF A PRODUCT RULE OF EXPONENTS

For any real numbers a and b and any integer n, the power of a product rule of exponents states that

$$(ab)^n = a^n b^n \tag{0.2.6}$$

Example 0.2.7: Using the Power of a Product Rule

Simplify each of the following products as much as possible using the power of a product rule. Write answers with positive exponents.

a. $(ab^2)^3$

b. $(2t)^{15}$

c. $(-2w^3)^3$

d. $\frac{1}{(-7z)^4}$

e. $(e^{-2}f^2)^7$

Solution

Use the product and quotient rules and the new definitions to simplify each expression.

a. $(ab^2)^3 = (a)^3 \times (b^2)^3 = a^{1\times3} \times b^{2\times3} = a^3b^6$

b. $(2t)^{15} = (2)^{15} \times (t)^{15} = 2^{15}t^{15} = 32,768t^{15}$

c. $(-2w^3)^3 = (-2)^3 \times (w^3)^3 = -8 \times w^{3\times3} = -8w^9$

d. $\frac{1}{(-7z)^4} = \frac{1}{(-7)^4 \times (z)^4} = \frac{1}{2401z^4}$

e. $(e^{-2}f^2)^7 = (e^{-2})^7 \times (f^2)^7 = e^{-2\times 7} \times f^{2\times 7} = e^{-14}f^{14} = \dfrac{f^{14}}{e^{14}}$

Try It 0.2.7

Simplify each of the following products as much as possible using the power of a product rule. Write answers with positive exponents.

a. $(g^2h^3)^5$
b. $(5t)^3$
c. $(-3y^5)^3$
d. $\dfrac{1}{(a^6b^7)^3}$
e. $(r^3s^{-2})^4$

Answers

a. $g^{10}h^{15}$ b. $125t^3$ c. $-27y^{15}$ d. $\dfrac{1}{a^{18}b^{21}}$ e. $\dfrac{r^{12}}{s^8}$

Finding the Power of a Quotient

To simplify the power of a quotient of two expressions, consider the expression below.

$$\left(\frac{2}{x}\right)^3 = \frac{2}{x}\cdot\frac{2}{x}\cdot\frac{2}{x} = \frac{2\cdot 2\cdot 2}{x\cdot x\cdot x} = \frac{2^3}{x^3}$$

Thus in general, the power of a quotient of factors is the quotient of the powers of the factors.

THE POWER OF A QUOTIENT RULE OF EXPONENTS

For any real numbers a and b, $b \neq 0$, and any integer n, the power of a quotient rule of exponents states that

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \tag{0.2.7}$$

Example 0.2.8: Using the Power of a Quotient Rule

Simplify each of the following quotients as much as possible using the power of a quotient rule. Write answers with positive exponents.

a. $\left(\dfrac{4}{z^{11}}\right)^3$
b. $\left(\dfrac{p}{q^3}\right)^6$
c. $\left(\dfrac{-1}{t^2}\right)^{27}$
d. $(j^3k^{-2})^4$
e. $(m^{-2}n^{-2})^3$

Solution

a. $\left(\dfrac{4}{z^{11}}\right)^3 = \dfrac{(4)^3}{(z^{11})^3} = \dfrac{64}{z^{11\times 3}} = \dfrac{64}{z^{33}}$

b. $\left(\dfrac{p}{q^3}\right)^6 = \dfrac{(p)^6}{(q^3)^6} = \dfrac{p^{1\times 6}}{q^{3\times 6}} = \dfrac{p^6}{q^{18}}$

c. $\left(\dfrac{-1}{t^2}\right)^{27} = \dfrac{(-1)^{27}}{(t^2)^{27}} = \dfrac{-1}{t^{2\times 27}} = \dfrac{-1}{t^{54}} = -\dfrac{1}{t^{54}}$

d. $(j^3k^{-2})^4 = \left(\dfrac{j^3}{k^2}\right)^4 = \dfrac{(j^3)^4}{(k^2)^4} = \dfrac{j^{3\times 4}}{k^{2\times 4}} = \dfrac{j^{12}}{k^8}$

e. $(m^{-2}n^{-2})^3 = \left(\frac{1}{m^2n^2}\right)^3 = \frac{(1)^3}{(m^2n^2)^3} = \frac{1}{(m^2)^3(n^2)^3} = \frac{1}{m^{2\times3}n^{2\times3}} = \frac{1}{m^6n^6}$

Try It 0.2.8

Simplify each of the following quotients as much as possible using the power of a quotient rule. Write answers with positive exponents.

a. $\left(\frac{b^5}{c}\right)^3$
b. $\left(\frac{5}{u^8}\right)^4$
c. $\left(\frac{-1}{w^3}\right)^{35}$
d. $(p^{-4}q^3)^8$
e. $(c^{-5}d^{-3})^4$

Answers

a. $\frac{b^{15}}{c^3}$ b. $\frac{625}{u^{32}}$ c. $\frac{-1}{w^{105}}$ d. $\frac{q^{24}}{p^{32}}$ e. $\frac{1}{c^{20}d^{12}}$

Simplifying Exponential Expressions

Recall that to simplify an expression means to rewrite it by combing terms or exponents; in other words, to write the expression more simply with fewer terms. The rules for exponents may be combined to simplify expressions.

Example 0.2.9: Simplifying Exponential Expressions

Simplify each expression and write the answer with positive exponents only.

a. $(6m^2n^{-1})^3$

b. $17^5 \times 17^{-4} \times 17^{-3}$

c. $\left(\frac{u^{-1}v}{v^{-1}}\right)^2$

d. $(-2a^3b^{-1})(5a^{-2}b^2)$

e. $(x^2\sqrt{2})^4(x^2\sqrt{2})^{-4}$

f. $\frac{(3w^2)^5}{(6w^{-2})^2}$

Solution

a.

$$\begin{aligned}(6m^2n^{-1})^3 &= (6)^3(m^2)^3(n^{-1})^3 && \text{The power of a product rule}\\ &= 6^3m^{2\times3}n^{-1\times3} && \text{The power rule}\\ &= 216m^6n^{-3} && \text{The power rule}\\ &= \frac{216m^6}{n^3} && \text{The negative exponent rule}\end{aligned}$$

b.

$$\begin{aligned} 17^5 \times 17^{-4} \times 17^{-3} &= 17^{5+(-4)+(-3)} && \text{The product rule} \\ &= 17^{-2} && \text{Simplify} \\ &= \frac{1}{17^2} \text{ or } \frac{1}{289} && \text{The negative exponent rule} \end{aligned}$$

To avoid making mistakes in subtracting negative numbers, it is easier to apply the Negative Exponent Rule before the Quotient Rule. Both approaches are illustrated below.

c.

$$\begin{aligned} \left(\frac{u^{-1}v}{v^{-1}}\right)^2 &= \frac{(u^{-1}v)^2}{(v^{-1})^2} && \text{The power of a quotient rule} \\ &= \frac{u^{-2}v^2}{v^{-2}} && \text{The power of a product rule} \end{aligned}$$

$= u^{-2}v^{2-(-2)}$	Quotient rule	$= \frac{v^2 v^2}{u^2}$	Negative Exponent Rule
$= u^{-2}v^4$	Simplify	$= \frac{v^{2+2}}{u^2}$	Product Rule
$= \frac{v^4}{u^2}$	Negative exponent rule	$= \frac{v^4}{u^2}$	Simplify

d.

$$\begin{aligned} \left(-2a^3b^{-1}\right)\left(5a^{-2}b^2\right) &= -2 \cdot a^3 \cdot b^{-1} \cdot 5 \cdot a^{-2} \cdot b^2 && \text{Associative law of multiplication} \\ &= -2 \cdot 5 \cdot a^3 \cdot a^{-2} \cdot b^{-1} \cdot b^2 && \text{Commutative law of multiplication} \\ &= (-2 \cdot 5) \cdot (a^3 \cdot a^{-2}) \cdot (b^{-1} \cdot b^2) && \text{Associative law of multiplication} \\ &= -10 \times a^{3+(-2)} \times b^{-1+2} && \text{The product rule} \\ &= -10ab && \text{Simplify} \end{aligned}$$

e.

$$\begin{aligned} \left(x^2\sqrt{2}\right)^4\left(x^2\sqrt{2}\right)^{-4} &= \left(x^2\sqrt{2}\right)^{4-4} && \text{The product rule} \\ &= \left(x^2\sqrt{2}\right)^0 && \text{Simplify} \\ &= 1 && \text{The zero exponent rule} \end{aligned}$$

f.

$$\begin{aligned} \frac{(3w^2)^5}{(6w^{-2})^2} &= \frac{(3)^5 \times (w^2)^5}{(6)^2 \times (w^{-2})^2} && \text{The power of a product rule} \\ &= \frac{3^5 w^{2\times 5}}{6^2 w^{-2\times 2}} && \text{The power rule} \\ &= \frac{243w^{10}}{36w^{-4}} && \text{Simplify} \\ &= \frac{243w^{10}w^4}{36} && \text{Negative exponent rule} \\ &= \frac{243w^{10+4}}{36} && \text{The product rule} \\ &= \frac{27w^{14}}{4} && \text{Reduce fraction} \end{aligned}$$

Try It 0.2.9x

Simplify each of the following exponential expressions. Write answers with positive exponents.

a. $(2uv^{-2})^{-3}$
b. $x^8 \cdot c^{-12} \cdot x$
c. $\left(\frac{e^2 f^{-3}}{f^{-1}}\right)^2$
d. $(9r^{-5}s^3)(3r^6s^{-4})$
e. $(\frac{4}{9}tw^{-2})^{-3}(\frac{4}{9}tw^{-2})^3$

f. $\frac{(2h^2k)^4}{(7h^{-1}k^2)^2}$

Answers

a. $\frac{v^6}{8u^3}$ b. $\frac{x^9}{c^{12}}$ c. $\frac{e^4}{f^4}$ d. $\frac{27r}{s}$ e. 1 f. $\frac{8h^{10}}{49}$

Working "Backwards"

All our previous examples had a constant as an exponent and a variable in the base, like x^2. Expressions like this are called power functions. In contrast, the base can be a constant and the exponent can have a variable in it, like 2^x. Expressions in this form are called exponential functions. When simplifying expressions with the variable in the exponent, we often use the Laws of Exponents "backwards". Some examples below illustrate this.

Example 0.2.10

Write the following exponential expressions so they use only one exponent, and that exponent is just x.

1. 4^{x+2}	2. 5^{x-3}	3. $6^x \cdot 7^x$	4. $\frac{9^x}{8^x}$	5. 4^{2x}

Solution

1. Using the Product Rule in reverse we obtain $4^{x+2} = 4^x \cdot 4^2 = 16 \cdot 4^x$.
2. Using the Product Rule in reverse we obtain $5^{x-3} = 5^x \cdot 5^{-3} = \frac{5^x}{5^3} = \frac{5^x}{125}$
3. Using the Power Rule in reverse we obtain $6^x \cdot 7^x = (6 \times 7)^x = 42^x$
4. Using the Power Rule in reverse we obtain $\frac{9^x}{8^x} = \left(\frac{9}{8}\right)^x = 1.125^x$
5. Using the Power Rule in reverse we obtain $4^{2x} = (4^2)^x = 16^x$. Notice the difference between this expression and 4^{x+2}!

In (3) and (4) we are using the 'like exponents rule" - if you have like exponents, then you multiply the bases and keep the exponent the same. Somewhat conversely, the Product Rule could be called the "like bases rule" - if you have like bases, then you add the exponents and keep the base the same.

Key Equations

Rules of Exponents For nonzero real numbers a and b and integers m and n	
Product rule	$a^m \cdot a^n = a^{m+n}$
Quotient rule	$\frac{a^m}{a^n} = a^{m-n}$
Power rule	$(a^m)^n = a^{m \cdot n}$
Zero exponent rule	$a^0 = 1$
Negative rule	$a^{-n} = \frac{1}{a^n}$
Power of a product rule	$(a \cdot b)^n = a^n \cdot b^n$
Power of a quotient rule	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Key Concepts

- Products of exponential expressions with the same base can be simplified by adding exponents.
- Quotients of exponential expressions with the same base can be simplified by subtracting exponents.
- Powers of exponential expressions with the same base can be simplified by multiplying exponents.
- An expression with exponent zero is defined as 1.
- An expression with a negative exponent is defined as a reciprocal.
- The power of a product of factors is the same as the product of the powers of the same factors.

- The power of a quotient of factors is the same as the quotient of the powers of the same factors.
- The rules for exponential expressions can be combined to simplify more complicated expressions.

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0.02e: Exercises - Whole number exponents

A: Zero and Negative Exponents

Exercise 0.2.1

★ **Simplify. (Assume all variables represent nonzero numbers.)**

1. $-5x^0$	3. $-2x^{-3}$	5. $10x^{-3}y^2$	7. $3x^{-2}y^2z^{-1}$
2. $3x^2y^0$	4. $(-2x)^{-2}$	6. $-3x^{-5}y^{-2}$	8. $-5x^{-4}y^{-2}z^2$

Answers to odd exercises.

1. -5	3. $-\frac{2}{x^3}$	5. $\frac{10y^2}{x^3}$	7. $\frac{3y^2}{x^2z}$

B: Product Rule

Exercise 0.2.2

★ **Simplify. (Assume all variables represent nonzero numbers.)**

11. $10^4 \cdot 10^7$	15. $a^4 \cdot a^{-5} \cdot a^2$	19. $-6x^2yz^3 \cdot 3xyz^4$	23. $(2x+3)^4(2x+3)^9$
12. $7^3 \cdot 7^2$	16. $b^{-8} \cdot b^3 \cdot b^4$	20. $2xyz^2\left(-4x^2y^2z\right)$	24. $(3y-1)^7(3y-1)^2$
13. $x^3 \cdot x^2$	17. $5x^2y \cdot 3xy^2$	21. $3x^ny^{2n} \cdot 5x^2y$	25. $(a+b)^3(a+b)^5$
14. $y^5 \cdot y^3$	18. $-10x^3y^2 \cdot 2xy$	22. $8x^{5n}y^n \cdot 2x^{2n}y$	26. $(x-2y)^7(x-2y)^3$

Answers to odd exercises.

11. 10^{11}	15. a	19. $-18x^3y^2z^7$	23. $(2x+3)^{13}$
13. x^5	17. $15x^3y^3$	21. $15x^{n+2}y^{2n+1}$	25. $(a+b)^8$

C: Quotient Rule

Exercise 0.2.3

★ **Simplify. (Assume all variables represent nonzero numbers.)**

31. $\frac{10^2 \cdot 10^4}{10^5}$	35. $\frac{a^8 \cdot a^{-3}}{a^{-6}}$	39. $\frac{40x^5y^3z}{4x^2y^2z}$	43. $\frac{x^{2n} \cdot x^{3n}}{x^n}$
32. $\frac{7^5 \cdot 7^9}{7^2}$	36. $\frac{b^{-10} \cdot b^4}{b^{-2}}$	40. $\frac{8x^2y^5z^3}{16x^2yz}$	44. $\frac{x^n \cdot x^{8n}}{x^{3n}}$
33. $\frac{a^8 \cdot a^6}{a^5}$	37. $\frac{25x^{-3}y^2}{5x^{-1}y^{-3}}$	41. $\frac{24a^8b^3(a-5b)^{10}}{8a^5b^3(a-5b)^2}$	
34. $\frac{b^4 \cdot b^{10}}{b^8}$	38. $\frac{-9x^{-1}y^3z^{-5}}{3x^{-2}y^2z^{-1}}$	42. $\frac{175m^9n^5(m+n)^7}{25m^8n(m+n)^3}$	

Answers to odd exercises.

31. 10
33. a^9 35. a^{11}
37. $\frac{5y^5}{x^2}$ 39. $10x^3y$
41. $3a^3(a-5b)^8$ 43. x^{4n}

D: Power Rule for Products

Exercise 0.2.4

★ **Simplify. (Assume all variables represent nonzero numbers.)**

51. $(x^5)^3$
52. $(y^4)^3$
53. $(x^4y^5)^3$
54. $(x^7y)^5$
55. $(-5x)^0$
56. $(3x^2y)^0$
57. $(x^2y^3z^4)^4$
58. $(xy^2z^3)^2$
59. $(-5x^2yz^3)^2$
60. $(-2xy^3z^4)^5$
61. $(x\cdot x^3\cdot x^2)^3$
62. $(y^2\cdot y^5\cdot y)^2$
63. $(-2x^4y^2z)^6$
64. $(-3xy^4z^7)^5$
65. $(-2a^2b^0c^3)^5$
66. $(-3a^4b^2c^0)^4$
67. $(-5x^{-3}y^2z)^{-3}$
68. $(-7x^2y^{-5}z^{-2})^{-2}$
69. $(x^2yz^5)^n$
70. $(xy^2z^3)^{2n}$
71. $\frac{a^2\cdot(a^4)^2}{a^3}$
72. $\frac{a\cdot a^3\cdot a^2}{(a^2)^3}$
73. $\frac{(9x^3y^2z^0)^2}{3xy^2}$
74. $\frac{(-5x^0y^5z)^3}{25y^2z^0}$

Answers to odd exercises.

51. x^{15}
53. $x^{12}y^{15}$
55. 1
57. $x^8y^{12}z^{16}$
59. $25x^4y^2z^6$
61. x^{18}
63. $64x^{24}y^{12}z^6$
65. $-32a^{10}c^{15}$
67. $-\frac{x^9}{125y^6z^3}$
69. $x^{2n}y^nz^{5n}$
71. a^7
73. $27x^5y^2$

E: Power Rule for Quotients of Products

Exercise 0.2.5

★ **Simplify. (Assume all variables represent nonzero numbers.)**

81. $\left(\frac{-3ab^2}{2c^3}\right)^3$
82. $\left(\frac{-10a^3b}{3c^2}\right)^2$
83. $\left(\frac{-2xy^4}{z^3}\right)^4$
84. $\left(\frac{-7x^9y}{z^4}\right)^3$
85. $\left(\frac{12x^3y^2z}{2x^7yz^8}\right)^3$
86. $\left(\frac{150xy^8z^2}{90x^7y^2z}\right)^2$
87. $\left(\frac{2x^{-3}z}{y^2}\right)^{-5}$
88. $\left(\frac{5x^5z^{-2}}{2y^{-3}}\right)^{-3}$
89. $\left(\frac{xy^2}{z^3}\right)^n$
90. $\left(\frac{2x^2y^3}{z}\right)^n$
91. $\left(\frac{-9a^{-3}b^4c^{-2}}{3a^3b^5c^{-7}}\right)^{-4}$
92. $\left(\frac{-15a^7b^5c^{-8}}{3a^{-6}b^2c^3}\right)^{-3}$

Answers to odd exercises:

81. $-\frac{27a^3b^6}{8c^9}$	83. $\frac{16x^4y^{16}}{z^{12}}$	85. $\frac{216y^3}{x^{12}z^{21}}$	87. $\frac{x^{15}y^{10}}{32z^5}$	89. $\frac{x^ny^{2n}}{z^{3n}}$	91. $\frac{a^{24}b^4}{81c^{20}}$

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0.03: Review - Radicals (Square Roots)

Evaluating Square Roots

When the square root of a number is squared, the result is the original number. Since $4^2 = 16$, the square root of 16 is 4.The square root function is the inverse of the squaring function just as subtraction is the inverse of addition. To undo squaring, we take the square root.

In general terms, if a is a positive real number, then the square root of a is a number that, when multiplied by itself, gives a. The square root could be positive or negative because multiplying two negative numbers gives a positive number. However, the $\sqrt{\ }$ symbol denotes just the non-negative result, or so-called **principal square root**. The square root obtained using a calculator is the principal square root.

The principal square root is the nonnegative number that when multiplied by itself equals a. The principal square root of a is written as $\sqrt{a}$. The symbol is called a radical, the term under the symbol is called the radicand, and the entire expression is called a **radical expression**.



Example 0.3.1

Does $\sqrt{25} = \pm 5$?

Solution

No. Although both 5^2 and $(-5)^2$ are 25, the radical symbol implies only a *nonnegative* root, the principal square root. The principal square root of 25 is $\sqrt{25} = 5$.

Definition: Principal Square Root

The **principal square root** of a is the nonnegative number that, when multiplied by itself, equals a. It is written as a radical expression $\sqrt{a}$, with the symbol called a ***radical***, over the term a, called the ***radicand***. $\sqrt{a}$.

Example 0.3.2: Evaluating Square Roots

Evaluate each expression.

a. $\sqrt{100}$
b. $\sqrt{\sqrt{16}}$
c. $\sqrt{25+144}$
d. $\sqrt{49}$-$\sqrt{81}$

Solution

a. $\sqrt{100} = 10$ because $10^2 = 100$
b. $\sqrt{\sqrt{16}} = \sqrt{4} = 2$ because $4^2 = 16$ and $2^2 = 4$
c. $\sqrt{25+144} = \sqrt{169} = 13$ because $13^2 = 169$
d. $\sqrt{49} - \sqrt{81} = 7 - 9 = -2$ because $7^2 = 49$ and $9^2 = 81$

Example 0.3.3:

For $\sqrt{25+144}$, can we find the square roots before adding?

Solution

No. $\sqrt{25}+\sqrt{144}=5+12=17$. This is not equivalent to $\sqrt{25+144}=\sqrt{169}=13$. The order of operations requires us to add the terms of the radicand together before finding the square root.

Try It 0.3.3

Evaluate each expression.

a. $\sqrt{25}$	b. $\sqrt{\sqrt{81}}$	c. $\sqrt{25-9}$	d. $\sqrt{36}+\sqrt{121}$

Answers

a. 5 b. 3 c.4 d. 17

Using the Product Rule to Simplify Square Roots

To simplify a square root, we rewrite it such that there are no perfect squares in the radicand. There are several properties of square roots that allow us to simplify complicated radical expressions. The first rule we will look at is the product rule for simplifying square roots, which allows us to separate the square root of a product of two numbers into the product of two separate rational expressions. For instance, we can rewrite $\sqrt{15}$ as $\sqrt{3}\times\sqrt{5}$. We can also use the product rule to express the product of multiple radical expressions as a single radical expression.

The Product Rule For Simplifying Square Roots

If a and b are nonnegative, the square root of the product ab is equal to the product of the square roots of a and b

$$\sqrt{ab}=\sqrt{a}\times\sqrt{b}$$

How to: Simplify the square root of a product.

1. Factor any perfect squares from the radicand.
2. Write the radical expression as a product of radical expressions.
3. Simplify.

Example 0.3.4: Using the Product Rule to Simplify Square Roots

Simplify the radical expression.

a. $\sqrt{300}$
b. $\sqrt{162a^5b^4}$

Solution

a. $\sqrt{100\times3}$ Factor perfect square from radicand.

$\sqrt{100}\times\sqrt{3}$ Write radical expression as product of radical expressions.

$10\sqrt{3}$ Simplify

b. $\sqrt{81a^4b^4\times2a}$ Factor perfect square from radicand

$\sqrt{81a^4b^4}\times\sqrt{2a}$ Write radical expression as product of radical expressions

$9a^2b^2\sqrt{2a}$ Simplify

Try It 0.3.4

Simplify $\sqrt{50x^2y^3z}$

Answer

$5|x||y|\sqrt{2yz}$

Notice the absolute value signs around x and y. That's because their value must be positive!

How to: Simplify the product of multiple radical expressions

1. Express the product of multiple radical expressions as a single radical expression.
2. Simplify.

Example 0.3.5: Using the Product Rule to Simplify the Product of Multiple Square Roots

Multiply. Simplify the radical expression.

a. $\sqrt{12} \times \sqrt{3}$

b. $\sqrt{6x^3y^3} \times \sqrt{2x^3}$.

Solution

a. **Express the product as a single radical expression:** $\sqrt{12 \times 3} = \sqrt{36} = 6$

b. Begin by writing as a single radical expression: $\sqrt{12x^6y^3}$.

Determine the square factors of $12, x^6$, and y^3.

$$\left.\begin{aligned} 12 &= 2^2 \cdot 3 \\ x^6 &= (x^3)^2 \\ y^3 &= y^2 \cdot y \end{aligned}\right\} \quad \textit{Square factors}$$

Make these substitutions, and then apply the product rule for radicals and simplify.

$$\begin{aligned} \sqrt{12x^6y^3} &= \sqrt{2^2 \cdot 3 \cdot (x^3)^2 \cdot y^2 \cdot y} && \textit{Apply the product rule for radicals.} \\ &= \sqrt{2^2} \cdot \sqrt{(x^3)^2} \cdot \sqrt{y^2} \cdot \sqrt{3y} && \textit{Simplify.} \\ &= 2 \cdot x^3 \cdot |y| \cdot \sqrt{3y} \\ &= 2|x^3y|\sqrt{3y} \end{aligned}$$

Notice the absolute value bars around x^3y. That is because the result of taking a square root is never negative.

Try It 0.3.5

Simplify $\sqrt{50x} \times \sqrt{2x}$ assuming $x > 0$.

Answer

$10|x|$

Using the Quotient Rule to Simplify Square Roots

Just as we can rewrite the square root of a product as a product of square roots, so too can we rewrite the square root of a quotient as a quotient of square roots, using the quotient rule for simplifying square roots.

THE QUOTIENT RULE FOR SIMPLIFYING SQUARE ROOTS

The square root of the quotient $\frac{a}{b}$ is equal to the quotient of the square roots of a and b, where $a \geq 0$ and $b > 0$.

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

How to: Simplify the radical of a quotient

1. Simplify the quotient radicand.
2. Write the simplified quotient radical expression as the quotient of two radical expressions.
3. Simplify the numerator and denominator.

Example 0.3.6: Using the Quotient Rule to Simplify Square Roots

Simplify the radical expression.

a. $\sqrt{\frac{5}{36}}$

b. $\sqrt{\frac{18a^5}{b^8}}$

Solution

a. Write as a quotient of two radical expressions and then simplify the denominator: $\frac{\sqrt{5}}{\sqrt{36}} = \frac{\sqrt{5}}{6}$

b. Begin by determining the square factors of 18, a^5, and b^8.

$$\left.\begin{aligned} 18 &= 2 \cdot 3^2 \\ a^5 &= a^2 \cdot a^2 \cdot a = (a^2)^2 \cdot a \\ b^8 &= b^4 \cdot b^4 \quad = (b^4)^2 \end{aligned}\right\} \text{ Square factors}$$

Make these substitutions, apply the product and quotient rules for radicals, and then simplify.

$$\begin{aligned} \sqrt{\frac{18a^5}{b^8}} &= \sqrt{\frac{2 \cdot 3^2 \cdot (a^2)^2 \cdot a}{(b^4)^2}} && \text{Apply the product and quotient rule for radicals.} \\ &= \frac{\sqrt{3^2} \cdot \sqrt{(a^2)^2} \cdot \sqrt{2a}}{\sqrt{(b^4)^2}} && \text{Simplify.} \\ &= \frac{3a^2\sqrt{2a}}{b^4} \end{aligned}$$

Try It 0.3.6

Simplify $\sqrt{\frac{2x^2}{9y^4}}$

Answer

$\frac{|x|\sqrt{2}}{3y^2}$ We do not need the absolute value signs for y^2 because that term will always be nonnegative.

How to: Simplify the quotient of multiple radical expressions

1. Express the quotient of multiple radical expressions as a quotient of a single radical expression.
2. Simplify the radical.

Example 0.3.7: Use the Quotient Rule to Simplify the Quotient of Two Square Roots

Divide. Simplify the radical expression.

a. $\frac{\sqrt{234x^{11}y}}{\sqrt{26x^7y}}$

b. $\frac{\sqrt{50x^6y^4}}{\sqrt{8x^3y}}$

Solution

a. First combine numerator and denominator into one radical expression

$$\frac{\sqrt{234x^{11}y}}{\sqrt{26x^7y}} = \sqrt{\frac{234x^{11}y}{26x^7y}} \quad \text{Write under one radical}$$

$$= \sqrt{9x^4} \quad \text{Simplify radicand}$$

$$= 3x^2 \quad \text{Simplify square root}$$

b. Write as a single square root and cancel common factors before simplifying.

$$\frac{\sqrt{50x^6y^4}}{\sqrt{8x^3y}} = \sqrt{\frac{50x^6y^4}{8x^3y}} \quad \textit{Apply the quotient rule for radicals and cancel.}$$

$$= \sqrt{\frac{25x^3y^3}{4}} \quad \textit{Simplify.}$$

$$= \frac{\sqrt{25x^3y^3}}{\sqrt{4}}$$

$$= \frac{5|xy|\sqrt{xy}}{2}$$

Try It 0.3.7

Simplify $\frac{\sqrt{9a^5b^{14}}}{\sqrt{3a^4b^5}}$

Answer

$b^4\sqrt{3ab}$

From this point on the assumption will be made that all variables represent non-negative real numbers. Therefore using absolute values when simplifying will not be necessary.

Adding and Subtracting Square Roots

We can add or subtract radical expressions only when they have the same radicand and when they have the same radical type such as square roots. For example, the sum of $\sqrt{2}$ and $3\sqrt{2}$ is $4\sqrt{2}$. However, it is often possible to simplify radical expressions, and that may change the radicand. The radical expression $\sqrt{18}$ can be written with a 2 in the radicand, as $3\sqrt{2}$, so $\sqrt{2}+\sqrt{18}=\sqrt{2}+3\sqrt{2}=4\sqrt{2}$

How to: Simplify a radical expression requiring addition or subtraction of square roots

1. Simplify each radical expression.
2. Add or subtract expressions with equal radicands.

Often, we will have to simplify before we can identify the like radicals within the terms. If the radicand and the index are not exactly the same, then the radicals are not similar and we cannot combine them.

Example 0.3.8: Add or Subtract Square Roots

Perform the indicated operation and simplify.

a. $4\sqrt{10}-5\sqrt{10}$
b. $5\sqrt{12}+2\sqrt{3}$.
c. $20\sqrt{72a^3b^4c}-14\sqrt{8a^3b^4c}$

Solution

a.
$$\begin{aligned}&=(4-5)\sqrt{10}\\&=-1\sqrt{10}\\&=-\sqrt{10}\end{aligned}$$

b. We can rewrite $5\sqrt{12}$ as $5\sqrt{4\times 3}$. According the product rule, this becomes $5\sqrt{4}\sqrt{3}$. The square root of $\sqrt{4}$ is 2 , so the expression becomes $5\times 2\sqrt{3}$, which is $10\sqrt{3}$. Now we can the terms have the same radicand so we can add.

$$10\sqrt{3}+2\sqrt{3}=12\sqrt{3}$$

c. Rewrite each term so they have equal radicands.

$$\begin{aligned}20\sqrt{72a^3b^4c}&=20\sqrt{9}\sqrt{4}\sqrt{2}\sqrt{a}\sqrt{a^2}\sqrt{(b^2)^2}\sqrt{c}\\&=20(3)(2)|a|b^2\sqrt{2ac}\\&=120|a|b^2\sqrt{2ac}\end{aligned}$$

$$\begin{aligned}14\sqrt{8a^3b^4c}&=14\sqrt{2}\sqrt{4}\sqrt{a}\sqrt{a^2}\sqrt{(b^2)^2}\sqrt{c}\\&=14(2)|a|b^2\sqrt{2ac}\\&=28|a|b^2\sqrt{2ac}\end{aligned}$$

Now the terms have the same radicand so we can subtract.

$$120|a|b^2\sqrt{2ac}-28|a|b^2\sqrt{2ac}=92|a|b^2\sqrt{2ac} \tag{0.3.1}$$

Try It 0.3.8

a. Add $\sqrt{5}+6\sqrt{20}$
b. Subtract $3\sqrt{80x}-4\sqrt{45x}$

Answers

a. $13\sqrt{5}$ b. 0

Example 0.3.9: Adding and Subtracting Square Roots

Simplify.

a. $10\sqrt{5}+6\sqrt{2}-9\sqrt{5}-7\sqrt{2}$
b. $\sqrt{32}-\sqrt{18}+\sqrt{50}$
c. $2a\sqrt{125a^2b}-a^2\sqrt{80b}+4\sqrt{20a^4b}$.

Solution

a.

$$\begin{aligned}&10\sqrt{5}+6\sqrt{2}-9\sqrt{5}-7\sqrt{2}\\&=10\sqrt{5}-9\sqrt{5}+6\sqrt{2}-7\sqrt{2}\\&=\sqrt{5}-\sqrt{2}\end{aligned}$$

We cannot simplify any further because $\sqrt{5}$ and $\sqrt{2}$ are not like radicals; the radicands are not the same.

Caution: It is important to point out that $\sqrt{5}-\sqrt{2}\neq\sqrt{5-2}$. We can verify this by calculating the value of each side with a calculator.

$$\sqrt{5}-\sqrt{2}\approx 0.82 \qquad \text{is not the same as} \qquad \sqrt{5-2}=\sqrt{3}\approx 1.73$$

b.

$$\begin{aligned}&\sqrt{32}-\sqrt{18}+\sqrt{50}\\&=\sqrt{16\cdot 2}-\sqrt{9\cdot 2}+\sqrt{25\cdot 2}\\&=4\sqrt{2}-3\sqrt{2}+5\sqrt{2}\\&=6\sqrt{2}\end{aligned}$$

At first glance, the radicals do not appear to be similar. However, after simplifying completely, we see that we can combine them.

c. **Step 1**: Simplify the radical expression.
Step 2: Combine all like radicals. Remember to add only the coefficients; the variable parts remain the same.

$$\begin{aligned}&2a\sqrt{125a^2b}-a^2\sqrt{80b}+4\sqrt{20a^4b}\\&=2a\sqrt{25\cdot 5\cdot a^2\cdot b}-a^2\sqrt{16\cdot 5\cdot b}+4\sqrt{4\cdot 5\cdot (a^2)^2b} && \textit{Factor.}\\&=2a\cdot 5\cdot a\sqrt{5b}-a^2\cdot 4\sqrt{5b}+4\cdot 2\cdot a^2\sqrt{5b} && \textit{Simplify.}\\&=10a^2\sqrt{5b}-4a^2\sqrt{5b}+8a^2\sqrt{5b} && \textit{Combine like terms.}\\&=14a^2\sqrt{5b}\end{aligned}$$

Try It 0.3.9

Simplify: $\sqrt{20}+\sqrt{27}-3\sqrt{5}-2\sqrt{12}$.

Answer

$-\sqrt{5}-\sqrt{3}$

Caution: Simplifying Radicals

Take careful note of the differences between products and sums within a radical. Assume both x and y are nonnegative.

Products	*Sums*
$\sqrt{x^2y^2}=xy$	$\sqrt{x^2+y^2}\neq x+y$

The property says that we can simplify radicals when the operation in the radicand is multiplication. There is no corresponding property for addition.

Simplifying Products of Expressions containing Square Roots

Often, there will be coefficients in front of the radicals.

Example 0.3.10.1x:

Multiply: $3\sqrt{6}\cdot 5\sqrt{2}$

Solution

Using the product rule for radicals and the fact that multiplication is commutative, we can multiply the coefficients and the radicands as follows.

$$\begin{aligned} 3\sqrt{6}\cdot 5\sqrt{2} &= 3\cdot 5\cdot\sqrt{6}\cdot\sqrt{2} && \textit{Multiplication is commutative.}\\ &= 15\cdot\sqrt{12} && \textit{Multiply the coefficients and the radicands.}\\ &= 15\sqrt{4\cdot 3} && \textit{Simplify.}\\ &= 15\cdot 2\cdot\sqrt{3}\\ &= 30\sqrt{3} \end{aligned}$$

Typically, the first step involving the application of the commutative property is not shown.

We will often find the need to subtract a radical expression with multiple terms. If this is the case, remember to apply the distributive property before combining like terms.

Example 0.3.10.2x: Use the distributive property with square roots

Simplify: $(5\sqrt{x}-4\sqrt{y})-(4\sqrt{x}-7\sqrt{y})$ **Solution:**

$$\begin{aligned} &= 5\sqrt{x}-4\sqrt{y}-4\sqrt{x}+7\sqrt{y} && \textit{Distribute.}\\ &= 5\sqrt{x}-4\sqrt{x}-4\sqrt{y}+7\sqrt{y}\\ &= \sqrt{x}+3\sqrt{y} \end{aligned}$$

Use the distributive property when multiplying rational expressions with more than one term.

Example 0.3.10.3x:

Multiply: $5\sqrt{2x}(3\sqrt{x}-\sqrt{2x})$.

Solution:

Apply the distributive property and multiply each term by $5\sqrt{2x}$.

$$\begin{aligned} 5\sqrt{2x}(3\sqrt{x}-\sqrt{2x}) &= 5\sqrt{2x}\cdot 3\sqrt{x}-5\sqrt{2x}\cdot\sqrt{2x} && \textit{Distribute.}\\ &= 15\sqrt{2x^2}-5\sqrt{4x^2} && \textit{Simplify.}\\ &= 15x\sqrt{2}-5\cdot 2x\\ &= 15x\sqrt{2}-10x \end{aligned}$$

The process for multiplying radical expressions with multiple terms is the same process used when multiplying polynomials. Apply the distributive property, simplify each radical, and then combine like terms.

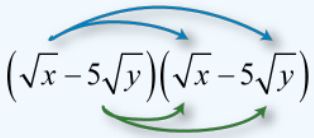
Example 0.3.10.4x:

Multiply: $(\sqrt{x}-5\sqrt{y})^2$.

Solution

$$(\sqrt{x}-5\sqrt{y})^2=(\sqrt{x}-5\sqrt{y})(\sqrt{x}-5\sqrt{y})$$

Begin by applying the distributive property.



$$= \sqrt{x}\cdot\sqrt{x} + \sqrt{x}(-5\sqrt{y}) + (-5\sqrt{y})\sqrt{x} + (-5\sqrt{y})(-5\sqrt{y})$$
$$= \sqrt{x^2} - 5\sqrt{xy} - 5\sqrt{xy} + 25\sqrt{y^2}$$
$$= x - 10\sqrt{xy} + 25y$$

The binomials $(a+b)$ and $(a-b)$ are called **conjugates**. When multiplying conjugate binomials the middle terms are opposites and their sum is zero.

Example 0.3.10.5x:

Multiply: $(\sqrt{10}+\sqrt{3})(\sqrt{10}-\sqrt{3})$.

Solution

Apply the distributive property, and then combine like terms.

$$\begin{aligned}(\sqrt{10}+\sqrt{3})(\sqrt{10}-\sqrt{3}) &= \sqrt{10}\cdot\sqrt{10} + \sqrt{10}(-\sqrt{3}) + \sqrt{3}(\sqrt{10}) + \sqrt{3}(-\sqrt{3}) \\ &= \sqrt{100} - \sqrt{30} + \sqrt{30} - \sqrt{9} \\ &= 10 - \sqrt{30} + \sqrt{30} - 3 \\ &= 10 - 3 \\ &= 7\end{aligned}$$

It is important to note that when multiplying conjugate radical expressions, we obtain a rational expression. This is true in general

$$\begin{aligned}(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y}) &= \sqrt{x^2} - \sqrt{xy} + \sqrt{xy} - \sqrt{y^2} \\ &= x - y\end{aligned}$$

Alternatively, using the formula for the difference of squares we have,

$$(a+b)(a-b) = a^2 - b^2 \qquad \textit{Difference of squares.}$$
$$\begin{aligned}(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y}) &= (\sqrt{x})^2 - (\sqrt{y})^2 \\ &= x - y\end{aligned}$$

Try It 0.3.10x

Multiply: $(3-2\sqrt{y})(3+2\sqrt{y})$. (Assume y is positive).

Answer

$9-4y$

Rationalizing Denominators

When an expression involving square root radicals is written in simplest form, it will not contain a radical in the denominator. We can remove radicals from the denominators of fractions using a process called rationalizing the denominator.

We know that multiplying by 1 does not change the value of an expression. We use this property of multiplication to change expressions that contain radicals in the denominator. To remove radicals from the denominators of fractions, multiply by the form of 1 that will eliminate the radical.

Monomial Denominators

For a denominator containing a single term, multiply by the radical in the denominator over itself. In other words, if the denominator is $b\sqrt{c}$, multiply by $\frac{\sqrt{c}}{\sqrt{c}}$.

How To: Rationalize the denominator of an expression with a monomial denominator

1. Multiply the numerator and denominator by the radical in the denominator.
2. Simplify.

Sometimes, we will find the need to reduce, or cancel, after rationalizing the denominator.

Example 0.3.11: Rationalize a Denominator Containing a Single Term

Write in simplest form (rationalize the denominator).

a. $\frac{2\sqrt{3}}{3\sqrt{10}}$ b. $\frac{\sqrt{2}}{\sqrt{5x}}$ c. $\frac{3a\sqrt{2}}{\sqrt{6ab}}$

Solution

a. The radical in the denominator is $\sqrt{10}$. So multiply the fraction by $\frac{\sqrt{10}}{\sqrt{10}}$. Then simplify.

$$\frac{2\sqrt{3}}{3\sqrt{10}} \times \frac{\sqrt{10}}{\sqrt{10}} = \frac{2\sqrt{30}}{30} = \frac{\sqrt{30}}{15}$$

b. The goal is to find an equivalent expression without a radical in the denominator. The radicand in the denominator determines the factors that you need to use to rationalize it. In this example, multiply by 1 in the form $\frac{\sqrt{5x}}{\sqrt{5x}}$.

$$\begin{aligned}\frac{\sqrt{2}}{\sqrt{5x}} &= \frac{\sqrt{2}}{\sqrt{5x}} \cdot \frac{\sqrt{5x}}{\sqrt{5x}} \\ &= \frac{\sqrt{10x}}{\sqrt{25x^2}} \qquad \textit{Simplify.} \\ &= \frac{\sqrt{10x}}{5x}\end{aligned}$$

c. In this example, we will multiply by 1 in the form $\frac{\sqrt{6ab}}{\sqrt{6ab}}$.

$$\begin{aligned}\frac{3a\sqrt{2}}{\sqrt{6ab}} &= \frac{3a\sqrt{2}}{\sqrt{6ab}} \cdot \frac{\sqrt{6ab}}{\sqrt{6ab}} \\ &= \frac{3a\sqrt{12ab}}{\sqrt{36a^2b^2}} \qquad \textit{Simplify.} \\ &= \frac{3a\sqrt{4 \cdot 3ab}}{6ab} \\ &= \frac{6a\sqrt{3ab}}{6ab} \qquad \textit{Cancel.} \\ &= \frac{\sqrt{3ab}}{b}\end{aligned}$$

Notice that b does not cancel in this example. Do not cancel factors inside a radical with those that are outside.

Try It 0.3.11

Write in simplest form (rationalize the denominator).

a. $\frac{12\sqrt{3}}{\sqrt{2}}$ b. $\sqrt{\frac{9x}{2y}}$

Answer

a. $6\sqrt{6}$ b. $\frac{3\sqrt{2xy}}{2y}$

Binomial Denominators

For a denominator containing the sum or difference of rational or irrational terms, multiply the numerator and denominator by the conjugate of the denominator, which is found by changing the sign that connects the two terms in the denominator. If the denominator is $a+b\sqrt{c}$, then the conjugate is $a-b\sqrt{c}$.

How to: Rationalize the denominator of an expression with a binomial denominator

1. Find the conjugate of the denominator.
2. Multiply the numerator and denominator by the conjugate.
3. Use the distributive property.
4. Simplify.

Example 0.3.12: Rationalizing a Denominator Containing Two Terms

Write in simplest form (rationalize the denominator).

a. $\frac{4}{1+\sqrt{5}}$	b. $\frac{1}{\sqrt{5}-\sqrt{3}}$	c. $\frac{\sqrt{10}}{\sqrt{2}+\sqrt{6}}$	d. $\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$

Solution

a. Begin by finding the conjugate of the denominator by writing the denominator and changing the sign. So the conjugate of $1+\sqrt{5}$ is $1-\sqrt{5}$. Then multiply the fraction by $\frac{1-\sqrt{5}}{1-\sqrt{5}}$.

$$\frac{4}{1+\sqrt{5}} \times \frac{1-\sqrt{5}}{1-\sqrt{5}}$$

$$\frac{4-4\sqrt{5}}{-4} \qquad \text{Use the distributive property}$$

$$-1+\sqrt{5} \qquad \text{Simplify}$$

b. In this example, the conjugate of the denominator is $\sqrt{5}+\sqrt{3}$. Therefore, multiply by 1 in the form $\frac{(\sqrt{5}+\sqrt{3})}{(\sqrt{5}+\sqrt{3})}$.

$$\begin{aligned}\frac{1}{\sqrt{5}-\sqrt{3}} &= \frac{1}{(\sqrt{5}-\sqrt{3})}\frac{(\sqrt{5}+\sqrt{3})}{(\sqrt{5}+\sqrt{3})} \quad \textit{Multiply by "1".}\\ &= \frac{\sqrt{5}+\sqrt{3}}{\sqrt{25}+\sqrt{15}-\sqrt{15}-\sqrt{9}} \quad \textit{Simplify.}\\ &= \frac{\sqrt{5}+\sqrt{3}}{5-3}\\ &= \frac{\sqrt{5}+\sqrt{3}}{2}\end{aligned}$$

Notice that the terms involving the square root in the denominator are eliminated by multiplying by the conjugate. We can use the property $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=a-b$ to expedite the process of multiplying the expressions in the denominator.

c. Multiply by 1 in the form $\frac{\sqrt{2}-\sqrt{6}}{\sqrt{2}-\sqrt{6}}$.

$$\frac{\sqrt{10}}{\sqrt{2}+\sqrt{6}} = \frac{(\sqrt{10})}{(\sqrt{2}+\sqrt{6})}\frac{(\sqrt{2}-\sqrt{6})}{(\sqrt{2}-\sqrt{6})} \quad \textit{Multiple by the conjugate.}$$

$$= \frac{\sqrt{20}-\sqrt{60}}{2-6} \quad \textit{Simplify.}$$

$$= \frac{\sqrt{4\cdot 5}-\sqrt{4\cdot 15}}{-4}$$

$$= \frac{2\sqrt{5}-2\sqrt{15}}{-4}$$

$$= \frac{2(\sqrt{5}-\sqrt{15})}{-4}$$

$$= \frac{\sqrt{5}-\sqrt{15}}{-2} = -\frac{\sqrt{5}-\sqrt{15}}{2} = \frac{-\sqrt{5}+\sqrt{15}}{2}$$

d. In this example, we will multiply by 1 in the form $\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}-\sqrt{y}}$.

$$\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}} = \frac{(\sqrt{x}-\sqrt{y})}{(\sqrt{x}+\sqrt{y})}\frac{(\sqrt{x}-\sqrt{y})}{(\sqrt{x}-\sqrt{y})} \quad \textit{Multiply by the conjugate of the denominator.}$$

$$= \frac{\sqrt{x^2}-\sqrt{xy}-\sqrt{xy}+\sqrt{y^2}}{x-y} \quad \textit{Simplify.}$$

$$= \frac{x-2\sqrt{xy}+y}{x-y}$$

Try It 0.3.12

Write in simplest form (rationalize the denominator).

a. $\frac{7}{2+\sqrt{3}}$

b. $\frac{2\sqrt{3}}{5-\sqrt{3}}$

Answer

a. $14-7\sqrt{3}$ b. $\frac{5\sqrt{3}+3}{11}$

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0.03e: Exercises - Square Roots

A: Simplify Square Roots

Exercise 0.3.1

★ Simplify. (Assume all variable expressions represent positive numbers, so absolute values is not needed.)

1. $\sqrt{9x^2}$	5. $\sqrt{4a^6}$	9. $\sqrt{49a^2}$	13. $\sqrt{180x^3}$	17. $\sqrt{45x^5y^3}$
2. $\sqrt{16y^2}$	6. $\sqrt{a^{10}}$	10. $\sqrt{64b^2}$	14. $\sqrt{150y^3}$	18. $\sqrt{50x^6y^4}$
3. $\sqrt{36a^4}$	7. $\sqrt{18a^4b^5}$	11. $\sqrt{x^2y^2}$	15. $\sqrt{49a^3b^2}$	19. $\sqrt{64r^2s^6t^5}$
4. $\sqrt{100a^8}$	8. $\sqrt{48a^5b^3}$	12. $\sqrt{25x^2y^2z^2}$	16. $\sqrt{4a^4b^3c}$	20. $\sqrt{144r^8s^6t^2}$

Answers to Odd Exercises:

1. $3x$	5. $2a^3$	9. $7a$	13. $6x\sqrt{5x}$	17. $3x^2y\sqrt{5xy}$
3. $6a^2$	7. $3a^2b^2\sqrt{2b}$	11. xy	15. $7ab\sqrt{a}$	19. $8rs^3t^2\sqrt{t}$

★ Simplify. Rationalize denominators. (Assume all variable expressions represent positive numbers, so absolute values is not needed.)

21. $\sqrt{(5x-4)^2}$	26. $\sqrt{9x^2+6x+1}$	31. $-3\sqrt{4x^2}$	36. $6a^2b\sqrt{9a^7b^2}$	40. $\sqrt{\frac{147m^9}{n^6}}$
22. $\sqrt{(3x-5)^4}$	27. $\sqrt{(x+1)^2}$	32. $7\sqrt{9y^2}$	37. $\sqrt{\frac{9x^3}{25y^2}}$	41. $\sqrt{\frac{2r^2s^5}{25t^4}}$
23. $\sqrt{x^2-6x+9}$	28. $\sqrt{(2x+3)^2}$	33. $-5x\sqrt{4x^2y}$	38. $\sqrt{\frac{4x^5}{9y^4}}$	42. $\sqrt{\frac{36r^5}{s^2t^6}}$
24. $\sqrt{x^2-10x+25}$	29. $\sqrt{4(3x-1)^2}$	34. $-3y\sqrt{16x^3y^2}$	39. $\sqrt{\frac{m^7}{36n^4}}$	
25. $\sqrt{4x^2+12x+9}$	30. $\sqrt{9(2x+3)^2}$	35. $12ab\sqrt{a^5b^3}$		

Answers to Odd Exercises:

21. $5x-4$ 23. $x-3$ 25. $2x+3$	27. $x+1$ 29. $2(3x-1)$ 31. $-6x$	33. $-10x^2\sqrt{y}$ 35. $12a^3b^2\sqrt{ab}$	37. $\frac{3x\sqrt{x}}{5y}$	39. $\frac{m^3\sqrt{m}}{6n^2}$	41. $\frac{rs^2\sqrt{2s}}{5t^2}$

B: Add and Subtract Square Roots of Constants.

Exercise 0.3.2

★ **Combine like terms**

45. $10\sqrt{3}-5\sqrt{3}$	50. $3\sqrt{10}-8\sqrt{10}-2\sqrt{10}$	55. $6\sqrt{5}-(4\sqrt{3}-3\sqrt{5})$
46. $15\sqrt{6}-8\sqrt{6}$	51. $\sqrt{6}-4\sqrt{6}+2\sqrt{6}$	56. $-12\sqrt{2}-(6\sqrt{6}+\sqrt{2})$
47. $9\sqrt{3}+5\sqrt{3}$	52. $5\sqrt{10}-15\sqrt{10}-2\sqrt{10}$	57. $(2\sqrt{5}-3\sqrt{10})-(\sqrt{10}+3\sqrt{5})$
48. $12\sqrt{6}+3\sqrt{6}$	53. $13\sqrt{7}-6\sqrt{2}-5\sqrt{7}+5\sqrt{2}$	58. $(-8\sqrt{3}+6\sqrt{15})-(\sqrt{3}-\sqrt{15})$
49. $4\sqrt{5}-7\sqrt{5}-2\sqrt{5}$	54. $10\sqrt{13}-12\sqrt{15}+5\sqrt{13}-18\sqrt{15}$	

Answers to Odd Exercises:

45. $5\sqrt{3}$	49. $-5\sqrt{5}$	53. $8\sqrt{7}-\sqrt{2}$	57. $-\sqrt{5}-4\sqrt{10}$
47. $14\sqrt{3}$	51. $-\sqrt{6}$	55. $9\sqrt{5}-4\sqrt{3}$	

★ **Simplify and combine like terms**

59. $\sqrt{75}-\sqrt{12}$	65. $\sqrt{45}-\sqrt{80}+\sqrt{245}-\sqrt{5}$	71. $3\sqrt{243}-2\sqrt{18}-\sqrt{48}$
60. $\sqrt{24}-\sqrt{54}$	66. $\sqrt{108}+\sqrt{48}-\sqrt{75}-\sqrt{3}$	72. $6\sqrt{216}-2\sqrt{24}-2\sqrt{96}$
61. $\sqrt{32}+\sqrt{27}-\sqrt{8}$	67. $4\sqrt{2}-(\sqrt{27}-\sqrt{72})$	73. $2\sqrt{18}-3\sqrt{75}-2\sqrt{98}+4\sqrt{48}$
62. $\sqrt{20}+\sqrt{48}-\sqrt{45}$	68. $-3\sqrt{5}-(\sqrt{20}-\sqrt{50})$	74. $2\sqrt{45}-\sqrt{12}+2\sqrt{20}-\sqrt{108}$
63. $\sqrt{28}-\sqrt{27}+\sqrt{63}-\sqrt{12}$	69. $2\sqrt{27}-2\sqrt{12}$	75. $(2\sqrt{363}-3\sqrt{96})-(7\sqrt{12}-2\sqrt{54})$
64. $\sqrt{90}+\sqrt{24}-\sqrt{40}-\sqrt{54}$	70. $3\sqrt{50}-4\sqrt{32}$	76. $(2\sqrt{288}+3\sqrt{360})-(2\sqrt{72}-7\sqrt{40})$

Answers to Odd Exercises:

59. $3\sqrt{3}$	63. $5\sqrt{7}-5\sqrt{3}$	67. $10\sqrt{2}-3\sqrt{3}$	71. $23\sqrt{3}-6\sqrt{2}$	75. $8\sqrt{3}-6\sqrt{6}$
61. $2\sqrt{2}+3\sqrt{3}$	65. $5\sqrt{5}$	69. $2\sqrt{3}$	73. $-8\sqrt{2}+\sqrt{3}$	

C: Add and Subtract Square Root Expressions.

Exercise 0.3.3

★ **Add. (Assume all radicands containing variable expressions are positive.)**

79. $\sqrt{2x}-4\sqrt{2x}$	83. $7x\sqrt{y}-3x\sqrt{y}+x\sqrt{y}$	87. $5\sqrt{xy}-(3\sqrt{xy}-7\sqrt{xy})$
80. $5\sqrt{3y}-6\sqrt{3y}$	84. $10y^2\sqrt{x}-12y^2\sqrt{x}-2y^2\sqrt{x}$	88. $-8a\sqrt{b}-(2a\sqrt{b}-4\sqrt{ab})$
81. $9\sqrt{x}+7\sqrt{x}$	85. $2\sqrt{ab}-5\sqrt{a}+6\sqrt{ab}-10\sqrt{a}$	89. $(3\sqrt{2x}-\sqrt{3x})-(\sqrt{2x}-7\sqrt{3x})$
82. $-8\sqrt{y}+4\sqrt{y}$	86. $-3x\sqrt{y}+6\sqrt{y}-4x\sqrt{y}-7\sqrt{y}$	90. $(\sqrt{y}-4\sqrt{2y})-(\sqrt{y}-5\sqrt{2y})$

Answers to Odd Exercises:

79. $-3\sqrt{2x}$	81. $16\sqrt{x}$	83. $5x\sqrt{y}$	85. $8\sqrt{ab}-15\sqrt{a}$	87. $9\sqrt{xy}$	89. $2\sqrt{2x}+6\sqrt{3x}$

★ Simplify and add. (Assume all radicands containing variable expressions are positive.)

91. $\sqrt{81b}+\sqrt{4b}$

92. $\sqrt{100a}+\sqrt{a}$

93. $\sqrt{9a^2b}-\sqrt{36a^2b}$

94. $\sqrt{50a^2}-\sqrt{18a^2}$

95. $\sqrt{49x}-\sqrt{9y}+\sqrt{x}-\sqrt{4y}$

96. $\sqrt{9x}+\sqrt{64y}-\sqrt{25x}-\sqrt{y}$

97. $7\sqrt{8x}-(3\sqrt{16y}-2\sqrt{18x})$

98. $2\sqrt{64y}-(3\sqrt{32y}-\sqrt{81y})$

99. $2\sqrt{9m^2n}-5m\sqrt{9n}+\sqrt{m^2n}$

100. $4\sqrt{18n^2m}-2n\sqrt{8m}+n\sqrt{2m}$

101. $\sqrt{4x^2y}-\sqrt{9xy^2}-\sqrt{16x^2y}+\sqrt{y^2x}$

102. $\sqrt{32x^2y^2}+\sqrt{12x^2y}-\sqrt{18x^2y^2}-\sqrt{27x^2y}$

103. $\left(\sqrt{9x^2y}-\sqrt{16y}\right)-\left(\sqrt{49x^2y}-4\sqrt{y}\right)$

104. $\left(\sqrt{72x^2y^2}-\sqrt{18x^2y}\right)-\left(\sqrt{50x^2y^2}+x\sqrt{2y}\right)$

105. $\sqrt{12m^4n}-m\sqrt{75m^2n}+2\sqrt{27m^4n}$

106. $5n\sqrt{27mn^2}+2\sqrt{12mn^4}-n\sqrt{3mn^2}$

107. $2\sqrt{27a^3b}-a\sqrt{48ab}-a\sqrt{144a^3b}$

108. $2\sqrt{98a^4b}-2a\sqrt{162a^2b}+a\sqrt{200b}$

Answers to Odd Exercises:

91. $11\sqrt{b}$	97. $20\sqrt{2x}-12\sqrt{y}$	103. $-4x\sqrt{y}$
93. $-3a\sqrt{b}$	99. $-8m\sqrt{n}$	105. $3m^2\sqrt{3n}$
95. $8\sqrt{x}-5\sqrt{y}$	101. $-2x\sqrt{y}-2y\sqrt{x}$	107. $2a\sqrt{3ab}-12a^2\sqrt{ab}$

D: Multiply and Divide Square Root Expressions.

Exercise 0.3.4

★ Multiply and simplify. (Assume all variables represent non-negative real numbers.)

111. $\sqrt{3}\cdot\sqrt{7}$

112. $\sqrt{2}\cdot\sqrt{5}$

113. $\sqrt{6}\cdot\sqrt{12}$

114. $\sqrt{10}\cdot\sqrt{15}$

115. $\sqrt{2}\cdot\sqrt{6}$

116. $\sqrt{5}\cdot\sqrt{15}$

117. $\sqrt{7}\cdot\sqrt{7}$

118. $\sqrt{12}\cdot\sqrt{12}$

119. $2\sqrt{5}\cdot7\sqrt{10}$

120. $3\sqrt{15}\cdot2\sqrt{6}$

121. $(2\sqrt{5})^2$

122. $(6\sqrt{2})^2$

123. $\sqrt{2x}\cdot\sqrt{2x}$

124. $\sqrt{5y}\cdot\sqrt{5y}$

125. $\sqrt{3a}\cdot\sqrt{12}$

126. $\sqrt{3a}\cdot\sqrt{2a}$

127. $4\sqrt{2x}\cdot3\sqrt{6x}$

128. $5\sqrt{10y}\cdot2\sqrt{2y}$

Answers to Odd Exercises:

111. $\sqrt{21}$	115. $2\sqrt{3}$	119. $70\sqrt{2}$	123. $2x$	127. $24x\sqrt{3}$
113. $6\sqrt{2}$	117. 7	121. 20	125. $6\sqrt{a}$	

★ **Distribute and simplify. (Assume all variables represent non-negative real numbers.)**

129. $\sqrt{5}(3-\sqrt{5})$	134. $\sqrt{15}(\sqrt{5}+\sqrt{3})$	139. $(\sqrt{2}-\sqrt{5})(\sqrt{3}+\sqrt{7})$	144. $(\sqrt{7}-\sqrt{2})^2$
130. $\sqrt{2}(\sqrt{3}-\sqrt{2})$	135. $\sqrt{x}(\sqrt{x}+\sqrt{xy})$	140. $(\sqrt{3}+\sqrt{2})(\sqrt{5}-\sqrt{7})$	145. $(2\sqrt{3}+\sqrt{2})(2\sqrt{3}-\sqrt{2})$
131. $3\sqrt{7}(2\sqrt{7}-\sqrt{3})$	136. $\sqrt{y}(\sqrt{xy}+\sqrt{y})$	141. $(2\sqrt{3}-4)(3\sqrt{6}+1)$	146. $(\sqrt{2}+3\sqrt{7})(\sqrt{2}-3\sqrt{7})$
132. $2\sqrt{5}(6-3\sqrt{10})$	137. $\sqrt{2ab}(\sqrt{14a}-2\sqrt{10b})$	142. $(5-2\sqrt{6})(7-2\sqrt{3})$	147. $(\sqrt{a}-\sqrt{2b})^2$
133. $\sqrt{6}(\sqrt{3}-\sqrt{2})$	138. $\sqrt{6ab}(5\sqrt{2a}-\sqrt{3b})$	143. $(\sqrt{5}-\sqrt{3})^2$	148. $(\sqrt{ab}+1)^2$

Answers to Odd Exercises:

129. $3\sqrt{5}-5$	135. $x+x\sqrt{y}$	141. $18\sqrt{2}+2\sqrt{3}-12\sqrt{6}-4$
131. $42-3\sqrt{21}$	137. $2a\sqrt{7b}-4b\sqrt{5a}$	143. $8-2\sqrt{15}$
133. $3\sqrt{2}-2\sqrt{3}$	139. $\sqrt{6}+\sqrt{14}-\sqrt{15}-\sqrt{35}$	145. 10
		147. $a-2\sqrt{2ab}+2b$

★ **Divide and simplify. (Assume all variables represent non-negative real numbers.)**

149. $\frac{\sqrt{75}}{\sqrt{3}}$	151. $\frac{\sqrt{72}}{\sqrt{75}}$	153. $\frac{\sqrt{90x^5}}{\sqrt{2x}}$	155. $\frac{\sqrt{96y^3}}{\sqrt{3y}}$	157. $\frac{\sqrt{363x^4y^9}}{\sqrt{3xy}}$
150. $\frac{\sqrt{360}}{\sqrt{10}}$	152. $\frac{\sqrt{90}}{\sqrt{98}}$	154. $\frac{\sqrt{96y^3}}{\sqrt{3y}}$	156. $\frac{\sqrt{162x^7y^5}}{\sqrt{2xy}}$	

Answers to Odd Exercises:

149. 5	151. $\frac{2\sqrt{6}}{5}$	153. $3x^2\sqrt{5}$	155. $4y\sqrt{2}$	157. $11xy^4\sqrt{x}$

E: Rationalize the Denominator.

Exercise 0.3.5

★ **Rationalize the denominator. (Assume all variables represent positive real numbers.)**

161. $\frac{1}{\sqrt{5}}$	163. $\frac{\sqrt{2}}{\sqrt{3}}$	165. $\frac{5}{2\sqrt{10}}$	167. $\frac{\sqrt{3}-\sqrt{5}}{\sqrt{3}}$	169. $\frac{1}{\sqrt{7x}}$	171. $\frac{a}{5\sqrt{ab}}$
162. $\frac{1}{\sqrt{6}}$	164. $\frac{\sqrt{3}}{\sqrt{7}}$	166. $\frac{3}{5\sqrt{6}}$	168. $\frac{\sqrt{6}-\sqrt{2}}{\sqrt{2}}$	170. $\frac{1}{\sqrt{3y}}$	172. $\frac{3b^2}{2\sqrt{3ab}}$

Answers to Odd Exercises:

161. $\frac{\sqrt{5}}{5}$	163. $\frac{\sqrt{6}}{3}$	165. $\frac{\sqrt{10}}{4}$	167. $\frac{3-\sqrt{15}}{3}$	169. $\frac{\sqrt{7x}}{7x}$	171. $\frac{\sqrt{ab}}{5b}$

★ **Rationalize the denominator. (Assume all variables represent positive real numbers.)**

173. $\dfrac{3}{\sqrt{10}-3}$

174. $\dfrac{2}{\sqrt{6}-2}$

175. $\dfrac{1}{\sqrt{5}+\sqrt{3}}$

176. $\dfrac{1}{\sqrt{7}-\sqrt{2}}$

177. $\dfrac{\sqrt{3}}{\sqrt{3}+\sqrt{6}}$

178. $\dfrac{\sqrt{5}}{\sqrt{5}+\sqrt{15}}$

179. $\dfrac{10}{5-3\sqrt{5}}$

180. $\dfrac{-2\sqrt{2}}{4-3\sqrt{2}}$

181. $\dfrac{\sqrt{3}+\sqrt{5}}{\sqrt{3}-\sqrt{5}}$

182. $\dfrac{\sqrt{10}-\sqrt{2}}{\sqrt{10}+\sqrt{2}}$

183. $\dfrac{2\sqrt{3}-3\sqrt{2}}{4\sqrt{3}+\sqrt{2}}$

184. $\dfrac{6\sqrt{5}+2}{2\sqrt{5}-\sqrt{2}}$

Answers to Odd Exercises:

173. $3\sqrt{10}+9$

175. $\dfrac{\sqrt{5}-\sqrt{3}}{2}$

177. $-1+\sqrt{2}$

179. $\dfrac{-5-3\sqrt{5}}{2}$

181. $-4-\sqrt{15}$

183. $\dfrac{15-7\sqrt{6}}{23}$

★ **Rationalize the denominator. (Assume all variables represent positive real numbers.)**

185. $\dfrac{x-y}{\sqrt{x}+\sqrt{y}}$

186. $\dfrac{x-y}{\sqrt{x}-\sqrt{y}}$

187. $\dfrac{x+\sqrt{y}}{x-\sqrt{y}}$

188. $\dfrac{x-\sqrt{y}}{x+\sqrt{y}}$

189. $\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$

190. $\dfrac{\sqrt{ab}+\sqrt{2}}{\sqrt{ab}-\sqrt{2}}$

191. $\dfrac{\sqrt{x}}{5-2\sqrt{x}}$

192. $\dfrac{1}{\sqrt{x}-y}$

193. $\dfrac{\sqrt{x}+\sqrt{2y}}{\sqrt{2x}-\sqrt{y}}$

194. $\dfrac{\sqrt{3x}-\sqrt{y}}{\sqrt{x}+\sqrt{3y}}$

195. $\dfrac{\sqrt{2x+1}}{\sqrt{2x+1}-1}$

196. $\dfrac{\sqrt{x+1}}{1-\sqrt{x+1}}$

197. $\dfrac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}$

198. $\dfrac{\sqrt{2x+3}-\sqrt{2x-3}}{\sqrt{2x+3}+\sqrt{2x-3}}$

Answers to Odd Exercises:

185. $\sqrt{x}-\sqrt{y}$

187. $\dfrac{x^2+2x\sqrt{y}+y}{x^2-y}$

189. $\dfrac{a-2\sqrt{ab}+b}{a-b}$

191. $\dfrac{5\sqrt{x}+2x}{25-4x}$

193. $\dfrac{x\sqrt{2}+3\sqrt{xy}+y\sqrt{2}}{2x-y}$

195. $\dfrac{2x+1+\sqrt{2x+1}}{2x}$

197. $x+\sqrt{x^2-1}$

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0.04: Review - Rational Exponents

Rational Exponents Defined

In a previous section, properties of **integer exponents** were examined. Most of these properties are also valid for **any real number exponent**. These properties are:

Rules of Exponents

For real numbers a, b, m, and n with $a \neq 0,\ b \neq 0$. Also, $a > 0$ and $b > 0$ or n and m in lowest terms, have odd denominators.

Zero exponent Rule	Negative Rule	Product Rule	Quotient Rule	Power Rule	Power of a product Rule	Power of a quotient Rule
$a^0 = 1$	$a^{-n} = \frac{1}{a^n}$	$a^m \cdot a^n = a^{m+n}$	$\frac{a^m}{a^n} = a^{m-n}$ $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$	$(a^m)^n = a^{m \cdot n}$	$(a \cdot b)^n = a^n \cdot b^n$	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

n^{th} Roots

From our previous work with square roots, if $\sqrt[2]{a} = b$ then $a = b^2$, so for square roots (a root with an index of 2) the radicand a of the square root $\sqrt[2]{a}$ cannot be negative. If the radicand a is negative, then the square root $\sqrt[2]{a}$ is said to be undefined. We also know that $(\sqrt{a})^2 = a$, for $a \geq 0$. Now consider what the meaning of $a^{1/2}$ should be. Using the Power Rule, $(a^{1/2})^2 = a^{(1/2)\cdot 2} = a^1 = a$. Combining these two different forms for a, it can be concluded that $a^{1/2} = \sqrt{a}$ for $a \geq 0$.

Now extend this idea to cube roots. Cube roots are roots with an index of 3. The cube root of a can be written $\sqrt[3]{a}$ or $a^{1/3}$. However, unlike square roots, the radicand a can be any real number (not only on-negative numbers) and its cube root will always be defined. This is because if $\sqrt[3]{a} = b$ then $a = b^3$ and a can be any real number because when a number is cubed, if it is negative, the result is negative and if it is positive the result is positive. These ideas can be extended in a similar fashion to any non-zero integer nth root.

Definition: Principal nth root of a

For any non-zero integer n, the **principal nth root of a** is written $\sqrt[n]{a}$ and has the same sign as a.
Furthermore, $\sqrt[n]{a} = a^{\frac{1}{n}}$ and $(\sqrt[n]{a})^n = a$. The denominator of a fractional exponent determines the index of an nth root.
When a is negative and n is even, the principal nth root is undefined.

$$(25)^{1/2} = \sqrt{25} = 5 \text{ because } (\sqrt{25})^2 = (5)^2 = 25$$
$$(-27)^{1/3} = \sqrt[3]{-27} = -3 \text{ because } (\sqrt[3]{-27})^3 = (-3)^3 = -27$$

Example 0.4.1: Simplifying n^{th} Roots

Simplify each of the following:

a. $\sqrt[5]{-32}$	b. $\sqrt[4]{4} \times \sqrt[4]{1024}$	c. $-\sqrt[3]{\frac{8x^6}{125}}$	d. $8\sqrt[4]{3} - \sqrt[4]{48}$

Solution

a. $\sqrt[5]{-32} = -2$ because $(-2)^5 = -32$

b. $\sqrt[4]{4} \times \sqrt[4]{1024} = \sqrt[4]{4 \times 1024} = \sqrt[4]{4096} = 8$ because $8^4 = 4096$

c. $-\sqrt[3]{\frac{8x^6}{125}} = \frac{-\sqrt[3]{8x^6}}{\sqrt[3]{125}} = \frac{-2x^2}{5}$

d. $8\sqrt[4]{3} - \sqrt[4]{48} = 8\sqrt[4]{3} - \sqrt[4]{2^4 \cdot 3} = 8\sqrt[4]{3} - 2\sqrt[4]{3} = 6\sqrt[4]{3}$

Try It: 0.4.1.

1. Simplify

a. $\sqrt[3]{-216}$	b. $\frac{3\sqrt[4]{80}}{\sqrt[4]{5}}$	d. $6\sqrt[3]{9000}+7\sqrt[3]{576}$

Answer

a. -6 b. 6 c. $88\sqrt[3]{9}$.

Writing Rational Exponential Expressions in Radical Form

An expression with a **rational exponent** is equivalent to a radical where the denominator is the index and the numerator is the exponent. Any radical expression can be written with a rational exponent, which we call **exponential form**.

Let m and n be positive integers with no common factor other than 1. Using the Power Rule, we have

$$(a^{1/n})^m = a^{(1/n)\cdot m} = a^{m/n}$$

Definition: Rational Exponents

Rational exponents are another way to express principal n^{th} roots. The general form for converting between a radical expression with a radical symbol and one with a rational exponent is

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$$

If a is negative and n is even, no meaning can be assigned to this expression.

It is important to note that, as long as the base is positive, it does not matter if we apply the power first or the root first. For example, we can apply the power before the nth root:

$$8^{2/3} = (8^2)^{1/3} = (64)^{1/3} = \sqrt[3]{64} = 4$$

Or we can apply the nth root before the power:

$$8^{2/3} = (8^{1/3})^2 = (\sqrt[3]{8})^2 = (2)^2 = 4$$

The results are the same, although when doing calculations by hand it is often easier to find the nth root first because the numbers are smaller.

How to: Given an expression with a rational exponent, write the expression as a radical.

1. Determine the power by looking at the numerator of the exponent.
2. Determine the root by looking at the denominator of the exponent.
3. Using the base as the radicand, raise the radicand to the power and use the root as the index.

Example 0.4.2: Writing Rational Exponents of Constants as Radicals

Rewrite as a radical.

a. $6^{1/2}$	b. $6^{1/3}$	c. $6^{2/5}$	d. $3^{3/4}$

Solution

a. $6^{1/2} = \sqrt[2]{6} = \sqrt{6}$	b. $6^{1/3} = \sqrt[3]{6}$	c. $6^{2/5} = \sqrt[5]{6^2} = \sqrt[5]{36}$	d. $3^{3/4} = \sqrt[4]{3^3} = \sqrt[4]{27}$

Example 0.4.3:

Rewrite as a radical and then simplify.

a. $16^{1/2}$	b. $16^{1/4}$	c. $9^{\frac{5}{2}}$	d. $(-8)^{2/3}$

Solution

a. $16^{1/2} = \sqrt{16} = \sqrt{4^2} = 4$	b. $16^{1/4} = \sqrt[4]{16} = \sqrt[4]{2^4} = 2$	c. $(\sqrt{9})^5 = (3)^5 = 243$	d. $(-8)^{2/3} = (\sqrt[3]{-8})^2 = (-2)^2 = 4$

Example 0.4.4:

Rewrite as a radical and then simplify.

a. $343^{\frac{2}{3}}$	b. $(12)^{5/3}$	c. $9^{\frac{5}{2}}$

Solution

a. The 2 tells us the power and the 3 tells us the root. $343^{\frac{2}{3}} = (\sqrt[3]{343})^2 = \sqrt[3]{343^2}$

We know that $\sqrt[3]{343} = 7$ because $7^3 = 343$. Because the cube root is easy to find, it is easiest to find the cube root before squaring for this problem. In general, it is easier to find the root first and then raise it to a power.

$$343^{\frac{2}{3}} = (\sqrt[3]{343})^2 = 7^2 = 49$$

b. Sometimes very large integers can be avoided by working with their prime factorization.

$$\begin{aligned}(12)^{5/3} &= \sqrt[3]{(12)^5} && \textit{Replace 12 with } 2^2 \cdot 3.\\ &= \sqrt[3]{(2^2 \cdot 3)^5} && \textit{Apply the rules for exponents.}\\ &= \sqrt[3]{2^{10} \cdot 3^5} && \textit{Simplify.}\\ &= \sqrt[3]{2^9 \cdot 2 \cdot 3^3 \cdot 3^2}\\ &= 2^3 \cdot 3 \cdot \sqrt[3]{2 \cdot 3^2}\\ &= 24\sqrt[3]{18}\end{aligned}$$

c. $\left(\frac{16}{9}\right)^{-\frac{1}{2}} = \left(\left(\frac{16}{9}\right)^{-1}\right)^{\frac{1}{2}} = \left(\frac{16^{-1}}{9^{-1}}\right)^{\frac{1}{2}} = \left(\frac{9}{16}\right)^{\frac{1}{2}} = \sqrt{\frac{9}{16}} = \frac{3}{4}$

Try It: 0.4.4.

Rewrite as a radical and then simplify.

a. $100^{3/2}$	b. $27^{2/3}$

Answer

a. $1,000$ b. 9

Operations on Rational Exponential Expressions

The same rules used for exponential operations on expressions with integer exponents can be used on expressions with rational exponents. The exponential form is often more useful than the radical form, but the radical form is often used with constants because it is more familiar. It is usually far more efficient to perform operations on rational exponents rather than radicals.

Example 0.4.5: Product and Quotient Rules

Simplify.

a. $7^{1/3} \cdot 7^{4/9}$	b. $5(2x^{\frac{3}{4}})(3x^{\frac{1}{5}})$	c. $\frac{x^{3/2}}{x^{2/3}}$

Solution

a. $7^{1/3} \cdot 7^{49} = 7^{1/3+49}$ *Apply the product rule $x^m \cdot x^n = x^{m+n}$.*
$= 7^{3/9+4/9}$
$= 7^{7/9}$

b. $5(2x^{\frac{3}{4}})(3x^{\frac{1}{5}}) = 30x^{\frac{3}{4}}\ x^{\frac{1}{5}}$ Multiply the coefficients
$= 30x^{\frac{3}{4}+\frac{1}{5}}$ Use properties of exponents
$= 30x^{\frac{19}{20}}$ Simplify

c. $\frac{x^{3/2}}{x^{2/3}} = x^{3/2-2/3}$ *Apply the quotient rule $\frac{x^m}{x^n} = x^{m-n}$.*
$= x^{9/6-4/6}$
$= x^{5/6}$

Example 0.4.6: Power Rule

Simplify.

1. $(64x^3)^{1/3}$	2. $(-32x^5y^{10})^{1/5}$	3. $(y^{3/4})^{2/3}$	4. $(81a^8b^{12})^{3/4}$	5. $(9x^4)^{-3/2}$

Solution

1. $(64x^3)^{1/3} = 64^{1/3}(x^3)^{1/3}$
$= \sqrt[3]{64}x^{3/3}$
$= 4x$

2. $(-32x^5y^{10})^{1/5} = (-32)^{1/5}(x^5)^{1/5}(y^{10})^{1/5}$
$= \sqrt[5]{(-2)^5}x^{5/5}y^{10/5}$
$= -2xy^2$

3. $\left(y^{3/4}\right)^{2/3} = y^{(3/4)(2/3)}$ *Apply the power rule $(x^m)^n = x^{m \cdot n}$.*
$= y^{6/12}$ *Multiply the exponents and reduce.*
$= y^{1/2}$

4. $(81a^8b^{12})^{3/4} = (3^4a^8b^{12})^{3/4}$ *Rewrite 81 as 3^4.*
$= (3^4)^{3/4}(a^8)^{3/4}(b^{12})^{3/4}$ *Apply the power rule for a product.*
$= 3^{4(3/4)}a^{8(3/4)}b^{12(3/4)}$ *Apply the power rule to each factor.*
$= 3^3a^6b^9$ *Simplify.*
$= 27a^6b^9$

5. $(9x^4)^{-3/2} = \frac{1}{(9x^4)^{3/2}}$ *Apply the definition of negative exponents $x^{-n} = \frac{1}{x^n}$.*
$= \frac{1}{(3^2x^4)^{3/2}}$ *Write 9 as 3^2 and apply the rules of exponents.*
$= \frac{1}{3^{2(3/2)}x^{4(3/2)}}$
$= \frac{1}{3^3 \cdot x^6}$
$= \frac{1}{27x^6}$

Try It: 0.4.7.

Simplify

a. $(8x)^{\frac{1}{3}}\left(14x^{\frac{6}{5}}\right)$

b. $\dfrac{\left(125a^{1/4}b^{6}\right)^{2/3}}{a^{1/6}}$

Answer

a. $28x^{\frac{23}{15}}$ b. $25b^{4}$

Writing Radical Expressions as Exponential Expressions

Sometimes a problem is stated in radical form. In order to effectively perform the operation or simplification, translation to exponential form first is a useful first step.

Example 0.4.8: Write Radicals as Exponential Expressions

Rewrite using rational exponents.

a. $\dfrac{4}{\sqrt[7]{a^2}}$

b. $\sqrt[5]{x^3}$

c. $\sqrt[6]{y^3}$

Solution

a. The power is 2 and the root is 7, so the rational exponent will be $\frac{2}{7}$. We get $\dfrac{4}{a^{\frac{2}{7}}} = 4a^{\frac{-2}{7}}$

b. Here the index is 5 and the power is 3. We can write $\sqrt[5]{x^3} = x^{3/5}$

c. Here the index is 6 and the power is 3. We can write $\sqrt[6]{y^3} = y^{3/6} = y^{1/2}$

Try It: 0.4.8.

Write $x\sqrt{(5y)^9}$ using a rational exponent.

Answer

$x(5y)^{\frac{9}{2}}$

To apply the product or quotient rule for radicals, the indices of the radicals involved must be the same. If the indices are different, then first rewrite the radicals in exponential form and then apply the rules for exponents.

Example 0.4.9: Use Rational Exponents to Simplify Radical Expressions

Multiply $\sqrt{2}\cdot\sqrt[3]{2}$

Solution

In this example, the index of each radical factor is different. Hence the product rule for radicals does not apply. Begin by converting the radicals into an equivalent form using rational exponents. Then apply the product rule for exponents.

$$\begin{aligned}\sqrt{2}\cdot\sqrt[3]{2} &= 2^{1/2}\cdot 2^{1/3} && \textit{Equivalents using rational exponents.}\\ &= 2^{1/2+1/3} && \textit{Apply the product rule for exponents.}\\ &= 2^{5/6} \quad \text{or } \sqrt[6]{2^5}\end{aligned}$$

Example 0.4.10:

Divide $\frac{\sqrt[3]{4}}{\sqrt[5]{2}}$

Solution

In this example, the index of the radical in the numerator is different from the index of the radical in the denominator. Hence the quotient rule for radicals does not apply. Begin by converting the radicals into an equivalent form using rational exponents and then apply the quotient rule for exponents.

$$\begin{aligned}\frac{\sqrt[3]{4}}{\sqrt[5]{2}} &= \frac{\sqrt[3]{2^2}}{\sqrt[5]{2}} \\ &= \frac{2^{2/3}}{2^{1/5}} && \textit{Equivalents using rational exponents.} \\ &= 2^{2/3-1/5} && \textit{Apply the quotient rule for exponents.} \\ &= 2^{7/15} && \text{or } \sqrt[15]{2^7}\end{aligned}$$

Example 0.4.11:

Simplify $\sqrt{\sqrt[3]{4}}$

Solution

Here the radicand of the square root is a cube root. After rewriting this expression using rational exponents, we will see that the power rule for exponents applies.

$$\begin{aligned}\sqrt{\sqrt[3]{4}} &= \sqrt{\sqrt[3]{2^2}} \\ &= \left(2^{2/3}\right)^{1/2} && \textit{Equivalents using rational exponents.} \\ &= 2^{(2/3)(1/2)} && \textit{Apply the power rule for exponents.} \\ &= 2^{1/3} && \text{or } \sqrt[3]{2}\end{aligned}$$

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0.04e: Exercises - Rational Exponents

A: Radical to Exponential Notation

Exercise 0.4.1

★ **Express using rational exponents.**

1. $\sqrt{10}$	3. $\sqrt[3]{3}$	5. $\sqrt[3]{5^2}$	7. $\sqrt[3]{49}$	9. $\sqrt[5]{x}$	11. $\sqrt[6]{x^7}$	13. $\frac{1}{\sqrt{x}}$
2. $\sqrt{6}$	4. $\sqrt[4]{5}$	6. $\sqrt[4]{2^3}$	8. $\sqrt[3]{9}$	10. $\sqrt[6]{x}$	12. $\sqrt[5]{x^4}$	14. $\frac{1}{\sqrt[3]{x^2}}$

Answers to odd exercises:

1. $10^{1/2}$	3. $3^{1/3}$	5. $5^{2/3}$	7. $7^{2/3}$	9. $x^{1/5}$	11. $x^{7/6}$	13. $x^{-1/2}$

B: Exponential to Radical Notation.

Exercise 0.4.2

★ **Express in radical form.**

15. $10^{1/2}$	17. $7^{2/3}$	19. $x^{3/4}$	21. $x^{-1/2}$	23. $(\frac{1}{x})^{-1/3}$	25. $(2x+1)^{2/3}$
16. $11^{1/3}$	18. $2^{3/5}$	20. $x^{5/6}$	22. $x^{-3/4}$	24. $(\frac{1}{x})^{-3/5}$	26. $(5x-1)^{1/2}$

Answers to odd exercises:

15. $\sqrt{10}$	17. $\sqrt[3]{49}$	19. $\sqrt[4]{x^3}$	21. $\frac{1}{\sqrt{x}}$	23. $\sqrt[3]{x}$	25. $\sqrt[3]{(2x+1)^2}$

C: Exponential to Radical Form; then Simplify.

Exercise 0.4.3

★ **Write as a radical and then simplify.**

27. $64^{1/2}$	31. $4^{-1/2}$	35. $8^{1/3}$	40. $(-64)^{1/3}$	45. $100,000^{1/5}$
28. $49^{1/2}$	32. $9^{-1/2}$	36. $125^{1/3}$	41. $16^{1/4}$	46. $(-32)^{1/5}$
29. $\left(\frac{1}{4}\right)^{1/2}$	33. $\left(\frac{1}{4}\right)^{-1/2}$	37. $\left(\frac{1}{27}\right)^{1/3}$	42. $625^{1/4}$	47. $\left(\frac{1}{32}\right)^{1/5}$
30. $\left(\frac{4}{9}\right)^{1/2}$	34. $\left(\frac{1}{16}\right)^{-1/2}$	38. $\left(\frac{8}{125}\right)^{1/3}$	43. $81^{-1/4}$	48. $\left(\frac{1}{243}\right)^{1/5}$
		39. $(-27)^{1/3}$	44. $16^{-1/4}$	

Answers to odd exercises:

27. 8 29. $\frac{1}{2}$	31. $\frac{1}{2}$ 33. 2	35. 2 37. $\frac{1}{3}$	39. -3 41. 2	43. $\frac{1}{3}$ 45. 10	47. $\frac{1}{2}$

★ Write as a radical and then simplify.

49. $9^{3/2}$ 50. $4^{3/2}$	51. $8^{5/3}$ 52. $27^{2/3}$	53. $16^{3/2}$ 54. $32^{2/5}$	55. $\left(\frac{1}{16}\right)^{3/4}$ 56. $\left(\frac{1}{81}\right)^{3/4}$	57. $(-27)^{2/3}$ 58. $(-27)^{4/3}$	59. $(-32)^{3/5}$ 60. $(-32)^{4/5}$

Answers to odd exercises:

49. 27	51. 32	53. 64	55. $\frac{1}{8}$	57. 9	59. -8

D: Exponential Operations. PRODUCTS and POWERS of Products

Exercise 0.4.4

★ Perform the operations and simplify. Leave answers in exponential form.

61. $5^{3/2} \cdot 5^{1/2}$	65. $y^{1/4} \cdot y^{2/5}$	69. $(8^{1/2})^{2/3}$	73. $(y^8)^{-1/2}$	77. $(2x^{1/3}y^{2/3})^3$
62. $3^{2/3} \cdot 3^{7/3}$	66. $x^{1/2} \cdot x^{1/4}$	70. $(3^6)^{2/3}$	74. $(y^6)^{-2/3}$	78. $(8x^{3/2}y^{1/2})^2$
63. $5^{1/2} \cdot 5^{1/3}$	67. $(u^{12}v^{18})^{\frac{1}{6}}$	71. $(x^{2/3})^{1/2}$	75. $(4x^2y^4)^{1/2}$	79. $(36x^4y^2)^{-1/2}$
64. $2^{1/6} \cdot 2^{3/4}$	68. $(r^9s^{12})^{\frac{1}{3}}$	72. $(y^{3/4})^{4/5}$	76. $(9x^6y^2)^{1/2}$	80. $(8x^3y^6z^{-3})^{-1/3}$

Answers to odd exercises:

61. 25	65. $y^{13/20}$	69. 2	73. $\frac{1}{y^4}$	77. $8xy^2$
63. $5^{5/6}$	67. u^2v^3	71. $x^{1/3}$	75. $2xy^2$	79. $\frac{1}{6x^2y}$

★ Perform the operations and simplify. Leave answers in exponential form.

81. $\left(27q^{\frac{3}{2}}\right)^{\frac{4}{3}}$	84. $\left(m^{\frac{4}{3}}n^{\frac{1}{2}}\right)^{\frac{3}{4}}$	87. $\left(4p^{\frac{1}{3}}q^{\frac{1}{2}}\right)^{\frac{3}{2}}$	90. $\left(81x^8y^{-4/3}z^{-4}\right)^{-3/4}$
82. $\left(64s^{\frac{3}{7}}\right)^{\frac{1}{6}}$	85. $\left(16u^{\frac{1}{3}}\right)^{\frac{3}{4}}$	88. $\left(9x^{\frac{2}{5}}y^{\frac{3}{5}}\right)^{\frac{5}{2}}$	91. $\left(100a^{-2/3}b^4c^{-3/2}\right)^{-1/2}$
83. $\left(a^{\frac{1}{3}}b^{\frac{2}{3}}\right)^{\frac{3}{2}}$	86. $\left(625n^{\frac{8}{3}}\right)^{\frac{3}{4}}$	89. $\left(16x^2y^{-1/3}z^{2/3}\right)^{-3/2}$	92. $\left(125a^9b^{-3/4}c^{-1}\right)^{-1/3}$

Answers to odd exercises:

81. $81q^2$

83. $a^{\frac{1}{2}}b$

85. $8u^{\frac{1}{4}}$

87. $8p^{\frac{1}{2}}q^{\frac{3}{4}}$

85. $8u^{\frac{1}{4}}$

87. $8p^{\frac{1}{2}}q^{\frac{3}{4}}$

89. $\dfrac{y^{1/2}}{64x^3z}$

91. $\dfrac{a^{1/3}c^{3/4}}{10b^2}$

E: Exponential Operations. QUOTIENTS and POWERS of Quotients

Exercise 0.4.5

★ Perform the operations and simplify. Leave answers in exponential form.

101. $\dfrac{5^{11/3}}{5^{2/3}}$

102. $\dfrac{2^{9/2}}{2^{1/2}}$

103. $\dfrac{2a^{2/3}}{a^{1/6}}$

104. $\dfrac{3b^{1/2}}{b^{1/3}}$

105. $\dfrac{r^{\frac{5}{2}}\cdot r^{-\frac{1}{2}}}{r^{-\frac{3}{2}}}$

106. $\dfrac{a^{\frac{3}{4}}\cdot a^{-\frac{1}{4}}}{a^{-\frac{10}{4}}}$

107. $\dfrac{c^{\frac{5}{3}}\cdot c^{-\frac{1}{3}}}{c^{-\frac{2}{3}}}$

108. $\dfrac{m^{\frac{7}{4}}\cdot m^{-\frac{5}{4}}}{m^{-\frac{2}{4}}}$

109. 113 $\dfrac{y^{1/2}y^{2/3}}{y^{1/6}}$

110. $\dfrac{x^{2/5}x^{1/2}}{x^{1/10}}$

111. $\dfrac{xy}{x^{1/2}y^{1/3}}$

112. $\dfrac{x^{5/4}y}{xy^{2/5}}$

113. $\dfrac{49a^{5/7}b^{3/2}}{7a^{3/7}b^{1/4}}$

114. $\dfrac{16a^{5/6}b^{5/4}}{8a^{1/2}b^{2/3}}$

Answers to odd exercises:

101. 125 | 103. $2a^{1/2}$ | 105. $r^{\frac{7}{2}}$ | 107. c^2 | 109. y | 111. $x^{1/2}y^{2/3}$ | 113. $7a^{2/7}b^{5/4}$

★ Perform the operations and simplify. Leave answers in exponential form.

115. $\left(\dfrac{a^{3/4}}{a^{1/2}}\right)^{4/3}$

116. $\left(\dfrac{b^{4/5}}{b^{1/10}}\right)^{10/3}$

117. $\left(\dfrac{4x^{2/3}}{y^4}\right)^{1/2}$

118. $\left(\dfrac{27x^{3/4}}{y^9}\right)^{1/3}$

118. $\left(\dfrac{27x^{3/4}}{y^9}\right)^{1/3}$

119. $\left(\dfrac{36s^{\frac{1}{5}}t^{-\frac{3}{2}}}{s^{-\frac{9}{5}}t^{\frac{1}{2}}}\right)^{\frac{1}{2}}$

120. $\left(\dfrac{27b^{\frac{2}{3}}c^{-\frac{5}{2}}}{b^{-\frac{7}{3}}c^{\frac{1}{2}}}\right)^{\frac{1}{3}}$

121. $\left(\dfrac{8x^{\frac{5}{3}}y^{-\frac{1}{2}}}{27x^{-\frac{4}{3}}y^{\frac{5}{2}}}\right)^{\frac{1}{3}}$

122. $\left(\dfrac{16m^{\frac{1}{5}}n^{\frac{3}{2}}}{81m^{\frac{9}{5}}n^{-\frac{1}{2}}}\right)^{\frac{1}{4}}$

123. $\dfrac{\left(9x^{2/3}y^6\right)^{3/2}}{x^{1/2}y}$

124. $\dfrac{\left(125x^3y^{3/5}\right)^{2/3}}{xy^{1/3}}$

125. $\dfrac{\left(27a^{1/4}b^{3/2}\right)^{2/3}}{a^{1/6}b^{1/2}}$

126. $\dfrac{\left(25a^{2/3}b^{4/3}\right)^{3/2}}{a^{1/6}b^{1/3}}$

Answers to odd exercises:

115. $a^{1/3}$ | 117. $\dfrac{2x^{1/3}}{y^2}$ | 119. $\dfrac{6s}{t}$ | 121. $\dfrac{2x}{3y}$ | 123. $27x^{1/2}y^8$ | 125. $9b^{1/2}$

F: Radical to Exponential Form Operations.

Exercise 0.4.6

★ **Rewrite in exponential form and then perform the operations and write the final answer in radical form.**

131. $\sqrt[3]{9} \cdot \sqrt[5]{3}$

132. $\sqrt{5} \cdot \sqrt[5]{25}$

133. $\sqrt{x} \cdot \sqrt[3]{x}$

134. $\sqrt{y} \cdot \sqrt[4]{y}$

135. $\sqrt[3]{x^2} \cdot \sqrt[4]{x}$

136. $\sqrt[5]{x^3} \cdot \sqrt[3]{x}$

137. $\dfrac{\sqrt[3]{100}}{\sqrt{10}}$

138. $\dfrac{\sqrt[5]{16}}{\sqrt[3]{4}}$

139. $\dfrac{\sqrt[3]{a^2}}{\sqrt{a}}$

140. $\dfrac{\sqrt[5]{b^4}}{\sqrt[3]{b}}$

141. $\dfrac{\sqrt[3]{x^2}}{\sqrt[5]{x^3}}$

142. $\dfrac{\sqrt[4]{x^3}}{\sqrt[3]{x^2}}$

143. $\sqrt{\sqrt[3]{16}}$

144. $\sqrt{\sqrt[3]{9}}$

145. $\sqrt[3]{\sqrt[3]{2}}$

Answers to odd exercises:

131. $\sqrt[15]{3^{13}}$

133. $\sqrt[6]{x^5}$

135. $\sqrt[12]{x^{11}}$

137. $\sqrt[6]{10}$

139. $\sqrt[6]{a}$

141. $\sqrt[15]{x}$

143. $\sqrt[5]{4}$

145. $\sqrt[15]{2}$

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0.05: Review - Factoring

Factor the Greatest Common Factor of a Polynomial

When we study fractions, we learn that the greatest common factor (GCF) of two numbers is the largest number that divides evenly into both numbers. For instance, 4 is the GCF of 16 and 20 because it is the largest number that divides evenly into both 16 and 20 The GCF of polynomials works the same way: $4x$ is the GCF of $16x$ and $20x^2$ because it is the largest polynomial that divides evenly into both $16x$ and $20x^2$.

When factoring a polynomial expression, our first step should be to check for a GCF. Look for the GCF of the coefficients, and then look for the GCF of the variables.

Definition: Greatest Common Factor

The *greatest common factor* (GCF) of polynomials is the largest polynomial that divides evenly into the polynomials.

How to: Given a polynomial expression, factor out the greatest common factor

1. Find the GCF of the expression
   a. Identify the GCF of the coefficients.
   b. Identify the GCF of the variables and of the variable expressions.
   c. Combine to find the GCF of the expression.
2. Determine what the GCF needs to be multiplied by to obtain each term in the expression.
3. Write the factored expression as the product of the GCF and the sum of the terms we need to multiply by.

Example 0.5.1: Factor out the GCF of constants and variables

Factor $6x^3y^3 + 45x^2y^2 + 21xy$.

Solution

Step 1. First, find the GCF of the expression. The GCF of 6, 45, and 21 is 3. The GCF of x^3 , x^2, and x is x . (Note that the GCF of a set of expressions in the form x^n will always be the exponent of lowest degree.) And the GCF of y^3, y^2, and y is y. Combine these to find the GCF of the polynomial, $3xy$.

Step 2. Next, determine what the GCF needs to be multiplied by to obtain each term of the polynomial. We find that

- $3xy(2x^2y^2) = 6x^3y^3$,
- $3xy(15xy) = 45x^2y^2$, and
- $3xy(7) = 21xy$.

Notice that for each term, What-the-GCF-needs-to-be-multiplied-by $= \dfrac{\text{A Term}}{\text{The GCF}}$

So, $\dfrac{6x^3y^3}{3xy} = 2x^2y^2$, $\dfrac{45x^2y^2}{3xy} = 15xy$, and $\dfrac{21xy}{3xy} = 7$

Step 3. Finally, write the factored expression as the product of the GCF and the sum of the terms we needed to multiply by.

$$(3xy)(2x^2y^2 + 15xy + 7)$$

Analysis. After factoring, multiply to confirm result: $(3xy)(2x^2y^2 + 15xy + 7) = 6x^3y^3 + 45x^2y^2 + 21xy$

Try It 0.5.1: Factor out the GCF of a common variable expression

Factor $x(b^2 - a) + 6(b^2 - a)$ by pulling out the GCF.

Answer

$(b^2 - a)(x + 6)$

Factor Four Terms by Grouping Pairs

Factoring by grouping is a technique that enables us to factor polynomials with four terms into a product of binomials. This involves an intermediate step where a common binomial factor will be factored out. For example,

$$\begin{aligned} 3x^3 - 12x^2 + 2x - 8 &= \underbrace{3x^3 - 12x^2}_{\text{group}} + \underbrace{2x - 8}_{\text{group}} && \text{Begin by grouping the first two terms and the last two terms} \\ &= 3x^2(x - 4) + 2(x - 4) && \text{factor out the GCF of each grouping} \\ &= 3x^2(x - 4) + 2(x - 4) && \text{A common binomial factor appears} \\ &= (x - 4)(3x^2 + 2) && \text{Factor out the common binomial} \end{aligned}$$

We can check by multiplying.

$$(x - 4)\left(3x^2 + 2\right) = 3x^3 + 2x - 12x^2 - 8 \checkmark$$

When factoring a polynomial expression, our first step should always be to check for a GCF. Look for the GCF of the coefficients, and then look for the GCF of the variables. Then if there are four terms in the polynomial, try factoring by grouping pairs.

Can every four term polynomial be factored by grouping?

No. Some four term polynomials can be factored by starting with a different pairing of the four terms; however some four term polynomials simply cannot be factored with grouping by pairs. Sometimes four term polynomials can be factored by grouping three terms together first. Some can be factored by determining the possible rational zeros the polynomial might have. And then there are some four term polynomials that simply are not factorable. These polynomials are said to be *prime*.

How to: Given a four term polynomial, factor it

1. Determine if there are any factors common to all four terms. If so, factor the GCF from each of the four terms.
2. Factor the GCF from the first two terms. Factor out the GCF from the last two terms. Sometimes the four terms can be rearranged in order for this to be possible to do.
3. If both expressions from step 2 produce the same binomial factor, then factor out this common binomial GCF.

Example 0.5.2x: Factor by Grouping Pairs

Factor

1. $24a^4 - 18a^3 - 20a + 15$
2. $xy - 2x^2y + x^3 - 2y^2$

Solution

1. $$\begin{aligned} 24a^4 - 18a^3 - 20a + 15 &= \underbrace{24a^4 - 18a^3}_{\text{group}} - \underbrace{20a + 15}_{\text{group}} && \text{group} \\ &= 6a^3(4a - 3) + 5(-4a + 3) && \text{Binomial factors are different} \\ &= 6a^3(4a - 3) - 5(4a - 3) && \text{The binomial factors are the same} \\ &= (4a - 3)(6a^3 - 5) && \text{Factor out the common binomial} \end{aligned}$$

2. $$\begin{aligned} xy - 2x^2y + x^3 - 2y^2 &= \underbrace{xy - 2x^2y}_{\text{group}} + \underbrace{x^3 - 2y^2}_{\text{group}} \\ &= xy(1 - 2x) + (x^3 - 2y^2) && \text{Binomial factors are different} \\ x^3 - 2x^2y + xy - 2y^2 &= \underbrace{x^3 - 2x^2y}_{\text{group}} + \underbrace{xy - 2y^2}_{\text{group}} && \text{Change the order of the terms and group} \\ &= x^2(x - 2y) + y(x - 2y) && \text{The binomial factors are the same} \\ &= (x - 2y)(x^2 + y) && \text{Factor out the common binomial} \end{aligned}$$

Analysis. After factoring, we can check our work by multiplying. Use the distributive property to confirm that $24a^4 - 18a^3 - 20a + 15 = (4a - 3)(6a^3 - 5)$ and $xy - 2x^2y + x^3 - 2y^2 = (x - 2y)(x^2 + y)$

Try It 0.5.2x

Factor $x^3 - x^2y - xy + y^2$

Answer

$(x - y)(x^2 - y)$

Factor a Trinomial with Leading Coefficient 1

Although we should always begin by looking for a GCF, pulling out the GCF is not the only way that polynomial expressions can be factored. The polynomial $x^2 + 5x + 6$ has a GCF of 1, but it can be written as the product of the factors $(x + 2)$ and $(x + 3)$.

Trinomials of the form $x^2 + bx + c$ can be factored by finding two numbers with a product of c and a sum of b. The trinomial $x^2 + 10x + 16$, for example, can be factored using the numbers 2 and 8 because the product of those numbers is 16 and their sum is 10. The trinomial can be rewritten as the product of $(x + 2)$ and $(x + 8)$.

FACTORING A TRINOMIAL WITH LEADING COEFFICIENT 1

A trinomial of the form $x^2 + bx + c$ can be written in factored form as $(x + p)(x + q)$ where $pq = c$ and $p + q = b$.

Can every trinomial be factored as a product of binomials?

No. Some polynomials cannot be factored. These polynomials are said to be *prime*.

How to: Given a trinomial in the form $x^2 + bx + c$, factor it

1. List factor pairs of c.
2. Find p and q, a pair of factors of c with a sum of b.
3. Write the factored expression $(x + p)(x + q)$.

Example 0.5.3: Factoring a Trinomial with Leading Coefficient 1

Factor $x^2 + 2x - 15$.

Solution

We have a trinomial with leading coefficient 1, $b = 2$, and $c = -15$. We need to find two numbers with a product of -15 and a sum of 2. In the table below, we list factor pairs until we find a pair with the desired sum.

Factors of −15	Sum of Factors
1,−15	−14
−1,15	14
3,−5	−2
−3,5	2

Now that we have identified p and q as -3 and 5, write the factored form as $(x - 3)(x + 5)$.

Analysis. We can check our work by multiplying. Use FOIL to confirm that $(x - 3)(x + 5) = x^2 + 2x - 15$.

Notice that the order of the factors does not matter. This is because multiplication is commutative. Therefore, $(x-3)(x+5)=(x+5)(x-3)$.

Try It 0.5.3

Factor x^2-7x+6.

Answer

$(x-6)(x-1)$

Factor a Trinomial using the Reverse FOIL Method

Trinomials with leading coefficients other than 1 are slightly more complicated to factor. Two different methods are **Reverse Foil** and **AC Grouping**. The choice of which method to use is largely a matter of personal preference. In some cases, one approach may be more efficient than the other, but the efficacy of one method over the other is not really predictable beforehand. The Reverse Foil Method will be illustrated below; the AC grouping method will follow.

Factor by using Reverse FOIL

The Reverse FOIL method is derived from assuming that the trinomial ax^2+bx+c can be factored into the form $(Sx+s)(Tx+t)$. When multiplied out, the factored form becomes $STx^2+Stx+sTx+st$, and for this expression to match the original trinomial means that we are seeking four numbers S, s, T, and t such that $ST=a$, $st=c$ and $St+sT=b$.

How to: Given a trinomial in the form ax^2+bx+c, factor by Reverse FOIL.

Determine if there are any factors common to all three terms. If so, factor the GCF from each of the three terms. This GCF should be negative if the high order term is negative. Start with ax^2+bx+c where the GCF has been factored out.

1. List all combinations of positive factor pairs, S and T, for which $ST=a$.
   Use these pairs of value to begin constructing possible templates for the final factored result: $(Sx\ldots)(Tx\ldots)$.
   List all permutations of positive factor pairs, s and t, for which $st=|c|$.
2. Find s and t to revise the template to look like: $(Sx\ldots s)(Tx\ldots t)$.
   If c is positive, look for a **sum** of products $|St+Ts|=|b|$.
   If c is negative, look for a **difference** of products $|St-Ts|=|b|$.
3. Determine the signs for s and t so that the sum of the inner and outer products is equal to the coefficient of the middle term: $sTx+Stx=bx$.

Example 0.5.4.1x: Factoring a Trinomial by Reverse FOIL

Factor $3x^2-7x-20$ by Reverse Foil.

Solution

Step 1	Step 2	

Factors of 3	Factors of 20	20 is negative so find a difference of 7	Proposed Factored Form
1,3	1,20 ✗ 2,10 ✗ 4,5 ✓ 5,4 10,2 20,1	$1\cdot 20=20$ $3\cdot 1=3$ $20-3\neq 7$ $1\cdot 10=10$ $3\cdot 2=6$ $10-6\neq 7$ $1\cdot 5=5$ $3\cdot 4=12$ $12-5=7$	$(1x.....)(3x.....)$ $(1x...4)(3x...5)$

Step 3. Determine the signs: Inner product + outer product = middle term: $-12x+5x=-7x$. Thus our final factored result is: $(1x-4)(3x+5)$

Analysis. We can check our work by multiplying. Use FOIL to confirm that $(3x+5)(x-4)=3x^2-7x-20$.

Example 0.5.4.2x: Factoring a Trinomial by Reverse FOIL

Factor $15x^2+17x+4$ by Reverse Foil.

Solution

Step 1		Step 2	
Factors of 15	Factors of 4	17 is positive so find a sum of 17	Proposed Factored Form
1,15	1,4 ✗ 2,2 ✗ 4,1 ✗	$1\cdot 1=1$ $15\cdot 4=60$ $1+60\neq 17$ $1\cdot 2=2$ $15\cdot 2=30$ $30+2\neq 17$ $1\cdot 4=4$ $15\cdot 1=15$ $4+15\neq 17$	$(1x.....)(15x.....)$ **Reject! Try other template.**
3,5	1,4 ✓ 2,2 4,1	$3\cdot 4=12$ $5\cdot 1=5$ $12+5=17$	$(3x.....)(5x.....)$ $(3x...1)(5x...4)$

Step 3. Determine the signs: Inner product + outer product = middle term: $+5x+12x=17x$. Thus our final factored result is: $(3x+1)(5x+4)$

Analysis. We can check our work by multiplying. Use FOIL to confirm that $(3x+1)(5x+4)=15x^2+17x+4$.

Try It 0.5.4x

Factor:

a. $2x^2+9x+9$

Answer a

$(2x+3)(x+3)$

b. $6x^2+x-1$

Answer b

$(3x-1)(2x+1)$

Factor a Trinomial using the AC Grouping Method

The other method trinomials with leading coefficients other than 1 are factored is by the **AC Grouping Method**. In this method, the middle x term is rewritten as two terms, and then the resulting four term expression is factored by grouping pairs. For example, the trinomial $2x^2+5x+3$ can be rewritten as $2x^2+2x+3x+3$. This is then factored by grouping to obtain $2x(x+1)+3(x+1)$. The GCF of $(x+1)$ is pulled out to finally obtain the factored expression $(x+1)(2x+3)$.

Factor by Grouping

To factor a trinomial in the form ax^2+bx+c by grouping, we find two numbers with a product of ac and a sum of b. We use these numbers to divide the x term into the sum of two terms and factor each portion of the expression separately, then factor out the GCF of the entire expression.

This method works because if we write the factored result as $(a_1x+c_1)(a_2x+c_2)$ and multiplied it out we would get $a_1a_2x^2+(a_1c_2+a_2c_1)x+c_1c_2$, where $a_1a_2=a$, $c_1c_2=c$. The *product* of the leading coefficients in each factor is $a_1a_2\cdot c_1c_2=ac$. The product of the two terms that comprise the x coefficient *sum* is $a_1c_2\cdot a_2c_1=ac$.

How to: Given a trinomial in the form ax^2+bx+c, factor by grouping.

1. List factor pairs of ac.
2. Find p and q, a pair of factors of ac with a sum of b.
3. Rewrite the original expression as $ax^2+px+qx+c$.
4. Factor the resulting four term expression using factoring by grouping.

Example 0.5.5: Factoring a Trinomial by Grouping

Factor $5x^2+7x-6$ by grouping.

Solution. We have a trinomial with $a=5$, $b=7$, and $c=-6$. First, determine $ac=-30$. We need to find two numbers with a product of -30 and a sum of 7. In the table below, we list factor pairs until we find a pair with the desired sum.

Step 1. List Factors of −30	Step 2. Find factor pair whose sum is 7
1,−30	1+ -30 = −29 ✗
−1,30	-1 + 30 = 29 ✗
2,−15	2 + -15 = −13 ✗
−2,15	-2 + 15 = 13 ✗
3,−10	3 + -10 = −7 ✗
−3,10	-3 + 10 = 7 ✓

Step 3. Rewrite $5x^2+7x-6$ as $5x^2-3x+10x-6$

Step 4. Factor by grouping.

$x(5x-3)+2(5x-3)$ Factor out the GCF of each part

$(5x-3)(x+2)$ Factor out the binomial GCF of the expression.

Analysis. We can check our work by multiplying. Use FOIL to confirm that $(5x-3)(x+2)=5x^2+7x-6$.

Try It 0.5.5

Factor:

a. $2x^2+9x+9$

Answer a

$(2x+3)(x+3)$

b. $6x^2+x-1$

Answer b

$(3x-1)(2x+1)$

Factor a Perfect Square Trinomial

A perfect square trinomial is a trinomial that can be written as the square of a binomial. Recall that when a binomial is squared, the result is the square of the first term added to twice the product of the two terms and the square of the last term.

$$(a+b)^2=(a+b)(a+b)=a^2+ab+ab+b^2=a^2+2ab+b^2 \quad \text{and}$$
$$(a-b)^2=(a-b)(a-b)=a^2-ab-ab+b^2=a^2-2ab+b^2$$

We can use these equations to factor any perfect square trinomial.

Perfect Square Trinomials

A perfect square trinomial can be written as the square of a binomial:

$$a^2+2ab+b^2=(a+b)^2$$
$$a^2-2ab+b^2=(a-b)^2$$

How to: Given a perfect square trinomial, factor it into the square of a binomial

1. Confirm that the first term a^2 and last term b^2 are perfect squares.
2. Confirm that the middle term is twice the product of ab.
3. Write the factored form as $(a+b)^2$ if the middle term is positive and $(a-b)^2$ if the middle term is negative.

Example 0.5.6: Factoring a Perfect Square Trinomial

Factor $25x^2+20x+4$.

Solution

Notice that $25x^2$ and 4 are perfect squares because $25x^2=(5x)^2$ and $4=2^2$. Then check to see if the middle term is twice the product of $5x$ and 2. The middle term is, indeed, twice the product: $2(5x)(2)=20x$. Therefore, the trinomial is a perfect square trinomial and can be written as $(5x+2)^2$.

Try It 0.5.6

Factor $49x^2-14x+1$.

Answer

$(7x-1)^2$

Factor a Difference of Squares

A difference of squares is a perfect square subtracted from a perfect square. Recall that when two factors containing the same terms but opposite signs is multiplied out, the middle terms cancel each other out, and all that is left is a difference of squares.

Furthermore, the order of the two factors does not matter.

$$(a-b)(a+b) = a^2 + ab - ab + b^2 = a^2 - b^2 \qquad \text{and}$$
$$(a+b)(a-b) = a^2 - ab + ab + b^2 = a^2 - b^2$$

Differences of Squares

A difference of squares can be rewritten as two factors containing the same terms but opposite signs.

$$a^2 - b^2 = (a-b)(a+b) \tag{0.5.1}$$

How to: Given a difference of squares, factor it into binomials

1. Confirm that there are only two terms, both are perfect squares, and they are being subtracted.
2. Write the factored form as $(a-b)(a+b)$.

Example 0.5.7.1: Factoring a Difference of Squares

Factor $9x^2 - 25$.

Solution Notice that $9x^2$ and 25 are perfect squares because $9x^2 = (3x)^2$ and $25 = 5^2$, so if the formula is used, $a = 3x$ and $b = 5$. The polynomial represents a difference of squares and can be rewritten as $(3x-5)(3x+5)$.

Try It 0.5.7

Factor $81y^2 - 100$.

Answer

$(9y-10)(9y+10)$

Is there a formula to factor the sum of squares?

No. A sum of squares cannot be factored if only real numbers are used.

Example 0.5.7.2x

Factor: $x^2 - (2x-1)^2$.

Solution

First, identify this expression as a difference of squares. Use $a = x$ and $b = (2x-1)$ in the formula for a difference of squares and then simplify.

$$\begin{aligned} x^2 - (2x-1)^2 &= [x - (2x-1)][x + (2x-1)] \\ &= (x - 2x + 1)(x + 2x - 1) \\ &= (-x+1)(3x-1) \\ &= -(x-1)(3x-1) \end{aligned}$$

When factoring, always check the resulting factors to see if they can be factored further.

Example 0.5.7.3x

Factor $x^4 - 81y^4$ completely .

Solution

First, identify what is being squared. To do this, recall the power rule for exponents, $(x^m)^n = x^{mn}$. Thus, $x^4 - 81y^4 = (x^2)^2 - (9y^2)^2$. Therefore, substitute $a = x^2$ and $b = 9y^2$ into the formula for difference of squares.

$$x^4 - 81y^4 = (x^2 - 9y^2)(x^2 + 9y^2)$$

At this point, notice that the factor $(x^2 - 9y^2)$ is itself a difference of two squares and thus can be further factored using $a = x^2$ and $b = 3y$. The factor $(x^2 + 9y^2)$ is prime and cannot be factored using real numbers.

$$\begin{aligned} x^4 - 81y^4 &= (x^2 - 9y^2)(x^2 + 9y^2) \\ &= (x - 3y)(x + 3y)(x^2 + 9y^2) \end{aligned}$$

Factor Four Terms by a 3 and 1 Grouping

Sometimes a four term polynomial is the result of a difference between a binomial square and another squared expression. Expressions that can be factored in this manner are characterized by having three terms that are perfect squares.

How to: Given a four term polynomial containing 3 perfect squares, factor.

Determine if there are any factors common to all three terms. If so, factor the GCF from each of the three terms.

1. Confirm that three of the four terms are perfect squares. Two are positive, one is negative.
2. Confirm that the two positive perfect squares and the non-perfect square term, can be factored as a binomial square.
3. Factor the difference of the binomial square result and the remaining negative square, as a difference of squares.

Example 0.5.8x: Factoring a Trinomial by 3 and 1 Grouping

Factor $9x^2 + 30x + 25 - 81$.

Solution. Identify three perfect squares, $9x^2$, 25, and 81.

$$\begin{aligned} 9x^2 + 30x + 25 - 81 &= (9x^2 + 30x + 25) - 81 && \text{Factor the 2 positive squares and the non-square} \\ &= (3x + 5)(3x + 5) - 81 && \text{As a binomial square} \\ &= (3x + 5)^2 - 9^2 && \text{This produces a difference of squares} \\ &= (3x + 5 - 9)(3x + 5 + 9) && \text{Use } a = (3x + 5) \text{ and } b = 9 \text{ in the difference of squares formula} \\ &= (3x - 4)(3x + 14) && \text{And simplify} \end{aligned}$$

Try It 0.5.8x

Factor $16x^2 - 24x + 9 - 144x^2$.

Answer

$-(8x + 3)(16x - 3)$

Factor the Sum and Difference of Cubes

Now, we will look at two new special products: the sum and difference of cubes. Although the sum of squares cannot be factored, the sum of cubes can be factored into a binomial and a trinomial. Similarly, the sum of cubes can be factored into a binomial and a trinomial, but with different signs.

Sum and Difference of Cubes

The sum of two cubes can be factored as

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2) \tag{0.5.2}$$

The difference of two cubes can be factored as

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2) \tag{0.5.3}$$

The acronym SOAP can be used to remember the signs when factoring the sum or difference of cubes. The first letter of each word relates to the signs: Same Opposite AlwaysPositive. For example, consider the following example.

$$x^3 - 2^3 = (x-2)(x^2+2x+4)$$

The sign of the first 2 is the Same as the sign between $x^3 - 2^3$. The sign of the $2x$ term is Opposite the sign between $x^3 - 2^3$. And the sign of the last term, 4, is Always Positive.

How to: Given a sum of cubes or difference of cubes, factor it

1. Confirm both terms are cubes, $a^3 + b^3$ or $a^3 - b^3$. Identify a and b.
2. For a sum of cubes, write the factored form as $(a+b)(a^2 - ab + b^2)$.
   For a difference of cubes, write the factored form as $(a-b)(a^2 + ab + b^2)$.

If the terms could be factored both as a difference of squares or a difference of cubes, factor as a difference of squares first.

Example 0.5.9: Factoring a Sum of Cubes

Factor $x^3 + 512$.

Solution

Notice that x^3 and 512 are cubes because $8^3 = 512$, so $a = x$ and $b = 8$. Rewrite the sum of cubes as $(x+8)(x^2 - 8x + 64)$.

Analysis. After writing the sum of cubes this way, we might think we should check to see if the trinomial portion can be factored further. However, the trinomial portion cannot be factored, so we do not need to check.

Try It 0.5.9

Factor the sum of cubes: $216a^3 + b^3$.

Answer

$(6a+b)(36a^2 - 6ab + b^2)$

Example 0.5.10: Factoring a Difference of Cubes

Factor $8x^3 - 125$.

Solution

Notice that $8x^3$ and 125 are cubes because $8x^3 = (2x)^3$ and $125 = 5^3$. Write the difference of cubes as $(2x-5)(4x^2 + 10x + 25)$.

Analysis. Just as with the sum of cubes, we will not be able to further factor the trinomial portion.

Try It 0.5.10

Factor the difference of cubes: $1000x^3 - 1$

Answer

$(10x-1)(100x^2 + 10x + 1)$

If a binomial is both a difference of squares and a difference cubes, then first factor it as difference of squares. This will result in a more complete factorization.

Example 0.5.11x

Factor $64x^6 - y^6$ completely.

Solution

This binomial is both a difference of squares and difference of cubes.

$$64x^6 - y^6 = (4x^2)^3 - (y^2)^3 \quad \textit{Difference of cubes}$$
$$64x^6 - y^6 = (8x^3)^2 - (y^3)^2 \quad \textit{Difference of squares}$$

When confronted with a binomial that is a difference of both squares and cubes, as this is, factor using difference of squares first.

$$\begin{aligned} 64x^6 - y^6 &= (8x^3)^2 - (y^3)^2 && \text{Factor first as a difference of squares} \\ &= (8x^3 - y^3)(8x^3 + y^3) \\ &= \underbrace{(8x^3 - y^3)}_{(2x-y)(4x^2+2xy+y^2)} \cdot \underbrace{(8x^3 + y^3)}_{(2x+y)(4x^2-2xy+y^2)} && \text{Then factor a sum and difference of cubes} \\ &= (2x - y)(4x^2 + 2xy + y^2)(2x + y)(4x^2 - 2xy + y^2) \end{aligned}$$

The trinomial factors are prime and the expression is completely factored.

As an exercise, factor the previous example as a difference of cubes first. The result is $(2x - y)(2x + y)(16x^4 + 4x^2y^2 + y^4)$. Notice in this case, the result is not as complete as the solution obtained above when the difference of squares factoring was done first.

Try It 0.5.11x

Factor: $a^6b^6 - 1$

Answer

$(ab + 1)(a^2b^2 - ab + 1)(ab - 1)(a^2b^2 + ab + 1)$

Factoring Expressions with Fractional or Negative Exponents

Expressions with fractional or negative exponents can be factored by pulling out a GCF. Look for the variable or exponent that is common to each term of the expression and pull out that variable or exponent raised to the lowest power. These expressions follow the same factoring rules as those with integer exponents. For instance, $2x^{\frac{1}{4}} + 5x^{\frac{3}{4}}$ can be factored by pulling out $x^{\frac{1}{4}}$ and rewritten as $x^{\frac{1}{4}}(2 + 5x^{\frac{1}{2}})$.

Example 0.5.12: Factoring an Expression with Fractional or Negative Exponents

Factor $3x(x + 2)^{-\frac{1}{3}} + 4(x + 2)^{\frac{2}{3}}$.

Solution

Factor out the term with the lowest value of the exponent. In this case, that would be $(x + 2)^{-\frac{1}{3}}$.

$$\begin{aligned} &(x + 2)^{-\frac{1}{3}}(3x + 4(x + 2)) && \text{Factor out the GCF} \\ &(x + 2)^{-\frac{1}{3}}(3x + 4x + 8) && \text{Simplify} \\ &(x + 2)^{-\frac{1}{3}}(7x + 8) \end{aligned}$$

Try It 0.5.12

Factor $2(5a - 1)^{\frac{3}{4}} + 7a(5a - 1)^{-\frac{1}{4}}$.

Answer

$(5a - 1)^{-\frac{1}{4}}(17a - 2)$

Factoring Summary

The following outlines a useful strategy for factoring polynomials.

USE A GENERAL STRATEGY FOR FACTORING POLYNOMIALS.

1. Is there a greatest common factor?
   Factor it out.
2. Is the polynomial a binomial, trinomial, or are there more than three terms?
   If it is a binomial:
   - Is it a sum?
     Of squares? Sums of squares do not factor.
     Of cubes? Use the sum of cubes pattern.
   - Is it a difference?
     Of squares? Factor as the product of conjugates.
     Of cubes? Use the difference of cubes pattern.

   If it is a trinomial:
   - Is it of the form x^2+bx+c? Undo FOIL.
   - Is it of the form ax^2+bx+c?
     If a and c are squares, check if it fits the trinomial square pattern.
     Use the Reverse FOIL or "ac" method.

   If it has four terms:
   - Use the grouping method. (pairs, or (if three terms are perfect squares), 3-and-1)
3. Check.
   Is it factored completely?
   Do the factors multiply back to the original polynomial?

Remember, a polynomial is completely factored if, other than monomials, its factors are **prime**!

Example 0.5.13.1x

Factor completely: $7x^3-21x^2-70x$.

Solution:

1. Is there a GCF? Yes, $7x$. Factor out the GCF. $7x(x^2-3x-10)$
2. In the parentheses, is a trinomial with leading coefficient 1, so "Undo" FOIL. $7x(x+2)(x-5)$
3. Is the expression factored completely? Yes. Neither binomial can be factored.

Check. $7x(x+2)(x-5) = 7x(x^2-5x+2x-10) = 7x(x^2-3x-10) = 7x^3-21x^2-70x$✓

Be careful when you are asked to factor a binomial as there are several options!

Example 0.5.13.2x

Factor completely: $24y^2-150$

Solution:

1. Is there a GCF? Yes, 6. Factor out the GCF. $6(4y^2-25)$
2. Two terms are in the parentheses.
   The binomial is a difference of squares. $6((2y)^2-(5)^2)$
   Write as a product of conjugates. $6(2y-5)(2y+5)$
3. Is the expression factored completely? Yes. Neither binomial can be factored.

Check. $6(2y-5)(2y+5) = 6(4y^2-25) = 24y^2-150$✓

The next example can be factored using several methods. Recognizing the trinomial squares pattern will make your work easier.

Example 0.5.13.3x

Factor completely: $4a^2 - 12ab + 9b^2$.

Solution:

1. Is there a GCF? No.
2. A trinomial with $a \neq 1$. But the first term is a perfect square.
   Is the last term a perfect square? Yes. $(2a)^2 - 12ab + (3b)^2$
   Does it fit the pattern, $a^2 - 2ab + b^2$? Yes. $(2a)^2 - 12ab + (3b)^2$ Check. $(2a - 3b)^2 = (2a)^2$
   $\searrow -2(2a)(3b) \swarrow$
   Write it as a binomial square. $(2a - 3b)^2$
3. Is the expression factored completely? Yes. The binomial cannot be factored.

$-2 \cdot 2a \cdot 3b + (3b)^2 = 4a^2 - 12ab + 9b^2$✓

Remember, sums of squares do not factor, but sums of cubes do!

Example 0.5.13.4x

Factor completely $12x^3y^2 + 75xy^2$.

Solution:

1. Is there a GCF? Yes, $3xy^2$. Factor out the GCF. $3xy^2(4x^2 + 25)$
2. In the parentheses is a binomial - a sum of squares Sums of squares cannot be factored.
3. Is the expression factored completely? Yes. Check. $3xy^2(4x^2 + 25) = 12x^3y^2 + 75xy^2$✓

When using the sum or difference of cubes pattern, being careful with the signs.

Example 0.5.13.5x

Factor completely: $24x^3 + 81y^3$.

Solution:

1. Is there a GCF? Yes, 3. Factor out the GCF $3(8x^3 + 27y^3)$
2. Two terms are in the parentheses.
   The binomial is a sum of cubes. $3((2x)^3 + (3y)^3)$
   Factor it using the sum of cubes formula. $3(2x + 3y)((2x)^2 - 2x \cdot 3y + (3y)^2)$
   $3(2x + 3y)(4x^2 - 6xy + 9y^2)$
3. Is the expression factored completely? Yes. Neither binomial nor trinomial can be factored.

Check. $3(2x + 3y)(4x^2 - 6xy + 9y^2)$
$= (6x + 9y)(4x^2 - 6xy + 9y^2) = 24x^3 - 36x^2y + 54xy^2 + 36x^2y - 54xy^2 + 81y^3 = 24x^3 + 81y^3$✓

Example 0.5.13.6x

Factor completely: $3x^5y - 48xy$.

Solution:

1. Is there a GCF? Yes, $3xy$. Factor out the GCF. $3xy(x^4 - 16)$
2. The binomial in parentheses is a difference of squares. $3xy\left((x^2)^2 - (4)^2\right)$
   Factor it as a product of conjugates $3xy(x^2 - 4)(x^2 + 4)$
3. Is the expression factored completely? NO!
2. The first binomial is a difference of squares. $3xy\left((x)^2 - (2)^2\right)(x^2 + 4)$
   Factor it as a product of conjugates. $3xy(x - 2)(x + 2)(x^2 + 4)$
3. Is the expression factored completely? Yes.

Check. $3xy(x - 2)(x + 2)(x^2 + 4) = 3xy(x^2 - 4)(x^2 + 4) = 3xy(x^4 - 16) = 3x^5y - 48xy$✓

Example 0.5.13.7x

Factor completely: $4x^2+8bx-4ax-8ab$.

Solution:

1. Is there a GCF? Yes, 4. Factor out the GCF. $4(x^2+2bx-ax-2ab)$
2. There are four terms. Factor each pair. $4\big(x(x+2b)-a(x+2b)\big)$
   Factor out the common binomial GCF $4(x+2b)(x-a)$
3. Is the expression factored completely? Yes.

Check. $4(x+2b)(x-a) = (4x+8b)(x-a) = 4x^2-4ax+8bx-8ab$✓

Example 0.5.13.8x: Remember to first look for a GCF

Factor completely: $40x^2y+44xy-24y$.

Solution:

1. Is there a GCF? Yes, $4y$. Factor out the GCF. $4y(10x^2+11x-6)$
2. Factor the trinomial with $a\neq 1$. $4y(5x-2)(2x+3)$
   Use the Reverse FOIL or AC Grouping Method
3. Is the expression factored completely? Yes.

Check. $4y(5x-2)(2x+3) = 4y(10x^2+11x-6) = 40x^2y+44xy-24y$✓

Example 0.5.13.9x: 4 terms as a difference of a binomial square and another square

Factor completely: $9x^2-12xy+4y^2-49$.

Solution:

1. Is there a GCF? No.
2. Four terms, use grouping. Grouping pairs doesn't work
   Three terms are squares. One square is negative. $9x^2-12xy+4y^2-49$
   Group the other 3 terms. $(3x)^2-12xy+(2y)^2-49$
   The trinomial fits the pattern, $a^2-2ab+b^2$ $\searrow\ -2(3x)(2y)\ \swarrow$
   Write the trinomial as a binomial square. $(3x-2y)^2-49$
2. Two terms - a difference of squares. $(3x-2y)^2-7^2$
   Write it as a product of conjugates. $((3x-2y)-7)((3x-2y)+7)$
3. Is the expression factored completely? Yes. $(3x-2y-7)(3x-2y+7)$

Check. $(3x-2y-7)(3x-2y+7)$
$= 9x^2-6xy-21x-6xy+4y^2+14y+21x-14y-49 = 9x^2-12xy+4y^2-49$✓

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0.05e: Exercises - Factoring

A: Factor out the GCF

Exercise 0.5.A

Factor out the GCF.

1. $12x^4 - 16x^3 + 4x^2$
2. $15x^5 - 10x^4 - 5x^3$
3. $20y^8 + 28y^6 + 40y^3$
4. $18y^7 - 24y^5 - 30y^3$
5. $2a^4b^3 - 6a^3b^2 + 8a^2b$
6. $28a^3b^3 - 21a^2b^4 - 14ab^5$
7. $2x^3y^5 - 4x^4y^4 + x^2y^3$
8. $3x^5y - 2x^4y^2 + x^3y^3$
9. $5x^2(2x+3) - 3(2x+3)$
10. $y^2(y-1) + 9(y-1)$
11. $9x^2(3x-1) + (3x-1)$
12. $7y^2(5y+2) - (5y+2)$
13. $x^{5n} - x^{3n} + x^n$
14. $y^{6n} - y^{3n} - y^{2n}$
15. $2x^4 - 12x^3 - 2x^2$
16. $18a^3b - 3a^2b^2 + 3ab^3$
17. $x^4y^3 - 3x^3y + x^2y$
18. $x^{3n} - x^{2n} - x^n$
19. $35y + 84$
20. $6y^2 + 12y - 6$
21. $18x^3 - 15x$
22. $15m^4 + 6m^2n$
23. $4x^3 - 12x^2 + 16x$
24. $-3x + 24$
25. $-3x^3 + 27x^2 - 12x$
26. $3x(x-1) + 5(x-1)$

Answers to odd exercises:

1. $4x^2(3x^2 - 4x + 1)$
3. $4y^3(5y^5 + 7y^3 + 10)$
5. $2a^2b(a^2b^2 - 3ab + 4)$
7. $x^2y^3(2xy^2 - 4x^2y + 1)$
9. $(2x+3)(5x^2 - 3)$
11. $(3x-1)(9x^2 + 1)$
13. $x^n(x^{4n} - x^{2n} + 1)$
15. $2x^2(x^2 - 6x - 1)$
17. $x^2y(x^2y^2 - 3x + 1)$
19. $7(5y + 12)$
21. $3x(6x^2 - 5)$
23. $4x(x^2 - 3x + 4)$
25. $-3x(x^2 - 9x + 4)$

B: Factor 4 term Polynomials.

Exercise 0.5.B

I. Factor by grouping.

31. $2x^3 + 3x^2 + 2x + 3$
32. $5x^3 + 25x^2 + x + 5$
33. $6x^3 - 3x^2 + 4x - 2$
34. $3x^3 - 2x^2 - 15x + 10$
35. $x^3 - x^2 - 3x + 3$
36. $6x^3 - 15x^2 - 2x + 5$
37. $2x^3 + 7x^2 - 10x - 35$
38. $3x^3 - x^2 + 24x - 8$
39. $14y^4 + 10y^3 - 7y - 5$
40. $5y^4 + 2y^3 + 20y + 8$
41. $x^{4n} + x^{3n} + 2x^n + 2$
42. $x^{5n} + x^{3n} + 3x^{2n} + 3$
43. $x^3 - x^2y + xy^2 - y^3$
44. $x^3 + x^2y - 2xy^2 - 2y^3$
45. $3x^3y^2 + 9x^2y^3 - x - 3y$
46. $2x^3y^3 - x^2y^3 + 2x - y$
47. $a^2b - 4ab^2 - 3a + 12b$
48. $a^2b + 3ab^2 + 5a + 15b$
49. $a^4 + a^2b^3 + a^2b + b^4$
50. $a^3b + 2a^2 + 3ab^4 + 6b^3$
51. $3ax + 10by - 5ay - 6bx$
52. $a^2x - 5b^2y - 5a^2y + b^2x$
53. $x^4y^2 - x^3y^3 + x^2y^4 - xy^5$
54. $2x^5y^2 + 4x^4y^2 + 18x^3y + 36x^2y$
55. $a^5b^2 + a^4b^4 + a^3b^3 + a^2b^5$
56. $3a^6b + 3a^5b^2 + 9a^4b^2 + 9a^3b^3$

Answers to odd exercises:

31. $(2x+3)(x^2+1)$
33. $(2x-1)(3x^2+2)$
35. $(x-1)(x^2-3)$
37. $(2x+7)(x^2-5)$
39. $(7y+5)(2y^3-1)$
41. $(x^n+1)(x^{3n}+2)$
43. $(x-y)(x^2+y^2)$
45. $(x+3y)(3x^2y^2-1)$
47. $(a-4b)(ab-3)$
49. $(a^2+b)(a^2+b^3)$
51. $(a-2b)(3x-5y)$
53. $xy^2(x-y)(x^2+y^2)$
55. $a^2b^2(a^2+b)(a+b^2)$

II. Factor by grouping.

57. $2x^3-x^2+2x-1$
58. $3x^3-x^2-6x+2$
59. $x^3-5x^2y+xy^2-5y^3$
60. $a^2b-a+ab^3-b^2$
61. $2x^4-4xy^3+2x^2y^2-4x^3y$
62. $x^4y^2-xy^5+x^3y^4-x^2y^3$
63. $ax-ay+bx-by$
64. $x^2y-xy^2+2x-2y$
65. $x^2+7x-3x-21$
66. $4x^2-16x+3x-12$
67. m^3+m^2+m+1
68. $5x-5y-y+x$
69. $9x^2+18x+9-121x^2$
70. $9x^2+30x+25-81x^2$
71. $16x^2+24x+9-144x^2$
72. $16x^2+32x+16-121x^2$
73. $25x^2+70x+49-121x^2$
74. $25x^2+80x+64-100x^2$
75. $9x^2+12x+4-144$
76. $9x^2+24x+16-144$
77. $16x^2+24x+9-36$
78. $16x^2+48x+36-81$
79. $25x^2+30x+9-81$
80. $25x^2+20x+4-100$

Answers to odd exercises:

57. $(x^2+1)(2x-1)$
59. $(x^2+y^2)(x-5y)$
61. $2x(x-2y)(x^2+y^2)$
63. $(a+b)(x-y)$
65. $(x-3)(x+7)$
67. $(m^2+1)(m+1)$
69. $(-8x+3)(14x+3)$
71. $(-8x+3)(16x+3)$
73. $(-6x+7)(16x+7)$
75. $(3x-10)(3x+14)$
77. $(4x-3)(4x+9)$
79. $(5x-6)(5x+12)$

C: Factor Binomials.

Exercise 0.5.C

I. Factor two term polynomials.

81. x^2-64
82. x^2-100
83. $9-4y^2$
84. $25-y^2$
85. x^2-81y^2
86. x^2-49y^2
87. a^2b^2-4
88. $1-9a^2b^2$
89. $a^2b^2-c^2$
90. $4a^2-b^2c^2$
91. x^4-64
92. $36-y^4$
93. $(2x+5)^2-x^2$
94. $(3x-5)^2-x^2$
95. $y^2-(y-3)^2$
96. $y^2-(2y+1)^2$
97. $(2x+5)^2-(x-3)^2$
98. $(3x-1)^2-(2x-3)^2$
99. x^4-16
100. $81x^4-1$
101. x^4y^4-1
102. x^4-y^4
103. x^8-y^8
104. y^8-1
105. $x^{2n}-y^{2n}$
106. $x^{2n}y^{2n}-4$
107. $x^{4n}-y^{4n}$
108. $x^{4n}y^{4n}-16$

Answers to odd exercises:

81. $(x+8)(x-8)$
83. $(3+2y)(3-2y)$
85. $(x+9y)(x-9y)$
87. $(ab+2)(ab-2)$
89. $(ab+c)(ab-c)$
91. $(x^2+8)(x^2-8)$
93. $(3x+5)(x+5)$
95. $3(2y-3)$
97. $(3x+2)(x+8)$
99. $(x^2+4)(x+2)(x-2)$
101. $(x^2y^2+1)(xy+1)(xy-1)$
103. $(x^4+y^4)(x^2+y^2)$ $(x+y)(x-y)$
105. $(x^n+y^n)(x^n-y^n)$
107. $(x^{2n}+y^{2n})(x^n+y^n)(x^n-y^n)$

II. Factor two term polynomials.

109. x^3-27
110. $8x^3-125$
111. $8y^3+27$
112. $64x^3+343$
113. x^3-y^3
114. x^3+y^3
115. $8a^3b^3+1$
116. $27a^3-8b^3$
117. x^3y^3-125
118. $216x^3+y^3$
119. $x^3+(x+3)^3$
120. $y^3-(2y-1)^3$
121. $(2x+1)^3-x^3$
122. $(3y-5)^3-y^3$
123. $x^{3n}-y^{3n}$
124. $x^{3n}+y^{3n}$
125. a^6+64
126. $64a^6-1$
127. x^6-y^6
128. x^6+y^6
129. $x^{6n}-y^{6n}$
130. $x^{6n}+y^{6n}$

Answers to odd exercises:

109. $(x-3)(x^2+3x+9)$
111. $(2y+3)(4y^2-6y+9)$
113. $(x-y)(x^2+xy+y^2)$
115. $(2ab+1)(4a^2b^2-2ab+1)$
117. $(xy-5)(x^2y^2+5xy+25)$
119. $(2x+3)(x^2+3x+9)$
121. $(x+1)(7x^2+5x+1)$
123. $(x^n-y^n)(x^{2n}+x^ny^n+y^{2n})$
125. $(a^2+4)(a^4-4a^2+16)$
127. $(x+y)(x^2-xy+y^2)$ $(x-y)(x^2+xy+y^2)$
129. 129. $(x^n+y^n)(x^{2n}-x^ny^n+y^{2n})$ $(x^n-y^n)(x^{2n}+x^ny^n+y^{2n})$

III. Factor two term polynomials.

131. $64x^2-1$
132. $9-100y^2$
133. x^2-36y^2
134. $4-(2x-1)^2$
135. a^3b^3+125
136. $64x^3-y^3$
137. $81x^4-y^4$
138. x^8-1
139. x^6-64y^6
140. $1-a^6b^6$
141. $81x^2-9x$
142. $64-16x$
143. $169m^2-n^2$
144. $25p^2-1$
145. $9-121y^2$
146. $20x^2-125$
147. $169n^3-n$
148. $6p^2q^2-54p^2$
149. $24p^2+54$
150. $49x^2-81y^2$
151. $16z^4-1$
152. $48m^4n^2-243n^2$
153. $a^2+6a+9-9b^2$
154. $x^2-16x+64-y^2$
155. a^3-125
156. b^3-216
157. $2m^3+54$
158. $81m^3+3$

Answers to odd exercises:

131. $(8x+1)(8x-1)$
133. $(x+6y)(x-6y)$
135. $(ab+5)\left(a^2b^2-5ab+25\right)$
137. $\left(9x^2+y^2\right)(3x+y)(3x-y)$
139. $(x+2y)\left(x^2-2xy+4y^2\right)$ $(x-2y)\left(x^2+2xy+4y^2\right)$
141. $9x(9x-1)$
143. $(13m+n)(13m-n)$
145. $(3+11y)(3-11y)$
147. $n(13n+1)(13n-1)$
149. $6(4p^2+9)$
151. $(2z-1)(2z+1)(4z^2+1)$
153. $(a+3-3b)(a+3+3b)$
155. $(a-5)(a^2+5a+25)$
157. $2(m+3)(m^2-3m+9)$

D: Factor Trinomials, $a=1$.

Exercise 0.5.D

I. Factor trinomials, $a=1$.

161. x^2+5x-6
162. x^2+5x+6
163. $x^2+4x-12$
164. $x^2+3x-18$
165. $x^2-14x+48$
166. $x^2-15x+54$
167. $x^2+11x-30$
168. $x^2-2x+24$
169. $x^2-18x+81$
170. $x^2-22x+121$
171. $x^2-xy-20y^2$
172. $x^2+10xy+9y^2$
173. $x^2y^2+5xy-50$
174. $x^2y^2-16xy+48$
175. $a^2-6ab-72b^2$
176. $a^2-21ab+80b^2$
177. $u^2+14uv-32v^2$
178. $m^2+7mn-98n^2$
179. $(x+y)^2-2(x+y)-8$
180. $(x-y)^2-2(x-y)-15$

Answers to odd exercises:

161. $(x-1)(x+6)$
163. $(x-2)(x+6)$
165. $(x-6)(x-8)$
167. Prime
169. $(x-9)^2$
171. $(x-5y)(x+4y)$
173. $(xy-5)(xy+10)$
175. $(a+6b)(a-12b)$
177. $(u-2v)(u+16v)$
179. $(x+y-4)(x+y+2)$

II. Factor trinomials, $a=1$.

181. x^4-7x^2-8
182. x^4+13x^2+30
183. x^4-8x^2-48
184. x^4+25x^2+24
185. y^4-20y^2+100
186. y^4+14y^2+49
187. $x^4+3x^2y^2+2y^4$
188. $x^4-8x^2y^2+15y^4$
189. $a^4b^4-4a^2b^2+4$
190. $a^4+6a^2b^2+9b^4$
191. x^6-18x^3-40
192. x^6+18x^3+45
193. $x^6-x^3y^3-6y^6$
194. $x^6+x^3y^3-20y^6$
195. $x^6y^6+2x^3y^3-15$
196. $x^6y^6+16x^3y^3+48$
197. $x^{2n}+12x^n+32$
198. $x^{2n}+41x^n+40$
199. $x^{2n}+2ax^n+a^2$
200. $x^{2n}-2ax^n+a^2$

Answers to odd exercises:

181. $\left(x^2-8\right)\left(x^2+1\right)$
183. $\left(x^2+4\right)\left(x^2-12\right)$
185. $\left(y^2-10\right)^2$
187. $\left(x^2+y^2\right)\left(x^2+2y^2\right)$
189. $\left(a^2b^2-2\right)^2$
191. $\left(x^3-20\right)\left(x^3+2\right)$
193. $\left(x^3+2y^3\right)\left(x^3-3y^3\right)$
195. $\left(x^3y^3-3\right)\left(x^3y^3+5\right)$
197. $\left(x^n+4\right)\left(x^n+8\right)$
199. $\left(x^n+a\right)^2$

E: Factor trinomials, $a \neq 1$

Exercise 0.5.E

I. Factor trinomials. (Leading coefficient is not one).

201. $3x^2+20x-7$
202. $2x^2-9x-5$
203. $6a^2+13a+6$
204. $4a^2+11a+6$
205. $6x^2+7x-10$
206. $4x^2-25x+6$
207. $24y^2-35y+4$
208. $10y^2-23y+12$
209. $14x^2-11x+9$
210. $9x^2+6x+8$
211. $4x^2-28x+49$
212. $36x^2-60x+25$
213. $27x^2-6x-8$
214. $24x^2+17x-20$
215. $6x^2+23xy-4y^2$
216. $10x^2-21xy-27y^2$
217. $8a^2b^2-18ab+9$
218. $12a^2b^2-ab-20$
219. $8u^2-26uv+15v^2$
220. $24m^2-26mn+5n^2$
221. $4a^2-12ab+9b^2$
222. $16a^2+40ab+25b^2$
223. $5(x+y)^2-9(x+y)+4$
224. $7(x-y)^2+15(x-y)-18$
225. $7x^4-22x^2+3$
226. $5x^4-41x^2+8$

Answers to odd exercises:

201. $(3x-1)(x+7)$
203. $(2a+3)(3a+2)$
205. $(6x-5)(x+2)$
207. $(8y-1)(3y-4)$
209. Prime
211. $(2x-7)^2$
213. $(9x+4)(3x-2)$
215. $(6x-y)(x+4y)$
217. $(4ab-3)(2ab-3)$
219. $(2u-5v)(4u-3v)$
221. $(2a-3b)^2$
223. $(x+y-1)(5x+5y-4)$
225. $(x^2-3)\,(7x^2-1)$

II. Factor trinomials. (Leading coefficient is not one).

227. $4y^6-3y^3-10$
228. $12y^6+4y^3-5$
229. $5a^4b^4-a^2b^2-18$
230. $21a^4b^4+5a^2b^2-4$
231. $6x^6y^6+17x^3y^3+10$
232. $16x^6y^6+46x^3y^3+15$
233. $8x^{2n}-10x^n-25$
234. $30x^{2n}-11x^n-6$
235. $36x^{2n}+12ax^n+a^2$
236. $9x^{2n}-12ax^n+4a^2$
237. $-3x^2+14x+5$
238. $-2x^2+13x-20$
239. $-x^2-10x+24$
240. $-x^2+8x+48$
241. $54-12x-2x^2$
242. $60+5x-5x^2$
243. $4x^3+16x^2+20x$
244. $2x^4-12x^3+14x^2$
245. $2x^3-8x^2y-24xy^2$
246. $6x^3-9x^2y-6xy^2$
247. $4a^3b-4a^2b^2-24ab^3$
248. $15a^4b-33a^3b^2+6a^2b^3$
249. $3x^5y+30x^3y^3+75xy^5$
250. $45x^5y^2-60x^3y^4+20xy^6$

Answers to odd exercises:

227. $(y^3-2)\,(4y^3+5)$
229. $(a^2b^2-2)\,(5a^2b^2+9)$
231. $(6x^3y^3+5)\,(x^3y^3+2)$
233. $(2x^n-5)\,(4x^n+5)$
235. $(6x^n+a)^2$
237. $-(x-5)(3x+1)$
239. $-(x-2)(x+12)$
241. $-2(x-3)(x+9)$
243. $4x\,(x^2+4x+5)$
245. $2x(x+2y)(x-6y)$
247. $4ab(a-3b)(a+2b)$
249. $3xy(x^2+5y^2)^2$

F: Factor Trinomials.

Exercise 0.5.F

I. Factor the following trinomials.

251. $x^2-8x-48$
252. $x^2-15x+54$
253. x^2-4x-6
254. $x^2-12xy+36y^2$
255. $x^2+20xy+75y^2$
256. $-x^2+5x+150$
257. $-2y^2+20y+48$
258. $28x^2+20x+3$
259. $150x^2-100x+6$
260. $24a^2-38ab+3b^2$
261. $27u^2-3uv-4v^2$
262. $16x^2y^2-78xy+27$
263. $16m^2+72mn+81n^2$
264. $4x^2-5x+20$
265. $25x^4-35x^2+6$
266. $2x^4+7x^2+3$
267. $x^6+3x^3y^3-10y^6$
268. $a^6-8a^3b^3+15b^6$
269. $x^{2n}-2x^n+1$
270. $6x^{2n}-x^n-2$
271. $25x^2+30x+9$
272. $36a^2-84ab+49b^2$
273. $40x^2+360x+810$
274. $5k^3-70k^2+245k$
275. $75u^4-30u^3v+3u^2v^2$
276. $k^2-16k+60$

Answers to odd exercises:

251. $(x-12)(x+4)$ 253. Prime 255. $(x+5y)(x+15y)$ 257. $-2(y-12)(y+2)$	259. $2(15x-1)(5x-3)$ 261. $(3u+v)(9u-4v)$ 263. $(4m+9n)^2$	265. $(5x^2-6)\,(5x^2-1)$ 267. $(x^3+5y^3)\,(x^3-2y^3)$ 269. $(x^n-1)^2$	271. $(5x+3)^2$ 273. $10(2x+9)^2$ 275. $3u^2(5u-v)^2$

II. Factor the following trinomials.

277. $m^2+3m-54$ 278. $x^2-3x-10$ 279. $x^2+12xy+35y^2$ 280. $r^2+3rs-28s^2$ 281. $a^2+4ab-21b^2$ 282. $p^2-5pq-36q^2$ 283. $m^2-5mn+30n^2$	284. x^3+5x^2-24x 285. $3y^3-21y^2+30y$ 286. $5x^4+10x^3-75x^2$ 287. $5y^2+14y+9$ 288. $8x^2+25x+3$ 289. $10y^2-53y-11$	290. $6p^2-19pq+10q^2$ 291. $-81a^2+153a+18$ 292. $2x^2+9x+4$ 293. $18a^2-9a+1$ 294. $15p^2+2p-8$ 295. $15x^2+6x-2$	296. $8a^2+32a+24$ 297. $3x^2+3x-36$ 298. $48y^2+12y-36$ 299. $18a^2-57a-21$ 300. $3n^4-12n^3-96n^2$

Answers to odd exercises:

277. $(m+9)(m-6)$ 279. $(x+5y)(x+7y)$ 281. $(a+7b)(a-3b)$	283. Prime 285. $3y(y-5)(y-2)$ 287. $(5y+9)(y+1)$	289. $(5y+1)(2y-11)$ 291. $-9(9a+1)(a-2)$ 293. $(3a-1)(6a-1)$	295. Prime 297. $3(x+4)(x-3)$ 299. $3(2a-7)(3a+1)$

G: Mixed Factoring Practice.

Exercise 0.5.*G*

I. Mixed practice. Factor completely.

301. $4-25x^2$ 302. $8x^3-y^3$ 303. $9x^2-12xy+4y^2$ 304. $30a^2-57ab-6b^2$ 305. $10a^2-5a-6ab+3b$ 306. $3x^3-4x^2+9x-12$ 307. x^2+4y^2 308. x^2-x+2 309. $15a^3b^2+6a^2b^3-3ab^4$	310. $54x^2-63x$ 311. $45x^3-20x$ 312. $12x^4-70x^3+50x^2$ 313. $-20x^2+32x-3$ 314. $-x^3y+9xy^3$ 315. $24a^4b^2+3ab^5$ 316. $64a^6b^6-1$ 317. $64x^2+1$	318. $x^3+x^2y-xy^2-y^3$ 319. $8y^3+16y^2-24y$ 320. $5y^3-15y^2-270y$ 321. $16x^3-36x$ 322. $27y^2-48$ 323. $4x^2+20xy+25y^2$ 324. $9x^2-24xy+16y^2$ 325. $50x^3y+72xy$	326. $250m^3+432n^3$ 327. $4a^5b-64ab$ 328. $7xy^5-7xy$ 329. $6x^2-12xc+6bx-12bc$ 330. $16x^2+24xy-4x-6y$ 331. $4p^2q-16pq+12q$ 332. $6pq^2-9pq-6p$ 333. $4x^2-12xy+9y^2-25$ 334. $16x^2-24xy+9y^2-64$

Answers to odd exercises:

301. $(2-5x)(2+5x)$ 303. $(3x-2y)^2$ 305. $(2a-1)(5a-3b)$ 307. Prime 309. $3ab^2\,(5a^2+2ab-b^2)$ 311. $5x(3x+2)(3x-2)$	313. $-(10x-1)(2x-3)$ 315. $3ab^2(2a+b)\,(4a^2-2ab+b^2)$ 317. Prime 319. $8y(y-1)(y+3)$ 321. $4x(2x-3)(2x+3)$	323. $(2x+5y)^2$ 325. $2xy(25x^2+36)$ 327. $4ab(a^2+4)(a-2)(a+2)$ 329. $6(x+b)(x-2c)$ 331. $4q(p-3)(p-1)$ 333. $(2x-3y-5)(2x-3y+5)$

II. Mixed practice. Factor completely.

335. $80a^2+120a^3$
336. $5m(m-1)+3(m-1)$
337. $x^2+13x+36$
338. $p^2+pq-12q^2$
339. $xy-8y+7x-56$
340. $40r^2+810$
341. $9s^2-12s+4$
342. $6x^2-11x-10$
343. $3x^2-75y^2$
344. $6u^2+3u-18$
345. x^3+125
346. $32x^5y^2-162xy^2$
347. $6x^4-19x^2+15$
348. $3x^3-36x^2+108x$
349. $24x^3+44x^2$
350. $24a^4-9a^3$
351. $16n^2-56mn+49m^2$
352. $6a^2-25a-9$
353. $5u^4-45u^2$
354. n^4-81
355. $64j^2+225$
356. $5x^2+5x-60$
357. b^3-64
358. m^3+125
359. $2b^2-2bc+5cb-5c^2$
360. $48x^5y^2-243xy^2$
361. $5q^2-15q-90$
362. $4u^5v+4u^2v^3$
363. $10m^4-6250$
364. $60x^2y-75xy+30y$
365. $16x^2+32x+16-121$
366. $9x^2+42x+49-9$
367. $16x^2+72x+81-144x^2$
368. $9x^2+12x+4-36x^2$

Answers to odd exercises:

335. $40a^2(2+3a)$
337. $(x+9)(x+4)$
339. $(x-8)(y+7)$
341. $(3s-2)^2$
343. $3(x+5y)(x-5y)$
345. $(x+5)(x^2-5x+25)$
347. $(3x^2-5)(2x^2-3)$
349. $4x^2(6x+11)$
351. $(4n-7m)^2$
353. $5u^2(u+3)(u-3)$
355. prime
357. $(b-4)(b^2+4b+16)$
359. $(2b+5c)(b-c)$
361. $5(q+3)(q-6)$
363. $10(m-5)(m+5)(m^2+25)$
365. $(4x-7)(4x+15)$
367. $(-8x+9)(16x+9)$

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0.06: Review - Rational Expressions

Rational Expressions

A rational expression is the quotient of two polynomials. Some examples are

$$\frac{3}{6x-5}, \quad \frac{x+1}{2}, \quad \frac{8-x^3}{x^2-6x+9}, \quad \text{and} \quad \frac{(x+3)}{(x+1)(x-4)}$$

The set of real numbers for which an expression is defined is the **domain** of the expression. Because rational expressions have denominators, and division by zero is undefined, the values of the variable that make the denominator zero are not included in the domain of the expression. These values of the variable are restrictions to the domain.

How to: find the domain of a rational expression

1. Set the denominator equal to zero and solve for x.
2. The solutions are the restrictions to the domain.

Example 0.6.1x: Simplifying Rational Expressions

Find the restrictions to the domain for the following rational functions

1. $\frac{x-8}{x-11}$
2. $\frac{x^2-1}{x^2+5x+4}$

Solution

1. **Answer: 11**. The expression $\frac{x-8}{x-11}$ is undefined when its denominator is zero. Solving $x-11=0$ produces $x=11$. Therefore the restriction to the domain of $\frac{x-8}{x-11}$ is $x=11$. Notice that the numerator CAN be zero, so $x=8$ is NOT a restricted value for the variable.
2. **Answer: -4 and -1**. The expression $\frac{x^2-1}{x^2+5x+4}$ is undefined when its denominator is zero. Solving $x^2+5x+4=0$ produces $(x+4)(x+1)=0$. Therefore the restricted values for x are -4 and -1. Again it should be pointed out that the numerator is completely ignored when determining the domain of an expression.

Try It 0.6.1x

Find the restrictions to the domain for the rational function $\frac{x-5}{x^2-36}$

Answer

6 and -6

Simplifying Rational Expressions

The quotient of two polynomial expressions is called a rational expression. We can apply the properties of fractions to rational expressions, such as simplifying the expressions by canceling common factors from the numerator and the denominator. To do this, we first need to factor both the numerator and denominator. Let's start with the rational expression shown.

$$\frac{x^2+8x+16}{x^2+11x+28}$$

We can factor the numerator and denominator to rewrite the expression.

$$\frac{(x+4)^2}{(x+4)(x+7)}$$

Then we can simplify that expression by canceling the common factor $(x+4)$ to obtain $\frac{x+4}{x+7}$

How to: Given a rational expression, simplify it

1. Factor the numerator and denominator.
2. Cancel any common factors. The domain of the simplified rational expression remains the same as the domain of the original expression, even if some cancellation is done.

Example 0.6.2: Simplifying Rational Expressions

Simplify $\frac{x^2-9}{x^2+4x+3}$ and find all the numbers that must be excluded from the domain of the simplified rational expression.

Solution

$$\frac{(x+3)(x-3)}{(x+3)(x+1)} \qquad \text{Factor the numerator and the denominator}$$

$$\frac{x-3}{x+1} \qquad \text{Cancel common factor } (x+3)$$

The restrictions to the domain of the simplified result $\frac{x-3}{x+1}$ are $x \neq -1$ and $x \neq -3$.

Can the x^2 term be cancelled in the last example?

No. A factor is an expression that is multiplied by another expression. The x^2 term is not a factor of the numerator or the denominator.

Try It 0.6.2

Simplify $\frac{x-6}{x^2-36}$

Answer

$$\frac{1}{x+6}$$

Multiplying Rational Expressions

Multiplication of rational expressions works the same way as multiplication of other fractions. Multiply the numerators to find the numerator of the product, and multiply the denominators to find the denominator of the product. Before multiplying, first factor the numerators and denominators. The final form of the simplified rational expression should be in factored form.

How to: Given two rational expressions, multiply them

1. Factor the numerator and denominator.
2. Multiply the numerators. (Keep as a product of factors - do not multiply out).
3. Multiply the denominators. (Keep as a product of factors - do not multiply out).
4. Simplify. (Leave in factored form!)

Example 0.6.3: Multiplying Rational Expressions

Multiply the rational expressions and show the product in simplest form:

$$\frac{(x+5)(x-1)}{3(x+6)} \times \frac{(2x-1)}{(x+5)}$$

Solution

$$\frac{(x+5)(x-1)}{3(x+6)} \times \frac{(2x-1)}{(x+5)}$$ Factor the numerator and denominator.

$$\frac{(x+5)(x-1)(2x-1)}{3(x+6)(x+5)}$$ Multiply numerators and denominators

$$\frac{(x-1)(2x-1)}{3(x+6)}$$ Cancel common factors to simplify

Try It 0.6.3

Multiply the rational expressions and show the product in simplest form:

$$\frac{x^2+11x+30}{x^2+5x+6} \times \frac{x^2+7x+12}{x^2+8x+16}$$

Answer

$$\frac{(x+5)(x+6)}{(x+2)(x+4)}$$

Dividing Rational Expressions

Division of rational expressions works the same way as division of other fractions. To divide a rational expression by another rational expression, multiply the first expression by the reciprocal of the second. Using this approach, we would rewrite $\frac{1}{x} \div \frac{x^2}{3}$ as the product $\frac{1}{x} \cdot \frac{3}{x^2}$. Once the division expression has been rewritten as a multiplication expression, we can multiply as we did before.

$$\frac{1}{x} \cdot \frac{3}{x^2} = \frac{3}{x^3}$$

How to: Given two rational expressions, divide them

1. Rewrite as the first rational expression multiplied by the reciprocal of the second.
2. Factor the numerators and denominators.
3. Multiply the numerators. (Keep as a product of factors - do not multiply out).
4. Multiply the denominators. (Keep as a product of factors - do not multiply out).
5. Simplify. (Leave in factored form!)

Example 0.6.4: Dividing Rational Expressions

Divide the rational expressions and express the quotient in simplest form:

$$\frac{2x^2+x-6}{x^2-1} \div \frac{x^2-4}{x^2+2x+1}$$

Solution

$$\frac{2x^2+x-6}{x^2-1} \div \frac{x^2-4}{x^2+2x+1}$$

$$\frac{2x^2+x-6}{x^2-1} \times \frac{x^2+2x+1}{x^2-4}$$ Rewrite as a multiplication problem

$$\frac{(2x-3)(x+2)}{(x-1)(x+1)} \times \frac{(x+1)(x+1)}{(x-2)(x+2)}$$ Factor the numerator and denominator.

$$\frac{(2x-3)(x+2)(x+1)(x+1)}{(x-1)(x+1)(x-2)(x+2)}$$ Multiply numerators and denominators

$$\frac{(2x-3)(x+1)}{(x-1)(x-2)}$$ Cancel common factors to simplify

Try It 0.6.4

Divide the rational expressions and express the quotient in simplest form:

$$\frac{9x^2-16}{3x^2+17x-28} \div \frac{3x^2-2x-8}{x^2+5x-14}$$

Answer

1

Adding and Subtracting Rational Expressions

Adding and subtracting rational expressions works just like adding and subtracting numerical fractions. To add fractions, we need to find a common denominator. Let's look at an example of fraction addition.

$$\begin{aligned}\frac{5}{24}+\frac{1}{40} &= \frac{5}{24}\cdot\frac{5}{5}+\frac{1}{40}\cdot\frac{3}{3}\\ &= \frac{25}{120}+\frac{3}{120}\\ &= \frac{28}{120} \qquad = \frac{7}{30}\end{aligned}$$

We have to rewrite the fractions so they share a common denominator before we are able to add. We must do the same thing when adding or subtracting rational expressions.

The easiest common denominator to use will be the **least common denominator**, or LCD. The LCD is the smallest multiple that the denominators have in common. To find the LCD of two rational expressions, we factor the expressions and multiply all of the distinct factors. For instance, if the factored denominators were $(x+3)(x+4)$ and $(x+4)(x+5)$, then the LCD would be $(x+3)(x+4)(x+5)$.

Once we find the LCD, we need to multiply each expression by the form of 1 that will change the denominator to the LCD. We would need to multiply the expression with a denominator of $(x+3)(x+4)$ by $\frac{x+5}{x+5}$ and the expression with a denominator of $(x+4)(x+5)$ by $\frac{x+3}{x+3}$.

How to: Given two rational expressions, add or subtract them

1. Factor the numerator and denominator.
2. Find the LCD of the expressions.
3. Multiply the expressions by a form of 1 that changes the denominators to the LCD. (Keep the denominator factored.)
4. Add or subtract the numerators. (Multiply out numerators only, combine like terms, then factor the result.)
5. Simplify. (Keep in factored form!)

Example 0.6.5: Adding Rational Expressions

Add the rational expressions:

$$\frac{5}{x}+\frac{6}{y}$$

Solution

First, we have to find the LCD. In this case, the LCD will be xy. We then multiply each expression by the appropriate form of 1 to obtain xy as the denominator for each fraction.

$$\frac{5}{x}\times\frac{y}{y}+\frac{6}{y}\times\frac{x}{x}$$
$$\frac{5y}{xy}+\frac{6x}{xy}$$

Now that the expressions have the same denominator, we simply add the numerators to find the sum.

$$\frac{6x+5y}{xy}$$

Analysis

Multiplying by $\frac{y}{y}$ or $\frac{x}{x}$ does not change the value of the original expression because multiplying an expression by 1 does not change the value of the original expression.

Example 0.6.6: Subtracting Rational Expressions

Subtract the rational expressions: $\frac{6}{x^2+4x+4}-\frac{2}{x^2-4}$

Solution

$$\frac{6}{(x+2)^2}-\frac{2}{(x+2)(x-2)}$$ Factor
$$\frac{6}{(x+2)^2}\times\frac{x-2}{x-2}-\frac{2}{(x+2)(x-2)}\times\frac{x+2}{x+2}$$ Multiply each fraction to get LCD as denominator
$$\frac{6(x-2)}{(x+2)^2(x-2)}-\frac{2(x+2)}{(x+2)^2(x-2)}$$ Multiply
$$\frac{6x-12-(2x+4)}{(x+2)^2(x-2)}$$ Apply distributive property (be careful here!!)
$$\frac{4x-16}{(x+2)^2(x-2)}$$ Subtract
$$\frac{4(x-4)}{(x+2)^2(x-2)}$$ Simplify

Do we have to use the LCD to add or subtract rational expressions?

No. Any common denominator will work, but it is easiest to use the LCD.

Try It 0.6.6

Subtract the rational expressions: $\frac{3}{x+5}-\frac{1}{x-3}$

Answer

$$\frac{2(x-7)}{(x+5)(x-3)}$$

Simplifying Complex Rational Expressions

A complex rational expression is a rational expression that contains additional rational expressions in the numerator, the denominator, or both. We can simplify complex rational expressions by rewriting the numerator and denominator as single rational expressions and dividing. The complex rational expression $\frac{a}{\frac{1}{b}+c}$ can be simplified by rewriting the numerator as the fraction $\frac{a}{1}$ and combining the expressions in the denominator as $\frac{1+bc}{b}$. We can then rewrite the expression as a multiplication problem using the reciprocal of the denominator. We get $\frac{a}{1}\cdot\frac{b}{1+bc}$, which is equal to $\frac{ab}{1+bc}$.

How to: Simplify a complex rational expression (Method I)

1. Combine the expressions in the numerator into a single rational expression by adding or subtracting.
2. Combine the expressions in the denominator into a single rational expression by adding or subtracting.
3. Rewrite as the numerator divided by the denominator.
4. Rewrite as multiplication.
5. Multiply.
6. Factor and simplify.

Example 0.6.7: Simplifying Complex Rational Expressions

Simplify: $\frac{y+\frac{1}{x}}{\frac{x}{y}}$

Solution

Begin by combining the expressions in the numerator into one expression.

$$y\times\frac{x}{x}+\frac{1}{x} \quad \text{Multiply by } \frac{x}{x} \text{ to get LCD as denominator}$$

$$\frac{xy}{x}+\frac{1}{x}$$

$$\frac{xy+1}{x} \quad \text{Add numerators}$$

Now the numerator is a single rational expression and the denominator is a single rational expression.

We can rewrite this as division, and then multiplication.

$$\frac{\frac{xy+1}{x}}{\frac{x}{y}}$$

$$\frac{xy+1}{x} \div \frac{x}{y}$$

$$\frac{xy+1}{x} \times \frac{y}{x} \quad \text{Rewrite as multiplication}$$

$$\frac{y(xy+1)}{x^2} \quad \text{Multiply}$$

Try It 0.6.7

Simplify: $\dfrac{\frac{x}{y}-\frac{y}{x}}{y}$

Answer

$$\frac{x^2-y^2}{xy^2} = \frac{(x-y)(x+y)}{xy^2}$$

Can a complex rational expression always be simplified?

Yes. We can always rewrite a complex rational expression as a simplified rational expression.

Method 2: Simplify Using the LCD

An alternative method for simplifying complex rational expressions involves clearing the fractions in the numerator and denominator by multiplying the rational expression by a special form of **1**. In this method, multiply the numerator and denominator by the least common denominator (LCD) of all the fractions present in both the numerator and the denominator.

How to: Simplify a complex rational expression (Method II)

1. Determine the LCD of all the fractions present in both the numerator and the denominator.
2. Multiply the complex fraction by a form of 1 composed of the fraction $\frac{\text{LCD}}{\text{LCD}}$. This effectively results in every term in the numerator and every term in the denominator getting multiplied by this LCD.
3. Factor and simplify the resulting rational expression.

Example 0.6.8x:

Simplify: $\dfrac{4-\frac{12}{x}+\frac{9}{x^2}}{2-\frac{5}{x}+\frac{3}{x^2}}$.

Solution

Step 1: Determine the LCD of all the fractions in the numerator and denominator. In this case, the denominators of the given fractions are $1, x$, and x^2. Therefore, the LCD is x^2.

Step 2: Multiply the numerator and denominator by the LCD. This step should clear the fractions in both the numerator and denominator.

$$\frac{4-\frac{12}{x}+\frac{9}{x^2}}{2-\frac{5}{x}+\frac{3}{x^2}} = \frac{\left(4-\frac{12}{x}+\frac{9}{x^2}\right)\cdot x^2}{\left(2-\frac{5}{x}+\frac{3}{x^2}\right)\cdot x^2}$$ *Multiply numerator and denominator.*

$$= \frac{4\cdot x^2-\frac{12}{x}\cdot x^2+\frac{9}{x^2}\cdot x^2}{2\cdot x^2-\frac{5}{x}\cdot x^2+\frac{3}{x^2}\cdot x^2}$$ *Distribute and then cancel.*

$$= \frac{4x^2-12x+9}{2x^2-5x+3}$$

This leaves us with a single algebraic fraction with a polynomial in the numerator and in the denominator.

Step 3: Factor the numerator and denominator completely.

$$= \frac{4x^2-12x+9}{2x^2-5x+3}$$
$$= \frac{(2x-3)(2x-3)}{(x-1)(2x-3)}$$

Cancel all common factors.

$$= \frac{(2x-3)\,\cancel{(2x-3)}}{(x-1)\,\cancel{(2x-3)}}$$
$$= \frac{2x-3}{x-1}$$

It is important to point out that multiplying the numerator and denominator by the same nonzero factor is equivalent to multiplying by 1 and does not change the problem.

Try It 0.6.8x

Simplify using the LCD: $\dfrac{\frac{1}{y^2}-\frac{1}{x^2}}{\frac{1}{y}+\frac{1}{x}}$.

Answer

$\dfrac{x-y}{xy}$

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0.06e: Exercises - Rational Expressions

A: Simplify Rational Expressions.

Exercise 0.6.1

★ Simplify each rational expression. Find all numbers that must be excluded from the domain of the simplified rational expression.

1. $\frac{25x^9}{5x^5}$
2. $\frac{64x^8}{16x^3}$
3. $\frac{x^2-64}{x^2+16x+64}$
4. $\frac{x^2+x-20}{x^2-25}$
5. $\frac{9-4x^2}{2x^2-5x+3}$
6. $\frac{x-3x^2}{9x^2-6x+1}$
7. $\frac{2x^2-8x-42}{2x^2+5x-3}$
8. $\frac{6x^2+5x-4}{3x^2+x-4}$
9. $\frac{x^3+x^2-x-1}{x^2+2x+1}$
10. $\frac{2x^3-5x^2-8x+20}{2x^2-9x+10}$
11. $\frac{66x(2x-5)}{18x^3(2x-5)^2}$
12. $\frac{26x^4(5x+2)^3}{20x^5(5x+2)}$
13. $\frac{x^2+5x+6}{x^2-5x-14}$
14. $\frac{x^2-8x+12}{x^2-2x-24}$
15. $\frac{1-x^2}{5x^2+x-6}$
16. $\frac{4-9x^2}{3x^2-8x+4}$
17. $\frac{4x^2+15x+9}{9-x^2}$
18. $\frac{6x^2+13x-5}{25-4x^2}$
19. $\frac{x^2-5x+4}{x^3-x^2-16x+16}$
20. $\frac{x^4+4x^2}{x^3+3x^2+4x+12}$

Answers to odd exercises:

1. $5x^4; x \neq 0$
3. $\frac{x-8}{x+8}; x \neq -8$
5. $-\frac{2x+3}{x-1}; x \neq 1, \frac{3}{2}$
7. $\frac{2(x-7)}{2x-1}; x \neq -3, \frac{1}{2}4$
9. $x-1; x \neq -1$
11. $\frac{11}{3x^2(2x-5)}; x \neq 0, \frac{5}{2}$
13. $\frac{x+3}{x-7}; x \neq -2, 7$
15. $-\frac{x+1}{5x+6}; x \neq -\frac{6}{5}, 1$
17. $-\frac{4x+3}{x-3}; x \neq \pm 3$
19. $\frac{1}{x+4}; x \neq 1, \pm 4$

B: Multiply or Divide Rational Expressions.

Exercise 0.6.2

★ Multiply or divide as indicated, state the restrictions, and simplify.

21. $\frac{14(x+12)^2}{5x^3} \cdot \frac{45x^4}{2(x+12)^3}$
22. $\frac{27x^6}{20(x-7)^3} \cdot \frac{(x-7)^5}{54x^7}$
23. $\frac{x^2-64}{36x^4} \cdot \frac{12x^3}{x^2+4x-32}$
24. $\frac{50x^5}{x^2+6x-27} \cdot \frac{x^2-81}{125x^3}$
25. $\frac{2x^2+7x+5}{3x^2} \cdot \frac{15x^3-30x^2}{2x^2+x-10}$
26. $\frac{3x^2+14x-5}{2x^2+11x+5} \cdot \frac{4x^2+4x+1}{6x^2+x-1}$
27. $\frac{x^2+4x-21}{5x^2+10x} \div \frac{x^2-6x+9}{x^2+9x+14}$
28. $\frac{x^2-49}{9x^2-24x+16} \div \frac{2x^2-13x-7}{6x^2-5x-4}$
29. $\frac{5x^2+x-6}{4x^2-7x-15} \div \frac{1-x^2}{4x^2+9x+5}$
30. $\frac{6x^2-8x-8}{4-9x^2} \div \frac{3x^2-4x-4}{9x^2+12x+4}$
31. $\frac{x^2+4x-12}{x^2-2x-15} \div \frac{2x^2-13x+18}{6x^2-31x+5}$
32. $\frac{8x^2+x-9}{25x^2-1} \div \frac{2x^2-x-1}{10x^2-3x-1}$
33. $\frac{1}{12ab} \cdot \frac{50a^2(a-b)^2}{a^2-b^2} \cdot \frac{6b}{a(a-b)}$
34. $\frac{b^2-a^2}{(a-b)^2} \cdot \frac{12a(a-b)}{36a^2b} \cdot \frac{9ab(a-b)}{a+b}$
35. $\frac{x^3+y^3}{5xy} \cdot \frac{x^2-y^2}{x^2-2xy+y^2} \cdot \frac{25x^2y}{(y+x)^2}$

Answers to odd exercises:

21. $\frac{63x}{x+12}; x \neq -12, 0$

23. $\frac{x-8}{3x(x-4)}; x \neq -8, 0, 4$

25. $5(x+1); x \neq -\frac{5}{2}, 0, 2$

27. $\frac{(x+7)^2}{5x(x-3)}; x \neq -7, -2, 0, 3$

29. $-\frac{5x+6}{x-3}; x \neq -\frac{5}{4}, -1, 1, 3$

31. $\frac{(x+6)(6x-1)}{(x+3)(2x-9)}; x \neq -3, \frac{1}{6}, 2, \frac{9}{2}, 5$

33. $\frac{25}{a+b}; a\backslash new0, \pm b, b \neq 0]$

35. $\frac{5x\left(x^2-xy+y^2\right)}{x-y}; x \neq 0, \pm y, y \neq 0$

★ **Multiply or divide as indicated, state the restrictions, and simplify.**

36. $\frac{3xy^2}{(2y+x)^2} \cdot \frac{2x^2+5xy+2y^2}{9x^2} \cdot \frac{x^3+8y^3}{6xy^2+3y^3}$

37. $\frac{2x+5}{x-3} \cdot \frac{x^2-9}{5x^4} \div \frac{2x^2+15x+25}{25x^5}$

38. $\frac{5x^2-15x}{9x^2-4} \cdot \frac{3x-2}{20x^3} \div \frac{x-3}{3x^2-x-2}$

39. $\frac{x^2+5x-50}{x^2+5x-14} \div \frac{x^2-25}{x^2-49} \cdot \frac{x-2}{x^2+3x-70}$

40. $\frac{x^2-x-56}{4x^2-4x-3} \div \frac{2x^2+11x-21}{25-9x^2} \cdot \frac{4x^2-12x+9}{3x^2-19x-40}$

41. $\frac{20x^2-8x-1}{6x^2+13x+6} \div \frac{1-100x^2}{3x^2-x-2} \cdot \frac{10x-1}{2x^2-3x+1}$

42. $\frac{12x^2-13x+1}{x^2+18x+81} \div (144x^2-1) \cdot \frac{x^2+14x+45}{12x^2-11x-1}$

Answers to odd exercises:

37. $\frac{5x(x+3)}{x+5}; x \neq 3, 0, -5, -\frac{5}{2}$

39. $\frac{1}{x+5}; \ x \neq \pm 7, 2, \pm 5, -10$

41. $-\frac{1}{2x+3}; \ x \neq -\frac{2}{3}, \pm\frac{1}{10}, 1, \frac{1}{2}$

C: Add or Subtract Rational Expressions.

Exercise 0.6.3

★ **Add or subtract and simplify. State the restrictions.**

51. $\frac{3x}{3x+4}+\frac{2}{3x+4}$

52. $\frac{3x}{2x-1}-\frac{2x+1}{2x-1}$

53. $\frac{x-2}{2x^2-11x-6}+\frac{x+3}{2x^2-11x-6}$

54. $\frac{4x-1}{3x^2+2x-5}-\frac{x-6}{3x^2+2x-5}$

55. $\frac{1}{x}-2x$

56. $\frac{4}{x^3}-\frac{1}{x}$

57. $\frac{1}{x-1}-5$

58. $\frac{1}{x+7}-1$

59. $\frac{1}{x-2}-\frac{1}{3x+4}$

60. $\frac{2}{5x-2}-\frac{x}{x+3}$

61. $\frac{1}{x^2}+\frac{1}{x-2}$

62. $\frac{2x}{x}+\frac{2}{x-2}$

63. $\frac{3x-7}{x(x-7)}+\frac{1}{7-x}$

64. $\frac{2}{8-x}+\frac{3x^2-1}{x^2(x-8)}$

65. $\frac{x-1}{x^2-25}-\frac{2}{x^2-10x+25}$

66. $\frac{x+1}{2x^2+5x-3}-\frac{x}{4x^2-1}$

67. $\frac{x}{x^2+4x}-\frac{2}{x^2+8x+16}$

68. $\frac{2x-1}{4x^2+8x-5}-\frac{3}{4x^2+20x+25}$

Answers to odd exercises:

51. $\frac{3x+2}{3x+4}; x \neq -\frac{4}{3}$

53. $\frac{1}{x-6}; x \neq -\frac{1}{2}, 6$

55. $\frac{1-2x^2}{x}; x \neq 0$

57. $\frac{-5x+6}{x-1}; x \neq 1$

59. $\frac{2(x+3)}{(x-2)(3x+4)}; x \neq -\frac{4}{3}, 2$

61. $\frac{(x-1)(x+2)}{x^2(x-2)}; x \neq 0, 2$

63. $\frac{2x-7}{x(x-7)}; x \neq 0, 7$

65. $\frac{x^2-8x-5}{(x+5)(x-5)^2}; x \neq \pm 5$

67. $\frac{x+2}{(x+4)^2}; x \neq 0, -4$

★ Add or subtract and simplify. State the restrictions.

69. $\frac{5-x}{7x+x^2} - \frac{x+2}{49-x^2}$

70. $\frac{2x}{4x^2+x} - \frac{x+1}{8x^2+6x+1}$

71. $\frac{x-1}{2x^2-7x-4} + \frac{2x-1}{x^2-5x+4}$

72. $\frac{2(x+3)}{3x^2-5x-2} + \frac{4-x}{3x^2+10x+3}$

73. $\frac{x^2}{4+2x^2} - \frac{2}{x^4+2x^2}$

74. $\frac{3x}{4x^4+6x^3} - \frac{2x^2}{6x^3+9x^2}$

75. $\frac{3x^2-12}{x^4-8x^2+16} - \frac{x^2+2}{4-x^2}$

76. $\frac{x^2}{2x^2+1} + \frac{6x^2-24}{2x^4-7x^2-4}$

77. $1 + \frac{3}{x} - \frac{5x-1}{x^2}$

78. $4 + \frac{2}{x} - \frac{6x-1}{x^2}$

79. $\frac{2x}{x-8} - \frac{1}{3x+1} - \frac{2x+9}{3x^2-23x-8}$

80. $\frac{4x}{x-2} - \frac{10}{3x+1} - \frac{19x+18}{3x^2-5x-2}$

81. $\frac{1}{x-1} + \frac{1}{(x-1)^2} - \frac{1}{x^2-1}$

82. $\frac{1}{x-2} - \frac{1}{x^2-4} + \frac{1}{(x-2)^2}$

83. $\frac{2x+1}{x-1} - \frac{3x}{2x^2-3x+1} + \frac{x+1}{x-2x^2}$

84. $\frac{5x^2}{2x^2+2x} - \frac{x^2-4x}{x^2-2x} + \frac{4+2x^2}{4+2x-2x^2}$

85. $\frac{x+2}{2x(3x-2)} + \frac{4x}{(x-2)(3x-2)} - \frac{3x+2}{2x(x-2)}$

86. $\frac{10x}{x(x-5)} - \frac{2x^2}{(2x-5)(x-5)} - \frac{5x}{x(2x-5)}$

Answers to odd exercises:

69. $\frac{7(5-2x)}{x(7+x)(7-x)}; x \neq -7, 0, 7$

71. $\frac{x(5x-2)}{(x-4)(x-1)(2x+1)}; x \neq -\frac{1}{2}, 1, 4$

73. $\frac{x^2-2}{2x^2}; x \neq 0$

75. $\frac{x^2+5}{(x+2)(x-2)}; x \neq \pm 2$

77. $\frac{(x-1)^2}{x^2}; x \neq 0$

79. $\frac{2x-1}{x-8}; x \neq -\frac{1}{3}, 8$

81. $\frac{x^2+1}{(x-1)^2(x+1)}; x \neq \pm 1$

83. $\frac{2x+1}{x}; x \neq 0, \frac{1}{2}, 1$

85. $0; x \neq 0, \frac{2}{3}, 2$

★ Add or subtract and simplify.

87. $x^{-2} + y^{-2}$

88. $x^{-2} + (2y)^{-2}$

89. $2x^{-1} + y^{-2}$

90. $x^{-2} - 4y^{-1}$

91. $16x^{-2} + y^2$

92. $xy^{-1} - yx^{-1}$

93. $3(x+y)^{-1} + x^{-2}$

94. $2(x-y)^{-2} - (x-y)^{-1}$

95. $a^{-2} - (a+b)^{-1}$

96. $(a-b)^{-1} - (a+b)^{-1}$

97. $x^{-n} + y^{-n}$

98. $xy^{-n} + yx^{-n}$

Answers to odd exercises:

87. $\frac{y^2+x^2}{x^2y^2}$	91. $\frac{x^2y^2+16}{x^2}$	95. $\frac{a+b-a^2}{a^2(a+b)}$
89. $\frac{x+2y^2}{xy^2}$	93. $\frac{3x^2+x+y}{x^2(x+y)}$	97. $\frac{x^n+y^n}{x^ny^n}$

D: Simplify Complex Rational Expressions.

Exercise 0.6.4

★ Simplify each complex rational expression.

101. $\dfrac{\frac{75x^2}{(x-3)^2}}{\frac{25x^3}{x-3}}$

102. $\dfrac{\frac{x+5}{36x^3}}{\frac{(x+5)^3}{9x^2}}$

103. $\dfrac{\frac{x^2-36}{32x^5}}{\frac{x-6}{4x^3}}$

104. $\dfrac{\frac{x-8}{56x^2}}{\frac{x^2-64}{7x^3}}$

105. $\dfrac{\frac{5x+1}{2x^2+x-10}}{\frac{25x^2+10x+1}{4x^2-25}}$

106. $\dfrac{\frac{4x^2-27x-7}{4x^2-1}}{\frac{x-7}{6x^2-x-1}}$

107. $\dfrac{\frac{x^2-4x-5}{2x^2+3x+1}}{\frac{x^2-10x+25}{2x^2+7x+3}}$

108. $\dfrac{\frac{5x^2+9x-2}{x^2+4x+4}}{\frac{10x^2+3x-1}{4x^2+7x-2}}$

109. $\dfrac{x^2}{\frac{1}{5}-\frac{3}{x}}$

110. $\dfrac{\frac{4}{x}-3}{2x^2}$

111. $\dfrac{\frac{1}{3}-\frac{1}{x}}{\frac{1}{9}-\frac{1}{x^2}}$

112. $\dfrac{\frac{2}{5}+\frac{1}{x}}{\frac{4}{25}-\frac{1}{x^2}}$

113. $\dfrac{\frac{1}{y^2}-36}{6-\frac{1}{y}}$

114. $\dfrac{\frac{1}{5}-\frac{1}{y}}{\frac{1}{y^2}-\frac{1}{25}}$

115. $\dfrac{1-\frac{6}{x}+\frac{8}{x^2}}{3-\frac{5}{x}-\frac{2}{x^2}}$

116. $\dfrac{2+\frac{13}{x}-\frac{7}{x^2}}{3+\frac{1}{x}-\frac{10}{x^2}}$

117. $\dfrac{9-\frac{12}{x}+\frac{4}{x^2}}{9-\frac{4}{x^2}}$

118. $\dfrac{4-\frac{25}{x^2}}{4-\frac{8}{x}-\frac{5}{x^2}}$

119. $\dfrac{\frac{1}{x}+\frac{5}{3x-1}}{\frac{2}{3x-1}-\frac{1}{x}}$

120. $\dfrac{\frac{2}{x-5}-\frac{1}{x}}{\frac{1}{x}-\frac{3}{x-5}}$

121. $\dfrac{\frac{1}{x+1}+\frac{2}{x-2}}{\frac{2}{x-3}-\frac{1}{x-2}}$

122. $\dfrac{\frac{4}{x+5}-\frac{1}{x-3}}{\frac{3}{x-3}+\frac{1}{2x-1}}$

123. $\dfrac{\frac{x-1}{3x-1}-\frac{1}{x+1}}{\frac{x-1}{x+1}-\frac{2}{x+1}}$

124. $\dfrac{\frac{x+1}{3x+5}-\frac{1}{x+3}}{\frac{2}{x+3}-\frac{x+1}{x+3}}$

125. $\dfrac{\frac{2x+3}{2x-3}+\frac{2x-3}{2x+3}}{\frac{2x+3}{2x-3}-\frac{2x-3}{2x+3}}$

126. $\dfrac{\frac{x-1}{x+1}-\frac{x+1}{x-1}}{\frac{x+1}{x-1}-\frac{x-1}{x+1}}$

127. $\dfrac{\frac{1}{2x+5}-\frac{1}{2x-5}+\frac{4x}{4x^2-25}}{\frac{1}{2x+5}+\frac{1}{2x-5}+\frac{4x}{4x^2-25}}$

128. $\dfrac{\frac{1}{3x-1}+\frac{1}{3x+1}}{\frac{3x}{3x-1}-\frac{1}{3x+1}-\frac{6x}{9x^2-1}}$

Answers to odd exercises:

101. $\frac{3}{x(x-3)}$

103. $\frac{x+6}{8x^2}$

105. $\frac{2x-5}{(x-2)(5x+1)}$

107. $\frac{x+3}{x-5}$

109. $\frac{5x^3}{x-15}$

111. $\frac{3x}{x+3}$

113. $-\frac{6y+1}{y}$

115. $\frac{x-4}{3x+1}$

117. $\frac{3x-2}{3x+2}$

119. $-\frac{8x-1}{x-1}$

121. $\frac{3x(x-3)}{(x+1)(x-1)}$

123. $\frac{x}{(3x-1)}$

125. $\frac{4x^2+9}{12x}$

127. $\frac{2x-5}{4x}$

★ **Simplify each complex rational expression.**

129. $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}}$

130. $\dfrac{\dfrac{1}{x}}{1-\dfrac{1}{1+\dfrac{1}{x}}}$

131. $\dfrac{\dfrac{1}{y}-\dfrac{1}{x}}{\dfrac{1}{y^2}-\dfrac{1}{x^2}}$

132. $\dfrac{\dfrac{2}{y}+\dfrac{1}{x}}{\dfrac{4}{y^2}-\dfrac{1}{x^2}}$

133. $\dfrac{\dfrac{1}{25y^2}-\dfrac{1}{x^2}}{\dfrac{1}{x}-\dfrac{1}{5y}}$

134. $\dfrac{16y^2-\dfrac{1}{x^2}}{\dfrac{1}{x}-4y}$

135. $\dfrac{\dfrac{1}{b}+\dfrac{1}{a}}{\dfrac{1}{b^3}+\dfrac{1}{a^3}}$

136. $\dfrac{\dfrac{1}{a}-\dfrac{1}{b}}{\dfrac{1}{b^3}-\dfrac{1}{a^3}}$

137. $\dfrac{\dfrac{x}{y}-\dfrac{y}{x}}{\dfrac{1}{y^2}-\dfrac{2}{xy}+\dfrac{1}{x^2}}$

138. $\dfrac{\dfrac{2}{y}-\dfrac{5}{x}}{4x-\dfrac{25y^2}{x}}$

139. $\frac{x^{-1}+y^{-1}}{y^{-2}-x^{-2}}$

140. $\frac{y^{-2}-25x^{-2}}{5x^{-1}-y^{-1}}$

141. $\frac{1-x^{-1}}{x-x^{-1}}$

142. $\frac{16-x^{-2}}{x^{-1}-4}$

143. $\frac{1-4x^{-1}-21x^{-2}}{1-2x^{-1}-15x^{-2}}$

144. $\frac{x^{-1}-4(3x^2)^{-1}}{3-8x^{-1}+16(3x^2)^{-1}}$

145. $\frac{(x-3)^{-1}+2x^{-1}}{x^{-1}-3(x-3)^{-1}}$

146. $\frac{(4x-5)^{-1}+x^{-2}}{x^{-2}+(3x-10)^{-1}}$

Answers to odd exercises:

129. $\frac{x+1}{2x+1}$

131. $\frac{xy}{x+y}$

133. $-\frac{x+5y}{5xy}$

135. $\frac{a^2b^2}{a^2-ab+b^2}$

137. $\frac{xy(x+y)}{x-y}$

139. $\frac{xy}{x-y}$

141. $\frac{1}{x+1}$

143. $\frac{x-7}{x-5}$

145. $-\frac{3(x-2)}{2x+3}$

E: Mixed Practice with Rational Expressions.

Exercise 0.6.5

★ **Perform the indicated operations. Simplify the result if possible. State the restrictions.**

151. $\dfrac{108x^3}{12x^2}$

152. $\dfrac{56x^2(x-2)^2}{8x(x-2)^3}$

153. $\dfrac{64-x^2}{2x^2-15x-8}$

154. $\dfrac{3x^2+28x+9}{81-x^2}$

155. $\dfrac{x^2-25}{5x^2}\cdot\dfrac{10x^2-15x}{2x^2+7x-15}$

156. $\dfrac{7x^2-41x-6}{(x-7)^2}\cdot\dfrac{49-x^2}{x^2+x-42}$

157. $\dfrac{28x^2(2x-3)}{4x^2-9}\div\dfrac{7x}{4x^2-12x+9}$

158. $\dfrac{x^2-10x+24}{x^2-8x+16}\div\dfrac{2x^2-13x+6}{x^2+2x-24}$

159. $\dfrac{5x-6}{x^2-36}-\dfrac{4x}{x^2-36}$

160. $\dfrac{2}{x}+5x$

161. $\dfrac{5}{x-5}+\dfrac{1}{2x}$

162. $\dfrac{x}{x-2}+\dfrac{3}{x+3}$

163. $\dfrac{7(x-1)}{4x^2-17x+15}-\dfrac{2}{x-3}$

164. $\dfrac{5}{x}-\dfrac{19x+25}{2x^2+5x}$

165. $\dfrac{x}{x-5}-\dfrac{2}{x-3}-\dfrac{5(x-3)}{x^2-8x+15}$

166. $\dfrac{3x}{2x-1}-\dfrac{x-4}{x+4}+\dfrac{12(2-x)}{2x^2+7x-4}$

167. $\dfrac{1}{t-1}+\dfrac{1}{(t-1)^2}-\dfrac{1}{t^2-1}$

168. $\dfrac{1}{t-1}-\dfrac{2t-5}{t^2-2t+1}-\dfrac{5t^2-3t-2}{(t-1)^3}$

169. $2x^{-1}+x^{-2}$

170. $(x-4)^{-1}-2x^{-2}$

171. $\dfrac{\frac{1}{7}+\frac{1}{x}}{\frac{1}{49}-\frac{1}{x^2}}$

172. $\dfrac{\frac{1}{100}-\frac{1}{x^2}}{\frac{1}{x}-\frac{1}{10}}$

173. $\dfrac{\frac{3}{x}-\frac{1}{x-5}}{\frac{5}{x+2}-\frac{2}{x}}$

174. $\dfrac{1-\frac{12}{x}+\frac{35}{x^2}}{1-\frac{25}{x^2}}$

175. $\dfrac{x-4x^{-1}}{2-5x^{-1}+2x^{-2}}$

176. $\dfrac{8x^{-1}+y^{-1}}{y^{-2}-64x^{-2}}$

Answers to odd exercises:

151. $9x;\ x\neq 0$

153. $-\dfrac{x+8}{2x+1};\ x\neq -\dfrac{1}{2}, 8$

155. $\dfrac{x-5}{x};\ x\neq -5, 0, \dfrac{3}{2}$

157. $\dfrac{4x(2x-3)^2}{2x+3};\ x\neq \pm\dfrac{3}{2}, 0$

159. $\dfrac{1}{x+6};\ x\neq \pm 6$

161. $\dfrac{11x-5}{2x(x-5)};\ x\neq 0, 5$

163. $-\dfrac{1}{4x-5};\ x\neq \dfrac{5}{4}, 3$

165. $\dfrac{x-5}{x-3};\ x\neq 3, 5$

167. $\dfrac{t^2+1}{(t+1)(t-1)^2};\ t\neq \pm 1$

169. $\dfrac{2x+1}{x^2};\ x\neq 0$

171. $\dfrac{7x}{x-7};\ x\neq 0, 7, -7$

173. $\dfrac{(x+2)(2x-15)}{(x-5)(3x-4)};$
$x\neq -2, 0, 5, \frac{4}{3}$

175. $\dfrac{x(x+2)}{2x-1}\ d\neq 0, 2, \frac{1}{2}$

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0.07: Review - Solving Linear Equations

Solving Basic Linear Equations

An **equation** is a statement indicating that two algebraic expressions are equal. A **linear equation with one variable**, x, is an equation that can be written in the standard form $ax+b=0$ where a and b are real numbers and $a \neq 0$. For example

$$3x-12=0$$

A **solution** to a linear equation is any value that can replace the variable to produce a true statement. The variable in the linear equation $3x-12=0$ is x and the solution is $x=4$. To verify this, substitute the value 4 in for x and check that you obtain a true statement.

$$\begin{aligned} 3x-12&=0 \\ 3(4)-12&=0 \\ 12-12&=0 \\ 0&=0 \quad \checkmark \end{aligned}$$

Alternatively, when an equation is equal to a constant, we may verify a solution by substituting the value in for the variable and showing that the result is equal to that constant. In this sense, we say that solutions "satisfy the equation."

Example 0.7.1:

Is $a=-\frac{1}{2}$ a solution to $-10a+5=25$?

Solution

Recall that when evaluating expressions, it is a good practice to first replace all variables with parentheses, and then substitute the appropriate values. By making use of parentheses, we avoid some common errors when working the order of operations.

$$-10a+5=-10(-\tfrac{1}{2})+5=5+5=10 \neq 25 \quad \times$$

Answer:

No, $a=-\frac{1}{2}$ does not satisfy the equation.

Developing techniques for solving various algebraic equations is one of our main goals in algebra. This section reviews the basic techniques used for solving linear equations with one variable. We begin by defining **equivalent equations** as equations with the same solution set.

$$\left.\begin{aligned} 3x-5&=16 \\ 3x&=21 \\ x&=7 \end{aligned}\right\} \quad \textit{Equivalent equations}$$

Here we can see that the three linear equations are equivalent because they share the same solution set, namely, $\{7\}$. To obtain equivalent equations, use the following **properties of equality**. Given algebraic expressions A and B and c is a number:

Addition or subtraction property of equality:	If $A=B$, then $A+c=B+c$ or If $A=B$, then $A-c=B-c$
Multiplication or division property of equality:	If $A=B$, then $cA=cB$ or If $A=B$, then $\frac{A}{c}=\frac{B}{c}$, $c \neq 0$

CAUTION: Multiplying or dividing both sides of an equation by 0 should be carefully avoided. Dividing by 0 is undefined and multiplying both sides by 0 results in the equation $0=0$.

We solve algebraic equations by isolating the variable with a coefficient of 1. If given a linear equation of the form $ax+b=c$, then we can solve it in two steps. First, use the appropriate equality property of addition or subtraction to isolate the variable term. Next, isolate the variable using the equality property of multiplication or division. Checking the solution in the following examples is left to the reader.

Example 0.7.2:

Solve: $7x-2=19$.

Solution

$$\begin{aligned} 7x-2&=19 \\ 7x-2\color{blue}{+2}&=19\color{blue}{+2} \quad \textit{Add 2 to both sides.} \\ 7x&=21 \\ \frac{7x}{\color{blue}{7}}&=\frac{21}{\color{blue}{7}} \quad \textit{Divide both sides by 7.} \\ x&=3 \quad \text{The solution set is: } \{3\} \end{aligned}$$

Example 0.7.3:

Solve: $56=8+12y$.

Solution

When no sign precedes a term, it is understood to be positive. In other words, think of this as $56=+8+12y$. Therefore, we begin by subtracting 8 on both sides of the equal sign.

$$\begin{aligned} 56\color{blue}{-8}&=8+12y\color{blue}{-8} \\ 48&=12y \\ \frac{48}{\color{blue}{12}}&=\frac{12y}{\color{blue}{12}} \\ 4&=y \quad \text{The solution set is: } \{4\} \end{aligned}$$

It does not matter on which side we choose to isolate the variable because the **symmetric property** states that $4=y$ is equivalent to $y=4$.

Example 0.7.4:

Solve: $\frac{5}{3}x+2=-8$.

Solution

Isolate the variable term using the addition property of equality, and then multiply both sides of the equation by the reciprocal of the coefficient $\frac{5}{3}$.

$$\begin{aligned} \frac{5}{3}x+2&=-8 \\ \frac{5}{3}x+2\color{blue}{-2}&=-8\color{blue}{-2} \quad \textit{Subtract 2 on both sides.} \\ \frac{5}{3}x&=-10 \\ \frac{3}{\color{blue}{5}}\cdot\frac{5}{3}x&=\frac{3}{\color{blue}{5}}\cdot\left(\frac{\overset{-2}{\cancel{-10}}}{\cancel{5}}\right) \quad \textit{Multiply both sides by } \frac{3}{5}. \\ 1x&=3\cdot(-2) \\ x&=-6 \quad \text{The solution set is: } \{-6\} \end{aligned} \tag{0.7.1}$$

In summary, to retain equivalent equations, we must perform the same operation on both sides of the equation.

Try It 0.7.4

Solve: $\frac{2}{3}x+\frac{1}{2}=-\frac{5}{6}$.

Answer

$x=-2$

General Guidelines for Solving Linear Equations

How To: Solve a Linear Equation

Step 1: Simplify each side of the equation separately using the order of operations and combining all like terms on the same side of the equal sign.

Step 2: Use the appropriate properties of equality (add or subtract) to combine like terms on opposite sides of the equal sign. The goal is to obtain the variable term on one side of the equation and the constant term on the other.

Step 3: Divide or multiply as needed to isolate the variable.

Step 4: Check to see if the answer solves the original equation.

At this point in our study of algebra the use of the properties of equality should seem routine. Therefore, displaying these steps in this text, usually in blue, becomes optional.

Example 0.7.5:

Solve: $-4a+2-a=1$.

Solution

First combine the like terms on the left side of the equal sign.

$$\begin{aligned} -4a+2-a&=1 && \textit{Combine same - side like terms.}\\ -5a+2&=1 && \textit{Subtract 2 on both sides.}\\ -5a&=-1 && \textit{Divide both sides by } -5.\\ a=\frac{-1}{-5}&=\frac{1}{5} && \text{The solution set is: } \left\{\frac{1}{5}\right\} \end{aligned}$$

Always use the original equation to check to see if the solution is correct.

$$\begin{aligned} -4a+2-a&=-4\left(\frac{1}{5}\right)+2-\frac{1}{5}\\ &=-\frac{4}{5}+\frac{2}{1}\cdot\frac{5}{5}-\frac{1}{5}\\ &=\frac{-4+10+1}{5}\\ &=\frac{5}{5}=1\ \checkmark \end{aligned}$$

Given a linear equation in the form $ax+b=cx+d$, we begin the solving process by combining like terms on opposite sides of the equal sign. To do this, use the addition or subtraction property of equality to place like terms on the same side so that they can be combined. In the examples that remain, the check is left to the reader.

Example 0.7.6:

Solve: $-2y-3=5y+11$.

Solution

Subtract $5y$ on both sides so that we can combine the terms involving y on the left side.

$$\begin{aligned} -2y-3-5y &= 5y+11-5y \\ -7y-3 &= 11 \end{aligned}$$

From here, solve using the techniques developed previously.

$$\begin{aligned} -7y-3 &= 11 && \textit{Add 3 to both sides.} \\ -7y &= 14 \\ y &= \frac{14}{-7} && \textit{Divide both sides by } -7. \\ y &= -2 && \text{The solution set is: } \{-2\} \end{aligned}$$

Solving will often require the application of the distributive property.

Example 0.7.7:

Solve: $-\frac{1}{2}(10x-2)+3=7(1-2x)$.

Solution

Simplify the linear expressions on either side of the equal sign first.

$$\begin{aligned} -\frac{1}{2}(10x-2)+3 &= 7(1-2x) && \textit{Distribute} \\ -5x+1+3 &= 7-14x && \textit{Combine same - side like terms.} \\ -5x+4 &= 7-14x && \textit{Combine opposite - side like terms.} \\ 9x &= 3 && \textit{Solve.} \\ x &= \frac{3}{9}=\frac{1}{3} && \text{The solution set is: } \left\{\tfrac{1}{3}\right\} \end{aligned}$$

Example 0.7.8:

Solve: $5(3-a)-2(5-2a)=3$.

Solution

Begin by applying the distributive property.

$$\begin{aligned} 5(3-a)-2(5-2a) &= 3 \\ 15-5a-10+4a &= 3 \\ 5-a &= 3 \\ -a &= -2 \end{aligned}$$

Here we point out that $-a$ is equivalent to $-1a$; therefore, we choose to divide both sides of the equation by -1.

$$\begin{aligned} -a &= -2 \\ \frac{-1a}{-1} &= \frac{-2}{-1} \\ a &= 2 && \text{The solution set is: } \{2\} \end{aligned}$$

Alternatively, we can multiply both sides of $-a=-2$ by negative one and achieve the same result.

$$\begin{aligned} -a &= -2 \\ (-1)(-a) &= (-1)(-2) \\ a &= 2 \end{aligned}$$

Try It 0.7.8

Solve: $6-3(4x-1)=4x-7$.

Answer

$x=1$

There are three different types of equations. Up to this point, we have been solving **conditional equations**. These are equations that are true for particular values. An **identity** is an equation that is true for all possible values of the variable. For example,

$$x=x \quad \textit{Identity}$$

has a solution set consisting of all real numbers, $\mathbb{R}$. A **contradiction** is an equation that is never true and thus has no solutions. For example,

$$x+1=x \quad \textit{Contradiction}$$

has no solution. We use the empty set, $\varnothing$ or $\{\}$, to indicate that there are no solutions.

If the end result of solving an equation is a true statement, like $0=0$, then the equation is an identity and any real number is a solution. If solving results in a false statement, like $0=1$, then the equation is a contradiction and there is no solution.

Example 0.7.9:

Solve: $4(x+5)+6=2(2x+3)$.

Solution

$$\begin{aligned} 4(x+5)+6&=2(2x+3)\\ 4x+20+6&=4x+6\\ 4x+26&=4x+6\\ 26&=6 \ \text{✗} \end{aligned}$$

Solving leads to a false statement; therefore, the equation is a contradiction and there is no solution.

Answer: $\varnothing$ or $\{\ \}$

Example 0.7.10:

Solve: $3(3y+5)+5=10(y+2)-y$.

Solution

$$\begin{aligned} 3(3y+5)+5&=10(y+2)-y\\ 9y+15+5&=10y+20-y\\ 9y+20&=9y+20\\ 9y&=9y\\ 0&=0 \ \text{✓} \end{aligned}$$

Solving leads to a true statement; therefore, the equation is an identity and any real number is a solution.

Answer: $\mathbb{R}$

The coefficients of linear equations may be any real number, even decimals and fractions. When this is the case it is possible to use the multiplication property of equality to clear the fractional coefficients and obtain integer coefficients in a single step. If given fractional coefficients, then multiply both sides of the equation by the least common multiple of the denominators (LCD).

Example 0.7.11:

Solve: $\frac{1}{3}x+\frac{1}{5}=\frac{1}{5}x-1$.

Solution

Clear the fractions by multiplying both sides by the least common multiple of the given denominators. In this case, it is the $LCD(3,5)=15$.

$$\begin{aligned} 15\cdot\left(\frac{1}{3}x+\frac{1}{5}\right) &= 15\cdot\left(\frac{1}{5}x-1\right) && \textit{Multiply both sides by } 15. \\ 15\cdot\frac{1}{3}x+15\cdot\frac{1}{5} &= 15\cdot\frac{1}{5}x-15\cdot 1 && \textit{Simplify.} \\ 5x+3 &= 3x-15 && \textit{Solve.} \\ 2x &= -18 \\ x &= \frac{-18}{2}=-9 && \text{The solution set is: } \{-9\} \end{aligned}$$

It is important to know that this technique only works for equations. *Do not try to clear fractions when simplifying expressions.* As a reminder:

Expression	Equation
$\frac{1}{2}x+\frac{5}{3}$	$\frac{1}{2}x+\frac{5}{3}=0$

We simplify expressions and solve equations. If you multiply an expression by **6**, you will change the problem. However, if you multiply both sides of an equation by **6**, you obtain an equivalent equation.

Incorrect way to SIMPLIFY an EXPRESSION	Correct way to SOLVE an EQUATION
$\frac{1}{2}x+\frac{5}{3}$ $\neq 6\cdot\left(\frac{1}{2}x+\frac{5}{3}\right)$ $=3x+10$ ✗	$\frac{1}{2}x+\frac{5}{3}=0$ $6\cdot\left(\frac{1}{2}x+\frac{5}{3}\right)=6\cdot 0$ $3x+10=0$ ✓ $\boxed{x=-\frac{10}{3}}$

The correct way to simplify an expression involves using the LCD to rewrite each term so that all denominators are the same. This approach can also be used to solve equations, although clearing the fractions (as illustrated above) is simpler.

Correct way to SIMPLIFY an EXPRESSION	Another Correct way to SOLVE an EQUATION
$\frac{1}{2}x+\frac{5}{3}$ $=\frac{3}{3}\cdot\frac{1}{2}x+\frac{2}{2}\cdot\frac{5}{3}$ $=\boxed{\frac{3}{6}x+\frac{10}{6}}$ or $=\boxed{\frac{1}{6}(3x+10)}$	$\frac{1}{2}x+\frac{5}{3}=0$ $\frac{3}{3}\cdot\frac{1}{2}x+\frac{2}{2}\cdot\frac{5}{3}=0$ $\frac{3}{6}x+\frac{10}{3}=0$ $\frac{1}{6}(3x+10)=0$ $6\cdot\frac{1}{6}(3x+10)=6\cdot 0$ $(3x+10)=0$ $\boxed{x=-\frac{10}{3}}$

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0.07e: Exercises - Linear Equations

A: Check a Solution

Exercise 0.7.1

★ **Determine whether or not the given value is a solution.**

1. $-5x+4=-1; x=-1$
2. $4x-3=-7; x=-1$
3. $3y-4=5; y=\frac{9}{3}$
4. $-2y+7=12; y=-\frac{5}{2}$
5. $3a-6=18-a; a=-3$
6. $5(2t-1)=2-t; t=2$
6. $ax-b=0; x=\dfrac{b}{a}$
7. $ax+b=2b; x=\dfrac{b}{a}$

Answers to odd exercises:

1. No 3. Yes 5. No 7. Yes

B: Solve Linear Equations (I)

Exercise 0.7.2

★ **Solve.**

11. $5x-3=27$
12. $6x-7=47$
13. $4x+13=35$
14. $6x-9=18$
15. $9a+10=10$
16. $5-3a=5$
17. $-8t+5=15$
18. $-9t+12=33$
19. $\dfrac{2}{3}x+\dfrac{1}{2}=1$
20. $\dfrac{3}{8}x+\dfrac{5}{4}=\dfrac{3}{2}$
21. $\dfrac{1-3y}{5}=2$
22. $\dfrac{2-5y}{6}=-8$
23. $7-y=22$
24. $6-y=12$
25. Solve for x : $ax-b=c$
26. Solve for x : $ax+b=0$

Answers to odd exercises:

11. 6 13. $\frac{11}{2}$ 15. 0 17. $-\frac{5}{4}$ 19. $\frac{3}{4}$ 21. -3 23. -15 25. $x=\frac{b+c}{a}$

C: Solve Linear Equations (II)

Exercise 0.7.3

★ **Solve.**

31. $6x-5+2x=19$
32. $7-2x+9=24$
33. $12x-2-9x=5x+8$
34. $16-3x-22=8-4x$
35. $5y-6-9y=3-2y+8$
36. $7-9y+12=3y+11-11y$
37. $3+3a-11=5a-8-2a$
38. $2-3a=5a+7-8a$
39. $\dfrac{1}{3}x-\dfrac{3}{2}+\dfrac{5}{2}x=\dfrac{5}{6}x+\dfrac{1}{4}$
40. $\dfrac{3}{8}+\dfrac{1}{5}x-\dfrac{3}{4}=\dfrac{3}{10}x-\dfrac{1}{4}$
41. $1.2x-0.5-2.6x=2-2.4x$
42. $1.59-3.87x=3.48-4.1x-0.51$
43. $5-10x=2x+8-12x$
44. $8x-3-3x=5x-3$
45. $5(y+2)=3(2y-1)+10$
46. $7(y-3)=4(2y+1)-21$
47. $7-5(3t-9)=22$
48. $10-5(3t+7)=20$
49. $5-2x=4-2(x-4)$
50. $2(4x-5)+7x=5(3x-2)$
51. $4(4a-1)=5(a-3)+2(a-2)$
52. $6(2b-1)+24b=8(3b-1)$
53. $\dfrac{2}{3}(x+18)+2=\dfrac{1}{3}x-13$
54. $\dfrac{2}{5}x-\dfrac{1}{2}(6x-3)=\dfrac{4}{3}$
55. $1.2(2x+1)+0.6x=4x$
56. $6+0.5(7x-5)=2.5x+0.3$
57. $5(y+3)=15(y+1)-10y$
58. $3(4-y)-2(y+7)=-5y$
59. $\dfrac{1}{5}(2a+3)-\dfrac{1}{2}=\dfrac{1}{3}a+\dfrac{1}{10}$
60. $\dfrac{3}{2}a=\dfrac{3}{4}(1+2a)-\dfrac{1}{5}(a+5)$
61. $6-3(7x+1)=7(4-3x)$
62. $6(x-6)-3(2x-9)=-9$
63. $\dfrac{3}{4}(y-2)+\dfrac{2}{3}(2y+3)=3$
64. $\dfrac{5}{4}-\dfrac{1}{2}(4y-3)=\dfrac{2}{5}(y-1)$
65. $-2(3x+1)-(x-3)=-7x+1$
66. $6(2x+1)-(10x+9)=0$

Answers to odd exercises:

31. 3 33. -5 35. $-\frac{17}{2}$ 37. $\mathbb{R}$ 39. $\frac{7}{8}$ 41. 2.5 43. $\varnothing$ 45. 3 47. 2 49. $\varnothing$
51. $-\frac{5}{3}$ 53. -81 55. 1.2 57. $\mathbb{R}$ 59. 0 61. $\varnothing$ 63. $\frac{6}{5}$ 65. $\mathbb{R}$

C: Solve Linear Formulas

Exercise 0.7.4

★ **Solve.**

67. Solve for $w: P = 2l + 2w$
68. Solve for $a: P = a + b + c$
69. Solve for $t: D = rt$
70. Solve for $w: V = lwh$
71. Solve for $b: A = \frac{1}{2}bh$
72. Solve for $a: s = \frac{1}{2}at^2$
73. Solve for $a: A = \frac{1}{2}h(a + b)$
74. Solve for $h: V = \frac{1}{3}\pi r^2 h$
75. Solve for $F: C = \frac{5}{9}(F - 32)$
76. Solve for $x: ax + b = c$

Answers to odd exercises:

67. $w = \frac{P-2l}{2}$ 69. $t = \frac{D}{r}$ 71. $b = \frac{2A}{h}$ 73. $a = \frac{2A}{h} - b$ 75. $F = \frac{9}{5}C + 32$.

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0.08: Review - Linear Inequalities in One Variable

Linear Inequalities

A **linear inequality** is a mathematical statement that relates a linear expression as either less than or greater than another. The following are some examples of linear inequalities, all of which are solved in this section:

$5x+7<22$	$-2(x+8)+6\geq 20$	$-2(4x-5)<9-2(x-2)$

A **solution to a linear inequality** is a real number that will produce a true statement when substituted for the variable. Linear inequalities have either infinitely many solutions or no solution. If there are infinitely many solutions, graph the solution set on a number line and/or express the solution using interval notation.

Example 0.8.1:

Are $x=-4$ and $x=6$ solutions to $5x+7<22$?

Solution

Substitute the values in for x, simplify, and check to see if we obtain a true statement.

Check $x=4$:	Check $x=6$:
$5(-4)+7<22$ $-20+7<22$ $-13<22$ ✓	$5(6)+7<22$ $30+7<22$ $37<22$ ✗

Answer: $x=-4$ is a solution and $x=6$ is not

All but one of the techniques learned for solving linear equations apply to solving linear inequalities. You may add or subtract any real number to both sides of an inequality, and you may multiply or divide both sides by any *positive* real number to create equivalent inequalities. For example:

$$\begin{aligned} 10 &> -5 \\ 10-7 &> -5-7 && \textit{Subtract 7 on both sides.} \\ 3 &> -12 \quad ✓ && \textit{True.} \\ 10 &> -5 \\ \tfrac{10}{5} &> \tfrac{-5}{5} && \textit{Divide both sides by 5.} \\ 2 &> -1 \quad ✓ && \textit{True.} \end{aligned}$$

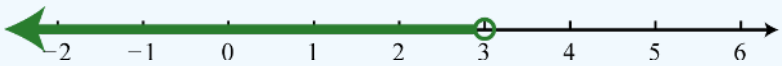
Subtracting 7 from each side and dividing each side by positive 5 results in an inequality that is true.

Example 0.8.2:

Solve and graph the solution set: $5x+7<22$.

Solution

$$\begin{aligned} 5x+7 &< 22 \\ 5x+7-7 &< 22-7 \\ 5x &< 15 \\ \tfrac{5x}{5} &< \tfrac{15}{5} \\ x &< 3 \end{aligned}$$



It is helpful to take a minute and choose a few values in and out of the solution set, substitute them into the original inequality, and then verify the results. As indicated, you should expect $x=0$ to solve the original inequality and that $x=5$ should not.

Check $x = 0$:	Check $x = 5$:
$5(0) + 7 < 22$ $7 < 22$ ✓	$5(5) + 7 < 22$ $25 + 7 < 22$ $32 < 22$ ✗

Checking in this manner gives us a good indication that we have solved the inequality correctly.

We can express this solution set in two ways: using set notation and interval notation.

$$\{x|x < 3\} \quad \textit{Set notation}$$
$$(-\infty, 3) \quad \textit{Interval notation}$$

In this text we will choose to present answers using interval notation.

Answer: $(-\infty,\ 3)$

When working with linear inequalities, a different rule applies when multiplying or dividing by a negative number. To illustrate the problem, consider the true statement $10 > -5$ and divide both sides by -5.

$$10 > -5$$
$$\frac{10}{-5} > \frac{-5}{-5} \quad \textit{Divide both sides by } -5.$$
$$-2 > 1 \quad \text{✗} \textit{False}$$

Dividing by -5 results in a false statement. To retain a true statement, the inequality must be reversed.

$$10 > -5$$
$$\frac{10}{-5} < \frac{-5}{-5} \quad \textit{Reverse the inequality.}$$
$$-2 < 1 \quad \text{✓} \textit{True}$$

The same problem occurs when multiplying by a negative number. This leads to the following new rule: *when multiplying or dividing by a negative number, reverse the inequality*. It is easy to forget to do this so take special care to watch for negative coefficients. In general, given algebraic expressions A and B, where c is a positive nonzero real number, we have the following **properties of inequalities**:

Addition property of inequalities:	If $A < B$, then $A+c < B+c$	
Subtraction property of inequalities:	If $A < B$, then $A-c < B-c$	
Multiplication property of inequalities:	If $A < B$, then $cA < cB$	If $A < B$, then $-cA > -cB$
Division property of inequalities:	If $A < B$, then $\frac{A}{c} < \frac{B}{c}$	If $A < B$, then $\frac{A}{-c} > \frac{B}{-c}$

We use these properties to obtain an **equivalent inequality**, one with the same solution set, where the variable is isolated. The process is similar to solving linear equations.

Example 0.8.3:

Solve and graph the solution set: $-2(x+8)+6 \geq 20$.

Solution

$$\begin{aligned} -2(x+8)+6 &\geq 20 && \textit{Distribute.} \\ -2x-16+6 &\geq 20 && \textit{Combine like terms.} \\ -2x-10 &\geq 20 && \textit{Solve for } x. \\ -2x &\geq 30 && \textit{Divide both sides by } -2. \\ \frac{-2x}{-2} &\leq \frac{30}{-2} && \textit{Reverse the inequality.} \\ x &\leq -15 \end{aligned}$$



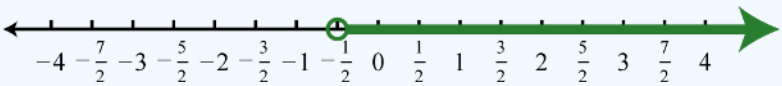
Answer: Interval notation $(-\infty, \ -15]$

Example 0.8.4:

Solve and graph the solution set: $-2(4x-5) < 9-2(x-2)$.

Solution

$$\begin{aligned} -2(4x-5) &< 9-2(x-2) \\ -8x+10 &< 9-2x+4 \\ -8x+10 &< 13-2x \\ -6x &< 3 \\ \frac{-6x}{-6} &> \frac{3}{-6} && \textit{Reverse the inequality.} \\ x &> -\frac{1}{2} \end{aligned}$$



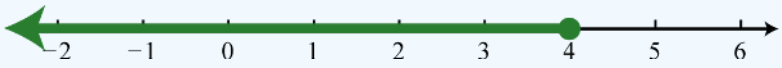
Answer: Interval notation $(-\frac{1}{2}, \infty)$

Example 0.8.5:

Solve and graph the solution set: $\frac{1}{2}x-2 \geq \frac{1}{2}(\frac{7}{4}x-9)+1$.

Solution

$$\begin{aligned} \frac{1}{2}x-2 &\geq \frac{7}{8}x-\frac{9}{2}+1 \\ \frac{1}{2}x-\frac{7}{8}x &\geq -\frac{7}{2}+2 \\ -\frac{3}{8}x &\geq -\frac{3}{2} \\ \left(-\frac{8}{3}\right)\left(-\frac{3}{8}x\right) &\leq \left(-\frac{8}{3}\right)\left(-\frac{3}{2}\right) && \textit{Reverse the inequality.} \\ x &\leq 4 \end{aligned}$$



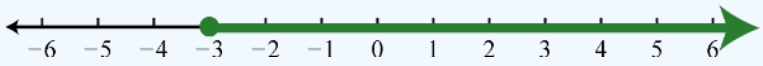
Answer: Interval notation: $(-\infty, \ 4]$

Try It 0.8.5

Solve and graph the solution set: $10-5(2x+3) \leq 25$

Answer

$[-3, \infty)$;



Compound Inequalities

Following are some examples of compound linear inequalities:

$-13 < 3x - 7 < 17$	$4x + 5 \leq -15$ or $6x - 11 > 7$

These **compound inequalities** are actually two inequalities in one statement joined by the word **and** or by the word **or**. For example,

$$-13 < 3x - 7 < 17$$

is a compound inequality because it can be decomposed as follows:

$$-13 < 3x - 7 \text{ and } 3x - 7 < 17$$

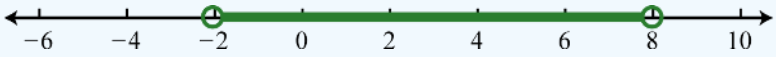
We can solve each inequality individually; the intersection of the two solution sets solves the original compound inequality. While this method works, there is another method that usually requires fewer steps. Apply the properties of this section to all three parts of the compound inequality with the goal of *isolating the variable in the middle* of the statement to determine the bounds of the solution set.

Example 0.8.6:

Solve and graph the solution set: $-13 < 3x - 7 < 17$.

Solution

$$\begin{aligned} -13 &< 3x - 7 < 17 \\ -13+7 &< 3x - 7+7 < 17+7 \\ -6 &< 3x < 24 \\ \tfrac{-6}{3} &< \tfrac{3x}{3} < \tfrac{24}{3} \\ -2 &< x < 8 \end{aligned}$$



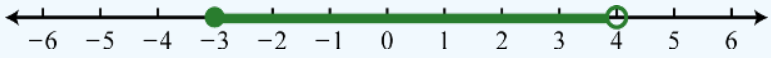
Answer: Interval notation: $(-2, 8)$

Example 0.8.7:

Solve and graph the solution set: $\frac{5}{6} \leq \frac{1}{3}(\frac{1}{2}x + 4) < 2$.

Solution

$$\frac{5}{6} \leq \frac{1}{3}\left(\frac{1}{2}x + 4\right) < 2$$
$$\frac{5}{6} \leq \frac{1}{6}x + \frac{4}{3} < 2$$
$$6 \cdot \left(\frac{5}{6}\right) \leq 6 \cdot \left(\frac{1}{6}x + \frac{4}{3}\right) < 6 \cdot (2)$$
$$5 \leq x + 8 < 12$$
$$5 - 8 \leq x + 8 - 8 < 12 - 8$$
$$-3 \leq x < 4$$



Answer: Interval notation $[-3, 4)$

It is important to note that when multiplying or dividing all three parts of a compound inequality by a negative number, you must reverse all of the inequalities in the statement. For example:

$$-10 < -2x < 20$$
$$\frac{-10}{-2} > \frac{-2x}{-2} > \frac{20}{-2}$$
$$5 > x > -10$$

The answer above can be written in an equivalent form, where smaller numbers lie to the left and the larger numbers lie to the right, as they appear on a number line.

$$-10 < x < 5$$

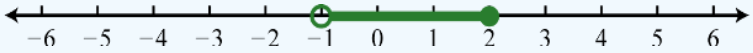
Use interval notation, write: $(-10, 5)$

Try It 0.8.7

Solve and graph the solution set: $-3 \leq -3(2x - 3) < 15$.

Answer

$(-1, 2]$;



"OR" Compound Inequalities

For compound inequalities with the word "*or*" you work both inequalities separately and then consider the union of the solution sets. Values in this union solve either inequality.

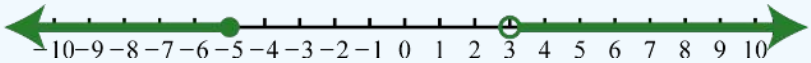
Example 0.8.8:

Solve and graph the solution set: $4x + 5 \leq -15$ or $6x - 11 > 7$.

Solution

Solve each inequality and form the union by combining the solution sets.

$$\begin{aligned} 4x + 5 &\leq -15 \\ 4x &\leq -20 \\ x &\leq -5 \end{aligned} \quad \text{or} \quad \begin{aligned} 6x - 11 &> 7 \\ 6x &> 18 \\ x &> 3 \end{aligned}$$



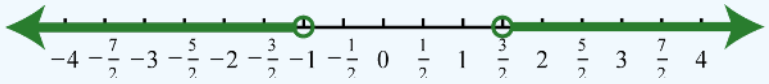
Answer: Interval notation $(-\infty, -5] \cup (3, \infty)$

Try It 0.8.8

Solve and graph the solution set: $5(x-3) < -20$ or $2(5-3x) < 1$.

Answer

$(-\infty, -1) \cup \left(\frac{3}{2}, \infty\right)$



"AND" Compound Inequalities

For compound inequalities with the word "AND" you work both inequalities separately and then consider the intersection of the solution sets. Values in this intersection are solutions to BOTH inequalities.

Example 0.8.9:

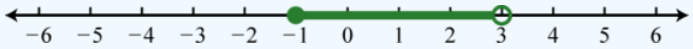
Solve and graph the solution set: $4x - 7 < 5$ and $8x + 9 \geq 1$.

Solution

Solve each inequality and form the union by combining the solution sets.

$$\begin{aligned} 4x - 7 &< 5 \\ 4x &< 12 \\ x &< 3 \end{aligned} \quad \text{and} \quad \begin{aligned} 8x + 9 &\geq 1 \\ 8x &\geq -8 \\ x &\geq -1 \end{aligned}$$

Answer: Interval notation $[-1, 3)$

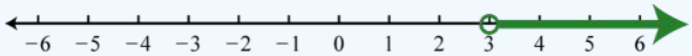


Try It 0.8.9

Solve and graph the solution set: $5x - 2(x-2) < 6x - 5$ and $3 - 4x \leq -1$.

Answer

The two solutions are $x > 3$ and $x \geq 1$. The intersection between the two sets is $x > 3$. In interval notation this is $(3, \infty]$



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0.08e: Exercises- Linear Inequalities

A: Check a solution

Exercise 0.8.1

★ **Determine whether or not the given value is a solution.**

1. $5x-1<-2; x=-1$
2. $-3x+1>-10; x=1$
3. $2x-3<-5; x=1$
4. $5x-7<0; x=2$
5. $9y-4\geq 5; y=1$
6. $-6y+1\leq 3; y=-1$
7. $12a+3\leq -2; a=-\frac{1}{3}$
8. $25a-2\leq -22; a=-\frac{4}{5}$
9. $-10<2x-5<-5; x=-\frac{1}{2}$
10. $3x+8<-2$ or $4x-2>5; x=2$

Answers to odd exercises:

1. Yes	3. No	5. Yes	7. No	9. Yes

B: Solve Linear Inequalities

Exercise 0.8.2

★ **Graph all solutions on a number line and provide the corresponding interval notation.**

11. $3x+5>-4$
12. $2x+1>-1$
13. $5-6y<-1$
14. $7-9y>43$
15. $6-a\leq 6$
16. $-2a+5>5$
17. $\frac{5x+6}{3}\leq 7$
18. $\frac{4x+11}{6}\leq \frac{1}{2}$
19. $\frac{1}{2}y+\frac{5}{4}\geq \frac{1}{4}$
20. $\frac{1}{12}y+\frac{2}{3}\leq \frac{5}{6}$
21. $2(3x+14)<-2$
22. $5(2y+9)>-15$
23. $5-2(4+3y)\leq 45$
24. $-12+5(5-2x)<83$
25. $6(7-2a)+6a\leq 12$
26. $2a+10(4-a)\geq 8$
27. $9(2t-3)-3(3t+2)<30$
28. $-3(t-3)-(4-t)>1$
29. $\frac{1}{2}(5x+4)+\frac{5}{6}x>-\frac{4}{3}$
30. $\frac{2}{5}+\frac{1}{6}(2x-3)\geq \frac{1}{15}$
31. $5x-2(x-3)<3(2x-1)$
32. $3(2x-1)-10>4(3x-2)-5x$
33. $-3y\geq 3(y+8)+6(y-1)$
34. $12\leq 4(y-1)+2(2y+1)$
35. $-2(5t-3)-4>5(-2t+3)$
36. $-7(3t-4)>2(3-10t)-t$
37. $\frac{1}{2}(x+5)-\frac{1}{3}(2x+3)>\frac{7}{6}x+\frac{3}{2}$
38. $-\frac{1}{3}(2x-3)+\frac{1}{4}(x-6)\geq \frac{1}{12}x-\frac{5}{4}$
39. $4(3x+4)\geq 3(6x+5)-6x$
40. $1-4(3x+7)<-3(x+9)-9x$
41. $6-3(2a-1)\leq 4(3-a)+1$
42. $12-5(2a+6)\geq 2(5-4a)-a$

Answers to odd exercises:

11. $(-3,\infty)$;



Figure 1.8.11

13. $(1,\infty)$;

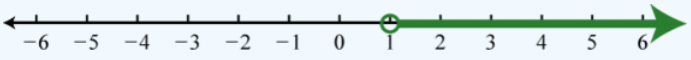

Figure 0.8e.13

15. $[0,\infty)$;



Figure 0.8e.15

17. $(-\infty,3]$;

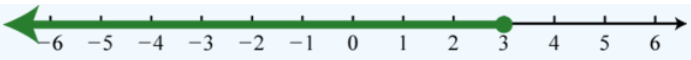

Figure 0.8e.17

19. $[-2, \infty)$;

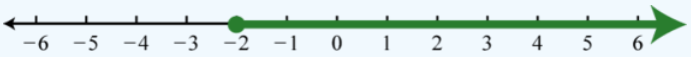

Figure 0.8e.19

21. $(-\infty, -5)$;

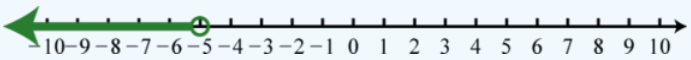

Figure 0.8e.21

23. $[-8, \infty)$;

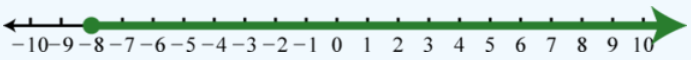

Figure 0.8e.23

25. $[5, \infty)$;

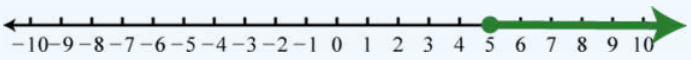

Figure 0.8e.25

27. $(-\infty, 7)$;



Figure 0.8e.27

29. $(-1, \infty)$;



Figure 0.8e.29

31. $(3, \infty)$;

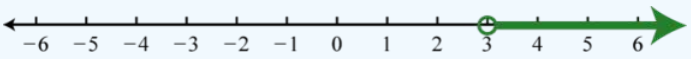

Figure 0.8e.31

33. $\left(-\infty, -\frac{3}{2}\right]$;

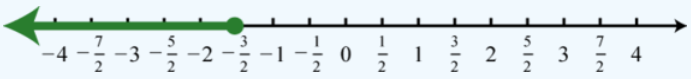

Figure 0.8e.33

35. $\emptyset$;

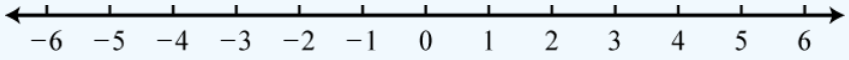

Figure 0.8e.35

37. $(-\infty, 0)$;



Figure 0.8e.37

39. $\mathbb{R}$;

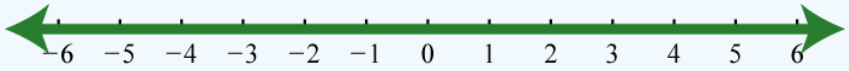

Figure 0.8e.39

41. $[-2, \infty)$;

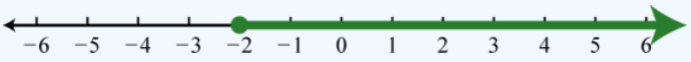

Figure 0.8e.41

C: Solve Compound Linear Inequalities

Exercise 0.8.3

★ **Graph all solutions on a number line and provide the corresponding interval notation.**

43. $-1 < 2x + 1 < 9$
44. $-4 < 5x + 11 < 16$
45. $-7 \le 6y - 7 \le 17$
46. $-7 \le 3y + 5 \le 2$
47. $-7 < \frac{3x+1}{2} \le 8$
48. $-1 \le \frac{2x+7}{3} < 1$
49. $-4 \le 11 - 5t < 31$
50. $15 < 12 - t \le 16$
51. $-\frac{1}{3} \le \frac{1}{6}a + \frac{1}{3} \le \frac{1}{2}$
52. $-\frac{1}{6} < \frac{1}{3}a + \frac{5}{6} < \frac{3}{2}$
53. $5x + 2 < -3$ or $7x - 6 > 15$
54. $4x + 15 \le -1$ or $3x - 8 \ge -11$
55. $8x - 3 \le 1$ or $6x - 7 \ge 8$
56. $6x + 1 < -3$ or $9x - 20 > -5$
57. $8x - 7 < 1$ or $4x + 11 > 3$
58. $10x - 21 < 9$ or $7x + 9 \ge 30$
59. $7 + 2y < 5$ or $20 - 3y > 5$
60. $5 - y < 5$ or $7 - 8y \le 23$
61. $15 + 2x < -15$ or $10 - 3x > 40$
62. $10 - \frac{1}{3}x \le 5$ or $5 - \frac{1}{2}x \le 15$
63. $9 - 2x \le 15$ and $5x - 3 \le 7$
64. $5 - 4x > 1$ and $15 + 2x \ge 5$
65. $7y - 18 < 17$ and $2y - 15 < 25$
66. $13y + 20 \ge 7$ and $8 + 15y > 8$
67. $5 - 4x \le 9$ and $3x + 13 \le 1$
68. $17 - 5x \ge 7$ and $4x - 7 > 1$
69. $9y + 20 \le 2$ and $7y + 15 \ge 1$
70. $21 - 6y \le 3$ and $-7 + 2y \le -1$
71. $-21 < 6(x - 3) < -9$
72. $0 \le 2(2x + 5) < 8$
73. $-15 \le 5 + 4(2y - 3) < 17$
74. $5 < 8 - 3(3 - 2y) \le 29$
75. $5 < 5 - 3(4 + t) < 17$
76. $-3 \le 3 - 2(5 + 2t) \le 21$
77. $-40 < 2(x + 5) - (5 - x) \le -10$
78. $-60 \le 5(x - 4) - 2(x + 5) \le 15$
79. $-\frac{1}{2} < \frac{1}{30}(x - 10) < \frac{1}{3}$
80. $-\frac{1}{5} \le \frac{1}{15}(x - 7) \le \frac{1}{3}$
81. $-1 \le \frac{a + 2(a-2)}{5} \le 0$
82. $0 < \frac{5 + 2(a-1)}{6} < 2$

Answers to odd exercises:

43. $(-1, 4)$;

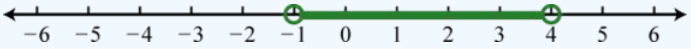

Figure 0.8e.43

45. $[0, 4]$;



Figure 0.8e.45

47. $(-5, 5]$;



Figure 0.8e.47

49. $(-4, 3]$;



Figure 0.8e.49

51. $[-4, 1]$;



Figure 0.8e.51

53. $(-\infty, -1) \cup (3, \infty)$;

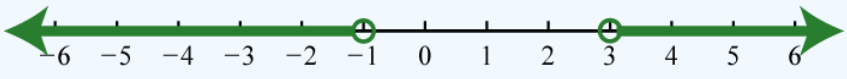

Figure 0.8e.53

55. $(-\infty, \frac{1}{2}] \cup [\frac{5}{2}, \infty)$;

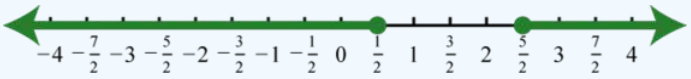

Figure 0.8e.55

57. $\mathbb{R}$;

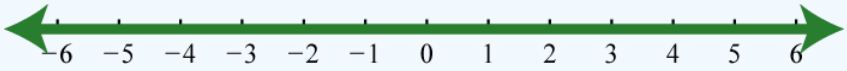

Figure 0.8e.57

59. $(-\infty, 5)$;

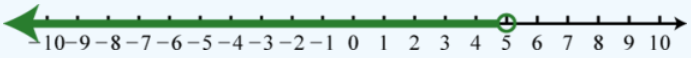

Figure 0.8e.59

61. $(-\infty, -10)$;

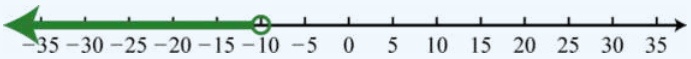

Figure 1.8.61

63. $[-3, 2]$;



Figure 0.8e.63

65. $(-\infty, 5)$;



Figure 0.8e.65

67. $\emptyset$;

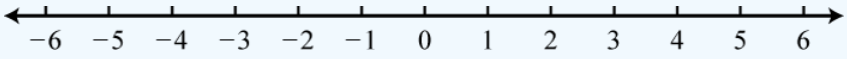

Figure 0.8e.67

69. -2;

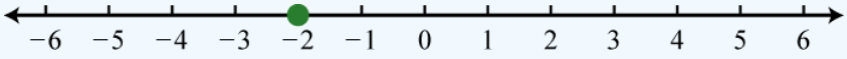

Figure 0.8e.69

71. $(-\frac{1}{2}, \frac{3}{2})$;

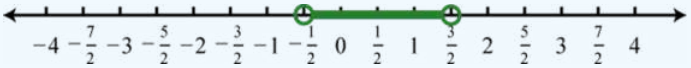

Figure 0.8e.71

73. $[-1, 3)$;

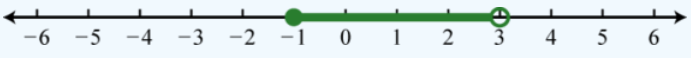

Figure 0.8e.73

75. $(-8, -4)$;

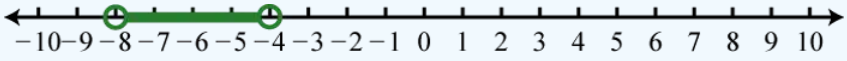

Figure 0.8e.75

77. $(-15, -5]$;

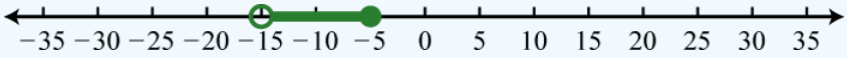

Figure 0.8e.77

79. $(-5, 20)$;

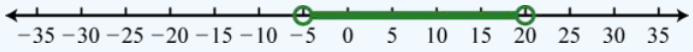

Figure 0.8e.79

81. $[-\frac{1}{3}, \frac{4}{3}]$;

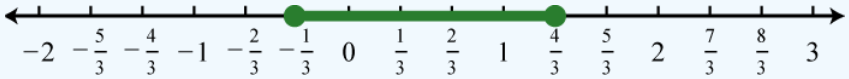

Figure 0.8e.81

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0.09: Review - Graphs of Linear Equations

Learning Objectives

- Find the slope of a line
- Graph a line given a point and the slope
- Graph a line using its slope and intercept
- Use slopes to identify parallel and perpendicular lines

Find the Slope of a Line

A. Find the slope of a line given a graph

When you graph linear equations, you may notice that some lines tilt up as they go from left to right and some lines tilt down. Some lines are very steep and some lines are flatter. In mathematics, the measure of the steepness of a line is called the *slope* of the line. We can assign a numerical value to the slope of a line by finding the ratio of the rise and run. The *rise* is the amount the vertical distance changes while the *run* measures the horizontal change, as shown in this illustration.

SLOPE OF A LINE

The slope of a line is $m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in vertical distance}}{\text{change in horizontal distance}}$.

The rise measures the vertical change and run measures the horizontal change.

When the **vertical** change is **positive**, the direction is **up**.
When the **vertical** change is **negative**, the direction is **down**.
When the **horizontal** change is **positive**, the direction is to the **right**.
When the **horizontal** change is **negative**, the direction is to the **left**.

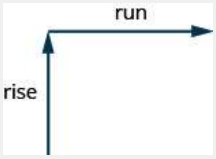


Figure 1 Slope

To find the slope of a line, locate two points on the line whose coordinates are integers. Draw a path from one point to the other consisting of a vertical line and a horizontal line. The vertical distance is called the *rise* and the horizontal distance is called the *run*.

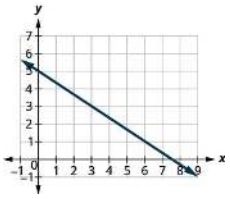
HOW TO FIND THE SLOPE OF A LINE FROM ITS GRAPH USING $m = \frac{\text{rise}}{\text{run}}$

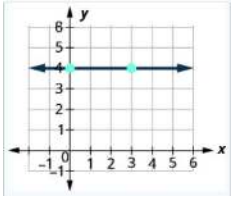
1. Locate two points on the line whose coordinates are integers
2. Starting with one point, sketch a vertical line and a horizontal line that goes from the first point to the second point. Count the vertical rise and the horizontal run.
   Rise is **positive** if you go **up** to get to the second point; rise is **negative** if you go **down** to get to the second point.
   Run is **positive** if you go **right** to get to the second point; run is **negative** if you go **left** to get to the second point.
   Remember, $\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$ and $a = \frac{a}{1}$ and $\frac{a}{b} = \frac{-a}{-b}$.
3. Take the ratio of rise to run to find the slope: $m = \frac{\text{rise}}{\text{run}}$.

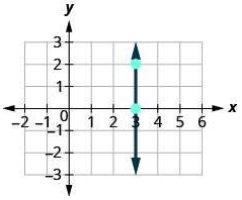
The slope of a **horizontal line** is zero (because the rise is **0**). The slope of a **vertical line** is undefined (because the run is **0**).

Example 0.9.1

Find the slope of the lines shown.

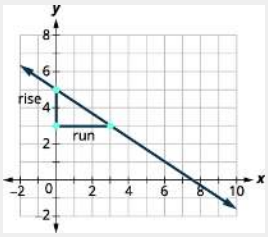
a. 

b. 

c. 

Solution

a.

Step 1. Locate two points on the graph whose coordinates are integers.	$(0,5)$ and $(3,3)$
Step 2. Starting at $(0,5)$, sketch a right triangle to $(3,3)$ as shown in this graph. Count the rise — since it goes down, the rise is negative. Count the run — since it goes up, the run is positive.	 The rise is -2 or down 2 units. The run is 3 or right 3 units.
Step 3. Use the slope formula. Substitute the values of the rise and run. Simplify.	$m = \frac{\text{rise}}{\text{run}}$ $m = \frac{-2}{3} = -\frac{2}{3}$

The slope of the line is $-\frac{2}{3}$. So y decreases by 2 units as x increases by 3 units.

b.

Step 1. Two points on the line are $(0,4)$ and $(3,4)$.
Step 2. The rise is 0; the run is 3.
Step 3. $m = \frac{\text{rise}}{\text{run}} = \frac{0}{3} = 0$
All horizontal lines have slope 0. When the y-coordinates are the same, the rise is 0

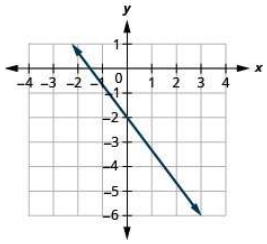
c.

Step 1. Two points on the line are $(3,0)$ and $(3,2)$.
Step 2. The rise is 2; the run is 0.
Step 3. $m = \frac{\text{rise}}{\text{run}} = \frac{2}{0} =$ undefined.
The slope is undefined since division by zero is undefined.
All vertical lines have an undefined slope. When the x-coordinates are the same, the run is 0.

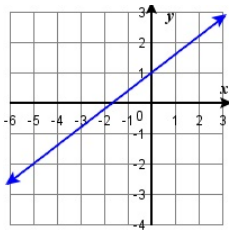
Try It 0.9.2

Find the slope of the lines shown.

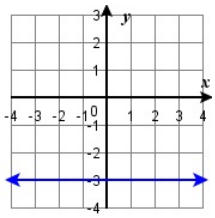
a.



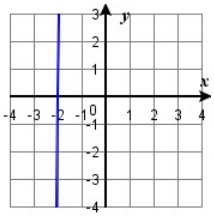
b.



c.



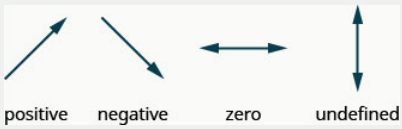
d.



Answer

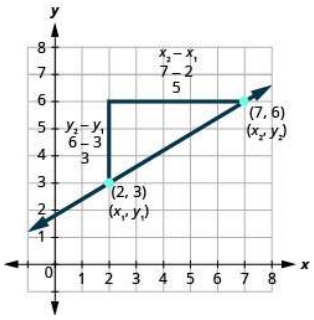
a. $-\frac{4}{3}$ or down 4 and right 3 because $-\frac{4}{3} = \frac{-4}{3}$. Alternatively, up 4 and left 3 because $-\frac{4}{3} = \frac{4}{-3}$.
b. $\frac{3}{5}$ or up 3 and right 5. Alternatively, down 3 and left 5 because $\frac{3}{5} = \frac{-3}{-5}$.
c. $\frac{0}{3}$ or zero. All horizontal lines have a slope of zero because the y-coordinates are the same so the rise is zero.
d. $\frac{3}{0}$ or undefined. All vertical lines have an undefined slope because the x-coordinates are the same so the run is zero.

QUICK GUIDE TO THE SLOPES OF LINES



B. Find the slope given 2 points on a line

There are occasions when the slope of a line between two points must be found but it isn't practical or possible to start with a graph and count out the rise and the run. Instead, a formula can be used to find the slope.



We have seen that an ordered pair (x, y) gives the coordinates of a point. But when we work with slopes, we use two points. To represent two points in general and to distinguish between the two points, subscripts are used.

Consider finding the slope of the line between the points $(2, 3)$ and $(7, 6)$, as shown in this graph.

On the graph, we count a rise of 3 (up 3) and a run of 5 (right 5) so $m = \frac{\text{rise}}{\text{run}} = \frac{3}{5}$.

Since we have two points, subscript notation is used to represent the two points so $(2, 3) = (x_1, y_1)$ and $(7, 6) = (x_2, y_2)$.

The rise of 3 can be found by subtracting the second y-coordinate minus first y-coordinate: $\text{rise} = 3 = 6 - 3 = y_2 - y_1$.

The run of 5 can be found by subtracting the second x-coordinate minus first x-coordinate: $\text{run} = 5 = 7 - 2 = x_2 - x_1$.

Thus, the slope $m = \frac{\text{rise}}{\text{run}} = \frac{3}{5}$ can more generally be expressed with the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$.

SLOPE OF A LINE BETWEEN TWO POINTS

The slope of the line between two points (x_1, y_1) and (x_2, y_2) is:

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{change in } y \text{ values}}{\text{change in } x \text{ values}} = \frac{\Delta y}{\Delta x}.$$

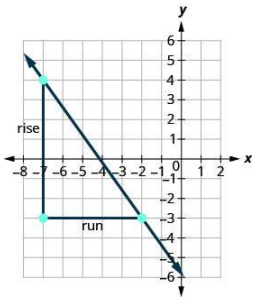
Example 0.9.3

Use the slope formula to find the slope of the line through the points $(-2,-3)$ and $(-7,4)$.

Solution

Choose (−2,−3) to be point #1 and (−7,4) to be point #2. (The same result is obtained if the designations for point #1 and point #2 are reversed).	$\begin{pmatrix} x_1, & y_1 \\ -2 & -3 \end{pmatrix} \quad \begin{pmatrix} x_2, & y_2 \\ -7 & 4 \end{pmatrix}$
Use the slope formula.	$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-(-3)}{-7-(-2)} = \frac{7}{-5} = -\frac{7}{5}$
Simplify	$m = \frac{7}{-5}$ up 7, left 5 $m = \frac{-7}{5}$ down 7, right 5

The graph verifies the result for the slope



Try It 0.9.4

Use the slope formula to find the slope of the line through the pair of points:

a. $(-3,4)$ and $(2,-1)$. b. $(-2,6)$ and $(-3,-4)$. c. $(-3,4)$ and $(2,4)$. d. $(2,6)$ and $(2,-1)$.

Answer

a. $\frac{-5}{5} = -1$ or down 1 right 1 because the slope is $-1 = \frac{-1}{1}$. Alternatively, up 1 left 1 because $-1 = \frac{1}{-1}$

b. $\frac{-10}{-1} = 10$ or up 10 right 1 because the slope is $10 = \frac{10}{1}$. Alternatively, down 10 left 1 because $10 = \frac{-10}{-1}$

c. $\frac{0}{5} = 0$ or a horizontal line

d. $\frac{-7}{0} =$ **undefined** or a vertical line

C. IDENTIFY slope and y-intercept from a linear equation

GENERAL FORM OF AN EQUATION OF A LINE

The **general form** of an equation of a line is $\boxed{Ax + By = C}$.

The general equation for a line is $ax + by = c$. This equation can be rewritten in the form, $y = mx + b$. Consider the equation $y = \frac{1}{2}x + 3$ written in this form and its graph.

In this graph, the red lines show us the rise is 1 and the run is 2. Substituting into the slope formula $m = \frac{\text{rise}}{\text{run}}$, the slope is $m = \frac{1}{2}$.

In this graph, the y-intercept is $(0, 3)$.

Comparing these values to the line written in the form, $y = mx + b$, we see that the slope is $m = \frac{1}{2}$ and y-intercept is $(0, 3)$.

In conclusion, the equation written in the form $y = mx + b$ is said to be in **slope–intercept form**. Written in this form, the **coefficient of the x term is the slope** and **the constant term is the y-coordinate of the y-intercept**.

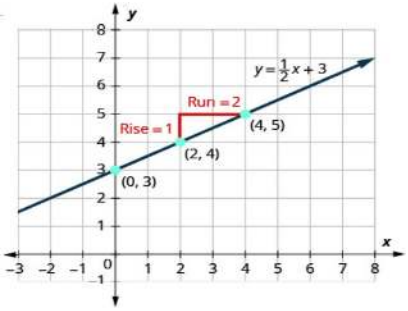


Figure 2: $y = \frac{1}{2}x + 3$

SLOPE INTERCEPT FORM OF AN EQUATION OF A LINE

The **slope–intercept form** of an equation of a line with slope m and y-intercept, $(0, b)$ is $\boxed{y = mx + b}$.

Two special situations are the following.

A **horizontal line**, $\boxed{y=b}$, has a slope $m=0$ and a y-intercept at $(0,b)$.

A **vertical line**, $\boxed{x=a}$, has an undefined slope and does not have a y-intercept.

Example 0.9.5

Identify the slope and y-intercept of the line from the equation:

a. $y=-\frac{4}{7}x-2$ b. $x+3y=9$ c. $x=8$ d. $y=-5$.

Solution

a. We compare our equation to the slope–intercept form of the equation.

Compare the slope–intercept form of the equation of the line with $y=mx+b$	$y=-\frac{4}{7}x-2$ $y=mx+b$
Identify the slope and the y-intercept.	slope is $m=-\frac{4}{7}$; y-intercept is $(0,-2)$

b. When an equation of a line is not given in slope–intercept form, our first step will be to solve the equation for y.

Write the equation in slope-intercept form. Subtract x from each side. Divide both sides by 3. Simplify.	$x+3y=9$ $3y=-x+9$ $\frac{3y}{3}=\frac{-x+9}{3}$ $y=-\frac{1}{3}x+3$
Compare the slope–intercept form of the equation of the line with $y=mx+b$.	$y=-\frac{1}{3}x+3$ $y=mx+b$
Identify the slope and the y-intercept.	slope, $m=-\frac{1}{3}$ or down 1, right 3;yy-intercept is $(0,3)$

c. $x=8$ This is a vertical line. Its slope is undefined. It does not have a y-intercept.

d. $y=-5$ This is a horizontal line which can also be written $y=0x-5$. It has slope 0. Its y-intercept is $(0,-5)$.

Try It 0.9.6

Identify the slope and y-intercept from the equation of the line.

a. $y=\frac{2}{5}x-1$ b. $x+4y=8$ c. $y=-\frac{4}{3}x+1$ d. $3x+2y=12$ e. $x=-4$ f. $y=7$

Answer

a. $m=\frac{2}{5}$ or up 2, right 5; y-intercept $(0,-1)$	b. $m=-\frac{1}{4}$ or down 1, right 4; y-intercept $(0,2)$
c. $m=-\frac{4}{3}$ or down 4 right 3; y-intercept $(0,1)$	d. $m=-\frac{3}{2}$ or down 3 right 2; y-intercept $(0,6)$
e. $x=-4$ is a vertical line. Slope is undefined. There is no y-intercept.	f. $y=7$ is a horizontal line. Slope is 0. Its y-intercept is $(0,7)$.

Graph a line

D. Graph a Line Given a Point and the Slope

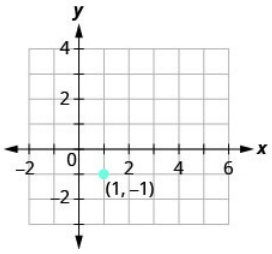
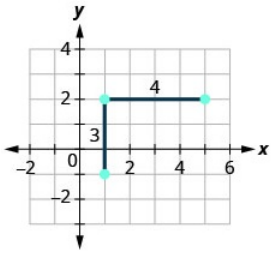
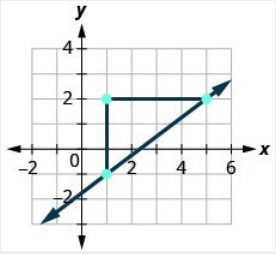
Up to now we have found the slope and y-intercept of a line from a graph and from a linear equation.

Now we will use these skills to graph a line after determining its slope and y-intercept. We will start by graphing a line with a stated point on the line and the slope of the line.

Example 0.9.7: How to graph a Line Given a Point and the Slope

Graph the line passing through the point $(1, -1)$ whose slope is $m = \frac{3}{4}$.

Solution

Step 1. Plot the given point, $(1, -1)$.	
Step 2. Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.	$m = \frac{3}{4} = \frac{\text{rise}}{\text{run}}$ rise = 3 and run = 4
Step 3. Starting at the given point $(1, -1)$, count out the rise (up 3), and run (right 4), to mark the second point.	
Step 4. Connect the points with a line.	

You can check your work by finding a third point. Since the slope is $m = \frac{3}{4}$, it can also be written as $m = \frac{-3}{-4}$ (negative divided by negative is positive!). Go back to $(1, -1)$ and count out the rise, -3 (down 3), and the run, -4 (left 4).

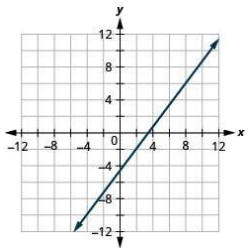
Try It 0.9.8

Graph the line described below.

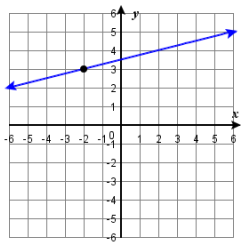
a. passing through the point $(2, -2)$ with the slope $m = \frac{4}{3}$.
b. passing through the point $(-2, 3)$ with the slope $m = \frac{1}{4}$.

Answer

a.



b.



E. Graph a Linear Equation Using its Slope and Intercept

We have graphed a line when given the slope and a point. We also know how to find the slope and y-intercept of a line from its equation. Therefore, we can use the y-intercept as the point, and then count out the slope from there.

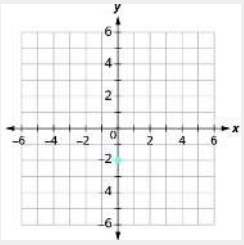
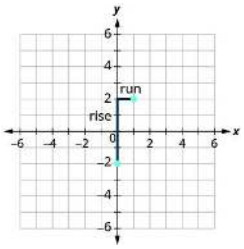
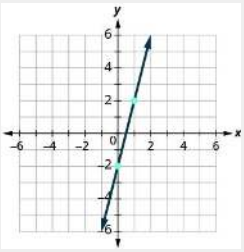
HOW TO GRAPH A LINE GIVEN A POINT AND THE SLOPE.

1. Write the equation in slope-intercept form $y = mx + b$, and identify the y-intercept $(0, b)$ and the slope m.
   - If the equation looks like $x = a$, the graph is a vertical line with undefined slope that intersects the x-axis at $(a, 0)$ and has no y-intercept.
2. Plot the y-intercept (on the y-axis).
3. Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run. Starting at the given point, count out the rise and run to mark the second point.
   If the rise is positive, go up; if the rise is negative, go down.
   If the run is positive, go right; if the run is negative, go left.
   Remember, $m = \frac{m}{1}$, and also $-\frac{u}{v} = \frac{-u}{v} = \frac{u}{-v}$, and $\frac{u}{v} = \frac{-u}{-v}$.
4. Connect the points with a line.

Example 0.9.9: How to Graph a Line Using its Slope and Intercept

Graph the line of the equation $y = 4x - 2$ using its slope and y-intercept.

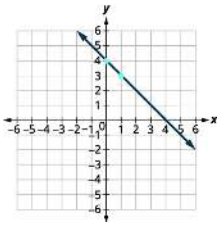
Solution

Step 1. Find the slope–intercept form of the equation and identify the slope and y-intercept.	$y = 4x - 2$ $y = mx + b$ $y = 4x + (-2)$ slope is $m = 4$ $b = -2$, so the y-intercept is $(0, -2)$
Step 2. Plot the y-intercept $(0, -2)$.	
Step 3. Use the slope formula $\frac{\text{rise}}{\text{run}}$ to identify the rise over the run. $m = 4 = \frac{\text{rise}}{\text{run}} = \frac{4}{1}$ so rise = 4 (up 4), run = 1 (right 1) Starting from the y-intercept at $(0, -2)$, count out the rise (up 4) and run (right 1) to mark the second point. Alternatively, $m = \frac{-4}{-1}$ so rise = -4 (down 4), run = -1 (left 1)	rise = 4 (up 4), run = 1 (right 1) 
Step 4. Connect the points with a line.	

Example 0.9.10

Graph the line of the equation $y = -x + 4$ using its slope and y-intercept.

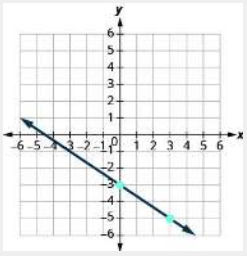
Solution

Step 1. Find the slope–intercept form of the equation and identify the slope and y-intercept.	$y = -x + 4$ slope is $m = -1$ y-intercept is $(0, 4)$
Step 2. Plot the y-intercept $(0, 4)$. **Step 3.** Identify the rise over the run. Starting from the y-intercept at $(0, 4)$, count out the rise (down 1) and run (right 1) to mark the second point.	$m = -1 = \frac{-1}{1}$ rise -1 or down 1, and run 1 or right 1 Alternatively, $m = -1 = \frac{1}{-1}$ rise 1 or up 1, and run -1 or left 1
Step 4. Connect the points with a line.	

Example 0.9.11

Graph the line of the equation $y = -\frac{2}{3}x - 3$ using its slope and y-intercept.

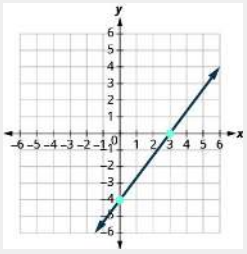
Solution

Step 1. The equation is in slope–intercept $y = mx + b$ form. Identify the slope and y-intercept.	$y = -\frac{2}{3}x - 3$ slope is $m = -\frac{2}{3}$; y-intercept is $(0, -3)$
Step 2. Plot the y-intercept.	
Step 3. Identify the rise and the run. Count out the rise and run to mark the second point. To fill in points to the left of the y-axis, use the equivalent slope, $\frac{2}{-3}$ or rise = up 2, run = left 3.	$m = \frac{-2}{3}$, so rise = down 2, run = right 3 Alternatively, $m = \frac{2}{-3}$, so rise = up 2, run = left 3
Step 4. Connect the points with a line. Draw the line.	

Example 0.9.12

Graph the line of the equation $4x - 3y = 12$ using its slope and y-intercept.

Solution

Step 1. Write the equation is in slope–intercept $y = mx + b$ form. Identify the slope and y-intercept.	$4x - 3y = 12$ $-3y = -4x + 12$ $-\frac{3y}{3} = \frac{-4x+12}{-3}$ $y = \frac{4}{3}x - 4$ slope is $m = \frac{4}{3}$ y-intercept is $(0, -4)$
Step 2. Plot the y-intercept.	
Step 3. Identify the rise and the run. Count out the rise and run to mark the second point.	rise = up 4, run = right 3 for slope $m = \frac{4}{3}$. Alternatively, rise = down 4, run = left 3 for slope $m = \frac{-4}{-3}$
Step 4. Connect the points with a line. Draw the line.	

Try It 0.9.14

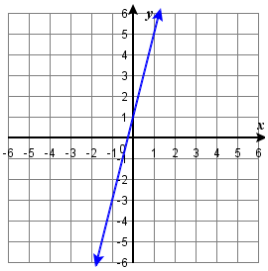
Use slope and y-intercept to graph the following lines.

a. $y = 4x + 1$ b. $y = 2x - 3$ c. $y = -x - 3$ d. $y = -x - 1$

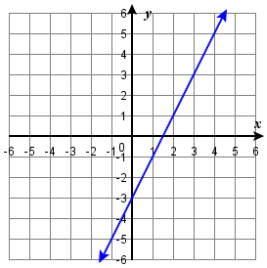
e. $y = -\frac{5}{2}x + 1$ f. $y = -\frac{3}{4}x - 2$ g. $2x - y = 6$ h. $3x - 2y = 8$

Answer

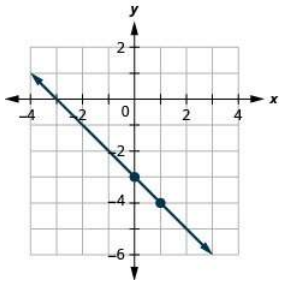
a.



b.



c.



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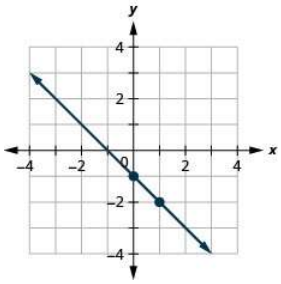
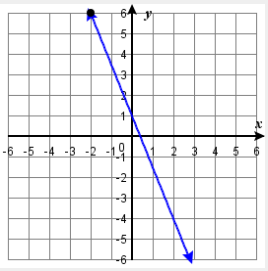
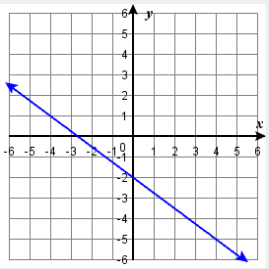


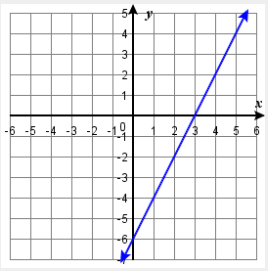
e.



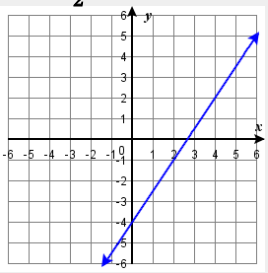
f.



g. $y = 2x - 6$



h. $y = \frac{3}{2}x - 4$



Example 0.9.15

Graph the following lines: a. $y = 5$ b. $4x - 5y = 20$ c. $x = -3$ d. $y = -\frac{5}{9}x + 8$

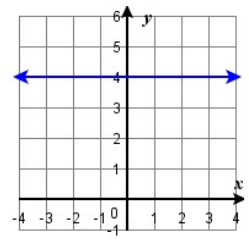
Solution

a. $y = 5$ This equation has only one variable, y. It could be rewritten as $y = 0x + 5$ which illustrates it has a slope of zero. Its graph is a horizontal line crossing the y-axis at 5.

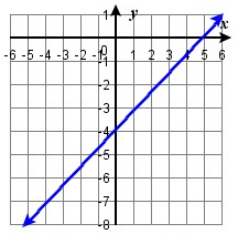
b. $4x - 5y = 20$ This equation is of the form $Ax + By = C$. Solve for y and write the equation in intercept form: $-5y = -4x + 20$ so $y = \frac{4}{5}x - 4$. Slope is $\frac{4}{5}$ and the y-intercept is $(0, -4)$, so plot point $(0, -4)$ and from there go up 4 units and right 5 units. Then plot another point. Etc. Then reverse direction and plot points to the left of the y-axis by starting at point $(0, -4)$ and from there go down 4 units and left 5 units (because $\frac{4}{5} = \frac{-4}{-5}$).

c. $x = -3$ There is only one variable, x. The graph is a vertical line crossing the x-axis at -3.

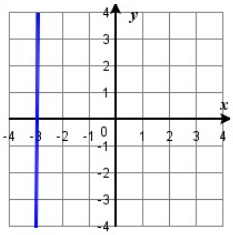
d. $y = -\frac{5}{9}x + 8$ Since this equation is in $y = mx + b$ form, the slope is $-\frac{5}{9}$ and the y-intercept is $(0, 8)$. Plot point $(0, 8)$ and from there go down 5 units and right 9 units. Then plot another point. Etc. Then reverse direction and plot points to the left of the y-axis by starting at point $(0, 8)$ and from there go up 5 units and left 9 units (because $-\frac{5}{9} = \frac{-5}{9} = \frac{5}{-9}$).



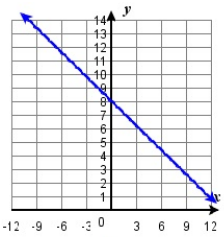
$y = 5$



$4x - 5y = 20$



$x = -3$



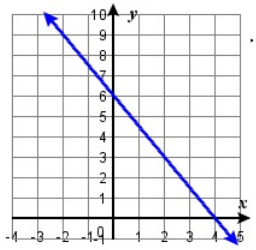
$y = -\frac{5}{9}x + 8$

Try It 0.9.16

Graph each line:	a. $3x + 2y = 12$	b. $y = 4$	c. $y = \frac{1}{5}x - 4$	d. $x = -7$
	e. $x = 6$	f. $y = -\frac{3}{4}x + 1$	g. $y = -8$	h. $4x - 3y = -1$

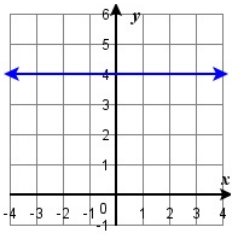
Answer

a.



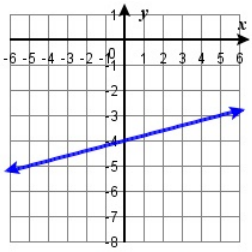
$3x + 2y = 12$

b.



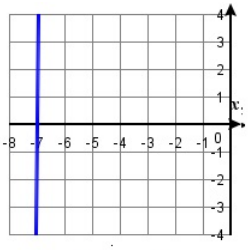
$y = 4$

c.



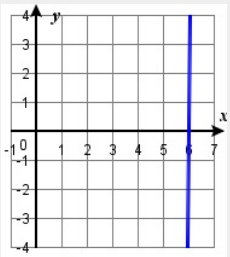
$y = \frac{1}{5}x - 4$

d.



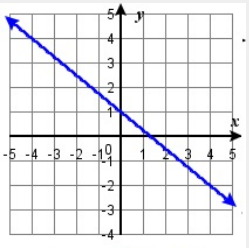
$x = -7$

e.



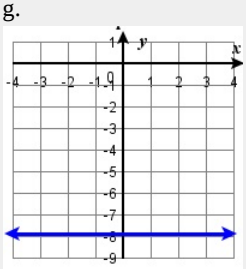
$x = 6$

f.



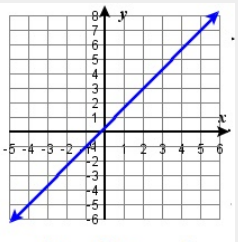
$y = -\frac{3}{4}x + 1$

g.



$y = -8$

h.



$4x - 3y = -1$

For (h) $y = \frac{4}{3}x + \frac{1}{3}$ so the y-intercept of 1/3 is not a "nice" point to graph. In this case find a value of x that will make y be a nice whole number. When $x = 2$ then $y = \frac{8}{3} + \frac{1}{3} = \frac{9}{3} = 3$, so use the point $(2, 3)$ to start with.

Use Slopes to Identify Parallel and Perpendicular Lines

Parallel lines

Two lines that have the same slope are called **parallel lines**.

We say this more formally in terms of the rectangular coordinate system.

Two lines that have the same slope and have different y-intercepts (i.e. never intersect) are parallel lines. Verify that both lines in Figure 3 to the right have the same slope, $m = \frac{2}{5}$, and different y-intercepts.

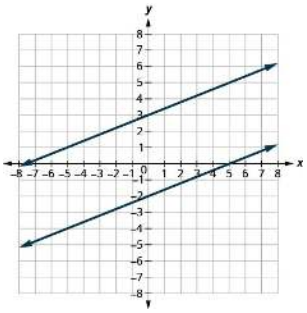


Figure 3 Parallel Lines

Two lines that have the same slope but also have the same y-intercept also could be said to be parallel since they don't intersect at just one point but rather share the same points everywhere along the line. Consequently, a line is parallel to itself.

What about vertical lines? The slope of a vertical line is undefined, so vertical lines don't quite fit in the definition above. We say that vertical lines that have different x-intercepts are parallel, like the lines shown in Figure 4 on the right.

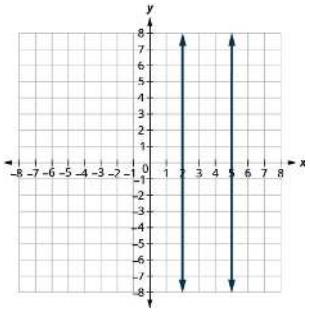


Figure 4 Vertical Lines

Vertical lines that have the same equation (and thus the same x-intercept) are also parallel because it can be said that a line is parallel to itself.

PARALLEL LINES

Parallel lines are lines in the same plane that have the same slope.

- If m_1 and m_2 are the slopes of two parallel lines then $m_1 = m_2$.

Parallel lines that have the same slope and different y-intercepts do not intersect.

Parallel vertical lines that have different x-intercepts do not intersect.

Lines that have the same equation represent the same line and could also be said to be parallel since a line is parallel to itself.

We can now just look at the slope–intercept form of the equations of lines and decide if the lines are parallel.

Example 0.9.17

Use slopes and y-intercepts to determine if the lines are parallel:

a. $3x - 2y = 6$ and $y = \frac{3}{2}x + 1$	b. $y = 2x - 3$ and $-6x + 3y = -9$	c. $y = -4$ and $y = 3$	d. $x = -2$ and $x = -5$

Solution

a.

Solve both equations for y and write in slope-intercept form $y = mx + b$.	$3x - 2y = 6$ $-2y = -3x + 6$ $\frac{-2y}{-2} = \frac{-3x+6}{-2}$ $y = \frac{3}{2}x - 3$	$y = \frac{3}{2}x + 1$
Identify the slope (m) and y-intercept of both lines.	$m = \frac{3}{2}$ y-intercept is $(0, -3)$	$m = \frac{3}{2}$ y-intercept is $(0, 1)$

The lines have the same slope and different y-intercepts and so they are parallel.

b.

Solve both equations for y and write in slope-intercept form $y = mx + b$.	$y = 2x - 3$	$-6x + 3y = -9$ $3y = 6x - 9$ $\frac{3y}{3} = \frac{6x-9}{3}$ $y = 2x - 3$
Identify the slope (m) and yy-intercept of both lines.	$m = 2$ y-intercept is $(0, -3)$	$m = 2$ y-intercept is $(0, -3)$

The lines have the same slope, but they also have the same y-intercepts. Their equations represent the same line so we say the lines are **coincident**. Since a line is parallel to itself, coincident lines are parallel lines.

c. $y = -4$ and $y = 3$

We recognize right away from the equations that these are horizontal lines, and so we know their slopes are both 0.
Since the horizontal lines cross the y-axis at $y = -4$ and at $y = 3$, we know the y-intercepts are $(0, -4)$ and $(0, 3)$.
The lines have the same slope and different y-intercepts and so they are parallel.

d. $x = -2$ and $x = -5$

We recognize right away from the equations that these are vertical lines, and so we know their slopes are undefined.
Since the vertical lines cross the x-axis at $x = -2$ and $x = -5$, we know the x-intercepts are $(-2, 0)$ and $(-5, 0)$.
The lines are vertical and have different x-intercepts and so they are parallel.

Try It 0.9.18

Use slopes and y-intercepts to determine if the lines are parallel:

a. $2x + 5y = 5$ and $y = -\frac{2}{5}x - 4$	b. $y = -\frac{1}{2}x - 1$ and $x + 2y = -2$	c. $4x - 3y = 6$ and $y = \frac{4}{3}x - 1$	d. $y = \frac{3}{4}x - 3$ and $3x - 4y = 12$
e. $y = 8$ and $y = -6$	f. $x = 1$ and $x = -5$	g. $y = 1$ and $y = -5$	h. $x = 8$ and $x = -6$

Answer

a. parallel b. the same line (parallel) c. parallel d. the same line (parallel)
e. parallel horizontal lines f. parallel vertical lines g. parallel horizontal lines h. parallel vertical lines

Perpendicular Lines

Let's look at the lines whose equations are $y = \frac{1}{4}x - 1$ and $y = -4x + 2$, shown in Figure 5 on the right.

These lines lie in the same plane and intersect in right angles. We call these lines perpendicular.

If we look at the slope of the first line, $m_1 = \frac{1}{4}$, and the slope of the second line, $m_2 = -4$, we can see that they are *negative reciprocals* of each other. If we multiply these two slopes together, their product is -1 because $m_1 \cdot m_2 = \frac{1}{4} \cdot (-4) = -1$.

This is always true for **perpendicular lines** and leads us to this definition.

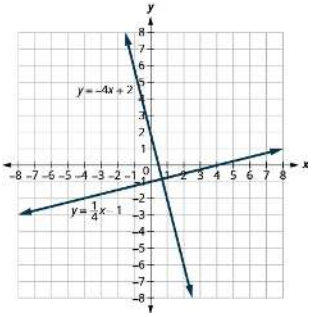


Figure 5. Perpendicular Lines

PERPENDICULAR LINES

Perpendicular lines are lines in the same plane that form a right angle.

- If m_1 and m_2 are the slopes of two perpendicular lines, then:
  - their slopes are negative reciprocals of each other, $m_1 = -\frac{1}{m_2}$.
  - the product of their slopes is -1, $m_1 \cdot m_2 = -1$.
- A vertical line and a horizontal line are always perpendicular to each other

We were able to look at the slope–intercept form of linear equations and determine whether or not the lines were parallel. We can do the same thing for perpendicular lines.

We find the slope–intercept form of the equation, and then see if the slopes are opposite reciprocals. If the product of the slopes is -1, the lines are perpendicular.

Example 0.9.19

Use slopes to determine if the lines are perpendicular:

a. $y = -5x - 4$ and $x - 5y = 5$ b. $7x + 2y = 3$ and $2x + 7y = 5$

Solution

a.	$y = -5x - 4$	$x - 5y = 5$
Solve both equations for y and write in slope-intercept form $y = mx + b$.		$-5y = -x + 5$ $\frac{-5y}{-5} = \frac{-x+5}{-5}$ $y = \frac{1}{5}x - 1$
Identify the slope (m) of both lines.	$m_1 = -5$	$m_2 = \frac{1}{5}$

The slopes are negative reciprocals of each other, so the lines are perpendicular. We check by multiplying the slopes, Since $-5(\frac{1}{5}) = -1$, it checks.

b.	$7x + 2y = 3$	$2x + 7y = 5$
Solve both equations for y and write in slope-intercept form $y = mx + b$.	$2y = -7x + 3$ $\frac{2y}{2} = \frac{-7x+3}{2}$ $y = -\frac{7}{2}x + \frac{3}{2}$	$7y = -2x + 5$ $\frac{7y}{7} = \frac{-2x+5}{7}$ $y = -\frac{2}{7}x + \frac{5}{7}$
Identify the slope (m) of both lines.	$m_1 = -\frac{7}{2}$	$m_2 = -\frac{2}{7}$

The slopes are reciprocals of each other, but they have the same sign. Since they are not negative reciprocals, the lines are not perpendicular.

Try It 0.9.20

Use slopes to determine if the lines are perpendicular:

a. $y = -3x + 2$ and $x - 3y = 4$	b. $5x + 4y = 1$ and $4x + 5y = 3$	c. $y = 2x - 5$ and $x + 2y = -6$	d. $2x - 9y = 3$ and $9x - 2y = 1$

Answer

a. perpendicular b. not perpendicular c. perpendicular d. not perpendicular

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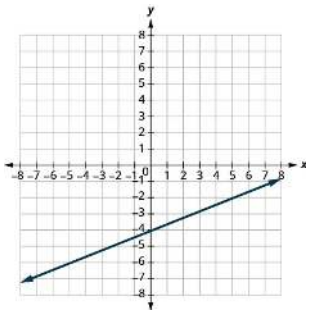
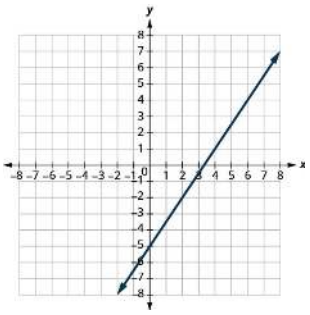
0.09e: Exercises - Graphs of Linear Equations

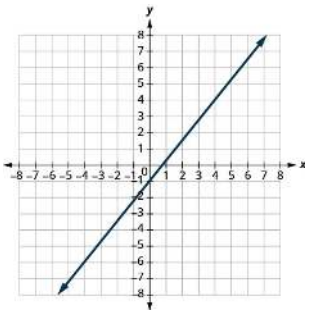
A: Find the slope of a line given its graph

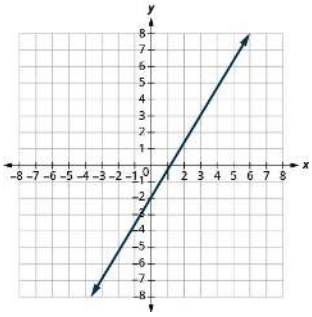
Exercise $0.9.A$: Find the Slope of a Line given its Graph

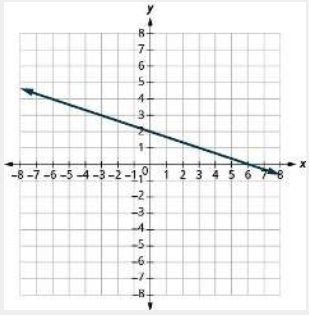
Find the slope of each line shown.

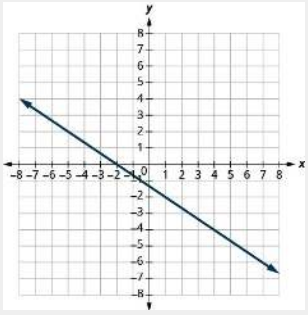
1. 

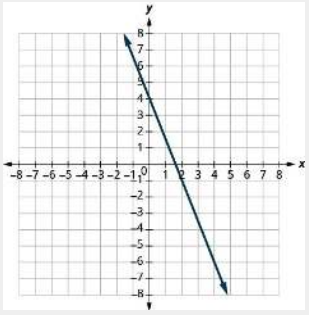
2. 

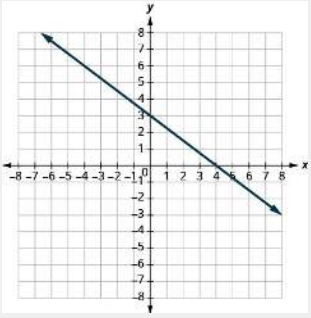
3. 

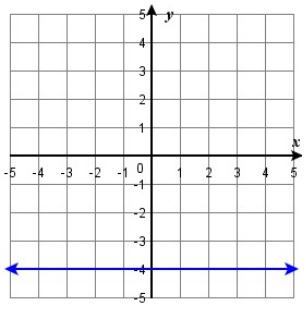
4. 

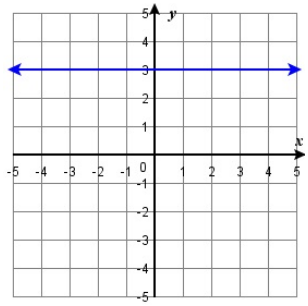
5. 

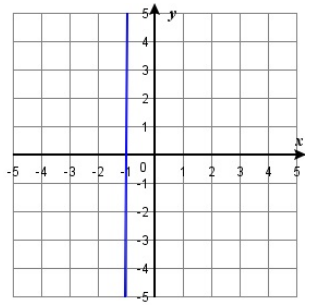
6. 

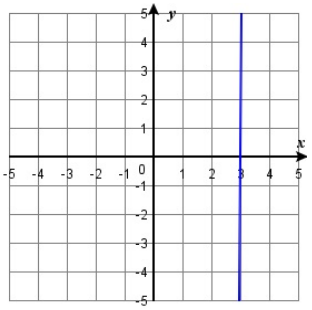
7. 

8. 

9. 

10. 

11. 

12. 

Answers to odd exercises.

1. $m = \frac{2}{5}$ 3. $m = \frac{5}{4}$ 5. $m = -\frac{1}{3}$ 7. $m = -\frac{5}{2}$ 9. $m = 0$ 11. m = undefined

★

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B: Calculate the slope of a line given 2 points on the line

Exercise 0.9.B: Calculate Slope given 2 points

Use the slope formula to find the slope of the line between each pair of points.

13. $(2,5),\ (4,0)$	14. $(3,6),\ (8,0)$	15. $(-3,3),\ (4,-5)$	16. $(-2,4),\ (3,-1)$
17. $(-1,-2),\ (2,5)$	18. $(-2,-1),\ (6,5)$	19. $(4,-5),\ (1,-2)$	20. $(3,-6),\ (2,-2)$

Answers to odd exercises.

13. $m=-\frac{5}{2}$ 15. $m=-\frac{8}{7}$ 17. $m=\frac{7}{3}$ 19. $m=-1$

⋆

⋆

C: Identify attributes given a linear equation

Exercise 0.9.C: Identify Slope and y-intercept given a linear equation

Identify the slope and y-intercept of each line.

29. $y=-7x+3$	30. $y=4x-10$	31. $3x+y=5$	32. $4x+y=8$
33. $6x+4y=12$	34. $8x+3y=12$	35. $5x-2y=6$	36. $7x-3y=9$
37. $y=3$	38. $y=-2$	39. $x=-5$	40. $x=4$

Use slopes and y-intercepts to determine if the lines are parallel, perpendicular, or neither.

41. $y=\frac{3}{4}x-3;\ 3x-4y=-2$	42. $3x-4y=-2;\ y=\frac{3}{4}x-3$	43. $2x-4y=6;\ x-2y=3$	44. $8x+6y=6;\ 12x+9y=12$
45. $x=5;\ x=-6$	46. $x=-3;\ x=-2$	47. $4x-2y=5;\ 3x+6y=8$	48. $8x-2y=7;\ 3x+12y=9$
49. $3x-6y=12;\ 6x-3y=3$	50. $9x-5y=4;\ 5x+9y=-1$	51. $7x-4y=8;\ 4x+7y=14$	52. $5x-2y=11;\ 5x-y=7$
53. $3x-2y=8;\ 2x+3y=6$	54. $2x+3y=5;\ 3x-2y=7$	55. $3x-2y=1;\ 2x-3y=2$	56. $2x+4y=3;\ 6x+3y=2$
57. $y=2;\ y=6$	58. $y=-1;\ y=2$		

Answers to odd exercises.

29. $m=-7;\ (0,3)$ 31. $m=-3;\ (0,5)$ 33. $m=-\frac{3}{2};\ (0,3)$ 35. $m=\frac{5}{2};\ (0,-3)$
37. $m=0;\ (0,3)$ 39. slope is undefined; no y-intercept
41. parallel 43. lines are the same 45. parallel vertical lines 47. perpendicular 49. neither 51. perpendicular
53. perpendicular 55. neither 57. parallel horizontal lines

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D: Graph a line given its attributes

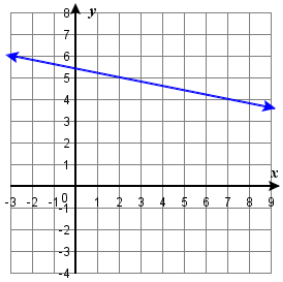
Exercise 0.9.D: Graph a line given its attributes

Graph each line with the given point and slope.

71. $(2,5);\ m=-\frac{1}{3}$	72. $(1,4);\ m=-\frac{1}{2}$	73. $(-1,-4);\ m=\frac{4}{3}$	74. $(-3,-5);\ m=\frac{3}{2}$
75. y-intercept: $(0,3);\ m=-\frac{2}{5}$	76. x-intercept: $(-2,0);\ m=\frac{3}{4}$	77. $(-4,2);\ m=4$	78. $(1,5);\ m=-3$

Answers to odd exercises.

71.



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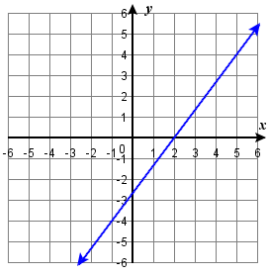

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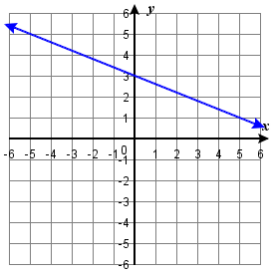

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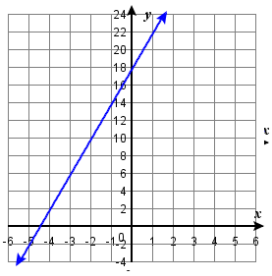

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E: Graph a linear equation using the slope-intercept method

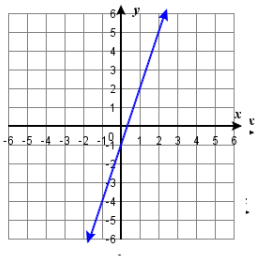
Exercise 0.9.E: Graph a linear equation

Graph the line of each equation using its slope and *y*-intercept.

83. $y=3x-1$	84. $y=2x-3$	85. $y=-x+3$	86. $y=-x-4$
87. $y=-\frac{2}{5}x-3$	88. $y=-\frac{3}{5}x+2$	89. $3x-2y=4$	90. $3x-4y=8$

Answers to odd exercises.

83.



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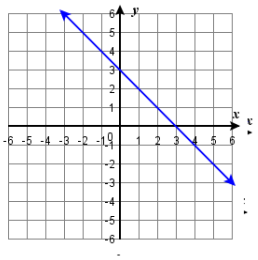

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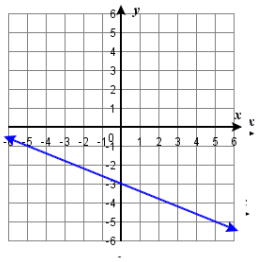
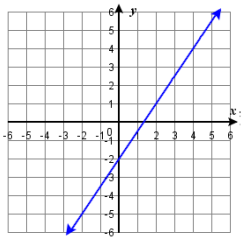


89.



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0.10: Review - Find the Equation of a Line

Learning Objectives

By the end of this section, you will be able to:

- Find an equation of the line given the graph of a line
- Find an equation of the line given the slope and y-intercept
- Find an equation of the line given the slope and a point
- Find an equation of the line given two points
- Find an equation of a line parallel to a given line
- Find an equation of a line perpendicular to a given line

Before you get started, take this readiness quiz.

1. Simplify: $\frac{2}{5}(x+15)$.
2. Simplify: $-3(x-(-2))$.
3. Solve for y: $y-3=-2(x+1)$.

EQUATIONS OF A LINE

The **general form** of an equation for *any* line is $\boxed{Ax+By=C}$.

The **slope-intercept form** of an equation of a line with slope m and y-intercept b is $\boxed{y=mx+b}$. Any line except a vertical line can be written in slope-intercept form.

A **horizontal line**, $\boxed{y=b}$, has a slope $m=0$ and a y-intercept at $(0,b)$.
A **vertical line**, $\boxed{x=a}$, has an undefined slope and does not have a y-intercept. It has an x-intercept at $(a,0)$.

We can easily determine the slope and intercept of a line if the equation is written in slope-intercept form, $y=mx+b$. Now we will do the reverse—we will start with the slope and y-intercept and use them to find the equation of the line.

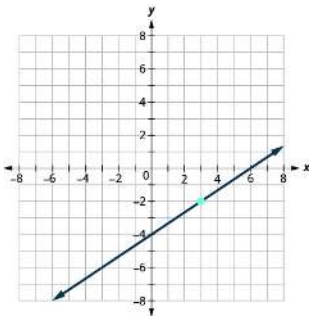
Find an Equation of a Line Given a Graph

Sometimes, the slope and intercept need to be determined from the graph.

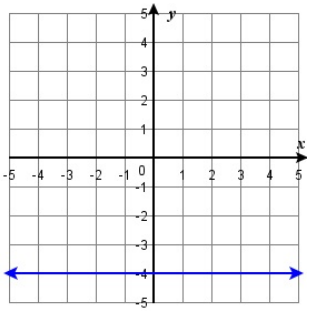
Example 0.10.1 Equation of a line given a graph

Find the equation of the line shown in the graph.

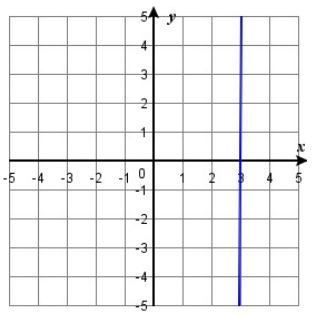
a.



b.



c.



a. Solution

We need to find the slope and y-intercept of the line from the graph so we can substitute the needed values into the slope-intercept form, $y=mx+b$.

To find the slope, we choose two points on the graph.

The y-intercept is $(0,-4)$ and the graph passes through $(3,-2)$.

Step 1. Find the slope, by counting the rise and run.	$m=\frac{\text{rise}}{\text{run}}=\frac{2}{3}$
Step 2. Find the y-intercept.	y-intercept is $(0,-4)$
Step 3. Substitute the values into $y=mx+b$.	$y=mx+b$ $y=\frac{2}{3}x-4$

b. Solution

The graph is a horizontal line, we know the form of the equation for the line is $y=b$. The y intercept is $(0,-4)$ so the equation is $y=-4$.

The slope of this line is zero because the y coordinates of all the points on the line are the same value of -4. Thus the rise, which is the numerator of the slope equation, is zero. The y-intercept is $(0,-4)$ so substitution in the equation $y=mx+b$ produces $y=0x+(-4)$ or $y=-4$.

c. Solution

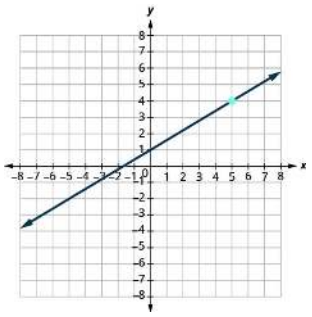
The graph is a vertical line, so we know the form of the equation for the line is $x=a$. The x intercept is $(3,0)$ so the equation is $x=3$.

The slope of this line is undefined because the x coordinates of all the points on the line are the same value of 3. Thus the run, which is the denominator of the slope equation, is zero, and division by zero is undefined, and thus the slope-intercept form of an equation for a line cannot be used.

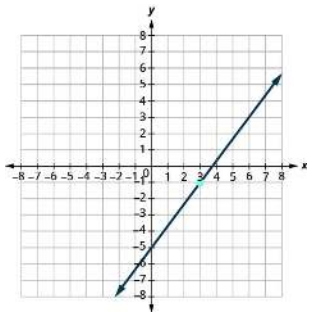
Try It 0.10.2

Find the equation of the following lines.

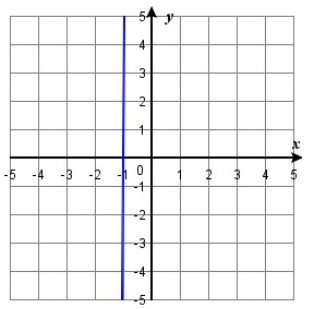
a.



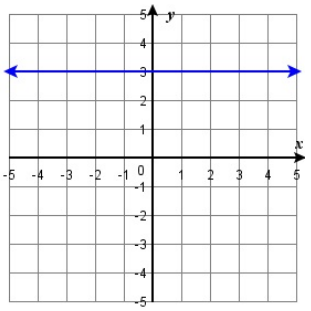
b.



c.



d.



Answer

a. $y=\frac{3}{5}x+1$ b. $y=\frac{4}{3}x-5$ c. $x=-1$ d. $y=3$

Find an Equation of the Line Given the Slope and y-Intercept

Sometimes we are told what the slope and y-intercept are.

Example 0.10.3 Equation of a line given the slope and y-intercept

Find the equation of a line with slope -9 and y-intercept $(0,-4)$.

Solution

Since we are given the slope and *y*-intercept of the line, we can substitute the needed values into the slope-intercept form, $y = mx + b$.

Step 1. State the slope.	$m = -9$
Step 2. State the *y*-intercept.	y-intercept is $(0, -4)$
Step 3. Substitute the values into $y = mx + b$. Simplify.	$y = mx + b$ $y = -9x + (-4)$ $y = -9x - 4$

Try It 0.10.4

Find the equation of a line with the following attributes.

a. slope $\frac{2}{5}$, *y*-intercept $(0, 4)$ b. slope -1, *y*-intercept $(0, -3)$

Answer

a. $y = \frac{2}{5}x + 4$ b. $y = -x - 3$

Find an Equation of the Line Given the Slope and a Point

Finding an equation of a line using the slope-intercept form of the equation works well when you are given the slope and *y*-intercept or when you read them off a graph. When given a point on the line instead of the *y*-intercept a slightly different approach must be used to obtain the equation of the line.

One approach directly uses the familiar **slope-intercept form** of an equation for a line $y = mx + b$. First substitute the values of the given slope m, and coordinates x and y of the given point into the formula $y = mx + b$. Solve for the value for b. Then rewrite the slope-intercept formula for a line with the now known values for m and b.

Alternatively, a different form of an equation for a line, called the **point-slope form**, can be used. The derivation of this equation begins by supposing we have a line that has slope m and that contains some specific point (x_1, y_1) and some other point, which we will just call (x, y). The slope of this line is $m = \frac{y - y_1}{x - x_1}$. Multiplying both sides of this equation by $(x - x_1)$ produces $m(x - x_1) = y - y_1$, which is the point-slope form of an equation for a line.

POINT-SLOPE FORM OF AN EQUATION OF A LINE

The **point-slope form** of an equation of a line with slope *m* and containing the point (x_1, y_1) is: $\boxed{y - y_1 = m(x - x_1)}$

How to: Find an equation of a line given the slope and a point.

Point-Slope Method	Slope-Intercept Method
1. Identify the slope, m. 2. Identify the point, (x_1, y_1). 3. Substitute values of m, x_1, and y_1 into the point-slope form, $y - y_1 = m(x - x_1)$. 4. Solve for y. Write in slope-intercept form, $y = mx + b$.	1. Identify the slope, m. 2. Identify the point, (x, y). 3. Substitute values of m, x, and y into the slope-intercept form $y = mx + b$. 4. Solve for b. Substitute values of m and b into the slope-intercept form $y = mx + b$.

Example 0.10.5 Equation of a line given the slope and a point

Find an equation of a line given the following slope and point on the line.

a. slope $m = -\frac{1}{3}$, point $(6, -4)$	b. slope $m = 0$, point $(-2, -6)$	c. Undefined slope, point $(1, 5)$

a. Solution

	Point-Slope Method	Slope-Intercept Method
Step 1. Identify the slope.	$m=-\frac{1}{3}$	$m=-\frac{1}{3}$
Step 2. Identify the point.	$\begin{pmatrix} x_1, & y_1 \\ 6 & -4 \end{pmatrix}$	$\begin{pmatrix} x, & y \\ 6 & -4 \end{pmatrix}$
Step 3. Substitute.	$y-y_1=m(x-x_1)$ $y-(-4)=-\frac{1}{3}(x-6)$	$y=mx+b$. $-4=-\frac{1}{3}(6)+b$
Step 4. Solve and write the equation in slope-intercept form.	$y+4=-\frac{1}{3}x+2$ $y=-\frac{1}{3}x-2$	$-4=-2+b$ $b=-2$ $y=-\frac{1}{3}x-2$

b. Solution

A line with zero slope is a horizontal line, so all the points on the line have the same y coordinate. Since $(-2,-6)$ is a point on the line, all y-coordinates are -6, so the equation is $y=-6$.

Alternatively, the Point-Slope method could be used to find the equation:

Step 1. Identify the slope. Every horizontal line has slope 0.	$m=0$
Step 2. Identify the point.	$\begin{pmatrix} x_1, & y_1 \\ -2 & -6 \end{pmatrix}$
Step 3. Substitute the values into the point-slope form.	$y-y_1=m(x-x_1)$ $y-(-6)=0(x--2)$
Step 4. Simplify. Write in slope-intercept form. Notice the result is in the form of a horizontal line, $y=a$	$y+6=0$ $y=-6$

Finally, if the Slope-Intercept Method were used instead, substitution in Step 3 yields $-6=0(-2)+b$ or $b=-6$. Putting the values of m and b into the equation $y=mx+b$ produces $y=0x+(-6)$ or $y=-6$.

c. Solution

A line with undefined slope is a vertical line, so all the points on the line have the same x coordinate. Since $(1,5)$ is a point on the line, all x-coordinates are x, so the equation is $x=1$.

Since the slope is undefined, neither the point-slope method nor the slope-intercept method would work.

Try It 0.10.6

Find an equation of a line given the following slope and point on the line.

a. Slope $m=-\frac{2}{5}$ and containing the point $(10,-5)$	b. Slope $m=-\frac{3}{4}$, and containing the point $(4,-7)$	c. Slope $m=0$, and containing the point $(-3,8)$	d. Slope $m=$ **undefined**, and containing the point $(-1,4)$

Answer

a. $y=-\frac{2}{5}x-1$ b. $y=-\frac{3}{4}x-4$ c. $y=8$ d. $x=-1$

Find an Equation of the Line Given Two Points

When we are not given the slope nor a point but rather just two points on the line, the value of the slope must be calculated.

How to: Find an equation of a line given two points.

1. Find the slope using the given points. $m = \frac{y_2 - y_1}{x_2 - x_1}$
2. Choose one point.
3. Substitute the values into the point-slope form: $y - y_1 = m(x - x_1)$ and solve for y.
   (Alternatively, use the the slope-intercept form $y = mx + b$ and substitute values for m, x, and y into it).
4. Solve for y. Write in slope-intercept form, $y = mx + b$.
   (Alternatively, if the slope-intercept form is used instead, solve for b and update the slope-intercept equation.)

Example 0.10.7 Equation of a line given two points

Find an equation of a line that contains the following points Write the equation in slope-intercept form if possible.

a. $(-3, -1)$ and $(2, -2)$ b. $(-3, 5)$ and $(-3, 4)$ c. $(2, 1)$ and $(5, 1)$

a. Solution

Step 1. Find the slope using the given points $(-3. -1)$ and $(2, -2)$.	$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - (-1)}{2 - (-3)} = \frac{-1}{5}$ $m = -\frac{1}{5}$
Step 2. Choose one point. (Either point can be chosen).	$\begin{pmatrix} x_1, & y_1 \\ 2 & -2 \end{pmatrix}$
Step 3. Substitute the values into the point-slope form.	$y - y_1 = m(x - x_1)$ $y - (-2) = -\frac{1}{5}(x - 2)$
Step 4. Solve for y and write the equation in slope-intercept form.	$y + 2 = -\frac{1}{5}x + \frac{2}{5}$ $y = -\frac{1}{5}x - \frac{8}{5}$

b. Solution

The first step will be to find the slope. $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 5}{-3 - (-3)} = \frac{-1}{0}$

The slope is undefined. This tells us it is a vertical line. Both of our points have an x-coordinate of -3. So our equation of the line is $x = -3$. Since there is no y, we cannot write it in slope-intercept form.

c. Solution

The first step will be to find the slope. $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 1)}{5 - 2} = \frac{0}{3}$

The slope is zero. This tells us it is a horizontal line. Both of our points have an y-coordinate of 1. So our equation of the line is $y = 1$.

Try It 0.10.8

Find the equation of a line containing the points

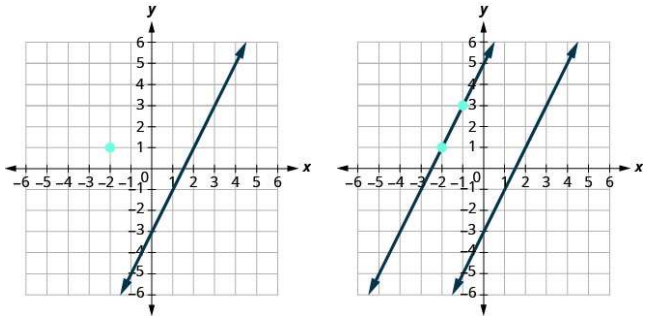
a. $(-2, -4)$ and $(1, -3)$. b. $(-4, -3)$ and $(1, -5)$. c. $(5, 1)$ and $(5, -4)$. d. $(4, -4)$ and $(3, -4)$.

Answer

a. $y = \frac{1}{3}x - \frac{10}{3}$ b. $y = -\frac{2}{5}x - \frac{23}{5}$ c. $x = 5$ d. $y = -4$

Find an Equation of a Line Parallel to a Given Line

Suppose we need to find an equation of a line that passes through a specific point and is parallel to a given line. We can use the fact that parallel lines have the same slope. So we will have a point and the slope—just what we need to use the point-slope equation.



First, let's look at this graphically.

Given the graph for $y = 2x - 3$, we want to graph a line parallel to this line and passing through the point $(-2, 1)$.

We know that parallel lines have the same slope. We'll use the notation $m_{\parallel}$ to represent the slope of a line parallel to a line with slope m. (Notice that the subscript $\parallel$ looks like two parallel lines.)

So the second line will have the same slope as $y = 2x - 3$. That slope is $m_{\parallel} = 2$.

The second (parallel) line will pass through $(-2, 1)$ and have slope $m = 2$.

To graph the line, we start at $(-2, 1)$ and count out the rise and run.

With $m = 2$ (or $m = \frac{2}{1}$), we count out the rise 2, the run 1, and draw the line, as shown in the graph.

Now let's see how to do this algebraically.

How to: Find an equation of a line parallel to a given line.

1. Identify the slope of the given line, and find the slope of the parallel line, $m_{\parallel}$. Use $m_{\parallel}$ for m in the substitution step below (Step 3).
2. Identify the point.
3. Substitute the values x_1, y_1, and m into the point-slope form: $y - y_1 = m(x - x_1)$.
   (Alternatively, substitute values for x, y, and m into the slope-intercept form $y = mx + b$).
4. Solve for y and write the equation in slope-intercept form.
   (Alternatively, if the slope-intercept form was used instead, solve for b and update the slope-intercept equation.)

Example 0.10.9 Equation of a line given a point and a parallel line

Find an equation of a line parallel to $y = 2x - 3$ that contains the point $(-2, 1)$. Write the equation in slope-intercept form.

Solution

Step 1. Identify the slope of the given line, $y = 2x - 3$. Find the slope of the parallel line. Parallel lines have the same slope.	Slope of the given line is $m = 2$ Slope of the line wanted is $m_{\parallel} = 2$
Step 2. Identify given point, $(-2, 1)$.	$\begin{pmatrix} x_1, & y_1 \\ -2 & 1 \end{pmatrix}$
Step 3. Substitute the values into the point-slope form.	$y - y_1 = m_{\parallel}(x - x_1)$ $y - (1) = 2(x - (-2))$
Step 4. Solve for y and write the equation in slope-intercept form.	$y - 1 = 2(x + 2)$ $y - 1 = 2x + 4$ $y = 2x + 5$

Try It 0.10.10

Find an equation of a line parallel to the following lines. Write the equation in slope-intercept form.

a. Parallel to line $y = 3x + 1$ that contains the point $(4, 2)$.

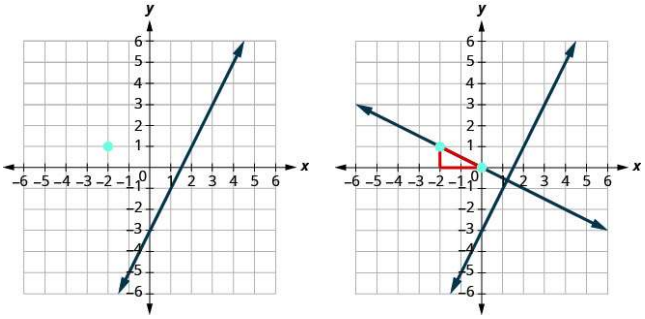
b. Parallel to the line $y = \frac{1}{2}x - 3$ that contains the point $(6, 4)$.

Answer

a. $y = 3x - 10$ b. $y = \frac{1}{2}x + 1$

Find an Equation of a Line Perpendicular to a Given Line

Now, let's consider perpendicular lines. Suppose we need to find a line passing through a specific point and which is perpendicular to a given line. We can use the fact that perpendicular lines have slopes that are negative reciprocals. We will again use the point-slope equation, like we did with parallel lines.



First, let's look at this graphically.

Given the graph of $y = 2x - 3$, we want to graph a line perpendicular to this line and passing through $(-2, 1)$.

We know that perpendicular lines have slopes that are negative reciprocals. We'll use the notation $m_\perp$ to represent the slope of a line perpendicular to a line with slope m. (Notice that the subscript $\perp$ looks like the right angles made by two perpendicular lines.)

So the second line will have a slope that is the negative reciprocal of the given line $y = 2x - 3$. The slope of the given line is $m = 2$ so the slope of the wanted line will be $m_\perp = -\frac{1}{2}$.

We now know the second (perpendicular) line will pass through $(-2, 1)$ and have slope $m_\perp = -\frac{1}{2}$.

To graph the line, we start at $(-2, 1)$ and count out the rise -1, the run 2 and draw the line.

Now, let's see how to do this algebraically.

How to: Find an equation of a line perpendicular to a given line.

1. Identify the slope of the given line and find the slope of the perpendicular line, $m_\perp$. Use $m_\perp$ for m in the substitution step below (Step 3).
2. Identify the point.
3. Substitute the values x_1, y_1, and m into the point-slope form: $y - y_1 = m(x - x_1)$.
   (Alternatively, substitute values for x, y, and m into the slope-intercept form $y = mx + b$).
4. Solve for y and write the equation in slope-intercept form.
   (Alternatively, if the slope-intercept form is used instead, solve for b and update the slope-intercept equation.)

Example 0.10.11 Equation of a line given a point and a perpendicular line

Find an equation of a line perpendicular to $y = 2x - 3$ that contains the point $(-2, 1)$.
Write the equation in slope-intercept form.

Solution

Step 1. Identify the slope of the given line, $y = 2x - 3$. Find the slope of the perpendicular line. The slopes of perpendicular lines are negative reciprocals.	$m = 2$ $m_\perp = -\frac{1}{2}$
Step 2. Identify the given point, $(-2, 1)$.	$\begin{pmatrix} x_1, & y_1 \\ -2 & 1 \end{pmatrix}$
Step 3. Substitute the values into the point-slope form.	$y - y_1 = m_\perp(x - x_1)$ $y - (1) = -\frac{1}{2}(x - (-2))$
Step 4. Solve for y and write the equation in slope-intercept form.	$y - 1 = -\frac{1}{2}(x + 2)$ $y - 1 = -\frac{1}{2}x - 1$ $y = -\frac{1}{2}x$

Try It 0.10.12

Find an equation of a line with the following attributes. Write the equation in slope-intercept form.

a. Perpendicular to the line $y = 3x + 1$ that contains the point $(4, 2)$.

b. Perpendicular to the line $y = \frac{1}{2}x - 3$ that contains the point $(6, 4)$.

Answer

a. $y = -\frac{1}{3}x + \frac{10}{3}$ b. $y = -2x + 16$

Find an Equation of a Line Perpendicular to a Given Horizontal or Vertical Line

Example 0.10.13 Equation of a line given a point and a perpendicular vertical line

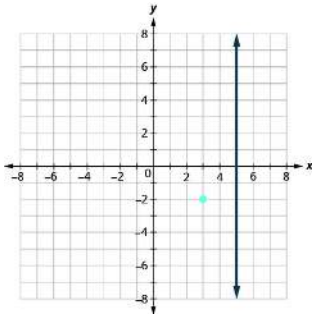
Find an equation of a line perpendicular to $x = 5$ that contains the point $(3, -2)$. Write the equation in slope-intercept form.

Solution Using characteristics of the given line.

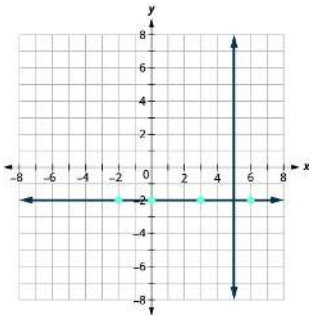
We want to find a line that is perpendicular to $x = 5$ that contains the point $(3, -2)$.

The graph of the equation $x = 5$ is a vertical line.

This graph shows the line $x = 5$ and the point $(3, -2)$.



We know every line perpendicular to a vertical line is horizontal, so we will sketch the horizontal line through $(3, -2)$. If we look at a few points on this horizontal line, we notice they all have y-coordinates of -2. So, the equation of the line perpendicular to the vertical line $x = 5$ is $y = -2$.



Solution Using characteristics of the given line.

Step 1. Identify the slope of the given line, $x = 5$. Find the slope of the perpendicular line.	The line $x = 5$ is vertical, so its perpendicular will be horizontal. $m_{\perp} = 0$
Step 2. Identify the given point, $(3, -2)$.	$\begin{pmatrix} x_1, & y_1 \\ 3 & -2 \end{pmatrix}$
Step 3. Substitute the values into the point-slope form.	$y - y_1 = m(x - x_1)$ $y - (-2) = 0(x - (3))$
Step 4. Simplify. Write in slope-intercept form.	$y + 2 = 0$ $y = -2$

Example 0.10.14 Equation of a line given a point and a perpendicular horizontal line

Find an equation of a line that is perpendicular to $y = -3$ that contains the point $(-3, 5)$.
Write the equation in slope-intercept form.

Solution

The line $y = -3$ is a horizontal line. Any line perpendicular to it must be vertical, in the form $x = a$. Since the perpendicular line is vertical and passes through $(-3, 5)$, every point on it has an x-coordinate of -3. The equation of the perpendicular line is $x = -3$.

Try It 0.10.15

Find an equation of a line with the following attributes. Write the equation in slope-intercept form when possible.

1. Perpendicular to the line $x = 4$ that contains the point $(4, -5)$.
2. Perpendicular to the line $x = 2$ that contains the point $(2, -1)$.
3. Perpendicular to the line $y = 1$ that contains the point $(-5, 1)$.
4. Perpendicular to the line $y = -5$ that contains the point $(-4, -5)$.

Answer

a. $y = -5$ or $y = 0x - 5$ b. $y = -1$ or $y = 0x - 1$ c. $x = -5$ d. $x = -4$

Key Concepts

- **How to determine the slope of a line if it is not given**
  - If given a graph of a line, choose two points on the graph and count the rise (up is positive) and the run (right is positive to get from one point to the other point. The slope is the ratio, $m = \frac{\text{Rise}}{\text{Run}}$.
  - If two points (x_1, y_1) and (x_2, y_2) on the line are given rather than the slope, the slope is $m = \frac{y_2 - y_1}{x_2 - x_1}$
  - If given a parallel line, identify the slope of the given line. The slope of the wanted line has the same slope.
  - If given a perpendicular line, identify the slope of the given line. The slope of the wanted line is the negative reciprocal of the given line.
  - If the slope of the given line is 0, the given line is horizontal and its equation is $y = b$.
  - If the slope of the given line is undefined, the given line is vertical and its equation is $x = c$.
- **Vertical lines ($x = c$) and horizontal lines ($y = b$) are perpendicular to each other.**
- **How to find an equation of a line using the point-slope equation.**
  1. Identify the slope, m.
  2. Identify a point (x_1, y_1) on the line. If given two points, either point may be chosen.
  3. Substitute the values of y_1, x_1, and m into the point-slope equation,$y - y_1 = m(x - x_1)$.
  4. Solve for y and write the equation in slope-intercept form, $y = mx + b$.

- **How to find an equation of a line using the slope-intercept equation.**
    1. Identify the slope, m.
    2. Identify a point (x, y) on the line. If given two points, either point may be chosen.
    3. Substitute the values of x, y, and m into the slope-intercept equation,$y = mx + b$.
    4. Solve for b. Substitute the values of m and b in the slope-intercept form, $y = mx + b$.

Glossary

point-slope form

The point-slope form of an equation of a line with slope *m* and containing the point (x_1, y_1) is $y - y_1 = m(x - x_1)$.

slope-intercept form

The slope-intercept form of an equation of a line with slope m and y-intercept $(0, b)$ is $y = mx + b$. The y-intercept is a point on the line where it crosses the y-axis.

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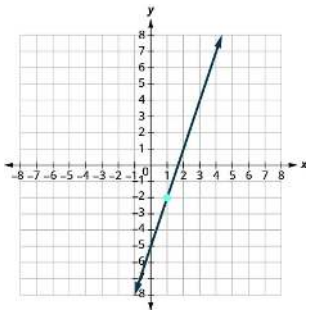
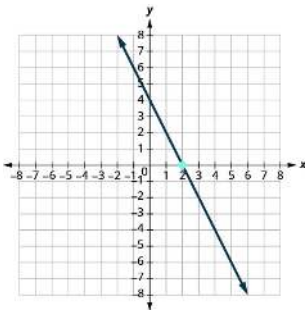
0.10e: Exercises - Find the Equation of a Line

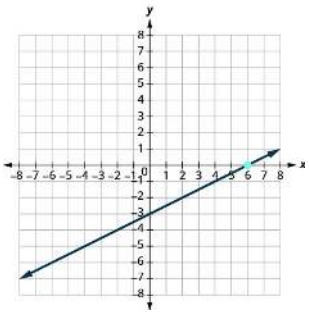
A: Construct a linear equation given a graph

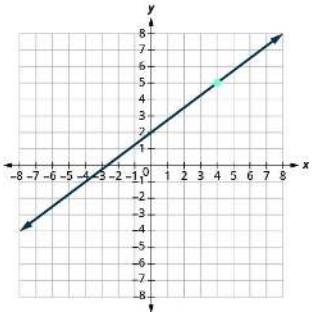
Exercise $0.10.A$: Construct a linear equation given its graph

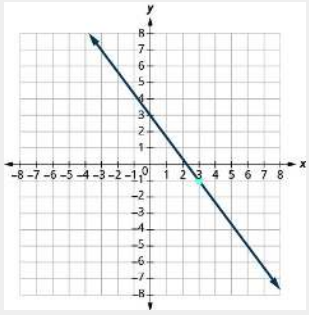
In the following exercises, find the equation of the line shown in each graph. Write the equation in slope-intercept form.

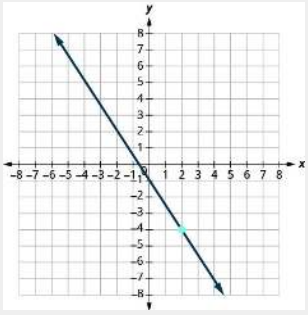
1. 

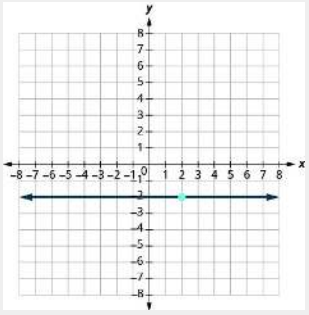
2. 

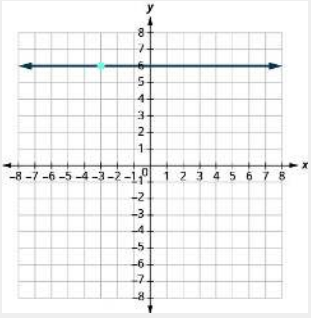
3. 

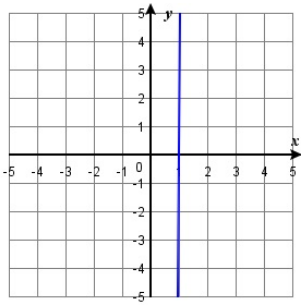
4. 

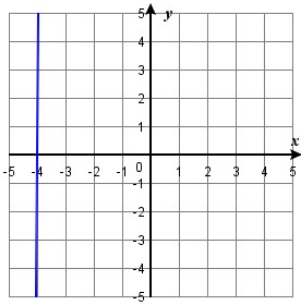
5. 

6. 

7. 

8. 

9. 

10. 

Answers to odd exercises.

1. $y = 3x - 5$ 3. $y = \frac{1}{2}x - 3$ 5. $y = -\frac{4}{3}x + 3$ 7. $y = -2$ 9. $x = 1$

B: Construct a linear equation given a point on the line and the slope

Exercise $0.10.B$: Construct a linear equation given attributes of the line

In the following exercises, find the equation of a line with given slope and y-intercept. Write the equation in slope-intercept form.

13. slope 3 and y-intercept $(0,5)$	14. slope 8 and y-intercept $(0,-6)$	15. slope -3 and y-intercept $(0,-1)$	16. slope -1 and y-intercept $(0,3)$
17. slope $\frac{1}{5}$ and y-intercept $(0,-5)$	18. slope $-\frac{3}{4}$ and y-intercept $(0,-2)$	19. slope 0 and y-intercept $(0,-1)$	20. slope -4 and y-intercept $(0,0)$

In the following exercises, find the equation of a line with given slope and containing the given point. Write the equation in slope-intercept form.

25. $m=\frac{5}{8}$, point $(8,3)$	26. $m=\frac{5}{6}$, point $(6,7)$	27. $m=-\frac{3}{5}$, point $(10,-5)$	28. $m=-\frac{3}{4}$, point $(8,-5)$
29. $m=-\frac{3}{2}$, point $(-4,-3)$	30. $m=-\frac{5}{2}$, point $(-8,-2)$	31. $m=-7$, point $(-1,-3)$	32. $m=-4$, point $(-2,-3)$
33. Horizontal line containing $(-2,5)$	34. Horizontal line containing $(-2,-3)$	35. Vertical line containing $(-1,-7)$	36. Vertical line containing $(4,-8)$

Answers to odd exercises.

13. $y=3x+5$ 15. $y=-3x-1$ 17. $y=\frac{1}{5}x-5$ 19. $y=-1$
25. $y=\frac{5}{8}x-2$ 27. $y=-\frac{3}{5}x+1$ 29. $y=-\frac{3}{2}x-9$ 31. $y=-7x-10$ 33. $y=5$ 35. $x=-1$

C: Construct a linear equation given two points on the line

Exercise 0.10.C: Construct a linear equation given two points on the line

In the following exercises, find the equation of a line containing the given points. Write the equation in slope-intercept form.

41. $(2,6)$ and $(5,3)$	42. $(4,3)$ and $(8,1)$	43. $(-3,-4)$ and $(5,-2)$	44. $(-5,-3)$ and $(4,-6)$
45. $(-1,3)$ and $(-6,-7)$	46. $(-2,8)$ and $(-4,-6)$	47. $(0,4)$ and $(2,-3)$	48. $(0,-2)$ and $(-5,-3)$
49. $(7,2)$ and $(7,-2)$	50. $(-2,1)$ and $(-2,-4)$	51. $(3,-4)$ and $(5,-4)$	52. $(-6,-3)$ and $(-1,-3)$

Answers to odd exercises.

41. $y=-x+8$ 43. $y=\frac{1}{4}x-\frac{13}{4}$ 45. $y=2x+5$ 47. $y=-\frac{7}{2}x+4$ 49. $x=7$ 51. $y=-4$

D: Construct a linear equation given a point on the line and a parallel or perpendicular line

Exercise 0.10.D: Construct a linear equation given attributes of the line

In the following exercises, find an equation of a line parallel to the given line and contains the given point. Write the equation in slope-intercept form.

55. line $y=4x+2$, point $(1,2)$	56. line $y=-3x-1$, point $(2,-3)$	57. line $2x-y=6$, point $(3,0)$	58. line $2x+3y=6$, point $(0,5)$
59. line $x=-4$, point $(-3,-5)$	60. line $x-2=0$, point $(1,-2)$	61. line $y=5$, point $(2,-2)$	62. line $y+2=0$, point $(3,-3)$

In the following exercises, find an equation of a line perpendicular to the given line and contains the given point. Write the equation in slope-intercept form.

67. line $y=-2x+3$, point $(2,2)$	68. line $y=-x+5$, point $(3,3)$	69. line $y=\frac{3}{4}x-2$, point $(-3,4)$	70. line $y=\frac{2}{3}x-4$, point $(2,-4)$
71. line $2x-3y=8$, point $(4,-1)$	72. line $4x-3y=5$, point $(-3,2)$	73. line $2x+5y=6$, point $(0,0)$	74. line $4x+5y=-3$, point $(0,0)$
75. line $x=3$, point $(3,4)$	76. line $x=-5$, point $(1,-2)$	77. line $x=7$, point $(-3,-4)$	78. line $x=-1$, point $(-4,0)$
79. line $y-3=0$, point$(-2,-4)$	80. line $y-6=0$, point $(-5,-3)$	81. line y-axis, point $(3,4)$	82. line y-axis, point $(2,1)$

Answers to odd exercises.

55. $y=4x-2$ 57. $y=2x-6$ 59. $x=-3$ 61. $y=-2$
67. $y=\frac{1}{2}x+1$ 69. $y=-\frac{4}{3}x$ 71. $y=-\frac{3}{2}x+5$ 73. $y=\frac{5}{2}x$
75. $y=4$ 77. $y=-4$ 79. $x=-2$ 81. $y=4$

E: Construct a linear equation (Mixed Practice)

Exercise 0.10.E: Construct a linear equation given two points on the line

In the following exercises, find the equation of each line. Write the equation in slope-intercept form.

87. Containing the points $(4,3)$ and $(8,1)$	88. Containing the points $(-2,0)$ and $(-3,-2)$	89. $m=\frac{1}{6}$, containing point $(6,1)$	90. $m=\frac{5}{6}$, containing point $(6,7)$
91. Parallel to the line $4x+3y=6$, containing point $(0,-3)$	92. Parallel to the line $2x+3y=6$, containing point $(0,5)$	93. $m=-\frac{3}{4}$, containing point $(8,-5)$	94. $m=-\frac{3}{5}$, containing point $(10,-5)$
95. Perpendicular to the line $y-1=0$, point $(-2,6)$	96. Perpendicular to the line y-axis, point $(-6,2)$	97. Parallel to the line $x=-3$, containing point $(-2,-1)$	98. Parallel to the line $x=-4$, containing point $(-3,-5)$
99. Containing the points $(-3,-4)$ and $(2,-5)$	100. Containing the points $(-5,-3)$ and $(4,-6)$	101. Perpendicular to the line $x-2y=5$, point $(-2,2)$	102. Perpendicular to the line $4x+3y=1$, point $(0,0)$

Answers to odd exercises.

87. $y=-\frac{1}{2}x+5$ 89. $y=\frac{1}{6}x$ 91. $y=-\frac{4}{3}x-3$ 93. $y=-\frac{3}{4}x+1$
95. $x=-2$ 97. $x=-2$ 99. $y=-\frac{1}{5}x-\frac{23}{5}$ 101. $y=-2x-2$

08/17/2025

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CHAPTER OVERVIEW

1: Equations and Inequalities

1.1: Solve Polynomial Equations by Factoring

1.1e: Exercises - Solve by Factoring

1.2: Square Root Property, Complete the Square, and The Quadratic Formula

1.2e: Exercises - SqRP, CTS, QF

1.3: Rational Equations

1.3e: Exercises - Rational Equations

1.4: Radical Equations

1.4e: Exercises - Radical Equations

1.5: Equations with Rational Exponents

1.5e: Exercises - Solve Equations with Rational Exponents

1.6: Equations Quadratic in Form

1.6e: Exercises - Quadratic in Form

1.7: Absolute Value Equations and Inequalities

1.7e: Exercises - Absolute Value

1.8: Variation - Constructing and Solving Equations

1.8e: Exercises - Variation

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1.1: Solve Polynomial Equations by Factoring

Reviewing General Factoring Strategies

We have learned various techniques for factoring polynomials with up to four terms. The challenge is to identify the type of polynomial and then decide which method to apply. The following outlines a general guideline for factoring polynomials.

General guidelines for factoring polynomials

Step 1: Check for common factors. If the terms have common factors, then factor out the greatest common factor (GCF).

Step 2: Determine the number of terms in the polynomial.

1. Factor four-term polynomials by grouping (either GCF of pairs, or binomial square then difference of squares).
2. Factor trinomials (3 terms) using "trial and error" or the AC method.
   - Possibly a **Binomial Square**, which has the form: $a^2+2ab+b^2=(a+b)^2$ or $a^2-2ab+b^2=(a-b)^2$
3. Factor binomials (2 terms) using the following special products:
   - **Difference of squares**:$a^2-b^2=(a-b)(a+b)$
   - **Sum of squares**: a^2+b^2 no general formula
   - **Difference of cubes**: $a^3-b^3=(a-b)(a^2+ab+b^2)$
   - **Sum of cubes**: $a^3+b^3=(a+b)(a^2-ab+b^2)$
   - If a binomial is both a difference of squares and a difference cubes, then first factor it as difference of squares. This will result in a more complete factorization.

Step 3: Look for factors that can be factored further.

Step 4: Check by multiplying.

Use Factoring to Solve Equations

We will first solve some equations by using the **Zero Factor Property**. The Zero Factor Property (also called the Zero Product Property) says that if the product of two quantities is zero, then at least one of those quantities is zero. The only way to get a product equal to zero is to multiply by zero itself.

Zero Factor Property

If $a\cdot b=0$, then either $a=0$ or $b=0$ or both.

For example, consider the equation $(x-3)(x-2)=0$. According to the Zero Factor Property, this product can only be zero if one of the factors is zero. For this equation, the factors are $(x-3)$ and $(x-2)$. **Factors** are the expressions that are multiplied together to form a **product**.

$$\begin{array}{c}(x-3)(x-2)=0\\ x-3=0 \quad \text{or} \quad x-2=0\\ x=3 \qquad\qquad x=2\\ \text{Solution Set: } \{3,2\}\end{array}$$

These proposed solutions can be checked by substituting back in the original equation.

$$\begin{array}{rcl}\text{Check } x=3 & \text{and} & \text{Check } x=2\\ (3-3)(3-2)=0 & & (2-3)(2-2)=0\\ (0)(1)=0 \ \checkmark & & (-1)(0)=0 \ \checkmark\end{array}$$

How to: Use the Zero Factor Property to Solve an Equation.

1. **ZERO**. Write the equation so one side of the equation is **zero**. Write the expression on the other side of the equal sign in order of descending powers of x with a *positive* coefficient on the term with highest exponent.
2. **FACTOR**. **Factor** the expression.

3. **PROPERTY**. Set each factor equal to zero and solve. (This is the **property** -- a factor that is zero will make the product of factors it is a part of also equal to zero). The solutions obtained are the values of x that will make the original equation a true statement.
4. Check by substituting solutions into the original equation.

Example 1.1.1 Factor out a GCF

Solve: $2x^2 = 8x$.

Solution. Notice that the first step requires one side of the equation to be made zero. If both sides of the equal sign instead were first divided by x, then only one solution $x = 4$ would have been found. Dividing by a variable expression can result in lost solutions!

$$2x^2 - 8x = 0$$
$$2x(x-4) = 0$$
$$2x = 0 \quad \text{or} \quad x - 4 = 0$$
$$x = 0 \qquad x = 4$$
$$\text{Solution Set: } \{0, 4\}$$

1. **Zero**. Make one side zero
2. **Factor**. Factor out the GCF
3. **Property**. Set each factor to 0 And solve

Example 1.1.2 Factor four terms by grouping pairs

Solve: $4x^3 - x^2 - 100x + 25 = 0$.

Solution

$$4x^3 - x^2 - 100x + 25 = 0$$
$$x^2(4x-1) - 25(4x-1) = 0$$
$$(4x-1)\left(x^2 - 25\right) = 0$$
$$(4x-1)(x-5)(x+5) = 0$$
$$4x - 1 = 0 \qquad x - 5 = 0 \qquad x + 5 = 0$$
$$4x = 1 \qquad x = 5 \qquad x = -5$$
$$x = \frac{1}{4}$$
$$\text{Solution Set: } \left\{\frac{1}{4}, 5, -5\right\}$$

1. **Zero**. One side already zero
2. **Factor**. Factor by grouping pairs
   Factor out the common binomial
   Factor a difference of squares
3. **Property**. Set each factor to 0 And solve

Example 1.1.3 Factor a trinomial (with a constant GCF and then $a = 1$)

Solve: $3x^2 = 12x + 63$.

Solution. This example highlights the essential first step of making one side of the equation zero before the Zero Factor Property is applied. Also, factoring produces three factors, but the first factor is a constant (3) which can never be equal to 0

$$3x^2 - 12x - 63 = 0$$
$$3(x^2 - 4x - 21) = 0$$
$$3(x-7)(x+3) = 0$$
$$3 = 0 \qquad x - 7 = 0 \qquad x + 3 = 0$$
$$3 \neq 0 \qquad x = 7 \qquad x = -3$$
$$\text{Solution Set: } \{7, -3\}$$

1. **Zero**. Make one side zero
2. **Factor**. Factor out the GCF
   Factor the trinomial
3. **Property**. Set each factor to 0 And solve

Example 1.1.4 Factor a trinomial (with $a \neq 1$)

Solve: $15x^2 + 3x - 8 = 5x - 7$.

Solution

$$
\begin{aligned}
15x^2+3x-8 &= 5x-7 \\
15x^2-2x-1 &= 0 \\
(3x-1)(5x+1) &= 0 \\
3x-1=0 \quad &\text{and} \quad 5x+1=0 \\
3x &= 1 \qquad\quad 5x=-1 \\
x &= \tfrac{1}{3} \qquad\quad x=-\tfrac{1}{5}
\end{aligned}
$$

Solution Set: $\left\{\frac{1}{3},-\frac{1}{5}\right\}$

1. **Zero**. Make one side zero
2. **Factor**. Factor the trinomial
3. **Property**. Set each factor to 0
   And solve

Example 1.1.5 Factor a trinomial (with $a \neq 1$)

Solve: $(3x-8)(x-1)=3x$.

Solution. This quadratic equation appears to be factored; hence it might be tempting to set each factor equal to $3x$. However, this would lead to incorrect results. We must first rewrite the equation equal to zero, so that we can apply the zero-product property

$$
\begin{aligned}
(3x-8)(x-1) &= 3x \\
3x^2-11x+8 &= 3x \\
3x^2-14x+8 &= 0 \\
(3x-2)(x-4) &= 0 \\
3x-2=0 \quad &\text{and} \quad x-4=0 \\
3x &= 2 \qquad\quad x=4 \\
x &= \tfrac{2}{3}
\end{aligned}
$$

Solution Set: $\left\{\frac{2}{3},4\right\}$

1. **Zero**. Make one side zero
2. **Factor**. Factor the trinomial
3. **Property**. Set each factor to 0
   And solve

Example 1.1.6 Factor a difference of squares

Solve: $169q^2=49$.

Solution

$$
\begin{aligned}
169q^2 &= 49 \\
169x^2-49 &= 0 \\
(13x-7)(13x+7) &= 0 \\
13x-7=0 \quad &\text{and} \quad 13x+7=0 \\
13x &= 7 \qquad\quad 13x=-7 \\
x &= \tfrac{7}{13} \qquad\quad x=-\tfrac{7}{13}
\end{aligned}
$$

Solution Set: $\left\{\frac{7}{13},-\frac{7}{13}\right\}$

1. **Zero**. Make one side zero
2. **Factor**. Factor a difference of squares
3. **Property**. Set each factor to 0
   And solve

Example 1.1.7 Factor a perfect square trinomial (with a variable GCF)

Solve: $9m^3 + 100m = 60m^2$

Solution

$$9m^3 + 100m = 60m^2$$
$$9m^3 - 60m^2 + 100m = 0 \quad \text{1. Zero. Make one side zero}$$
$$m(9m^2 - 60m + 100) = 0 \quad \text{2. Factor. Factor out the GCF}$$
$$m(3m-10)^2 = 0 \quad \text{Factor the Perfect Square trinomial}$$
$$m = 0 \quad \text{and} \quad 3m - 10 = 0 \quad \text{3. Property. Set each factor to 0}$$
$$m = 10 \quad \text{And solve}$$
$$m = \tfrac{10}{3}$$

Solution Set: $\left\{0, \tfrac{10}{3}\right\}$

Example 1.1.8 Factor four terms (the difference of a binomial square and a monomial square)

Solve: $4x^2 + 36x + 81 = 100x^2$.

Solution

$$4x^2 + 36x + 81 = 100x^2$$
$$4x^2 + 36x + 81 - 100x^2 = 0 \quad \text{1. Zero. Make one side zero}$$
$$4x^2 + 36x + 81 - 100x^2 \quad \text{Observe three terms are perfect squares (2 positive, 1 negative)}$$
$$(2x+9)(2x+9) - 100x^2 = 0 \quad \text{2a. Factor. Factor the perfect square binomial}$$
$$(2x+9)^2 - (10x)^2 = 0$$
$$(2x+9-10x)(2x+9+10x) = 0 \quad \text{2b. Factor. Factor the difference of squares}$$
$$-8x + 9 = 0 \quad \text{and} \quad 12x + 9 = 0 \quad \text{3. Property. Set each factor to 0}$$
$$8x = 9 \qquad 12x = -9 \quad \text{And solve}$$
$$x = \frac{9}{8} \qquad x = -\frac{9}{12}$$

Solution Set: $\left\{\frac{9}{8}, -\frac{3}{4}\right\}$

Factoring using the sum or difference of cubes formula to solve an equation will be discussed in the next section that includes the Quadratic Formula and the Complete the Square technique.

Try It 1.1.9

Solve each equation by factoring.

a. $(3m-2)(2m+1) = 0$	d. $(2m+1)(m+3) = 12m$	g. $64x^2 + 225 = 240x$
b. $3c^2 = 10c - 8$	e. $123b = -6 - 60b^2$	h. $3x^3 - 2x^2 - 12x + 8 = 0$
c. $25p^2 = 49$	f. $8x^3 = 24x^2 - 18x$	i. $9x^2 + 16 = 27x + 64$

Answer

a. $m = \frac{2}{3},\ m = -\frac{1}{2}$	d. $m = 1,\ m = \frac{3}{2}$	g. $x = \frac{15}{8}$
b. $c = 2,\ c = \frac{4}{3}$	e. $b = -2,\ b = -\frac{1}{20}$	h. $x = 2,\ x = -2,\ x = \frac{2}{3}$
c. $p = \frac{7}{5}, p = -\frac{7}{5}$	f. $x = 0,\ x = \frac{3}{2}$	i. $x = \frac{4}{3}\ x = -\frac{9}{4}$

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1.1e: Exercises - Solve by Factoring

A: Solve by Factoring

Exercise 1.1e.A

Solve by factoring (Zero Factor Property, GCF, Difference of Squares).

1. $(3a-10)(2a-7)=0$	6. $(3y+5)^2=0$	11. $4x^3=36x^2$	16. $625=x^2$
2. $(5b+1)(6b+1)=0$	7. $8(2x+1)(3x-5)=0$	12. $16x^2=8x$	17. $16y^2=81$
3. $6m(12m-5)=0$	8. $4(5x-1)(2x+3)=0$	13. $7a^2+14a=7a$	16. $64p^2=225$
4. $2x(6x-3)=0$	9. $x^2+4x=0$	14. $2x^2-25x=7x^2$	19. $121n^4=36n^2$
5. $(2x-1)^2=0$	10. $3x^2+2x=0$	15. $49m^2=144$	20. $100y^4=9y^2$

Answers to Odd Exercises:

1. $a=\frac{10}{3},\ a=\frac{7}{2}$ 3. $m=0,\ m=\frac{5}{12}$ 5. $x=\frac{1}{2}$	7. $x=-\frac{1}{2},\ x=\frac{5}{3}$ 9. $x=0,\ x=-4$ 11. $x=0,\ x=9$ 13. $a=-1,\ a=0$	15. $m=\frac{12}{7},\ m=-\frac{12}{7}$ 17. $y=-\frac{9}{4},\ y=\frac{9}{4}$ 19. $n=0,\ n=-\frac{6}{11},\ n=\frac{6}{11}$

★

Solve each trinomial equation by factoring.

21. $x^4-5x^2+4=0$	28. $4x^2+100=40x$	35. $(x+4)(x-2)=16$
22. $4x^4-37x^2+9=0$	29. $x^3+36x=12x^2$	36. $(x+1)(x-7)=9$
23. $x^2-15x+50=0$	30. $m^3-2m^2=-m$	37. $(x+1)(x-3)=-4x$
24. $x^2+10x-24=0$	31. $3y^3+48y=24y^2$	38. $(y-3)(y+2)=4y$
25. $x^2+36=13x$	32. $2y^3+2y^2=12y$	39. $(x+2)(x-8)=2(x-14)$
26. $n^2=5-6n$	33. $(x+6)(x-3)=-8$	40. $(x+4)(x-6)=2(x+4)$
27. $3y^2-18y=-27$	34. $(p-5)(p+3)=-7$	

Answers to Odd Exercises:

21. $x=\pm1,\ x=\pm2$ 23. $x=5,\ x=10$ 25. $a=4,\ a=9$	27. $y=3$ 29. $x=0\ x=6$	31. $x=0,\ x=4$ 33. $x=2,\ x=-5$ 35. $x=-6,\ x=4$	37. $x=1,\ x=-3$ 39. $x=2,\ x=6$

★

Solve each trinomial equation by factoring.

41. $3x^2+2x-5=0$

42. $2x^2+9x+7=0$

43. $5a^2-26a=24$

44. $4b^2+7b=-3$

45. $4m^2=17m-15$

46. $4x^2-13x=-3$

47. $20x^2-60x=-45$

48. $18b^2+60b+50=0$

49. $15x^2-10x=40$

50. $14y^2-77y=-35$

51. $18x^2-9=-21x$

52. $16y^2+12=-32y$

53. $6x^2-5x-2=30x+4$

54. $6x^2-9x+15=20x-13$

55. $5x^2-39x+12=4(x-3)$

56. $4x^2+5x-5=15(3-2x)$

57. $(6x+1)(x+1)=6$

58. $(2x-1)(x-4)=39$

59. $(3x-2)(x+4)=12x$

60. $(2y-3)(3y-1)=8y$

Answers to Odd Exercises:

41. $x=-\frac{5}{3},\ x=1$
43. $a=-\frac{4}{5},\ a=6$
45. $m=\frac{5}{4},\ m=3$
47. $x=\frac{3}{2}$
49. $x=2,\ x=-\frac{4}{3}$
51. $x=-\frac{3}{2},\ x=\frac{1}{3}$
53. $x=-\frac{1}{6},\ x=6$
55. $x=\frac{3}{5},\ x=8$
57. $x=-\frac{5}{3},\ x=\frac{1}{2}$
59. $x=-\frac{4}{3},\ x=2$

★

Solve each polynomial equation by factoring.

61. $4x^3-14x^2-30x=0$

62. $9x^3+48x^2-36x=0$

63. $-10x^3-28x^2+48x=0$

64. $-2x^3+15x^2+50x=0$

65. $16p^3=24p^2-9p$

66. $36x^3+24x^2=-4x$

67. $\frac{1}{10}x^2-\frac{7}{15}x-\frac{1}{6}=0$

68. $\frac{2}{3}x^2-\frac{7}{3}x-\frac{5}{4}=0$

69. $\frac{1}{4}-\frac{4}{9}x^2=0$

70. $\frac{2}{25}x^2=\frac{1}{2}$

71. $\frac{1}{3}x^3-\frac{3}{4}x=0$

72. $\frac{1}{2}x^3-\frac{1}{50}x=0$

73. $2x^3-x^2-72x+36=0$

74. $x^3-3x^2-x+3=0$

75. $45x^3-9x^2-5x+1=0$

76. $4x^3-32x^2-9x+72=0$

77. $16x^2+24x+9=144x^2$

78. $25x^2-80x+64=100x^2$

79. $9x^2+12x+4=144$

80. $16x^2-40x+25=36$

Answers to Odd Exercises:

61. $x=0,\ x=-\frac{3}{2},\ x=5$
63. $x=-4,\ x=0,\ x=\frac{6}{5}$
65. $p=0,\ p=\frac{3}{4}$
67. $x=-\frac{1}{3},\ x=5$
69. $x=\pm\frac{3}{4}$
71. $x=0,\ x=\pm\frac{3}{2}$
73. $x=\pm6,\ x=\frac{1}{2}$
75. $x=\pm\frac{1}{3},\ x=\frac{1}{5}$
77. $x=3/8,\ x=-3/16$
79. $x=10/3,\ x=-14/3$

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1.2: Square Root Property, Complete the Square, and The Quadratic Formula

Solve Quadratic Equations by Using the Square Root Property

A quadratic equation in **standard form** is $ax^2+bx+c=0$ where a, b, and c are real numbers and $a \neq 0$. Quadratic equations can have two real solutions, one real solution, or no real solution—in which case there will be two complex solutions.

Reviewing a few definitions pertaining to complex numbers:

- A **complex number** is a number in the form $a+bi$ where a and b are real numbers. The real part is a and the imaginary part is bi.
- The **imaginary unit** is denoted as i and is defined as $i=\sqrt{-1}$ and so $i^2=-1$.
- The **square root of a negative number** should always be written in its complex number form: $\sqrt{-b}=i\sqrt{b}$

When it is factorable, a quadratic equation can be solved by factoring. In this section, an approach to solving the special case $ax^2+c=0$ where there is no linear term (i.e. $b=0$), is examined. As an example, $4x^2-9=0$ can be solved by factoring as follows:

$$\begin{aligned} 4x^2-9&=0 \\ (2x+3)(2x-3)&=0 \\ 2x+3=0 \quad &\text{or} \quad 2x-3=0 \\ 2x=-3 \quad &\quad 2x=3 \\ x=-\tfrac{3}{2} \quad &\quad x=\tfrac{3}{2} \end{aligned}$$

The solution set is $\left\{\pm\frac{3}{2}\right\}$. Here we use $\pm$ to write the two solutions in a more compact form.

An alternative method that can be used to more easily solve this equation comes from first isolating x^2 and then taking the square root of both sides of the equation. Besides its simplicity, this method allows us to solve equations that do not factor. Continuing the above example to illustrate this alternative approach:

$$\begin{aligned} 4x^2-9&=0 & \\ 4x^2&=9 & \text{First, isolate the squared term} \\ x^2&=\frac{9}{4} & \\ \sqrt{x^2}&=\pm\sqrt{\frac{9}{4}} & \text{Take the square root of both sides of the equal sign. Always remember the } \pm \text{ !!!} \\ x&=\pm\frac{3}{2} & \text{Simplify the radical} \end{aligned}$$

In summary, when there is no linear term in a quadratic equation, one method to solve it is to use the **square root property**. In this approach, the x^2 term (or more generally the **squared term**) is isolated first, and then the square root of both sides of the equal sign is taken.

The Square Root Property

Given an algebraic expression u, and a nonzero real number k, then the equation $u^2=k$ has exactly two solutions.

If $u^2=k$, then $u=\pm\sqrt{k}$ which is equivalent to $u=\sqrt{k}$ and $u=-\sqrt{k}$

How to: Use the Square Root Property to Solve an Equation.

This method can only be used on equations that have a squared expression and a constant term, but no linear term.

1. Isolate the squared expression u^2 on one side of the equal sign. Here u could simply be a single variable like x, or it could be an expression involving x.
2. Take the square root of both sides of the equation, putting a $\pm$ sign before the expression on the side opposite the squared expression.
3. Simplify the numbers on the side with the $\pm$ sign.

Example 1.2.1:

Solve: $9x^2 - 8 = 0$.

Solution

$$
\begin{aligned}
9x^2 - 8 &= 0 && \text{The Square Root Property can be used} \\
9x^2 &= 8 && \text{Isolate the squared term} \\
x^2 &= \frac{8}{9} && \\
x &= \pm\sqrt{\frac{8}{9}} && \text{Square root both sides. Remember the } \pm \text{ !!!} \\
&= \pm\frac{2\sqrt{2}}{3} && \text{Simplify.} \quad \textbf{Solution Set: } \left\{\pm\frac{2\sqrt{2}}{3}\right\}
\end{aligned}
$$

For completeness, these two solutions can be checked.

Check $x = -\frac{2\sqrt{2}}{3}$	Check $x = \frac{2\sqrt{2}}{3}$
$9x^2 - 8 = 0$	$9x^2 - 8 = 0$
$9\left(-\frac{2\sqrt{2}}{3}\right)^2 - 8 = 0$	$9\left(\frac{2\sqrt{2}}{3}\right)^2 - 8 = 0$
$9\left(\frac{4(2)}{9}\right) - 8 = 0$	$9\left(\frac{4(2)}{9}\right) - 8 = 0$
$8 - 8 = 0$	$8 - 8 = 0$
$0 = 0$ ✓	$0 = 0$ ✓

Sometimes quadratic equations have no real solution. In this case, the solutions will be complex numbers.

Example 1.2.2:

Solve: $x^2 + 31 = 6$.

Solution

$$
\begin{aligned}
x^2 + 31 &= 6 && \text{The Square Root Property can be used} \\
x^2 &= -25 && \text{Isolate the squared term} \\
x &= \pm\sqrt{-25} && \text{Square root both sides. Remember the } \pm \text{ !!!} \\
&= \pm\sqrt{-1}\cdot\sqrt{25} && \text{Simplify.} \\
&= \pm i \cdot 5 && \\
&= \pm 5i && \textbf{Solution Set: } \{\pm 5i\}
\end{aligned}
$$

The Square Root Property can also be used on equations that have a squared expression rather than just a simple squared variable.

Example 1.2.3:

Solve $(x+5)^2 = 9$.

Solution

$$
\begin{aligned}
(x+5)^2 &= 9 && \text{The Square Root Property can be used. The squared term is already isolated.} \\
x + 5 &= \pm\sqrt{9} && \text{Square root both sides. Remember the } \pm \text{ !!!} \\
x + 5 &= \pm 3 && \text{Simplify.} \\
x &= -5 \pm 3 &&
\end{aligned}
$$

At this point, separate the "plus or minus" into two equations and solve each individually.

$$
\begin{aligned}
x &= -5 + 3 \quad \text{or} \quad x = -5 - 3 \\
x &= -2 \qquad\qquad x = -8
\end{aligned}
$$

Solution Set: -2, -8

In addition to fewer steps, this method allows us to solve equations that do not factor.

Example 1.2.4:

Solve: $2(x-2)^2-1=4$.

Solution

$$\begin{aligned} 2(x-2)^2-1&=4 && \text{The Square Root Property can be used}\\ 2(x-2)^2&=5 && \text{Isolate the squared term}\\ (x-2)^2&=\frac{5}{2}\\ x-2&=\pm\sqrt{\frac{5}{2}} && \text{Square root both sides. Remember the } \pm \text{ !!!}\\ x&=2\pm\frac{\sqrt{5}}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} && \text{Rationalize the denominator.}\\ x&=2\pm\frac{\sqrt{10}}{2} && \text{Simplify.}\\ x&=\frac{4\pm\sqrt{10}}{2} && \textbf{Solution Set: } \left\{\frac{4-\sqrt{10}}{2},\frac{4+\sqrt{10}}{2}\right\} \end{aligned}$$

Example 1.2.5: Solving a Quadratic Equation Using the Square Root Property

Solve $25x^4-9=7$.

Solution

First, isolate the x^4 term. Then take the square root of both sides.

$$\begin{aligned} 25x^4-9&=7 && \text{The Square Root Property can be used}\\ 25x^4&=16 && \text{Isolate the squared term}\\ (x^2)^2&=\frac{16}{25}\\ x^2&=\pm\frac{16}{25} && \text{Square root both sides. Remember the } \pm \text{ !!!}\\ x^2&=\pm\frac{4}{5} && \text{Simplify.}\\ x^2&=\cancel{\text{and}}\ \frac{4}{5}\ \cancel{\text{and}}\ -\frac{4}{5} && \text{Isolate the squared term}\\ x&=\cancel{x=}\pm\sqrt{\frac{4}{5}}\sqrt{\frac{-4}{5}} && \text{Square root both sides. Remember the } \pm \text{ !!!}\\ x&=\cancel{x=}\frac{\sqrt{4}}{\sqrt{5}}\frac{\sqrt{-4}}{\sqrt{5}} && \text{Simplify.}\\ x&=\cancel{x=}\frac{2}{\sqrt{5}}\frac{\sqrt{-4}}{\sqrt{5}}\frac{\sqrt{5}}{\sqrt{5}}\\ x&=\cancel{x=}\frac{2\sqrt{5}2i\sqrt{5}}{5} && \textbf{Solution}\\ & && \textbf{Set: } \left\{\frac{2\sqrt{5}}{5},-\frac{2\sqrt{5}}{5},\frac{2i\sqrt{5}}{5},-\frac{2i\sqrt{5}}{5}\right\} \end{aligned} \tag{1.2.1}$$

Try It 1.2.6

Solve.

a) $2x^2+3=0$	b) $3(x-4)^2=15$	c) $2(3x-1)^2+9=0$

Answer

a) $\pm\frac{\sqrt{6}}{2}i$	b) $x=4\pm\sqrt{5}$	c) $\frac{1}{3}\pm\frac{\sqrt{2}}{2}i$

Completing the Square

Not all quadratic equations can be factored or can be solved in their original form using the square root property. In these cases, we may use a method for solving a **quadratic equation** known as **completing the square**. Using this method, we add or subtract terms to both sides of the equation until we have a perfect square trinomial on one side of the equal sign. We then apply the square root property. To complete the square, the leading coefficient, a, must equal 1. If it is not, then divide the entire equation by a before beginning the complete the square process. Then, we can use the following procedures to solve a quadratic equation by completing the square.

We will use the example $x^2+6x+1=0$ to illustrate each step. This is a quadratic equation that cannot be factored and with $a=1$.

$x^2+6x+1=0$ Put the constant on the right of the equal sign; the x and x^2 terms on the other side

$x^2+6x=-1$ Isolate the x and x^2 terms.

$x^2+6x+\square=-1+\square$ Add a constant $\square$ to create a trinomial that is also a binomial square: $(x+\bigcirc)^2$

$(x+\bigcirc)^2=-1+\square$ To accomplish this, $(x+\bigcirc)^2=x^2+2\bigcirc x+\bigcirc^2=x^2+6x+\square$ The binomial square constant, $\bigcirc$, is always HALF the x coefficient

$(x+3)^2=-1+9$ The number added to both sides of the equal sign, $\square$, is the square of that constant: $\square=\bigcirc^2$

$(x+3)^2=8$ Now use the square root property to solve the resulting equation.

$\sqrt{(x+3)^2}=\pm\sqrt{8}$

$x+3=\pm2\sqrt{2}$

$x=-3\pm2\sqrt{2}$ Solution Set: $\{-3+2\sqrt{2},-3-2\sqrt{2}\}$

How to: Use Completing the Square to Solve an Equation.

1. The coefficient of the x^2 term MUST be 1. If it is not 1, divide both sides of the equal sign by the coefficient of the x^2 term to make it 1.
2. Isolate the variable (x and x^2) terms on one side of the equal sign.
3. Add a constant to both sides of the equal sign that combined with the (x and x^2) terms will create a trinomial that is a perfect square binomial.
    1. Add the square of half the coefficient of x to both sides of the equation.
    2. Factor the resulting trinomial into a binomial square.
4. Use the Square Root Property to solve for x. Remember the $\pm$ sign !!

This technique can be used to solve ANY quadratic equation, whereas factoring works only some of the time.

Example 1.2.7:

Solve by completing the square: $x^2-8x-2=0$.

Solution

$x^2-8x-2=0$	The leading coefficient is 1
$x^2-8x=2$	Isolate the variable terms
$x^2-8x+4^2=2+4^2$	Add a constant - the square of half the coefficient of x
$x^2-8x+16=2+16$	
$(x-4)^2=18$	Factor.
$\sqrt{(x-4)^2}=\pm\sqrt{18}$	Solve using the square root property
$x-4=\pm3\sqrt{2}$	
$x=4\pm3\sqrt{2}$	Solution Set: $\{4+3\sqrt{2},4-3\sqrt{2}\}$

Example 1.2.8:

Solve by completing the square: $x^2 - 10x + 26 = 0$.

Solution

$x^2 - 10x + 26 = 0$	The leading coefficient is 1
$x^2 - 10x = -26$	Isolate the variable terms
$x^2 - 10x + 25 = -26 + 25$	Add a constant - the square of half the coefficient of x, $(-5)^2 = 25$
$(x-5)^2 = -1$	Factor into a binomial square
$\sqrt{(x-5)^2} = \pm\sqrt{-1}$	Solve using the square root property
$x - 5 = \pm i$	
$x = 5 \pm i$	**Solution Set:** $\{5+i, 5-i\}$

Example 1.2.9:

Solve by completing the square: $x^2 + 2x - 48 = 0$.

Solution

$x^2 + 2x - 48 = 0$	The leading coefficient is 1
$x^2 + 2x = 48$	Isolate the variable terms
$x^2 + 2x + 1 = 48 + 1$	Add a constant - the square of half the coefficient of x, $(1)^2 = 1$
$(x+1)(x+1) = 49$	Factor into a binomial square
$(x+1)^2 = 49$	Solve using the square root property
$x + 1 = \pm\sqrt{49}$	
$x + 1 = \pm 7$	
$x = -1 \pm 7$	$x = -1 + 7$ or $x = -1 - 7$ **Solution Set:** $\{6, -8\}$

When the coefficient of x is not divisible by 2, the constant added in the complete the square process will be a fraction.

Example 1.2.10:

Solve by completing the square: $x^2 + 3x + 4 = 0$.

Solution

$x^2 + 3x + 4 = 0$	The leading coefficient is 1
$x^2 + 3x = -4$	Isolate the variable terms
$x^2 + 3x + \frac{9}{4} = -4 + \frac{9}{4}$	Add a constant - the square of half the coefficient of x, $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$
$\left(x + \frac{3}{2}\right)\left(x + \frac{3}{2}\right) = \frac{-16}{4} + \frac{9}{4}$	Factor into a binomial square
$\left(x + \frac{3}{2}\right)^2 = \frac{-7}{4}$	Solve using the square root property
$x + \frac{3}{2} = \pm\sqrt{\frac{-1 \cdot 7}{4}}$	
$x + \frac{3}{2} = \pm\frac{i\sqrt{7}}{2}$	
$x = -\frac{3}{2} \pm \frac{\sqrt{7}}{2}i$	**Solution Set:** $\left\{-\frac{3}{2} \pm \frac{\sqrt{7}}{2}i\right\}$

So far, all of the examples have had a leading coefficient of 1. If this is not the case, remove it. This can be done by dividing both sides of the equal sign by the leading coefficient before completing the square.

Example 1.2.11:

Solve by completing the square: $3x^2 - 12x + 17 = 0$.

Solution

$$3x^2-12x+17=0$$ The leading coefficient is NOT 1

$$\frac{3x^2-12x+17}{3}=\frac{0}{3}$$ Divide by the coefficient to remove it from the x^2 term

$$\frac{3x^2}{3}-\frac{12x}{3}+\frac{17}{3}=0$$

$$x^2-4x=\frac{-17}{3}$$ Isolate the variable terms

$$x^2-4x+4=\frac{-17}{3}+4$$ Add a constant - the square of half the coefficient of x, $\left(\frac{4}{2}\right)^2=4$

$$(x-2)(x-2)=\frac{-17}{3}+\frac{12}{3}$$ Factor into a binomial square

$$(x-2)^2=\frac{-5}{3}$$

$$x-2=\pm\sqrt{\frac{-5}{3}}$$ Solve using the square root property

$$x=2\pm\frac{i\sqrt{5}}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}$$

$$x=\frac{6}{3}\pm\frac{i\sqrt{15}}{3}$$ **Solution Set:** $\left\{\frac{6\pm i\sqrt{15}}{3}\right\}$

Example 1.2.12:

Solve by completing the square: $2x^2+5x-1=0$.

Solution

$$2x^2+5x-1=0$$ The leading coefficient is NOT 1

$$\frac{2x^2+5x-1}{2}=\frac{0}{2}$$ Divide by the coefficient to remove it from the x^2 term

$$\frac{2x^2}{2}+\frac{5x}{2}-\frac{1}{2}=0$$

$$x^2+\frac{5}{2}x=\frac{1}{2}$$ Isolate the variable terms

$$x^2+\frac{5}{2}x+\frac{25}{16}=\frac{1}{2}+\frac{25}{16}$$ Add a constant - the square of half the coefficient of x, $\left(\frac{5}{4}\right)^2=\frac{25}{16}$

$$\left(x+\frac{5}{4}\right)\left(x+\frac{5}{4}\right)=\frac{8}{16}+\frac{25}{16}$$ Factor into a binomial square

$$\left(x+\frac{5}{4}\right)^2=\frac{33}{16}$$

$$x+\frac{5}{4}=\pm\sqrt{\frac{33}{16}}$$ Solve using the square root property

$$x+\frac{5}{4}=\pm\frac{\sqrt{33}}{4}$$

$$x=-\frac{5}{4}\pm\frac{\sqrt{33}}{4}$$ **Solution Set:** $\left\{\frac{-5\pm\sqrt{33}}{4}\right\}$

Try It 1.2.13

Solve by completing the square.

a) $x^2-6x=13$

b) $x^2-2x-17=0$

c) $3x^2-2x+1=0$

Answer

a) $x=3\pm\sqrt{22}$

b) $x=1\pm3\sqrt{2}$

c) $x=\frac{1}{3}\pm\frac{\sqrt{2}}{3}i$

The Quadratic Formula

The fourth method of solving a **quadratic equation** is by using the **quadratic formula**, a formula that will solve all quadratic equations. Although the quadratic formula works on any quadratic equation in standard form, it is easy to make errors in substituting the values into the formula. Pay close attention when substituting, and use parentheses when inserting a negative number.

We can derive the quadratic formula by **completing the square**. Given $ax^2+bx+c=0, a\neq 0$, we will complete the square as follows:

$$\begin{aligned}
ax^2+bx+c&=0 && \text{The leading coefficient is NOT 1}\\
x^2+\frac{b}{a}x+\frac{c}{a}&=0 && \text{Divide by the coefficient to remove it from the } x^2 \text{ term}\\
x^2+\frac{b}{a}x&=-\frac{c}{a} && \text{Isolate the variable terms}\\
x^2+\frac{b}{a}x+\frac{b^2}{4a^2}&=-\frac{c}{a}\cdot\frac{4a}{4a}+\frac{b^2}{4a^2} && \text{Add a constant - the square of half the coefficient of } x, \left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}\\
\left(x+\frac{b}{2a}\right)^2&=\frac{b^2-4ac}{4a^2} && \text{Factor into a binomial square}\\
x+\frac{b}{2a}&=\pm\sqrt{\frac{b^2-4ac}{4a^2}} && \text{Solve using the square root property}\\
x&=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{\sqrt{4a^2}} && \textbf{Solution Set: } \left\{\frac{-b\pm\sqrt{b^2-4ac}}{2a}\right\}
\end{aligned}$$

The Quadratic Formula

Written in standard form, $ax^2+bx+c=0$ where a, b, and c are real numbers and $a\neq 0$, any quadratic equation can be solved using the **quadratic formula**:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

How to: Use the Quadratic Formula to Solve an Equation.

1. Make sure the equation is in standard form: $ax^2+bx+c=0$.
2. Make note of the values of the coefficients and constant term, a, b, and c.
3. Carefully substitute the values noted in step 2 into the equation. To avoid needless errors, use parentheses around each number input into the formula.
4. Calculate and solve.

Example 1.2.14:

Solve using the quadratic formula: $2x^2-7x-15=0$

Solution

The coefficients are: $a=2, b=-7, c=-15$. Substitute these values into the quadratic formula and simplify.

$$\begin{aligned}
x&=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\\
&=\frac{-(-7)\pm\sqrt{(-7)^2-4(2)(-15)}}{2(2)}\\
&=\frac{7\pm\sqrt{49+120}}{4}\\
&=\frac{7\pm\sqrt{169}}{4}\\
&=\frac{7\pm 13}{4}
\end{aligned}$$

Solution Set: $\left\{-\frac{3}{2},5\right\}$ because $x=\frac{7-13}{4}=\frac{-6}{4}=-\frac{3}{2}$ and $x=\frac{7+13}{4}=\frac{20}{4}=5$

Example 1.2.15:

Solve using the quadratic formula: $3x^2+6x-2=0$.

Solution

The coefficients are: $a=3 \quad b=6 \quad c=-2$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\ &= \frac{-(6) \pm \sqrt{(6)^2-4(3)(-2)}}{2(3)} \\ &= \frac{-6 \pm \sqrt{36+24}}{6} \\ &= \frac{-6 \pm \sqrt{60}}{6} \\ &= \frac{-6 \pm 2\sqrt{15}}{6} = \frac{\cancel{2}(-3 \pm \sqrt{15})}{\cancel{6}} = \frac{-3 \pm \sqrt{15}}{3} \\ &\text{or} \quad = \frac{-6}{6} \pm \frac{2\sqrt{15}}{6} = -1 \pm \frac{\sqrt{15}}{3} \end{aligned}$$

Two ways the solution set can be written are: $\left\{\frac{-3 \pm \sqrt{15}}{3}\right\}$ or $\left\{-1 \pm \frac{\sqrt{15}}{3}\right\}$

Sometimes terms are missing. When this is the case, use 0 as the coefficient. Also make sure the terms are written in descending powers of x so the correct values of the parameters $a, b,$ and c are used in the formula.

Example 1.2.16:

Solve using the quadratic formula: $45-x^2=0$

Solution

This equation is equivalent to $-1x^2+0x+45=0$ so $a=-1 \quad b=0 \quad c=45$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\ &= \frac{-(0) \pm \sqrt{(0)^2-4(-1)(45)}}{2(-1)} \\ &= \frac{0 \pm \sqrt{0+180}}{-2} \\ &= \frac{\pm\sqrt{36(5)}}{-2} \\ &= \frac{\pm 6\sqrt{5}}{2} \\ &= \pm 3\sqrt{5} \quad \text{Solution Set: } \{\pm 3\sqrt{5}\} \end{aligned}$$

Often solutions to quadratic equations are not real.

Example 1.2.17:

Solve using the quadratic formula: $x^2-4x+29=0$.

Solution

The coefficients are: $a=1 \quad b=-4 \quad c=29$

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(29)}}{2(1)} \\
&= \frac{4 \pm \sqrt{16 - 116}}{2} \\
&= \frac{4 \pm \sqrt{-100}}{2} \\
&= \frac{4 \pm 10i}{2} \\
&= \frac{4}{2} \pm \frac{10i}{2} \\
&= 2 \pm 5i \qquad \text{Solution Set: } \{2 \pm 5i\}
\end{aligned}
$$

Example 1.2.18:

Use the quadratic formula to solve $x^2 + x + 2 = 0$.

Solution

The coefficients are: $a = 1,\ b = 1,$ and $c = 2$. Substitute these values into the quadratic formula.

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(1) \pm \sqrt{(1)^2 - 4(1)(2)}}{2(1)} \\
&= \frac{-1 \pm \sqrt{1 - 8}}{2} \\
&= \frac{-1 \pm \sqrt{-7}}{2} \\
&= \frac{-1 \pm i\sqrt{7}}{2} \qquad \text{Solution Set: } \left\{ \frac{-1 \pm i\sqrt{7}}{2} \right\}
\end{aligned}
$$

Try It 1.2.20

Solve the quadratic equation using the quadratic formula: $9x^2 + 3x - 2 = 0$.

Answer

$x = -\frac{2}{3}, x = \frac{1}{3}$

Solving Cubic Equations

If multiple roots and complex roots are counted, then the **fundamental theorem of algebra** implies that every polynomial with one variable will have as many roots as its degree. For example, we expect $f(x) = x^3 - 8$ to have three roots. In other words, the equation

$$x^3 - 8 = 0$$

should have three solutions. To find them one might first think of trying to extract the cube roots just as we did with square roots,

$$\begin{aligned} x^3 - 8 &= 0 \\ x^3 &= 8 \\ x &= \sqrt[3]{8} \\ x &= 2 \end{aligned}$$

As you can see, this leads to one solution, the real cube root. There should be two others; let's try to find them.

Example 1.2.19:

Find the set of all roots: $f(x) = x^3 - 8$.

Solution

Notice that the expression $x^3 - 8$ is a difference of cubes and recall that $a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$. Here $a = x$ and $b = 2$ and we can write

$$\begin{aligned} x^3 - 8 &= 0 \\ (x-2)\left(x^2 + 2x + 4\right) &= 0 \end{aligned}$$

Next apply the zero-product property and set each factor equal to zero. After setting the factors equal to zero we can then solve the resulting equation using the appropriate methods.

$$\begin{aligned} x - 2 = 0 \quad \text{or} \quad & x^2 + 2x + 4 = 0 \\ & x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ & = \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(4)}}{2(1)} \\ x = 2 \qquad & = \frac{-2 \pm \sqrt{-12}}{2} \\ & = \frac{-2 \pm 2i\sqrt{3}}{2} \\ & = -1 \pm i\sqrt{3} \end{aligned}$$

Solution Set: $\{2, -1 \pm i\sqrt{3}\}$

Using this method, we were able to obtain the set of all three roots $\{2, -1 \pm i\sqrt{3}\}$, one real and two complex.

Sometimes factoring will produce a quadratic factor that needs to be solved using the quadratic formula or complete the square.

Example 1.2.21:

Solve $2x^4 + 2000x = 0$

Solution:

Factor first. $2x^4 + 2000x = 2x(x^3 + 1000) = 2x(x+10)(x^2 - 10x + 100)$

Use the zero factor property and complete the square on: $2x(x+10)(x^2 - 10x + 100) = 0$

$$\begin{array}{lll} 2x = 0 & (x+10) = 0 & (x^2 - 10x + 100) = 0 \\ x = 0 & x = -10 & x^2 - 10x + \square = -100 + \square \\ & & (x-5)^2 = -100 + 25 \\ & & x - 5 = \pm\sqrt{-75} \\ & & x = 5 \pm 5i\sqrt{3} \end{array}$$

The solution set is $\{0, -10, 5 \pm 5i\sqrt{3}\}$

When an expression is both a difference of squares and a sum or difference of cubes, if factoring as a difference of squares is done first, a more complete factorization is obtained. For example, if given the equation $64x^6 - 1 = 0$ to solve, when it is first factored as a difference of squares as $(8x^3 - 1)(8x^3 + 1)$, and then as a difference of cubes, the solutions obtained are $\pm\frac{1}{2}$ and $\pm\frac{1}{4}(1 \pm i\sqrt{3})$. In contrast, if it is first factored as a difference of cubes as $(4x^2 - 1)(16x^4 + 4x^2 + 1)$, the solutions eventually obtained are $\pm\frac{1}{2}$ and $\pm\frac{1}{4}\sqrt{-2 \pm 2i\sqrt{3}}$.

The Discriminant

The **quadratic formula** not only generates the solutions to a quadratic equation, it tells us about the nature of the solutions when we consider the **discriminant**, or the expression under the radical, $b^2 - 4ac$. The discriminant tells us whether the solutions are real numbers or complex numbers, and how many solutions of each type to expect. The table below relates the value of the discriminant to the solutions of a quadratic equation.

Value of Discriminant	$b^2 - 4ac = 0$	$b^2 - 4ac > 0$, and is a perfect square	$b^2 - 4eac > 0$, and is not a perfect square	$b^2 - 4ac < 0$
Results	One rational solution (double solution)	Two rational solutions	Two irrational solutions	Two complex solutions

The Discriminant

For $ax^2 + bx + c = 0$, where a, b, and c are real numbers, the **discriminant** is the expression under the radical in the quadratic formula: $b^2 - 4ac$. It tells us whether the solutions are real numbers or complex numbers and how many solutions of each type to expect.

Example 1.2.22: Using the Discriminant to Find the Nature of the Solutions to a Quadratic Equation

Use the discriminant to find the nature of the solutions to the following quadratic equations:

a. $x^2 + 4x + 4 = 0$
b. $8x^2 + 14x + 3 = 0$
c. $3x^2 - 5x - 4 = 0$
d. $3x^2 - 10x + 15 = 0$

Solution. Calculate the discriminant $b^2 - 4ac$ for each equation and state the expected type of solutions.

a. $x^2 + 4x + 4 = 0$
$b^2 - 4ac = (4)^2 - 4(1)(4) = 0$ There will be one rational double solution.

b. $8x^2 + 14x + 3 = 0$
$b^2 - 4ac = (14)^2 - 4(8)(3) = 100$ As 100 is a perfect square, there will be two rational solutions.

c. $3x^2 - 5x - 4 = 0$
$b^2 - 4ac = (-5)^2 - 4(3)(-4) = 73$ As 73 is positive but not a perfect square, there will be two irrational solutions.

d. $3x^2 - 10x + 15 = 0$
$b^2 - 4ac = (-10)^2 - 4(3)(15) = -80$ There will be two complex solutions.

1.2: Square Root Property, Complete the Square, and The Quadratic Formula is shared under a not declared license and was authored, remixed, and/or curated by LibreTexts.

1.2e: Exercises - SqRP, CTS, QF

A: Square Root Property

Exercise 1.2e.A

★ Solve each equation by the square root property

1. $x^2 = 81$	5. $x^2 = 12$	9. $2t^2 = 1$	13. $x^2 - 40 = 0$
2. $x^2 = 1$	6. $x^2 = 18$	10. $3t^2 = 2$	14. $x^2 - 24 = 0$
3. $y^2 = \frac{1}{9}$	7. $16x^2 = 9$	11. $x^2 - 16 = 0$	15. $x^2 + 1 = 0$
4. $y^2 = \frac{1}{16}$	8. $4x^2 = 25$	12. $x^2 - 36 = 0$	16. $x^2 + 100 = 0$

Answers to Odd Exercises:

1. ± 9 3. $\pm\frac{1}{3}$	5. $\pm 2\sqrt{3}$ 7. $\pm\frac{3}{4}$	9. $\pm\frac{\sqrt{2}}{2}$ 11. $-4, 4$	13. $\pm 2\sqrt{10}$ 15. $\pm i$

★ Solve each equation by the square root property

21. $9y^2 - 1 = 0$	25. $8x^2 + 1 = 0$	29. $y^2 + 6 = 2$	33. $3x^2 + 25 = 1$
22. $4y^2 - 25 = 0$	26. $12x^2 + 5 = 0$	30. $y^2 + 8 = 7$	34. $2x^2 + 81 = 31$
23. $5x^2 - 1 = 0$	27. $x^2 - \frac{4}{9} = 0$	31. $x^2 - 5 = 3$	35. $5y^2 + 7 = 9$
24. $6x^2 - 5 = 0$	28. $x^2 - \frac{9}{25} = 0$	32. $t^2 - 14 = 4$	36. $3x^2 + 4 = 5$

Answers to Odd Exercises:

21. $-\frac{1}{3}, \frac{1}{3}$ 23. $\pm\frac{\sqrt{5}}{5}$	25. $\pm\frac{\sqrt{2}}{4}i$ 27. $\pm\frac{2}{3}$	29. $\pm 2i$ 31. $\pm 2\sqrt{2}$	33. $\pm 2i\sqrt{2}$ 35. $\pm\frac{\sqrt{10}}{5}$

★ Solve each equation by the square root property

41. $(x-2)^2 - 1 = 0$	46. $(x+9)^2 - 36 = 0$	51. $4(y-2)^2 - 9 = 0$	56. $5(2x-1)^2 + 2 = 0$
42. $(x+1)^2 - 4 = 0$	47. $(x-5)^2 - 20 = 0$	52. $9(y+1)^2 - 4 = 0$	57. $3\left(y - \frac{2}{3}\right)^2 - \frac{3}{2} = 0$
43. $(u-5)^2 - 25 = 0$	48. $(x+1)^2 - 28 = 0$	53. $4(3x+1)^2 - 27 = 0$	58. $2\left(3y - \frac{1}{3}\right)^2 - \frac{5}{2} = 0$
44. $(u+2)^2 - 4 = 0$	49. $(3t+2)^2 + 6 = 0$	54. $9(2x-3)^2 - 8 = 0$	59. $-3(t-1)^2 + 12 = 0$
45. $(x+7)^2 - 4 = 0$	50. $(3t-5)^2 + 10 = 0$	55. $2(3x-1)^2 + 3 = 0$	60. $-2(t+1)^2 + 8 = 0$

Answers to Odd Exercises:

41. $1, 3$ 43. $0, 10$ 45. $-9, -5$	47. $5 \pm 2\sqrt{5}$ 49. $-\frac{2}{3} \pm \frac{\sqrt{6}}{3}i$	51. $\frac{1}{2}, \frac{7}{2}$ 53. $\frac{-2 \pm 3\sqrt{3}}{6}$ 55. $\frac{1}{3} \pm \frac{\sqrt{6}}{6}i$	57. $\frac{4 \pm 3\sqrt{2}}{6}$ 59. $-1, 3$

B: Complete the Square

Exercise $1.2e.B$

★ Determine the constant that should be added to the binomial and then complete the square

61. $x^2 - 2x + ? = (x - ?)^2$ 62. $x^2 - 4x + ? = (x - ?)^2$ 63. $x^2 + 10x + ? = (x + ?)^2$	64. $x^2 + 12x + ? = (x + ?)^2$ 65. $x^2 + 7x + ? = (x + ?)^2$ 66. $x^2 + 5x + ? = (x + ?)^2$	67. $x^2 - x + ? = (x - ?)^2$ 68. $x^2 - \frac{1}{2}x + ? = (x - ?)^2$	69. $x^2 + \frac{2}{3}x + ? = (x + ?)^2$ 70. $x^2 + \frac{4}{5}x + ? = (x + ?)^2$

Answers to Odd Exercises:

61. $x^2 - 2x + 1 = (x - 1)^2$ 63. $x^2 + 10x + 25 = (x + 5)^2$	65. $x^2 + 7x + \frac{49}{4} = \left(x + \frac{7}{2}\right)^2$ 67. $x^2 - x + \frac{1}{4} = \left(x - \frac{1}{2}\right)^2$	69. $x^2 + \frac{2}{3}x + \frac{1}{9} = \left(x + \frac{1}{3}\right)^2$

★ Solve each equation by completing the square

71. $x^2 + 2x = 8$ 72. $x^2 - 8x = -15$ 73. $y^2 + 2y = 24$ 74. $y^2 - 12y = -11$ 75. $x^2 - 4x - 1 = 15$ 76. $x^2 - 12x + 8 = -10$	77. $x(x + 1) - 11(x - 2) = 0$ 78. $(x + 1)(x + 7) - 4(3x + 2) = 0$ 79. $2y^2 - y - 1 = 0$ 80. $2y^2 + 7y - 4 = 0$ 81. $x^2 + 6x - 1 = 0$ 82. $x^2 + 8x + 10 = 0$ b	83. $x^2 - 2x - 7 = 0$ 84. $x^2 - 6x - 3 = 0$ 85. $y^2 - 2y + 4 = 0$ 86. $y^2 - 4y + 9 = 0$ 87. $t^2 + 10t - 75 = 0$ 88. $t^2 + 12t - 108 = 0$

Answers to Odd Exercises:

71. $-4, 2$ 73. $-6, 4$	75. $2 \pm 2\sqrt{5}$ 77. $5 \pm \sqrt{3}$	79. $-\frac{1}{2}, 1$ 81. $-3 \pm \sqrt{10}$	83. $1 \pm 2\sqrt{2}$ 85. $1 \pm i\sqrt{3}$	87. $-15, 5$

★ **Solve each equation by completing the square**

91. $t^2+3t=28$

92. $t^2-7t=-10$

93. $x^2+x-1=0$

94. $x^2+x-3=0$

95. $y^2+3y-2=0$

96. $y^2+5y-3=0$

97. $x^2+3x+5=0$

98. $x^2+x+1=0$

99. $y^2=(2y+3)(y-1)-2(y-1)$

100. $(2y+5)(y-5)-y(y-8)=-24$

101. $x^2-7x+\frac{11}{2}=0$

102. $x^2-9x+\frac{3}{2}=0$

103. $t^2-\frac{1}{2}t-1=0$

104. $t^2-\frac{1}{3}t-2=0$

105. $u^2-\frac{2}{3}u-\frac{1}{3}=0$

106. $u^2-\frac{4}{5}u-\frac{1}{5}=0$

Answers to Odd Exercises:

91. $-7, 4$
93. $\frac{-1\pm\sqrt{5}}{2}$
95. $\frac{-3\pm\sqrt{17}}{2}$
97. $-\frac{3}{2}\pm\frac{\sqrt{11}}{2}i$
99. $\frac{1\pm\sqrt{5}}{2}$
101. $\frac{7\pm3\sqrt{3}}{2}$
103. $\frac{1\pm\sqrt{17}}{4}$
105. $-\frac{1}{3}, 1$

★ **Solve each equation by completing the square**

111. $2x^2-4x+10=0$

112. $6x^2-24x+42=0$

113. $4x^2-8x-1=0$

114. $2x^2-4x-3=0$

115. $3x^2+6x+1=0$

116. $5x^2+10x+2=0$

117. $4x^2-12x-15=0$

118. $2x^2+3x-2=0$

119. $3x^2-x-2=0$

120. $2x^2+4x-43=0$

121. $3x^2+2x-3=0$

122. $5x^2+2x-5=0$

123. $2x^2-x-2=0$

124. $2x^2+3x-1=0$

125. $3u^2+2u+2=0$

126. $3u^2-u+1=0$

127. $2y^2-y-1=0$

128. $2y^2+7y-4=0$

129. $(t+2)^2=3(3t+1)$

130. $(3t+2)(t-4)-(t-8)=1-10t$

131. $(2x-1)^2=2x$

132. $(3x-2)^2=5-15x$

133. $(2x+1)(3x+1)=9x+4$

134. $(3x+1)(4x-1)=17x-4$

135. $9x(x-1)-2(2x-1)=-4x$

136. $(6x+1)^2-6(6x+1)=0$

Answers to Odd Exercises:

111. $1\pm2i$
113. $\frac{2\pm\sqrt{5}}{2}$
115. $\frac{-3\pm\sqrt{6}}{3}$
117. $\frac{3\pm2\sqrt{6}}{2}$
119. $1, -\frac{2}{3}$
121. $\frac{-1\pm\sqrt{10}}{3}$
123. $\frac{1\pm\sqrt{17}}{4}$
125. $\frac{-1\pm i\sqrt{5}}{3}$
127. $-\frac{1}{2}, 1$
129. $\frac{5\pm\sqrt{21}}{2}$
131. $\frac{3\pm\sqrt{5}}{4}$
133. $\frac{2\pm\sqrt{22}}{6}$
135. $\frac{1}{3}, \frac{2}{3}$

C: Quadratic Formula

Exercise 1.2e.C

★ **Solve each equation using the Quadratic Formula**

141. $x^2 - 6x - 16 = 0$

142. $x^2 - 3x - 18 = 0$

143. $2x^2 + 7x - 4 = 0$

144. $3x^2 + 5x - 2 = 0$

145. $-x^2 + 9x - 20 = 0$

146. $-2x^2 - 3x + 5 = 0$

147. $16y^2 - 24y + 9 = 0$

148. $4y^2 - 20y + 25 = 0$

149. $x^2 - 5x + 1 = 0$

150. $x^2 - 7x + 2 = 0$

151. $x^2 + 8x + 5 = 0$

152. $x^2 - 4x + 2 = 0$

153. $5u^2 - 2u + 1 = 0$

154. $8u^2 - 20u + 13 = 0$

155. $-y^2 + 16y - 62 = 0$

156. $-y^2 + 14y - 46 = 0$

157. $-2t^2 + 4t + 3 = 0$

158. $-4t^2 + 8t + 1 = 0$

Answers to Odd Exercises:

141. $-2, 8$

143. $-4, \frac{1}{2}$

145. $4, 5$

147. $\frac{3}{4}$

149. $\frac{5 \pm \sqrt{21}}{2}$

151. $-4 \pm \sqrt{11}$

153. $\frac{1}{5} \pm \frac{2}{5}i$

155. $8 \pm \sqrt{2}$

157. $\frac{2 \pm \sqrt{10}}{2}$

★ **Solve each equation using the Quadratic Formula**

161. $4y^2 - 9 = 0$

162. $9y^2 - 25 = 0$

163. $5t^2 - 6t = 0$

164. $t^2 + 6t = 0$

165. $x^2 - 18 = 0$

166. $x^2 - 12 = 0$

167. $x^2 + 12 = 0$

168. $x^2 + 20 = 0$

169. $3x^2 + 2 = 0$

170. $5x^2 + 3 = 0$

171. $y^2 = 2y - 10$

172. $y^2 = 4y - 13$

173. $2x^2 - 10x = 1$

174. $2x^2 - 4x = 3$

175. $3x^2 + 2 = x$

176. $4x^2 + 1 = 3x$

Answers to Odd Exercises:

161. $\pm\frac{3}{2}$

163. $0, \frac{6}{5}$

165. $\pm 3\sqrt{2}$

167. $\pm 2i\sqrt{3}$

169. $\pm\frac{i\sqrt{6}}{3}$

171. $1 \pm 3i$

173. $\frac{5 \pm 3\sqrt{3}}{2}$

175. $\frac{1}{6} \pm \frac{\sqrt{23}}{6}i$

★ **Solve each equation using the Quadratic Formula**

181. $\frac{1}{2}y^2 + 5y + \frac{3}{2} = 0$

182. $3y^2 + \frac{1}{2}y - \frac{1}{3} = 0$

183. $2x^2 - \frac{1}{2}x + \frac{1}{4} = 0$

184. $3x^2 - \frac{2}{3}x + \frac{1}{3} = 0$

185. $1.2x^2 - 0.5x - 3.2 = 0$

186. $0.4x^2 + 2.3x + 1.1 = 0$

187. $2.5x^2 - x + 3.6 = 0$

188. $-0.8x^2 + 2.2x - 6.1 = 0$

Answers to Odd Exercises:

181. $-5 \pm \sqrt{22}$

183. $\frac{1}{8} \pm \frac{\sqrt{7}}{8}i$

185. $x \approx -1.4$ or $x \approx 1.9$

187. $x \approx 0.2 \pm 1.2i$

★ **Solve each equation using the Quadratic Formula**

191. $(x+2)^2+9=0$	195. $-2y^2=3(y-1)$	199. $(x+5)(x-1)=2x+1$	203. $3t(t-2)+4=0$
192. $(x-4)^2+1=0$	196. $3y^2=5(2y-1)$	200. $(x+7)(x-2)=3(x+1)$	204. $5t(t-1)=t-4$
193. $(2x+1)^2-2=0$	197. $(t+1)^2=2t+7$	201. $2x(x-1)=-1$	205. $(2x+3)^2=16x+4$
194. $(3x+1)^2-5=0$	198. $(2t-1)^2=73-4t$	202. $x(2x+5)=3x-5$	206. $(2y+5)^2-12(y+1)=0$

Answers to Odd Exercises:

191. $-2\pm 3i$ 193. $\frac{-1\pm\sqrt{2}}{2}$	195. $\frac{-3\pm\sqrt{33}}{4}$ 197. $\pm\sqrt{6}$	199. $-1\pm\sqrt{7}$ 201. $\frac{1}{2}\pm\frac{1}{2}i$	203. $1\pm\frac{\sqrt{3}}{3}i$ 205. $\frac{1}{2}\pm i$

D: Factor and Solve (Sum and Difference of Cubes)

Exercise 1.2e.D

★ **Solve.**

211. $x^3=8$	213. $64x^3=-27$	215. $8x^3=125$	217. $2x^4=54x$	219. $729x^6-1=0$
212. $x^3=1$	214. $27x^3=-64$	216. $125x^3=8$	218. $3x^4+192x=0$	220. $x^6=64$

Answers to Odd Exercises:

211. $\{2,-1\pm i\sqrt{3}\}$ 213. $\{-\frac{3}{4},\frac{3}{8}\pm\frac{3\sqrt{3}}{8}i\}$	215. $\{\frac{5}{2},-\frac{5}{4}\pm\frac{5\sqrt{3}}{4}i\}$ 217. $\{0,\ 3,\ -\frac{3}{2}\pm\frac{3\sqrt{3}}{2}i\}$	219. $\{\pm\frac{1}{3},\frac{1}{6}\pm\frac{\sqrt{3}}{6}i,-\frac{1}{6}\pm\frac{\sqrt{3}}{6}i\}$

E: The Discriminant

Exercise 1.2e.E

★ **Calculate the discriminant and use it to determine the number and type of solutions. Do not solve.**

221. $x^2-x+1=0$	224. $3x^2-1x-2=0$	228. $2x^2+3x=0$	232. $-x^2-3x+4=0$
222. $x^2+2x+3=0$	225. $3x^2-1x+2=0$	229. $4x^2-5x=0$	233. $-x^2-5x+3=0$
223. $x^2-2x-3=0$	226. $9y^2+2=0$	230. $\frac{1}{2}x^2-2x+\frac{5}{2}=0$	234. $25t^2+30t+9=0$
224. $x^2-5x-5=0$	227. $9y^2-2=0$	231. $\frac{1}{2}x^2-x-\frac{1}{2}=0$	235. $9t^2-12t+4=0$

Answers to Odd Exercises:

221. -3; two complex solutions 223. 16; two rational solutions 225. -23; two complex solutions	227. 72; two irrational solutions 229. 25; two rational solutions 231. 2; two irrational solutions	233. 37; two irrational solutions 235. 0; one rational solution

★

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1.3: Rational Equations

A **rational expression** is the quotient of two polynomials.

A **rational equation** is an equation containing one or more rational expressions.

Rational expressions typically contain a variable in the denominator. For this reason, care must be taken to ensure that the denominator is not 0 by making note of restrictions and checking that no solution makes a denominator zero in the original equation. Solving rational equations involves clearing fractions by multiplying both sides of the equation by the least common denominator (LCD).

How to: Solve a Rational Equation.

1. Factor all denominators to determine the LCD.
   Note the restrictions to x. These are the values of x that make the denominator zero. They are values that x can't be because a fraction with a zero denominator is undefined.
2. Multiply both sides of the equal sign by the LCD. Every term in the equation is multiplied by the LCD. This results in the fractions getting "cleared" from the equation.
3. Solve the resulting equation
4. Check for extraneous solutions. Check each solution to confirm the value does not make the denominator in the original equation equal to zero. If a solution is found to make a denominator in the original equation zero, it must be rejected as a solution.

Example 1.3.1:

Solve: $\frac{1}{x}+\frac{2}{x^2}=\frac{x+9}{2x^2}$.

Solution

Both sides of the equation are multiplied by the LCD, which is $2x^2$. Note the restriction on x is $x \neq 0$.

$$2x^2 \cdot \left(\frac{1}{x}+\frac{2}{x^2}\right) = 2x^2 \cdot \left(\frac{x+9}{2x^2}\right) \qquad \textit{Multiply both sides by the LCD.}$$

$$2x^2 \cdot \frac{1}{x} + 2x^2 \cdot \frac{2}{x^2} = 2x^2 \cdot \frac{x+9}{2x^2} \qquad \textit{Distribute.}$$

$$2x+4 = x+9 \qquad \textit{Simplify and then solve.}$$

$$x = 5$$

The answer must always be checked that it is not a restricted value. In other words, answers must be checked so that when substituted back into the original equation, no division by zero occurs.

The solution set is $\{5\}$.

After multiplying both sides of the previous example by the LCD, we were left with a linear equation to solve. This is not always the case; sometimes we will be left with quadratic equation.

Example 1.3.2:

Solve: $\frac{3(x+2)}{x-4}-\frac{x+4}{x-2}=\frac{x-2}{x-4}$.

Solution

In this example, there are two restrictions, $x \neq 4$ and $x \neq 2$. Begin by multiplying both sides by the LCD, $(x-2)(x-4)$.

$$(x-2)(x-4)\cdot\left(\frac{3(x+2)}{x-4}-\frac{x+4}{x-2}\right)=(x-2)(x-4)\cdot\left(\frac{x-2}{x-4}\right)$$
$$\cancel{(x-2)}\,\cancel{(x-4)}\cdot\frac{3(x+2)}{\cancel{x-4}}-\cancel{(x-2)}\,(x-4)\cdot\frac{x+4}{\cancel{x-2}}=(x-2)\,\cancel{(x-4)}\cdot\frac{x-2}{\cancel{x-4}}$$
$$3(x+2)(x-2)-(x+4)(x-4)=(x-2)(x-2)$$
$$3\left(x^2-4\right)-\left(x^2-16\right)=x^2-2x-2x+4$$
$$3x^2-12-x^2+16=x^2-4x+4$$
$$2x^2+4=x^2-4x+4$$

To solve, rewrite the quadratic equation in standard form, factor, and then set each factor equal to 0.

$$2x^2+4=x^2-4x+4$$
$$x^2+4x=0$$
$$x(x+4)=0$$
$$x=0 \text{ or } x+4=0$$
$$x=-4$$

These values are checked, confirming they are in the domain of the original equation.

The solution set is $\{0,-4\}$.

Up to this point, all of the possible solutions have solved the original equation. However, this may not always be the case. Multiplying both sides of an equation by variable factors may lead to **extraneous solutions**, which are solutions that do not solve the original equation. A complete list of steps for solving a rational equation is outlined in the following example.

Example 1.3.3:

Solve: $\frac{2x}{3x+1}=\frac{1}{x-5}-\frac{4(x-1)}{3x^2-14x-5}$.

Solution

Step 1: Factor all denominators and determine the LCD.

$$\frac{2x}{3x+1}=\frac{1}{x-5}-\frac{4(x-1)}{3x^2-14x-5}$$
$$\frac{2x}{(3x+1)}=\frac{1}{(x-5)}-\frac{4(x-1)}{(3x+1)(x-5)}$$

The LCD is $(3x+1)(x-5)$.

Identify the restrictions. In this case, $x\neq-\frac{1}{3}$ and $x\neq5$.

Step 2: Multiply both sides of the equation by the LCD. Distribute carefully and then simplify.

$$(3x+1)(x-5)\cdot\frac{2x}{(3x+1)}=(3x+1)(x-5)\cdot\left(\frac{1}{(x-5)}-\frac{4(x-1)}{(3x+1)(x-5)}\right)$$
$$\cancel{(3x+1)}\,(x-5)\cdot\frac{2x}{\cancel{(3x+1)}}=(3x+1)\,\cancel{(x-5)}\cdot\frac{1}{\cancel{(x-5)}}-\cancel{(3x+1)}\,\cancel{(x-5)}\cdot\frac{4(x-1)}{\cancel{(3x+1)}\,\cancel{(x-5)}}$$
$$2x(x-5)=(3x+1)-4(x-1)$$

Step 3: Solve the resulting equation. Here the result is a quadratic equation. Rewrite it in standard form, factor, and then set each factor equal to 0.

$$2x(x-5)=(3x+1)-4(x-1)$$
$$2x^2-10x=3x+1-4x+4$$
$$2x^2-10x=-x+5$$
$$2x^2-9x-5=0$$
$$(2x+1)(x-5)=0$$
$$2x+1=0 \quad \text{or} \quad x-5=0$$
$$2x=-1 \qquad\qquad x=5$$
$$x=-\frac{1}{2}$$

Step 4: Check for extraneous solutions. Here 5 is a restriction and so 5 produces a zero denominator in the original equation and so is an extraneous solution. Therefore it is not included in the solution set.

The solution set is $\left\{-\frac{1}{2}\right\}$.

If this process produces a solution that happens to be a restriction, always disregard it as a solution.

Try It 1.3.3

Solve: $\frac{4(x-3)}{36-x^2}=\frac{1}{6-x}+\frac{2x}{6+x}$.

Answer

$-\frac{3}{2}$

Sometimes all potential solutions are extraneous, in which case we say that there is no solution to the original equation. In the next two examples, we demonstrate two ways in which rational equation can have no solutions.

Example 1.3.4:

Solve: $1+\frac{5x+22}{x^2+3x-4}=\frac{x+4}{x-1}$

Solution

To identify the LCD, first factor the denominators.

$$1+\frac{5x+22}{x^2+3x-4}=\frac{x+4}{x-1}$$
$$1+\frac{5x+22}{(x+4)(x-1)}=\frac{x+4}{(x-1)}$$

Multiply both sides by the LCD, $(x+4)(x-1)$, distributing carefully.

$$(x+4)(x-1)\cdot\left(1+\frac{5x+22}{(x+4)(x-1)}\right)=(x+4)(x-1)\cdot\frac{x+4}{(x-1)}$$
$$(x+4)(x-1)\cdot 1+(x+4)(x-1)\cdot\frac{(5x+22)}{(x+4)(x-1)}=(x+4)(x-1)\cdot\frac{(x+4)}{(x-1)}$$
$$(x+4)(x-1)+(5x+22)=(x+4)(x+4)$$
$$x^2-x+4x-4+5x+22=x^2+4x+4x+16$$
$$x^2+8x+18=x^2+8x+16$$
$$18=16 \quad \textit{False}$$

The equation is a contradiction and thus has **no solution**. The solution set is { } or equivalently, $\varnothing$

Example 1.3.5:

Solve: $\frac{3x}{2x-3}-\frac{3(4x+3)}{4x^2-9}=\frac{x}{2x+3}$.

Solution

First, factor the denominators.

$$\frac{3x}{(2x-3)}-\frac{3(4x+3)}{(2x+3)(2x-3)}=\frac{x}{(2x+3)}$$

Take note that the restrictions on the domain are $x\neq\pm\frac{3}{2}$. To clear the fractions, multiply by the LCD, $(2x+3)(2x-3)$.

$$\frac{3x\cdot(2x+3)(2x-3)}{(2x-3)}-\frac{3(4x+3)\cdot(2x+3)(2x-3)}{(2x+3)(2x-3)}=\frac{x\cdot(2x+3)(2x-3)}{(2x+3)}$$

$$3x(2x+3)-3(4x+3)=x(2x-3)$$
$$6x^2+9x-12x-9=2x^2-3x$$
$$6x^2-3x-9=2x^2-3x$$
$$4x^2-9=0$$
$$(2x+3)(2x-3)=0$$

$$2x+3=0 \quad \text{or} \quad 2x-3=0$$
$$2x=-3 \qquad\qquad 2x=3$$
$$x=-\frac{3}{2} \qquad\qquad x=\frac{3}{2}$$

Both of these values are restrictions of the original equation; hence both are extraneous, and the solution set is $\varnothing$

Rational Expressions and Rational Equations

It is important to point out that this technique for clearing algebraic fractions only works for equations. Do not try to clear algebraic fractions when simplifying expressions. As a reminder, an example of each is provided below.

	Expression	Equation	
	$\frac{1}{x}+\frac{x}{2x+1}$	$\frac{1}{x}+\frac{x}{2x+1}=0$	

Expressions are to be simplified and equations are to be solved. If we multiply the expression by the LCD, $x(2x+1)$, we obtain another expression that is not equivalent.

Incorrect	Correct if an Expression	Correct if an Equation
$\frac{1}{x}+\frac{x}{2x+1}$ $\neq x(2x+1)\cdot\left(\frac{1}{x}+\frac{x}{2x+1}\right)$ $=2x+1+x^2$ ✗	$\frac{1}{x}+\frac{x}{2x+1}$ $=\frac{1}{x}\frac{2x+1}{2x+1}+\frac{x}{2x+1}\frac{x}{x}$ $=\frac{2x+1+x^2}{x(2x+1)}$ $=\frac{(x+1)^2}{x(2x+1)}$	$\frac{1}{x}+\frac{x}{2x+1}=0$ $x(2x+1)\cdot(\frac{1}{x}+\frac{x}{2x+1})=x(2x+1)\cdot 0$ $2x+1+x^2=0$ $x^2+2x+1=0$ ✓

Rational equations are sometimes expressed using negative exponents.

Example 1.3.6:

Solve: $6+x^{-1}=x^{-2}$.

Solution:

Begin by removing the negative exponents.

$$6+x^{-1}=x^{-2}$$
$$6+\frac{1}{x}=\frac{1}{x^2}$$

Here we can see the restriction, $x\neq 0$. Next, multiply both sides by the LCD, x^2.

$$x^2\cdot\left(6+\frac{1}{x}\right)=x^2\cdot\left(\frac{1}{x^2}\right)$$
$$x^2\cdot 6+x^2\cdot\frac{1}{x}=x^2\cdot\frac{1}{x^2}$$
$$6x^2+x=1$$
$$6x^2+x-1=0$$
$$(3x-1)(2x+1)=0$$

$$3x-1=0 \quad \text{or} \quad 2x+1=0$$
$$3x=1 \qquad\qquad 2x=-1$$
$$x=\frac{1}{3} \qquad\qquad x=-\frac{1}{2}$$

Neither solution is a restricted value and will not produce division by zero in the original equation, so the solution set is $\left\{-\frac{1}{2},\frac{1}{3}\right\}$

Proportions

A **proportion** is a statement of equality of two ratios, $\frac{a}{b}=\frac{c}{d}$. This proportion is often read "a is to b as c is to d." Given any nonzero real numbers a,b,c, and d that satisfy a proportion, multiply both sides by the product of the denominators to obtain the following:

$$\frac{a}{b}=\frac{c}{d}$$
$$bd\cdot\frac{a}{b}=bd\cdot\frac{c}{d}$$
$$ad=bc$$

This shows that cross products are equal, and is commonly referred to as **cross multiplication**.

$$\text{If } \frac{a}{b}=\frac{c}{d} \text{ then } ad=bc$$

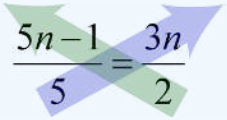
Cross multiply to solve proportions where terms are unknown.

Example 1.3.7:

Solve: $\frac{5n-1}{5}=\frac{3n}{2}$.

Solution

When cross multiplying, be sure to group $5n-1$.



$$(5n-1)\cdot 2=5\cdot 3n$$

Apply the distributive property in the next step.

$$
\begin{aligned}
(5n-1)\cdot 2 &= 5\cdot 3n \\
10n-2 &= 15n \quad \textit{Distribute.} \\
-2 &= 5n \quad \textit{Solve.} \\
\frac{-2}{5} &= n
\end{aligned}
$$

Neither fraction can ever have a denominator of zero, so there are no extraneous solutions and the solution set is $\{-\frac{2}{5}\}$

Cross multiplication can be used as an alternate method for solving rational equations, but in order to employ this technique, each side of the equation must be simplified to a single algebraic fraction and then cross multiplication can be done.

Example 1.3.8:

Solve: $\frac{1}{2}-\frac{4}{x}=-\frac{x}{8}$.

Solution

Obtain a single algebraic fraction on the left side by subtracting the equivalent fractions with a common denominator.

$$
\begin{aligned}
\frac{1}{2}\cdot\frac{x}{x}-\frac{4}{x}\cdot\frac{2}{2} &= -\frac{x}{8} \\
\frac{x}{2x}-\frac{8}{2x} &= -\frac{x}{8} \\
\frac{x-8}{2x} &= -\frac{x}{8}
\end{aligned}
$$

Note that $x\neq 0$, cross multiply, and then solve for x.

$$
\begin{aligned}
\frac{x-8}{2x} &= \frac{-x}{8} \\
8(x-8) &= -x\cdot 2x \\
8x-64 &= -2x^2 \\
2x^2+8x-64 &= 0 \\
2\left(x^2+4x-32\right) &= 0 \\
2(x-4)(x+8) &= 0
\end{aligned}
$$

Next, set each variable factor equal to zero.

$$
\begin{aligned}
x-4=0 \quad &\text{or} \quad x+8=0 \\
x=4 \quad &\qquad\quad x=-8
\end{aligned}
$$

The restrictions to x are $x\neq 0$, so both solutions are legitimate. The solution set is $\{-8,4\}$

Try It 1.3.3

Solve: $\frac{2(2x-5)}{x-1}=-\frac{x-4}{2x-5}$.

Answer

$-\frac{3}{2}2,3$

A note of caution should accompany proposed use of cross multiplication because it sometimes results in producing a more complicated equation to solve.

Example 1.3.9:

Solve $\frac{2x+3}{x^2-1}=\frac{x+1}{2x-2}$.

Solution

Method 1. Using cross multiplication produces: $4x^2+2x-6=x^3+x^2-x-1$. Zeroing one side of the equal sign and combining like terms produces $0=x^3-3x^2-3x+5$ which is not an equation for which a solution can easily be found.

Method 2. Starting with determining the LCD, which is $2(x^2-1)$ or $2(x-1)(x+1)$, and multiplying both sides of the equal sign by that LCD produces $4x+6=x^2+2x+1$. Simplification produces the equation $5=x^2-2x$ which is easily solved. The solution set is $\{1\pm\sqrt{6}\}$.

In order to avoid complications like this, denominators should be examined before performing cross multiplication, and any common factors in the denominators should be canceled first. In the above example, the two denominators are $(x-1)(x+1)$ and $2(x-1)$. If the common factor in both denominators, $(x-1)$, is removed (by multiplying both sides of the equal sign by $(x-1)$, then when cross multiplication is done, the result would be a quadratic $2(2x+3)=(x+1)(x+1)$ which is more easily solved than a cubic.

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1.3e: Exercises - Rational Equations

A: Rational Expression or Equation?

Exercise 1.3e.A

★ Simplify or solve, whichever is appropriate

1. $\frac{1}{x}+\frac{2}{x-3}=-\frac{2}{3}$
2. $\frac{1}{x-3}-\frac{3}{4}=\frac{1}{x}$
3. $\frac{x-2}{3x-1}-\frac{2-x}{x}$
4. $\frac{5}{2}+\frac{x}{2x-1}-\frac{1}{2x}$
5. $\frac{x-1}{3x}+\frac{2}{x+1}-\frac{5}{6}$
6. $\frac{x-1}{3x}+\frac{2}{x+1}=\frac{5}{6}$
7. $\frac{2x+1}{2x-3}+2=\frac{1}{2x}$
8. $5-\frac{3x+1}{2x}+\frac{1}{x+1}$

Answers to Odd Exercises:

1. Solve; $-3, \frac{3}{2}$
3. Simplify; $\frac{(4x-1)(x-2)}{x(3x-1)}$
5. Simplify; $-\frac{(x-2)(3x-1)}{6x(x+1)}$
7. Solve; $\frac{1}{2}$

B: Solve Rational Equations

Exercise 1.3e.B

★ Solve

11. $\frac{3}{x}+2=\frac{1}{3x}$
12. $5-\frac{1}{2x}=-\frac{1}{x}$
13. $\frac{7}{x^2}+\frac{3}{2x}=\frac{1}{x^2}$
14. $\frac{4}{3x^2}+\frac{1}{2x}=\frac{1}{3x^2}$
15. $\frac{1}{6}+\frac{2}{3x}=\frac{7}{2x^2}$
16. $\frac{1}{12}-\frac{1}{3x}=\frac{1}{x^2}$
17. $2+\frac{3}{x}+\frac{7}{x(x-3)}=0$
18. $\frac{20}{x}-\frac{x+44}{x(x+2)}=3$
19. $\frac{2x}{2x-3}+\frac{4}{x}=\frac{x-18}{x(2x-3)}$
20. $\frac{2x}{x-5}+\frac{1}{x}=\frac{9x+5}{x(x-5)}$
21. $\frac{4}{4x-1}-\frac{1}{x-1}=\frac{2}{4x-1}$
22. $\frac{5}{2x-3}-\frac{1}{x+3}=\frac{2}{2x-3}$

Answers to Odd Exercises:

11. $-\frac{4}{3}$
13. -4
15. $-7, 3$
17. $-\frac{1}{2}, 2$
19. $-2, -\frac{3}{2}$
21. $-\frac{1}{2}$

★ Solve

23. $\frac{4x}{x-3}+\frac{4}{x^2-2x-3}=-\frac{1}{x+1}$
24. $\frac{x}{x-2}-\frac{2}{x+4}=\frac{12}{x^2+2x-8}$
25. $\frac{x}{x-8}-\frac{8}{x-1}=\frac{56}{x^2-9x+8}$
26. $\frac{2x}{x-1}+\frac{9}{3x-1}+\frac{11}{3x^2-4x+1}=0$
27. $\frac{3x}{x-2}-\frac{14}{2x^2-x-6}=\frac{2}{2x+3}$
28. $\frac{x}{x-4}-\frac{4}{x-5}=-\frac{4}{x^2-9x+20}$
29. $\frac{2x}{5+x}-\frac{1}{5-x}=\frac{2x}{x^2-25}$
30. $\frac{2x}{2x+3}-\frac{1}{2x-3}=\frac{6}{9-4x^2}$
31. $1+\frac{1}{x+1}=\frac{8}{x-1}-\frac{16}{x^2-1}$
32. $1-\frac{1}{3x+5}=\frac{2x}{3x-5}-\frac{2(6x+5)}{9x^2-25}$

Answers to Odd Exercises:

23. $-\frac{1}{4}$	25. $\varnothing$	27. $-2, \frac{5}{6}$	29. $\frac{1}{2}$	31. 6

★ **Solve**

33. $2x^{-1} = 2x^{-2} - x^{-1}$

34. $3 + x(x+1)^{-1} = 2(x+1)^{-1}$

35. $x^{-2} - 64 = 0$

36. $1 - 4x^{-2} = 0$

37. $x - (x+2)^{-1} = -2$

38. $2x - 9(2x-1)^{-1} = 1$

39. $2x^{-2} + (x-12)^{-1} = 0$

40. $-2x^{-2} + 3(x+4)^{-1} = 0$

Answers to Odd Exercises:

33. $\frac{2}{3}$	35. $\pm\frac{1}{8}$	37. $-3, -1$	39. $-6, 4$

★ **Solve**

41. $\frac{5}{n} = -\frac{3}{n-2}$

42. $\frac{2n-1}{2n} = -\frac{1}{2}$

43. $-3 = \frac{5n+2}{3n}$

44. $\frac{n+1}{2n-1} = \frac{1}{3}$

45. $\frac{x+2}{x-5} = \frac{x+4}{x-2}$

46. $\frac{x+1}{x+5} = \frac{x-5}{x}$

47. $\frac{2x+1}{6x-1} = \frac{x+5}{3x-2}$

48. $\frac{6(2x+3)}{4x-1} = \frac{3x}{x+2}$

49. $\frac{3(x+1)}{1-x} = \frac{x+3}{x+1}$

50. $\frac{8(x-2)}{x+1} = \frac{5-x}{x-2}$

51. $\frac{x+3}{x+7} = \frac{x+3}{3(5-x)}$

52. $\frac{x+1}{x+4} = \frac{-8(x+4)}{x+7}$

Answers to Odd Exercises:

41. $\frac{5}{4}$	43. $-\frac{1}{7}$	45. -16	47. $\frac{1}{10}$	49. $-2, 0$	51. $-3, 2$

★ **Solve**

53. $\frac{x}{x-2} - \frac{3}{x+8} = \frac{x+2}{x+8} + \frac{5(x+3)}{x^2+6x-16}$

54. $\frac{2x}{x-10} + \frac{1}{x-3} = \frac{x+3}{x-10} + \frac{x^2-5x+5}{x^2-13x+30}$

55. $\frac{5}{x^2+9x+18} + \frac{x+3}{x^2+7x+6} = \frac{5}{x^2+4x+3}$

56. $\frac{1}{x^2+4x-60} + \frac{x-6}{x^2+16x+60} = \frac{1}{x^2-36}$

57. $\frac{4}{x^2+10x+21} + \frac{2(x+3)}{x^2+6x-7} = \frac{x+7}{x^2+2x-3}$

58. $\frac{x-1}{x^2-11x+28} + \frac{x-1}{x^2-5x+4} = \frac{x-4}{x^2-8x+7}$

59. $\frac{5}{x^2+5x+4} + \frac{x+1}{x^2+3x-4} = \frac{5}{x^2-1}$

60. $\frac{1}{x^2-2x-63} + \frac{x-9}{x^2+10x+21} = \frac{1}{x^2-6x-27}$

61. $\frac{4}{x^2-4} + \frac{2(x-2)}{x^2-4x-12} = \frac{x+2}{x^2-8x+12}$

62. $\frac{x+2}{x^2-5x+4} + \frac{x+2}{x^2+x-2} = \frac{x-1}{x^2-2x-8}$

Answers to Odd Exercises:

53. $\varnothing$	55. $-8, 2$	57. 5	59. $-6, 4$	61. 10

★

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1.4: Radical Equations

Radical Equations

A **radical equation** is any equation that contains one or more radicals with a variable in the radicand. Following are some examples of radical equations, all of which will be solved in this section:

$\sqrt{2x-1}=3$	$\sqrt[3]{4x^2+7}-2=0$	$\sqrt{x+2}-\sqrt{x}=1$

The **squaring property of equality** states that if given real numbers a and b that are equal, the equality is retained if both numbers are squared. For example.

Given $-3=-3$, then squaring both quantities is also a true statement: $(-3)^2=(-3)^2$ because $9=9$ ✓

The converse, on the other hand, is not necessarily true,

Given $9=9$ which could be written $(-3)^2=(3)^2$ does not produce an equality if the squaring operation is removed: $-3\neq 3$ ✗

This is important because we will use this property to solve radical equations. Because the converse of the squaring property of equality is not necessarily true, solutions to the squared equation may not be solutions to the original. Hence squaring both sides of an equation introduces the possibility of **extraneous solutions**, which are solutions that do not solve the original equation. For example, to find the solution to $\sqrt{x}=-5$, the technique used is to square both sides of the equal sign,$(\sqrt{x})^2=(5)^2$ which produces the solution $x=25$. However, checking this solution produces $\sqrt{25}=5$ which contradicts the original problem statement, $\sqrt{x}=-5$.

For this reason, answers that result from squaring both sides of an equation must ALWAYS be checked.

How to: Solve a Radical Equation.

1. **Isolate a radical**. Put ONE radical on one side of the equal sign and put everything else on the other side.
2. **Eliminate the radical**. Raise both sides of the equal sign to the power that matches the index on the radical. This means square both sides if it is a square root; cube both sides if it is a cube root; etc. It is this step that can introduce extraneous roots if both sides are raised to an even power!!
3. **Solve**. If the equation still contains radicals, repeat steps 1 and 2. If there are no more radicals, solve the resulting equation.
4. **Check for extraneous solutions**. Check each solution to confirm the value produces a true statement when substituted back into the original equation.

Example 1.4.1:

Solve: $\sqrt{3x+1}=4$.

Solution

$$\begin{aligned}\sqrt{3x+1}&=4\\(\sqrt{3x+1})^2&=(4)^2 \quad \textit{Square both sides.}\\3x+1&=16 \quad \textit{Solve.}\\3x&=15\\x&=5\end{aligned}$$

Next, we must check.

$$\begin{aligned}\sqrt{3(5)+1}&=4\\\sqrt{15+1}&=4\\\sqrt{16}&=4\\4&=4\ \checkmark\end{aligned}$$

The solution set is $\{5\}$.

Example 1.4.2:

Solve $\sqrt{x-3}=x-5$.

Solution

$$\begin{aligned}\sqrt{x-3}&=x-5\\(\sqrt{x-3})^2&=(x-5)^2 \qquad \textit{Square both sides.}\\x-3&=x^2-10x+25\end{aligned}$$

The resulting quadratic equation can be solved by factoring.

$$\begin{aligned}x-3&=x^2-10x+25\\0&=x^2-11x+28\\0&=(x-4)(x-7)\end{aligned}$$

$$\begin{aligned}x-4=0 \quad &\text{or} \quad x-7=0\\x=4 \quad &\qquad\quad x=7\end{aligned}$$

Checking the solutions after squaring both sides of an equation is not optional. Use the original equation when performing the check.

Check $x=4$	*Check* $x=7$
$\sqrt{x-3}=x-5$ $\sqrt{4-3}=4-5$ $\sqrt{1}=-1$ $1=-1$ ✗	$\sqrt{x-3}=x-5$ $\sqrt{7-3}=7-5$ $\sqrt{4}=2$ $2=2$ ✓

After checking, you can see that $x=4$ is an extraneous solution; it does not solve the original radical equation. Disregard that answer. The solution set consequently is just $\{7\}$.

In the previous two examples, notice that the radical is isolated on one side of the equation. Typically, this is not the case. The steps for solving radical equations involving square roots are outlined in the following example.

Example 1.4.3:

Solve: $\sqrt{2x-1}+2=x$.

Solution

Step 1: Isolate the square root.

$$\begin{aligned}\sqrt{2x-1}+2&=x\\\sqrt{2x-1}&=x-2\end{aligned}$$

Step 2: Square both sides.

$$\begin{aligned}(\sqrt{2x-1})^2&=(x-2)^2\\2x-1&=x^2-4x+4\end{aligned}$$

Step 3: Solve the resulting equation.

$$\begin{aligned}2x-1&=x^2-4x+4\\0&=x^2-6x+5\\0&=(x-1)(x-5)\end{aligned}$$

$$\begin{aligned}x-1=0 \quad &\text{or} \quad x-5=0\\x=1 \quad &\qquad\quad x=5\end{aligned}$$

Step 4: Check the solutions in the original equation. Squaring both sides introduces the possibility of extraneous solutions; hence the check is required.

Check $x=1$	*Check* $x=5$
$\sqrt{2x-1}+2=x$	$\sqrt{2x-1}+2=x$
$\sqrt{2(1)-1}+2=1$	$\sqrt{2(5)-1}+2=5$
$\sqrt{1}+2=1$	$\sqrt{9}+2=5$
$1+2=1$	$3+2=5$
$3=1$ ✗	$5=5$ ✓

After checking, we can see that $x=1$ is an extraneous solution; it does not solve the original radical equation. This leaves $\{5\}$ as the solution set.

Sometimes there is more than one solution to a radical equation.

Example 1.4.4:

Solve: $2\sqrt{2x+5}-x=4$.

Solution

Begin by isolating the term with the radical.

$$\begin{aligned} 2\sqrt{2x+5}-x&=4 \qquad \textit{Add } x \textit{ to both sides.}\\ 2\sqrt{2x+5}&=x+4 \end{aligned}$$

Despite the fact that the term on the left side has a coefficient, we still consider it to be isolated. Recall that terms are separated by addition or subtraction operators.

$$\begin{aligned} 2\sqrt{2x+5}&=x+4\\ (2\sqrt{2x+5})^2&=(x+4)^2 \qquad \textit{Square both sides.}\\ 4(2x+5)&=x^2+8x+16 \end{aligned}$$

Solve the resulting quadratic equation.

$$\begin{aligned} 4(2x+5)&=x^2+8x+16\\ 8x+20&=x^2+8x+16\\ 0&=x^2-4\\ 0&=(x+2)(x-2)\\ x+2=0 \quad &\text{or } x-2=0\\ x=-2 \quad &\qquad x=2 \end{aligned}$$

Since we squared both sides, we must check our solutions.

Check $x=-2$	*Check* $x=2$
$2\sqrt{2x+5}-x=4$	$2\sqrt{2x+5}-x=4$
$2\sqrt{2(-2)+5}-(-2)=4$	$2\sqrt{2(2)+5}-(2)=4$
$2\sqrt{-4+5}+2=4$	$2\sqrt{4+5}-2=4$
$2\sqrt{1}+2=4$	$2\sqrt{9}-2=4$
$2+2=4$	$6-2=4$
$4=4$ ✓	$4=4$ ✓

After checking, we can see that both are solutions to the original equation. The solution set is $\{\pm 2\}$.

Sometimes both of the possible solutions are extraneous.

Example 1.4.5:

Solve: $\sqrt{4-11x}-x+2=0$.

Solution

$$\begin{aligned}\sqrt{4-11x}-x+2&=0 && \textit{Isolate the radical.}\\ \sqrt{4-11x}&=x-2\\ (\sqrt{4-11x})^2&=(x-2)^2 && \textit{Square both sides.}\\ 4-11x&=x^2-4x+4 && \textit{Solve.}\\ 0&=x^2+7x\\ 0&=x(x+7)\\ x=0 \text{ or } x+7&=0\\ x&=-7\end{aligned}$$

Since we squared both sides, we must check our solutions.

Check $x=0$	*Check* $x=-7$
$\sqrt{4-11x}-x+2=0$ $\sqrt{4-11(0)}-0+2=0$ $\sqrt{4}+2=0$ $2+2=0$ $4=0$ ✗	$\sqrt{4-11x}-x+2=0$ $\sqrt{4-11(-7)}-(-7)+2=0$ $\sqrt{4+77}+7+2=0$ $\sqrt{81}+9=0$ $9+9=0$ $18=0$ ✗

Since both possible solutions are extraneous, the equation has no solution and the solution set is $\{\ \}$.

The squaring property of equality extends to any positive integer power n. Given real numbers a and b, the **power property of equality** states: **If** $a=b$, **then** $a^n=b^n$. This, and the fact that $(\sqrt[n]{a})^n=\sqrt[n]{a^n}=a$, when a is nonnegative, is used to solve radical equations with indices greater than 2.

Example 1.4.6:

Solve $\sqrt[3]{4x^2+7}-2=0$.

Solution

$$\begin{aligned}\sqrt[3]{4x^2+7}-2&=0 && \textit{Isolate the radical.}\\ \sqrt[3]{4x^2+7}&=2\\ \left(\sqrt[3]{4x^2+7}\right)^3&=(2)^3 && \textit{Cube both sides.}\\ 4x^2+7&=8 && \textit{Solve.}\\ 4x^2-1&=0\\ (2x+1)(2x-1)&=0\\ 2x+1=0 \quad &\text{or} \quad 2x-1=0\\ 2x=-1 \quad &\quad\ \ 2x=1\\ x=-\tfrac{1}{2} \quad &\quad\ \ x=\tfrac{1}{2}\end{aligned}$$

Check $x=-\frac{1}{2}$	Check $x=\frac{1}{2}$
$\sqrt[3]{4x^2+7}-2=0$	$\sqrt[3]{4x^2+7}-2=0$
$\sqrt[3]{4\left(-\frac{1}{2}\right)^2+7}-2=0$	$\sqrt[3]{4\left(\frac{1}{2}\right)^2+7}-2=0$
$\sqrt[3]{4\cdot\frac{1}{4}+7}-2=0$	$\sqrt[3]{4\cdot\frac{1}{4}+7}-2=0$
$\sqrt[3]{8}-2=0$	$\sqrt[3]{1+7}-2=0$
$2-2=0$	$\sqrt[3]{8}-2=0$
$0=0$ ✓	$2-2=0$
	$0=0$ ✓

The solution set is $\left\{\pm\frac{1}{2}\right\}$.

Try It 1.4.6

Solve: $x-3\sqrt{3x+1}=3$

Answer

The solution is **33**. (The other proposed solution, $x=0$ was rejected.)

It may be the case that the equation has more than one term that consists of radical expressions.

Example 1.4.7:

Solve: $\sqrt{5x-3}=\sqrt{4x-1}$.

Solution

Both radicals are considered isolated on separate sides of the equation.

$$\begin{aligned}\sqrt{5x-3}&=\sqrt{4x-1}\\(\sqrt{5x-3})^2&=(\sqrt{4x-1})^2 \quad \textit{Square both sides.}\\5x-3&=4x-1 \quad \textit{Solve.}\\x&=2\end{aligned}$$

Check $x=2$.

$$\begin{aligned}\sqrt{5x-3}&=\sqrt{4x-1}\\\sqrt{5(2)-3}&=\sqrt{4(2)-1}\\\sqrt{10-3}&=\sqrt{8-1}\\\sqrt{7}&=\sqrt{7}\ \checkmark\end{aligned}$$

The solution set is $\{2\}$.

Example 1.4.8:

Solve: $\sqrt[3]{x^2+x-14}=\sqrt[3]{x+50}$.

Solution

Eliminate the radicals by cubing both sides.

$$\begin{aligned}\sqrt[3]{x^2+x-14}&=\sqrt[3]{x+50}\\\left(\sqrt[3]{x^2+x-14}\right)^3&=(\sqrt[3]{x+50})^3 \quad \textit{Cube both sides.}\\x^2+x-14&=x+50 \quad \textit{Solve.}\\x^2-64&=0\\(x+8)(x-8)&=0\end{aligned}$$

$$x+8=0 \quad \text{or} \quad x-8=0$$
$$x=-8 \qquad\qquad x=8$$

Check $x=-8$	*Check* $x=8$
$\sqrt[3]{x^2+x-14}=\sqrt[3]{x+50}$ $\sqrt[3]{(-8)^2+(-8)-14}=\sqrt[3]{(-8)+50}$ $\sqrt[3]{64-8-14}=\sqrt[3]{42}$ $\sqrt[3]{42}=\sqrt[3]{42}$ ✓	$\sqrt[3]{x^2+x-14}=\sqrt[3]{x+50}$ $\sqrt[3]{(8)^2+(8)-14}=\sqrt[3]{(8)+50}$ $\sqrt[3]{64+8-14}=\sqrt[3]{58}$ $\sqrt[3]{58}=\sqrt[3]{58}$ ✓

(1.4.1)

Answer:

The solution set is $\{\pm 8\}$.

It may not be possible to isolate a radical on both sides of the equation. When this is the case, isolate the radicals, one at a time, and apply the squaring property of equality multiple times until only a polynomial remains.

Example 1.4.9:

Solve: $\sqrt{x+2}-\sqrt{x}=1$

Solution

Isolate one of the radicals.

$$\sqrt{x+2}-\sqrt{x}=1$$
$$\sqrt{x+2}=\sqrt{x}+1$$

Square both sides. Be careful to apply the distributive property correctly to the right side.

$$(\sqrt{x+2})^2=(\sqrt{x}+1)^2$$
$$x+2=(\sqrt{x}+1)(\sqrt{x}+1)$$
$$x+2=\sqrt{x^2}+\sqrt{x}+\sqrt{x}+1$$
$$x+2=x+2\sqrt{x}+1$$

Now the equation contains only one radical. Isolate it and square both sides again.

$$x+2=x+2\sqrt{x}+1$$
$$1=2\sqrt{x}$$
$$(1)^2=(2\sqrt{x})^2$$
$$1=4x$$
$$\frac{1}{4}=x$$

Check to see if $x=\frac{1}{4}$ satisfies the original equation $\sqrt{x+2}-\sqrt{x}=1$

$$\sqrt{\tfrac{1}{4}+2}-\sqrt{\tfrac{1}{4}}=1$$
$$\sqrt{\tfrac{9}{4}}-\tfrac{1}{2}=1$$
$$\tfrac{3}{2}-\tfrac{1}{2}=1$$
$$\tfrac{2}{2}=1$$
$$1=1 \checkmark$$

The solution set is $\{\frac{1}{4}\}$.

Note

Observe that $(A+B)^2 \neq A^2+B^2$, even though $(A \cdot B)^2 = A^2 \cdot B^2$!!

For example,

$$(\sqrt{x+2})^2-(\sqrt{x})^2=(1)^2$$

Incorrect!

This is a common mistake and leads to an incorrect result. When squaring both sides of an equation with multiple terms, we must take care to apply the distributive property correctly.

Example 1.4.10:

Solve: $\sqrt{2x+10}-\sqrt{x+6}=1$

Solution

Isolate one of the radicals.

$$\begin{aligned}\sqrt{2x+10}-\sqrt{x+6}&=1\\ \sqrt{2x+10}&=\sqrt{x+6}+1\end{aligned}$$

Square both sides. Take care to apply the distributive property CORRECTLY to the right side.

$$\begin{aligned}(\sqrt{2x+10})^2&=(\sqrt{x+6}+1)^2\\ 2x+10&=x+6+2\sqrt{x+6}+1\\ 2x+10&=x+7+2\sqrt{x+6}\end{aligned}$$

At this point we have one term that contains a radical. Isolate it and square both sides again.

$$\begin{aligned}2x+10&=x+7+2\sqrt{x+6}\\ x+3&=2\sqrt{x+6}\\ (x+3)^2&=(2\sqrt{x+6})^2\\ x^2+6x+9&=4(x+6)\\ x^2+6x+9&=4x+24\\ x^2+2x-15&=0\\ (x-3)(x+5)&=0\end{aligned}$$

$$x-3=0 \quad \text{or} \quad x+5=0$$
$$x=3 \qquad\qquad x=-5$$

Check $x=3$	*Check* $x=-5$
$\sqrt{2x+10}-\sqrt{x+6}=1$	$\sqrt{2x+10}-\sqrt{x+6}=1$
$\sqrt{2(3)+10}-\sqrt{3+6}=1$	$\sqrt{2(-5)+10}-\sqrt{-5+6}=1$
$\sqrt{16}-\sqrt{9}=1$	$\sqrt{0}-\sqrt{1}=1$
$4-3=1$	$0-1=1$
$1=1$ ✓	$-1=1$ ✗

The solution set is $\{3\}$.

Try It 1.4.10

Solve: $\sqrt{4x+21}-\sqrt{2x+22}=1$

Answer

The solution is 7 (The other proposed solution, $x=-3$ was rejected.)

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1.4e: Exercises - Radical Equations

A: Radical Equations (I)

Exercise 1.4e.A

★ **Solve.**

1. $\sqrt{x}=7$
2. $\sqrt{x}=4$
3. $\sqrt{x}+8=9$
4. $\sqrt{x}-4=5$
5. $\sqrt{x}+7=4$
6. $\sqrt{x}+3=1$
7. $5\sqrt{x}-1=0$
8. $3\sqrt{x}-2=0$
9. $\sqrt{3x+1}=2$
10. $\sqrt{5x-4}=4$
11. $\sqrt{7x+4}+6=11$
12. $\sqrt{3x-5}+9=14$
13. $2\sqrt{x-1}-3=0$
14. $3\sqrt{x+1}-2=0$
15. $\sqrt{x+1}=\sqrt{x}+1$
16. $\sqrt{2x-1}=\sqrt{2x}-1$
17. $\sqrt{4x-1}=2\sqrt{x}-1$
18. $\sqrt{4x-11}=2\sqrt{x}-1$
19. $\sqrt{x+8}=\sqrt{x}-4$
20. $\sqrt{25x-1}=5\sqrt{x}+1$
21. $\sqrt[3]{x}=3$
22. $\sqrt[3]{x}=-4$
23. $\sqrt[3]{2x+9}=3$
24. $\sqrt[3]{4x-11}=1$
25. $\sqrt[3]{5x+7}+3=1$
26. $\sqrt[3]{3x-6}+5=2$
27. $4-2\sqrt[3]{x+2}=0$
28. $6-3\sqrt[3]{2x-3}=0$
29. $\sqrt[5]{3(x+10)}=2$
30. $\sqrt[5]{4x+3}+5=4$

Answers to odd exercises:

1. 49 3. 1 5. $\varnothing$ 7. $\frac{1}{25}$ 9. 1 11. 3 13. $\frac{13}{4}$ 15. 0 17. $\frac{1}{4}$ 19. $\varnothing$ 21. 27 23. 9 25. -3 27. 6 29. $\frac{2}{3}$.

B: Radical Equations (II)

Exercise 1.4e.B

★ **Solve.**

31. $\sqrt{8x+11}=3\sqrt{x+1}$
32. $2\sqrt{3x-4}=\sqrt{2(3x+1)}$
33. $\sqrt{2(x+10)}=\sqrt{7x-15}$
34. $\sqrt{5(x-4)}=\sqrt{x+4}$
35. $\sqrt[3]{5x-2}=\sqrt[3]{4x}$
36. $\sqrt[3]{9(x-1)}=\sqrt[3]{3(x+7)}$
37. $\sqrt[3]{3x+1}=\sqrt[3]{2(x-1)}$
38. $\sqrt[3]{9x}=\sqrt[3]{3(x-6)}$
39. $\sqrt[5]{3x-5}=\sqrt[5]{2x+8}$
40. $\sqrt[5]{x+3}=\sqrt[5]{2x+5}$

Answers to odd exercises:

31. 2 33. 7 35. 2 37. -3 39. 13.

C: Radical Equations (III)

Exercise 1.4e.C

★ **Solve.**

41. $\sqrt{4x+21}=x$
42. $\sqrt{8x+9}=x$
43. $\sqrt{4(2x-3)}=x$
44. $\sqrt{3(4x-9)}=x$
45. $2\sqrt{x-1}=x$
46. $3\sqrt{2x-9}=x$
47. $\sqrt{9x+9}=x+1$
48. $\sqrt{3x+10}=x+4$
49. $\sqrt{x-1}=x-3$
50. $\sqrt{2x-5}=x-4$
51. $\sqrt{16-3x}=x-6$
52. $\sqrt{7-3x}=x-3$
53. $3\sqrt{2x+10}=x+9$
54. $2\sqrt{2x+5}=x+4$
55. $3\sqrt{x-1}-1=x$
56. $2\sqrt{2x+2}-1=x$
57. $\sqrt{10x+41}-5=x$
58. $\sqrt{6(x+3)}-3=x$
59. $\sqrt{8x^2-4x+1}=2x$
60. $\sqrt{18x^2-6x+1}=3x$
61. $5\sqrt{x+2}=x+8$
62. $4\sqrt{2(x+1)}=x+7$
63. $\sqrt{x^2-25}=x$
64. $\sqrt{x^2+9}=x$
65. $3+\sqrt{6x-11}=x$
66. $2+\sqrt{9x-8}=x$
67. $\sqrt{4x+25}-x=7$
68. $\sqrt{8x+73}-x=10$
69. $2\sqrt{4x+3}-3=2x$
70. $2\sqrt{6x+3}-3=3x$
71. $2x-4=\sqrt{14-10x}$
72. $3x-6=\sqrt{33-24x}$
73. $\sqrt[3]{x^2-24}=1$
74. $\sqrt[3]{x^2-54}=3$
75. $\sqrt[3]{x^2+6x}+1=4$
76. $\sqrt[3]{x^2+2x}+5=7$
77. $\sqrt[3]{25x^2-10x-7}=-2$
78. $\sqrt[3]{9x^2-12x-23}=-3$
79. $\sqrt[3]{4x^2-1}-2=0$
80. $4\sqrt[3]{x^2}-1=0$
81. $\sqrt[5]{x(2x+1)}-1=0$
82. $\sqrt[5]{3x^2-20x}-2=0$

Answers to odd exercises:

41. 7 43. 2, 6 45. 2 47. −1, 8 49. 5 51. ∅ 53. −3, 3 55. 2, 5 57. −4, 4 59. $\frac{1}{2}$ 61. 2, 7 63. ∅ 65. 10
67. −6, −4 69. $-\frac{1}{2}, \frac{3}{2}$ 71. ∅ 73. −5, 5 75. −9, 3 77. $\frac{1}{5}$ 79. $-\frac{3}{2}, \frac{3}{2}$ 81. $-1, \frac{1}{2}$

D: Radical Equations (IV)

Exercise 1.4e.D

★ **Solve.**

83. $\sqrt{2x^2-15x+25}=\sqrt{(x+5)(x-5)}$
84. $\sqrt{x^2-4x+4}=\sqrt{x(5-x)}$
85. $\sqrt[3]{2\left(x^2+3x-20\right)}=\sqrt[3]{(x+3)^2}$
86. $\sqrt[3]{3x^2+3x+40}=\sqrt[3]{(x-5)^2}$
87. $\sqrt{2x-5}+\sqrt{2x}=5$
88. $\sqrt{4x+13}-2\sqrt{x}=3$
89. $\sqrt{8x+17}-2\sqrt{2-x}=3$
90. $\sqrt{3x-6}-\sqrt{2x-3}=1$
91. $\sqrt{2(x-2)}-\sqrt{x-1}=1$
92. $\sqrt{2x+5}-\sqrt{x+3}=2$
93. $\sqrt{2(x+1)}-\sqrt{3x+4}-1=0$
94. $\sqrt{6-5x}+\sqrt{3-3x}-1=0$
95. $\sqrt{x-2}-1=\sqrt{2(x-3)}$
96. $\sqrt{14-11x}+\sqrt{7-9x}=1$
97. $\sqrt{x+1}=\sqrt{3}-\sqrt{2-x}$
98. $\sqrt{2x+9}-\sqrt{x+1}=2$

Answers to odd exercises:

83. 5, 10 85. −7, 7 87. $\frac{9}{2}$ 89. 1 91. 10 93. ∅ 95. 3 97. −1, 2.

E: Radical Equations (V)

Exercise 1.4e.E

★ **Solve.**

99. $x^{1/2}-10=0$
100. $x^{1/2}-6=0$
101. $x^{1/3}+2=0$
102. $x^{1/3}+4=0$
103. $(x-1)^{1/2}-3=0$
104. $(x+2)^{1/2}-6=0$
105. $(2x-1)^{1/3}+3=0$
106. $(3x-1)^{1/3}-2=0$
107. $(4x+15)^{1/2}-2x=0$
108. $(3x+2)^{1/2}-3x=0$
109. $(2x+12)^{1/2}-x=6$
110. $(4x+36)^{1/2}-x=9$
111. $2(5x+26)^{1/2}=x+10$
112. $3(x-1)^{1/2}=x+1$
113. $x^{1/2}+(3x-2)^{1/2}=2$
114. $(6x+1)^{1/2}-(3x)^{1/2}=1$
115. $(3x+7)^{1/2}+(x+3)^{1/2}-2=0$
116. $(3x)^{1/2}+(x+1)^{1/2}-5=0$
117. $\sqrt{3x+7}+\sqrt{x+1}=2$
118. $\sqrt{2x+4}-\sqrt{x+3}=1$
119. $\sqrt{2x}-\sqrt{x+1}=1$

Answers to odd exercises:

99. 100 101. −8 103. 10 105. −13 107. $\frac{5}{2}$ 109. −6, −4 111. −2, 2 113. 1 115. −2 117. {-1} 119. {8}

★

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1.5: Equations with Rational Exponents

We have solved linear equations, rational equations, radical equations, and quadratic equations using several methods. However, there are many other types of equations, such as equations involving rational exponents, polynomial equations, absolute value equations, equations in quadratic form, and some rational equations that can be transformed into quadratics. Solving any equation, however, employs the same basic algebraic rules.

Solving Equations Involving Rational Exponents

Rational exponents are exponents that are fractions, where the numerator is a power and the denominator is a root. For example, $16^{\frac{1}{2}}$ is another way of writing $\sqrt{16}$; $8^{\frac{1}{3}}$ is another way of writing $\sqrt[3]{8}$. The ability to work with rational exponents is a useful skill, as it is highly applicable in calculus.

Equations in which a variable expression is raised to a rational exponent can be solved by raising both sides of the equation to the reciprocal of the exponent. The reason the expression is raised to the reciprocal of its exponent is because the product of a number and its reciprocal is one. Therefore the exponent on the variable expression becomes one and is thus eliminated.

Definition: Rational Exponents

A *rational exponent* indicates a power in the numerator and a root in the denominator. There are multiple ways of writing an expression, a variable, or a number with a rational exponent:

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

Example 1.5.1: Evaluate a Number Raised to a Rational Exponent

Evaluate $8^{\frac{2}{3}}$

Solution. It does not matter whether the root or the power is done first because $8^{\frac{2}{3}} = (8^2)^{\frac{1}{3}} = (8^{\frac{1}{3}})^2$. Since the cube root of 8 is easy to find, $8^{\frac{2}{3}}$ can be evaluated as $\left(8^{\frac{1}{3}}\right)^2 = (2)^2 = 4$.

Try It 1.5.1

Evaluate $64^{-\frac{1}{3}}$

Answer

$\frac{1}{4}$

Howto: Solve an Equation with Rational Exponents.

1. **Isolate** the expression with the rational exponent
2. Raise both sides of the equation to the **reciprocal power**.
   - If the numerator of the reciprocal power is an even number, the solution must be checked because the solution involves the squaring process which can introduce extraneous roots.
   - If the denominator of the reciprocal power is an even number, this is equivalent to taking an even root so +/- must be included.

Example 1.5.2: Solve an Equation Containing a Variable Raised to a Rational Exponent

Solve the equation in which a variable is raised to a rational exponent: $x^{\frac{3}{4}} = 8$.

Solution The exponent on x is removed by raising both sides of the equation to a power that is the reciprocal of $\frac{3}{4}$. The reciprocal of $\frac{3}{4}$ is $\frac{4}{3}$. The numerator of this exponent we are applying is an even number, which means that both sides are being raised to an even power.

$$\begin{aligned} x^{\frac{3}{4}} &= 8 \\ \left(x^{\frac{3}{4}}\right)^{\frac{4}{3}} &= (8)^{\frac{4}{3}} \\ x &= (8^{1/3})^4 \\ &= (2)^4 \\ &= 16 \end{aligned}$$

It is necessary to check our result because the solution involved raising both sides of the equation to an even power. Raising both sides of an equation to an even power can introduce "extraneous" roots. Therefore our answer must be checked: $16^{\frac{3}{4}} = (16^{\frac{1}{4}})^3 = 2^3 = 8$. ✓ The solution set is $\{16\}$.

Example 1.5.3

Solve $x^{\frac{5}{4}} + 36 = 4$.

Solution

$$\begin{aligned} x^{\frac{5}{4}} &= -32 \\ (x^{\frac{5}{4}})^{\frac{4}{5}} &= (-32)^{\frac{4}{5}} \\ x &= (\sqrt[5]{-32})^4 \\ x &= (-2)^4 \\ x &= 16 \end{aligned}$$

It is necessary in this case to check our result because the solution involved raising both sides of the equation to an even power. Raising both sides of an equation to an even power can introduce "extraneous" roots. $16^{\frac{5}{4}} + 36 = (\sqrt[4]{16})^5 + 36 = 2^5 + 36 = 32 + 36 = 68 \neq 4$. Therefore, the solution $x = 16$ must be rejected. Therefore this problem has no solution. The solution set is $\{\quad\}$.

Example 1.5.4

Solve $x^{\frac{4}{3}} = 81$

Solution. The solution involves raising both sides of the equal sign to the power of $\frac{3}{4}$. Because the denominator is an even number, that means tha we are actually taking the even root of a quantity, which could be either a positive or negative value.

$$\begin{aligned} (x^{\frac{4}{3}})^{\frac{3}{4}} &= \pm 81^{\frac{3}{4}} \\ x &= \pm(\sqrt[4]{81})^3 \\ x &= \pm(3)^3 \\ x &= \pm 27 \end{aligned}$$

No checking is required in this example because the process did not involve raising both sides of the equation to an even power. The even number was in the denominator, not the numerator of the reciprocal power. The solution set is $\{-27, 27\}$.

Example 1.5.5

Solve $(x+5)^{\frac{2}{3}} = 64$

Solution. Notice here that the reciprocal power has an even denominator which represents taking the square root of both sides of the equation. This requires using $\pm$ in the solution process.

$$\begin{aligned} ((x+5)^{\frac{2}{3}})^{\frac{3}{2}} &= \pm(64^{\frac{3}{2}}) \\ x+5 &= \pm(\sqrt{64})^3 \\ x+5 &= \pm(8)^3 \end{aligned}$$

$$x+5=\pm 512$$
$$x=-5+512 \text{ and } x=-5-512$$
$$x=509 \text{ and } x=-517$$

The solution does not need to be checked! Solution Set: $\{509,-517\}$

Try It 1.5.6

Solve the equation

a. $(x-4)^{\frac{2}{3}}=25$	b. $(x+5)^{\frac{3}{2}}=8$	c. $(x+12)^{\frac{3}{2}}=8$

Answer

a. $\{129,-121\}$ b. $\{-1\}$ c. $\{\}$

Example 1.5.7: Solve an Equation involving Rational Exponents and Factoring

Solve $3x^{\frac{3}{4}}=x^{\frac{1}{2}}$.

Solution

This equation involves rational exponents as well as factoring rational exponents. Let us take this one step at a time. First, put the variable terms on one side of the equal sign and set the equation equal to zero.

$$3x^{\frac{3}{4}}-\left(x^{\frac{1}{2}}\right)=x^{\frac{1}{2}}-\left(x^{\frac{1}{2}}\right)$$
$$3x^{\frac{3}{4}}-x^{\frac{1}{2}}=0$$

Now, it looks like we should factor the left side, but what do we factor out? We can always factor the term with the lowest exponent. The factor with the lowest exponent is $x^{1/2}$, so $x^{3/4}$ needs to be rewritten as a product involving $x^{1/2}$.

$$3x^{\frac{3}{4}}-x^{\frac{1}{2}}=0$$
$$3x^{(\frac{1}{2}+\frac{1}{4})}-x^{\frac{1}{2}}=0$$
$$3x^{\frac{1}{2}}x^{\frac{1}{4}}-x^{\frac{1}{2}}=0$$
$$x^{\frac{1}{2}}\left(3x^{\frac{1}{4}}-1\right)=0$$

Now we have two factors and can use the zero factor theorem.

$$x^{\frac{1}{2}}\left(3x^{\frac{1}{4}}-1\right)=0$$

$$x^{\frac{1}{2}}=0 \quad \Big| \quad 3x^{\frac{1}{4}}-1=0$$
$$x=0 \quad \Big| \quad 3x^{\frac{1}{4}}=1$$
$$\Big| \quad x^{\frac{1}{4}}=\frac{1}{3}$$
$$\Big| \quad \left(x^{\frac{1}{4}}\right)^4=\left(\frac{1}{3}\right)^4$$
$$\Big| \quad x=\frac{1}{81}$$

The solution set is $\left\{0,\frac{1}{81}\right\}$.

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1.5e: Exercises - Solve Equations with Rational Exponents

A: Solve Equations with Rational Exponents I

Exercise 1.5e.A

★ **Solve.**

1. $x^{1/2}-10=0$
2. $x^{1/2}-6=0$
3. $x^{1/3}+2=0$
4. $x^{1/3}+4=0$
5. $(x-1)^{1/2}-3=0$
6. $(x+2)^{1/2}-6=0$
7. $(2x-1)^{1/3}+3=0$
8. $(3x-1)^{1/3}-2=0$
9. $(4x+15)^{1/2}-2x=0$
10. $(3x+2)^{1/2}-3x=0$
11. $(2x+12)^{1/2}-x=6$
12. $(4x+36)^{1/2}-x=9$
13. $2(5x+26)^{1/2}=x+10$
14. $3(x-1)^{1/2}=x+1$
15. $x^{1/2}+(3x-2)^{1/2}=2$
16. $(6x+1)^{1/2}-(3x)^{1/2}=1$
17. $(3x+7)^{1/2}+(x+3)^{1/2}-2=0$
18. $(3x)^{1/2}+(x+1)^{1/2}-5=0$

Answers to odd exercises:

1. $\{100\}$ 3. $\{-8\}$ 5. $\{10\}$ 7. $\{-13\}$ 9. $\{\frac{5}{2}\}$ 11. $\{-6,-4\}$ 13. $\{-2,2\}$ 15. $\{1\}$ 17. $\{-2\}$.

B: Solve Equations with Rational Exponents II

Exercise 1.5e.B

★ **Solve**

21. $(x-6)^{\frac{5}{7}}=32$
22. $(x+3)^{\frac{3}{7}}+2=10$
23. $(x-1)^{\frac{7}{3}}=128$
24. $(x+2)^{\frac{7}{6}}+128=0$
25. $(x-2)^{\frac{3}{5}}-4=4$
26. $(5x+7)^{\frac{3}{5}}=8$
27. $8x^{\frac{5}{3}}-24=0$
28. $(x+1)^{\frac{5}{3}}=32$
29. $3x^{\frac{1}{3}}+5=17$
30. $7x^{\frac{3}{7}}+9=65$
31. $x^{\frac{4}{3}}+11=92$
32. $x^{\frac{2}{3}}+1=65$
33. $(x-4)^{\frac{2}{3}}=16$
34. $(x+5)^{\frac{2}{3}}=4$
35. $8(3x-1)^{\frac{2}{3}}=200$
36. $5x^{\frac{2}{3}}=45$
37. $(7x-3)^{\frac{2}{5}}=4$
38. $(7x-8)^{\frac{2}{3}}=4(x-5)^{\frac{2}{3}}$
39. $3x^{\frac{5}{2}}=96$
40. $x^{\frac{3}{2}}-2=6$
41. $2x^{\frac{3}{2}}=54$
42. $6x^{\frac{5}{2}}-12=0$
43. $6x^{\frac{3}{2}}-141=1917$
44. $5x^{\frac{3}{2}}-3=4997$
45. $-2(x-5)^{\frac{3}{4}}+48=-202$
46. $2x^{\frac{5}{4}}=686$
47. $(4x+5)^{\frac{1}{2}}=x-4$
48. $x^3=(4x-3)^{\frac{3}{2}}$
49. $x^{\frac{5}{2}}=16x^{\frac{1}{2}}$
50. $(x^2-x-4)^{\frac{3}{4}}-2=6$
51. $(x^2-3x+3)^{\frac{3}{2}}-1=0$

Answers to odd exercises:

21. $\{134\}$ 23. $\{9\}$ 25. $\{34\}$ 27. $\{3^{\frac{3}{5}}\}$ 29. $\{64\}$ 31. $\{27,-27\}$ 33. $\{-60,68\}$ 35. $\{42,\ -\frac{124}{3}\}$ 37. $\{5,\frac{-29}{7}\}$ 39. $\{4\}$ 41. $\{9\}$ 43. $\{49\}$ 45. $\{630\}$ 47. $\{11\}$ 49. $\{0,4,-4\}$ 51. $\{1,2\}$.

★

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1.6: Equations Quadratic in Form

Solving Equations in Quadratic Form

Definition: Quadratic in Form

Equations in quadratic form are equations with three terms. The term with the highest exponent (called the leading term) has a power that is a multiple of **2**, the middle term has an exponent that is one-half the exponent of the leading term, and the third term is a constant. An equation that is quadratic in form can be written in the form $au^2+bu+c=0$ where u represents an algebraic expression.

A few examples of these equations include:

$$x^4-5x^2+4=0 \overset{u=x^2}{\Longrightarrow} u^2-5u+4=0$$
$$x^6+7x^3-8=0 \overset{u=x^3}{\Longrightarrow} u^2+7u-8=0$$
$$x^{\frac{2}{3}}+4x^{\frac{1}{3}}+2=0 \overset{x^{\frac{1}{3}}}{\Longrightarrow} u^2+4u+2=0$$
$$\left(\frac{t+2}{t}\right)^2+8\left(\frac{t+2}{t}\right)+7=0 \overset{u=\frac{t+2}{t}}{\Longrightarrow} u^2+8u+7=0$$
$$x-3\sqrt{x}-10=0 \overset{u=\sqrt{x}}{\Longrightarrow} u^2-3u-10=0$$
$$3y^{-2}+7y^{-1}-6=0 \overset{u=y^{-1}}{\Longrightarrow} 3u^2+7u-6=0$$

In each example, doubling the exponent of the middle term equals the exponent on the leading term.
If an equation can be expressed in quadratic form, then it can be solved by any of the techniques used to solve ordinary quadratic equations. For example, consider the following fourth-degree polynomial equation,

$$x^4-4x^2-32=0$$

If we let $u=x^2$ then $u^2=\left(x^2\right)^2=x^4$ and we can write

$$\begin{array}{c} x^4-4x^2-32=0 \Rightarrow \left(x^2\right)^2-4\left(x^2\right)-32=0 \\ \downarrow \quad \downarrow \\ u^2 \quad - \quad 4u \quad -32=0 \end{array}$$

This substitution transforms the equation into a familiar quadratic equation in terms of u which, in this case, can be solved by factoring.

$$\begin{array}{c} u^2-4u-32=0 \\ (u-8)(u+4)=0 \\ u=8 \quad \text{or} \quad u=-4 \end{array}$$

Since $u=x^2$ we can back substitute and then solve for x.

$$\begin{array}{ll} u=8 \quad \text{or} & u=-4 \\ \downarrow & \downarrow \\ x^2=8 & x^2=-4 \\ x=\pm\sqrt{8} & x=\pm\sqrt{-4} \\ x=\pm 2\sqrt{2} & x=\pm 2i \end{array}$$

Therefore, the equation $x^4-4x^2-32=0$ has four solutions $\{\pm 2\sqrt{2}, \pm 2i\}$, two real and two complex. This technique is called a **u-substitution**, and can often be used to solve equations that are quadratic in form.

How To: Howto: Solve an equation that is quadratic in form.

1. Confirm the equation is quadratic in form and rewrite it.
   a. The equation is quadratic in form if the exponent on the leading term is double the exponent on the middle term.
   b. Substitute u for the variable portion of the middle term and rewrite the equation in the form $au^2+bu+c=0$.
2. Solve using one of the usual methods for solving a quadratic.

3. Back substitute. Replace u with the original term.
4. Solve the resulting equations for the original variable. Check for domain restrictions or extraneous solutions if necessary.

Example 1.6.1: Solving a Fourth-degree Equation in Quadratic Form

Solve this fourth-degree equation: $3x^4 - 2x^2 - 1 = 0$.

Solution

Step 1. This equation fits the main criteria, that the power on the leading term is double the power on the middle term. Next, we will make a substitution for the variable term in the middle. Let $u = x^2$. Rewrite the equation in u.

$$3u^2 - 2u - 1 = 0$$

Step 2. Now solve the quadratic.

$$3u^2 - 2u - 1 = 0$$
$$(3u + 1)(u - 1) = 0$$

$$\begin{array}{l|l} 3u + 1 = 0 & u - 1 = 0 \\ 3u = -1 & u = 1 \\ u = -\frac{1}{3} & \end{array}$$

Step 3. Replace u with the original term.

$$x^2 = -\frac{1}{3} \qquad x^2 = 1$$

Step 4. Solve for the original variable

$$\begin{array}{l|l} x^2 = -\frac{1}{3} & x^2 = 1 \\ x = \pm i\sqrt{\frac{1}{3}} & x = \pm 1 \end{array}$$

Neither the original equation nor the solution process pose any restrictions to x so the solution set is: $\left\{ \pm i\sqrt{\frac{1}{3}}, \pm 1 \right\}$

Example 1.6.2: Solving an Equation in Quadratic Form Containing a Binomial

Solve the equation in quadratic form: $(x + 2)^2 + 11(x + 2) - 12 = 0$.

Solution

This equation contains a binomial in place of the single variable. The tendency is to expand what is presented. However, recognizing that it fits the criteria for being in quadratic form makes all the difference in the solving process. First, make a substitution, letting $u = x + 2$. Then rewrite the equation in u.

$$u^2 + 11u - 12 = 0$$

Solve for u using the zero-factor property

$$(u + 12)(u - 1) = 0$$
$$\begin{array}{l|l} u + 12 = 0 & u - 1 = 0 \\ u = -12 & u = 1 \end{array}$$

Replace u with the original expression and solve for x

$$\begin{array}{l|l} x + 2 = -12 & x + 2 = 1 \\ x = -14 & x = -1 \end{array}$$

Neither the original equation nor the solution process pose any restrictions to x so the solution set is: $\{-14, -1\}$.

Example 1.6.3: Solving an Equation in Quadratic Form Containing a Radical

Solve: $x-2\sqrt{x}-8=0$.

Solution

This is a radical equation that can be written in quadratic form. If we let $u=\sqrt{x}$ then $u^2=(\sqrt{x})^2=x$ and we can write

$$\begin{array}{c} x-2\sqrt{x}-8=0 \\ \downarrow \quad \downarrow \\ u^2 \ -2u-8=0 \end{array}$$

Solve for u.

$$\begin{array}{c} u^2-2u-8=0 \\ (u-4)(u+2)=0 \\ u=4 \text{ or } u=-2 \end{array}$$

Back substitute $u=\sqrt{x}$ and solve for x.

$$\begin{array}{cc} \sqrt{x}=4 & \text{or} \quad \sqrt{x}=-2 \\ (\sqrt{x})^2=(4)^2 & (\sqrt{x})^2=(-2)^2 \\ x=16 & x=4 \end{array}$$

The original equation poses the restriction that $x \geq 0$ which neither solution obtained violates, but recall that squaring both sides of an equation introduces the possibility of extraneous solutions. Therefore we must check our potential solutions.

Illegal pream-token ( )

Because $x=4$ is extraneous, there is only one solution, $x=16$.
Solution Set: $\{16\}$.

Example 1.6.4: Solving an Equation in Quadratic Form Containing Rational Exponents

Solve: $x^{2/3}-3x^{1/3}-10=0$.

Solution

If we let $u=x^{1/3}$ then $u^2=\left(x^{1/3}\right)^2=x^{2/3}$ and we can write

$$\begin{array}{c} x^{2/3}-3x^{1/3}-10=0 \\ \downarrow \quad \downarrow \\ u^2 \ -3u-10=0 \end{array}$$

Solve for u.

$$\begin{array}{c} u^2-3u-10=0 \\ (u-5)(u+2)=0 \\ u=5 \quad \text{or} \quad u=-2 \end{array}$$

Back substitute $u=x^{1/3}$ and solve for x.

$$\begin{array}{cc} x^{1/3}=5 & \text{or} \quad x^{1/3}=-2 \\ \left(x^{1/3}\right)^3=(5)^3 & \left(x^{1/3}\right)^3=(-2)^3 \\ x=125 & x=-8 \end{array}$$

The domain of the original equation does not have any restrictions. Furthermore, the solution process involved cubing, not squaring. Squaring can introduce extraneous solutions, but cubing does not, so a check is not necessary.

Solution Set: $\{-8,125\}$.

Example 1.6.5: Solving an Equation in Quadratic Form Containing Negative Exponents

Solve: $3y^{-2}+7y^{-1}-6=0$

Solution

If we let $u=y^{-1}$, then $u^2=(y^{-1})^2=y^{-2}$ and we can write

$$\begin{array}{c} 3y^{-2}+7y^{-1}-6=0 \\ \downarrow \quad \downarrow \\ 3u^2+7u-6=0 \end{array}$$

Solve for u.

$$\begin{array}{c} 3u^2+7u-6=0 \\ (3u-2)(u+3)=0 \\ u=\frac{2}{3} \quad \text{or} \quad u=-3 \end{array}$$

Back substitute $u=y^{-1}$ and solve for y.

$$\begin{array}{cc} y^{-1}=\frac{2}{3} \text{ or } y^{-1}=-3 \\ \frac{1}{y}=\frac{2}{3} \quad \frac{1}{y}=-3 \\ y=\frac{3}{2} \quad y=-\frac{1}{3} \end{array}$$

The original equation is a rational equation where $y\neq 0$. In this case, the solutions are not restrictions.

The solution set is $\left\{-\frac{1}{3},\frac{3}{2}\right\}$.

Example 1.6.6: Solving an Equation in Quadratic Form Containing Rational Expressions

Solve: $\left(\frac{t+2}{t}\right)^2+8\left(\frac{t+2}{t}\right)+7=0$

Solution

If we let $u=\frac{t+2}{t}$, then $u^2=\left(\frac{t+2}{t}\right)^2$ and we can write

$$\begin{array}{c} \left(\frac{t+2}{t}\right)^2+8\left(\frac{t+2}{t}\right)+7=0 \\ \downarrow \quad \downarrow \\ u^2+\quad 8u \quad +7=0 \end{array}$$

Solve for u.

$$\begin{array}{c} u^2+8u+7=0 \\ (u+1)(u+7)=0 \\ u=-1 \quad \text{or} \quad u=-7 \end{array}$$

Back substitute $u=\frac{t+2}{t}$, and solve for t.

$$\begin{array}{rlrl} \frac{t+2}{t} & =-1 \quad \text{or} & \frac{t+2}{t} & =-7 \\ t+2 & =-t & t+2 & =-7t \\ 2t & =-2 & 8t & =-2 \\ t & =-1 & t & =-\frac{1}{4} \end{array}$$

The domain restriction to the equation is $t\neq 0$, so both solutions work and the solution set is $\left\{-1,-\frac{1}{4}\right\}$.

Example 1.6.7: Solving an Equation in Quadratic Form Containing a Radical Rational Expression

Solve: $\sqrt{\frac{x+10}{x-6}}+\sqrt{\frac{x-6}{x+10}}-\frac{34}{15}=0$

Solution

If we let $u=\sqrt{\frac{x+10}{x-6}}$, then $\frac{1}{u}=\sqrt{\frac{x-6}{x+10}}$ and we can write

$$\begin{array}{c}\sqrt{\frac{x+10}{x-6}}+\sqrt{\frac{x-6}{x+10}}-\frac{34}{15}=0\\ \downarrow \quad \downarrow \\ u+\frac{1}{u}-\frac{34}{15}=0\end{array}$$

Solve for u. First multiply by $15u$

$$\begin{aligned}15u^2+15-34u&=0\\ 15u^2-34u+15&=0\\ (3u-5)(5u-3)&=0\\ u=\tfrac{5}{3} \quad &\text{or} \quad u=\tfrac{3}{5}\end{aligned}$$

Back substitute $u=\sqrt{\frac{x+10}{x-6}}$ and solve for x.

$$\begin{aligned}\sqrt{\frac{x+10}{x-6}}&=\frac{5}{3} \qquad \text{or} \qquad \sqrt{\frac{x+10}{x-6}}=\frac{3}{5}\\ \frac{x+10}{x-6}&=\frac{25}{9} \qquad\qquad \frac{x+10}{x-6}=\frac{9}{25}\\ 9x+90&=25x-150 \qquad 25x+250=9x-54\\ 240&=16x \qquad\qquad 16x=-304\\ x&=15 \qquad\qquad x=-19\end{aligned}$$

Check. The restrictions to the domain of the original equation are $x>6$ and $x<-10$ and do not apply to the solutions found, but the solution process involved squaring, so the answers must be checked.

Using $x=15$, we obtain $\sqrt{\frac{x+10}{x-6}}=\sqrt{\frac{15+10}{15-6}}=\sqrt{\frac{25}{9}}=\frac{5}{3}$ ✓

Using $x=-19$, we obtain $\sqrt{\frac{x+10}{x-6}}=\sqrt{\frac{-19+10}{-19-6}}=\sqrt{\frac{-9}{-25}}=\frac{3}{5}$ ✓

The solution set is $\{15,-19\}$.

So far all of the examples were of equations that factor. As we know, not all quadratic equations factor. If this is the case, we use the quadratic formula.

Example 1.6.8: Solving a Fourth-degree Equation in Unfactorable Quadratic Form

Solve $x^4-10x^2+23=0$.

Solution

If we let $u=x^2$, then $u^2=\left(x^2\right)^2=x^4$ and we can write

$$\begin{array}{c}x^4-10x^2+23=0\\ \downarrow \quad \downarrow \\ u^2-10u+23=0\end{array}$$

This equation does not factor; therefore, use the quadratic formula to find the solutions for u. Here $a=1, b=-10$, and $c=23$.

$$\begin{aligned} u &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\ &= \frac{-(-10) \pm \sqrt{(-10)^2-4(1)(23)}}{2(1)} \\ &= \frac{10 \pm \sqrt{8}}{2} \\ &= \frac{10 \pm 2\sqrt{2}}{2} \\ &= 5 \pm \sqrt{2} \end{aligned}$$

Therefore, $u = 5 \pm \sqrt{2}$. Now back substitute $u = x^2$ and solve for x.

$$\begin{array}{cc} u = 5-\sqrt{2} & \text{or} \quad u = 5+\sqrt{2} \\ \downarrow & \downarrow \\ x^2 = 5-\sqrt{2} & x^2 = 5+\sqrt{2} \\ x = \pm\sqrt{5-\sqrt{2}} & x = \pm\sqrt{5+\sqrt{2}} \end{array}$$

There are no domain restrictions nor potential extraneous solutions so the solution set is $\{\pm\sqrt{5+\sqrt{2}}, \pm\sqrt{5-\sqrt{2}}\}$

If approximate values to the nearest hundredth are desired, the four approximate solutions are

$$\begin{array}{cc} x = -\sqrt{5-\sqrt{2}} \approx -1.89 & x = -\sqrt{5+\sqrt{2}} \approx -2.53 \\ x = \sqrt{5-\sqrt{2}} \approx 1.89 & x = \sqrt{5+\sqrt{2}} \approx 2.53 \end{array}$$

Try It 1.6.6

Solve using substitution:

a. $x^4 - 8x^2 - 9 = 0$	b. $(x-5)^2 - 4(x-5) - 21 = 0$	c. $12x^{-2} - 16x^{-1} + 5 = 0$

Answers

a. $x = -3, 3, -i, i$	b. $x = 2, x = 12$	c. $\{\frac{6}{5}, 2\}$

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1.6e: Exercises - Quadratic in Form

A: Quadratic in Form Polynomial Equations

Exercise 1.6e.A

★ Solve quadratic in form 4th degree equations.

1. $x^4+x^2-72=0$	9. $x^4-8x^2+14=0$	17. $2x^4-5x^2+3=0$
2. $x^4-17x^2-18=0$	10. $9x^4-30x^2+1=0$	18. $3x^4-14x^2+8=0$
3. $x^4-13x^2+36=0$	11. $4x^4-16x^2+13=0$	19. $2x^4-7x^2+3=0$
4. $4x^4-17x^2+4=0$	12. $x^4-9x^2+18=0$	20. $x^4-2x^2-35=0$
5. $y^4-14y^2+46=0$	13. $x^4-7x^2+12=0$	21. $x^4-11x^2+18=0$
6. $x^4-6x^2+6=0$	14. $x^4+5x^2-36=0$	22. $6x^4-23x^2-225=0$
7. $x^4-6x^2+7=0$	15. $x^4-13x^2-30=0$	23. $12x^4-31x^2+9=0$
8. $x^4-12x^2+31=0$	16. $4x^4-5x^2+1=0$	24. $x^4+19x^2+48=0$

Answers to odd exercises:

1. $\pm 2\sqrt{2}, \pm 3i$ 3. $\pm 2, \pm 3$ 5. $\pm\sqrt{7-\sqrt{3}}, \pm\sqrt{7+\sqrt{3}}$ 7. $\pm 1.26, \pm 2.10$ or $\pm\sqrt{3\pm\sqrt{2}}$	9. $\pm 1.61, \pm 2.33$ or $\pm\sqrt{4\pm\sqrt{2}}$ 11. $\pm 1.06, \pm 1.69$ or $\pm\sqrt{2\pm\frac{1}{2}\sqrt{3}}$ 13. $x=\pm\sqrt{3}, x=\pm 2$ 15. $x=\pm\sqrt{15}, x=\pm\sqrt{2}i$	17. $x=\pm 1, x=\frac{\pm\sqrt{6}}{2}$ 19. $x=\pm\sqrt{3}, x=\pm\frac{\sqrt{2}}{2}$ 21. $\{\pm 3, \pm\sqrt{2}\}$ 23. $\{\pm\frac{3}{2}.\pm\frac{\sqrt{3}}{3}\}$

★ Solve quadratic in form polynomial equations.

31. $x^6+7x^3-8=0$	35. $(x-3)^2-5(x-3)-36=0$	39. $\left(x^2+1\right)^2-5\left(x^2+1\right)+4=0$
32. $x^6+28x^3+27=0$	36. $(x+2)^2-3(x+2)-54=0$	40. $\left(x^2-4\right)^2-4\left(x^2-4\right)+3=0$
33. $81y^4-1=0$	37. $(3y+2)^2+(3y+2)-6=0$	41. $2\left(x^2-5\right)^2-5\left(x^2-5\right)+2=0$
34. $x^6+16x^3+64=0$	38. $(5y-1)^2+3(5y-1)-28=0$	42. $2\left(x^2-5\right)^2-7\left(x^2-5\right)+6=0$

43. $(x^2-2x+3)^2-5(x^2-2x+3)+6=0$	44. $(x^2+x-2)^2-5(x^2+x-2)-50=0$

Answers to odd exercises

31. $-2, 1, 1\pm i\sqrt{3}, -\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$ 33. $\pm\frac{1}{3}, \pm\frac{i}{3}$ 35. $x=12,\ x=-1$	37. $x=-\frac{5}{3}, x=0$ 39. $x=0, x=\pm\sqrt{3}$	41. $x=\pm\sqrt{7}, x=\pm\frac{\sqrt{22}}{2}$ 43. $\{0, 1, 1, 2\}$

B. Quadratic in Form Radical and Exponential Equations

Exercise 1.6e.B

★ **Solve quadratic in form radical equations.**

51. $x+2\sqrt{x}-3=0$	58. $x+4\sqrt{x}-21=0$	65. $x-8x^{\frac{1}{2}}+7=0$
52. $x-\sqrt{x}-2=0$	59. $6x+\sqrt{x}-2=0$	66. $2x-7x^{\frac{1}{2}}=15$
53. $x-5\sqrt{x}+6=0$	60. $6x+\sqrt{x}-1=0$	67. $x^{1/2}-3x^{1/4}+2=0$
54. $x-6\sqrt{x}+5=0$	61. $10x-17\sqrt{x}+3=0$	68. $x+5\sqrt{x}-50=0$
55. $x-\sqrt{x}-20=0$	62. $12x+5\sqrt{x}-3=0$	69. $6x^{1/4}-x^{1/2}-9=0$
56. $x-8\sqrt{x}+15=0$	63. $x+2\sqrt{x}-15=0$	70. $x^{1/3}+x^{1/6}-6=0$
57. $x+6\sqrt{x}-16=0$	64. $x-10\sqrt{x}+21=0$	71. $2x^{1/3}-3x^{1/6}+1=0$
		72. $x^{1/3}-x^{1/6}-2=0$

Answers to odd exercises:

51. 1 53. $4, 9$ 55. $x=25$	57. $x=4$ 59. $x=\frac{1}{4}$	61. $x=\frac{1}{25}, x=\frac{9}{4}$ 63. $\{9\}$	65. $x=1, x=49$ 67. $1, 16$	69. $\{81\}$ 71. $\frac{1}{64}, 1$

★ **Solve quadratic in form radical equations.**

81. $x^{2/3}+5x^{1/3}+6=0$	89. $6x^{\frac{2}{3}}-x^{\frac{1}{3}}=12$	97. $x^{2/3}-2x^{1/3}-6=0$
82. $x^{2/3}-2x^{1/3}-35=0$	90. $3x^{\frac{2}{3}}-10x^{\frac{1}{3}}=8$	98. $9x^{2/3}-33x^{1/3}+28=0$
83. $4x^{2/3}-4x^{1/3}+1=0$	91. $8x^{\frac{2}{3}}-43x^{\frac{1}{3}}+15=0$	99. $6x^{2/3}+11x^{1/3}-10=0$
84. $3x^{2/3}-2x^{1/3}-1=0$	92. $20x^{\frac{2}{3}}-23x^{\frac{1}{3}}+6=0$	100. $y^{2/3}-9=0$
85. $8x^{2/3}+7x^{1/3}-1=0$	93. $x^{\frac{2}{3}}+9x^{\frac{1}{3}}+8=0$	101. $30y^{2/3}-15y^{1/3}=0$
86. $x^{\frac{2}{3}}-3x^{\frac{1}{3}}=28$	94. $x^{2/3}+12x^{1/3}+35=0$	102. $x^{4/3}-2x^{2/3}+1=0$
87. $x^{\frac{2}{3}}+4x^{\frac{1}{3}}=12$	95. $x^{2/3}-3x^{1/3}-18=0$	103. $x^{4/3}-13x^{2/3}+36=0$
88. $x^{\frac{2}{3}}-11x^{\frac{1}{3}}+30=0$	96. $x^{2/3}-x^{1/3}+4=0$	104. $x^{4/3}-5x^{2/3}-6=0$

Answers to odd exercises:

81. $-27, -8$; 83. $\frac{1}{8}$ 85. $-1, \frac{1}{512}$	87. $x=8, x=-216$ 89. $x=\frac{27}{8}, x=-\frac{64}{27}$ 91. $x=\frac{27}{512}, x=125$	93. $x=-1, x=-512$ 95. $\{216, -27\}$ 97. $\{22\pm10\sqrt{7}\}$	99. $\{\frac{8}{27}, -\frac{125}{8}\}$ 101. $0, \frac{1}{8}$ 103. $\{\pm8, \pm27\}$

C: Quadratic in Form Negative Exponential Equations

Exercise 1.6e.C

★ Solve quadratic in form negative exponential equations.

111. $5x^{-2}+9x^{-1}-2=0$

112. $3x^{-2}+8x^{-1}-3=0$

113. $8x^{-2}+14x^{-1}-15=0$

114. $9x^{-2}-24x^{-1}+16=0$

115. $2y^{-2}-y^{-1}-1=0$

117. $10x^{-2}-19x^{-1}-2=0$

118. $6x^{-2}+13x^{-1}+5=0$

119. $15x^{-2}-26x^{-1}+8=0$

120. $8x^{-2}-2x^{-1}-3=0$

121. $15x^{-2}-4x^{-1}-4=0$

124. $4y^{-2}-9=0$

125. $16y^{-2}+4y^{-1}=0$

126. $2x^{-2/3}-3x^{-1/3}-2=0$

127. $4x^{-1}-17x^{-1/2}+4=0$

128. $3x^{-1}-8x^{-1/2}+4=0$

Answers to odd exercises:

111. $-\frac{1}{2},5$
113. $-\frac{2}{5},\frac{4}{3}$
115. $-2,1$
117. $x=-10,x=\frac{1}{2}$
119. $x=\frac{3}{4},x=\frac{5}{2}$.
121. $x=\frac{3}{2},x=-\frac{5}{2}$
125. $y=-4$
127. $\frac{1}{16},16$

D: Quadratic in Form Rational Equations

Exercise 1.6e.D: Quadratic in Form Rational Equations

★ Solve quadratic in form rational equations.

131. 17. $\left(\frac{x-3}{x}\right)^2-2\left(\frac{x-3}{x}\right)-24=0$

132. $\left(\frac{2x+1}{x}\right)^2+9\left(\frac{2x+1}{x}\right)-36=0$

133. $2\left(\frac{x}{x+1}\right)^2-5\left(\frac{x}{x+1}\right)-3=0$

134. $3\left(\frac{x}{3x-1}\right)^2+13\left(\frac{x}{3x-1}\right)-10=0$

135. $12\left(\frac{x}{2x-3}\right)^2-11\left(\frac{x}{2x-3}\right)+2=0$

136. $5\left(\frac{1}{x+2}\right)^2-3\left(\frac{1}{x+2}\right)-2=0$

137. $\left(\frac{2x}{x-5}\right)^2+\frac{6x}{x-5}-18=0$

138. $\frac{x^2+2}{x-6}+\frac{x-6}{x^2+2}+\frac{13}{6}=0$

139. $\frac{x^2+2}{x^2-2}+\frac{x^2-2}{x^2+2}-\frac{170}{77}=0$

140. $\sqrt{\frac{x+10}{x-6}}+\sqrt{\frac{x-6}{x+10}}-\frac{34}{15}=0$

141. $\sqrt{\frac{x+7}{x-5}}-\sqrt{\frac{x-5}{x+7}}-\frac{3}{2}=0$

Answers to odd exercises:

131. $\pm\frac{3}{5}$
133. $-\frac{3}{2},-\frac{1}{3}$
135. $-\frac{3}{2},6$
137. $\{15,\frac{15}{4}\}$
139. $\{\pm3,\ \pm3i\}$
141. $\{9\}$

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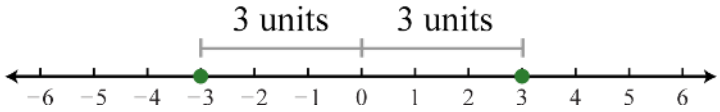
1.7: Absolute Value Equations and Inequalities

Solving an Absolute Value Equation

Recall that the **absolute value** of a real number a, denoted $|a|$, is defined as the distance between zero (the origin) and the graph of that real number on the number line. For example, $|-3| = 3$ and $|3| = 3$.



In addition, the absolute value of a real number can be defined algebraically as a piecewise function.

$$|a| = \begin{cases} a \text{ if } a \geq 0 \\ -a \text{ if } a < 0 \end{cases}$$

Given this definition, $|3| = 3$ and $|-3| = -(-3) = 3$.Therefore, the equation $|x| = 3$ has two solutions for x, namely $\{\pm 3\}$.

Next, To solve an equation such as $|2x - 6| = 8$, notice that the absolute value will be equal to 8 if the quantity inside the absolute value bars is 8 or -8. This leads to two different equations can be solved independently.

$$\begin{array}{r|l} |2x-6| = 8 & \\ 2x - 6 = 8 & 2x - 6 = -8 \\ 2x = 14 & 2x = -2 \\ x = 7 & x = -1 \end{array}$$

Knowing how to solve problems involving absolute value functions is useful. For example, we may need to identify numbers or points on a line that are at a specified distance from a given reference point.

Definition is a green note

Definition: Absolute Value Equations

The absolute value of any algebraic expression u, is written as $|u|$ and is never negative, although the argument u inside the absolute value bars can be either positive or negative.
For any real number c, the absolute value equation $|u| = c$ has the following properties

- If $c < 0$, then $|u| = c$ has no solution.
- If $c = 0$, then $|u| = c$ has one solution.
- If $c > 0$, then $|u| = c$ has two solutions.

Howto: Solve an Absolute Value Equation.

1. Isolate the absolute value expression $|u|$ on one side of the equal sign, producing an equation of the form $|u| = c$
2. If $c > 0$, write and solve two equations: $u = c$ and $u = -c$.
   If $c = 0$, solve the single equation: $u = 0$
   If $c < 0$, the equation has no solution.

Example 1.7.1: Solving Absolute Value Equations

Solve the following absolute value equations:

a. $|6x + 4| = 8$
b. $|3x + 4| + 7 = 1$
c. $|2x + 3| - 1 = 3$

d. $2|5x-1|-3=9$.
e. $|-5x+10|+3=3$

Solution:

a. The absolute value expression is isolated already, so rewrite it as two separate equations and solve them.

$$\begin{array}{r|r} |6x+4|=8 & \\ 6x+4=8 & 6x+4=-8 \\ 6x=4 & 6x=-12 \\ x=\dfrac{2}{3} & x=-2 \end{array}$$

The solution set is $\left\{\frac{2}{3},-2\right\}$

b. Isolate the absolute value expression in the equation.

$$|3x+4|+7=1$$
$$|3x+4|=-6$$

An absolute value cannot be negative. Therefore the equation does not have a solution.
The solution set is $\{\ \}$

c. Isolate the absolute value expression in the equation.

$$|2x+3|-1=3$$
$$|2x+3|=4$$

Rewrite the absolute value as two separate equations and solve them.

$$\begin{array}{r|r} 2x+3=-4 & 2x+3=4 \\ 2x=-7 & 2x=1 \\ -\frac{7}{2} & x=\frac{1}{2} \end{array}$$

The solution set is $\{-\frac{7}{2},\frac{1}{2}\}$

d. Isolate the absolute value expression in the equation.

$$2|5x-1|-3=9$$
$$2|5x-1|=12$$
$$|5x-1|=6$$

Rewrite the absolute value as two separate equations and solve them.

$$\begin{array}{r|r} 5x-1=-6 & 5x-1=6 \\ 5x=-5 & 5x=7 \\ x=-1 & x=\frac{7}{5} \end{array}$$

The solution set is $\{-1,\frac{7}{5}\}$

e. Isolate the absolute value expression in $|-5x+10|+3=3$.

$$|-5x+10|+3=3$$
$$|-5x+10|=0$$

The equation is set equal to zero, so we have to write only one equation.

$$-5x+10=0$$
$$-5x=-10$$
$$x=2$$

The solution set is $\{2\}$

Try It 1.7.1

Solve.

a. $|1-4x|+8=13$.
b. $2-7|x+4|=-12$.

Answer

a, $x=-1, x=\frac{3}{2}$ b. $-6,-2$

If two absolute value expressions are equal, then the arguments can either be the exact same value, or be the exact same value but with opposite signs.

Example 1.7.2:

Solve: $|2x-5|=|x-4|$.

Solution

Set $2x-5$ equal to $\pm(x-4)$ and then solve each linear equation.

$$\begin{array}{cc} |2x-5|=|x-4| & \\ 2x-5=-(x-4) \text{ or } & 2x-5=+(x-4) \\ 2x-5=-x+4 & 2x-5=x-4 \\ 3x=9 & x=1 \\ x=3 & \end{array}$$

The solution set is $\{1,3\}$

Try It 1.7.2

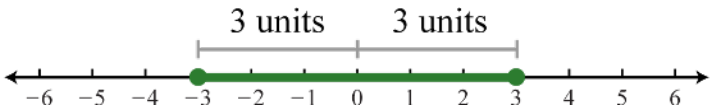
Solve: $|x+10|=|3x-2|$.

Answer

$-2,6$

Absolute Value Inequalities

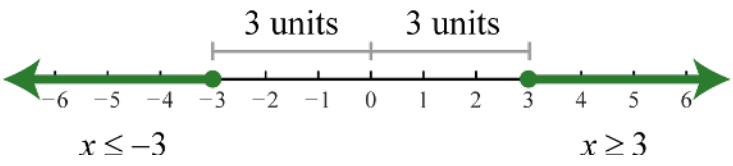
Consider the solutions to the inequality $|x| \leq 3$. The absolute value of a number represents the distance from the origin. Therefore, this inequality describes all numbers whose distance from zero is less than or equal to 3. We can graph this solution set by shading all such numbers.



This solution set is expressed using set notation or interval notation as follows:

$$\{x|-3 \leq x \leq 3\} \textit{Set Notation}$$
$$[-3,3] \quad \textit{Interval Notation}$$

In contrast, examine solutions to the inequality $|x| \geq 3$. This inequality describes all numbers whose distance from the origin is greater than or equal to 3. On a graph, we can shade all such numbers.



This solution set is expressed using using set notation and interval notation as follows:

$$\{x|x \leq -3 \text{ or } x \geq 3\} \quad \textit{Set Notation}$$
$$(-\infty, -3] \cup [3, \infty) \quad \textit{Interval Notation}$$

How To: Howto: Solve a Linear Absolute Value Inequality.

Given an algebraic expression u and a positive number c,

1. Isolate the absolute value expression $|u|$ on left side of the inequality symbol
2. The solution of $|u| < c$ is a single interval composed of the numbers that satisfy the compound inequality $-c < u < c$.
3. The solution of $|u| > c$ is a combination of two separate intervals that satisfy the two separate inequalities $u < -c$ **or** $u > c$. NOTE that $|u| > c$ CANNOT be written as $-c < u > c$.
4. These rules are valid if $<$ is replaced with $\leq$ and $>$ is replaced with $\geq$.

Example 1.7.3:

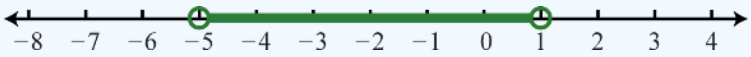
Solve and graph the solution set: $|x + 2| < 3$.

Solution

The absolute value expression is already isolated. Bound the argument $x + 2$ by -3 and 3 in a compound inequality and solve.

$$|x + 2| < 3$$
$$-3 < x + 2 < 3$$
$$-3 \textcolor{blue}{-2} < x + 2 \textcolor{blue}{-2} < 3 \textcolor{blue}{-2}$$
$$-5 < x < 1$$

Here we use open dots to indicate strict inequalities on the graph as follows.



Using interval notation, $(-5, 1)$.

Example 1.7.4:

Solve: $4|x + 3| - 7 \leq 5$.

Solution

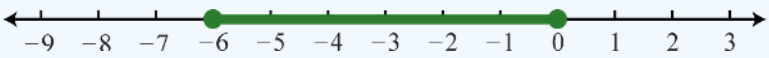
Isolate the absolute value expression.

$$4|x + 3| - 7 \leq 5$$
$$4|x + 3| \leq 12$$
$$|x + 3| \leq 3$$

Rewrite the absolute value inequality as a compound inequality and solve.

$$-3 \leq x + 3 \leq 3$$
$$-3 \textcolor{blue}{-3} \leq x + 3 \textcolor{blue}{-3} \leq 3 \textcolor{blue}{-3}$$
$$-6 \leq x \leq 0$$

Shade the solutions on a number line and present the answer in interval notation. Here we use closed dots to indicate inclusive inequalities on the graph as follows:



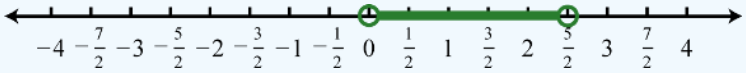
Using interval notation, $[-6, 0]$

Try It 1.7.4

Solve and graph the solution set: $3+|4x-5|<8$.

Answer

Interval notation: $(0, \frac{5}{2})$



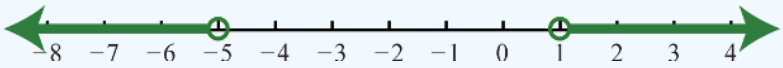
Example 1.7.5:

Solve and graph the solution set: $|x+2|>3$.

Solution

The absolute value expression is already isolated. The argument $x+2$ must be less than -3 or greater than 3.

$$|x+2|>3$$
$$x+2<-3 \quad \text{or} \quad x+2>3$$
$$x<-5 \qquad\qquad x>1$$



Using interval notation, $(-\infty,-5)\cup(1,\infty)$.

Example 1.7.6:

Solve: $13-2|4x-7|\leq 3$.

Solution

Isolate the absolute value expression. Notice in the last step that division by -2 changes the inequality symbol.

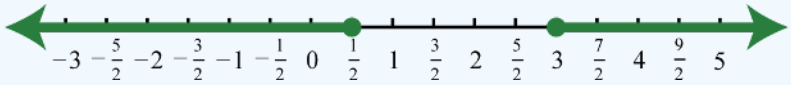
$$13-2|4x-7|\leq 3$$
$$-2|4x-7|\leq -10$$
$$|4x-7|\geq 5$$

Next, apply the theorem and rewrite the absolute value inequality as a compound inequality with two separate intervals and solve.

Illegal pream-token (4)

$$4x\leq 2 \qquad 4x\geq 12$$
$$x\leq \tfrac{2}{4} \qquad x\geq \tfrac{12}{4}$$
$$x\leq \tfrac{1}{2} \qquad x\geq 3$$

Shade the solutions on a number line and present the answer using interval notation.



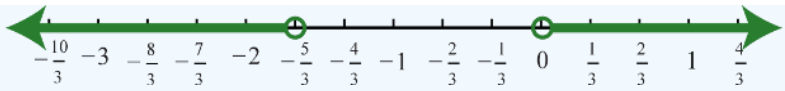
Using interval notation, $(-\infty,\frac{1}{2}]\cup[3,\infty)$

Try It 1.7.6

Solve and graph: $3|6x+5|-2>13$.

Answer

Using interval notation, $\left(-\infty,-\frac{5}{3}\right)\cup(0,\infty)$



Up to this point, the solution sets of linear absolute value inequalities have consisted of a single bounded interval or two unbounded intervals. This is not always the case.

Example 1.7.7:

Solve and graph: $|2x-1|+5>2$.

Solution

Begin by isolating the absolute value.

$$|2x-1|+5>2$$
$$|2x-1|>-3$$

Notice that we have an absolute value greater than a negative number. For any real number x the absolute value of the argument will always be positive. Hence, any real number will solve this inequality.



All real numbers, $\mathbb{R}$.

Example 1.7.8:

Solve and graph: $|x+1|+4\leq 3$.

Solution

Begin by isolating the absolute value.

$$|x+1|+4\leq 3$$
$$|x+1|\leq -1$$

In this case, we can see that the isolated absolute value is to be less than or equal to a negative number. Again, the absolute value will always be positive; hence, we can conclude that there is no solution.

Answer: $\varnothing$ or $\{\ \}$

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1.7e: Exercises - Absolute Value

A: Absolute Value Equations (I)

Exercise 1.7e.A

★ Solve the following absolute value equations.

1. $|x-5|=8$
2. $|x-2|=4$
3. $|x+4|=3$
4. $|x+2|=11$
5. $|x+2|-3=4$
6. $|4-x|+5=12$
7. $2|x-7|+5=9$
8. $3|x+5|=6$
9. $3|x-4|-4=8$
10. $4|x-1|+2=10$
11. $3|x-4|+2=11$
12. $3|4x-5|-4=11$
13. $3|x+2|-5=4$
14. $|2x-3|-4=1$
15. $|3x-5|-1=6$
16. $|5x-4|-3=8$
17. $|4x-3|-5=2$
18. $-2|x-3|+8=-4$
19. $-2|3-2x|=-6$
20. $|3x-4|+5=7$
21. $|4x+7|+2=5$
22. $|34x-3|+7=2$
23. $|35x-2|+5=2$
24. $|12x+5|+4=1$
25. $|4x-1|-3=0$
26. $|14x+3|+3=1$

Answers to odd exercises:

1. $x=-3$ or $x=13$
3. $x=-7$ or $x=-1$
5. $x=-9$ or $x=5$
7. $x=5$ or $x=9$
9. $x=8,\ x=0$
11. $x=7,\ x=1$
13. $x=1,\ x=-5$
15. $x=4,\ x=-\frac{2}{3}$
17. $x=-1,\ x=\frac{5}{2}$
19. $x=0$ or $x=3$
21. $x=-1,\ x=-\frac{5}{2}$
23. no solution
25. $x=1,\ x=-\frac{1}{2}$

★ Solve the following absolute value equations.

31. $-3\left|\frac{x}{2}-4\right|+4=-5$
32. $\left|\frac{2}{3}x-4\right|-11=3.$
33. $\left|\frac{x}{3}-\frac{1}{4}\right|=\frac{1}{12}$
34. $\left|\frac{x}{4}-\frac{1}{2}\right|=\frac{2}{3}$
35. $\left|\frac{3}{4}x-5\right|-9=4$
36. $\left|\frac{5}{6}x+6\right|=8$
37. $|x+2|=\frac{1}{3}x+5$
38. $|x-3|=5-\frac{1}{2}x$
39. $|x-2|=\frac{1}{3}x+2$
40. $|x+4|=\frac{1}{3}x+4$
41. $|4x+3|=|2x+1|$
42. $|3x-2|=|2x-3|$
43. $|6-x|=|3-2x|$
44. $|6x-5|=|2x+3|$
45. $|5x-1|=|2x+3|$
46. $|7x-3|=|3x+7|$
47. $|6x-5|=|3x+4|$
48. $3|x+2|-5=|x+2|+7$
49. $4-3|4-x|=2|4-x|-1$

Answers to odd exercises:

31. $x=14,\ x=2$
33. $x=\frac{1}{2},\ x=1$
35. $x=\frac{-32}{3},\ x=24$
37. $x=\frac{9}{2},\ x=\frac{-21}{4}$
39. $x=0,\ x=6$
41. $x=-1,\ x=-\frac{2}{3}$
43. $x=-3,\ x=3$
45. $x=-\frac{2}{7},\ x=\frac{4}{3}$
47. $x=3, x=\frac{1}{9}$
49. $x=3,\ x=5$

B: Absolute Value Linear Inequalities (I)

Exercise 1.7e.B: Absolute Value Linear Inequalities I

★ Solve. State the solution in interval notation and graph the solution set on the number line.

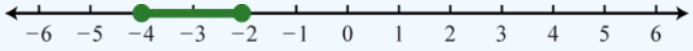
51. $|x| < 5$
52. $|x| \leq 2$
53. $|x+3| \leq 1$
54. $|x-7| < 8$
55. $|x-5| < 0$
56. $|x+8| < -7$
57. $|2x-3| \leq 5$
58. $|3x-9| < 27$
59. $|5x-3| \leq 0$
60. $|10x+5| < 25$
61. $\left|\frac{1}{3}x - \frac{2}{3}\right| \leq 1$
62. $\left|\frac{1}{12}x - \frac{1}{2}\right| \leq \frac{3}{2}$
63. $|x| \geq 5$
64. $|x| > 1$
65. $|x+2| > 8$
66. $|x-7| \geq 11$
67. $|x+5| \geq 0$
68. $|x-12| > -4$
69. $|2x-5| \geq 9$
70. $|2x+3| \geq 15$
71. $|4x-3| > 9$
72. $|3x-7| \geq 2$
73. $\left|\frac{1}{7}x - \frac{3}{14}\right| > \frac{1}{2}$
74. $\left|\frac{1}{2}x + \frac{5}{4}\right| > \frac{3}{4}$

Answers to odd exercises:

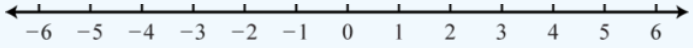
51. $(-5, 5)$;



53. $[-4, -2]$;



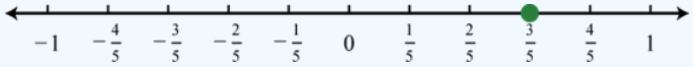
55. $\emptyset$;



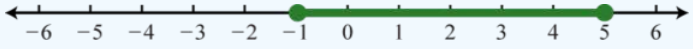
57. $[-1, 4]$;



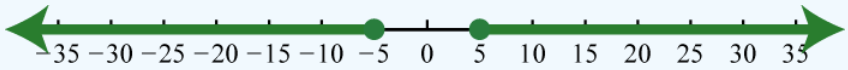
59. $\{\frac{3}{5}\}$;



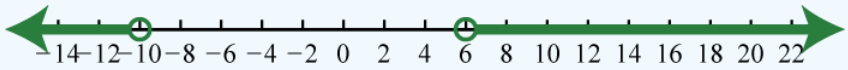
61. $[-1, 5]$;



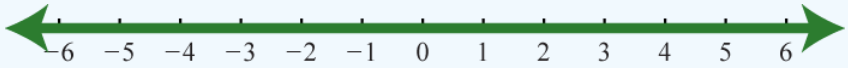
63. $(-\infty, -5] \cup [5, \infty)$;



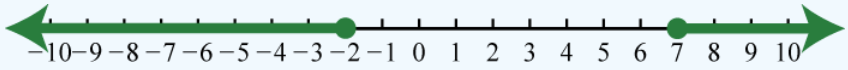
65. $(-\infty, -10) \cup (6, \infty)$;



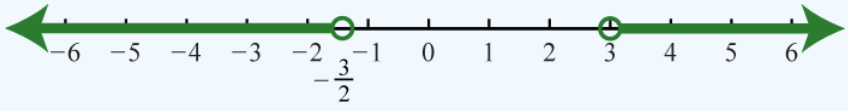
67. $\mathbb{R}$;



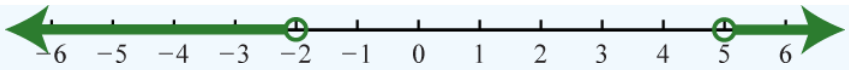
69. $(-\infty, -2] \cup [7, \infty)$;



71. $\left(-\infty, -\frac{3}{2}\right) \cup (3, \infty)$;



73. $(-\infty, -2) \cup (5, \infty)$;



C: Absolute Value Linear Inequalities (II)

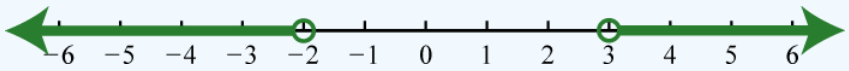
Exercise 1.7e.C: Absolute Value Linear Inequalities II

★ Solve. State the solution in interval notation and graph the solution set on the number line.

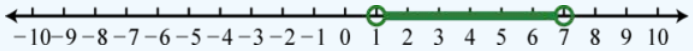
81. $|3(2x-1)| > 15$
82. $|3(x-3)| \le 21$
83. $-5|x-4| > -15$
84. $-3|x+8| \le -18$
85. $6-3|x-4| < 3$
86. $5-2|x+4| \le -7$
87. $1+|2x+5| > 12$
88. $2+|3x-7| \le 9$
89. $|2x+25|-4 \ge 9$
90. $|3(x-3)|-8 < -2$
91. $2|9x+5|+8 > 6$
92. $3|4x-9|+4 < -1$
93. $5|4-3x|-10 \le 0$
94. $6|1-4x|-24 \ge 0$
95. $3-2|x+7| > -7$
96. $9-7|x-4| < -12$
97. $|5(x-4)+5| > 15$
98. $|3(x-9)+6| \le 3$
99. $7-|-4+2(3-4x)| > 5$
100. $9-|6+3(2x-1)| \ge 8$
101. $12+4|2x-1| \le 12$
102. $3-6|3x-2| \ge 3$
103. $\frac{1}{2}|2x-1|+3 < 4$
104. $2\left|\frac{1}{2}x+\frac{2}{3}\right|-3 \le -1$
105. $\left|\frac{1}{3}(x+2)-\frac{7}{6}\right|-\frac{2}{3} \le -\frac{1}{6}$
106. $\left|\frac{1}{10}(x+3)-\frac{1}{2}\right|+\frac{3}{20} > \frac{1}{4}$
107. $\frac{3}{2}-\left|2-\frac{1}{3}x\right| < \frac{1}{2}$
108. $\frac{5}{4}-\left|\frac{1}{2}-\frac{1}{4}x\right| < \frac{3}{8}$

Answers to odd exercises:

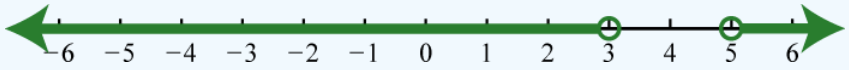
81. $(-\infty,-2)\cup(3,\infty)$;



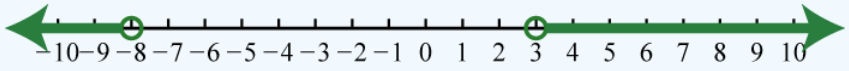
83. $(1,7)$;



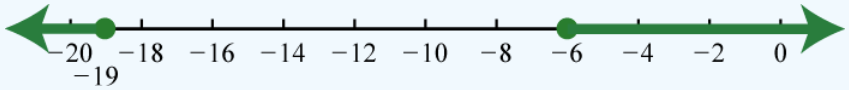
85. $(-\infty,3)\cup(5,\infty)$;



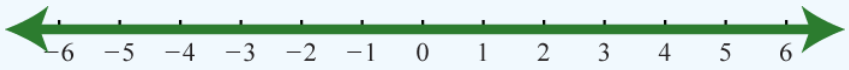
87. $(-\infty,-8)\cup(3,\infty)$;



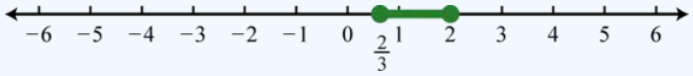
89. $(-\infty,-19]\cup[-6,\infty)$;



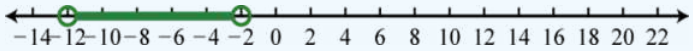
91. $\mathbb{R}$;



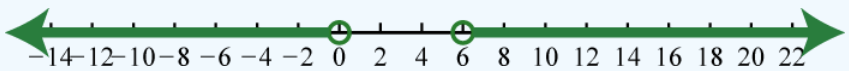
93. $\left[\frac{2}{3},2\right]$;



95. $(-12,-2)$;



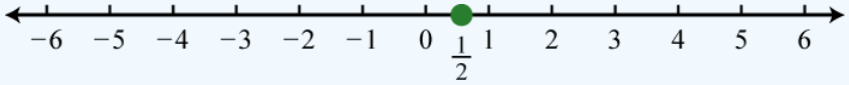
97. $(-\infty,0)\cup(6,\infty)$;



99. $(0, \frac{1}{2})$;



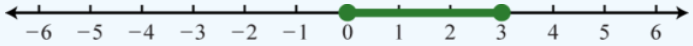
101. $\frac{1}{2}$;



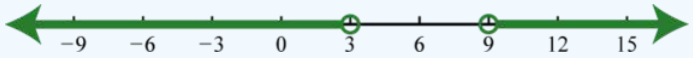
103. $(-\frac{1}{2}, \frac{3}{2})$;



105. $[0, 3]$;



107. $(-\infty, 3) \cup (9, \infty)$;



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1.8: Variation - Constructing and Solving Equations

Solving Problems involving Direct, Inverse, and Joint variation

Certain relationships occur so frequently in applied situations that they are given special names. Variation equations show how one quantity changes in relation to other quantities. The relationship between the quantities can be described as **direct**, **inverse**, or **joint variation**.

Direct Variation

Many real-world problems encountered in the sciences involve two types of functional relationships. The first type of functional relationship can be explored using the fact that the distance s in feet an object falls from rest, without regard to air resistance, can be approximated using the following formula:

$$s = 16t^2$$

Here t represents the time in seconds the object has been falling. For example, after 2 seconds the object will have fallen $s = 16(2)^2 = 16 \cdot 4 = 64$ feet.

Time t in seconds	0	1	2	3	4
Distance in feet $s = 16t^2$	0	16	64	144	256

In this example, we can see that the distance varies over time as the product of a constant 16 and the square of the time t. This relationship is described as **direct variation** and 16 is called the **constant of variation** or the **constant of proportionality**.

Definition: Direct Variation ($y = kx$)

Direct variation is a relationship where quantities behave in a like manner. If one increases, so does the other. If one decreases, so does the other.

For two quantities x and y, this relationship is described as "y **varies directly as** x" or "y **is directly proportional to** x".

The equation that describes this relationship is $y = kx$, where k is a non-zero constant called the **constant of variation** or the **proportionality constant**.

Howto: Solve a Variation Problem.

1. Translate the given English statement containing the words **varies** or **proportional**, into a model equation.
2. Substitute a given set of values into the equation and solve for k, the constant of variation.
3. Rewrite the equation obtained in step 1 as a formula with a value for k found in step 2 defined. Make note of the units used for each variable in the formula.
4. Use the equation from step 3, and another set of values (with one value missing) to solve for the unknown quantity.

Example 1.8.1: Direct Variation

An object's weight on Earth varies directly to its weight on the Moon. If a man weighs 180 pounds on Earth, then he will weigh 30 pounds on the Moon. Set up an algebraic equation that expresses the weight on Earth in terms of the weight on the Moon and use it to determine the weight of a woman on the Moon if she weighs 120 pounds on Earth.

Solution

Step 1. Translate "the weight on Earth varies directly to the weight on the Moon." $E = kM$

Step 2. Find k using "If a man weighs 180 pounds on Earth, then he will weigh 30 pounds on the Moon." $E = 180$ pounds, $M = 30$ pounds

$$\begin{aligned} E &= kM \quad \text{Model equation} \\ 180 &= k \cdot 30 \\ \tfrac{180}{30} &= k \\ 6 &= k \end{aligned}$$

Step 3. The formula is $E = 6M$, where E is the weight on Earth in pounds and M is the weight on the moon in pounds.

Step 4. Answer the question: "determine the weight of a woman on the Moon if she weighs 120 pounds on Earth." $E = 120$ pounds, find M

$$\begin{aligned} E &= 6M \quad \text{Formula:} \quad E \text{ pounds on Earth} \\ & \qquad\quad\;\; M \text{ pounds on the Moon} \\ 120 &= 6M \\ \tfrac{120}{6} &= M \\ 20 &= M \end{aligned}$$

Answer:

The woman weighs 20 pounds on the Moon.

Indirect Variation

The second functional relationship can be explored using the model that relates the intensity of light I to the square of the distance from its source d.

$$I = \frac{k}{d^2}$$

Here k represents some constant. A foot-candle is a measurement of the intensity of light. One foot-candle is defined to be equal to the amount of illumination produced by a standard candle measured one foot away. For example, a 125-Watt fluorescent growing light is advertised to produce 525 foot-candles of illumination. This means that at a distance $d = 1$ foot, $I = 525$ foot-candles and we have:

$$\begin{aligned} 525 &= \frac{k}{(1)^2} \\ 525 &= k \end{aligned}$$

Using $k = 525$ we can construct a formula which gives the light intensity produced by the bulb:

$$I = \frac{525}{d^2}$$

Distance t in feet	1	2	3	4	5
Light Intensity $I = \frac{525}{d^2}$	525	131.25	58.33	32.81	21

Here d represents the distance the growing light is from the plants. In the chart above, we can see that the amount of illumination fades quickly as the distance from the plants increases.

This type of relationship is described as an **inverse variation**. We say that I is **inversely proportional** to the square of the distance d, where 525 is the constant of proportionality.

Definition: Indirect Variation ($y = \frac{k}{x}$)

Indirect variation is a relationship between quantities where if one increases, the other decreases.

For two quantities x and y, this relationship is described as "y **varies indirectly as** x" or "y **is inversely proportional to** x".

The equation that describes this relationship is $y = \dfrac{k}{x}$, where k is a non-zero constant called the **constant of variation** or the **proportionality constant**.

Example 1.8.2: Indirect Variation

The weight of an object varies inversely as the square of its distance from the center of Earth. If an object weighs 100 pounds on the surface of Earth (approximately $4,000$ miles from the center), how much will it weigh at $1,000$ miles above Earth's surface?

Solution

Step 1. Translate "w varies inversely as the square of d" $w = \frac{k}{d^2}$

Step 2. Find k using "An object weighs 100 pounds on the surface of Earth, approximately $4,000$ miles from the center". $w = 100$ when $d = 4,000$

$$(4,000)^2 \cdot 100 = (4,000)^2 \cdot \frac{k}{(4,000)^2}$$
$$1,600,000,000 = k$$
$$1.6 \times 10^9 = k$$

Step 3. The formula is $w = \frac{1.6 \times 10^9}{d^2}$, where w is the weight of the object in pounds and d is the distance of the object from the center of the Earth in miles.

Step 4. Answer the question: "how much will it weigh at $1,000$ miles above Earth's surface?"
Since the object is $1,000$ miles above the surface, the distance of the object from the center of Earth is $d = 4,000 + 1,000 = 5,000$ miles

$$\begin{aligned} y &= \frac{1.6 \times 10^9}{(5,000)^2} \\ &= \frac{1.6 \times 10^9}{25,000,000} \\ &= \frac{1.6 \times 10^9}{2.5 \times 10^9} \\ &= 0.64 \times 10^2 \\ &= 64 \end{aligned}$$

Answer:

The object will weigh 64 pounds at a distance $1,000$ miles above the surface of Earth.

Joint Variation

Lastly, we define relationships between multiple variables.

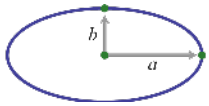
Definition: Joint Variation and Combined Variation

Joint variation is a relationship in which one quantity is proportional to the product of two or more quantities.

Combined variation exists when combinations of direct and/or inverse variation occurs

Example 1.8.3: Joint Variation

The area of an ellipse varies jointly as a, half of the ellipse's major axis, and b, half of the ellipse's minor axis as pictured. If the area of an ellipse is $300\pi\ cm^2$, where $a = 10$ cm and $b = 30$ cm, what is the constant of proportionality? Give a formula for the area of an ellipse.



Solution

Step 1. If we let A represent the area of an ellipse, then we can use the statement "area varies jointly as a and b" to write

$$A = kab$$

Step 2. To find the constant of variation k, use the fact that the area is 300π when $a = 10$ and $b = 30$.

$$\begin{aligned} 300\pi &= k(10)(30) \\ 300\pi &= 300k \\ \pi &= k \end{aligned}$$

Step 3. Therefore, the formula for the area of an ellipse is

$$A = \pi ab$$

Answer:

The constant of proportionality is π and the formula for the area of an ellipse is $A = ab\pi$.

Try It 1.8.3: Combined Variation

Given that y varies directly as the square of x and inversely with z, where $y = 2$ when $x = 3$ and $z = 27$, find y when $x = 2$ and $z = 16$.

Answer

$\frac{3}{2}$

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1.8e: Exercises - Variation

A: Translate Words into a Formula

Exercise 1.8e.A

★ Translate each of the following sentences into a mathematical formula.

1. The distance D an automobile can travel is directly proportional to the time t that it travels at a constant speed.

2. The extension of a hanging spring d is directly proportional to the weight w attached to it.

3. An automobile's braking distance d is directly proportional to the square of the automobile's speed v.

4. The volume V of a sphere varies directly as the cube of its radius r.

5. The volume V of a given mass of gas is inversely proportional to the pressure p exerted on it.

6. Every particle of matter in the universe attracts every other particle with a force F that is directly proportional to the product of the masses m_1 and m_2 of the particles, and it is inversely proportional to the square of the distance d between them.

7. Simple interest I is jointly proportional to the annual interest rate r and the time t in years a fixed amount of money is invested.

8. The time t it takes an object to fall is directly proportional to the square root of the distance d it falls.

Answers to odd exercises:

1. $D = kt$	3. $d = kv^2$	5. $V = \frac{k}{p}$	7. $I = krt$

B: Translate Words and Find a Formula

Exercise 1.8e.B

★ Construct a mathematical model given the following:

9. y varies directly as x, and $y = 30$ when $x = 6$.

10. y varies directly as x, and $y = 52$ when $x = 4$.

11. y is directly proportional to x, and $y = 12$ when $x = 3$.

12. y is directly proportional to x, and $y = 120$ when $x = 20$.

13. y is directly proportional to x, and $y = 3$ when $x = 9$.

14. y is directly proportional to x, and $y = 21$ when $x = 3$.

15. y varies inversely as x, and $y = 2$ when $x = \frac{1}{8}$.

16. y varies inversely as x, and $y = \frac{3}{2}$ when $x = \frac{1}{9}$.

17. y is jointly proportional to x and z, where $y = 2$ when $x = 1$ and $z = 3$.

18. y is jointly proportional to x and z, where $y = 15$ when $x = 3$ and $z = 7$.

19. y varies jointly as x and z, where $y = \frac{2}{3}$ when $x = \frac{1}{2}$ and $z = 12$.

20. y varies jointly as x and z, where $y = 5$ when $x = \frac{3}{2}$ and $z = \frac{2}{9}$.

21. y varies directly as the square of x, where $y = 45$ when $x = 3$.

22. y varies directly as the square of x, where $y = 3$ when $x = \frac{1}{2}$.

23. y is inversely proportional to the square of x, where $y = 27$ when $x = \frac{1}{3}$.

24. y is inversely proportional to the square of x, where $y = 9$ when $x = \frac{2}{3}$.

25. y varies jointly as x and the square of z, where $y = 6$ when $x = \frac{1}{4}$ and $z = \frac{2}{3}$.

26. y varies jointly as x and z and inversely as the square of w, where $y = 5$ when $z = 1, z = 3$, and $w = \frac{1}{2}$.

27. y varies directly as the square root of x and inversely as the square of z, where $y = 15$ when $x = 25$ and $z = 2$.

28. y varies directly as the square of x and inversely as z and the square root of w, where $y = 14$ when $x = 4, w = 9$ and $z = 2$.

Answers to odd exercises:

9. $y = 5x$ 11. $y = 4x$	13. $y = \frac{1}{3}x$ 15. $y = \frac{1}{4x}$	17. $y = \frac{2}{3}xz$ 19. $y = \frac{1}{9}xz$	21. $y = 5x^2$ 23. $y = \frac{3}{x^2}$	25. $y = 54xz^2$ 27. $y = \frac{12\sqrt{x}}{z^2}$

C: Direct variation problems

Exercise 1.8e.C: Direct variation problems

★ **Solve applications involving variation.**

29. Revenue in dollars is directly proportional to the number of branded sweatshirts sold. The revenue earned from selling **25** sweatshirts is **$318.75**. Determine the revenue if **30** sweatshirts are sold.

30. The sales tax on the purchase of a new car varies directly as the price of the car. If an **$18,000** new car is purchased, then the sales tax is **$1,350**. How much sales tax is charged if the new car is priced at **$22,000**?

31. The price of a share of common stock in a company is directly proportional to the earnings per share (EPS) of the previous **12** months. If the price of a share of common stock in a company is $22.55, and the EPS is published to be **$1.10**, determine the value of the stock if the EPS increases by **$0.20**.

32. The distance traveled on a road trip varies directly with the time spent on the road. If a **126**-mile trip can be made in **3** hours, then what distance can be traveled in **4** hours?

33. The circumference of a circle is directly proportional to its radius. The circumference of a circle with radius **7** centimeters is measured as 14π centimeters. What is the constant of proportionality?

34. The area of circle varies directly as the square of its radius. The area of a circle with radius **7** centimeters is determined to be 49π square centimeters. What is the constant of proportionality?

35. The surface area of a sphere varies directly as the square of its radius. When the radius of a sphere measures **2** meters, the surface area measures 16π square meters. Find the surface area of a sphere with radius **3** meters.

36. The volume of a sphere varies directly as the cube of its radius. When the radius of a sphere measures **3** meters, the volume is 36π cubic meters. Find the volume of a sphere with radius **1** meter.

37. With a fixed height, the volume of a cone is directly proportional to the square of the radius at the base. When the radius at the base measures **10** centimeters, the volume is **200** cubic centimeters. Determine the volume of the cone if the radius of the base is halved.

38. The distance d an object in free fall drops varies directly with the square of the time t that it has been falling. If an object in free fall drops **36** feet in **1.5** seconds, then how far will it have fallen in **3** seconds?

★ Hooke's law suggests that the extension of a hanging spring is directly proportional to the weight attached to it. The constant of variation is called the spring constant.



Robert Hooke (1635-1703)

39. A hanging spring is stretched **5** inches when a **20**-pound weight is attached to it. Determine its spring constant.

40. A hanging spring is stretched **3** centimeters when a **2**-kilogram weight is attached to it. Determine the spring constant.

41. If a hanging spring is stretched 3 inches when a 2-pound weight is attached, how far will it stretch with a 5-pound weight attached?

42. If a hanging spring is stretched 6 centimeters when a 4-kilogram weight is attached to it, how far will it stretch with a 2-kilogram weight attached?

★ The braking distance of an automobile is directly proportional to the square of its speed.

43. It takes 36 feet to stop a particular automobile moving at a speed of 30 miles per hour. How much breaking distance is required if the speed is 35 miles per hour?

44. After an accident, it was determined that it took a driver 80 feet to stop his car. In an experiment under similar conditions, it takes 45 feet to stop the car moving at a speed of 30 miles per hour. Estimate how fast the driver was moving before the accident.

45. The period T of a pendulum is directly proportional to the square root of its length L. If the length of a pendulum is 1 meter, then the period is approximately 2 seconds. Approximate the period of a pendulum that is 0.5 meter in length.

46. The time t it takes an object to fall is directly proportional to the square root of the distance d it falls. An object dropped from 4 feet will take $\frac{1}{2}$ second to hit the ground. How long will it take an object dropped from 16 feet to hit the ground?

Answers to odd exercises:

29. $\$382.50$	35. 36π square meters	41. 7.5 inches
31. $\$26.65$	37. 50 cubic centimeters	43. 49 feet
33. 2π	39. $\frac{1}{4}$ inches/pound	45. 1.4 seconds

D: Inverse Variation Problems

Exercise 1.8e.D

47. To balance a seesaw, the distance from the fulcrum that a person must sit is inversely proportional to his weight. If a 72-pound boy is sitting 3 feet from the fulcrum, how far from the fulcrum must a 54-pound boy sit to balance the seesaw?

48. The current I in an electrical conductor is inversely proportional to its resistance R. If the current is $\frac{1}{4}$ ampere when the resistance is 100 ohms, what is the current when the resistance is 150 ohms?

49. The amount of illumination I is inversely proportional to the square of the distance d from a light source. If 70 foot-candles of illumination is measured 2 feet away from a lamp, what level of illumination might we expect $\frac{1}{2}$ foot away from the lamp?

50. The amount of illumination I is inversely proportional to the square of the distance d from a light source. If 40 foot-candles of illumination is measured 3 feet away from a lamp, at what distance can we expect 10 foot-candles of illumination?

★ Boyle's law states that if the temperature remains constant, the volume V of a given mass of gas is inversely proportional to the pressure p exerted on it.



Robert Boyle (1627-1691)

51. A balloon is filled to a volume of 216 cubic inches on a diving boat under 1 atmosphere of pressure. If the balloon is taken underwater approximately 33 feet, where the pressure measures 2 atmospheres, then what is the volume of the balloon?

52. A balloon is filled to 216 cubic inches under a pressure of 3 atmospheres at a depth of 66 feet. What would the volume be at the surface, where the pressure is 1 atmosphere?

Answers to odd exercises:

47. 4 feet	49. $1,120$ foot-candles	51. 108 cubic inches

E: Joint and Combined Variation

Exercise 1.8e.E

53. The number of men, represented by y, needed to lay a cobblestone driveway is directly proportional to the area A of the driveway and inversely proportional to the amount of time t allowed to complete the job. Typically, 3 men can lay $1,200$ square feet of cobblestone in 4 hours. How many men will be required to lay $2,400$ square feet of cobblestone in 6 hours?

54. The volume of a right circular cylinder varies jointly as the square of its radius and its height. A right circular cylinder with a 3-centimeter radius and a height of 4 centimeters has a volume of 36π cubic centimeters. Find a formula for the volume of a right circular cylinder in terms of its radius and height.

★ **Newton's universal law of gravitation states that every particle of matter in the universe attracts every other particle with a force F that is directly proportional to the product of the masses m_1 and m_2 of the particles and inversely proportional to the square of the distance d between them. The constant of proportionality is called the gravitational constant.**	*Sir Isaac Newton (1643-1727)* 

55. If two objects with masses 50 kilograms and 100 kilograms are $\frac{1}{2}$ meter apart, then they produce approximately 1.34×10^{-6} newtons (N) of force. Calculate the gravitational constant.

56. Use the gravitational constant from the previous exercise to write a formula that approximates the force F in newtons between two masses m_1 and m_2, expressed in kilograms, given the distance d between them in meters.

57. Calculate the force in newtons between Earth and the Moon, given that the mass of the Moon is approximately 7.3×10^{22} kilograms, the mass of Earth is approximately 6.0×10^{24} kilograms, and the distance between them is on average 1.5×10^{11} meters.

58. Calculate the force in newtons between Earth and the Sun, given that the mass of the Sun is approximately 2.0×10^{30} kilograms, the mass of Earth is approximately 6.0×10^{24} kilograms, and the distance between them is on average 3.85×10^{8} meters.

59. If y varies directly as the square of x, then how does y change if x is doubled?

60. If y varies inversely as square of t, then how does y change if t is doubled?

61. If y varies directly as the square of x and inversely as the square of t, then how does y change if both x and t are doubled?

Answers to odd exercises:

53. 4 men 55. $6.7 \times 10^{-11}\text{Nm}^2/\text{kg}^2$	57. $1.30 \times 10^{15}\text{N}$ 59. y changes by a factor of 4	61. y remains unchanged

F: More Variation Problems

Exercise 1.8e.F

★ **Solve the following variation problems.**

71. The number of calories, c, burned varies directly with the amount of time, t, spent exercising. Arnold burned 312 calories in 65 minutes exercising. How many calories would he burn if he exercises for 90 minutes?

72. The number of gallons of gas a car uses varies directly with the number of miles driven. Driving 469.8 miles used 14.5 gallons of gas. How many gallons of gas would the car use if driven 1000 miles?

73. The weight of a liquid varies directly as its volume. A liquid that weighs 24 pounds has a volume of 4 gallons. If a liquid has volume 13 gallons, what is its weight?

74. The maximum load a beam will support varies directly with the square of the diagonal of the beam's cross-section. A beam with diagonal 4" will support a maximum load of 75 pounds. What is the maximum load that can be supported by a beam with diagonal 8"?

75. The area of a circle varies directly as the square of the radius. A circular pizza with a radius of 6 inches has an area of 113.1 square inches. What is the area of a pizza with a radius of 9 inches?

76. The fuel consumption (mpg) of a car varies inversely with its weight. A car that weighs 3100 pounds gets 26 mpg on the highway. What would be the fuel consumption of a car that weighs 4030 pounds?

77. A car's value varies inversely with its age. If a two-year-old car is worth $20,000, what will be the value of the car when it is 5 years old?

78. The number of hours it takes for ice to melt varies inversely with the air temperature. Suppose a block of ice melts in 2 hours when the temperature is 65 degrees. How many hours would it take for the same block of ice to melt if the temperature was 78 degrees?

79. The force needed to break a board varies inversely with its length. Richard uses 24 pounds of pressure to break a 2-foot long board. How many pounds of pressure is needed to break a 5-foot long board?

80. For people with roughly the same build, the weight of the person varies as the cube of their height. If a person 65 inches high weighs 125 pounds, how much would a person 75 inches high with a similar build be expected to weigh?

81. The fuel consumption (mpg) of a car varies inversely with its weight. A Ford Focus weighs 3000 pounds and gets 28.7 mpg on the highway. What would the fuel consumption be for a Ford Expedition that weighs 5,500 pounds? Round to the nearest tenth.

82. A person's BMI (body mass index) varies directly as their weight and inversely as the square of their height. Given a person who weighs 180 pounds and is 60 inches tall has a BMI of 35.2, what is the BMI for someone who is 150 pounds and 68 inches tall?

83. The maximum load L that a cylindrical column with a circular cross section can hold varies directly as the fourth power of the diameter d and inversely as the square of the height h. If an 8.0 m column that is 2.0 m in diameter will support 64 tons, how many tons can be supported by a column 12.0 m high and 3.0 m in diameter?

84. The heat loss per hour through a glass window varies directly with the difference in temperature between the inside and outside temperatures and inversely as the thickness of the glass. A 0.3 cm thick window loses 2.4 BTU per hour when the outside temperature is 50 degrees Fahrenheit and the inside temperature is 70 degrees Fahrenheit. What will the heat loss be for a 1.5 cm thick window when the outside temperature is 30 degrees Fahrenheit and the inside temperature is 70 degrees Fahrenheit?

85. The heat loss of a glass window varies jointly as the window's area and the difference between the outside and inside temperatures. A window 3 feet wide by 6 feet long loses 1200 Btu per hour when the temperature outside is 20 degrees colder than the temperature inside. Find the heat loss through a glass window that is 6 feet wide by 9 feet long when the temperature outside is 10 degrees colder that the temperature inside.

86. Sound intensity varies inversely as the square root of the distance from the sound source. If you are in a movie theater and you change your seat to one that is twice as far from the speakers, how does the new sound intensity compare to that of your original seat?

87. The number of hours h that it takes p people to assemble m machines varies directly as the number of machines and inversely as the number of people. If four people can assemble 12 machines in four hours, how many people are needed to assemble 36 machines in eight hours?

88. The amount of time t needed to build a wall varies directly as the number of bricks b need and inversely as the number of workers w. If it takes 18 hours for six workers to make a wall composed of 2400 bricks, how long would it take to build a wall of 4500 bricks with 10 workers?

Answers to odd exercises:

71. 432 calories	73. 78 pounds	75. 254.5 square inches	77. $8,000	79. 9.6 pounds
81. 15.6 mpg	83. 144 tons	85. 1800 BTU per hour	87. 6 people	

*

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CHAPTER OVERVIEW

2: Functions and Their Graphs

2.1: Functions and Function Notation

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2.2e: Exercises - Attributes of Functions

2.3: Transformations of Functions

2.3e: Exercises - Transformations

2.4: Function Compilations - Piecewise, Combinations, and Composition

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2.5: One-to-One and Inverse Functions

2.5e: Exercises Inverse Functions

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2.1: Functions and Function Notation

Learning Objectives

- Determine whether a relation represents a function algebraically and graphically
- Use function notation to evaluate functions given a formula or a graph
- Us a graph or formula to find input function values that will produce a given output
- Evaluate and simplify difference quotients

Determine Whether a Relation Represents a Function

A relation is a set of ordered pairs. The set of the first components of each ordered pair is called the domain and the set of the second components of each ordered pair is called the range. Consider the following set of ordered pairs. The first numbers in each pair are the first five natural numbers. The second number in each pair is twice that of the first.

$$\{(1,2),(2,4),(3,6),(4,8),(5,10)\}$$

The domain is $\{1,2,3,4,5\}$. The range is $\{2,4,6,8,10\}$.

Note that each value in the domain is also known as an **input** value, or **independent variable**, and is often labeled with the lowercase letter x. Each value in the range is also known as an output value, or **dependent variable**, and is often labeled lowercase letter y.

A function f is a relation that assigns each value in the domain of a function to a single value in the range to a value in the domain. In other words, no x-value is assigned to more than one y-value. For our example that relates the first five **natural numbers** to numbers double their values, this relation is a function because each element in the domain, {1, 2, 3, 4, 5}, is paired with exactly one element in the range, $\{2,4,6,8,10\}$.

Consider the set of ordered pairs that relates the terms “even” and “odd” to the first five natural numbers. It would appear as

$$\{(\text{odd},1),(\text{even},2),(\text{odd},3),(\text{even},4),(\text{odd},5)\}$$

Notice that each element in the domain, {even, odd} is not paired with exactly one element in the range, $\{1,2,3,4,5\}$. For example, the term “odd” corresponds to three values from the domain, $\{1,3,5\}$, and the term “even” corresponds to two values from the range, $\{2,4\}$. This violates the definition of a function, so this relation is not a function.

Figure Runaway argument for \eatSpaces? compares relations that are functions and not functions.

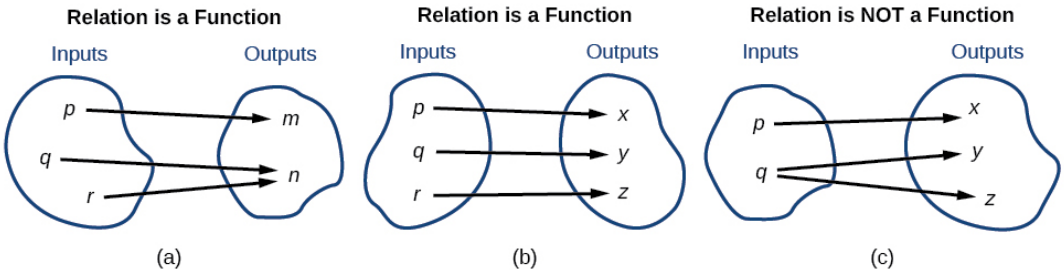


Figure Runaway argument for \eatSpaces?: (a) This relationship is a function because each input is associated with a single output. Note that input q and r both give output n. (b) This relationship is also a function. In this case, each input is associated with a single output. (c) This relationship is not a function because input q is associated with two different outputs.

Definition: Function, Domain, Range

A **function** is a relation in which each possible input value leads to exactly one output value. We say “the output is a function of the input.”

The **input** values make up the **domain**, and the **output** values make up the **range**.

Often in equations, the input values are x values, the output values are y values and "y is a function of x."

Howto: Determine whether a relationship between two quantities is a function.

1. Identify the input values.
2. Identify the output values.
3. If each input value leads to only one output value, then "*the output is a function of the input*", and the relationship is classified as a function. If any input value leads to two or more outputs, do not classify the relationship as a function.

Practical Examples of Functions

Example 2.1.1: Determining If Menu Price Lists Are Functions

The coffee shop menu, shown in Figure **2.1.1** consists of items and their prices.

a. Is price a function of the item?
b. Is the item a function of the price?



Figure **2.1.1**: A menu of donut prices where a plain donut is $1.49 and a jelly donut and chocolate donut are $1.99.

Solution

a. Let's begin by considering the input as the items on the menu. The output values are then the prices. See below.



Figure **2.1.1*a***: A menu of donut prices where a plain donut is $1.49 and a jelly donut and chocolate donut are $1.99.

Each item on the menu has only one price, so the price is a function of the item.

a. Two items on the menu have the same price. If we consider the prices to be the input values and the items to be the output, then the same input value could have more than one output associated with it. See below.

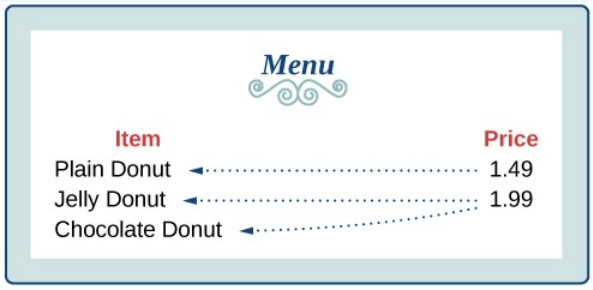


Figure **2.1.1*b***: Association of the prices to the donuts.

Therefore, the item is a not a function of price.

Example 2.1.2: Determining If Class Grade Rules Are Functions

In a particular math class, the overall percent grade corresponds to a grade point average. Is grade point average a function of the percent grade? Is the percent grade a function of the grade point average? Table 2.1.2 shows a possible rule for assigning grade points.

Table 2.1.2: Class grade points.

Percent grade	0–56	57–61	62–66	67–71	72–77	78–86	87–91	92–100
Grade point average	0.0	1.0	1.5	2.0	2.5	3.0	3.5	4.0

Solution

For any percent grade earned, there is just one associated grade point average, so the grade point average is a function of the percent grade. In other words, if we input the percent grade, the output is a specific grade point average.

In the grading system given, there is a range of percent grades that correspond to the same grade point average. For example, students who receive a grade point average of 3.0 could have a variety of percent grades ranging from 78 all the way to 86. Thus, percent grade is not a function of grade point average.

Try It 2.1.2

The table below lists the five greatest baseball players of all time in order of rank.

Player	Rank
Babe Ruth	1
Willie Mays	2
Ty Cobb	3
Walter Johnson	4
Hank Aaron	5

a. Is the rank a function of the player name?
b. Is the player name a function of the rank?

Answer a

Yes

Answer b

yes. (Note: If two players had been tied for, say, 4th place, then the name would not have been a function of rank.)

Determine if an Equation is a Function

In order to be a function, each element in the domain can correspond to just a single value in the range. When there exists an element in the domain that corresponds to two (or more) different values in the range, the relation is not a function. In the case of equations, if an equation is solved for y and the result is that for some value of x, multiple different values of y can be obtained from that single value for x, then the equation does not define y as a function of x.

Howto: Determine if an equation defines a function.

Given an equation in two variables, for example, x and y.

1. Solve the equation for y.
2. If more than one value of y can be obtained for a given x, then the equation does not define y as a function of x. One example is if an equation is obtained that looks like $y = \pm \ldots$.

Example 2.1.3A

Express the relationship $2n + 6p = 12$ as a function of n, if possible.

Solution: To express the relationship as a function of n, the equation needs to be rewritten in the form p=[expression involving n].

$$\begin{aligned} 2n + 6p &= 12 \\ 6p &= 12 - 2n && \text{Subtract 2n from both sides.} \\ p &= \frac{12 - 2n}{6} && \text{Divide both sides by 6 and simplify.} \\ p &= \frac{12}{6} - \frac{2n}{6} \\ p &= 2 - \frac{1}{3}n \end{aligned}$$

From this result, we can see that for each value of n there is one and only one value for p, so therefore the equation defines p as a function of n

Example 2.1.3B

Does the equation $x^2 + y^2 = 1$ represent a function of x?

Solution First we subtract x^2 from both sides. Then solve for y.

$$y^2 = 1 - x^2$$

$$y = \pm\sqrt{1 - x^2}$$

$$= +\sqrt{1 - x^2} \text{ and } -\sqrt{1 - x^2}$$

Since more than a single value for y can be obtained for a given value of x, the equation is not a function of x

Try It 2.1.3

Do the following equations represent functions of x?

a. $x - 8y^3 = 0$	b. $\lvert y + 3 \rvert = 4 - x$	c. $x + 8 = \sqrt{y + 1}$

a. Answer

$y = \frac{\sqrt[3]{x}}{2}$

y is a function of x because every value chosen for x will generate just a single value for y.

b. Answer

y is not a function of x because there is at least one value chosen for x that will generate multiple different values for y. Example: If $x = 0$ then $y = 1$ or $y = -7$.

c. Answer

$y = (x + 8)^2 - 1$

y is a function of x because every value chosen for x will generate just a single value for y.

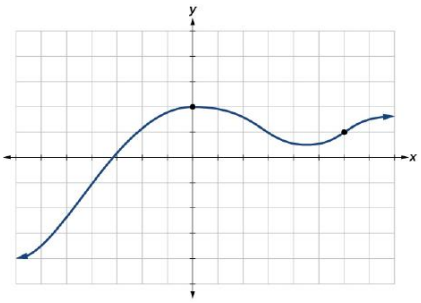
Are there relationships expressed by a equation that do represent a function but which still cannot be represented by an algebraic formula ?

Yes, this can happen. For example, given the equation $x = y + 2^y$, if we want to express y as a function of x, there is no simple algebraic formula involving only x that equals y. However, each x does determine a unique value for y, and there are mathematical procedures by which y can be found to any desired accuracy. In this case, we say that the equation gives an implicit (implied) rule for y as a function of x, even though the formula cannot be written explicitly.

Determine if a graph represents a function

Functions can be represented by equations; they can also be represented as a graph. Graphs display a great many input-output pairs in a small space. The visual information they provide often makes relationships easier to understand. By convention, graphs are typically constructed with the input values along the horizontal axis and the output values along the vertical axis.

The most common graphs name the input value x and the output y, and we say y is a function of x, or $y = f(x)$ when the function is named f. The graph of the function is the set of all points (x, y) in the plane that satisfies the equation $y = f(x)$. If the function is defined for only a few input values, then the graph of the function is only a few points, where the x-coordinate of each point is an input value and the y-coordinate of each point is the corresponding output value. For example, the black dots on the graph of a polynomial in the figure below tell us that $f(0) = 2$ and $f(6) = 1$. However, the set of all points (x, y) satisfying $y = f(x)$ is a curve. The curve shown includes $(0, 2)$ and $(6, 1)$ because the curve passes through those points

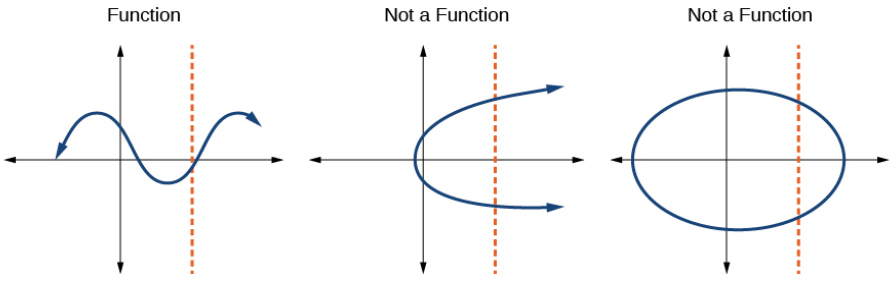


Graph of a polynomial.

Definition: The Vertical Line Test

The Vertical Line Test is a test used to determine if a graph represents a function. If there is no vertical line that crosses a graph in more than one point, the graph is a function.

The vertical line test can be used to determine whether a graph represents a function. If we can draw any vertical line that intersects a graph more than once, then the graph does not define a function because a function has only one output value for each input value. These ideas are illustrated in the figure below.



Three graphs visually showing what is and is not a function.

Howto: Use the vertical line test to determine if a graph represents a function.

1. Inspect the graph to see if any vertical line drawn would intersect the curve more than once.
2. If there is any such line, the graph does not represent a function.

Example 2.1.4: Applying the Vertical Line Test

Which of the graphs in Figure 2.1.4 represent(s) a function?

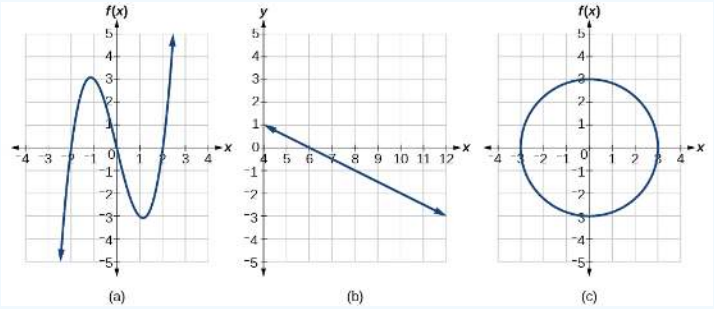


Figure 2.1.4: Graph of a polynomial (a), a downward-sloping line (b), and a circle (c).

Solution

If any vertical line intersects a graph more than once, the relation represented by the graph is not a function. Notice that any vertical line would pass through only one point of the two graphs shown in parts (a) and (b) of Figure 2.1.4. From this we can conclude that these two graphs represent functions. The third graph does not represent a function because, at most x-values, a vertical line would intersect the graph at more than one point, as shown in Figure 2.1.4c.

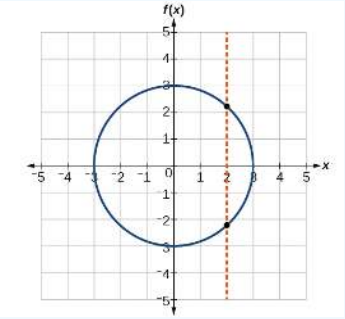
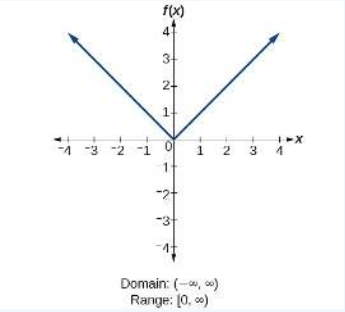


Figure 2.1.4c: Graph of a circle.

Try It 2.1.4

Does the graph of an absolute value function in the figure below represent a function?



Graph of an absolute value function.

Answer

yes

Function Notation and Evaluation

Once we determine that a relationship is a function, we need to display and define the functional relationships so that we can understand and use them, and sometimes also so that we can program them into computers. There are various ways of representing functions. A standard function notation is one representation that facilitates working with functions.

To represent "height is a function of age," we start by identifying the descriptive variables h for height and a for age. The letters f, g, and h are often used to represent functions just as we use x, y, and z to represent numbers and A, B, and C to represent sets.

h is f of a	We name the function f; height is a function of age.
$h = f(a)$	We use parentheses to indicate the function input.
$f(a)$	We name the function f ; the expression is read as " f of a."

Remember, we can use any letter to name the function; the notation $h(a)$ shows us that h depends on a. The value a must be put into the function h to get a result. The parentheses indicate that age is input into the function; they do not indicate multiplication.

We can also give an algebraic expression as the input to a function. For example $f(a + b)$ means "first add a and b, and the result is the input for the function f." The operations must be performed in this order to obtain the correct result.

Definition: Function Notation

The notation $y = f(x)$ defines a function named f. This is read as "y is a function of x." The letter x represents the input value, or independent variable. The letter y, or $f(x)$, represents the output value, or dependent variable.

Example 2.1.5A: Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month.

Solution

Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month.

The number of days in a month is a function of the name of the month, so if we name the function f, we write $\text{days} = f(\text{month})$ or $d = f(m)$. The name of the month is the input to a "rule" that associates a specific number (the output) with each input.

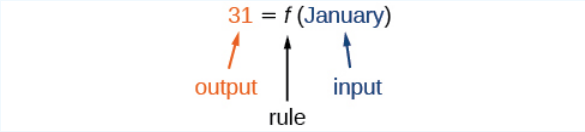


Figure 2.1.5: The function $31 = f(January)$ where 31 is the output, f is the rule, and January is the input.

For example, $f(\text{March}) = 31$, because March has 31 days. The notation $d = f(m)$ reminds us that the number of days, d (the output), is dependent on the name of the month, m (the input).

Analysis

Note that the inputs to a function do not have to be numbers; function inputs can be names of people, labels of geometric objects, or any other element that determines some kind of output. However, most of the functions we will work with in this book will have numbers as inputs and outputs.

Example 2.1.5B: Interpreting Function Notation

A function $N = f(y)$ gives the number of police officers, N, in a town in year y. What does $f(2005) = 300$ represent?

Solution

When we read $f(2005) = 300$, we see that the input year is 2005. The value for the output, the number of police officers (N), is 300. Remember, $N = f(y)$. The statement $f(2005) = 300$ tells us that in the year 2005 there were 300 police officers in the town.

Try It 2.1.5

Use function notation to express the weight of a pig in pounds as a function of its age in days d.

Answer

$w = f(d)$

Instead of a notation such as $y = f(x)$, could we use something like $y = y(x)$, meaning "y is a function of x?"

Yes, this is often done, especially in applied subjects that use higher math, such as physics and engineering. However, in exploring math itself we like to maintain a distinction between a function such as f, which is a rule or procedure, and the output y we get by applying f to a particular input x. This is why we usually use notation such as $y = f(x)$, $P = W(d)$, and so on.

Evaluate a Function Given a Formula

When we know an input value and want to determine the corresponding output value for a function, we evaluate the function. Evaluating will always produce one result because each input value of a function corresponds to exactly one output value.

When we have a function in formula form, it is usually a simple matter to evaluate the function. For example, the function $f(x) = 5 - 3x^2$ can be evaluated by squaring the input value, multiplying by 3, and then subtracting the product from 5.

Howto: Evaluate a function given its formula.

Given the formula for a function, evaluate.

1. Replace the input variable in the formula with the value provided.
2. Calculate the result.

Example 2.1.6A: Evaluate a polynomial function

Evaluate $f(x) = x^2 + 3x - 4$ at a. 2 b. a c. $a + 5$

Solution

Replace the x in the function with each specified value.

a. Because the input value is a number, 2, we can use simple algebra to simplify.

$$\begin{aligned} f(2) &= (2)^2 + 3(2) - 4 \\ &= 4 + 6 - 4 \\ &= 6 \end{aligned}$$

b. In this case, the input value is a letter so we cannot simplify the answer any further.

$$f(a) = (a)^2 + 3(a) - 4$$

c. With an input value of $a + 5$, we must use the distributive property.

$$\begin{aligned} f(a+5) &= (a+5)(a+5)^2 + 3(a+5) - 4 \\ &= a^2 + 2a(5) + 5^2 + 3a + 15 - 4 \\ &= a^2 + 13a + 36 \end{aligned}$$

Example 2.1.6*B*

For the function $g(x) = 3x - 5$, evaluate the function. a. $g(h^2+1)$ b. $g(x+2)$

Solution

a.

$$\begin{aligned} g(h^2+1) &= 3(h^2+1) - 5 && \text{To evaluate } g(h^2+1), \text{ substitute } h^2+1 \text{ for } x \\ &= 3h^2 + 3 - 5 && \text{Simplify.} \\ &= 3h^2 - 2 \end{aligned}$$

b.

$$\begin{aligned} g(x+2) &= 3(x+2) - 5 && \text{To evaluate } g(x+2), \text{ substitute } x+2 \text{ for } x \\ &= 3x + 6 - 5 && \text{Simplify.} \\ &= 3x + 1 \end{aligned}$$

Example 2.1.6*C*: Evaluate a function containing radicals

For the function $f(x) = \sqrt{2x-1}$, find a. $f(5)$ b. $f(-2)$ c. $f(8x^2+5)$

Solution:

a.

$$\begin{aligned} f(5)(5) &= \sqrt{2\cdot(5)-1} && \text{To evaluate } f(5), \text{ substitute 5 for } x \\ &= \sqrt{9} && \text{Simplify.} \\ &= 3 && \text{Take the square root.} \end{aligned}$$

b.

$$\begin{aligned} f(-2) &= \sqrt{2(-2)-1} && \text{To evaluate } f(-2), \text{ substitute } -2 \text{ for } x \\ &= \sqrt{-5} && \text{Simplify.} \\ &= i\sqrt{5} && \text{Take the square root.} \\ & && \text{This function does not have a real number value at } x = 2. \end{aligned}$$

c.

$$\begin{aligned} f(5) &= \sqrt{2(8x^2+(5))-1} && \text{To evaluate } f(8x^2+5), \text{ substitute } 8x^2+5 \text{ for } x \\ &= \sqrt{16x^2+10-1} && \text{Simplify.} \\ &= \sqrt{16x^2+9} && \text{This expression cannot be further simplified!!} \end{aligned}$$

Example 2.1.6*D*: Evaluate a rational function

For the function $f(x) = \dfrac{2-x}{x^2-1}$, find a. $f(4)$ b. $f(1)$ c. $f(x+3)$

Solution:

a.

$$\begin{aligned} f(4) &= \frac{2-(4)}{(4)^2-1} && \text{To evaluate } f(4), \text{ substitute 4 for } x \\ &= \frac{-2}{15} && \text{Simplify.} \end{aligned}$$

b.

$$\begin{aligned} f(1) &= \frac{2-1}{(1)^2-1} && \text{To evaluate } f(1), \text{ substitute 1 for } x \\ &= \frac{1}{0} && \text{Simplify.} \\ &= \text{Undefined} && \text{This function is undefined at } x=1. \end{aligned}$$

c.

$$\begin{aligned} f(x+3) &= \frac{2-(x+3)}{(x+3)^2-1} && \text{To evaluate } f(x+3), \text{ substitute } (x+3) \text{ for } x \\ &= \frac{2-x-3}{x^2+6x+9-1} && \text{Simplify.} \\ &= \frac{-x-1}{x^2+6x+8} && \text{Simplify (factor)} \\ &= -\frac{x+1}{(x+2)(x+4)} \end{aligned}$$

Try It 2.1.6: Evaluate functions

Given $f(x)=\sqrt{3x-2}$, $g(x)=4x-7$ and $h(x)=\dfrac{|x-3|}{x^2+x^3+1}$, evaluate the following. Simplify.

a. $f(6)$ b. $f(0)$ c. $g(m^2)$ d. $g(x-3)$ e. $h(2)$ f. $h(-x)$

Answer

a. $f(6)=4$ b. $f(0)=i\sqrt{2}$ c. $4m^2-7$ d. $4x-19$ e. $h(2)=\dfrac{1}{13}$ f. $h(-x)=\dfrac{|x+3|}{x^2-x^3+1}$

Evaluate a Function Given a Graph

Evaluating a function using a graph requires finding the corresponding output value for a given input value. We find the output value by looking at the graph. Solving a function equation using a graph requires finding all instances of the given output value on the graph and observing the corresponding input value(s).

Example 2.1.7: Reading Function Values from a Graph

Given the graph in Figure 2.1.7, a. Evaluate $f(2)$. b. For what values of x is $f(x)=4$?

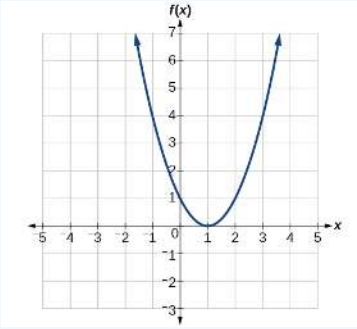
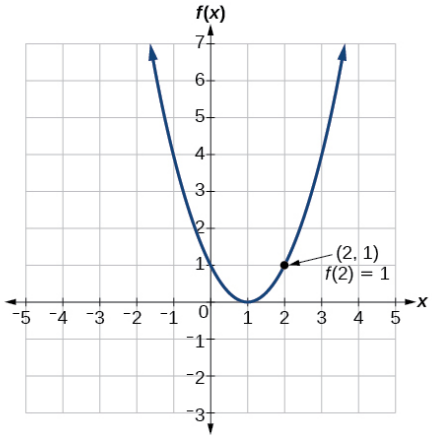


Figure 2.1.7: Graph of a positive parabola centered at $(1,0)$.

Solution

a. To evaluate $f(2)$, locate the point on the curve where $x=2$, then read the y-coordinate of that point. The point has coordinates $(2,1)$, so $f(2)=1$. See Figure 2.1.7*a*.

b. To find what values of x have a y—coordinate of 4, examine where the horizontal line $y = 4$ intersects the graph. We locate two points of the curve with output value 4: $(-1, 4)$ and $(3, 4)$. This means $f(-1) = 4$ and $f(3) = 4$ and the two x values that have a y-coordinate of 4 are $x = -1$ or $x = 3$. These points also represent the two solutions to $f(x) = 4$:See Figure 2.1.7*b*.



2.1.7*a*: Graph of a positive parabola centered at $(1, 0)$ with the labeled point $(2, 1)$ where $f(2) = 1$.

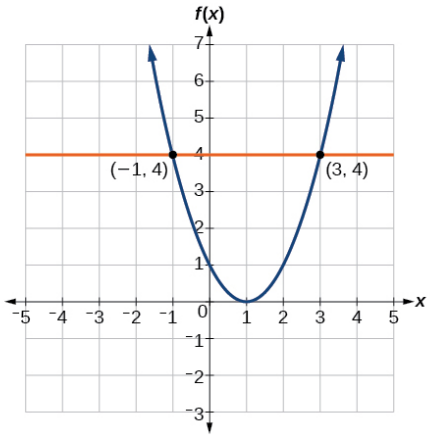


Figure 2.1.7*b*: Graph of an upward-facing parabola with a vertex at $(0, 1)$ and labeled points at $(-1, 4)$ and $(3, 4)$. A line at $y = 4$ intersects the parabola at the labeled points.

Try It 2.1.7

Given the graph in Figure 2.1.7, for what values of x is $f(x) = 1$?

Answer

$x = 0$ or $x = 2$

Find a function's input value given its output value and its formula

When we know an output value and want to determine the input values that would produce that output value algebraically, we set the output equal to the function's formula and solve for the input. Solving can produce more than one solution because different input values can produce the same output value.

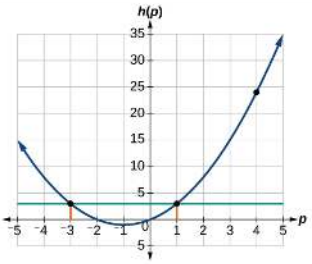
Example 2.1.8: Solving Functions

Given the function $h(p) = p^2 + 2p$, solve for $h(p) = 3$.

Solution

$$
\begin{aligned}
h(p) &= 3 \\
p^2 + 2p &= 3 && \text{Substitute the original function} \\
p^2 + 2p - 3 &= 0 && \text{Subtract 3 from each side.} \\
(p+3)(p-1) &= 0 && \text{Factor.} \\
p + 3 = 0 \;\big|\; & p - 1 = 0 && \text{Use zero factor property to solve} \\
p = -3 \;\big|\; & p = 1
\end{aligned}
$$

This gives us two solutions. The output $h(p) = 3$ when the input is either $p = 1$ or $p = -3$. We can also verify by graphing as shown in Figure 2.1.8. The graph verifies that $h(1) = h(-3) = 3$.



p	-3	-2	0	1	4
$h(p)$	3	0	0	3	24

Figure 2.1.8: Graph of $h(p) = p^2 + 2p$

Try It 2.1.8

Given the function $g(m) = \sqrt{m-4}$, solve $g(m) = 2$.

Answer

$m = 8$

Difference Quotients

An important concept in calculus involves at looking at a quantity called the **difference quotient** which measures the **average rate of change** of a function over an interval, and is used to find the slope of a function at a point on its graph.

Example 2.1.9A: Evaluate the difference quotient

Evaluate $\dfrac{f(a+h)-f(a)}{h}$ given $f(x) = x^2 + 3x - 4$

Solution

Replace the x in the function with each specified value.
Evaluate $f(a+h)$. The distributive property must be used.

$$\begin{aligned} f(a+h) &= (a+h)^2 + 3(a+h) - 4 \\ &= a^2 + 2ah + h^2 + 3a + 3h - 4 \end{aligned}$$

Evaluate $f(a)$

$$f(a) = a^2 + 3a - 4$$

Now combine the results and simplify.

$$\begin{aligned} \frac{f(a+h)-f(a)}{h} &= \frac{(a^2 + 2ah + h^2 + 3a + 3h - 4) - (a^2 + 3a - 4)}{h} \\ &= \frac{(2ah + h^2 + 3h)}{h} \\ &= \frac{h(2a + h + 3)}{h} \qquad \text{Factor out } h. \\ &= 2a + h + 3 \qquad \text{Simplify.} \end{aligned}$$

Example 2.1.9B: Evaluating the Difference Quotient

1. Evaluate the difference quotient $\dfrac{f(x+h)-f(x)}{h}$ for the following functions. In all cases simplify so that the denominator does not have a factor of h.

a. $f(x) = 2x^2 - 3x$

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{(2(x+h)^2-3(x+h))-(2x^2-3)}{h} && \text{Substitute.}\\
&= \frac{(2x^2+4xh+2h^2-3x-3h)-2x^2+3x}{h} && \text{Multiply out numerator.}\\
&= \frac{4xh+2h^2-3h}{h} && \text{Combine like terms.}\\
&= 4x+2h-3 && \text{Cancel}
\end{aligned}$$

b. $f(x)=\frac{x}{x-2}$. Simplify the complex fraction.

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{\frac{x+h}{x+h-2}-\frac{x}{x-2}}{h}\\
&= \frac{\frac{x+h}{x+h-2}-\frac{x}{x-2}}{h}\cdot\frac{(x+h-2)(x-2)}{(x+h-2)(x-2)} && \text{Multiply by a convenient 1}\\
&= \frac{(x+h)(x-2)-x(x+h-2)}{h(x+h-2)(x-2)} && \text{Multiply.}\\
&= \frac{x^2-2x+hx-2h-x^2-xh+2x}{h(x+h-2)(x-2)} && \text{Multiply out numerator.}\\
&= \frac{-2h}{h(x+h-2)(x-2)} && \text{Combine like terms.}\\
&= \frac{-2}{(x+h-2)(x-2)} && \text{Cancel}
\end{aligned}$$

c. $f(x)=\sqrt{5x+1}$. Rationalize the numerator.

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{\sqrt{5(x+h)+1}-\sqrt{5x+1}}{h}\\
&= \frac{\sqrt{5(x+h)+1}-\sqrt{5x+1}}{h} && \text{Rationalize the numerator}\\
&\quad\cdot\frac{\sqrt{5(x+h)+1}+\sqrt{5x+1}}{\sqrt{5(x+h)+1}+\sqrt{5x+1}}\\
&= \frac{5x+5h+1-(5x+1)}{h(\sqrt{5(x+h)+1}+\sqrt{5x+1})} && \text{Multiply out numerator}\\
&= \frac{5h}{h(\sqrt{5(x+h)+1}+\sqrt{5x+1})} && \text{Simplify}\\
&= \frac{5}{\sqrt{5(x+h)+1}+\sqrt{5x+1}} && \text{Cancel}
\end{aligned}$$

Other formulas used for the difference quotient are: $\frac{f(x)-f(a)}{x-a}$ and $\frac{f(a+h)-f(a)}{h}$

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2.1e: Exercises - Functions and Function Notation

A: Concepts

Exercise 2.1e.A

1) What is the difference between a relation and a function?
2) What is the difference between the input and the output of a function?
3) Why does the vertical line test tell us whether the graph of a relation represents a function?
4) How can you determine if a relation is a one-to-one function?
5) Why does the horizontal line test tell us whether the graph of a function is one-to-one?

Answers to odd exercises:

1. A relation is a set of ordered pairs. A function is a special kind of relation in which no two ordered pairs have the same first coordinate.

3. When a vertical line intersects the graph of a relation more than once, that indicates that for that input there is more than one output. At any particular input value, there can be only one output if the relation is to be a function.

5. When a horizontal line intersects the graph of a function more than once, that indicates that for that output there is more than one input. A function is one-to-one if each output corresponds to only one input.

B: Determine if a set of points is a function

Exercise 2.1e.B

★ **Determine whether the relation represents a function.**

7) $\{(3,4),(4,5),(5,6)\}$ 8) $\{(-1,-1),(-2,-2),(-3,-3)\}$ 9) $\{(10,4),(-1,1),(-2,4),(4,9)\}$	10) $\{(2,5),(7,11),(15,8),(7,9)\}$ 11) $\{(1,4),(1,5),(1,6)\}$ 12) $\{(4,1),(5,1),(6,1)\}$	13) $\{(a,b),(b,c),(c,c)\}$ 14) $\{(a,b),(c,d),(a,c)\}$

Answers to odd exercises:

7. function 9. function 11. not a function 13. function.

C: Determine if an Equation is a function

Exercise 2.1e.C

★ **Determine whether the relation represents y as a function of x.**

15) $x+y=25$ 16) $5x+2y=10$ 16.1) $\lvert y\rvert+x=2$ 16.2) $\lvert x\rvert-y=-3$ 17) $x=y^2$	18) $y=x^2$ 19) $3x^2+y=14$ 20) $2x+y^2=6$ 21) $y=-2x^2+40x$ 22) $x=y^2-8y+9$	23) $y^2=x^2$ 24) $y^3=x^2$ 25) $y=x^3$ 26) $x=y^3$	27) $y=\frac{1}{x}$ 28) $2xy=1$ 29) $y=\frac{3x+5}{7x-1}$ 30) $x=\frac{3y+5}{7y-1}$	31) $y=\pm\sqrt{1-x}$ 32) $x=\pm\sqrt{1-y}$ 33) $y=\sqrt{1-x^2}$ 34) $x=\sqrt{1-y^2}$

Answers to odd exercises:

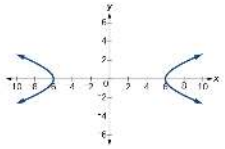
15. function 17. not a function 19. function 21. function 23. not a function
25. function 27. function 29. function 31. not a function 33. function

D: Determine if a graph is a function (Vertical Line Test)

Exercise 2.1e.D

★ **Use the vertical line text to identify graphs in which y is a function of x**

35)



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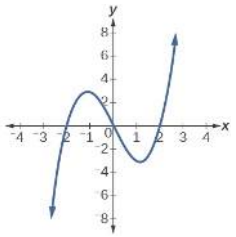

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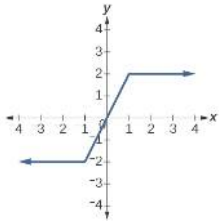

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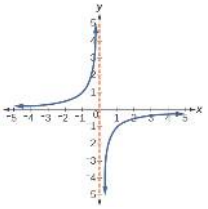

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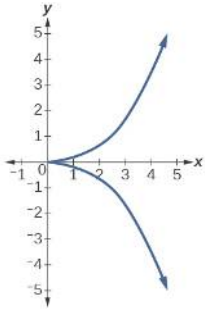

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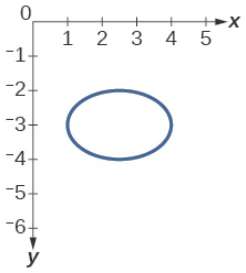

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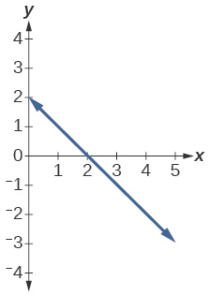

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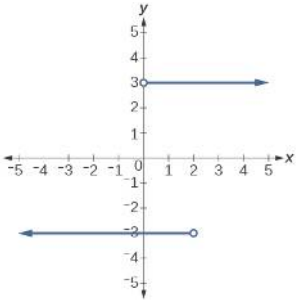

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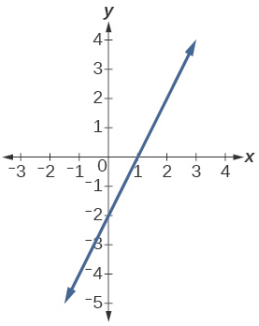

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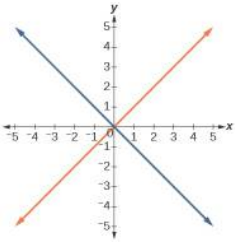

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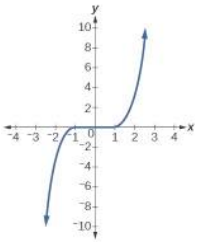

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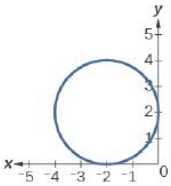

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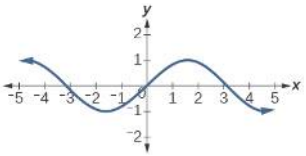

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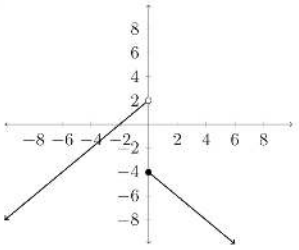


Answers to odd exercises:

35. not a function 37. function 39. not a function 41. function 43. function 45. function 47. function

E: Obtain Function Values from a formula

Exercise 2.1e.E

★ **Evaluate the function f at the values a. $f(-2)$, b. $f(-1)$, c. $f(0)$, d. $f(1)$, and e. $f(2)$.**

49) $f(x) = 8 - 3x$	51) $f(x) = 3 + \sqrt{x+3}$	53) $f(x) = 3^x$
50) $f(x) = 4 - 2x$	52) $f(x) = 8x^2 - 7x + 3$	54) $f(x) = \dfrac{x-2}{x+3}$

★ **Evaluate the function f at the values a. $f(-3)$, b. $f(2)$, c. $f(-a)$, d. $-f(a)$, e. $f(a+h)$.**

55) $f(x) = 2x - 5$	57) $f(x) = \sqrt{2-x} + 5$	59) $f(x) = \lvert x-1 \rvert - \lvert x+1 \rvert$	61) $f(x) = \dfrac{x^2-1}{x+3}$
56) $f(x) = -5x^2 + 2x - 1$	58) $f(x) = \sqrt{x^2+16} - 3$	60) $f(x) = \dfrac{\lvert x+3 \rvert}{x-3}$	62) $f(x) = \dfrac{1-x}{5x+2}$

★ For the following exercises, find the values listed below for each function, if they exist, then simplify.

a. $f(0)$, b. $f(1)$, c. $f(3)$, d. $f(-x)$, e. $f(a)$, f. $f(a+h)$

63) $f(x)=5x-2$ 64) $f(x)=4x^2-3x+1$	65) $f(x)=\dfrac{2}{x}$ 66) $f(x)=\lvert x-7\rvert+8$	67) $f(x)=\sqrt{6x+5}$ 68) $f(x)=\dfrac{x-2}{3x+7}$	69) $f(x)=9$

Answers to odd exercises:

49. a. $f(-2)=14$; b. $f(-1)=11$; c. $f(0)=8$; d. $f(1)=5$; e. $f(2)=2$
51. a. $f(-2)=4$; b. $f(-1)=4.414$; c. $f(0)=4.732$; d. $f(1)=5$; e. $f(2)=5.236$
53. a. $f(-2)=\frac{1}{9}$; b. $f(-1)=\frac{1}{3}$; c. $f(0)=1$; d. $f(1)=3$; e. $f(2)=9$
55. a. $f(-3)=-11$; b. $f(2)=-1$; c. $f(-a)=-2a-5$; d. $-f(a)=-2a+5$; e. $f(a+h)=2a+2h-5$
57. a. $f(-3)=\sqrt{5}+5$; b. $f(2)=5$; c. $f(-a)=\sqrt{2+a}+5$; d. $-f(a)=-\sqrt{2-a}-5$;
    e. $f(a+h)=\sqrt{2-a-h}+5$
59. a. $f(-3)=2$; b. $f(2)=1-3=-2$; c. $f(-a)=\lvert -a-1\rvert-\lvert -a+1\rvert$;
    d. $-f(a)=-\lvert a-1\rvert+\lvert a+1\rvert$; e. $f(a+h)=\lvert a+h-1\rvert-\lvert a+h+1\rvert$
61. a. $f(-3)=$ undefined; b. $f(2)=\frac{3}{5}$; c. $f(-a)=\frac{a^2-1}{3-a}$; d. $-f(a)=-\frac{a^2-1}{a+3}$; e. $f(a+h)=\frac{a^2+2ah+h^2-1}{a+h+3}$
63) a. -2 b. 3 c. 13 d. $-5x-2$ e. $5a-2$ f. $5a+5h-2$
65) a. Undefined b. 2 c. $\frac{2}{3}$ d. $-\frac{2}{x}$ e. $\frac{2}{a}$ f. $\frac{2}{a+h}$
67) a. $\sqrt{5}$ b. $\sqrt{11}$ c. $\sqrt{23}$ d. $\sqrt{-6x+5}$ e. $\sqrt{6a+5}$ f. $\sqrt{6a+6h+5}$
69) a. 9 b. 9 c. 9 d. 9 e. 9 f. 9

F: Determine a function's input or output

Exercise 2.1e.F

★ Determine the following function values.

71) Given the relation $3r+2t=18$. a. Write the relation as a function $r=f(t)$. b. Evaluate $f(-3)$. c. Solve $f(t)=2$. 72) Given the function $k(t)=2t-1$: a. Evaluate $k(2)$. b. Solve $k(t)=7$.	73) Given the function $f(x)=8-3x$: a. Evaluate $f(-2)$. b. Solve $f(x)=-1$. 74) Given the function $p(c)=c^2+c$: a. Evaluate $p(-3)$. b. Solve $p(c)=2$.	75) Given the function $f(x)=x^2-3x$: a. Evaluate $f(5)$. b. Solve $f(x)=4$. 76) Given the function $f(x)=\sqrt{x+2}$: a. Evaluate $f(7)$. b. Solve $f(x)=4$.

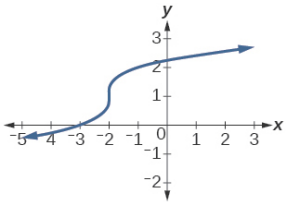
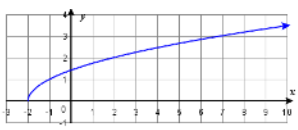
Answers to odd exercises:

71. a. $f(t)=6-\frac{2}{3}t$; b. $f(-3)=8$; c. $t=6$ 73. a. $f(-2)=14$; b. $x=3$ 75. a. $f(5)=10$; b. $x=-1$ or $x=4$

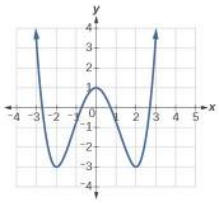
G: Obtain Function Values from a Graph

Exercise 2.1e.G

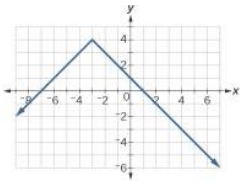
★ Use the graph of f to find each indicated function value.

79) Given the following graph, • Evaluate $f(0)$. • Solve for $f(x)=0$.		80) Given the following graph, • Evaluate $f(-1)$. • Solve for $f(x)=3$.	

81) Given the following graph,
- Evaluate $f(0)$.
- Solve for $f(x) = -3$.



82) Given the following graph,
- Evaluate $f(4)$.
- Solve for $f(x) = 1$.



Answers to odd exercises:

79. a. $f(0) \approx 2.3$; b. $f(x) = 0$, $x = -3$ 81. a. $f(0) = 1$; b. $f(x) = -3$, $x = -2$ or $x = 2$

H: Evaluate Difference Quotients

Exercise 2.1e.H

★ **Find and simplify the difference quotient $\frac{f(x+h)-f(x)}{h}$ for the given function. Simplify so h is not a factor in the denominator. This may involve simplifying complex fractions or rationalizing the numerator.**

83. $f(x) = 2x - 5$
84. $f(x) = -3x + 5$
85. $f(x) = 6$
86. $f(x) = 3x^2 - x$
87. $f(x) = -x^2 + 2x - 1$
88. $f(x) = 4x^2$
89. $f(x) = x - x^2$
90. $f(x) = x^3 + 1$
91. $f(x) = mx + b$ where $m \neq 0$
92. $f(x) = ax^2 + bx + c$ where $a \neq 0$
93. $f(x) = \frac{2}{x}$
94. $f(x) = \frac{3}{1-x}$
95. $f(x) = \frac{1}{x^2}$
96. $f(x) = \frac{2}{x+5}$
97. $f(x) = \frac{1}{4x-3}$
98. $f(x) = \frac{3x}{x+1}$
99. $f(x) = \frac{x}{x-9}$
100. $f(x) = \frac{x^2}{2x+1}$
101. $f(x) = \sqrt{x-9}$
102. $f(x) = \sqrt{2x+1}$
103. $f(x) = \sqrt{-4x+5}$
104. $f(x) = \sqrt{4-x}$
105. $f(x) = \sqrt{ax+b}$, where $a \neq 0$.
106. $f(x) = x\sqrt{x}$
107. $f(x) = \sqrt[3]{x}$.
HINT: $(a-b)(a^2+ab+b^2) = a^3 - b^3$
108. $g(x) = 5 - x^2$

109) Given the function $g(x) = x^2 + 2x$, evaluate $\frac{g(x)-g(a)}{x-a}$, $x \neq a$ and simplify.

Answers to odd exercises:

83. 2
85. 0
87. $-2x - h + 2$
89. $-2x - h + 1$
91. m
93. $\frac{-2}{x(x+h)}$
95. $\frac{-(2x+h)}{x^2(x+h)^2}$
97. $\frac{-4}{(4x-3)(4x+4h-3)}$
99. $\frac{-9}{(x-9)(x+h-9)}$
101. $\frac{1}{\sqrt{x+h-9}+\sqrt{x-9}}$
103. $\frac{-4}{\sqrt{-4x-4h+5}+\sqrt{-4x+5}}$
105. $\frac{a}{\sqrt{ax+ah+b}+\sqrt{ax+b}}$
107. $\frac{1}{(x+h)^{2/3}+(x(x+h))^{1/3}+x^{2/3}}$
109. $\frac{g(x)-g(a)}{x-a} = x + a + 2$, $x \neq a$

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2.2: Attributes of Functions

Domain and Range

A domain is the largest set of real numbers for which the value of an expression is a real number. Values that must be excluded from a domain include real numbers that cause division by zero and real numbers that result in taking an even root of a negative number. Often the domain is not explicitly stated. However, it is useful to be able to determine what the domain of an expression is.

Domain of a Function

The following steps can be taken to determine what the domain of a function defined by an equation actually is.

How to: Find the Domain of a Function Defined by an Equation.

Given a function, $f(x)$, start with the domain as the set of all real numbers.

1. **Radicals with even indexes**. If the equation has a radical with an even index (like square root has an index of 2), restrict the domain to only those values of x for which the expression inside the radical (called the **radicand**) is greater than or equal to zero. To obtain all the intervals that x must be restricted to, solve the **inequality**, **radicand** ≥ 0, for each radical with an even index. The result of all these restrictions is the intersection of these intervals.
2. **Denominators**. If the equation has a denominator, exclude from the domain any values of x that would make the denominator zero. To obtain all these restrictions, solve the **equation**, **denominator** $\neq 0$, for each denominator containing variables. Denominators include each expression found below a fraction bar in the equation. Every one of these excluded values must then be removed from the result obtained in step 1.
3. The result obtained in step 2 is the domain of the equation. Write it in interval form.

Example 2.2.1: Find the Domain of a Polynomial Function

Find the domain of the function $f(x) = x^2 - 1$.

Solution

- We start with a domain of all real numbers.
- Step 1. The function has no radicals with even indices, so no restrictions to the domain are introduced in this step.
- Step 2. The function has no denominators, so no restrictions to the domain are introduced in this step.
- Step 3. Thus there are no restrictions on the domain of this function. The domain is the set of real numbers. In interval form, the domain of f is $(-\infty, \infty)$.

Example 2.2.2: Find the Domain of a Rational Function

Find the domain of the function $f(x) = \dfrac{x+1}{2-x}$.

Solution

- We start with a domain of all real numbers.
- Step 1. The function has no radicals with even indices, so no restrictions to the domain are introduced in this step.
- Step 2. The function has a denominator, so the domain is restricted such that $2 - x \neq 0$. Solving this **equation** we obtain the restriction to the domiain: $x \neq 2$. In other words the domain is $x < 2$ or $x > 2$.
- Step 3. The domain is all real numbers where $x < 2$ or $x > 2$. We can use a symbol known as the union, $\cup$, to combine the two sets. In interval notation, the domain is $(-\infty, 2) \cup (2, \infty)$.

The root functions $f(x) = x^{1/n}$, which can be written as radicals with an index of n, have different characteristics depending on whether n is odd or even. For all even integers $n \geq 2$, the domain of $f(x) = x^{1/n}$ is the interval $[0, \infty)$. For all odd integers $n \geq 1$, the

domain of $f(x) = x^{1/n}$ is the set of all real numbers. This is the reason why domains of radical expressions with even indices are restricted and why domains of radical expressions with odd indices have no restrictions.

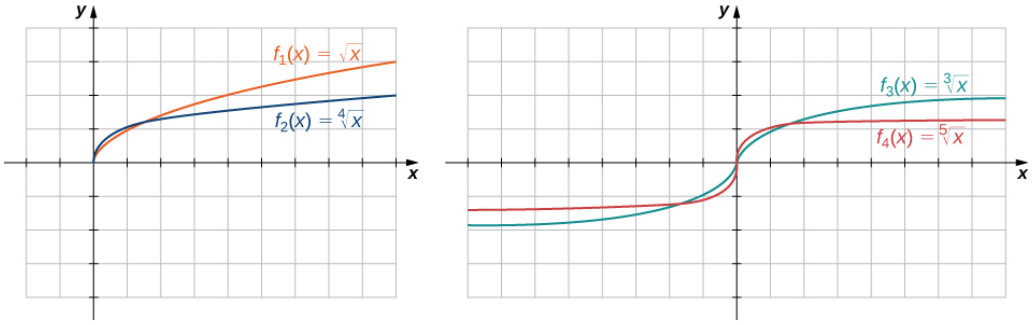


Figure 2.2.7: (a) If n is even, the domain of $f(x) = \sqrt[n]{x}$ is $[0, \infty)$. (b) If n is odd, the domain of $f(x) = \sqrt[n]{x}$ is $(-\infty, \infty)$.

Example 2.2.3: Find the Domain of a Radical Function

Find the domain of the function $f(x) = \sqrt{7 - x}$.

Solution

- We start with a domain of all real numbers.
- Step 1. The function has a square root, which is a radical with an even index, so the domain is restricted to values of x such that $7 - x \geq 0$. Solving this **inequality** we obtain $-x \geq -7$ or $x \leq 7$. At this point the domain has been restricted to all real numbers less than or equal to 7.
- Step 2. The function has no denominators, so no further restrictions to the domain are introduced in this step.
- Step 3. The domain is the set of real numbers less than or equal to 7. In interval form, the domain of f is $(-\infty, 7]$.

Example 2.2.4: Find Domains of Functions

For each of the following functions, determine the domain of the function.

a. $f(x) = \dfrac{3}{x^2 - 1}$

b. $f(x) = \dfrac{2x + 5}{3x^2 + 4}$

c. $f(x) = \sqrt[3]{2x - 1}$

Solution

a. The function has a denominator and you cannot divide by zero, so the domain is the set of values x such that $x^2 - 1 \neq 0$. Therefore, the domain is $\{x | x \neq \pm 1\}$.

b. The function has a denominator and you cannot divide by zero, so the domain is the set of values x such that $3x^2 + 4 \neq 0$. Solving this inequality, we obtain $x^2 \neq -\frac{4}{3}$. Since $x^2 \geq 0$ for all real numbers x, the inequality $x^2 \neq -\frac{4}{3}$ will be true for any real number chosen for x. Therefore, the domain is $(-\infty, \infty)$.

c. The function does not have an even index (it has a cube root, which has an odd index of 3, and is defined for all real numbers). The function also does not have a denominator. So the domain is the interval $(-\infty, \infty)$.

Try It 2.2.5

Find the domain of the following functions.

a. $f(x)=5-x+x^3$	b. $f(x)=\dfrac{1+4x}{2x-1}$	c. $f(x)=\sqrt{4-3x}$
a. Answer $(-\infty,\infty)$	**b. Answer** $(-\infty,\frac{1}{2})\cup(\frac{1}{2},\infty)$	**c. Answer** $(-\infty,4/3]$

Example 2.2.6: More Domains of Functions

For each of the following functions, determine the domain of the function.

a. $f(x)=\sqrt{x-3}+\sqrt{8-x}$

Solution:

$x-3\geq 0$	AND	$8-x\geq 0$	Radicands of square roots must be greater than or equal to zero
$x\geq 3$		$x\leq 8$	The domain is the intersection of these intervals
			The domain is: $[3,8]$

b. $f(x)=\dfrac{x+1}{x-4}+\dfrac{5x}{x+6}$

Solution:

$x-4\neq 0$	AND	$x+6\neq 0$	Denominators cannot be equal to zero
$x\neq 4$		$x\neq -6$	The domain does not include these values
			The domain is: $(\infty,-6)\cup(-6,4)\cup(4,\infty)$

c. $f(x)=\dfrac{x-9}{\sqrt{x+2}}$

Solution:

$x+2\geq 0$			Radicands of square roots must be greater than or equal to zero
		$\sqrt{x+2}\neq 0$	Denominators cannot be equal to zero
$x\geq -2$	AND	$x\neq -2$	Both are restrictions to the domain
			The domain is: $(-2,\infty)$

d. $f(x)=\dfrac{1+\frac{x+2}{x-3}}{4-\sqrt{x+5}}$

Solution:

$x+5\geq 0$					Step 1. Radicands of even roots must be ≥ 0
$x\geq -5$					Resulting Restricted Interval
		$x-3\neq 0$	AND	$4-\sqrt{x+5}\neq 0$	Step 2. Denominators cannot be equal to zero
		$x\neq 3$	AND	$4\neq\sqrt{x+5}$	Simplify
				$16\neq x+5$	Simplify (square both sides)
		$x\neq 3$	AND	$x\neq 11$	Restrictions due to denominators
$x\geq -5$	AND	$x\neq 3$	AND	$x\neq 11$	All 3 restrictions must be true
					The domain is: $(-5,3)\cup(3,11)\cup(11,\infty)$

Try It 2.2.7

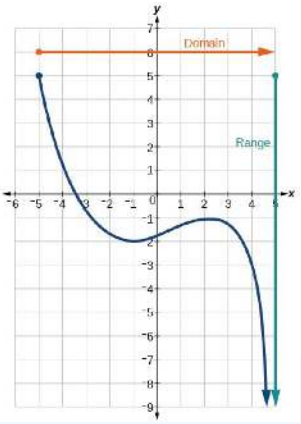
Find the domain for each of the following functions: $f(x)=(5-2x)/(x^2+2)$ and $g(x)=\sqrt{5x-1}$.

Answer

The domain of f is $(-\infty,\infty)$. The domain of g is $[1/5,\infty)$

Domain and Range of a Graph

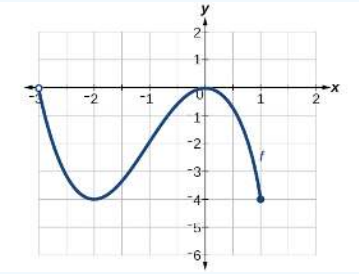
Another way to identify the domain and range of functions is by using graphs. Because the domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the x-axis. The range is the set of possible output values, which are shown on the y-axis. Keep in mind that if the graph continues beyond the portion of the graph we can see, the domain and range may be greater than the visible values. This is illustrated in the figure at the right.



We can observe that the graph extends horizontally from −5 to the right without bound, so the domain is $[-5, \infty)$. The vertical extent of the graph is all range values 5 and below, so the range is $(-\infty, 5]$. Note that the domain and range are always written from smaller to larger values, or from left to right for domain, and from the bottom of the graph to the top of the graph for range.

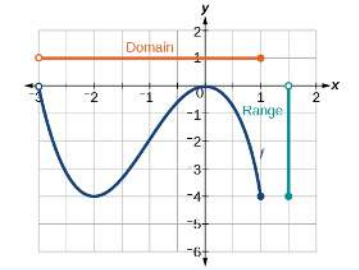
Example 2.2.8: Find Domain and Range from a Graph

Find the domain and range of the function f whose graph is shown below



Solution

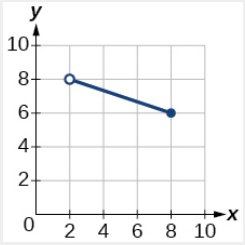
We can observe that the horizontal extent of the graph is –3 to 1, so the domain of f is $(-3, 1]$. The vertical extent of the graph is between 0 and –4, so the range is $[-4, 0]$. (Because (0,0) is a point on the graph, $y = 0$ needs to be included in the range). This is illustrated below.



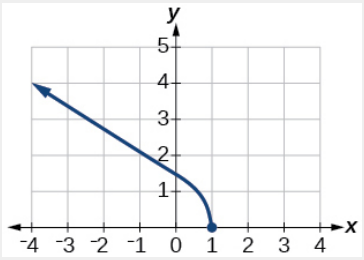
Try It 2.2.8

Find the domain and range of the functions illustrated in the graphs below.

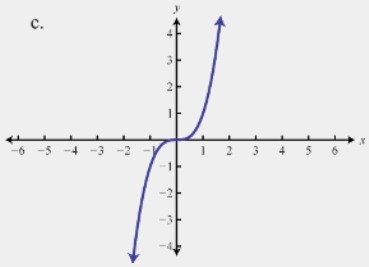
a)



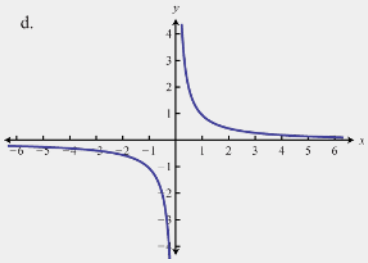
b)



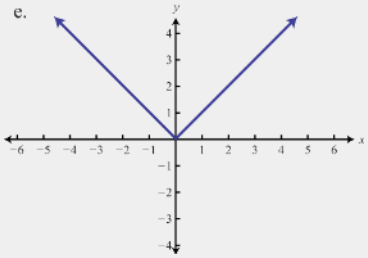
c)



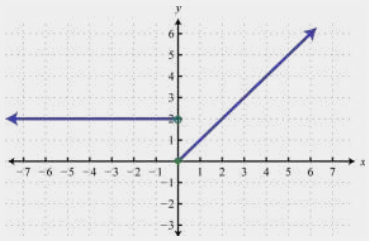
d) Asymptotes at x=0 and y=0



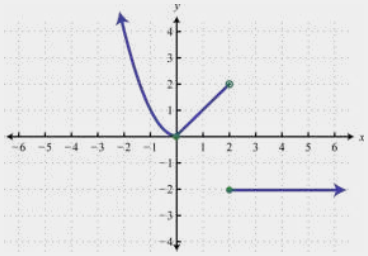
e)



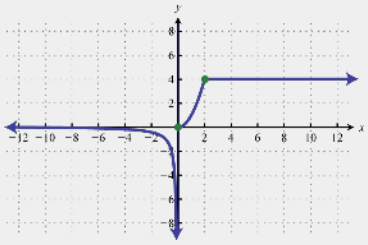
f) Open circle at (0,2)



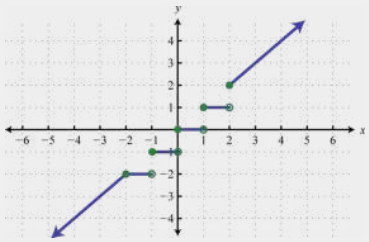
g) Open circles at (2,2) and (2,-2)



h) Asymptote at $x = 0$, filled circle at (0,0)



i) Each horizontal segment ends in an open circle



a. Answer

Domain: $(2, 8]$
Range: $[6, 8)$

b. Answer

Domain: $(-\infty, 1]$
Range: $[0, \infty)$

c. Answer

Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

d. Answer

Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$

e. Answer

Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

f. Answer

Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

g. Answer

Domain: $(-\infty, 2) \cup (2, \infty)$
Range: $[-2, -2] \cup [0, \infty)$

h. Answer

Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

i. Answer

Domain: $(-\infty, \infty)$
Range: $(-\infty, -2] \cup [-1, -1] \cup [0, 0] \cup [1, 1] \cup [2, \infty)$

Construct a Graph from a Table of Values

One way to describe relations is with equations. Visualizations of equations can be done using graphs.

The Fundamental Graphing Principle

The graph of an equation is the set of points which satisfy the equation. That is, a point (x, y) is on the graph of an equation if and only if x and y satisfy the equation.

Here, `$x$ and $y$ satisfy the equation' means `x and y make the equation true'. If the equation to be graphed contains both x and y, then the equation defines the relationship between the two variables. The points (x, y) that we graph are the pairs of x and y which make the equation true.

Example 2.2.9: Determine if a point is on a graph

Determine whether or not $(2, -1)$ is on the graph of $x^2 + y^3 = 1$

Solution. We substitute $x = 2$ and $y = -1$ into the equation to see if the equation is satisfied.

$$\begin{aligned} (2)^2 + (-1)^3 &\stackrel{?}{=} 1 \\ 3 &\neq 1 \end{aligned}$$

Hence, $(2, -1)$ is **not** on the graph of $x^2 + y^3 = 1$ because substituting $x = 2$ and $y = -1$ into the equation results in a statement that is **not** true.

Example 2.2.10: Construct a graph using a table of values

Graph $x^2 + y^3 = 1$

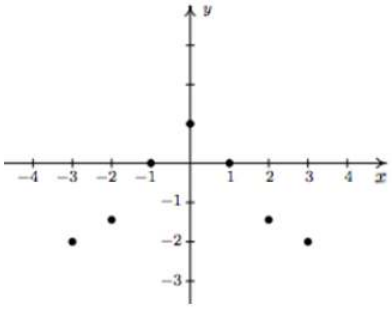
Solution. Step 1. To efficiently generate points on the graph of this equation, we first solve for y

$$\begin{aligned} x^2 + y^3 &= 1 \\ y^3 &= 1 - x^2 \\ \sqrt[3]{y^3} &= \sqrt[3]{1 - x^2} \\ y &= \sqrt[3]{1 - x^2} \end{aligned}$$

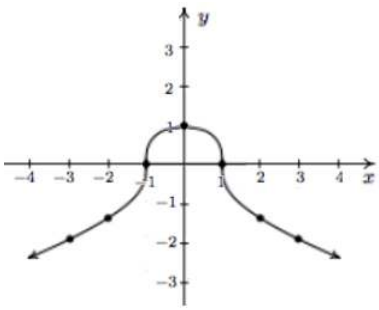
Step 2. Choose values for x and calculate corresponding values for y. Plot the resulting point (x, y). For example, substituting $x = -3$ into the equation yields $y = \sqrt[3]{1 - x^2} = \sqrt[3]{1 - (-3)^2} = \sqrt[3]{-8} = -2$, so the point $(-3, -2)$ is on the graph. Continuing in this manner, we generate a table of points which are on the graph of the equation. These points are then plotted in the plane as shown below.

x	y	(x, y)
-3	-2	$(-3, -2)$
-2	$-\sqrt[3]{3}$	$(-2, -\sqrt[3]{3})$
-1	0	$(-1, 0)$
0	1	$(0, 1)$
1	0	$(1, 0)$
2	$-\sqrt[3]{3}$	$(2, -\sqrt[3]{3})$
3	-2	$(3, -2)$

Table of values



Plotted Points



Connected 'Dots'

These points constitute only a small sampling of the points on the graph of this equation. To get a better idea of the shape of the graph, we could plot more points until we feel comfortable `connecting the dots'. Doing so would result in a curve similar to the one pictured above on the far right.

It should be mentioned here that it is entirely possible to choose a value for x which does not correspond to a point on the graph. For example, if we were plotting points for the equation $y = \sqrt{1-(x-2)^2}$ and substituted $x = 0$ into the equation, we would obtain $y = \sqrt{1-(0-2)^2} = \sqrt{1-4} = \sqrt{-3}$, which is not a real number. This means there are no points on the graph with an x-coordinate of 0. When this happens, we move on and try another point. This is a drawback of the `plug-and-plot' approach to graphing equations. Soon we will develop techniques which allow us to graph entire families of equations quickly (and without the use of a calculator!)

Zeros and Intercepts

Of all of the points on the graph of an equation, the places where the graph touches or crosses the axes hold special significance. These are called the intercepts of the graph. Intercepts come in two distinct varieties: x-intercepts and y-intercepts. They are defined below.

Definition: Zeros and Intercepts

Zeros are all the real and imaginary values of x that are solutions to the equation $y = 0$. Zeros are typically listed in set notation.

x-intercepts are the points on the x axis where a graph touches or crosses the x axis. They are all the real number values of x for which $y = 0$. The y coordinate corresponding to an x-intercept is always zero. x-intercepts are written as coordinate points in the form $(a, 0)$.

y-intercepts are the points on the y axis where a graph touches or crosses the y axis. They are all the real number values of y for which $x = 0$. The x coordinate corresponding to an y-intercept is always zero. Functions can have at most one y-intercept. y -intercepts are written as coordinate points in the form $(0, b)$.

In our previous example the graph had two x-intercepts, $(-1, 0)$ and $(1, 0)$, and one y-intercept, $(0, 1)$. The graph of an equation can have any number of intercepts, including none at all!

Zeros and Intercepts from a Graph

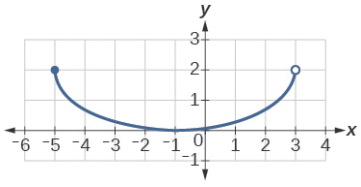
How to: Find Zeros and Intercepts from a graph.

- x-intercepts are the points $(a, 0)$ where the graph touches or crosses the x-axis
- y-intercepts are the points $(0, b)$ where the graph touches or crosses the y-axis
- Only real number zeros are visible on a graph. Imaginary zeros are not visible on a graph.

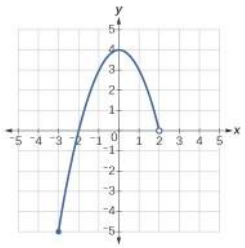
Example 2.2.11: Zeros and Intercepts on a Graph

For each function graphed below, find the zeros, x-intercepts, and y-intercept.

a.



b.



Solution. Graphs only show real number zeros. A graph cannot supply any information about the existence of imaginary zeros.

a. visible zeros: { -1 }, x-intercepts: (-1, 0), y-intercept: (0, .1) approximately
b. visible zeros: { -2 }, x-intercepts: (-2, 0), y-intercept: (0, 4)

Zeros and Intercepts from an Equation

Since x-intercepts lie on the x-axis, we can find them by setting $y = 0$ in the equation. Similarly, since y-intercepts lie on the y-axis, we can find them by setting $x = 0$ in the equation. Keep in mind, intercepts are *points* and therefore must be written as ordered pairs. To summarize,

How to: Find Zeros and Intercepts given an Equation.

Given an equation involving x and y

- Zeros are written as the set $\{z_1, z_2, \dots\}$ of all the solutions obtained for x when $y = 0$ is substituted into the equation.
- x-intercepts have the form $(x, 0)$. Set $y = 0$ in the equation and solve for x. The x-intercepts are the real number solutions found. Imaginary number solutions and solutions that are not in the domain are NOT included.
- y-intercepts have the form $(0, y)$. Set $x = 0$ in the equation and solve for y. The y-intercepts are the real number solutions found.

Example 2.2.12: Find Zeros and Intercepts given an Equation

Find the x- and y-intercepts (if any) of the graph of $(x-2)^2 + y^2 = 1$.

Solution

To look for x-intercepts, we set $y = 0$ and solve to find the zeros:

$$\begin{aligned} (x-2)^2 + y^2 &= 1 \\ (x-2)^2 + 0^2 &= 1 \\ (x-2)^2 &= 1 \\ \sqrt{(x-2)^2} &= \sqrt{1} \quad \text{extract square roots} \\ x - 2 &= \pm 1 \\ x &= 2 \pm 1 \\ x &= 3, 1 \end{aligned}$$

We get two answers for x so the zeros are $\{1,\ 3\}$, which correspond to two x-intercepts: $(1, 0)$ and $(3, 0)$.

Turning our attention to y-intercepts, we set $x = 0$ and solve:

$$\begin{aligned} (x-2)^2 + y^2 &= 1 \\ (0-2)^2 + y^2 &= 1 \\ 4 + y^2 &= 1 \\ y^2 &= -3 \end{aligned}$$

Since there is no real number which squares to a negative number, we are forced to conclude that the graph has no y-intercepts.

Zeros and Intercepts from a Function

How to: Find Zeros and Intercepts given a Function.

Given a function $f(x)$

- Zeros are written as the set $\{z_1, z_2, \dots\}$ of all the solutions obtained for the equation $f(x) = 0$. Results that are not in the domain are not considered to be solutions.
- x-intercepts have the form $(x, 0)$. x-intercepts include only the subset of zeros that are real numbers. Imaginary zeros are not included.
- The y-intercept is the point $(0, f(0))$. A function has at most one y-intercept.

Example 2.2.13: Find Zeros and Intercepts given a Function

For each function below, (a) Find all the zeros and x intercepts of f, and (b) Find the y-intercept (if any).

1. $f(x) = -4x + 2$.
2. $f(x) = \sqrt{x+3} + 1$.

Solution

1. (a) To find the zeros, solve $f(x) = 0$: $-4x + 2 = 0 \longrightarrow 2 = 4x \longrightarrow x = 1/2$.
   The zeros of f are $\{1/2\}$. The x-intercept is $(1/2, 0)$.

   (b) The y-intercept is given by $(0, f(0))$. $f(0) = -4(0) + 2 = 2$. Therefore the y-intercept is $(0, 2)$.

2. (a) To find the zeros, solve $\sqrt{x+3} + 1 = 0$. This equation implies $\sqrt{x+3} = -1$.
   Since $\sqrt{x+3} \geq 0$ for all x, this equation has no solutions, and therefore f has no zeros and no x-intercepts.

   (b) The y-intercept is given by $(0, f(0)) = (0, \sqrt{3} + 1)$.

Example 2.2.14

For each function below, find the zeros, x-intercepts, and y-intercept.

a. $f(x) = 1 - x + \sqrt{2x+6}$

Solution: Solve: $1 - x + \sqrt{2x+6} = 0$

Zeros: $\rightarrow \sqrt{2x+6} = x - 1 \rightarrow 2x + 6 = x^2 - 2x + 1$ Zeros: $\{5\}$

$\rightarrow x^2 - 4x - 5 = 0 \rightarrow (x-5)(x+1) = 0 \rightarrow x = 5, x = -1$ x-intercepts: $(5,0)$

$x = -1$ must be rejected because $\sqrt{2(-1)+6} \neq -1 - 1$

y-intercept: $f(0) = 1 - 0 + \sqrt{0+6} + 1$ y-intercept: $(0, 1 + \sqrt{6})$

In this example notice that because the process of solving involved squaring, the solutions ($x = -1$ and $x = 5$) had to be checked.

b. $g(x) = \sqrt{8x-1}$

Solution:

Zeros: Solve: $\sqrt{8x-1} = 0$ Zeros: $\{\frac{1}{8}\}$

$8x - 1 = 0$ x-intercepts: $(\frac{1}{8}, 0)$

$8x = 1 \rightarrow x = 1/8$

y-intercept: $f(0) = \sqrt{0-1} = i$ y-intercept: none

c. $g(x) = \dfrac{3}{x}$

Solution:

Zeros: Solve: $\dfrac{3}{x} = 0$ Zeros: None

$(x) \cdot \dfrac{3}{x} = 0 \cdot (x)$ x-intercepts: None

$3 = 0 \rightarrow$ No solution

y-intercept: $f(0) = \dfrac{3}{0} =$ Undefined y-intercept: None

d. $h(x) = \dfrac{x^2+9}{x^2+4}$

Solution:

Zeros: Solve: $\dfrac{x^2+9}{x^2+4} = 0$ Zeros: $\{3i, -3i\}$

$(x^2+4) \cdot \dfrac{x^2+9}{x^2+4} = 0 \cdot (x^2+4)$ x-intercepts: None

$x^2 + 9 = 0 \rightarrow x^2 = -9 \rightarrow x = \pm\sqrt{-9}$

y-intercept: $f(0) = \dfrac{0+9}{0+4} = \dfrac{9}{4}$ y-intercept: $\left(0, \dfrac{9}{4}\right)$

Try It 2.2.15: Find zeros and Intercepts given a function

Find the zeros of $f(x) = x^3 - 5x^2 + 6x$.

Answer

$x = 0, 2, 3$

Symmetry

The graphs of certain functions have symmetry properties that help us understand the function and the shape of its graph.

Definition: Symmetry

- **Symmetry with respect to the x-axis**. For every point (x, y) on the graph, point $(x, -y)$ is also on the graph.
- **Symmetry with respect to the y-axis**. For every point (x, y) on the graph, point $(-x, y)$ is also on the graph.
- **Symmetry with respect to the origin**. For every point (x, y) on the graph, point $(-x, -y)$ is also on the graph.

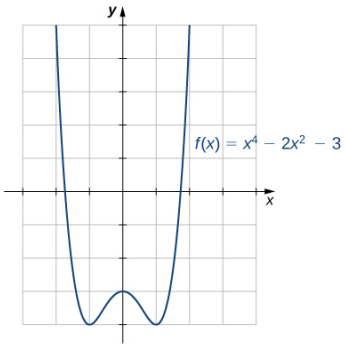
Symmetry from a Graph

Consider the function $f(x) = x^4 - 2x^2 - 3$ shown in the figure **2.2.16a** below. If we take the part of the curve that lies to the right of the y-axis and flip it over the y-axis, it lies exactly on top of the curve to the left of the y-axis. In this case, we say the function has **symmetry about the y-axis**.

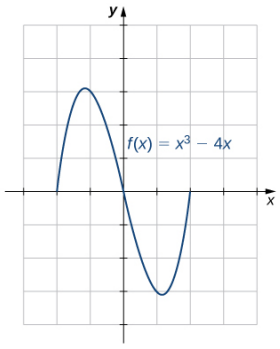
On the other hand, consider the function $f(x) = x^3 - 4x$ shown in Figure **2.2.16b**. If we take the graph and rotate it $180°$ about the origin, the new graph will look exactly the same. In this case, we say the function has **symmetry about the origin**.

If we take the part of the curve in in Figure **2.2.16c** that is above the x-axis and flip it over the x-axis, it will lie exactly on top of the curve that is below the x-axis. In this case, the graph has **symmetry about the x-axis**. Unlike the other two examples, the graph is not that of a function, since it fails the vertical line test. Graphs with x-axis symmetry are never functions.

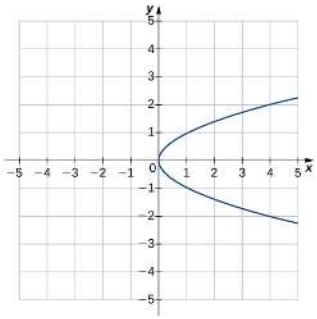
Figure **2.2.16**: Illustration of various types of symmetry



(a) Symmetry about the y-axis



(b) Symmetry about the origin

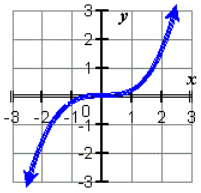


(c) Symmetry about the x-axis

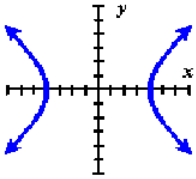
Example 2.2.16

Determine the type(s) of symmetry for each graph.

1.



2.



Solution: 1. Origin symmetry, 2. x-axis symmetry, y-axis symmetry, and origin symmetry

Symmetry of an Equation

We have defined symmetry from a graphical perspective, but the ideas can also be applied to equations. For example, the graph we obtained for $x^2+y^3=1$ in Example **2.2.10** appears to be symmetric about the y-axis. To actually show this algebraically, we assume (x,y) is a generic point on the graph of the equation. That is, we assume $x^2+y^3=1$ is true. The point symmetric to (x,y) about the y-axis is $(-x,y)$ To show that the graph is symmetric about the y-axis, we need to show that $(-x,y)$ satisfies the equation $x^2+y^3=1$ too. Substituting $(-x,y)$ into the equation gives

$$\begin{aligned}(-x)^2+(y)^3 &\stackrel{?}{=} 1\\ x^2+y^3 &\stackrel{\checkmark}{=} 1\end{aligned}$$

Since we are assuming the original equation $x^2+y^3=1$ is true, we have shown that $(-x,y)$ satisfies the equation (since it leads to a true result) and hence is on the graph.

In this way, we can check whether the graph of a given equation possesses x-axis, y-axis, or origin symmetry.

How to: Test the Symmetry of an Equation

Given an equation in x and y, perform each of the following substitutions to assess the symmetry of the graph of the given equation.

- **y-axis symmetry** exists if substituting $-x$ for x results in an equivalent equation.
- **x-axis symmetry** exists if substituting $-y$ for y results in an equivalent equation.
- **origin symmetry** exists if substituting both $-x$ for x and $-y$ for y results in an equivalent equation.

It is possible for the graph of an equation to have all three types of symmetry.

Example 2.2.17: Find Symmetry given an Equation

Determine the symmetry of the graphs for each equation.

a. $9x^2=4y^2+18xy+27$
b. $|y^3+2y|=4|x^2-5|$

Solution

a. $9x^2=4y^2+18xy+27$ **Answer**: The graph of the equation has origin symmetry:

1. Test for y-axis symmetry. $9(-x)^2=4y^2+18(-x)y+27 \rightarrow 9x^2=4y^2-18xy+27$, which is not an equivalent equation, so the graph of the equation does not have y-axis symmetry.
2. Test for x-axis symmetry. $9x^2=4(-y)^2+18x(-y)+27 \rightarrow 9x^2=4y^2-18xy+27$, which is not an equivalent equation, so the graph of the equation does not have x-axis symmetry.
3. Test for origin symmetry. $9(-x)^2=4(-y)^2+18(-x)(-y)+27 \rightarrow 9x^2=4y^2+18xy+27$, which is an equivalent equation, so the graph of the equation has origin symmetry.

b. $|y^3+2y|=4|x^2-5|$ **Answer**: The graph of the equation has y-axis, x-axis and origin symmetry:

1. Test for y-axis symmetry. $|y^3+2y|=4|(-x)^2-5| \rightarrow |y^3+2y|=4|x^2-5|$, which is an equivalent equation, so the graph of the equation has y-axis symmetry.
2. Test for x-axis symmetry. $|(-y)^3+2(-y)|=4|x^2-5| \rightarrow |-y^3-2y|=4|x^2-5| \rightarrow |-(y^3+2y)|=4|x^2-5| \rightarrow |y^3+2y|=4|x^2-5|$, which is an equivalent equation, so the graph of the equation has x-axis symmetry.
3. Test for origin symmetry produces an equivalent equation, so the graph of the equation has origin symmetry: $|(-y)^3+2(-y)|=4|(-x)^2-5| \rightarrow |-y^3-2y|=4|x^2-5| \rightarrow |-(y^3+2y)|=4|x^2-5| \rightarrow |y^3+2y|=4|x^2-5|$

Note that in a situation where an equation has been shown to have two of the three types of symmetry, then it must have all three types of symmetry.

Try It 2.2.17: Test an equation for symmetry

Test the equation $(x-2)^2+y^2=1$ for symmetry.

Answer

The graph is symmetric about the x-axis. This means we can cut our `plug and plot' time to graph the equation in half: whatever happens below the x-axis is reflected above the x-axis, and vice-versa.

Symmetry of a Function; Odd and Even Functions

Sometimes we are given a function in the form $y=f(x)$. Looking at Figure 2.2.16a, we see that since f is symmetric about the y-axis, if the point (x,y) is on the graph, the point $(-x,y)$ is also on the graph. In other words, $f(x)=y$ and $f(-x)=y$ so $f(-x)=f(x)$. If a function f has this property, we say f is an **even function**. All even functions have y-axis symmetry. The function $f(x)=x^2$ is even because $f(-x)=(-x)^2=x^2=f(x)$.

In contrast, looking at Figure 2.2.16b, if a function f is symmetric about the origin, then whenever the point (x,y) is on the graph, the point $(-x,-y)$ is also on the graph. In other words, $f(x)=y$ and $f(-x)=-y$. Since $f(x)=y$ can be rewritten $-f(x)=-y$, we can conclude (f(−x)=−f(x)\). If f has this property, we say f is an **odd function**. All odd functions have symmetry about the origin. The function $f(x)=x^3$ is odd because $f(-x)=(-x)^3=-x^3=-f(x)$.

Definition: Even and Odd Functions

- If $f(x)=f(-x)$ for all x in the domain of f, then f is an ***even* function**. An even function is **symmetric about the y-axis**.
- If $f(-x)=-f(x)$ for all x in the domain of f, then f is an ***odd* function**. An odd function is **symmetric about the origin**.

Looking at these definitions, it is clear that being able to simplify various functions that have an argument of $-x$ is important. The practice exercise below illustrates some of these needed skills.

Try It 2.2.18

Simplify the following expressions if possible. Write the final result as an expression that does not have any parentheses.

a. $(-x)$ b. $(-x)^2$	c. $(-x)^3$ d. $(-x)^4$	e. $(-x)^5$ f. $\lvert(-x)\rvert$	g. $\sqrt{(-x)}$ h. $\sqrt[3]{(-x)}$	i. $\sqrt[4]{(-x)}$ j. $\sqrt[5]{(-x)}$

Answer

a. $-x$ b. x^2 c. $-x^3$ d. x^4 e. $-x^5$ f. $|x|$ g. $\sqrt{-x}$ h. $-\sqrt[3]{x}$ i. $\sqrt[4]{-x}$ j. $-\sqrt[5]{x}$

How to: Determine if a Function is Even or Odd, and what Kind of Symmetry it Has.

1. Evaluate $f(-x)$ and simplify it.
2. Compare the result with $f(x)$ and $-f(x)$
   a. If $f(x)=f(-x)$ then f is an ***even* function** and has **y-axis symmetry**.

b. If $f(-x) = -f(x)$ then f is an ***odd* function** and has **origin symmetry**.

Any **function** (except the function $f(x) = 0$) cannot be both even and odd.
Any **function** (except the function $f(x) = 0$) can never have x-axis symmetry.

Example 2.2.19: Even and Odd Functions

Determine whether each of the following functions is even, odd, or neither, and state the type of symmetry it has.

a. $f(x) = -5x^4 + 7x^2 - 2$
b. $f(x) = 2x^5 - 4x + 5$
c. $f(x) = \dfrac{3x}{x^2+1}$

Solution

To determine whether a function is even or odd, we evaluate $f(-x)$ and compare it to $f(x)$ and $-f(x)$.

a. $f(-x) = -5(-x)^4 + 7(-x)^2 - 2 = -5x^4 + 7x^2 - 2 = f(x)$. Therefore, f is even, and symmetric about the y-axis.
b. $f(-x) = 2(-x)^5 - 4(-x) + 5 = -2x^5 + 4x + 5$. Now, $f(-x) \neq f(x)$. Furthermore, noting that $-f(x) = -2x^5 + 4x - 5$, we see that $f(-x) \neq -f(x)$. Therefore, f is neither even nor odd, and so has neither y-axis nor origin symmetry.
c. $f(-x) = \dfrac{3(-x)}{(-x)2+1} = \dfrac{-3x}{x^2+1} = -\dfrac{3x}{x^2+1} = -f(x)$. Therefore, f is odd, and is symmetric about the origin.

Try It 2.2.19

Determine whether $f(x) = 4x^3 - 5x$ is even, odd, or neither.

Answer

$f(x)$ is odd.

More Attributes Observable in a Graph

Increasing, Decreasing, or Constant

As part of exploring how functions change, we can identify intervals over which the function is changing in specific ways. We say that a function is increasing on an interval if the function values increase as the input values increase within that interval. Similarly, a function is decreasing on an interval if the function values decrease as the input values increase over that interval.

Definition: Increasing and Decreasing

- A function f is an **increasing function** on an open interval (a, b) where $b > a$, if $f(b) > f(a)$ for every a and b in the interval.
- A function f is a **decreasing function** on an open interval (a, b) where $b > a$, if $f(b) < f(a)$ for every a and b in the interval.
- A function f is a **constant function** on an open interval (a, b) where $b > a$, if $f(b) = f(a)$ for every a and b in the interval.

Figure **2.2.20** shows examples of increasing and decreasing intervals on a function.

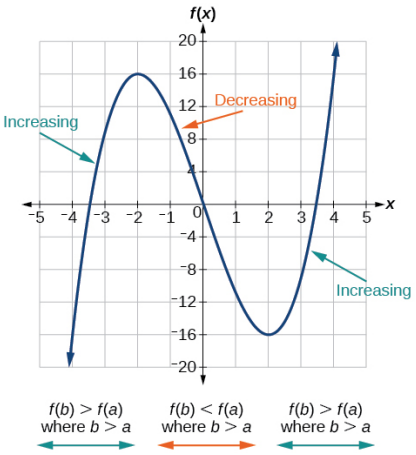
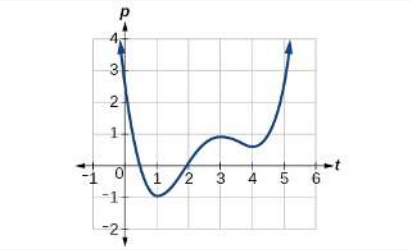


Figure 2.2.20: The function $f(x) = x^3 - 12x$ is increasing on $(-\infty, -2)$ and $(2, \infty)$ and is decreasing on $(-2, 2)$.

Example 2.2.20: Find Increasing and Decreasing Intervals on a Graph.

Given the function $p(t)$ in the figure below, identify the intervals on which the function appears to be increasing.



Solution

The function is not constant on any interval. The function is increasing where it slants upward as we move to the right and decreasing where it slants downward as we move to the right. The function appears to be increasing from $t = 1$ to $t = 3$ and from $t = 4$ on. In interval notation, we would say the function appears to be increasing on the interval $(1, 3)$ and the interval $(4, \infty)$.

Analysis. Notice in this example that we used open intervals (intervals that do not include the endpoints), because the function is neither increasing nor decreasing at $t = 1$, $t = 3$, and $t = 4$. These points are the local extrema (two minima and a maximum) instead.

Local Maxima and Minima

While some functions are increasing (or decreasing) over their entire domain, many others are not. A value of the input where a function changes from increasing to decreasing (as we go from left to right, that is, as the input variable increases) is called a **local maximum**. If a function has more than one, we say it has local maxima. Similarly, a value of the input where a function changes from decreasing to increasing as the input variable increases is called a **local minimum**. The plural form is “local minima.” Together, local maxima and minima are called **local extrema**, or local extreme values, of the function. (The singular form is “extremum.”) The term "**relative extrema**", rather than "local extrema" is often used.

Clearly, a function is neither increasing nor decreasing on an interval where it is constant. A function is also neither increasing nor decreasing at extrema. Note that we have to speak of local extrema, because any given local extremum as defined here is not necessarily the highest maximum or lowest minimum in the function’s entire domain.

For the function whose graph is shown in Figure 2.2.21, the local maximum is 16, and it occurs at $x = -2$. The local minimum is −16 and it occurs at $x = 2$.

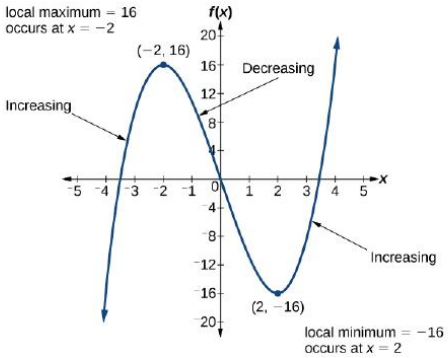


Figure 2.2.21: Graph of a polynomial that shows the increasing and decreasing intervals and local minimum/maximum

To locate the local maxima and minima from a graph, we need to observe the graph to determine where the graph attains its highest and lowest points, respectively, **within an open interval.** Like the summit of a roller coaster, the graph of a function is higher at a local maximum than at nearby points **on both sides**. The graph will also be lower at a local minimum than at neighboring points. Figure 2.2.22 illustrates these ideas for a local maximum.

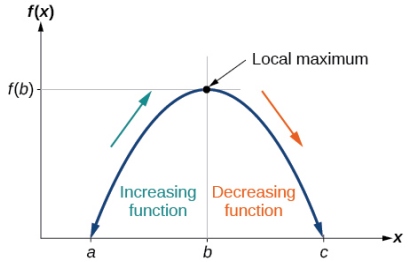


Figure 2.2.22: Definition of a local maximum

These observations lead us to a formal definition of local extrema.

Definition: Local Minima and Local Maxima

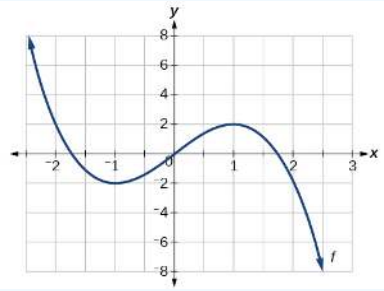
A function f **has a local maximum** at a point b in an open interval (a, c) if $f(b)$ is greater than or equal to $f(x)$ for every point x (x does not equal b) in the interval. The **maximum is** at $x = b$; the **maximum** is $y = f(b)$.

A function f **has a local minimum** at a point b in an open interval (a, c) if $f(b)$ is less than or equal to $f(x)$ for every x (x does not equal b) in the interval. The **minimum is** at $x = b$; the **minimum** is $y = f(b)$.

Local extrema are also called **relative** extrema.

Example 2.2.23: Find Local Maxima and Minima from a Graph

For the function f whose graph is shown in the figure below, find all local maxima and minima.



Solution

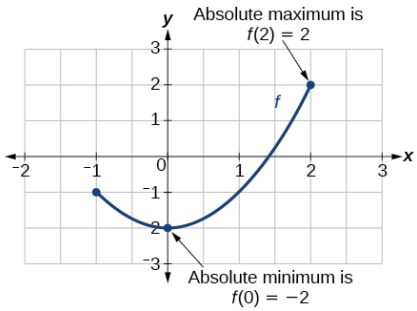
Observe the graph of f. The graph attains a local maximum at $x = 1$ because it is the highest point in an open interval around $x = 1$. The local maximum is the y-coordinate at $x = 1$, which is 2.

https://math.libretexts.org/@go/page/34871

The graph attains a local minimum at $x = -1$ because it is the lowest point in an open interval around $x = -1$. The local minimum is the y-coordinate at $x = -1$, which is -2.

Absolute Maxima and Minima

There is a difference between locating the highest and lowest points on a graph in a region around an open interval (locally) and locating the highest and lowest points on the graph for the entire domain. The y-coordinates (output) at the highest and lowest points are called the **absolute maximum** and **absolute minimum**, respectively. To locate absolute maxima and minima from a graph, we need to observe the graph to determine where the graph attains it highest and lowest points on the domain of the function.



Graph of a segment of a parabola with an absolute minimum at (0, -2) and absolute maximum at (2, 2).

Not every function has an absolute maximum or minimum value. The function $f(x) = x^3$ is one such function.

Definition: Absolute Minima and Absolute Maxima

- The **absolute maximum** of f at $x = c$ is $f(c)$ where $f(c) \geq f(x)$ for all x in the domain of f.
- The **absolute minimum** of f at $x = d$ is $f(d)$ where $f(d) \leq f(x)$ for all x in the domain of f.

Absolute extrema are also called **global** extrema.

Example 2.2.24: Find Absolute Maxima and Minima from a Graph

For the function f shown in Figure 2.2.24, find all absolute maxima and minima.

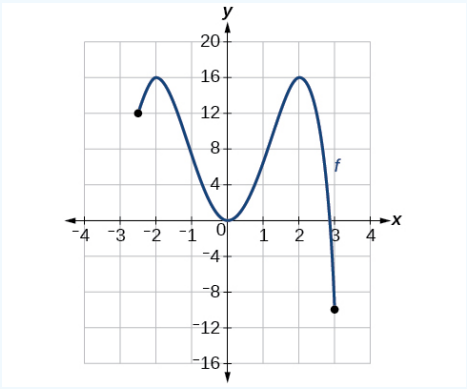


Figure 2.2.24: Graph of a polynomial.

Solution

Observe the graph of f. The graph attains an absolute maximum in two locations, $x = -2$ and $x = 2$, because at these locations, the graph attains its highest point on the domain of the function. The absolute maximum is the y-coordinate at $x = -2$ and $x = 2$, which is 16.

The graph attains an absolute minimum at $x = 3$, because it is the lowest point on the domain of the function's graph. The absolute minimum is the y-coordinate at $x = 3$, which is -10.

We can see the difference between local and global extrema in Figure 2.2.25 below. In this example, $y = 5$ is **both** a local maximum and a global maximum. The graph does **NOT** have a global minimum because $y = -\infty$ is not a finite real number value and it ($-\infty$) is not in the range of f since the range is $(-\infty, 5]$ which does not include $-\infty$!

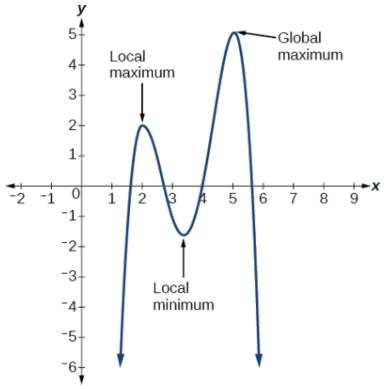


Figure 2.2.25: Graph of an even-degree polynomial that denotes the local maximum and minimum and the global maximum.

Do all polynomial functions have a global minimum or maximum?

No. Only polynomial functions of even degree have a global minimum or maximum. For example, $f(x) = x$ has neither a global maximum nor a global minimum.

Discontinuities

A function has discontinuities at the places where there are breaks in the graph of f. Discontinuities take on several different appearances. We classify types of discontinuities as removable discontinuities, infinite discontinuities, or jump discontinuities. Intuitively, a **removable discontinuity** is a discontinuity for which there is a hole in the graph, a **jump discontinuity** is a noninfinite discontinuity for which the sections of the function do not meet up, and an **infinite discontinuity** is a discontinuity located at a vertical asymptote.

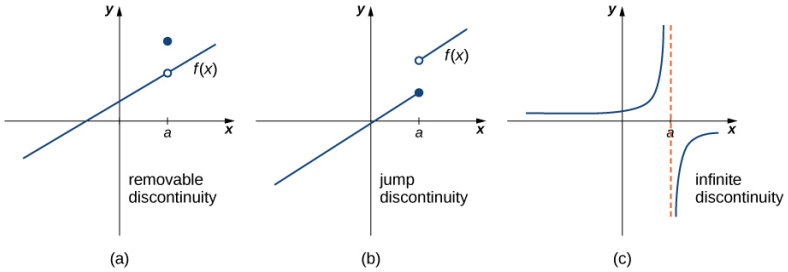


Figure 2.2.26: Discontinuities are classified as (a) removable, (b) jump, or (c) infinite.

Figure 2.2.26 illustrates the differences in these types of discontinuities. Although these terms provide a handy way of describing three common types of discontinuities, keep in mind that not all discontinuities fit neatly into these categories.

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2.2e: Exercises - Attributes of Functions

A: Concepts

Exercise 2.2e.A

1) Why does the domain differ for different functions?
2) How do we determine the domain of a function defined by an equation?
3) Explain why the domain of $f(x) = \sqrt[3]{x}$ is different from the domain of $f(x) = \sqrt{x}$.
4) When describing sets of numbers using interval notation, when do you use a parenthesis and when do you use a bracket?
5) How do you graph a piecewise function?

Answers to Odd Exercises:

1. The domain of a function depends upon what values of the independent variable make the function undefined or imaginary.

3. There is no restriction on x for $f(x) = \sqrt[3]{x}$ because you can take the cube root of any real number. So the domain is all real numbers, $(-\infty, \infty)$. When dealing with the set of real numbers, you cannot take the square root of negative numbers. So x-values are restricted for $f(x) = \sqrt{x}$ to nonnegative numbers and the domain is $[0, \infty)$.

5. Graph each formula of the piecewise function over its corresponding domain. Use the same scale for the x-axis and y-axis for each graph. Indicate inclusive endpoints with a solid circle and exclusive endpoints with an open circle. Use an arrow to indicate $-\infty$ or ∞. Combine the graphs to find the graph of the piecewise function.

B: Find the Domain given an Equation

Exercise 2.2e.B

★ Find the domain of each function. State the domain in interval notation.

7) $f(x) = 5 - 2x^2$

8) $f(x) = -2x(x-1)(x-2)$

9) $f(x) = 3\sqrt{x-2}$

10) $f(x) = 3 - \sqrt{6-2x}$

11) $f(x) = \sqrt{4-3x}$

12) $f(x) = \sqrt{x^2+4}$

13. $f(x) = \sqrt{12-3x} + \sqrt{x-1}$

14. $f(x) = \sqrt[3]{1-2x}$

15) $f(x) = \sqrt[3]{x-1}$

16) $f(x) = \dfrac{9}{x-6}$

17) $f(x) = \dfrac{1}{x^2-x-6}$

18) $f(x) = \dfrac{x}{x}$

19) $f(x) = \dfrac{3x+1}{4x+2}$

20) $f(x) = \dfrac{x-3}{x^2+9x-22}$

21) $f(x) = \dfrac{x^2-9x}{x^2-81}$

22) $f(x) = \dfrac{2x^3-250}{x^2-2x-15}$

23) $f(x) = \dfrac{\sqrt{x+4}}{x-4}$

24) $f(x) = \dfrac{x}{x-5} + \dfrac{\sqrt{x-3}}{2x-7}$

25) $f(x) = \dfrac{5}{\sqrt{x-3}}$

26) $f(x) = \dfrac{2x+1}{\sqrt{5-x}}$

27) $f(x) = \dfrac{\sqrt{x-6}}{\sqrt{x-4}}$

28) $f(x) = \dfrac{\sqrt{x-4}}{\sqrt{6-x}}$

30) $f(x) = \sqrt{x^2-6x+8}$

30.1) $f(x) = \sqrt{x^2(x-1)(x+3)}$

30.2) $f(x) = \sqrt{\dfrac{(x-1)}{(x+2)(x+5)^2}}$

30.3) $f(x) = \sqrt{\dfrac{(x-1)^2}{(x-2)^2(x+7)}}$

Answers to Odd Exercises:

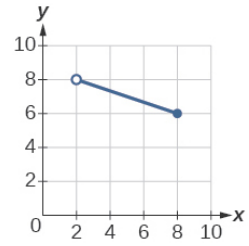
7. $(-\infty, \infty)$
9. $[2, \infty)$
11. $\left(-\infty, \frac{4}{3}\right]$
13. $[1, 4]$
15. $(-\infty, \infty)$
17. $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$
19. $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, \infty\right)$
21. $(-\infty, -9) \cup (-9, 9) \cup (9, \infty)$
23. $[-4, 4) \cup (4, \infty)$
25. $(3, \infty)$
27. $[6, \infty)$

C: Find the Domain, Range, and Intercepts given a Continuous Graph

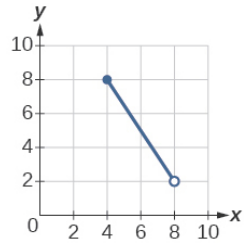
Exercise 2.2e.C

★ For each function whose graph is illustrated below, (a) Determine the domain and range and state results in interval notation, and (b) State the x- and y-intercepts.

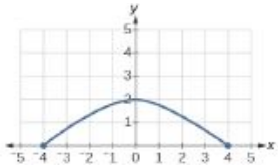
31)



32)



33)



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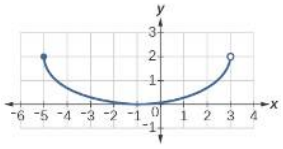
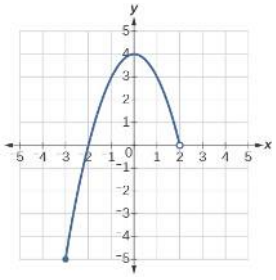
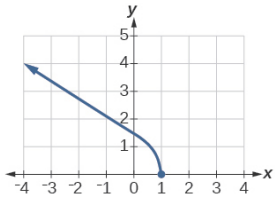


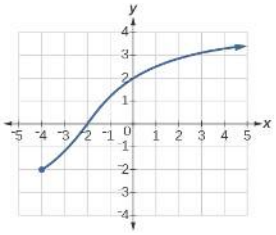
36)



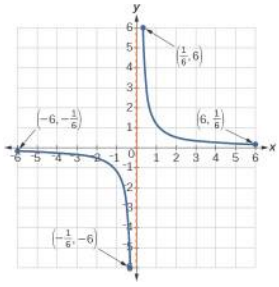
37)



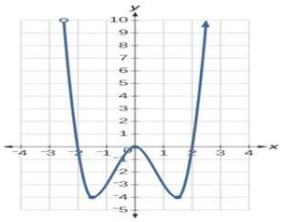
38)



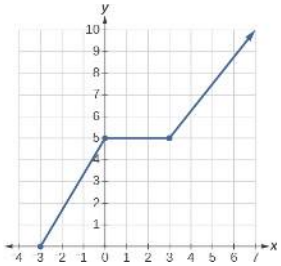
39)



40)



41)



Answers to Odd Exercises:

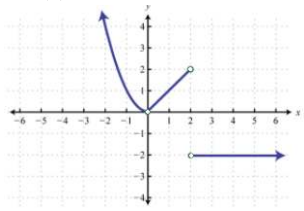
31. (a) domain: $(2, 8]$; range: $[6, 8)$;
    (b) no intercepts
33. (a) domain: $[-4, 4]$; range: $[0, 2]$;
    (b) intercepts: $(-4, 0)$, $(4, 0)$, $(0, 2)$
35. (a) domain: $[-5, 3)$; range: $[0, 2]$;
    (b) intercepts: $(-1, 0)$, $(0, \approx .1)$
37. (a) domain: $(-\infty, 1]$, range: $[0, \infty)$;
    (b) intercepts: $(1, 0)$, $(0, 1.5)$
39. (a) domain: $[-6, -\frac{1}{6}] \cup [\frac{1}{6}, 6]$;
    range: $[-6, -\frac{1}{6}] \cup [\frac{1}{6}, 6]$;
    (b) no intercepts
41. (a) domain: $[-3, \infty)$; range: $[0, \infty)$;
    (b) intercepts: $(-3, 0)$, $(0, 5)$

D: Find Domain and Range given a Piecewise Graph

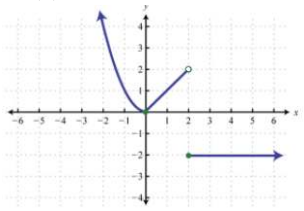
Exercise 2.2e.D

★ **Find the domain and range of the following graphs.**

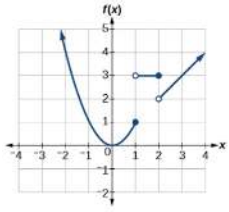
48. (a)



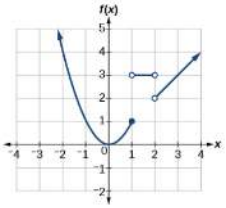
(b)



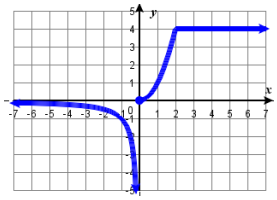
49. (a)



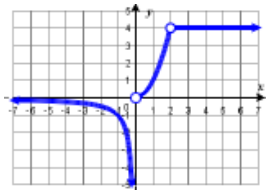
(b)



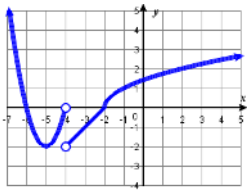
50. (a)



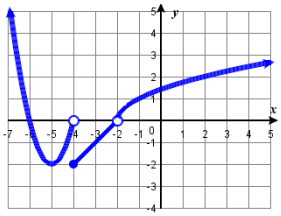
(b)



51. (a)



(b)



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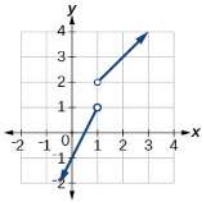
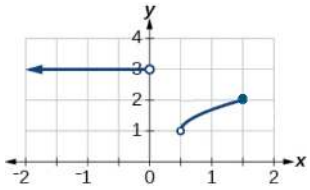


53.



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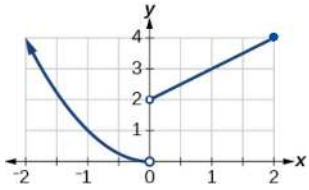

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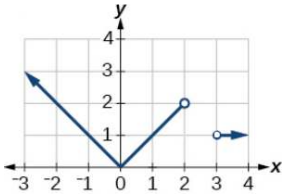

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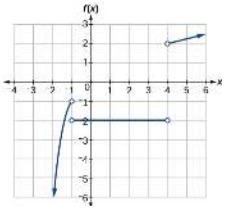

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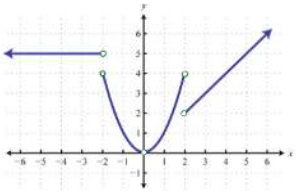


Answers to Odd Exercises:

49. (a) domain: $(-\infty, \infty)$, range: $[0, \infty)$ (b) domain: $(-\infty, 2) \cup (2, \infty)$, range: $[0, \infty)$

51. (a) domain: $(-\infty, -4) \cup (-4, \infty)$, range: $[-2, \infty)$ (b) domain: $(-\infty, -2) \cup (-2, \infty)$, range: $[-2, \infty)$

53. domain: $(-\infty, 0) \cup (1/2, 3/2]$, range: $(1, 2] \cup [3, 3]$ 55. domain: $(-\infty, 2) \cup (3, \infty)$, range: $[0, \infty)$

57. domain: $(-\infty, -2) \cup (-2, 0) \cup (0, 2) \cup (2, \infty)$, range: $(0, \infty)$

E: Find Zeros and Intercepts given a Function

Exercise 2.2e.E: Find Zeros and Intercepts for a Function

★ **Find the zeros and intercepts of each function below.**

63) $f(x) = 9x^2 - 12x + 4$
64) $p(x) = 64x^2 - 1$
65) $f(x) = (x+5)^2 + 1$
66) $f(x) = (3x-4)^2 + 7$
67) $g(x) = -(x+1)^4 + 16$
68) $f(x) = (2x-3)^2 - 4$
68.1) $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{4}{3}x + 2$
69.1) $f(x) = 3x^3 - 2x^2 + 27x - 18$

69.2) $f(x) = \dfrac{4x^2 - 9}{2x - 3}$
70) $f(x) = \dfrac{3x^2 - 2x - 1}{x^2 - 1}$
71) $f(x) = \dfrac{1}{x} - 3$
72) $f(x) = \dfrac{x}{x^2 - 16}$
73) $f(x) = \dfrac{1}{x+5}$
74) $h(x) = \dfrac{3}{x^2 + 4}$

75) $f(x) = \sqrt{2x-3} - 1$
76) $f(x) = 3\sqrt{x-7} - 6$
77) $f(x) = 2\sqrt{x+2} - 8$
78) $f(x) = -1 + \sqrt{x+2}$
79) $f(x) = \sqrt[3]{x-8} - 2$
80) $f(x) = 2\sqrt[3]{x-1} + 6$
81) $g(x) = \sqrt{\dfrac{7}{x-5}}$
82) $f(x) = 4|x^2 + 5|$

Answers to Odd Exercises:

63). zeros: {2/3}; x-intercepts: (2/3,0); y-intercepts: (0,4) 65. $\{-5 \pm i\}$; no x intercepts; (0, 26)
67). $\{ \pm 2, \pm 2i \}$; (2,0), (-2,0); (0,15) 69.1. zeros: $\{\pm 3i, \frac{2}{3}$; x-intercepts: (3/2,0); y-intercepts: (0,-18)
69.2). {-3/2}; (-3/2,0); (0,3) 71). {1/3}; (1/3, 0); no y-intercepts 73). no zeros; no x-intercepts; (0, $\frac{1}{5}$)
75). {2}; (2,0); no y-intercept 77). {14}; (14,0); (0, $2\sqrt{2} - 8$)
79). {16}; (16,0); (0, -4) 81). no zeros; no x-intercepts; no y-intercept.

F: Find Zeros, Intercepts, and Symmetry given an Equation

Exercise 2.2e.F: Find Zeros, Intercepts and Symmetry for a graph given its equation

★ **For each equation below (a) Find the zeros and x- and y-intercept(s) of the graph, if any exist and (b) Test for symmetry.**

83. $y = x^2 + 1$
84. $y = x^2 - 2x - 8$
85. $y = x^3 - x$
86. $y = \frac{x^3}{4} - 3x$
87. $y = \sqrt{x-2}$
88. $y = 2\sqrt{x+4} - 2$
89. $3x - y = 7$
90. $3x - 2y = 10$
91. $(x+2)^2 + y^2 = 16$
92. $x^2 - y^2 = 1$
93. $4y^2 - 9x^2 = 36$
94. $x^3 y = -4$

Answers to odd exercises.

83. (a) The graph has no x-intercepts, y-intercept: $(0,1)$, zeros: $\{\pm i\}$.
(b) x-axis symmetry: no. y-axis symmetry: yes. Origin symmetry: no.

85. x-intercepts: $(-1,0), (0,0), (1,0)$, y-intercept: $(0,0)$, zeros: $\{-1,\ 0,\ 1\}$.
(b) x-axis symmetry: no. y-axis symmetry: no. Origin symmetry: yes.

87. x-intercept: $(2,0)$, The graph has no y-intercepts, zeros: $\{2\}$.
(b) x-axis symmetry: no. y-axis symmetry: no. Origin symmetry: no.

89. x-intercept: $(\frac{7}{3},0)$, y-intercept: $(0,-7)$, zeros: $\{\frac{7}{3}\}$.
(b) x-axis symmetry: no. y-axis symmetry: no. Origin symmetry: no.

91. x-intercepts: $(-6,0), (2,0)$, y-intercepts: $\left(0, \pm 2\sqrt{3}\right)$, zeros: $\{2, -6\}$.
(b) x-axis symmetry: yes. y-axis symmetry: no. Origin symmetry: no.

93. The graph has no x-intercepts, y-intercepts: $(0, \pm 3)$, zeros: $\{\pm 2i\}$.
(b) x-axis symmetry: yes. y-axis symmetry: yes. Origin symmetry: yes.

G: Find Symmetry for an Equation or a Function

Exercise 2.2e.G

★ Determine the symmetry for the following equations

97. $y = 2x + 5$
98. $y = x^2 + 3$
99. $x = y^2 - 6x$
100. $7x + 3y^3 = 5$
101. $y = |x| - 2$
102. $x = |y| + 3$
103. $(x+1)^2 = |y - 2xy|$
104. $x^2y^2 + 3xy = 1$
105. $y = \dfrac{3x}{x^2+9}$
106. $x = \dfrac{y^2-4}{2y}$

Answers to Odd Exercises:

97) no symmetry 99) x-axis symmetry 101) y-axis symmetry 103) x-axis symmetry 105) origin symmetry

★ Given the following functions (a) Determine the symmetry of the function and (b) State if the function is even or odd.

108. $f(x) = (x-2)^2$
109. $f(x) = 3x^4$
110. $h(x) = 2x - x^3$
111. $g(x) = \sqrt{x}$
112. $f(x) = x\sqrt{1-x^2}$
113. $f(x) = x^2\sqrt{x^2+16}$
114. $f(x) = \sqrt{x^3+2x}$
115. $f(x) = \sqrt[3]{2x^2+1}$
116. $f(x) = \sqrt[3]{x}$
117. $h(x) = \dfrac{1}{x} + 3x$
118. $f(x) = \dfrac{x}{|3x|}$
118.1 $f(x) = |x^3 + 2x|$

Answers to Odd Exercises:

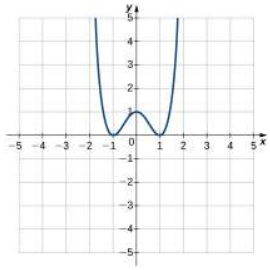
109) (a) y-axis symmetry, (b) even function 111) (a) neither y-axis or origin symmetry, (b) neither even nor odd 113) (a) y-axis symmetry, (b) even function 115) (a) y-axis symmetry, (b) even function 117) (a) origin symmetry, (b) odd function,

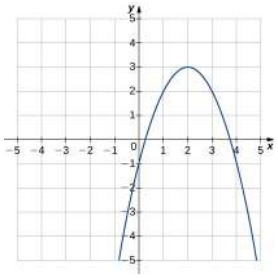
H: Attributes from Graphs

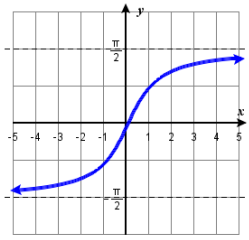
Exercise 2.2e.H

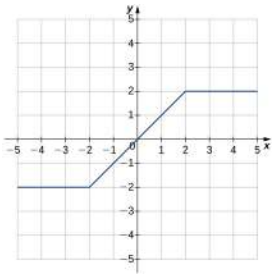
★ Determine the following for each graph below. These graphs do not have arrows, but assume that a graph continues at both ends if it extends beyond the given grid.

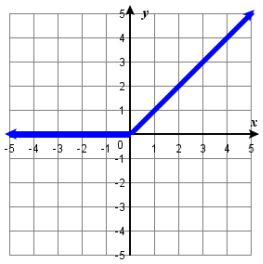
a. **Domain and range**
b. **x -intercept, if any (estimate where necessary)**
c. **y-Intercept, if any (estimate where necessary)**
d. **The intervals for which the function is increasing (only if a function)**
e. **The intervals for which the function is decreasing (only if a function)**
f. **The intervals for which the function is constant (only if a function)**
g. **Symmetry about any axis and/or the origin**
h. **Whether the function is even, odd, or neither (only if a function)**

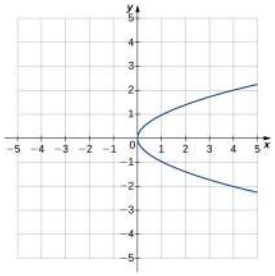
121) 

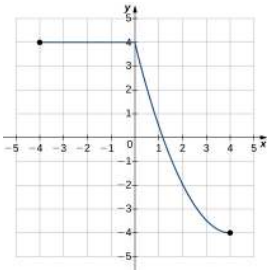
122) 

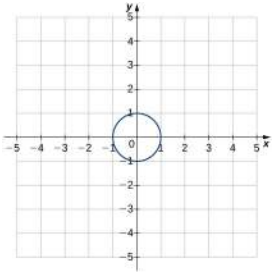
123) 

124) 

125) 

126) 

127) 

128) 

Answers to Odd Exercises:

121) Solution: Function; a. Domain: $(-\infty,\infty)$, range: $[0,\infty)$ b. $(-1,0)$, $(1,0)$ c. $(0,1)$ d. $-1<x<0$ and $x>1$ e. $x<-1$ and $0<x<1$ f. Not constant g. y-axis h. Even

123) Solution: Function; a. Domain: $(-\infty,\infty)$, range: $(-\frac{\pi}{2},\frac{\pi}{2})$ b. $(0,0)$ c. $(0,0)$ d. all real numbers e. None f. Not constant g. Origin h. Odd

125) Solution: Function; a. Domain: $(-\infty,\infty)$, range: $[0,\infty)$ b. $\{(x,0)|\ x\leq 0\}$ c. $(0,0)$ d. $x>0$ e. Not decreasing f. $x<0$ g. No Symmetry h. Neither

127) Solution: Function; a. Domain: $[-4,4]$, range: $[-4,4]$ b. $(\approx 1.2,0)$ c. $(0,4)$ d. Not increasing e. $0<x<4$ f. $-4<x<0$ g. No Symmetry h. Neither

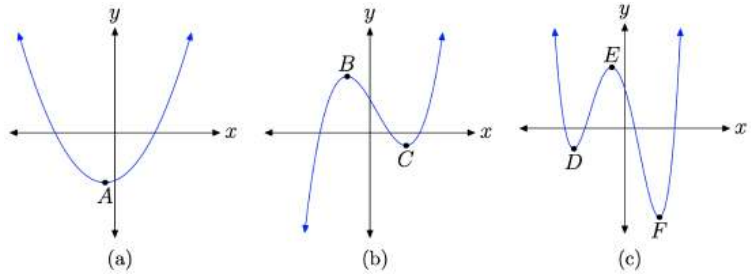
I: Extrema and Continuity given a Graph

Exercise 2.2e.I

★ **Find the following attributes in the graphs below.**
Assume that graphs do NOT extend beyond the given grid unless arrows are drawn. For minima and maxima, state the coordinates of the point or the label on the point as appropriate. Estimate if needed.

a) Local minima **b) Absolute minima** **c) Local maxima** **d) Absolute maxima**
e) Places where the function is discontinuous

131)



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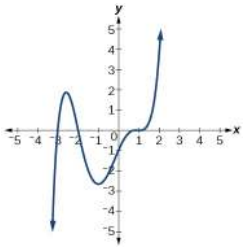
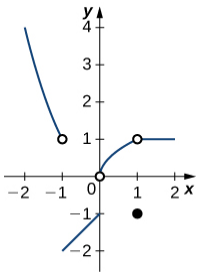


Assume that graphs do NOT extend beyond the given grid unless arrows are drawn. Estimate if needed.

133)



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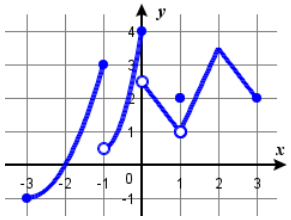
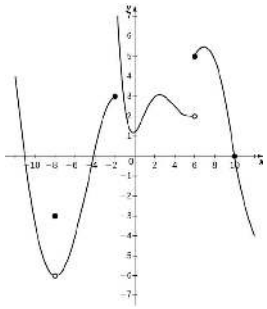


135)



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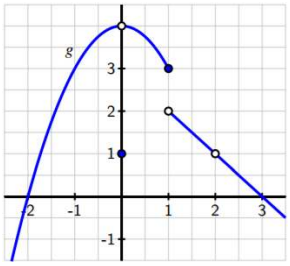

Answers to Odd Exercises:

131a) (a) A, (b) A, (c) none, (d) none (e) none; 131b) (a) C, (b) none, (c) B, (d) none, (e) none
131c) (a) D,F, (b) F, (c) E, (d) none, (e) none
133) (a) (1,-1), (b) (-1,-2); (c) none, (d) (-2,4); (e) x=-1, x=0, x=1
135) (a) (0, 1.2), (b) none; (c) (-8, -3), (2,3), (7, 5.5) (d) (-2,7) (e) x=-8, x=-2, x=6.

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2.3: Transformations of Functions

Basic Toolkit Functions

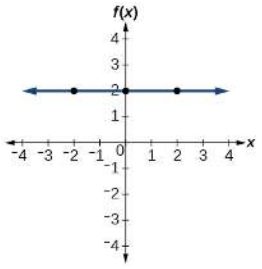
In this text, we will be exploring functions—the shapes of their graphs, their unique characteristics, their algebraic formulas, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of building-block elements. We call these our "toolkit functions," which form a set of basic functions for which we know the graph, formula, and special properties. Some of these functions are programmed to individual buttons on many calculators. For these definitions we will use x as the input variable and $y = f(x)$ as the output variable.

We will see these toolkit functions, combinations of toolkit functions, their graphs, and their transformations frequently throughout this book. It is very helpful if we can recognize these toolkit functions and their features quickly by name, formula, graph, and basic table properties. The graphs and sample table values for each function are shown below.

Table 2.3.1: Toolkit Functions

Constant Function

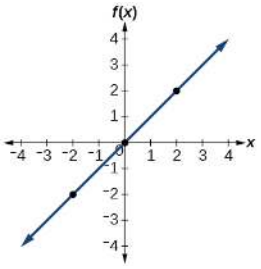
$f(x) = c$ where c is a constant



x	$f(x)$
−2	2
0	2
2	2

Identity Function

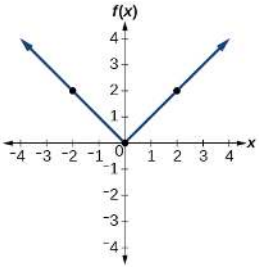
$f(x) = c$ where c is a constant



x	$f(x)$
−2	−2
0	0
2	2

Absolute Value Function

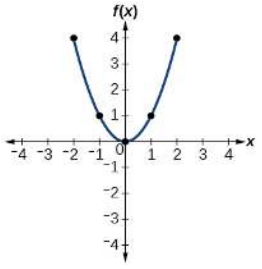
$f(x) = |x|$



x	$f(x)$
−2	2
0	0
2	2

Quadratic Function

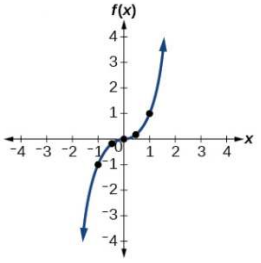
$f(x) = x^2$



x	$f(x)$
−2	4
−1	1
0	0
1	1
2	4

Cubic

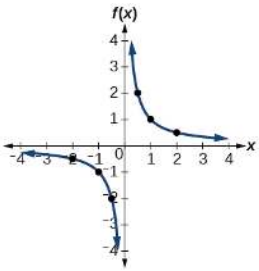
$f(x) = x^3$



x	$f(x)$
−2	−8
−1	−1
−0.5	−0.125
0	0
0.5	0.125
1	1
2	8

Reciprocal Function

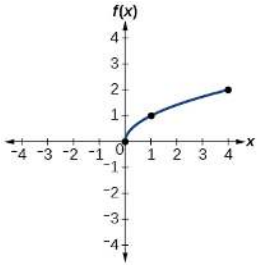
$f(x) = \frac{1}{x}$



x	$f(x)$
−2	−0.5
−1	−1
−0.5	−2
0.5	2
1	1
2	0.5

Square Root Function

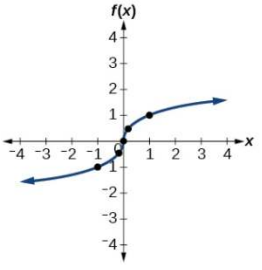
$f(x) = \sqrt{x}$



x	$f(x)$
0	0
1	1
4	2

Cube Root Function

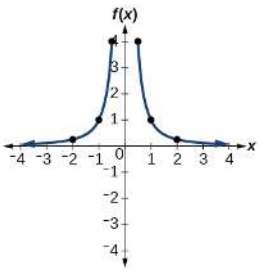
$f(x) = \sqrt[3]{x}$



x	$f(x)$
−8	−2
−1	−1
−0.125	−0.5
0	0
0.125	0.5
1	1
8	2

Reciprocal Squared Function

$f(x) = \frac{1}{x^2}$

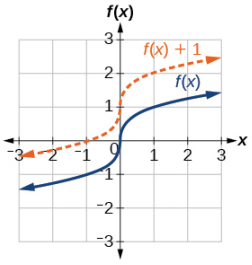


x	$f(x)$
−2	0.25
−1	1
−0.5	4
0.5	4
1	1
2	0.25

It is important to recognize these basic functions and to be familiar with their graphs. These functions are often encountered in slightly modified forms that are a consequence of shifts, reflections, compressions or stretches of the original basic graph. In this type of situation, it is not necessary to graph by plotting points. Instead, a much easier graphing technique called transformations can be used. This section explains why this technique works and how to use it.

Vertical Shifts

One simple kind of transformation involves shifting the entire graph of a function up, down, right, or left. The simplest shift is a vertical shift, moving the graph up or down, because this transformation involves adding a positive or negative constant to the function. In other words, we add the same constant to the output value of the function regardless of the input. For a function $g(x) = f(x) + k$, the function $f(x)$ is shifted vertically k units. The figure on the right is an example of a vertical shift by $k = 1$ of the cube root function $f(x) = \sqrt[3]{x}$.



To help you visualize the concept of a vertical shift, consider that $y = f(x)$. Therefore, $f(x) + k$ is equivalent to $y + k$. Every unit of y is replaced by $y + k$, so the y-value increases or decreases depending on the value of k. The result is a shift upward or downward.

Vertical Shift

Given a function $f(x)$, a new function $g(x) = f(x) + k$, where k is a constant, is a **vertical shift** of the function $f(x)$. All the output values change by k units. If k is positive, the graph shifts up. If k is negative, the graph shifts down.

Example 2.3.1: Adding a Constant to a Function

To regulate temperature in a green building, airflow vents near the roof open and close throughout the day. Figure **2.3.1e** shows the area of open vents V (in square feet) throughout the day in hours after midnight, t. During the summer, the facilities manager decides to try to better regulate temperature by increasing the amount of open vents by 20 square feet throughout the day and night. Sketch a graph of this new function.

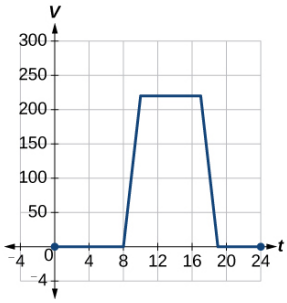


Figure **2.3.1e**: V(t)

Solution

We can sketch a graph of this new function by adding 20 to each of the output values of the original function. This will have the effect of shifting the graph vertically up, as shown in Figure **2.3.1s**.

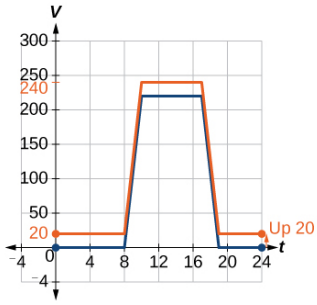


Figure **2.3.1s**: V(t)+20

Notice that in Figure **2.3.1s**, for each input value, the output value has increased by 20, so if we call the new function $S(t)$, we could write

$$S(t) = V(t) + 20$$

This notation tells us that, for any input value of t, $S(t)$ can be found by evaluating the function V at the same input and then adding 20 to the result. This defines S as a transformation of the function V, in this case a vertical shift up 20 units. Notice that, with a vertical shift, the input values stay the same and only the output values change. See the table below.

t	0	8	10	17	19	24
$V(t)$	0	0	220	220	0	0
$S(t)=V(t)+20$	20	20	240	240	20	20

How to: Given a tabular function, create a new row to represent a vertical shift.

1. Identify the output row or column.
2. Determine the **magnitude** of the shift.
3. Add the shift to the value in each output cell. Add a positive value for up or a negative value for down.

Example 2.3.2: Shifting a Tabular Function Vertically

A function $f(x)$ is given in the table to the right.
Create a table for the function $g(x)=f(x)-3$.

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x)=f(x)-3$ tells us that we can find the output values of g by subtracting 3 from the output values of f. For example:

$f(x)=1$ **Original function**
$g(x)=f(x)-3$ **Given Transformation**
$g(2)=f(2)-3$
$=1-3$
$=-2$

Subtracting 3 from each $f(x)$ value, we can complete a table of values for $g(x)$ as shown in the table on the right.

x	2	4	6	8
$f(x)$	1	3	7	11
$g(x)=f(x)-3$	-2	0	4	8

Analysis. As with the earlier vertical shift, notice the input values stay the same and only the output values change.

Try It 2.3.3

The function $h(t)=-4.9t^2+30t$ gives the height h of a ball (in meters) thrown upward from the ground after t seconds. Suppose the ball was instead thrown from the top of a 10 meter building. Relate this new height function $b(t)$ to $h(t)$, and then find a formula for $b(t)$.

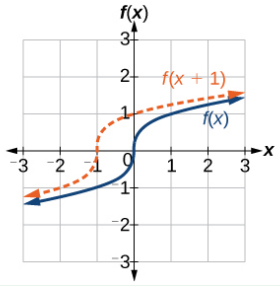
Answer

$b(t)=h(t)+10=-4.9t^2+30t+10$

Horizontal Shifts

We just saw that the vertical shift is a change to the output, or outside, of the function. We will now look at how changes to input, on the inside of the function, change its graph and meaning. A shift to the input results in a movement of the graph of the function left or right in what is known as a **horizontal shift**, shown in the figure at the right. (*The figure illustrates the horizontal shift of the function* $f(x)=\sqrt[3]{x}$. *Note that the argument* $x+1$ *shifts the graph to the left, that is, towards negative values of* x.

For example, if $f(x) = x^2$, then $g(x) = (x-2)^2$ is a new function. Each input is reduced by 2 prior to squaring the function. *The result is that the graph is shifted 2 units to the right, because we would need to increase the prior input by 2 units to yield the same output value as given in f.*



How to: Horizontal Shift

Given a function f, a new function $g(x) = f(x+p)$, where p is a constant that produces a **horizontal shift** of the function f. If p is positive, the graph will shift *left*. If p is negative, the graph will shift *right*.

Example 2.3.4: Adding a Constant to an Input

Returning to our building airflow example from Figure 2.3.1e, suppose that in autumn the facilities manager decides that the original venting plan starts too late, and wants to begin the entire venting program 2 hours earlier. Sketch a graph of the new function.

Solution

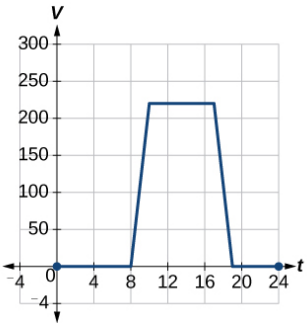


Figure 2.3.1e: V(t)

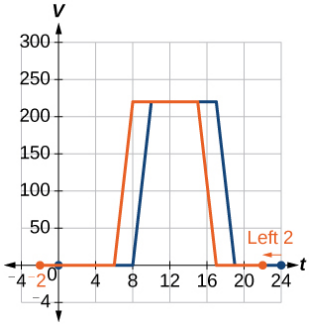


Figure 2.3.4s: F(t) = V(t+2)

We can set $V(t)$ to be the original program and $F(t)$ to be the revised program.

$$V(t) = \text{ the original venting plan}$$

$$F(t) = \text{ starting 2 hrs sooner}$$

In the new graph, at each time, the airflow is the same as the original function V was 2 hours later. For example, in the original function V, the airflow starts to change at 8 a.m., whereas for the function F, the airflow starts to change at 6 a.m. The comparable function values are $V(8) = F(6)$. See Figure 2.3.4s. Notice also that the vents first opened to 220ft^2 at 10 a.m. under the original plan, while under the new plan the vents reach 220ft^2 at 8 a.m., so $V(10) = F(8)$.

In both cases, we see that, because $F(t)$ starts 2 hours sooner, $h = -2$. That means that the same output values are reached when $F(t) = V(t-(-2)) = V(t+2)$.

Analysis

Note that $V(t+2)$ has the effect of shifting the graph to the left.

Horizontal changes or "inside changes" affect the domain of a function (the input) instead of the range and often seem counterintuitive. The new function $F(t)$ uses the same outputs as $V(t)$, but matches those outputs to inputs 2 hours earlier than those of $V(t)$. Said another way, we must add 2 hours to the input of V to find the corresponding output for F : $F(t) = V(t+2)$.

How to: Given a tabular function, create a new row to represent a horizontal shift.

1. Identify the input row or column.
2. Determine the **magnitude** of the shift.
3. Add the shift to the value in each input cell.

Example 2.3.5: Shifting a Tabular Function Horizontally

A function $f(x)$ is given in Table 2.3.5e.
Create a table for the function $g(x) = f(x-3)$.

Table 2.3.5e

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = f(x-3)$ tells us that the output values of g are the same as the output value of f when the input value is 3 less than the original value. For example, we know that $f(2) = 1$. To get the same output from the function g, we will need an input value that is 3 larger. We input a value that is 3 larger for $g(x)$ because the function takes 3 away before evaluating the function f.

$$\begin{aligned} g(5) &= f(5-3) \\ &= f(2) \\ &= 1 \end{aligned}$$

We continue with the other values to create Table 2.3.5s.

Table 2.3.5s

x	5	7	9	11
$x-3$	2	4	6	8
$f(x-3)$	1	3	7	11
$g(x)$	1	3	7	11

The result is that the function $g(x)$ has been shifted to the right by 3. Notice the output values for $g(x)$ remain the same as the output values for $f(x)$, but the corresponding input values, x for g, have shifted to the right by 3. Specifically, 2 shifted to 5, 4 shifted to 7, 6 shifted to 9, and 8 shifted to 11.

Analysis

Figure 2.3.5t represents both of the functions. We can see the horizontal shift in each point.

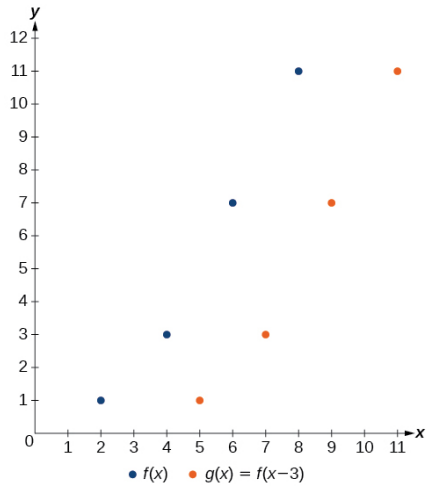


Figure 2.3.5t: Graph of the points from Table 2.3.5s for $f(x)$ and $g(x) = f(x-3)$

Example 2.3.6: Construct an equation of a translated toolkit function from a graph

Figure 2.3.6e represents a transformation of the toolkit function $f(x) = x^2$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

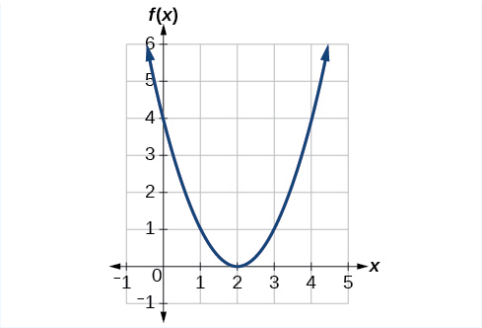


Figure **2.3.6e**: Graph of a parabola

Solution

Notice that the graph is identical in shape to the $f(x) = x^2$ function, but the x-values are shifted to the right 2 units. The vertex used to be at $(0, 0)$, but now the vertex is at $(2, 0)$. The graph is the basic quadratic function shifted 2 units to the right, so

$$g(x) = f(x - 2)$$

Notice how we must input the value $x = 2$ to get the output value $y = 0$; the x-values must be 2 units larger because of the shift to the right by 2 units. We can then use the definition of the $f(x)$ function to write a formula for $g(x)$ by evaluating $f(x - 2)$.

$$f(x) = x^2$$
$$g(x) = f(x - 2)$$
$$g(x) = f(x - 2) = (x - 2)^2$$

Analysis

To determine whether the shift is $+2$ or -2, consider a single reference point on the graph. For a quadratic, looking at the vertex point is convenient. In the original function, $f(0) = 0$. In our shifted function, $g(2) = 0$. To obtain the output value of 0 from the function f, we need to decide whether a plus or a minus sign will work to satisfy $g(2) = f(x - 2) = f(0) = 0$. For this to work, we will need to subtract 2 units from our input values.

Example 2.3.7: Interpreting Horizontal versus Vertical Shifts

The function $G(m)$ gives the number of gallons of gas required to drive m miles. Interpret $G(m) + 10$ and $G(m + 10)$

Solution

$G(m) + 10$ can be interpreted as adding 10 to the output, gallons. This is the gas required to drive m miles, plus another 10 gallons of gas. The graph would indicate a vertical shift.

$G(m + 10)$ can be interpreted as adding 10 to the input, miles. So this is the number of gallons of gas required to drive 10 miles more than m miles. The graph would indicate a horizontal shift.

Try It 2.3.8

Given the function $f(x) = \sqrt{x}$, graph the original function $f(x)$ and the transformation $g(x) = f(x + 2)$ on the same axes. Is this a horizontal or a vertical shift? Which way is the graph shifted and by how many units?

Answer

The graphs of $f(x)$ and $g(x)$ are shown below. The transformation is a horizontal shift. The function is shifted to the left by 2 units.

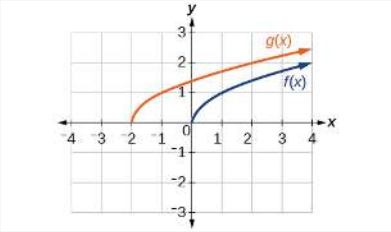


Figure **2.3.8**

Combining Vertical and Horizontal Shifts

Now that we have two transformations, we can combine them together. Vertical shifts are outside changes that affect the output (y–) axis values and shift the function up or down. Horizontal shifts are inside changes that affect the input (x–) axis values and shift the function left or right. Combining the two types of shifts will cause the graph of a function to shift up or down and right or left.

How to: Given a function and both a vertical and a horiontal shift, sketch the graph.

1. Given a function f and a new function $g(x) = f(x + p) + k$, identify the vertical and horizontal shifts from the formula.
2. The vertical shift results from the constant k added to the output. Move the graph up for a positive constant and down for a negative constant.
3. The horizontal shift results from the constant p added to the input. Move the graph left for a positive constant and right for a negative constant.
4. Apply the shifts to the graph in either order.

Example 2.3.9: Graphing Combined Vertical and Horizontal Shifts

Given $f(x) = |x|$, sketch a graph of $h(x) = f(x + 1) - 3$.

Solution

The function f is our toolkit absolute value function. We know that this graph has a V shape, with the point at the origin. The graph of h has transformed f in two ways: $f(x + 1)$ is a change on the inside of the function, giving a horizontal shift left by 1, and the subtraction by 3 in $f(x + 1) - 3$ is a change to the outside of the function, giving a vertical shift down by 3. The transformation of the graph is illustrated in Figure **2.3.9**.

Let us follow one point of the graph of $f(x) = |x|$.

The point $(0,0)$ is transformed first by shifting left 1 unit: $(0,0) \rightarrow (-1,0)$
The point $(-1,0)$ is transformed next by shifting down 3 units: $(-1,0) \rightarrow (-1,-3)$

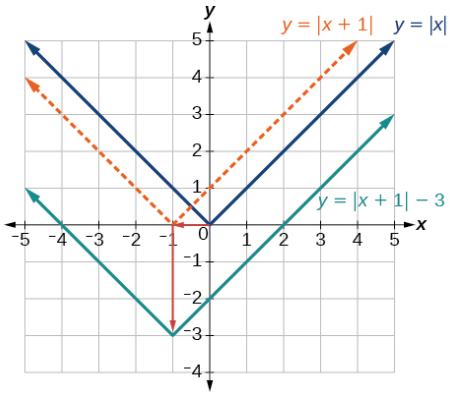
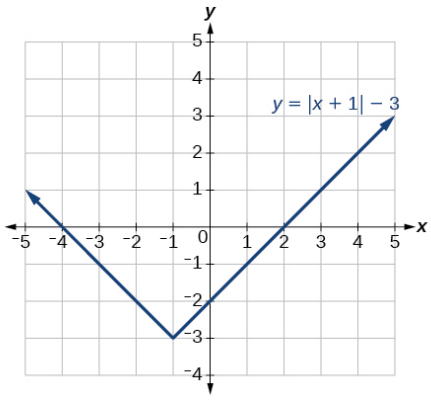


Figure **2.3.9**: Graph of an absolute function, $y = |x|$, and how it was transformed to $y = |x+1|-3$



The final function $h(x) = |x+1|-3$.x

Try It 2.3.10

Given $f(x) = |x|$, sketch a graph of $h(x) = f(x-2)+4$.

Answer

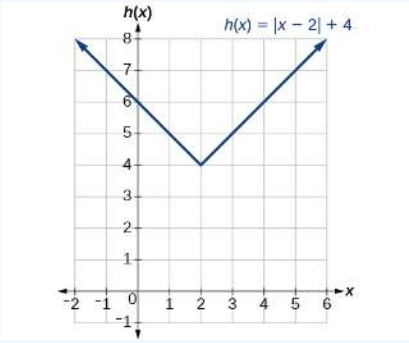


Figure **2.3.10**

Example 2.3.11: Construct an equation of a translated toolkit function from a graph

Write a formula for the graph shown in Figure **2.3.11**, which is a transformation of the toolkit square root function.

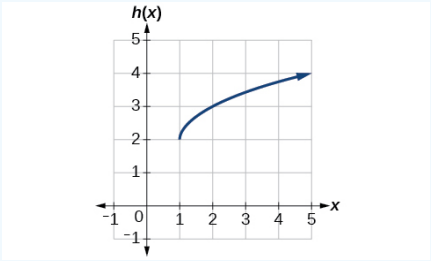


Figure **2.3.11**: Graph of a square root function transposed right one unit and up 2.

Solution

The graph of the toolkit function starts at the origin, so this graph has been shifted 1 to the right and up 2. In function notation, we could write that as

$$h(x) = f(x-1) + 2$$

Using the formula for the square root function, we can write

$$h(x) = \sqrt{x-1} + 2$$

Analysis

Note that this transformation has changed the domain and range of the function. This new graph has domain $[1, \infty)$ and range $[2, \infty)$.

Try It 2.3.12

Write a formula for a transformation of the reciprocal function $f(x) = \frac{1}{x}$ that shifts the function's graph one unit to the right and one unit up.

Answer

$$g(x) = \frac{1}{x-1} + 1$$

Graphing Functions Using Reflections about the Axes

Another transformation that can be applied to a function is a reflection over the x- or y-axis. A **vertical reflection** reflects a graph vertically across the x-axis, while a **horizontal reflection** reflects a graph horizontally across the y-axis. The reflections are shown in Figure **2.3.13.**

Notice that the vertical reflection produces a new graph that is a mirror image of the base or original graph about the x-axis. The horizontal reflection produces a new graph that is a mirror image of the base or original graph about the y-axis.

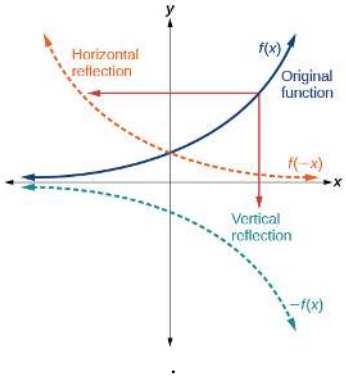


Figure **2.3.13**: Graph of the vertical and horizontal reflection of a function.

Reflections

Given a function $f(x)$, a new function $g(x) = -f(x)$ is a **vertical reflection** of the function $f(x)$, sometimes called a reflection about (or over, or through) the x-axis.

Given a function $f(x)$, a new function $g(x) = f(-x)$ is a **horizontal reflection** of the function $f(x)$, sometimes called a reflection about the y-axis.

How to: Given a function, reflect the graph vertically or horizontally.

1. Multiply all outputs by –1 for a vertical reflection. The new graph is a reflection of the original graph about the x-axis.
2. Multiply all inputs by –1 for a horizontal reflection. The new graph is a reflection of the original graph about the y-axis.

Example 2.3.13: Reflecting a Graph Horizontally and Vertically

Given the square root function $s(t) = \sqrt{t}$

a. State the function $V(t)$ that is the vertical reflection of $s(t)$, and graph it.
b. State the function $H(t)$ that is the horizontal reflection of $s(t)$, and graph it.

Solution

a. Reflecting the graph vertically means that each output value will be reflected over the horizontal t-axis. Because each output value is the opposite of the original output value, we can write $V(t) = -s(t)$ or $V(t) = -\sqrt{t}$. Notice that this is an outside change, or vertical shift, that affects the output $s(t)$ values, so the negative sign belongs outside of the function. The graph is shown in Figure **2.3.13***a* below.

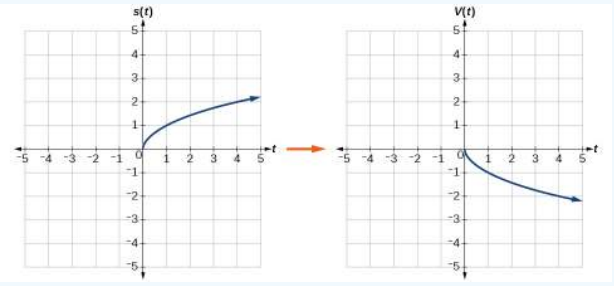


Figure **2.3.13***a*: Graph of the vertical reflection of the square root function.

b. Reflecting horizontally means that each input value will be reflected over the vertical axis. Because each input value is the opposite of the original input value, we can write $H(t) = s(-t)$ or $H(t) = \sqrt{-t}$. Notice that this is an inside change or horizontal change that affects the input values, so the negative sign is on the inside of the function. The graph is shown in Figure **2.3.13*b*** below.

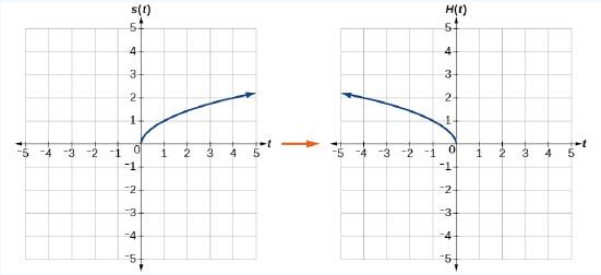


Figure **2.3.13*b***: *Horizontal reflection of the square root function*

Note that these transformations can affect the domain and range of the functions. While the original square root function has domain $[0, \infty)$ and range $[0, \infty)$, the vertical reflection gives the $V(t)$ function the range $(-\infty, 0]$ and the horizontal reflection gives the $H(t)$ function the domain $(-\infty, 0]$.

Try It 2.3.14

Reflect the graph of $f(x) = |x - 1|$ (a) vertically and (b) horizontally, and state the function corresponding to these transformations

Answer

a. $V(t) = -|x - 1|$

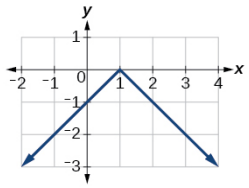


Figure **2.3.14*a***: Graph of a vertically reflected absolute function.

b. $H(t) = |-x - 1| = |x + 1|$

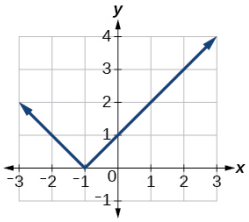


Figure **2.3.14*b***: Graph of an absolute function translated one unit left.

Example 2.3.15: Reflecting a Tabular Function Horizontally and Vertically

A function $f(x)$ is given as Table **2.3.15**.
Create a table for the functions below.
a. $g(x) = -f(x)$
b. $h(x) = f(-x)$

Table **2.3.15**

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

a. For $g(x)$, the negative sign outside the function indicates a vertical reflection, so the x-values stay the same and each output value will be the opposite of the original output value. See Table **2.3.15***a* below.

Table **2.3.15***a*

x	2	4	6	8
$g(x) = -f(x)$	-1	-3	-7	-11

b. For $h(x)$, the negative sign inside the function indicates a horizontal reflection, so each input value will be the opposite of the original input value and the $h(x)$ values stay the same as the $f(x)$ values. See Table **2.3.15***b* below.

Table **2.3.15***b*

x	-2	-4	-6	-8
$h(x) = f(-x)$	1	3	7	11

Try It 2.3.16

A function $f(x)$ is given as Table **2.3.16**. Create a table for the functions below.

a. $g(x) = -f(x)$

b. $h(x) = f(-x)$

Table **2.3.16**

x	-2	0	2	4
$f(x)$	5	10	15	20

Answer

a. For $g(x)$, the negative sign outside the function indicates a vertical reflection, so the x-values stay the same and each output value will be the opposite of the original output value. See Table **2.3.16***a* below.

Table **2.3.16***a*

x	-2	0	2	4
$g(x) = -f(x)$	-5	-10	-15	-20

b. For $h(x)$, the negative sign inside the function indicates a horizontal reflection, so each input value will be the opposite of the original input value and the $h(x)$ values stay the same as the $f(x)$ values. See Table **2.3.16***b* below.

Table **2.3.16***b*

x	2	0	-2	-4
$h(x) = f(-x)$	5	10	15	20

Example 2.3.17: Applying a Learning Model Equation

A common model for learning has an equation similar to $k(t) = -2^{-t} + 1$, where k is the percentage of mastery that can be achieved after t practice sessions. This is a transformation of the function $f(t) = 2^t$ shown in Figure **2.3.17**. Sketch a graph of $k(t)$.

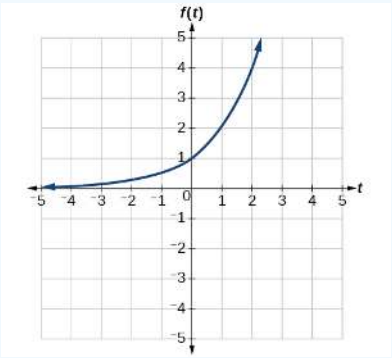


Figure **2.3.17**: Graph of $k(t)$

Solution

This equation combines three transformations into one equation.

- A horizontal reflection: $f(-t) = 2^{-t}$
- A vertical reflection: $-f(-t) = -2^{-t}$
- A vertical shift: $-f(-t) + 1 = -2^{-t} + 1$

We can sketch a graph by applying these transformations one at a time to the original function. Let us follow two points through each of the three transformations. We will choose the points $(0, 1)$ and $(1, 2)$.

- First, we apply a horizontal reflection (x coordinates are negated): $(0, 1)$ $(-1, 2)$.
- Then, we apply a vertical reflection (y coordinates are negated): $(0, -1)$ $(-1, -2)$.
- Finally, we apply a vertical shift (up 1 unit so y coordinates are increased by one): $(0, 0)$ $(-1, -1)$.

This means that the original points, $(0, 1)$ and $(1, 2)$ become $(0, 0)$ and $(-1, -1)$ after we apply the transformations.

In Figure 2.3.17s, the first graph results from a horizontal reflection. The second results from a vertical reflection. The third results from a vertical shift up 1 unit.

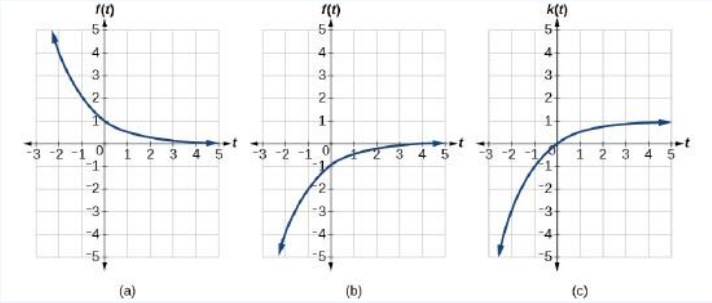


Figure 2.3.17s: Graphs of all the transformations.

Analysis. As a model for learning, this function would be limited to a domain of $t \geq 0$, with corresponding range $[0, 1)$.

Try It 2.3.18

Given the toolkit function $f(x) = x^2$, graph $g(x) = -f(x)$ and $h(x) = f(-x)$. Take note of any surprising behavior for these functions.

Answer

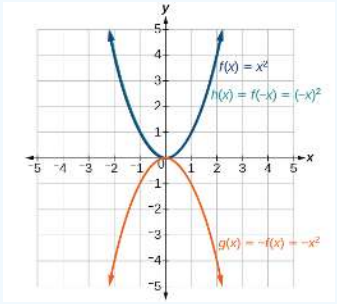


Figure 2.3.18: Graph of x^2 and its reflections.

Notice: $g(x) = f(-x)$ looks the same as $f(x)$.

Stretches and Compressions

Adding a constant to the inputs or outputs of a function changes the position of a graph with respect to the axes, but it does not affect the shape of a graph. We now explore the effects of multiplying the inputs or outputs by some quantity.

We can transform the inside (input values) of a function or we can transform the outside (output values) of a function. Each change has a specific effect that can be seen graphically.

Vertical Stretches and Compressions

When we multiply a function by a positive constant, we get a function whose graph is stretched or compressed vertically in relation to the graph of the original function. If the constant is greater than 1, we get a **vertical stretch**; if the constant is between 0 and 1, we get a **vertical compression**. Figure 2.3.19 shows a function multiplied by constant factors 2 and 0.5 and the resulting vertical stretch and compression.

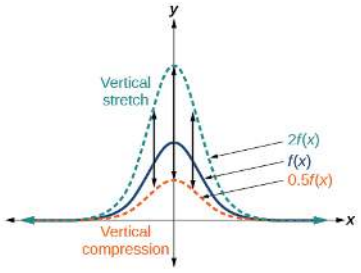


Figure 2.3.19: Vertical stretch and compression

Vertical Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = af(x)$, where a is a constant, is a **vertical stretch** or **vertical compression** of the function $f(x)$.

- If $a > 1$, then the graph will be stretched.
- If $0 < a < 1$, then the graph will be compressed.
- If $a < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection.

How to: Given a function, graph its vertical stretch.

1. Given a function f and a new function $g(x) = af(x + p) + k$, identify the value of the vertical stretch factor a.
2. Multiply all range (y) values by a
    a. If $a > 1$, the graph is stretched by a factor of a.
    b. If $0 < a < 1$, the graph is compressed by a factor of a.
    c. If $a < 0$, the graph is either stretched or compressed and is also reflected about the x-axis.

Example 1.5.20: Graphing a Vertical Stretch

A function $P(t)$ models the population of fruit flies. The graph is shown in Figure 2.3.20.

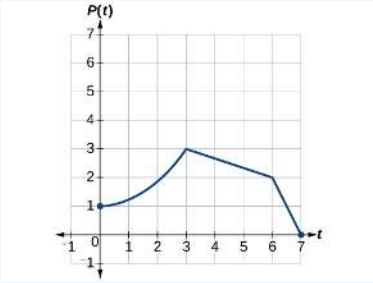


Figure 2.3.20: Graph to represent the growth of the population of fruit flies.

A scientist is comparing this population to another population, Q, whose growth follows the same pattern, but is twice as large. Sketch a graph of this population.

Solution

Because the population is always twice as large, the new population's output values are always twice the original function's output values. Graphically, this is shown in Figure **2.3.20s** .

TIf we choose four reference points, $(0,1)$, $(3,3)$, $(6,2)$ and $(7,0)$ we will multiply all of the outputs by 2.

The following shows where the new points for the new graph will be located. The corresponding graph is on the right.

$$(0,1) \to (0,2)$$
$$(3,3) \to (3,6)$$
$$(6,2) \to (6,4)$$
$$(7,0) \to (7,0)$$

Symbolically, the relationship is written as

$$Q(t) = 2P(t)$$

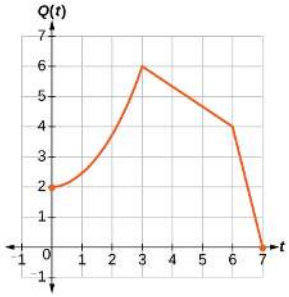


***Figure* 2.3.20s:** *Graph of the population function doubled.*

This means that for any input t, the value of the function Q is twice the value of the function P. Notice that the effect on the graph is a vertical stretching of the graph, where every point doubles its distance from the horizontal axis. The input values, t, stay the same while the output values are twice as large as before.

How to: Given a tabular function, create a table for a vertical stretch or compression.

1. Given a function f and a new function $g(x) = af(x+p) + k$, identify the value of the vertical stretch factor a.
2. Multiply all of the output values by a.

Example 2.3.21: Finding a Vertical Compression of a Tabular Function

A function f is given as Table **2.3.21**.
Create a table for the function $g(x) = \frac{1}{2}f(x)$.

Table **2.3.21**

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = \frac{1}{2}f(x)$ tells us that the output values of g are half of the output values of f with the same inputs. For example, we know that $f(4) = 3$. Then

$$g(4) = \frac{1}{2}f(4) = \frac{1}{2}(3) = \frac{3}{2}$$

We do the same for the other values to produce Table **2.3.21s**.

Table **2.3.21s**

x	2	4	6	8
$g(x)$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{7}{2}$	$\frac{11}{2}$

Analysis

The result is that the function $g(x)$ has been compressed vertically by $\frac{1}{2}$. Each output value is divided in half, so the graph is half the original height.

Try It 2.3.22

A function f is given as Table 2.3.22.
Create a table for the function $g(x) = \frac{3}{4}f(x)$.

Table 2.3.22

x	2	4	6	8
$f(x)$	12	16	20	0

Answer

Table 2.3.22s

x	2	4	6	8
$g(x)$	9	12	15	0

Example 2.3.23: Construct an equation of a stretched toolkit function from a graph

The graph in Figure 2.3.23 is a transformation of the toolkit function $f(x) = x^3$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

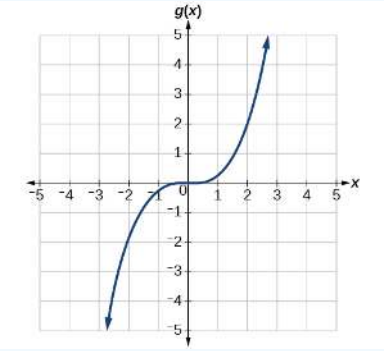


Figure 2.3.23: Graph of a transformation of $f(x) = x^3$

When trying to determine a vertical stretch or shift, it is helpful to look for a point on the graph that is relatively clear. In this graph, it appears that the function is a vertical stretch of the basic cubing function, so the general form of g is $g(x) = ax^3$ and also, $g(2) = 2$.

Putting these two pieces of information together we can solve for a: $2 = a(2)^3 \longrightarrow 2 = 8a \longrightarrow a = \frac{1}{4}$ and thus $g(x) = \frac{1}{4}x^3$.

Another approach to finding the formula for g is using the fact that with the basic cubic function at the same input, $f(2) = 2^3 = 8$. Based on that, it appears that the outputs of g are $\frac{1}{4}$ the outputs of the function f because $g(2) = \frac{1}{4}f(2)$. From this we can fairly safely conclude that $g(x) = \frac{1}{4}f(x)$. Then we can write a formula for g by using the definition of the function f to obtain $g(x) = \frac{1}{4}f(x) = \frac{1}{4}x^3$.

Try It 2.3.24

Write the formula for the function that we get when we stretch the identity toolkit function by a factor of 3, and then shift it down by 2 units.

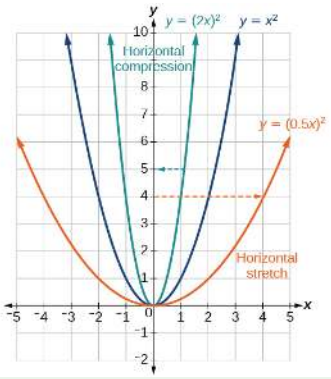
Answer

$g(x) = 3x - 2$

Horizontal Stretches and Compressions

Now we consider changes to the inside of a function. When we multiply a function's input by a positive constant, we get a function whose graph is stretched or compressed horizontally in relation to the graph of the original function. If the constant is between 0 and 1, we get a **horizontal stretch**; if the constant is greater than 1, we get a **horizontal compression** of the function.

Given a function $y = f(x)$, the form $y = f(bx)$ results in a horizontal stretch or compression. Consider the function $y = x^2$. Observe the graph above that illustrates vertical stretch and compression of x^2. The graph of $y = (0.5x)^2$ is a horizontal stretch of the graph of the function $y = x^2$ by a factor of 2. The graph of $y = (2x)^2$ is a horizontal compression of the graph of the function $y = x^2$ by a factor of $\frac{1}{2}$.



How to: Horizontal Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = f(bx)$, where b is a constant, is a **horizontal stretch** or **horizontal compression** of the function $f(x)$.

- If $b > 1$, then the graph will be compressed by $\frac{1}{b}$.
- If $0 < b < 1$, then the graph will be stretched by $\frac{1}{b}$.
- If $b < 0$, then there will be combination of a horizontal stretch or compression with a horizontal reflection.

How to: Given a description of a function, sketch a horizontal compression or stretch.

1. Write a formula to represent the function.
2. Set $g(x) = f(bx)$ where $b > 1$ for a compression or $0 < b < 1$ for a stretch.

Example 2.3.25: Graphing a Horizontal Compression

Suppose a scientist is comparing a population of fruit flies to a population that progresses through its lifespan twice as fast as the original population. In other words, this new population, R, will progress in 1 hour the same amount as the original population, P, does in 2 hours, and in 2 hours, it will progress as much as the original population does in 4 hours. Sketch a graph of this population.

Solution

Symbolically, we could write

$$\begin{aligned} R(1) &= P(2), \\ R(2) &= P(4), \qquad \text{and in general,} \\ R(t) &= P(2t). \end{aligned}$$

See Figure 2.3.25 for a graphical comparison of the original population and the compressed population.

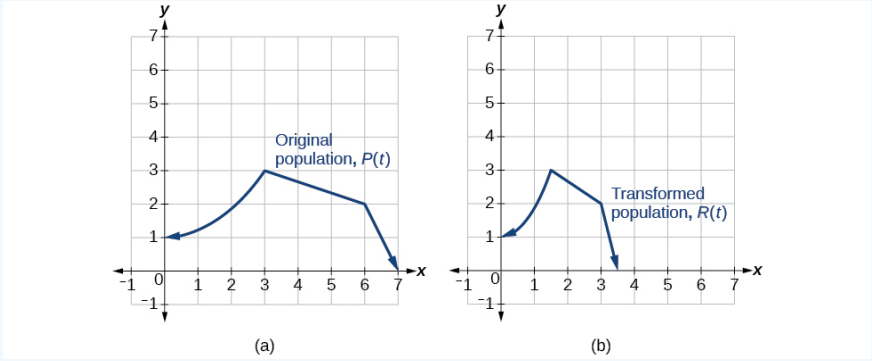


Figure 2.3.25: *(a) Original population graph (b) Compressed population graph*

Analysis

Note that the effect on the graph is a horizontal compression where all input values are half of their original distance from the vertical axis.

Example 2.3.26: Finding a Horizontal Stretch for a Tabular Function

A function $f(x)$ is given as Table 2.3.26. Create a table for the function $g(x) = f(\frac{1}{2}x)$.

Table 2.3.26

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = f(\frac{1}{2}x)$ tells us that the output values for g are the same as the output values for the function f at an input half the size. Notice that we do not have enough information to determine $g(2)$ because $g(2) = f(\frac{1}{2} \cdot 2) = f(1)$, and we do not have a value for $f(1)$ in our table. Our input values to g will need to be twice as large to get inputs for f that we can evaluate. For example, we can determine $g(4)$.

$$g(4) = f(\frac{1}{2} \cdot 4) = f(2) = 1$$

We do the same for the other values to produce Table 2.3.26s. Figure 2.3.26g shows the graphs of both of these sets of points.

Table 2.3.26s for $g(x) = f(\frac{1}{2}x)$

x	4	8	12	16
$\frac{1}{2}x$	2	4	6	8
$g(x)$	1	3	7	11

Analysis. Because each input value has been doubled, the result is that the function $g(x)$ has been stretched horizontally by a factor of 2.

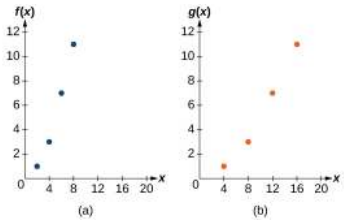


Figure 2.3.26g: Graph of the previous table.

Example 2.3.27: Recognizing a Horizontal Compression on a Graph

Relate the function $g(x)$ to $f(x)$ in Figure 2.3.27.

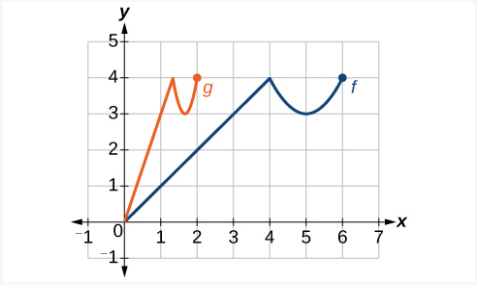


Figure 2.3.27: Graph of $f(x)$ being vertically compressed to $g(x)$.

Solution

The graph of $g(x)$ looks like the graph of $f(x)$ horizontally compressed. Because $f(x)$ ends at (6,4) and $g(x)$ ends at (2,4), we can see that the x-values have been compressed by $\frac{1}{3}$, because $6(\frac{1}{3}) = 2$. We might also notice that $g(2) = f(6)$ and $g(1) = f(3)$. Either way, we can describe this relationship as $g(x) = f(3x)$. This is a horizontal compression by $\frac{1}{3}$.

Analysis

Notice that the coefficient needed for a horizontal stretch or compression is the reciprocal of the stretch or compression. So to stretch the graph horizontally by a scale factor of 4, we need a coefficient of $\frac{1}{4}$ in our function: $f(\frac{1}{4}x)$. This means that the

input values must be four times larger to produce the same result, requiring the input to be larger, causing the horizontal stretching.

Try It 2.3.28

Write a formula for the toolkit square root function horizontally stretched by a factor of 3.

Answer

$g(x) = f(\frac{1}{3}x)$, so using the square root function we get $g(x) = \sqrt{\frac{1}{3}x}$

Performing a Sequence of Transformations

When combining transformations, it is very important to consider the order of the transformations. For example, vertically shifting by 3 and then vertically stretching by 2 does not create the same graph as vertically stretching by 2 and then vertically shifting by 3, because when we shift first, both the original function and the shift get stretched, while only the original function gets stretched when we stretch first.

When we see an expression such as $2f(x) + 3$, which transformation should we start with? The answer here follows nicely from the order of operations. Given the output value of $f(x)$, we first multiply by 2, causing the vertical stretch, and then add 3, causing the vertical shift. In other words, multiplication before addition.

Horizontal transformations are a little trickier to think about. When we write $g(x) = f(2x + 3)$, for example, we have to think about how the inputs to the function g relate to the inputs to the function f. Suppose we know $f(7) = 12$. What input to g would produce that output? In other words, what value of x will allow $g(x) = f(2x + 3) = 12$? We would need $2x + 3 = 7$. To solve for x, we would first subtract 3, resulting in a horizontal shift, and then divide by 2, causing a horizontal compression.

This format ends up being very difficult to work with, because it is usually much easier to horizontally stretch a graph before shifting. We can work around this by factoring inside the function.

$$f(bx + p) = f(b(x + \tfrac{p}{b}))$$

Let's work through an example.

$$f(x) = (2x + 4)^2$$

We can factor out a 2.

$$f(x) = (2(x + 2))^2$$

Now we can more clearly observe a horizontal shift to the left 2 units and a horizontal compression. Factoring in this way allows us to horizontally stretch first and then shift horizontally.

How to: Combine Transformations

- For vertical transformations in the form: $\overset{1}{\downarrow}f(x) + \overset{2}{\downarrow}k$ $\begin{cases} 1^{st} \text{ vertically stretch by a factor of } a \\ 2^{nd} \text{ vertically shift by } k \text{ units.} \end{cases}$
- For horizontal transformations in the form: $f(\overset{2}{\downarrow}bx + \overset{1}{\downarrow}p)$: $\begin{cases} 1^{st} \text{ horizontally shift by } -p \text{ units} \\ 2^{nd} \text{ horizontally stretch by a factor of } \frac{1}{b}. \end{cases}$
- For horizontal transformations in the form: $f(\overset{1}{\downarrow}b(x + \overset{2}{\downarrow}p))$ $\begin{cases} 1^{st} \text{ horizontally stretch by a factor of } \frac{1}{b}. \\ 2^{nd} \text{ horizontally shift by } -p \text{ units} \end{cases}$

Horizontal and vertical transformations are independent. It does not matter whether horizontal or vertical transformations are performed first.

Example 2.3.29: Finding a Triple Transformation of a Tabular Function

Given Table 2.3.29 for the function $f(x)$,
create a table of values for the function $g(x) = 2f(3x) + 1$.

Table 2.3.29

x	6	12	18	24
$f(x)$	10	14	15	17

Solution

There are three steps to this transformation, and we will work from the inside out. Starting with the horizontal transformations, $f(3x)$ is a horizontal compression by $\frac{1}{3}$, which means we multiply each x-value by $\frac{1}{3}$. See Table 2.3.29a.

Table 2.3.29a

x	2	4	6	8
$f(3x)$	10	14	15	17

Looking now to the vertical transformations, we start with the vertical stretch, which will multiply the output values by 2. We apply this to the previous transformation. See Table 2.3.29b.

Table 2.3.29b

x	2	4	6	8
$2f(3x)$	20	28	30	34

Finally, we can apply the vertical shift, which will add 1 to all the output values. See Table 2.3.29c.

Table 2.3.29c for $g(x) = 2f(3x) + 1$

x	2	4	6	8
$2f(3x) + 1$	21	29	31	35

Example 2.3.30: Finding a Triple Transformation of a Graph

Use the graph of $f(x)$ in Figure 2.3.30 to sketch a graph of $k(x) = f\left(\frac{1}{2}x + 1\right) - 3$.

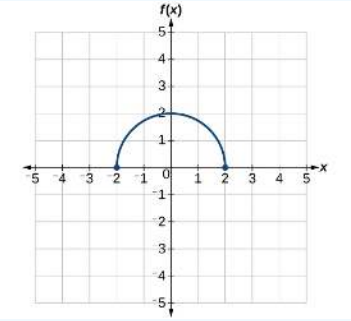


Figure 2.3.30: Graph of a half-circle.

Solution

Step 1. To simplify, let's start by factoring out the inside of the function.

$$f\left(\frac{1}{2}x + 1\right) - 3 = f\left(\frac{1}{2}(x + 2)\right) - 3$$

By factoring the inside, we can first horizontally stretch by 2, as indicated by the $\frac{1}{2}$ on the inside of the function. Remember that twice the size of 0 is still 0, so the point $(0, 2)$ remains at $(0, 2)$ while the point $(2, 0)$ will stretch to $(4, 0)$. See Figure 2.3.30a.

Step 2. Next, we horizontally shift left by 2 units, as indicated by $x + 2$. See Figure 2.3.30b.

Step 3. Last, we vertically shift down by 3 to complete our sketch, as indicated by the −3 on the outside of the function. See Figure **2.3.30***c*.

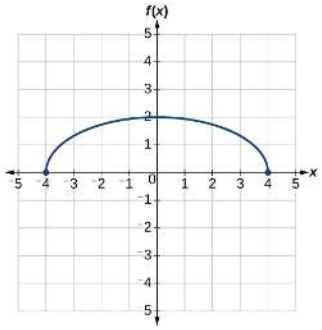


Figure **2.3.30***a*: Graph of a vertically stretch half-circle.

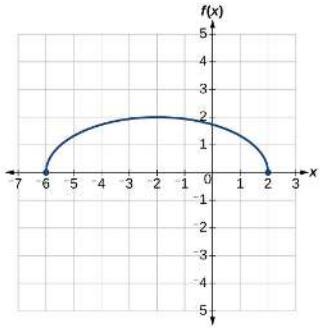


Figure **2.3.30***b*: Graph of a vertically stretch and translated half-circle.

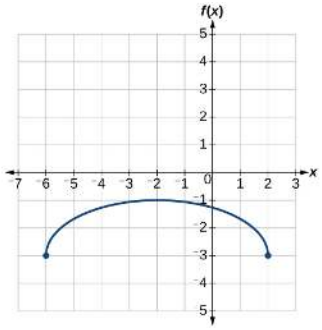


Figure **2.3.30***c*: Graph of a vertically stretch and translated half-circle.

Key Equations

- Vertical shift $g(x) = f(x) + k$ (up for $k > 0$)
- Horizontal shift $g(x) = f(x - h)$(right) for $h > 0$
- Vertical reflection $g(x) = -f(x)$
- Horizontal reflection $g(x) = f(-x)$
- Vertical stretch $g(x) = af(x), \quad a > 1$
- Vertical compression $g(x) = af(x), \quad 0 < a < 1$
- Horizontal stretch $g(x) = f(bx), \quad 0 < b < 1$
- Horizontal compression $g(x) = f(bx), \quad b > 1$

Key Concepts

- A function can be shifted vertically by adding a constant to the output.
- A function can be shifted horizontally by adding a constant to the input.
- Vertical and horizontal shifts are often combined.
- A vertical reflection reflects a graph about the x-axis. A graph is reflected vertically by multiplying the output by –1.
- A horizontal reflection reflects a graph about the y-axis. A graph is reflected horizontally by multiplying the input by –1.
- A graph can be reflected both vertically and horizontally.
- A function presented in tabular form can also be reflected by multiplying the values in the input and output rows or columns accordingly.
- A function presented as an equation can be reflected by applying transformations one at a time.
- A function can be compressed or stretched vertically by multiplying the output by a constant.
- A function can be compressed or stretched horizontally by multiplying the input by a constant.
- The order in which different transformations are applied does affect the final function. Both vertical and horizontal transformations must be applied in the order given. However, a vertical transformation may be combined with a horizontal transformation in any order.

Glossary

horizontal compression
a transformation that compresses a function’s graph horizontally, by multiplying the input by a constant b>1

horizontal reflection
a transformation that reflects a function’s graph across the y-axis by multiplying the input by −1

horizontal shift
a transformation that shifts a function’s graph left or right by adding a positive or negative constant to the input

horizontal stretch
a transformation that stretches a function’s graph horizontally by multiplying the input by a constant 0<b<1

vertical compression
a function transformation that compresses the function’s graph vertically by multiplying the output by a constant 0<a<1

vertical reflection
a transformation that reflects a function’s graph across the x-axis by multiplying the output by −1

vertical shift
a transformation that shifts a function’s graph up or down by adding a positive or negative constant to the output

vertical stretch
a transformation that stretches a function’s graph vertically by multiplying the output by a constant a>

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2.3e: Exercises - Transformations

A: Concepts

Exercise $2.3e.A$

1) When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal shift from a vertical shift?
2) When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal stretch from a vertical stretch?
3) When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal compression from a vertical compression?
4) When examining the formula of a function that is the result of multiple transformations, how can you tell a reflection with respect to the x-axis from a reflection with respect to the y-axis?
5) How can you determine whether a function is odd or even from the formula of the function?

Answers to Odd Exercises:

1. A horizontal shift results when a constant is added to or subtracted from the input. A vertical shifts results when a constant is added to or subtracted from the output.

3. A horizontal compression results when a constant greater than 1 is multiplied by the input. A vertical compression results when a constant between 0 and 1 is multiplied by the output.

5. For a function f, substitute (−x) for (x) in $f(x)$. Simplify. If the resulting function is the same as the original function, $f(-x) = f(x)$, then the function is even. If the resulting function is the opposite of the original function, $f(-x) = -f(x)$, then the original function is odd. If the function is not the same or the opposite, then the function is neither odd nor even.

B: Describe transformations of a function written in function notation

Exercise $2.3e.B$

★ Describe how the graph of the function is a transformation of the graph of the original function f.

6) $y = f(x-49)$	10) $y = f(x)+5$	14) $y = f(x-2)+3$	18) $g(x) = 6f(x)$	22) $g(x) = f\left(\frac{1}{5}x\right)$
7) $y = f(x+43)$	11) $y = f(x)+8$	15) $y = f(x+4)-1$	19) $g(x) = 4f(x)$	23) $g(x) = f\left(\frac{1}{3}x\right)$
8) $y = f(x+3)$	12) $y = f(x)-2$	16) $g(x) = f(-x)$	20) $g(x) = f(2x)$	24) $g(x) = -f(3x)$
9) $y = f(x-4)$	13) $y = f(x)-7$	17) $g(x) = -f(x)$	21) $g(x) = f(5x)$	25) $g(x) = 3f(-x)$

Answers to Odd Exercises:

7. The graph of $f(x+43)$ is a horizontal shift to the left 43 units of the graph of f.

9. The graph of $f(x-4)$ is a horizontal shift to the right 4 units of the graph of f.

11. The graph of $f(x)+8$ is a vertical shift up 8 units of the graph of f.

13. The graph of $f(x)-7$ is a vertical shift down 7 units of the graph of f.

15. The graph of $f(x+4)-1$ is a horizontal shift left 4 units and vertical shift down 1 unit of the graph of f.

17. The graph of g is a vertical reflection (across the x-axis) of the graph of f.

19. The graph of g is a vertical stretch by a factor of 4 of the graph of f.

21. The graph of g is a horizontal compression by a factor of $\frac{1}{5}$ of the graph of f.

23. The graph of g is a horizontal stretch by a factor of 3 of the graph of f.

25. The graph of g is a horizontal reflection across the y-axis and a vertical stretch by a factor of 3 of the graph of f.

Exercise 2.3e.C

★ **Begin by graphing the basic quadratic function $f(x) = x^2$. State the transformations needed to apply to f to graph the function below. Then use transformations to graph the function.**

27. $g(x) = x^2 + 1$
28. $g(x) = x^2 - 4$
29. $g(x) = (x-5)^2$
30. $g(x) = (x+1)^2$
31. $g(x) = (x-5)^2 + 2$
32. $g(x) = (x+2)^2 - 5$
33. $f(t) = (t+1)^2 - 3$
34. $f(x) = -(x+2)^2$
35. $f(x) = -x^2 + 6$
36. $g(x) = -2x^2$
37. $g(x) = 4(x+1)^2 - 5$
38. $g(x) = 5(x+3)^2 - 2$
39. $h(x) = \frac{1}{2}(x-1)^2$
40. $h(x) = \frac{1}{3}(x+2)^2$
41. $f(x) = (-\frac{1}{2}x - 3)^2 + 1$
42. $g(x) = (-2x+3)^2 - 4$

★ **Begin by graphing the square root function $f(x) = \sqrt{x}$. State the transformations needed to apply to f to graph the function below. Then use transformations to graph the function.**

43. $g(x) = \sqrt{x} - 5$
44. $g(x) = \sqrt{x-5}$
45. $g(x) = \sqrt{x-2} + 1$
46. $g(x) = \sqrt{x+2} + 3$
47. $a(x) = \sqrt{-x+4}$
48. $m(t) = 3 - \sqrt{t+2}$
49. $h(x) = \sqrt{-x} + 2$
50. $g(x) = -\sqrt{x} + 2$
51. $g(x) = -\frac{1}{2}\sqrt{x-3}$
52. $h(x) = -\sqrt{x-2} + 1$
53. $f(x) = 4\sqrt{x-1} + 2$
54. $f(x) = -5\sqrt{x+2}$
55. $k(x) = \sqrt{2x+5} - 1$
56.1 $a(x) = \sqrt{\frac{1}{3}x - 4}$
56.2 $b(x) = \sqrt{3-x} + 2$

★ **Begin by graphing the absolute value function $f(x) = |x|$. State the transformations needed to apply to f to graph the function below. Then use transformations to graph the function.**

57. $h(x) = |x+4|$
58. $h(x) = |x-4|$
59. $h(x) = |x-1| - 3$
60. $h(x) = |x+2| - 5$
61. $g(x) = -|x-1|$
62. $h(x) = |x-1| + 4$
63. $f(x) = -3|x|$
64. $f(x) = -|x| - 3$
65. $h(x) = -2|x-4| + 3$
66. $n(x) = \frac{1}{3}|x-2|$
67. $h(x) = |-3x+4| - 2$
68. $g(x) = |\frac{1}{3}x - 2| + 1$

★ **Begin by graphing the standard cubic function $f(x) = x^3$. State the transformations needed to apply to f to graph the function below. Then use transformations to graph the function.**

69. $h(x) = (x-2)^3$
70. $h(x) = x^3 + 4$
71. $h(x) = (x-1)^3 - 4$
72. $h(x) = (x+1)^3 + 3$
73. $g(x) = -(x+2)^3$
74. $k(x) = (x-2)^3 - 1$
75. $g(x) = -x^3 + 4$
76. $m(x) = \frac{1}{2}x^3$
77. $g(x) = -\frac{1}{4}(x+3)^3 - 1$
78. $q(x) = \left(\frac{1}{4}x\right)^3 + 1$
79. $p(x) = \left(\frac{1}{3}x\right)^3 - 3$

★ **Begin by graphing the appropriate parent function : the basic cube root function $f(x) = \sqrt[3]{x}$, constant function $f(x) = 0$, or linear function $f(x) = x$. Then use transformations of this graph to graph the given function.**

81. $g(x) = \sqrt[3]{x} - 1$
82. $g(x) = \sqrt[3]{x-1}$
83. $g(x) = \sqrt[3]{x-2} + 6$
84. $g(x) = \sqrt[3]{x+8} - 4$
84.1 $g(x) = \sqrt[3]{-x+3} - 2$
84.1 $g(x) = -\sqrt[3]{x-1} + 2$
85. $g(x) = -2\sqrt[3]{x+3} + 4$
86. $g(x) = \sqrt[3]{-2x-5} - 1$
87. $f(x) = x + 3$
88. $h(x) = -2x + 1$
89. $g(x) = -4$

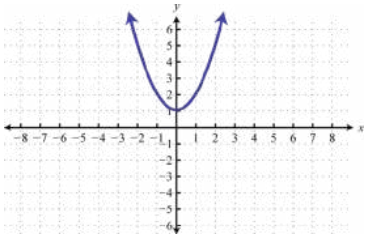
★ **Begin by graphing the basic reciprocal function $f(x) = \frac{1}{x}$. State the transformations needed to apply to f to graph the function below. Then use transformations to graph the function.**

91. $f(x) = \dfrac{1}{x-2}$
92. $f(x) = \dfrac{1}{x+3}$
93. $f(x) = \dfrac{1}{x} + 5$
94. $f(x) = \dfrac{1}{x} - 3$
95. $f(x) = \dfrac{1}{x+1} - 2$
96. $f(x) = \dfrac{1}{x-3} + 3$
97. $f(x) = -\dfrac{1}{x+2}$
98. $f(x) = -\dfrac{1}{x}$
99. $p(x) = -\dfrac{1}{x+1} + 2$
100.1 $a(x) = \dfrac{2}{x-3} - 5$
100.2 $b(x) = \dfrac{1}{2x+6} + 4$

Answers to Odd Numbered Exercises for the Squaring Function:

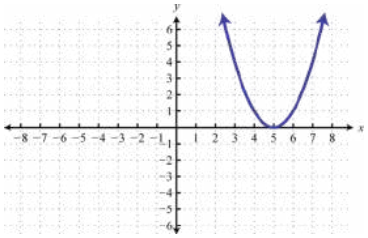
Squaring Function

27. $y = x^2$; Shift up 1 unit; domain: $\mathbb{R}$; range: $[1, \infty)$



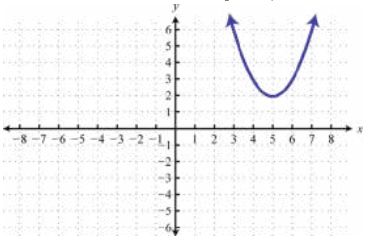
$g(x) = x^2 + 1$

29. $y = x^2$; Shift right 5 units; domain: $\mathbb{R}$; range: $[0, \infty)$



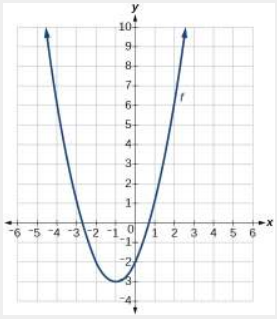
$g(x) = (x - 5)^2$

31. $y = x^2$; Shift right 5 units and up 2 units; domain: $\mathbb{R}$; range: $[2, \infty)$



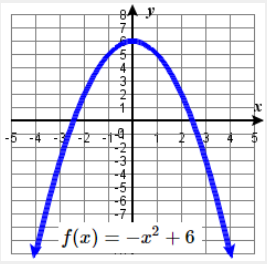
$g(x) = (x - 5)^2 + 2$

33. Shift left 1 unit and down 3 units;

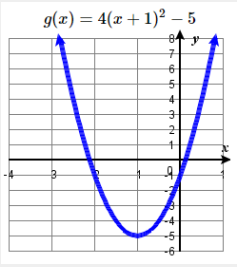


$f(t) = (t + 1)^2 - 3$

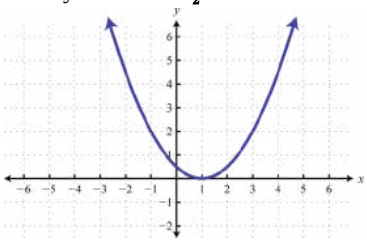
#35 Reflect over x-axis, up 6 units.



37 $f(x) = x^2$ is shifted to the left 1 unit, stretched vertically by a factor of 4, and shifted down 5 units.

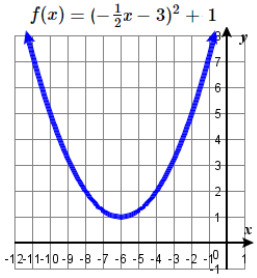


39. Shift right 1 unit, and vertically shrink by a factor of $\frac{1}{2}$



$h(x) = \frac{1}{2}(x - 1)^2$

#41 Shift right 3, reflect over y-axis, horizontally stretch by a factor of 2, up 1 units.

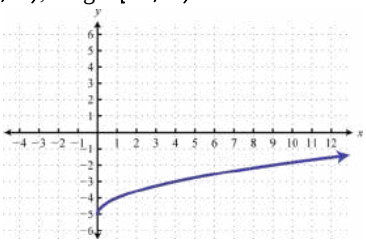


for # 41, if $f(x) = (-\frac{1}{2}x - 3)^2 + 1$ is rewritten as
$f(x) = (-\frac{1}{2}(x + 6))^2 + 1$,
then the transformations would be horizontal stretch by a factor of 2, reflect in y-axis (no change), left 6, up 1.

Answers to Odd Numbered Exercises for the Square Root Function

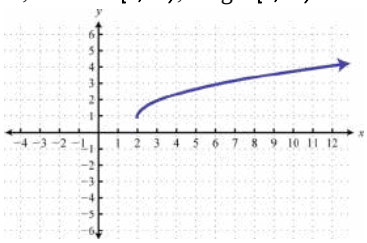
Square Root Function

43. $y = \sqrt{x}$; Shift down 5 units; domain: $[0, \infty)$; range: $[-5, \infty)$



$g(x) = \sqrt{x} - 5$

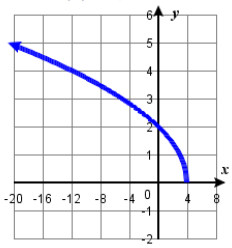
45. $y = \sqrt{x}$; Shift right 2 units and up 1 unit; domain: $[2, \infty)$; range: $[1, \infty)$



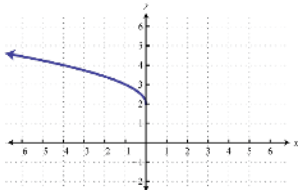
$g(x) = \sqrt{x-2} + 1$

47 The graph of $f(x) = \sqrt{x}$ is shifted left 4 units and then reflected across the y-axis.

$a(x) = \sqrt{-x+4}$

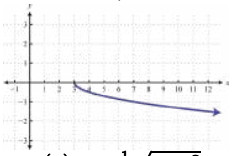


49. Reflect over y-axis, up 2



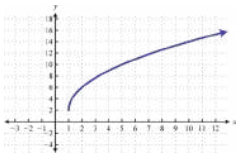
$h(x) = \sqrt{-x} + 2$

51. Right 3, Reflect over x axis, Vertically compressed by a factor of $1/2$.



$g(x) = -\frac{1}{2}\sqrt{x-3}$

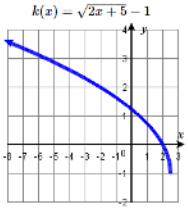
53. Right 1, Vertically stretched by a factor of 4, up 2



$f(x) = 4\sqrt{x-1} + 2$

#55 Horizontal compression by $1/2$, shift left 2.5, down 1 unit.

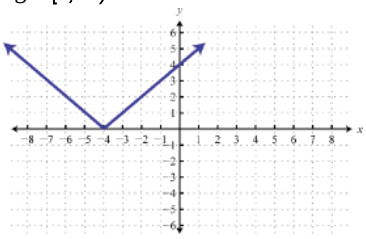
$k(x) = \sqrt{2x+5} - 1$



Answers to Odd Numbered Exercises for the Absolute Value Function:

Absolute Value Function

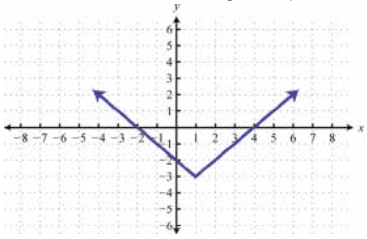
57. $y = |x|$; Shift left 4 units; domain: $\mathbb{R}$; range: $[0, \infty)$



$h(x) = |x + 4|$

.

59. $y = |x|$; Shift right 1 unit and down 3 units; domain: $\mathbb{R}$; range: $[-3, \infty)$



$h(x) = |x - 1| - 3$

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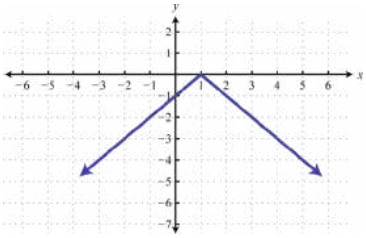
61. Right 1, Reflect over x-axis



$g(x) = -|x - 1|$

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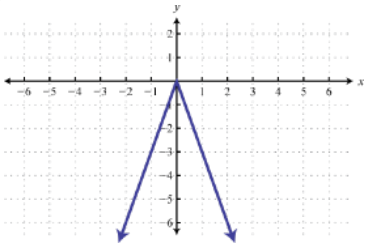
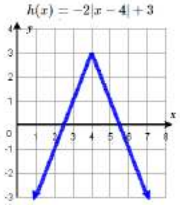
63. Reflect over x-axis, vertically stretch by a factor of 3

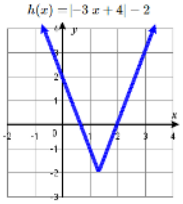


$f(x) = -3|x|$

65 The graph of $f(x) = |x|$ is shifted horizontally 4 units to the right, stretched vertically by a factor of 2, reflected across the horizontal axis, then shifted up 3 units.



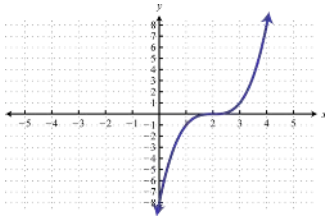
67. $h(x) = |-3(x - \frac{4}{3})| - 2 \longrightarrow$ Horizontally compress by a factor of $\frac{1}{3}$, right $\frac{4}{3}$, down 2



Answers to Odd Numbered Exercises for the Cubing Function

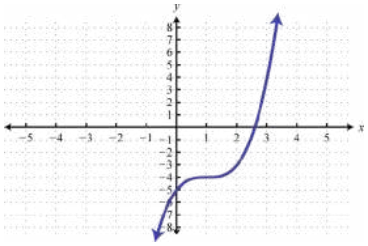
Cubing Function

69. $y = x^3$; Shift right 2 units; domain: $\mathbb{R}$; range: $\mathbb{R}$



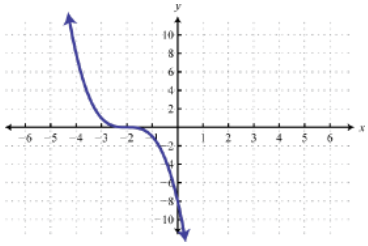
$h(x) = (x-2)^3$

71. $y = x^3$; Shift right 1 unit and down 4 units; domain: $\mathbb{R}$; range: $\mathbb{R}$



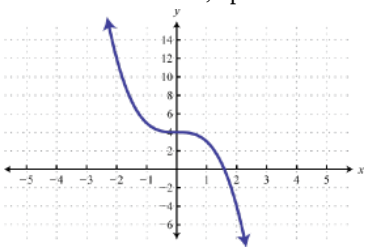
$h(x) = (x-1)^3 - 4$

73. Left 2 units, reflect over x-axis



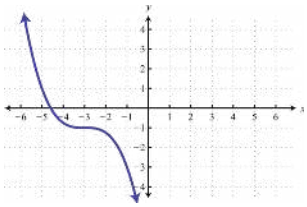
$g(x) = -(x+2)^3$

75. Reflect over x-axis, up 4 units



$g(x) = -x^3 + 4$

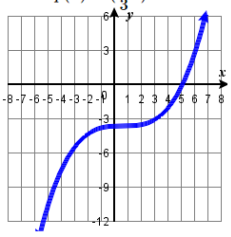
77. Left 3 units, reflect over x-axis, vertically shrink by a factor of $\frac{1}{4}$, down 1 unit



$g(x) = -\frac{1}{4}(x+3)^3 - 1$

79. Stretch horizontally by a factor of 3 and shift vertically downward by 3 units.

$p(x) = (\frac{1}{3}x)^3 - 3$

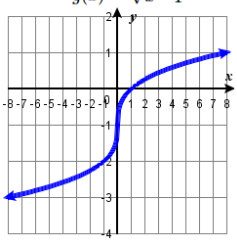


Answers to Odd Numbered Exercises for the Cube Root, Linear and Constant Functions

Cube Root, Linear, Constant Functions

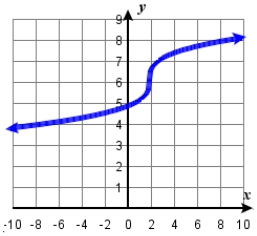
81. $y = \sqrt[3]{x}$; Shift down 1

$g(x) = \sqrt[3]{x} - 1$



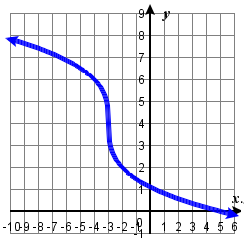
83. $y = \sqrt[3]{x}$; Shift up 6 units and right 2 units; domain: $\mathbb{R}$; range: $\mathbb{R}$

$g(x) = \sqrt[3]{x-2} + 6$

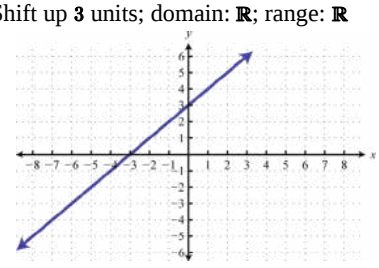


85. $y = \sqrt[3]{x}$; Left 3, reflect over x-axis, vertically stretch by a factor of 2, up 4.

$g(x) = -2\sqrt[3]{x+3} + 4$

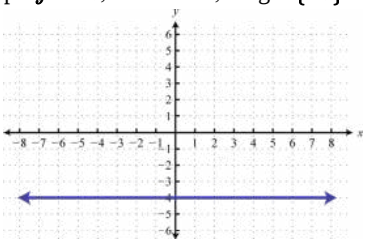


87. $y = x$; Shift up 3 units; domain: $\mathbb{R}$; range: $\mathbb{R}$



$f(x) = x + 3$

89. Basic graph $y = -4$; domain: $\mathbb{R}$; range: $\{-4\}$

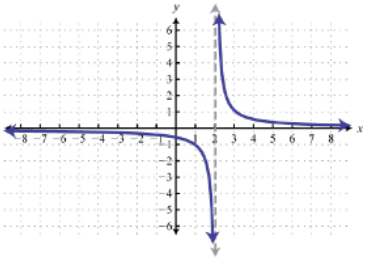


$g(x) = -4$

Answers to Odd Numbered Exercises for the Reciprocal Function:

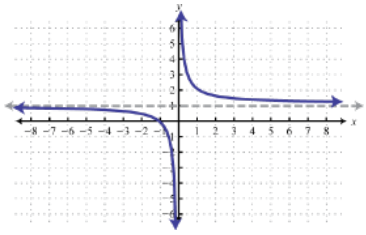
Reciprocal Function

91. $y = \frac{1}{x}$; Shift right 2 units; domain: $(-\infty, 2) \cup (2, \infty)$; range: $(-\infty, 0) \cup (0, \infty)$



$f(x) = \frac{1}{x-2}$

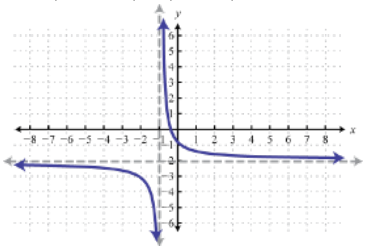
93. $y = \frac{1}{x}$; Shift up 5 units; domain: $(-\infty, 0) \cup (0, \infty)$; range: $(-\infty, 1) \cup (1, \infty)$



$f(x) = \frac{1}{x} + 5$

★

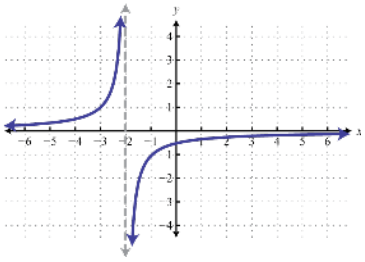
95. $y = \frac{1}{x}$; Shift left 1 unit and down 2 units; domain: $(-\infty, -1) \cup (-1, \infty)$; range: $(-\infty, -2) \cup (-2, \infty)$



$f(x) = \frac{1}{x+1} - 2$

97

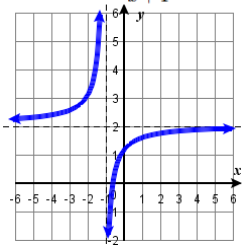
97. Left 2 units, reflect over x-axis



$f(x) = -\frac{1}{x+2}$

#99 Left 1 unit, reflect over x-axis, up 2 units.

$p(x) = \frac{-1}{x+1} + 2$

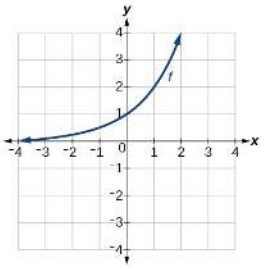


D: Graph Transformations of a Graph

Exercise 2.3e.D

★ Use the graph of $f(x)$ shown in the Figure below to sketch a graph of each transformation of $f(x)$.

Given the graph of $f(x)$ on the right, sketch the graph for the following transformations of f

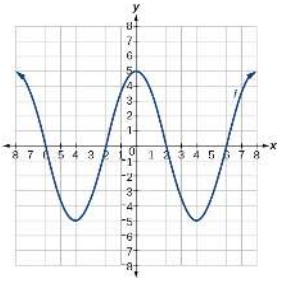


101. $h(x) = 2^x - 3$

102. a) $g(x) = 2^x + 1$

b) $w(x) = 2^x - 1$

Given the graph of $f(x)$ on the right, sketch the graph for the following transformations of f



103. a) $g(x) = -f(x)$

b) $g(x) = f(x-2)$

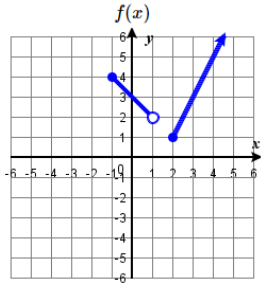
104. a) $g(x) = f(x) - 2$

b) $g(x) = f(x+1)$

106. Given the graph of $f(x)$ below, sketch the graph for the following transformations of f

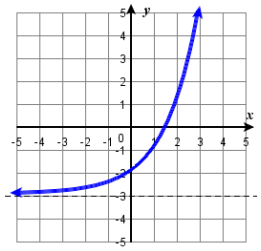
a. $a(x) = f(x) + 1$
b. $b(x) = f(x+1)$
c. $c(x) = f(x) + 2$
d. $d(x) = -f(x)$
e. $e(x) = f(1-x) - 2$
f. $f(x) = 2f(x)$
g. $g(x) = -f(x)$
h. $h(x) = \frac{1}{2}f(x+2) + 3$

i. $i(x) = f(x) - 1$
j. $j(x) = f(x-1)$
k. $k(x) = f(x) - 2$
l. $l(x) = f(-x)$
m. $m(x) = -f(x-1) + 2$
n. $n(x) = f(2x)$
o. $o(x) = f(-x)$
p. $p(x) = f(\frac{1}{2}x - 2) - 3$

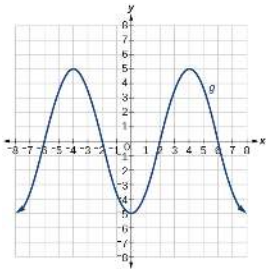


Answers to Odd Exercises:

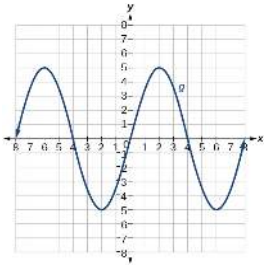
101



103. a



103. b



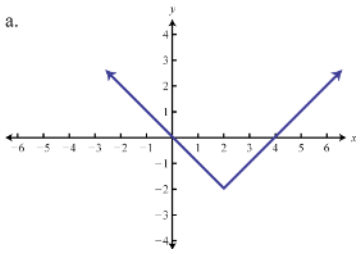
E: Match transformations of functions with graphs

Exercise 2.3e.E

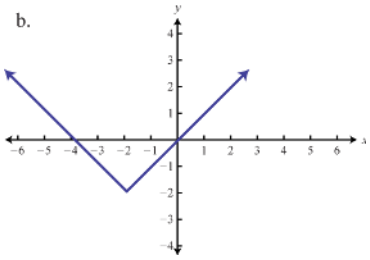
★ **Match the graph to the function definition.**

107. $f(x)=\sqrt{x+4}$
108. $f(x)=|x-2|-2$
109. $f(x)=\sqrt{x+1}-1$
110. $f(x)=|x-2|+1$
111. $f(x)=\sqrt{x+4}+1$
112. $f(x)=|x+2|-2$

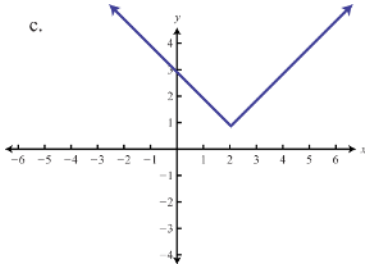
a.



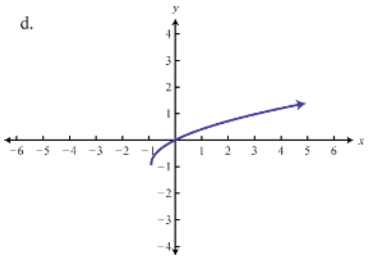
b.



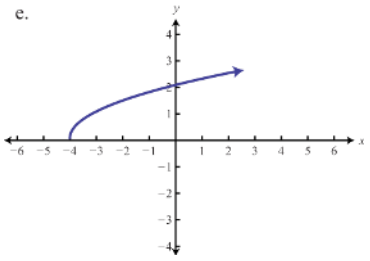
c.



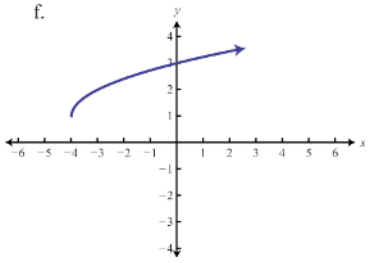
d.



e.



f.



Match the graph to the given function definition.

113. $f(x) = -3\|x\|$ 114. $f(x) = -(x+3)^2 - 1$	115. $f(x) = -\|x+1\| + 2$ 116. $f(x) = -x^2 + 1$	117. $f(x) = -\frac{1}{3}\|x\|$ 118. $f(x) = -(x-2)^2 + 2$

a.

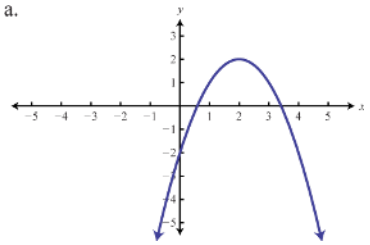


Figure (a)

b.

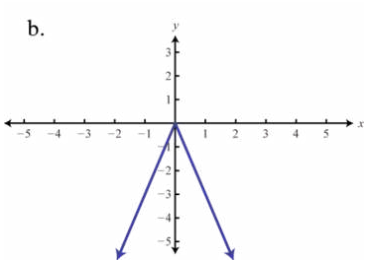


Figure (b)

★

c.

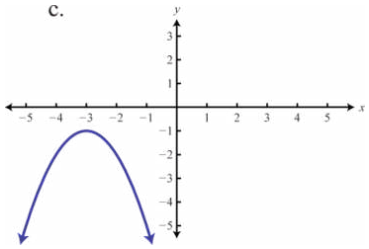


Figure (c)

d.

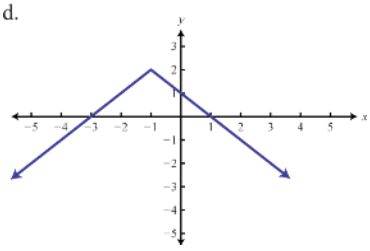


Figure (d)

★

e.

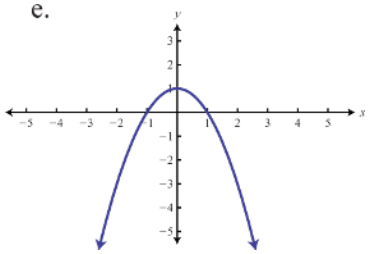


Figure (e)

f.

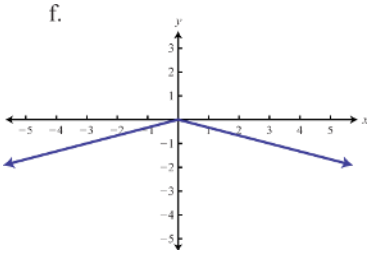


Figure (f)

★

Answers to Odd Exercises:

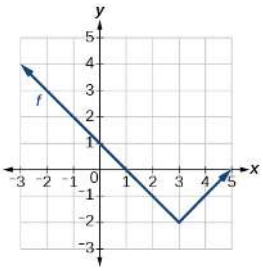
part 1 answers 107e, 109d, 111f, part 2 Answers: 113.b, 115.d, 117.f

F: Construct equations from graphs of transformed basic functions

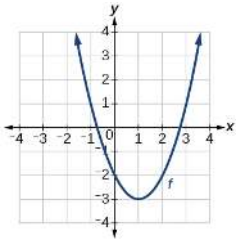
Exercise 2.3e.F

★ **Write an equation for each graphed function by using transformations of the graphs of one of the toolkit functions.**

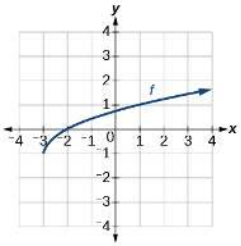
119.



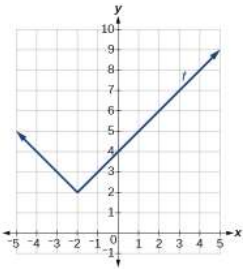
120.



121.



122.



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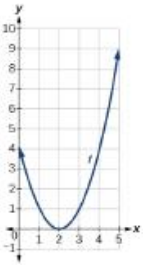
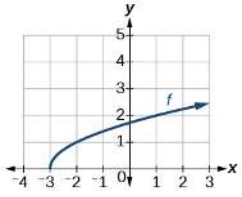


124.



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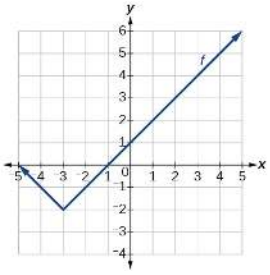
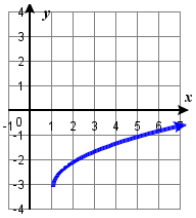


126.



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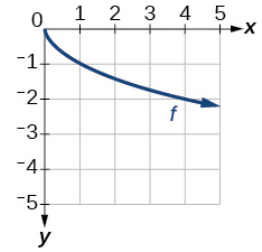
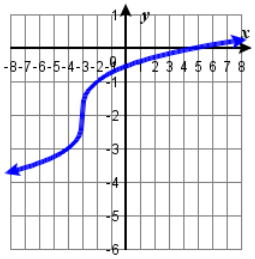
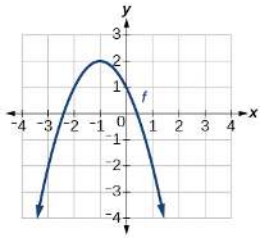


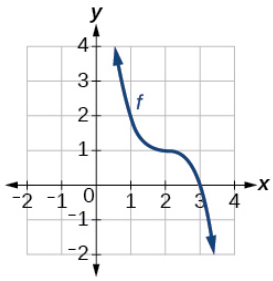
128.



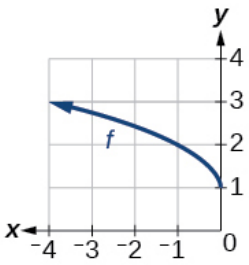
129.



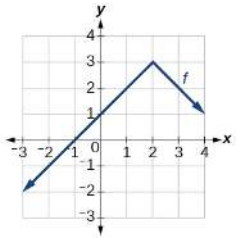
130. (a)



130. (b)



130. (c)



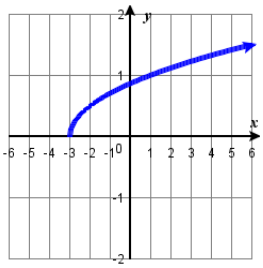
Answers to Odd Exercises:

119. $f(x) = \lvert x-3\rvert - 2$	123. $f(x) = (x-2)^2$	127. $f(x) = -\sqrt{x}$
121. $f(x) = \sqrt{x+3} - 1$	125. $f(x) = \lvert x+3\rvert - 2$	129. $f(x) = -(x+1)^2 + 2$

★ **Write an equation that represents the function whose graph is given.**

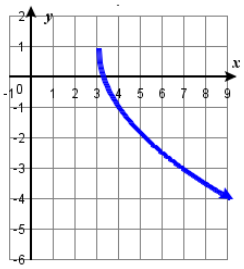
131.

Figure #131



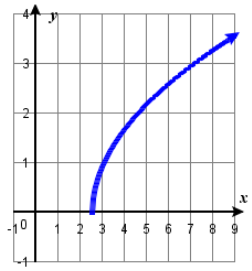
132.

Figure #132



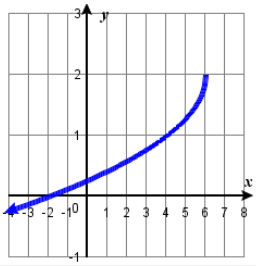
133.

Figure #133



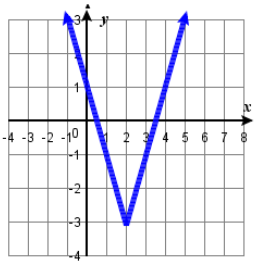
134.

Figure #134



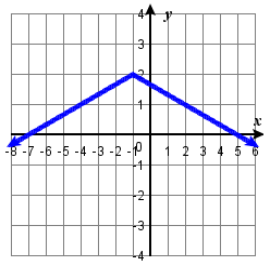
135.

Figure #135



136.

Figure #136



137.

Figure #137

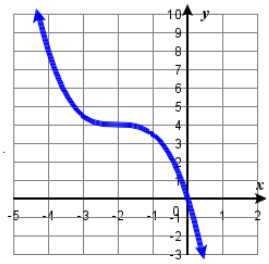


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Figure #138

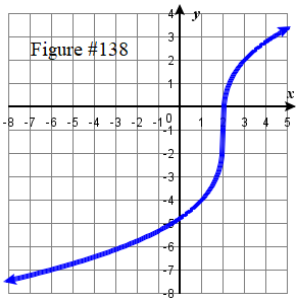

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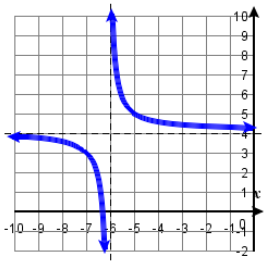
Figure #139



Answers to Odd Exercises:

131. $f(x)=\frac{1}{2}\sqrt{x+3}$	133. $f(x)=\sqrt{(2x-5)}$ 135. $f(x)=2\lvert x-2\rvert-3$	137. $f(x)=-\frac{1}{2}(x+2)^3+4$	139. $f(x)=\frac{1}{x+6}+4$

G: Construct a formula from a description

Exercise 2.3e.G

★ **Write a formula for the function with the following transformations**

141. Write a formula for the function obtained when the graph of $f(x)=|x|$ is shifted down 3 units and right 1 unit.

142. Write a formula for the function obtained when the graph of $f(x)=\frac{1}{x}$ is shifted down 4 units and right 3 units.

143. Write a formula for the function obtained when the graph of $f(x)=\frac{1}{x^2}$ is shifted up 2 units and left 4 units.

144. Write a formula for the function obtained when the graph of $f(x)=\sqrt{x}$ is shifted up 1 unit and left 2 units.

145. The graph of $f(x)=|x|$ is reflected over the y-axis and horizontally compressed by a factor of $\frac{1}{4}$.

146. The graph of $f(x) = \sqrt{x}$ is reflected over the x-axis and horizontally stretched by a factor of 2.

147. The graph of $f(x) = \dfrac{1}{x^2}$ is vertically compressed by a factor of $\dfrac{1}{3}$, then shifted left 2 units and down 3 units.

148. The graph of $f(x) = \dfrac{1}{x}$ is vertically stretched by a factor of 8, then shifted to the right 4 units and up 2 units.

149. The graph of $f(x) = x^2$ is vertically compressed by a factor of $\dfrac{1}{2}$, then shifted to the right 5 units and up 1 unit.

150. The graph of $f(x) = x^2$ is horizontally stretched by a factor of 3, then shifted left 4 units and down 3 units.

Answers to Odd Exercises:

141. $g(x) = \lvert x-1\rvert - 3$ 143. $g(x) = \dfrac{1}{(x+4)^2} + 2$	145. $g(x) = \lvert -4x\rvert$ 147. $g(x) = \dfrac{1}{3(x+2)^2} - 3$	149. $g(x) = \dfrac{1}{2}(x-5)^2 + 1$

H: Construct equations from transformations of tabular values

Exercise 2.3e.H

★ Given tabular representations for the functions f, g, and h, write $g(x)$ and $h(x)$ as transformations of $f(x)$.

155. Tabular representations for the functions f, g, and h are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

x:	-2	-1	0	1	2
$f(x)$:	-2	-1	-3	1	2

x:	-1	0	1	2	3
$g(x)$:	-2	-1	-3	1	2

x:	-2	-1	0	1	2
$h(x)$:	-1	0	-2	2	3

156. Tabular representations for the functions f, g, and h are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

x:	-2	-1	0	1	2
$f(x)$:	-1	-3	4	2	1

x:	-3	-2	-1	0	1
$g(x)$:	-1	-3	4	2	1

x:	-2	-1	0	1	2
$h(x)$:	-2	-4	3	1	0

Answers to Odd Exercises:

155. $g(x) = f(x-1)$, $h(x) = f(x) + 1$.

I: Identify Increasing/decreasing Intervals

Exercise 2.3e.I

★ Use transformations to determine the interval(s) on which the function is increasing and decreasing.

151. $g(x) = 5(x+3)^2 - 2$	152. $f(x) = 4(x+1)^2 - 5$	153. $k(x) = -3\sqrt{x} - 1$	154. $a(x) = \sqrt{-x+4}$

Answers to Odd Exercises:

151. decreasing on $(-\infty, -3)$ and increasing on $(-3, \infty)$	153. decreasing on $(0, \infty)$

★

2.3e: Exercises - Transformations is shared under a not declared license and was authored, remixed, and/or curated by LibreTexts.

2.4: Function Compilations - Piecewise, Combinations, and Composition

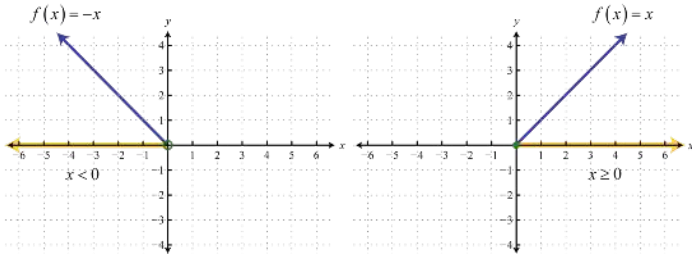
Piecewise-Defined Functions

A **piecewise function** is a function whose definition changes depending on the value of its argument. The function is defined by different formulas for different parts of its domain.

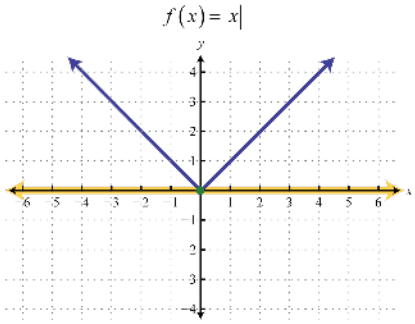
For example, we can write the absolute value function $f(x) = |x|$ as a piecewise function:

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

In this case, the definition used depends on the sign of the x-value. If the x-value is positive, $x \geq 0$, then the function is defined by $f(x) = x$. And if the x-value is negative, $x < 0$, then the function is defined by $f(x) = -x$. The graphs of these two pieces are shown below.



The graph of the absolute function itself is the combination of these two pieces on the same rectangular coordinate plane. The resulting graph is illustrated below.



How to: Given a piecewise function sketch a graph .

1. Indicate on the x-axis the boundaries defined by the intervals on each piece of the domain. Each equation defines a graph for a column of the Cartesian coordinate plane.
2. For each piece of the domain, graph on that interval using the corresponding equation pertaining to that piece. Do not graph two functions in the same interval because the graph would no longer be a graph of a function because it would fail the Vertical Line Test.

Example 2.4.1: Graph a 2-piece function

Graph: $g(x) = \begin{cases} x^2 & \text{if } x < 0 \\ \sqrt{x} & \text{if } x \geq 0 \end{cases}$.

Solution

In this case, we graph the squaring function over negative x-values and the square root function over positive x-values.

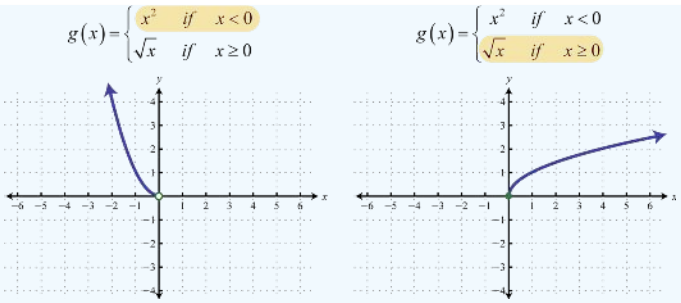


Figure 2.4.1 Separate graph of each piece

Notice the open dot used at the origin for the squaring function and the closed dot used for the square root function. This was determined by the inequality that defines the domain of each piece of the function. The entire function consists of each piece graphed on the same coordinate plane.

Answer:

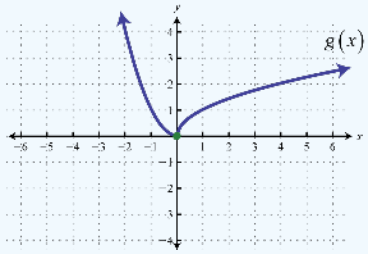


Figure 2.4.1 Graph of piecewise function

Example 2.4.2

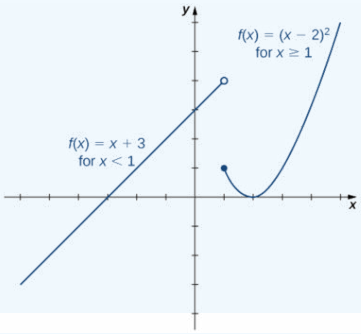
Sketch a graph of the following piecewise-defined function:

$$f(x)=\begin{cases} x+3, & x<1 \\ (x-2)^2, & x\geq 1 \end{cases}$$

Solution

Graph the linear function $y=x+3$ on the interval $(-\infty,1)$ and graph the quadratic function $y=(x-2)^2$ on the interval $[1,\infty)$. Since the value of the function at $x=1$ is given by the formula $f(x)=(x-2)^2$, we see that $f(1)=1$. To indicate this on the graph, we draw a closed circle at the point $(1,1)$. The value of the function is given by $f(x)=x+2$ for all $x<1$, but not at $x=1$. To indicate this on the graph, we draw an open circle at $(1,4)$.

The figure at the right illustrates this piecewise-defined function that is linear for $x<1$ and quadratic for $x\geq 1$.



Try It 2.4.3

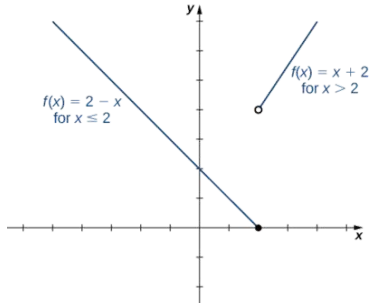
Graph the following piecewise functions.

a. $f(x)=\begin{cases} 2-x, x\leq 2 \\ x+2, x>2 \end{cases}$.

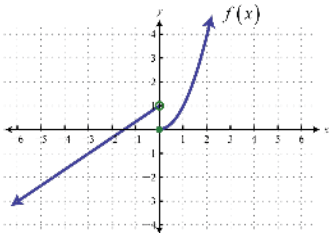
b. $f(x)=\begin{cases} \frac{2}{3}x+1 & \text{if } x<0 \\ x^2 & \text{if } x\geq 0 \end{cases}$.

Answer

a.



b.



Example 2.4.4: Graph a 3-piece function

Sketch a graph of the function.

$$f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ 3 & \text{if } 1 < x \le 2 \\ x & \text{if } x > 2 \end{cases}$$

Solution

Each of the component functions is from our library of toolkit functions, so we know their shapes. We can imagine graphing each function and then limiting the graph to the indicated domain. At the endpoints of the domain, we draw open circles to indicate where the endpoint is not included because of a less-than or greater-than inequality; we draw a closed circle where the endpoint is included because of a less-than-or-equal-to or greater-than-or-equal-to inequality.

Figure 2.4.4 shows the three components of the piecewise function graphed on separate coordinate systems.

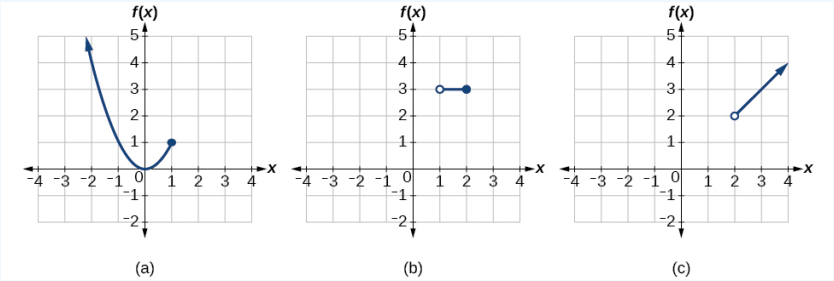


Figure 2.4.4: *Graph of each part of the piece-wise function f(x)*

(a) $f(x) = x^2$ if $x \le 1$; (b) $f(x) = 3$ if $1 < x \le 2$; (c) $f(x) = x$ if $x > 2$

Now that we have sketched each piece individually, we combine them in the same coordinate plane. See Figure 2.4.4s.

Analysis

Note that the graph does pass the vertical line test even at $x = 1$ and $x = 2$ because the points $(1, 3)$ and $(2, 2)$ are not part of the graph of the function, though $(1, 1)$ and $(2, 3)$ are.

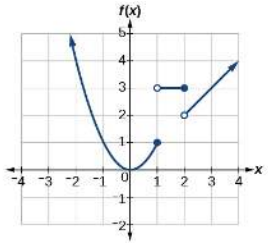


Figure 2.4.4s: Graph of the entire function.

Example 2.4.5:

Graph: $f(x) = \begin{cases} x^3 & \text{if } x < 0 \\ x & \text{if } 0 \le x \le 4 \\ 6 & \text{if } x > 4 \end{cases}$.

Solution

In this case, graph the cubing function over the interval $(-\infty, 0)$. Graph the identity function over the interval $[0, 4]$. Finally, graph the constant function $f(x) = 6$ over the interval $(4, \infty)$. And because $f(x) = 6$ where $x > 4$, we use an open dot at the point $(4, 6)$. Where $x = 4$, we use $f(x) = x$ and thus $(4, 4)$ is a point on the graph as indicated by a closed dot.

Answer:

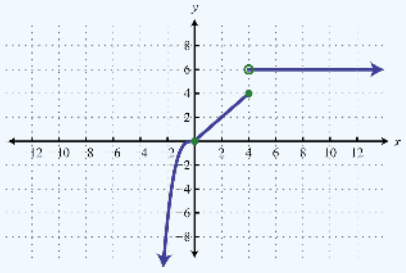


Figure 2.4.5s

Try It 2.4.6

Graph the following piecewise function.

$$f(x) = \begin{cases} x^3 & \text{if } x < -1 \\ -2 & \text{if } -1 < x < 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$$

Answer

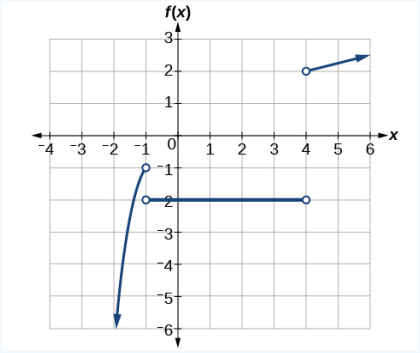


Figure **2.4.6***s*

Can more than one formula from a piecewise function be applied to a value in the domain?

No. Each value corresponds to one equation in a piecewise formula.

Evaluating Piecewise Functions

When evaluating piecewise functions, the value in the domain determines the appropriate definition to use.

Example 2.4.7:

Given the function h, find $h(-5), h(0)$, and $h(3)$.

$$h(t) = \begin{cases} 7t+5 & \text{if } t < 0 \\ -16t^2 + 32t & \text{if } t \geq 0 \end{cases}$$

Solution

Use $h(t) = 7t + 3$ where t is negative, as indicated by $t < 0$.

$$\begin{aligned} h(t) &= 7t + 5 \\ h(-5) &= 7(-5) + 5 \\ &= -35 + 5 \\ &= -30 \end{aligned}$$

Where t is greater than or equal to zero, use $h(t) = -16t^2 + 32t$.

$$\begin{aligned} h(0) &= -16(0) + 32(0) \\ &= 0 + 0 \\ &= 0 \end{aligned} \qquad \begin{aligned} h(3) &= 16(3)^2 + 32(3) \\ &= -144 + 96 \\ &= -48 \end{aligned}$$

Answer:

$h(-5) = -30, h(0) = 0$, and $h(3) = -48$

Example 2.4.8: Working with a Piecewise Function

A cell phone company uses the function below to determine the cost, C, in dollars for g gigabytes of data transfer.

$$C(g) = \begin{cases} 25 & \text{if } 0 < g < 2 \\ 25 + 10(g-2) & \text{if } g \geq 2 \end{cases}$$

Find the cost of using 1.5 gigabytes of data and the cost of using 4 gigabytes of data.

Solution

To find the cost of using 1.5 gigabytes of data, $C(1.5)$, we first look to see which part of the domain our input falls in. Because 1.5 is less than 2, we use the first formula.

$$C(1.5) = \$25$$

To find the cost of using 4 gigabytes of data, C(4), we see that our input of 4 is greater than 2, so we use the second formula.

$$C(4) = 25 + 10(4-2) = \$45$$

Analysis

The function is represented in Figure 2.4.8. We can see where the function changes from a constant to a shifted and stretched identity at $g = 2$. We plot the graphs for the different formulas on a common set of axes, making sure each formula is applied on its proper domain.

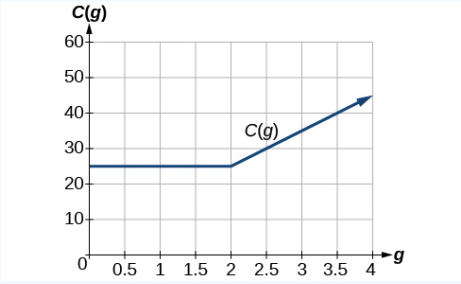


Figure 2.4.8

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The greatest Integer Function

The **greatest integer function**, denoted $f(x) = [[x]]$ assigns the greatest integer less than or equal to any real number in its domain. For example,

$$\begin{aligned} f(2.7) &= [[2.7]] &&= 2 \\ f(\pi) &= [[\pi]] &&= 3 \\ f(0.23) &= [[0.23]] &&= 0 \\ f(-3.5) &= [[-3.5]] &&= -4 \end{aligned}$$

This function associates any real number with the greatest integer less than or equal to it and should not be confused with rounding off.

Example 2.4.9: Greatest Integer Function

Graph: $f(x) = [[x]]$.

Solution

If x is any real number, then $y =$ is the greatest integer less than or equal to x.

$$\begin{aligned} &\vdots \\ -1 \le x < 0 &\Rightarrow y = [[x]] = -1 \\ 0 \le x < 1 &\Rightarrow y = [[x]] = 0 \\ 1 \le x < 2 &\Rightarrow y = [[x]] = 1 \\ &\vdots \end{aligned}$$

Using this, we obtain the following graph.

Answer:

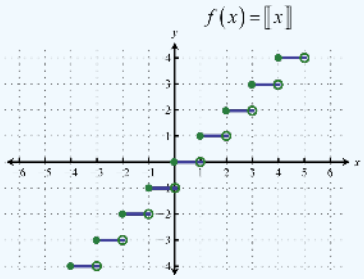


Figure 2.4.9

The domain of the greatest integer function consists of all real number $\mathbb{R}$ and the range consists of the set of integers $\mathbb{Z}$. This function is often called the **floor function** and has many applications in computer science.

Writing Piecewise-Defined Functions

How to: Given a piecewise function, write the formula and identify the domain for each interval.

1. Identify the intervals for which different rules apply.
2. Determine formulas that describe how to calculate an output from an input in each interval.
3. Use braces and if-statements to write the function.

Example 2.4.10: Writing a Piecewise Function

A museum charges $5 per person for a guided tour with a group of 1 to 9 people or a fixed $50 fee for a group of 10 or more people. Write a **function** relating the number of people, n, to the cost, C.

Solution

Two different formulas will be needed. For n-values under 10, $C = 5n$. For values of n that are 10 or greater, $C = 50$.

$$C(n) = \begin{cases} 5n & \text{if } n < 10 \\ 50 & \text{if } n \ge 10 \end{cases}$$

Analysis

The function is represented in Figure 2.4.10. The graph is a diagonal line from $n = 0$ to $n = 10$ and a constant after that. In this example, the two formulas agree at the meeting point where $n = 10$, but not all piecewise functions have this property.

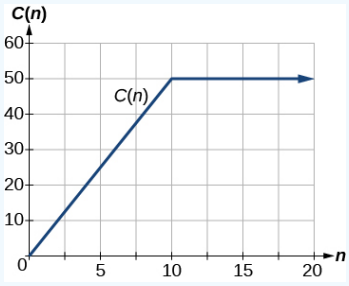


Figure 2.4.10

Example 2.4.11: Writing a Greatest Integer Piecewise-Defined Function

In a big city, drivers are charged variable rates for parking in a parking garage. They are charged $10 for the first hour or any part of the first hour and an additional $2 for each hour or part thereof up to a maximum of $30 for the day. The parking garage is open from 6 a.m. to 12 midnight.

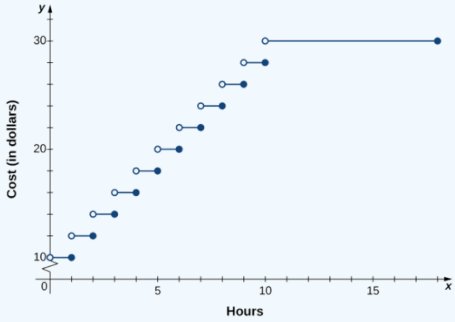
a. Write a piecewise-defined function that describes the cost C to park in the parking garage as a function of hours parked x.
b. Sketch a graph of this function $C(x)$.

Solution

1.Since the parking garage is open 18 hours each day, the domain for this function is $\{x \mid 0 < x \leq 18\}$. The cost to park a car at this parking garage can be described piecewise by the function

$$C(x) = \begin{cases} 10, & 0 < x \leq 1 \\ 12, & 1 < x \leq 2 \\ 14, & 2 < x \leq 3 \\ 16, & 3 < x \leq 4 \\ \vdots & \\ 30, & 10 < x \leq 18 \end{cases}.$$

2.The graph of the function consists of several horizontal line segments.



Try It 2.4.12

The cost of mailing a letter is a function of the weight of the letter. Suppose the cost of mailing a letter is 49¢ for the first ounce and 21¢ for each additional ounce. Write a piecewise-defined function describing the cost C as a function of the weight x for $0 < x \leq 3$, where C is measured in cents and x is measured in ounces.

Hint

The piecewise-defined function is constant on the intervals (0,1], (1,2],….

Answer

$$C(x) = \begin{cases} 49, & 0 < x \leq 1 \\ 70, & 1 < x \leq 2 \\ 91, & 2 < x \leq 3 \end{cases}$$

Algebraic Combinations of Functions

Functions have their own arithmetic which is consistent with the arithmetic of real numbers. The following definitions allow us to add, subtract, multiply and divide functions using the arithmetic we already know for real numbers.

Definition: Function Arithmetic

Suppose f and g are functions and x is in both the domain of f and the domain of g. The domain of the combination function is the intersection of the two domains.

- The **sum** of f and g, denoted $f+g$, is the function defined by the formula
$$(f+g)(x) = f(x) + g(x)$$
- The **difference** of f and g, denoted $f-g$, is the function defined by the formula
$$(f-g)(x) = f(x) - g(x)$$
- The **product** of f and g, denoted fg, is the function defined by the formula
$$(fg)(x) = f(x) \cdot g(x)$$
- The **quotient** of f and g, denoted $\frac{f}{g}$, is the function defined by the formula
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \qquad \text{provided } g(x) \neq 0$$

In other words, to add two functions, we add their outputs; to subtract two functions, we subtract their outputs, and so on. Note that while the formula $(f+g)(x) = f(x) + g(x)$ looks suspiciously like some kind of distributive property, it is nothing of the sort; the addition on the left hand side of the equation is *function* addition, and we are using this equation to *define* the output of the new function $f+g$ as the sum of the real number outputs from f and g.

How to: Find a Combination Function and its Domain.

1. Write the formula for the combination function.
2. Substitute the given functions.
3. Use this UN-simplified expression to find the domain of the combination function.
4. Simplify the expression. If in the simplification process, a factor like for example $(x-c)$ is canceled, then the final result should also include a statement like $\ldots, x \neq c$, indicating the restriction to the combination function that is no longer apparent in its formula .

Example 2.4.13: Performing Algebraic Operations on Functions

Given $f(x) = x - 1$ and $g(x) = x^2 - 1$. Determine the compound function, simplify the result, and find the domain of the compound function.

a. $(g-f)(x)$
b. $\left(\frac{g}{f}\right)(x)$

Solution. Begin by writing the formula for the combination function, and then substitute the given functions.

a. $(g-f)(x) = g(x) - f(x)$

$= (x^2 - 1) - (x - 1)$ This expression has no domain restrictions

$= x^2 - x$ Now simplify

$(g-f)(x) = x(x-1)$

The domain of $(g-f)(x)$ is $(-\infty, \infty)$

b. $\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)}$

$= \frac{x^2-1}{x-1}$ This expression the domain restriction $x \neq 1$

$= \frac{(x+1)(x-1)}{x-1}$ Now simplify

$\left(\frac{g}{f}\right)(x) = x + 1, \quad x \neq 1$

The domain of $\left(\frac{g}{f}\right)(x)$ is $(-\infty, 1) \cup (1, \infty)$. Note: For $\left(\frac{g}{f}\right)(x)$, the condition $x \neq 1$ is necessary because when $x = 1$, and the compound function is constructed from the beginning, the denominator is equal to 0, which makes the function undefined.

Try It 2.4.14

Given $f(x) = x - 1$ and $g(x) = x^2 - 1$ Find and simplify the functions below and determine their domain.

a. $(fg)(x)$
b. $(f-g)(x)$

Answer

a. $(fg)(x) = f(x)g(x) = (x-1)(x^2-1) = x^3 - x^2 - x + 1$ Domain is $(-\infty, \infty)$

b. $(f-g)(x) = f(x) - g(x) = (x-1) - (x^2-1) = x - x^2$ Domain is $(-\infty, \infty)$

Example 2.4.15:

Let $f(x) = 6x^2 - 2x$ and $g(x) = 3 - \frac{1}{x}$.

a. Find $(f+g)(-1)$
b. Find $(fg)(2)$
c. Find the domain of $g-f$ then find and simplify a formula for $(g-f)(x)$.
d. Find the domain of $\left(\frac{g}{f}\right)$ then find and simplify a formula for $\left(\frac{g}{f}\right)(x)$.

Solution

a. To find $(f+g)(-1)$ we first find $f(-1) = 8$ and $g(-1) = 4$. By definition, we have that $(f+g)(-1) = f(-1) + g(-1) = 8 + 4 = 12$.

b. To find $(fg)(2)$, we first need $f(2)$ and $g(2)$. Since $f(2) = 20$ and $g(2) = \frac{5}{2}$, our formula yields $(fg)(2) = f(2) \cdot g(2) = (20)\left(\frac{5}{2}\right) = 50$.

c. To find the domain of $g-f$ the formula for $(g-f)(x)$ needs to be analyzed before simplifying. Because x is in the denominator, x cannot be 0.

$$(g-f)(x) = g(x) - f(x) = \left(3 - \frac{1}{x}\right) - (6x^2 - 2x)$$

So the domain is $(-\infty, 0) \cup (0, \infty)$.

Next, the formula for $(g-f)(x)$ must be simplified. In this case, we get common denominators and attempt to reduce the resulting fraction. Doing so, we get

$$\begin{aligned}(g-f)(x) &= g(x)-f(x)\\ &= \left(3-\frac{1}{x}\right)-(6x^2-2x) = 3-\frac{1}{x}-6x^2+2x\\ &= \frac{3x}{x}-\frac{1}{x}-\frac{6x^3}{x}+\frac{2x^2}{x} \qquad \text{get common denominators}\\ &= \frac{3x-1-6x^3+2x^2}{x} = \frac{-6x^3+2x^2+3x-1}{x}\\ &= \frac{-2x^2(3x-1)+1(3x-1)}{x} = \frac{(3x-1)(-2x^2+1)}{x}\\ &= \frac{-(3x-1)(2x^2-1)}{x}\end{aligned}$$

d. As in the previous example, write the formula, substitute the values of the functions and then examine the un-simplified result to determine the domain of the combination function.

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{3-\frac{1}{x}}{6x^2-2x}$$

We see immediately from the 'little' denominator that $x \neq 0$. To keep the 'big' denominator away from 0, we solve $6x^2-2x=0$, or $2x(3x-1)=0$, and get $x=0$ or $x=\frac{1}{3}$. Hence, as before, we find the domain of $\frac{g}{f}$ to be $(-\infty,0) \cup (0,\frac{1}{3}) \cup (\frac{1}{3},\infty)$.

Next, the formula for $\left(\frac{g}{f}\right)(x)$ must be simplified.

$$\begin{aligned}\left(\frac{g}{f}\right)(x) &= \frac{\left(3-\frac{1}{x}\right)}{2x(3x-1)} \cdot \frac{\frac{x}{1}}{x/1} \qquad \text{factor and clear complex fraction}\\ &= \frac{3x-1}{2x^2(3x-1)} = \frac{1}{2x^2}\end{aligned}$$

Our final answer then is $\left(\frac{g}{f}\right)(x) = \frac{1}{2x^2}, \quad x \neq \frac{1}{3}$ and the domain is $(-\infty,0) \cup (0,\frac{1}{3}) \cup (\frac{1}{3},\infty)$.

Please note the importance of finding the domain of a function **before** simplifying its expression. If we waited to find the domain of $\frac{g}{f}$ until **after** simplifying, we'd just have the formula $\frac{1}{2x^2}$ to go by, and we would (incorrectly!) state the domain as $(-\infty,0) \cup (0,\infty)$, since the other troublesome number, $x=\frac{1}{3}$, was canceled away.

Composition of Functions

Suppose we wanted to calculate how much it costs to heat a house on a particular day of the year. The cost to heat a house will depend on the average daily temperature, and the average daily temperature depends on the particular day of the year. Notice how we have just defined two relationships: The cost depends on the temperature, and the temperature depends on the day. Using descriptive variables, we can notate these two functions.

The first function, $C(T)$, gives the cost C of heating a house when the average daily temperature is T degrees Celsius, and the second, $T(d)$, gives the average daily temperature on day d of the year in some city. If we wanted to determine the cost of heating the house on the 5th day of the year, we could do this by linking our two functions together, an idea called composition of functions. Using the function $T(d)$, we could evaluate $T(5)$ to determine the average daily temperature on the 5 th day of the year. We could then use that temperature as the input to the $C(T)$ function to find the cost to heat the house on the 5 th day of the year: $C(T(5))$.

Definition: Composition of Functions

The **composition of the function f with g** is denoted by $(f \circ g)(x)$, read "f of g of x", and is defined by the equation

$$(f \circ g)(x) = f(g(x))$$

In general, $f \circ g$ and $g \circ f$ are different functions.

It is important to realize that the product of functions $(fg)(x)$ is not the same as the function composition $(f \circ g)(x)$.

The **domain of the composite function** $f \circ g$ is the set of all x such that

1. x is in the domain of g and ← so that the first (inner) operation is defined
2. $g(x)$ is in the domain of f. ← so that the final resulting operation is defined

Example 2.4.16: Interpreting Composite Functions

The function $c(s)$ gives the number of calories burned completing s sit-ups, and $s(t)$ gives the number of sit-ups a person can complete in t minutes. Interpret $c(s(3))$.

Solution

The inside expression in the composition is $s(3)$. Because the input to the s-function is time, $t = 3$ represents 3 minutes, and $s(3)$ is the number of sit-ups completed in 3 minutes.

Using $s(3)$ as the input to the function $c(s)$ gives us the number of calories burned during the number of sit-ups that can be completed in 3 minutes, or simply the number of calories burned in 3 minutes (by doing sit-ups).

Note that it is not important that the same variable be used for the output of the inside function and the input to the outside function. However, it is essential that the units on the output of the inside function match the units on the input to the outside function, if the units are specified.

Example 2.4.17: Investigating the Order of Function Composition

Suppose $f(x)$ gives miles that can be driven in x hours and $g(y)$ gives the gallons of gas used in driving y miles. Which of these expressions is meaningful: $f(g(y))$ or $g(f(x))$?

Solution

The function $y = f(x)$ is a function that produces the number of miles driven given the number of hours driven:

$$\text{number of miles } = f(\text{number of hours})$$

The function $g(y)$ is a function that produces the number of gallons used given the number of miles driven:

$$\text{number of gallons } = g(\text{number of miles})$$

Consider the composition $f(g(y))$. The expression $g(y)$ takes miles as the input and a number of gallons as the output. The function $f(x)$ requires a number of hours as the input. Trying to input a number of gallons into f does not make sense. The expression $f(g(y))$ is meaningless.

Consider the composition $g(f(x))$. The expression $f(x)$ takes hours as input and a number of miles driven as the output. The function $g(y)$ requires a number of miles as the input. Using $f(x)$ (miles driven) as an input value for $g(y)$, where gallons of gas depends on miles driven, does make sense. The expression $g(f(x))$ makes sense, and will yield the number of gallons of gas used, g, driving a certain number of miles, $f(x)$, in x hours.

Try It 2.4.18

In a department store you see a sign that says 50% off clearance merchandise, so final cost C depends on the clearance price, p, according to the function $C(p)$. Clearance price, p, depends on the original discount, d, given to the clearance item, $p(d)$. Interpret $C(p(d))$.

Answer

The final cost, C, depends on the clearance price, p, which is based on the original discount, d. (Or the original discount d, determines the clearance price and the final cost is half of the clearance price.)

Try It 2.4.19

The gravitational force on a planet a distance r from the sun is given by the function $G(r)$. The acceleration of a planet subjected to any force F is given by the function $a(F)$. Form a meaningful composition of these two functions, and explain what it means.

Answer

A gravitational force is still a force, so $a(G(r))$ makes sense as the acceleration of a planet at a distance r from the Sun (due to gravity), but $G(a(F))$ does not make sense.

Evaluating Composite Functions

When working with functions given as tables, graphs, and formulas we can look up values for the functions using a provided table, graph, or formula. Start evaluation with the inside function. Then use the output of the inside function as the input to the outside function. To remember this, always work from the inside out - just like with order of operations.

Evaluate Composite Functions Using Tables

When working with functions given as tables, we read input and output values from the table entries and always work from the inside to the outside. We evaluate the inside function first and then use the output of the inside function as the input to the outside function.

Example 2.4.20: Using a Table to Evaluate a Composite Function

Using the tables below, evaluate $f(g(3))$ and $g(f(3))$.

x	$f(x)$		x	$g(x)$
1	6		1	3
2	8		2	5
3	3		3	2
4	1		4	7

Solution

To evaluate $f(g(3))$, start from the inside with the input value 3. Then evaluate the inside expression $g(3)$ using the table that defines the function $g: g(3) = 2$. Use that result as the input to the function f, so $g(3)$ is replaced by 2 to get $f(2)$. Then, using the table that defines the function f, we find that $f(2) = 8$.

Step 1: $g(3) = 2$, Step 2: $f(g(3)) = f(2) = 8$

To evaluate $g(f(3))$, first evaluate the inside expression $f(3)$ using the first table: $f(3) = 3$. Then, using the table for g, we can evaluate

Step 1: $f(3) = 3$, Step 2: $g(f(3)) = g(3) = 2$

Try It 2.4.21

Use the table in Example 2.4.20 above to evaluate $f(g(1))$, $g(f(4))$ and $f(g(4))$.

Answer

$f(g(1)) = f(3) = 3$, $g(f(4)) = g(1) = 3$, and $f(g(4)) = f(7) =$ undefined

Notice the composition is not a commutative operation!

Evaluate Composite Functions Using Graphs

When we are given individual functions as graphs, the procedure for evaluating composite functions is similar to the process we use for evaluating tables. We read the input and output values, but this time, from the x- and y-axes of the graphs.

How to: Use graphs of individual functions to evaluate a composite function.

1. Locate the given input to the inner function on the x-axis of its graph.
   Read off the output of the inner function from the y-axis of its graph.
2. Locate the inner function output on the x-axis of the graph of the outer function.
   Read the output of the outer function from the y-axis of its graph. This is the output of the composite function.

Example 2.4.22: Using a Graph to Evaluate a Composite Function

Using Figure 2.4.22, evaluate $f(g(1))$.

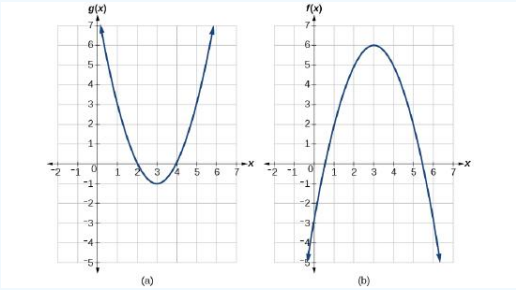


Figure 2.4.22: Two graphs of a positive and negative parabola.

Solution

To evaluate $f(g(1))$, we start with the inside evaluation. See Figure 2.4.22.

Step 1. $f(g(1)) = f(3)$: We evaluate $g(1)$ using the graph of $g(x)$, finding the input of 1 on the x-axis and finding the output value of the graph at that input. Here, $g(1) = 3$. We use this value as the input to the function f.

Step 2. $f(3) = 6$: We can then evaluate the composite function by looking to the graph of $f(x)$, finding the input of 3 on the x-axis and reading the output value of the graph at this input. Here, $f(3) = 6$, so $f(g(1)) = 6$.

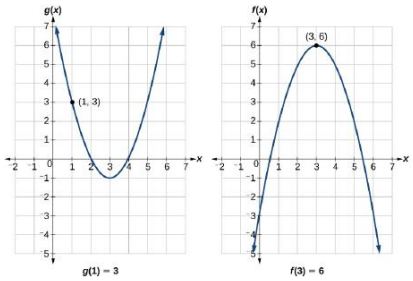


Figure 2.4.22a: Two graphs of a positive parabola $g(x)$ and a negative parabola $f(x)$. The following points are plotted: $g(1) = 3$ and $f(3) = 6$.

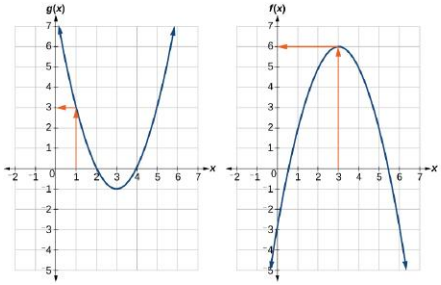


Figure 2.4.22b: This diagram shows how the graphs can be marked to trace the path from the input value to the output value.

Try It 2.4.23

Using Figure 2.4.22, evaluate $g(f(2))$.

Answer

$g(f(2)) = g(5) = 3$

Evaluate Composite Functions Using Formulas

When evaluating a composite function where we have either created or been given formulas, the rule of working from the inside out remains the same. The input value to the outer function will be the output of the inner function, which may be a numerical

value, a variable name, or a more complicated expression.

How to: Evaluate a Composition $(f \circ g)(c)$.

1. Write the definition for the composition function: $(f \circ g)(c) = f(g(c))$.
2. Substitute c in the function definition for g to evaluate the inside function $g(c)$. Let k be this result: $(f \circ g)(c) = f(g(c)) = f(k)$.
3. Substitute k in the function definition for f to evaluate the outside function

Example 2.4.24: Evaluate a Composition of Functions Expressed as Formulas with a Numerical Input

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate $(f \circ h)(1))$.

Solution

Rewrite $(f \circ h)(1))$ using the definition: $(f \circ h)(1)) = f(h(1))$

The inside expression is $h(1)$. The argument of h is 1, so substitute 1 for x in the formula for h and evaluate it.

$$h(1) = 3(1) + 2$$
$$h(1) = 5$$

Now $f(h(1)) = f(5)$. The argument of f is 5 so substitute 5 for t in the formula for f and evaluate it.

$$f(h(1)) = f(5)$$
$$f(h(1)) = 5^2 - 5$$
$$f(h(1)) = 20$$

Analysis

It makes no difference what the input variables t and x were called in this problem because we evaluated for specific numerical values.

Example 2.4.25: Evaluating a Composition of Functions Expressed as Formulas with a Numerical Input

Given $f(x) = x^2 - 4x$, $g(x) = 2 - \sqrt{x+3}$, and $h(x) = \dfrac{2x}{x+1}$ evaluate the following.

a. $(g \circ f)(1)$ b. $(f \circ g)(1)$ c. $(g \circ g)(6)$ d. $(g \circ f)(2)$ e. $(g \circ h)(-1)$

Solution

a. Use the definition: $(g \circ f)(1) = g(f(1))$.
Evaluate the inner function: $f(1) = -3$
Evaluate the outer function using this result: $g(f(1)) = g(-3) = 2$

b. Use the definition to write $(f \circ g)(1) = f(g(1))$.
Evaluate the inner function: $g(1) = 0$,
Evaluate the outer function using this result: $f(g(1)) = f(0) = 0$

c. Use the definition $(g \circ g)(6) = g(g(6))$.
Evaluate the inner function: $g(6) = -1$
Evaluate the outer function using this result: $g(g(6)) = g(-1) = 2 - \sqrt{2}$

d. Use the definition: $(g \circ f)(2) = g(f(2))$
Evaluate the inner function: $f(2) = (2)^2 - 4(2) = 4 - 8 = -4$
Evaluate the outer function using this result: $g(f(2)) = g(-4) = 2 - \sqrt{-4+3} = 2 - \sqrt{-1} = 2 - i$. This result is not a real number, so we say $(g \circ f)(2)$ is undefined.

e. Use the definition: $(g \circ h)(-1) = g(h(-1))$

Evaluate the inner function: $h(-1) = \dfrac{2(-1)}{(-1)+1} = \dfrac{-2}{0} = \textbf{undefined}$

Evaluate the outer function: $g(h(-1)) = g(\textbf{undefined})$ cannot be evaluated! So $(g \circ h)(-1) = \textbf{undefined}$.

Try It 2.4.26

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate

a. $h(f(2))$
b. $h(f(-2))$

Answers:

a. 8 b. 20

Finding the Formula and Domain of a Composite Function

While we can compose the functions for each individual input value, it is sometimes helpful to find a single formula that will calculate the result of a composition $(f \circ g)(x)$. To do this, we will extend our idea of function evaluation. Recall that, when we evaluate a function with a formula like $f(t) = t^2 - t$, we substitute the value inside the parentheses (called the argument) into the formula wherever we see the input variable.

The **domain of a composite function** such as $f \circ g$ is dependent on the domain of g and the domain of f. It is important to know when we can apply a composite function and when we cannot, that is, to know the domain of a function such as $f \circ g$. If we write the composite function for an input x as $f(g(x))$, we can see right away that x must be a member of the domain of g in order for the expression to be meaningful, because otherwise we cannot complete the inner function evaluation. Secondly, the outer function evaluation, $f(g(x))$, must also be defined. Thus the restrictions to the domain of $f \circ g$ consist of not only the values of x that make $g(x)$ undefined, but also the values of x that make the expression $f(g(x))$ undefined.

How to: Determine a Function Composition $(f \circ g)(x)$ and its Domain.

1. Write the definition for the composition function: $(f \circ g)(x) = f(g(x))$.
2. Substitute the function definition for g in for the argument of f.
   Determine the domain of g because the argument to f must be defined during the composition process.
3. Substitute the argument expression $g(x)$ into the function definition for f to determine the formula for $(f \circ g)(x)$.
4. Simplify the formula for $(f \circ g)(x)$.
5. The domain of $(f \circ g)(x)$ combines both the domain restrictions to the inner function and the domain restrictions apparent in the resulting function. It is important to include the domain restrictions to g because sometimes they get lost in the simplification process (step 4).

Example 2.4.27: Finding a composition and its domain

Given $f(x) = 5x - 1$ and $g(x) = \dfrac{4}{3x-2}$ find $(f \circ g)(x)$ and its domain.

Solution: 1. $(f \circ g)(x) = f(g(x))$

2. $f(g(x)) = f\left(\dfrac{4}{3x-2}\right)$. The domain of $g(x)$ consists of all real numbers except $x = \frac{2}{3}$, since that input value would cause us to divide by 0.

3. $f\left(\dfrac{4}{3x-2}\right) = 5\left(\dfrac{4}{3x-2}\right) - 1$. Notice in this form the domain restriction $x \neq \frac{2}{3}$

4. $5\left(\dfrac{4}{3x-2}\right) - 1 = 5\left(\dfrac{4}{3x-2}\right) - \dfrac{3x-2}{3x-2} = \dfrac{20-3x+2}{3x-2} = \dfrac{22-3x}{3x-2}$

5. The domain restriction of the simplified expression is still $x \neq \frac{2}{3}$.

Answer: $(f \circ g)(x) = \dfrac{22-3x}{3x-2}$ and its domain is $(-\infty, \frac{2}{3}) \cup (\frac{2}{3}, \infty)$

Example 2.4.28

Given $f(x) = \sqrt{x+2}$ and $g(x) = \sqrt{3-x}$ find $(f \circ g)(x)$ and its domain.

Solution: 1. $(f \circ g)(x) = f(g(x))$

2. $f(g(x)) = f(\sqrt{3-x})$. The domain restriction for g is $3 - x \geq 0 \longrightarrow 3 \geq x \longrightarrow x \leq 3$
3. $f(\sqrt{3-x}) = \sqrt{\sqrt{3-x}+2}$
4. This expression cannot be simplified.
5. The domain restriction for this expression is $\sqrt{3-x}+2 \geq 0$. As long as $\sqrt{3-x}$ is a real number, $\sqrt{3-x} \geq 0$ so $\sqrt{3-x}+2 \geq 2$. The expression $\sqrt{3-x}$ is a real number whenever $3 - x \geq 0$ or $x \leq 3$.

Answer: $(f \circ g)(x) = \sqrt{\sqrt{3-x}+2}$ and its domain is $(-\infty, 3]$.

Example 2.4.29

Let $f(x) = x^2 - 4x$, $g(x) = 2 - \sqrt{x+3}$, and $h(x) = \dfrac{2x}{x+1}$. Find and simplify the indicated composite functions. State the domain of each.

a. $(f \circ g)(x)$
b. $(h \circ g)(x)$
c. $(h \circ h)(x)$

Solution

a. 1. $(f \circ g)(x) = f(g(x))$

2. $f(g(x)) = f(2 - \sqrt{x+3})$. The domain of g is $x + 3 \geq 0 \longrightarrow x \geq -3$

3. $f(2-\sqrt{x+3}) = (2-\sqrt{x+3})^2 - 4(2-\sqrt{x+3})$
4. $= 4 - 4\sqrt{x+3} + (\sqrt{x+3})^2 - 8 + 4\sqrt{x+3}$
$= 4 + x + 3 - 8$
$= x - 1$

5. The resulting expression has no domain restrictions, but the domain restriction found for g still applies to the composition process.

Answer: $(f \circ g)(x) = x - 1$ with a domain of $[-3, \infty)$

b. 1. $(h \circ g)(x) = h(g(x))$

2. $h(g(x)) = h(2 - \sqrt{x+3})$. The domain of g here is $x + 3 \geq 0 \longrightarrow x \geq -3$.

3. $h(2-\sqrt{x+3}) = \dfrac{2(2-\sqrt{x+3})}{(2-\sqrt{x+3})+1}$
4. $= \dfrac{4-2\sqrt{x+3}}{3-\sqrt{x+3}}$ **Simplify**
$= \dfrac{4-2\sqrt{x+3}}{3-\sqrt{x+3}} \cdot \dfrac{3+\sqrt{x+3}}{3+\sqrt{x+3}}$ **Rationalize denominator**
$= \dfrac{12+4\sqrt{x+3}-6\sqrt{x+3}-2(x+3)}{9-(x+3)}$ **Multiply out and simplify**
$= \dfrac{6-2x-2\sqrt{x+3}}{6-x}$

5. The resulting expression has two domain restrictions: $x + 3 \geq 0$ and $6 - x \neq 0$. Therefore, $x \geq -3$ and $x \neq 6$. Both of these restrictions apply to the composition process. The domain restriction to g found earlier is repeated here.

Answer: $(h \circ g)(x) == \dfrac{4-2\sqrt{x+3}}{3-\sqrt{x+3}}$ with a domain of $[-3,6) \cup (6,\infty)$.

c. 1. $(h \circ h)(x) = h(h(x))$.

2. $h(h(x)) = h\left(\dfrac{2x}{x+1}\right)$. The domain of h here is $x+1 \neq 0 \longrightarrow x \neq -1$.

3. $h\left(\dfrac{2x}{x+1}\right) = \dfrac{2\left(\dfrac{2x}{x+1}\right)}{\left(\dfrac{2x}{x+1}\right)+1}$

4. $= \dfrac{\left(\dfrac{4x}{x+1}\right)}{\left(\dfrac{2x}{x+1}\right)+1} \cdot \dfrac{x+1}{x+1}$ **Simplify complex fraction**

$= \dfrac{\left(\dfrac{4x}{x+1}\right)\cdot(x+1)}{\left(\dfrac{2x}{x+1}\right)\cdot(x+1)+1\cdot(x+1)}$

$= \dfrac{4x}{2x+x+1} = \dfrac{4x}{3x+1}$

5. The resulting expression has one domain restriction, $3x+1 \neq 0$. Therefore, $x \neq \frac{1}{3}$. The domain of the composition process includes not only this domain but also the domain restriction to h found earlier: $x \neq -1$

Answer: $(h \circ h)(x) = \dfrac{4x}{3x+1}$ and its domain is $(-\infty,-1) \cup \left(-1,-\frac{1}{3}\right) \cup \left(-\frac{1}{3},\infty\right)$.

Try It 2.4.30

Find and simplify $(f \circ g)(x)$ and find its domain given

a. $f(x) = \dfrac{1}{x-2}$ and $g(x) = \sqrt{x+4}$

b. $f(x) = \dfrac{4}{3x-2}$ and $g(x) = \dfrac{1}{x-1}$

Answers

a. $(f \circ g)(x) = \dfrac{2+\sqrt{x+4}}{x}$ with domain $[-4,0) \cup (0,\infty)$

b. $(f \circ g)(x) = \dfrac{4(x-1)}{5-2x}$ with domain $(-\infty,1) \cup (1,2.5) \cup (2.5,\infty)$

Decomposing a Composite Function into its Component Functions

In some cases, it is necessary to decompose a complicated function. In other words, we can write it as a composition of two simpler functions. There may be more than one way to **decompose a composite function**, so we may choose the decomposition that appears to be most expedient.

Example 2.4.31: Decomposing a Function

Write $f(x) = \sqrt{5-x^2}$ as the composition of two functions.

Solution

We are looking for two functions, g and h, so $f(x) = g(h(x))$. To do this, we look for a function inside a function in the formula for $f(x)$. As one possibility, we might notice that the expression $5-x^2$ is the inside of the square root. We could then decompose the function as

$$h(x) = 5-x^2 \text{ and } g(x) = \sqrt{x}$$

We can check our answer by recomposing the functions.

$$g(h(x)) = g(5 - x^2) = \sqrt{5 - x^2}$$

Try It 2.4.32

Write $f(x) = \dfrac{4}{3 - \sqrt{4 + x^2}}$ as the composition of two functions.

Answer

Possible answers:

$g(x) = \sqrt{4 + x^2}$
$h(x) = \dfrac{4}{3 - x}$
$f = h \circ g$

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2.4e: Exercises - Piecewise Functions, Combinations, Composition

A: Concepts

Exercise 2.4e.A

1) How does one find the domain of the quotient of two functions, $\frac{f}{g}$?

2) What is the composition of two functions, $f \circ g$?

3) If the order is reversed when composing two functions, can the result ever be the same as the answer in the original order of the composition? If yes, give an example. If no, explain why not.

4) How do you find the domain for the composition of two functions, $f \circ g$?

5) How do you graph a piecewise function?

Answers 1-5:

1. Find the numbers that make the function in the denominator g equal to zero, and check for any other domain restrictions on f and g, such as an even-indexed root or zeros in the denominator

3. Yes. Sample answer: Let $f(x) = x + 1$ and $g(x) = x - 1$. Then $f(g(x)) = f(x - 1) = (x - 1) + 1 = x$ and $g(f(x)) = g(x + 1) = (x + 1) - 1 = x$. So $f \circ g = g \circ f$.

5. Graph each formula of the piecewise function over its corresponding domain. Use the same scale for the x-axis and y-axis for each graph. Indicate inclusive endpoints with a solid circle and exclusive endpoints with an open circle. Use an arrow to indicate endpoints of $-\infty$ or ∞.Combine the graphs to find the graph of the piecewise function

Piecewise Functions

B. Evaluate Piecewise Functions

Exercise 2.4e.B

Given function f, evaluate $f(-3)$, $f(-2)$, $f(-1)$, and $f(0)$.

6. $f(x) = \begin{cases} x+1 & \text{if } x < -2 \\ -2x-3 & \text{if } x \geq -2 \end{cases}$

7. $f(x) = \begin{cases} 1 & \text{if } x \leq -3 \\ 0 & \text{if } x > -3 \end{cases}$

8. $f(x) = \begin{cases} -2x^2+3 & \text{if } x \leq -1 \\ 5x-7 & \text{if } x > -1 \end{cases}$

Given function f, evaluate $f(-1)$, $f(0)$, $f(2)$, and $f(4)$.

9. $f(x) = \begin{cases} 7x+3 & \text{if } x < 0 \\ 7x+6 & \text{if } x \geq 0 \end{cases}$

10. $f(x) = \begin{cases} x^2-2 & \text{if } x < 2 \\ 4+\lvert x-5 \rvert & \text{if } x \geq 2 \end{cases}$

11. $f(x) = \begin{cases} 5x & \text{if } x < 0 \\ 3 & \text{if } 0 \geq x \leq 2 \\ x^2 & \text{if } x > 3 \end{cases}$

Write the domain for each piecewise function in interval notation.

12. $f(x) = \begin{cases} x+1 & \text{if } x < -2 \\ -2x-3 & \text{if } x \geq -2 \end{cases}$

13. $f(x) = \begin{cases} x^2-2 & \text{if } x < 1 \\ -x^2+2 & \text{if } x > 1 \end{cases}$

14. $f(x) = \begin{cases} x^2-3 & \text{if } x < 0 \\ -3x^2 & \text{if } x \geq 2 \end{cases}$

15. Find $f(-5), f(0)$, and $f(3)$ given $f(x) = \begin{cases} x^2 & \text{if } x \leq 0 \\ x+2 & \text{if } x > 0 \end{cases}$

16. Find $f(-3), f(0)$, and $f(2)$ given $f(x) = \begin{cases} x^3 & \text{if } x < 0 \\ 2x-1 & \text{if } x \geq 0 \end{cases}$

17. Find $g(-1), g(1)$, and $g(4)$ given $g(x) = \begin{cases} 5x-2 & \text{if } x < 1 \\ \sqrt{x} & \text{if } x \geq 1 \end{cases}$

18. Find $g(-3), g(-2)$, and $g(-1)$ given $g(x) = \begin{cases} x^3 \text{ if } x \leq -2 \\ |x| \text{ if } x > -2 \end{cases}$

19. Find $h(-2), h(0)$, and $h(4)$ given $h(x) = \begin{cases} -5 & \text{if } x < 0 \\ 2x-3 & \text{if } 0 \leq x < 2 \\ x^2 & \text{if } x \geq 2 \end{cases}$

20. Find $h(-5), h(4)$, and $h(25)$ given $h(x) = \begin{cases} -3x \text{ if } x \leq 0 \\ x^3 \text{ if } 0 < x \leq 4 \\ \sqrt{x} \text{ if } x > 4 \end{cases}$

21. Find $f(-2), f(0)$, and $f(3)$ given $f(x) = [[x - 0.5]]$

22. Find $f(-1.2), f(0.4)$, and $f(2.6)$ given $f(x) = [[2x]] + 1$

Answers to Odd Exercises:

7. $f(-3) = 1$; $f(-2) = 0$; $f(-1) = 0$; $f(0) = 0$
9. $f(-1) = -4$; $f(0) = 6$; $f(2) = 20$; $f(4) = 34$
11. $f(-1) = -5$; $f(0) = 3$; $f(2) = 3$; $f(4) = 16$
13. domain: $(-\infty, 1) \cup (1, \infty)$
15. $f(-5) = 25, f(0) = 0$, and $f(3) = 5$
17. $g(-1) = -7, g(1) = 1$, and $g(4) = 2$
19. $h(-2) = -5, h(0) = -3$, and $h(4) = 16$
21. $f(-2) = -3, f(0) = -1$, and $f(3) = 2$

C: Graph Piecewise Functions

Exercise 2.4e.C

★ **Graph two-part piecewise functions.**

23. $h(x) = \begin{cases} x^2+2 & \text{if } x < 0 \\ x+2 & \text{if } x \geq 0 \end{cases}$

24. $h(x) = \begin{cases} x^2-3 \text{ if } x < 0 \\ \sqrt{x}-3 \text{ if } x \geq 0 \end{cases}$

25. $h(x) = \begin{cases} x^3-1 & \text{if } x < 0 \\ |x-3|-4 & \text{if } x \geq 0 \end{cases}$

26. $h(x) = \begin{cases} x^3 & \text{if } x < 0 \\ (x-1)^2-1 & \text{if } x \geq 0 \end{cases}$

27. $h(x) = \begin{cases} x^2-1 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0 \end{cases}$

28. $h(x) = \begin{cases} x+2 & \text{if } x < 0 \\ (x-2)^2 & \text{if } x \geq 0 \end{cases}$

29. $g(x) = \begin{cases} 2 & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$

30. $g(x) = \begin{cases} x^2 & \text{if } x < 0 \\ 3 & \text{if } x \geq 0 \end{cases}$

31. $h(x) = \begin{cases} x & \text{if } x < 0 \\ \sqrt{x} & \text{if } x \geq 0 \end{cases}$

32. $h(x) = \begin{cases} |x| \text{ if } x < 0 \\ x^3 \text{ if } x \geq 0 \end{cases}$

33. $f(x) = \begin{cases} |x| \text{ if } x < 2 \\ 4 \text{ if } x \geq 2 \end{cases}$

34. $f(x) = \begin{cases} x & \text{if } x < 1 \\ \sqrt{x} & \text{if } x \geq 1 \end{cases}$

35. $g(x) = \begin{cases} x^2 \text{ if } x \leq -1 \\ x \text{ if } x > -1 \end{cases}$

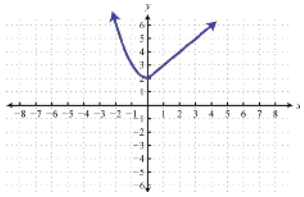
36. $g(x) = \begin{cases} -3 \text{ if } x \leq -1 \\ x^3 \text{ if } x > -1 \end{cases}$

37. $h(x) = \begin{cases} 0 \text{ if } x \leq 0 \\ \frac{1}{x} \text{ if } x > 0 \end{cases}$

38. $h(x) = \begin{cases} \frac{1}{x} \text{ if } x < 0 \\ x^2 \text{ if } x \geq 0 \end{cases}$

Answers to Odd Exercises:

23.



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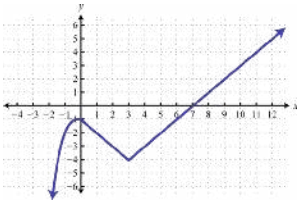

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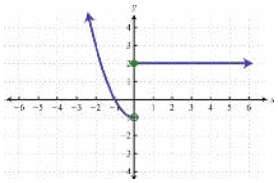

Figure 2.4.27

29.

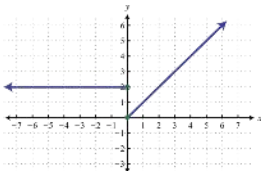

Figure 2.4.29

31.

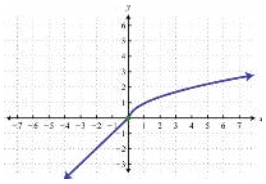

Figure 2.4.31

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33.

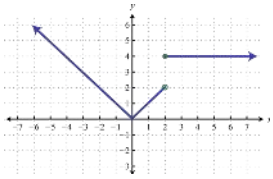

Figure 2.4.33

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35.

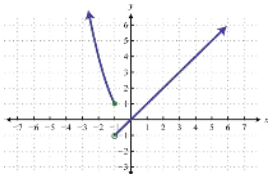

Figure 2.4.35

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37.

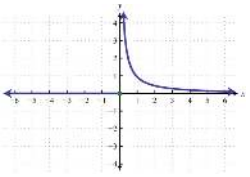

Figure 2.4.37

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★ Graph 3 or more part piecewise functions.

39. $h(x)=\begin{cases}(x+10)^2-4 & \text{if } x<-8\\ x+4 & \text{if } -8\le x<-4\\ \sqrt{x+4} & \text{if } x\ge -4\end{cases}$

40. $f(x)=\begin{cases}x+10 & \text{if } x\le -10\\ |x-5|-15 & \text{if } -10<x\le 20\\ 10 & \text{if } x>20\end{cases}$

41. $f(x)=\begin{cases}x^2 & \text{if } x<0\\ x & \text{if } 0\le x<2\\ -2 & \text{if } x\ge 2\end{cases}$

42. $f(x)=\begin{cases}x & \text{if } x<-1\\ x^3 & \text{if } -1\le x<1\\ 3 & \text{if } x\ge 1\end{cases}$

43. $g(x)=\begin{cases}5 & \text{if } x<-2\\ x^2 & \text{if } -2\le x<2\\ x & \text{if } x\ge 2\end{cases}$

44. $g(x)=\begin{cases}x & \text{if } x<-3\\ |x| & \text{if } -3\le x<1\\ \sqrt{x} & \text{if } x\ge 1\end{cases}$

45. $h(x)=\begin{cases}\frac{1}{x} \text{ if } x<0\\ x^2 \text{ if } 0\le x<2\\ 4 \text{ if } x\ge 2\end{cases}$

46. $h(x)=\begin{cases}0 \text{ if } x<0\\ x^3 \text{ if } 0<x\le 2\\ 8 \text{ if } x>2\end{cases}$

47. $f(x)=[[x+0.5]]$

48. $f(x)=[[x]]+1$

49. $f(x)=[[0.5x]]$

50. $f(x)=2[[x]]$

Answers to Odd Exercises:

39.

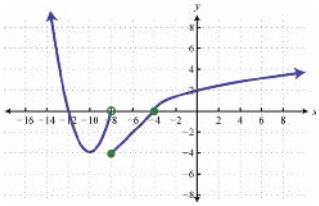


Figure 2.4.39

41.

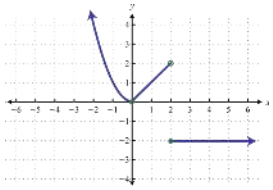


Figure 2.4.41

x

43.

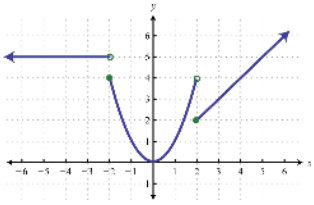


Figure 2.4.43

45.

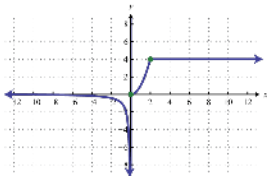


Figure 2.4.45

x

47.

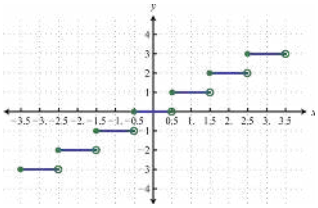


Figure 2.4.47

49.

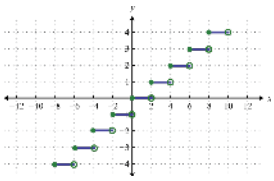


Figure 2.4.49

x

D: Graph Piecewise Functions and find their domain

Exercise 2.4e.D

★ For each of the following, (a) graph the piecewise function, and (b) state its domain in interval notation.

51. $f(x)=\begin{cases}2x-1 & \text{if } x<1\\ 1+x & \text{if } x\geq 1\end{cases}$

52. $f(x)=\begin{cases}x+1 & \text{if } x<-2\\ -2x-3 & \text{if } x\geq -2\end{cases}$

53. $f(x)=\begin{cases}3 & \text{if } x<0\\ \sqrt{x} & \text{if } x\geq 0\end{cases}$

54. $f(x)=\begin{cases}x+1 & \text{if } x<0\\ x-1 & \text{if } x>0\end{cases}$

55. $f(x)=\begin{cases}x^2 & \text{if } x<0\\ x+2 & \text{if } x\geq 0\end{cases}$

56. $f(x)=\begin{cases}x^2 & \text{if } x<0\\ 1-x & \text{if } x>0\end{cases}$

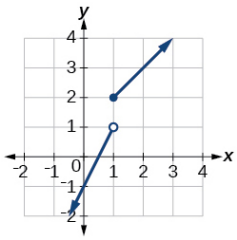
57. $f(x)=\begin{cases}|x| & \text{if } x<2\\ 1 & \text{if } x\geq 2\end{cases}$

58. $f(x)=\begin{cases}x+1 & \text{if } x<1\\ x^3 & \text{if } x\geq 1\end{cases}$

Answers to Odd Exercises:

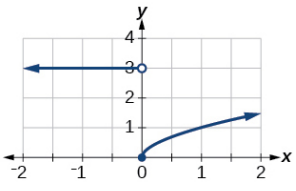
51.

domain: $(-\infty,\infty)$



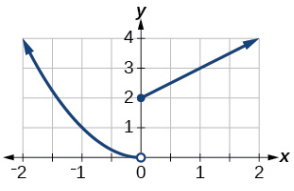
53.

domain: $(-\infty,\infty)$



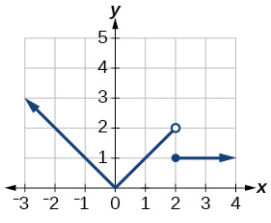
55.

domain: $(-\infty,\infty)$



57.

domain: $(-\infty,\infty)$



E: Graph Piecewise Functions and evaluate them

Exercise 2.4e.E

★ For each of the piecewise-defined functions, (a) sketch the graph, and (b) evaluate at the given values of the independent variable.

59. $f(x)=\begin{cases}x^2-3, & x\leq 0\\ 4x-3, & x>0\end{cases}$ Find $f(-4)$, $f(0)$, $f(2)$

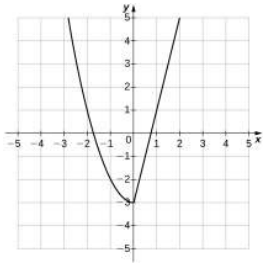
60. $f(x)=\begin{cases}4x+3, & x\leq 0\\ -x+1, & x>0\end{cases}$ Find $f(-3)$, $f(0)$, $f(2)$

61. $g(x)=\begin{cases}\frac{3}{x-2}, & x\neq 2\\ 4, & x=2\end{cases}$ Find $g(0)$, $g(-4)$, $g(2)$

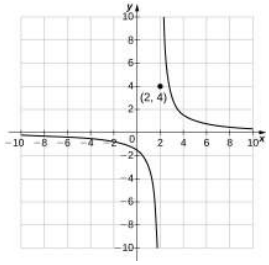
62. $h(x)=\begin{cases}x+1, & x\leq 5\\ 4, & x>5\end{cases}$ Find $h(0)$, $h(\pi)$, $h(5)$

Answers to Odd Exercises:

59. $f(-4) = 13,\ f(0) = -3,\ f(2) = 5$



61. a. $g(0) = \frac{-3}{2},\ g(-4) = \frac{-1}{2},\ g(2) = 4$



F: Construct the equation for a piecewise function given a graph

Exercise 2.4e.F

★ (a) Evaluate piecewise function values from a graph. (b) Construct a piecewise function corresponding to the graph.

63. Find $f(-4), f(-2)$, and $f(0)$.

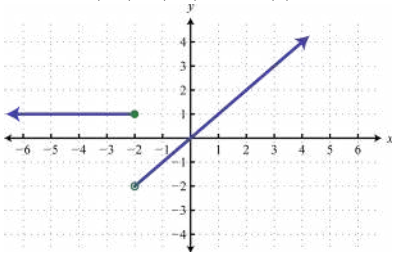


Figure 2.4e.63

64. Find $f(-3), f(0)$, and $f(1)$.

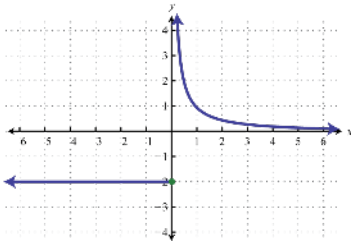


Figure 2.4e.64

65. Find $f(0), f(2)$, and $f(4)$.

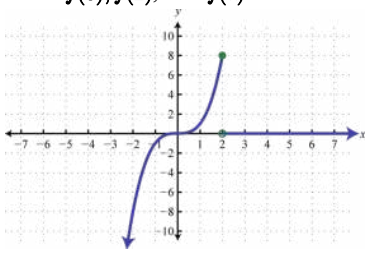


Figure 2.4e.65

66. Find $f(-5), f(-2)$, and $f(2)$.

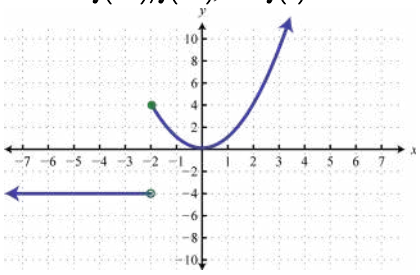


Figure 2.4e.66

67. Find $f(-3), f(-2)$, and $f(2)$.

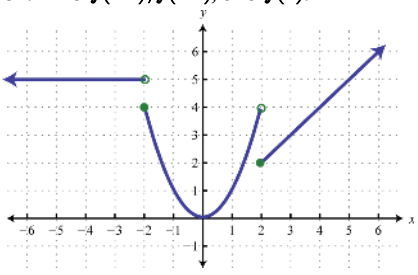


Figure 2.4e.67

68. Find $f(-3), f(0)$, and $f(4)$.

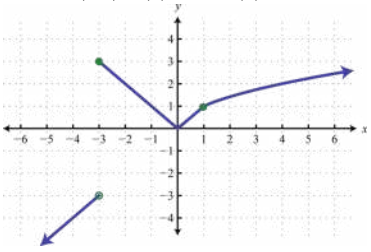


Figure 2.4e.68

69. Find $f(-2), f(0)$, and $f(2)$.

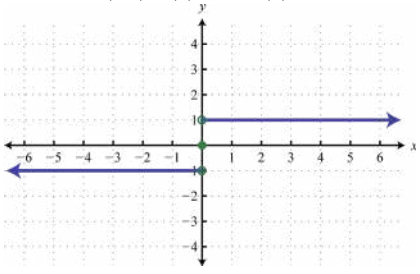


Figure 2.4e.69

70. Find $f(-3), f(1)$, and $f(2)$.

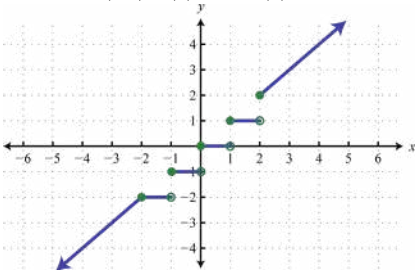


Figure 2.4e.70

Answers to Odd Exercises:

63. $f(-4)=1, f(-2)=1$, and $f(0)=0$ $\quad f(x)=\begin{cases}1, & x\le -2\\ x, & x>-2\end{cases}$)

65. $f(0)=0, f(2)=8$, and $f(4)=0$ $\quad f(x)=\begin{cases}-2, & x\le 0\\ \frac{1}{x}, & x>0\end{cases}$

67. $f(-3)=5, f(-2)=4$, and $f(2)=2$ $\quad f(x)=\begin{cases}5, & x<-2\\ x^2, & -2\le x<2\\ x, & x\ge 2\end{cases}$

69. $f(-2)=-1, f(0)=0$, and $f(2)=1$ $\quad f(x)=\begin{cases}-1, & x<0\\ 0, & x=0\\ 1, & x>0\end{cases}$

G: Simplify Combination Functions and Find their Domains

Exercise 2.4e.G

★ **For each pair of functions f and g given below, find and simplify the combination functions $f+g$, $f-g$, fg, and $\frac{f}{g}$. State the domain of each combination functions in interval notation.**

71. $f(x)=x^2+2x,\ g(x)=6-x^2$
.
72. $f(x)=-3x^2+x,\ g(x)=5$.

73. $f(x)=2x^2+4x,\ g(x)=\frac{1}{2x}$.
74. $f(x)=\frac{1}{x-4},\ g(x)=\frac{1}{6-x}$.

75. $f(x)=3x^2,\ g(x)=\sqrt{x-5}$.
76. $f(x)=\sqrt{x},\ g(x)=|x-3|$

Answers to Odd Exercises:

71. $(f+g)(x)=2x+6$, domain: $(-\infty,\infty)$
$(f-g)(x)=2x^2+2x-6$, domain: $(-\infty,\infty)$
$(fg)(x)=-x^4-2x^3+6x^2+12x$, domain: $(-\infty,\infty)$
$\left(\frac{f}{g}\right)(x)=\frac{x^2+2x}{6-x^2}$, domain: $(-\infty,-\sqrt{6})\cup(\sqrt{6},\sqrt{6})\cup(\sqrt{6},\infty)$

73. $(f+g)(x)=\frac{4x^3+8x^2+1}{2x}$, domain: $(-\infty,0)\cup(0,\infty)$
$(f-g)(x)=\frac{4x^3+8x^2-1}{2x}$, domain: $(-\infty,0)\cup(0,\infty)$
$(fg)(x)=x+2$, domain: $(-\infty,0)\cup(0,\infty)$
$\left(\frac{f}{g}\right)(x)=4x^3+8x^2$, domain: $(-\infty,0)\cup(0,\infty)$

75. $(f+g)(x)=3x^2+\sqrt{x-5}$, domain: $[5,\infty)$
$(f-g)(x)=3x^2-\sqrt{x-5}$, domain: $[5,\infty)$
$(fg)(x)=3x^2\sqrt{x-5}$, domain: $[5,\infty)$
$\left(\frac{f}{g}\right)(x)=\frac{3x^2}{\sqrt{x-5}}$, domain: $(5,\infty)$

Composition

H: Evaluate a Composition from Tables

Exercise 2.4e.H

★ **Use the function values for f and g shown in the table below to evaluate each expression.**

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	7	6	5	8	4	0	2	1	9	3
$g(x)$	9	5	6	2	1	8	7	3	4	0

78. $f(g(8))$ 79. $f(g(5))$ 80. $g(f(5))$ 81. $g(f(3))$ 82. $f(f(4))$ 83. $f(f(1))$ 84. $g(g(2))$ 85. $g(g(6))$

★ **Use the function values for f and g shown in the table below to evaluate each expression.**

x	-3	-2	-1	0	1	2	3
$f(x)$	11	9	7	5	3	1	-1
$g(x)$	-8	-3	0	1	0	-3	-8

86. $(f \circ g)(1)$ 87. $(f \circ g)(2)$ 88. $(g \circ f)(2)$ 89. $(g \circ f)(3)$ 90. $(g \circ g)(1)$ 91. $(f \circ f)(3)$

Answers to Odd Exercises:

79. 9	81. 4	83. 2	85. 3	87. 11	89. 0	91. 7

I: Evaluate a Composition from Graphs

Exercise 2.4e.I

★ **Use graphs to evaluate the following compositions.**

92. $(f \circ g)(3)$

92.1 $(f \circ g)(6)$

93. $(f \circ g)(1)$

94. $(g \circ f)(1)$

95. $(g \circ f)(0)$

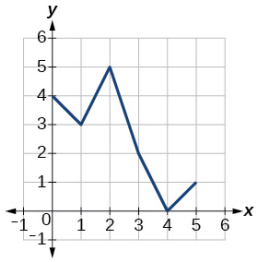
96. $(f \circ f)(5)$

97. $(f \circ f)(4)$

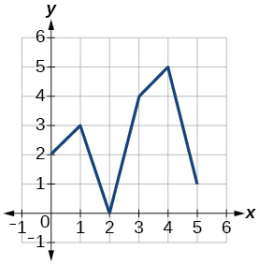
98. $(g \circ g)(2)$

99. $(g \circ g)(0)$

f



g



★ **Use graphs to evaluate the following compositions.**

100. $g(f(1))$

101. $g(f(2))$

102. $f(g(4))$

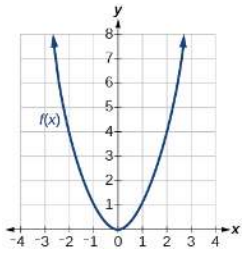
103. $f(g(1))$

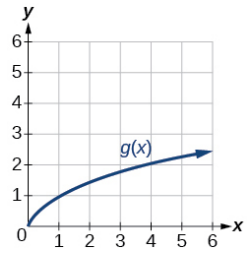
104. $f(h(2))$

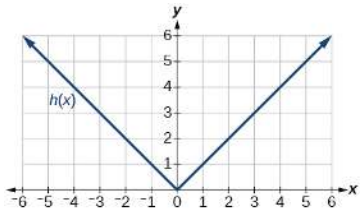
105. $h(f(2))$

106. $f(g(h(4)))$

107. $f(g(f(-2)))$







Answers to Odd Exercises:

93. 2	95. 5	97. 4	99. 0	101. 2	103. 1	105. 4	107. 4

J: Evaluate a Composition from Formulas

Exercise 2.4e.J

★ **Use the given pair of functions to find the following values if they exist.**

a. $(g \circ f)(0)$	b. $(f \circ g)(-1)$	c. $(f \circ f)(2)$	d. $(g \circ f)(-3)$	e. $(f \circ g)\left(\frac{1}{2}\right)$	f. $(f \circ f)(-2)$

111. $f(x) = x^2,\ g(x) = 2x + 1$

112. $f(x) = 4 - x,\ g(x) = 1 - x^2$

113. $f(x) = 4 - 3x,\ g(x) = |x|$

114. $f(x) = |x - 1|,\ g(x) = x^2 - 5$

115. $f(x) = 4x + 5,\ g(x) = \sqrt{x}$

116. $f(x) = \sqrt{3 - x},\ g(x) = x^2 + 1$

117. $f(x) = 6 - x - x^2,\ g(x) = x\sqrt{x + 10}$

118. $f(x) = \sqrt[3]{x + 1},\ g(x) = 4x^2 - x$

119. $f(x) = \dfrac{3}{1 - x},\ g(x) = \dfrac{4x}{x^2 + 1}$

120. $f(x) = \dfrac{x}{x + 5},\ g(x) = \dfrac{2}{7 - x^2}$

121. $f(x) = \dfrac{2x}{5 - x^2},\ g(x) = \sqrt{4x + 1}$

122. $f(x) = \sqrt{2x + 5},\ g(x) = \dfrac{10x}{x^2 + 1}$

Answers to Odd Exercises

111. $f(x)=x^2$, $g(x)=2x+1$:
a. $(g\circ f)(0)=1$
b. $(f\circ g)(-1)=1$
c. $(f\circ f)(2)=16$
d. $(g\circ f)(-3)=19$
e. $(f\circ g)\left(\frac{1}{2}\right)=4$
f. $(f\circ f)(-2)=16$

113. $f(x)=4-3x$, $g(x)=|x|$:
a. $(g\circ f)(0)=4$
b. $(f\circ g)(-1)=1$
c. $(f\circ f)(2)=10$
d. $(g\circ f)(-3)=13$
e. $(f\circ g)\left(\frac{1}{2}\right)=\frac{5}{2}$
f. $(f\circ f)(-2)=-26$

115. $f(x)=4x+5$, $g(x)=\sqrt{x}$:
a. $(g\circ f)(0)=\sqrt{5}$
b. $(f\circ g)(-1)$ is not real
c. $(f\circ f)(2)=57$
d. $(g\circ f)(-3)$ is not real
e. $(f\circ g)\left(\frac{1}{2}\right)=5+2\sqrt{2}$
f. $(f\circ f)(-2)=-7$

117. $f(x)=6-x-x^2$, $g(x)=x\sqrt{x+10}$
a. $(g\circ f)(0)=24$
b. $(f\circ g)(-1)=0$
c. $(f\circ f)(2)=6$
d. $(g\circ f)(-3)=0$
e. $(f\circ g)\left(\frac{1}{2}\right)=\frac{27-2\sqrt{42}}{8}$
f. $(f\circ f)(-2)=-14$

119. $f(x)=\frac{3}{1-x}$, $g(x)=\frac{4x}{x^2+1}$:
a. $(g\circ f)(0)=\frac{6}{5}$
b. $(f\circ g)(-1)=1$
c. $(f\circ f)(2)=\frac{3}{4}$
d. $(g\circ f)(-3)=\frac{48}{25}$
e. $(f\circ g)\left(\frac{1}{2}\right)=-5$
f. $(f\circ f)(-2)$ is undefined

121. $f(x)=\frac{2x}{5-x^2}$, $g(x)=\sqrt{4x+1}$:
a. $(g\circ f)(0)=1$
b. $(f\circ g)(-1)$ is not real
c. $(f\circ f)(2)=-\frac{8}{11}$
d. $(g\circ f)(-3)=\sqrt{7}$
e. $(f\circ g)\left(\frac{1}{2}\right)=\sqrt{3}$
f. $(f\circ f)(-2)=\frac{8}{11}$

K: Simplify a Composition and Find its Domain

Exercise 2.4e.K: Find and simplify the Equation for a Composition

★ Find and simplify (a) $(f\circ g)(x)$, and (b) $(g\circ f)(x)$. State the domain for (c) $(f\circ g)(x)$ and for (d) $(g\circ f)(x)$.

127. $f(x)=x^5$, $g(x)=x+1$

128. $f(x)=|x|$, $g(x)=5x+1$

129. $f(x)=2x+3$, $g(x)=x^2-9$

130. $f(x)=4x+8$, $g(x)=7-x^2$

131. $f(x)=5x+7$, $g(x)=4-2x^2$

132. $f(x)=2x^2+1$, $g(x)=3x+5$

133. $f(x)=2x^2+1$, $g(x)=3x-5$

134. $f(x)=x^2-x+1$, $g(x)=3x-5$

135. $f(x)=x^2-4$, $g(x)=|x|$

Answers to Odd Exercises

127. a. $(f\circ g)(x)=(x+1)^5$, domain: $(-\infty,\infty)$ b. $(g\circ f)(x)=x^5+1$. domain: $(-\infty,\infty)$
129. a. $(f\circ g)(x)=2x^2-15$, domain: $(-\infty,\infty)$ b. $(g\circ f)(x)=4x^2+12x$, domain: $(-\infty,\infty)$
131. a. $(f\circ g)(x)=27-10x^2$, domain: $(-\infty,\infty)$ b. $(g\circ f)(x)=-50x^2-140x-94$. domain: $(-\infty,\infty)$
133. a. $(f\circ g)(x)=2(3x-5)^2+1$, domain: $(-\infty,\infty)$ b. $(g\circ f)(x)=6x^2-2$. domain: $(-\infty,\infty)$
135. a. $(f\circ g)(x)=x^2-4$, domain: $(-\infty,\infty)$ b. $(g\circ f)(x)=|x^2-4|$, domain: $(-\infty,\infty)$

★ **Find and simplify (a) $(f\circ g)(x)$, and (b) $(g\circ f)(x)$. State the domain for (c) $(f\circ g)(x)$ and for (d) $(g\circ f)(x)$.**

137. $f(x)=3x-5,\ g(x)=\sqrt{x}$

138. $f(x)=\sqrt{x}+2,\ g(x)=x^2+3$

139. $f(x)=|x+1|,\ g(x)=\sqrt{x}$

140. $f(x)=|x|,\ g(x)=\sqrt{4-x}$

141. $f(x)=x^2+2,\ \ g(x)=\sqrt{x-2}$

142. $f(x)=x^2+1,\ g(x)=\sqrt{x+2}$

143. $f(x)=x^2-x-1,\ g(x)=\sqrt{x-5}$

144. $f(x)=3-x^2,\ g(x)=\sqrt{x+1}$

145. $f(x)=\dfrac{1}{\sqrt{x}},\ \ g(x)=x^2-4$

146. $f(x)=\dfrac{1}{\sqrt{x}},\ \ g(x)=x^2-9$

147. $f(x)=\sqrt{x+4},\ g(x)=12-x^3$

148. $f(x)=x^3+1$ and $g(x)=\sqrt[3]{x-1}$

Answers to Odd Exercises

137. a. $(f\circ g)(x)=3\sqrt{x}-5$, domain: $[0,\infty)$ b. $(g\circ f)(x)=\sqrt{3x-5}$. domain: $[\frac{5}{3},\infty)$
139. a. $(f\circ g)(x)=\sqrt{x}+1$, domain: $[0,\infty)$ b. $(g\circ f)(x)=\sqrt{|x+1|}$, domain: $(-\infty,\infty)$
141. a. $(f\circ g)(x)=x$, domain: $[2,\infty)$ b. $(g\circ f)(x)=|x|$. domain: $(-\infty,\infty)$
143. a. $(f\circ g)(x)=x-6-\sqrt{x-5}$, d: $[5,\infty)$ b. $(g\circ f)(x)=\sqrt{x^2-x-6}$. d: $(-\infty,-2]\cup[3,\infty)$
145. a. $(f\circ g)(x)=\frac{1}{\sqrt{x^2-4}}$, domain: $(-\infty,-2)\cup(2,\infty)$ b. $(g\circ f)(x)=\frac{1}{x}-4$. domain: $(0,\infty)$
147. a. $(f\circ g)(x)=\sqrt{16-x^3}$, d: $(-\infty,2\sqrt[3]{2}]$ b. $(g\circ f)(x)=12-(x+4)\sqrt{x+4}$. d: $[-4,\infty)$

★ **Find and simplify (a) $(f\circ g)(x)$, and (b) $(g\circ f)(x)$. State the domain for (c) $(f\circ g)(x)$ and for (d) $(g\circ f)(x)$.**

149. $f(x)=\dfrac{1}{x},\ \ g(x)=x-3$

150. $f(x)=\frac{1}{x+2},\ g(x)=4x+3$

151. $f(x)=3x-1,\ g(x)=\dfrac{1}{x+3}$

152. $f(x)=\dfrac{1}{x-6},\ g(x)=\dfrac{7}{x}+6$

153. $f(x)=\dfrac{1}{x-4},\ g(x)=\dfrac{2}{x}+4$

154. $f(x)=\dfrac{3x}{x-1},\ g(x)=\dfrac{x}{x-3}$

155. $f(x)=\dfrac{x}{2x+1},\ g(x)=\dfrac{2x+1}{x}$

156. $f(x)=\dfrac{1-x}{x},\ \ g(x)=\dfrac{1}{1+x^2}$

157. $f(x)=\dfrac{1}{x},\ \ g(x)=\sqrt{x-1}$

158. $f(x)=\sqrt{2-4x},\ \ g(x)=-\dfrac{3}{x}$

159. $f(x)=\sqrt[3]{x},\ g(x)=\dfrac{x+1}{x^3}$

160. $f(x)=\dfrac{2x}{x^2-4},\ g(x)=\sqrt{1-x}$

Answers to Odd Exercises:

149. a. $(f\circ g)(x)=\frac{1}{x-3}$, domain: $(-\infty,3)\cup(3,\infty)$ b. $(g\circ f)(x)=\frac{1}{x}-3$. domain: $(-\infty,0)\cup(0,\infty)$
151. a. $(f\circ g)(x)=-\frac{x}{x+3}$, d: $(-\infty,-3)\cup(-3,\infty)$ b. $(g\circ f)(x)=\frac{1}{3x+2}$, d: $(-\infty,-\frac{2}{3})\cup(-\frac{2}{3},\infty)$
153. a. $(f\circ g)(x)=\dfrac{x}{2}$, domain: $(-\infty,0)\cup(0,\infty)$ b. $(g\circ f)(x)=2x-4$. domain: $(-\infty,4)\cup(4,\infty)$
155. a. $(f\circ g)(x)=\frac{2x+1}{5x+2}$, d: $(-\infty,-\frac{2}{5})\cup(-\frac{2}{5},0)\cup(0,\infty)$
b. $(g\circ f)(x)=\frac{4x+1}{x}$, domain: $(-\infty,-\frac{1}{2})\cup(-\frac{1}{2},0),\cup(0,\infty)$
157. a. $(f\circ g)(x)=\frac{1}{\sqrt{x-1}}$, domain: $(1,\infty)$ b. $(g\circ f)(x)=\sqrt{\frac{1}{x}-1}$. domain: $(0,1]$
159. a. $(f\circ g)(x)=\dfrac{\sqrt[3]{x+1}}{x}$, d: $(-\infty,0)\cup(0,\infty)$ b. $(g\circ f)(x)=\frac{\sqrt[3]{x}+1}{x}$. d: $(-\infty,0)\cup(0,\infty)$

★ **(a) Find and simplify $(f \circ f)(x)$\ and (b) state the domain of the composition.**

163. $f(x) = 2x + 3$
164. $f(x) = x^2 - x + 1$
165. $f(x) = x^2 - 4$
166. $f(x) = 3x - 5$
167. $f(x) = |x + 1|$
168. $f(x) = 3 - x^2$
169. $f(x) = |x|$
170. $f(x) = x^2 - x - 1$
171. $f(x) = 3x - 1$
172. $f(x) = \dfrac{3x}{x-1}$
173. $f(x) = \dfrac{x}{2x+1}$
174. $f(x) = \dfrac{2x}{x^2-4}$

★ **Given $f(x) = -2x$, $g(x) = \sqrt{x}$ and $h(x) = |x|$, find and simplify expressions for the following functions and state the domain of each using interval notation.**

175. $(h \circ g \circ f)(x)$
176. $(h \circ f \circ g)(x)$
177. $(g \circ f \circ h)(x)$
178. $(g \circ h \circ f)(x)$
179. $(f \circ h \circ g)(x)$
180. $(f \circ g \circ h)(x)$

Answers to Odd Exercises:

163. $(f \circ f)(x) = 4x + 9$, domain: $(-\infty, \infty)$
165. $(f \circ f)(x) = x^4 - 8x^2 + 12$, domain: $(-\infty, \infty)$
167. $(f \circ f)(x) = ||x+1|+1| = |x+1|+1$, domain: $(-\infty, \infty)$
169. $(f \circ f)(x) = ||x|| = |x|$, domain: $(-\infty, \infty)$
171. $(f \circ f)(x) = 9x - 4$, domain: $(-\infty, \infty)$
173. $(f \circ f)(x) = \frac{x}{4x+1}$, d: $(-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, -\frac{1}{4}) \cup (-\frac{1}{4}, \infty)$
175. $(h \circ g \circ f)(x) = |\sqrt{-2x}| = \sqrt{-2x}$, domain: $(-\infty, 0]$
177. $(g \circ f \circ h)(x) = \sqrt{-2|x|}$, domain: $\{0\}$
179. $(f \circ h \circ g)(x) = -2|\sqrt{x}| = -2\sqrt{x}$, domain: $[0, \infty)$

L: Decomposition

Exercise 2.4e.L

★ **Find functions $f(x)$ and $g(x)$ so the given function can be expressed as $f(g(x))$.**

185. $h(x) = \sqrt{\dfrac{2x-1}{3x+4}}$
186. $h(x) = (x+2)^2$
187. $h(x) = (x-5)^3$
188. $h(x) = \dfrac{3}{x-5}$
189. $h(x) = \dfrac{4}{(x+2)^2}$
190. $h(x) = 4 + \sqrt[3]{x}$
191. $h(x) = \sqrt[3]{\dfrac{1}{2x-3}}$
192. $h(x) = \dfrac{1}{(3x^2-4)^{-3}}$
193. $h(x) = \sqrt[4]{\dfrac{3x-2}{x+5}}$
194. $h(x) = \left(\dfrac{8+x^3}{8-x^3}\right)^4$
195. $h(x) = \sqrt{2x+6}$
196. $h(x) = (5x-1)^3$
197. $h(x) = \sqrt[3]{x-1}$
198. $h(x) = |x^2+7|$
199. $h(x) = \dfrac{1}{(x-2)^3}$
200. $h(x) = \left(\dfrac{1}{2x-3}\right)^2$
201. $p(x) = (2x+3)^3$
202. $P(x) = (x^2-x+1)^5$
203. $h(x) = \sqrt{2x-1}$
204. $H(x) = |7-3x|$
205. $r(x) = \dfrac{2}{5x+1}$
206. $R(x) = \dfrac{7}{x^2-1}$
207. $q(x) = \dfrac{|x|+1}{|x|-1}$
208. $Q(x) = \dfrac{2x^3+1}{x^3-1}$
209. $v(x) = \dfrac{2x+1}{3-4x}$
210. $w(x) = \dfrac{x^2}{x^4+1}$

Answers to Odd Exercises:

185. sample: $f(x) = \sqrt{x}$, $g(x) = \frac{2x-1}{3x+4}$
187. sample: $f(x) = x^3$, $g(x) = x - 5$
189: sample: $f(x) = \frac{4}{x}$, $g(x) = (x+2)^2$
191. sample: $f(x) = \sqrt[3]{x}$, $g(x) = \frac{1}{2x-3}$
193. sample: $f(x) = \sqrt[4]{x}$, $g(x) = \frac{3x-2}{x+5}$

195. sample: $f(x)=\sqrt{x}, \quad g(x)=2x+6$
197. sample: $f(x)=\sqrt[3]{x}, \quad g(x)=(x-1)$
199. sample: $f(x)=x^3, \quad g(x)=\frac{1}{x-2}$
201. Let $g(x)=2x+3$ and $f(x)=x^3$, then $p(x)=(f\circ g)(x)$.
203. Let $g(x)=2x-1$ and $f(x)=\sqrt{x}$, then $h(x)=(f\circ g)(x)$.
205. Let $g(x)=5x+1$ and $f(x)=\frac{2}{x}$, then $r(x)=(f\circ g)(x)$.
207. Let $g(x)=|x|$ and $f(x)=\frac{x+1}{x-1}$, then $q(x)=(f\circ g)(x)$.
209. Let $g(x)=2x$ and $f(x)=\frac{x+1}{3-2x}$, then $v(x)=(f\circ g)(x)$.

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2.5: One-to-One and Inverse Functions

Learning Objectives

- Understand the concept of a one-to-one function.
- Determine the conditions for when a function has an inverse.
- Use the horizontal line test to recognize when a function is one-to-one.
- Find the inverse of a given function.
- Draw the graph of an inverse function.

One-to-one functions

Some functions have a given output value that corresponds to two or more input values. For example, on a menu there might be five different items that all cost $7.99. If the domain of a function is all of the items listed on the menu and the range is the prices of the items, then there are five different input values that all result in the same output value of $7.99.

Example 2.5.1: Determining Whether a Relationship Is a One-to-One Function

Is the area of a circle a function of its radius? If yes, is the function one-to-one?

Solution

A circle of radius r has a unique area measure given by $A = \pi r^2$, so for any input, r, there is only one output, A. The area is a function of radius$r$.

If the function is one-to-one, every output value for the area, must correspond to a unique input value, the radius.

- For any given radius, only one value for the area is possible. Any area measure A is given by the formula $A = \pi r^2$.
- For any given area, only one value for the radius can be produced. It is not possible that a circle with a different radius would have the same area. Any radius measure r is given by the formula $r = \pm\sqrt{\frac{A}{\pi}}$. Because areas and radii are positive numbers, there is exactly one solution: $\sqrt{\frac{A}{\pi}}$.

Every radius corresponds to just one area and every area is associated with just one radius.
So the area of a circle is a one-to-one function of the circle's radius.

Try It 2.5.1

At a bank, a printout is made at the end of the day, listing each bank account number and its balance.

a. Is the ending balance a function of the bank account number?
b. Is the ending balance a one-to-one function of the bank account number?

Answer

a. yes, because each bank account has a single balance at any given time;
b. no, because each bank account corresponds to just one balance, but each balance does not correspond to just one bank account (the same balance can belong to two different accounts)..

Definition: One-to-One Functions

A **one-to-one function** is a particular type of **function** in which for each output value y there is exactly one input value x that is associated with it. In other words, a function is one-to-one if each output y corresponds to precisely one input x.

It's easiest to understand this definition by looking at mapping diagrams and graphs of some example functions.

Example 2.5.2: Definition of 1-1 functions

Consider the two functions h and k defined according to the mapping diagrams in **Figure 1**. In **Figure 1**(a), there are two values in the domain that are both mapped onto 3 in the range. Hence, the function h is not one-to-one. On the other hand, in **Figure 1**(b), for each output in the range of k, there is only one input in the domain that gets mapped onto it. Therefore, k is a one-to-one function.

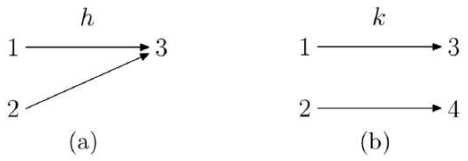


Figure 2. Mapping diagrams help to determine if a function is one-to-one.

Identifying 1-1 functions graphically

When examining a graph of a function, if a horizontal line (which represents a single value for y), intersects the graph of a function in more than one place, then for each point of intersection, you have a different value of x associated with the same value of y. Thus the y value does NOT correspond to just precisely one input, and the graph is NOT that of a one-to-one function. This idea is the idea behind the Horizontal Line Test.

How To: Howto: Use the horizontal line test to determine if a given graph represents a 1-1 function.

1. Confirm the graph is a function by using the vertical line test. (a 1-1 function **must** be a function)
2. Inspect the graph to see if any horizontal line drawn would intersect the curve more than once.
3. If there is any such line, then the function is not one-to-one, but if every horizontal line intersects the graph in at most one point, then the function represented by the graph is **one-to-one**.

[m] Example 2.5.3: HLT

Determine (a) whether each graph is the graph of a function and, if so, (b) whether it is one-to-one.

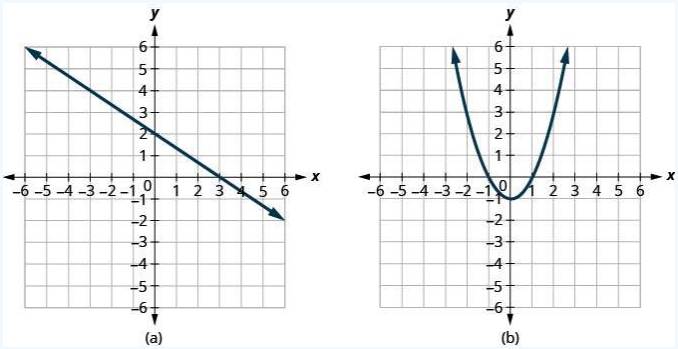
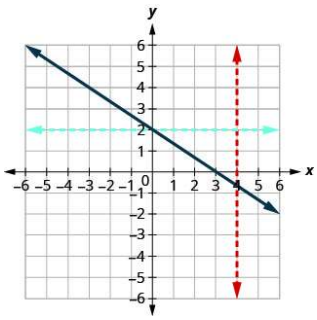
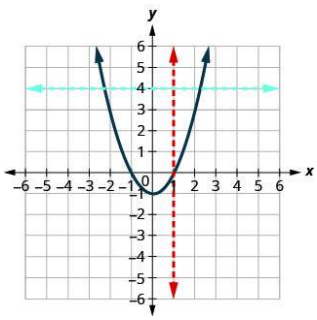


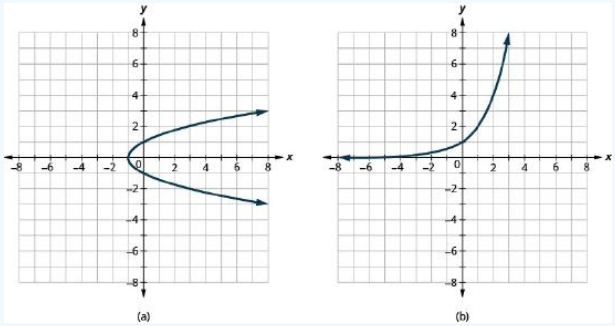
Figure 3

Solution:

	
Since any vertical line intersects the graph in at most one point, the graph is the graph of a function. Since any horizontal line intersects the graph in at most one point, the graph is the graph of a one-to-one function.	Since any vertical line intersects the graph in at most one point, the graph is the graph of a function. The horizontal line shown on the graph intersects it in two points. This graph does not represent a one-to-one function.

Try It 2.5.3: HLT

Determine whether each graph is the graph of a function and, if so, whether it is one-to-one.



Answer

a. Not a function -- so not a one-to-one function
b. One-to-one function

Note: 1-1 function attributes

It follows from the horizontal line test that if f is a strictly increasing function, then f is one-to-one. Likewise, every strictly decreasing function is also one-to-one.

One-to-One Attribute Confirmed Algebraically

Algebraic Definition: One-to-One Functions

If a function f is one-to-one and a and b are in the domain of f then

if $a \neq b$ then $f(a) \neq f(b)$	Two different x values always produce different y values
if $f(a) = f(b)$ then $a = b$	No value of y corresponds to more than one value of x

Example 2.5.4: Confirm 1-1 algebraically

Show algebraically that $f(x) = (x+2)^2$ is not one-to-one

Solution

$$\begin{array}{rcl} \text{If } f(a) & = & f(b) \text{ then ...} \\ (a+2)^2 & = & (b+2)^2 \\ \sqrt{(a+2)^2} & = & \pm\sqrt{(b+2)^2} \\ a+2=b+2 & or & a+2=-(b+2) \\ a=b & or & a=-b-4 \end{array}$$

Since we have shown that when $f(a) = f(b)$ we do not always have the outcome that $a = b$ then we can conclude the f is not one-to-one.

Implications of One-to-one Attribute when Solving Equations

One of the ramifications of being a one-to-one function f is that when solving an equation $f(u) = f(v)$ then this equation can be solved more simply by just solving $u = v$. This is commonly done when log or exponential equations must be solved. Another implication of this property we have already seen when we encounter extraneous roots when square root equations are solved by squaring. This is because the solutions to $g(x) = x^2$ are not necessarily the solutions to $f(x) = \sqrt{x}$ because g is not a one-to-one function.

One-to-One Functions and Inverses

Let's look at a one-to one function, f, represented by the ordered pairs $\{(0,5),(1,6),(2,7),(3,8)\}$. For each x-value, f adds 5 to get the y-value. To 'undo' the addition of 5, we subtract 5 from each y-value and get back to the original x-value. We can call this "taking the inverse of f" and name the function f^{-1}.

WARNING: Notation

f^{-1} does **not** mean $\frac{1}{f}$

The -1 exponent is just notation in this context. When applied to a function, it stands for the inverse of the function, not the reciprocal of the function.

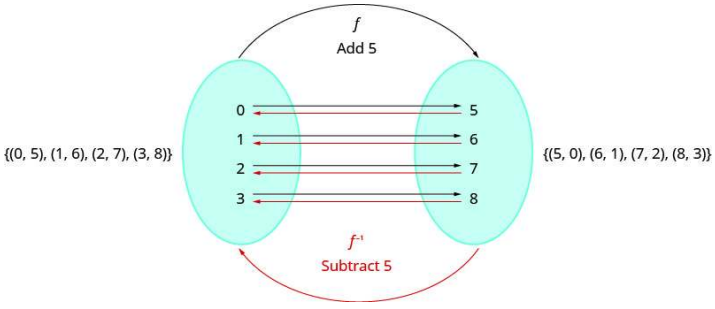


Figure 5

Note: One-to-one functions and Inverses

A function must be one-to-one in order to have an inverse.

Consider the function h illustrated in Figure 2(a). h is not one-to-one. If we reverse the arrows in the mapping diagram for a non one-to-one function like h in Figure 2(a), then the resulting relation will not be a function, because 3 would map to both 1 and 2. In contrast, if we reverse the arrows for a one-to-one function like k in Figure 2(b) or f in the example above, then the resulting relation IS a function which undoes the effect of the original function. Thus in order for a function to have an inverse, it must be a one-to-one function and conversely, every one-to-one function has an inverse function.

Inverse Functions

Verification of inverse functions

When we began our discussion of an inverse function, we talked about how the inverse function 'undoes' what the original function did to a value in its domain in order to get back to the original x-value. This is shown diagrammatically below.

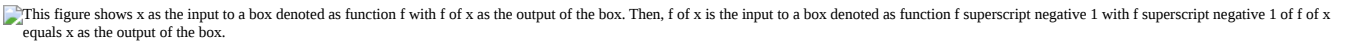


Figure 6

Putting these concepts into an algebraic form, we come up with the definition of an inverse function

Definition: Inverse Functions

$f^{-1}(f(x)) = x$, for all x in the domain of f

$f\left(f^{-1}(x)\right) = x$, for all x in the domain of f^{-1}

We can use this property to verify that two functions are inverses of each other.

Example 2.5.6: Verify Inverses of linear functions

Verify that $f(x) = 5x - 1$ and $g(x) = \frac{x+1}{5}$ are inverse functions.

Solution:

The functions are inverses of each other if $g(f(x)) = x$ and $f(g(x)) = x$.

	$g(f(x)) \stackrel{?}{=} x$		$f(g(x)) \stackrel{?}{=} x$
Substitute $5x-1$ for $f(x)$.	$g(5x-1) \stackrel{?}{=} x$	Substitute $\frac{x+1}{5}$ for $g(x)$.	$f\left(\frac{x+1}{5}\right) \stackrel{?}{=} x$
Find $g(5x-1)$ where $g(x) = \frac{x+1}{5}$	$\frac{(5x-1)+1}{5} \stackrel{?}{=} x$	Find $f(\frac{x+1}{5})$ where $f(x) = 5x - 1$	$5\left(\frac{x+1}{5}\right) - 1 \stackrel{?}{=} x$
Simplify.	$\frac{5x}{5} \stackrel{?}{=} x$ $x = x$ ✓	Simplify.	$x + 1 - 1 \stackrel{?}{=} x$ $x = x$ ✓

Since both $g(f(x)) = x$ and $f(g(x)) = x$ are true, the functions $f(x) = 5x - 1$ and $g(x) = \frac{x+1}{5}$ are inverse functions of each other.

Try It 2.5.6a

1. Verify that the functions are inverse functions. $f(x) = 4x - 3$ and $g(x) = \frac{x+3}{4}$.
2. Verify that the functions are inverse functions. $f(x) = 2x + 6$ and $g(x) = \frac{x-6}{2}$

Answer

1. $g(f(x)) = x$, and $f(g(x)) = x$, so they are inverses.
2. $g(f(x)) = x$, and $f(g(x)) = x$, so they are inverses.

Try It 2.5.6b

Show that $f(x) = \frac{x+5}{3}$ and $f^{-1}(x) = 3x - 5$ are inverses.

Answer

1. $f^{-1}(f(x)) = f^{-1}(\frac{x+5}{3}) = 3(\frac{x+5}{3}) - 5 = (x-5)+5 = x$
2. $f(f^{-1}(x)) = f(3x-5) = \frac{(3x-5)+5}{3} = \frac{3x}{3} = x$

Notice the inverse operations are in reverse order of the operations from the original function.

Example 2.5.7: Verify Inverses of Rational Functions

Show that $f(x) = \frac{1}{x+1}$ and $f^{-1}(x) = \frac{1}{x} - 1$ are inverses, for $x \neq 0, -1$.

Solution

We must show that $f^{-1}(f(x)) = x$ for all x in the domain of f

$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}\left(\frac{1}{x+1}\right) \\ &= \frac{1}{\frac{1}{x+1}} - 1 \\ &= (x+1) - 1 \\ &= x \qquad \text{for all } x \neq -1, \text{ the domain of } f \end{aligned}$$

and $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}.

$$\begin{aligned} f(f^{-1}(x)) &= f(\frac{1}{x-1}) \\ &= \frac{1}{\left(\frac{1}{x-1}\right)+1} \\ &= \frac{1}{\frac{1}{x}} \\ &= x \qquad \text{for all } x \neq 0, \text{ the domain of } f^{-1} \end{aligned}$$

Therefore, $f(x) = \frac{1}{x+1}$ and $f^{-1}(x) = \frac{1}{x} - 1$ are inverses.

Example 2.5.8: Verify Inverses for Power Functions

If $f(x) = x^3$ (the cube function) and $g(x) = \frac{1}{3}x$, is $g = f^{-1}$?

Solution

$$f(g(x)) = \frac{x^3}{27} \neq x$$

No, the functions are not inverses. The correct inverse to the cube is, of course, the cube root $\sqrt[3]{x} = x^{\frac{1}{3}}$, that is, the one-third is an exponent, not a multiplier.

Try It 2.5.8

1. If $f(x) = x^3 - 4$ and $g(x) = \sqrt[3]{x+4}$, is $g = f^{-1}$?
2. If $f(x) = (x-1)^3$ and $g(x) = \sqrt[3]{x} + 1$, is $g = f^{-1}$?

Answer

1. yes 2. yes

Inverses of Ordered Pairs

Definition: Inverse of a Function Defined by Ordered Pairs.

If $f(x)$ is a one-to-one function whose ordered pairs are of the form (x, y), then its inverse function $f^{-1}(x)$ is the set of ordered pairs (y, x).

In the next example we will find the inverse of a function defined by ordered pairs.

Example 2.5.9: Inverse of Ordered Pairs

Find the inverse of the function $\{(0,3),(1,5),(2,7),(3,9)\}$. Determine the domain and range of the inverse function.

Solution:

This function is one-to-one since every x-value is paired with exactly one y-value.

To find the inverse we reverse the x-values and y-values in the ordered pairs of the function.

Function	$\{(0,3),(1,5),(2,7),(3,9)\}$
Inverse Function	$\{(3,0),(5,1),(7,2),(9,3)\}$
Domain of Inverse Function	$\{3,5,7,9\}$
Range of Inverse Function	$\{0,1,2,3\}$

Try It 2.5.9

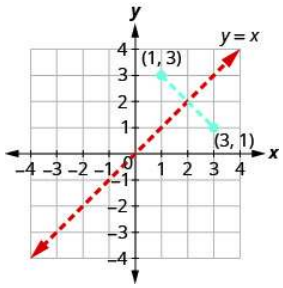
1. Find the inverse of $\{(0,4),(1,7),(2,10),(3,13)\}$. Determine the domain and range of the inverse function.
2. Find the inverse of $\{(-1,4),(-2,1),(-3,0),(-4,2)\}$. Determine the domain and range of the inverse function.

Answer

1. Inverse function: $\{(4,0),(7,1),(10,2),(13,3)\}$. Domain: $\{4,7,10,13\}$. Range: $\{0,1,2,3\}$.
2. Inverse function: $\{(4,-1),(1,-2),(0,-3),(2,-4)\}$. Domain: $\{0,1,2,4\}$. Range: $\{-4,-3,-2,-1\}$.

Inverses from Graphs

Since every point on the graph of a function $f(x)$ is a mirror image of a point on the graph of $f^{-1}(x)$, we say the graphs are mirror images of each other through the line $y = x$. We will use this concept to graph the inverse of a function in the next example.



Graph an Inverse

We just noted that if $f(x)$ is a one-to-one function whose ordered pairs are of the form (x, y), then its inverse function $f^{-1}(x)$ is the set of ordered pairs (y, x).

So if a point (a, b) is on the graph of a function $f(x)$, then the ordered pair (b, a) is on the graph of $f^{-1}(x)$. The Figure on the right illustrates this.

The distance between any two pairs (a, b) and (b, a) is cut in half by the line $y = x$. So we say the points are mirror images of each other through the line $y = x$.

Example 2.5.10a: Graph Inverses

Given the graph of $f(x)$ in Figure 2.5.10a, sketch a graph of $f^{-1}(x)$.

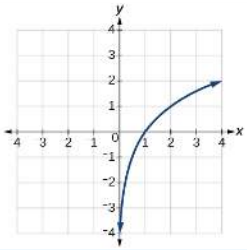
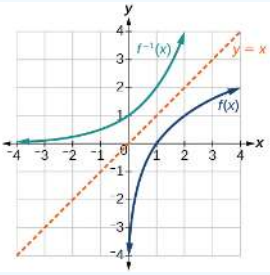


Figure 2.5.10a

Solution. The graph of $f(x)$ is a one-to-one function, so we will be able to sketch an inverse. Note that the graph shown has an apparent domain of $(0, \infty)$ and range of $(-\infty, \infty)$, so the inverse will have a domain of $(-\infty, \infty)$ and range of $(0, \infty)$.

If we reflect this graph over the line $y = x$, the point $(1, 0)$ reflects to $(0, 1)$ and the point $(4, 2)$ reflects to $(2, 4)$. Sketching the inverse on the same axes as the original graph gives the graph illustrated in the Figure to the right. The graph clearly shows the graphs of the two functions are reflections of each other across the identity line $y = x$.



Example 2.5.10b: Graph Inverses

Graph, on the same coordinate system, the inverse of the one-to one function shown.

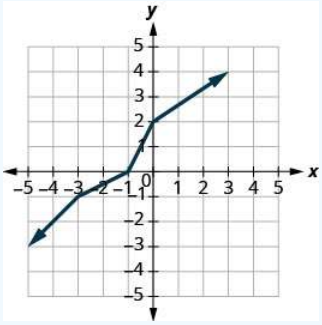
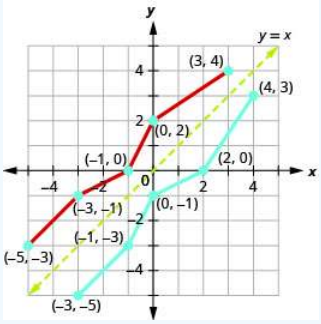


Figure 10b

Solution:

We can use points on the graph to find points on the inverse graph. Some points on the graph are: $(-5, -3), (-3, -1), (-1, 0), (0, 2), (3, 4)$.

So, the inverse function will contain the points: $(-3, -5), (-1, -3), (0, -1), (2, 0), (4, 3)$.



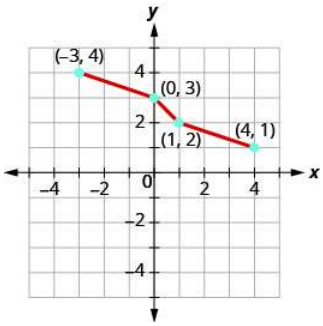
Graph of function illustrated in Figure 10b and its inverse

Notice how the graph of the original function and the graph of the inverse functions are mirror images through the line $y = x$.

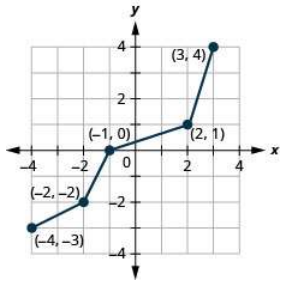
Try It 2.5.10

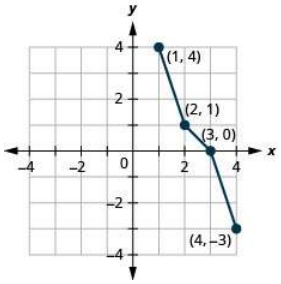
Graph, on the same coordinate system, the inverse of the one-to one function.

The graph shows a line from (negative 3, negative 4) to (negative 2, negative 2) then to (0, negative 1), then to (1, 2) and then to (4, 3). The graph shows a line from (negative 3, 4) to (0, 3) then to (1, 2) and then to (4, 1).



Answer





Obtain Function Values for an Inverse

How To: Given the graph of a function, evaluate its inverse at specific points.

1. Find the desired x coordinate of f^{-1} on the y-axis of the given graph of f.
2. Read the corresponding y coordinate of f^{-1} from the x-axis of the given graph of f.

Example 2.5.12: Evaluating a Function and Its Inverse from a Graph at Specific Points

A function $g(x)$ is given in Figure 2.5.12. Find $g(3)$ and $g^{-1}(3)$.

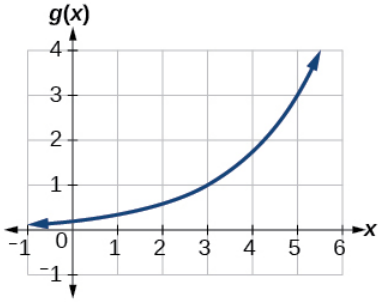


Figure 2.5.12: Graph of $g(x)$

Solution. To evaluate $g(3)$, we find 3 on the x-axis and find the corresponding output value on the y-axis. The point $(3,1)$ tells us that $g(3)=1$.

To evaluate $g^{-1}(3)$, recall that by definition $g^{-1}(3)$ means the value of x for which $g(x)=3$. By looking for the output value 3 on the vertical axis, we find the point $(5,3)$ on the graph, which means $g(5)=3$, so by definition, $g^{-1}(3)=5$. See Figure 2.5.12s below.

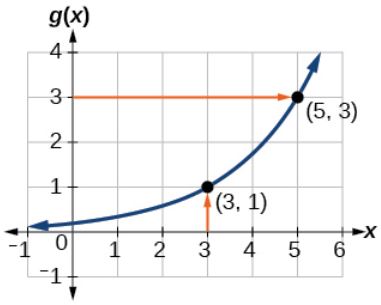


Figure 2.5.12s: Graph of $g(x)$.

Try It 2.5.12

Using the graph in Figure 2.5.12, (a) find $g^{-1}(1)$, and (b) estimate $g^{-1}(4)$.

Answer

(a) 3 (b) 5.6

Inverses from Equations

We have found inverses of function defined by ordered pairs and from a graph. We will now look at how to find an inverse using an algebraic equation. The method uses the idea that if $f(x)$ is a one-to-one function with ordered pairs (x, y), then its inverse function $f^{-1}(x)$ is the set of ordered pairs (y, x).

If we reverse the x and y in the function and then solve for y, we get our **inverse function**.

We summarize the steps below.

Howto: Find the Inverse of a One-to-One Function.

Make sure that f is one-to-one. If f is not one-to-one it does NOT have an inverse. (Alternatively, the proposed inverse could be found and then it would be necessary to confirm the two are functions and indeed inverses).

1. Substitute y for $f(x)$.
2. Interchange the variables x and y.
3. Solve for y.
4. Substitute $f^{-1}(x)$ for y.

Inverses of Linear Functions

Example 2.5.13: Inverses of a Linear Function

Find the inverse of $f(x) = 4x + 7$.

Solution:

The graph of function f is a line and so it is one-to-one.

Step 1. Substitute y for $f(x)$.	Replace $f(x)$ with y.	$f(x) = 4x + 7$ $y = 4x + 7$
Step 2: Interchange the variables x and y.	Replace x with y and then y with x.	$x = 4y + 7$
Step 3: Solve for y.	Subtract 7 from each side. Divide by 4.	$x - 7 = 4y$ $\frac{x-7}{4} = y$
Step 4: Substitute $f^{-1}(x)$ for y.	Replace y with $f^{-1}(x)$.	$\frac{x-7}{4} = f^{-1}(x)$

Try It 2.5.13

1. Find the inverse of the function $f(x) = 5x - 3$.
2. Find the inverse of the function $f(x) = 8x + 5$.

Answer

1. $f^{-1}(x) = \frac{x+3}{5}$ 2. $f^{-1}(x) = \frac{x-5}{8}$

Inverses of Odd Power Functions

Example 2.5.14a

Find the inverse of $f(x) = 2x^5 + 3$.

Solution. A check of the graph shows that f is one-to-one (this is left for the reader to verify).

STEP 1: Write the formula in xy-equation form: $y = 2x^5 + 3$.

STEP 2: Interchange x and y: $x = 2y^5 + 3$.

STEP 3: Solve for y:

$$x = 2y^5 + 3.$$
$$\rightarrow x - 3 = 2y^5$$
$$\rightarrow \frac{x-3}{2} = y^5$$
$$\rightarrow \sqrt[5]{\frac{x-3}{2}} = y$$

STEP 4: Thus, $f^{-1}(x) = \sqrt[5]{\frac{x-3}{2}}$

Example 2.5.14b: Finding the Inverse of a Cubic Function

Find the inverse of the function $f(x) = 5x^3 + 1$.

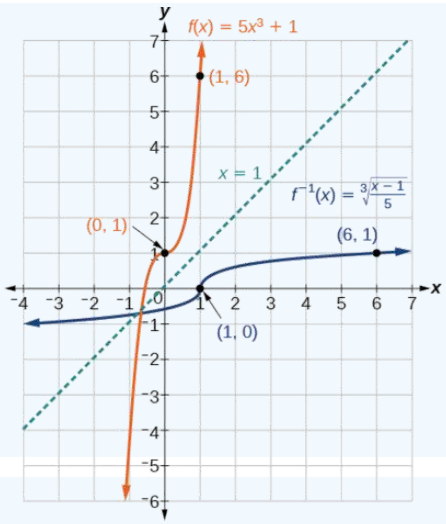
Solution

This is a transformation of the basic cubic toolkit function, and based on our knowledge of that function, we know it is one-to-one. Solve for the inverse by switching x and y and solving for y.

$$y = 5x^3 + 1$$
$$x = 5y^3 + 1$$
$$x - 1 = 5y^3$$

$$\frac{x-1}{5} = y^3$$

$$f^{-1}(x) = \sqrt[3]{\frac{x-1}{5}}$$



Analysis

Look at the graph of f and f^{-1}. Notice that one graph is the reflection of the other about the line $y = x$. This is always the case when graphing a function and its inverse function.

Also, since the method involved interchanging x and y, notice corresponding points in the accompanying figure. If (a, b) is on the graph of f, then (b, a) is on the graph of f^{-1}. Since $(0, 1)$ is on the graph of f, then $(1, 0)$ is on the graph of f^{-1}. Similarly, since $(1, 6)$ is on the graph of f, then $(6, 1)$ is on the graph of f^{-1} .

Try It 2.5.14

Find the inverse function of $f(x) = \sqrt[3]{x+4}$.

Answer

$f^{-1}(x) = x^3 - 4$

Domain and Range of Inverse Functions

Notice that that the ordered pairs of f and f^{-1} have their x-values and y-values reversed. Therefore we can indirectly determine the domain and range of a function and its inverse.

Note: Domain and Range of f and f^{-1}

The domain of f is the range of f^{-1} and the domain of f^{-1} is the range of f.

How To: Given a function, find the domain and range of its inverse.

1. If the function is one-to-one, write the range of the original function as the domain of the inverse, and write the domain of the original function as the range of the inverse.
2. If the domain of the original function needs to be restricted to make it one-to-one, then this restricted domain becomes the range of the inverse function.

Inverses of Radical Functions

Example 2.5.15: Inverse of radical functions

Find the inverse of $f(x) = \sqrt[5]{2x-3}$. State the domain and range of both the function and its inverse function.

Solution:

$$f(x) = \sqrt[5]{2x-3}$$

Substitute y for $f(x)$. (Notice here that the domain of f is all real numbers.)

$$y = \sqrt[5]{2x-3}$$

Interchange the variables x and y.

$$x = \sqrt[5]{2y-3}$$

Solve for y.

$$(x)^5 = (\sqrt[5]{2y-3})^5$$
$$x^5 = 2y-3$$
$$x^5+3 = 2y$$
$$\frac{x^5+3}{2} = y$$

Substitute $f^{-1}(x)$ for y.

$$f^{-1}(x) = \frac{x^5+3}{2}$$

Verify that the functions are inverses.

$$f^{-1}(f(x)) \stackrel{?}{=} x \qquad f(f^{-1}(x)) \stackrel{?}{=} x$$
$$f^{-1}(\sqrt[5]{2x-3}) \stackrel{?}{=} x \qquad f\left(\tfrac{x^5+3}{2}\right) \stackrel{?}{=} x$$
$$\frac{(\sqrt[5]{2x-3})^5+3}{2} \stackrel{?}{=} x \qquad \sqrt[5]{2\left(\frac{x^5+3}{2}\right)-3} \stackrel{?}{=} x$$
$$\frac{2x-3+3}{2} \stackrel{?}{=} x \qquad \sqrt[5]{x^5+3-3} \stackrel{?}{=} x$$
$$\frac{2x}{2} \stackrel{?}{=} x \qquad \sqrt[5]{x^5} \stackrel{?}{=} x$$
$$x = x \qquad x = x$$

Domain and Range.	$f(x) = \sqrt[5]{2x-3}$	Domain of f: $(-\infty, \infty)$	Range of f: $(-\infty, \infty)$
	$f^{-1}(x) = \frac{x^5+3}{2}$	Domain of f^{-1}: $(-\infty, \infty)$	Range of f^{-1}: $(-\infty, \infty)$

Try It 2.5.15

1. Find the inverse of the function $f(x) = \sqrt[5]{3x-2}$.
2. Find the inverse of the function $f(x) = \sqrt[4]{6x-7}$. State the domain and range of f and its inverse.

Answer

1. $f^{-1}(x) = \frac{x^5+2}{3}$
2. $f^{-1}(x) = \frac{x^4+7}{6}$

$f(x) = \sqrt[4]{6x-7}$	Domain of f: $[\frac{7}{6}, \infty)$	Range of f: $(-\infty, \infty)$
$f^{-1}(x) = \frac{x^4+7}{6}$	Domain of f^{-1}: $(-\infty, \infty)$	Range of f^{-1}: $[\frac{7}{6}, \infty)$

If we want to find the inverse of a radical function, we will need to restrict the domain of the answer if the range of the original function is limited.

Example 2.5.16: Solving to Find an Inverse with Square Roots

Find the inverse of the function $f(x) = 2 + \sqrt{x-4}$.

Solution

$$y = 2 + \sqrt{x-4}$$
$$x = 2 + \sqrt{y-4}$$
$$x-2 = \sqrt{y-4} \qquad \text{Before squaring, } x-2 \geq 0 \text{ so } x \geq 2$$
$$(x-2)^2 = y-4$$
$$y = (x-2)^2+4$$

So $f^{-1}(x) = (x-2)^2 + 4,\ x \geq 2$. Since the domain restriction $x \geq 2$ is not apparent from the formula, it should always be specified in the function definition.

The domain of f is $[4, \infty)$ so the range of f^{-1} is also $[4, \infty)$.
Since the domain of f^{-1} is $x \geq 2$ or $[2, \infty)$, the range of f is also $[2, \infty)$.

Analysis

The formula we found for $f^{-1}(x) = (x-2)^2 + 4$ looks like it would be valid for all real x. However, BOTH f^{-1} and f must be one-to-one functions and $y = (x-2)^2 + 4$ is a parabola which clearly is not one-to-one. This is where the subtlety of the restriction to x comes in during the solving for y. Recall that squaring can introduce extraneous solutions and that is precisely what happened here - after squaring, x had no apparent restrictions, but before squaring, $x - 2$ could not be negative. Thus, $x \geq 2$ defines the domain of f^{-1}. Also observe this domain of f^{-1} is exactly the range of f.

Try It 2.5.16

What is the inverse of the function $f(x) = 2 - \sqrt{x}$? State the domains of both the function and the inverse function.

Answer

$f^{-1}(x) = (2-x)^2,\ x \leq 2$; domain of f: $[0, \infty)$; domain of f^{-1}: $(-\infty, 2]$

Try It 2.5.17

What is the inverse of the function $f(x) = \sqrt{2x+3}$? State the domain and range of both the function and the inverse function.

Answer

$f(x) = \sqrt{2x+3}$	Domain of f: $[-\frac{3}{2}, \infty)$	Range of f: $[0, \infty)$
$f^{-1}(x) = \frac{x^2-3}{2},\ x \geq 0$	Domain of f^{-1}: $[0, \infty)$	Range of f^{-1}: $[-\frac{3}{2}, \infty)$

Inverses of Rational Functions

Example 2.5.18

Find the inverse of $f(x) = \frac{5}{7+x}$.

Solution

A check of the graph shows that f is one-to-one (this is left for the reader to verify).

STEP 1: Write the formula in xy-equation form: $y = \frac{5}{7+x}$.

STEP 2: Interchange x and y : $x = \frac{5}{7+y}$.

STEP 3: Solve for y :

$$x = \frac{5}{7+y}$$
$$\rightarrow x(7+y) = 5$$
$$\rightarrow 7 + y = \frac{5}{x}$$
$$y = \frac{5}{x} - 7 = \frac{5-7x}{x}$$

STEP 4: Thus, $f^{-1}(x) = \frac{5-7x}{x}$

Example 2.5.19: Solving to Find an Inverse Function

Find the inverse of the function $f(x) = \dfrac{2}{x-3} + 4$.

Solution

$$
\begin{aligned}
y &= \frac{2}{x-3+4} && \text{Set up an equation.}\\
x &= \frac{2}{y-3+4} && \text{Switch variables.}\\
x-4 &= \frac{2}{y-3} && \text{Subtract 4 from both sides.}\\
y-3 &= \frac{2}{x-4} && \text{Multiply both sides by } y-3 \text{ and divide by } x-4.\\
y &= \frac{2}{x-4}+3 && \text{Add 3 to both sides.}
\end{aligned}
$$

So $f^{-1}(x) = \dfrac{2}{x-4} + 3$.

$f(x) = \dfrac{2}{x-3} + 4$	Domain of f: $(-\infty, 3] \cup [3, \infty)$	Range of f: $(-\infty, 4] \cup [4, \infty)$
$f^{-1}(x) = \dfrac{2}{x-4} + 3$	Range of f^{-1}: $(-\infty, 4] \cup [4, \infty)$	Domain of f^{-1}: $(-\infty, 3] \cup [3, \infty)$

Example 2.5.20

This example is a bit more complicated: find the inverse of the function $f(x) = \dfrac{5x+2}{x-3}$.

Solution

A check of the graph shows that f is one-to-one (this is left for the reader to verify).

STEP 1: Write the formula in xy-equation form: $y = \dfrac{5x+2}{x-3}$.

STEP 2: Interchange \)x\) and y : $x = \dfrac{5y+2}{y-3}$.

STEP 3: Solve for y :

$$x = \tfrac{5y+2}{y-3}$$

$$\rightarrow x(y-3) = 5y+2$$

$$\rightarrow xy - 3x = 5y+2$$

This equation is linear in y. Isolate the terms containing the variable y on one side of the equation, factor, then divide by the coefficient of y.

$$xy - 3x = 5y + 2$$

$$xy - 5y = 3x + 2$$

$$y(x-5) = 3x+2$$

$$y = \frac{3x+2}{x-5}$$

STEP 4: Thus, $f^{-1}(x) = \dfrac{3x+2}{x-5}$.

Inverse of a domain restricted quadratic

Example 2.5.21

Find the inverse function for $h(x) = x^2$

Solution

According to the horizontal line test, the function $h(x) = x^2$ is certainly not one-to-one. However, if we only consider the right half or left half of the function, by restricting the domain to either the interval $[0, \infty)$ or $(-\infty, 0]$, then the function is one-to-one, and therefore would have an inverse.

We will choose to restrict the domain of h to the left half of the parabola as illustrated in **Figure 21**(a) and find the inverse for the function $f(x) = x^2$, $x \leq 0$. In this case, the procedure still works, provided that we carry along the domain condition in all of the steps.

The graph in **Figure 21**(a) passes the horizontal line test, so the function $f(x) = x^2$, $x \leq 0$, for which we are seeking an inverse, is one-to-one.

Step 1: Write the formula in xy-equation form: $y = x^2$, $x \leq 0$

Step 2: Interchange x and y: $x = y^2$, $y \leq 0$. Note how x and y must also be interchanged in the domain condition.

Step 3: Solve for y: $y = \pm\sqrt{x}$, $y \leq 0$

> Now there are two choices for y, one positive and one negative, but the condition $y \leq 0$ tells us that the negative choice is the correct one. Thus, the last statement is equivalent to $y = -\sqrt{x}$.

Step 4: Thus, $f^{-1}(x) = -\sqrt{x}$. The graph of f^{-1} is shown in **Figure 21**(b), and the graphs of both f and f^{-1} are shown in **Figure 21**(c) as reflections across the line y = x.

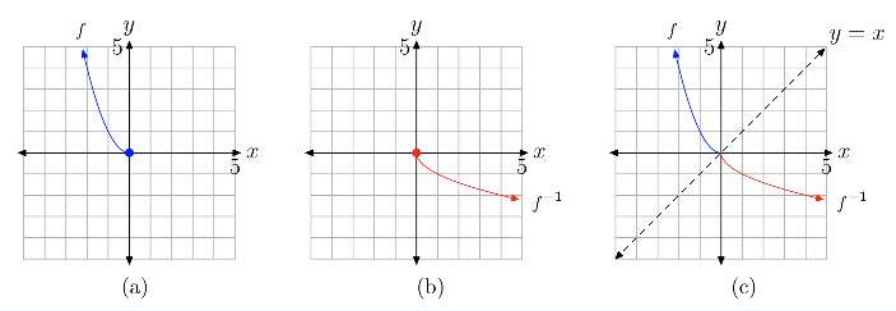


Figure 21. Domain of a Quadratic is Restricted to a part that is 1-1 before an inverse can be found. Inverse functions are reflections across the line $y = x$.

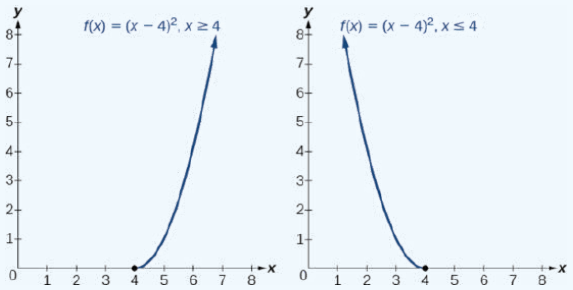
Example 2.5.22: Restricting the Domain to Find the Inverse of a Polynomial Function

Find the inverse function of f:

1. $f(x) = (x-4)^2$, $x \geq 4$
2. $f(x) = (x-4)^2$, $x \leq 4$

Solution

The original function $f(x) = (x-4)^2$ is not one-to-one, but the function can be restricted to a domain of $x \geq 4$ or $x \leq 4$ on which it is one-to-one (These two possibilities are illustrated in the figure to the right.) To find the inverse, start by replacing $f(x)$ with the simple variable y. (We will choose which domain restriction is being used at the end).



$y = (x-4)^2$ Interchange x and y.

$x = (y-4)^2$ Take the square root.

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$$\pm\sqrt{x} = y - 4 \text{ Add } 4 \text{ to both sides.}$$

$$4 \pm \sqrt{x} = y \text{ so } y = \begin{cases} 4 + \sqrt{x} & \longrightarrow y \geq 4 \\ 4 - \sqrt{x} & \longrightarrow y \leq 4 \end{cases}$$

This expression for y is not a function. We need to go back and consider the domain of the original domain-restricted function we were given in order to determine the appropriate choice for y and thus for f^{-1}.

1. Given the function $f(x) = (x-4)^2$, $x \geq 4$, the domain of f is restricted to $x \geq 4$, so the range of f^{-1} needs to be the same. Therefore, $y \geq 4$, and we must use the + case for the inverse:

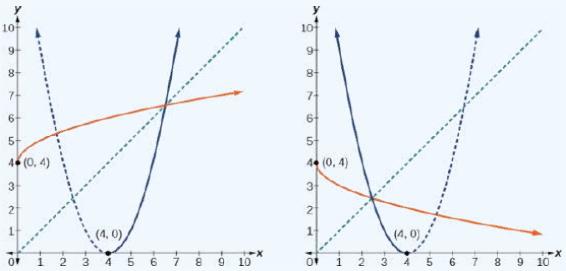
$$f^{-1}(x) = 4 + \sqrt{x}$$

2. Given the function $f(x) = (x-4)^2$, $x \leq 4$, the domain of f is restricted to $x \leq 4$, so the range of f^{-1} needs to be the same. Therefore, $y \leq 4$, and we must use the – case for the inverse:

$$f^{-1}(x) = 4 - \sqrt{x}$$

Analysis

On the graphs in the figure to the right, we see the original function graphed on the same set of axes as its inverse function. Notice that together the graphs show symmetry about the line $y = x$. The coordinate pair $(4, 0)$ is on the graph of f and the coordinate pair $(0, 4)$ is on the graph of f^{-1}. For any coordinate pair, if (a, b) is on the graph of f, then (b, a) is on the graph of f^{-1}. Finally, observe that the graph of f intersects the graph of f^{-1} on the line $y = x$. Points of intersection for the graphs of f and f^{-1} will always lie on the line $y = x$.



Example 2.5.23: Finding the Inverse of a Quadratic Function When the Restriction Is Not Specified

Restrict the domain and then find the inverse of $f(x) = x^2 - 4x + 1$.

Solution

We can see this is a parabola that opens upward. Because the graph will be decreasing on one side of the vertex and increasing on the other side, we can restrict this function to a domain on which it will be one-to-one by limiting the domain to one side of the vertex. The x-coordinate of the vertex can be found from the formula $x = \frac{-b}{2a} = \frac{-(-4)}{2 \cdot 1} = 2$. Therefore, we will choose to restrict the domain of f to $x \geq 2$.

To find the inverse, we start by replacing $f(x)$ with a simple variable, y, switching x and y, and then solving for y. Solving for y turns out to be a bit complicated because there is both a y^2 term and a y term in the equation. The approach is to use either Complete the Square or the Quadratic formula to obtain an expression for y.

$$y = x^2 - 4x + 1, x \geq 2 \text{ Interchange } x \text{ and } y.$$

$$x = y^2 - 4y + 1, \ y \geq 2 \text{ Solve for } y \text{ using Complete the Square !}$$

$$x - 1 = y^2 - 4y, \ y \geq 2 \text{ Isolate the } y \text{ terms.}$$

$$x - 1 + 4 = y^2 - 4y + 4, \ y \geq 2 \text{ Add the square of half the } y \text{ coefficient.}$$

$$x + 3 = (y-2)^2 \text{ Take the square root.}$$

$$\pm\sqrt{x+3} = y - 2 \text{ Add 2 to both sides.}$$

$$2 \pm \sqrt{x+3} = y \text{ Rename the function.}$$

$$f^{-1}(x) = 2 \pm \sqrt{x+3}$$

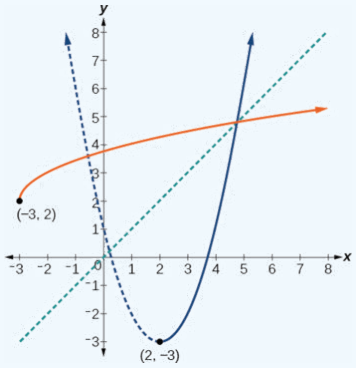
$$f^{-1}(x) = \begin{cases} 2 + \sqrt{x+3} & \geq 2 \\ 2 - \sqrt{x+3} & \leq 2 \end{cases}$$

Now we need to determine which case to use. Because we restricted our original function to a domain of $x \geq 2$, the outputs of the inverse are $y \geq 2$ so we must use the + case

$$f^{-1}(x) = 2 + \sqrt{x+3}$$

Analysis

Notice that we arbitrarily decided to restrict the domain on $x \geq 2$. We could just as easily have opted to restrict the domain to $x \leq 2$, in which case $f^{-1}(x) = 2 - \sqrt{x+3}$. Observe the original function graphed on the same set of axes as its inverse function in the figure on the right. Notice that both graphs show symmetry about the line $y = x$. The coordinate pair $(2, -3)$ is on the graph of f and the coordinate pair $(-3, 2)$ is on the graph of f^{-1}. Observe from the graph of both functions on the same set of axes that



$$\text{domain of } f = \text{range of } f^{-1} = [2, \infty)$$

and

$$\text{domain of } f^{-1} = \text{range of } f = [-3, \infty).$$

Finally, observe that the graph of f intersects the graph of f^{-1} along the line $y = x$.

Try It 2.5.24

Find the inverse of the function $f(x) = x^2 + 1$, on the domain $x \geq 0$.

Answer

$f^{-1}(x) = \sqrt{x-1}$

Can more than one formula from a piecewise function be applied to a value in the domain?

Yes. If $f = f^{-1}$, then $f(f(x)) = x$, and we can think of several functions that have this property. The identity function does, and so does the reciprocal function, because $1/(1/x) = x$. Any function $f(x) = c - x$, where c is a constant, is also equal to its own inverse.

Key Concepts

- **Horizontal Line Test:** If every horizontal line, intersects the graph of a function in at most one point, it is a one-to-one function.
- **Inverse of a Function Defined by Ordered Pairs:** If $f(x)$ is a one-to-one function whose ordered pairs are of the form (x, y), then its inverse function $f^{-1}(x)$ is the set of ordered pairs (y, x).
- **Inverse Functions:** For every x in the domain of one-to-one functions f and f^{-1}, $f^{-1}(f(x)) = x$ and $f\left(f^{-1}(x)\right) = x$
- **How to Find the Inverse of a One-to-One Function:**
  1. Substitute y for $f(x)$.
  2. Interchange the variables x and y.
  3. Solve for y.
  4. Substitute $f^{-1}(x)$ for y.
  5. Verify that the functions are inverses.

Glossary

one-to-one function

A function is one-to-one if each value in the range has exactly one element in the domain. For each ordered pair in the function, each y-value is matched with only one x-value.

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2.5e: Exercises Inverse Functions

A: Concepts

Exercise 2.5e.A

1) Describe why the horizontal line test is an effective way to determine whether a function is one-to-one?

2) Why do we restrict the domain of the function $f(x) = x^2$ to find the function's inverse?

3) Can a function be its own inverse? Explain.

4) Are one-to-one functions either always increasing or always decreasing? Why or why not?

5) How do you find the inverse of a function algebraically?

Answers to Odd Exercises:

1. Each output of a function must have exactly one output for the function to be one-to-one. If any horizontal line crosses the graph of a function more than once, that means that y-values repeat and the function is not one-to-one. If no horizontal line crosses the graph of the function more than once, then no y-values repeat and the function is one-to-one.

3. Yes. For example, $f(x) = \frac{1}{x}$ is its own inverse.

5. Given a function $y = f(x)$, solve for x in terms of y. Interchange the x and y. Solve the new equation for y. The expression for y is the inverse, $y = f^{-1}(x)$.

B: Horizontal Line Test

Exercise 2.5e.B)

★ **For the following exercises, use the horizontal line test to determine whether each of the given graphs is one-to-one.**

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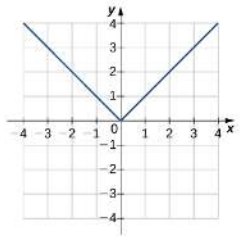

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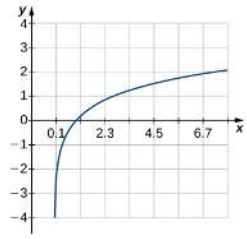

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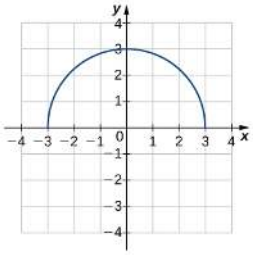

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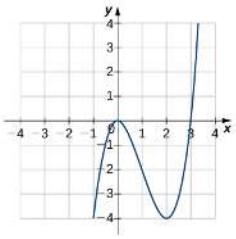

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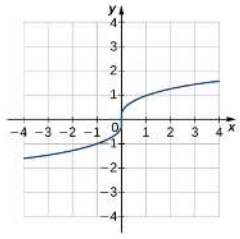

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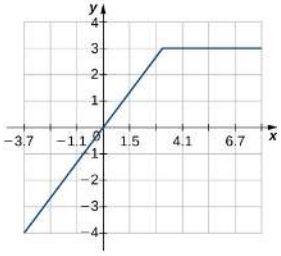

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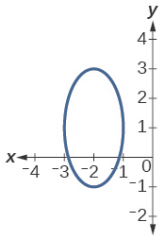

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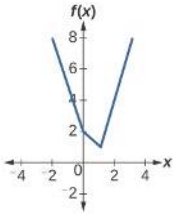

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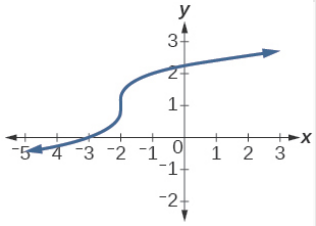

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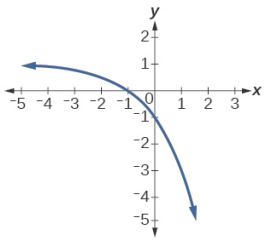

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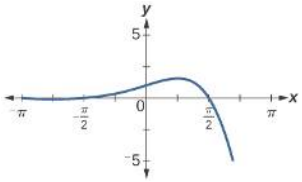

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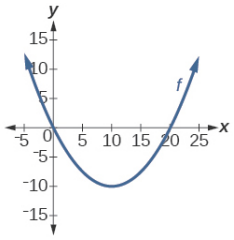


Answers to Odd Exercises:

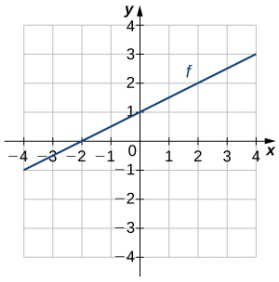
7. Not one-to-one, 9. Not one-to-one, 11. One-to-one, 13. Not one-to-one, 15. One-to-one, 17. Not one-to-one

C: Graphs of Inverse Functions

Exercise 2.5e.C

★ **For the following exercises, use the graph of f to sketch the graph of its inverse function.**

19)



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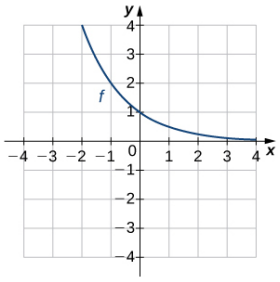
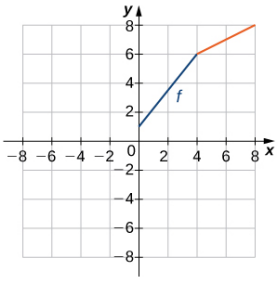


21)



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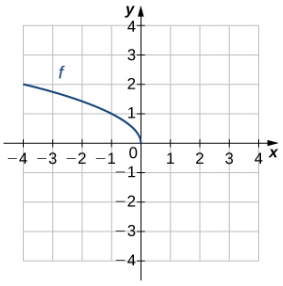


★ **Use the graph of the one-to-one function shown in the Figure to answer the following questions.**

23) Find $f(0)$

.

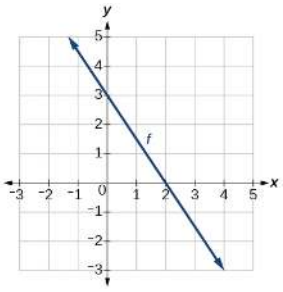
24) Solve $f(x) = 0$

.

25) Find $f^{-1}(0)$

.

26) Solve $f^{-1}(x) = 0$.



27) Sketch the graph of f^{-1}

.

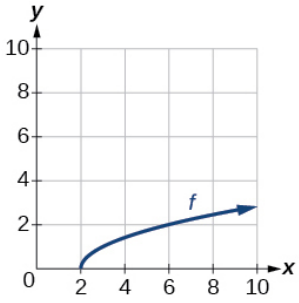
28) Find $f(6)$ and $f^{-1}(2)$

.

29) If the complete graph of f is shown, find the domain of f

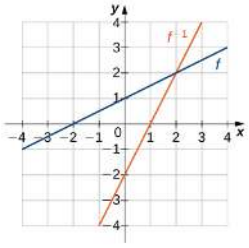
.

30) If the complete graph of f is shown, find the range of f

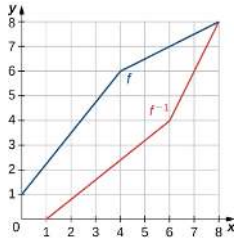


Answers to Odd Exercises:

19)

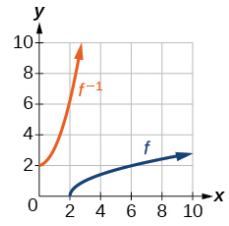


21)



23. 3
25. 2
29. $[2, \infty)$

27.



D: Inverse Function Values

Exercise 2.5e.D

★ **Find values of a function from its inverse and vice versa.**

31) If $f(6)=7$, find $f^{-1}(7)$.	32.1) If $f(4)=5$, find $f^{-1}(4)$.	33) If $f^{-1}(-4)=-8$, find $f(-8)$.
32) If $f(3)=2$, find $f^{-1}(2)$.	32.2) If $f^{-1}(9)=3$, find $f(9)$.	34) If $f^{-1}(-2)$=−1, find $f(-1)$.

★ **Given the table of values for f, determine the following.**

35) Find $f(1)$.
36) Solve $f(x)=3$.
37) Find $f^{-1}(4)$.
38) Solve $f^{-1}(x)=7$.
39) Solve $f^{-1}(x)=11$.
40) Find $f^{-1}(12)$.

x	1	4	7	12	16
$f(x)$	12	11	9	4	3

Answers to Odd Exercises:

31. 6 33. −4, 35. 12, 37. 12, 39. Undefined

E: Verify Two Functions are Inverses

Exercise 2.5e.E

★ **For the following exercises, use composition to determine which pairs of functions are inverses.**

41) $f(x)=8x,\ g(x)=\frac{x}{8}$
42) $f(x)=\frac{2}{3}x+2,\ g(x)=\frac{3}{2}x+3$
43) $f(x)=5x-7,\ g(x)=\frac{x+5}{7}$
44) $f(x)=8x+3,\ g(x)=\frac{x-3}{8}$
45) $f(x)=\frac{1}{x-1}, x\neq 1,\ g(x)=\frac{1}{x}+1, x\neq 0$
46) $f(x)=-3x+5,\quad g(x)=\frac{x-5}{-3}$
47) $f(x)=\frac{x}{2+x},\ g(x)=\frac{2x}{1-x}$
48) $f(x)=x^3+1,\ g(x)=(x-1)^{1/3}$
49) $f(x)=x^2+2x+1, x\geq -1,\ g(x)=-1+\sqrt{x}, x\geq 0$
50) $f(x)=\sqrt{4-x^2}, 0\leq x\leq 2,\ g(x)=\sqrt{4-x^2}, 0\leq x\leq 2$

Answers to Odd Exercises:

41) inverses. 43) not inverses. 45) inverses. 47) inverses 49) inverses

F: Find inverses of linear and rational functions

Exercise 2.5e.F

★ **Find the formula for the inverse function $f^{-1}(x)$.**

51) $f(x)=x+3$

52) $f(x)=x+5$

53) $f(x)=2-x$

54) $f(x)=3-x$

55) $f(x)=7x-9$

56) $f(x)=6x-4$

57) $f(x)=-5x+2$

58) $f(x)=6x+8$

59) $f(x)=\dfrac{3}{x+2}$

60) $f(x)=\dfrac{x}{x-2}$

61) $f(x)=\dfrac{x-4}{x+2}$

62) $f(x)=\dfrac{2x-7}{x+6}$

63) $f(x)=\dfrac{2x+3}{5x+4}$

64) $f(x)=\dfrac{2x+6}{x-3}$

65) $f(x)=\dfrac{9x-3}{9x+7}$

66) $f(x)=\dfrac{3x+7}{2x+8}$

67) $f(x)=\dfrac{4x+2}{4x+3}$

68) $f(x)=\dfrac{8x-7}{3x-6}$

69) $f(x)=\dfrac{4x-1}{2x+2}$

70) $f(x)=-\dfrac{9x-3}{7x+6}$

Answers to Odd Exercises:

51. $f^{-1}(x)=x-3$ 53. $f^{-1}(x)=2-x$
55. $f^{-1}(x)=\frac{x+9}{7}$ 57. $f^{-1}(x)=-\frac{x-2}{5}$ 59. $f^{-1}(x)=\frac{3}{x}-2$ 61. $f^{-1}(x)=-2\frac{x+2}{x-1}$
63. $f^{-1}(x)=\frac{-4x+3}{5x-2}$ 65. $f^{-1}(x)=-\frac{7x+3}{9x-9}$ 67. $f^{-1}(x)=-\frac{3x-2}{4x-4}$ 69. $f^{-1}(x)=-\frac{2x+1}{2x-4}$

G: Find inverses of odd degree power and root functions

Exercise 2.5e.G

★ **Find the formula for the inverse function $f^{-1}(x)$.**

71) $f(x)=x^3+1$

72) $f(x)=x^3-27$

73) $f(x)=(x-4)^3$

74) $y=(x+8)^3+3$

75) $f(x)=5x^3-5$

76) $f(x)=4x^5-9$

77) $f(x)=3x^5-9$

78) $f(x)=5x^7+4$

79) $f(x)=9x^9+5$

80) $f(x)=4x^7-3$

81) $f(x)=\sqrt[3]{x-4}$

82) $f(x)=\sqrt[3]{3x+1}$

83) $f(x)=\sqrt[3]{x}+5$

84) $f(x)=\sqrt[3]{x}-8$

85) $f(x)=\sqrt[3]{-6x-4}$

86) $f(x)=\sqrt[3]{9x-7}$

87) $f(x)=\sqrt[7]{-3x-5}$

88) $f(x)=\sqrt[7]{8x-3}$

89) $f(x)=\sqrt[3]{6x+7}$

90) $f(x)=\sqrt[9]{8x+2}$

Answers to Odd Exercises:

71. $f^{-1}(x)=\sqrt[3]{x-1}$ 73. $f^{-1}(x)=4+\sqrt[3]{x}$ 75. $\sqrt[3]{\frac{x+5}{5}}$ 77. $\sqrt[5]{\frac{x+9}{3}}$ 79. $\sqrt[9]{\frac{x-5}{9}}$
81. $f^{-1}(x)=x^3+4$ 83. $f^{-1}(x)=(x-5)^3$ 85. $-\frac{x^3+4}{6}$ 87. $-\frac{x^7+5}{3}$ 89. $\frac{x^3-7}{6}$

H: Find inverses of even degree power and root functions

Exercise 2.5e.H]

★ **Find $f^{-1}(x)$ for each function below. In #105-108 also state the restrictions for x in f^{-1}.**

91) $f(x)=x^4,\ x\le 0$	97) $f(x)=x^4+3,\ x\le 0$	103) $f(x)=x^2-8x+3,\ x\le 4$
92) $f(x)=x^4,\ x\ge 0$	98) $f(x)=x^4-5,\ x\ge 0$	104) $f(x)=x^2+2x+50,\ x\ge -1$
93) $f(x)=x^2-1,\ x\le 0$	99) $f(x)=(x-1)^2,\ x\ge 1$	105) $f(x)=\sqrt{x-1}$
94) $f(x)=x^2+2,\ x\ge 0$	100) $f(x)=(x+3)^2,\ x\ge -3$	106) $f(x)=\sqrt{x+2}$
95) $f(x)=x^2-4,\ x\ge 0$	101) $f(x)=(x-1)^2,\ x\le 1$	107) $f(x)=\sqrt{x}+9$
96) $f(x)=x^2+11,\ x\le 0$	102) $f(x)=(x+2)^2,\ x\ge -2$	108) $f(x)=\sqrt{x}-1$

Answers to Odd Exercises:

91. $f^{-1}(x)=-\sqrt[4]{x}$ 93. $f^{-1}(x)=-\sqrt{x+1}$ 95. $f^{-1}(x)=\sqrt{x+4}$	97. $f^{-1}(x)=-\sqrt[4]{x-3}$ 99. $f^{-1}(x)=1+\sqrt{x}$ 101. $f^{-1}(x)=-\sqrt{x}+1$	103. $f^{-1}(x)=4-\sqrt{x+13}$ 105. $f^{-1}(x)=x^2+1,\ x\ge 0$ 107. $f^{-1}(x)=(x-9)^2,\ x\ge 9$

I: Find the inverse and its domain and range

Exercise 2.5e.I

★ **Find a domain on which f is one-to-one and non-decreasing. Then find the inverse of f.**

111) $f(x)=(x+7)^2$	112) $f(x)=(x-6)^2$	113) $f(x)=x^2-5$	114) $f(x)=3(x-4)^2+1$

Answers to Odd Exercises

111. domain of $f(x)$: $[-7,\infty)$; $f^{-1}(x)=\sqrt{x}-7$ 113. domain of $f(x)$: $[0,\infty)$; $f^{-1}(x)=\sqrt{x+5}$

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CHAPTER OVERVIEW

3: Polynomial and Rational Functions

In this chapter, we will learn about these concepts and discover how mathematics can be used in such applications.

3.1: Graphs of Quadratic Functions

3.1e: Exercises - Quadratic Functions

3.2: Circles

3.2e: Circle Exercises.

3.3: Power Functions and Polynomial Functions

3.3e: Exercises - Polynomial End Behaviour

3.4: Graphs of Polynomial Functions

3.4e: Exercises - Polynomial Graphs

3.5: Dividing Polynomials

3.5e: Exercises - Division of Polynomials

3.6: Zeros of Polynomial Functions

3.6e: Exercises - Zeroes of Polynomial Functions

3.7: The Reciprocal Function

3.7e: Exercises for the reciprocal function

3.8: Polynomial and Rational Inequalities

3.8e: Exercises - Polynomial and Rational Inequalities

3.9: Rational Functions

3.9e: Exercises - Rational Functions

Contributors

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3.1: Graphs of Quadratic Functions

Graphs of Quadratic Functions

A **Quadratic Function** is any function defined by a polynomial whose greatest exponent is two. That means it can be written in the form $f(x) = ax^2 + bx + c$, with the restrictions that the parameters a, b, and c are real numbers and a canNOT be zero.

The graph of any quadratic function is a U-shaped curve called a **parabola**. There are certain key features that are important to recognize on a graph and to calculate from an equation.

Key features of a parabola

1. The **orientation** of a parabola is that it either opens up or opens down
2. The **vertex** is the lowest or highest point on the graph
3. The **axis of symmetry** is the vertical line that goes through the vertex, dividing the parabola into two equal parts. If h is the x-coordinate of the vertex, then the equation for the axis of symmetry is $x = h$.
4. The **maximum or minimum** value of a parabola is the y-coordinate of the vertex.
5. The y**-intercept** is the point at which the parabola crosses the y-axis.
6. The x**-intercepts** are the points at which the parabola crosses the x-axis.
6. The **domain** of a parabola is all real numbers, $(-\infty, \infty)$.
7. The **range** of a parabola starts or ends with the value of the y-coordinate of the vertex.

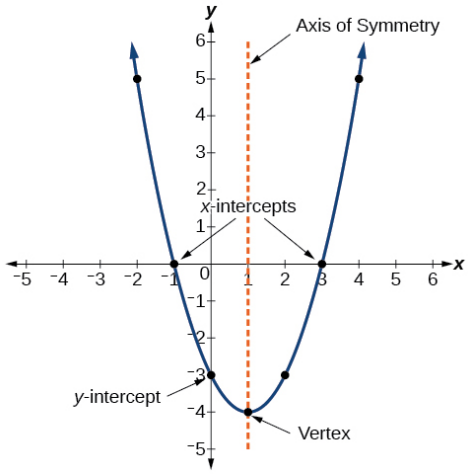
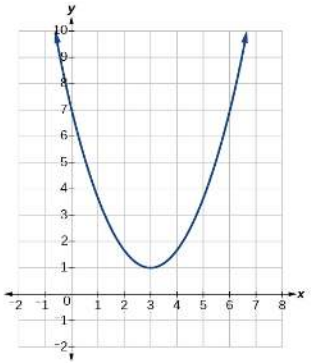


Figure 3.1.1: Graph illustrating features of a parabola .

Example 3.1.1: Identify Features of a Parabola from a graph

Determine features of th parabola illustrated below.



Solution.
Orientation: opens up
Vertex: $(3, 1)$
Axis of symmetry: $x = 3$
Minimum Value: $y = 1$
y-intercept: $(0, 7)$
x-intercept: none
Domain: $(-\infty, \infty)$
Range: $[1, \infty)$

Forms of a Quadratic Function

There are two important forms of a quadratic function

Definitions: Forms of Quadratic Functions

A quadratic function is a polynomial function of degree two. The graph of a quadratic function is a parabola.

- The **general form** of a quadratic function is $f(x) = ax^2 + bx + c$ with real number parameters a, b, and c and $a \neq 0$.
- The **standard form** or **vertex form** of a quadratic function is $f(x) = a(x-h)^2 + k$ with real number parameters a, h, and k and $a \neq 0$. NOTICE the minus sign in front of the parameter h!!

The graph of a quadratic in standard form, $f(x) = a(x-h)^2 + k$, is a graph of $y = x^2$ that has been shifted horizontally h units and vertically k units. Thus the vertex, originally at at $(0,0)$, is located at the point (h,k) in the graph of f.

A formula for the location of the vertex for a quadratic in general form can be found by equating the two forms for a quadratic.

$$a(x-h)^2 + k = ax^2 + bx + c$$
$$ax^2 - 2ahx + (ah^2 + k) = ax^2 + bx + c$$

This is an identity, so it is true for ALL values of x: the coefficients for the x^2 terms must be the same, the coefficients for the x terms must be the same, and the constant terms must be the same. Equating the x coefficients we get

$$-2ah = b, \text{ so } h = -\frac{b}{2a}.$$

This is a formula for the x-coordinate of the vertex. The y-coordinate of the vertex is $y = f(h) = k$.

Features of the graph of a quadratic function depend on the parameter values a, b, c or a, h, k used in its equation. How features of the parabola for a quadratic function can be obtained is summarized below.

How to: Find Features of a Parabola Given a Quadratic Equation

- **Orientation**
  - $a > 0$, the parabola opens **up** $\overset{+}{\cup}\overset{+}{}$
  - $a < 0$, the parabola opens **down** $\overset{-}{\cap}\overset{-}{}$
- The **vertex** is located at (h,k).
  - If the function is in general form, calculate h and k: $h = \frac{-b}{2a}$, $\quad k = f(h) = f(\frac{-b}{2a})$.
- The **axis of symmetry**, $x = h$ is the equation of the vertical line through the vertex.
- The **maximum or minimum value** depends on the y-coordinate, k, of the vertex and the orientation of the parabola
  - If (\a > 0\), the parabola opens up and the vertex is the lowest point on the graph with a **minimum** value k.
  - If (\a < 0\), the parabola opens down and the vertex is the highest point on the graph with a **maximum** value k.
- The **domain** is always $\mathbb{R}$ or $(-\infty, \infty)$.
  The **range** depends on the y-coordinate, k, of the vertex and the orientation of the parabola
  - If $a > 0$, the vertex is the lowest point on the graph so the **range** of the function is $[k, \infty)$.
  - If $a < 0$, the vertex is the highest point on the graph so the **range** of the function is $(-\infty, k]$.
- **Intercepts** are the points where the parabola crosses the axes.
  - The x-intercepts are the points $(s, 0)$, where s is a real solution to $f(x) = 0$
  - The y-intercept is the point $(0, f(0))$.

Orientation

When the quadratic term, ax^2, is positive, the parabola opens upward, and when the quadratic term is negative, the parabola opens downward.

Example 3.1.2: Find the orientation of a parabola

Determine whether each parabola opens upward or downward:

a. $f(x) = -3x^2 + 2x - 4$

a. Solution:

Find the value of a.

$$f(x) = ax^2 + bx + c$$
$$f(x) = -3x^2 + 2x - 4$$
$$a = -3$$

Since the a is negative, the parabola will open downward.

b. $f(x) = 6(x+1)^2 - 11$

b. Solution:

Find the value of a.

$$f(x) = a(x-h)^2 + k$$
$$f(x) = 6(x+1)^2 - 11$$
$$a = 6$$

Since the a is positive, the parabola will open upward.

Try It 3.1.2

Determine whether the graph of each function is a parabola that opens upward or downward:

a. $f(x) = 2x^2 + 5x - 2$
b. $f(x) = -3(x-4)^2 + 7$

Answer

a. up b. down

c. $f(x) = -2x^2 - 2x - 3$
d. $f(x) = 5(x+1)^2 + 4$

Answer

c. down d. up

Vertex and Axis of Symmetry

When given a quadratic in standard form $f(x) = a(x-h)^2 + k$, the vertex and axis of symmetry is easily found once the parameters h and k have been identified. (Notice the sign on h!!) The vertex is (h, k) and the axis of symmetry is the vertical line $x = h$.

When given a quadratic in general form: $f(x) = ax^2 + bx + c$, more computation is required. After identifying parameters a and b, calculate $h = -\frac{b}{2a}$. Then find the corresponding y coordinate for that point on the graph: $y = f(h) = k$. Once h and k have been determined, the vertex is at (h, k) and the axis of symmetry is the vertical line $x = h$.

Example 3.1.3a: Find the Vertex from the General Form of the Quadratic Equation

For the graph of $f(x) = 3x^2 - 6x + 2$ find (a) the axis of symmetry and (b) the vertex.

Solution:

a. Identify the equation parameters — $a = 3, b = -6, c = 2$

The axis of symmetry is the vertical line $x = -\frac{b}{2a}$

Substitute the values a and b into the formula — $x = -\frac{-6}{2 \cdot 3} = 1$

The axis of symmetry is the line $x = 1$

b. The vertex is a point on the line of symmetry, so — The x coordinate of the vertex is $x = 1$

The y coordinate will be $f(1)$ — $f(1) = 3(1)^2 - 6(1) + 2$

Simplify — $f(1) = 3 - 6 + 2$

The result is the y coordinate of the vertex. — $f(1) = -1$

The vertex is $(1, -1)$

Example 3.1.3b: Find the Vertex from the Standard Form of the Quadratic Equation

For the graph of $f(x) = 6(x-3)^2 + 4$ find:

a. the axis of symmetry
b. the vertex

Solution:

a.	Identify the equation parameters	$a=6, h=3, k=4$
	The axis of symmetry is the vertical line $x=h$	
	Substitute.	The axis of symmetry is the line $x=3$
b.	Use the equation parameters	$a=6, h=3, k=4$
	The vertex is the point (h,k)	The vertex is the point $(3,4)$

Try It 3.1.3

For the following quadratic functions find a. the axis of symmetry and b. the vertex

$f(x)=2x^2-8x+1$

Answer

a. $x=2$ b. $(2,-7)$

$f(x)=2(x-1)^2-5$

Answer

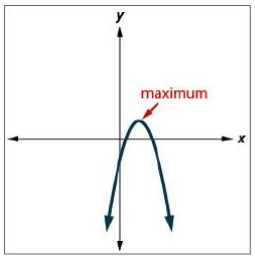
a. $x=1$ b. $(1,-5)$

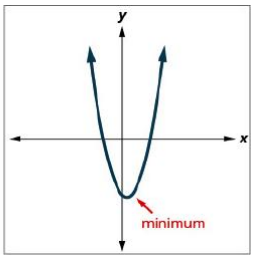
Minimum or Maximum Value of a Quadratic Function

Knowing that the **vertex** of a parabola is the lowest or highest point of the parabola gives us an easy way to determine the minimum or maximum value of a quadratic function.

The **y-coordinate of the vertex** of the graph of a quadratic function is

- *the minimum* value of the quadratic equation if the parabola opens *upward*.
- *the maximum* value of the quadratic equation if the parabola opens *downward*.





Example 3.1.4

Find the minimum or maximum value of the quadratic function $f(x)=x^2+2x-8$.

Solution:

Identify the equation parameters	$a=1, b=2, c=-8$
Since a is positive, the parabola opens upward.	The quadratic equation has a minimum.
State the formula for the axis of symmetry	$x=-\frac{b}{2a}$
Substitute and simplify.	$x=-\frac{2}{2\cdot 1}=-1$ The axis of symmetry is the line $x=-1$
	The x coordinate of the vertex is $x=-1$
The y coordinate will be $f(-1)$	$f(-1)=(-1)^2-2(-1)-8$
Simplify	$f(-1)=1-2-8$
The result is the y coordinate of the vertex.	$f(-1)=-9$ The vertex is $(-1,-9)$

Since the parabola has a minimum, the y-coordinate of the vertex is the minimum y-value of the quadratic equation. The minimum value of the quadratic is -9 and it occurs when $x=-1$.

Try It 3.1.4

Find the maximum or minimum value of the quadratic function

a. $f(x)=x^2-8x+12$.
b. $f(x)=-4(x-2)^2+5$.

Answers

a. The minimum value of the quadratic function is -4 and it occurs when $x = 4$.

b. The maximum value of the quadratic function is 5 and it occurs when $x = 2$.

Domain and Range

Any number can be the input value, x, to a quadratic function. Therefore, the domain of any quadratic function is all real numbers. Because parabolas have a maximum or a minimum point, the range is restricted. Since the vertex of a parabola will be either a maximum or a minimum, the range will consist of all y-values greater than or equal to the y-coordinate at the vertex or less than or equal to the y-coordinate at the vertex, depending on whether the parabola opens up or down.

How to: Given a quadratic function, find the domain and range.

1. The **domain** of any quadratic function is always $\mathbb{R}$ or $(-\infty, \infty)$.
2. Determine the maximum or minimum value of the parabola, k
   a. If the function is in the form $f(x) = a(x-h)^2 + k$, then the value of k is readily visible as one of the parameters.
   b. If the function is in the form $f(x) = ax^2 + bx + c$, the vertex must be determined and the value for k is the y coordinate of the vertex.
3. Determine whether a is positive or negative.
   a. If a is positive, the parabola has a minimum value of k and the **range** of the function is $[k, \infty)$.
   b. If a is negative, the parabola has a maximum value of k and the **range** of the function is $(-\infty, k]$.

Example 3.1.5: Find the Domain and Range of a Quadratic Function

Find the domain and range of $f(x) = -5x^2 + 9x - 1$.

Solution

As with any quadratic function, the domain is all real numbers.

Because a is negative, the parabola opens downward and has a maximum value.
The maximum value must be determined. Begin by finding the x-value of the vertex.

$$h = -\frac{b}{2a} = -\frac{9}{2(-5)} = \frac{9}{10}$$

The maximum value is given by $f(h)$.

$$f\left(\tfrac{9}{10}\right) = -5\left(\tfrac{9}{10}\right)^2 + 9\left(\tfrac{9}{10}\right) - 1 = \tfrac{61}{20}$$

The range is $f(x) \leq \frac{61}{20}$, or $\left(-\infty, \frac{61}{20}\right]$.

Try It 3.1.5

Find the domain and range of $f(x) = 2\left(x - \frac{4}{7}\right)^2 + \frac{8}{11}$.

Answer

The domain is all real numbers. The range is $f(x) \geq \frac{8}{11}$, or $\left[\frac{8}{11}, \infty\right)$.

Intercepts

The y-intercept is the point where the graph crosses the y axis. All points on the y-axis have an x coordinate of zero, so the y-intercept of a quadratic is found by evaluating the function $f(0)$.

The x-intercepts are the points where the graph crosses the x-axis. All points on the x-axis have a y coordinate of zero, so the x-intercept of a quadratic can be found by solving the equation $f(x) = 0$. Notice in Figure 3.1.6 that the number of x-intercepts can vary depending upon the location of the graph.

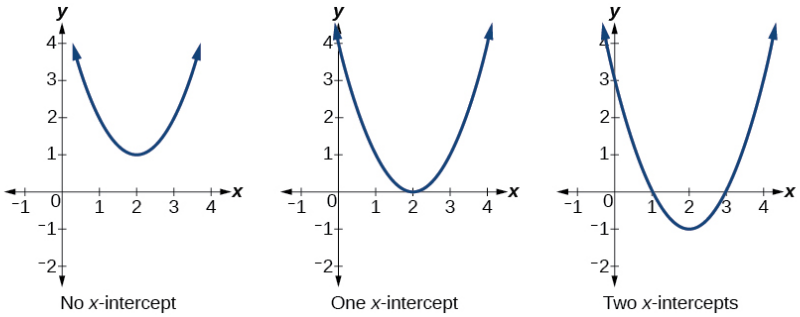


Figure **3.1.6**: Number of x-intercepts of a parabola.

How To: Given a quadratic function $f(x)$, find the y- and x-intercepts.

1. Evaluate $f(0)$ to find the y-intercept. The y intercept is written in the form of a coordinate point $0, f(0)$.
2. Solve the quadratic equation $f(x) = 0$ to find the x-intercepts. Each real number solution x_i is written as an x-intercept in the form $x_i, 0$.

Example 3.1.6: Finding the y- and x-intercepts of a General Form Quadratic

Find the y- and x-intercepts of the quadratic $f(x) = 3x^2 + 5x - 2$.

Solution

Find the y-intercept by evaluating $f(0)$.

$$f(0) = 3(0)^2 + 5(0) - 2 = -2$$

So the y-intercept is at $(0, -2)$.

For the x-intercepts, find all solutions of $f(x) = 0$.

$$0 = 3x^2 + 5x - 2$$

In this case, the quadratic can be factored easily, providing the simplest method for solution. Typically, quadratics in general form, like this example, are usually solved using factoring, or failing that, using the quadratic formula or complete the square.

$$0 = (3x - 1)(x + 2)$$

$$0 = 3x - 1 \qquad 0 = x + 2$$
$$x = \frac{1}{3} \quad \text{or} \quad x = -2$$

So the x-intercepts are at $(\frac{1}{3}, 0)$ and $(-2, 0)$.

Example 3.1.7:

Find the y- and x-intercepts of the quadratic $f(x) = x^2 + x + 2$.

Solution

Find the y-intercept by evaluating $f(0)$.

$$f(0) = (0)^2 + (0) + 2 = 2$$

So the y-intercept is at $(0,\ 2)$.

For the x-intercepts, find all solutions of $f(x) = 0$ or $x^2 + x + 2 = 0$. Clearly this does not factor, so employ the quadratic formula.

The quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and for this equation, $a = 1$, $b = 1$, and $c = 2$. Substituting these values into the formula produces

$$x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (2)}}{2 \cdot 1}$$
$$= \frac{-1 \pm \sqrt{1-8}}{2} = \frac{-1 \pm \sqrt{-7}}{2} = \frac{-1 \pm i\sqrt{7}}{2}$$

Since the solutions are imaginary, there are no x-intercepts.

Example 3.1.8: Find the y- and x-intercepts of a Standard Form Quadratic

Find the y- and x-intercepts of the quadratic $f(x) = -2(x+3)^2 + 5$.

Solution

Find the y-intercept by evaluating $f(0)$. Notice that the quantity inside the parentheses ($(0+3) = (3)$) is evaluated FIRST!!!

$$\begin{aligned} f(0) &= -2(0+3)^2 + 5 \\ &= -2(3)^2 + 5 \\ &= -2(9) + 5 \\ &= -18 + 5 = -13 \end{aligned}$$

So the y-intercept is at $(0, -13)$.

For the x-intercepts, find all solutions of $f(x) = 0$. Solving a quadratic equation given in standard form, like in this example, is most efficiently accomplished by using the Square Root Property

$$\begin{aligned} 0 &= -2(x+3)^2 + 5 \\ 2(x+3)^2 &= 5 \\ (x+3)^2 &= \frac{5}{2} \\ x+3 &= \pm\sqrt{\frac{5}{2}} \\ x &= -3 \pm \sqrt{2.5} \end{aligned}$$

So the x-intercepts are at $(-3+\sqrt{2.5}, 0)$ and $(-3-\sqrt{2.5}, 0)$.

Try It 3.1.8

Find the y- and x-intercepts for the function $g(x) = 13 + x^2 - 6x$.

Answer

y-intercept at $(0, 13)$, No x-intercepts

Graph a Quadratic Function

Details on how to find features of a quadratic function have been covered. Now, these features will be used to sketch a graph.

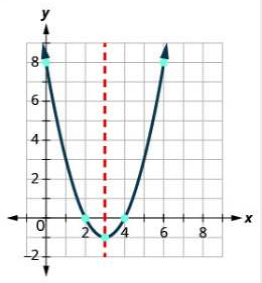
How To: Graph a quadratic function in the form $f(x) = ax^2 + bx + c$

1. Determine whether the parabola opens upward ($a > 0$) or downward ($a < 0$).
2. Find the equation of the axis of symmetry, $x = h$ where $h = -\frac{b}{2a}$.
3. Find the vertex, (h, k), where $k = f(h)$.
4. Find the y-intercept, $f(0)$. Find the point symmetric to the y-intercept across the axis of symmetry.
5. Find the x-intercepts. (Set $f(x) = 0$ and solve for x using factoring, QF or CTS). Find additional points if needed.
6. Graph the parabola.

Example 3.1.9 How to Graph a General Form Quadratic Function Using Properties

Graph $f(x) = x^2 - 6x + 8$ by using its properties.

Solution:

Step 1: Determine whether the parabola opens upward or downward.	Look a in the equation $f(x) = x^2 - 6x + 8$ Since a is positive, the parabola opens up.	$f(x) = x^2 - 6x + 8$ $a = 1,\ b = -6,\ c = 8$ **The parabola opens upward.**
Step 2: Find the axis of symmetry.	$f(x) = x^2 - 6x + 8$ The axis of symmetry is the line $x = -\frac{b}{2a}$.	Axis of Symmetry $x = -\frac{b}{2a}$ $x = -\frac{(-6)}{2 \cdot 1}$ $x = 3$ **The axis of symmetry is the line $x = 3$.**
Step 3: Find the vertex.	The vertex is on the axis of symmetry. Substitute $x = 3$ into the function.	Vertex $f(x) = x^2 - 6x + 8$ $f(3) = (3)^2 - 6(3) + 8$ $f(3) = -1$ **The vertex is $(3, -1)$.**
Step 4: Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.	Find $f(0)$. Use the axis of symmetry to find a point symmetric to the y-intercept. The y-intercept is 3 units left of the axis of symmetry, $x = 3$. A point 3 units to the right of the axis of symmetry has $x = 6$.	y-intercept $f(x) = x^2 - 6x + 8$ $f(0) = (0)^2 - 6(0) + 8$ $f(0) = 8$ **The y-intercept is $(0, 8)$.** Point symmetric to y-intercept: **The point is $(6, 8)$.**
Step 5: Find the x-intercepts. Find additional points if needed.	Solve $f(x) = 0$. Solve this quadratic equation by factoring.	x-intercepts $f(x) = x^2 - 6x + 8$ $0 = x^2 - 6x + 8$ $0 = (x - 2)(x - 4)$ $x = 2$ or $x = 4$ **The x-intercepts are $(2, 0)$ and $(4, 0)$.**
Step 6: Graph the parabola.	We graph the vertex, intercepts, and the point symmetric to the y-intercept. We connect these 5 points to sketch the parabola.	

Try It 3.1.9

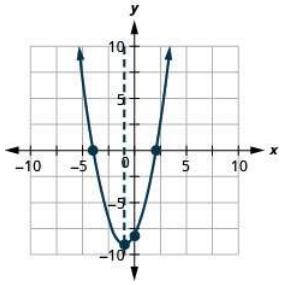
https://math.libretexts.org/@go/page/44432

Graph the following quadratic functions by using its properties.

a. $f(x) = x^2 + 2x - 8$

a. Answer

opens up, vertex: $(-1, -9)$, axis: $x = -1$,
intercepts: $(0, -8), (-4, 0), (2, 0)$, symm. pt: $(-2, -8)$



b. $f(x) = x^2 - 8x + 12$

b. Answer

opens up, vertex: $(4, -4)$, axis: $x = 4$,
intercepts: $(0, 12), (2, 0), (6, 0)$, symm.pt: $(8, 12)$

This figure shows an upward-opening parabola graphed on the x y-coordinate plane. The x-axis of the plane runs from negative 10 to 10. The y-axis of the plane runs from negative 10 to 15. The axis of symmetry, x equals 4, is graphed as a dashed line. The parabola has a vertex at (4, negative 4). The y-intercept of the parabola is the point (0, 12). The x-intercepts of the parabola are the points (2, 0) and (6, 0).

How To: How to: Graph a quadratic function in the form $f(x) = a(x - h)^2 + k$

1. Determine whether the parabola opens upward $(a > 0)$ or downward $(a < 0)$.
2. Find the equation of the axis of symmetry, $x = h$.
3. Find the vertex, (h, k).
4. Find the y-intercept, $f(0)$. Find the point symmetric to the y-intercept across the axis of symmetry.
5. Find the x-intercepts. (Use the square root property to solve $a(x - h)^2 + h = 0$. Find additional points if needed.
6. Graph the parabola.

Example 3.1.10: How to Graph a Vertex Form Quadratic Using Properties

Graph the function $f(x) = 2(x + 1)^2 + 3$ by using its properties

Solution:

Step 1: Determine whether the parabola opens upward or downward.	Identify the constants a, h, k. Since $a = 2$, the parabola opens upward.	$a = 2,\ h = -1,\ k = 3$ **The parabola opens upward.**
Step 2: Find the axis of symmetry.	The axis of symmetry is $x = h$.	**The axis of symmetry is the line $x = -1$.**
Step 3: Find the vertex.	The vertex is (h, k).	**The vertex is $(-1, 3)$.**
Step 4: Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.	Find the y-intercept by finding $f(0)$. The y-intercept is 1 units right of the axis of symmetry, $x = -1$. A point 1 units to the left of the axis of symmetry has $x = -2$.	$f(0) = 2(0+1)^2 + 3 = 2(1) + 3 = 5$ **The y-intercept is $(0, 5)$.** **Point symmetric to y-intercept is $(-2, 5)$**
Step 5: Find the x-intercepts. Find additional points if needed.	Solve $f(x) = 0$. Use the Square Root Property. This equation has imaginary solutions, so there are no x-intercepts	$2(x+1)^2 + 3 = 0$ $2(x+1)^2 = -3$ $(x+1)^2 = -3/2$ $x + 1 = \pm\sqrt{(}-3/2)$ $x = -1 \pm i\sqrt{(}1.5)$ **No x intercepts**
Step 6: Graph the parabola.	Graph the vertex, intercepts, and the point symmetric to the y-intercept. Connect these points to sketch the parabola. Two more points: $f(1) = 2(1+1)^2 + 3 = 2(2^2) + 3 = 11$ Therefore, $(1, 11)$ is on the graph. By symmetry, the point $(-3, 11)$ is also on the graph	

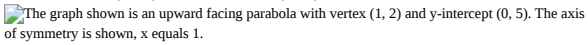
Try It 3.1.10

Graph the following functions using properties

a. $f(x) = 3(x-1)^2 + 2$

a. Answer

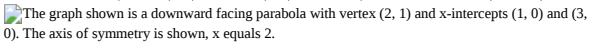
opens up, vertex: $(1, 2)$, axis: $x = 1$,
intercepts: $(0, 5)$, symm. pt: $(2, 5)$


The graph shown is an upward facing parabola with vertex (1, 2) and y-intercept (0, 5). The axis of symmetry is shown, x equals 1.

b. $f(x) = -2(x-2)^2 + 1$

b. Answer

opens down, vertex: $(2, 1)$, axis: $x = 2$,
intercepts: $(0, -7), (\approx 1.3, 0), (\approx 2.7, 0)$, symm. pt: $(4, -7)$


The graph shown is a downward facing parabola with vertex (2, 1) and x-intercepts (1, 0) and (3, 0). The axis of symmetry is shown, x equals 2.

Rewrite Quadratics into Vertex Form

As the above examples illustrate, it is often easier to graph a quadratic equation that is in vertex form, rather than in general form. This is particularly true when trying to find x-intercepts for equations that don't easily factor. There are two different approaches for transforming an equation in general form into an equation in standard (or vertex) form. One method uses the formulas for h and k. The other method uses Complete the Square. Both will be illustrated below.

Formula method

How to: Rewrite $y = ax^2 + bx + c$ into vertex form - formula method.

1. Identify constants a and b.
2. Substitute a and b into the formula: $h = -\frac{b}{2a}$.
3. Substitute $x = h$ into the general form of the quadratic function to find k.
4. Rewrite the quadratic in vertex (standard) form using h and k. The vertex form of the function is $f(x) = a(x-h)^2 + k$.

Example 3.1.11: Formula method of rewriting into vertex form

Rewrite the quadratic function $f(x) = 2x^2 + 4x - 4$ into vertex form.

Solution

Step 1. Values of the parameters in the general form are $a = 2$, $b = 4$, and $c = -4$.

Step 2. Solve for h.

$$h = -\frac{b}{2a} = -\frac{4}{2(2)} = -1$$

Step 3. Use the value found for h to find k.

$$k = f(h) = f(-1) = 2(-1)^2 + 4(-1) - 4 = -6$$

Step 4. The vertex form of the function is:

$$f(x) = a(x-h)^2 + k$$
$$f(x) = 2(x+1)^2 - 6$$

Try It 3.1.11

Find the vertex form for the function $g(x) = 13 + x^2 - 6x$.

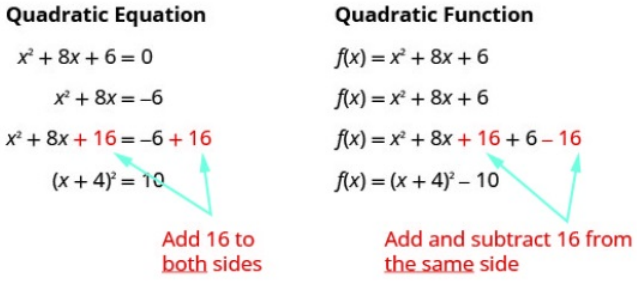
Answer

$g(x) = (x-3)^2 + 4$

Complete the square method

Another way of transforming $f(x) = ax^2 + bx + c$ into the form $f(x) = a(x-h)^2 + k$ is by completing the square. The latter form is known as the vertex form or standard form. This approach will also be used when circles are studied.

We must be careful to both add and subtract the number to the SAME side of the function to complete the square. We cannot add the number to both sides as we did when we completed the square with quadratic equations.



Different ways to complete the square.

When we complete the square in a function with a coefficient of x^2 that is not one, we have to factor that coefficient from the x-terms. We do not factor it from the constant term. It is often helpful to move the constant term a bit to the right to make it easier to

focus only on the x-terms.

Once we get the constant we want to complete the square, we must remember to multiply it by the coefficient that was part of the x^2 term before we then subtract it.

How to: Rewrite $y = ax^2 + bx + c$ into vertex form - complete the square method.

1. Separate the x terms from the constant.
2. If the coefficient of x^2 is not 1, factor it out from the x^2 and x terms.
3. Find the CTS constant needed to complete the square on the x^2 and x terms.
4. Add the CTS constant to the x^2 and x terms. Subtract the CTS constant (multiplied by the coefficient of x^2 if not 1)
5. Write the trinomial as a binomial square and combine constants outside the binomial square to arrive at the standard form of the function.

Example 3.1.12: CTS method of rewriting into vertex form

Rewrite $f(x) = -3x^2 - 6x - 1$ in the $f(x) = a(x-h)^2 + k$ form by completing the square.

Solution:

Step 1. Separate the x terms from the constant.	$f(x) = -3x^2 - 6x - 1$ $f(x) = -3x^2 - 6x \quad\quad - 1$
Step 2. Factor the coefficient of $x^2, -3$.	$f(x) = -3(x^2 + 2x) \quad\quad - 1$
Step 3. Prepare to complete the square.	$f(x) = -3(x^2 + 2x \quad\quad) - 1$
Take half of 2 and then square it to complete the square $(\frac{1}{2} \cdot 2)^2 = 1$	$f(x) = -3(x^2 + 2x + \square) - 1 + 3\square$
Step 4. The constant 1 completes the square in the parentheses, but the parentheses is multiplied by -3. So we are really adding -3. We must then add 3 to not change the value of the function.	
Step 5. Rewrite the trinomial as a square and combine the constants. The function is now in the $f(x) = a(x-h)^2 + k$ form.	$f(x) = -3(x+1)^2 + 2$

Try It 3.1.12

Rewrite the following functions in the $f(x) = a(x-h)^2 + k$ form by completing the square.

a. $f(x) = -4x^2 - 8x + 1$

a. Answer

$f(x) = -4(x+1)^2 + 5$

b. $f(x) = 2x^2 - 8x + 3$

b. Answer

$f(x) = 2(x-2)^2 - 5$

Obtain the Equation of a Quadratic Function from a Graph

So far we have started with a function and then found its graph.

Now we are going to reverse the process. Starting with the graph, we will find the function.

How to: Write a quadratic function in a general form

Given a graph of a quadratic function, write the equation of the function in general form.

1. Identify the horizontal shift of the parabola; this value is h. Identify the vertical shift of the parabola; this value is k.
2. Substitute the values of the horizontal and vertical shift for h and k. in the function $f(x) = a(x–h)^2 + k$.
3. Substitute the values x and $f(x)$ of any point on the graph other than the vertex into the equation from step 2.
4. Solve for the parameter a. The result is the quadratic equation in standard (vertex) form.
5. Expand and simplify to obtain the general form for the quadratic.

Example 3.1.13: Writing the Equation of a Quadratic Function from the Graph

Write an equation for the quadratic function g in Figure 3.1.13 in standard (vertex) form and then rewrite the result into general form.

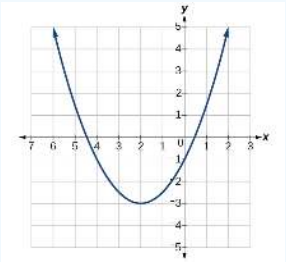


Figure 3.1.13: Graph of a parabola with its vertex at $(-2, -3)$.

Solution

Since it is quadratic, we start with the $g(x) = a(x - h)^2 + k$ form. (Observe the minus sign in front of h!) The vertex, (h, k), is $(-2, -3)$ so $h = -2$ and $k = -3$. Substituting theses values we obtain $g(x) = a(x + 2)^2–3$.

Substituting the coordinates of a point on the curve, such as $(0, -1)$, we can solve for the parameter a.

$$\begin{aligned} -1 &= a(0 + 2)^2 - 3 \\ 2 &= 4a \\ a &= \frac{1}{2} \end{aligned}$$

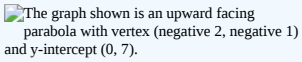
Using the values found for parameters a, h, and k, write the standard form of the equation: $g(x) = \frac{1}{2}(x + 2)^2–3$.

To write this in general polynomial form, we can expand the formula and simplify terms.

$$\begin{aligned} g(x) &= \frac{1}{2}(x + 2)^2 - 3 \\ &= \frac{1}{2}(x + 2)(x + 2) - 3 \\ &= \frac{1}{2}(x^2 + 4x + 4) - 3 \\ &= \frac{1}{2}x^2 + 2x + 2 - 3 \\ g(x) &= \frac{1}{2}x^2 + 2x - 1 \end{aligned}$$

Example 3.1.14

Determine the quadratic function whose graph is shown.



Solution:

Since it is quadratic, we start with the $f(x) = a(x - h)^2 + k$ form.

The vertex, (h, k), is $(-2, -1)$ so $h = -2$ and $k = -1$.

$$f(x) = a(x - (-2))^2 - 1 \quad \longrightarrow \quad f(x) = a(x + 2)^2 - 1$$

To find a, we use the y-intercept, $(0, 7)$. So $x = 0$ and $f(0) = 7$.

$$7 = a(0 + 2)^2 - 1$$

Solve for a.

$$\begin{aligned} 7 &= 4a - 1 \\ 8 &= 4a \\ 2 &= a \end{aligned}$$

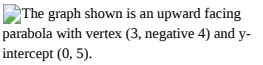
Write the function.

$$f(x) = 2(x + 2)^2 - 1$$

Try It 3.1.14

Write the quadratic function in the form $f(x) = a(x - h)^2 + k$ for each graph.

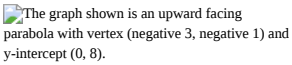
a.



a. Answer

$f(x) = (x - 3)^2 - 4$

b.



b. Answer

$f(x) = (x + 3)^2 - 1$

Try It 3.1.15

A coordinate grid has been superimposed over the quadratic path of a basketball in Figure **3.1.15**. Find an equation for the path of the ball. Does the shooter make the basket?

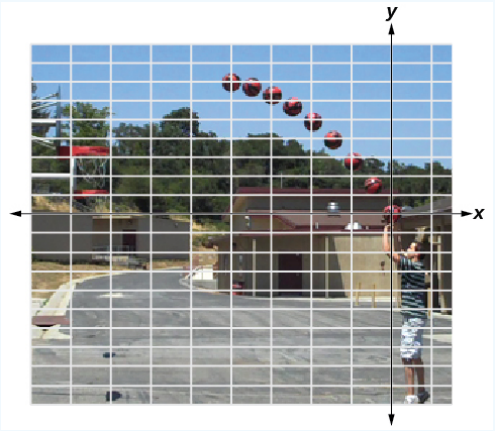


Figure* 3.1.15: Stop motioned picture of a boy throwing a basketball into a hoop to show the parabolic curve it makes. (credit: modification of work by Dan Meyer)*

Answer

The path passes through the origin and has vertex at $(-4, 7)$, so $h(x) = -\frac{7}{16}(x + 4)^2 + 7$. To make the shot, $h(-7.5)$ would need to be about 4 but $h(-7.5) \approx 1.64$; so, he doesn't make it.

Key Equations

- general form of a quadratic function: $f(x) = ax^2 + bx + c$
- the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- standard form of a quadratic function: $f(x) = a(x - h)^2 + k$

Key Concepts

- A polynomial function of degree two is called a quadratic function.
- The graph of a quadratic function is a parabola. A parabola is a U-shaped curve that can open either up or down.
- The axis of symmetry is the vertical line passing through the vertex. The x-intercepts are the points at which the parabola crosses the x-axis. The y-intercept is the point at which the parabola crosses the y-axis.
- Quadratic functions are often written in general form. Standard or vertex form is useful to easily identify the vertex of a parabola. Either form can be written from a graph.
- The vertex can be found from an equation representing a quadratic function. .
- The domain of a quadratic function is all real numbers. The range varies with the function.
- A quadratic function's minimum or maximum value is given by the y-value of the vertex.
- Some quadratic equations must be solved by using the quadratic formula.

Glossary

axis of symmetry

a vertical line drawn through the vertex of a parabola around which the parabola is symmetric; it is defined by $x = -\frac{b}{2a}$.

general form of a quadratic function

the function that describes a parabola, written in the form $f(x) = ax^2 + bx + c$, where $a, b,$ and c are real numbers and a≠0.

standard form of a quadratic function

the function that describes a parabola, written in the form $f(x) = a(x - h)^2 + k$, where (h, k) is the vertex.

vertex

the point at which a parabola changes direction, corresponding to the minimum or maximum value of the quadratic function

vertex form of a quadratic function

another name for the standard form of a quadratic function

zeros

in a given function, the values of x at which $y = 0$, also called roots

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3.1e: Exercises - Quadratic Functions

A: Concepts

Exercise 3.1e.A

1) Explain the advantage of writing a quadratic function in standard form.

2) How can the vertex of a parabola be used in solving real world problems?

3) Explain why the condition of $a \neq 0$ is imposed in the definition of the quadratic function.

4) What is another name for the standard form of a quadratic function?

5) What two algebraic methods can be used to find the horizontal intercepts of a quadratic function?

Answers to Odd Examples:

1. When written in that form, the vertex can be easily identified.

3. If $a = 0$ then the function becomes a linear function.

5. If possible, we can use factoring. Otherwise, we can use the quadratic formula.Add texts here.

B: Parabola Orientation

Exercise 3.1e.B

★ Determine if the parabola opens up or down.

7. a. $f(x) = -2x^2 - 6x - 7$ b. $f(x) = 6x^2 + 2x + 3$	9. a. $f(x) = -3x^2 + 5x - 1$ b. $f(x) = 2x^2 - 4x + 5$	11. $y = x^2 - 9x + 20$ 12. $y = x^2 - 12x + 32$ 13. $y = -2x^2 + 5x + 12$
8. a. $f(x) = 4x^2 + x - 4$ b. $f(x) = -9x^2 - 24x - 16$	10. a. $f(x) = x^2 + 3x - 4$ b. $f(x) = -4x^2 - 12x - 9$	14. $y = -6x^2 + 13x - 6$ 15. $y = 64 - x^2$ 16. $y = -3x + 9x^2$

Answers to Odd Examples:

7. a. down b. up	9. a. down b. up	11. Upward	13. Downward	15. Downward

C: Vertex and Axis of Symmetry

Exercise 3.1e.C

★ Determine the vertex.

17. $y = -(x-5)^2 + 3$ 18. $y = -2(x-1)^2 + 7$	19. $y = 5(x+1)^2 + 6$ 20. $y = 3(x+4)^2 + 10$	21. $y = -5(x+8)^2 - 1$ 22. $y = (x+2)^2 - 5$

★ Find the vertex and the axis of symmetry.

23. $f(x) = x^2 + 8x - 1$ 24. $f(x) = x^2 + 10x + 25$ 25. $f(x) = -x^2 + 2x + 5$ 26. $f(x) = -2x^2 - 8x - 3$	27. $y = -x^2 + 10x - 34$ 28. $y = -x^2 - 6x + 1$ 29. $y = -4x^2 + 12x - 7$ 30. $y = -9x^2 + 6x + 2$	31. $y = 4x^2 - 1$ 32. $y = x^2 - 16$

Answers to Odd Examples

17. $(5,3)$ 19. $(-1,6)$ 21. $(-8,-1)$ 23. Vertex: $(-4,-17)$, Axis of symmetry: $x=-4$	25. Vertex: $(1,6)$, Axis of symmetry: $x=1$ 27. Vertex: $(5,-9)$; axis of symmetry: $x=5$ 29. Vertex: $(\frac{3}{2},2)$; axis of symmetry: $x=\frac{3}{2}$ 31. Vertex: $(0,-1)$; axis of symmetry: $x=0$

D: Domain and Range

Exercise $3.1e.D$

★ Use the vertex of the graph of the quadratic function and the direction the graph opens to find the domain and range of the function.

33) Vertex $(1,-2)$, opens up.	34) Vertex $(-1,2)$ opens down.	35) Vertex $(-5,11)$, opens down.	36) Vertex $(-100,100)$, opens up.

★ Given the following quadratic functions, determine the domain and range.

37. $f(x)=3x^2+30x+50$ 38. $f(x)=5x^2-10x+1$ 39. $g(x)=-2x^2+4x+1$	40. $g(x)=-7x^2-14x-9$ 41. $f(x)=x^2+x-1$ 42. $f(x)=-x^2+3x-2$	43) $f(x)=(x-3)^2+2$ 44) $f(x)=-2(x+3)^2-6$ 45) $f(x)=x^2+6x+4$	46) $f(x)=2x^2-4x+2$ 47) $k(x)=3x^2-6x-9$

Answers to Odd Examples:

33. Domain is $(-\infty,\infty)$. Range is $[-2,\infty)$. 35. Domain is $(-\infty,\infty)$ Range is $(-\infty,11]$. 37. Domain: $(-\infty,\infty)$; range:$[-25,\infty)$ 39. Domain: $(-\infty,\infty)$; range: $(-\infty,3]$	41. Domain: $(-\infty,\infty)$; range: $[-\frac{5}{4},\infty)$ 43. Domain is $(-\infty,\infty)$. Range is $[2,\infty)$. 45. Domain is $(-\infty,\infty)$. Range is $[-5,\infty)$. 47. Domain is $(-\infty,\infty)$. Range is $[-12,\infty)$.

E: Minimum or maximum Value

Exercise $3.1e.E$

★ In the following exercises, find the maximum or minimum value of each function.

49. $f(x)=2x^2+x-1$ 50. $y=-4x^2+12x-5$ 51. $y=x^2-6x+15$ 52. $y=-x^2+4x-5$ 53. $y=-9x^2+16$ 54. $y=4x^2-49$	55. $y=-x^2-6x+1$ 56. $y=-x^2-4x+8$ 57. $y=25x^2-10x+5$ 58. $y=16x^2-24x+7$ 59. $y=-x^2$ 60. $y=1-9x^2$	61. . $y=20x-10x^2$ 62. $y=12x+4x^2$ 63. $y=3x^2-4x-2$ 64. $y=6x^2-8x+5$ 65. $y=x^2-5x+1$ 66. $y=1-x-x^2$

★ Determine whether there is a minimum or maximum value to each quadratic function. Find the value and the axis of symmetry.

68. $y(x)=2x^2+10x+12$ 69. $f(x)=2x^2-10x+4$ 70. $f(x)=-x^2+4x+3$	71. $f(x)=4x^2+x-1$ 72. $h(t)=-4t^2+6t-1$	73) $f(x)=\frac{1}{2}x^2+3x+1$ 74) $f(x)=-\frac{1}{3}x^2-2x+3$

Answers to Odd Examples:

49. The minimum value is $-\frac{9}{8}$ when $x=-\frac{1}{4}$.
51. The minimum value is 6 when $x=3$.
53. The maximum value is 16 when $x=0$.
55. Maximum: $y=10$
57. Minimum: $y=4$ when $x=\frac{1}{5}$
59. Maximum: $y=0$
61. Maximum: $y=10$
63. Minimum: $y=-\frac{10}{3}$
65. Minimum: $y=-\frac{21}{4}$
69. Minimum is $-\frac{17}{2}$ and occurs at $\frac{5}{2}$.
    Axis of symmetry is $x=\frac{5}{2}$.
71. Minimum is $-\frac{17}{16}$ and occurs at $-\frac{1}{8}$.
    Axis of symmetry is $x=-\frac{1}{8}$.
73. Minimum is $-\frac{7}{2}$ and occurs at -3.
    Axis of symmetry is $x=-3$.

F: Intercepts

Exercise 3.1e.F

★ **Determine the x- and y-intercepts of each function.**

75. $f(x)=x^2+7x+6$
76. $f(x)=x^2+10x-11$
77. $f(x)=x^2+8x+12$
78. $f(x)=x^2+5x+6$
79. $f(x)=-x^2+8x-19$
80. $f(x)=-3x^2+x-1$
81. $f(x)=x^2+6x+13$
82. $f(x)=x^2+8x+12$
83. $f(x)=4x^2-20x+25$
84. $f(x)=-x^2-14x-49$
85. $f(x)=-x^2-6x-9$
86. $f(x)=4x^2+4x+1$
87. $y=x^2+4x-12$
88. $y=x^2-13x+12$
89. $y=2x^2+5x-3$
90. $y=3x^2-4x-4$
91. $y=-5x^2-3x+2$
92. $y=-6x^2+11x-4$
93. $y=4x^2-27$
94. $y=9x^2-50$
95. $y=x^2-x+1$
96. $y=x^2-6x+4$
97) $g(x)=x(x-4)-20$
98) $g(x)=x(x-2)-10$

Answers to Odd Examples:

75. y-intercept: $(0,6)$; x-intercept(s): $(-1,0),(-6,0)$
77. y-intercept: $(0,12)$; x-intercept(s): $(-2,0),(-6,0)$
79. y-intercept: $(0,-19)$; x-intercept(s): none
81. y-intercept: $(0,13)$; x-intercept(s): none
83. y-intercept: $(0,25)$; x-intercept(s): $(\frac{5}{2},0)$
85. y-intercept: $(0,9)$; x-intercept(s): $(-3,0)$
87. x-intercepts: $(-6,0),(2,0)$; y-intercept: $(0,-12)$
89. x-intercepts: $(-3,0),(\frac{1}{2},0)$; y-intercept: $(0,-3)$
91. x-intercepts: $(-1,0),(\frac{2}{5},0)$; y-intercept: $(0,2)$
93. x-intercepts: $\left(-\frac{3\sqrt{3}}{2},0\right),\left(\frac{3\sqrt{3}}{2},0\right)$; y-intercept: $(0,-27)$
95. x-intercepts: none; y-intercept: $(0,1)$
97. x-intercepts: $2+2\sqrt{6},2-2\sqrt{6}$; y-intercept: $(0,-20)$

G: Graph Quadratic Functions

Exercise 3.1e.G

★ **Sketch a graph of the quadratic function and give the vertex, axis of symmetry, and intercepts.**

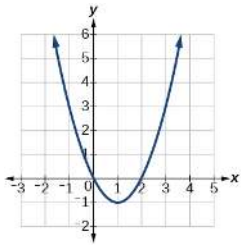
99) $f(x)=x^2-2x$
100) $f(x)=x^2-6x-1$
101) $f(x)=x^2-5x-6$
102) $f(x)=x^2-7x+3$
103) $f(x)=-2x^2+5x-8$
105) $f(x)=4x^2-12x-3$

★ **Sketch each quadratic function below**

107. $f(x)=x^2-10x$
108. $f(x)=x^2+8x$
109. $f(x)=x^2-9$
110. $f(x)=x^2-25$
111. $f(x)=1-x^2$
112. $f(x)=4-x^2$
113. $f(x)=x^2-2$
114. $f(x)=x^2-3$
115. $f(x)=-2x^2+3$
116. $f(x)=-2x^2-1$
117. $f(x)=x^2-1$
118. $f(x)=x^2+1$
121. n/a
122. $f(x)=\frac{1}{3}x^2-3$
123. n/a
124. $f(x)=5x^2+2$

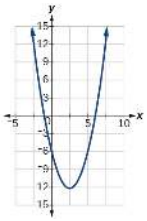
Answers to Odd Examples:

99.



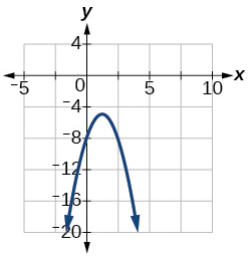
Vertex $(1,-1)$,
Axis of symmetry is $x=1$.
Intercepts: $(0,0)$, $(2,0)$

101.



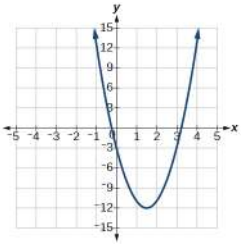
Vertex $\left(\frac{5}{2},\frac{-49}{4}\right)$,
Axis of symmetry is $x=\frac{5}{2}$.
Intercepts: $(0,-6)$, $(-1,0)$, $(6,0)$

103.



Vertex $\left(\frac{5}{4},-\frac{39}{8}\right)$,
Axis of symmetry is $x=\frac{5}{4}$.
Intercepts: $(0,-8)$

105.



Vertex $\left(\frac{3}{2},-12\right)$,
Axis of symmetry is $x=\frac{3}{2}$,
Intercepts: $\left(\frac{3}{2}\pm\sqrt{3},0\right)$, $(0,-3)$

107.

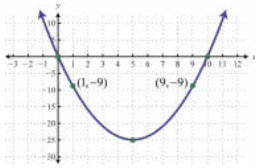


Figure 107

109.

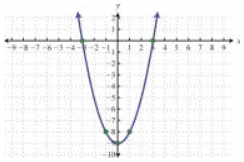


Figure 109

111.

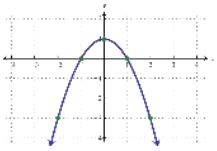


Figure 111

113.

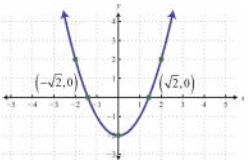


Figure 113

115.

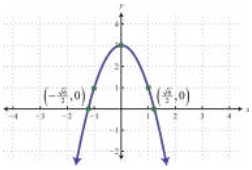


Figure 115

117.

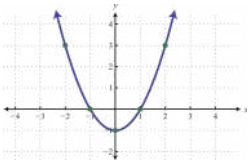


Figure 117

★ Sketch each quadratic function below

125. $f(x)=-x^2+2x-7$
126. $f(x)=-x^2+2x-4$
127. $f(x)=x^2-2x-8$
128. $f(x)=-x^2-2x+15$
129. $f(x)=x^2+3x+4$
130. $f(x)=-x^2+3x-4$
131. $f(x)=x^2+4x+3$
132. $f(x)=x^2+4x-12$
133. $f(x)=-x^2-4x+2$
134. $f(x)=-x^2+4x-5$
135. $f(x)=x^2+6x+5$
136. $f(x)=x^2-6x+8$
137. $f(x)=x^2-6x+15$
138. $f(x)=x^2-6x+6$

Answers to Odd Examples:

125.

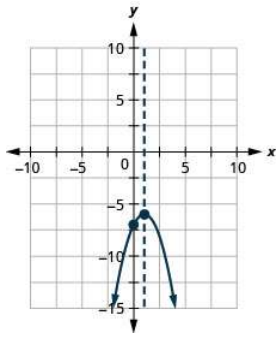


Figure 125

127.

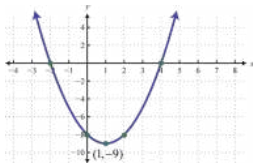


Figure 127

129.

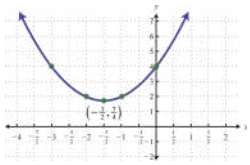


Figure 129

131.

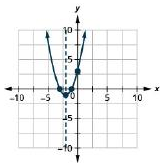


Figure 133

133.



This figure shows a downward-opening parabola on the x y-coordinate plane. It has a vertex of (negative 2, 6), y-intercept of (0, 2), and axis of symmetry shown at x equals negative 2.

Figure 133

135.

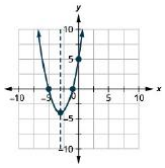


Figure 135

137.



This figure shows an upward-opening parabolas on the x y-coordinate plane. It has a vertex of (3, 6), y-intercept of (0, 10), and axis of symmetry shown at x equals 3.

Figure 137

★ Sketch each quadratic function below

145. $f(x) = 2x^2 - 4x + 1$
146. $f(x) = 3x^2 - 6x - 1$
147. $f(x) = -2x^2 + 8x - 10$
148. $f(x) = -2x^2 - 4x - 5$
149. $f(x) = 5x^2 - 10x + 8$
150. $f(x) = 3x^2 - 12x + 7$
151. $f(x) = 3x^2 + 18x + 20$
152. $f(x) = -3x^2 + 6x + 1$
153. $f(x) = -4x^2 + 12x - 9$
154. $f(x) = -4x^2 - 6x - 2$
155. $f(x) = -4x^2 + 4x - 3$
156. $f(x) = -4x^2 - 4x + 3$
157. $f(x) = -2x^2 + 6x - 3$
158. $f(x) = 9x^2 + 12x + 4$
159. $f(x) = 2x^2 + 4x - 3$
160. $f(x) = 3x^2 + 2x - 2$

Answers to Odd Examples:

145.

This figure shows an upward-opening parabola on the x y-coordinate plane. It has a vertex of (1, negative 1), y-intercept of (0, 1), and axis of symmetry shown at x equals 1.

Figure 145

147.

This figure shows a downward-opening parabola on the x y-coordinate plane. It has a vertex of (2, negative 2), y-intercept of (0, negative 10), and axis of symmetry shown at x equals 2.

Figure 147

149.

This figure shows an upward-opening parabola on the x y-coordinate plane. It has a vertex of (1, 3), y-intercept of (0, 8), and axis of symmetry shown at x equals 1.

Figure 149

151.

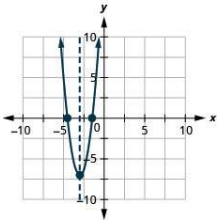


Figure 151

153.

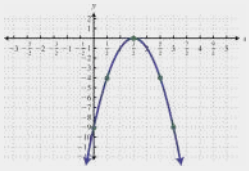


Figure 153

155.

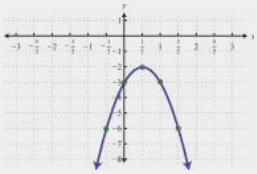


Figure 155

157.

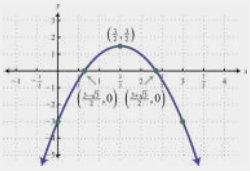


Figure 157

159.

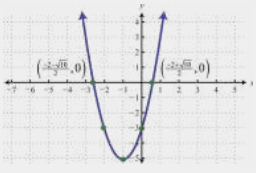


Figure 117

★ Sketch each quadratic function below

165. $f(x) = (x-1)^2$
166. $f(x) = (x+1)^2$
167. $f(x) = (x-1)^2+5$
169. $f(x) = (x-3)^2+4$
169. $f(x) = (x-4)^2-9$
170. $f(x) = (x-6)^2-2$
171. $f(x) = (x+2)^2+1$
172. $f(x) = (x+3)^2-1$
173. $f(x) = (x-4)^2-3$
174. $f(x) = (x+5)^2-2$
175. $f(x) = -2(x-4)^2+22$
176. $f(x) = 2(x+3)^2-13$
177. $f(x) = -2(x+1)^2+8$
178. $f(x) = -2(x-5)^2-3$
179. $f(x) = -4(x-1)^2-2$
180. $f(x) = -3(x+2)^2+12$
181. $f(x) = -5(x-1)^2$
182. $f(x) = -(x+2)^2$

Answers to Odd Examples:

165.

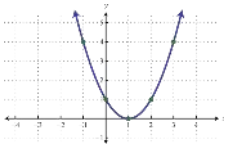


Figure 165

167.

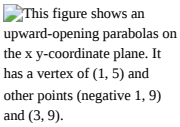


Figure 167

169.

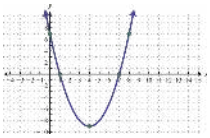


Figure 169

171.

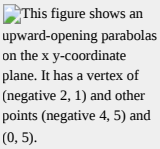


Figure 171

173.

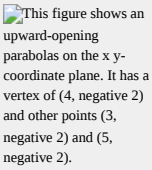


Figure 173

175.

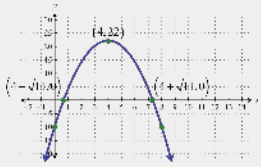


Figure 175

177.

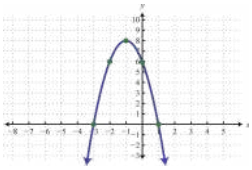


Figure 177

179.

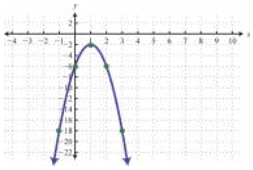


Figure 179

181.

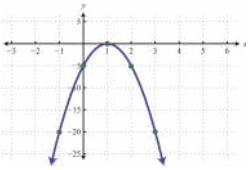


Figure 181

H: Convert to vertex form

Exercise 3.1e.H

★ Rewrite in Standard (vertex) form $y = a(x-h)^2+k$ and determine the vertex.

185. $y = x^2-14x+24$
186. $y = x^2-12x+40$
187. $y = x^2+4x-12$
188. $y = x^2+6x-1$
189. $y = 2x^2-12x-3$
190. $y = 3x^2-6x+5$
191. $y = -x^2+16x+17$
192. $y = -x^2+10x$
193. $f(x) = -x^2-4x+2$
194. $f(x) = x^2-12x+32$
195. $g(x) = x^2+2x-3$
196. $f(x) = x^2-x$
197. $f(x) = x^2+5x-2$
198. $h(x) = 2x^2+8x-10$
199. $k(x) = 3x^2-6x-9$
200. $f(x) = 2x^2-6x$
201. $f(x) = 3x^2-5x-1$
203. $f(x) = -3x^2-12x-5$
204. $f(x) = 2x^2-12x+7$
205. $f(x) = 3x^2+6x-1$
206. $f(x) = -4x^2-16x-9$
207. $f(x) = 5x^2-10x+8$
208. $f(x) = 3x^2-6x-1$
209. $f(x) = 2x^2-4x+1$
210. $f(x) = -16x^2+24x+6$

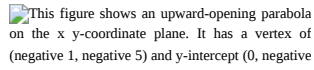
Answers to Odd Examples:

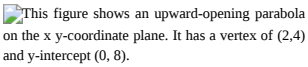
185. $y=(x-7)^2-25$; vertex: $(7,-25)$
187. $y=(x+2)^2-16$; vertex: $(-2,-16)$
189. $y=2(x-3)^2-21$; vertex: $(3,-21)$
191. $y=-(x-8)^2+81$; vertex: $(8,81)$
193. $f(x)=-(x+2)^2+6$; vertex: $(-2,6)$
195. $g(x)=(x+1)^2-4$, Vertex $(-1,-4)$
197. $f(x)=\left(x+\frac{5}{2}\right)^2-\frac{33}{4}$, Vertex $\left(-\frac{5}{2},-\frac{33}{4}\right)$
199. $k(x)=3(x-1)^2-12$, Vertex $(1,-12)$
201. $f(x)=3\left(x-\frac{5}{6}\right)^2-\frac{37}{12}$, Vertex $\left(\frac{5}{6},-\frac{37}{12}\right)$
203. $f(x)=-3(x+2)^2+7$; vertex: $(-2,7)$
205. $f(x)=3(x+1)^2-4$; vertex: $(-1,-4)$
207. $f(x)=5(x-1)^2+3$; vertex: $(1,3)$
209. $f(x)=2(x-1)^2-1$; vertex: $(1,-1)$

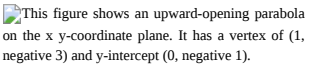
I: Convert to vertex form

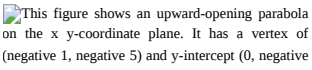
Exercise 3.1e.I

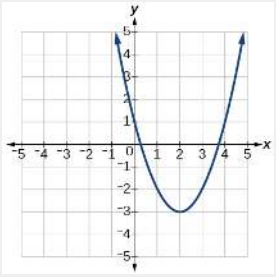
★ In the following exercises, write the quadratic function in $f(x)=a(x-h)^2+k$ form whose graph is shown.

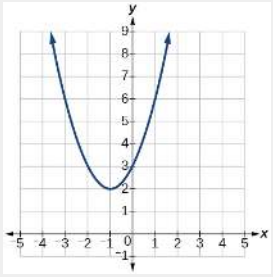
211.

This figure shows an upward-opening parabola on the x y-coordinate plane. It has a vertex of (negative 1, negative 5) and y-intercept (0, negative 4).

212.

This figure shows an upward-opening parabola on the x y-coordinate plane. It has a vertex of (2,4) and y-intercept (0, 8).

213.

This figure shows an upward-opening parabola on the x y-coordinate plane. It has a vertex of (1, negative 3) and y-intercept (0, negative 1).

214.

This figure shows an upward-opening parabola on the x y-coordinate plane. It has a vertex of (negative 1, negative 5) and y-intercept (0, negative 3).

215.


216.


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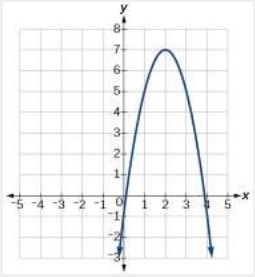

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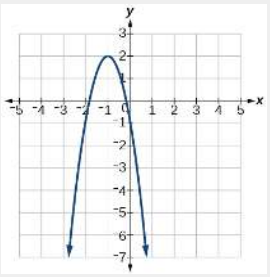

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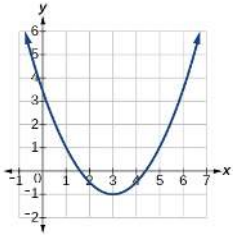
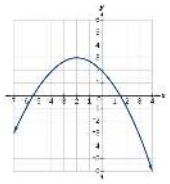
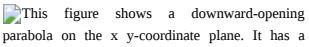


220.


221.

This figure shows a downward-opening parabola on the x y-coordinate plane. It has a vertex of (4, 0), y-intercept of (0, negative 16), and axis of symmetry shown at x equals 4.

Answers to Odd Examples:

211. $f(x)=(x+1)^2-5$
213. $f(x)=2(x-1)^2-3$
215. $f(x)=x^2-4x+1$
217. $f(x)=-2x^2+8x-1$
219. $f(x)=\frac{1}{2}x^2-3x+\frac{7}{2}$
221. $f(x)=-(x-4)^2+0$

J: Construct an equation from points

Exercise 3.1e.J

★ Use the vertex (h,k) and a point on the graph (x,y) to find the general form of the equation of the quadratic function.

223) $(h,k)=(2,0),\ (x,y)=(4,4)$
224) $(h,k)=(-2,-1),\ (x,y)=(-4,3)$
225) $(h,k)=(0,1),\ (x,y)=(2,5)$
226) $(h,k)=(2,3),\ (x,y)=(5,12)$
227) $(h,k)=(-5,3),\ (x,y)=(2,9)$
228) $(h,k)=(3,2),\ (x,y)=(10,1)$
229) $(h,k)=(0,1),\ (x,y)=(1,0)$
230) $(h,k)=(1,0),\ (x,y)=(0,1)$

★ Write the equation of the quadratic function that contains the given point and has the same shape as the given function.

231) Contains $(1,1)$ and has shape of $f(x)=2x^2$. Vertex is on the y-axis.

232) Contains $(-1,4)$ and has the shape of $f(x)=2x^2$. Vertex is on the y-axis.

233) Contains $(2,3)$ and has the shape of $f(x)=3x^2$. Vertex is on the y-axis.

234) Contains $(1,-3)$ and has the shape of $f(x)=-x^2$. Vertex is on the y-axis.

235) Contains $(4,3)$ and has the shape of $f(x)=5x^2$. Vertex is on the y-axis.

236) Contains $(1,-6)$ has the shape of $f(x)=3x^2$. Vertex has x-coordinate of -1.

Answers to Odd Examples:

223. $f(x)=x^2-4x+4$
225. $f(x)=x^2+1$
227. $f(x)=\frac{6}{49}x^2+\frac{60}{49}x+\frac{297}{49}$
229. $f(x)=-x^2+1$
231. $f(x)=2x^2-1$
233. $f(x)=3x^2-9$
235. $f(x)=5x^2-77$

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3.2: Circles

Circles

Recall from geometry that the definition of a circle is the set of points that are equidistant from a fixed point.

If we wanted to find an equation to represent a circle with a radius of r centered at a point (h, k), we notice that the distance between any point (x, y) on the circle and the center point is always the same: r. Noting this, we can use our distance formula to write an equation for the radius:

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

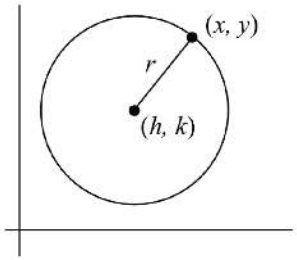
Squaring both sides of the equation gives us the standard equation for a circle.

Definition: The Standard Equation for a Circle

The **equation of a circle** centered at the point (h, k) with radius r can be written as

$$(x-h)^2 + (y-k)^2 = r^2$$

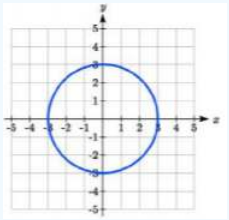
Notice that a circle does not pass the vertical line test - it is not a function and it is not possible to write y as a function of x or vice versa.



Obtain the equation of a circle given a graph

Example 3.2.1

Write an equation for the circle graphed here.



Solution

This circle is centered at the origin, the point (0, 0). By measuring horizontally or vertically from the center out to the circle, we can see the radius is 3. Using this information in our formula gives:

$$(x-0)^2 + (y-0)^2 = 3^2$$

simplified, this gives

$$x^2 + y^2 = 9$$

Obtain the equation of a circle given a description.

Example 3.2.2

Write an equation for a circle centered at the point (-3, 2) with radius 4.

Solution

Using the equation from above, $h = -3$, $k = 2$, and the radius $r = 4$. Using these in our formula,

$$(x-(-3))^2+(y-2)^2=4^2$$

simplified, this gives

$$(x+3)^2+(y-2)^2=16$$

Example 3.2.3

Write an equation for a circle with center (−2,3) and radius 5.

Solution.

Here, $(h,k)=(-2,3)$ and $r=5$, so we get

$$\begin{aligned}(x-(-2))^2+(y-3)^2 &= (5)^2\\ (x+2)^2+(y-3)^2 &= 25\end{aligned}$$

Exercise 3.2.4

Write an equation for a circle centered at (4, -2) with radius 6.

Answer

$$(x-4)^2+(y+2)^2=36$$

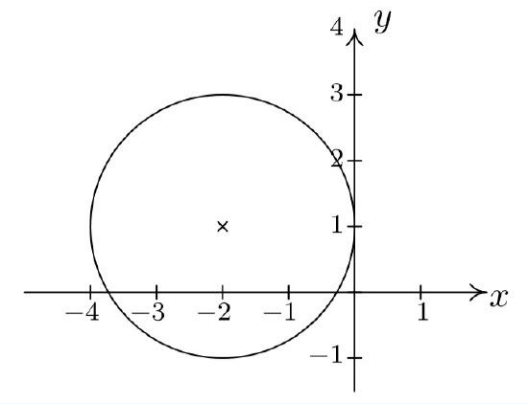
Graph a circle given an equation.

Example 3.2.5:

Graph $(x+2)^2+(y-1)^2=4$. Find the center and radius.

Solution

From the standard form of a circle we have that $x+2$ is $x-h$, so $h=-2$ and $y-1$ is $y-k$ so $k=1$. This tells us that our center is $(-2,1)$. Furthermore, $r^2=4$, so $r=2$. Thus we have a circle centered at $(-2,1)$ with a radius of 2. Graphing gives us



□

Graph a circle given an equation that is not in standard form.

If we were to expand the equation in the previous example and gather up like terms, instead of the easily recognizable $(x+2)^2+(y-1)^2=4$, we'd be contending with $x^2+4x+y^2-2y+1=0$. If we're given such an equation, we can complete the square in each of the variables to see if it fits the form of the Standard Equation of a circle by following the steps given below.

To Write the Equation of a Circle in Standard Form

1. Group the same variables together on one side of the equation and position the constant on the other side.
2. Complete the square on each variable.
3. Divide both sides by the coefficient of the squares. (For circles, the coefficients will be the same.)

Example 3.2.6:

Complete the square to find the center and radius of $3x^2 - 6x + 3y^2 + 4y - 4 = 0$.

Solution

$$\begin{aligned} 3x^2 - 6x + 3y^2 + 4y - 4 &= 0 \\ 3x^2 - 6x + 3y^2 + 4y &= 4 && \text{add 4 to both sides} \\ 3\left(x^2 - 2x\right) + 3\left(y^2 + \frac{4}{3}y\right) &= 4 && \text{factor out leading coefficients} \\ 3\left(x^2 - 2x + \underline{1}\right) + 3\left(y^2 + \frac{4}{3}y + \underline{\underline{\frac{4}{9}}}\right) &= 4 + 3\underline{(1)} + 3\underline{\underline{\left(\frac{4}{9}\right)}} && \text{complete the square in } x, y \\ 3(x-1)^2 + 3\left(y + \frac{2}{3}\right)^2 &= \frac{25}{3} && \text{factor} \\ (x-1)^2 + \left(y + \frac{2}{3}\right)^2 &= \frac{25}{9} && \text{divide both sides by 3} \end{aligned}$$

From the standard form of the equation for a circle, we identify $x - 1$ as $x - h$, so $h = 1$, and $y + \frac{2}{3}$ as $y - k$, so $k = -\frac{2}{3}$. Hence, the center is $(h, k) = \left(1, -\frac{2}{3}\right)$. Furthermore, we see that $r^2 = \frac{25}{9}$ so the radius is $r = \frac{5}{3}$.

It is possible to obtain equations like $(x-3)^2 + (y+1)^2 = 0$ or $(x-3)^2 + (y+1)^2 = -1$, neither of which describes a circle. (Do you see why not?) The reader is encouraged to think about what, if any, points lie on the graphs of these two equations.

Ellipses

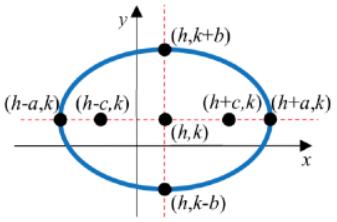
An ellipse is an oval-shaped circle. In this section we will simplistically regard an ellipse as a squashed circle with a horizontal radius a and a vertical radius b. The vertical radius can be larger or smaller than the horizontal radius. When the horizontal and vertical radii are the same length, the shape is a circle, rather than an ellipse. The formula for an ellipse centered at the origin is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. When we shift the graph right h units and up k units by replacing x with x–h and y with y–k, similar to what we did when we learned transformations, we obtain the standard form of the ellipse.

Definition: EQUATION OF AN ELLIPSE CENTERED AT (h, k) IN STANDARD FORM

The standard form for the equation of an ellipse centered at (h, k) is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

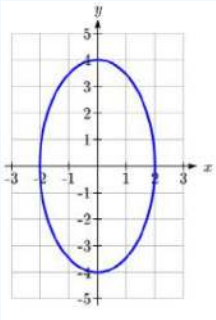
where a is the length of the horizontal "radius", and b is the length of the vertical "radius".
The four endpoints of the radii are called **vertices**.
The horizontal diameter or **horizontal axis** is $2a$
The vertical diameter or **vertical axis** is $2b$.



Obtain the equation of an ellipse given a graph

Example 3.2.7

Find the standard form of the equation for the ellipse graphed here.



Solution

The center is at (0,0). From the graph we can see the horizontal radius is $a = 2$ and the vertical radius is $b = 4$.

The equation will be

$$\frac{x^2}{2^2} + \frac{y^2}{4^2} = 1 \text{ or } \frac{x^2}{4} + \frac{y^2}{16} = 1$$

Obtain the equation of an ellipse given a description.

Example 3.2.8

Find the standard form of the equation for an ellipse centered at (0,0) with horizontal radius of 16 and vertical radius of 8.

Solution

Since the center is at (0,0) , the horizontal radius is $a = 14$ and the vertical radius is $b = 8$. The formula for the ellipse is then

$$\frac{x^2}{16^2} + \frac{y^2}{8^2} = 1 \text{ or } \frac{x^2}{256} + \frac{y^2}{64.} = 1$$

Exercise 3.2.9

Find the standard form of the equation for an ellipse centered at the origin with horizontal radius of 10 and vertical radius of 3.

Answer

$a = 10$ and $b = 3$.

$$\frac{x^2}{100} + \frac{y^2}{9} = 1$$

Graph an ellipse given an equation.

Example 3.2.10

Put the equation of the ellipse $9x^2 + y^2 = 9$ in standard form. Find the lengths of the horizontal and vertical radii and sketch the graph.

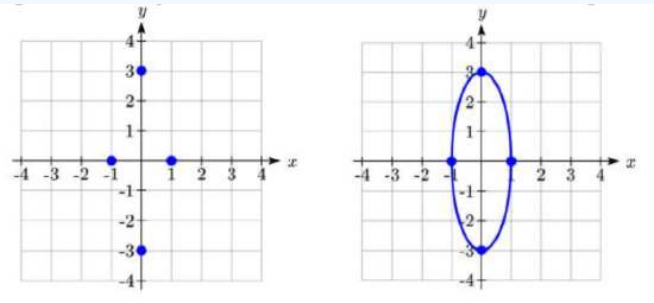
Solution

The standard equation has a 1 on the right side, so this equation can be put in standard form by dividing by 9:

$$\frac{x^2}{1} + \frac{y^2}{9} = 1$$

The horizontal radius is $\sqrt{1}=1$, the vertical radius is $\sqrt{9}=3$, and the center of the ellipse is at the origin.

To sketch the graph we locate the center and plot the location of the radii. Then we sketch the ellipse.



Exercise 3.2.11

Find the center and horizontal and vertical radii for the ellipse $(x-4)^2+\frac{(y+2)^2}{4}=1$ and sketch the graph.

Answer

Center (4, -2) with $a=1$, $b=2$.
Vertices at (4 ± 1, -2) = (3,-2) and (5,-2) and (4, -2 ± 2) = (4, 0) and (4, -4). (Graph unavailable)

Graph an ellipse given an equation that is not in standard form.

Sometimes we are given the equation of an ellipse that is not in standard form. In this situation, completing the squre must be done twice to obtain the equation in standard form.

Example 3.2.12

Put the equation of the ellipse $x^2+2x+4y^2-24y=-33$ in standard form. Find the center and vertices, and sketch the graph.

Solution

To rewrite this in standard form, we will need to complete the square, twice.

Looking at the x terms, x^2+2x, we like to have something of the form $(x+n)^2$. Notice that if we were to expand this, we'd get $x^2+2nx+n^2$, so in order for the coefficient on x to match, we'll need $(x+1)^2=x^2+2x+1$. However, we don't have $a+1$ on the left side of the equation to allow this factoring. To accommodate this, we will add 1 to both sides of the equation, which then allows us to factor the left side as a perfect square:

$$x^2+2x+1+4y^2-24y=-33+1$$

$$(x+1)^2+4y^2-24y=-32$$

Repeating the same approach with the y terms, first we'll factor out the 4.

$$4y^2-24y=4(y^2-6y)$$

Now we want to be able to write $4\left(y^2-6y\right)$ as

$$4(y+n)^2=4\left(y^2+2ny+n^2\right)$$

For the coefficient of y to match, n will have to -3, giving $4(y-3)^2=4\left(y^2-6y+9\right)=4y^2-24y+36$.

To allow this factoring, we can add 36 to both sides of the equation.

$$(x+1)^2+4y^2-24y+36=-32+36$$

$$(x+1)^2+4\left(y^2-6y+9\right)=4$$

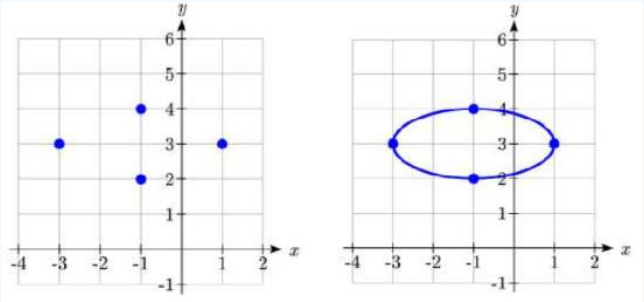
$$(x+1)^2+4(y-3)^2=4$$

Dividing by 4 gives the standard form of the equation for the ellipse

$$\frac{(x+1)^2}{4}+\frac{(y-3)^2}{1}=1$$

The center is at (h, k) = (-1, 3). The value of $a=\sqrt{4}=2$ and the value of $b=\sqrt{1}=1$.

To sketch the graph we locate the center and then locate the endpoints of the radii. Then we sketch the ellipse.



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3.2e: Circle Exercises.

A: Write the Equation for a Circle from a Description

Exercise 3.2e.A

1. Write an equation of the circle centered at (8 , -10) with radius 8.

2. Write an equation of the circle centered at (-9, 9) with radius 16.

Answer

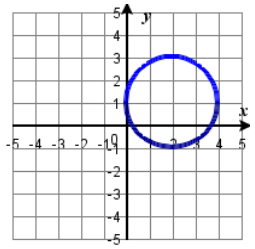
1. $(x-8)^2+(y+10)^2=64$

B: Construct an equation for a circle from a graph

Exercise 3.2e.B

★ **State the center and radius of the circle graphed below and construct the equation for it.**

3.



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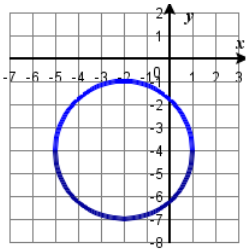

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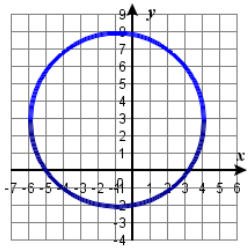

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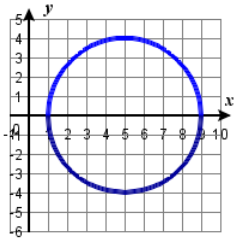


Answers to Odd Numbered Problems:

3. Center $(2,1)$, radius $r=2$, $(x-2)^2+(y+1)^2=4$

5. Center $(-1,3)$, radius $r=5$, $(x+1)^2+(y-3)^2=25$

C: Graph a circle given an equation

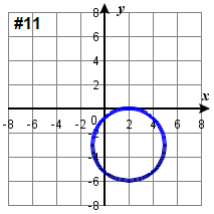
Exercise 3.2e.C

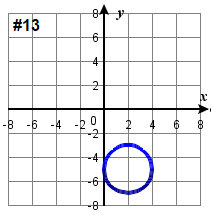
★ **State the center and radius of the circle described by the equation and graph it.**

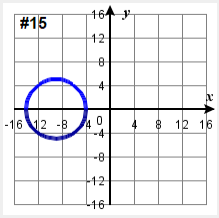
11. $(x-2)^2+(y+3)^2=9$
12. $(x+1)^2+(y-2)^2=16$
13. $(x-2)^2+(y+5)^2=4$
14. $(x+1)^2+(y+5)^2=100$
15. $(x+9)^2+y^2=25$
16. $(x-4)^2+(y+2)^2=9$
17. $(x+4)^2+(y-5)^2=42$
18. $(x-3)^2+(y-6)^2=20$
19. $\left(x+\frac{5}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{30}{4}$
20. $(x-1)^2+(y-5)^2=5$
21. $\left(x+\frac{1}{2}\right)^2+\left(y-\frac{3}{5}\right)^2=\frac{161}{100}$
22. $(x-3)^2+(y-5)^2=65$
23. $x^2+(y-3)^2=0$
24. $x^2+(y-72)^2=4096$

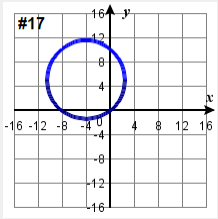
Answers to Odd Numbered Problems:

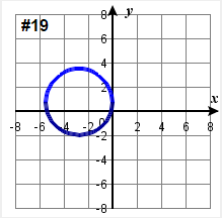
11. Center $(2,-3)$, radius $r=3$
13. Center $(2,-5)$, radius $r=2$
15. Center $(-9,0)$, radius $r=5$
17. Center $(-4,5)$, radius $r=\sqrt{42}$
19. Center $\left(-\frac{5}{2},\frac{1}{2}\right)$, radius $r=\frac{\sqrt{30}}{2}$
21. Center $\left(-\frac{1}{2},\frac{3}{5}\right)$, radius $r=\frac{\sqrt{161}}{10}$
23. This is not a circle.

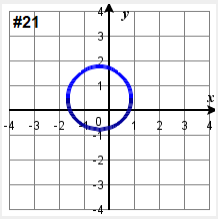












D: Graph a circle given an equation in non-standard form

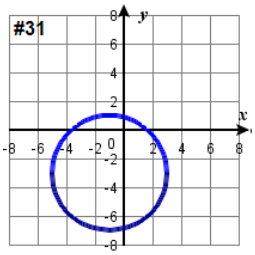
Exercise 3.2e.D

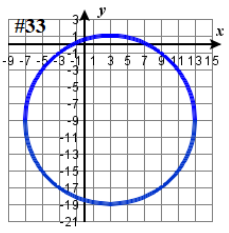
★ **Complete the square and write the equation in standard form. Then give the center and radius of each circle and graph it.**

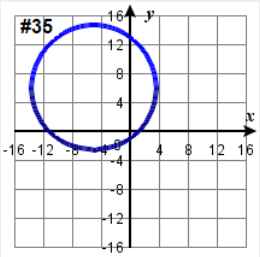
31. $x^2+y^2+2x+6y-6=0$
32. $x^2+y^2+4x+2y-4=0$
33. $x^2+y^2-6x+18y-10=0$
34. $x^2+y^2-8x+4y-5=0$
35. $x^2+y^2+10x-12y=13$
36. $x^2+y^2+12x-8y=5$
37. $x^2+y^2-14x-20y=20$
38. $x^2+y^2-16x-10y=11$
39. $x^2+y^2+x+14y+\frac{53}{4}=0$
40. $x^2+y^2+16x+5y=\frac{25}{4}$
41. $x^2+y^2+2y=3$
42. $x^2+y^2+7x-\frac{7}{4}=0$

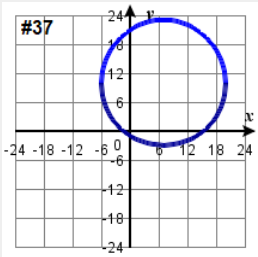
Answers to Odd Numbered Problems:

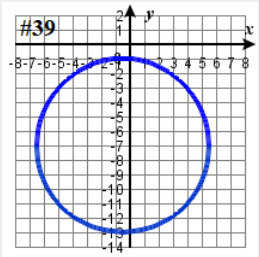
31. $(x+1)^2+(y+3)^2=16$, Center: $(-1,-3)$, Radius: 4
33. $(x-3)^2+(y+9)^2=100$, Center: $(-3,9)$, R: 10
35. $(x+5)^2+(y-6)^2=74$, Center: $(-5,6)$, R: $\sqrt{74}$
37. $(x-7)^2+(y-10)^2=169$, Center: $(7,10)$, Radius: 13
39. $(x+0.5)^2+(y+7)^2=36$, Center: $(-\frac{1}{2},-7)$, Radius: 6
41. $x^2+(y+1)^2=4$, Center: $(0,-1)$, Radius = 2

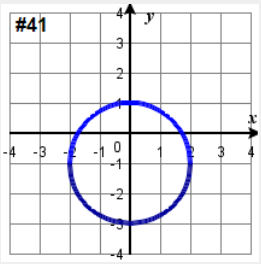








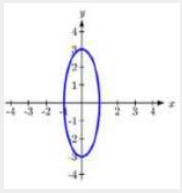
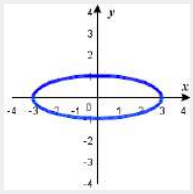
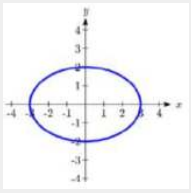
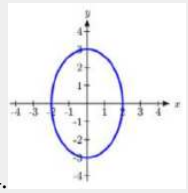




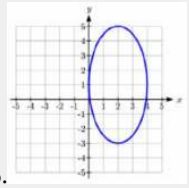
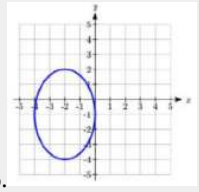
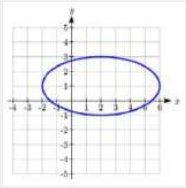
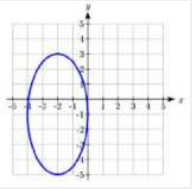
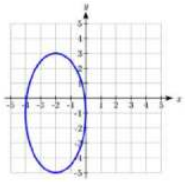
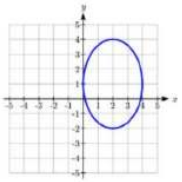
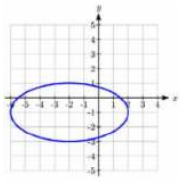
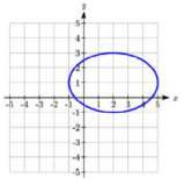
E: Match equations with graphs of ellipses

Exercise 3.2e.E

★ **Match each graph with one of the equations A–D.**

A. $\frac{x^2}{4}+\frac{y^2}{9}=1$	B. $\frac{x^2}{9}+\frac{y^2}{4}=1$	C. $\frac{x^2}{9}+y^2=1$	D. $x^2+\frac{y^2}{9}=1$
41. 	42. 	43. 	44. 

★ **Match each graph to equations A-H.**

A. $\frac{(x-2)^2}{4}+\frac{(y-1)^2}{9}=1$ B. $\frac{(x-2)^2}{4}+\frac{(y-1)^2}{16}=1$	C. $\frac{(x-2)^2}{16}+\frac{(y-1)^2}{4}=1$ D. $\frac{(x-2)^2}{9}+\frac{(y-1)^2}{4}=1$	E. $\frac{(x+2)^2}{4}+\frac{(y+1)^2}{9}=1$ F. $\frac{(x+2)^2}{4}+\frac{(y+1)^2}{16}=1$	G. $\frac{(x+2)^2}{16}+\frac{(y+1)^2}{4}=1$ H. $\frac{(x+2)^2}{9}+\frac{(y+1)^2}{4}=1$
45. 	46. 	47. 	49. 
49. 	50. 	51. 	52. 

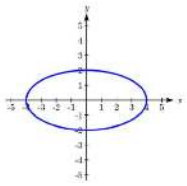
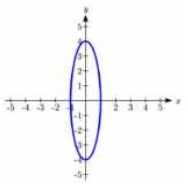
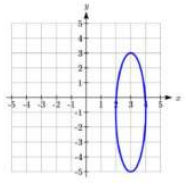
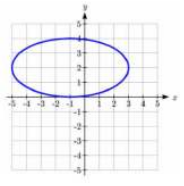
Answers to Odd Numbered Problems:

41. D 43. B 45. B 47. C 49. F 51. G

F: Write the equation for an ellipse given a graph

Exercise 3.2e.F

★ **Write an equation for the graph.**

53. 	54. 	55. 	56. 

★ **Find the standard form of the equation for an ellipse satisfying the given conditions.**

57. Center (0,0), horizontal radius = **32**, vertical radius = **7**
58. Center (0,0), horizontal radius = **9**, vertical radius = **18**
59. Center (0,0), horizontal radius = **2**, vertical radius = **3**
60. Center (0,0), horizontal radius = **4**, vertical radius = **4**
61. Center (-4, 3), horizontal radius = **4**, vertical radius = **5**
62. Center (1, -2), horizontal radius = **7**, vertical radius = **8**

Answers to Odd Numbered Problems:

53. $\frac{x^2}{16}+\frac{y^2}{4}=1$
55. $(x-3)^2+\frac{(y+1)^2}{16}=1$
57. $\frac{x^2}{1024}+\frac{y^2}{49}=1$
59. $\frac{x^2}{4}+\frac{y^2}{9}=1$
61. $\frac{(x+4)^2}{16}+\frac{(y-3)^2}{25}=1$

G: Graph an ellipse given an equation

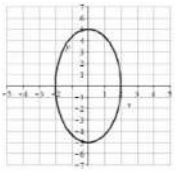
Exercise 3.2e.G: Graph an Ellipse from an Equation in Standard Form

★ **Find the center, and horizontal and vertical radii. Sketch the graph.**

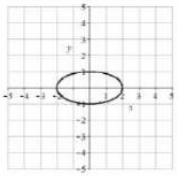
63. $\frac{x^2}{4}+\frac{y^2}{25}=1$
64. $\frac{x^2}{16}+\frac{y^2}{4}=1$
65. $\frac{x^2}{4}+y^2=1$
66. $x^2+\frac{y^2}{25}=1$
67. $x^2+25y^2=25$
68. $16x^2+y^2=16$
69. $16x^2+9y^2=144$
70. $16x^2+25y^2=400$
71. $9x^2+y^2=18$
72. $x^2+4y^2=12$

Answers to Odd Numbered Problems:

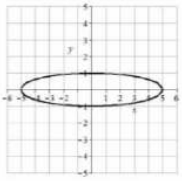
63. Center $(0,0)$, Horiz. radius $a=2$, Vert. radius $b=5$



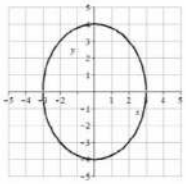
65. Center $(0,0)$, Horiz. radius $a=2$, Vert. radius $b=1$



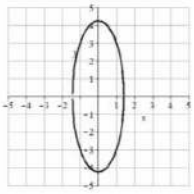
67. Center $(0,0)$, Horiz. radius $a=5$, Vert. radius $b=1$



69. Center $(0,0)$, Horiz. radius $a=3$, Vert. radius $b=4$



71. Center $(0,0)$, Horiz. radius $a=\sqrt{2}$, Vert. radius $b=3\sqrt{2}$



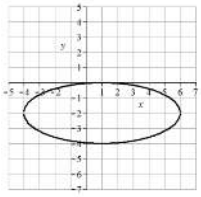
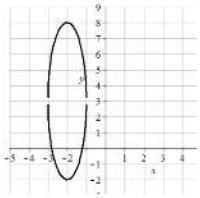
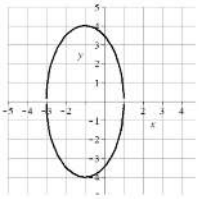
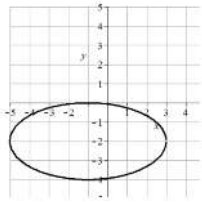
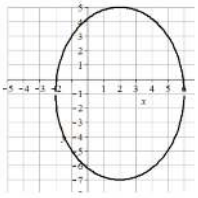
H: Graph an ellipse given an equation in non-standard form

Exercise 3.2e.H

★ **Find the center, and horizontal and vertical radii. Sketch the graph.**

73. $\frac{(x-1)^2}{25}+\frac{(y+2)^2}{4}=1$
74. $\frac{(x+5)^2}{16}+\frac{(y-3)^2}{36}=1$
75. $(x+2)^2+\frac{(y-3)^2}{25}=1$
76. $\frac{(x-1)^2}{25}+(y-6)^2=1$
77. $4x^2+8x+4+y^2=16$
78. $x^2+4y^2+16y+16=36$
79. $x^2+2x+4y^2+16y=-1$
80. $4x^2+16x+y^2-8y=4$
81. $9x^2-36x+4y^2+8y=104$
82. $4x^2+8x+9y^2+36y=-4$

Answers to Odd Numbered Problems:

73. Center (1, -2), Horiz. radius $a = 5$, Vert. radius $b = 2$ 	75. Center (-2, 3), Horiz. radius $a = 1$, Vert. radius $b = 5$ 	77. Center (-1, 0), Horiz. radius $a = 2$, Vert. radius $b = 4$ 	79. Center (-1, -2), Horiz. radius $a = 4$, Vert. radius $b = 2$ 	81. Center (2, -1), Horiz. radius $a = 4$, Vert. radius $b = 6$ 

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3.3: Power Functions and Polynomial Functions

Learning Objectives

- Identify power functions and polynomial functions.
- Identify end behavior of power functions and polynomial functions.
- Identify the degree and leading coefficient of polynomial functions.

Identifying Power Functions

A power function is a function with *a single term* that is the product of a real number (called a coefficient), and a variable raised to a fixed real number. For example, look at functions for the area of a circle or volume of a sphere with radius r.

$$A(r) = \pi r^2 \qquad V(r) = \frac{4}{3}\pi r^3$$

Both of these are examples of power functions because they consist of a coefficient, π or $\frac{4}{3}\pi$, multiplied by a variable r raised to a fixed power.

Definition: Power Function

A **power function** is a function that can be represented in the form

$$f(x) = kx^p \tag{3.3.1}$$

where parameters k and p are real numbers, k is the **coefficient** and p is the **power** of the function.

Is $f(x) = 2^x$ a power function?

No. The variable in a power function is in the base. This function has the variable in the exponent instead, so it is called an exponential function, not a power function.

Example 3.3.1: Identify Equations of Power Functions

Which of the following functions are power functions?

$f(x) = 1$	Constant function
$f(x) = x$	Identity function
$f(x) = x^2$	Quadratic function
$f(x) = x^3$	Cubic function
$f(x) = \frac{1}{x}$	Reciprocal function
$f(x) = \frac{1}{x^2}$	Reciprocal squared function
$f(x) = \sqrt{x}$	Square root function
$f(x) = \sqrt[3]{x}$	Cube root function

Solution

All of the listed functions are power functions because they can be written in the form of Equation 3.3.1.

- The constant and identity functions are power functions because they can be written as $f(x) = x^0$ and $f(x) = x^1$ respectively.
- The quadratic and cubic functions are power functions with whole number powers $f(x) = x^2$ and $f(x) = x^3$.
- The **reciprocal** and reciprocal squared functions are power functions with negative whole number powers because they can be written as $f(x) = x^{-1}$ and $f(x) = x^{-2}$.
- The square and **cube root** functions are power functions with fractional powers because they can be written as $f(x) = x^{1/2}$ or $f(x) = x^{1/3}$.

Try It 3.3.1

Which functions are power functions?

- $f(x) = 2x^2 \cdot 4x^3$
- $g(x) = -x^5 + 5x^3 - 4x$
- $h(x) = \dfrac{2x^5 - 1}{3x^2 + 4}$

Answer

$f(x)$ is a power function because it can be written as $f(x) = 8x^5$. The other functions are not power functions.

Identifying End Behavior of Power Functions

When describing power functions, what happens as numbers get very large is important. To describe the behavior as numbers become larger and larger, we use the idea of infinity. We use the symbol ∞ for positive infinity and $-\infty$ for negative infinity. When we say that "x approaches infinity," which can be symbolically written as $x \to \infty$, we are describing a behavior; we are saying that x is increasing without bound. The behavior of the graph of a function as the magnitude of the input values gets very large ($x \to -\infty$ or $x \to \infty$) is referred to as the **end behavior** of the function. We can use words or symbols to describe end behavior.

Figure **3.3.2** shows the graphs of $f(x) = x^2$, $g(x) = x^4$ and and $h(x) = x^6$, which are all power functions with even, whole-number powers. Notice that these graphs have similar shapes, very much like that of the quadratic function in the toolkit. However, as the power increases, the graphs flatten somewhat near the origin and become steeper away from the origin.

With these even-power functions, as the input increases or decreases without bound, the output values become very large, positive numbers. Equivalently, we could describe this behavior by saying that as x approaches positive or negative infinity, the $f(x)$ values increase without bound. In symbolic form, we could write

$$\text{as } x \to \pm\infty, f(x) \to \infty \quad \text{or} \quad \lim_{x \to \pm\infty} (f(x) = \infty$$

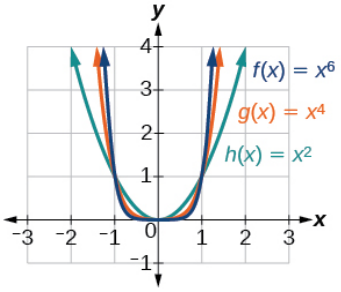


Figure **3.3.2**: Even-power functions

This example and the one below illustrate that functions of the form $f(x) = x^n$ reveal symmetry of one kind or another. First, in Figure **3.3.2** above we see that even functions of the form $f(x) = x^n$, n even, are symmetric about the y-axis. In Figure **3.3.3** below we see that odd functions of the form $f(x) = x^n$, n odd, are symmetric about the origin.

Figure **3.3.3** shows the graphs of $f(x) = x^3$, $g(x) = x^5$, and $h(x) = x^7$, which are all power functions with odd, whole-number powers. Notice that these graphs look similar to the cubic function in the toolkit. Again, as the power increases, the graphs flatten near the origin and become steeper away from the origin.

For these odd power functions, as x approaches negative infinity, $f(x)$ decreases without bound. As x approaches positive infinity, $f(x)$ increases without bound. In symbolic form we write

$$\text{as } x \to -\infty,\ f(x) \to -\infty \quad \text{or} \quad \lim_{x \to -\infty} f(x) = -\infty$$

$$\text{as } x \to \infty,\ f(x) \to \infty \quad \text{or} \quad \lim_{x \to \infty} f(x) = \infty$$

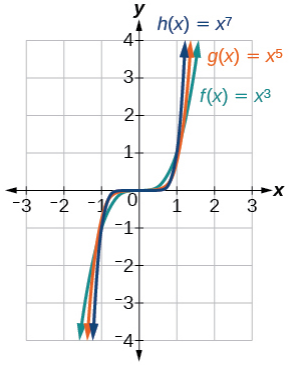


Figure **3.3.3**: Odd-power function

Figure **3.3.4** below shows the end behavior of power functions in the form $f(x) = kx^n$ where n is a non-negative integer depending on the power n and the constant coefficient k.

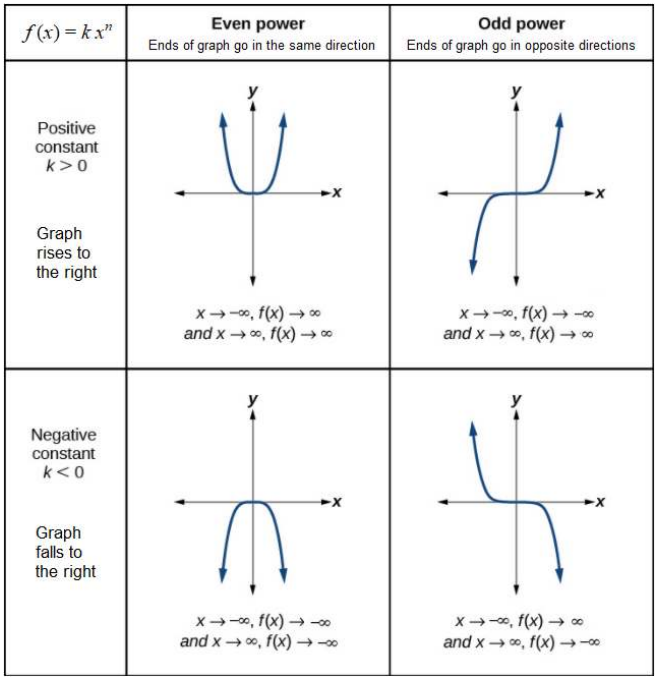


Figure **3.3.4** End Behaviour of Power Functions

HowTo: Find the end behaviour of a power function $f(x) = kx^n$, n is a non-negative integer.

1. Determine whether the constant coefficient k is positive or negative.
   - If positive, the graph **rises on the right**
   - If negative, the graph **falls on the right**
2. Determine whether the power n is even or odd.
   - If even, the ends of the graph go in the **same direction**
   - If odd, the ends of the graph go in **opposite directions**
3. The end behaviour analysis can be confirmed by looking at Figure **3.3.4**.

Example 3.3.2: Identify End Behavior for an Equation of a Power Function

Describe the end behavior of the graph of $f(x) = x^8$.

Solution

Because the coefficient of the power function is 1 which is positive

- The graph rises on the right.
- More formally, as x increases without bound, the output (value of $f(x)$ increases without bound.
- Symbolically we write as $x \to \infty$, $f(x) \to \infty$.

Because the exponent of the power function is 8 (an even number)

- Both ends of the graph go in the same direction (up)
- As x decreases without bound, $f(x)$ increases without bound.
- As $x \to -\infty$, $f(x) \to \infty$.

The graph of the function is shown in Figure **3.3.5**.

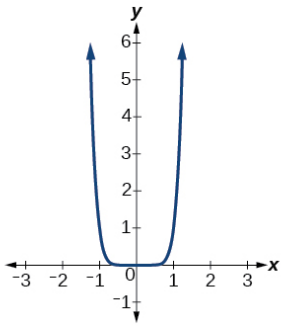


Figure **3.3.5**: Graph of $f(x) = x^8$.

https://math.libretexts.org/@go/page/34891

Example 3.3.3:

Describe the end behavior of the graph of $f(x) = -x^9$.

Solution

The exponent of the power function is 9 (an odd number).
The coefficient is –1 (negative).
Because the coefficient (-1) is negative

- The graph falls on the right.
- As x increases without bound, $f(x)$ decreases without bound.
- As $x \to \infty$, $f(x) \to -\infty$.

Because the exponent of the power function is 9 (an odd number)

- The ends of the graph go in opposite directions (so the graph goes up on the left)
- As x decreases without bound, $f(x)$ increases without bound.
- As $x \to -\infty$, $f(x) \to \infty$.

The graph of the function is shown in Figure 3.3.6.

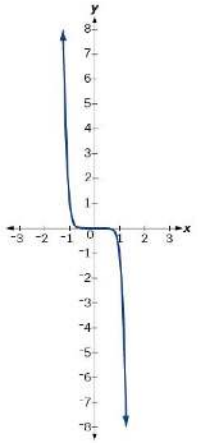


Figure 3.3.6: Graph of $f(x) = -x^9$.

Try It 3.3.3

Describe in words and symbols the end behavior of $f(x) = -5x^4$.

Answer

Coefficient -5 is negative, so the graph falls on the right; as $x \to \infty$, $f(x) \to -\infty$
Power 4 is even, so the ends of the graph go in the same direction (down); as $x \to -\infty$, $f(x) \to -\infty$

Identifying Polynomial Functions

A polynomial function consists of either zero or the sum of a finite number of non-zero terms, each of which is the product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power.

Definition: Polynomial Functions

Let n be a non-negative integer. A *polynomial function* is a function that can be written in the form

$$f(x) = a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0 \tag{3.3.2}$$

This is called the **general form** of a polynomial function. Each a_i is a **coefficient** and can be any real number. Each product $a_i x^i$ is a **term** of a polynomial function. Each **exponent** is a non-negative integer.

Example 3.3.4: Identify Equations of Polynomial Functions

Which of the following are polynomial functions?

- $f(x) = 2x^3 \cdot 3x + 4$
- $g(x) = -x(x^2 - 4)$
- $h(x) = 5\sqrt{x} + 2$

Solution

The first two functions are examples of polynomial functions because they can be written in the form of Equation 3.3.2, where the powers are non-negative integers and the coefficients are real numbers.

- $f(x)$ can be written as $f(x) = 6x^4 + 4$.
- $g(x)$ can be written as $g(x) = -x^3 + 4x$.
- $h(x)$ cannot be written in this form and is therefore not a polynomial function.

Identifying Polynomial Graphs

Polynomial functions have graphs that do not have sharp corners; these types of graphs are called **smooth**. Graphs of polynomial functions also have no breaks. Curves with no breaks are called **continuous**. The graphs of polynomial functions are both continuous and smooth.

Example 3.3.5: Recognize Graphs of Polynomial Functions

Which of the graphs in Figure **3.3.2** represents a polynomial function?

Solution

- The graphs of f and h are graphs of polynomial functions. They are smooth and **continuous**.
- The graphs of g and k are graphs of functions that are not polynomials. The graph of function g has a sharp corner. The graph of function k is not continuous.

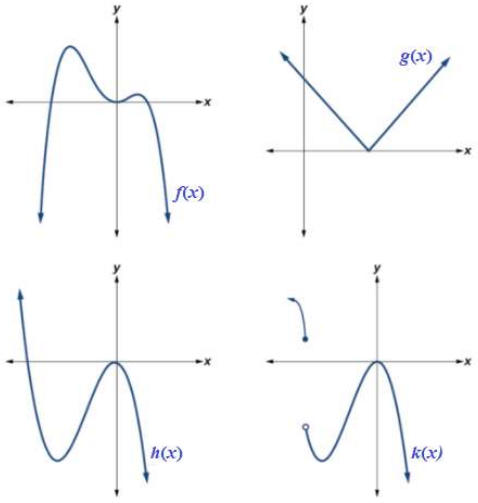


***Figure* 3.3.2**

Do all polynomial functions have as their domain all real numbers?

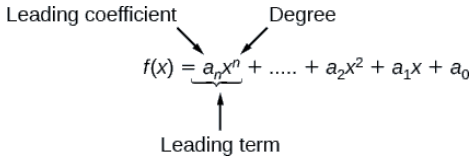
Do all polynomial functions have as their domain all real numbers?

- Yes. Any real number is a valid input for a polynomial function.

Identifying the Degree and Leading Coefficient of a Polynomial Function

Terminology of Polynomial Functions

- **degree** is the highest power of the variable in the polynomial.
- **leading term** is the term containing the highest power of the variable.
- **leading coefficient** is the coefficient of the leading term.
- **general form** is when terms are arranged in descending order of power.



How To: Given a polynomial function, identify the degree and leading coefficient

1. Find the highest power of x. This is the **degree** of the polynomial.
2. Identify the term containing the highest power of x. This is the **leading term**.
3. Identify the coefficient of the leading term. This is the **leading coefficient**.

Example 3.3.6: Identify the Degree and Leading Coefficient of a Polynomial Function

Identify the degree, leading term, and leading coefficient of the following polynomial functions.

$$f(x) = 3 + 2x^2 - 4x^3$$

$$g(t) = 5t^5 - 2t^3 + 7t$$

$$h(p) = 6p - p^3 - 2$$

Solution

- For the function $f(x)$, the highest power of x is 3, so the degree is 3. The leading term is the term containing that degree, $-4x^3$. The leading coefficient is the coefficient of that term, −4.
- For the function $g(t)$, the highest power of t is 5, so the degree is 5. The leading term is the term containing that degree, $5t^5$. The leading coefficient is the coefficient of that term, 5.
- For the function $h(p)$, the highest power of p is 3, so the degree is 3. The leading term is the term containing that degree, $-p^3$; the leading coefficient is the coefficient of that term, −1.

Try It 3.3.7

Identify the degree, leading term, and leading coefficient of the polynomial $f(x) = 4x^2 - x^6 + 2x - 5$.

Answer

The degree is 6. The leading term is $-x^6$. The leading coefficient is −1.

Identifying End Behavior of Polynomial Equations

Knowing the degree of a polynomial function is useful in helping us predict its end behavior. To determine its end behavior, look at the leading term of the polynomial function. Because the power of the leading term is the highest, that term will grow significantly faster than the other terms as x grows without bound, so its behavior will dominate the graph. For any polynomial, the end behavior of the polynomial will match the end behavior of the term of highest degree. Therefore, the graphs of all polynomial functions will eventually rise or fall without bound when going far to the left or far to the right.

HowTo: Given a polynomial function, find its end behaviour

1. Identify the leading term of the polynomial
2. Examine the coefficient of the leading term to determine the end behaviour of the graph on the right.
   - If positive, the graph **rises on the right**
   - If negative, the graph **falls on the right**
3. Examine the degree of the leading term to determine how the ends of the graph behave relative to each other.
   - If even, the ends of the graph go in the **same direction**
   - If odd, the ends of the graph go in **opposite directions**

Graph		Ends go in the same direction	Ends go in opposite directions
	Leading Term	even power	odd power
Rises Right	positive coefficient	↖ ··· ↗	↙ ··· ↗
Falls Right	negative coefficient	↙ ··· ↘	↖ ··· ↘

Schematic Diagram of Polynomial End Behaviour

Because the end behaviour of a polynomial is determined by its leading term, which is a power function, the end behaviour of a polynomial is the same as that of a power function, as illustrated in Figure **3.3.4**.

Example 3.3.8: Given an Equation of a Polynomial Function, State its End Behavior

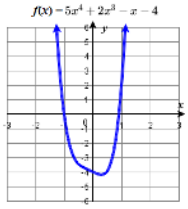
Given the following polynomial functions, determine its leading term, degree, and end behavior.

(a) $5x^4+2x^3-x-4$ (b) $3x^5-4x^4+2x^2+1$ (c) $-2x^6-x^5+3x^4+x^3$ (d) $-6x^3+7x^2+3x+1$

Solution

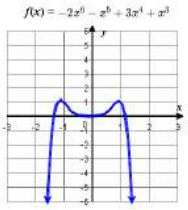
(a) Polynomial Function: $5x^4+2x^3-x-4$
Leading Term: $5x^4$ (degree 4)
- positive so graph rises to the right and
- even degree so ends go in the same direction: ↖ ... ↗

Graph:



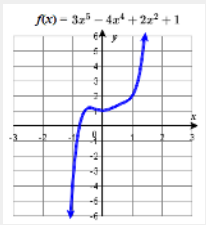
(c) Polynomial Function: $-2x^6-x^5+3x^4+x^3$
Leading Term: $-2x^6$ (degree 6)
- negative so graph falls to the right and
- even degree so ends go in the same direction: ↙ ··· ↘

Graph:



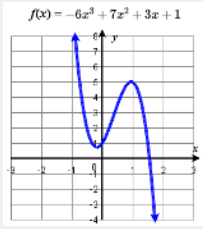
(b) Polynomial Function: $3x^5-4x^4+2x^2+1$
Leading Term: $3x^5$ (degree 5)
- positive so graph rises to the right and
- odd degree so ends go in opposite directions: ↙ ... ↗

Graph:



(d) Polynomial Function: $-6x^3+7x^2+3x+1$
Leading Term: $-6x^3$ (degree 3)
- negative so graph falls to the right and
- odd degree so ends go in opposite directions: ↖ ... ↘

Graph:



How to: Given a factored polynomial, determine its leading term

1. Multiply the high order terms of each of the factors.
2. The result is the leading term of the polynomial.

Example 3.3.9

Given the function $f(x) = -3x^2(x-1)(x+4)$, determine its leading term, degree, and end behavior.

Solution

One could begin by expanding $f(x)$ to obtain the general form of the polynomial, $f(x) = -3x^2(x-1)(x+4)$. HOWEVER, this involves a lot of unnecessary work because the only term of interest is the leading term. The leading term of $f(x) = -3x^2(x-1)(x+4) = -3x^2(x-1)(x+4)$ is the product of the leading terms in each of the factors. So in this example, the leading term is simply the product $-3x^2(x)(x) = -3x^4$. Thus the leading coefficient is negative (–3), so the graph falls to the right ... ↘ and the degree of the polynomial is 4, which is an even number, so the ends of the graph go in the same direction (↙ ... ↘), or more formally, as $x \to -\infty, f(x) \to -\infty$ and as $x \to \infty, f(x) \to -\infty$

Try It 3.3.10

Given the function $f(x) = 0.2(x-2)(4x+1)^3(x-5)^3$, determine its leading term, degree, and end behavior.

Answer

The leading term is $0.2(x)(4x)^3(x)^3 = 12.8x^7$, so it is a degree 7 polynomial. As x approaches positive infinity, $f(x)$ increases without bound; as x approaches negative infinity, $f(x)$ decreases without bound. (rises to the right, ends in opposite directions: ↙ ... ↗)

Intercepts of Polynomials

How to: Given a polynomial function, determine its intercepts

1. Determine the y-intercept by setting $x = 0$ and finding y.
2. If the polynomial function is factorable, factor it into a product of linear and irreducible quadratic factors.
3. Set each factor equal to zero and solve. The x-intercepts are the solutions (or zeros) that are real numbers.
4. A polynomial of degree n has n zeros. Some of these zeros can have the same value; some can be imaginary.

Example 3.3.11: Find Intercepts of a Factored Polynomial

Given the polynomial function $f(x) = (x-2)(x+1)(x-4)$, determine the y-and x-intercepts, the leading term, and the end behaviour.

Solution

The y-intercept occurs when the input is zero, so substitute 0 for x.

$$f(0) = (0-2)(0+1)(0-4) = (-2)(1)(-4) = 8$$

The y-intercept is $(0, 8)$.

The x-intercepts occur when the output is zero.

$$0 = (x-2)(x+1)(x-4)$$

$$\begin{array}{lllllll} x-2=0 & \text{or} & x+1=0 & \text{or} & x-4=0 \\ x=2 & \text{or} & x=-1 & \text{or} & x=4 \end{array}$$

The x-intercepts are $(2, 0)$,$(-1, 0)$, and $(4, 0)$.

The leading term is $(x)(x)(x) = x^3$. Therefore the graph rises to the right (positive coefficient) and falls to the left (odd degree so ends go in opposite directions): ↙ ⋯ ↗

Example 3.3.12

Find the y- and x-intercepts of $g(x) = (x-2)^2(2x+3)$. Find the leading term and describe the end behaviour.

Solution

The y-intercept can be found by evaluating $g(0)$.

$$g(0) = (0-2)^2(2(0)+3) = 12$$

So the y-intercept is $(0, 12)$.

The x-intercepts can be found by solving $g(x) = 0$.

$$(x-2)^2(2x+3) = 0$$

$$\begin{aligned} (x-2)^2 = 0 \quad &\text{or} \quad (2x+3) = 0 \\ x-2 = 0 \quad &\text{or} \quad x = -\frac{3}{2} \\ x = 2 \quad & \end{aligned}$$

So the x-intercepts are $(2, 0)$ and $\left(-\frac{3}{2}, 0\right)$.

The leading term is $(x)^2(2x) = 2x^3$. Therefore the graph rises to the right (positive coefficient) and falls to the left (odd degree so ends go in opposite directions): ↙ ⋯ ↗

Example 3.3.13: Find x-intercepts of a Polynomial Function by Factoring

Find the y- and x-intercepts for $f(x) = x^3 - 5x^2 - x + 5$.

Solution

The y-intercept is $(0, 5)$ because $f(0) = 5$.

Find solutions for $f(x) = 0$ by factoring.

$$\begin{aligned} x^3 - 5x^2 - x + 5 &= 0 \quad \text{Factor by grouping.} \\ x^2(x-5) - (x-5) &= 0 \quad \text{Factor out the common factor.} \\ (x^2-1)(x-5) &= 0 \quad \text{Factor the difference of squares.} \\ (x+1)(x-1)(x-5) &= 0 \quad \text{Set each factor equal to zero.} \end{aligned}$$

$$\begin{aligned} x+1 = 0 \quad &\text{or} \quad x-1 = 0 \quad \text{or} \quad x-5 = 0 \\ x = -1 \quad &\quad\quad x = 1 \quad\quad\quad x = 5 \end{aligned}$$

There are three x-intercepts: $(-1, 0)$, $(1, 0)$, and $(5, 0)$.

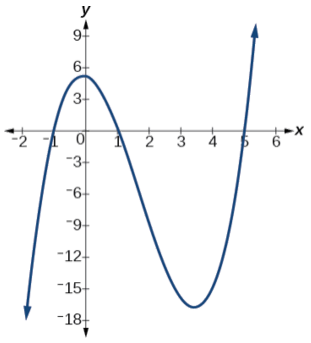


Figure 3.3.13: Graph of $f(x)$.

Try It 3.3.14

Given the polynomial function $f(x) = 2x^3 - 6x^2 - 20x$, determine the y- and x-intercepts. Find the leading term and describe the end behaviour.

Answer

y-intercept $(0, 0)$; x-intercepts $(0, 0)$,$(-2, 0)$, and $(5, 0)$. The leading term is $2x^3$. The curve rises to the right (positive coefficient) and falls to the left (odd degree so ends go in opposite directions): ↙ ⋯ ↗

Try It 3.3.15

Find the y- and x-intercepts of the function $f(x) = x^4 + 5x^3 - 4x^2 - 20x$. Find the leading term and describe the end behaviour.

Answer

- y-intercept $(0, 0)$;
- x-intercepts $(0, 0)$, $(-2, 0)$, $(2, 0)$, and $(-5, 0)$.
- The leading term is x^4, so the curve rises both on the left and on the right: ↖ ⋯ ↗

Example 3.3.16: Given a graph of a polynomial, find a possible degree for it

Describe the end behavior and characteristics of the leading term and degree of the polynomial function in the figure below

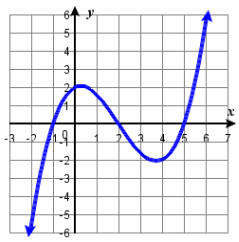


Figure **3.3.16**.

Solution

1. The graph rises to the right ⋯ ↗ so the l**eading coefficient must be positive**.
- As the input values x get very large, the output values $f(x)$ increase without bound.
- **As** $x \to \infty, f(x) \to \infty$
2. The ends go in opposite directions (the graph falls to the left) so the degree of the **polynomial must be odd**.
- As the input values x get very small, the output values $f(x)$ decrease without bound.
- **as** $x \to -\infty, f(x) \to -\infty$
3. Because there are 3 x-intercepts, the degree is at least three.

Try It 3.3.17

Describe the end behavior, and characteristics of the leading term and degree of the polynomial function in the Figure below.

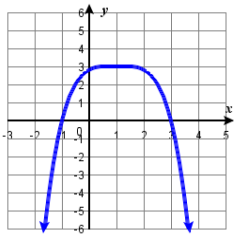


Figure TryIt **3.3.17**

Answer

1. The graph falls to the right ⋯ ↘ so **the leading coefficient must be negative.**
   - As $x \to \infty$, $f(x) \to -\infty$;
2. The ends go in the same direction (down) so **the degree of the polynomial must be even**.
   - As $x \to -\infty$, $f(x) \to -\infty$
3. Because there are 2 x-intercepts, the degree must be at least 2.

Key Equations

- general form of a polynomial function: $f(x) = a_n x^n + a_{n-1} x^{n-1} \ldots + a_2 x^2 + a_1 x + a_0$

Key Concepts

- A power function is a variable base raised to a number power.
- The behavior of a graph as the input decreases beyond bound and increases beyond bound is called the end behavior.
- The end behavior depends on whether the power is even or odd and the sign of the leading term.
- A polynomial function is the sum of terms, each of which consists of a transformed power function with positive whole number power.

- The degree of a polynomial function is the highest power of the variable that occurs in a polynomial. The term containing the highest power of the variable is called the leading term. The coefficient of the leading term is called the leading coefficient.
- The end behavior of a polynomial function is the same as the end behavior of the power function that corresponds to the leading term of the function.

Glossary

- **coefficient** a nonzero real number multiplied by a variable raised to an exponent
- **continuous function** a function whose graph can be drawn without lifting the pen from the paper because there are no breaks in the graph
- **degree** the highest power of the variable that occurs in a polynomial
- **end behavior** the behavior of the graph of a function as the input decreases without bound and increases without bound
- **leading coefficient** the coefficient of the leading term
- **leading term** the term containing the highest power of the variable
- **polynomial function** a function that consists of either zero or the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power.
- **power function** a function that can be represented in the form $f(x) = kx^p$ where k is a constant, the base is a variable, and the exponent, p, is a constant
- **smooth curve** a graph with no sharp corners
- **term of a polynomial function** any $a_i x^i$ of a polynomial function in the form $f(x) = a_n x^n + a_{n-1} x^{n-1} \ldots + a_2 x^2 + a_1 x + a_0$

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3.3e: Exercises - Polynomial End Behaviour

A: Concepts

Exercise 3.3e.A

1) Explain the difference between the coefficient of a power function and its degree.

2) If a polynomial function is in factored form, what would be a good first step in order to determine the degree of the function?

3) In general, explain the end behavior of a power function with odd degree if the leading coefficient is positive.

4) What can we conclude if, in general, the graph of a polynomial function exhibits the following end behavior? As $x \to -\infty, f(x) \to -\infty$ and as $x \to \infty, f(x) \to -\infty$.

Answers to odd exercises:

1. The coefficient of the power function is the real number that is multiplied by the variable raised to a power. The degree is the highest power appearing in the function.

3. As x decreases without bound, so does $f(x)$. As x increases without bound, so does $f(x)$.

B: Identify Power Functions and Polynomials

Exercise 3.3e.B

★ Identify the function as a power function, a polynomial function, or neither. If neither, explain.

5) $f(x) = (x^2)^3$ 6) $f(x) = x^5$	7) $f(x) = x - x^4$ 8) $f(x) = \dfrac{x^2}{x^2 - 1}$	9) $f(x) = 3^{x+1}$ 10) $f(x) = 2x(x+2)(x-1)^2$

Answers to odd exercises:

5. Power function, 7. Polynomial, 9. Neither - this is an exponential function because the variable is in the exponent

C: Degree and Leading Coefficient of a Polynomial

Exercise 3.3e.C

★ (a) Find the degree and leading coefficient for the given polynomial. (b) State the end behaviour.

11) $7 - 2x^2$ 12) $-3x^4$	13) $x(4 - x^2)(2x + 1)$ 14) $-2x^2 - 3x^5 + x - 6$	15) $x^2(2x - 3)^2$ 16) $-3(2x + 5)^2(1 - x)^3$

Answers to odd exercises:

11. Degree 2, Coefficient = -2 ↙ ⋯ ↘	13. Degree 4, Coefficient = -2 ↙ ⋯ ↘	15. Degree 4, Coefficient = 4 or ↖ ⋯ ↗

D: Polynomial End Behaviour

Exercise 3.3e.D

★ **Determine the end behavior of the functions.**

17) $f(x) = x^4$
18) $f(x) = x^3$
19) $f(x) = -x^4$
20) $f(x) = -x^9$
21) $f(x) = -2x^4 - 3x^2 + x - 1$
22) $f(x) = 3x^2 + x - 2$
23) $f(x) = x^2(2x^3 - x + 1)$
24(a) $f(x) = (2 - x)^7$
24(b) $f(x) = 3x(2x - 3)^4$
25) $f(x) = x^4 - 5x^2$
26) $f(x) = -x^3$
27) $f(x) = (x - 1)(x - 2)(3 - x)$
28) $f(x) = x^2(1 - x)^2$
29) $f(x) = \frac{x^5}{10} - x^4$

Answers to odd exercises:

17. As $x \to \pm\infty$, $f(x) \to \infty$, or $\nwarrow \cdots \nearrow$
19. As $x \to \pm\infty$, $f(x) \to -\infty$, or $\swarrow \cdots \searrow$
21. As $x \to \pm\infty$, $f(x) \to -\infty$, or $\swarrow \cdots \searrow$
23. As $x \to \infty$, $f(x) \to \infty$, as $x \to -\infty$, $f(x) \to -\infty$ or $\swarrow \cdots \nearrow$
25. As $x \to \pm\infty$, $f(x) \to \infty$, or $\nwarrow \cdots \nearrow$
27. as $x \to -\infty$, $f(x) \to \infty$, as $x \to \infty$, $f(x) \to -\infty$ or $\nwarrow \cdots \searrow$
29. As $x \to \infty$, $f(x) \to \infty$, as $x \to -\infty$, $f(x) \to -\infty$ or $\swarrow \cdots \nearrow$

E: Polynomial Intercepts

Exercise 3.3e.E

★ **Find the intercepts of the functions.**

30) $C(t) = 2(t - 4)(t + 1)(t - 6)$
31) $f(t) = 2(t - 1)(t + 2)(t - 3)$
32) $g(n) = -2(3n - 1)(2n + 1)$
33) $f(x) = x^4 - 16$
34) $f(x) = x^3 + 27$
35) $f(x) = x(x^2 - 2x - 8)$
36) $f(x) = (x + 3)(4x^2 - 1)$
37) $C(t) = 3(t + 2)(t - 3)(t + 5)$
38) $C(t) = 4t(t - 2)^2(t + 1)$
39) $C(t) = 2t(t - 3)(t + 1)^2$
40) $C(t) = 2t^4 - 8t^3 + 6t^2$
41) $C(t) = 4t^4 + 12t^3 - 40t^2$
42) $f(x) = x^4 - x^2$
43) $f(x) = x^3 + x^2 - 20x$
44) $f(x) = x^3 + 6x^2 - 7x$
45) $f(x) = x^3 + x^2 - 4x - 4$
46) $f(x) = x^3 + 2x^2 - 9x - 18$
47) $f(x) = 2x^3 - x^2 - 8x + 4$
48) $f(x) = x^6 - 7x^3 - 8$
49) $f(x) = 2x^4 + 6x^2 - 8$
50) $f(x) = x^3 - 3x^2 - x + 3$
51) $f(x) = x^6 - 2x^4 - 3x^2$
52) $f(x) = x^6 - 3x^4 - 4x^2$
53) $f(x) = x^5 - 5x^3 + 4x$

★ **Determine the intercepts and the end behavior.**

55) $f(x) = x^3(x - 2)$
56) $f(x) = x(x - 3)(x + 3)$
57) $f(x) = x(14 - 2x)(10 - 2x)$
58) $f(x) = x(14 - 2x)(10 - 2x)^2$
59) $f(x) = x^3 - 16x$
60) $f(x) = x^3 - 27$
61) $f(x) = x^4 - 81$
62) $f(x) = -x^3 + x^2 + 2x$
63) $f(x) = x^3 - 2x^2 - 15x$
64) $f(x) = x^3 - 0.01x$

Answers to odd exercises:

31. $(0, 12)$; $(1, 0)$, $(-2, 0)$, $(3, 0)$
33. $(0, -16)$; $(2, 0)$, $(-2, 0)$.
35. $(0, 0)$, $(4, 0)$, $(-2, 0)$.
37. $(0, 90)$; $(-2, 0)$, $(3, 0)$, $(-5, 0)$
39. $(0, 0)$, $(3, 0)$, $(-1, 0)$
41. $(0, 0)$, $(-5, 0)$, $(2, 0)$
43. $(0, 0)$, $(-5, 0)$, $(4, 0)$
45. $(0, -4)$; $(2, 0)$, $(-2, 0)$, $(-1, 0)$
47. $(0, 4)$; $(-2, 0)$, $(2, 0)$, $(\frac{1}{2}, 0)$
49. $(0, -8)$; $(1, 0)$, $(-1, 0)$
51. $(0, 0)$, $(\sqrt{3}, 0))$, $(-\sqrt{3}, 0)$
53. $(0, 0)$, $(1, 0)$, $(-1, 0)$, $(2, 0)$, $(-2, 0)$

55. $(0, 0)$; $(0, 0)$, $(2, 0)$. As $x \to \pm\infty$, $f(x) \to \infty$
57. $(0, 0)$; $(0, 0)$, $(5, 0)$, $(7, 0)$. As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to \infty$
59. $(0, 0)$; $(-4, 0)$, $(0, 0)$, $(4, 0)$. As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to \infty$
61. $(0, -81)$; $(3, 0)$, $(-3, 0)$. As $x \to \pm\infty$, $f(x) \to \infty$
63. $(0, 0)$; $(-3, 0)$, $(0, 0)$, $(5, 0)$. As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to \infty$

*

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3.4: Graphs of Polynomial Functions

Learning Objectives

- Identify zeros and their multiplicities.
- Use factoring to find zeros of polynomial functions.
- Understand the relationship between degree and turning points.
- Graph polynomial functions.

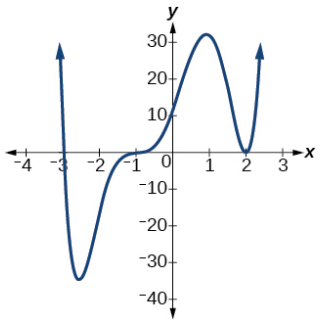
We have already explored the local behavior (the location of x- and y-intercepts) for quadratics, a special case of polynomials. In this section we will explore the local behavior of polynomials in general.

Identify Zeros and Their Multiplicities from a Graph

Graphs behave differently at various x-intercepts. Sometimes, the graph will cross over the horizontal axis at an intercept. Other times, the graph will touch the horizontal axis and bounce off. Suppose, for example, we graph the function

$$f(x) = (x+3)(x-2)^2(x+1)^3$$

Notice in the figure to the right illustrates that the behavior of this function at each of the x-intercepts is different.



The x-intercept -3 is the solution of equation $(x+3) = 0$. The graph passes directly through the x-intercept at $x = -3$. The factor is linear (has a degree of 1), so the behavior near the intercept is like that of a line—it passes directly through the intercept. We call this a single zero because the zero corresponds to a single factor of the function.

The x-intercept 2 is the repeated solution of equation $(x-2)^2 = 0$. The graph touches the axis at the intercept and changes direction. The factor is quadratic (degree 2), so the behavior near the intercept is like that of a quadratic—it bounces off of the horizontal axis at the intercept. The factor is repeated, that is, $(x-2)^2 = (x-2)(x-2)$, so the solution, $x = 2$, appears twice. The number of times a given factor appears in the factored form of the equation of a polynomial is called the **multiplicity**. The zero associated with this factor, $x = 2$, has multiplicity 2 because the factor $(x-2)$ occurs twice.

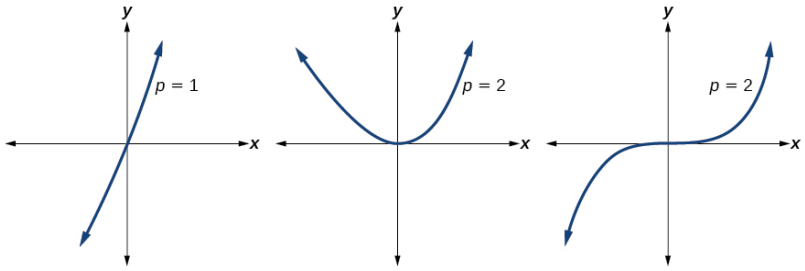
The x-intercept -1 is the repeated solution of factor $(x+1)^3 = 0$. The graph passes through the axis at the intercept, but flattens out a bit first. This factor is cubic (degree 3), so the behavior near the intercept is like that of a cubic—with the same S-shape near the intercept as the toolkit function $f(x) = x^3$. We call this a triple zero, or a zero with multiplicity 3.

Graphical Behavior of Polynomials at x-intercepts

If a polynomial contains a factor of the form $(x-h)^p$, the behavior near the x-intercept is determined by the power p. We say that $x = h$ is a zero of **multiplicity** p.

- The graph of a polynomial function will touch the x-axis at zeros with even multiplicities.
- The graph will cross the x-axis at zeros with odd multiplicities.
- The higher the multiplicity, the flatter the curve is at the zero.
- The sum of the multiplicities is the degree of the polynomial function.

For **zeros** with even multiplicities, the graphs touch or are tangent to the x-axis. For zeros with odd multiplicities, the graphs cross or intersect the x-axis. See the figure below for examples of graphs of polynomial functions with a zero of multiplicity 1, 2, and 3. The graphs clearly show that the higher the multiplicity, the flatter the graph is at the zero.



Single zero | Zero with multiplicity 2 | Zero with multiplicity 3

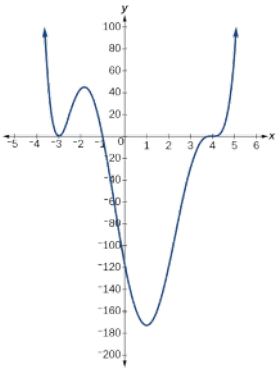
For higher even powers, such as 4, 6, and 8, the graph will still touch and bounce off of the horizontal axis but, for each increasing even power, the graph will appear flatter as it approaches and leaves the x-axis. For higher odd powers, such as 5, 7, and 9, the graph will still cross through the horizontal axis, but for each increasing odd power, the graph will appear flatter as it approaches and leaves the x-axis.

How to: Given a graph of a polynomial function, identify the zeros and their mulitplicities

1. If the graph crosses the x-axis at a zero, it is a zero with odd multiplicity.
2. If the graph touches and bounces off of the x-axis, it is a zero with even multiplicity.
3. The higher the multiplicity of the zero, the flatter the graph gets at the zero.

Example 3.4.1: Find Zeros and Their Multiplicities From a Graph

Use the graph of the function of degree 6 in the figure below to identify the zeros of the function and their possible multiplicities.



Graph of a polynomial function with degree 6.

Solution

Starting from the left, the first zero occurs at $x = -3$. The graph touches the x-axis, so the multiplicity of the zero must be even. The zero of −3 has multiplicity 2.

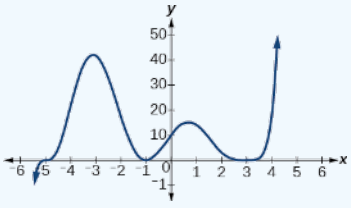
The next zero occurs at $x = -1$. The graph looks almost linear at this point. This is a single zero of multiplicity 1.

The last zero occurs at $x = 4$. The graph crosses the x-axis, so the multiplicity of the zero must be odd. Since the graph is flat around this zero, the multiplicity is likely 3 (rather than 1).

The polynomial function is of degree 6 so the sum of the multiplicities must be at least $2 + 1 + 3$ or 6.

Try It 3.4.2

Use the graph of the function in the figure below to identify the zeros of the function and their possible multiplicities.



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Graph of a polynomial function.

Answer

The zero at -5 is odd. Since the curve is somewhat flat at -5, the zero likely has a multiplicity of 3 rather than 1.
The zero at -1 has even multiplicity of 2.
The zero at 3 has even multiplicity. Since the curve is flatter at 3 than at -1, the zero more likely has a multiplicity of 4 rather than 2.

Find Zeros and their Multiplicities from a Polynomial Equation

Recall that if f is a polynomial function, the values of x for which $f(x) = 0$ are called **zeros** of f. If the equation of the polynomial function can be factored, we can set each factor equal to zero and solve for the zeros.

How to: Given an equation of a polynomial function, identify the zeros and their multiplicities

1. Factor the polynomial as a product of linear factors (of the form $(ax + b)$), and irreducible quadratic factors (of the form $(ax^2 + bx + c)$. When irreducible quadratic factors are set to zero and solved for x, imaginary solutions are produced. Write each repeated factor in exponential form.
2. If the exponent on a linear factor is even, its corresponding zero has even multiplicity equal to the value of the exponent and the graph will touch the x-axis and turn around at this zero.
3. If the exponent on a linear factor is odd, its corresponding zero has odd multiplicity equal to the value of the exponent, and the graph will cross the x-axis at this zero.
4. The sum of the multiplicities plus the number of imaginary zeros is equal to the degree of the polynomial.

Example 3.4.3: Find zeros and their multiplicity from a factored polynomial

Find the zeros and their multiplicity for the following polynomial functions.

a) $f(x) = x(x + 1)^2(x + 2)^3$

b) $f(x) = x^2(x^2 - 3x)(x^2 + 4)(x^2 - x - 6)(x^2 - 7)$

Solution.

a) This polynomial is already in factored form. All factors are linear factors.

- Starting from the left, the first factor is x, so a zero occurs at $x = 0$. The exponent on this factor is 1 which is an odd number. Therefore the zero of 0 has odd multiplicity of 1, and the graph will cross the x-axis at this zero.
- The next factor is $(x + 1)^2$, so a zero occurs at $x = -1$. The exponent on this factor is 2 which is an even number. Therefore the zero of -1 has even multiplicity of 2, and the graph will touch and turn around at this zero.
- The last factor is $(x + 2)^3$, so a zero occurs at $x = -2$. The exponent on this factor is 3 which is an odd number. Therefore the zero of -2 has odd multiplicity of 3, and the graph will cross the x-axis at this zero.

b) This polynomial is partly factored. All the zeros can be found by setting each factor to zero and solving

- The factor $x^2 = x \cdot x$ which when set to zero produces two identical solutions, $x = 0$ and $x = 0$
- The factor $(x^2 - 3x) = x(x - 3)$ when set to zero produces two solutions, $x = 0$ and $x = 3$
- The factor $(x^2 + 4)$ when set to zero produces two imaginary solutions, $x = 2i$ and $x = -2i$. Since these solutions are imaginary, this factor is said to be an irreducible quadratic factor.
- The factor $(x^2 - x - 6) = (x - 3)(x + 2)$ when set to zero produces two solutions, $x = 3$ and $x = -2$
- The factor $(x^2 - 7)$ when set to zero produces two irrational solutions, $x = \pm\sqrt{7}$

Now we need to count the number of occurrences of each zero thereby determining the multiplicity of each real number zero.

- The solution $x = 0$ occurs 3 times so the zero of 0 has multiplicity 3 or odd multiplicity.
- The solution $x = 3$ occurs 2 times so the zero of 3 has multiplicity 2 or even multiplicity.

- The real number solutions $x = -2$, $x = \sqrt{7}$ and $x = -\sqrt{7}$ each occur 1 time so these zeros have multiplicity 1 or odd multiplicity.
- The imaginary solutions $x = 2i$ and $x = -2i$ each occur 1 time so these zeros have multiplicity 1 or odd multiplicity but since these are imaginary numbers, they are not x-intercepts.

Try It 3.4.4

Find the zeros and their multiplicity for the polynomial $f(x) = x^4 - x^3 - x^2 + x$.

Answer

Zeros -1 and 0 have odd multiplicity of 1. Zero 1 has even multiplicity of 2

Graph Polynomial Functions

We can use what we have learned about multiplicities, end behavior, and intercepts to sketch graphs of polynomial functions. Let us put this all together and look at the steps required to graph polynomial functions.

How to: Given a polynomial function, sketch the graph

1. Determine the end behavior by examining the leading term.
2. Find the intercepts and use the multiplicities of the zeros to determine the behavior of the polynomial at the x-intercepts.
3. Use the end behavior and the behavior at the intercepts to sketch a graph.
4. Optionally ...
   - Check for symmetry. If the function is an even function, its graph is symmetrical about the y-axis, that is, $f(-x) = f(x)$. If a function is an odd function, its graph is symmetrical about the origin, that is, $f(-x) = -f(x)$.
   - Ensure that the number of turning points does not exceed one less than the degree of the polynomial.
   - Use technology to check the graph.

Example 3.4.5: Sketch the Graph of a Polynomial Function

Sketch a graph of $f(x) = -2(x+3)^2(x-5)$.

Solution

Step 1. The leading term, if this polynomial were multiplied out, would be $-2x^3$, so the end behavior is that of a vertically reflected cubic, with the the graph falling to the right and going in the opposite direction (up) on the left: ↖ ⋯ ↘ See Figure 3.4.5*a*.

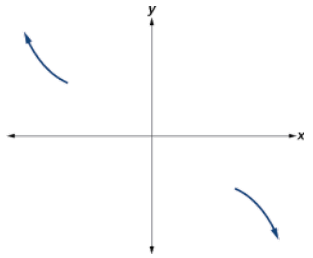


Figure 3.4.5*a*: Illustration of the end behaviour of the polynomial.

Step 2.
This graph has two x-intercepts. At $x = -3$, the factor is squared, indicating a multiplicity of 2. The graph will bounce at this x-intercept. At $x = 5$, the function has a multiplicity of one, indicating the graph will cross through the axis at this intercept.
The y-intercept is found by evaluating $f(0)$.

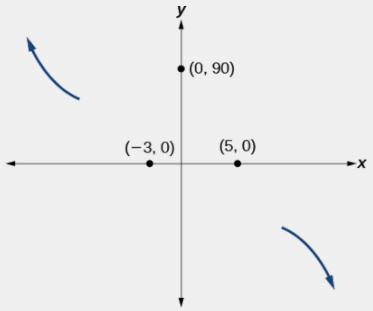


Figure 3.4.5*b*: The graph crosses at x-intercept $(5, 0)$ and bounces at $(-3, 0)$. The y-intercept is $(0, 90)$.

Step 3. Connect the end behaviour lines with the intercepts.

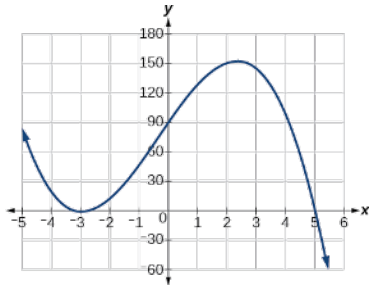


Figure 3.4.5*c*: The complete graph of the polynomial function $f(x) = -2(x+3)^2(x-5)$.

Example 3.4.6

Sketch a graph of $f(x) = x^2(x^2-1)(x^2-2)$. State the end behaviour, the y-intercept, and x-intercepts and their multiplicity.

Solution

Step 1. The leading term is the product of the high order terms of each factor: $(x^2)(x^2)(x^2) = x^6$.
The leading term is positive so the curve rises on the right.
The degree of the leading term is even, so both ends of the graph go in the same direction (up). ↖ ⋯ ↗

Step 2. The y-intercept can be found by evaluating $f(0)$. So $f(0) = 0^2(0^2-1)(0^2-2) = (0)(-1)(-2) = 0$.

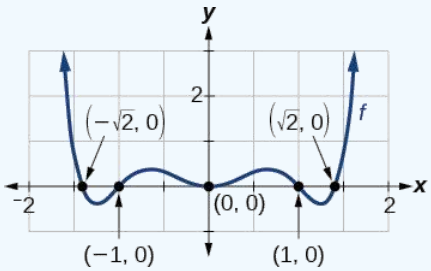
The x-intercepts can be found by solving $f(x) = 0$. Set each factor equal to zero.

$$\begin{array}{lll} x^2 = 0 & (x^2-1) = 0 & (x^2-2) = 0 \\ x^2 = 0 \quad \text{or} & x^2 = 1 \quad \text{or} & x^2 = 2 \\ x = 0,\ x = 0 & x = \pm 1 & x = \pm\sqrt{2} \end{array}$$

.

This gives us five x-intercepts: $(0,0)$, $(1,0)$, $(-1,0)$, $(\sqrt{2},0)$, and $(-\sqrt{2},0)$.
The x-intercepts $(1,0)$, $(-1,0)$, $(\sqrt{2},0)$, and $(-\sqrt{2},0)$ all have odd multiplicity of 1, so the graph will cross the x-axis at those intercepts.
The x-intercept $(0,0)$ has even multiplicity of 2, so the graph will stay on the same side of the x-axis at 2. (The graph is said to be tangent to the x- axis at 2 or to "bounce" off the x-axis at 2).

Step 3. The polynomial is an even function because $f(-x) = f(x)$, so the graph is symmetric about the y-axis. The graph appears below.



Graph of $f(x) = x^2(x^2-1)(x^2-2)$.

Example 3.4.7

Sketch a graph of the polynomial function $f(x) = x^4 - 4x^2 - 45$. State the end behaviour, the y-intercept, and x-intercepts and their multiplicity.

Solution

Step 1. The leading term is x^4.
The leading term is positive so the curve rises on the right.
The degree of the leading term is even, so both ends of the graph go in the same direction (up). ↖ ··· ↗

Step 2. The y-intercept occurs when the input is zero.

$$f(0) = (0)^4 - 4(0)^2 - 45 = -45$$

The y-intercept is $(0,-45)$.

The x-intercepts occur when the output is zero. To determine when the output is zero, we will need to factor the polynomial.

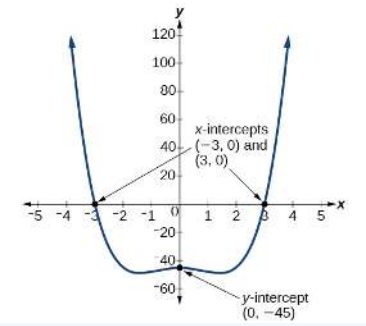
$$\begin{aligned} f(x) &= x^4 - 4x^2 - 45 \\ &= (x^2-9)(x^2+5) \\ &= (x-3)(x+3)(x^2+5) \end{aligned}$$

$$0 = (x-3)(x+3)(x^2+5)$$

$$\begin{array}{ccccccc} x-3=0 & \text{or} & x+3=0 & \text{or} & x^2+5=0 \\ x=3 & \text{or} & x=-3 & \text{or} & \text{(no real solution)} \end{array}$$

The x-intercepts $(3,0)$ and $(-3,0)$ all have odd multiplicity of 1, so the graph will cross the x-axis at those intercepts.

Step 3. The polynomial is an even function because $f(-x) = f(x)$, so the graph is symmetric about the y-axis. The graph appears below. The imaginary zeros are not x-intercepts, but the graph below shows they do contribute to "wiggles" (truning points) in the graph of the function.



Graph of $f(x) = x^4 - 4x^2 - 45$.

Try It 3.4.8

Sketch a graph of $f(x) = \frac{1}{6}(x-1)^3(x+3)(x+2)$.

Answer

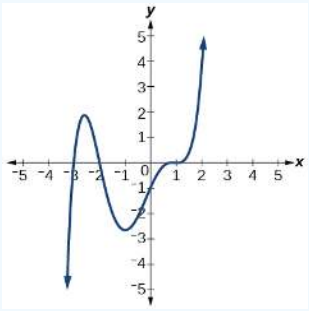


Figure 3.4.8: Graph of $f(x) = \frac{1}{6}(x-1)^3(x+3)(x+2)$

Turning Points

Definition: Turning Points

A turning point is a point of the graph where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising).

A polynomial of degree n will have, at most, n x-intercepts and $n-1$ turning points.

The degree of a polynomial function helps us to determine the number of x-intercepts and the number of turning points. A polynomial function of nth degree is the product of n factors, so it will have at most n roots or zeros. When the zeros are real numbers, they appear on the graph as x-intercepts. The graph of the polynomial function of degree n can have at most n–1 turning points. This means the graph has at most one fewer turning points than the degree of the polynomial or one fewer than the number of factors.

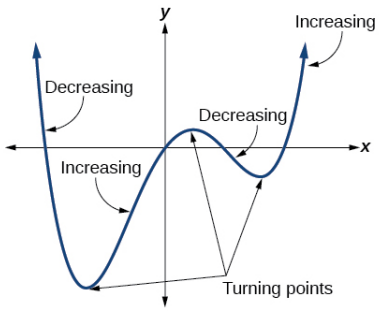


Figure 3.4.9: Graph of $f(x) = x^4 - x^3 - 4x^2 + 4x$, a 4th degree polynomial function with 3 turning points

The maximum number of turning points of a polynomial function is always one less than the degree of the function.

Example 3.4.9: Find the Maximum Number of Turning Points of a Polynomial Function

Find the maximum number of turning points of each polynomial function.

a. $f(x) = -x^3 + 4x^5 - 3x^2 + 1$
b. $f(x) = -(x-1)^2(1+2x^2)$

Solution

a. $f(x) = -x^3 + 4x^5 - 3x^2 + 1$

First, rewrite the polynomial function in descending order: $f(x) = 4x^5 - x^3 - 3x^2 + 1$

Identify the degree of the polynomial function. This polynomial function is of degree 5.

The maximum number of turning points is $5 - 1 = 4$.

b. $f(x) = -(x-1)^2(1+2x^2)$

First, identify the leading term of the polynomial function if the function were expanded: multiply the leading terms in each factor together.

$$\begin{aligned} f(x) &= -(x-1)^2(1+2x^2) \\ &= -1(x-1)^2(1+2x^2) \\ \textbf{High order term} &= -1(x)^2(2x^2) \\ &= -2x^4 \end{aligned}$$

Then, identify the degree of the polynomial function. This polynomial function is of degree 4.

The maximum number of turning points is $4 - 1 = 3$.

Example 3.4.10: Find the Maximum Number of Intercepts and Turning Points of a Polynomial

Without graphing the function, determine the maximum number of x-intercepts and turning points for $f(x) = -3x^{10} + 4x^7 - x^4 + 2x^3$.

Solution

The polynomial has a degree of n=**10**, so there are at most 10 x-intercepts and at most 9 turning points.

Try It 3.4.11

Without graphing the function, determine the maximum number of x-intercepts and turning points for $f(x) = 108 - 13x^9 - 8x^4 + 14x^{12} + 2x^3$

Answer

There are at most 12 x-intercepts and at most 11 turning points.

Example 3.4.12: Drawing Conclusions about a Polynomial Function from the Factors

Given the function $f(x) = -4x(x+3)(x-4)$, determine the y-intercept and the number, location and multiplicity of x-intercepts, and the maximum number of turning points.

Solution

The y-intercept is found by evaluating $f(0)$.

$$f(0) = -4(0)(0+3)(0-4) = 0$$

The y-intercept is $(0,0)$.

The x-intercepts are found by determining the zeros of the function.

$$\begin{aligned} 0 &= -4x(x+3)(x-4) \\ x = 0 \quad &\text{or} \quad x+3=0 \quad \text{or} \quad x-4=0 \\ x = 0 \quad &\text{or} \quad x=-3 \quad \text{or} \quad x=4 \end{aligned}$$

The three x-intercepts $(0,0)$,$(-3,0)$, and $(4,0)$ all have odd multiplicity of 1.

The degree is 3 so the graph has at most 2 turning points.

Try It 3.4.13

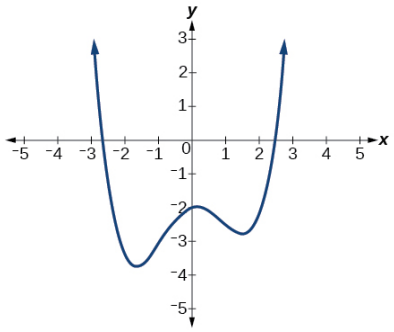
Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$, determine the local behavior.

Answer

The function is a 3rd degree polynomial with three x-intercepts $(2,0)$, $(-1,0)$, and $(5,0)$ all have multiplicity of 1, the y-intercept is $(0,2)$, and the graph has at most 2 turning points.

Example 3.4.14: Drawing Conclusions about a Polynomial Function from the Graph

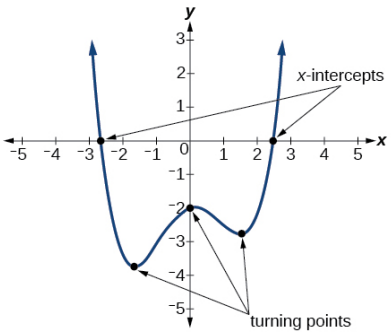
What can we conclude about the degree of the polynomial and the leading coefficient represented by the graph shown below based on its intercepts and turning points?



Graph of a Polynomial.

Solution

- The end behavior of the graph tells us this is the graph of an even-degree polynomial (ends go in the same direction), with a positive leading coefficient (rises right).
- The graph has 2 x-intercepts each with odd multiplicity, suggesting a degree of 2 or greater.
- The graph has 3 turning points, suggesting a degree of 4 or greater.

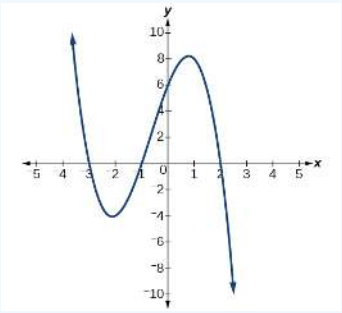


Graph of an even-degree polynomial.

Conclusion: the degree of the polynomial is even and at least 4.

Try It 3.4.15

What can we conclude about the polynomial represented by the graph shown below based on its intercepts and turning points?



A polynomial function

Answer

The end behavior indicates an odd-degree polynomial function (ends in opposite direction), with a negative leading coefficient (falls right). There are 3 x-intercepts each with odd multiplicity, and 2 turning points, so the degree is odd and at least 3.

Write a Formula for a Polynomial given its Graph

Now that we know how to find zeros of polynomial functions, we can use them to write formulas based on graphs. Because a **polynomial function** written in factored form will have an x-intercept where each factor is equal to zero, we can form a function that will pass through a set of x-intercepts by introducing a corresponding set of factors.

Note: Factored Form of Polynomials

If a polynomial of lowest degree p has horizontal intercepts at $x = x_1, x_2, \ldots, x_n$, then the polynomial can be written in the factored form: $f(x) = a(x - x_1)^{p_1}(x - x_2)^{p_2} \cdots (x - x_n)^{p_n}$ where the powers p_i on each factor can be determined by the behavior of the graph at the corresponding intercept, and the stretch factor a can be determined given a value of the function other than the x-intercept.

How to: Given a graph of a polynomial function, write a formula for the function.

1. Identify the x-intercepts of the graph to find the factors of the polynomial.
2. Examine the behavior of the graph at the x-intercepts to determine the multiplicity of each factor.
3. Find the polynomial of least degree containing all the factors found in the previous step.
4. Use any other point on the graph (the y-intercept may be easiest) to determine the stretch factor.

Example 3.4.16: Writing a Formula for a Polynomial Function from the Graph

Construct the factored form of a possible equation for each graph given below.

(a)

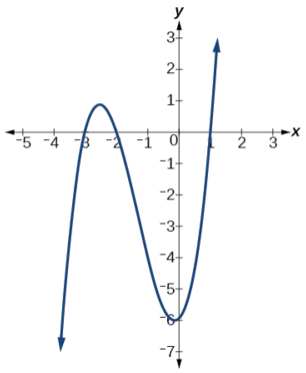


Figure 3.4.16a: Graph of $h(x)$.

Solution

Looking at the graph of this function, as shown in Figure 3.4.16, it appears that there are x-intercepts at $x = -3, -2,$ and 1.

Each x-intercept corresponds to a zero of the polynomial function and each zero yields a factor, so we can now write the polynomial in factored form.

$$h(x) = a(x+3)(x+2)(x-1)$$

The stretch factor a can be found by using another point on the graph, like the y-intercept, $(0, -6)$.

$$\begin{aligned} f(0) &= a(0+3)(0+2)(0-1) \\ -6 &= a(-6) \\ a &= 1 \end{aligned}$$

Thus $h(x) = (x+3)(x+2)(x-1)$.

(b)

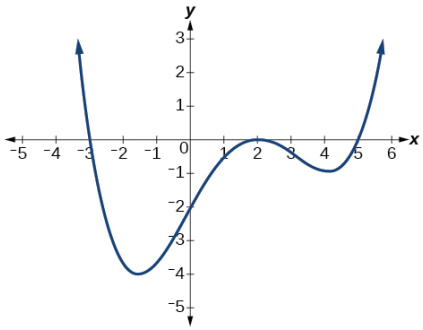


Figure 3.4.16b.

Solution

This graph has three x-intercepts: $x = -3,\ 2,$ and 5. The y-intercept is located at $(0, 2)$. At $x = -3$ and $x = 5$, the graph passes through the axis linearly, suggesting the corresponding factors of the polynomial will be linear. At $x = 2$, the graph bounces at the intercept, suggesting the corresponding factor of the polynomial will be second degree (quadratic). Together, this gives us

$$f(x) = a(x+3)(x-2)^2(x-5) \qquad (3.4.1)$$

To determine the stretch factor, we utilize another point on the graph. We will use the y-intercept $(0, -2)$, to solve for a.

$$\begin{aligned} f(0) &= a(0+3)(0-2)^2(0-5) \\ -2 &= a(0+3)(0-2)^2(0-5) \\ -2 &= -60a \\ a &= \frac{1}{30} \end{aligned}$$

The graphed polynomial appears to represent the function $f(x) = \frac{1}{30}(x+3)(x-2)^2(x-5)$.

Try It 3.4.17: Construct a formula for a polynomial given a graph

Given the graph shown in Figure 3.4.21, write a formula for the function shown.

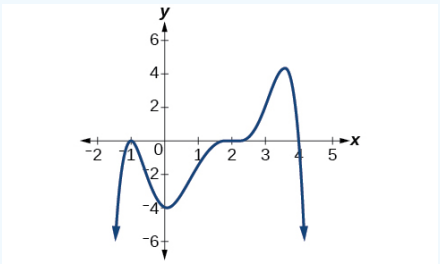


Figure 3.4.18.

Answer

https://math.libretexts.org/@go/page/34892

$$f(x) = -\frac{1}{8}(x-2)^3(x+1)^2(x-4)$$

Try It 3.4.18: Construct a formula for a polynomial given a description

Write a formula for a polynomial of degree 5, with zeros of multiplicity 2 at $x = 3$ and $x = 1$, a zero of multiplicity 1 at $x = -3$, and vertical intercept at (0, 9)

Answer

$$f(x) = \frac{1}{3}(x-1)^2(x-3)^2(x+3)$$

Key Concepts

- Polynomial functions of degree 2 or more are smooth, continuous functions.
- To find the zeros of a polynomial function, if it can be factored, factor the function and set each factor equal to zero.
- Another way to find the x-intercepts of a polynomial function is to graph the function and identify the points at which the graph crosses the x-axis.
- The multiplicity of a zero determines how the graph behaves at the x-intercepts.
- The graph of a polynomial will cross the horizontal axis at a zero with odd multiplicity.
- The graph of a polynomial will touch the horizontal axis at a zero with even multiplicity.
- The end behavior of a polynomial function depends on the leading term.
- The graph of a polynomial function changes direction at its turning points.
- A polynomial function of degree n has at most $n-1$ turning points.
- To graph polynomial functions, find the zeros and their multiplicities, determine the end behavior, and ensure that the final graph has at most $n-1$ turning points.

Glossary

multiplicity
the number of times a given factor appears in the factored form of the equation of a polynomial; if a polynomial contains a factor of the form $(x-h)^p$, $x = h$ is a zero of multiplicity p.

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3.4e: Exercises - Polynomial Graphs

A: Concepts

Exercise 3.4e.A

1) What is the difference between an x-intercept and a zero of a polynomial function f?

2) If a polynomial function of degree n has n distinct zeros, what do you know about the graph of the function?

3) What is the relationship between the degree of a polynomial function and the maximum number of turning points in its graph? .

4) Explain how the factored form of the polynomial helps us in graphing it.

5) If the graph of a polynomial just touches the x-axis and then changes direction, what can we conclude about the factored form of the polynomial?

Answers to odd exercises:

1. The x-intercept is where the graph of the function crosses the x-axis, and the zero of the function is the input value for which $f(x) = 0$.

3. The maximum number of turning points is one less than the degree of the polynomial.

5. There will be a factor raised to an even power.

B: Multiplicity from an Equation

Exercise 3.4e.B

★ Find the zeros and give the multiplicity of each.

6) $f(x) = (x+2)^3(x-3)^2$
7) $f(x) = x^2(2x+3)^5(x-4)^2$
8) $f(x) = x^3(x-1)^3(x+2)$
9) $f(x) = x^2(x^2+4x+4)$
10) $f(x) = (2x+1)^3(9x^2-6x+1)$
11) $f(x) = (3x+2)^5(x^2-10x+25)$
12) $f(x) = x(4x^2-12x+9)(x^2+8x+16)$
13) $f(x) = x^6 - x^5 - 2x^4$
14) $f(x) = 3x^4 + 6x^3 + 3x^2$
15) $f(x) = 4x^5 - 12x^4 + 9x^3$
16) $f(x) = 2x^4(x^3-4x^2+4x)$
17) $f(x) = 4x^4(9x^4-12x^3+4x^2)$

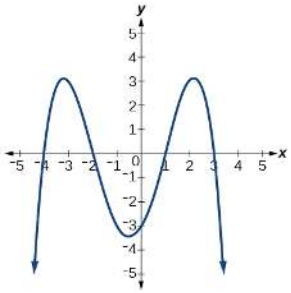
Answers to odd exercises:

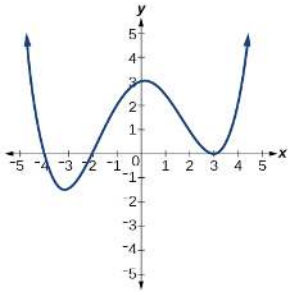
7. 0 and 4 with multiplicity 2, $-\frac{3}{2}$ with multiplicity 5
9. 0 with multiplicity 2, -2 with multiplicity 2
11. $-\frac{2}{3}$ with multiplicity 5, 5 with multiplicity 2
13. 0 with multiplicity 4, 2 and -1 with multiplicity 1
15. $\frac{3}{2}$ with multiplicity 2, 0 with multiplicity 3
17. 0 with multiplicity 6, $\frac{2}{3}$ with multiplicity 2

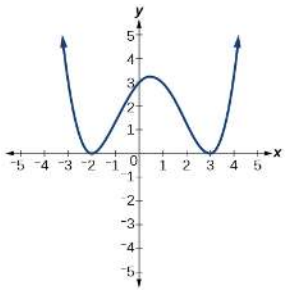
C: Multiplicity from a Graph

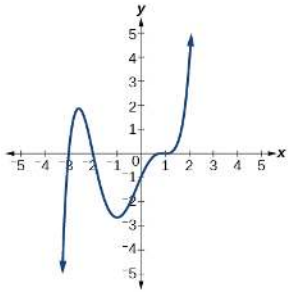
Exercise 3.4e.C

★ **Use the graph to identify zeros and multiplicity.**

19) 

20) 

21) 

22) 

Answers to odd exercises:

19. $-4, -2, 1, 3$ with multiplicity 1	21. $-2, 3$ each with multiplicity 2

D: Graph polynomials

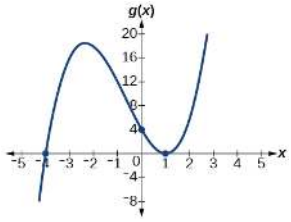
Exercise $3.4e.D$

★ **Graph the polynomial functions. State the x- and y- intercepts, multiplicity, and end behavior.**

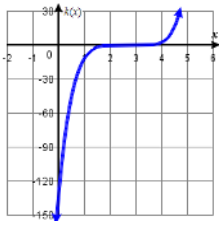
24) $f(x) = (x+3)^2(x-2)$	29) $n(x) = -3x(x+2)(x-4)$	35. $P(x) = (x-1)(x-2)(x-3)(x-4)$
25) $g(x) = (x+4)(x-1)^2$	30. $a(x) = x(x+2)^2$	36. $q(x) = (x+5)^2(x-3)^4$
26) $h(x) = (x-1)^3(x+3)^2$	31. $g(x) = x(x+2)^3$	37. $h(x) = x^2(x-2)^2(x+2)^2$
27) $k(x) = (x-3)^3(x-2)^2$	32. $f(x) = -2(x-2)^2(x+1)$	38. $h(t) = (3-t)(t^2+1)$
28) $m(x) = -2x(x-1)(x+3)$	33. $g(x) = (2x+1)^2(x-3)$	39. $Z(b) = b(42-b^2)$
	34. $f(x) = x^3(x+2)^2$	

Answers to odd exercises:

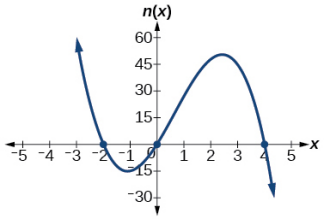
25. x-intercepts, $(1,0)$ with multiplicity 2, $(-4,0)$ with multiplicity 1, y- intercept $(0,4)$. As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to \infty$.



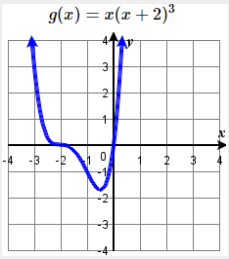
27. x-intercepts $(3,0)$ with multiplicity 3, $(2,0)$ with multiplicity 2, y- intercept $(0,-108)$. As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to \infty$.



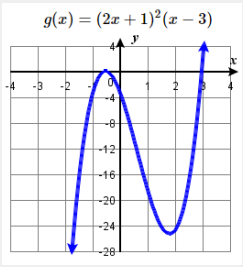
29. x-intercepts $(0,0),(-2,0),(4,0)$ with multiplicity 1, y-intercept (0, 0). As $x \to -\infty$, $f(x) \to \infty$, as $x \to \infty$, $f(x) \to -\infty$.



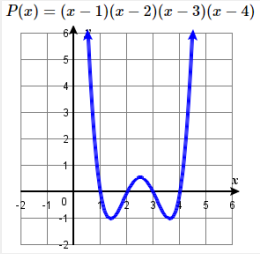
31. $(-2,0)$ multiplicity 3,
$(0,0)$ multiplicity 1,
y-intercept $(0,0)$,
end behaviour: ↖ ··· ↗



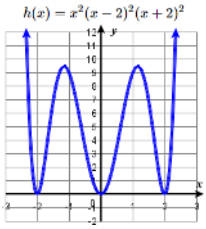
33. $(-\frac{1}{2},0)$ multiplicity 2,
$(3,0)$ multiplicity 1,
y-intercept $(0,-3)$,
end behaviour: ↙ ··· ↗



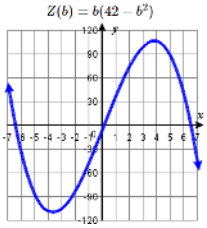
35. $(1,0)$, $(2,0)$, $(3,0)$, $(4,0)$
all multiplicity 1,
y-intercept $(0,24)$,
end behaviour: ↖ ··· ↗



37. $(-2,0)$, $(2,0)$, $(0,0)$
all multiplicity 2, y-intercept $(0,0)$,
end behaviour: ↖ ··· ↗



39. $(\sqrt{42},0)$, $(-\sqrt{42},0)$, $(0,0)$
all multiplicity 1, y-intercept $(0,0)$,
end behaviour: ↖ ··· ↘

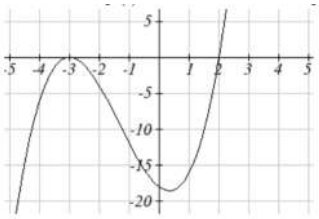


★ Graph the polynomial functions. State the x- and y- intercepts, multiplicity, and end behavior.

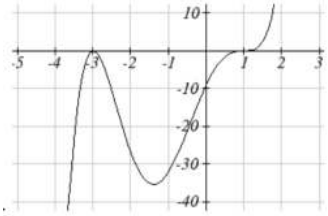
41. $f(x) = (x+3)^2(x-2)$
42. $g(x) = (x+4)(x-1)^2$
43. $h(x) = (x-1)^3(x+3)^2$
44. $k(x) = (x-3)^3(x-2)^2$
45. $m(x) = -2x(x-1)(x+3)$
46. $n(x) = -3x(x+2)(x-4)$
47. $f(x) = 9x - x^3$
48. $f(x) = 8 + x^3$
49. $f(x) = x^4 - 25x^2$
50. $f(x) = 16 - x^4$
51. $f(x) = -x^4 + 2x^3 + 8x^2$
52. $f(x) = x^3 + 7x^2 - 9x$
53. $f(x) = 2x^3 + 12x^2 - 8x - 48$
54. $f(x) = 4x^4 + 10x^3 - 4x^2 - 10x$

Answers to odd exercises:

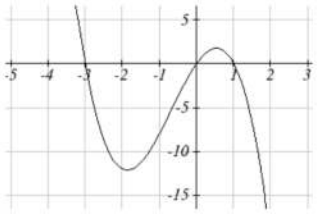
41. $(-3,0)$ multiplicity 2,
$(2,0)$ multiplicity 1,
y-intercept$(0,-18)$, ↙ ··· ↗



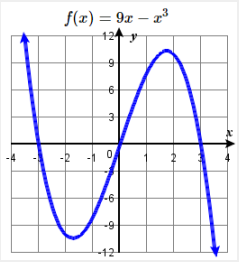
43. $(-3,0)$ multiplicity 2,
$(1,0)$ multiplicity 3,
y-intercept$(0,-9)$, ↙ ··· ↗



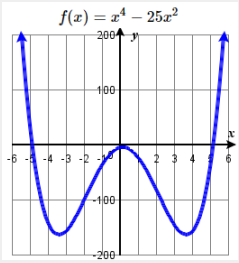
45. $(-3,0)$, $(0,0)$, $(1,0)$
all multiplicity 1,
y-intercept$(0,0)$, ↖ ··· ↘



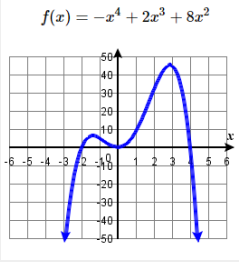
47. $(-3,0)$, $(0,0)$, $(3,0)$
all multiplicity 1,
y-intercept $(0,0)$, ↖ ··· ↘



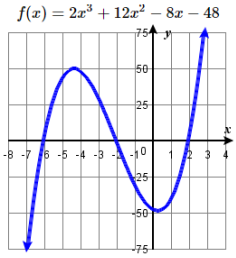
49. $(-5,0)$, $(5,0)$ both multiplicity 1,
$(0,0)$ multiplicity 2,
y-intercept $(0,0)$, ↖ ··· ↗



51. $(-2,0)$, $(4,0)$ both multiplicity 1,
$(0,0)$ multiplicity 2,
y-intercept $(0,0)$, ↙ ··· ↘



53. $(-6,0)$, $(-2,0)$, $(2,0)$
all multiplicity 1,
y-intercept $(0,-48)$, ↙ ··· ↗

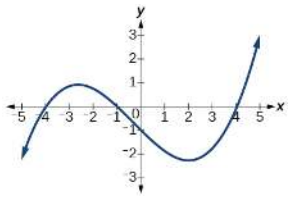


E: Polynomial Degree from a Graph

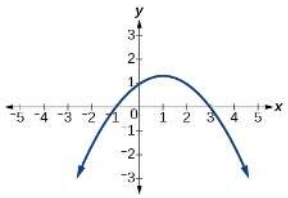
Exercise $3.4e.E$

★ **Determine the least possible degree of the polynomial function shown.**

61)



62)



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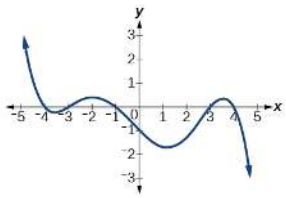
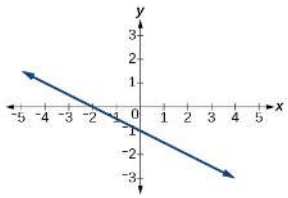
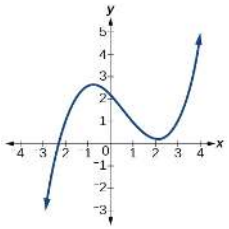


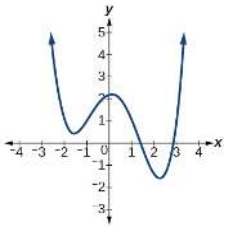
64)



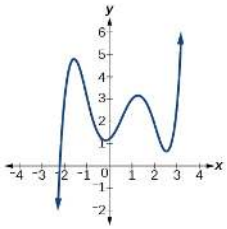
65)



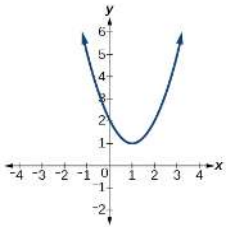
66)



67)



68)



Answers to odd exercises:

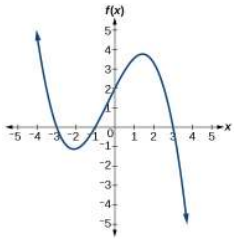
61. 3, 63. 5, 65. 3, 67. 5

F: Construct an Equation from a graph

Exercise 3.4e.F

★ **Use the graphs to write the formula for the polynomial function of least degree.**

69)



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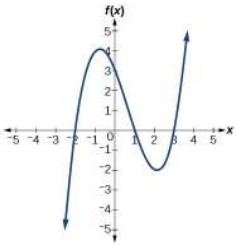
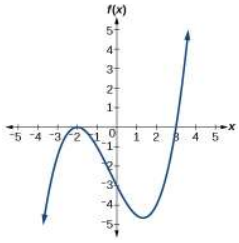
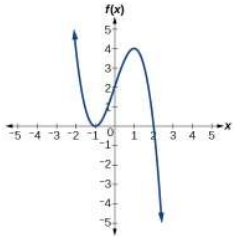


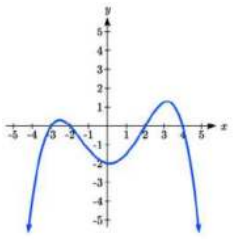
71)



72)



73.



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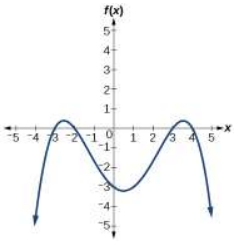
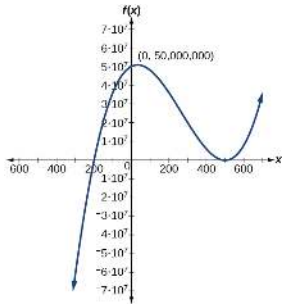


Answers to odd exercises:

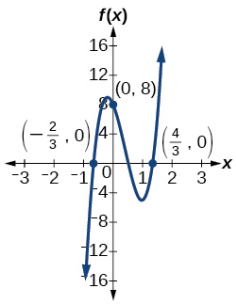
69. $f(x) = -(x+3)(x+1)(x-3)$ or $f(x) = -\frac{2}{9}(x+3)(x+1)(x-3)$
71. $f(x) = (x+2)^2(x-3)$ or $f(x) = \frac{1}{4}(x+2)^2(x-3)$
73. $f(x) = -(x+3)(x+2)(x-2)(x-4)$ or $f(x) = -\frac{1}{24}(x+3)(x+2)(x-2)(x-4)$

★ **Use the graphs to write a formula for the polynomial function of least degree.**

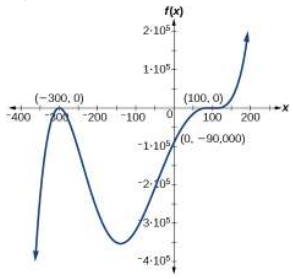
75)



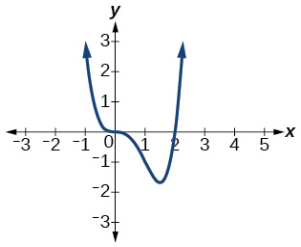
76)



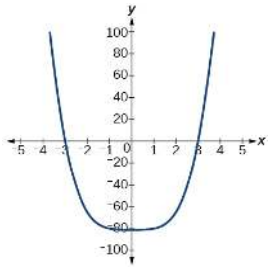
77)



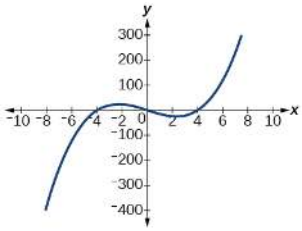
78)



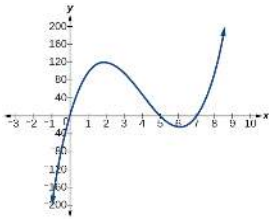
79)



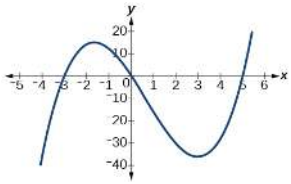
80)



81)



82)

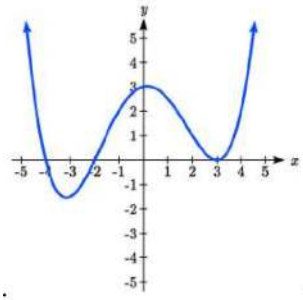


Answers to odd exercises:

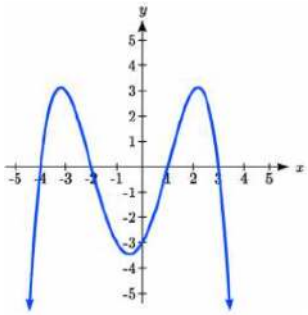
75. $f(x) = (x-500)^2(x+200)$ 77. $f(x) = (x+300)^2(x-100)^3$
79. $f(x) = (x+3)(x-3)(x^2+10)$ 81. $f(x) = 4x(x-5)(x-7)$

★ **Use the graphs to write a formula for the polynomial function of least degree.**

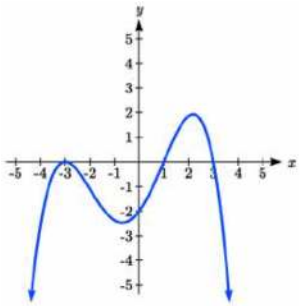
83.



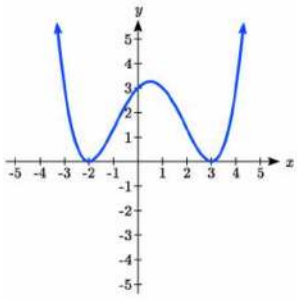
84(a).



84(b).



85.



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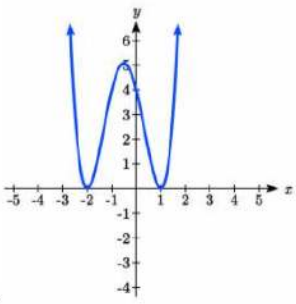

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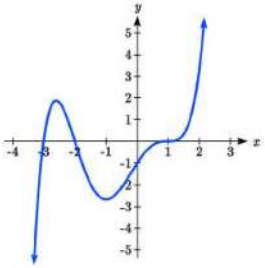

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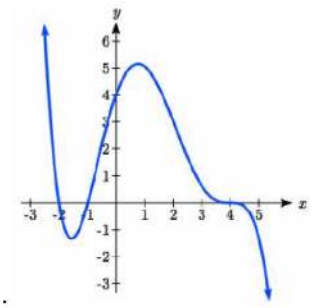

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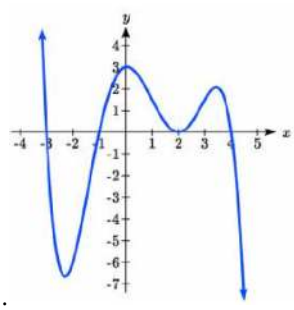

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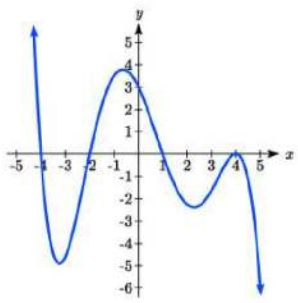


Answers to odd exercises:

83. $y=\frac{1}{24}(x+4)(x+2)(x-3)^2$

85. $y=\frac{1}{12}(x+2)^2(x-3)^2$

87. $y=\frac{1}{6}(x+3)(x+2)(x-1)^3$

89. $y=-\frac{1}{16}(x+3)(x+1)(x-2)^2(x-4)$

G: Construct an Equation from a Description

Exercise 3.4e.G

★ Use the information about the graph of a polynomial function to determine the function. Assume the leading coefficient is 1 or −1. There may be more than one correct answer.

91) The y-intercept is $(0,-4)$. The x-intercepts are $(-2,0),(2,0)$. Degree is 2. End behavior: ↖ ··· ↗

92) The y-intercept is $(0,9)$. The x-intercepts are $(-3,0),(3,0)$. Degree is 2. End behavior: ↙ ··· ↘

93) The y-intercept is $(0,0)$. The x-intercepts are $(0,0),(2,0)$. Degree is 3. End behavior: ↙ ··· ↘

94) The y-intercept is $(0,1)$. The x-intercept is $(1,0)$. Degree is 3. End behavior: ↖ ··· ↘

95) The y-intercept is $(0,1)$. There is no x-intercept. Degree is 4. End behavior: ↖ ··· ↗

★ Use the given information about the polynomial graph to write the equation.

97) Degree 3. Zeros at $x=-2, x=1$, and $x=3$. y-intercept at $(0,-4)$

98) Degree 3. Zeros at $x=-5$, $x=-2$, and $x=1$. y-intercept at $(0,6)$

99) Degree 5. Roots of multiplicity 2 at $x=3$ and $x=1$. Root of multiplicity 1 at $x=-3$. y-intercept at $(0,9)$

100) Degree 4. Root of multiplicity 2 at $x=4$. Roots of multiplicity 1 at $x=1$ and $x=-2$. y-intercept at $(0,-3)$

101) Degree 5. Double zero at $x=1$. Triple zero at $x=3$. Passes through the point $(2,15)$

102) Degree 3. Zeros at $x=4$, $x=3$, and $x=2$. y-intercept at $(0,-24)$

103) Degree 3. Zeros at $x=-3$, $x=-2$ and $x=1$. y-intercept at $(0,12)$

104) Degree 5. Roots of multiplicity 2 at $x=-3$ and $x=2$. Root of multiplicity 1 at $x=-2$. y-intercept at $(0,4)$.

105) Degree 4. Roots of multiplicity 2 at $x=\frac{1}{2}$. Roots of multiplicity 1 at $x=6$ and $x=-2$. y-intercept at $(0,18)$

106) Double zero at $x=-3$. Triple zero at $x=0$. Passes through the point $(1,32)$.

107. Degree 3. Zeros at x = -2, x = 1, and x = 3. Vertical intercept at (0, -4)

108. Degree 3. Zeros at x = -5, x = -2, and x = 1. Vertical intercept at (0, 6)

109. Degree 5. Roots of multiplicity 2 at x = 3 and x = 1. Root of multiplicity 1 at x = -3. Vertical intercept at (0, 9)

110. Degree 4. Root of multiplicity 2 at $x = 4$. Roots of multiplicity 1 at $x = 1$ and $x = -2$.
Vertical intercept at (0, -3)

111. Degree 5. Double zero at $x = 1$. Triple zero at $x = 3$. Passes through the point (2, 15)

112. Degree 5. Single zero at $x = -2$ and $x = 3$. Triple zero at $x = 1$. Passes through the point (2, 4)

Answers to odd exercises:

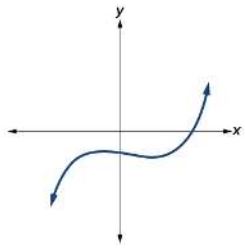
91. $f(x) = x^2 - 4$
93. $f(x) = x^3 - 4x^2 + 4x$
95. $f(x) = x^4 + 1$
97. $f(x) = -\frac{2}{3}(x+2)(x-1)(x-3)$
99. $f(x) = \frac{1}{3}(x-3)^2(x-1)^2(x+3)$
101. $f(x) = -15(x-1)^2(x-3)^3$
103. (f(x)=−2(x+3)(x+2)(x−1)\)
105. $f(x) = -\frac{3}{2}(2x-1)^2(x-6)(x+2)$
107. $y = -\frac{2}{3}(x+2)(x-1)(x-3)$
109. $y = \frac{1}{3}(x-1)^2(x-3)^2(x+3)$
111. $y = -15(x-1)^2(x-3)^3$

H: Turning Points

Exercise 3.4e.H

★ **Determine whether the graph of the function provided is a graph of a polynomial function. If so, determine the number of turning points and the least possible degree for the function.**

115)



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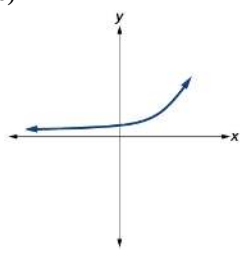

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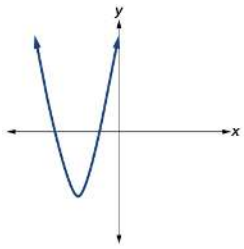

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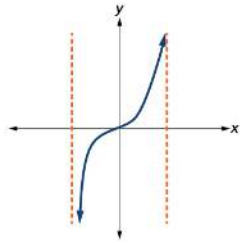

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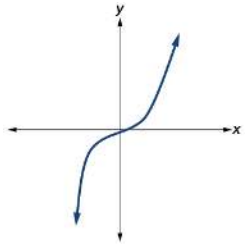

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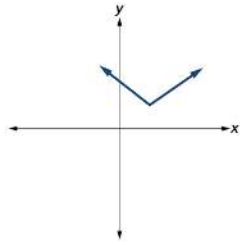

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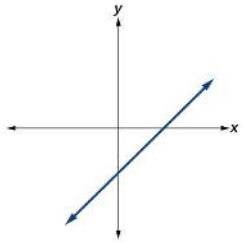

Answers to odd exercises:

115. Yes. Number of turning points is **2**. Least possible degree is **3**.

117. Yes. Number of turning points is **1**. Least possible degree is **2**.

119. Yes. Number of turning points is **0**. Least possible degree is **1**.

212. Yes. Number of turning points is **0**. Least possible degree is **1**.

★

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3.5: Dividing Polynomials

Learning Objectives

- Use long division to divide polynomials.
- Use synthetic division to divide polynomials.

Using Long Division to Divide Polynomials

We are familiar with the **long division** algorithm for ordinary arithmetic. We begin by dividing into the digits of the dividend that have the greatest place value. We divide, multiply, subtract, include the digit in the next place value position, and repeat. For example, let's divide 178 by 3 using long division.

Long Division

$$\begin{array}{r} 59 \\ 3\overline{)178} \\ \underline{-15}\downarrow \\ 28 \\ \underline{-27} \\ 1 \end{array}$$

Step 1: $5 \times 3 = 15$ and $17 - 15 = 2$
Step 2: Bring down the 8
Step 3: $9 \times 3 = 27$ and $28 - 27 = 1$

Answer: $59\,R\,1$ or $59\frac{1}{3}$

Another way to look at the solution is as a sum of parts. This should look familiar, since it is the same method used to check division in elementary arithmetic.

$$\begin{aligned} \textbf{dividend} &= (\textbf{divisor}\cdot\textbf{quotient}) + \textbf{remainder} \\ 178 &= (3\cdot 59) + 1 \\ &= 177 + 1 \\ &= 178 \end{aligned}$$

We call this the **Division Algorithm** and will discuss it more formally after looking at an example.

Division of polynomials that contain more than one term has similarities to long division of whole numbers. We can write a polynomial dividend as the product of the divisor and the quotient added to the remainder. The terms of the polynomial division correspond to the digits (and place values) of the whole number division. This method allows us to divide two polynomials. For example, if we were to divide $2x^3 - 3x^2 + 4x + 5$ by $x + 2$ using the long division algorithm, it would look like this:

$$\begin{array}{rl} 2x^2 - 7x + 18 & \text{Step 1. Divide: } \frac{2x^3}{x} \quad \text{Step 4. Divide: } \frac{-7x^2}{x} = -7x \quad \text{Step 7. Divide: } \frac{18x}{x} = 18 \\ x + 2\,\overline{)2x^3 - 3x^2 + 4x + 5} & \text{Original problem} \\ \underline{-(2x^3 + 4x^2)} & \text{Step 2. Multiply: } 2x^2(x+2) \\ -7x^2 + 4x & \text{Step 3. Subtract and Bring Down} \\ \underline{-(-7x^2 - 14x)} & \text{Step 5. Multiply: } -7x(x+2) \\ 18x + 5 & \text{Step 6. Subtract and Bring Down} \\ \underline{-(18x + 36)} & \text{Step 8. Multiply: } 18(x+2) \\ -31 & \text{Step 9. Subtract. This is the remainder} \end{array}$$

The result is: $\dfrac{2x^3 - 3x^2 + 4x + 5}{x + 2} = 2x^2 - 7x + 18 - \dfrac{31}{x + 2}$

or $2x^3 - 3x^2 + 4x + 5 = (x + 2)(2x^2 - 7x + 18) - 31$

We can identify the dividend, the **divisor**, the **quotient**, and the **remainder**.

Writing the result in this manner illustrates the Division Algorithm.

The Division Algorithm

The Division Algorithm states that, given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)q(x) + r(x)$$

$q(x)$ is the quotient and $r(x)$ is the remainder. The remainder is either equal to zero or has degree strictly less than $d(x)$.
If $r(x) = 0$, then $d(x)$ divides evenly into $f(x)$. This means that, in this case, both $d(x)$ and $q(x)$ are factors of $f(x)$.

How to: Given a polynomial and a binomial, use long division to divide the polynomial by the binomial

1. Set up the division problem.
2. Determine the first term of the quotient by dividing the leading term of the dividend by the leading term of the divisor.
3. Multiply the answer by the divisor and write it below the like terms of the dividend.
4. Subtract the bottom **binomial** from the top binomial.
5. Bring down the next term of the dividend.
6. Repeat steps 2–5 until the degree of the remainder is less than the degree of the divisor.
7. If the remainder is non-zero, express as a fraction using the divisor as the denominator.

Example 3.5.1: Using Long Division to Divide a Second-Degree Polynomial

Divide $5x^2 + 3x - 2$ by $x + 1$.

Solution

$$
\begin{array}{rl}
5x - 2 & \text{Step 1. Divide: } \frac{5x}{x} \qquad \text{Step 4. Divide: } \frac{-2}{x} = -7x \\
x + 1 \,\overline{)\,5x^2 + 3x - 2} & \text{Original problem} \\
-(5x^2 + 5x) & \text{Step 2. Multiply: } 5x(x+1) \\
-2x - 2 & \text{Step 3. Subtract and Bring Down} \\
-(-2x - 2) & \text{Step 5. Multiply: } -2(x+1) \\
0 & \text{Step 6. Subtract. There is no remainder}
\end{array}
$$

The quotient is $5x - 2$. The remainder is 0. The result is $\dfrac{5x^2 + 3x - 2}{x + 1} = 5x - 2$

or $\quad 5x^2 + 3x - 2 = (x + 1)(5x - 2)$

Analysis

This division problem had a remainder of 0. This tells us that the dividend is divided evenly by the divisor, and that the divisor is a factor of the dividend.

3.5.2: Using Long Division to Divide a Third-Degree Polynomial

Divide $6x^3 + 11x^2 - 31x + 15$ by $3x - 2$.

Solution

$$
\begin{array}{rl}
2x^2 + 5x - 7 & \\
3x - 2 \,\overline{)\,6x^3 + 11x^2 - 31x + 1} & 6x^3 \text{ divided by } 3x \text{ is } 2x^2. \\
-(6x^3 - 4x^2) & \text{Multiply } x + 2 \text{ by } 2x^2. \\
15x^2 - 31x & \text{Subtract. Bring down the next term. } 15x^2 \text{ divided by } 3x \text{ is } 5x. \\
-(15x^2 + 10x) & \text{Multiply } 3x - 2 \text{ by } 5x. \\
21x + 15 & \text{Subtract. Bring down the next term. } -21x \text{ divided by } 3x \text{ is } -7. \\
-(-21x + 14) & \text{Multiply } 3x - 2 \text{ by } -7. \\
1 & \text{Subtract. The remainder is 1.}
\end{array}
$$

There is a remainder of 1. We can express the result as: $\dfrac{6x^3 + 11x^2 - 31x + 15}{3x - 2} = 2x^2 + 5x - 7 + \dfrac{1}{3x - 2}$

Analysis

We can check our work by using the Division Algorithm to rewrite the solution. Then multiply.

$$(3x - 2)(2x^2 + 5x - 7) + 1 = 6x^3 + 11x^2 - 31x + 15$$

Notice, as we write our result,

- the dividend is $6x^3+11x^2-31x+15$
- the divisor is $3x-2$
- the quotient is $2x^2+5x-7$
- the remainder is 1

Try It 3.5.2

Divide $16x^3-12x^2+20x-3$ by $4x+5$.

Solution

$$4x^2-8x+15-\frac{78}{4x+5}$$

The divisor can also be a higher degree polynomial.

Example 3.5.3

Divide.

$$x^2+3x+1\overline{)\;2x^4+7x^3+4x^2-2x-1}$$

Solution

$$\begin{array}{rl}
2x^2+x-1 & \text{Step 1. Divide: } \frac{2x^4}{x^2} \quad \text{Step 4. Divide: } \frac{x^3}{x^2} \quad \text{Step 7. Divide: } \frac{-x^2}{x^2} \\
x^2+3x+1\,/\,2x^4+7x^3+4x^2-2x-1 & \text{Original problem} \\
-(2x^4+6x^3+2x^2) & \text{Step 2. Multiply: } 2x^2(x^2+3x+1) \\
x^3+2x^2-2x & \text{Step 3. Subtract and Bring Down} \\
-(x^3+3x^2+x) & \text{Step 5. Multiply: } x(x^2+3x+1) \\
-x^2-3x-1 & \text{Step 6. Subtract and Bring Down} \\
-(-x^2-3x-1) & \text{Step 8. Multiply: } -1(x^2+3x+1) \\
0 & \text{Step 9. Subtract. There is no remainder}
\end{array}$$

Because x^2+3x+1 divides evenly into $2x^4+7x^3+4x^2-2x-1$ we have a zero remainder.

The final answer is $\dfrac{2x^4+7x^3+4x^2-2x-1}{x^2+3x+1}=2x^2+x-1$

Example 3.5.4

Divide. $\dfrac{3x^4-8x^3+19x^2-15x+10}{x^2-x+4}$

Solution

$$\begin{array}{rl}
3x^2-5x+2 & \text{Step 1. Divide: } \frac{3x^4}{x^2} \quad \text{Step 4. Divide: } \frac{-5x^3}{x^2} \quad \text{Step 7. Divide: } \frac{2x^2}{x^2} \\
x^2-x+4\,/\,3x^4-8x^3+19x^2-15x+10 & \text{Original problem} \\
-(3x^4-3x^3+12x^2) & \text{Step 2. Multiply: } x^2(x^2-x+4) \\
-5x^3+7x^2-15x & \text{Step 3. Subtract and Bring Down} \\
-(-5x^3+5x^2-20x) & \text{Step 5. Multiply: } x(x^2-x+4) \\
2x^2+5x+10 & \text{Step 6. Subtract and Bring Down} \\
-(2x^2-2x+8) & \text{Step 8. Multiply: } 2(x^2-x+4) \\
7x+2 & \text{Step 9. Subtract. This is the remainder}
\end{array}$$

Thus $3x^4-8x^3+19x^2-15x+10=(x^2-x+4)(3x^2-5x+2)+(7x+2)$

The final Answer is: $\dfrac{3x^4-8x^3+19x^2-15x+10}{x^2-x+4}=3x^2-5x+2+\dfrac{7x+2}{x^2-x+4}$

Using Synthetic Division to Divide Polynomials

As we've seen, long division of polynomials can involve many steps and be quite cumbersome. **Synthetic division** is a shorthand method of dividing polynomials for the special case of dividing by a linear factor whose leading coefficient is 1.

To illustrate the process, recall the example at the beginning of the section.

Divide $2x^3 - 3x^2 + 4x + 5$ by $x + 2$ using the long division algorithm.

The final form of the process looked like this:

$$\begin{array}{r} 2x^2 + x + 18 \\ x+2\overline{)\,2x^3 - 3x^2 + 4x + 5} \\ \underline{-(2x^3 + 4x^2)} \\ -7x^2 + 4x \\ \underline{-(-7x^2 - 14x)} \\ 18x + 5 \\ \underline{-(18x + 36)} \\ -31 \end{array}$$

There is a lot of repetition in the table. If we don't write the variables but, instead, line up their coefficients in columns under the division sign and also eliminate the partial products, we already have a simpler version of the entire problem.

$$\begin{array}{r|rrrr} 2 & 2 & -3 & 4 & 5 \\ & -2 & -4 & & \\ \hline & & -7 & 14 & \\ & & & 18 & -36 \\ \hline & & & & -31 \end{array}$$

Synthetic division carries this simplification even a few more steps. Collapse the table by moving each of the rows up to fill any vacant spots. Also, instead of dividing by 2, as we would in division of whole numbers, then multiplying and subtracting the middle product, we change the sign of the "divisor" to –2, multiply and add. The process starts by bringing down the leading coefficient.

$$\begin{array}{r|rrrr} -2 & 2 & -3 & 4 & 5 \\ & & -4 & 14 & -36 \\ \hline & 2 & -7 & 18 & -31 \end{array}$$

We then multiply it by the "divisor" and add, repeating this process column by column, until there are no entries left. The bottom row represents the coefficients of the quotient; the last entry of the bottom row is the remainder. In this case, the quotient is $2x^2 - 7x + 18$ and the remainder is –31.The process will be made more clear in the next example.

Synthetic Division

Synthetic division is a shortcut that can be used when the divisor is a binomial in the form $x - k$. In *synthetic division*, only the coefficients are used in the division process.

How to: Given two polynomials, use synthetic division to divide

1. Write k for the divisor.
2. Write the coefficients of the dividend.
3. Bring the lead coefficient down.
4. Multiply the lead coefficient by k. Write the product in the next column.
5. Add the terms of the second column.
6. Multiply the result by k. Write the product in the next column.
7. Repeat steps 5 and 6 for the remaining columns.
8. Use the bottom numbers to write the quotient. The number in the last column is the remainder and has degree 0, the next number from the right has degree 1, the next number from the right has degree 2, and so on.

Example 3.5.5: Using Synthetic Division to Divide a Second-Degree Polynomial

Use synthetic division to divide $5x^2 - 3x - 36$ by $x - 3$.

Solution

Begin by setting up the synthetic division. Write k and the coefficients.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ \hline & & & \end{array}$$

Bring down the lead coefficient. Multiply the lead coefficient by k.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ & & 15 & \\ \hline & 5 & & \end{array}$$

Continue by adding the numbers in the second column. Multiply the resulting number by k. Write the result in the next column. Then add the numbers in the third column.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ & & 15 & 36 \\ \hline & 5 & 12 & 0 \end{array}$$

The result is $5x + 12$. The remainder is 0. So $x - 3$ is a factor of the original polynomial.

Analysis

Just as with long division, we can check our work by multiplying the quotient by the divisor and adding the remainder.

$$(x - 3)(5x + 12) + 0 = 5x^2 - 3x - 36 \tag{3.5.1}$$

Example 3.5.6: Using Synthetic Division to Divide a Third-Degree Polynomial

Use synthetic division to divide $4x^3 + 10x^2 - 6x - 20$ by $x + 2$.

Solution

The binomial divisor is $x + 2$ so $k = -2$. Add each column, multiply the result by –2, and repeat until the last column is reached.

$$\begin{array}{r|rrrr} -2 & 4 & 10 & -6 & -20 \\ & & -8 & -4 & 20 \\ \hline & 4 & 2 & -10 & 0 \end{array}$$

The result is $4x^2 + 2x - 10$.

The remainder is 0. Thus, $x + 2$ is a factor of $4x^3 + 10x^2 - 6x - 20$.

Example 3.5.7: Using Synthetic Division to Divide a Fourth-Degree Polynomial

Use synthetic division to divide $-9x^4 + 10x^3 + 7x^2 - 6$ by $x - 1$.

Solution

Notice there is no x-term. We will use a zero as the coefficient for that term.

$$\begin{array}{r|rrrrr} 1 & -9 & 10 & 7 & 0 & -6 \\ & & -9 & 1 & 8 & 8 \\ \hline & -9 & 1 & 8 & 8 & 2 \end{array}$$

The result is $-9x^3 + x^2 + 8x + 8 + \dfrac{2}{x - 1}$.

3.5.7

Use synthetic division to divide $3x^4+18x^3-3x+40$ by $x+7$.

Solution

$3x^3-3x^2+21x-150+\frac{1,090}{x+7}$

Key Equations

Division Algorithm $f(x)=d(x)q(x)+r(x)$ where $q(x)\neq 0$

Key Concepts

- Polynomial long division can be used to divide a polynomial by any polynomial with equal or lower degree.
- The Division Algorithm tells us that a polynomial dividend can be written as the product of the divisor and the quotient added to the remainder.
- Synthetic division is a shortcut that can be used to divide a polynomial by a binomial in the form x−k.
- Polynomial division can be used to solve application problems, including area and volume.

Glossary

Division Algorithm

given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that $f(x)=d(x)q(x)+r(x)$ where $q(x)$ is the quotient and $r(x)$ is the remainder. The remainder is either equal to zero or has degree strictly less than $d(x)$.

synthetic division

a shortcut method that can be used to divide a polynomial by a binomial of the form $x-k$

Contributors

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3.5e: Exercises - Division of Polynomials

A: Concepts

Exercise 3.5e.A

1) If division of a polynomial by a binomial results in a remainder of zero, what can be conclude?

2) If a polynomial of degree n is divided by a binomial of degree 1, what is the degree of the quotient?

Answers to odd exercises:

1. The binomial is a factor of the polynomial.

B: Perform Polynomial Long Division

Exercise 3.5e.B

★ Use long division to divide. Also specify the quotient and the remainder.

3) $(x^2+5x-1)\div(x-1)$
4) $(2x^2-9x-5)\div(x-5)$
5) $(3x^2+23x+14)\div(x+7)$
6) $(4x^2-10x+6)\div(4x+2)$
7) $(6x^2-25x-25)\div(6x+5)$
8) $(-x^2-1)\div(x+1)$
9) $(2x^2-3x+2)\div(x+2)$
10) $(x^3-126)\div(x-5)$
11) $(3x^2-5x+4)\div(3x+1)$
12) $(x^3-3x^2+5x-6)\div(x-2)$
13) $(2x^3+3x^2-4x+15)\div(x+3)$

★ Divide.

14) $\dfrac{x^3-5x^2+x+15}{x-3}$
15) $\dfrac{y^3-4y^2+6y-4}{y-2}$
16) $\dfrac{x^5+3x+2}{x^3+2x+1}$
17) $\dfrac{2z^3+5z+8}{z+1}$
18) $\dfrac{3x^5-4x^3+3x^2+12x-10}{x^2+2x-1}$
19) $\dfrac{2y^5-3y^4-y^2+y+4}{y^2+1}$
20) $\dfrac{3y^3-4y^2-3}{y^2+5y+2}$
21) $\dfrac{5x^4-3x^2+2}{x^2-3x+5}$

Answers to odd exercises:

3. $x+6+\dfrac{5}{x-1}$, quotient : $x+6$, remainder : 5
5. $3x+2$, quotient : $3x+2$, remainder : 0
7. $x-5$, quotient : $x-5$, remainder : 0
9. $2x-7+\dfrac{16}{x+2}$, quotient : $2x-7$, remainder : 16
11. $x-2+\dfrac{6}{3x+1}$, quotient : $x-2$, remainder : 6
13. $2x^2-3x+5$, quotient : $2x^2-3x+5$, remainder : 0
15. y^2-2y+2
17. $2z^2-2z+7+\dfrac{1}{z+1}$
19. $2y^3-3y^2-2y+2+\dfrac{3y+2}{y^2+1}$
21. $5x^2+15x+17+\dfrac{-24x-83}{x^2-3x+5}$

C: Use Long Division to Rewrite a Polynomial

Exercise 3.5e.C

★ Use polynomial long division to perform the indicated division. Write the polynomial dividend in the form $p(x) = d(x)q(x)+r(x)$.

23. $(4x^2+3x-1)\div(x-3)$
24) $\dfrac{x^3-4x^2-3x-10}{x^2+x+2}$
25. $(2x^3-x+1)\div(x^2+x+1)$
26) $\dfrac{2x^3-3x^2+7x-3}{x^2-x+3}$
27. $(5x^4-3x^3+2x^2-1)\div(x^2+4)$
28) $\dfrac{x^4+2x^3-x^2+x+6}{x+2}$
29. $(-x^5+7x^3-x)\div(x^3-x^2+1)$
30) $\dfrac{x^4+x^3+5x^2+3x+6}{x^2+x-1}$
31. $(9x^3+5)\div(2x-3)$
32) $\dfrac{x^4+2x^3+4x^2+3x+2}{x^2+x+2}$
33. $(4x^2-x-23)\div(x^2-1)$
34) $\dfrac{2x^4+3x^3+3x^2-5x-3}{2x^2-x-1}$

Answers to odd exercises:

23. $4x^2+3x-1=(x-3)(4x+15)+44$

25. $2x^3-x+1=\left(x^2+x+1\right)(2x-2)+(-x+3)$

27. $5x^4-3x^3+2x^2-1=\left(x^2+4\right)\left(5x^2-3x-18\right)+(12x+71)$

29. $-x^5+7x^3-x=\left(x^3-x^2+1\right)\left(-x^2-x+6\right)+\left(7x^2-6\right)$

31. $9x^3+5=(2x-3)\left(\frac{9}{2}x^2+\frac{27}{4}x+\frac{81}{8}\right)+\frac{283}{8}$

33. $4x^2-x-23=\left(x^2-1\right)(4)+(-x-19)$

D: Perform Synthetic Division

Exercise $3.5e.D$

★ Use synthetic division to divide. Also state the quotient and remainder.

35) $\frac{4x^3-33}{x-2}$

36) $\frac{2x^3+25}{x+3}$

37) $\frac{3x^3+2x-5}{x-1}$

38) $\frac{-4x^3-x^2-12}{x+4}$

39) $\frac{x^4-22}{x+2}$

★For the exercises below, use synthetic division to find the quotient.

40) $(3x^3-2x^2+x-4)\div(x+3)$
41) $(2x^3-6x^2-7x+6)\div(x-4)$
42) $(6x^3-10x^2-7x-15)\div(x+1)$
43) $(4x^3-12x^2-5x-1)\div(2x+1)$
44) $(9x^3-9x^2+18x+5)\div(3x-1)$
45) $(3x^3-2x^2+x-4)\div(x+3)$
46) $(-6x^3+x^2-4)\div(2x-3)$
47) $(2x^3+7x^2-13x-3)\div(2x-3)$
48) $(3x^3-5x^2+2x+3)\div(x+2)$
49) $(4x^3-5x^2+13)\div(x+4)$
50) $(x^3-3x+2)\div(x+2)$
51) $(x^3-21x^2+147x-343)\div(x-7)$
52) $(x^3-15x^2+75x-125)\div(x-5)$
53) $(9x^3-x+2)\div(3x-1)$
54) $(6x^3-x^2+5x+2)\div(3x+1)$
55) $(x^4+x^3-3x^2-2x+1)\div(x+1)$
56) $(x^4-3x^2+1)\div(x-1)$
57) $(x^4+2x^3-3x^2+2x+6)\div(x+3)$
58) $(x^4-10x^3+37x^2-60x+36)\div(x-2)$
59) $(x^4-8x^3+24x^2-32x+16)\div(x-2)$
60) $(x^4+5x^3-3x^2-13x+10)\div(x+5)$
61) $(x^4-12x^3+54x^2-108x+81)\div(x-3)$
62) $(4x^4-2x^3-4x+2)\div(2x-1)$
63) $(4x^4+2x^3-4x^2+2x+2)\div(2x+1)$

Answers to odd exercises:

35. $4x^2+8x+16+\frac{-1}{x-2}$,
Quotient : $4x^2+8x+16$, remainder : -1
37. $3x^2+3x+5$,
Quotient : $3x^2+3x+5$, remainder : 0
39. $x^3-2x^2+4x-8+\frac{-6}{x+2}$,
Quotient : x^3-2x^2+4x-8, remainder : -6
41. $2x^2+2x+1+\frac{10}{x-4}$
43. $2x^2-7x+1-\frac{2}{2x+1}$
45. $3x^2-11x+34-\frac{106}{x+3}$
47. x^2+5x+1
49. $4x^2-21x+84-\frac{323}{x+4}$
51. $x^2-14x+49$
53. $3x^2+x+\frac{2}{3x-1}$
55. x^3-3x+1
57. x^3-x^2+2
59. $x^3-6x^2+12x-8$
61. $x^3-9x^2+27x-27$
63. $2x^3-2x+2$

E: Use Synthetic Division to Rewrite a Polynomial

Exercise $3.5e.E$

★ **For the exercises below, use synthetic division to determine whether the first expression is a factor of the second. If it is, write the second expression as a product of two factors.**

64) $x-2,\ 4x^3-3x^2-8x+4$
65) $x-2,\ 3x^4-6x^3-5x+10$
66) $x+3,\ -4x^3+5x^2+8$
67) $x-2,\ 4x^4-15x^2-4$
68) $x-\frac{1}{2},\ 2x^4-x^3+2x-1$
69) $x+\frac{1}{3},\ 3x^4+x^3-3x+1$

★ **In the exercises below, use synthetic division to perform the indicated division. Write the polynomial in the form $p(x)=d(x)q(x)+r(x)$.**

70. $(3x^2-2x+1)\div(x-1)$
71. $(x^2-5)\div(x-5)$
72. $(3-4x-2x^2)\div(x+1)$
73. $(4x^2-5x+3)\div(x+3)$
74. $(x^3+8)\div(x+2)$
75. $(4x^3+2x-3)\div(x-3)$
76. $(18x^2-15x-25)\div\left(x-\frac{5}{3}\right)$
77. $(4x^2-1)\div\left(x-\frac{1}{2}\right)$
78. $(2x^3+x^2+2x+1)\div\left(x+\frac{1}{2}\right)$
79. $(3x^3-x+4)\div\left(x-\frac{2}{3}\right)$
80. $(2x^3-3x+1)\div\left(x-\frac{1}{2}\right)$
81. $(4x^4-12x^3+13x^2-12x+9)\div\left(x-\frac{3}{2}\right)$
82. $(x^4-6x^2+9)\div\left(x-\sqrt{3}\right)$
83. $(x^6-6x^4+12x^2-8)\div\left(x+\sqrt{2}\right)$

Answers to odd exercises:

65. Yes $(x-2)(3x^3-5)$
67. Yes $(x-2)(4x^3+8x^2+x+2)$
69. No
71. $(x^2-5)=(x-5)(x+5)+20$
73. $(4x^2-5x+3)=(x+3)(4x-17)+54$
75. $(4x^3+2x-3)=(x-3)(4x^2+12x+38)+111$
77. $(4x^2-1)=\left(x-\frac{1}{2}\right)(4x+2)+0$
79. $(3x^3-x+4)=\left(x-\frac{2}{3}\right)\left(3x^2+2x+\frac{1}{3}\right)+\frac{38}{9}$
81. $(4x^4-12x^3+13x^2-12x+9)=\left(x-\frac{3}{2}\right)(4x^3-6x^2+4x-6)+0$
83. $(x^6-6x^4+12x^2-8)=\left(x+\sqrt{2}\right)\left(x^5-\sqrt{2}x^4-4x^3+4\sqrt{2}x^2+4x-4\sqrt{2}\right)+0$

F: Synthetic Division with Complex Numbers

Exercise 3.5e.F

★ **For the exercises below, use synthetic division to determine the quotient involving a complex number.**

85) $\frac{x+1}{x-i}$
86) $\frac{x^2+1}{x-i}$
87) $\frac{x+1}{x+i}$
88) $\frac{x^2+1}{x+i}$
89) $\frac{x^3+1}{x-i}$

★ **Find the remainder.**

90) $(x^4-9x^2+14)\div(x-2)$
91) $(3x^3-2x^2+x-4)\div(x+3)$
92) $(x^4+5x^3-4x-17)\div(x+1)$
93) $(-3x^2+6x+24)\div(x-4)$
94) $(5x^5-4x^4+3x^3-2x^2+x-1)\div(x+6)$
95) $(x^4-1)\div(x-4)$
96) $(3x^3+4x^2-8x+2)\div(x-3)$
97) $(4x^3+5x^2-2x+7)\div(x+2)$

Answers to odd exercises:

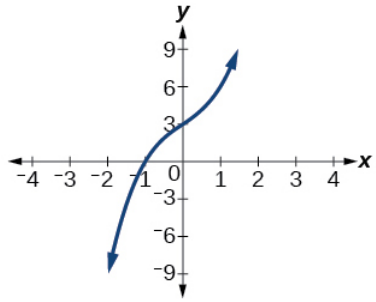
85. $1+\frac{1+i}{x-i}$
87. $1+\frac{1-i}{x+i}$
89. $x^2+ix-1+\frac{1-i}{x-i}$
91. $-106,\ f(2)=-106$
93. $0,\ f(4)=0$
95. $255,\ f(4)=255$
97. $-1,\ f(-2)=-1$

G: Construct a polynomial from a graph and a given Factor

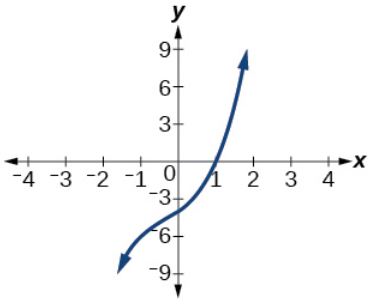
Exercise 3.5e.G

★ Use the graph of the third-degree polynomial and one factor to write the factored form of the polynomial suggested by the graph. The leading coefficient is one.

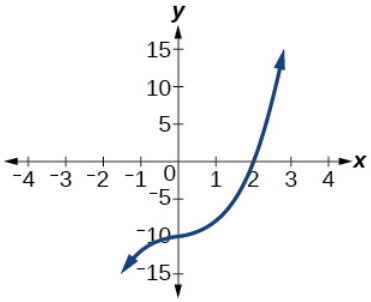
98) Factor is $x^2 - x + 3$



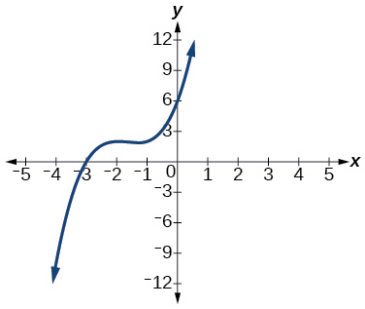
99) Factor is $(x^2 + 2x + 4)$



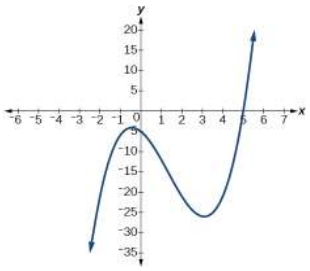
100) Factor is $x^2 + 2x + 5$



101) Factor is $x^2 + 2x + 2$



102) Factor is $x^2 + x + 1$



Answers to odd exercises:

99. $(x - 1)(x^2 + 2x + 4)$	101. $(x + 3)(x^2 + 2x + 2)$

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3.6: Zeros of Polynomial Functions

Learning Objectives

- Evaluate a polynomial using the Remainder Theorem.
- Use the Factor Theorem to solve a polynomial equation.
- Use the Rational Zero Theorem to find rational zeros.
- Find zeros of a polynomial function.
- Theorems to simplify search for zeros: Lower & Upper Bound Theorem, Intermediate Value Theorem, Descartes' Rule of Signs
- Use the Linear Factorization Theorem to find polynomials with given zeros.

In this section, we will discuss a variety of tools for writing polynomial functions and solving polynomial equations.

Evaluate a Polynomial Using the Remainder Theorem

In the last section, we learned how to divide polynomials. We can now use polynomial division to evaluate polynomials using the **Remainder Theorem**. If the polynomial is divided by $x–k$, the remainder may be found quickly by evaluating the polynomial function at k, that is, $f(k)$. Let's walk through the proof of the theorem.

Recall that **the Division Algorithm** states that, given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that $f(x) = d(x)q(x) + r(x)$.

If the divisor, $d(x)$, is $x - k$, this takes the form $f(x) = (x - k)q(x) + r$. Since the divisor $x - k$ is linear, the remainder will be a constant, r. And, if we evaluate $f(x)$ for $x = k$, we have

$$\begin{aligned} f(k) &= (k - k)q(k) + r \\ &= 0 \cdot q(k) + r \& = r \end{aligned}$$

In other words, $f(k)$ is the remainder obtained by dividing $f(x)$ by $x - k$.

The Remainder Theorem

If a polynomial $f(x)$ is divided by $x - k$, then the remainder is the value $f(k)$.

How to: Use the Remainder Theorem to evaluate $f(x)$ at $x = k$.

1. Use synthetic division to divide the polynomial by $x - k$.
2. The remainder is the value $f(k)$.

Example 3.6.1: Use the Remainder Theorem to Evaluate a Polynomial

Use the Remainder Theorem to evaluate $f(x) = 6x^4 - x^3 - 15x^2 + 2x - 7$ at $x = 2$.

Solution

To find the remainder using the Remainder Theorem, use synthetic division to divide the polynomial by $x - 2$.

Erroneous nesting of equation structures

The remainder is 25. Therefore, $f(2) = 25$.

Analysis

We can check our answer by evaluating $f(2)$.

$$\begin{aligned} f(x) &= 6x^4 - x^3 - 15x^2 + 2x - 7 \\ f(2) &= 6(2)^4 - (2)^3 - 15(2)^2 + 2(2) - 7 = 25 \end{aligned}$$

Try It 3.6.1

Use the Remainder Theorem to evaluate $f(x) = 2x^5 - 3x^4 - 9x^3 + 8x^2 + 2$ at $x = -3$.

Answer

$f(-3) = -412$

Use the Factor Theorem to Solve a Polynomial Equation

The **Factor Theorem** is another theorem that helps us analyze polynomial equations. It tells us how the zeros of a polynomial are related to the factors. Recall the definition of a zero, k, of f, is $f(k) = 0$ and that the Division Algorithm states

$$f(x) = (x - k)q(x) + r$$

If k is a zero of $f(x)$, then $f(k) = 0$. Therefore $f(k) = (k - k)q(k) + r = 0$ and thus $r = 0$. Because the remainder r of $f(k)$ is 0, then $f(x) = (x - k)q(x) + 0$ or $f(x) = (x - k)q(x)$. Notice, written in this form, $x - k$ is a factor of $f(x)$. We can conclude if k is a zero of $f(x)$, then $x - k$ is a factor of $f(x)$.

Conversely, if $x - k$ is a factor of $f(x)$, then the remainder r of the Division Algorithm $f(x) = (x - k)q(x) + r$ is 0. Therefore $f(x) = (x - k)q(x) + r = (x - k)q(x) + 0 = (x - k)q(x)$. Evaluating f when $x = k$ we obtain $f(k) = 0$. Since $f(k) = 0$ we can conclude k is a zero of $f(x)$.

This pair of implications is the Factor Theorem. As we will soon see, a polynomial of degree n in the complex number system will have n zeros. We can use the Factor Theorem to completely factor a polynomial into the product of n factors. Once the polynomial has been completely factored, we can easily determine the zeros of the polynomial.

THE FACTOR THEOREM

According to the Factor Theorem, **k is a zero** of $f(x)$ if and only if **$(x - k)$ is a factor** of $f(x)$.

How to: Factor a third-degree polynomial, given one of its factors.

1. Use synthetic division to divide the polynomial by $(x - k)$.
2. Confirm that the remainder is 0.
3. Write the polynomial as the product of $(x - k)$ and the quadratic quotient.
4. If possible, factor the quadratic and write the polynomial as the product of three linear factors. Otherwise, write it as the product of a linear factor and an irreducible quadratic factor.

Example 3.6.2: Use the Factor Theorem to Solve a Polynomial Equation

Show that $(x + 2)$ is a factor of $x^3 - 6x^2 - x + 30$. Find the remaining factors. Use the factors to determine the zeros of the **polynomial**.

Solution

We can use synthetic division to show that $(x + 2)$ is a factor of the polynomial.

Erroneous nesting of equation structures

The remainder is zero, so $(x + 2)$ is a factor of the polynomial. We can use the Division Algorithm to write the polynomial as the product of the divisor and the quotient:

$$x^3 - 6x^2 - x + 30 = (x + 2)(x^2 - 8x + 15)$$

We can factor the quadratic factor to write the polynomial as a **product of linear factors**

$$x^3 - 6x^2 - x + 30 = (x + 2)(x - 3)(x - 5)$$

By the Factor Theorem, the **zeros** of $x^3 - 6x^2 - x + 30$ are –2, 3, and 5.

Try It 3.6.2

Use the Factor Theorem to find the zeros of $f(x) = x^3 + 4x^2 - 4x - 16$ given that $(x - 2)$ is a factor of the polynomial.

Answer

The zeros are 2, –2, and –4.

Use the Rational Zero Theorem to Find Rational Zeros

Another use for the Remainder Theorem is to test whether a rational number is a zero for a given polynomial. But first we need a pool of rational numbers to test. The **Rational Zero Theorem** helps us to narrow down the number of possible rational zeros using the ratio of the factors of the constant term and the factors of the leading **coefficient** of the polynomial

Consider a quadratic function with two zeros $x = \frac{2}{5}$ and $x = \frac{3}{4}$. By the Factor Theorem, these zeros have factors associated with them. Let us set each factor equal to 0, and then construct the original quadratic function.

$x - \frac{2}{5} = 0$ or $x - \frac{3}{4} = 0$	Set each factor equal to 0.
$5x - 2 = 0$ or $4x - 3 = 0$	Multiply both sides of the equation to eliminate fractions.
$f(x) = (5x - 2)(4x - 3)$	Create the quadratic function, multiplying the factors.
$f(x) = 20x^2 - 23x + 6$	Expand the polynomial.
$f(x) = (5 \cdot 4)x^2 - 23x + (2 \cdot 3)$	

Notice that two factors of the constant term, 6, are the two numerators from the original rational roots: 2 and 3. Similarly, two of the factors from the leading coefficient, 20, are the two denominators from the original rational roots: 5 and 4.

We can infer that the numerators of the rational roots will always be factors of the constant term and the denominators will be factors of the leading coefficient. This is the essence of the Rational Zero Theorem; it is a means to give us a pool of possible rational zeros.

THE RATIONAL ZERO THEOREM

If the polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ has **integer coefficients**, then every rational zero of $f(x)$ has the form $\frac{p}{q}$ where p is a factor of the constant term a_0 and q is a factor of the leading coefficient a_n.

When the leading coefficient is 1, the possible rational zeros are the factors of the constant term.

How to: Use the Rational Zero Theorem to find all a polynomial's rational zeros.

1. Determine all factors of the constant term and all factors of the leading coefficient.
2. Determine all possible values of $\frac{p}{q}$, where p is a factor of the constant term and q is a factor of the leading coefficient. Be sure to include both positive and negative candidates.
3. Determine which possible zeros are actual zeros by using synthetic division. If the remainder is zero, a zero has been found. If the zero found is z, then f can be rewritten as a product of $(x - z)$ and a quotient. The quotient is a polynomial that is one degree less than f.
4. Repeat the search for zeros, **but this time search for a zero of the quotient rather than f**. The search for zeros is repeated until the quotient obtained is a quadratic or linear factor. When searching for zeros, remember that the same value for a zero can occur more than once, and that if a value is found to not be a zero for one quotient, it will not be a zero for any of the quotients.

Example 3.6.3: List All Possible Rational Zeros

List all possible rational zeros of $f(x) = 2x^4 - 5x^3 + x^2 - 4$.

Solution:

The only possible rational zeros of $f(x)$ are quotients of factors of the constant term, –4, and the leading coefficient, 2.

The constant term is –4; the factors of –4 are $p = \pm 1, \pm 2, \pm 4$.

The leading coefficient is 2; the factors of 2 are $q = \pm 1, \pm 2$.

If any of the four real zeros are rational zeros, then they will be of one of the following factors of –4 divided by one of the factors of 2.

$$\frac{p}{q} = \pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{4}{1} \qquad \frac{p}{q} = \pm\frac{1}{2}, \pm\frac{2}{2}, \pm\frac{4}{2}$$

Note that $\frac{2}{2} = 1$ and $\frac{4}{2} = 2$, which have already been listed. So we can shorten our list.

$$\frac{p}{q} = \frac{\text{Factors of the constant term}}{\text{Factors of the leading coefficient}} = \pm 1, \pm 2, \pm 4, \pm\frac{1}{2}$$

Example 3.6.4: Use the Rational Zero Theorem to Find Rational Zeros

Use the Rational Zero Theorem to find the rational zeros of $f(x) = 2x^3 + x^2 - 4x + 1$.

Solution. The Rational Zero Theorem tells us that all possible rational zeros have the form $\frac{p}{q}$ where p is a factor of 1 and q is a factor of 2.

$$\frac{p}{q} = \frac{\text{factor of constant term}}{\text{factor of coefficient}} = \frac{\text{factor of } 1}{\text{factor of } 2}$$

The factors of 1 are ±1 and the factors of 2 are ±1 and ±2. The possible values for $\frac{p}{q}$ are ±1 and $\pm\frac{1}{2}$. These are the possible rational zeros for the function. We can determine which of the possible zeros are actual zeros by using synthetic division and trying each of these values as the divisor.

Erroneous nesting of equation structures

The remainder is zero, so $f(x)$ can be factored as $f(x) = (x-1)(2x^2+3x-1)$. The zeros of the quadratic expression can be found by using the quadratic formula. The zeros are $\frac{-3\pm\sqrt{17}}{4}$, which are irrational numbers, so the only rational zero of f is 1.

Try It 3.6.4

Use the Rational Zero Theorem to find the rational zeros of $f(x) = x^3 - 5x^2 + 2x + 1$.

Answer

There are no rational zeros. All the zeros are irrational numbers or imaginary numbers rather than rational numbers.

Find all the Zeros of a Polynomial Function

The Rational Zero Theorem helps us to narrow down the list of possible rational zeros for a polynomial function. Once we have done this, we can use **synthetic division** repeatedly to determine all of the **zeros** of a polynomial function.

How to: Use synthetic division to find a polynomial's zeroes.

1. Use the Rational Zero Theorem to list all possible rational zeros of the function f.
2. Go through the list of possible rational zeros by performing synthetic division on $\frac{f(x)}{x-c}$, where c is in the list of possible rational zeros. When the remainder is 0, then a zero has been found: c.
3. Rewrite f as a product of $x-c$ and a quotient. (The quotient will always be one degree less than f). Repeat steps 1 and 2 on the quotient (rather than on f) until the new quotient is a quadratic.
4. Find the zeros of the quadratic function. Two possible methods for solving quadratics are factoring and using the quadratic formula.

Example 3.6.5: Find the Zeros of a Polynomial Function with Repeated Real Zeros

Find the zeros of $f(x) = 4x^3 - 3x - 1$.

Solution:

Step 1. The Rational zeros are $\pm 1, \pm\frac{1}{2}$, and $\pm\frac{1}{4}$ because by the Rational Zero Theorem, if $\frac{p}{q}$ is a zero of $f(x)$, then

$$\frac{p}{q} = \frac{\text{factor of constant term}}{\text{factor of coefficient}} = \frac{\text{factor of } -1}{\text{factor of } 4} = \frac{\pm 1}{\pm 1, \pm 2, or \pm 4}$$

Step 2. Use synthetic division to test each possible zero until we find one that gives a remainder of 0. Let's begin with 1.

1	4	0	−3	−1
		4	4	1
	4	4	1	0

Step 3. Dividing by $(x-1)$ gives a remainder of 0, so 1 is a zero of the function. The polynomial can be written as $(x-1)(4x^2+4x+1)$

Step 4. The quotient is a quadratic. The quadratic can be factored. It is a perfect square. $f(x)$ can be written as $f(x) = (x-1)(2x+1)^2$. We already know that 1 is a zero. The other zero will have a multiplicity of 2 because the factor is squared. To find the other zero, we can set the factor equal to 0.

$$2x + 1 = 0 \quad \longrightarrow \quad x = -\frac{1}{2}$$

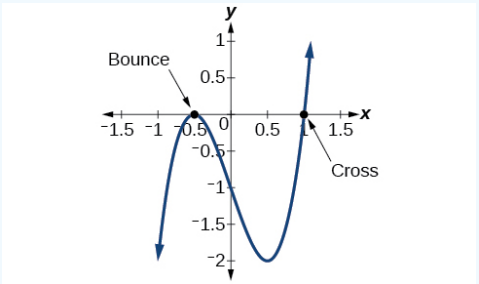
The zeros of the function are 1 with odd multiplicity of 1 and $-\frac{1}{2}$ with even multiplicity of 2.

Analysis

Look at the graph of the function f shown on the right.

Notice, at $x = -0.5$, the graph bounces off the x-axis, indicating the even multiplicity $(2,\ 4,\ 6\ldots)$ for the zero −0.5.

At $x = 1$, the graph crosses the x-axis, indicating the odd multiplicity (1, 3, 5 ...) for the zero $x = 1$



Example 3.6.6: Find zeros of a degree 4 polynomial

Find the zeros of $f(x) = x^4 + 2x^3 - 3x^2 - 4x + 4$.

Solution

Step 1. Possible rational zeros of f are $\pm 1 \pm 2 \pm 4$

Step 2. Use synthetic division with each candidate in this list until a remainder of zero is found. Below shows that a zero $x = 1$ has been found.

Erroneous nesting of equation structures

Step 3. $f(x) = (x - 1)(Q(x))$. The quotient polynomial, $Q(x) = x^3 + 3x^2 - 4$, is a third degree polynomial, so another round of searching for a zero needs to be done, this time on the quotient, $Q(x) = x^3 + 3x^2 - 4$

Step 1. Possible rational zeros of Q are (still) $\pm 1, \pm 2 \pm 4$

Step 2. Use synthetic division with each candidate in this list until a remainder of zero is found. It doesn't hurt to try 1 again. Below shows that another zero $x = 1$ has been found.

Erroneous nesting of equation structures

Step 3. $Q(x) = (x + 1)(R(x))$, so $f(x) = (x - 1)(Q(x)) \longrightarrow f(x) = (x - 1)(x - 1)(R(x))$. The quotient polynomial, $R(x) = x^2 + 4x + 4$, is a quadratic.

Step 4. Setting this quadratic to zero gives $x^2 + 4x + 4 = 0$, or $(x + 2)(x + 2) = 0$, which gives another zero -2 with a multiplicity of two.

In conclusion, the original polynomial was degree four, so we expect to find four zeros. The zeros are 1 and -2, both with multiplicity 2.

The end behaviour graph rises to the right and also rises to the left. The y-intercept is $(0, 4)$. A sketch of the function could now be produced.

Theorems to Simplify the Search for Zeros

Particularly when the rational zero theorem creates a long list of possible zeros, it is helpful to look out for some shortcuts in the search for zeros. The following Theorems are helpful in this regard.

Upper and Lower Bound Theorem

Upper and Lower Bounds Theorem

Suppose f is a polynomial of degree $n \geq 1$.

- If $c > 0$ is synthetically divided into f and all of the numbers in the final line of the synthetic division tableau have the same signs, then c is an upper bound for the real zeros of f. That is, there are no real zeros greater than c.
- If $c < 0$ is synthetically divided into f and the numbers in the final line of the synthetic division tableau alternate signs, then c is a lower bound for the real zeros of f. That is, there are no real zeros less than c.

If the number 0 occurs in the final line of the division tableau in either of the above cases, it can be treated as $(+)$ or $(-)$ as needed.

"Proof." Rewrite f so that its leading coefficient is positive. This will guarantee all successive quotients will have positive leading terms. Furthermore, the zeros for f and $-f$ are the same.

Part 1. Suppose $c > 0$ and the resulting line in the synthetic division tableau for $\frac{f(x)}{x-c}$ contains all nonnegative numbers. This means $f(x) = (x - c)q(x) + r$, where the coefficients of the quotient polynomial q and the remainder constant r are all nonnegative. For any value b larger than c,

$$f(b) = (b-c)q(b) + r = \text{positiveNumber} \times \text{positiveNumber} + \text{positiveNumber} > 0.$$

Therefore no zero for f can be greater than c.

Part 2. Suppose $c < 0$ and the resulting line in the synthetic division tableau for $\frac{f(x)}{x-c}$ contains alternating positive and negative numbers, starting with a positive leading coefficient. Consider any constant b less than c.

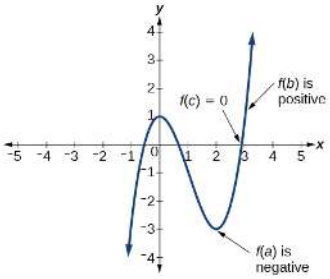
- Multiplying the coefficients of q that are alternating in sign, with powers of a negative number b that are also alternating in sign, produces terms of the quotient that all have the same sign.
- The function is equal to $f(b) = (b - c)q(b) + r$ and the factor $(b - c)$ is negative.
    - If q is even degree, then r is negative (because the signs of the coefficients are alternating), $q(b)$ is positive (because the terms start with Misplaced '#'), and so $f(b)$ would always be negative for any $b < c$.
    - If q is odd degree, then r is positive, $q(b)$ is negative, and so $f(b)$ would always be positive for any $b < c$.

In either case, there are no zeros to the left of c on the number line.

Intermediate Value Theorem

In some situations, we may know two points on a graph but not the zeros. If those two points are on opposite sides of the x-axis, we can confirm that there is a zero between them. Consider a polynomial function f whose graph is smooth and continuous.

The **Intermediate Value Theorem** states that for two numbers a and b in the domain of f, if $a < b$ and $f(a) \neq f(b)$, then the function f takes on every value between $f(a)$ and $f(b)$. We can apply this theorem to a special case that is useful in graphing polynomial functions. If a point on the graph of a **continuous** function f at $x = a$ lies above the x-axis and another point at $x = b$ lies below the x-axis, there must exist a third point between $x = a$ and $x = b$ where the graph crosses the x-axis. Call this point $(c, f(c))$. This means that we are assured there is a solution c where $f(c) = 0$.



In other words, the **Intermediate Value Theorem** tells us that when a polynomial function changes from a negative value to a positive value (or vice versa), the function must cross the x-axis. The figure on the right shows that there is a zero between a and b.

Intermediate Value Theorem

Let f be a polynomial function. The **Intermediate Value Theorem** states that if $f(a)$ and $f(b)$ have opposite signs, then there exists at least one value c between a and b for which $f(c) = 0$.

Example 3.6.7: Use the Intermediate Value Theorem

Show that the function $f(x) = x^3 - 5x^2 + 3x + 6$ has at least two real zeros between $x = 1$ and $x = 4$.

Solution

As a start, evaluate $f(x)$ at the integer values $x = 1,\ 2,\ 3,$ and 4.

x	1	2	3	4
$f(x)$	5	0	-3	2

One zero occurs at $x = 2$. Also, since $f(3)$ is negative and $f(4)$ is positive, by the Intermediate Value Theorem, there must be at least one real zero between 3 and 4. Therefore, there are at least two real zeros between $x = 1$ and $x = 4$.

Analysis

We can also see on the graph of the function in Figure 3.6.19 that there are two real zeros between $x = 1$ and $x = 4$.

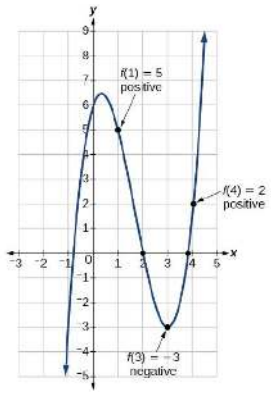


Figure 3.6.19

Try It 3.6.7

Show that the function $f(x) = 7x^5 - 9x^4 - x^2$ has at least one real zero between $x = 1$ and $x = 2$.

Answer

Because f is a polynomial function and since $f(1)$ is negative and $f(2)$ is positive, there is at least one real zero between $x = 1$ and $x = 2$.

The following example describes how using these two theorems can be incorporated into a search for zeros.

Example 3.6.8: FInd all the zeros of a 4th degree polynomial

Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$.

Solution

1. Possible rational zeros are ± 1, ± 3, $\pm \frac{1}{2}$, and $\pm \frac{3}{2}$.

Try our positive rational zeros, starting with the smallest, $\frac{1}{2}$. Since the remainder isn't zero, $\frac{1}{2}$ isn't a zero. Sadly, the final line in the division tableau has both positive and negative numbers, so $\frac{1}{2}$ is not an upper bound. The only information we get from this division is courtesy of the Remainder Theorem which tells us $f\left(\frac{1}{2}\right) = -\frac{45}{8}$ so the point $\left(\frac{1}{2}, -\frac{45}{8}\right)$ is on the graph of f.

Continue to our next possible zero, 1. As before, the only information we can glean from this is that $(1, -4)$ is on the graph of f. When we try our next possible zero, $\frac{3}{2}$, we get that it is not a zero. We also see that it is an upper bound on the zeros of f, since all of the numbers in the final line of the division tableau are positive. This means there is no point trying our last possible rational zero, 3.

The Intermediate Value Theorem indicates there is a zero between 1 and $\frac{3}{2}$, since $f(1) < 0$ and $f\left(\frac{3}{2}\right) > 0$. Since there is no number in the list of rational zeros between 1 and $\frac{3}{2}$, the zero must be an irrational number.

$$\begin{array}{r|ccccc} \frac{1}{2} & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & 1 & \frac{5}{2} & \frac{3}{4} & -\frac{21}{8} \\ \hline & 2 & 5 & \frac{3}{2} & -\frac{21}{4} & \boxed{-\frac{45}{8}} \end{array} \qquad \begin{array}{r|ccccc} 1 & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & 2 & 6 & 5 & -1 \\ \hline & 2 & 6 & 5 & -1 & \boxed{-4} \end{array} \qquad \begin{array}{r|ccccc} \frac{3}{2} & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & 3 & \frac{21}{2} & \frac{57}{4} & \frac{99}{8} \\ \hline & 2 & 7 & \frac{19}{2} & \frac{33}{4} & \boxed{\frac{75}{8}} \end{array}$$

Examine negative real zeros. Try the largest possible zero, $-\frac{1}{2}$. Synthetic division shows it is not a zero, nor is it a lower bound (since the numbers in the final line of the division tableau do not alternate), so we proceed to -1. This division shows -1 is a

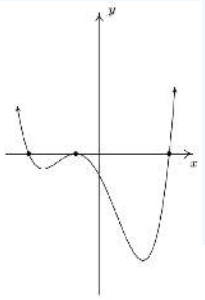
zero. Try -1 again, and it works once more. At this point, we have taken f, a fourth degree polynomial, and performed two successful divisions. Our quotient polynomial is quadratic, so look at it to find the remaining zeros.

$$\begin{array}{c|ccccc} -\frac{1}{2} & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & -1 & -\frac{3}{2} & \frac{5}{4} & \frac{19}{8} \\ \hline & 2 & 3 & -\frac{5}{2} & -\frac{19}{4} & \boxed{-\frac{5}{8}} \end{array} \qquad \begin{array}{c|ccccc} -1 & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & -2 & -2 & 3 & 3 \\ \hline -1 & 2 & 2 & -3 & -3 & \boxed{0} \\ & \downarrow & -2 & 0 & 3 & \\ \hline & 2 & 0 & -3 & \boxed{0} & \end{array}$$

Setting the quotient polynomial equal to zero yields $2x^2 - 3 = 0$, so that $x^2 = \frac{3}{2}$, or $x = \pm\frac{\sqrt{6}}{2}$.

Thus -1 is a zero of multiplicity 2 and zeros $\pm\frac{\sqrt{6}}{2}$ both have multiplicity 1.

We know the end behavior of $y = f(x)$ resembles that of its leading term $y = 2x^4$. This means that the graph enters the scene in Quadrant II and exits in Quadrant I. Since $\pm\frac{\sqrt{6}}{2}$ are zeros of odd multiplicity, we have that the graph crosses through the x-axis at the points $\left(-\frac{\sqrt{6}}{2}, 0\right)$ and $\left(\frac{\sqrt{6}}{2}, 0\right)$. Since -1 is a zero of multiplicity 2, the graph of $y = f(x)$ touches and rebounds off the x-axis at $(-1, 0)$. Putting this together, we get the sketch in the figure to the right.



Descartes' Rule of Signs

There is a straightforward way to determine the possible numbers of positive and negative real zeros for any polynomial function. If the polynomial is written in descending order, **Descartes' Rule of Signs** tells us of a relationship between the number of sign changes in $f(x)$ and the number of positive real zeros. For example, the polynomial function below has one sign change.

$$f(x) = x^4 + x^3 + x^2 + x - 1$$

This tells us that the function must have 1 positive real zero.

There is a similar relationship between the number of sign changes in $f(-x)$ and the number of negative real zeros.

$$f(-x) = (-x)^4 + (-x)^3 + (-x)^2 + (-x) - 1$$
$$f(-x) = +x^4 - x^3 + x^2 - x - 1$$

In this case, $f(-x)$ has 3 sign changes. This tells us that $f(x)$ could have 3 or 1 negative real zeros.

DESCARTES' RULE OF SIGNS

According to *Descartes' Rule of Signs*, if we let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial function with real coefficients:

- The number of positive real zeros is either equal to the number of sign changes of $f(x)$ or is less than the number of sign changes by an even integer.
- The number of negative real zeros is either equal to the number of sign changes of $f(-x)$ or is less than the number of sign changes by an even integer.

Example 3.6.9: Use Descartes' Rule of Signs

Use Descartes' Rule of Signs to determine the possible numbers of positive and negative real zeros for $f(x) = -x^4 - 3x^3 + 6x^2 - 4x - 12$.

Solution:

Begin by determining the number of sign changes.

$$f(x) = -x^4 - 3x^3 + 6x^2 - 4x - 12$$

There are two sign changes, so there are either 2 or 0 positive real roots. Next, we examine $f(-x)$ to determine the number of negative real roots.

$$f(-x) = -(-x)^4 - 3(-x)^3 + 6(-x)^2 - 4(-x) - 12$$
$$f(-x) = -x^4 + 3x^3 + 6x^2 + 4x - 12$$

$$f(-x) = -x^4 + 3x^3 + 6x^2 + 4x - 12$$

Again, there are two sign changes, so there are either 2 or 0 negative real roots. There are four possibilities, as we can see in Table **3.6.9**.

Table **3.6.9**

Positive Real Zeros	Negative Real Zeros	Complex Zeros	Total Zeros
2	2	0	4
2	0	2	4
0	2	2	4
0	0	4	4

Analysis We can confirm the numbers of positive and negative real roots by examining a graph of the function. See Figure **3.6.9**. We can see from the graph that the function has 0 positive real roots and 2 negative real roots.

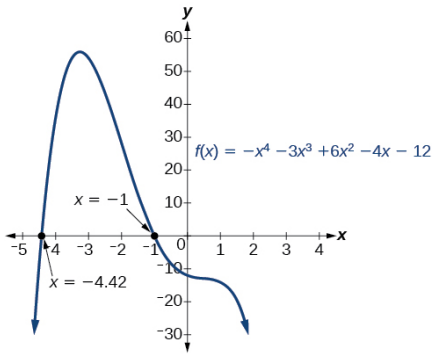


Figure **3.6.9**.

Try It 3.6.9

Use Descartes' Rule of Signs to determine the maximum possible numbers of positive and negative real zeros for $f(x) = 2x^4 - 10x^3 + 11x^2 - 15x + 12$. Use a graph to verify the numbers of positive and negative real zeros for the function.

Answer

There must be 4, 2, or 0 positive real roots and 0 negative real roots. The graph shows that there are 2 positive real zeros and 0 negative real zeros.

Use the Fundamental Theorem of Algebra to find an interval where a zero exists

Now that we can find rational zeros for a polynomial function, we will look at a theorem that discusses the number of complex zeros of a polynomial function. The **Fundamental Theorem of Algebra** tells us that every polynomial function has at least one complex zero. This theorem forms the foundation for solving polynomial equations.

Suppose f is a polynomial function of degree four, and $f(x) = 0$. The Fundamental Theorem of Algebra states that there is at least one complex solution, call it c_1. By the Factor Theorem, we can write $f(x)$ as a product of $x - c_1$ and a polynomial quotient. Since $x - c_1$ is linear, the polynomial quotient will be of degree three. Now we apply the Fundamental Theorem of Algebra to the third-degree polynomial quotient. It will have at least one complex zero, call it c_2. So we can write the polynomial quotient as a product of $x - c_2$ and a new polynomial quotient of degree two. Continue to apply the Fundamental Theorem of Algebra until all of the zeros are found. There will be four of them and each one will yield a factor of $f(x)$.

THE FUNDAMENTAL THEOREM OF ALGEBRA

The *Fundamental Theorem of Algebra* states that, if $f(x)$ is a polynomial of degree $n > 0$, then $f(x)$ has at least one complex zero. We can use this theorem to argue that, if $f(x)$ is a polynomial of degree $n > 0$, and a is a non-zero real number, then $f(x)$ has exactly n linear factors

$$f(x) = a(x - c_1)(x - c_2)\dots(x - c_n)$$

where c_1, c_2,..., c_n are complex numbers. Therefore, $f(x)$ has n roots if we allow for multiplicities.

Does every polynomial have at least one imaginary zero?

No. Real numbers are a subset of complex numbers, but not the other way around. A complex number is not necessarily imaginary. Real numbers are also complex numbers.

Example 3.6.10: Find the Zeros of a Polynomial Function with Complex Zeros

Find the zeros of $f(x) = 3x^3 + 9x^2 + x + 3$.

Solution:

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of 3 and q is a factor of 3.

$$\frac{p}{q} = \frac{\text{factor of constant term}}{\text{factor of coefficient}} = \frac{\text{factor of } 3}{\text{factor of } 3}$$

The factors of 3 are ±1 and ±3. The possible values for $\frac{p}{q}$, and therefore the possible rational zeros for the function, are ±3, ±1, and $\pm\frac{1}{3}$. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with –3.

$$\begin{array}{r|cccc} -3 & 3 & 9 & 1 & 3 \\ & & -9 & 0 & -3 \\ \hline & 3 & 0 & 1 & 0 \end{array}$$

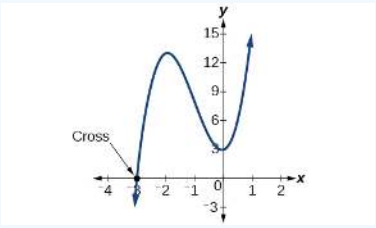
The remainder is 0, so –3 is a zero of the function and the polynomial can be written as $(x+3)(3x^2+1)$

The other zeros can be found by setting the quadratic equal to 0 and solving for x.

$$3x^2 + 1 = 0 \quad \longrightarrow \quad x^2 = -\frac{1}{3} \quad \longrightarrow \quad x = \pm\sqrt{-\frac{1}{3}} \quad \longrightarrow \quad x = \pm\frac{i\sqrt{3}}{3}$$

Analysis

Look at the graph of the function f in the figure to the right. Notice that, at $x = -3$, the graph crosses the x axis, indicating an odd multiplicity (1) for the zero $x = -3$. Also note the presence of the two turning points. This means that, since there is a 3rd degree polynomial, we are looking at the maximum number of turning points. So, the end behavior of increasing without bound to the right and decreasing without bound to the left will continue. Thus, all the x-intercepts for the function are shown. So either the multiplicity of $x = -3$ is 1 and there are two complex solutions, which is what we found, or the multiplicity at $x = -3$ is three. Either way, our result is correct.



Try It 3.6.10

Find the zeros of $f(x) = 2x^3 + 5x^2 - 11x + 4$.

Answer

The zeros are -4, $\frac{1}{2}$, and 1

Use the Linear Factorization Theorem to Find Polynomials with Given Zeros

A vital implication of the **Fundamental Theorem of Algebra**, as we stated above, is that a polynomial function of degree n will have n zeros in the set of complex numbers, if we allow for multiplicities. This means that we can factor the polynomial function into n factors.

The Linear Factorization Theorem

The *Linear Factorization Theorem* states that a polynomial function will have the same number of factors as its degree, and that each factor will be in the form $(x-c)$, where c is a complex number.

Let f be a polynomial function with **real** coefficients, and suppose $a+bi$, $b \neq 0$, is a zero of $f(x)$. Then, by the Factor Theorem, $x-(a+bi)$ is a factor of $f(x)$. For f to have real coefficients, $x-(a-bi)$ must also be a factor of $f(x)$. This is true because any factor other than $x-(a-bi)$, when multiplied by $x-(a+bi)$, will leave imaginary components in the product. Only multiplication with conjugate pairs will eliminate the imaginary parts and result in real coefficients. In other words, if a polynomial function f with real coefficients has a complex zero $a+bi$, then the complex conjugate $a-bi$ must also be a zero of $f(x)$. This is called the **Complex Conjugate Theorem**.

COMPLEX CONJUGATE THEOREM

If the polynomial function f has **real** coefficients and a complex zero in the form $a+bi$, then the complex conjugate of the zero, $a-bi$, is also a zero.

If the polynomial function f has **rational** coefficients and a zero in the form $a+c\sqrt{b}$, and b is not a perfect square, then the conjugate $a-c\sqrt{b}$, is also a zero.

How to: Construct a polynomial function given its zeroes.

1. Use the zeros to construct the linear factors of a general polynomial, $y=p(x)$.
2. To find the specific polynomial, $y=a\cdot p(x)$ that goes through point $(c,f(c))$, substitute $(c,f(c))$ into the function to determine the value of a.
3. Multiply out the result and simplify.

Example 3.6.11: Use the Linear Factorization Theorem to Find a Polynomial with Given Zeros

Find a fourth degree polynomial with real coefficients that has zeros of –3, 2, i, such that $f(-2)=100$.

Solution:

Because $x=i$ is a zero, by the Complex Conjugate Theorem $x=-i$ is also a zero. The polynomial must have factors of $(x+3),(x-2),(x-i)$, and $(x+i)$. So the lowest degree polynomial with these four zeros is $f(x)=(x+3)(x-2)(x-i)(x+i)$.

Now we are looking for a specific polynomial for which $f(-2)=100$. Substitute $x=-2$ and $f(-2)=100$ into $f(x)=a(x+3)(x-2)(x-i)(x+i)$ and solve for a.

$$\begin{aligned} 100 &= a(-2+3)(-2-2)(-2-i)(-2+i) \\ 100 &= a(1)(-4)(4-i^2) \\ 100 &= a(-4)(4+1) \\ 100 &= a(-20) \\ -5 &= a \end{aligned}$$

So the polynomial function is $f(x)=-5(x^4+x^3-5x^2+x-6)$ or $f(x)=-5x^4-5x^3+25x^2-5x+30$

Analysis. We found that both i and $-i$ were zeros, but only one of these zeros needed to be given. If i is a zero of a polynomial with real coefficients, then $-i$ must also be a zero of the polynomial because $-i$ is the complex conjugate of i .

If $2+3i$ is a zero of a polynomial with real coefficients, would $2-3i$ also need to be a zero?

Yes. When any complex number with an imaginary component is given as a zero of a polynomial with real coefficients, the conjugate must also be a zero of the polynomial.

Try it 3.6.11

Find a third degree polynomial with real coefficients that has zeros of 5 and $-2i$ such that $f(1) = 10$.

Answer

$f(x) = -\frac{1}{2}x^3 + \frac{5}{2}x^2 - 2x + 10$

Key Concepts

- To find $f(k)$, determine the remainder of the polynomial $f(x)$ when it is divided by $x - k$. This is known as the Remainder Theorem.
- According to the Factor Theorem, k is a zero of $f(x)$ if and only if $(x - k)$ is a factor of $f(x)$.
- According to the Rational Zero Theorem, each rational zero of a polynomial function with integer coefficients will be equal to a factor of the constant term divided by a factor of the leading coefficient.
- When the leading coefficient is 1, the possible rational zeros are the factors of the constant term.
- Synthetic division can be used to find the zeros of a polynomial function.
- According to the Fundamental Theorem, every polynomial function with degree greater than 0 has at least one complex zero.
- Allowing for multiplicities, a polynomial function will have the same number of factors as its degree. Each factor will be in the form $(x - c)$, where c is a complex number.
- The number of positive real zeros of a polynomial function is either the number of sign changes of the function or less than the number of sign changes by an even integer.
- The number of negative real zeros of a polynomial function is either the number of sign changes of $f(-x)$ or less than the number of sign changes by an even integer.

Glossary

Descartes' Rule of Signs

a rule that determines the maximum possible numbers of positive and negative real zeros based on the number of sign changes of $f(x)$ and $f(-x)$

Factor Theorem

k is a zero of polynomial function $f(x)$ if and only if $(x - k)$ is a factor of $f(x)$

Fundamental Theorem of Algebra

a polynomial function with degree greater than 0 has at least one complex zero

Intermediate Value Theorem

If two points of a polynomial are on opposite sides of the x-axis, there is at least one zero between them.

Linear Factorization Theorem

allowing for multiplicities, a polynomial function will have the same number of factors as its degree, and each factor will be in the form $(x - c)$, where c is a complex number

Rational Zero Theorem

the possible rational zeros of a polynomial function have the form $\frac{p}{q}$ where p is a factor of the constant term and q is a factor of the leading coefficient.

Remainder Theorem

if a polynomial $f(x)$ is divided by $(x - k)$, then the remainder is equal to the value $f(k)$

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3.6e: Exercises - Zeroes of Polynomial Functions

A: Concepts

Exercise $3.6e.A$: Concepts

1) Describe a use for the Remainder Theorem.

2) Explain why the Rational Zero Theorem does not guarantee finding zeros of a polynomial function.

3) What is the difference between rational and real zeros?

4) If Descartes' Rule of Signs reveals a 0 or 1 change of signs, what specific conclusion can be drawn?

5) If synthetic division reveals a zero, why should we try that value again as a possible solution?

Answers to odd exercises:

1. The theorem can be used to evaluate a polynomial.

3. Rational zeros can be expressed as fractions whereas real zeros include irrational numbers.

5. Polynomials can have repeated zeros, so the fact that number is a zero doesn't preclude it being a zero again.

B: Use the Remainder Theorem to Evaluate a Polynomial

Exercise $3.6e.B$: Use the Remainder Theorem

★ Use synthetic division to evaluate $p(c)$ and write $p(x)$ in the form $p(x)=(x-c)q(x)+r$.

6. $p(x)=2x^2-x+1,\ c=4$
7. $p(x)=4x^2-33x-180,\ c=12$
8. $p(x)=2x^3-x+6,\ c=-3$
9. $p(x)=x^3+2x^2+3x+4,\ c=-1$
10. $p(x)=3x^3-6x^2+4x-8,\ c=2$
11. $p(x)=8x^3+12x^2+6x+1,\ c=-\frac{1}{2}$
12. $p(x)=2x^4+x^3-4x^2+10x-7,\ c=\frac{3}{2}$
13. $p(x)=x^4-3x^3-20x^2-24x-8,\ c=7$
14. $p(x)=x^4-5x^2-8x-12,\ c=3$
15. $p(x)=x^4-5x^3+x^2+5,\ c=2$

Answers to odd exercises:

7. $p(12)=0,\ p(x)=(x-12)(4x+15)$
9. $p(-1)=2,\ p(x)=(x+1)(x^2+x+2)+2$
11. $p\left(-\frac{1}{2}\right)=0,\ p(x)=(2x+1)(4x^2+4x+1)$
13. $p(7)=216,\ p(x)=(x-7)(x^3+4x^2+8x+32)+216$
15. $p(2)=-15,\ p(x)=(x-2)(x^3-3x^2-5x-10)-15$

C: Given one zero or factor, find all Real Zeros, and factor a polynomial

Exercise $3.6e.C$: Use the Factor Theorem given one zero or factor

★ Given a polynomial and one of its factors, find the rest of the real zeros and write the polynomial as a product of linear and irreducible quadratic factors.

17) $f(x)=2x^3+x^2-5x+2$; Factor: $(x+2)$
18) $f(x)=3x^3+x^2-20x+12$; Factor: $(x+3)$
19) $f(x)=2x^3+3x^2+x+6$; Factor: $(x+2)$
20) $f(x)=-5x^3+16x^2-9$; Factor: $(x-3)$
21) $f(x)=x^3+3x^2+4x+12$; Factor: $(x+3)$
22) $f(x)=4x^3-7x+3$; Factor: $(x-1)$
23) $f(x)=2x^3+5x^2-12x-30$; Factor: $(2x+5)$
24) $f(x)=2x^3-9x^2+13x-6$; Factor: $(x-1)$

Answers to odd exercises:

17. $-2, 1, \frac{1}{2};\ f(x)=(x+2)(x-1)(2x-1)$
19. $-2;\ f(x)=(x+2)(2x^2-x+3)$
21. $-3;\ f(x)=(x+3)(x^2+4)$
23. $-\frac{5}{2},\ \sqrt{6},\ -\sqrt{6};\ f(x)=(2x+5)(x-\sqrt{6})(x+\sqrt{6})$

★ Given a polynomial and c, one of its zeros, find the rest of the real zeros and write the polynomial as a product of linear and irreducible quadratic factors. It is possible some factors are repeated.

25. $p(x) = x^3 - 24x^2 + 192x - 512,\ c = 8$
26. $p(x) = 3x^3 + 4x^2 - x - 2,\ c = \frac{2}{3}$
27. $p(x) = 2x^3 - 3x^2 - 11x + 6,\ c = \frac{1}{2}$
28. $p(x) = x^3 + 2x^2 - 3x - 6,\ c = -2$
29. $p(x) = 2x^3 - x^2 - 10x + 5,\ c = \frac{1}{2}$
30. $p(x) = 4x^4 - 28x^3 + 61x^2 - 42x + 9,\ c = \frac{1}{2}$
31. $p(x) = x^5 + 2x^4 - 12x^3 - 38x^2 - 37x - 12,\ c = -1$
32. $p(x) = 2x^5 + 7x^4 - 18x^2 - 8x + 8,\ c = \frac{1}{2}$
33. $p(x) = 3x^5 + 2x^4 - 15x^3 - 10x^2 + 12x + 8,\ c = -\frac{2}{3}$

Answers to odd exercises:

25. zeros: 8; $p(x) = (x-8)^3$
27. zeros: $\frac{1}{2}, -2, 3$; $p(x) = (2x-1)(x+2)(x-3)$
29. zeros: $\frac{1}{2}, \pm\sqrt{5}$; $p(x) = (2x-1)(x+\sqrt{5})(x-\sqrt{5})$
31. zeros: $-1, -3, 4$; $p(x) = (x+1)^3(x+3)(x-4)$
33. zeros: $-2,\ -1,\ -\frac{2}{3},\ 1,\ 2$
;
$p(x) = (x+2)(x+1)(x-1)(x-2)(3x+2)$

D: List all Possible Rational Zeros

Exercise 3.6e.D: Use the Rational Zero Theorem

★ Use the Rational Zeros Theorem to list all possible rational zeros for each given function.

35) $f(x) = 2x^3 + 3x^2 - 8x + 5$
36) $f(x) = 3x^3 + 5x^2 - 5x + 4$
37) $f(x) = 6x^4 - 10x^2 + 13x + 1$
38) $f(x) = 4x^5 - 10x^4 + 8x^3 + x^2 - 8$
39) $f(x) = x^3 - 2x^2 - 5x + 6$
40) $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
41) $f(x) = x^3 - 7x^2 + x - 7$
42) $f(x) = x^3 + 4x^2 - 11x + 6$
43) $f(x) = x^4 - 9x^2 - 4x + 12$
44) $f(x) = -2x^3 + 19x^2 - 49x + 20$
45) $f(x) = -17x^3 + 5x^2 + 34x - 10$
46) $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$
47) $f(x) = 3x^3 + 3x^2 - 11x - 10$
48) $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$

Answers to odd exercises:

35) $\pm 5, \pm 1, \pm\frac{1}{2}, \pm\frac{5}{2}$
37) $\pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}$
39) $\pm 1,\ \pm 2,\ \pm 3,\ \pm 6$
41) $\pm 1,\ \pm 7$
43) $\pm 1,\ \pm 2,\ \pm 3,\ \pm 4,\ \pm 6,\ \pm 12$
45) $\pm 1,\ \pm 2,\ \pm 5,\ \pm 10,\ \pm\frac{1}{17}, \pm\frac{2}{17}, \pm\frac{5}{17}, \pm\frac{10}{17}$
47) $\pm 1,\ \pm 2,\ \pm 5,\ \pm 10,\ \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{5}{3}, \pm\frac{10}{3}$

E: Find all Zeros that are Rational

Exercise 3.6e.E: Find all zeros that are rational

★ Use the Rational Zero Theorem to find all real number zeros.

49) $x^3 - 3x^2 - 10x + 24 = 0$
50) $2x^3 + 7x^2 - 10x - 24 = 0$
51) $x^3 + 2x^2 - 9x - 18 = 0$
52) $x^3 + 5x^2 - 16x - 80 = 0$
53) $x^3 - 3x^2 - 25x + 75 = 0$
54) $2x^3 - 3x^2 - 32x - 15 = 0$
55) $2x^3 + x^2 - 7x - 6 = 0$
56) $2x^3 - 3x^2 - x + 1 = 0$
57) $3x^3 - x^2 - 11x - 6 = 0$
58) $x^4 - 2x^3 - 7x^2 + 8x + 12 = 0$
59) $4x^3 - 3x + 1 = 0$
60) $4x^4 + 4x^3 - 25x^2 - x + 6 = 0$
61) $x^4 + 2x^3 - 9x^2 - 2x + 8 = 0$
62) $x^4 + 2x^3 - 4x^2 - 10x - 5 = 0$
63) $-5x^4 + 4x^3 - 19x^2 + 16x + 4 = 0$

Answers to odd exercises:

49) $-3,\ 2,\ 4$
51) $-2,\ 3,\ -3$
53) $3, -5, 5$
55) $2,\ -1,\ -\frac{3}{2}$
57) $-\frac{2}{3},\ \frac{1\pm\sqrt{13}}{2}$
59) $-1,\ \frac{1}{2}$
61) $1,\ 2,\ -1,\ -4$
63) $-\frac{1}{5},\ 1$

★ Find the real zeros of the polynomial. State the multiplicity of each real zero.

65. $f(x) = x^3 - 2x^2 - 5x + 6$
66. $f(x) = x^3 + 4x^2 - 11x + 6$
67. $f(x) = x^4 - 9x^2 - 4x + 12$
68. $f(x) = -17x^3 + 5x^2 + 34x - 10$
69. $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$
70. $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$
71. $f(x) = 2x^3 + 7x^2 + 4x - 4$
72. $f(x) = -2x^4 - 3x^3 + 10x^2 + 12x - 8$

Answers to odd exercises:

65. $x = 1$ (mult. 1), $x = 3$ (mult. 1), $x = -2$ (mult. 1)
67. $x = -2$ (mult. 2), $x = 1$ (mult. 1), $x = 3$ (mult. 1)
69. $x = \frac{1}{2}$ (mult. 2), $x = -\frac{1}{3}$ (mult. 2)
71. $x = -2$ (mult. 2), $x = \frac{1}{2}$ (mult. 1)

F: Find all zeros (both real and imaginary)

Exercise $3.6e.F$: Find all zeros

★ Use the Rational Zero Theorem to find all complex solutions (real and non-real).

72.5) $x^3 + x^2 + x + 1 = 0$
73) $x^3 - 8x^2 + 25x - 26 = 0$
74) $x^3 + 13x^2 + 57x + 85 = 0$
75) $3x^3 - 4x^2 + 11x + 10 = 0$
76) $x^4 + 2x^3 + 22x^2 + 50x - 75 = 0$
77) $2x^3 - 3x^2 + 32x + 17 = 0$

Answers to odd exercises:

73. $2, 3 + 2i, 3 - 2i$
75. $-\frac{2}{3}, 1 + 2i, 1 - 2i$
77. $-\frac{1}{2}, 1 + 4i, 1 - 4i$

G: Find all zeros and sketch

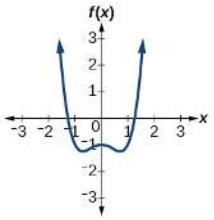
Exercise $3.6e.G$: Find all zeros and sketch

★ Determine the end behaviour, all the real zeros, their multiplicity, and y-intercept. Sketch the function. (Use synthetic division to find a rational zero. Use the quotient to find the next zero).

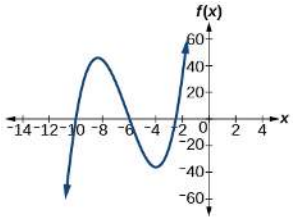
78) $f(x) = x^3 - 1$
79) $f(x) = x^4 - x^2 - 1$
80) $f(x) = x^3 - 2x^2 - 5x + 6$
81) $f(x) = 2x^3 + 37x^2 + 200x + 300$
82) $f(x) = x^4 + 2x^3 - 12x^2 + 14x - 5$
83) $f(x) = 2x^4 - 5x^3 - 5x^2 + 5x + 3$
84) $f(x) = x^3 - 2x^2 - 16x + 32$
85) $f(x) = -x^3 - 7x^2 - 8x + 16$
86) $f(x) = -x^4 - 4x^3 + 3x^2 + 10x - 8$
87. $f(x) = x^4 - 6x^3 + 8x^2 + 6x - 9$
88. $f(x) = x^4 + 4x^3 - 5x^2 - 36x - 36$
89. $f(x) = x^5 - x^4 - 5x^3 + x^2 + 8x + 4$
90. $f(x) = x^4 + 2x^3 + 6x - 9$

Answers to odd exercises:

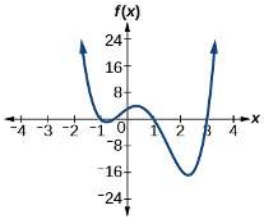
79. zeros (odd multiplicity): $\pm\sqrt{\frac{1+\sqrt{5}}{2}}$, 2 imaginary zeros, y-intercept $(0,1)$



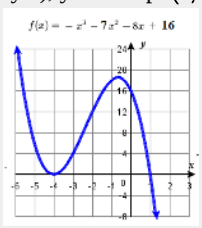
81. zeros (odd multiplicity): $\{-10,-6,\frac{-5}{2}\}$; y-intercept: $(0,300)$



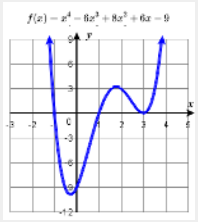
83. zeros (odd multiplicity); $\{-1,1,3,\frac{-1}{2}\}$, y-intercept $(0,3)$.



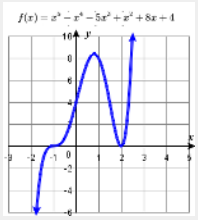
85. zeros; -4 (multiplicity 2), 1 (multiplicity 1), y-intercept $(0,16)$.



87. odd multiplicity zeros: $\{1,-1\}$; even multiplicity zero: $\{3\}$; y-intercept $(0,-9)$.



89. odd multiplicity zero: $\{-1\}$, even multiplicity zero $\{2\}$. y-intercept $(0,4)$.



H: Given zeros, construct a polynomial function

Exercise 3.6e.H: Given zeros, construct a polynomial function

★ **Construct a polynomial function of least degree possible using the given information. You may leave the polynomial in factored form.**

91) A lowest degree polynomial with real coefficients and zero $3i$

92) A lowest degree polynomial with rational coefficients and zeros: 2 and $\sqrt{6}$

93) A lowest degree polynomial with integer coefficients and Real roots: -1 (with multiplicity 2), and 1.

94) A lowest degree polynomial with integer coefficients and Real roots: -2, and $\frac{1}{2}$ (with multiplicity 2)

95) A lowest degree polynomial with integer coefficients and Real roots:$-\frac{1}{2},0,\frac{1}{2}$

96) A lowest degree polynomial with integer coefficients and Real roots: $-4,-1,1,4$

97) A lowest degree polynomial with integer coefficients and Real roots: $-1,1,3$

98. A lowest degree polynomial with real coefficients and zeros: -2 and $-5i$

99. A lowest degree polynomial with real coefficients and zeros: 4 and $2i$.

100. The solutions to $p(x)=0$ are $x=\pm 3$ and $x=6$. The leading term of $p(x)$ is $7x^4$.
The point $(-3,0)$ is a local minimum on the graph of $y=p(x)$.

101. The solutions to $p(x)=0$ are $x=\pm 3$, $x=-2$, and $x=4$, The leading term of $p(x)$ is $-x^5$.
The point $(-2,0)$ is a local maximum on the graph of $y=p(x)$.

102. p is degree 4. as $x\to\infty$, $p(x)\to-\infty$ p has exactly three x-intercepts: $(-6,0)$, $(1,0)$ and $(117,0)$.
The graph of $y=p(x)$ crosses through the x-axis at $(1,0)$.

103. Find a quadratic polynomial with **integer** coefficients which has $x=\frac{3}{5}\pm\frac{\sqrt{29}}{5}$ as its real zeros.

Answers to odd exercises:

91. $f(x) = (x^2 + 9)$
93. $f(x) = (x + 1)^2(x - 1)$
95. $f(x) = x(2x + 1)(2x - 1)$
97. $f(x) = (x + 1)(x - 1)(x - 3)$
99. $p(x) = (x - 4)(x - 2i)(x + 2i) = x^3 - 4x^2 + 4x - 16$
101. $p(x) = -(x + 2)^2(x - 3)(x + 3)(x - 4)$
103. $p(x) = 5x^2 - 6x - 4$

I: Use Intermediate Value Theorem

Exercise $3.6e.I$: Intermediate Value Theorem

★ Use the Intermediate Value Theorem to confirm the polynomial f has at least one zero within the given interval.

104) $f(x) = x^3 - 9x$, between $x = -4$ and $x = -2$.
105) $f(x) = x^3 - 9x$, between $x = 2$ and $x = 4$.
106) $f(x) = x^5 - 2x$, between $x = 1$ and $x = 2$.
107) $f(x) = -x^4 + 4$, between $x = 1$ and $x = 3$.
108) $f(x) = -2x^3 - x$, between $x = -1$ and $x = 1$.
109) $f(x) = x^3 - 100x + 2$, between $x = 0.01$ and $x = 0.1$

Answers to odd exercises:

105. $f(2) = -10,\ f(4) = 28$.
Sign change confirms.
107. $f(1) = 3,\ f(3) = -77$.
Sign change confirms.
109. $f(0.01) = 1.000001,\ f(0.1) = -7.999$.
Sign change confirms.

x

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3.7: The Reciprocal Function

Learning Objectives

- Use arrow notation to describe asymptotic behaviour.
- Use transformations to graph rational functions

Using Arrow Notation

We have seen the graphs of the basic **reciprocal function** and the squared reciprocal function from our study of toolkit functions. Examine these graphs, as shown in Figure **3.7.1**, and notice some of their features.

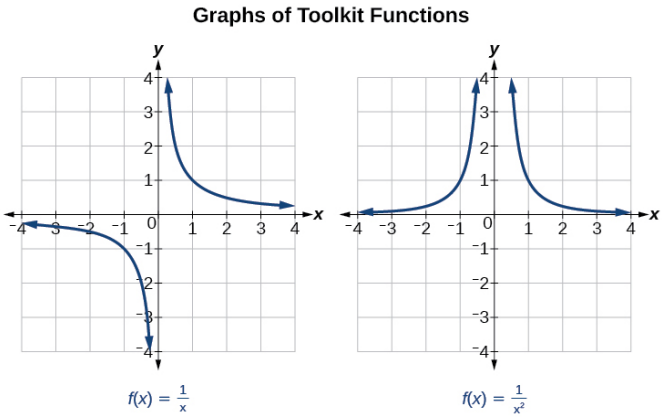


Figure **3.7.1**

Several things are apparent if we examine the graph of $f(x) = \dfrac{1}{x}$.

- On the left branch of the graph, the curve approaches the x-axis $(y = 0)$ as $x \to -\infty$.
- As the graph approaches $x = 0$ from the left, the curve drops, but as we approach zero from the right, the curve rises.
- Finally, on the right branch of the graph, the curves approaches the x-axis $(y = 0)$ as $x \to \infty$.

To summarize, we use **arrow notation** to show that x or $f(x)$ is approaching a particular value in the table below.

Symbol	Meaning
$x \to a^-$	x approaches a from the left $x < a$ but close to a
$x \to a^+$	x approaches a from the right $x > a$ but close to a
$x \to \infty$	x approaches infinity ("increases without bound")
$x \to -\infty$	x approaches negative infinity ("decreases without bound")

Symbol	Meaning
$f(x) \to \infty$	the output approaches infinity ("increases without bound")
$f(x) \to -\infty$	the output approaches negative infinity ("decreases without bound")
$f(x) \to a$	the output approaches a

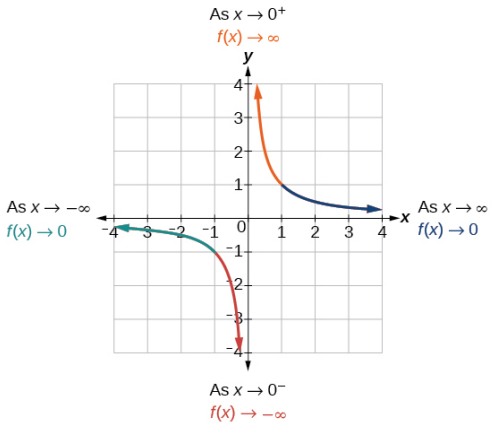


Figure **3.7.2**. Illustration of arrow notation used for Vertical Asymptote $x = 0$ and Horizontal Asymptote $y = 0$

Local Behavior of $f(x) = \frac{1}{x}$

Let's begin by looking at the reciprocal function, $f(x) = \frac{1}{x}$. We cannot divide by zero, which means the function is undefined at $x = 0$; so zero is not in the domain. As the input values approach zero from the left side (becoming very small, negative values), the

function values decrease without bound (in other words, they approach negative infinity).

Table **3.7.1**: Illustration of arrow notation - As $x \to 0^-$, $f(x) \to -\infty$

x	–0.1	–0.01	–0.001	–0.0001
$f(x) = \frac{1}{x}$	–10	–100	–1000	–10,000

As the input values approach zero from the right side (becoming very small, positive values), the function values increase without bound (approaching infinity).

Table **3.7.2**: Illustration of arrow notation - As $x \to 0^+$, $f(x) \to \infty$.

x	0.1	0.01	0.001	0.0001
$f(x) = \frac{1}{x}$	10	100	1000	10,000

See Figure **3.7.3** for how this behaviour appears on a graph.

Definition: VERTICAL ASYMPTOTE

A **vertical asymptote** of a graph is a vertical line $x = a$ where the graph tends toward positive or negative infinity as the inputs approach a.

In arrow notation this is written

$$\text{As } x \to a,\ f(x) \to \infty, \text{ or as } x \to a,\ f(x) \to -\infty.$$

This can also be written in limit notation as:

$$\lim_{x \to a} f(x) \to \infty, \text{ or as } \lim_{x \to a} f(x) \to -\infty$$

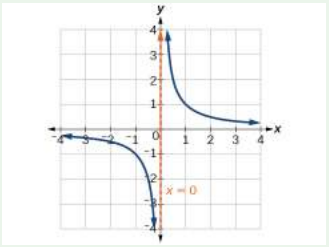


Figure **3.7.3**: Example of a Vertical Asymptote, $x = 0$

End Behavior of $f(x) = \frac{1}{x}$

As the values of x approach infinity, the function values approach 0. As the values of x approach negative infinity, the function values approach 0. See Figure **3.7.4**) for how this behaviour appears on a graph.. Symbolically, using arrow notation

$$\text{As } x \to \infty,\ f(x) \to 0, \text{ and as } x \to -\infty,\ f(x) \to 0.$$

Based on this overall behavior and the graph, we can see that the function approaches 0 but never actually reaches 0; it seems to level off as the inputs become large. This behavior creates a **horizontal asymptote**, a horizontal line that the graph approaches as the input increases or decreases without bound. In this case, the graph is approaching the horizontal line $y = 0$.

Definition: HORIZONTAL ASYMPTOTE

A **horizontal asymptote** of a graph is a horizontal line $y = b$ where the graph approaches the line as the inputs increase or decrease without bound.

In arrow notation this is written

$$\text{As } x \to \infty,\ f(x) \to b \quad \text{or} \quad x \to -\infty,\ f(x) \to b.$$

This can also be written in limit notation as:

$$\lim_{x \to \infty} f(x) \to b, \text{ or } \lim_{x \to -\infty} f(x) \to b$$

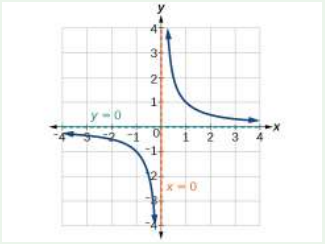
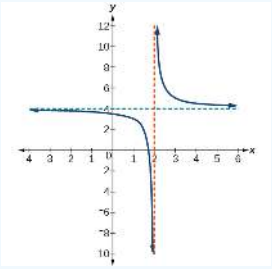


Figure **3.7.4**: Example of a Horizontal Asymptote, $y = 0$

Example 3.7.1: Using Arrow Notation.

Use arrow notation to describe the end behavior and local behavior of the function graphed in below.



Solution

Local Behaviour. Notice that the graph is showing a vertical asymptote at $x = 2$, which tells us that the function is undefined at $x = 2$.

$$\text{As } x \to 2^-,\ f(x) \to -\infty, \text{ and as } x \to 2^+,\ f(x) \to \infty.$$

End behaviour. And as the inputs decrease without bound, the graph appears to be leveling off at output values of 4, indicating a horizontal asymptote at $y = 4$. As the inputs increase without bound, the graph levels off at 4.

$$\text{As } x \to \infty,\ f(x) \to 4 \text{ and as } x \to -\infty,\ f(x) \to 4.$$

Try It 3.7.1

Use arrow notation to describe the end behavior and local behavior for the reciprocal squared function.

Answer

End behavior: as $x \to \pm\infty$, $f(x) \to 0$;

Local behavior: as $x \to 0$, $f(x) \to \infty$ (there are no x- or y-intercepts)

Graphing using Transformations

When graphing vertical and horizontal shifts of the reciprocal function, the order in which horizontal and vertical translations are applied does not affect the final graph.

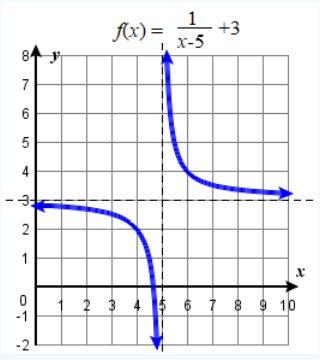
Example 3.7.2:

Sketch the graph of $g(x) = \dfrac{1}{x-5} + 3$.

Solution

Begin with the reciprocal function and identify the translations.

$$y = \frac{1}{x} \qquad \textit{Basic function}$$
$$y = \frac{1}{x-5} \qquad \textit{Horizontal shift right 5 units}$$
$$y = \frac{1}{x-5} + 3 \qquad \textit{Vertical shift up 3 units}$$



Start the graph by first drawing the vertical and horizontal asymptotes. Then use the location of the asymptotes to sketch in the rest of the graph.

It is easiest to graph translations of the reciprocal function by writing the equation in the form $y = \pm\frac{1}{x+c} + d$.

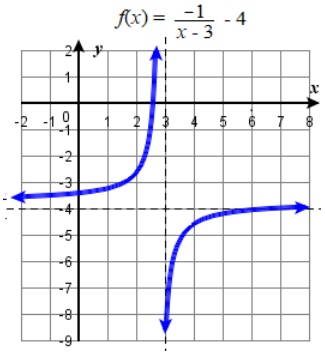
Example 3.7.3:

Sketch the graphs of $f(x) = \frac{-1}{x-3} - 4$ and $g(x) = \frac{1}{-x-2} + 1$.

Solution

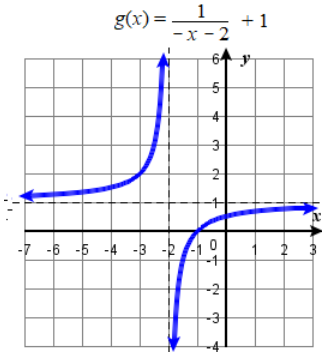
$$\begin{aligned} f(x) &= \frac{-1}{x-3} - 4 \\ &= -\frac{1}{x-3} \end{aligned}$$

To graph f, start with the parent function $y = \frac{1}{x}$, shift right 3 units, reflect over the x-axis, and shift down 4 units.



$$\begin{aligned} g(x) &= \frac{1}{-x-2} + 1 \\ &= \frac{1}{-(x+2)} + 1 \\ &= -\frac{1}{x+2} + 1 \end{aligned}$$

To graph g, start with the parent function $y = \frac{1}{x}$, shift left 2 units, reflect over the x-axis, and shift up 1 unit.



When a rational function consists of a linear numerator and linear denominator, it is actually just a translation of the reciprocal function. To see how to graph the function using transformations, long division or synthetic division on the original function must be done to obtain a more user friendly form of the equation.

Example 3.7.4: Use Transformations to Graph a Rational Function.

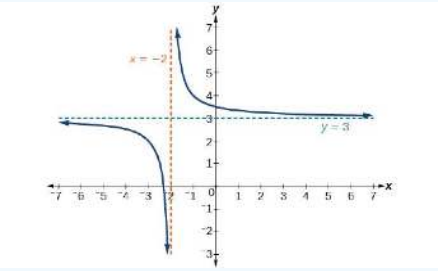
Sketch a graph of the function $f(x) = \frac{3x+7}{x+2}$. Identify the horizontal and vertical asymptotes of the graph, if any.

Solution

Use long division or synthetic division to obtain an equivalent form of the function, $f(x) = \frac{1}{x+2} + 3$.

Written in this form, it is clear the graph is that of the reciprocal function shifted two units left and three units up.

The graph of the shifted function is displayed to the right.



Local Behaviour. Notice that this function is undefined at $x = -2$, and the graph also is showing a vertical asymptote at $x = -2$.

$$\text{As } x \to -2^-,\ f(x) \to -\infty, \text{ and as } x \to -2^+,\ f(x) \to \infty.$$

End Behaviour. As the inputs increase and decrease without bound, the graph appears to be leveling off at output values of 3, indicating a horizontal asymptote at $y = 3$.

$$\text{As } x \to \pm\infty,\ f(x) \to 3.$$

Analysis. Notice that horizontal and vertical asymptotes are shifted left 2 and up 3 along with the function.

Try It 3.7.5: Graph and construct an equation from a description

State the transformations to perform on the graph of $y = \frac{1}{x}$ needed to graph $f(x) = \frac{18 - 14x}{x + 32}$.

Answer

$f(x) = -\frac{1}{x + 32} + 14$. Shift left 32 units, reflect over the x-axis, and shift up 14 units.

Try It 3.7.6: Graph and construct an equation from a description

Construct the equation, sketch the graph, and find the horizontal and vertical asymptotes of the reciprocal squared function that has been shifted right 3 units and down 4 units.

Answer

The function and the asymptotes are shifted 3 units right and 4 units down.

As $x \to 3$, $f(x) \to \infty$, and as $x \to \pm\infty$, $f(x) \to -4$.

The function is $f(x) = \frac{1}{(x-3)^2} - 4$.

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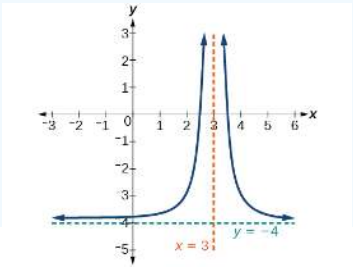

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3.7e: Exercises for the reciprocal function

A: Graph translations of $\frac{1}{x}$

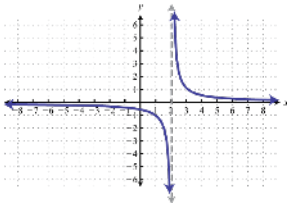
Exercise 3.7e.A

★ Graph the given function. Identify the translations on $y = \frac{1}{x}$ used to sketch the graph. Then state the domain and range.

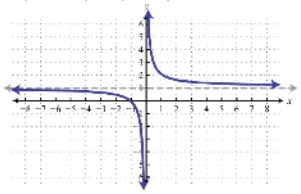
1. $f(x) = \frac{1}{x-2}$
2. $f(x) = \frac{1}{x+3}$
3. $f(x) = \frac{1}{x} + 5$
4. $f(x) = \frac{1}{x} - 3$
5. $f(x) = \frac{1}{x+1} - 2$
6. $f(x) = \frac{1}{x-3} + 4$

Answers to odd exercises:

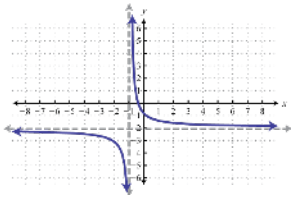
1. Shift right 2 units;
domain: $(-\infty, 2) \cup (2, \infty)$;
range: $(-\infty, 0) \cup (0, \infty)$



3. Shift up 5 units;
domain: $(-\infty, 0) \cup (0, \infty)$;
range: $(-\infty, 1) \cup (1, \infty)$



5. Shift left 1 unit and down 2 units;
domain: $(-\infty, -1) \cup (-1, \infty)$;
range: $(-\infty, -2) \cup (-2, \infty)$

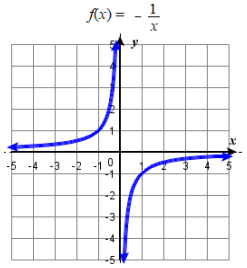


★ Use the transformations to graph the following functions.

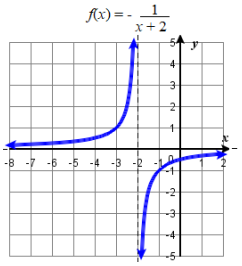
7. $f(x) = -\frac{1}{x}$
8. $f(x) = \frac{-1}{x+1} + 2$
9. $f(x) = \frac{-1}{x+2}$
10. $f(x) = \frac{1}{-x}$
11. $f(x) = \frac{1}{-x+2}$
12. $f(x) = \frac{1}{-x-1} + 2$

Answers to odd exercises:

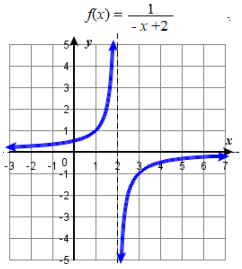
7. Reflect over x-axis;
domain: $(-\infty, 0) \cup (0, \infty)$;
range: $(-\infty, 0) \cup (0, \infty)$



9. Shift left 2 units,
Reflect over x-axis;
domain: $(-\infty, -2) \cup (-2, \infty)$;
range: $(-\infty, 0) \cup (0, \infty)$



11. Shift left 2 units,
Reflect over y-axis;
domain: $(-\infty, -2) \cup (-2, \infty)$;
range: $(-\infty, 0) \cup (0, \infty)$

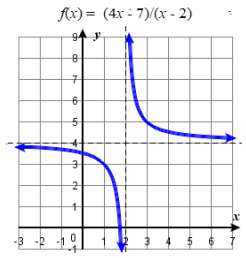


★ Graph using translations of $\frac{1}{x}$ by first using division to rewrite the function.

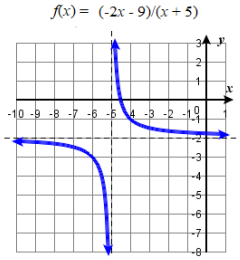
13. $f(x) = \frac{4x-7}{x-2}$
14. $f(x) = \frac{4-x}{x-3}$
15. $f(x) = \frac{-2x-9}{x+5}$
16. $f(x) = \frac{2x+5}{x+3}$
17. $f(x) = \frac{3-4x}{x-1}$
18. $f(x) = \frac{x+1}{x+2}$

Answers to odd exercises:

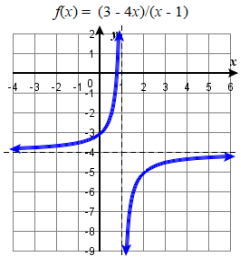
13. $f(x)=4+\frac{1}{x-2}$
Right 2, up 4



15. $f(x)=-2+\frac{1}{x+5}$
Left 5, down 2



17. $f(x)=-4-\frac{1}{x-1}$
Right 1, reflect over x-axis, down 4



★ Graph using translations of $\frac{1}{x}$ by first using division to rewrite the function.

19. $f(x)=\frac{x-3}{x+2}$

20. $f(x)=\frac{x+2}{x-4}$

21. $f(x)=\frac{5-x}{x+1}$

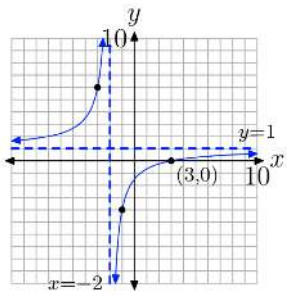
22. $f(x)=\frac{x+2}{4-x}$

23. $f(x)=\frac{2x-5}{x+1}$

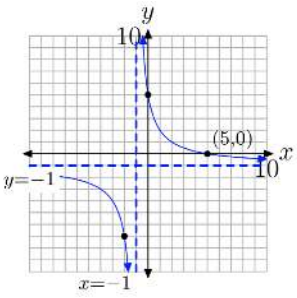
24. $f(x)=\frac{2x+5}{3-x}$

Answers to odd exercises:

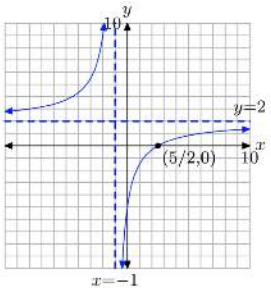
19. $f(x)=1-5\frac{1}{x+2}$
Left 2, Reflect over x-axis, y -> 5y, up 1



21. $f(x)=-1+6\frac{1}{x+1}$
Left 1, y -> 6y, down 1



23. $f(x)=2-7\frac{1}{x+1}$
Left 1, reflect over x-axis, y -> 7y, up 2



B: Construct a graph from a verbal description

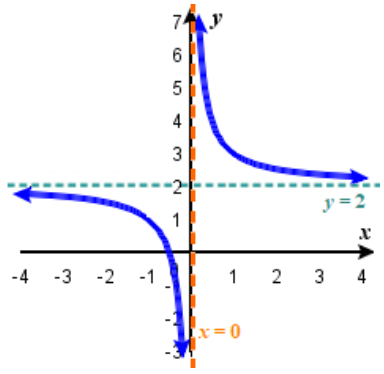
Exercise 3.7e.B

★ Use the given transformation to graph the function. Note the vertical and horizontal asymptotes.

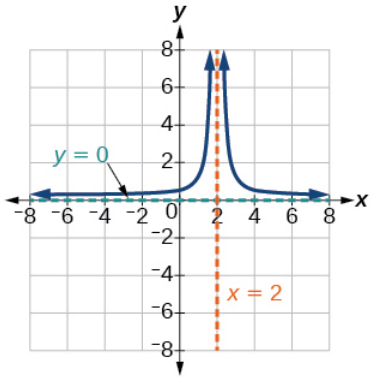
31. The reciprocal function shifted up two units.
32. The reciprocal function shifted down one unit and left three units.
33. The reciprocal squared function shifted to the right **2** units.
34. The reciprocal squared function shifted down **2** units and right **1** unit.

Answers to odd exercises.

31. V.A. $x = 0$, H.A. $y = 2$



33. V.A. $x = 2$, H.A. $y = 0$



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3.8: Polynomial and Rational Inequalities

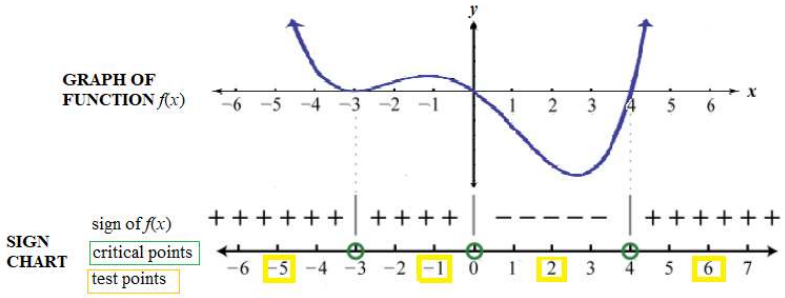
Skills to Develop

- Solve polynomial inequalities.
- Solve rational inequalities.

Solving Polynomial Inequalities

A **polynomial inequality** is a mathematical statement that relates a polynomial expression as either less than or greater than another. We can use **sign charts** to solve polynomial inequalities with one variable.

Graphs are helpful in providing a visualization to the solutions of polynomial inequalities. Examine the graph below to see the relationship between a graph of a polynomial and its corresponding sign chart.



In this graph, the x-intercepts, -3, 0 and 4, are **critical points**, which are the **only** places where the graph may possibly change from being above the x-axis (where we say that $f(x) > 0$)), to below the x-axis (where we say that $f(x) < 0$). Within each interval between two adjacent critical points, the graph is either always above the x-axis, or always below the x-axis. Thus, finding the critical points of a polynomial inequality plays a fundamental role in solving a polynomial inequality.

How to: Solve a Polynomial Inequality.

1. **Step 1**: Rewrite the inequality so there is a zero on the right side of the inequality. The expression on the left side designate as $f(x)$.
2. **Step 2**: Find the **critical numbers**. Critical numbers for polynomial functions are the real number solutions to $f(x) = 0$. Draw a number line with the critical numbers labelled. Draw an open circle at each critical number if the inequality uses " <" or ">"; draw a filled in circle at each critical number if the inequality uses " $\leq$" or "$\geq$".
3. **Step 3**: Create a **sign chart**. The critical numbers partition the number line into regions. Choose a test value for each region, including one to the left of all the critical values and one to the right of all the critical values. For each test value, v, determine if $f(v)$ is positive or negative and record the result on the sign chart for each region.
4. **Step 4**: Use the sign chart to find the set of all values of x for which the inequality is true. Write the answer in interval notation.

Example 3.8.1

Solve $x(x+3)^2(x-4) < 0$.

Solution

Step 2. Begin by finding the critical numbers. For a polynomial inequality in standard form, with zero on one side, the critical numbers are the roots. Because $f(x) = x(x+3)^2(x-4)$ is given in its factored form the roots are apparent. Here the roots are: $0, -3$, and 4. Because of the strict inequality, plot them using open dots on a number line.

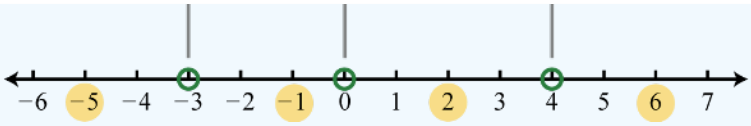


Figure 3.7.1 Critical Numbers

Step 3. In this case, the critical numbers partition the number line into four regions. Test values in each region to determine if f is positive or negative. Here we choose test values $-5, -1, 2$, and 6. Remember that we are only concerned with the sign ($+$ or $-$) of the result.

$$\begin{aligned} f(-5) &= (-5)(-5+3)^2(-5-4) = (-)(-)^2(-) && = +Positive \\ f(-1) &= (-1)(-1+3)^2(-1-4) = (-)(+)^2(-) && = +Positive \\ f(2) &= (2)(2+3)^2(2-4) = (+)(+)^2(-) && = -Negative \\ f(6) &= (6)(6+3)^2(6-4) = (+)(+)^2(+) && = +Positive \end{aligned}$$

After testing values we can complete a sign chart.

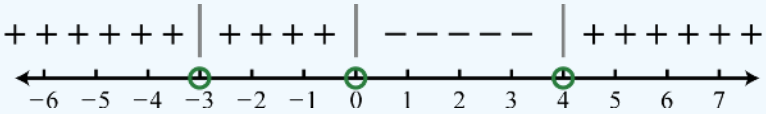


Figure 3.7.2 Sign Chart

Step 4. The question asks us to find the values where $f(x) < 0$, or where the function is negative. From the sign chart we can see that the function is negative for x-values in between 0 and 4.

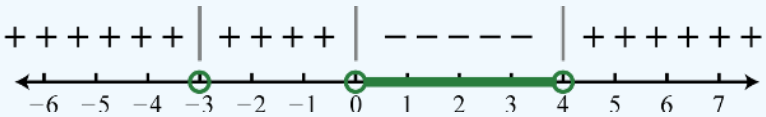


Figure 3.7.3 Solution Set on Number Line

We can express this solution set in two ways:

$$\begin{aligned} \{x|0 < x < 4\} \quad & Set\ notation \\ (0,4) \quad & Interval\ notation \end{aligned}$$

In this sectionwe will continue to present solution sets using interval notation. **Answer**: $(0, 4)$

Graphs can help us to visualize solutions of polynomial inequalities. Below is the graph of the function in the above example. Compare the graph to its corresponding sign chart. Notice that the sign chart is positive when the graph is above the x-axis and negative when the graph is below the x-axis. The graph crosses or touches the x-axis at the critical points.

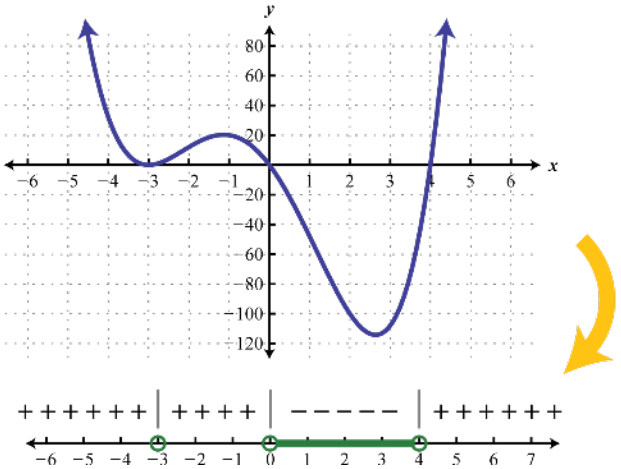


Figure 3.7.4 Polynomial with Sign Chart and Solution Set

Certainly it may not be the case that the polynomial is factored nor that it has zero on one side of the inequality. To model a function using a sign chart, all of the terms should be on one side and zero on the other. The general steps for solving a polynomial inequality are listed in the following example.

Example 3.8.2

Solve: $2x^4 > 3x^3 + 9x^2$.

Solution

Step 1: Obtain zero on one side of the inequality. In this case, subtract to obtain a polynomial on the left side in standard form.

$$2x^4 > 3x^3 + 9x^2$$
$$2x^4 - 3x^3 - 9x^2 > 0$$

Step 2: Find the critical numbers. Here we can find the zeros by factoring.

$$2x^4 - 3x^3 - 9x^2 = 0$$
$$x^2\left(2x^2 - 3x - 9\right) = 0$$
$$x^2(2x + 3)(x - 3) = 0$$

There are three solutions, hence, three critical numbers $-\frac{3}{2}, 0$, and 3. The strict inequality indicates that we should use open dots.

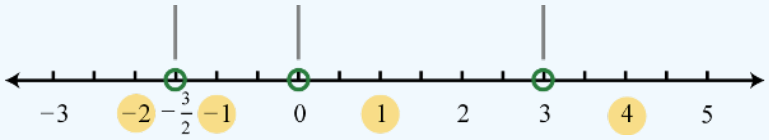


Figure 3.7.5 Critical Numbers

Step 3: Create a sign chart. In this case use $f(x) = x^2(2x + 3)(x - 3)$ and test values $-2, -1, 1$, and 4 to determine the sign of the function in each interval.

$$\begin{aligned} f(-2) &= (-2)^2[2(-2) + 3](-2-3) &&= (-)^2(-)(-) = + \\ f(-1) &= (-1)^2[2(-1) + 3](-1-3) &&= (-)^2(+)(-) = - \\ f(1) &= (1)^2[2(1) + 3](1-3) &&= (+)^2(+)(-) = - \\ f(4) &= (4)^2[2(4) + 3](4-3) &&= (+)^2(+)(+) = + \end{aligned}$$

With this information we can complete the sign chart.

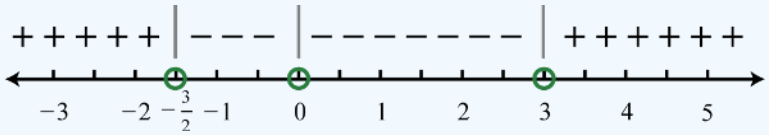


Figure 3.7.6 Sign Chart

Step 4: Use the sign chart to answer the question. Here the solution consists of all values for which $f(x) > 0$. Shade in the values that produce positive results and then express this set in interval notation.



Figure 3.7.7 Solution Set on Number Line

Answer: $\left(-\infty, -\frac{3}{2}\right) \cup (3, \infty)$

Example 3.8.3

Solve: $x^3 + x^2 \leq 4(x+1)$.

Solution

Step 1. Begin by rewriting the inequality in standard form, with zero on one side.

$$\begin{aligned} x^3 + x^2 &\leq 4(x+1) \\ x^3 + x^2 &\leq 4x + 4 \\ x^3 + x^2 - 4x - 4 &\leq 0 \end{aligned}$$

Step 2. Next find the critical numbers of $f(x) = x^3 + x^2 - 4x - 4$:

$$\begin{aligned} x^3 + x^2 - 4x - 4 &= 0 \quad \textit{Factor by grouping.} \\ x^2(x+1) - 4(x+1) &= 0 \\ (x+1)\left(x^2 - 4\right) &= 0 \\ (x+1)(x+2)(x-2) &= 0 \end{aligned}$$

The critical numbers are $-2, -1$, and 2. Because of the inclusive inequality ($\leq$) we will plot them using closed dots.

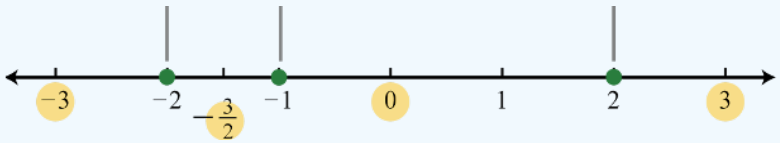


Figure 3.7.8 Critical Numbers

Step 3. Use test values, -3, $-\frac{3}{2}$, 0, and 3 to create a sign chart.

$$\begin{aligned} f(-3) &= (-3+1)(-3+2)(-3-2) &&= (-)(-)(-) = - \\ f(-\tfrac{3}{2}) &= (-\tfrac{3}{2}+1)(-\tfrac{3}{2}+2)(-\tfrac{3}{2}-2) &&= (-)(+)(-) = + \\ f(0) &= (0+1)(0+2)(0-2) &&= (+)(+)(-) = - \\ f(3) &= (3+1)(3+2)(3-2) &&= (+)(+)(+) = + \end{aligned}$$

And we have

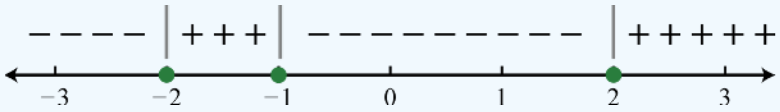


Figure 3.7.9 Sign Chart

Step 4. Use the sign chart to shade in the values that have negative results ($f(x) \leq 0$).

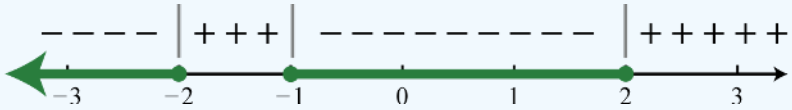


Figure 3.7.10 Solution Set on Number Line

Answer: $(-\infty, -2] \cup [-1, 2]$

3.8.3

Solve $-3x^4 + 12x^3 - 9x^2 > 0$.

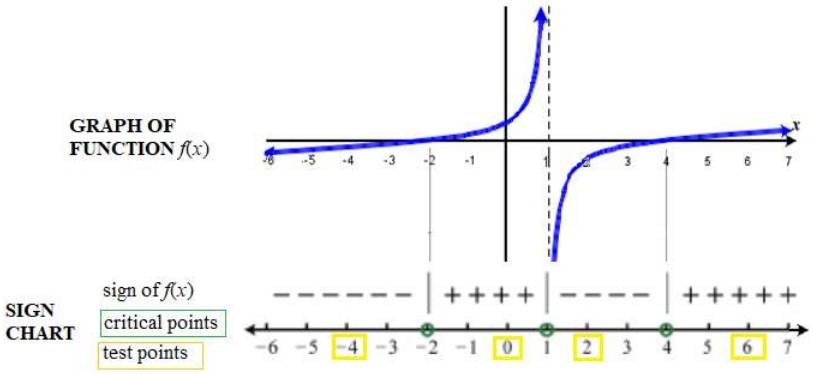
Answer

$(1, 3)$

Solving Rational Inequalities

A **rational inequality** is a mathematical statement that relates a rational expression as either less than or greater than another. Because rational functions have restrictions to the domain we must take care when solving rational inequalities. In addition to the zeros, we will include the restrictions to the domain of the function in the set of critical numbers.

Graphs are helpful in providing a visualization to the solutions of rational inequalities. Examine the graph below to see the relationship between a graph of a rational function and its corresponding **sign chart**. It is very similar to the sign chart for polynomials except additionally, vertical asymptotes are included in the list of critical points.



Both the x- intercepts, -2 and 4, and the vertical asymptote $x = 1$ are **critical points**. The vertical asymptote occurs at $x = 1$ because that is the value that makes the denominator zero, and thus makes the rational expression undefined. Again, these critical points are the only places where the graph may possibly change from being above the x-axis (where $f(x) > 0$)), to below the x-axis (where $f(x) < 0$). Within each interval between two adjacent critical points, the graph is either always above the x-axis, or always below the x-axis. Thus, finding the critical points of a rational inequality plays a fundamental role in solving it.

Howto: Solve a Rational Inequality.

1. **Step 1**: Rewrite the inequality so there is a zero on the right side of the inequality. Write the expression on the left as a single algebraic fraction. Designate this fraction as $f(x)$.
2. **Step 2**: Find the critical numbers. For rational inequalities these critical numbers come from two sources.
   a. Solve $f((x) = 0$. Find values that make the numerator of $f(x)$ zero. These critical points are the x-intercepts. If the inequality uses "<" or ">", draw an open circle at each of these critical numbers; if the inequality uses " ≤" or "≥", draw a filled in circle at each of these critical numbers .
   b. Solve $f((x) = undefined$. Find values that make the denominator of $f(x)$ zero. These critical points are where the vertical asymptotes are. These values are not part of the domain of $f(x)$. Therefore these critical values are never part of the solution set. Always draw an open circle at each of these critical numbers.
3. **Step 3**: Create a sign chart. The critical numbers partition the number line into regions. Choose a test value for each region, including one to the left of all the critical values and one to the right of all the critical values. For each test value, v, determine if $f(v)$ is positive or negative and record the result on the sign chart for each region.
4. **Step 4**: Use the sign chart to find the set of all values of x for which the inequality is true. Write the answer in interval notation.

Example 3.8.4

Solve: $\frac{(x-4)(x+2)}{(x-1)} \geq 0$

Solution

Step 2. The zeros of a rational function occur when the numerator is zero and the values that produce zero in the denominator are the restrictions. In this case,

Roots (Numerator)	Restriction(Denominator)
$x-4=0$ or $x+2=0$	$x-1=0$
$x=4 \quad x=-2$	$x=1$

Therefore the critical numbers are $-2, 1$, and 4. Because of the inclusive inequality ($\geq$) use a closed dot for the roots $-2, 4$ and always use an open dot for restrictions 1. Restrictions are never included in the solution set.

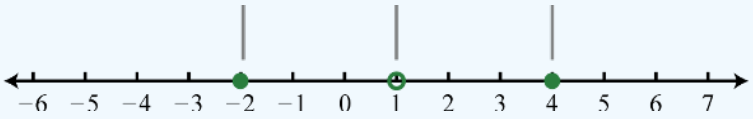


Figure 3.7.11 Critical Numbers

Step 3. Use test values $x=-4, 0, 2, 6$.

$$f(-4)=\frac{(-4-4)(-4+2)}{(-4-1)} \quad =\frac{(-)(-)}{(-)}=-$$
$$f(0)=\frac{(0-4)(0+2)}{(0-1)} \quad =\frac{(-)(+)}{(-)}=+$$
$$f(2)=\frac{(2-4)(2+2)}{(2-1)} \quad =\frac{(-)(+)}{(+)}=-$$
$$f(6)=\frac{(6-4)(6+2)}{(6-1)} \quad =\frac{(+)(+)}{(+)}=+$$

And then complete the sign chart.

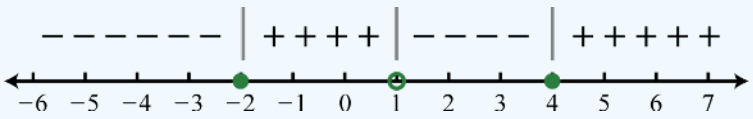


Figure 3.7.12 Sign Chart

Step 4. The question asks us to find the values for which $f(x) \geq 0$, in other words, positive or zero. Shade in the appropriate regions and present the solution set in interval notation.

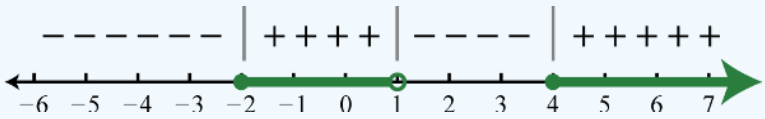


Figure 3.7.13 Solution Set on Number Line

Answer: $[-2, 1) \cup [4, \infty)$

Once again the graph of this function below provides a visualization of what the sign chart means. As in the case with the polynomial inequality, the sign chart is positive when the graph is above the x-axis and negative when the graph is below the x-axis. The graph (potentially) changes from being above the x-axis to below the x-axis at the critical values. Unlike the polynomial, where critical values are only where the graph crosses or touches the x-axis, for rational functions, critical values also exist at the vertical asymptotes of the graph.

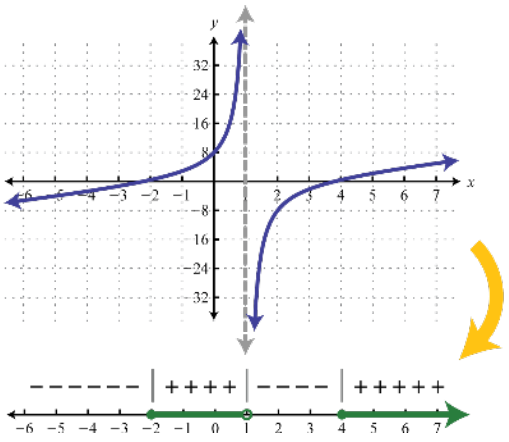


Figure 3.7.14 Rational Function with Sign Chart and Solution Set

Notice that the restriction $x = 1$ corresponds to a vertical asymptote which bounds regions where the function changes from positive to negative. While not included in the solution set, the restriction is a critical number. Before creating a sign chart we must ensure the inequality has a zero on one side. The general steps for solving a rational inequality are outlined in the following example.

Example 3.8.5

Solve $\frac{7}{x+3} < 2$.

Solution

Step 1: Begin by obtaining zero on the right side.

$$\frac{7}{x+3} < 2$$
$$\frac{7}{x+3} - 2 < 0$$

Step 2: Determine the critical numbers. The critical numbers are the zeros and restrictions. Begin by simplifying to a single algebraic fraction.

$$\frac{7}{x+3} - \frac{2}{1} < 0$$
$$\frac{7-2(x+3)}{x+3} < 0$$
$$\frac{7-2x-6}{x+3} < 0$$
$$\frac{-2x+1}{x+3} < 0$$

Next find the critical numbers. Set the numerator and denominator equal to zero and solve.

Root	Restriction
$-2x+1=0$	
$-2x=-1$	$x+3=0$
$x=\frac{1}{2}$	$x=-3$

In this case, the strict inequality indicates that we should use an open dot for the root. Any restriction always uses an open dot.

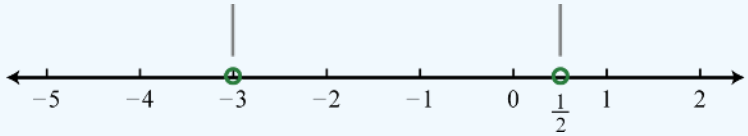


Figure 3.7.15 Critical Numbers

Step 3: Create a sign chart. Choose test values $-4, 0$, and 1.

$$f(-4) = \frac{-2(-4)+1}{-4+3} = \frac{+}{-} = -$$
$$f(0) = \frac{-2(0)+1}{0+3} = \frac{+}{+} = +$$
$$f(1) = \frac{-2(1)+1}{1+3} = \frac{-}{+} = -$$

And we have

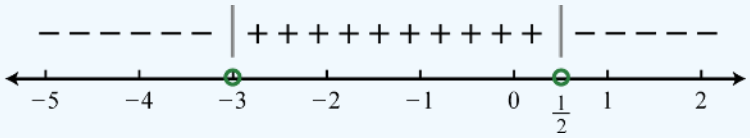


Figure 3.7.16 Sign Chart

Step 4: Use the sign chart to answer the question. In this example we are looking for the values for which the function is negative, $f(x) < 0$. Shade the appropriate values and then present your answer using interval notation.

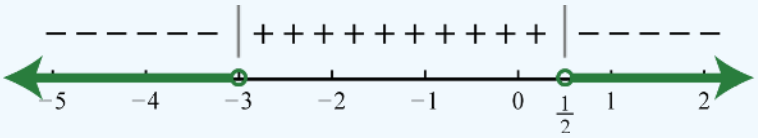


Figure 3.7.17 Solution Set on Number Line

Answer:

$(-\infty, -3) \cup \left(\frac{1}{2}, \infty\right)$

Example 3.8.6

Solve: $\frac{1}{x^2-4} \leq \frac{1}{2-x}$.

Solution

Step 1. Begin by obtaining zero on the right side.

$$\frac{1}{x^2-4} \leq \frac{1}{2-x}$$
$$\frac{1}{x^2-4} - \frac{1}{2-x} \leq 0$$

Step 2. Next simplify the left side to a single algebraic fraction.

$$\frac{1}{x^2-4} - \frac{1}{2-x} \leq 0$$
$$\frac{1}{(x+2)(x-2)} - \frac{1}{-(x-2)} \leq 0$$
$$\frac{1}{(x+2)(x-2)} + \frac{1(x+2)}{(x-2)(x+2)} \leq 0$$
$$\frac{1+x+2}{(x+2)(x-2)} \leq 0$$
$$\boxed{f(x) = \frac{x+3}{(x+2)(x-2)} \leq 0}$$

The critical numbers are $-3, -2$, and 2. Note that ± 2 are restrictions and thus we will use open dots when plotting them on a number line. Because of the inclusive inequality we will use a closed dot at the root -3.

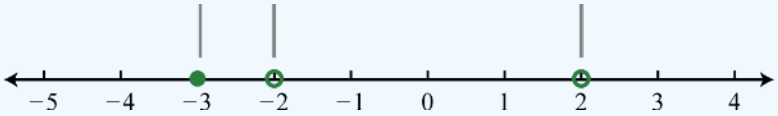


Figure 3.7.18 Critical Numbers

Step 3. Choose test values $-4, -2\frac{1}{2} = -\frac{5}{2}, 0$, and 3.

$$f(-4) = \frac{-4+3}{(-4+2)(-4-2)} = \frac{(-)}{(-)(-)} = -$$
$$f\left(-\frac{5}{2}\right) = \frac{-\frac{5}{2}+3}{(-\frac{5}{2}+2)(-\frac{5}{2}-2)} = \frac{(+)}{(-)(-)} = +$$
$$f(0) = \frac{0+3}{(0+2)(0-2)} = \frac{(+)}{(+)(-)} = -$$
$$f(3) = \frac{3+3}{(3+2)(3-2)} = \frac{(+)}{(+)(+)} = +$$

Construct a sign chart.

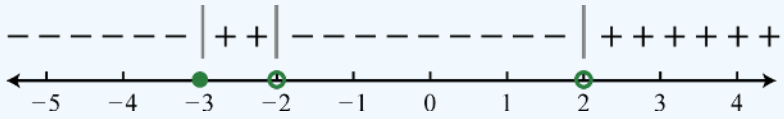


Figure 3.7.19 Sign Chart

Step 4. Answer the question; in this case, find x where $f(x) \leq 0$.

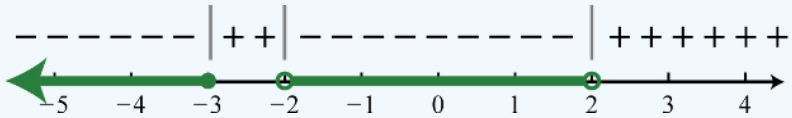


Figure 3.7.20 Solution Set on Number Line

Answer: $(-\infty, -3] \cup (-2, 2)$

Try it 3.8.3

Solve $\frac{2x^2}{2x^2+7x-4} \geq \frac{x}{x+4}$.

Answer

$(-4, 0] \cup \left(\frac{1}{2}, \infty\right)$

Key Takeaways

- When a polynomial inequality is in standard form, with zero on one side, the roots of the polynomial are the critical numbers. Create a sign chart that models the function and then use it to answer the question.
- When a rational inequality is written as a single algebraic fraction, with zero on one side, the roots as well as the restrictions are the critical numbers. The values that produce zero in the numerator are the roots, and the values that produce zero in the denominator are the restrictions. Always use open dots for restrictions, regardless of the given inequality, because restrictions are not part of the domain. Create a sign chart that models the function and then use it to answer the question.

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3.8e: Exercises - Polynomial and Rational Inequalities

A: Concepts

Exercise 3.8e.A

1. Does the sign chart for any given polynomial or rational function always alternate? Explain and illustrate your answer with some examples.

2. Write down your own steps for solving a rational inequality and illustrate them with an example. Do your steps also work for a polynomial inequality? Explain.

Answer 1:

1. Answer may vary

B: Solve Polynomial Inequalities

Exercise 3.8e.B

★ Solve each polynomial inequality and graph the solution set on a real number line. Express each solution set in interval notation.

3. $x(x+1)(x-3) \geq 0$
4. $x(x-1)(x+4) \geq 0$
5. $(x+2)(x-5)^2 < 0$
6. $(x-4)(x+1)^2 \geq 0$
7. $(2x-1)(x+3)(x+2) \leq 0$
8. $(3x+2)(x-4)(x-5) \geq 0$
9. $x(x+2)(x-5)^2 < 0$
10. $x(2x-5)(x-1)^2 > 0$
11. $x(4x+3)(x-1)^2 \geq 0$
12. $(x-1)(x+1)(x-4)^2 < 0$
13. $(x+5)(x-10)(x-5)^2 \geq 0$
14. $(3x-1)(x-2)(x+2)^2 \leq 0$
15. $-4x(4x+9)(x-8)^2 > 0$
16. $-x(x-10)(x+7)^2 > 0$
17. $x^3+2x^2-24x \geq 0$
18. $x^3-3x^2-18x \leq 0$
19. $4x^3-22x^2-12x < 0$
20. $9x^3+30x^2-24x > 0$
21. $12x^4+44x^3 > 80x^2$
22. $6x^4+12x^3 < 48x^2$
23. $x\left(x^2+25\right) < 10x^2$
24. $x^3 > 12x(x-3)$
25. $x^4-5x^2+4 \leq 0$
26. $x^4-13x^2+36 \geq 0$
27. $x^4 > 3x^2+4$
28. $4x^4 < 3-11x^2$
29. $9x^3-3x^2-81x+27 \leq 0$
30. $2x^3+x^2-50x-25 \geq 0$
31. $x^3-3x^2+9x-27 > 0$
32. $3x^3+5x^2+12x+20 < 0$

Answers 3-31:

3. $[-1,0] \cup [3,\infty)$
5. $(-\infty,-2)$
7. $(-\infty,-3] \cup \left[-2,\frac{1}{2}\right]$
9. $(-2,0)$
11. $\left(-\infty,-\frac{3}{4}\right] \cup [0,\infty)$
13. $(-\infty,-5] \cup [5,5] \cup [10,\infty)$
15. $\left(-\frac{9}{4},0\right)$
17. $[-6,0] \cup [4,\infty)$
19. $\left(-\infty,-\frac{1}{2}\right) \cup (0,6)$
21. $(-\infty,-5) \cup \left(\frac{4}{3},\infty\right)$
23. $(-\infty,0)$
25. $[-2,-1] \cup [1,2]$
27. $(-\infty,-2) \cup (2,\infty)$
29. $(-\infty,-3] \cup \left[\frac{1}{3},3\right]$
31. $(3,\infty)$

C: Solve Rational Inequalities

Exercise 3.8e.C

★ Solve each rational inequality and graph the solution set on a real number line. Express each solution set in interval notation.

33. $\frac{x}{x-3} \geq 0$

34. $\frac{x-5}{x} \geq 0$

35. $\frac{(x-3)(x+1)}{x} < 0$

36. $\frac{(x+5)(x+4)}{(x-2)} < 0$

37. $\frac{(2x+1)(x+5)}{(x-3)(x-5)} \leq 0$

38. $\frac{(3x-1)(x+6)}{(x-1)(x+9)} \geq 0$

39. $\frac{(x-8)(x+8)}{-2x(x-2)} \geq 0$

40. $\frac{(2x+7)(x+4)}{x(x+5)} \leq 0$

41. $\frac{x^2}{(2x+3)(2x-3)} \leq 0$

42. $\frac{(x-4)^2}{-x(x+1)} > 0$

43. $\frac{-5x(x-2)^2}{(x+5)(x-6)} \geq 0$

44. $\frac{(3x-4)(x+5)}{x(x-4)^2} \geq 0$

45. $\frac{1}{(x-5)^4} > 0$

46. $\frac{1}{(x-5)^4} < 0$

47. $\frac{x^2-11x-12}{x+4} < 0$

48. $\frac{x^2-10x+24}{x-2} > 0$

49. $\frac{x^2+x-30}{2x+1} \geq 0$

50. $\frac{2x^2+x-3}{x-3} \leq 0$

51. $\frac{3x^2-4x+1}{x^2-9} \leq 0$

52. $\frac{x^2-16}{2x^2-3x-2} \geq 0$

53. $\frac{x^2-12x+20}{x^2-10x+25} > 0$

54. $\frac{x^2+15x+36}{x^2-8x+16} < 0$

55. $\frac{8x^2-2x-1}{2x^2-3x-14} \leq 0$

56. $\frac{4x^2-4x-15}{x^2+4x-5} \geq 0$

57. $\frac{1}{x+5} + \frac{5}{x-1} > 0$

58. $\frac{5}{x+4} - \frac{1}{x-4} < 0$

59. $\frac{1}{x+7} > 1$

60. $\frac{1}{x-1} < -5$

61. $x \geq \frac{30}{x-1}$

62. $x \leq \frac{1-2x}{x-2}$

63. $\frac{1}{x-1} \leq \frac{2}{x}$

64. $\frac{3}{x+1} > -\frac{1}{x}$

65. $\frac{4}{x-3} \leq \frac{1}{x+3}$

66. $\frac{2x-9}{x} + \frac{49}{x-8} < 0$

67. $\frac{x}{2(x+2)} - \frac{1}{x+2} \leq \frac{12}{x(x+2)}$

68. $\frac{1}{2x+1} - \frac{9}{2x-1} \geq 2$

69. $\frac{3x}{x^2-4} - \frac{2}{x-2} < 0$

70. $\frac{x}{2x+1} + \frac{4}{2x^2-7x-4} < 0$

71. $\frac{x+1}{2x^2+5x-3} \geq \frac{x}{4x^2-1}$

72. $\frac{x^2-14}{2x^2-7x-4} \leq \frac{5}{1+2x}$

Answers 33-71:

33. $(-\infty, -0] \cup (3, \infty)$
35. $(-\infty, -1) \cup (0, 3)$
37. $[-5, -\frac{1}{2}] \cup (3, 5)$
39. $[-8, 0) \cup (2, 8]$
41. $(-\frac{3}{2}, \frac{3}{2})$
43. $(-\infty, -5) \cup [0, 6)$
45. $(-\infty, 5) \cup (5, \infty)$
47. $(-\infty, -4) \cup (-1, 12)$
49. $[-6, -\frac{1}{2}) \cup [5, \infty)$
51. $(-3, \frac{1}{3}] \cup [1, 3)$
53. $(-\infty, 2) \cup (10, \infty)$
55. $(-2, -\frac{1}{4}] \cup [\frac{1}{2}, \frac{7}{2})$
57. $(-5, -4) \cup (1, \infty)$
59. $(-7, -6)$
61. $[-5, 1) \cup [6, \infty)$
63. $(0, 1) \cup [2, \infty)$
65. $(-\infty, 5] \cup (-3, 3)$
67. $[-4, -2) \cup (0, 6]$
69. $(-\infty, -2) \cup (2, 4)$
71. $(-3, -\frac{1}{2}) \cup (\frac{1}{2}, \infty)$

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3.9: Rational Functions

Learning Objectives

- Find the domains of rational functions.
- Identify vertical asymptotes and "holes".
- Identify end behaviour asymptotes and points where the function intersects them.
- Graph rational functions and Construct a rational function from a graph

Domains of Rational Functions

A *vertical asymptote* represents a value at which a rational function is undefined, so that value is not in the domain of the function. A reciprocal function cannot have values in its domain that cause the denominator to equal zero. In general, to find the domain of a rational function, we need to determine which inputs would cause division by zero.

Definition: DOMAIN OF A RATIONAL FUNCTION

The domain of a rational function includes all real numbers except those that cause the denominator to equal zero.

How To: Given a rational function, find the domain.

1. Set the denominator equal to zero.
2. Solve to find the x-values that cause the denominator to equal zero.
3. The domain is all real numbers except those found in Step 2.

Example 3.9.1: Finding the Domain of a Rational Function

Find the domain of $f(x) = \dfrac{x+3}{x^2-9}$.

Solution

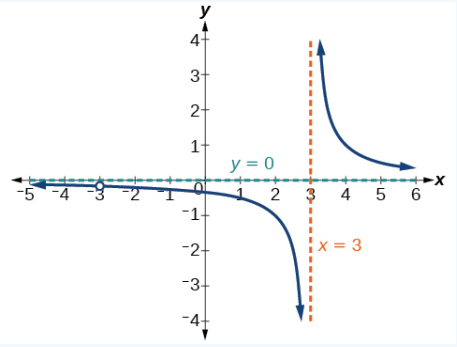
Begin by setting the denominator equal to zero and solving.

$$x^2 - 9 = 0$$
$$x^2 = 9$$
$$x = \pm 3$$

The denominator is equal to zero when $x = \pm 3$. The domain of the function is all real numbers except $x = \pm 3$.

Analysis

A graph of this function, as shown in Figure 3.9.1, confirms that the function is not defined when $x = \pm 3$.



There is a vertical asymptote at $x = 3$ and a hole in the graph at $x = -3$. We will discuss these types of holes in greater detail later in this section.

Figure 3.9.1

Try It 3.9.1

Find the domain of $f(x) = \dfrac{4x}{5(x-1)(x-5)}$.

Answer

The domain is all real numbers except $x = 1$ and $x = 5$.

Vertical Asymptotes of Rational Functions

By looking at the graph of a rational function, we can investigate its local behavior and easily see whether there are asymptotes. We may even be able to approximate their location. Even without the graph, however, we can still determine whether a given rational function has any asymptotes, and calculate their location.

Vertical Asymptotes

The vertical asymptotes of a rational function may be found by examining the factors of the denominator that are not common to the factors in the numerator. Vertical asymptotes occur at the zeros of such factors.

How To: Given a rational function, identify any vertical asymptotes of its graph

1. Factor the numerator and denominator.
2. Note any restrictions in the domain of the function.
3. Reduce the expression by canceling common factors in the numerator and the denominator.
4. Note any values that cause the denominator to be zero in this simplified version. These are where the vertical asymptotes occur.
5. Note any restrictions in the domain where asymptotes do not occur. These are removable discontinuities, or "holes."

Example 3.9.2: Identifying Vertical Asymptotes

Find the vertical asymptotes of the graph of $k(x) = \dfrac{5+2x^2}{2-x-x^2}$.

Solution

First, factor the numerator and denominator.

$$k(x) = \frac{5+2x^2}{2-x-x^2}$$

$$= \frac{5+2x^2}{(2+x)(1-x)}$$

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$(2+x)(1-x) = 0$$

$$x = -2,\ x = 1$$

Neither $x = -2$ nor $x = 1$ are zeros of the numerator, so the two values indicate two vertical asymptotes. The graph in Figure 3.9.2 confirms the location of the two vertical asymptotes.

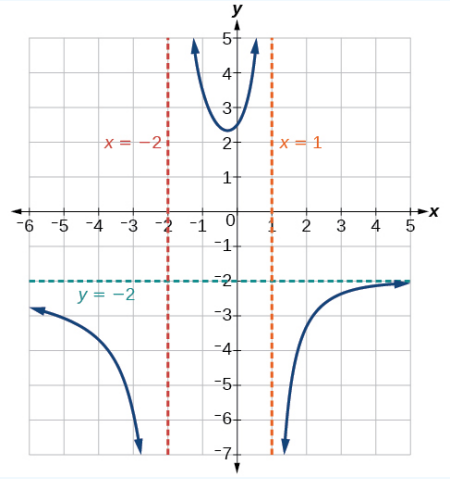


Figure 3.9.2.

Removable Discontinuities

Occasionally, a graph will contain a hole: a single point where the graph is not defined, indicated by an open circle. We call such a hole a *removable discontinuity*. For example, the function $f(x) = \dfrac{x^2-1}{x^2-2x-3}$ may be re-written by factoring the numerator and the denominator.

$$f(x) = \frac{(x+1)(x-1)}{(x+1)(x-3)}$$

Notice that $x+1$ is a common factor to the numerator and the denominator. The zero of this factor, $x = -1$, is the location of the removable discontinuity. Notice also that $(x–3)$ is not a factor in both the numerator and denominator. The zero of this factor, $x = 3$,

is the vertical asymptote. See Figure 3.9.3.1. [Note that removable discontinuities may not be visible when we use a graphing calculator, depending upon the window selected.]

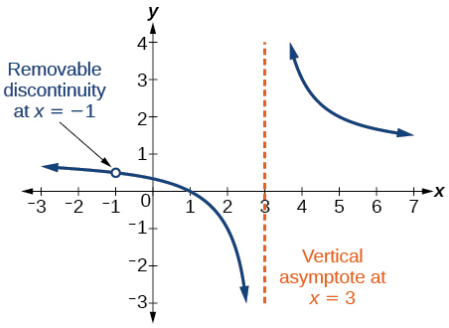


Figure 3.9.3.1.

REMOVABLE DISCONTINUITIES OF RATIONAL FUNCTIONS

A **removable discontinuity** occurs in the graph of a rational function at $x = a$ if a is a zero for a factor in the denominator that is common with a factor in the numerator. We factor the numerator and denominator and check for common factors. If we find any, we set the common factor equal to 0 and solve. This is the location of the removable discontinuity. This is true if the multiplicity of this factor is greater than or equal to that in the denominator. If the multiplicity of this factor is greater in the denominator, then there is still an asymptote at that value.

Example 3.9.3: Identifying Vertical Asymptotes and Removable Discontinuities for a Graph

Find the vertical asymptotes and removable discontinuities of the graph of $k(x) = \dfrac{x-2}{x^2-4}$.

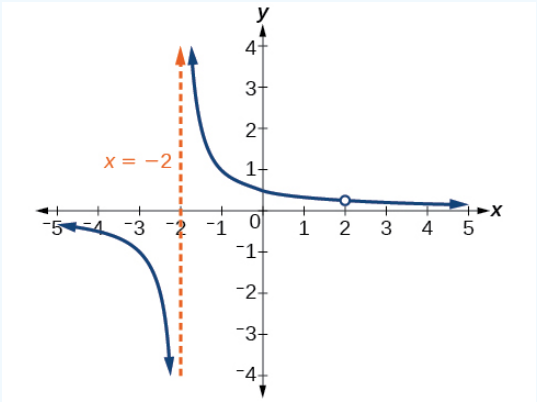
Solution

Factor the numerator and the denominator.

$$k(x) = \frac{x-2}{(x-2)(x+2)}$$

Notice that there is a common factor in the numerator and the denominator, $x–2$. The zero for this factor is $x = 2$. This is the location of the removable discontinuity.

Notice that there is a factor in the denominator that is not in the numerator, $x + 2$. The zero for this factor is $x = -2$. The vertical asymptote is $x = -2$. See Figure 3.9.3.2.



The graph of this function will have the vertical asymptote at $x = -2$, but at $x = 2$ the graph will have a hole.

Figure 3.9.3.2.

Try It 3.9.3

Find the vertical asymptotes and removable discontinuities of the graph of $f(x) = \dfrac{x^2-25}{x^3-6x^2+5x}$.

Answer

Removable discontinuity at $x = 5$. Vertical asymptotes: $x = 0$, $x = 1$.

End Behaviour Asymptotes of Rational Functions

While vertical asymptotes describe the behavior of a graph as the *output* gets very large or very small, horizontal asymptotes help describe the behavior of a graph as the *input* gets very large or very small. Recall that a polynomial's end behavior will mirror that of the leading term. Likewise, a rational function's end behavior will mirror that of the ratio of the function that is the ratio of the leading terms.

There are three distinct outcomes when checking for horizontal asymptotes:

Case 1: If the degree of the denominator > degree of the numerator, there is a **horizontal asymptote** at $y = 0$.

Example: $f(x) = \dfrac{4x+2}{x^2+4x-5}$

In this case, the end behavior is $f(x) \approx \dfrac{4x}{x^2} = \dfrac{4}{x}$. This tells us that, as the inputs increase or decrease without bound, this function will behave similarly to the function $g(x) = \dfrac{4}{x}$, and the outputs will approach zero, resulting in a horizontal asymptote at $y = 0$. See **Figure 3.9.4*a***. Note that this graph crosses the horizontal asymptote.

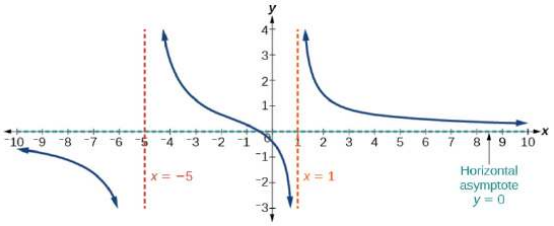


Figure **3.9.4*a***: Horizontal asymptote $y = 0$ occurs when the degree of the numerator is < degree of the denominator.

Case 2: If the degree of the denominator < degree of the numerator by one, we get a slant asymptote.

Example: $f(x) = \dfrac{3x^2-2x}{x-1}$

In this case, the end behavior is $f(x) \approx \dfrac{3x^2}{x} = 3x$. This tells us that as the inputs increase or decrease without bound, this function will behave similarly to the function $g(x) = 3x$. As the inputs grow large, the outputs will grow and not level off, so this graph has no horizontal asymptote. However, the graph of $g(x) = 3x$ looks like a diagonal line, and since f will behave similarly to g, it will approach a line close to $y = 3x$. This line is a slant asymptote.

To find the equation of the slant asymptote, divide $\dfrac{3x^2-2x}{x-1}$. The quotient is $3x+1$, and the remainder is 1. The slant asymptote is the graph of the line $g(x) = 3x+1$. See Figure **3.9.4*b***.

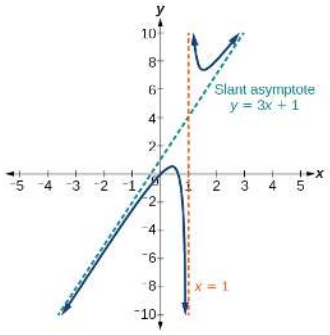


Figure **3.9.4*b***: Slant asymptote occurs when the degree of the numerator = 1+ degree of the denominator.

Case 3: If the degree of the denominator = degree of the numerator, there is a horizontal asymptote at $y = \frac{a_n}{b_n}$, where a_n and b_n are respectively the leading coefficients of the numerator and denominator of the rational function.

Example: $f(x) = \dfrac{3x^2+2}{x^2+4x-5}$

In this case, the end behavior is $f(x) \approx \dfrac{3x^2}{x^2} = 3$. This tells us that as the inputs grow large, this function will behave like the function $g(x) = 3$, which is a horizontal line. As $x \to \pm\infty$, $f(x) \to 3$, resulting in a horizontal asymptote at $y = 3$. See Figure **3.9.4*c***. Note that this graph crosses the horizontal asymptote.

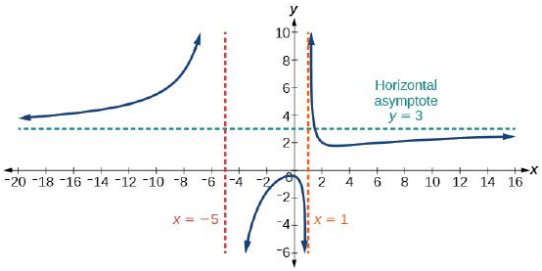


Figure **3.9.4*c***: Horizontal asymptote $y = a/b$ occurs when the degree of the numerator = degree of the denominator.

Notice that, although the graph of a rational function may have many vertical asymptotes, the graph will have at most one horizontal (or slant) asymptote.

It should be noted that, if the degree of the numerator is larger than the degree of the denominator by more than one, the **end behavior** of the graph will mimic the behavior of the reduced end behavior fraction. For instance, if we had the function

$$f(x) = \frac{3x^5 - x^2}{x + 3}$$

with end behavior

$$f(x) \approx \frac{3x^5}{x} = 3x^4$$

the end behavior of the graph would look similar to that of an even polynomial with a positive leading coefficient.

$$x \to \pm\infty, f(x) \to \infty$$

HORIZONTAL ASYMPTOTES OF RATIONAL FUNCTIONS

The **horizontal asymptote** of a rational function can be determined by looking at the degrees of the numerator and denominator.

- Degree of numerator is less than degree of denominator: horizontal asymptote at $y = 0$.
- Degree of numerator is greater than degree of denominator by one: no horizontal asymptote; slant asymptote.
- Degree of numerator is equal to degree of denominator: horizontal asymptote at ratio of leading coefficients.

Notice that, while the graph of a rational function will never cross a **vertical asymptote**, the graph may or may not cross a horizontal or slant asymptote.

Intersection of Rational Functions with End Behaviour Asymptotes

To find the location of any **points of intersection** with the graph of a rational function and its end behaviour asymptote, solve a system of two equations consisting of the Reduced Equation $R(x)$ and the equation of the End Behaviour Asymptote, $EBA(x)$. The End Behaviour Asymptote could be either a horizontal asymptote (in the form $y = c$), or a slant asymptote (in the form $y = mx + b$). The Reduced Equation is used to make calculations simpler and to reduce obtaining solutions that are not in the domain of the original rational function.

- Solve $R(x) = EBA(x)$.
- Each real number solution $x = c$ in the domain of the rational function is the x-coordinate of a point of intersection.
- The y-coordinate of the point of intersection can be calculated by substituting the solution into either original equation. Using $EBA(c)$ is the more convenient choice.

Example 3.9.4a: Identifying Horizontal and Slant Asymptotes and intersection with the rational function

For the functions listed, identify the horizontal or slant asymptote. Then find any points of intersection between the rational function and its End Behaviour Asymptote.

1. $g(x) = \frac{6x^3 - 10x}{2x^3 + 5x^2}$
2. $h(x) = \frac{x^2 - 4x + 1}{x + 2}$
3. $k(x) = \frac{x^2 + 4x}{x^3 - 8}$

Solution

1. $g(x) = \frac{6x^3 - 10x}{2x^3 + 5x^2}$.
   Both numerator and denominator have the same degree (**3**).
   Therefore, we can find the horizontal asymptote by taking the ratio of the leading terms. There is a horizontal asymptote at $y = \frac{6}{2}$ or $y = 3$.
   To find if $g(x)$ intersects with this horizontal asymptote, $g(x)$ must first of all be reduced. $R(x) = \frac{6x^2 - 10}{2x^2 + 5x}$
   Then the equation $R(x) = EBA(x)$ must be solved:

$$\frac{6x^2-10}{2x^2+5x}=3 \rightarrow\ 6x^2-10=3(2x^2+5x)\ \rightarrow\ 6x^2-10=6x^2+15x\ \rightarrow\ -10=15x\ \rightarrow\ x=-\frac{2}{3}$$

Finally the y-coordinate of the intersection point is $EBA\left(-\frac{2}{3}\right)=3$ so the point of intersection is $\left(-\frac{2}{3},3\right)$

2. $h(x)=\frac{x^2-4x+1}{x+2}$.

The degree of the numerator is one more than the degree of the denominator.

Therefore there is a slant asymptote which can be found by dividing $\frac{x^2-4x+1}{x+2}$.

−2	1	−4	1
		−2	12
	1	−6	13

The quotient is $x-6$ and the remainder is 13, so the equation of the slant asymptote is the line $y=x-6$.

Since $h(x)$ is already reduced, $R(x)=h(x)$. To find where $h(x)$ intersects with its horizontal asymptote, the equation $R(x)=EBA(x)$ must be solved:

$$\frac{x^2-4x+1}{x+2}=x-6\ \rightarrow\ x^2-4x+1=(x+2)(x-6)\ \rightarrow\ x^2-4x+1=x^2-4x-12\rightarrow\ 1=-12$$

The result is a false statement, so there is no point where the function intersects its End Behaviour Asymptote.

3. $k(x)=\frac{x^2+4x}{x^3-8}$.

The degree of the numerator is less than the degree of the denominator ($2<3$), so there is a horizontal asymptote $y=0$.

Since $k(x)$ is already reduced, $R(x)=k(x)$. To find where $k(x)$ intersects with its horizontal asymptote, the equation $R(x)=EBA(x)$ must be solved:

$\frac{x^2+4x}{x^3-8}=0\ \rightarrow\ x^2+4x=0\ \rightarrow\ x(x+4)=0$ so $x=0$ and $x=-4$.

The y-coordinate of both these intersection points is 0 so the points of intersection are $(0,0)$ and $(-4,0)$

Example 3.9.4.2 Identifying Horizontal Asymptotes

In the sugar concentration problem earlier, we created the equation $C(t)=\frac{5+t}{100+10t}$.

Find the horizontal asymptote and interpret it in context of the problem.

Solution

Both the numerator and denominator are linear (degree 1). Because the degrees are equal, there will be a horizontal asymptote at the ratio of the leading coefficients. In the numerator, the leading term is t, with coefficient 1. In the denominator, the leading term is 10t, with coefficient 10. The horizontal asymptote will be at the ratio of these values:

$$t\rightarrow\infty,\ C(t)\rightarrow\frac{1}{10}$$

This function will have a horizontal asymptote at $y=\frac{1}{10}$.

This tells us that as the values of t increase, the values of C will approach $\frac{1}{10}$. In context, this means that, as more time goes by, the concentration of sugar in the tank will approach one-tenth of a pound of sugar per gallon of water or $\frac{1}{10}$ pounds per gallon.

Example 3.9.4.3: Identifying Horizontal and Vertical Asymptotes

Find the horizontal and vertical asymptotes of the function $f(x)=\frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$

Solution

First, note that this function has no common factors, so there are no potential removable discontinuities.

The function will have vertical asymptotes when the denominator is zero, causing the function to be undefined. The denominator will be zero at $x=1,-2,$and 5, indicating vertical asymptotes at these values.

The numerator has degree 2, while the denominator has degree 3. Since the degree of the denominator is greater than the degree of the numerator, the denominator will grow faster than the numerator, causing the outputs to tend towards zero as the

inputs get large, and so as $x \to \pm\infty$, $f(x) \to 0$. This function will have a horizontal asymptote at $y = 0$. See Figure 3.9.4.3.

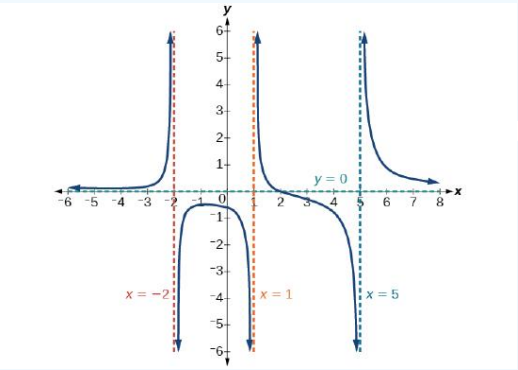


Figure 3.9.4.3.

Try It 3.9.4

Find the vertical and horizontal asymptotes of the function:

$$f(x) = \frac{(2x-1)(2x+1)}{(x-2)(x+3)}$$

Answer

Vertical asymptotes at $x = 2$ and $x = -3$

horizontal asymptote at $y = 4$.

Intercepts of Rational Functions

INTERCEPTS OF RATIONAL FUNCTIONS

A *rational function* will have a y-intercept at $f(0)$, if the function is defined at zero. A rational function will not have a y-intercept if the function is not defined at zero.

Likewise, a rational function will have x-intercepts at the inputs that cause the output to be zero. Since a fraction is only equal to zero when the numerator is zero, x-intercepts can only occur when the numerator of the rational function is equal to zero.

Example 3.9.5: Finding the Intercepts of a Rational Function

Find the intercepts of $f(x) = \dfrac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$.

Solution

We can find the y-intercept by evaluating the function at zero

$$f(0) = \frac{(0-2)(0+3)}{(0-1)(0+2)(0-5)} = -\frac{6}{10} = -\frac{3}{5} = -0.6$$

The x-intercepts will occur when the function is equal to zero. Notice the function is zero when the numerator is zero.

$$0 = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$$
$$0 = (x-2)(x+3)$$
$$x = 2, x = -3$$

The y-intercept is $(0, -0.6)$, the x-intercepts are $(2, 0)$ and $(-3, 0)$. See Figure 3.9.5.

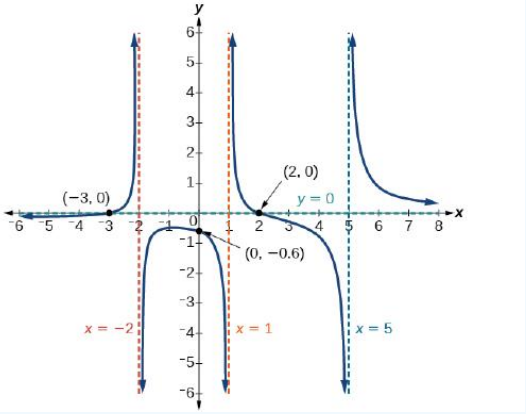


Figure 3.9.5.

Try It 3.9.5

Given the reciprocal squared function that is shifted right 3 units and down 4 units, write this as a rational function. Then, find the x- and y-intercepts and the horizontal and vertical asymptotes.

Answer

For the transformed reciprocal squared function, we find the rational form.

$$f(x)=\frac{1}{(x-3)^2}-4=\frac{1-4(x-3)^2}{(x-3)^2}=\frac{1-4(x^2-6x+9)}{(x-3)(x-3)}=\frac{-4x^2+24x-35}{x^2-6x+9}$$

Because the numerator is the same degree as the denominator we know that as $x\to\pm\infty$, $f(x)\to -4$; so $y=-4$ is the horizontal asymptote. Next, we set the denominator equal to zero, and find that the vertical asymptote is $x=3$, because as $x\to 3$, $f(x)\to\infty$. We then set the numerator equal to 0 and find the x-intercepts are at $(2.5,0)$ and $(3.5,0)$. Finally, we evaluate the function at 0 and find the y-intercept to be at $(0,-\frac{35}{9})$.

Construct Graphs of Rational Functions

In Example **3.9.10**, we see that the numerator of a rational function reveals the *x*-intercepts of the graph, whereas the denominator reveals the vertical asymptotes of the graph. As with polynomials, factors of the numerator may have integer powers greater than one. Fortunately, the effect on the shape of the graph at those intercepts is the same as we saw with polynomials.

The vertical asymptotes associated with the factors of the denominator will mirror one of the two toolkit reciprocal functions. When the degree of the factor in the denominator is odd, the distinguishing characteristic is that on one side of the vertical asymptote the graph heads towards positive infinity, and on the other side the graph heads towards negative infinity. See Figure **3.9.6*a***.

In contrast, when the degree of the factor in the denominator is even, the distinguishing characteristic is that the graph either heads toward positive infinity on both sides of the vertical asymptote or heads toward negative infinity on both sides. See Figure **3.9.6*b***.

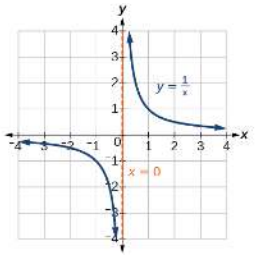


Figure **3.9.6*a***: Odd multiplicity

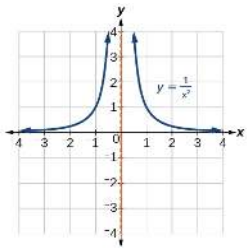


Figure **3.9.6*b***: Even multiplicity

For example, the graph of $f(x)=\dfrac{(x+1)^2(x-3)}{(x+3)^2(x-2)}$ is shown in Figure **3.9.6*c***.

At the x-intercept $x=-1$ corresponding to the $(x+1)^2$ factor of the numerator, the graph "bounces", consistent with the quadratic nature of the factor.

- At the x-intercept $x=3$ corresponding to the $(x-3)$ factor of the numerator, the graph passes through the axis as we would expect from a linear factor.
- At the vertical asymptote $x=-3$ corresponding to the $(x+3)^2$ factor of the denominator, the graph heads towards positive infinity on both sides of the asymptote, consistent with the behavior of the function $f(x)=\dfrac{1}{x^2}$.
- At the vertical asymptote $x=2$, corresponding to the $(x-2)$ factor of the denominator, the graph heads towards positive infinity on the left side of the asymptote and towards negative infinity on the right side, consistent with the behavior of the function $f(x)=\dfrac{1}{x}$.

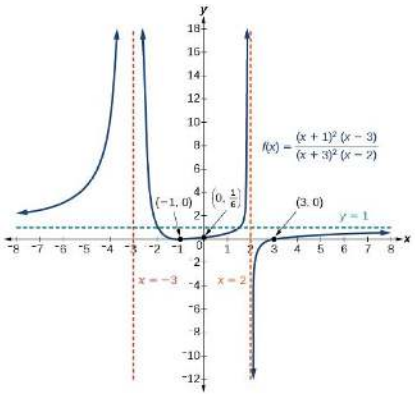


Figure **3.9.6*c***.

How To: Howto: Given a rational function, sketch a graph.

1. Evaluate the function at 0 to find the y-intercept.
2. Factor the numerator and denominator.
3. For factors in the numerator not common to the denominator, determine where each factor of the numerator is zero to find the x-intercepts.

4. Find the multiplicities of the x-intercepts to determine the behavior of the graph at those points.
5. For factors in the denominator, note the multiplicities of the zeros to determine the local behavior. For those factors not common to the numerator, find the vertical asymptotes by setting those factors equal to zero and then solve.
6. For factors in the denominator common to factors in the numerator, find the removable discontinuities by setting those factors equal to 0 and then solve.
7. Compare the degrees of the numerator and the denominator to determine the horizontal or slant asymptotes.
8. Sketch the graph.

Example 3.9.6: Graphing a Rational Function

Sketch a graph of $f(x) = \frac{(x+2)(x-3)}{(x+1)^2(x-2)}$.

Solution

We can start by noting that the function is already factored, saving us a step.

Next, we will find the intercepts. Evaluating the function at zero gives the y-intercept:

$$f(0) = \frac{(0+2)(0-3)}{(0+1)^2(0-2)} = 3$$

To find the x-intercepts, we determine when the numerator of the function is zero. Setting each factor equal to zero, we find x-intercepts at $x = -2$ and $x = 3$. At each, the behavior will be linear (multiplicity 1), with the graph passing through the intercept. We have a y-intercept at $(0, 3)$ and x-intercepts at $(-2, 0)$ and $(3, 0)$.

To find the vertical asymptotes, we determine when the denominator is equal to zero. This occurs when $x + 1 = 0$ and when $x - 2 = 0$, giving us vertical asymptotes at $x = -1$ and $x = 2$. There are no common factors in the numerator and denominator. This means there are no removable discontinuities.

Finally, the degree of denominator is larger than the degree of the numerator, telling us this graph has a horizontal asymptote at $y = 0$.

To sketch the graph, we might start by plotting the three intercepts. Since the graph has no x-intercepts between the vertical asymptotes, and the y-intercept is positive, we know the function must remain positive between the asymptotes, letting us fill in the middle portion of the graph as shown in Figure 3.9.6*a*.

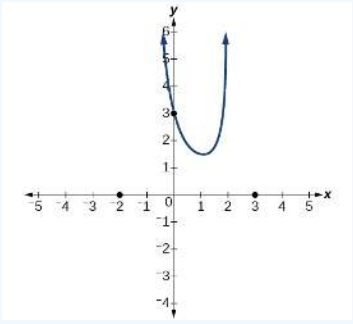


Figure 3.9.6*a*.

The factor associated with the vertical asymptote at $x = -1$ was squared, so we know the behavior will be the same on both sides of the asymptote. The graph heads toward positive infinity as the inputs approach the asymptote on the right, so the graph will head toward positive infinity on the left as well.

For the vertical asymptote at $x = 2$, the factor was not squared, so the graph will have opposite behavior on either side of the asymptote. See Figure 3.9.6*b*. After passing through the x-intercepts, the graph will then level off toward an output of zero, as indicated by the horizontal asymptote.

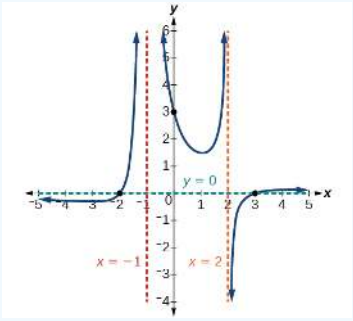


Figure 3.9.6*b*.

Try It 3.9.6

Given the function $f(x) = \frac{(x+2)^2(x-2)}{2(x-1)^2(x-3)}$, use the characteristics of polynomials and rational functions to describe its behavior and sketch the function.

Answer

Horizontal asymptote at $y = \frac{1}{2}$. Vertical asymptotes at $x = 1$ and $x = 3$. y-intercept at $(0, \frac{4}{3})$.

x-intercepts at $(2, 0)$ and $(-2, 0)$. $(-2, 0)$ is a zero with multiplicity 2, and the graph bounces off the x-axis at this point. $(2, 0)$ is a single zero and the graph crosses the axis at this point.

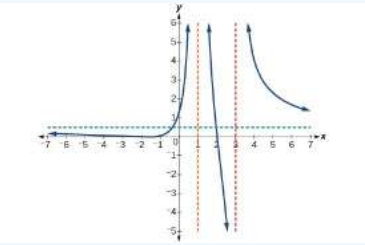


Figure 3.9.6.

Construct Equations for Rational Functions

Now that we have analyzed the equations for rational functions and how they relate to a graph of the function, we can use information given by a graph to write the function. A rational function written in factored form will have an *x*-intercept where each factor of the numerator is equal to zero. (An exception occurs in the case of a removable discontinuity.) As a result, we can form a numerator of a function whose graph will pass through a set of *x*-intercepts by introducing a corresponding set of factors. Likewise, because the function will have a vertical asymptote where each factor of the denominator is equal to zero, we can form a denominator that will produce the vertical asymptotes by introducing a corresponding set of factors.

WRITING RATIONAL FUNCTIONS FROM INTERCEPTS AND ASYMPTOTES

If a **rational function** has x-intercepts at $x = x_1, x_2, \ldots, x_n$, vertical asymptotes at $x = v_1, v_2, \ldots, v_m$, and no $x_i =$ any v_j, then the function can be written in the form:

$$f(x) = a\frac{(x - x_1)^{p_1}(x - x_2)^{p_2} \cdots (x - x_n)^{p_n}}{(x - v_1)^{q_1}(x - v_2)^{q_2} \cdots (x - v_m)^{q_n}}$$

where the powers p_i or q_i on each factor can be determined by the behavior of the graph at the corresponding intercept or asymptote, and the stretch factor a can be determined given a value of the function other than the x-intercept or by the horizontal asymptote if it is nonzero.

Given a graph of a rational function, write the function.

1. Determine the factors of the numerator. Examine the behavior of the graph at the *x*-intercepts to determine the zeroes and their multiplicities. (This is easy to do when finding the "simplest" function with small multiplicities—such as 1 or 3—but may be difficult for larger multiplicities—such as 5 or 7, for example.)
2. Determine the factors of the denominator. Examine the behavior on both sides of each vertical asymptote to determine the factors and their powers.
3. Use any clear point on the graph to find the stretch factor.

Example 3.9.7: Writing a Rational Function from Intercepts and Asymptotes

Write an equation for the rational function shown in Figure 3.9.7e.

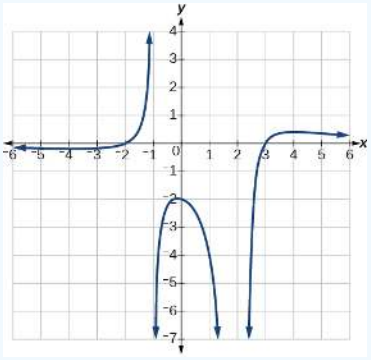


Figure 3.9.7e.

Solution

The graph appears to have x-intercepts at $x = -2$ and $x = 3$. At both, the graph passes through the intercept, suggesting linear factors. The graph has two vertical asymptotes. The one at $x = -1$ seems to exhibit the basic behavior similar to $\frac{1}{x}$, with the graph heading toward positive infinity on one side and heading toward negative infinity on the other. The asymptote at $x = 2$ is exhibiting a behavior similar to $\frac{1}{x^2}$, with the graph heading toward negative infinity on both sides of the asymptote. See Figure **3.9.7*s***.

We can use this information to write a function of the form

$$f(x) = a\frac{(x+2)(x-3)}{(x+1)(x-2)^2}$$

To find the stretch factor, we can use another clear point on the graph, such as the *y*-intercept $(0, -2)$.

$$-2 = a\frac{(0+2)(0-3)}{(0+1)(0-2)^2}$$
$$-2 = a\frac{-6}{4}$$
$$a = \frac{-8}{-6} = \frac{4}{3}$$

This gives us a final function of $f(x) = \frac{4(x+2)(x-3)}{3(x+1)(x-2)^2}$.

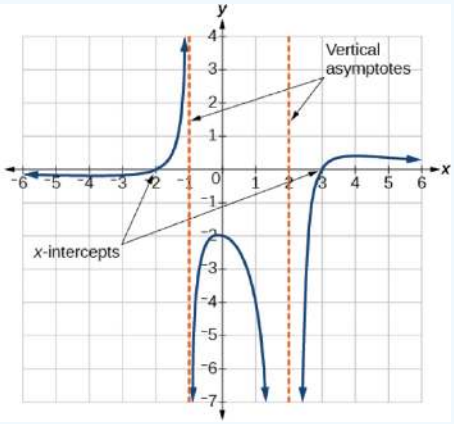


Figure **3.9.7*s***.

Key Equations

Rational Function $f(x) = \frac{P(x)}{Q(x)} = \frac{a_p x^p + a_{p-1}x^{p-1} + \ldots + a_1 x + a_0}{b_q x^q + b_{q-1}x^{q-1} + \ldots + b_1 x + b_0},\ Q(x) \neq 0$

Key Concepts

- We can use arrow notation to describe local behavior and end behavior of the toolkit functions $f(x) = \frac{1}{x}$ and $f(x) = \frac{1}{x^2}$.
- A function that levels off at a horizontal value has a horizontal asymptote. A function can have more than one vertical asymptote.
- Application problems involving rates and concentrations often involve rational functions.
- The domain of a rational function includes all real numbers except those that cause the denominator to equal zero.
- The vertical asymptotes of a rational function will occur where the denominator of the function is equal to zero and the numerator is not zero.
- A removable discontinuity might occur in the graph of a rational function if an input causes both numerator and denominator to be zero..
- A rational function's end behavior will mirror that of the ratio of the leading terms of the numerator and denominator functions.
- Graph rational functions by finding the intercepts, behavior at the intercepts and asymptotes, and end behavior.
- If a rational function has x-intercepts at $x = x_1, x_2, \ldots, x_n$, vertical asymptotes at $x = v_1, v_2, \ldots, v_m$, and no $x_i =$ any v_j, then the function can be written in the form $f(x) = a\frac{(x-x_1)^{p_1}(x-x_2)^{p_2}\cdots(x-x_n)^{p_n}}{(x-v_1)^{q_1}(x-v_2)^{q_2}\cdots(x-v_m)^{q_n}}$

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3.9e: Exercises - Rational Functions

A: Conceptual Questions

Exercise 3.9e.A

1. What is the fundamental difference in the algebraic representation of a polynomial function and a rational function?

2. What is the fundamental difference in the graphs of polynomial functions and rational functions?

3. If the graph of a rational function has a removable discontinuity, what must be true of the functional rule?

4. Can a graph of a rational function have no vertical asymptote? If so, how?

5. Can a graph of a rational function have no x-intercepts? If so, how?

Answers to odd exercises.

1. The rational function will be represented by a quotient of polynomial functions.
3. The numerator and denominator must have a common factor.
5. Yes. The numerator of the formula of the functions would have only complex roots and/or factors common to both the numerator and denominator.

B: Find Domain of Rational Functions

Exercise 3.9e.B

★ Find the domain of the rational functions.

7. $f(x)=\frac{x+1}{x^2-1}$	8. $f(x)=\frac{x-1}{x+2}$	9. $f(x)=\frac{x^2+4x-3}{x^4-5x^2+4}$	10. $f(x)=\frac{x^2+4}{x^2-2x-8}$

Answers to odd exercises.

7. All reals except that $x \neq -1, 1$
9. All reals except that $x \neq -1, -2, 1, 2$

C: Find domain, vertical asymptotes, holes, and horizontal asymptotes

Exercise 3.9e.C

★ For each function, find (a) the domain, vertical asymptotes and holes, and (b) the horizontal asymptote and points of intersection of the HA with the function.

11. $f(x)=\frac{2x^2}{3x^2+1}$	14. $f(x)=\frac{x+2}{x^2-9}$	17. $f(x)=\frac{3x-4}{x^3-16x}$	20. $f(x)=\frac{2x-4}{x-6}$
12. $f(x)=\frac{4}{x-1}$	15. $f(x)=\frac{x}{x^2+5x-36}$	18. $f(x)=\frac{x^3-x}{x^3+9x^2+14x}$	21. $f(x)=\frac{4-2x}{3x-1}$
13. $f(x)=\frac{2x-2}{5x^2-3x-2}$	16. $f(x)=\frac{3+x}{x^3-27}$	19. $f(x)=\frac{x+5}{x^2-25}$	

★ Identify the removable discontinuity ("hole").

25. $f(x)=\frac{x^2-4}{x-2}$	27. $f(x)=\frac{x^2+x-6}{x-2}$	28. $f(x)=\frac{2x^2+5x-3}{x+3}$	29. $f(x)=\frac{x^3+x^2}{x+1}$
26. $f(x)=\frac{x^3+1}{x+1}$			

Answers to odd exercises.

11. Domain is all reals; no VA, No holes; H.A. at $y=\frac{2}{3}$, No intersection;
13. Domain is all reals $x\neq 1,-\frac{2}{5}$, V.A. at $x=-\frac{2}{5}$; Hole at $(1,\frac{2}{7})$; H.A. at $y=0$; No intersection;
15. Domain is all reals $x\neq 4,-9$, V.A. at $x=4,-9$, No holes; H.A. at $y=0$; Intersection at $(0,0)$;
17. Domain is all reals $x\neq 0,4,-4$, V.A. at $x=0,4,-4$, No holes; H.A. at $y=0$; Intersection at $(\frac{4}{3},0)$;
19. Domain is all reals $x\neq 5,-5$, V.A. at $x=5$, Hole at $(-5,\frac{-1}{10})$; H.A. at $y=0$; No intersection;
21. Domain is all reals $x\neq\frac{1}{3}$, V.A. at $x=\frac{1}{3}$, No holes; H.A. at $y=-\frac{2}{3}$; No intersection;
25. $(2,4)$ 27. $(2,5)$ 29. $(-1,1)$

D: Describe local and end behaviour

Exercise 3.9e.D

★ Describe the local and end behavior of the functions.

31. $f(x)=\dfrac{x}{2x+1}$
32. $f(x)=\dfrac{2x}{x-6}$
33. $f(x)=\dfrac{-2x}{x-6}$
34. $f(x)=\dfrac{x^2-4x+3}{x^2-4x-5}$
35. $f(x)=\dfrac{2x^2-32}{6x^2+13x-5}$

Answers to odd exercises.

31. Local behavior: $x\to-\frac{1}{2}^+, f(x)\to-\infty, x\to-\frac{1}{2}^-, f(x)\to\infty$
End behavior: $x\to\pm\infty, f(x)\to\frac{1}{2}$
33. Local behavior: $x\to 6^+, f(x)\to-\infty, x\to 6^-, f(x)\to\infty$
End behavior: $x\to\pm\infty, f(x)\to -2$
35. Local behavior: $x\to-\frac{5}{2}^-,\ f(x)\to-\infty,\quad x\to-\frac{5}{2}^+,\ f(x)\to\infty$
$x\to\frac{1}{3}^-,\ f(x)\to\infty,\quad x\to\frac{1}{3}^+,\ f(x)\to-\infty,$ End behavior: $x\to\pm\infty, f(x)\to\frac{1}{3}$

E: Find Slant Asymptote and its intersection with the function

Exercise 3.9e.E

★ For each function, (a) Find the slant asymptote of the functions, (b) Determine any points of intersection between the function and this asymptote.

36. $f(x)=\dfrac{24x^2+6x}{2x+1}$
37. $f(x)=\dfrac{4x^2-10}{2x-4}$
38. $f(x)=\dfrac{81x^2-18}{3x-2}$
39. $f(x)=\dfrac{6x^3-5x}{3x^2+4}$
40. $f(x)=\dfrac{x^2+5x+4}{x-1}$
40.1 $f(x)=\dfrac{x^3-3x^2-x+3}{x^2-2x-8}$

Answers to odd exercises.

37. $y=2x+4$ No point of intersection 39. $y=2x$ Intersection at $(0,0)$

F: Find Intercepts

Exercise 3.9e.F

★ Find the x- and y-intercepts for the functions.

43. $f(x)=\dfrac{x}{x^2-x}$
44. $f(x)=\dfrac{x+5}{x^2+4}$
45. $f(x)=\dfrac{x^2+x+6}{x^2-10x+24}$
46. $f(x)=\dfrac{x^2+8x+7}{x^2+11x+30}$
48. $f(x)=\dfrac{94-2x^2}{3x^2-12}$

Answers to odd exercises.

43. none 45. x -intercepts none, y -intercept $\left(0, \frac{1}{4}\right)$

G: Identify attributes and sketch

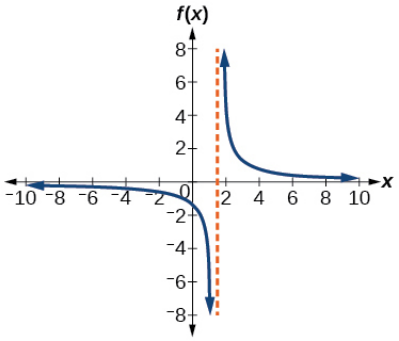
Exercise 3.9e.G

★ Find the (a) vertical asymptotes, (b) coordinates of any holes, (c) end behaviour asymptote (horizontal or slant), (d) coordinates of any points of intersection of the function with the end behaviour asymptote, (e) coordinates of any x-intercepts, (f) coordinates of the y-intercept. Use that information to sketch a graph.

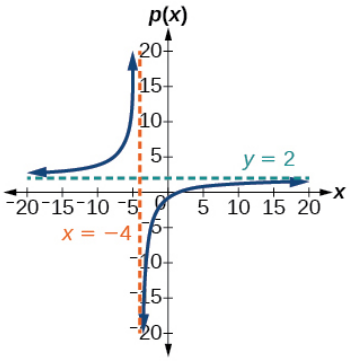
51. $f(x) = \dfrac{4}{2x-3}$
52. $f(x) = \dfrac{2}{x+1}$
53. $p(x) = \dfrac{2x-3}{x+4}$
54. $q(x) = \dfrac{x-5}{3x-1}$
55. $s(x) = \dfrac{4}{(x-2)^2}$
56. $r(x) = \dfrac{5}{(x+1)^2}$
57. $f(x) = \dfrac{x+2}{(x-1)(x-4)}$
58. $f(x) = \dfrac{2}{(x-1)(x+2)}$
59. $f(x) = \dfrac{(x-4)(x+3)^2}{(x+1)(x-4)(x-1)^2}$
60. $f(x) = \dfrac{(x-2)(x+4)}{(x-2)(x-3)(x+1)^2}$
61. $f(x) = \dfrac{(x-3)(x+6)}{(x+2)(x+4)(x-1)(x+6)}$
62. $f(x) = \dfrac{(x+1)(x-3)(-1)(x+2)}{(x+1)(x-4)(x+3)}$
63. $f(x) = \dfrac{3x^2-14x-5}{3x^2+8x-16}$
64. $g(x) = \dfrac{2x^2+7x-15}{3x^2-14+15}$
65. $a(x) = \dfrac{x^2+2x-3}{x^2-1}$
66. $b(x) = \dfrac{x^2-x-6}{x^2-4}$
67. $h(x) = \dfrac{2x^2+x-1}{x-4}$
68. $k(x) = \dfrac{2x^2-3x-20}{x-5}$
69. $w(x) = \dfrac{(x-1)(x+3)(x-5)}{(x+2)^2(x-4)}$
70. $z(x) = \dfrac{(x+2)^2(x-5)}{(x-3)(x+1)(x+4)}$
71. $f(x) = \dfrac{x^2+2}{x}$
72. $f(x) = \dfrac{x^2+x+1}{x+1}$
73. $f(x) = \dfrac{x^2-3x+2}{x-3}$
74. $f(x) = \dfrac{x^3+4x^2-x-4}{x^2+2x-8}$
75. $f(x) = \dfrac{x^2+3x+2}{x-1}$
76. $f(x) = \dfrac{x^2+3x+2}{x+3}$
77. $f(x) = \dfrac{x^2+4x+3}{x+4}$
78. $f(x) = \dfrac{x^2-3x}{x-4}$
79. $f(x) = \dfrac{x^2-4x+3}{x-2}$
80. $f(x) = \dfrac{x^2-x-6}{x-1}$
81. $f(x) = \dfrac{x^2+2x-8}{x+1}$
82. $f(x) = \dfrac{x^3-x^2-20x}{x^2+3x}$
83. $f(x) = \dfrac{4x-12}{x^3-3x^2-4x+12}$
84. $f(x) = \dfrac{2}{x^2-7x+10}$
85. $f(x) = \dfrac{x^2+9}{x^2-16}$
86. $f(x) = \dfrac{x^2-4}{x^2+2x-15}$
87. $f(x) = \dfrac{x^2-x}{x^2+7x+10}$
88. $f(x) = \dfrac{x^2+4x+3}{x^2-5x+4}$
89. $f(x) = \dfrac{x^3-x^2-x+1}{2x^2-11x+12}$
90. $f(x) = \dfrac{x^3-2x^2-x+2}{3x^2-11x+6}$
91. $f(x) = \dfrac{x-1}{x^2-7x+10}$
92. $f(x) = \dfrac{x+2}{x^2+x-12}$
93. $f(x) = \dfrac{x^2-1}{x^3-x^2-4x+4}$
94. $f(x) = \dfrac{x^2+7x+12}{x^3+9x^2+20x}$
95. $f(x) = \dfrac{x^2+4x+3}{x^2+6x+8}$
96. $f(x) = \dfrac{x^2-2x-3}{x^2+2x}$
97. $f(x) = \dfrac{x-3}{x^2+x-2}$
98. $f(x) = \dfrac{x^2-6x+9}{x^2+x-6}$
99. $f(x) = \dfrac{x^2+x-12}{x^2-4}$
100. $f(x) = \dfrac{x-3}{x^3-2x^2-4x+8}$
101. $f(x) = \dfrac{(x^3+3x^2+9x)}{x^2+x-2}$
102. $f(x) = \dfrac{x^4+x^2}{x^3-4x}$

Answers to odd exercises #51-67. More graphs at the end of this section

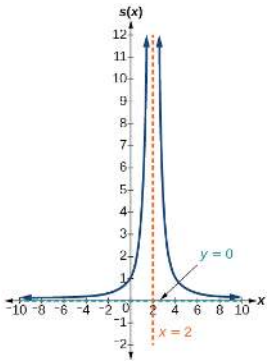
51. $\left(\frac{3}{2}, \infty\right)$



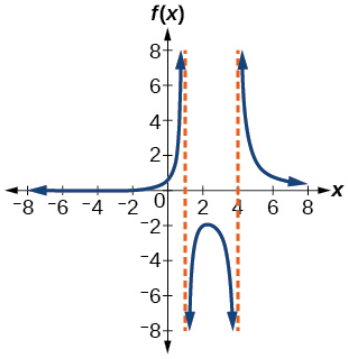
53. V.A. $x = -4$, H.A. $y = 2$; $\left(\frac{3}{2}, 0\right)$; $\left(0, -\frac{3}{4}\right)$



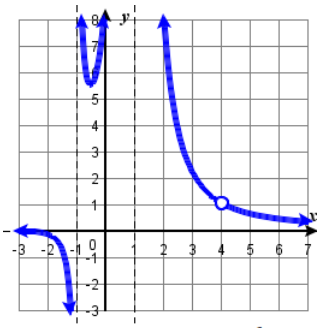
55. V.A. $x = 2$, H.A. $y = 0$, $(0, 1)$



57. $(-2, 1) \cup (4, \infty)$

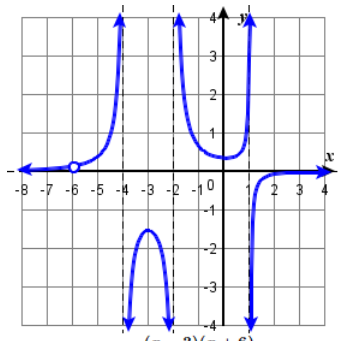


59.



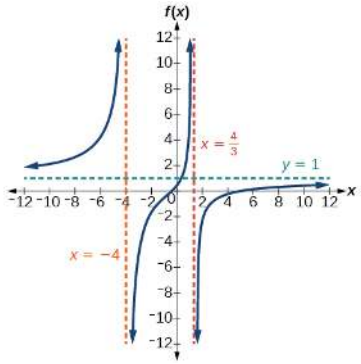
$$f(x) = \frac{(x-4)(x+3)^2}{(x+1)(x-4)(x-1)^2}$$

61.

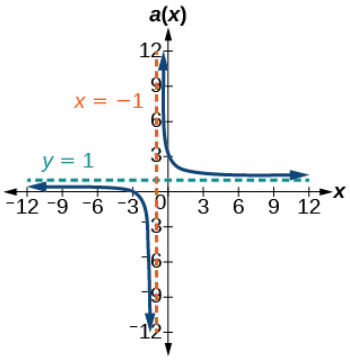


$$f(x) = \frac{(x-3)(x+6)}{(x+2)(x+4)(x-1)(x+6)}$$

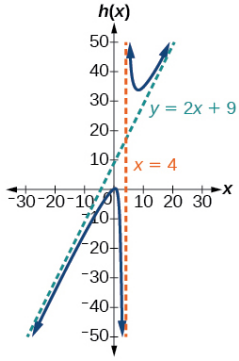
63. V.A. $x = -4$, $x = \frac{4}{3}$, H.A. $y = 1$; $(5, 0)$; $\left(-\frac{1}{3}, 0\right)$; $\left(0, \frac{5}{16}\right)$



65. V.A. $x = -1$, H.A. $y = 1$; $(-3, 0)$; $(0, 3)$



67. V.A. $x = 4$, S.A. $y = 2x + 9$; $(-1, 0)$; $\left(\frac{1}{2}, 0\right)$; $\left(0, \frac{1}{4}\right)$



H: Construct an Equation from a Description

Exercise 3.9e.H

★ **Write an equation for a rational function with the given characteristics.**

105. Vertical asymptotes at $x = 5$ and $x = -5$, x-intercepts at $(2, 0)$ and $(-1, 0)$, y -intercept at $(0, 4)$

106. Vertical asymptotes at $x = -4$ and $x = -1$, x -intercepts at $(1, 0)$ and $(5, 0)$, y -intercept at $(0, 7)$

107. Vertical asymptotes at $x = -4$ and $x = -5$, x -intercepts at $(4, 0)$ and $(-6, 0)$, Horizontal asymptote at $y = 7$

108. Vertical asymptotes at $x = -3$ and $x = 6$, x -intercepts at $(-2, 0)$ and $(1, 0)$, Horizontal asymptote at $y = -2$

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109. Vertical asymptote at $x = -1$, Double zero at $x = 2$, y -intercept at $(0, 2)$
110. Vertical asymptote at $x = 3$, Double zero at $x = 1$, y -intercept at $(0, 4)$

Answers to odd exercises.

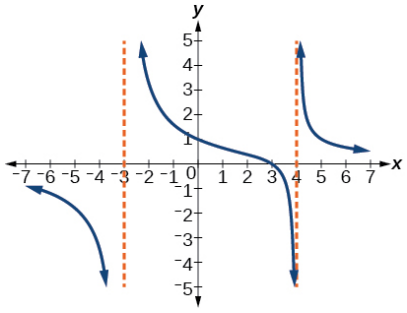
105. $y = \dfrac{50(x-2)(x+1)}{(x-5)(x+5)}$ 107. $y = \dfrac{7(x+6)(x-4)}{(x+5)(x+4)}$ 109. $y = \dfrac{(x-2)^2}{2(x+1)}$

I: Construct an Equation from a Graph

Exercise 3.9e.I

★ **Use the graphs to write an equation for the function.**

111.



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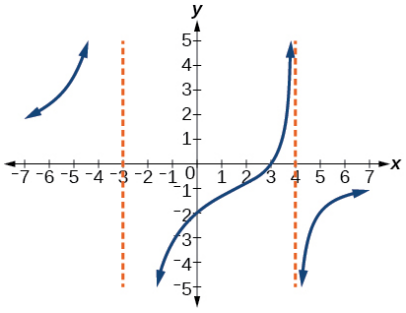
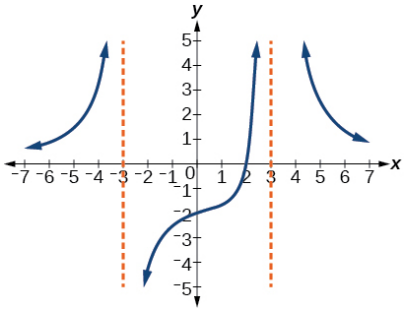
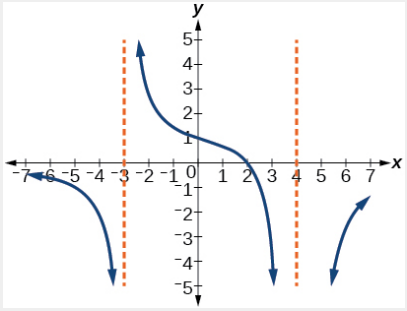


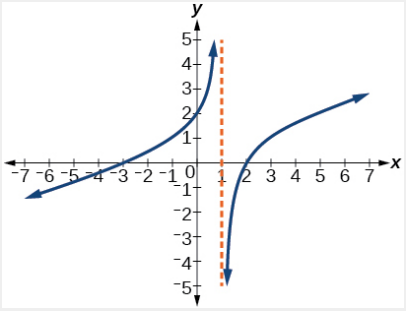
113.



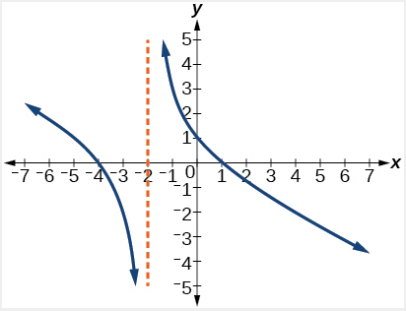
114.



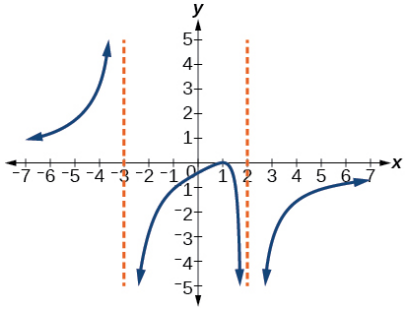
115.



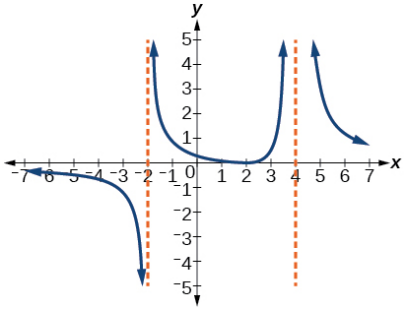
116.



117.



118.



Answers to odd exercises.

111. $y = \dfrac{4(x-3)}{(x-4)(x+3)}$ 113. $y = \dfrac{27(x-2)}{(x+3)(x-3)^2}$

115. $y = \dfrac{(x+3)(x-2)}{3(x-1)}$ 117. $y = \dfrac{-6(x-1)^2}{(x+3)(x-2)^2}$

Exercise 3.9e.J

★ **Make tables to show the behavior of the function near the vertical asymptote and horizontal asymptote**

121. $f(x)=\frac{1}{x-2}$	122. $f(x)=\frac{x}{x-3}$	123. $f(x)=\frac{2x}{x+4}$	124. $f(x)=\frac{2x}{(x-3)^2}$	125. $f(x)=\frac{x^2}{x^2+2x+1}$

Answers to odd exercises.

121.

Vertical asymptote $x=2$

x	2.01	2.001	2.0001	1.99	1.999
y	100	1,000	10,000	–100	–1,000

Horizontal asymptote $y=0$

x	10	100	1,000	10,000	100,000
y	.125	.0102	.001	.0001	.00001

123.

Vertical asymptote $x=-4$

x	–4.1	–4.01	–4.001	–3.99	–3.999
y	82	802	8,002	–798	–7998

Horizontal asymptote $y=2$

x	10	100	1,000	10,000	100,000
y	1.4286	1.9331	1.992	1.9992	1.999992

125.

Vertical asymptote $x=-1$

x	–.9	–.99	–.999	–1.1	–1.01
y	81	9,801	998,001	121	10,201

Horizontal asymptote $y=1$

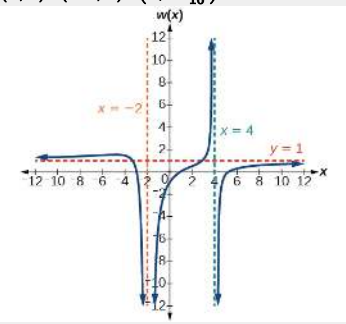
x	10	100	1,000	10,000	100,000
y	.82645	.9803	.998	.9998	.99998

Exercise $3.9e.A$

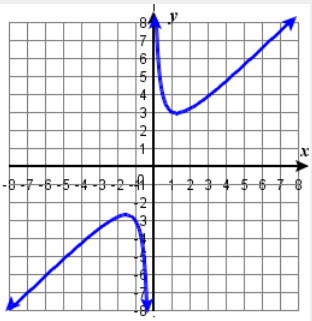
Graphs of odd-numbered exercises #71-101 in section G

Additional answers to odd exercises.

69. V.A. $x=-2$, $x=4$, H.A. $y=1$, $(1,0)$; $(5,0)$; $(-3,0)$; $\left(0,-\frac{15}{16}\right)$

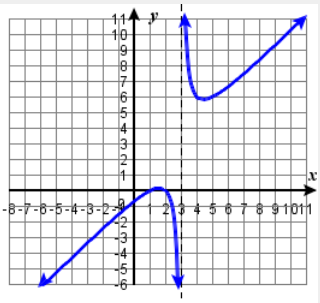


71.



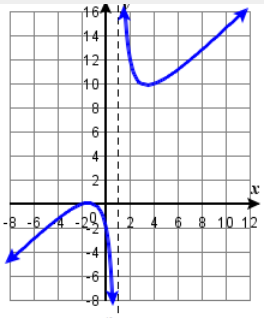
$$f(x)=\frac{x^2+2}{x}$$

73.



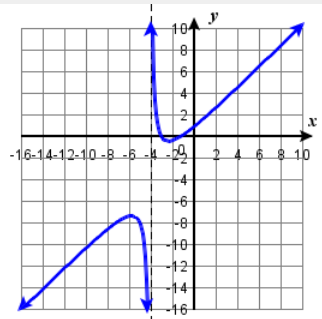
$$f(x)=\frac{x^2-3x+2}{x-3}$$

75.



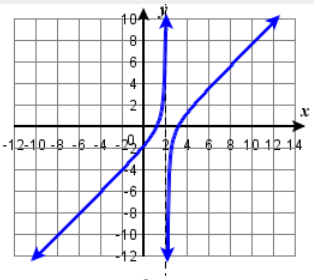
$$f(x)=\frac{x^2+3x+2}{x-1}$$

77.



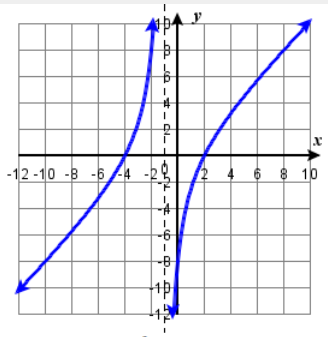
$$f(x)=\frac{x^2+4x+3}{x+4}$$

79.



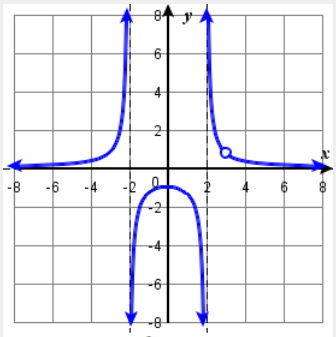
$$f(x)=\frac{x^2-4x+3}{x-2}$$

81.



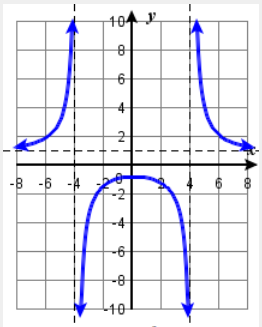
$$f(x)=\frac{x^2+2x-8}{x+1}$$

83.



$$f(x)=\frac{4x-12}{x^3-3x^2-4x+12}$$

85.



$$f(x)=\frac{x^2+9}{x^2-16}$$

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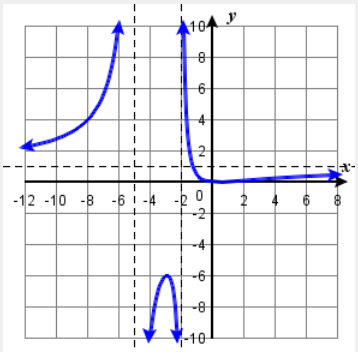
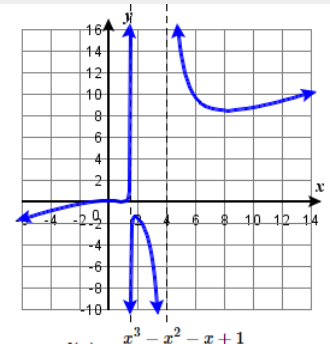
87.



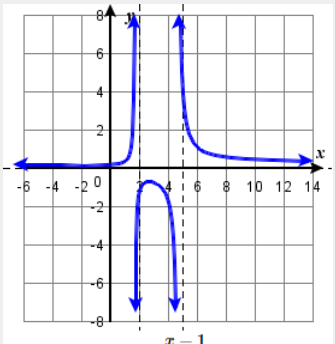
$$f(x)=\frac{x^2-x}{x^2+7x+10}$$

89.



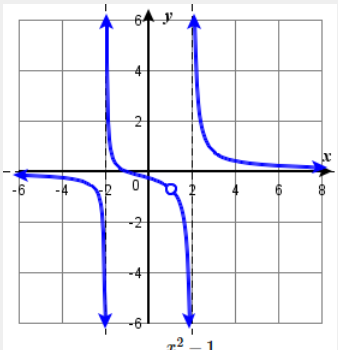
$$f(x)=\frac{x^3-x^2-x+1}{2x^2-11x+12}$$

91.



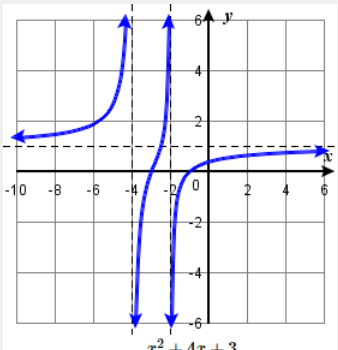
$$f(x)=\frac{x-1}{x^2-7x+10}$$

93.



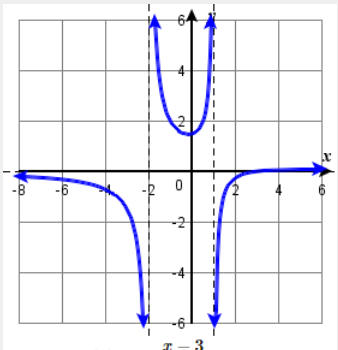
$$f(x)=\frac{x^2-1}{x^3-x^2-4x+4}$$

95.



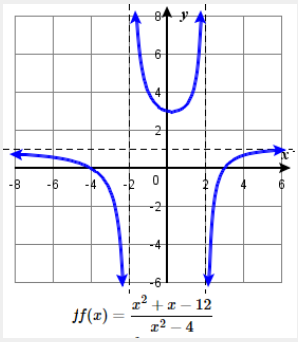
$$f(x)=\frac{x^2+4x+3}{x^2+6x+8}$$

97.



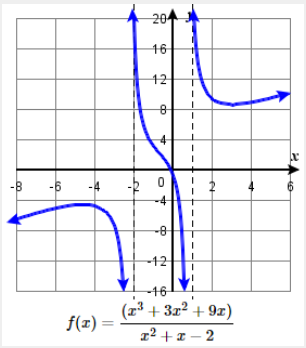
$$f(x)=\frac{x-3}{x^2+x-2}$$

99.



$$ff(x)=\frac{x^2+x-12}{x^2-4}$$

101.



$$f(x)=\frac{(x^3+3x^2+9x)}{x^2+x-2}$$

★

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CHAPTER OVERVIEW

4: Exponential and Logarithmic Functions

In this chapter, we will explore exponential functions, which can be used for, among other things, modeling growth patterns such as those found in bacteria. We will also investigate logarithmic functions, which are closely related to exponential functions. Both types of functions have numerous real-world applications when it comes to modeling and interpreting data.

- 4.1: Exponential Functions
  - 4.1e: Exercises - Exponential Functions
- 4.2: Graphs of Exponential Functions
  - 4.2e: Exercises - Graphs of Exponential Functions
- 4.3: Logarithmic Functions
  - 4.3e: Exercises - Logarithm Functions
- 4.4: Graphs of Logarithmic Functions
  - 4.4e: Exercises - Graphs of Logarithmic Functions
- 4.5: Logarithmic Properties
  - 4.5e: Exercises - Properties of Logarithms
- 4.6: Exponential and Logarithmic Equations
  - 4.6e: Exercises - Exponential and Logarithmic Equations
- 4.7: Exponential and Logarithmic Models
  - 4.7e: Exercises - Exponential Applications

Thumbnail: The functions $y = e^x$ and $y = \ln(x)$ are inverses of each other, so their graphs are symmetric about the line $y = x$. Image used with permission (CC BY-SA; OpenStax).

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4.1: Exponential Functions

Learning Objectives

- Identify and evaluate exponential functions.
- Construct equations that model exponential growth.
- Use compound interest formulas and continuous growth and decay models.

India is the second most populous country in the world with a population of about 1.25 billion people in 2013. The population is growing at a rate of about $.2\%$ each year. If this rate continues, the population of India will exceed China's population by the year 2031. When populations grow rapidly, we often say that the growth is "exponential," meaning that something is growing very rapidly. To a mathematician, however, the term *exponential growth* has a very specific meaning. In this section, we will take a look at *exponential functions*, which model this kind of rapid growth.

Identify Exponential Functions

When exploring linear growth, we observed a constant rate of change - a constant number by which the output increased for each unit increase in input (i.e. $f(x+1)=f(x)+c$). For example, in the linear equation $f(x)=3x+4$, the **difference** in consecutive outputs, each time the input increases by 1, is always a constant, 3 (notice 3 is the slope of the line). The scenario in the India population example is different - it exhibits exponential growth. In this situation, the **ratio** between consecutive outputs, each time 1 year goes by, is always a constant (i.e. $f(x+1)=f(x)\times c$). In other words, each year, the number of people increases by a fixed percentage of the population.

Exponential Functions

What exactly does it mean to *grow exponentially*? What does the word *double* have in common with *percent increase*? These words are often tossed around and appear frequently in the media.

- **Percent change** refers to a *change* based on a *percent* of the original amount.
- **Exponential growth** refers to a *percent* increase of the original amount over time. The amount of *increase* is based on a constant multiplicative rate of change over equal increments of time.
- **Exponential decay** refers to a *percent* decrease of the original amount over time. The amount of *decrease* is based on a constant multiplicative rate of change over equal increments of time.

For us to gain a clear understanding of **exponential growth**, let us contrast exponential growth with **linear growth**. We will construct two functions. Both start with an input of 0. The first function is exponential. Every time the input is increased by 1 we will **double** the corresponding consecutive output. Thus the **ratio** between consecutive outputs will always be 2. The second function is linear. Every time the input is increased by 1 we will **add 2** to the corresponding consecutive outputs. Thus the **difference** between consecutive output will always be 2. (Table $4.1.1$).

From Table $4.1.1$ we can infer that for these two functions, exponential growth dwarfs linear growth.

- **Exponential growth** refers to the original value from the range increasing by the *same percentage* over equal increments found in the domain.
- **Linear growth** refers to the original value from the range increasing by the *same amount* over equal increments found in the domain.

Table $4.1.1$.

x	0	1	2	3	4	5	6
$f(x)=2^x$	1	2	4	8	16	32	64
$g(x)=2x$	0	2	4	6	8	10	12

The next value, $f(x+1)$ is 2 **times** more than the previous value $f(x)$.
The next value, $g(x+1)$ is an **additional** 2 more than the previous value $g(x)$.

Clearly, the difference between "the same percentage" and "the same amount" is quite significant. For exponential growth, over equal increments, the constant multiplicative rate of change resulted in **multiplying** the output by 2 whenever the input increased by one. For linear growth, the constant additive rate of change over equal increments resulted in **adding** 2 to the output whenever the input was increased by one.

General form of the exponential function

The general form of the **exponential function** is $f(x) = ab^x$, where a is any nonzero number, b is a positive real number not equal to 1.

- The base, b, is a constant called the **growth factor**, with $b > 0$ and $b \neq 1$.
  - If $b > 1$, the function grows at a rate proportional to its size.
  - If $0 < b < 1$, the function decays at a rate proportional to its size.
- a is a constant called the **initial value**. It is called this because the value of f is a when $x = 0$
  - the y-intercept is $(0, a)$.
  - the **range** is all positive real numbers $(0, \infty)$ if $a > 0$ and all negative real numbers $(-\infty, 0)$ if $a < 0$.
- An exponential function has a **horizontal asymptote** at $y = 0$
- The independent variable x is in the exponent.
  - The exponent x can be any real number, so the **domain** of an exponential function is $(-\infty, \infty)$

The base of an exponential function **must be positive**. The reason for this restriction ensures the outputs will be real numbers. Observe what happens if the base is not positive:

Let $b = -9$ and $x = \frac{1}{2}$. Then $f(x) = f\left(\frac{1}{2}\right) = (-9)^{\frac{1}{2}} = \sqrt{-9}$, which is not a real number.

The base of an exponential function **cannot be** 1. The reason for this restriction is because base 1 results in the constant function. Observe what happens if the base is 1:

Let $b = 1$. Then $f(x) = 1^x = 1$ for any value of x. The horizontal line $y = 1$ is not an exponential function!

Example 4.1.1: Identify Exponential Functions

Which of the following equations are *not* exponential functions?

• $f(x) = 4^{3(x-2)}$	• $g(x) = x^3$	• $h(x) = \left(\frac{1}{3}\right)^x$	• $j(x) = (-2)^x$

Solution

By definition, an exponential function has a constant as a base and an independent variable as an exponent. Thus, $g(x) = x^3$ does not represent an exponential function because the base is an independent variable. Functions like $g(x) = x^3$ in which the variable is in the base and the exponent is a constant are called power functions.

Recall that the base b of an exponential function is always a positive constant, and $b \neq 1$. Thus, $j(x) = (-2)^x$ does not represent an exponential function because the base, -2, is less than 0.

Try It 4.1.1

Which of the following equations represent exponential functions?

• $f(x) = 2x^2 - 3x + 1$	• $g(x) = 0.875^x$	• $h(x) = 1.75x + 2$	• $j(x) = 1095.6^{-2x}$

Answer

$g(x) = 0.875^x$ and $j(x) = 1095.6^{-2x}$ represent exponential functions.

Evaluate Exponential Functions

To evaluate an exponential function with the form $f(x) = b^x$, we simply substitute x with the given value, and calculate the resulting power. For example: Let $f(x) = 2^x$. What is $f(3)$?

$$\begin{aligned} f(3) &= 2^3 && \text{Substitute } x = 3 \\ &= 8 && \text{Evaluate the power} \end{aligned}$$

To evaluate an exponential function with a form other than the basic form, it is important to follow the order of operations. For example: Let $f(x) = 30(2)^x$. What is $f(3)$?

$$\begin{aligned} f(3) &= 30(2)^3 && \text{Substitute } x = 3 \\ &= 30(8) && \text{Simplify the power first} \\ &= 240 && \text{Multiply} \end{aligned}$$

Note that if the order of operations were not followed, the result would be incorrect: $f(3) = 30(2)^3 \neq 60^3 = 216,000$

Example 4.1.2: Evaluate Exponential Functions

Let $f(x) = 5(3)^{x+1}$. Evaluate $f(2)$ without using a calculator.

Solution: Follow the order of operations. Be sure to pay attention to the parentheses.

$$\begin{aligned} f(2) &= 5(3)^{2+1} && \text{Substitute } x = 2 \\ &= 5(3)^3 && \text{Add the exponents} \\ &= 5(27) && \text{Simplify the power} \\ &= 135 && \text{Multiply} \end{aligned}$$

Try It 4.1.2

Let $f(x) = 8(1.2)^{x-5}$. Evaluate $f(3)$ using a calculator. Round to four decimal places.

Answer

5.5556

Exponential Growth Defined

Because the output of exponential functions increases very rapidly, the term "exponential growth" is often used in everyday language to describe anything that grows or increases rapidly. However, exponential growth can be defined more precisely in a mathematical sense. If the growth rate is proportional to the amount present, the function models exponential growth.

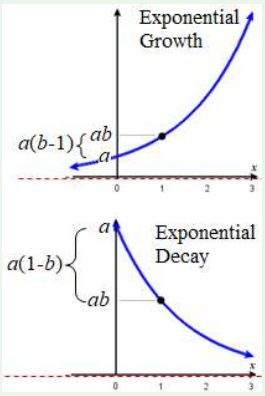
EXPONENTIAL GROWTH

A function that models exponential growth grows by a rate proportional to the amount present. For any real number a and x, and any positive real number b such that $b \neq 1$, an exponential growth function has the form

$$f(x) = ab^x$$

where

- a is the **initial** or **starting value** of the function. In other words, $f(0) = a$.
- b is the **growth factor** or growth multiplier per unit x. In other words, $f(x + 1) = b \cdot f(x)$.



- $b-1$ is the **growth rate**. It is the difference between outputs of consecutive values of x. In other words, $f(x+1)=f(x)+(b-1)\cdot f(x)$. If negative, there is exponential decay; if positive, there is exponential growth.

In more general terms, we have an *exponential function*, in which a constant base is raised to a variable exponent.

To differentiate between linear and exponential functions, let's consider two companies, A and B. Company A has 100 stores and expands by opening 50 new stores a year, so its growth can be represented by the function $A(x)=100+50x$. Company B has 100 stores and expands by increasing the number of stores by 50% each year, so its growth can be represented by the function $B(x)=100(1+0.5)^x$.

A few years of growth for these companies are illustrated below.

Table 4.1.3

Year, x	Stores, Company A	Stores, Company B
0	$100+50(0)=100$	$100(1+0.5)^0=100$
1	$100+50(1)=150$	$100(1+0.5)^1=150$
2	$100+50(2)=200$	$100(1+0.5)^2=225$
3	$100+50(3)=250$	$100(1+0.5)^3=337.5$
x	$A(x)=100+50x$	$B(x)=100(1+0.5)^x$

The graphs comparing the number of stores for each company over a five-year period are shown in Figure 4.1.2. We can see that, with exponential growth, the number of stores increases much more rapidly than with linear growth.

Notice that the domain for both functions is $[0,\infty)$,and the range for both functions is $[100,\infty)$. After year 1, Company B always has more stores than Company A.

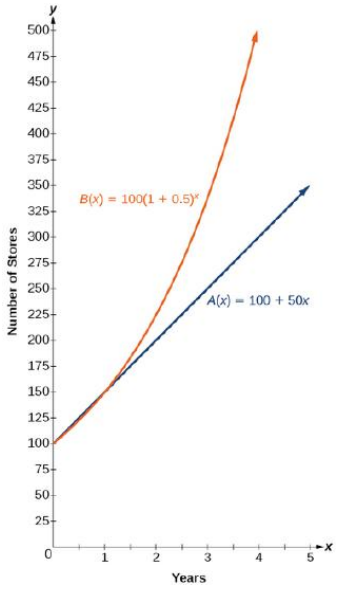


Figure 4.1.2: The graph shows the numbers of stores Companies A and B opened over a five-year period.

Now we will turn our attention to the function representing the number of stores for Company B, $B(x)=100(1+0.5)^x$. In this **exponential function**, 100 represents the **initial number** of stores, 0.50 represents the **growth rate** (the percent difference (as a decimal) between consecutive outputs), and $1+0.5=1.5$ represents the **growth factor** (the growth multiplier per unit time). Generalizing further, we can write this function as $B(x)=100(1.5)^x$, where 100 is the **initial value**, 1.5 is called the ***base,*** and x is called the ***exponent***.

Example 4.1.3: Construct an Exponential Model Given the Growth Rate

At the beginning of this section, we learned that the population of India was about 1.25 billion in the year 2013, with an annual growth rate of about 1.2%. This situation is represented by the growth function $P(t)=1.25(1.012)^t$, where t is the number of years since 2013. To the nearest thousandth, what will the population of India be in 2031?

Solution

To estimate the population in 2031, we evaluate the models for $t=18$, because 2031 is 18 years after 2013. Rounding to the nearest thousandth,

$$P(18)=1.25(1.012)^{18}\approx 1.549$$

There will be about 1.549 billion people in India in the year 2031.

Try It 4.1.3

The population of China was about 1.39 billion in the year 2013, with an annual growth rate of about 0.6%. This situation is represented by the growth function $P(t) = 1.39(1.006)^t$, where t is the number of years since 2013. To the nearest thousandth, what will the population of China be for the year 2031? How does this compare to the population prediction we made for India in Example 4.1.3 ?

Answer

About 1.548 billion people; by the year 2031, India's population will exceed China's by about 0.001 billion, or 1 million people.

Construct Equations that Model Exponential Growth

In the previous examples, we were given an exponential function, which we then evaluated for a given input. Sometimes we are given information about an exponential function without knowing the function explicitly. We must use the information to first write the form of the function, then determine the constants a and b, and evaluate the function.

How to: Given two data points, write an exponential model

1. If one of the data points has the form $(0, a)$, then a is the **initial value**. Using a, substitute the second point into the equation $f(x) = a(b)^x$, and solve for the **growth factor**, b. The **growth rate** is $b - 1$.
2. If neither of the data points have the form $(0, a)$, substitute both points into two equations with the form $f(x) = a(b)^x$. Solve the resulting system of two equations in two unknowns to find a and b.
3. Using the a and b found in the steps above, write the exponential function in the form $f(x) = a(b)^x$.

Example 4.1.4: Construct an Exponential Model when the Growth Rate is Not Given

In 2006, 80 deer were introduced into a wildlife refuge. By 2012, the population had grown to 180 deer. The population was growing exponentially. Write an algebraic function $N(t)$ representing the population (N) of deer over time t.

Solution

Choose the independent variable t be the number of years after 2006. This is done so that we can give ourselves an initial value for the function: when $t = 0$, $a = 80$. Thus, the information given in the problem can be written as input-output pairs: (0, 80) and (6, 180). We can now substitute the second point into the equation $N(t) = 80b^t$ to find b:

$$\begin{aligned} N(t) &= 80b^t \\ 180 &= 80b^6 && \text{Substitute using point } (6, 180) \\ \frac{9}{4} &= b^6 && \text{Divide and write in lowest terms} \\ b &= \left(\frac{9}{4}\right)^{\frac{1}{6}} && \text{Isolate b using properties of exponents} \\ b &\approx 1.1447 && \text{Round to 4 decimal places} \end{aligned}$$

To avoid rounding errors, do not round any intermediate calculations!. ONLY round the final answer. If the precision of the answer is not stated, give it to four digits.

The **growth factor**, b, is 1.1447.
The **growth rate** is $1.1447 - 1 = .1447$ or 14.47%

The exponential model for the population of deer is $N(t) = 80(1.1447)^t$. (Note that this exponential function models short-term growth. As the inputs gets large, the predicted output will get so large that the model may not be useful.)
We can graph our model to observe the population growth of deer in the refuge over time. Notice that the graph in Figure 4.1.3 passes through the initial points given in the problem, $(0, 80)$ and $(6, 180)$. We can also see that the domain for the function is $[0, \infty)$,and the range for the function is $[80, \infty)$.

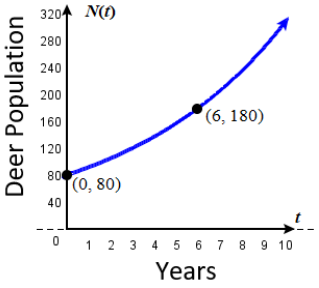


Figure 4.1.3: Graph showing the population of deer over time, $N(t) = 80(1.1447)^t$, t years after 2006

Try It 4.1.4

A wolf population is growing exponentially. In 2011, 129 wolves were counted. By 2013, the population had reached 236 wolves. What two points can be used to derive an exponential equation modeling this situation? Write the equation representing the population N of wolves over time t.

Answer

$(0, 129)$ and $(2, 236)$; $N(t) = 129(1.3526)^t$

Example 4.1.5: Construct an Exponential Model When the Initial Value is Not Known

Find an exponential function that passes through the points $(-2, 6)$ and $(2, 1)$.

Solution

Because we don't have the initial value, we substitute both points into an equation of the form $f(x) = ab^x$, and then solve the system for a and b.

- Substituting $(-2, 6)$ gives $6 = ab^{-2}$
- Substituting $(2, 1)$ gives $1 = ab^2$

Use the first equation to solve for a in terms of b:

$$6 = ab^{-2}$$
$$\frac{6}{b^{-2}} = a \quad \text{Divide}$$
$$a = 6b^2 \quad \text{Use properties of exponents to rewrite the denominator}$$

Substitute a in the second equation, and solve for b:

$$1 = ab^2 = (6b^2)b^2 = 6b^4 \quad \text{Substitute a}$$
$$b = \left(\frac{1}{6}\right)^{\frac{1}{4}} \quad \text{Round 4 decimal places rewrite the denominator}$$
$$b \approx 0.6389$$

Use the value of b in the first equation to solve for the value of a. The exact value for b should be used here to avoid round off errors.

$$a = 6b^2 = 6\left(\frac{1}{6}\right)^{\frac{1}{2}} \approx 2.4495$$

Thus, the equation is $f(x) = 2.4495(0.6389)^x$.

We can graph our model to check our work. Notice that the graph in Figure 4.1.4 passes through the initial points given in the problem, $(-2, 6)$ and $(2, 1)$. The graph is an example of an exponential decay function.

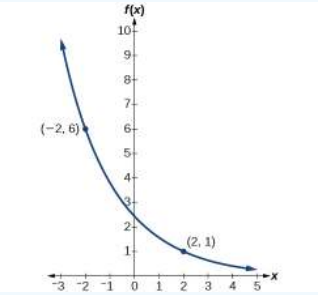


Figure 4.1.4: The graph of $f(x) = 2.4495(0.6389)^x$ models exponential decay.

Try It 4.1.5

Given the two points $(1, 3)$ and $(2, 4.5)$, find the equation of the exponential function that passes through these two points.

Answer

$f(x) = 2(1.5)^x$

Do two points always determine a unique exponential function?

Yes, provided the two points are either both above the x-axis or both below the x-axis and have different x-coordinates. But keep in mind that we also need to know that the graph is, in fact, an exponential function. Not every graph that looks exponential really is exponential. We need to know the graph is based on a model that shows the same percent growth with each unit increase in x, which in many real world cases involves time.

How to: Given the graph of an exponential function, write its equation

1. First, identify two points on the graph. Choose the y-intercept as one of the two points whenever possible. Try to choose points that are as far apart as possible to reduce round-off error.
2. If one of the data points is the y-intercept $(0, a)$, then a is the initial value. Using a, substitute the second point into the equation $f(x) = a(b)^x$, and solve for b
3. If neither of the data points have the form $(0, a)$, substitute both points into two equations with the form $f(x) = a(b)^x$. Solve the resulting system of two equations in two unknowns to find a and b.
4. Write the exponential function, $f(x) = a(b)^x$.

Example 4.1.6: Construct an Exponential Function Given Its Graph

Find an equation for the exponential function graphed in Figure 4.1.5.

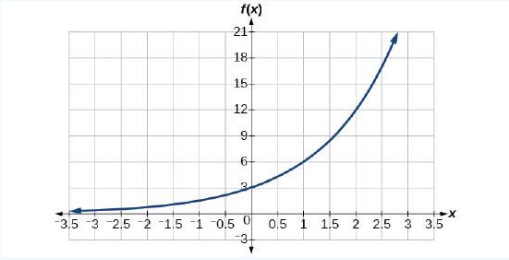


Figure 4.1.5

Solution

We can choose the y-intercept of the graph, $(0, 3)$, as our first point. This gives us the initial value, $a = 3$. Next, choose a point on the curve some distance away from $(0, 3)$ that has integer coordinates. One such point is $(2, 12)$.

$y = ab^x$	Write the general form of an exponential equation
$y = 3b^x$	Substitute the initial value 3 for a
$12 = 3b^2$	Substitute in 12 for y and 2 for x
$4 = b^2$	Divide by 3
$b = \pm 2$	Take the square root

Because we restrict ourselves to positive values of b, we will use $b = 2$. Substitute a and b into the standard form to yield the equation $f(x) = 3(2)^x$.

Try It 4.1.6

Find an equation for the exponential function graphed in Figure 4.1.6.

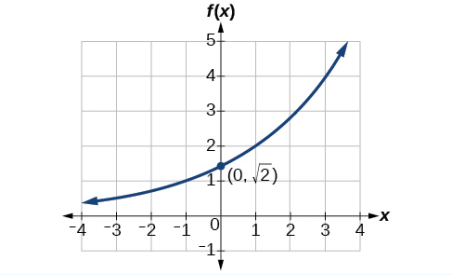


Figure 4.1.6

Answer

$f(x) = \sqrt{2}(\sqrt{2})^x$. Answers may vary due to round-off error. The answer should be very close to $1.4142(1.4142)^x$.

The Compound-Interest Formula

Savings instruments in which earnings are continually reinvested, such as mutual funds and retirement accounts, use **compound interest**. The term *compounding* refers to interest earned not only on the original value, but on the accumulated value of the account.

The **annual percentage rate (APR)** of an account, also called the **nominal rate**, is the **yearly interest rate** earned by an investment account. The term *nominal* is used when the compounding occurs a number of times other than once per year. In fact, when interest is compounded more than once a year, the effective interest rate ends up being *greater* than the nominal rate! This is a powerful tool for investing.

We can calculate the compound interest using the compound interest formula, which is an exponential function of the variables time t, principal P, yearly interest rate r, and number of compounding periods in a year n:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

For example, observe Table 4.1.4, which shows the result of investing $\$1,000$ at 10% for one year. Notice how the value of the account increases as the compounding frequency increases.

Table 4.1.4

Frequency	Value after 1 year	(effective interest rate)
Annually	$1100	10%
Semiannually	$1102.50	10.250%
Quarterly	$1103.81	10.381%
Monthly	$1104.71	10.471%
Daily	$1105.16	10.516%

Definition: THE COMPOUND INTEREST FORMULA

Compound interest can be calculated using the formula

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} \tag{4.1.1}$$

where

- $A(t)$ is the current value of the account,
- t is measured in years,
- P is the starting amount of the account, often called the principal, or more generally present value,
- r is the yearly interest rate expressed as a decimal, often called APR or nominal rate, and
- n is the number of compounding periods in one year.

Example 4.1.7: Calculate Compound Interest

If we invest $\$3,000$ in an investment account paying 3% interest compounded quarterly, how much will the account be worth in 10 years?

Solution

Because we are starting with $\$3,000$, $P = 3000$. Our interest rate is 3%, so $r = 0.03$. Because we are compounding quarterly, we are compounding 4 times per year, so $n = 4$. We want to know the value of the account in 10 years, so we are looking for $A(10)$, the value when $t = 10$.

$$\begin{aligned} A(t) &= P\left(1 + \frac{r}{n}\right)^{nt} && \text{Use the compound interest formula} \\ A(10) &= 3000\left(1 + \frac{0.03}{4}\right)^{(4)\cdot(10)} && \text{Substitute using given values} \\ &\approx \$4045.05 && \text{Round to two decimal places} \end{aligned}$$

The account will be worth about $\$4,045.05$ in 10 years.

Try It 4.1.7

An initial investment of $\$100,000$ at 12% interest is compounded weekly (use 52 weeks in a year). What will the investment be worth in 30 years?

Answer

about $\$3,644,675.88$

Example 4.1.8: Use the Compound Interest Formula to Solve for the Principal

A 529 Plan is a college-savings plan that allows relatives to invest money to pay for a child's future college tuition; the account grows tax-free. Lily wants to set up a 529 account for her new granddaughter and wants the account to grow to $\$40,000$ over 18

years. She believes the account will earn 6% compounded semi-annually (twice a year). To the nearest dollar, how much will Lily need to invest in the account now?

Solution

The nominal interest rate is 6%, so $r = 0.06$. Interest is compounded twice a year, so $k = 2$.

We want to find the initial investment, P, needed so that the value of the account will be worth \$40,000 in 18 years. Substitute the given values into the compound interest formula, and solve for P.

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} \quad \text{Use the compound interest formula}$$

$$40,000 = P\left(1 + \frac{0.06}{2}\right)^{2(18)} \quad \text{Substitute using given values } A, r, n, t$$

$$40,000 = P(1.03)^{36} \quad \text{Simplify}$$

$$\frac{40,000}{(1.03)^{36}} = P \quad \text{Isolate } P$$

$$P \approx \$13,801 \quad \text{Divide and round to the nearest dollar}$$

Lily will need to invest \$13,801 to have \$40,000 in 18 years.

Try It 4.1.8

In Example 4.1.9, how much, to the nearest dollar, would Lily need to invest if the account is compounded quarterly?

Answer

\$13,693

The Constant e

As we saw earlier, the amount earned on an account increases as the compounding frequency increases. Table 4.1.5 shows that the increase from annual to semi-annual compounding is larger than the increase from monthly to daily compounding. This might lead us to ask whether this pattern will continue.

Examine the value of \$1 invested at 100% interest for 1 year, compounded at various frequencies, listed in Table 4.1.5.

Table 4.1.5

Frequency	$A(t) = \left(1 + \frac{1}{n}\right)^n$	Value
Annually	$\left(1 + \frac{1}{1}\right)^1$	\$2
Semiannually	$\left(1 + \frac{1}{2}\right)^2$	\$2.25
Quarterly	$\left(1 + \frac{1}{4}\right)^4$	\$2.441406
Monthly	$\left(1 + \frac{1}{12}\right)^{12}$	\$2.613035
Daily	$\left(1 + \frac{1}{365}\right)^{365}$	\$2.714567
Hourly	$\left(1 + \frac{1}{8760}\right)^{8760}$	\$2.718127
Once per minute	$\left(1 + \frac{1}{525600}\right)^{525600}$	\$2.718279
Once per second	$\left(1 + \frac{1}{31536000}\right)^{31536000}$	\$2.718282

These values appear to be approaching a limit as n increases without bound. In fact, as n gets larger and larger, the expression $\left(1+\frac{1}{n}\right)^n$ approaches a number used so frequently in mathematics that it has its own name: the letter e. This value is an irrational number, which means that its decimal expansion goes on forever without repeating. Its approximation to six decimal places is $e \approx 2.718282$.

Definition: THE CONSTANT e

The letter e represents an irrational number

$$e = \left(1+\frac{1}{n}\right)^n \quad \text{as } n \text{ increases without bound}$$

The letter e is used as a base for many real-world exponential models. To work with base e, we use the approximation, $e \approx 2.718282$. The constant was named by the Swiss mathematician Leonhard Euler (1707–1783) who first investigated and discovered many of its properties.

Example 4.1.9: Use a Calculator to Find Powers of e

Calculate $e^{3.14}$. Round to five decimal places.

Solution

On a calculator, press the button labeled [e^x]. The window shows [$e^($]. Type 3.14 and then close parenthesis, [)]. Press [ENTER]. Rounding to 5 decimal places, $e^{3.14} \approx 23.10387$. Caution: Many scientific calculators have an "Exp" button, which is used to enter numbers in scientific notation. It is not used to find powers of e.

Try It 4.1.9

Use a calculator to find $e^{-0.5}$. Round to five decimal places.

Answer

$e^{-0.5} \approx 0.60653$

Continuous Growth or Decay

So far we have worked with rational bases for exponential functions. For most real-world phenomena, however, e is used as the base for exponential functions. Exponential models that use e as the base are called continuous growth or decay models. We see these models in finance, computer science, and most of the sciences, such as physics, toxicology, and fluid dynamics.

Definition: THE CONTINUOUS GROWTH/DECAY FORMULA

For all real numbers t, a and r, continuous growth or decay is represented by the formula

$$A(t) = ae^{rt} \tag{4.1.2}$$

where

- a is the **initial value**,
- r is the continuous **growth rate** per unit time,
- t is the elapsed time.

If $r > 0$, then the formula represents **continuous growth**. If $r < 0$, then the formula represents **continuous decay**.

For business applications, the continuous growth formula is called the **continuous compounding** formula and takes the form

$$A(t) = Pe^{rt} \tag{4.1.3}$$

where

- P is the principal or the initial amount invested,

- r is the growth or interest rate per unit time,
- t is the period or term of the investment.

How to: Construct and evaluate a continuous growth or decay function $A(t) = ae^{rt}$

1. Use the information in the problem to determine a, the initial value of the function.
2. Use the information in the problem to determine the growth rate r.
   - If the problem refers to continuous growth, then $r > 0$.
   - If the problem refers to continuous decay, then $r < 0$.
3. Use the information in the problem to determine the time t.
4. Substitute the given information into the continuous growth formula and evaluate $A(t)$.

Example 4.1.10: Calculate Continuous Growth

A person invested $\$1,000$ in an account earning a nominal 10% per year compounded continuously. How much was in the account at the end of one year?

Solution

Since the account is growing in value, this is a continuous compounding problem with growth rate $r = 0.10$. The initial investment was $\$1,000$, so $P = 1000$. We use the continuous compounding formula to find the value after $t = 1$ year:

$$\begin{aligned} A(t) &= Pe^{rt} && \text{Use the continuous compounding formula} \\ &= 1000(e)^{0.1} && \text{Substitute known values for } P, r, t \\ &\approx 1105.17 && \text{Use a calculator to approximate} \end{aligned}$$

The account is worth $\$1,105.17$ after one year.

Try It 4.1.10

A person invests $\$100,000$ at a nominal 12% interest per year compounded continuously. What will be the value of the investment in 30 years?

Answer

$\$3,659,823.44$

Example 4.1.11: Calculate Continuous Decay

Radon-222 decays at a continuous rate of 17.3% per day. How much will 100 mg of Radon-222 decay to in 3 days?

Solution

Since the substance is decaying, the rate, 17.3%, is negative. So, $r = -0.173$. The initial amount of Radon-222 was 100 mg, so $a = 100$. We use the continuous decay formula to find the value after $t = 3$ days:

$$\begin{aligned} A(t) &= ae^{rt} && \text{Use the continuous growth formula} \\ &= 100e^{-0.173(3)} && \text{Substitute known values for } a, r, t \\ &\approx 59.5115 && \text{Use a calculator to approximate} \end{aligned}$$

So 59.5115 mg of Radon-222 will remain.

Try It 4.1.11

Using the data in Example $4.1.12$, how much Radon-222 will remain after one year?

Answer

$3.77E-26$ (This is calculator notation for the number written as 3.77×10^{-26} in scientific notation. While the output of an exponential function is never zero, this number is so close to zero that for all practical purposes we can accept zero as the answer.)

Key Equations

definition of the exponential function	$f(x) = b^x$, where $b > 0$, $b \neq 1$
definition of exponential growth/decay	$f(x) = ab^x$, where $b > 0$, $b \neq 1$
compound interest formula	$A(t) = P(1 + \frac{r}{n})^{nt}$, where $A(t)$ is the account value at time t t is the number of years P is the initial investment, often called the principal r is the annual percentage rate (APR), or nominal rate n is the number of compounding periods in one year
continuous growth formula	$A(t) = ae^{rt}$, where t is the number of unit time periods of growth a is the starting amount (in the continuous compounding formula a is replaced with P, the principal) e is the mathematical constant, $e \approx 2.718282$

Key Concepts

- An exponential function is defined as a function with a positive constant other than 1 raised to a variable exponent.
- An exponential model can be found when the growth rate and initial value are known.
- An exponential model can be found when the two data points from the model are known.
- An exponential model can be found using two data points from the graph of the model.
- The value of an account at any time t can be calculated using the compound interest formula when the principal, annual interest rate, and compounding periods are known.
- The initial investment of an account can be found using the compound interest formula when the value of the account, annual interest rate, compounding periods, and life span of the account are known.
- The number e is a mathematical constant often used as the base of real world exponential growth and decay models. Its decimal approximation is $e \approx 2.718282$.
- Scientific and graphing calculators have the key $[e^x]$ or $[exp(x)]$ for calculating powers of e.
- Continuous growth or decay models are exponential models that use e as the base. Continuous growth and decay models can be found when the initial value and growth or decay rate are known.

Contributors

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4.1e: Exercises - Exponential Functions

A: Concepts

Exercise 4.1e.A

1. Explain why the values of an increasing exponential function will eventually overtake the values of an increasing linear function.

2. Given a formula for an exponential function, is it possible to determine whether the function grows or decays exponentially just by looking at the formula? Explain.

3. The Oxford Dictionary defines the word nominal as a value that is “stated or expressed but not necessarily corresponding exactly to the real value.” Develop a reasonable argument for why the term nominal rate is used to describe the annual percentage rate of an investment account that compounds interest.

4. Why is $b = 1$ excluded as a base in the definition of exponential functions? Explain.

5. Explain why an exponential function of the form $y = b^x$ can never be negative.

Answers to odd exercises:

1. Linear functions have a constant rate of change. Exponential functions increase based on a percent of the original.

3. When interest is compounded, the percentage of interest earned to principal ends up being greater than the annual percentage rate for the investment account. Thus, the annual percentage rate does not necessarily correspond to the real interest earned, which is the very definition of nominal.

5. Since the base b must always be positive, raising a positive number to any power will always be positive

B: Identify Exponential Functions

Exercise 4.1e.B

★ For the following exercises, identify whether the statement represents an exponential function. Explain.

6) The average annual population increase of a pack of wolves is 25.

7) A population of bacteria decreases by a factor of $\frac{1}{8}$ every 24 hours.

8) The value of a coin collection has increased by 3.25% annually over the last 20 years.

9) For each training session, a personal trainer charges his clients $\$5$ less than the previous training session.

10) The height of a projectile at time t is represented by the function $h(t) = -4.9t^2 + 18t + 40$

★ For the following exercises, determine whether the table could represent a function that is linear, exponential, or neither. If it appears to be exponential, find a function that passes through the points.

11(a)

x	1	2	3	4
$f(x)$	70	40	10	-20

11(b)

x	1	2	3	4
$m(x)$	80	61	42.9	25.61

11(c)

x	1	2	3	4
$g(x)$	3	6	12	24

12(a)

x	1	2	3	4
$h(x)$	70	49	34.3	24.01

12(b)

x	1	2	3	4
$f(x)$	10	20	40	80

12(c)

x	1	2	3	4
$g(x)$	-3.25	2	7.25	12.5

14. Determine whether the following are exponential functions or not

(a) $.381^{x-1}$	(b) $(-2.7)^x$	(c) $\frac{2}{5^x}$	(d) x^4+3x^2	(e) $\frac{3}{6-x}$	(f) x^2	(g) 1^x	(h) 3^x

Answers to odd exercises:

7. exponential; the population decreases by a proportional rate.

9. not exponential; the charge decreases by a constant amount each visit, so the statement represents a linear function.

11. (a) Linear, (b) Neither, (c) Exponential

C: Evaluate Exponential Functions

Exercise 4.1e.C

★ For the following exercises, evaluate each function. Give an exact answer.

17. $f(x)=3^x$ for $f(-2), f(0)$, and $f(2)$.
18. $f(x)=10^x$ for $f(-1), f(0)$, and $f(1)$.
19. $g(x)=\left(1-\frac{2}{3}\right)^x$ for $g(-1), g(0)$, and $g(3)$.
20. $g(x)=\left(1-\frac{1}{4}\right)^x$ for $g(-2), g(-1)$, and $g(0)$.
21. $h(x)=(1+8)^{-x}$ for $h(-1), h(0)$, and $h\left(\frac{1}{2}\right)$.
22. $h(x)=(1+3)^{-x}$ for $h(-1), h\left(-\frac{1}{2}\right)$, and $h(0)$.
23. $f(x)=-2^x+1$ for $f(-1), f(0)$, and $f(3)$.
24. $f(x)=2-3^x$ for $f(-1), f(0)$, and $f(2)$.
25. $g(x)=10^{-x}+20$ for $g(-2), g(-1)$, and $g(0)$.
26. $g(x)=1-2^{-x}$ for $g(-1), g(0)$, and $g(1)$.

Answers to odd exercises:

17. $f(-2)=\frac{1}{9}, f(0)=1, f(2)=9$
19. $g(-1)=3, g(0)=1, g(3)=\frac{1}{27}$
21. $h(-1)=9, h(0)=1, h\left(\frac{1}{2}\right)=\frac{1}{3}$
23. $f(-1)=\frac{1}{2}, f(0)=0, f(3)=-7$
25. $g(-2)=120, g(-1)=30, g(0)=21$

★ For the following exercises, evaluate each function. Round answers to two decimal places, if necessary.

27) $f(x)=-4^{2x+3}$ for $f(-1)$
28) $f(x)=2(5)^x$ for $f(-3)$
29) $f(x)=-2e^{x-1}$, for $f(-1)$
30) $f(x)=e^x$, for $f(3)$
31) $f(x)=1.2e^{2x}-0.3$, for $f(3)$
32) $f(x)=2.7(4)^{-x+1}+1.5$, for $f(-2)$
33) $g(x)=\frac{1}{3}(7)^{x-2}$ for $g(6)$.
34) $f(x)=-\frac{3}{2}(3)^{-x}+\frac{3}{2}$, for $f(2)$
35) $h(x)=-\frac{1}{2}\left(\frac{1}{2}\right)^x+6$ for $h(-7)$.
36) $f(x)=4(2)^{x-1}-2$ for $f(5)$.
37) $f(x)=5^x$ a. $x=3$ b. $x=\frac{1}{2}$ c. $x=\sqrt{2}$
38) $f(x)=(1-0.7)^x$ a. $x=-1$ b. $x=4$ c. $x=-1.5$
39) $f(x)=10^x$ a. $x=-2$ b. $x=4$ c. $x=\frac{5}{3}$
40) $f(x)=e^x$ a. $x=2$ b. $x=-3.2$ c. $x=\pi$
41. $f(x)=2^x+5$ for $f(2.5)$.
42. $f(x)=3^x-10$ for $f(3.2)$.
43. $g(x)=4^x$ for $g(\sqrt{2})$.
44. $g(x)=5^x-1$ for $g(\sqrt{3})$.
45. $h(x)=10^x$ for $h(\pi)$.
46. $h(x)=10^x+1$ for $h\left(\frac{\pi}{3}\right)$.
47. $f(x)=10^{-x}-2$ for $f(1.5)$.
48. $f(x)=5^{-x}+3$ for $f(1.3)$.
49. $f(x)=\left(1-\frac{1}{3}\right)^x+1$ for $f(-2.7)$.
50. $f(x)=\left(1-\frac{2}{5}\right)^{-x}-1$ for $f(1.4)$.

Answers to odd exercises:

27. $f(-1)=-4$
29. $f(-1)\approx -0.27$
31. $f(3)\approx 483.81$
33. $g(6)=800+\frac{1}{3}\approx 800.33$
35. $h(-7)=-58$
37: $a.\ 125$ $b.\ 2.24$ $c.\ 9.74$
39: $a.\ 0.01$ $b.\ 10,000$ $c.\ 46.42$
41. 10.66
43. 7.10
45. 1385.46
47. -1.97
49. 3.99

★ Find $f(-1), f(0)$, and $f(\frac{3}{2})$ for the given function. Use a calculator where appropriate to approximate to the nearest hundredth.

51. $f(x) = e^x + 2$	53. $f(x) = 5 - 3e^x$	55. $f(x) = 1 + e^{-x}$	57. $f(x) = e^{-2x} + 2$
52. $f(x) = e^x - 4$	54. $f(x) = e^{-x} + 3$	56. $f(x) = 3 - 2e^{-x}$	58. $f(x) = e^{-x^2} - 1$

Answers to odd exercises:

51. $f(-1) \approx 2.37, f(0) = 3, f\left(\frac{3}{2}\right) \approx 6.48$	55. $f(-1) \approx 3.72, f(0) = 2, f\left(\frac{3}{2}\right) \approx 1.22$
53. $f(-1) \approx 3.90, f(0) = 2, f\left(\frac{3}{2}\right) \approx -8.45$	57. $f(-1) \approx 9.39, f(0) = 3, f\left(\frac{3}{2}\right) \approx 2.05$

D: Exponential Growth - Identify Initial Value, Growth Factor and Growth Rate

Exercise 4.1e.D

★ For the following exercises, determine whether the equation represents exponential growth, exponential decay, or neither. Explain. State the growth factor and growth rate.

59) $y = 220(1.06)^x$	60) $y = 300(1-t)^5$	61) $y = 11,701(0.97)^t$	62) $y = 16.5(1.025)^{\frac{1}{x}}$

★ For the following exercises, determine whether the equation represents continuous growth, continuous decay, or neither. Explain. State the growth rate and the growth factor.

63) $y = 3742(e)^{0.75t}$	64) $y = 150(e)^{\frac{3.25}{t}}$	65) $y = 2.25(e)^{-2t}$

Answers to odd exercises:

59. exponential growth; The growth factor, 1.06 is greater than 1. The growth rate of 6% is positive.

61. exponential decay; The decay factor, 0.97 is between 0 and 1. The growth rate of -3% is negative.

63. continuous growth; the growth rate $.75$, is greater than 0. The growth factor is $e^{.75}$

65. continuous decay; the growth rate -2, is less than 0. The decay factor is e^{-2}.

★ For the following exercises, consider this scenario. For each year t, the population of a forest of trees is represented by the function $A(t) = 115(1.025)^t$. In a neighboring forest, the population of the same type of tree is represented by the function $B(t) = 82(1.029)^t$. (Round answers to the nearest whole number.)

67) Which forest's population is growing at a faster rate?

68) Which forest had a greater number of trees initially? By how many?

69) Assuming the population growth models continue to represent the growth of the forests, which forest will have a greater number of trees after 20 years? By how many?

70) Assuming the population growth models continue to represent the growth of the forests, which forest will have a greater number of trees after 100 years? By how many?

71) Discuss the above results from the previous four exercises. Assuming the population growth models continue to represent the growth of the forests, which forest will have the greater number of trees in the long run? Why? What are some factors that might influence the long-term validity of the exponential growth model?

Answers to odd exercises:

67. The forest represented by the function $B(t) = 82(1.029)^t$. The growth rate is 2.9%. The growth factor is 1.029.

69. After $t = 20$ years, forest A will have 43 more trees than forest B.

71. Answers will vary. Sample response: For a number of years, the population of forest A will increasingly exceed forest B, but because forest B actually grows at a faster rate, the population will eventually become larger than forest A and will remain that way as long as the population growth models hold. Some factors that might influence the long-term validity of the exponential growth model are drought, an epidemic that culls the population, and other environmental and biological factors.

E: Solve Exponential Applications Given a Formula

Exercise 4.1e.E

73. The population of a certain small town is growing according to the function $P(t) = 12,500(1.02)^t$ where t represents time in years since the last census. Use the function to determine the population on the day of the census (when $t = 0$) and estimate the population in 6 years from that time.

74. The population of a certain small town is decreasing according to the function $P(t) = 22,300(0.982)^t$ where t represents time in years since the last census. Use the function to determine the population on the day of the census (when $t = 0$) and estimate the population in 6 years from that time.

75. The decreasing value, in dollars, of a new car is modeled by the formula $V(t) = 28,000(0.84)^t$ where t represents the number of years after the car was purchased. Use the formula to determine the value of the car when it was new ($t = 0$) and the value after 4 years.

76. The number of unique visitors to the college website can be approximated by the formula $N(t) = 410(1.32)^t$ where t represents the number of years after 1997 when the website was created. Approximate the number of unique visitors to the college website in the year 2020.

77. If left unchecked, a new strain of flu virus can spread from a single person to others very quickly. The number of people affected can be modeled using the formula $P(t) = e^{0.22t}$ where t represents the number of days the virus is allowed to spread unchecked. Estimate the number of people infected with the virus after 30 days and after 60 days.

78. If left unchecked, a population of 24 wild English rabbits can grow according to the formula $P(t) = 24e^{0.19t}$ where the time t is measured in months. How many rabbits would be present $3\frac{1}{2}$ years later?

79. The population of a certain city in 1975 was $65,000$ people and was growing exponentially at an annual rate of 1.7%. At the time, the population growth was modeled by the formula $P(t) = 65,000e^{0.017t}$ where t represented the number of years since 1975. In the year 2000, the census determined that the actual population was $104,250$ people. What population did the model predict for the year 2000 and what was the actual error?

80. Because of radioactive decay, the amount of a 10 milligram sample of Iodine-131 decreases according to the formula $A(t) = 10e^{-0.087t}$ where t represents time measured in days. How much of the sample remains after 10 days?

81. The number of cells in a bacteria sample is approximated by the logistic growth model $N(t) = \frac{1.2\times10^5}{1+9e^{-0.32t}}$ where t represents time in hours. Determine the initial number of cells and then determine the number of cells 6 hours later.

82. The market share of a product, as a percentage, is approximated by the formula $P(t) = \frac{100}{2+e^{-0.44t}}$ where t represents the number of months after an aggressive advertising campaign is launched. By how much can we expect the market share to increase after the first three months of advertising?

Answers to odd exercises:

73. Initial population: $12,500$; Population 6 years later: $14,077$

75. New: $\$28,000$; In 4 years: $\$13,940.40$

77. After 30 days: 735 people; After 60 days: $540,365$ people

79. Model: $99,423$ people; error: $4,827$ people

81. Initially there are $12,000$ cells and 6 hours later there are $51,736$ cells.

F: Solve Exponential Application problems given a rate

Exercise 4.1e.F

83) The fox population in a certain region has an annual growth rate of 9% per year. In the year 2012, there were $23,900$ fox counted in the area. What is the equation that models this growth? What is the fox population predicted to be in the year 2020?

Answers to odd exercises:

83. $23900(1.09)^t$, $47,622$ fox

G: Exponential Application problems find a rate

Exercise 4.1e.G

84) A scientist begins with 100 milligrams of a radioactive substance that decays exponentially. After 35 hours, 50mg of the substance remains. How many milligrams will remain after 54 hours?

85) In the year 1985, a house was valued at $\$110,000$. By the year 2005, the value had appreciated to $\$145,000$. What was the annual growth rate between 1985 and 2005? Assume that the value continued to grow by the same percentage. What was the value of the house in the year 2010?

86) A car was valued at $\$38,000$ in the year 2007. By 2013, the value had depreciated to $\$11,000$ If the car's value continues to drop by the same percentage, what will it be worth by 2017?

Answers to odd exercises:

85. Growth factor $b = 1.0139$, Growth rate = 1.39%; $\$155,368.09$

H: Construct the Formula $f(x) = a(b)^x$ given Two Points

Exercise 4.1e.H

★ For the following exercises, find the formula for an exponential function $y = a(b^x)$ that passes through the two points given.

88) $(0,6)$ and $(3,750)$ 89) $(0,2000)$ $(2,20)$	90) $(-1,\frac{3}{2})$ and $(3,24)$ 91) $(-2,6)$ $(3,1)$	92) $(3,1)$ and $(5,4)$

Answers to odd exercises:

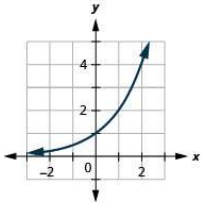
89. $f(x) = 2000(0.1)^x$	91. $f(x) = (\frac{1}{6})^{-\frac{3}{6}}(\frac{1}{6})^{\frac{x}{6}} \approx 2.93(0.699)^x$

I: Construct a formula $f(x) = a(b)^x$ given a Graph

Exercise 4.1e.I

★ Find an equation $y = ab^x$ for the exponential functions graphed below.

93.



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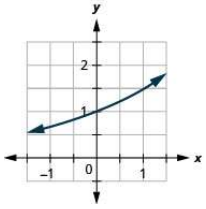

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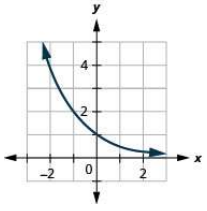

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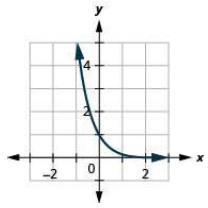	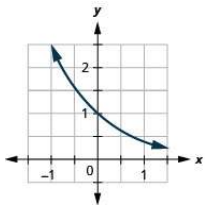	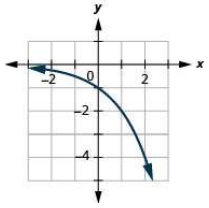

Answers to odd exercises:

93. $f(x) = 2^x$	95. $f(x) = \left(\frac{1}{2}\right)^x$	97. $f(x) = (0.4)^x$

J: Compound Interest

Exercise 4.1e.J

★ **For the following exercises, use the compound interest formula, $A(t) = P(1 + \frac{r}{n})^{nt}$**

After a certain number of years, the value of an investment account is represented by the equation $10,250(1 + \frac{0.04}{12})^{120}$.

100) What is the value of the account?

101) What was the initial deposit made to the account in the previous exercise?

102) How many years had the account from the previous exercise been accumulating interest?

An account is opened with an initial deposit of $\$6,500$ and earns 3.6% interest compounded semi-annually.

103) What will the account be worth in 20years?

104) How much more would the account in the previous exercise be worth if the interest were compounding weekly?

105) Solve the compound interest formula for the principal, P.

106) Use the formula found in the previous exercise to calculate the initial deposit of an account that is worth $\$14,472.74$ after earning 5.5% interest compounded monthly for 5 years. (Round to the nearest dollar.)

107) How much more would the account in the previous two exercises be worth if it were earning interest for 5 more years?

108) Use properties of rational exponents to solve the compound interest formula for the interest rate, r.

109) Use the formula found in the previous exercise to calculate the interest rate for an account that was compounded semi-annually, had an initial deposit of $\$9,000$ and was worth $\$13,373.53$ after 10 years.

110) Use the formula found in the previous exercise to calculate the interest rate for an account that was compounded monthly, had an initial deposit of $\$5,500$, and was worth $\$38,455$ after 30 years.

111) Repeat the previous exercise to find the APY formula of an account that compounds daily. Use the results from this and the previous exercise to develop a function $I(n)$ for the APY of any account that compounds n times per year.

112) The annual percentage yield (APY) of an investment account is a representation of the actual interest rate earned on a compounding account. It is based on a compounding period of one year. Show that the APY of an account that compounds monthly can be found with the formula $APY = (1 + \frac{r}{12})^{12} - 1$

113) Jamal wants to save $\$54,000$ for a down payment on a home. How much will he need to invest in an account with 8.2% APR, compounding daily, in order to reach his goal in 5 years?

114) Kyoko has $\$10,000$ that she wants to invest. Her bank has several investment accounts to choose from, all compounding daily. Her goal is to have $\$15,000$ by the time she finishes graduate school in 6 years. To the nearest hundredth of a percent,

what should her minimum annual interest rate be in order to reach her goal? (Hint: solve the compound interest formula for the interest rate.)

115. Jim invested $\$750$ in a 3-year CD that earns 4.2% annual interest that is compounded monthly. How much will the CD be worth at the end of the 3-year term?

116. Jose invested $\$2,450$ in a 4-year CD that earns 3.6% annual interest that is compounded semi-annually. How much will the CD be worth at the end of the 4-year term?

117. Jane has her $\$5,350$ savings in an account earning $3\frac{5}{8}\%$ annual interest that is compounded quarterly. How much will be in the account at the end of 5 years?

118. Bill has $\$12,400$ in a regular savings account earning $4\frac{2}{3}\%$ annual interest that is compounded monthly. How much will be in the account at the end of 3 years?

119. If $\$85,200$ is invested in an account earning 5.8% annual interest compounded quarterly, then how much interest is accrued in the first 3 years?

120. If $\$124,000$ is invested in an account earning 4.6% annual interest compounded monthly, then how much interest is accrued in the first 2 years?

Answers to odd exercises:

101. $\$10,250$ 103. $\$13,268.58$ 105. $P = A(t) \cdot (1 + \frac{r}{n})^{-nt}$ 107. $\$4,569.10$ 109. 4%

111. $APY = \frac{A(t)-a}{a} = \frac{a(1+\frac{r}{365})^{365(1)}-a}{a} = \frac{a\left[(1+\frac{r}{365})^{365}-1\right]}{a} = (1+\frac{r}{365})^{365} - 1 \quad I(n) = (1+\frac{r}{n})^{n} - 1$

113. $\$35,838.76$ 115. $\$850.52$ 117. $\$6,407.89$ 119. $\$16,066.13$

K: Continuous Interest

Exercise 4.1e.K

121. Bill invested $\$1,400$ in a 3-year CD that earns 4.2% annual interest that is compounded continuously. How much will the CD be worth at the end of the 3-year term?

122. Brooklyn invested $\$2,850$ in a 5-year CD that earns 5.3% annual interest that is compounded continuously. How much will the CD be worth at the end of the 5-year term?

123. Omar has his $\$4,200$ savings in an account earning $4\frac{3}{8}\%$ annual interest that is compounded continuously. How much will be in the account at the end of $2\frac{1}{2}$ years?

124. Nancy has her $\$8,325$ savings in an account earning $5\frac{7}{8}\%$ annual interest that is compounded continuously. How much will be in the account at the end of $5\frac{1}{2}$ years?

125. If $\$12,500$ is invested in an account earning 3.8% annual interest compounded continuously, then how much interest is accrued in the first 10 years?

126. If $\$220,000$ is invested in an account earning 4.5% annual interest compounded continuously, then how much interest is accrued in the first 2 years?

Answers to odd exercises:

121. $\$1,588.00$	123. $\$4,685.44$	125. $\$5,778.56$

L: Continuous and Compounded Interest

Exercise 4.1e.L

127) Suppose an investment account is opened with an initial deposit of $\$12,000$ earning 7.2% interest compounded monthly. How much will the account be worth after 30 years?

128) How much more would the account from the previous exercise be worth after 30 years if it were compounded continuously instead?

129) Alyssa opened a retirement account with 7.25% APR in the year 2000. Her initial deposit was $\$13,500$. How much will the account be worth in 2025 if interest compounds monthly? How much more would she make if interest compounded continuously?

130) An investment account with an annual interest rate of 7% was opened with an initial deposit of $\$4,000$ Compare the values of the account after 9 years when the interest is compounded annually, quarterly, monthly, and continuously.

Answers to odd exercises:

127. $\$103,384.23$	129. $\$82,247.78$; $\$449.75$

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4.2: Graphs of Exponential Functions

Learning Objectives

- Graph exponential functions and their transformations.
- Review Laws of Exponents

Exponential functions are used for many real-world applications such as finance, forensics, computer science, and most of the life sciences. Working with an equation that describes a real-world situation gives us a method for making predictions. Seeing their graphs gives us another layer of insight for predicting future events.

Graph Basic Exponential Functions

Exponential growth is modelled by functions of the form $f(x) = b^x$ where the base is greater than one. Exponential decay occurs when the base is between zero and one. We'll use the functions $f(x) = 2^x$ and $g(x) = \left(\frac{1}{2}\right)^x$ to get some insight into the behaviour of

graphs that model exponential growth and decay. In each table of values below, observe how the output values change as the input increases by 1.

Exponential Growth: $f(x) = 2^x$.

Table 4.2.1: Exponential Growth

x	-3	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Each output value is the product of the previous output and the base, 2. We call the base 2 the *constant ratio*. In fact, for any exponential function with the form $f(x) = ab^x$, b is the constant ratio of the function. This means that as the input increases by 1, the output value $f(x+1)$ will be the product of the base and the previous output, $bf(x) = b \cdot ab^x = ab^{x+1} = f(x+1)$, regardless of the value of a.

Notice from the table that

- the output values are positive for all values of x;
- as x increases, the output values increase without bound;
- as x decreases, the output values grow smaller, approaching zero.

The domain of $f(x) = 2^x$ is all real numbers, the range is $(0, \infty)$, and the horizontal asymptote is $y = 0$.

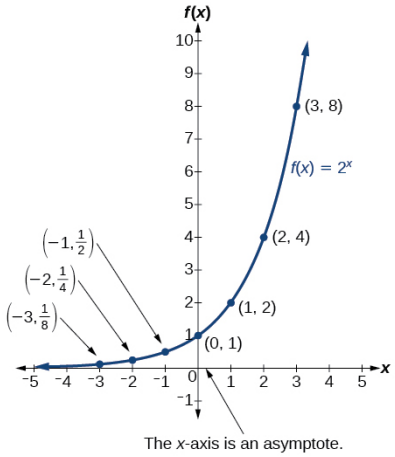


Figure 4.2.1: Graph of the exponential growth function $f(x) = 2^x$. Notice that the graph gets close to the x-axis, but never touches it.

Exponential Decay: $g(x) = \left(\frac{1}{2}\right)^x$.

Table 4.2.2: Exponential Decay

x	-3	-2	-1	0	1	2	3
$g(x) = \left(\frac{1}{2}\right)$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

Again, as the input is increases by 1, each output value is the product of the previous output and the base, $\frac{1}{2}$, so the constant ratio of the function is $\frac{1}{2}$. Notice from the table that

- the output values are positive for all values of x;
- as x increases, the output values grow smaller, approaching zero; and
- as x decreases, the output values grow without bound.

The domain of g is all real numbers, the range is $(0, \infty)$, and the horizontal asymptote is $y = 0$.

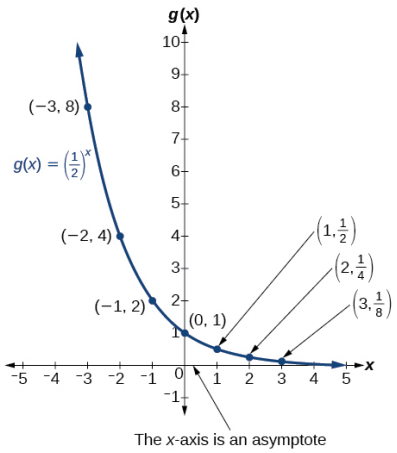


Figure 4.2.2: Graph of the exponential decay function, $g(x) = \left(\frac{1}{2}\right)^x$.

Notice that $g(x) = \left(\frac{1}{2}\right)^x = (2^{-1})^x = f(-x)$. Thus the graph of g is simply a reflection over the y-axis of the graph of f.

CHARACTERISTICS OF THE GRAPH OF THE PARENT FUNCTION $f(x) = b^x$

An exponential function with the form $f(x) = b^x, b > 0, b \neq 1$, has these characteristics:

- one-to-one function
- horizontal asymptote: $y = 0$
- x-intercept: none
- y-intercept: $(0, 1)$ and key point $(1, b)$
- domain: $(-\infty, \infty)$
- range: $(0, \infty)$
- increasing if $b > 1$ - "exponential growth"
- decreasing if $0 < b < 1$ - "exponential decay"

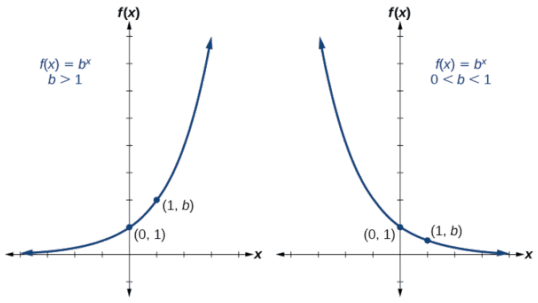


Figure 4.2.3: Graphs of exponential growth and decay functions.

How to: Graph a basic exponential function of the form $y = b^x$

1. Draw and label the horizontal asymptote, $y = 0$.
2. Create a table of points and use it to plot at least 3 points, including the y-intercept $(0, 1)$ and key point $(1, b)$.
3. Draw a smooth curve that goes through the points and approaches the horizontal asymptote.
4. State the domain $(-\infty, \infty)$, the range $(0, \infty)$, and the horizontal asymptote, $y = 0$.

Example 4.2.1: Sketch the Graph of an Exponential Function of the Form $f(x) = b^x$

Sketch a graph of $f(x) = 0.25^x$. State the domain, range, and asymptote.

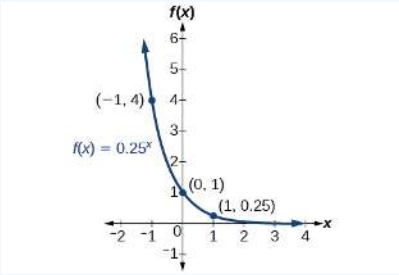
Solution

Before graphing, identify the behavior and create a table of points for the graph.

Table 4.2.3

x	-3	-2	-1	0	1	2	3
$f(x) = 0.25^x$	64	16	4	1	0.25	\(0.0625\0	0.015625

- Since $b = 0.25$ is between zero and one, we know the function is decreasing. The left tail of the graph will increase without bound, and the right tail will approach the asymptote $y = 0$.
- Create a table of points as in Table 4.2.3.
- Plot the asymptote, and the y-intercept $(0, 1)$, along with two other points. We can use $(-1, 4)$ and $(1, 0.25)$.



Draw a smooth curve connecting the points as shown to the right.

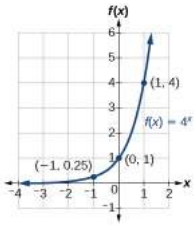
The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

Try It 4.2.1

Sketch the graph of $f(x) = 4^x$.
State the domain, range, and asymptote

Answer

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

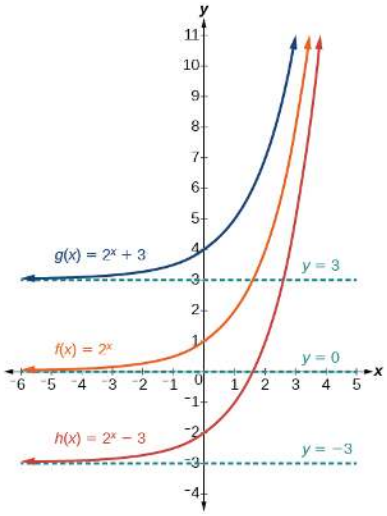


Graph Transformations of Exponential Functions

Transformations of exponential graphs behave similarly to those of other functions. Just as with other parent functions, we can apply the four types of transformations—shifts, reflections, stretches, and compressions—to the parent function $f(x) = b^x$ without loss of general shape. For instance, just as the quadratic function maintains its parabolic shape when shifted, reflected, stretched, or compressed, the exponential function also maintains its general shape regardless of the transformations applied.

Vertical Shifts

The first transformation occurs when we add a constant d to the parent function $f(x) = b^x$, giving us a vertical shift d units in the same direction as the sign. For example, if we begin by graphing a parent function, $f(x) = 2^x$, we can then graph two vertical shifts alongside it, using $d = 3$: the upward shift, $g(x) = 2^x + 3$ and the downward shift, $h(x) = 2^x - 3$. Both shifts are shown in the figure to the right.

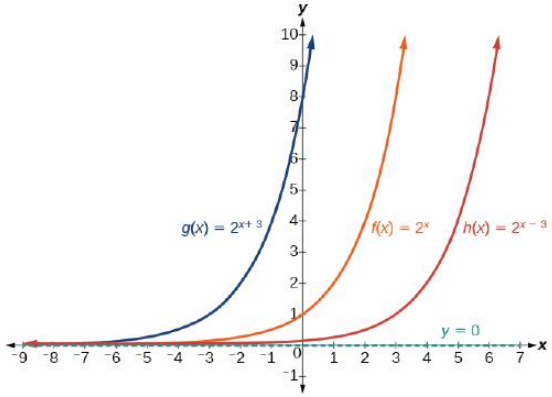


Observe the results of shifting $f(x) = 2^x$ vertically:

- The domain, $(-\infty, \infty)$ remains unchanged.
- When the function is shifted up 3 units to $g(x) = 2^x + 3$:
  - The y-intercept shifts up 3 units to $(0, 4)$.
  - The asymptote shifts up 3 units to $y = 3$.
  - The range becomes $(3, \infty)$.
- When the function is shifted down 3 units to $h(x) = 2^x - 3$:
  - The y-intercept shifts down 3 units to $(0, -2)$.
  - The asymptote also shifts down 3 units to $y = -3$.
  - The range becomes $(-3, \infty)$.

Horizontal Shifts

The next transformation occurs when we add a constant c to the input of the parent function $f(x) = b^x$, giving us a horizontal shift c units in the *opposite* direction of the sign. For example, if we begin by graphing the parent function $f(x) = 2^x$, we can then graph two horizontal shifts alongside it, using $c = 3$: the shift left, $g(x) = 2^{x+3}$, and the shift right, $h(x) = 2^{x-3}$. Both horizontal shifts are shown in the figure to the right.



Observe the results of shifting $f(x) = 2^x$ horizontally:

- The domain, $(-\infty, \infty)$, remains unchanged.
- The asymptote, $y = 0$, remains unchanged.
- The y-intercept shifts such that:

- When the function is shifted left 3 units to $g(x) = 2^{x+3}$, the y-intercept becomes $(0, 8)$. This is because $2^{x+3} = (8)2^x$, so the initial value of the function is 8.
- When the function is shifted right 3 units to $h(x) = 2^{x-3}$, the y-intercept becomes $(0, \frac{1}{8})$. Again, see that $2^{x-3} = (\frac{1}{8})2^x$, so the initial value of the function is $\frac{1}{8}$.

SHIFTS OF THE PARENT FUNCTION $y = b^x$

For any constants c and d, the function $f(x) = b^{x+c} + d$ shifts the parent function $y = b^x$

- Vertically d units, in the *same* direction of the sign of d.
  - If $d > 0$ the parent function is shifted **up** d units
  - If $d < 0$ the parent function is shifted **down** d units
  - The new y-coordinates are equal to $y + d$
  - The **horizontal asymptote** becomes $y = d$, and occurs when the exponential part of the function, b^{x+c} approaches zero.
  - The **range** becomes (d, ∞).
- Horizontally c units, in the *opposite* direction of the sign of c.
  - If $c > 0$ the parent function is shifted **left** c units
  - If $c < 0$ the parent function is shifted **right** c units
  - The new x-coordinates are equal to $x - c$.;
- The new *y*-intercept, at $(0, 1)$ in the parent function, occuring at $f(0)$, becomes $(0, b^c + d)$.
- The domain, $(-\infty, \infty)$, remains unchanged.

How to: Graph an exponential function of the form $f(x) = b^{x+c} + d$

1. Draw the horizontal asymptote $y = d$.
2. Identify the shift as $(-c, d)$. Shift the graph of $f(x) = b^x$ left c units if c is positive, and right c units if c is negative.
3. Shift the graph of $f(x) = b^x$ up d units if d is positive, and down d units if d is negative.
4. State the domain, $(-\infty, \infty)$, the range, (d, ∞), and the horizontal asymptote $y = d$.

Example 4.2.2: Graphing a Shift of an Exponential Function

Graph $f(x) = 2^{x+1} - 3$. State the domain, range, and asymptote.

Solution

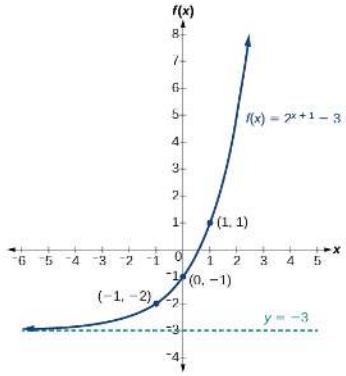
We have an exponential equation of the form $f(x) = b^{x+c} + d$, with $b = 2$, $c = 1$, and $d = -3$.

Draw the horizontal asymptote $y = d$, so draw $y = -3$.

Identify the shift as $(-c, d)$, so the shift is $(-1, -3)$.

Shift the graph of $f(x) = b^x$ left 1 units and down 3 units.

The domain is $(-\infty, \infty)$; the range is $(-3, \infty)$; the horizontal asymptote is $y = -3$.

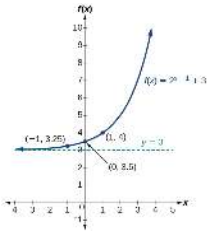


Try It 4.2.2

Graph $f(x) = 2^{x-1} + 3$. State domain, range, and asymptote.

Answer

The domain is $(-\infty, \infty)$; the range is $(3, \infty)$; the horizontal asymptote is $y = 3$



Reflections

In addition to shifting, compressing, and stretching a graph, we can also reflect it about the x-axis or the y-axis. When we multiply the parent function $f(x) = b^x$ by -1, we get a reflection about the x-axis. When we multiply the input by -1, we get a reflection about the y-axis. For example, if we begin by graphing the parent function $f(x) = 2^x$, we can then graph the two reflections alongside it. The reflection about the x-axis, $g(x) = -2^x$, is illustrated below in the graph on the left, and the reflection about the y-axis $h(x) = 2^{-x}$, is shown in the graph on the right.

REFLECTIONS OF THE PARENT FUNCTION $f(x) = b^x$

The function $f(x) = -b^x$

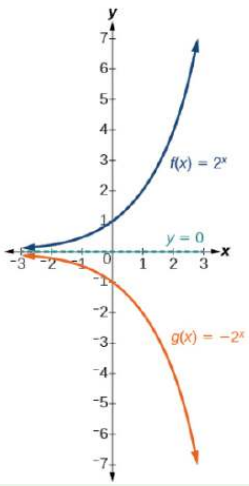
- reflects the parent function $f(x) = b^x$ about the x-axis.
- has a y-intercept of $(0, -1)$.
- has a range of $(-\infty, 0)$
- has a horizontal asymptote at $y = 0$ and domain of $(-\infty, \infty)$, which are unchanged from the parent function.

The function $f(x) = -b^x + d$ has both a vertical shift and reflection about the x-axis. In this situation, always do the vertical shift LAST.

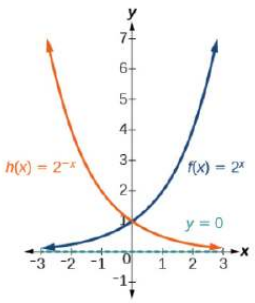
The function $f(x) = b^{-x}$

- reflects the parent function $f(x) = b^x$ about the y-axis.
- has a y-intercept of $(0, 1)$, a horizontal asymptote at $y = 0$, a range of $(0, \infty)$, and a domain of $(-\infty, \infty)$, which are unchanged from the parent function.

Reflection about the x-axis



Reflection about the y-axis



The function $f(x) = b^{-x+c}$ has both a horizontal shift and reflection about the y-axis. In this situation, always do the horizontal shift FIRST.

Example 4.2.3: Construct an Equation for a Reflected Exponential Function

Find and graph the equation for a function, $g(x)$, that reflects $f(x) = (\frac{1}{4})^x$ about the x-axis. State its domain, range, and asymptote.

Solution

Since we want to reflect the parent function $f(x) = (\frac{1}{4})^x$ about the x-axis, we multiply $f(x)$ by -1 to get, $g(x) = -f(x) = -(\frac{1}{4})^x$. Next we create a table of points below.

x	-2	-1	0	1	2
$g(x) = -(\frac{1}{4})^x$	-16	-4	-1	-0.25	-0.0625

Plot the y-intercept, $(0, -1)$, along with two other points. We can use $(-1, -4)$ and $(1, -0.25)$. Draw a smooth curve connecting the points. The domain is $(-\infty, \infty)$; the range is $(-\infty, 0)$; the horizontal asymptote is $y = 0$.

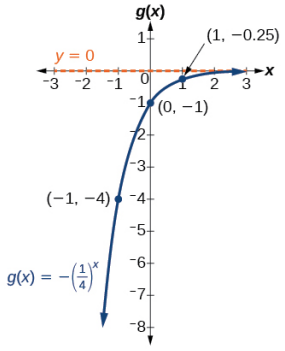


Figure 4.2.11. Graph of $g(x) = -(\frac{1}{4})^x$.

Try It 4.2.3

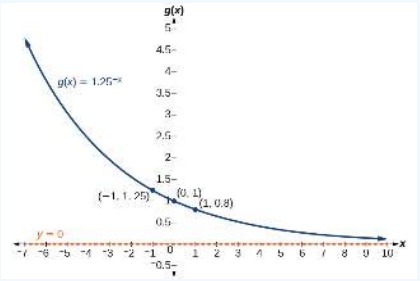
Find and graph the equation for a function, $g(x)$, that reflects $f(x) = 1.25^x$ about the y-axis. State its domain, range, and asymptote.

Answer

$g(x) = f(-x) = 1.25^{-x}$

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

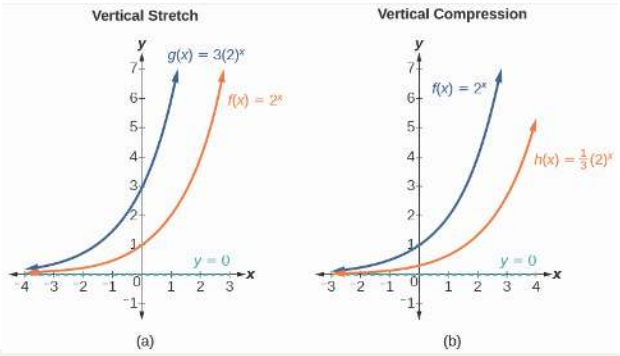
Included in the graph is the horizontal asymptote $y = 0$, and the points for $g(-1) = 1.25$, $g(0) = 1$, and $g(1) = .8$.



Vertical Stretches or Compressions

While horizontal and vertical shifts involve adding constants to the input or to the function itself, a **stretch** or **compression** occurs when we multiply the parent function $f(x) = b^x$ by a constant $|a| > 0$.

For example, if we begin by graphing the parent function $f(x) = 2^x$, we can then graph the stretch, using $a = 3$, to get $g(x) = 3(2)^x$ as shown in Figure (a), and the compression, using $a = \frac{1}{3}$, to get $h(x) = \frac{1}{3}(2)^x$ as shown on the right in Figure (b).



VERTICAL STRETCHES AND COMPRESSIONS OF THE PARENT FUNCTION $y = b^x$

For any factor $a \neq 0$, the function $f(x) = a(b)^x$

- is stretched vertically by a factor of a if $|a| > 1$.
- is compressed vertically by a factor of a if $0 < |a| < 1$.
- The new y-coordinates are equal to ay. This would include vertical reflection if present.
- has a y-intercept of $(0, a)$.
- has a horizontal asymptote at $y = 0$, a range of $(0, \infty)$, and a domain of $(-\infty, \infty)$, which are unchanged from the parent function.

If a vertically stretched, compressed and/or reflected function also has a vertical shift, like $g(x) = a(b)^x + d$, then the vertical shift, (d units up or down), must be done AFTER performing the vertical stretching, compression, and/or reflection.

Example 4.2.4: Graphing the Vertical Stretch of an Exponential Function

Sketch a graph of $f(x) = 4\left(\frac{1}{2}\right)^x$. State the domain, range, and asymptote.

Solution

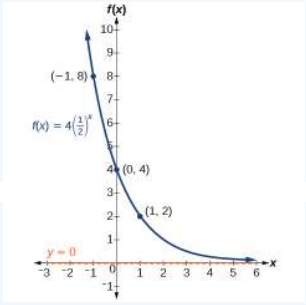
Before graphing, identify the behavior and key points on the graph.

- Since $b = \frac{1}{2}$ is between zero and one, the left tail of the graph will increase without bound as x decreases, and the right tail will approach the x -axis as x increases.
- Since $a = 4$, the graph of the parent function $y = \left(\frac{1}{2}\right)^x$ will be stretched vertically by a factor of 4.
- Create a table of points of the parent function as shown in Table 4.2.4. Then multiply all the y coordinates by 4.

Table 4.2.4

x	-3	-2	-1	0	1	2	3
$y = \left(\frac{1}{2}\right)^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
$f(x) = 4\left(\frac{1}{2}\right)^x$	32	16	8	4	2	1	0.5

- Plot the y-intercept, $(0, 4)$, along with two other points. We can use $(-1, 8)$ and $(1, 2)$.
- Draw a smooth curve connecting the points, as shown in the figure on the right.



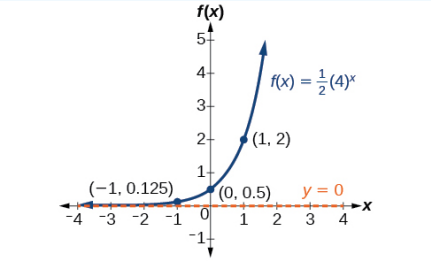
The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

Try It 4.2.4

Sketch the graph of $f(x) = \frac{1}{2}(4)^x$. State the domain, range, and asymptote.

Answer

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.



Combining Transformations of Exponential Functions

Now that we have worked with each type of translation for the exponential function, we can summarize them.

How to: GRAPH TRANSFORMATIONS OF EXPONENTIAL FUNCTIONS

A transformation of an exponential function has the form $f(x) = ab^{mx+c} + d$

The transformations to the parent function, $y = b^x, b > 1$, needed to obtain f are given below. The order these are done is important.

1. **FIRST**. Shift horizontally c units. $x \to (x - c)$.
   - If $c > 0$ shift left c units.
   - If $c < 0$ shift right c units..
2. Horizontal reflection, compression and/or stretching. $x \to \frac{1}{m}x$.
   - If m is negative, reflect over the y-axis. $(x \to -x)$.
   - If $|m| > 1$ shrink the graph horizontally by a factor of $\frac{1}{|m|}$.
   - If $0 < |m| < 1$ stretch the graph horizontally by a factor of $\frac{1}{|m|}$.
3. Vertical reflection, compression and/or stretching. $y \to ay$.
   - If a is negative, reflect over the x-axis. $(y \to -y)$.
   - If $|a| > 0$ stretch the graph vertically by a factor of $|a|$.
   - If $0 < |a| < 1$compress the graph vertically by a factor of $|a|$.
4. **LAST**. Shift vertically d units. $y \to y + d$.
   - If $d > 0$ shift up d units.
   - If $d < 0$ shift down d units.

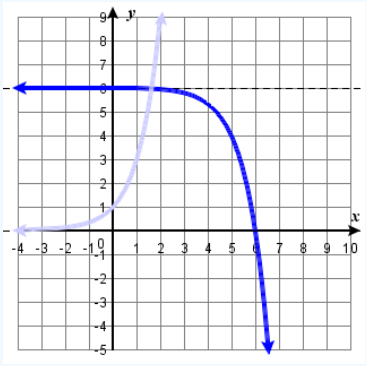
If the exponential function is written in the form $f(x) = ab^{m(x+c)} + d$ then reverse the order of steps 1 and 2 -- do reflecting and stretching first, then do the horizontal shift.

Example 4.2.5

Sketch the graphs of $g(x) = -2 \cdot 3^{x-5} + 6$ and $f(x) = 3 \cdot 2^{-x+1} - 4$ and the corresponding original parent function. State the domain, range, and horizontal asymptote of the transformation.

Solution

The basic parent function for $g(x) = -2 \cdot 3^{x-5} + 6$ is $y = 3^x$.



The transformations needed to obtain the graph of $g(x)$ from the graph of y are:

1. Shift right 5 units ($x \to x + 5$)
2. Reflect over the x-axis ($y \to -y$)
3. Vertically stretch by a factor of 2 ($y \to 2y$)
4. Shift up 6 units ($y \to y - 4$)

The graph of $g(x)$ and its parent function is on the right.

The domain is $(-\infty, \infty)$; the range is $(-\infty, 6)$; the horizontal asymptote is $y = 6$.

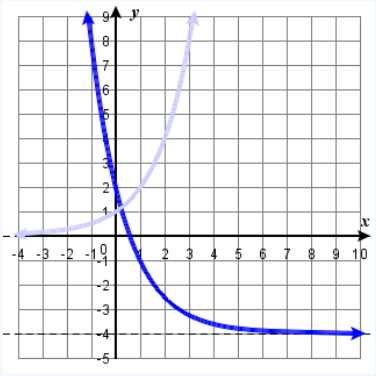
If tables are used to graph the function, coordinate points for the parent function appear in the table below

Parent function: x	HA	-1	0	1	2
$y = 3^x$	0	$\frac{1}{3}$	1	3	9

Corresponding coordinate points for the transformation $g(x) = -2 \cdot 3^{x-5} + 6$ would be

New x is: $x + 5$	HA	4	5	6	7
New y is: $-2y + 6$	6	$5\frac{1}{3}$	4	0	-12

The basic parent function for $f(x) = 3 \cdot 2^{-x+1} - 4$ is $y = 2^x$.



The transformations needed to obtain the graph of $f(x)$ from the graph of y are:

1. Shift left 1 unit ($x \to x - 1$)
2. Reflect over the y-axis ($x \to -x$)
3. Vertically stretch by a factor of 3 ($y \to 3y$)
4. Shift down 4 units ($y \to y - 4$)

If instead we re-wrote the function in the form: $f(x) = 3 \cdot 2^{-(x-1)} - 4$, the transformations would be

1. Reflect over the y-axis ($x \to -x$)
2. Shift right 1 unit ($x \to x + 1$)
3. Vertically stretch by a factor of 3 ($y \to 3y$)
4. Shift down 4 units ($y \to y - 4$)

The graph of $f(x)$ and its parent function is on the right.

The domain is $(-\infty, \infty)$; the range is $(\infty, -4)$; the horizontal asymptote is $y = -4$.

If tables are used to graph the function, coordinate points for the parent function appear in the table below

Parent function: x	HA	-1	0	1	2
$y = 2^x$	0	$\frac{1}{2}$	1	2	4

Corresponding coordinate points for the transformation $f(x) = 3 \cdot 2^{-x+1} - 4$ would be

New x is: $-x + 1$	HA	2	1	0	-1
New y is: $3y - 4$	-4	$-2\frac{1}{2}$	-1	2	8

Example 4.2.6

Sketch the graph of $f(x) = 2^{\frac{1}{2}x+4} - 3$. State the domain, range, and horizontal asymptote of the transformation.

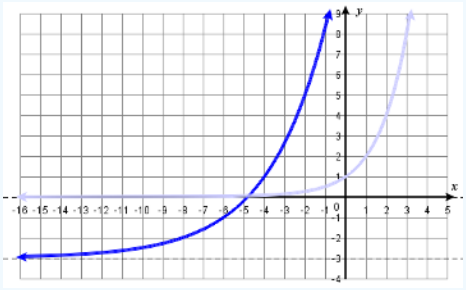
Solution

The basic parent function is $y = 2^x$.

The transformations needed to obtain the graph of $f(x)$ from the graph of y are:

1. Shift left 4 units ($x \to x - 4$)
2. Horizontally stretch by a factor of 2 ($x \to 2x$)
3. Shift down 3 units ($y \to y - 3$)

The graph of $f(x)$ and its parent function is on the right.



The domain is $(-\infty, \infty)$; the range is $(-3, \infty)$; the horizontal asymptote is $y = -3$.

If tables are used to graph the function, coordinate points for the parent function appear in the table below

Parent function: x	HA	-1	0	1	2
$y = 2^x$	0	$\frac{1}{2}$	1	2	4

Corresponding coordinate points for the transformation would be

New x is: $2(x-4)$	HA	-10	-8	-6	-4
New y is: $y-3$	-3	-2.5	-2	-1	1

Note that if the function were rewritten in the form: $f(x) = 2^{\frac{1}{2}(x+8)} - 3$ then the transformations indicated by this version of the function would be

1. Horizontally stretch by a factor of 2 ($x \to 2x$)
2. Shift left 8 units ($x \to x - 8$)
3. Shift down 3 units ($y \to y - 3$)

Try It 4.2.7

Sketch the graph of $f(x) = 2^{-2x-4} + 3$. State the domain, range, and horizontal asymptote of the transformation.

Answer

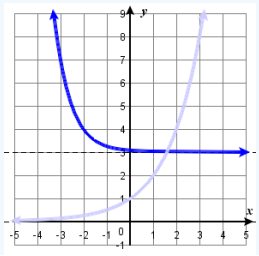
The domain is $(-\infty, \infty)$; the range is $(3, \infty)$; the horizontal asymptote is $y = 3$.

As written the transformations done on the parent function $y = 2^x$ would be

right 4, reflect over y axis, $x \to \frac{1}{2}x$, up 3.

However, if rewritten as $f(x) = 2^{-2(x+2)} + 3$, the transformations would be

reflect over y axis, $x \to \frac{1}{2}x$, left 2, up 3.



Construct an Exponential Equation from a Description

Example 4.2.8: Write a Function from a Description

Write the equation for the function described below. Give the horizontal asymptote, the domain, and the range.

$f(x) = e^x$ is vertically stretched by a factor of 2, reflected across the *y*-axis, and then shifted up 4 units.

Solution

We want to find an equation of the general form $f(x) = ab^{x+c} + d$. We use the description provided to find a, b, c, and d.

- We are given the parent function $f(x) = e^x$, so $b = e$.

- The function is vertically stretched by a factor of 2, so $a = 2$.
- The function is reflected about the y-axis. We replace x with $-x$ to get: e^{-x}.
- The graph is shifted vertically 4 units, so $d = 4$.

Substituting in the general form we get,

$$\begin{aligned} f(x) &= ab^{x+c} + d \\ &= 2e^{-x+0} + 4 \\ &= 2e^{-x} + 4 \end{aligned}$$

The domain is $(-\infty, \infty)$; the range is $(4, \infty)$; the horizontal asymptote is $y = 4$.

Try It 4.2.8

Write the equation for function described below. Give the horizontal asymptote, the domain, and the range.

$f(x) = e^x$ is compressed vertically by a factor of $\frac{1}{3}$, reflected across the x-axis and then shifted down 2 units.

Answer

$f(x) = -\frac{1}{3}e^x - 2$; the domain is $(-\infty, \infty)$; the range is $(-\infty, 2)$; the horizontal asymptote is $y = 2$.

Exponent Properties

When learning about exponents, we are typically given expressions that have a whole number exponent and a base that is a variable, like x^2. These type of expressions are technically called power functions. In contrast, the base can be a constant and the exponent can have a variable in it, like 2^x. Expressions in this form are called exponential functions. When simplifying expressions with the variable in the exponent, we often use the Laws of Exponents "backwards". A review of properties of exponents from a different perspective is briefly discussed here. Competency with exponents is a very useful skill to have. The table below summarizes these properties.

Rules of Exponents
For nonzero real numbers a and b and integers m and n

	Product Rule	Quotient Rule	Power Rule
Like Bases Rule	$a^m \cdot a^n = a^{m+n}$	$\frac{a^m}{a^n} = a^{m-n}$	$(a^m)^n = a^{m \cdot n}$
Reverse of Like Bases Rule	$a^{m+n} = a^m \cdot a^n$	$a^{m-n} = \frac{a^m}{a^n}$	$a^{m \cdot n} = (a^m)^n$
Like Exponents Rule	$a^n \cdot b^n = (a \cdot b)^n$	$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$	$\frac{1}{b^n} = \left(\frac{1}{b}\right)^n$

Example 4.2.9

Rewrite the following exponential expressions such that the result uses only one exponent, x. Simplify.

1. 2^{x+3}	2. 3^{x-2}	3. 5^{3-x}	4. $2^x \cdot 11^x$	5. 5^{3+2x}	6. $\frac{4^x}{12^x}$	7. $\frac{6^{3x}}{9^{2x}}$	8. $\frac{8^{x-1}}{2^{2x-5}}$

Solution

1. $2^{x+3} = 2^x 2^3 = 2^x \cdot 8 = 8(2^x)$
2. $3^{x-2} = 3^x 3^{-2} = \frac{3^x}{3^2} = \frac{3^x}{9}$
3. $5^{3-x} = 5^3 5^{-x} = \frac{5^3}{5^x} = \frac{125}{5^x} = 125\left(\frac{1}{5}\right)^x$
4. $2^x \cdot 11^x = (2 \cdot 11)^x = 22^x$
5. $5^{3+2x} = 5^3 5^{2x} = 125(5^2)^x = 125(25^x)$
6. $\frac{4^x}{12^x} = \left(\frac{4}{12}\right)^x = \left(\frac{1}{3}\right)^x$
7. $\frac{6^{3x}}{9^{2x}} = \frac{(6^3)^x}{(9^2)^x} = \left(\frac{6^3}{9^2}\right)^x = \left(\frac{8}{3}\right)^x$
8. $\frac{8^{x-1}}{2^{2x-5}} = \frac{8^x 8^{-1}}{2^{2x} 2^{-5}} = \frac{8^x}{2^{2x}} \cdot \frac{2^5}{8} = \frac{2^5}{8} \cdot \frac{8^x}{4^x} = \frac{32}{8} \cdot \left(\frac{8}{4}\right)^x$

   $= 4(2^x)$

4.2.9

Rewrite the following exponential expressions such that the result uses only one exponent, x. Simplify.

1. $3^{x+1} \cdot 4^{-2x}$
2. $\frac{2^{3x+4}}{7^{2x+1}}$

Answer

1. $3\left(\frac{3}{16}\right)^x$
2. $\frac{16}{7}\left(\frac{8}{49}\right)^x$

Try It 4.2.9b

Use exponent properties to describe the transformations needed to graph the function $f(x) = 5 \cdot 3^{-4x+2} - 6$ from a parent function.

Answer

$f(x) = 5 \cdot 3^{-4x+2} - 6 = f(x) = 5 \cdot 3^{-4x} \cdot 3^2 - 6 = f(x) = 5 \cdot 3^2 \cdot (3^4)^{-x} - 6 = f(x) = 45 \cdot 81^{-x} - 6$

Parent function is now $y = 81^x$

Reflect over the y-axis ($x \to -x$)

Vertically stretch by a factor of 45 ($y \to 45y$)

Shift down 6 units ($y \to y - 6$)

Key Concepts

General Form for the Translation of the Parent Function $f(x) = b^x$	$f(x) = ab^{x+c} + d$ or $f(x) = ab^{-x+c} + d$

- The graph of the function $f(x) = b^x$ has a y-intercept at $(0, 1)$,domain $(-\infty, \infty)$,range $(0, \infty)$, and horizontal asymptote $y = 0$.
- If $b > 1$,the function is increasing. The left tail of the graph will approach the asymptote $y = 0$, and the right tail will increase without bound.
- If $0 < b < 1$, the function is decreasing. The left tail of the graph will increase without bound, and the right tail will approach the asymptote $y = 0$.
- The equation $f(x) = b^x + d$ represents a vertical shift of the parent function $f(x) = b^x$.
- The equation $f(x) = b^{x+c}$ represents a horizontal shift of the parent function $f(x) = b^x$.
- The equation $f(x) = ab^x$, where $a > 0$, represents a vertical stretch if $|a| > 1$ or compression if $0 < |a| < 1$ of the parent function $f(x) = b^x$.
- When the parent function $f(x) = b^x$ is multiplied by -1, the result, $f(x) = -b^x$, is a reflection about the x-axis. When the input is multiplied by -1, the result, $f(x) = b^{-x}$, is a reflection about the y-axis.
- All translations of the exponential function can be summarized by the general equation $f(x) = ab^{mx+c} + d$.
- Using the general equation $f(x) = ab^{x+c} + d$, we can write the equation of a function given its description.

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4.2e: Exercises - Graphs of Exponential Functions

A: Concepts

Example 4.2e.A

1) What role does the horizontal asymptote of an exponential function play in telling us about the end behavior of the graph?

2) What is the advantage of knowing how to recognize transformations of the graph of a parent function algebraically?

3) Explore and discuss the graphs of $F(x) = (b)^x$ and $G(x) = \left(\frac{1}{b}\right)^x$. Then make a conjecture about the relationship between the graphs of the functions b^x and $\left(\frac{1}{b}\right)^x$ for any real number $b > 0$.

4) Prove the conjecture made in the previous exercise.

5) Explore and discuss the graphs of $f(x) = 4^x$, $g(x) = 4^{x-2}$, and $h(x) = \left(\frac{1}{16}\right)4^x$. Then make a conjecture about the relationship between the graphs of the functions b^x and $\left(\frac{1}{b^n}\right)b^x$ for any real number n and real number $b > 0$.

6) Prove the conjecture made in the previous exercise.

For the following exercises, match each function with one of the graphs in Figure at the right.

7) $f(x) = 2(0.69)^x$

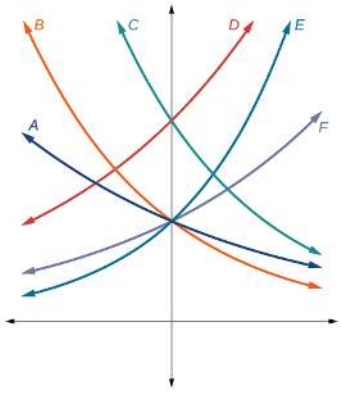
8) $f(x) = 2(1.28)^x$

9) $f(x) = 2(0.81)^x$

10) $f(x) = 4(1.28)^x$

11) $f(x) = 2(1.59)^x$

12) $f(x) = 4(0.69)^x$



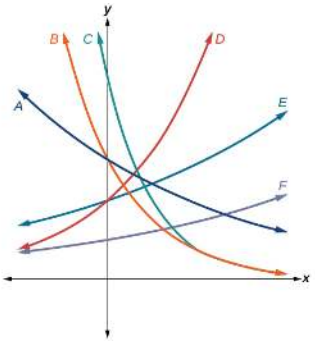
For the following exercises, use the graphs shown in Figure at the right. All have the form $f(x) = ab^x$.

13) Which graph has the largest value for b?

14) Which graph has the smallest value for b?

15) Which graph has the largest value for a?

16) Which graph has the smallest value for a?\



Answers to odd exercises:

1. An asymptote is a line that the graph of a function approaches, as x either increases or decreases without bound. The horizontal asymptote of an exponential function tells us the limit of the function's values as the independent variable gets either extremely large or extremely small.

3. The graph of $G(x) = \left(\frac{1}{b}\right)^x$ is the refelction about the y-axis of the graph of $F(x) = (b)^x$; For any real number $b > 0$ and function $f(x) = (b)^x$, the graph of $\left(\frac{1}{b}\right)^x$ is the the reflection about the y-axis, $F(-x)$.

5. $g(x)$ and $h(x)$ are the same graph and are a horizontal shift to the right of the graph of $f(x)$; For any real number n, real number $b > 0$, and function $f(x) = b^x$, the graph of $\left(\frac{1}{b^n}\right) b^x$ is the horizontal shift $f(x - n)$.

7. B, 9. A, 11. E, 13. D, 15. C

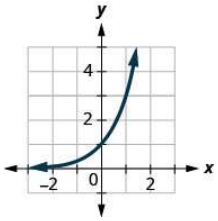
B: Match Graphs With Equations

Exercise $4.2e.B$

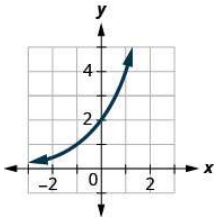
★ In the following exercises, match the graphs to one of the following functions:

a. 2^x	b. 2^{x+1}	c. 2^{x-1}	d. $2^x + 2$	e. $2^x - 2$	f. 3^x

17.



18.



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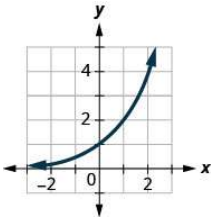

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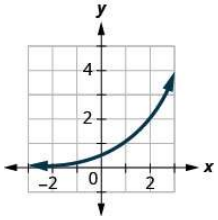
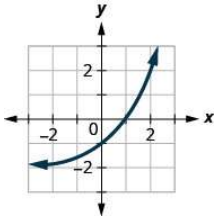
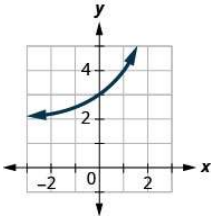


21.



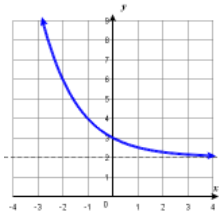
22.



★ For the following exercises, match the exponential equation to the correct graph.

a. $y = 4^{-x}$	b. $y = 3^{x-1}$	c. $y = 2^{x+1}$	d. $y = \left(\frac{1}{2}\right)^x + 2$	e. $y = -3^{-x}$	f. $y = 1 - 5^x$

23)



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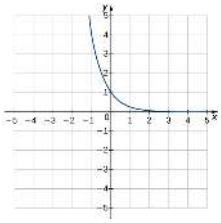

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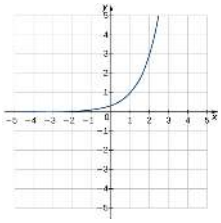
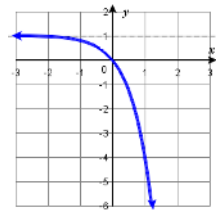
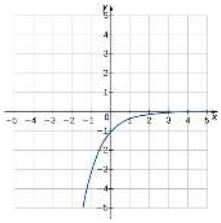


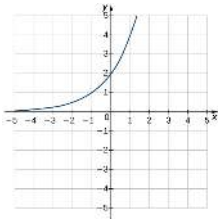
26)



27)



28)



Answers to odd exercises:

17. f, 19. a, 21. e; 23. d, 25. b, 27. e

C: Graph Basic Exponential Functions

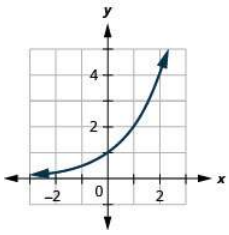
Exercise 4.2e.C

★ In the following exercises, graph each exponential function.

29. $f(x) = 2^x$	33. $f(x) = (1.5)^x$	37. $f(x) = \left(\frac{1}{6}\right)^x$	41. $f(x) = 4^x$
30. $g(x) = 3^x$	34. $g(x) = (2.5)^x$	38. $g(x) = \left(\frac{1}{7}\right)^x$	42. $g(x) = 5^x$
31. $f(x) = 6^x$	35. $f(x) = \left(\frac{1}{2}\right)^x$	39. $f(x) = (0.4)^x$	43. $f(x) = \left(\frac{1}{4}\right)^x$
32. $g(x) = 7^x$	36. $g(x) = \left(\frac{1}{3}\right)^x$	40. $g(x) = (0.6)^x$	44. $h(x) = \left(\frac{1}{5}\right)^x$

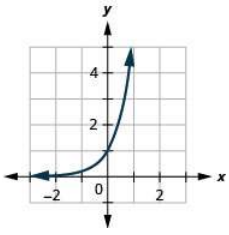
Answers to odd exercises:

29.



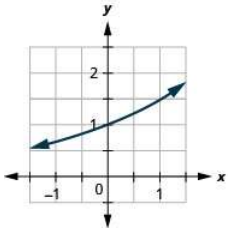
$f(x) = 2^x$

31.



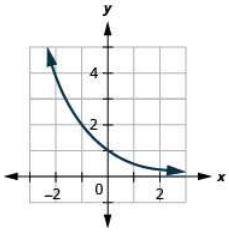
$f(x) = 6^x$

33.



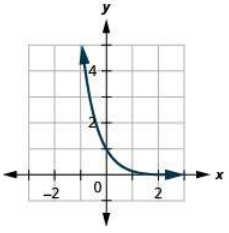
$f(x) = (1.5)^x$

35.



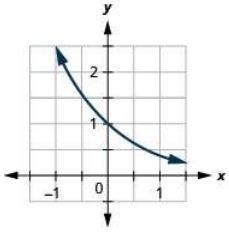
$f(x) = \left(\frac{1}{2}\right)^x$

37.



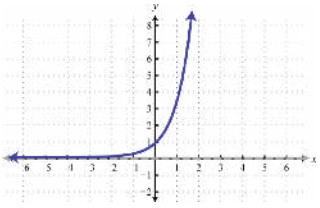
$f(x) = \left(\frac{1}{6}\right)^x$

39.



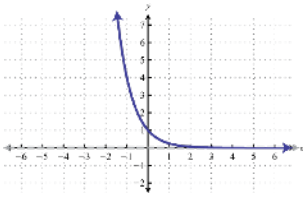
$f(x) = (0.4)^x$

41.



$f(x) = 4^x$

43.



$f(x) = \left(\frac{1}{4}\right)^x$

D: Graph Shifts of Exponential Functions

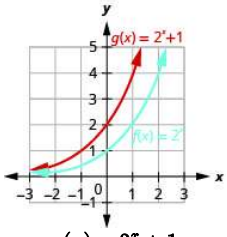
Exercise 4.2e.D

★ In the following exercises, use transformations to graph each exponential function. State the transformations that must be done to the parent function in order to obtain the graph.

45. $g(x)=2^x+1$	49. $g(x)=3^x+2$	53. $f(x)=2^x+3$	57. $h(x)=2^{x-3}-2$	61. $f(x)=4^x+2$
46. $g(x)=2^x-1$	50. $f(x)=3^x-6$	54. $f(x)=2^x-3$	58. $g(x)=3^{x-1}+3$	62. $f(x)=4^x-1$
47. $g(x)=2^{x-2}$	51. $g(x)=4^{x-1}$	55. $f(x)=3^{x+2}$	59. $f(x)=3^{x+1}-4$	63. $f(x)=5^{x+1}+2$
48. $g(x)=2^{x+2}$	52. $f(x)=4^{x+2}$	56. $f(x)=3^{x+1}$	60. $h(x)=3^{x-2}+4$	64. $f(x)=10^{x-4}+2$

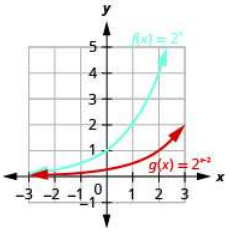
Answers to odd exercises:

45. Up 1



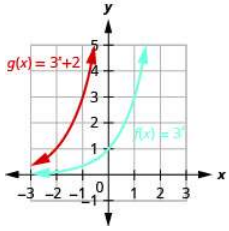
$g(x)=2^x+1$

47. Right 2



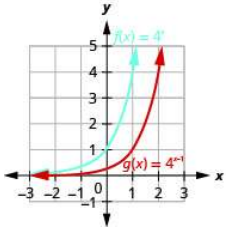
$g(x)=2^{x-2}$

49. Up 2



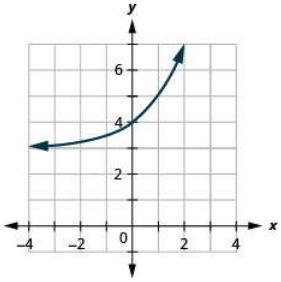
$g(x)=3^x+2$

51. Right 1



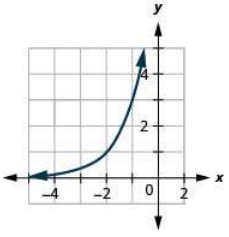
$g(x)=4^{x-1}$

53. Up 3



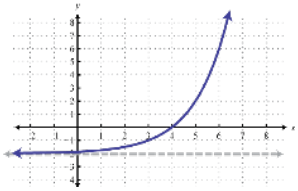
$f(x)=2^x+3$

55. Left 2



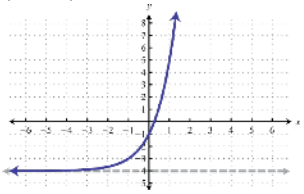
$f(x)=3^{x+2}$

57. Right 3, Down 2; Domain: $(-\infty,\infty)$; Range: $(-2,\infty)$



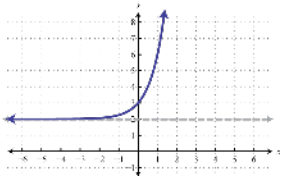
$h(x)=2^{x-3}-2$

59. Left 1, Down 4; Domain: $(-\infty,\infty)$; Range: $(-4,\infty)$



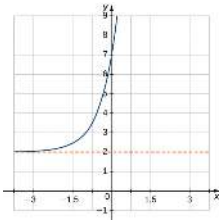
$f(x)=3^{x+1}-4$

61. up 2; Domain: $(-\infty,\infty)$; Range: $(2,\infty)$



$f(x)=4^x+2$

63. Left 1, up 2



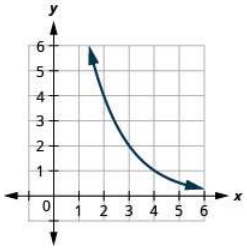
63. $f(x)=5^{x+1}+2$

★ In the following exercises, use transformations to graph each exponential function. State the transformations that must be done to the parent function in order to obtain the graph.

65. $f(x)=\left(\frac{1}{2}\right)^{x-4}$	67. $f(x)=\left(\frac{1}{4}\right)^x-2$	69. $f(x)=e^x-3$	72. $f(x)=e^{x-2}$	75. $f(x)=e^{x-2}+1$
66. $f(x)=\left(\frac{1}{2}\right)^x-3$	68. $h(x)=\left(\frac{1}{3}\right)^x+2$	70. $f(x)=e^x+2$	73. $f(x)=e^{x+1}$	76. $f(x)=e^{x+2}-1$
		71. $f(x)=e^x+1$	74. $f(x)=e^{x-3}$	

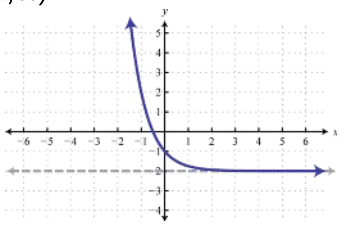
Answers to odd exercises:

65. Right 4



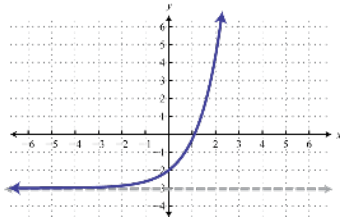
$f(x) = \left(\frac{1}{2}\right)^{x-4}$

67. Down 2; Domain: $(-\infty, \infty)$; Range: $(-2, \infty)$



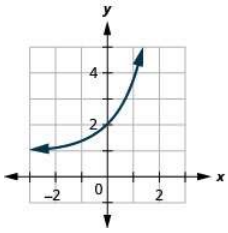
$f(x) = \left(\frac{1}{4}\right)^{x} - 2$

69. Down 3. Domain: $(-\infty, \infty)$; Range: $(-3, \infty)$



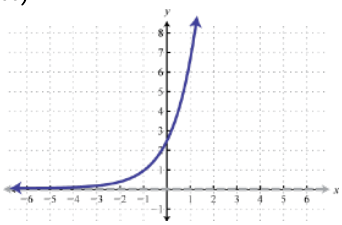
$f(x) = e^{x} - 3$

71. Up 1



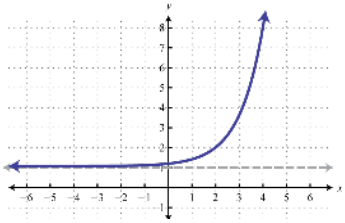
$f(x) = e^{x} + 1$

73. Left 1; Domain: $(-\infty, \infty)$; Range: $(0, \infty)$



$f(x) = e^{x+1}$

75. Right 2, up 1; Domain: $(-\infty, \infty)$; Range: $(1, \infty)$



$f(x) = e^{x-2} + 1$

E: Graph Reflections and Stretches of Exponential Functions

Exercise 4.2e.E

★ In the following exercises, graph each exponential function. State the transformations that must be done to the parent function in order to obtain the graph and state its domain, range, and horizontal asymptote.

77. $f(x) = -2^{x}$
78. $f(x) = 2^{-x-1} - 1$
79. $g(x) = 2^{-x} - 3$
80. $g(x) = 3^{-x} + 1$
81. $f(x) = 6 - 10^{-x}$
82. $g(x) = 5 - 4^{-x}$
83. $f(x) = 5 - 2^{x}$
84. $f(x) = 3 - 3^{x}$
84.1 $f(x) = 3 \cdot 2^{x-4} - 5$
84.2 $y = 5 - \frac{1}{2} \cdot 4^{x-3}$
84.3 $f(x) = 3^{x/2+3} - 4$
84.4 $f(x) = 5 - 2^{-x/3+1}$
85) $f(x) = 1 - 2^{-x}$
86) $f(x) = 2^{-x}$
87. $g(x) = -e^{x}$
88. $g(x) = e^{-x}$
89. $h(x) = -e^{x+1}$
90. $h(x) = -e^{x} + 3$
91) $f(x) = e^{-x} - 1$

★ For the following exercises, graph each set of functions on the same axes.

93) $f(x) = 3\left(\frac{1}{4}\right)^{x}, g(x) = 3(2)^{x}, h(x) = 3(4)^{x}$

94) $f(x) = \frac{1}{4}(3)^{x}, g(x) = 2(3)^{x}, h(x) = 4(3)^{x}$

★ For the following exercises, describe the end behavior of the graphs of the functions and the transformations needed to do on the parent graph to obtain the graph of the function.

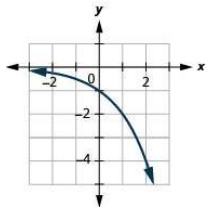
95) $f(x) = -5(4)^{x} - 1$

96) $f(x) = 3\left(\frac{1}{2}\right)^{x} - 2$

97) $f(x) = 3(4)^{-x} + 2$

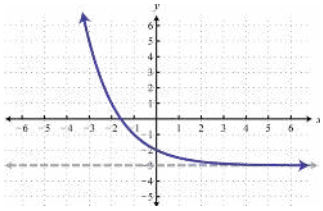
Answers to odd exercises:

77. Reflect over x-axis
HA: y = 0
Domain: $(-\infty, \infty)$; Range: $(-\infty, 0)$



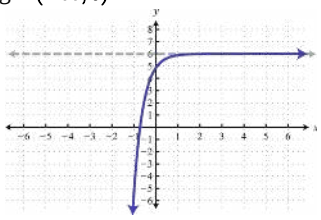
$f(x) = -2^x$

79. Reflect over y-axis, down 3;
HA: y = -3
Domain: $(-\infty, \infty)$; Range: $(-3, \infty)$



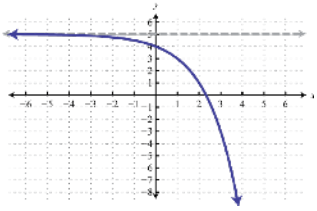
$g(x) = 2^{-x} - 3$

81. Reflect over y-axis, reflect over x-axis, up 6; HA: y = 6; Domain: $(-\infty, \infty)$; Range: $(-\infty, 6)$



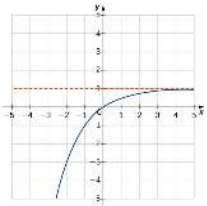
$f(x) = 6 - 10^{-x}$

83. Reflect over x-axis, up 5; HA: y = 5; Domain: $(-\infty, \infty)$; Range: $(-\infty, 5)$



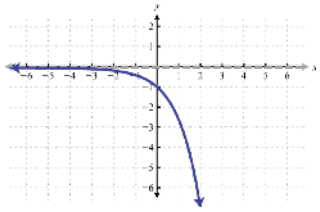
$f(x) = 5 - 2^x$

85. Reflect over y-axis, repfect over x-axis up 1; HA: y=1; Domain: all real numbers, range: $(-\infty, 1)$



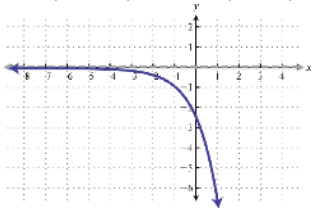
$f(x) = 1 - 2^{-x}$

87. Reflect over x-axis; HA: y = 0; Domain: $(-\infty, \infty)$; Range: $(-\infty, 0)$



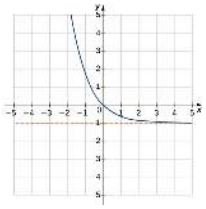
$g(x) = -e^x$

89. Left 1, Reflect over x-axis; HA: y=0; Domain: $(-\infty, \infty)$; Range: $(-\infty, 0)$



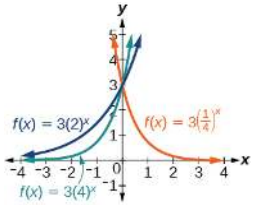
$h(x) = -e^{x+1}$

91. Reflect over y-axis, down 1; HA: y = -1; Domain: all real numbers, range: $(-1, \infty)$



$f(x) = e^{-x} - 1$

93.



95. As $x \to \infty, f(x) \to -\infty$; As $x \to -\infty, f(x) \to -1$; Reflect over x-axis, y's stretched by a factor of 5, down 1
97. As $x \to \infty, f(x) \to 2$; As $x \to -\infty, f(x) \to \infty$; Reflect over y-axis, y's get stretched by a factor of 3, up 2.

F: Construct an Equation Given a Description

Exercise 4.2e.F

★ For the following exercises, start with the graph of $f(x) = 4^x$. Then write a function that results from the given transformation.

98) Shift $f(x)$ 4 units upward	100) Shift $f(x)$ 2 units left	102) Reflect $f(x)$ about the x-axis
99) Shift $f(x)$ 3 units downward	101) Shift $f(x)$ 5 units right	103) Reflect $f(x)$ about the y-axis

★ For the following exercises, (a) graph the function, (b) graph its reflection about the y-axis and construct the function for the reflected graph, and (c) give the y-intercept.

104) $f(x) = 3\left(\frac{1}{2}\right)^x$	105) $g(x) = -2(0.25)^x$	106) $h(x) = 6(1.75)^{-x}$

★ For the following exercises, (a) graph the function, (b) graph its reflection about the x-axis and construct the function for the reflected graph.

107) $f(x) = \frac{1}{2}(4)^x$	108) $f(x) = 3(0.75)^x - 1$	109) $f(x) = -4(2)^x + 2$

★ For the following exercises, (a) construct the equation of the exponential function given a description. (b) State its y-intercept, domain, and range.

111) The graph of $f(x) = 10^x$ is reflected about the x-axis and shifted upward 7 units. What is the equation of the new function, $g(x)$? State its y-intercept, domain, and range.

112) The graph of $f(x) = \left(\frac{1}{2}\right)^{-x}$ is reflected about the y-axis and compressed vertically by a factor of $\left(\frac{1}{5}\right)$ What is the equation of the new function, $g(x)$ State its y-intercept, domain, and range.

113) The graph of $f(x) = 3^x$ is reflected about the y-axis and stretched vertically by a factor of 4. What is the equation of the new function, $g(x)$. State its y-intercept, domain, and range.

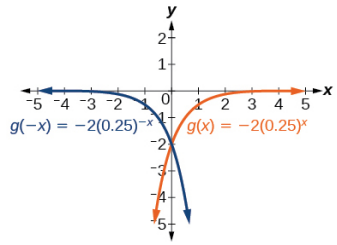
114) The graph of $f(x) = (1.68)^x$ is shifted right 3 units, stretched vertically by a factor of 2, reflected about the x-axis, and then shifted downward 3 units. What is the equation of the new function, $g(x)$? State its y-intercept (to the nearest thousandth), domain, and range.

115) The graph of $f(x) = 2\left(\frac{1}{4}\right)^{x-1}$ is shifted left 2 units, stretched vertically by a factor of 4, reflected about the x-axis, and then shifted downward 4 units. What is the equation of the new function, $g(x)$? State the y-intercept, domain, and range of $g(x)$.

Answers to odd exercises:

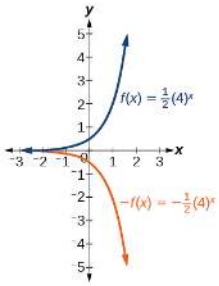
99. $f(x) = 4^x - 3$ 101. $f(x) = 4^{x-5}$ 103. $f(x) = 4^{-x}$

105.

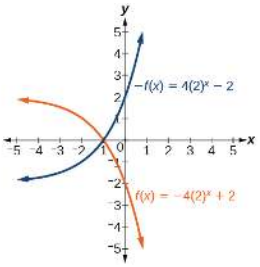


y-intercept: $(0, -2)$

107



109



111. $g(x) = -10^x + 7$; y-intercept: $(0, 6)$; Domain: all real numbers; Range: all real numbers less than 7.

113. $g(x) = 4(3)^{-x}$ $(0, 4)$ Domain: all real numbers; Range: all real numbers greater than 0.

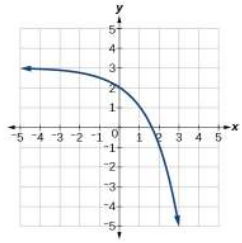
115. $g(x) = -8\left(\frac{1}{4}\right)^{x+1} - 4$; y-intercept: $(0, -6)$; Domain: $(-\infty, \infty)$; Range: $(-\infty, -4)$

G: Construct an Equation Given a Graph

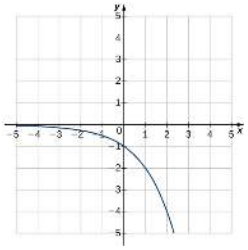
Exercise 4.2e.G

★ Write an equation for each of the functions graphed below. The graphs #117-123 are transformations of the parent function $f(x) = 2^x$. The parent functions for #124-127 must be determined.

117.



118.



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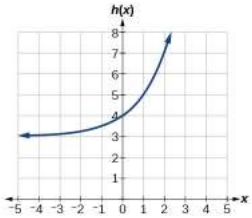

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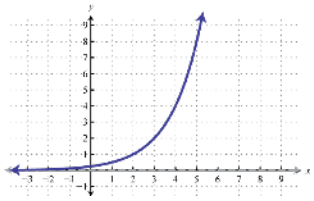
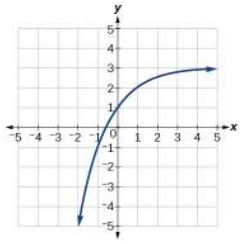
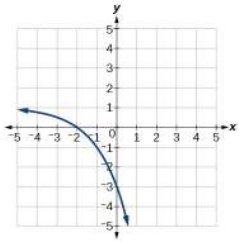


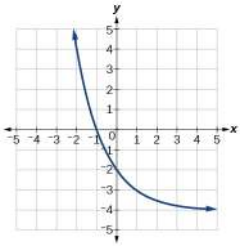
121.



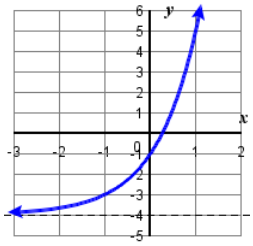
122.



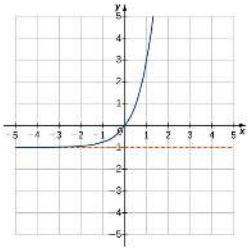
123.



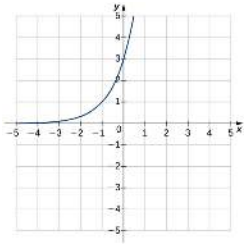
124.



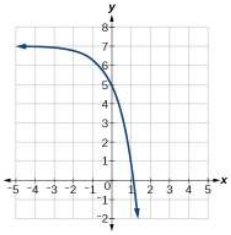
125.



126.



127. (Hint: this graph has been stretched)



Answer to odd exercises:

117. $y = -2^x + 3$	121. $y = -2^{-x+1} + 3$	125. $y = 4^x - 1$
119. $y = 2^x + 3$	123. $y = 2^{-x+1} - 4$	127. $y = -2(3)^x + 7$

H: Practice with Exponent Properties

Exercise 4.2e.H

★ Write the following with only one exponent, x. Simplify.

131. 2^{x+3}	137. 5^{2x}	143. 2^{3x+2}	149. $\frac{3^x}{5^x}$	152. $\frac{9^x}{6^{2x}}$	155. $\frac{6^{x+2}}{4^{2x+3}}$
132. 7^{2+x}	138. 4^{2x}	144. 3^{2x+3}			
133. 4^{x-3}	139. $5^x \cdot 7^x$	145. 4^{3x-2}	150. $\frac{4^x}{w^x}$	153. $\frac{3^{x+1}}{2^{2x-3}}$	
134. 9^{x-2}	140. $9^x \cdot 4^x$	146. 2^{5x-3}			156. $\frac{5^{2x-1}}{3^{3x-2}}$
135. 4^{4-x}	141. $2^x \cdot 3^{2x}$	147. $6^{x+1} \cdot 4^{x-2}$	151. $\frac{2^{4x}}{8^x}$	154. $\frac{8^{x-1}}{4^{3x+2}}$	
136. 6^{2-x}	142. $x^2 \cdot y^2$	148. $8^{2x+1} \cdot 7^{x-1}$			

Answers to odd exercises:

131. $8(2^x)$ 133. $\frac{4^x}{64}$ 135. $\frac{256}{4^x} = 256\left(\frac{1}{4}\right)^x$ 137. 25^x 139. 35^x 141. 18^x 143. $4(8^x)$
145. $\frac{64^x}{16}$ 147. $\frac{3}{8}24^x$ 149. $\left(\frac{3}{5}\right)^x$ 151. 2^x 153. $24\left(\frac{3}{4}\right)^x$ 155. $\frac{9}{16}\left(\frac{3}{8}\right)^x$

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4.3: Logarithmic Functions

Learning Objectives

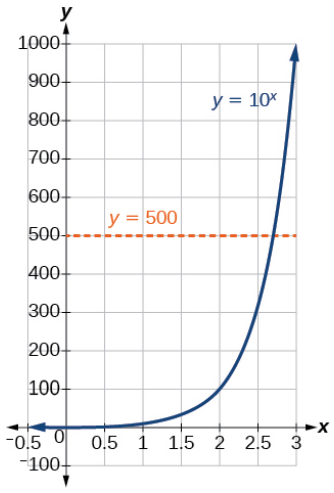
- Convert from logarithmic to exponential form.
- Convert from exponential to logarithmic form.
- Evaluate logarithms.
- Use common logarithms.
- Use natural logarithms.

In 2010, a major earthquake struck Haiti, destroying or damaging over 285,000 homes. One year later, another, stronger earthquake devastated Honshu, Japan, destroying or damaging over 332,000 buildings. Even though both caused substantial damage, the earthquake in 2011 was 100 times stronger than the earthquake in Haiti. How do we know? The magnitudes of earthquakes are measured on a scale known as the Richter Scale. The Haitian earthquake registered a 7.0 on the Richter Scale whereas the Japanese earthquake registered a 9.0.

The Richter Scale is a base-ten logarithmic scale. In other words, an earthquake of magnitude 8 is not twice as great as an earthquake of magnitude 4. It is $10^{8-4} = 10^4 = 10,000$ times as great! In this lesson, we will investigate the nature of the Richter Scale and the base-ten function upon which it depends.

Logarithmic Functions Defined

In order to analyze the magnitude of earthquakes or compare the magnitudes of two different earthquakes, we need to be able to convert between logarithmic and exponential form. For example, suppose the amount of energy released from one earthquake were 500 times greater than the amount of energy released from another. We want to calculate the difference in magnitude. The equation that represents this problem is $10^x = 500$, where x represents the difference in magnitudes on the Richter Scale. How would we solve for x?



We have not yet learned a method for solving exponential equations. None of the algebraic tools discussed so far is sufficient to solve $10^x = 500$. We know that $10^2 = 100$ and $10^3 = 1000$, so it is clear that x must be some value between 2 and 3, since $y = 10^x$ is increasing. We can examine a graph, as in the figure to the right, to better estimate the solution.

Estimating from a graph, however, is imprecise. To find an algebraic solution, we must introduce a new function. Observe that the graph of $y = 10^x$ passes the horizontal line test. The exponential function $y = b^x$ is one-to-one, so its inverse, $x = b^y$ is also a function. As is the case with all inverse functions, we simply interchange x and y and solve for y to find the inverse function. To represent y as a function of x, we use a logarithmic function of the form $y = \log_b(x)$. The base b **logarithm** of a number is the exponent by which we must raise b to get that number.

We read a logarithmic expression as, "The logarithm with base b of x is equal to y," or, simplified, "log base b of x is y." We can also say, "b raised to the power of y is x," because logs are exponents. For example, the base 2 logarithm of 32 is 5, because 5 is the exponent we must apply to 2 to get 32. Since $2^5 = 32$, we can write $\log_2 32 = 5$. We read this as "log base 2 of 32 is 5."

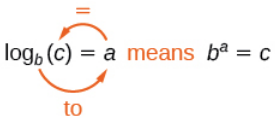
We can express the relationship between logarithmic form and its corresponding exponential form as follows:

$$\log_b(x) = y \Leftrightarrow b^y = x, \qquad b > 0,\ b \neq 1$$

Note that the base b is always positive.



Because logarithm is a function, it is most correctly written as $\log_b(x)$, using parentheses to denote function evaluation, just as we would with $f(x)$. However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written without parentheses, as $\log_b x$. Note that many calculators require parentheses around the x. We can illustrate the notation of logarithms as follows:



Notice that, comparing the logarithm function and the exponential function, the input and the output are switched. This means $y = \log_b(x)$ and $y = b^x$ are inverse functions.

DEFINITION OF THE LOGARITHMIC FUNCTION

A logarithm base b of a positive number x satisfies the following definition.

$$y = \log_b(x) \quad \text{is equivalent to} \quad b^y = x \quad \text{for } x > 0,\ b > 0,\ b \neq 1$$

- We read $\log_b(x)$ as "log base b of x." It is equivalent to saying "the exponent on b to get x."
- The logarithm y is the exponent to which b must be raised to get x.

Also, since the logarithmic and exponential functions switch the x and y values, the domain and range of the exponential function are interchanged for the logarithmic function. Therefore, for the parent logarithm function $y = \log_b(x)$,

- The domain of the logarithm function with base b is $(0, \infty)$.
- The range of the logarithm function with base b is $(-\infty, \infty)$.

Can we take the logarithm of a negative number?

No. Because the base of an exponential function is always positive, no power of that base can ever be negative. We can never take the logarithm of a negative number. Also, we cannot take the logarithm of zero. Calculators may output a log of a negative number when in complex mode, but the log of a negative number is not a real number.

Convert from Logarithmic to Exponential Form

How to: Given an equation in logarithmic form $\log_b(x) = y$, convert it to exponential form

1. Examine the equation $p = \log_b a$ and identify the base b, the power p, and the log argument a.
2. Rewrite $\log_b a = p$ as $b^p = a$.

Notice the result of taking the log of something is an exponent; the result of exponentiation is a log argument

Example 4.3.1: Convert from Logarithmic Form to Exponential Form

Write the following logarithmic equations in exponential form.

a. $\log_6(\sqrt{6}) = \frac{1}{2}$	b. $\log_3(9) = 2$

Solution

First, identify the values of b, p, and a. Then, write the equation in the form $b^y = x$.

a. $\log_6(\sqrt{6}) = \frac{1}{2}$ Here, the base $b = 6$, the power $p = \frac{1}{2}$, and the argument $a = \sqrt{6}$.

Therefore, the equation $\log_6(\sqrt{6}) = \frac{1}{2}$ is equivalent to $6^{\frac{1}{2}} = \sqrt{6}$

b. $\log_3(9) = 2$ Here, the base $b = 3$, the power $p = 2$, and the argument $a = 9$.

Therefore, the equation $\log_3(9) = 2$ is equivalent to $3^2 = 9$

Try It 4.3.1

Write the following logarithmic equations in exponential form.

a. $\log_{10}(1,000,000) = 6$

Answer a

$\log_{10}(1,000,000) = 6$ is equivalent to $10^6 = 1,000,000$

b. $\log_5(25) = 2$

Answer b

$\log_5(25) = 2$ is equivalent to $5^2 = 25$

Convert from Exponential to Logarithmic Form

How to: Given an equation in exponential form $b^x = y$, convert it to logarithmic form

1. Examine the equation $b^p = a$ and identify the base b, the power p, and the answer a (which will be the log argument).
2. Rewrite $b^p = a$ as $\log_b a = p$.

Example 4.3.2: Convert from Exponential Form to Logarithmic Form

Write the following exponential equations in logarithmic form.

a. $2^3 = 8$

b. $5^2 = 25$

c. $10^{-4} = \frac{1}{10,000}$

Solution

First, identify the values of b, p, and a. Then, write the equation in the form $x = \log_b(y)$.

a. $2^3 = 8$ Here, $b = 2$, $p = 3$,and $a = 8$.

Therefore, the equation $2^3 = 8$ is equivalent to $\log_2(8) = 3$.

b. $5^2 = 25$ Here, $b = 5$, $p = 2$,and $a = 25$.

Therefore, the equation $5^2 = 25$ is equivalent to $\log_5(25) = 2$.

c. $10^{-4} = \frac{1}{10,000}$ Here, $b = 10$, $p = -4$,and $a = \frac{1}{10,000}$.

Therefore, the equation $10^{-4} = \frac{1}{10,000}$ is equivalent to $\log_{10}\left(\frac{1}{10,000}\right) = -4$.

Try It 4.3.2

Write the following exponential equations in logarithmic form.

a. $3^2 = 9$

Answer a

$3^2 = 9$ is equivalent to $\log_3(9) = 2$

b. $5^3 = 125$

Answer b

$5^3 = 125$ is equivalent to $\log_5(125) = 3$

c. $2^{-1} = \frac{1}{2}$

Answer c

$2^{-1} = \frac{1}{2}$ is equivalent to $\log_2\left(\frac{1}{2}\right) = -1$

Evaluate Logarithms

Knowing the squares, cubes, and roots of numbers allows us to evaluate many logarithms mentally. For example, consider $\log_2 8$. We ask, "To what exponent must 2 be raised in order to get 8?" Because we already know $2^3 = 8$, it follows that $\log_2 8 = 3$.

Now consider solving $\log_7 49$ and $\log_3 27$ mentally.

- We ask, "To what exponent must 7 be raised in order to get 49?" We know $7^2 = 49$. Therefore, $\log_7 49 = 2$
- We ask, "To what exponent must 3 be raised in order to get 27?" We know $3^3 = 27$. Therefore, $\log_3 27 = 3$

Even some seemingly more complicated logarithms can be evaluated without a calculator. For example, let's evaluate $\log_{\frac{2}{3}} \frac{4}{9}$ mentally.

- We ask, "To what exponent must $\frac{2}{3}$ be raised in order to get $\frac{4}{9}$?"

  We know $2^2 = 4$ and $3^2 = 9$, so $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$. Therefore, $\log_{\frac{2}{3}}\left(\frac{4}{9}\right) = 2$.

How to: Given a logarithm of the form $y = \log_b(x)$, evaluate it mentally

1. Rewrite the argument x as a power of b: $b^y = x$.
2. Use previous knowledge of powers of b identify y by asking, "To what exponent should b be raised in order to get x?"

Example 4.3.3: Evaluate Logarithms Mentally

Solve $y = \log_4(64)$ without using a calculator.

Solution

First we rewrite the logarithm in exponential form: $4^y = 64$.

Next, we ask, "To what exponent must 4 be raised in order to get 64?"

We know $4^3 = 64$. Therefore, $\log_4(64) = 3$

Try It 4.3.3

Solve $y = \log_{121}(11)$ without using a calculator.

Answer

$\log_{121}(11) = \frac{1}{2}$ (recalling that $\sqrt{121} = (121)^{\frac{1}{2}} = 11$)

Example 4.3.4: Evaluate the Logarithm of a Reciprocal

Evaluate $y = \log_3\left(\frac{1}{27}\right)$ without using a calculator.

Solution

First rewrite the logarithm in exponential form: $3^y = \frac{1}{27}$.

Next ask, "To what exponent must 3 be raised in order to get $\frac{1}{27}$?"

We know $3^3 = 27$, so $\frac{1}{3^3} = \frac{1}{27}$. Therefore, $\log_3\left(\frac{1}{27}\right) = -3$.

Try It 4.3.4

Evaluate $y = \log_2\left(\frac{1}{32}\right)$ without using a calculator.

Answer

$\log_2\left(\frac{1}{32}\right) = -5$

Common Logarithms

Sometimes we may see a logarithm written without a base. In this case, we assume that the base is 10. In other words, the expression $\log(x)$ means $\log_{10}(x)$. We call a base 10 logarithm a *common logarithm*. Common logarithms are used to measure the Richter Scale mentioned at the beginning of the section. Scales for measuring the brightness of stars and the pH of acids and bases also use common logarithms.

DEFINITION OF THE COMMON LOGARITHM

A common logarithm is a logarithm with base 10. We write $\log_{10}(x)$ simply as $\log(x)$. The common logarithm of a positive number x satisfies the following definition.

$$y = \log(x) \quad \text{is equivalent to} \quad 10^y = x \quad \text{for } x > 0$$

We read $\log(x)$ as "log base 10 of x." The logarithm y is the exponent to which 10 must be raised to get x.

How to: Given a common logarithm of the form $y = \log(x)$, evaluate it mentally

1. Rewrite the argument x as a power of 10: $10^y = x$.
2. Use previous knowledge of powers of 10 to identify y by asking, "To what exponent must 10 be raised to get x?"

Example 4.3.5: Find the Value of a Common Logarithm Mentally

Evaluate $y = \log(1000)$ without using a calculator.

Solution

First we rewrite the logarithm in exponential form: $10^y = 1000$.

Next, we ask, "To what exponent must 10 be raised in order to get 1000?" We know $10^3 = 1000$

Therefore, $\log(1000) = 3$.

Try It 4.3.5

Evaluate $y = \log(1,000,000)$.

Answer

$\log(1,000,000) = 6$

How to: Given a common logarithm with the form $y = \log(x)$, evaluate it using a calculator

1. Press **[LOG]**.
2. Enter the value given for x, followed by **[ )]**.
3. Press **[ENTER]**.

Example 4.3.6: Find the Value of a Common Logarithm Using a Calculator

Evaluate $y = \log(321)$ to four decimal places using a calculator.

Solution

Press **[LOG]**. Enter 321, followed by **[ )]**. Press **[ENTER]**. Rounding to four decimal places, $\log(321) \approx 2.5065$.

Analysis

Note that $10^2 = 100$ and that $10^3 = 1000$ and 321 is between 100 and 1000. This gives us the following:

$$100 < 321 < 1000$$
$$\log(100) < \log(321) < \log(1000)$$
$$2 < 2.5065 < 3$$

Try It 4.3.6

Evaluate $y = \log(123)$ to four decimal places using a calculator.

Answer

$\log(123) \approx 2.0899$

Example 4.3.7: Application of Common logarithms in a Real-World Exponential Model

The amount of energy released from one earthquake was 500 times greater than the amount of energy released from another. The equation $10^x = 500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

Solution

We begin by rewriting the exponential equation in logarithmic form.

$$10^x = 500 \longleftrightarrow \log(500) = x$$

Next we evaluate the logarithm using a calculator:

Press **[LOG]**. Enter **500**, followed by **[ )]**. Press **[ENTER]**. To the nearest thousandth, $\log(500) \approx 2.699$.

The difference in magnitudes was about 2.699.

Try It 4.3.7

The amount of energy released from one earthquake was $8,500$ times greater than the amount of energy released from another. The equation $10^x = 8500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

Answer

The difference in magnitudes was about 3.929.

Natural Logarithms

The most frequently used base for logarithms is e. Base e logarithms are important in calculus and some scientific applications; they are called **natural logarithms**. The base e logarithm, $\log_e(x)$, has its own notation, $\ln(x)$. Most values of $\ln(x)$ can be found only using a calculator. The major exception is that, because the logarithm of 1 is always 0 in any base, $\ln 1 = 0$. For other natural logarithms, we can use the $\ln$ key that can be found on most scientific calculators. We can also find the natural logarithm of any power of e using the inverse property of logarithms.

DEFINITION OF THE NATURAL LOGARITHM

A natural logarithm is a logarithm with base e. We write $\log_e(x)$ simply as $\ln(x)$. The natural logarithm of a positive number x satisfies the following definition.

$$y = \ln(x) \quad \text{is equivalent to} \quad e^y = x \quad \text{for } x > 0$$

We read $\ln(x)$ as "the natural logarithm of x."

The logarithm y is the exponent to which e must be raised to get x.

Since the functions $y = e^x$ and $y = \ln(x)$ are inverse functions, $\ln(e^x) = x$ for all x and $e^{\ln(x)} = x$ for $x > 0$.

How to: Given a natural logarithm with the form $y = \ln(x)$, evaluate it using a calculator

1. Press **[LN]**.
2. Enter **the value** given for x, followed by **[ )]**.
3. Press **[ENTER]**.

Example 4.3.8: Evaluate a Natural Logarithm Using a Calculator

Evaluate $y = \ln(500)$ to four decimal places using a calculator.

Solution

Press **[LN]**. Enter **500**, followed by **[ )]**. Press **[ENTER]**. Rounding to four decimal places, $\ln(500) \approx 6.2146$

Try It 4.3.8

Evaluate $\ln(-500)$.

Answer

It is not possible to take the logarithm of a negative number in the set of real numbers.

Key Equations

Definition of the logarithmic function	For $x > 0$, $b > 0$, $b \neq 1$, $y = \log_b(x)$ if and only if $b^y = x$.
Definition of the common logarithm	For $x > 0$, $y = \log(x)$ if and only if $10^y = x$.
Definition of the natural logarithm	For $x > 0$, $y = \ln(x)$ if and only if $e^y = x$.

Key Concepts

- Exponential and logarithmic functions are inverses of each other.
- Logarithmic equations can be written in an equivalent exponential form, using the definition of a logarithm.
- Exponential equations can be written in their equivalent logarithmic form using the definition of a logarithm.
- Logarithmic functions with base b can be evaluated mentally using previous knowledge of powers of b.
- Common logarithms can be evaluated mentally using previous knowledge of powers of 10.
- When common logarithms cannot be evaluated mentally, a calculator can be used.
- Real-world exponential problems with base 10 can be rewritten as a common logarithm and then evaluated using a calculator.
- Natural logarithms can be evaluated using a calculator.

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4.3e: Exercises - Logarithm Functions

A: Concepts

Exercise 4.3e.A

1) What is a base b logarithm? Discuss the meaning by interpreting each part of the equivalent equations $b^y = x$ and $\log_b x = y$ for $b > 0, b \neq 1$

2) How is the logarithmic function $f(x) = \log_b x$ related to the exponential function $g(x) = b^x$? What is the result of composing these two functions?

3) How can the logarithmic equation $\log_b x = y$ be solved for x using the properties of exponents?

4) Discuss the meaning of the common logarithm. What is its relationship to a logarithm with base b, and how does the notation differ?

5) Discuss the meaning of the natural logarithm. What is its relationship to a logarithm with base b, and how does the notation differ?

6) Is $f(x) = 0$ in the range of the function $f(x) = \log(x)$? If so, for what value of x? Verify the result.

7) Is $x = 0$ in the domain of the function $f(x) = \log x$? If so, what is the value of the function when $x = 0$? Verify the result.

Answers to odd exercises:

1. A logarithm is an exponent. Specifically, it is the exponent to which a base b is raised to produce a given value. In the expressions given, the base b has the same value. The exponent, y, in the expression b^y can also be written as the logarithm, $\log_b x = y$, and the value of x is the result of raising b to the power of y.

3. Since the equation of a logarithm is equivalent to an exponential equation, the logarithm can be converted to the exponential equation $b^y = x$, and then properties of exponents can be applied to solve for x .

5. The natural logarithm is a special case of the logarithm with base b in that the natural log always has base e. Rather than notating the natural logarithm as $\log_e(x)$, the notation used is $\ln(x)$.

7. No, the function has no defined value for $x = 0$. To verify, suppose $x = 0$ is in the domain of the function $f(x) = \log(x)$. Then there is some number n such that $n = \log(0)$. Rewriting as an exponential equation gives: $10^n = 0$, which is impossible since no such real number n exists. Therefore, $x = 0$ is not the domain of the function $f(x) = \log(x)$.

B: Convert from log to exponential form

Exercise 4.3e.B

★ For the following exercises, rewrite each equation in exponential form.

8) $\log_3 81 = 4$	14. $3 = \log_4 64$	20. $5 = \ln x$	26. $4 = \log_x 81$	32) $\log_4(q) = m$
9) $\log_8 2 = \frac{1}{3}$	15. $6 = \log_2 64$	21) $\ln 1 = 0$	27. $5 = \log_x 32$	33) $\log_{15}(a) = b$
10) $\log_5 1 = 0$	16. $0 = \log_{12} 1$	22) $\ln(\frac{1}{e^3}) = -3$	28) $\log_y(x) = -11$	34) $\log_{16}(y) = x$
11) $\log_5 25 = 2$	17. $0 = \log_7 1$	23. $x = \ln 43$	29) $\log_{13}(142) = a$	35) $\log_a(b) = c$
12) $\log 0.1 = -1$	18. $1 = \log_3 3$	24. $-4 = \log \frac{1}{10,000}$	30) $\log_y(137) = x$	36) $\log(v) = t$
13) $\log_9 3 = 0.5$	19. $1 = \log_9 9$	25. $3 = \log 1,000$	31) $\log_x(64) = y$	37) $\ln(w) = n$

Answers to odd exercises:

9: $8^{1/3} = 2$	15. $64 = 2^6$	21: $e^0 = 1$	27. $32 = x^5$	33. $15^b = a$
11: $5^2 = 25$	17. $1 = 7^0$	23. $43 = e^x$	29. $13^a = 142$	35. $a^c = b$
13. $9^{0.5} = 3$	19. $9 = 9^1$	25. $1,000 = 10^3$	31. $x^y = 64$	37. $e^n = w$

C: Convert from exponential to log form

Exercise 4.3e.C

★ For the following exercises, write the equation in equivalent logarithmic form.

38. $2^3 = 8$
39. $4^{-2} = \frac{1}{16}$
40. $10^2 = 100$
41. $9^0 = 1$
42. $(\frac{1}{3})^3 = \frac{1}{27}$
43. $4^{-3/2} = 0.125$
44. $\sqrt[3]{64} = 4$
45. $n^4 = 103$
46. $4^2 = 16$
47. $2^5 = 32$
48. $3^3 = 27$
49. $5^3 = 125$
50. $b^3 = 45$
51. $9^y = 150$
52. $10^3 = 1000$
53. $10^{-2} = \frac{1}{100}$
58. $(\frac{1}{4})^2 = \frac{1}{16}$
59. $(\frac{1}{3})^4 = \frac{1}{81}$
60. $3^{-2} = \frac{1}{9}$
61. $4^{-3} = \frac{1}{64}$
62. $x^{\frac{1}{2}} = \sqrt{3}$
63. $x^{\frac{1}{3}} = \sqrt[3]{6}$
64. $32^x = \sqrt[4]{32}$
65. $17^x = \sqrt[5]{17}$
66. $e^x = 6$
67. $e^3 = x$
68. $e^x = y$
69. $e^k = h$
70. $m^{-7} = n$
71. $10^a = b$
72. $4^x = y$
73. $19^x = y$
74. $x^{-\frac{10}{13}} = y$
75. $y^x = \frac{39}{100}$
76. $(\frac{7}{5})^m = n$
77. $c^d = k$

Answers to odd exercises:

39: $\log_4(\frac{1}{16}) = -2$
41: $\log_9 1 = 0$
43: $\log_4 0.125 = -\frac{3}{2}$
45. $\log_n(103) = 4$
47. $\log_2 32 = 5$
49. $\log_5 125 = 3$
51: $\log_9 150 = y$
53. $\log \frac{1}{100} = -2$
59. $\log_{\frac{1}{3}} \frac{1}{81} = 4$
61. $\log_4 \frac{1}{64} = -3$
63. $\log_x \sqrt[3]{6} = \frac{1}{3}$
65. $\log_{17} \sqrt[5]{17} = x$
67. $\ln x = 3$
69. $k = \ln(h)$
71. $\log(b) = a$
73. $\log_{19}(y) = x$
75. $\log_y (\frac{39}{100}) = x$
77. $\log_c(k) = d$

D: Evaluate logarithms using the definition

Exercise 4.3e.D

★ In the following exercises, find the exact value of each logarithm without using a calculator.

78. $\log_3 243$
79. $\log_3 9$
80. $\log_4 4$
81. $\log_5 1$
82. $\log_5 625$
83. $\log_6 36$
84. $\log_7 49$
85. $\log_{25} 5$
86. $\log_8 2$
87. $\log_{27} 3$
88. $\log_{16} 4$
89. $\log_4 4^{10}$
90. $\log_9 9^5$
91) $6 \log_8(4)$
92. $\log_2 (\frac{1}{64})$
93. $\log_2 (\frac{1}{16})$
94) $\log_2 (\frac{1}{8}) + 4$
95. $\log_3 \frac{1}{27}$
96. $\log_3 (\frac{1}{9})$
97. $\log_4 (\frac{1}{2})$
98. $\log_4 \frac{1}{16}$
100. $\log_5 (\frac{1}{125})$
101. $\log_9 \frac{1}{81}$
102) $\log_6(\sqrt{6})$
103. $\log_5 \sqrt[3]{5}$
104. $\log_2 \sqrt{2}$
105. $\log_7 \left(\frac{1}{\sqrt{7}}\right)$
106. $\log_9 \left(\frac{1}{\sqrt[3]{9}}\right)$
107. $\log_{1/9} 1$
108. $\log_{1/3} 27$
109. $\log_{1/2} 4$
110. $\log_{\frac{1}{2}} 2$
111. $\log_{\frac{1}{2}} 4$
112. $\log_{3/4} (\frac{9}{16})$
113. $\log_{2/3} (\frac{2}{3})$
114. $\log_{27} (\frac{1}{3})$
115. $\log_{3/5} (\frac{5}{3})$

Answers to odd exercises:

79. 2
81. 0
83. 2
85. $\frac{1}{2}$
87. $\frac{1}{3}$
89. 10
91. 4
93. -4
95. -3
97. $-\frac{1}{2}$
101. -2
103. $\frac{1}{3}$
106. $-\frac{1}{2}$
107. 0
109. -2
111. -2
113. 1
115. -1

★

E: Evaluate common and natural logarithms without a calculator

Exercise 4.3e.E

★ **For the following exercises, use the definition of common and natural logarithms to evaluate without using a calculator.**

117. $\log 1000$	121. $\log 0.1$	125) $2\log(100^{-3})$	129. $\ln\left(\frac{1}{e}\right)$	133. $\ln e^4$
118. $\log 100$	122) $2\log(.0001)$	126) $10^{\log(32)}$	130. $\ln\left(\frac{1}{e^5}\right)$	134) $\ln(e^{-5.03})$
119) $\log(10,000)$	123) $\log(0.001)$	127) $e^{\ln(1.06)}$	131) $25\ln\left(e^{\frac{2}{5}}\right)$	135) $\ln(1)$
120) $\log(100^8)$	124) $\log(1)+7$	128) $e^{\ln(10.125)}+4$	132) $\ln\left(e^{\frac{1}{3}}\right)$	136) $\ln\left(e^{-0.225}\right)-3$

Answers to odd exercises:

117. 3	12`. -1	125. -12	129. -1	133. 4
119. 4	123. -3	127. 1.06	131. 10	135. 0

F: Use a calculator to evaluate logs

Exercise 4.3e.F

★ **For the following exercises, evaluate each expression using a calculator. Round to the nearest thousandth.**

139. $\log 162$	141. $\log 0.025$	143. $\ln(25)$	145. $\ln(0.125)$	147) $\ln(15)$	149) $\log(\sqrt{2})$
140. $\log e$	142. $\log 0.235$	144. $\ln(100)$	146. $\ln(0.001)$	148) $\ln\left(\frac{4}{5}\right)$	150) $\ln(\sqrt{2})$

Answers to odd exercises:

139. 2.210	141. -1.602	143. 3.219	145. -2.079	147. 2.708	149. 0.151

G: Solve log equations by converting to exponential form first

Exercise 4.3e.G

★ **In the following exercises, find the value of x in each logarithmic equation without using a calculator by first converting the logarithmic equation to exponential form.**

151) $\log_2(x)=-3$	159. $\log_3(x)=4$	167) $\log_9(x)=\frac{1}{2}$	175. $\log_{1/4}(x)=-2$	183. $\log_{1/8}\left(\frac{1}{64}\right)=x$
152. $\log_2(x)=-6$	160. $\log_3\left(\frac{1}{27}\right)=x$	168) $\log_{18}(x)=2$	176. $\log_{2/5}(x)=2$	184. $\log_{4/9}\left(\frac{2}{3}\right)=x$
153) $\log_2(x)=6$	161. $\log_5(x)=-3$	169. $\log_{12}(x)=0$	177. $\log_{1/9}(x)=\frac{1}{2}$	185. $\log_{\frac{1}{3}}\frac{1}{9}=x$
154. $\log_2(x)=5$	162) $\log_5(x)=2$	170) $\log(x)=3$	178. $\log_{1/4}(x)=\frac{3}{2}$	186. $\log_{\frac{1}{4}}\frac{1}{16}=x$
155. $\log_2(x)=8$	163. $\log_5(x)=3$	171) $\ln(x)=2$	179. $\log_{1/3}(x)=-1$	187. $\log_{\frac{1}{9}}81=x$
156. $\log_2\sqrt[5]{2}=x$	164. $\log_7(x)=-1$	172. $\ln(x)=9$	180. $\log_{1/5}(x)=0$	188. $\log_{\frac{1}{4}}64=x$
157) $\log_3(x)=3$	165) $\log_6(x)=-3$	173. $\ln(x)=\frac{1}{5}$	181. $\log 10^{12}=x$	189. $\log_x 121=2$
158. $\log_3(x)=-5$	166. $\log_6(x)=-2$		182. $\ln e=x$	190. $\log_x 49=2$
				191. $\log_x 64=3$
				192. $\log_x 27=3$

Answers to odd exercises:

151. $x=2^{-3}=\frac{1}{8}$	159. 81	167. $x=9^{\frac{1}{2}}=3$	175. 16	183. 2
153. 64	161. $\frac{1}{125}$	168. 1	177. $\frac{1}{3}$	185. $x=2$
155. 256	163. $x=125$	171. $x=e^2$	179. 3	187. $x=-2$
157. $x=3^3=27$	165. $x=6^{-3}=\frac{1}{216}$	173. $\sqrt[5]{e}$	181. 12	189. $x=11$
				191. $x=4$

★

H: Solve log equations by converting then using a calculator

Exercise 4.3e.H

★ **Find x. Round off to the nearest hundredth.**

195. $\log x = 2.5$ 196. $\log x = 1.8$	197. $\log x = -1.22$ 198. $\log x = -0.8$	199. $\ln x = 3.1$ 200. $\ln x = 1.01$	201. $\ln x = -0.69$ 202. $\ln x = -1$

Answers to odd exercises:

195. **316.23**	197. **0.06**	199. **22.20**	201. **0.50**

I: Extensions

Exercise 4.3e.I

203) Is there a number x such that $\ln x = 2$? If so, what is that number? Verify the result.

204) Is the following true: $\frac{\log_3(27)}{\log_4\left(\frac{1}{64}\right)} = -1$? Verify the result.

205) Is the following true: $\frac{\ln(e^{1.725})}{\ln(1)} = 1.725$ Verify the result.

206) The exposure index EI for a 35 millimeter camera is a measurement of the amount of light that hits the film. It is determined by the equation $EI = \log_2\left(\frac{f^2}{t}\right)$ where f is the "f-stop" setting on the camera, and t is the exposure time in seconds. Suppose the f-stop setting is 8 and the desired exposure time is 2seconds. What will the resulting exposure index be?

207) Refer to the previous exercise. Suppose the light meter on a camera indicates an EI of -2 , and the desired exposure time is 16 seconds. What should the f-stop setting be?

208) The intensity levels I of two earthquakes measured on a seismograph can be compared by the formula $\log\left(\frac{I_1}{I_2}\right) = M_1 - M_2$ where M is the magnitude given by the Richter Scale. In August 2009, an earthquake of magnitude 6.1 hit Honshu, Japan. In March 2011, that same region experienced yet another, more devastating earthquake, this time with a magnitude of 9.0. How many times greater was the intensity of the 2011 earthquake? Round to the nearest whole number.

Answers to odd exercises:

203. Yes. Suppose there exists a real number x such that $\ln x = 2$. Rewriting as an exponential equation gives $x = e^2$ which is a real number. To verify, let $x = e^2$. Then, by definition, $\ln(x) = \ln\left(e^2\right) = 2$.

205. No; $\ln(1) = 0$, so $\frac{\ln(e^{1.725})}{\ln(1)} = 1.725$ is undefined.

207. 2

★

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4.4: Graphs of Logarithmic Functions

Learning Objectives

- Graph basic logarithmic functions and transformations of those functions
- Algebraically find the domain and vertical asymptote of a logarithmic function
- Find an equation of a logarithmic function given its graph

In a previous section, it was shown how creating a graphical representation of an exponential model provides some insight in predicting future events. Logarithmic graphs provide similar insight but in reverse because every logarithmic function is the inverse of an exponential function. This section illustrates how logarithm functions can be graphed, and for what values a logarithmic function is defined.

Graphs of Basic Logarithmic Functions

To graph a logarithmic function $y = log_b(x)$, it is easiest to convert the equation to its exponential form, $x = b^y$. Generally, when graphing a function, various x-values are chosen and each is used to calculate the corresponding y-value. In contrast, for this method, it is the y-values that are chosen and the corresponding x-values that are then calculated.

Example 4.4.1

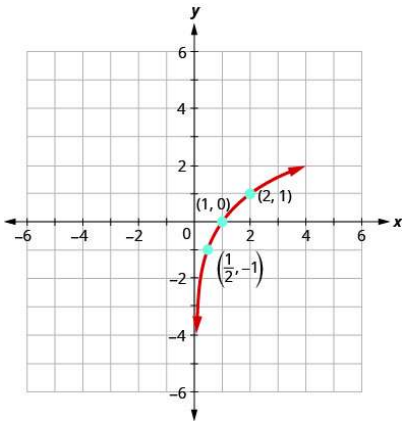
Graph $y = \log_2(x)$.

Solution:

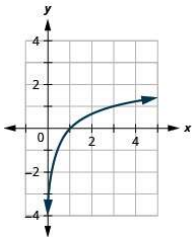
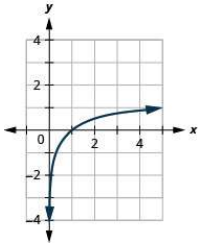
To graph the function, we will first rewrite the logarithmic equation, $y = \log_2(x)$, in exponential form, $2^y = x$.

We will use point plotting to graph the function. It will be easier to start with values of y and then get x.

y	$2^y = x$	(x, y)
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$(\frac{1}{4}, 2)$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$(\frac{1}{2}, -1)$
0	$2^0 = 1$	$(1, 0)$
1	$2^1 = 2$	$(2, 1)$
2	$2^2 = 4$	$(4, 2)$
3	$2^3 = 8$	$(8, 3)$



Try It 4.4.1

(a) Graph: $y = \log_3(x)$.	**Answer (a)**	(b) Graph: $y = \log_5(x)$.	**Answer (b)**
			

The graphs of $y = \log_2(x), y = \log_3(x)$, and $y = \log_5(x)$ (all log functions with $b > 1$), are similar in shape and also:

- All graphs contains the key point $(1, 0)$ because $0 = log_b(1)$ means $b^0 = (1)$ which is true for any b.
- All graphs contains the key point $(b, 1)$ because $1 = \log_b(b)$ means $b^1 = (b)$ which is true for any b.
- All graphs contain the key point $\left(\frac{1}{b}, -1\right)$ because $-1 = \log_b(\frac{1}{b})$ means $b^{-1} = (\frac{1}{b})$, which is true for any b.
- All graphs approach the y-axis very closely but never touch it. This line $x = 0$, the y-axis, is a vertical asymptote.
- The graphs never touch the y-axis so the domain is all positive numbers, written $(0, \infty)$ in interval notation.
- All the graphs have the same range - the set of all real numbers, written in interval notation as $(-\infty, \infty)$.

Our next example looks at the graph of $y = \log_b(x)$ when $0 < b < 1$.

Example 4.4.2

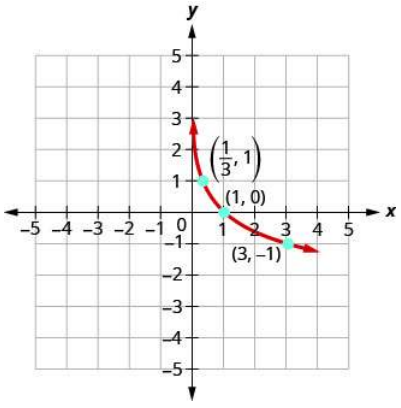
Graph $y = \log_{\frac{1}{3}}(x)$.

Solution:

To graph the function, we will first rewrite the logarithmic equation, $y = \log_{\frac{1}{3}}(x)$, in exponential form, $(\frac{1}{3})^y = x$.

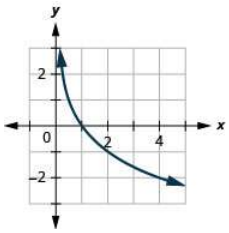
We will use point plotting to graph the function. It will be easier to start with values of y and then get x.

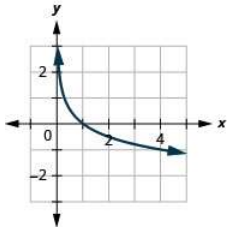
y	$(\frac{1}{3})^y = x$	(x, y)
-2	$(\frac{1}{3})^{-2} = 3^2 = 9$	$(9, -2)$
-1	$(\frac{1}{3})^{-1} = 3^1 = 3$	$(3, -1)$
0	$(\frac{1}{3})^0 = 1$	$(1, 0)$
1	$(\frac{1}{3})^1 = \frac{1}{3}$	$(\frac{1}{3}, 1)$
2	$(\frac{1}{3})^2 = \frac{1}{9}$	$(\frac{1}{9}, 2)$
3	$(\frac{1}{3})^3 = \frac{1}{27}$	$(\frac{1}{27}, 3)$



Try It 4.4.2

(a) Graph: $y = \log_{\frac{1}{2}}(x)$.	**Answer (a)**	(b) Graph: $y = \log_{\frac{1}{4}}(x)$.	**Answer (b)**





The graphs of $y = \log_{\frac{1}{2}}(x), y = \log_{\frac{1}{3}}(x)$ and $y = \log_{\frac{1}{4}}(x)$ are similar.

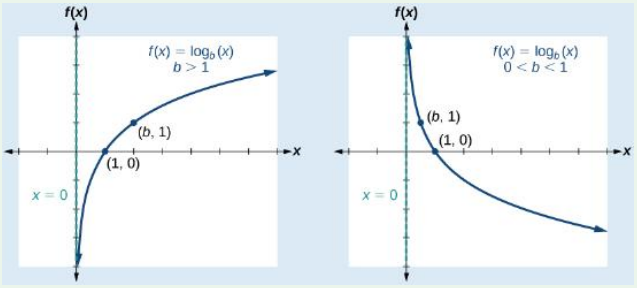
- The graphs of all have the same basic shape. This is because all the log functions have a fractional base $0 < b < 1$.
- All graphs contain the vertical asymptote $x = 0$ and key points $(1, 0),\ (b, 1),\ (\frac{1}{b}, -1)$, just like when $b > 1$.
- The domain and range are also the same as when $b > 1$. The domain is $(0, \infty)$, the range is $(-\infty, \infty)$ and the y-axis is the vertical asymptote.

We summarize these properties in the chart below.

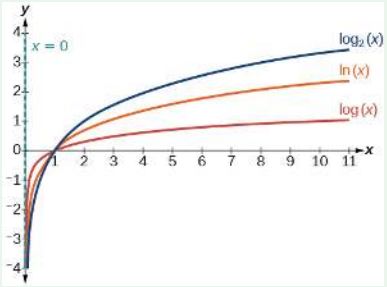
CHARACTERISTICS OF THE GRAPH OF THE PARENT FUNCTION, $f(x) = log_b(x)$

For any real number x and constant $b > 0$, $b \neq 1$, we can see the following characteristics in the graph of $f(x) = \log_b(x)$:

- one-to-one function
- vertical asymptote: $x = 0$
- key points: x-intercept: $(1, 0),\ (b, 1)$ and $(\frac{1}{b}, -1)$
- y-intercept: none
- domain: $(0, \infty)$
- range: $(-\infty, \infty)$
- increasing if $b > 1$
- decreasing if $0 < b < 1$



The diagram on the right illustrates the graphs of three logarithmic functions with different bases, all greater than 1. It shows how changing the base b in $f(x) = \log_b(x)$ can affect the graphs. Observe that the graphs compress vertically as the value of the base increases. (Note: recall that the function $\ln(x)$ has base $e \approx 2.718$.)



The family of logarithmic functions includes the parent function $y = \log_b(x)$ along with all its transformations: shifts, stretches, compressions, and reflections. When graphing transformations, we always begin with graphing the parent function $y = \log_b(x)$. Below is a summary of how to graph parent log functions.

How to: Graph the parent logarithmic function $f(x) = \log_b(x)$.

1. Graph the landmarks of the logarithmic function
   - Draw and label the vertical asymptote, $x = 0$.

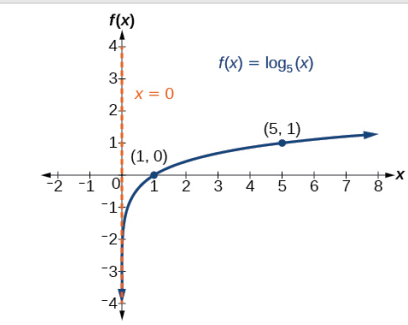
- Plot the keypoints: the x-intercept, $(1,0)$, $(b,1)$, and $\left(\frac{1}{b},-1\right)$

2. Obtain additional points if they are needed by rewriting $f(x)=\log_b x$ in exponential form as $b^y=x$. Choose small y values (like 2, 3 and -1), calculate the corresponding value for x, and plot the point on the graph.
3. Draw a smooth curve through the points.
4. State the domain, $(0,\infty)$, the range, $(-\infty,\infty)$, and the vertical asymptote, $x=0$.

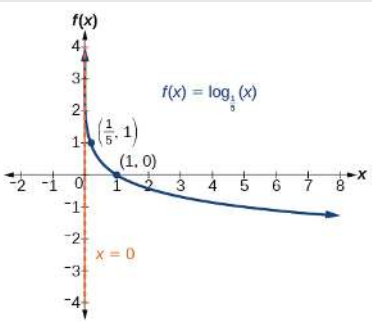
Try It 4.4.3

Graph $f(x)=\log_5(x)$ and $f(x)=\log_{\frac{1}{5}}(x)$. State the domain, range, and asymptote.

Answer



Landmarks are the vertical asymptote $x=0$ and key points $(1,0)$, $(5,1)$, and $\left(\frac{1}{5},-1\right)$. The domain is $(0,\infty)$, the range is $(-\infty,\infty)$, and the vertical asymptote is $x=0$.



Landmarks are the vertical asymptote $x=0$ and key points $(1,0)$, $\left(\frac{1}{5},-1\right)$ and $(5,1)$. The domain is $(0,\infty)$, the range is $(-\infty,\infty)$, and the vertical asymptote is $x=0$.

Graph Transformations of Logarithmic Functions

Transformations of logarithmic graphs behave similarly to those of other parent functions. We can shift, stretch, compress, and reflect the parent function $y=\log_b(x)$ without loss of basic shape. The general outline of the process appears below. Then illustrations of each type of transformation are described in detail. Finally, a summary of the steps involved in graphing a function with multiple transformations appears at the end of this section.

How to: Graph a logarithmic function $f(x)$ using transformations.

1. Determine the parent function of $f(x)$ and graph the parent function $y=\log_b(x)$ and its asymptote.
2. Identify the transformations on the graph of y needed to obtain the graph of $f(x)$.
3. Use transformations to graph $f(x)$ and its asymptote.

Vertical Shifts

When a constant d is added to the parent function $f(x)=\log_b(x)$, the result is a vertical shift d units in the direction of the sign on d. To visualize vertical shifts, we can observe the general graph of the parent function $f(x)=\log_b(x)$ alongside the shift up, $g(x)=\log_b(x)+d$ and the shift down, $h(x)=\log_b(x)-d$.

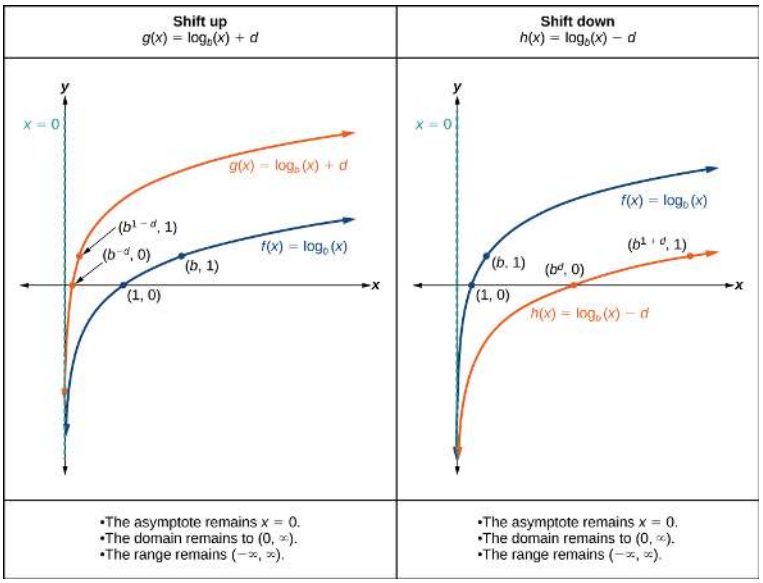


Figure 4.4.4

VERTICAL SHIFTS OF THE PARENT FUNCTION $y = \log_b(x)$

For any constant d, the function $f(x) = \log_b(x) + d$

- shifts the parent function $y = \log_b(x)$ up d units if $d > 0$.
- shifts the parent function $y = \log_b(x)$ down d units if $d < 0$.

The new y coordinates are equal to $y + d$.

Example 4.4.4: Graph a Vertical Shift of the Parent Function $y = \log_b(x)$

Sketch a graph of $f(x) = \log_3(x) - 2$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution

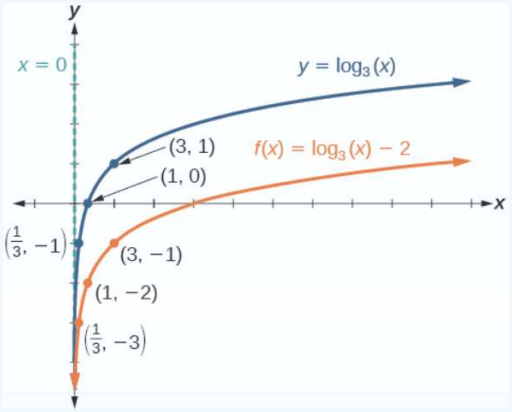
Step 1. Graph the parent function $y = \log_3(x)$. Landmarks are: vertical asymptote $x = 0$, and key points: x-intercept $(1, 0)$, $(3, 1)$ and $(\frac{1}{3}, -1)$. Additional points using $3^y = x$ are $(9, 2)$ and $(27, 3)$.

Step 2. Transformation on the graph of y needed to obtain the graph of $f(x)$ is: shift down 2 units.

Step 3. Shifting down 2 units means the new y coordinates are found by subtracting 2 from the old y coordinates. Therefore,

- The vertical asymptote for the translated function f is still $x = 0$.
- The key points for the translated function f are $(1, -2)$, $(3, -1)$, and $(\frac{1}{3}, -3)$. Additional points are $9, 0)$ and $27, 1)$.

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.



Try It 4.4.4

Sketch a graph of $f(x) = \log_2(x) + 2$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

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Answer

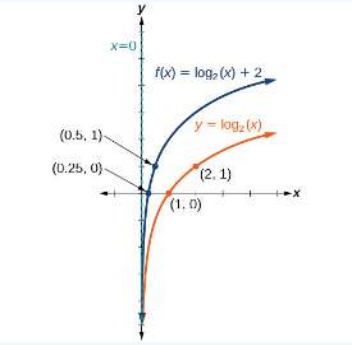

Figure **4.4.4***b* A graph of $f(x) = \log_2(x) + 2$

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

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Horizontal Shifts

When a constant c is added to the input of the parent function $f(x) = \log_b(x)$, the result is a horizontal shift c units in the *opposite* direction of the sign on c. To visualize horizontal shifts, we can observe the general graph of the parent function $f(x) = \log_b(x)$ and for $c > 0$ alongside the shift left, $g(x) = \log_b(x + c)$, and the shift right, $h(x) = \log_b(x - c)$. See Figure 4.4.5.

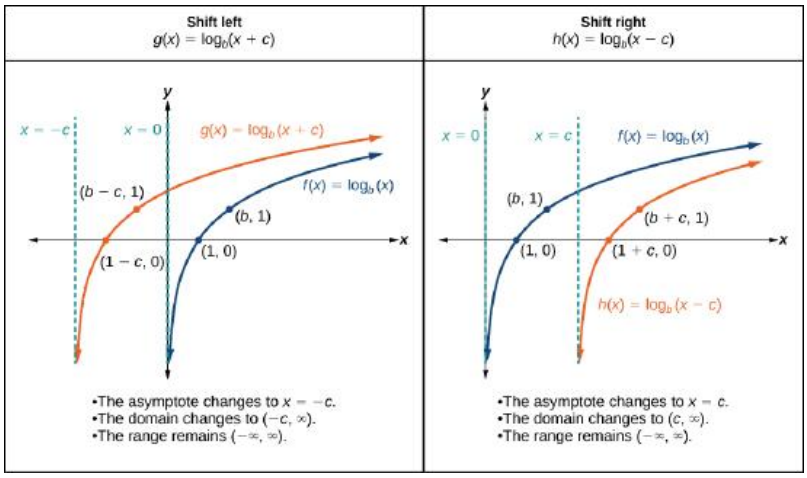


Figure 4.4.5

HORIZONTAL SHIFTS OF THE PARENT FUNCTION $y = \log_b(x)$

For any constant c, the function $f(x) = \log_b(x + c)$

- shifts the parent function $y = \log_b(x)$ left c units if $c > 0$.
- shifts the parent function $y = \log_b(x)$ right c units if $c < 0$.

The new x coordinates are equal to $x - c$.

Example 4.4.5: Graph a Horizontal Shift of the Parent Function $y = \log_b(x)$

Sketch the horizontal shift $f(x) = \log_3(x - 2)$ alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

Solution

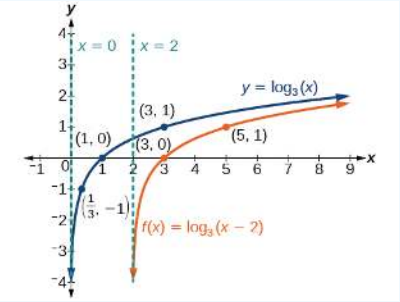
Step 1. Graph the parent function $y = \log_3(x)$. Landmarks are: vertical asymptote $x = 0$, and key points: x-intercept, $(1, 0)$, $(3, 1)$ and $(\frac{1}{3}, -1)$

Step 2. Transformation on the graph of y needed to obtain the graph of $f(x)$ is: shift right 2 units.

Step 3. Shifting right 2 units means the new x coordinates are found by adding 2 to the old x coordinates. Therefore,

- The vertical asymptote for the translated function f is $x = 0 + 2)$ or $x = 2$.
- The key points for the translated function f are $(3, 0)$, $(5, 1)$, and $(\frac{7}{3}, -1)$.

The domain is $(2, \infty)$, the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 2$.



Try It 4.4.5

Sketch a graph of $f(x) = \log_3(x + 4)$ alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

Answer

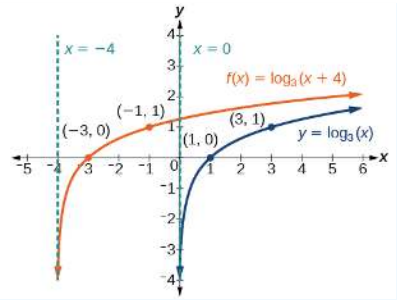
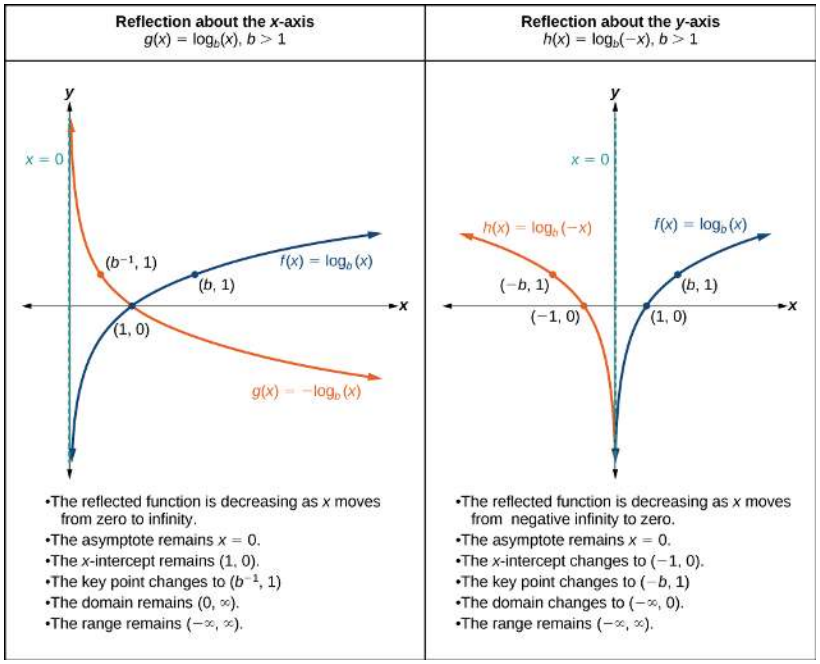


Figure 4.4.5b A graph of $f(x) = \log_3(x + 4)$

The domain is $(-4, \infty)$, the range $(-\infty, \infty)$, and the asymptote $x = -4$.

Reflections

When the parent function $f(x) = \log_b(x)$ is multiplied by -1,the result is a reflection about the x-axis. When the *input* is multiplied by -1, the result is a reflection about the y-axis. To visualize reflections, we restrict $b > 1$, and observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the reflection about the x-axis, $g(x) = -\log_b(x)$ and the reflection about the y-axis, $h(x) = \log_b(-x)$.



REFLECTIONS OF THE PARENT FUNCTION $y = log_b(x)$

$f(x) = -\log_b(x)$ reflects the parent function about the x-axis. Domain, range and vertical asymptote are unchanged.

$f(x) = \log_b(-x)$ reflects the parent function about the y-axis. Domain is changed.

Example 4.4.6: Graphing a Reflection of a Logarithmic Function

Sketch a graph of $f(x) = \log(-x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution

Step 1. Graph the parent function $y = \log(x)$. Landmarks are: vertical asymptote $x = 0$, and key points: $\left(\frac{1}{10}, -1\right)$, $(1, 0)$, and $(10, 1)$.

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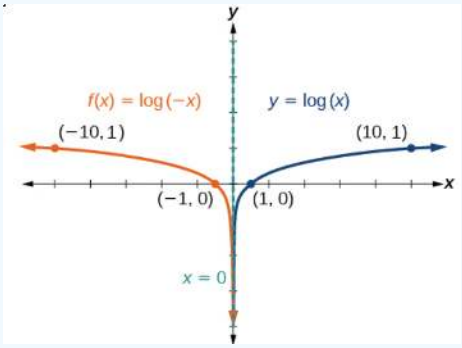
Step 2. Transformation on the graph of y needed to obtain the graph of $f(x)$ is: reflection of the parent graph about the y-axis.

Step 3. The reflection about the y-axis is accomplished by multiplying all the x-coordinates by −1. Therefore,

- The vertical asymptote for the translated function f remains $x = 0$.
- The key points for the translated function f are $(-\frac{1}{10}, -1)$, $(-1, 0)$, and $(-10, 1)$.

The domain is $(-\infty, 0)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.



Try It 4.4.6

Graph $f(x) = -\log(-x)$. State the domain, range, and asymptote.

Answer

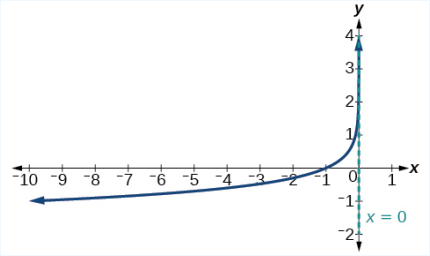


Figure 4.4.6*b* Graph of $f(x) = -\log(-x)$.

The domain is $(-\infty, 0)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Vertical Stretches and Compressions

When the parent function $f(x) = \log_b(x)$ is multiplied by a constant $a > 0$, the result is a vertical stretch or compression of the original graph. To visualize stretches and compressions, we set $a > 1$ and observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the vertical stretch, $g(x) = a\log_b(x)$ and the vertical compression, $h(x) = \frac{1}{a}\log_b(x)$.

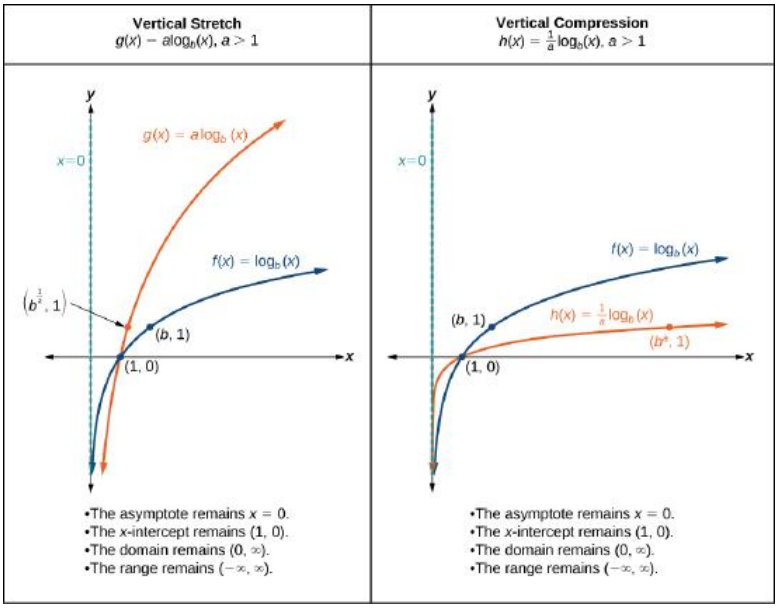


Figure 4.4.7

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VERTICAL STRETCHES AND COMPRESSIONS OF THE PARENT FUNCTION $y = log_b(x)$

For any constant $a \neq 0$, the function $f(x) = a\log_b(x)$

- stretches the parent function $y = \log_b(x)$ vertically by a factor of a if $|a| > 1$.
- compresses the parent function $y = \log_b(x)$ vertically by a factor of a if $0 < |a| < 1$.

The new y coordinates are equal to ay. (This would also include vertical reflection if present).

Example 4.4.7: Graph a Stretch or Compression of the Parent Function $y = log_b(x)$

Sketch a graph of $f(x) = 2\log_4(x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution

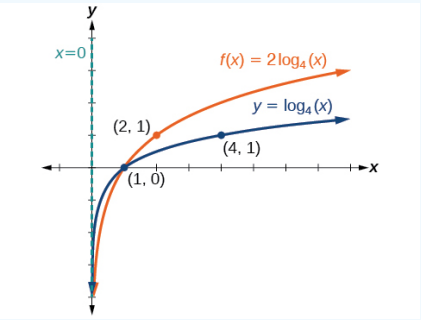
Step 1. Graph the parent function $y = \log_4(x)$. Landmarks are: vertical asymptote $x = 0$, and key points: $\left(\frac{1}{4}, -1\right)$, $(1, 0)$, and $(4, 1)$.

Step 2. Transformation on the graph of y needed to obtain the graph of $f(x)$ is: stretch the function $f(x) = \log_4(x)$ by a factor of 2.

Step 3. A vertical stretch by a factor of 2 means the new y coordinates are found by multiplying the y coordinates by 2. Therefore,

- The vertical asymptote for the translated function f is still $x = 0$.
- The key points for the translated function f are $\left(\frac{1}{4}, -2\right)$, $(1, 0)$, and $(4, 2)$.

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.



Try It 4.4.7

Sketch a graph of $f(x) = \frac{1}{2}\log_4(x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Answer

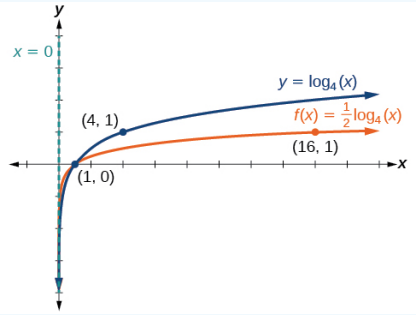


Figure 4.4.7b A graph of $f(x) = \frac{1}{2}\log_4(x)$

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Horizontal Stretches and Compressions

When the *input* of the parent function $f(x) = \log_b(x)$ is multiplied by m, the result is a stretch or compression of the original graph. To visualize horizontal stretches and compressions, compare the graph of the parent function $f(x) = \log_b(x)$ with the graph of $g(x) = \log_b(mx)$. If $f(p) = \log_b(p) = q$, then in order to obtain the same y value for g, the argument in g must be equal to that of f. Therefore, the argument on g must be $\frac{p}{m}$ because then $g(\frac{p}{m} = \log_b(m\frac{p}{m}) = \log_b(p) = q$. Thus in order for g to have the same output value as f, the input to g must be the original input value to f, multiplied by the factor $\frac{1}{m}$.

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VERTICAL STRETCHES AND COMPRESSIONS OF THE PARENT FUNCTION $y = log_b(x)$

For any constant $m \neq 0$, the function $f(x) = \log_b(mx)$

- compresses the parent function $y = \log_b(x)$ vertically by a factor of $\frac{1}{m}$ if $|m| > 1$.
- stretches the parent function $y = \log_b(x)$ vertically by a factor of $\frac{1}{m}$ if $0 < |m| < 1$.

The new x coordinates are equal to $\frac{1}{m}x$. (This would also include horizontal reflection if present).

Example 4.4.8: Graph a Stretch or Compression of the Parent Function $y = log_b(x)$

Sketch a graph of $f(x) = \log_2(4x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

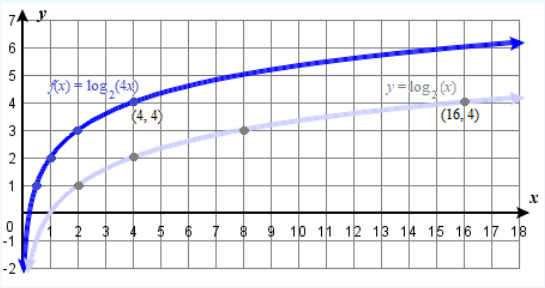
Solution

Step 1. Graph the parent function $y = \log_2(x)$. Landmarks are: vertical asymptote $x = 0$, and key points: $(1, 0)$, $(4, 2)$, and $(16, 4)$.

Step 2. Transformation on the graph of y needed to obtain the graph of $f(x)$ is: horizontally shrink the function $f(x) = \log_2(x)$ by a factor of $\frac{1}{4}$.

Step 3. A vertical stretch by a factor of $\frac{1}{4}$ means the new x coordinates are found by multiplying the x coordinates by $\frac{1}{4}$. Therefore,

- The vertical asymptote for the translated function f is still $x = 0$.
- The key points for the translated function f are $(\frac{1}{4}, 0)$, $(1, 2)$, and $(4, 4)$.



The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Try It 4.4.8

Sketch a graph of $f(x) = \log_2(\frac{1}{4}x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Answer

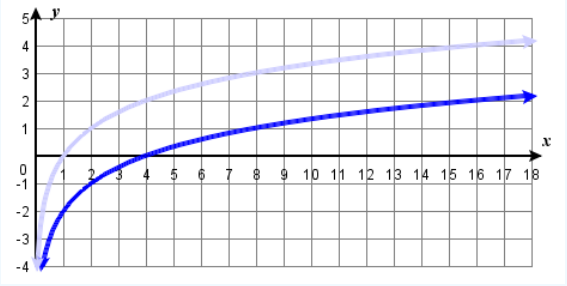


Figure 4.4.8b A graph of $f(x) = \log_2(\frac{1}{4}x)$

Transformation: $x \to 4x$. Some key points of graph of f include $(4, 0)$, $(8, 1)$, and $(16, 2)$.
The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Combine a Horizontal Shift and a Vertical Stretch

Example 4.4.9: Combine a Shift and a Stretch

Sketch a graph of $f(x) = 5\log(x+2)$. State the domain, range, and asymptote.

Solution

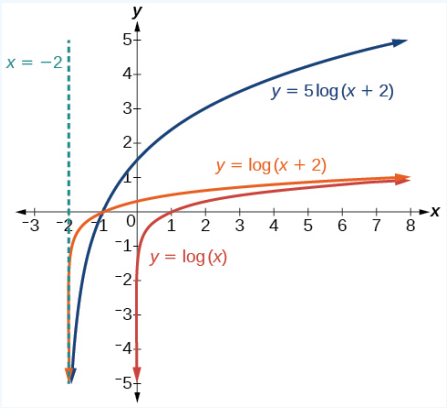
Step 1. Graph the parent function is $y = \log(x)$. Landmarks are: vertical asymptote $x = 0$, and key points: $\left(\frac{1}{10}, -1\right)$, $(1, 0)$, and $(10, 1)$.

Step 2. Transformations on the graph of y needed to obtain the graph of $f(x)$ are: move left 2 units (subtract 2 from all the x-coordinates), then vertically stretch by a factor of 5 (multiply all y-coordinates by 5). (Since these two transformations operate perpendicularly to each other, the order they are done does not matter, but it is a good idea to do all transformations in a prescribed order in order to establish a routine that will always work).

Step 3. 1. $x \to x - 2$, 2. $y \to 5y$. Therefore,

- The vertical asymptote for the translated function f will be shifted to $x = -2$.
- The key points for the translated function f are $\left(-1\frac{9}{10}, -5\right)$, $(-1, 0)$, and $(8, 5)$.

The domain is $(-2, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = -2$.



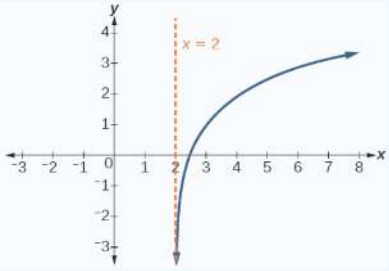
Try It 4.4.9

Sketch a graph of the function $f(x) = 3\log(x-2) + 1$. State the domain, range, and asymptote.

Answer

Graph of the function $f(x) = 3\log(x-2) + 1$ is on the right.

The domain is $(2, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 2$.



Summary of Transformations of the Logarithmic Function

Now that we have worked with each type of translation for the logarithmic function, we can summarize how to graph logarithmic functions that have undergone multiple transformations of their parent function.

How To: Graph a logarithmic function using translations

Transformations of the parent logarithmic function, $y = \log_b(x)$, have the form $f(x) = a\log_b(mx + c) + d$

- Horizontal transformations must be done in a particular order
  1. **FIRST**, shift horizontally to the left c units if $c > 0$ or to the right if $c < 0$.
  2. Then, if the coefficient of x is negative, the graph of the parent function is reflected about the y-axis.
  3. If $m \neq 1$ then the graph if stretched or shrunk horizontally by a factor of $\frac{1}{m}$.
     - If p is the x-coordinate of a point on the parent graph, then its new value is $\frac{(p-c)}{m}$
     - If the function has the form $f(x) = a\log_b(m(x + c)) + d$ then do the stretching or reflecting **FIRST**, and then the horizontal shift.

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- Vertical transformations must be done in a particular order
  1. First, stretching or compression and reflection about the x-axis is done
     - stretched vertically by a factor of $|a|$ if $|a| > 0$.
     - compressed vertically by a factor of $|a|$ if $0 < |a| < 1$.
     - reflected about the x-axis when $a < 0$.
  2. **LAST**, shift vertically up d units if $d > 0$ or down if $d < 0$.
  - If p is the y-coordinate of a point on the parent graph, then its new value is $ap + d$
- The range is always $(-\infty, \infty)$
  - If the coefficient of x was positive, the domain is $(-c, \infty)$, and the vertical asymptote is $x = -c$.
  - If the coefficient of x was negative, the domain is $(-\infty, c)$, and the vertical asymptote is $x = c$.

Find the Domain and Asymptote of a Logarithmic Function

Previously, the domain and vertical asymptote were determined by graphing a logarithmic function. It is also possible to determine the domain and vertical asymptote of any logarithmic function algebraically. Here we will take a look at the domain (the set of input values) for which the logarithmic function is defined, and its vertical asymptote.

Recall that the exponential function is defined as $y = b^x$ for any real number x and constant $b > 0$, $b \neq 1$, where

- The domain of y is $(-\infty, \infty)$.
- The range of y is $(0, \infty)$.

In the last section we learned that the logarithmic function $y = \log_b(x)$ is the inverse of the exponential function $y = b^x$. So, as inverse functions:

- The domain of $y = \log_b(x)$ is the range of $y = b^x$: $(0, \infty)$.
- The range of $y = \log_b(x)$ is the domain of $y = b^x$: $(-\infty, \infty)$.

When exponential functions are graphed, certain transformations can change the *range* of $y = b^x$. Similarly, applying transformations to the parent function $y = \log_b(x)$ can change the *domain*. When finding the domain of a logarithmic function, therefore, it is important to remember that the domain consists *only of positive real numbers*. That is, **the argument of the logarithmic function must be greater than zero**.

For example, consider $f(x) = \log_4(2x - 3)$. This function is defined for any values of x such that the argument, in this case $2x - 3$,is greater than zero. To find the domain, we set up an inequality and solve for x:

$$\begin{aligned} 2x - 3 &> 0 && \text{Show the argument greater than zero} \\ 2x &> 3 && \text{Add 3} \\ x &> 1.5 && \text{Divide by 2} \end{aligned}$$

In interval notation, the domain of $f(x) = \log_4(2x - 3)$ is $(1.5, \infty)$.

How To: Given a logarithmic function, identify the domain

1. Set up an inequality showing the argument greater than zero.
2. Solve for x.
3. Write the domain in interval notation.

The location of the asymptote of a logarithmic equation is always **at the boundary of its domain**. Therefore the vertical asymptote of a logarithmic function can be obtained by setting its argument to zero and solving for x.

How To: Given a logarithmic function, find the vertical asymptote algebraically

1. Set up an inequality showing the argument of the logarithmic function equal to zero.
2. Solve for x.

3. The result is the equation for the logarithmic function's vertical asymptote.

Example 4.4.10: Identifying the Domain of a Logarithmic Shift

What is the domain of $f(x) = \log_2(x+3)$?

Solution

The logarithmic function is defined only when the input is positive, so this function is defined when $x+3>0$. Solving this inequality,

$$\begin{aligned} x+3 &> 0 && \text{The input must be positive} \\ x &> -3 && \text{Subtract 3} \end{aligned}$$

The domain of $f(x) = \log_2(x+3)$ is $(-3,\infty)$. The vertical asymptote is $x=-3$

Try It 4.4.10

What is the domain of $f(x) = \log_5(x-2)+1$?

Answer

Domain is $(2,\infty)$. The vertical asymptote is $x=2$.

Example 4.4.11: Identifying the Domain of a Logarithmic Shift and Reflection

What is the domain of $f(x) = \log(5-2x)$? What is the equation for its vertical asymptote?

Solution

The logarithmic function is defined only when the input is positive, so this function is defined when $5–2x>0$. Solving this inequality,

$$\begin{aligned} 5-2x &> 0 && \text{The input must be positive} \\ -2x &> -5 && \text{Subtract 5} \\ x &< \frac{5}{2} && \text{Divide by -2 and switch the inequality} \end{aligned}$$

The domain of $f(x) = \log(5-2x)$ is $\left(-\infty, \frac{5}{2}\right)$. The vertical asymptote is $x=\frac{5}{2}$.

Try It 4.4.11

What is the domain of $f(x) = \log(x-5)+2$? What is the equation for its vertical asymptote?

Answer

$(5,\infty)$ The vertical asymptote is $x=5$.

Example 4.4.12: Finding the Vertical Asymptote of a Logarithm Graph

What is the vertical asymptote of $f(x) = -2\log_3(x+4)+5$?

Solution

The vertical asymptote is at $x=-4$.

Analysis

The coefficient, the base, and the upward translation do not affect the asymptote. The shift of the curve 4 units to the left shifts the vertical asymptote to$x=-4$.

Try It 4.4.12

What is the vertical asymptote of $f(x) = 3 + \ln(x - 1)$?

Answer

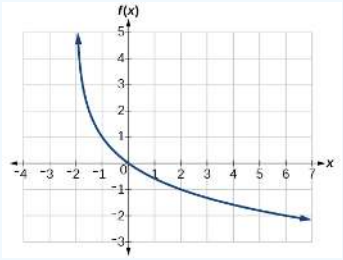
$x = 1$

Find the Equation of a Logarithmic Function given its Graph

In the discussion of transformations, a factor that contributes to horizontal stretching or shrinking was included. However, it is always possible to construct an equivalent equation for a transformation of a logarithmic equation that does not have a horizontal stretching of shrinking component to it. Therefore, when constructing an equation that corresponds to a graph, it is safe to assume the general logarithmic equation to be found is of the form $f(x) = a\log_b(x + c) + d$ or $f(x) = a\log_b(-x + c) + d$.

Example 4.4.13: Finding the Equation from a Graph

Find a possible equation for the common logarithmic function graphed below.



Solution

Two slightly different approaches will be givene here. The general form of the common logarithmic function is $f(x) = a\log(\pm x + c) + d$, or if a base B logarithm is used instead, the general form would be $f(x) = a\log_B(\pm x + c) + d$.

This graph has a vertical asymptote at $x = -2$ and has not been horizontally reflected. Therefore the argument of the logarithmic function must be $(x + 2)$. The graph has been vertically reflected so we know the parameter a is negative. We do not know yet the vertical shift or the vertical stretch. Thus, so far we know that the equation will have form:

$$f(x) = -a\log(x + 2) + d \text{ or } f(x) = -a\log_B(x + 2) + d$$

Method 1. In this approach, the general form of the function used will be $f(x) = -a\log_B(x + 2) + d$.

Recall that $\log_B(1) = 0$. Therefore, when $x + 2 = 1$ (or when $x = -1$), then $y = d$. From the graph we see that when $x = -1$, $y = 1$. Therefore $d = 1$. Now the equation looks like

$$f(x) = -a\log_B(x + 2) + 1.$$

Recall that $\log_B(B) = 1$. Therefore, when $x + 2 = B$, $y = -a + 1$. Another point observed to be on the graph is $(2, -1)$. Substituting these values for x and y in this pair of equations, we can get values for B and a: $2 + 2 = B$ and $-1 = -a + 1$. Thus $B = 4$ and $a = 2$, and the final form of the equation is obtained:

$$f(x) = -2\log_4(x + 2) + 1.$$

Method 2. In this approach, the general form of the function used will be $f(x) = -a\log(x + 2) + d$ instead.

It appears the graph passes through the points $(-1, 1)$ and $(2, -1)$. Substituting $(-1, 1)$,

$1 = -a\log(-1 + 2) + d$	Substitute$(-1, 1)$
$1 = -a\log(1) + d$	Arithmetic
$1 = d$	Because $\log(1) = 0$

Thus the equation now looks like $f(x) = -a\log(x + 2) + 1$. Next, substituting in $(2, -1)$,

$$\begin{aligned} -1 &= -a\log(2+2)+1 && \text{Substitute } (2,-1) \\ -2 &= -a\log(4) && \text{Arithmetic} \\ a &= \frac{2}{\log(4)} && \text{Solve for a} \end{aligned}$$

Now the equation is $f(x) = -\frac{2}{\log(4)}\log(x+2)+1$. Furthermore, $\frac{\log(x+2)}{\log(4)} = \log_4(x+2)$ by The Change of Base Formula, so the equation can be written as $f(x) = -2\log_4(x+2)+1$.

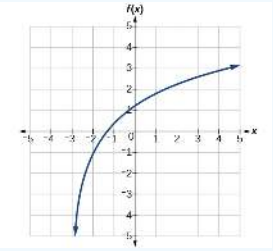
Analysis

We can verify this answer by calculating various values of our $f(x)$ and comparing with corresponding points on the graph.

x	−1	0	1	2	3	4	5	6	7	8
$f(x)$	1	0	−0.58	−1	−1.32	−1.59	−1.81	-2	-2.17	-2.32

Try It 4.4.13

Give the equation of the natural logarithm graphed below.



Answer

$f(x) = 2\ln(x+3)-1$

Key Equations

General Form for the Transformation of the Parent Logarithmic Function $f(x) = \log_b(x)$ is $f(x) = a\log_b(\pm x + c) + d$

Key Concepts

- Landmarks on the graph of the parent function $f(x) = \log_b(x)$ are: vertical asymptote $x = 0$, and keypoints x-intercept $(1, 0)$, $(b, 1)$, and $\left(\frac{1}{b}, -1\right)$. Domain is $(0, \infty)$ and range is $(-\infty, \infty)$. The function is increasing if $b > 1$; the function is decreasing if $0 < b < 1$.
- The equation $f(x) = \log_b(x + c)$ shifts the parent function $y = \log_b(x)$ horizontally: left c units if $c > 0$, right c units if $c < 0$.
- The equation $f(x) = \log_b(x) + d$ shifts the parent function $y = \log_b(x)$ vertically: up d units if $d > 0$, down d units if $d < 0$.
- For any constant $a > 0$, the equation $f(x) = a\log_b(x)$
  - stretches the parent function $y = \log_b(x)$ vertically by a factor of a if $|a| > 1$.
  - compresses the parent function $y = \log_b(x)$ vertically by a factor of a if $|a| < 1$
- The equation $f(x) = -\log_b(x)$ represents a reflection of the parent function about the x-axis. The equation $f(x) = \log_b(-x)$ represents a reflection of the parent function about the y-axis.
- To find the domain of a logarithmic function, set up an inequality showing the argument greater than zero, and solve for x. The vertical asymptote, $x = v$ is along the border of this domain.
- The general equation $f(x) = a\log_b(\pm x + c) + d$ can be used to write the equation of a logarithmic function given its graph.

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4.4e: Exercises - Graphs of Logarithmic Functions

A: Concepts

Exercise 4.4e.A: Concepts

1) The inverse of every logarithmic function is an exponential function and vice-versa. What does this tell us about the relationship between the coordinates of the points on the graphs of each?

2) What type(s) of translation(s), if any, affect the range of a logarithmic function?

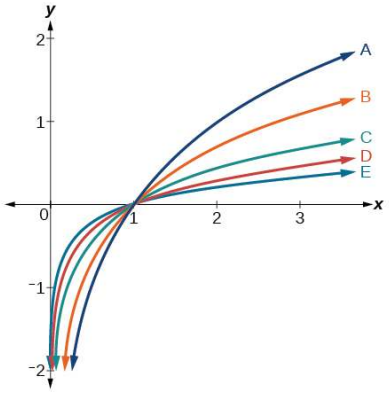
3) What type(s) of translation(s), if any, affect the domain of a logarithmic function?

4) Consider the general logarithmic function $f(x) = \log_b(x)$. Why can't x be zero?

5) Does the graph of a general logarithmic function have a horizontal asymptote? Explain.

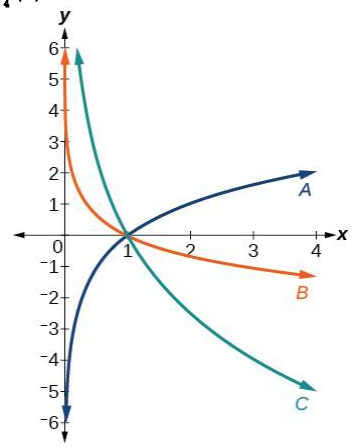
For the following exercises, match each function in the Figure below with the letter corresponding to its graph.

6) $d(x) = \log(x)$
7) $f(x) = \ln(x)$
8) $g(x) = \log_2(x)$
9) $h(x) = \log_5(x)$
10) $j(x) = \log_{25}(x)$



Graph for #6 - #10

12) $g(x) = \log_2(x)$
13) $f(x) = \log_{\frac{1}{3}}(x)$
14) $h(x) = \log_{\frac{3}{4}}(x)$



Graph for #11 - #13

Answers to odd exercises:

1. Since the functions are inverses, their graphs are mirror images about the line $y - x$ So for every point (a, b) on the graph of a logarithmic function, there is a corresponding point (b, a) on the graph of its inverse exponential function.

3. Shifting the function right or left and reflecting the function about the y-axis will affect its domain.

5. No. A horizontal asymptote would suggest a limit on the range, and the range of any logarithmic function in general form is all real numbers.

7. B, 9. C, 13. B

B: Find Domain and Range

Exercise 4.4e.B: Find Domain and Range

★ **For the following exercises, state the domain and range of the function.**

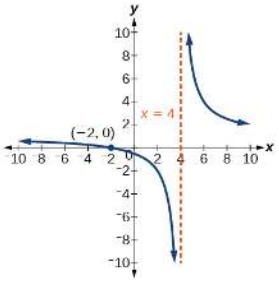
16. $f(x) = \log_3(x + 4)$
17. $h(x) = \ln\left(\frac{1}{2} - x\right)$
18. $g(x) = \log_5(2x + 9) - 2$
19. $h(x) = \ln(4x + 17) - 5$
20. $f(x) = \log_2(12 - 3x) - 3$

★ **State the domain of each function**

21. $f(x)=\ln\left(\frac{x+2}{x-4}\right)$
22.1 $f(x)=\log_2\left((x^2-1)(x+1)\right)$
22.2 $f(x)=\log_3\left((x^2-9)(x+4)\right)$
22.3 $f(x)=\log_3\left(\frac{(x+1)^2}{(x-3)}\right)$
22.4 $f(x)=\log_2\left(\frac{x}{(x-2)^2}\right)$

Answers to odd exercises:

17. Domain: $\left(-\infty,\frac{1}{2}\right)$; Range: $(-\infty,\infty)$
19. Domain: $\left(-\frac{17}{4},\infty\right)$; Range: $(-\infty,\infty)$
21. Recall that the argument of a logarithmic function must be positive, so we determine where $\frac{x+2}{x-4}>0$. From the graph of the function $f(x)=\frac{x+2}{x-4}$, note that the graph lies above the x-axis on the interval $(-\infty,-2)$ and again to the right of the vertical asymptote, that is $(4,\infty)$. Therefore, the domain is $(-\infty,-2)\cup(4,\infty)$.



Graph for #21

C: Find Asymptote

Exercise 4.4e.C: Find Asymptote

★ **For the following exercises, state the domain and the vertical asymptote of the function.**

23. $f(x)=\log_b(x-5)$
24. $g(x)=\ln(3-x)$
25. $f(x)=\log(3x+1)$
26. $f(x)=3\log(-x)+2$
27. $g(x)=-\ln(3x+9)-7$
28. $f(x)=\log(x+2)$
29. $f(x)=\log(x-5)$
31. $f(x)=\ln(3-x)$
32. $f(x)=\ln(5-x)$
33. $f(x)=\log(3x+1)$
34. $f(x)=\log(2x+5)-3$
34.1. $f(x)=2\log(-x)+1$
35. $f(x)=3\log(-x)+2$

Answers to odd exercises:

23. Domain: $(5,\infty)$; V. A. $x=5$
25. Domain: $(-\frac{1}{3},\infty)$; V. A. $x=-\frac{1}{3}$
27. Domain: $(-3,\infty)$; V. A. $x=-3$
29. Domain: $x>5$ V. A. @ $x=5$
31. Domain: $x<3$ V. A. @ $x=3$
33. Domain: $x>-\frac{1}{3}$ V. A. @ $x=-\frac{1}{3}$
35. Domain: $x<0$ V. A. @ $x=0$

D: State End Behaviour

Exercise 4.4e.D: State End Behaviour

★ **For the following exercises, state the domain, vertical asymptote, and end behavior of the function.**

36. $f(x)=\ln(2-x)$
37. $f(x)=\log\left(x-\frac{3}{7}\right)$
38. $h(x)=-\log(3x-4)+3$
39. $g(x)=\ln(2x+6)-5$
40. $f(x)=\log_3(15-5x)+6$

Answers to odd exercises:

37. Domain: $\left(\frac{3}{7},\infty\right)$, Vertical asymptote: $x=\frac{3}{7}$, End behavior: as $x\to\left(\frac{3}{7}\right)^+$, $f(x)\to-\infty$ and as $x\to\infty, f(x)\to\infty$
39. Domain: $(-3,\infty)$, Vertical asymptote: $x=-3$, End behavior: as $x\to-3^+$, $f(x)\to-\infty$ and as $x\to\infty, f(x)\to\infty$

E: Find Intercepts

Exercise 4.4e.E: Find Intercepts

★ For the following exercises, state (a) the domain and range, and (b) x- and y-intercepts, if they exist or write NONE.

41. $h(x) = \log_4(x-1)+1$ 42. $f(x) = \log(5x+10)+3$	43. $g(x) = \ln(-x)-2$ 44. $f(x) = \log_2(x+2)-5$	45. $h(x) = 3\ln(x)-9$ 46. $f(x) = \log\left(x^2+4x+4\right)$

Answers to odd exercises:

41. Domain: $(1,\infty)$ Range: $-\infty,\infty$ Vertical asymptote: $x=1$ x-intercept: $\left(\frac{5}{4},0\right)$ y-intercept: NONE	43. Domain: $(-\infty,0)$ Range: $-\infty,\infty$ Vertical asymptote: $x=0$ x-intercept: $(-e^2,0)$ y-intercept: NONE	45. Domain: $(0,\infty)$ Range: $-\infty,\infty$ Vertical asymptote: $x=0$ x-intercept: $(e^3,0)$ y-intercept: NONE

F: Graph Basic Logarithmic Functions

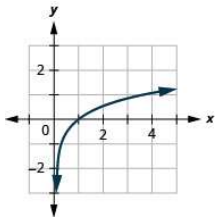
Exercise 4.4e.F: Graph Basic Logarithmic Functions

★ In the following exercises, graph each logarithmic function (or functions).

47. $y=\log_4(x)$ 48. $y=\log_2(x)$ 49. $y=\log_7(x)$ 50. $y=\log_6(x)$	51. $y=\log_{2.5}(x)$ 52. $y=\log_{1.5}(x)$ 53. $y=\log_{\frac{1}{6}}(x)$ 54. $y=\log_{\frac{1}{3}}(x)$	55. $y=\log_{0.6}(x)$ 56. $y=\log_{0.4}(x)$ 57. $f(x)=\log_{1/2}(x)$ 58. $f(x)=\log(x)$, and $g(x)=\ln(x)$

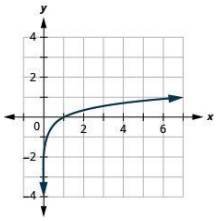
Answers to odd exercises:

47.



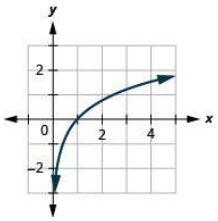
$y=\log_4 x$

49.



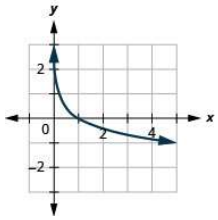
$y=\log_7 x$

51.



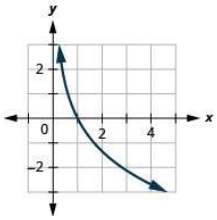
$y=\log_{2.5} x$

53.



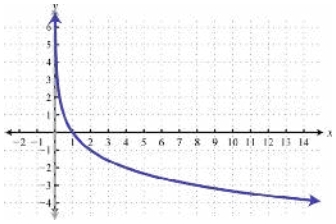
$y=\log_{\frac{1}{6}} x$

55.



$y=\log_{0.6} x$

57.



$f(x)=\log_{1/2} x$

G: Graph Vertical and Horizontal Shifts of Basic Log Functions

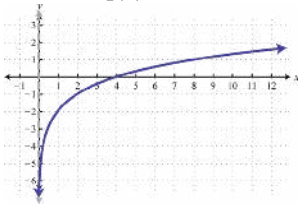
Exercise 4.4e.G: Graph Vertical and Horizontal Shifts of Basic Log Functions

★ **In the following exercises, graph each function using transformations. State the parent function and the transformations needed to be made on the parent function in order to obtain the graph of the translated function.**

62. $f(x) = \log(x) - 1$
63. $f(x) = \log_2(x) - 2$
64. $f(x) = \log_3(x) + 3$
65. $f(x) = \log(x) + 5$
66. $f(x) = \log_{1/3}(x) + 2$
67. $f(x) = 3 + \ln(x)$
68. $f(x) = \log_3(x - 2)$
69. $f(x) = \log_2(x + 2)$
70. $f(x) = \log_3(x + 4)$
71. $f(x) = \log_2(x + 1)$
72. $h(x) = \log_4(x + 2)$
73. $f(x) = \log_{1/2}(x - 2)$
74. $f(x) = \ln(x - 1)$
75. $f(x) = \ln(x - 3)$
76. $f(x) = \ln(x + 5)$
77. $f(x) = \ln(x + 1)$
78. $f(x) = \log(x - 5) + 10$
79. $f(x) = \log_2(x - 2) + 4$
80. $f(x) = \log_3(x + 1) - 2$
81. $f(x) = \log(x + 4) - 8$
82. $f(x) = \log_{1/3}(x + 1) - 1$
83. $f(x) = \ln(x - 2) + 4$

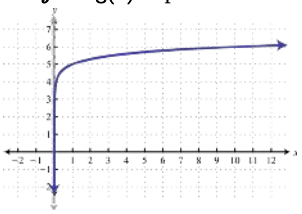
Answers to odd exercises:

63. $y = \log_2(x)$. Down 2



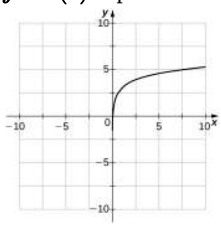
$f(x) = \log_2 x - 2$

65. $y = \log(x)$. Up 5



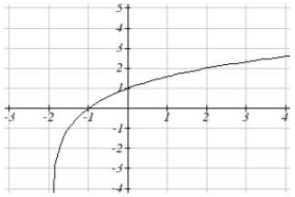
$f(x) = \log x + 5$

67. $y = \ln(x)$. Up 3



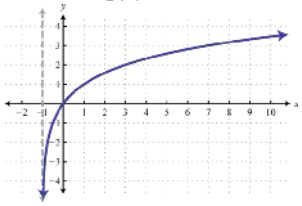
$f(x) = 3 + \ln(x)$

69. $y = \log_2(x)$. Left 2



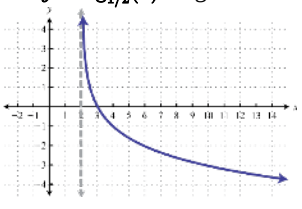
$f(x) = \log_2(x + 2)$

71. $y = \log_2(x)$. Left 1



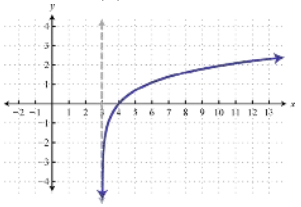
$f(x) = \log_2(x + 1)$

73. $y = \log_{1/2}(x)$. Right 2



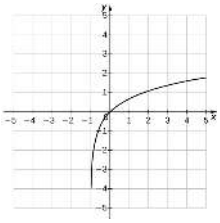
$f(x) = \log_{1/2}(x - 2)$

75. $y = \ln(x)$. Right 3



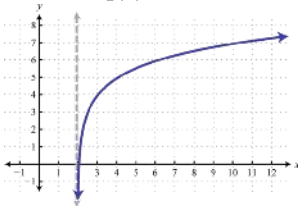
$f(x) = \ln(x - 3)$

77. $y = \ln(x)$. Left 1



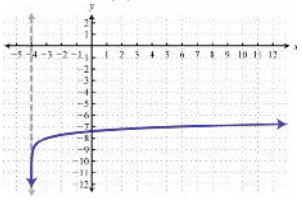
$f(x) = ln(x + 1)$

79. $y = \log_2(x)$. Right 2, up 4



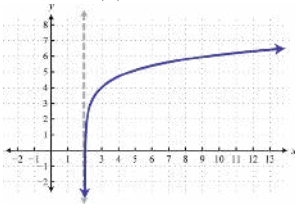
$f(x) = \log_2(x - 2) + 4$

81. $y = \log(x)$. Left 4, down 8



$f(x) = \log(x + 4) - 8$

83. $y = \ln(x)$. Right 2, up 4



$f(x) = \ln(x - 2) + 4$

H: Graph Reflections and Transformations of Log Functions

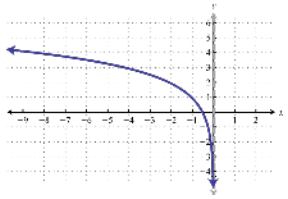
Exercise 4.4e.H: Graph Reflections and Transformations of Log Functions

★ **In the following exercises, graph each function using transformations. State the parent function and the transformations needed to be made on the parent function in order to obtain the graph of the translated function.**

84. $f(x) = 2 - \log_3(-x)$
85. $f(x) = \log_2(-x) + 1$
86. $f(x) = -\log_3(x + 3)$
87. $f(x) = -\log_2(x) + 1$
88. $f(x) = 1 - \log_4(-x)$
89. $f(x) = 2 + \log_4(x)$
90. $f(x) = 1 + \log_{1/2}(-x)$
91. $f(x) = 1 - \log_{1/3}(x - 2)$
92. $f(x) = 1 - \ln(x)$
93. $f(x) = 2 - \ln(x)$
94. $f(x) = -\ln(x - 1)$
95. $f(x) = \ln(-x)$
96. $f(x) = -\log(x - 1) + 2$
97. $f(x) = -\log(x + 2)$
98. $f(x) = 2\log_3(5 - x) - 1$
99. $f(x) = \log(6 - 3x) + 1$
100. $f(x) = \log_2(4x + 16) + 4$
102. $f(x) = 4\log_3(x - 2)$ -5
103. $f(x) = 2\log(x)$
104. $h(x) = -4\log_2(x + 1) - 3$

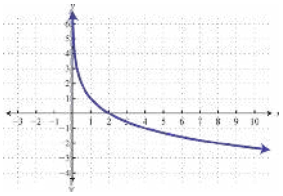
Answers to odd exercises:

85. $f(x) = \log_2(x)$. Reflect over y-axis, up 1



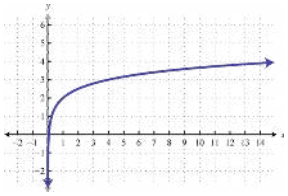
$f(x) = \log_2(-x) + 1$

87. $f(x) = \log_2(x)$. Reflect over x-axis, up 1



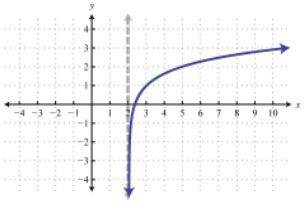
$f(x) = -\log_2 x + 1$

89. $f(x) = \log_4(x)$. Up 2



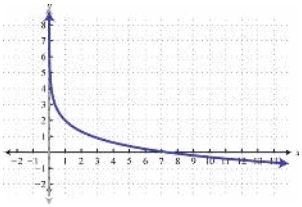
$f(x) = 2 + \log_4 x$

91. $f(x) = \log_{1/3}(x)$. Right 2, reflect over x-axis, up 1



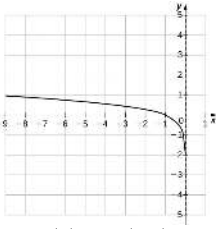
$f(x) = 1 - \log_{1/3}(x - 2)$

93. $f(x) = \ln(x)$. Reflect over x-axis, up 2



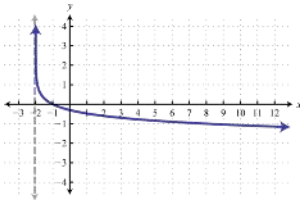
$f(x) = 2 - \ln x$

95. $f(x) = \ln(x)$. Reflect over y-axis



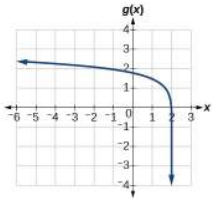
$f(x) = \ln(-x)$

97. $f(x) = \log(x)$. Left 2, reflect over x-axis



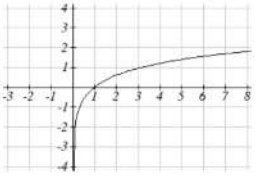
$f(x) = -\log(x + 2)$

99. $f(x) = \log(x)$. Left 6, reflect over y-axis, horizontally shrunk by a factor of 1/3, up 1



$f(x) = \log(6 - 3x) + 1$

103. $f(x) = \log(x)$. Vertically stretched by a factor of 2



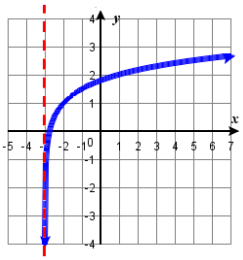
$f(x) = 2\log(x)$

I: Find the Equation for the Graph of a Logarithmic Function (no stretching)

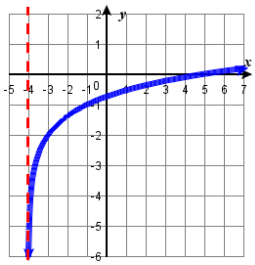
Exercise 4.4e.I: Find the Equation for the Graph of a Logarithmic Function

★ **For the following exercises, write a logarithmic equation corresponding to the graph shown.**

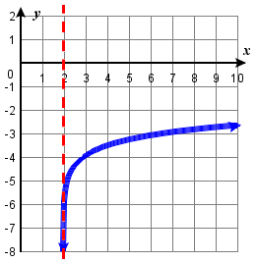
105.



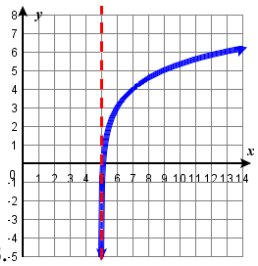
106.



107. ($(7,-3)$ is on the graph).

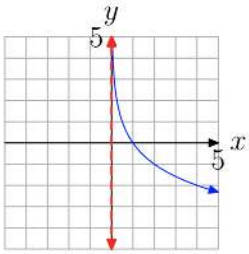


108.

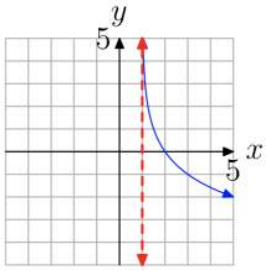


★ **For the following exercises, write a logarithmic equation corresponding to the graph shown.**

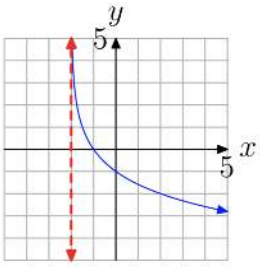
109.



110.

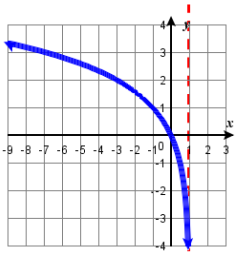


111.

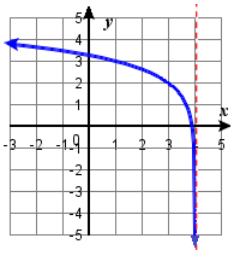


★ **For the following exercises, write a logarithmic equation corresponding to the graph shown.**

113.



114.



Answers to odd exercises:

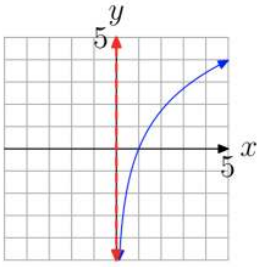
105. $f(x)=\log_4(x+3)+1$
107. $f(x)=\log_5(x-2)-4$
109. $y=-\log_2(x)$
111. $y=-\log_2(x+2)$
113. $f(x)=\log_2(-(x-1))$

I: Find the Equation for the Graph of a Logarithmic Function (vertical stretching, no vertical shifting)

Exercise 4.4e.I: Find the Equation for the Graph of a Logarithmic Function

★ **Write a logarithmic equation corresponding to the graph shown.**

120.



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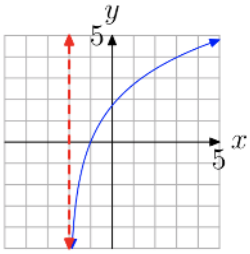

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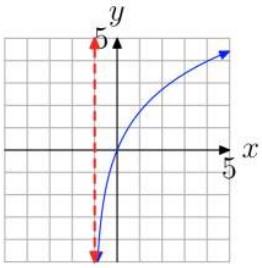
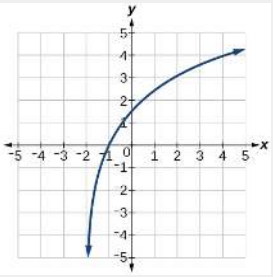
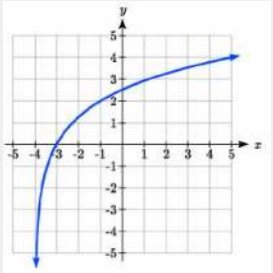


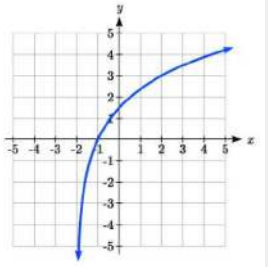
123.



124.

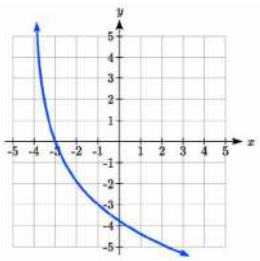


125.

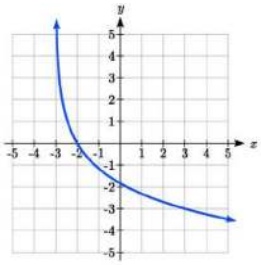


★ **Write a logarithmic equation corresponding to the graph shown.**

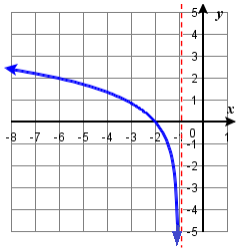
129. $(-1,-3)$ is on the graph



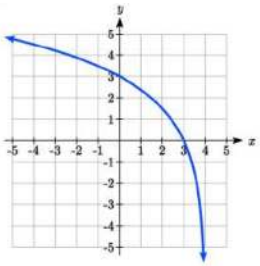
130.



131. $(-6,2)$ is on the graph

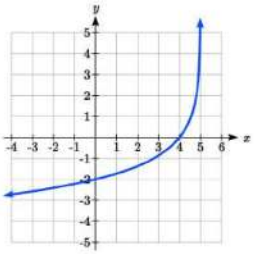


132.

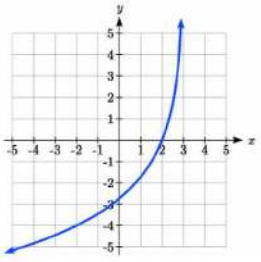


★ **Write a logarithmic equation corresponding to the graph shown.**

135.



136.



Answers to odd exercises:

121. $y = 4\log_5(x+2)$
123. $f(x) = 3\log_4(x+2)$
125. $y = 3\log_4(x+2)$
129. $y = -3\log_3(x+4)$
131. $y = 2\log_5(-x-1)$
135. $y = -2\log_5(-(x-5))$

⋆

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4.5: Logarithmic Properties

Learning Objectives

- Use the product, quotient, and power rules for logarithms.
- Expand logarithmic expressions.
- Condense logarithmic expressions.
- Use the change-of-base formula for logarithms.

In chemistry, the pH scale is used as a measure of the acidity or alkalinity of a substance. Substances with a pH less than 7 are considered acidic, and substances with a pH greater than 7 are said to be alkaline. To determine whether a solution is acidic or alkaline, we find its pH, which is a measure of the number of active positive hydrogen ions in the solution. The pH is defined by the following formula, where a is the concentration of hydrogen ion in the solution

$$pH = -\log([H^+]) = \log\left(\frac{1}{[H^+]}\right)$$

The equivalence of these expressions is one of the logarithm properties we will examine in this section.

Basic Properties

Recall that the logarithmic and exponential functions "undo" each other. This means that logarithms have similar properties to exponents. Some important properties of logarithms are given here.

PROPERTIES OF ONE FOR LOGARITHMS

$$\log_b(1) = 0 \qquad \log_b(b) = 1 \qquad \ln(1) = 0 \qquad \ln(e) = 1$$

These properties are easy to prove by writing the log equation in its exponential form.

$$\log_b(1) = 0 \longleftrightarrow b^0 = (1)\checkmark \qquad \ln(1) = 0 \longleftrightarrow \log_e(1) = 0 \longleftrightarrow e^0 = (1)\checkmark$$

$$\log_b(b) = 1 \longleftrightarrow b^1 = (b)\checkmark \qquad \ln(e) = 1 \longleftrightarrow \log_e(e) = 1 \longleftrightarrow e^1 = (e)\checkmark$$

Previously, a logarithm was evaluated by rewriting it in exponential form. But recognizing and applying log properties saves time!

Example 4.5.1

Evaluate using the properties of logarithms:

a. $\log_8(1)$
b. $\log_6(6)$

Solution:

a. Use the property, $\log_b(1) = 0$: the log of 1, regardless of the base used is always zero $\longrightarrow \log_8(1) = 0$

Previous approach: rewrite in exponential form: $\log_8(1) = x \longleftrightarrow 8^x = 1 \longleftrightarrow x = 0 \longleftrightarrow \log_8(1) = 0$

b. Use the property, $\log_b(b) = 1$. Thus $\log_6(6) = 1$

Try It 4.5.1

Evaluate using properties of logarithms:

a. $\log_{13}(1)$ b. $\log_9(9)$

Answer

a. **0** b. **1**

c. $\log_5(1)$ d. $\log_7(7)$

Answer

c. **0** d. **1**

INVERSE PROPERTIES FOR LOGARITHMS

$$\log_b(b^x) = x \qquad b^{\log_b(x)} = x \qquad \ln(e^x) = x \qquad e^{\ln(x)} = x$$

The rational behind these properties is the fact that the functions $f(x) = \log_b(x)$ and $g(x) = b^x$ are inverse functions. As inverse functions f and g have the property that $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$.

$$(f \circ g)(x) = x \leftrightarrow f(g(x)) = x \leftrightarrow f(b^x) = x \leftrightarrow \log_b(b^x) = x \checkmark$$

$$(g \circ f)(x) = x \leftrightarrow g(f(x)) = x \leftrightarrow g(\log_b(x)) = x \leftrightarrow b^{\log_b(x)} = x \checkmark$$

Example 4.5.2

Evaluate using properties of logarithms:

a. $4^{\log_4(9)}$

b. $\log_3(3^5)$

Solution:

a. Use the property, $b^{\log_b(x)} = x$. Therefore, $4^{\log_4(9)} = 9$

b. Use the property, $\log_b(b^x) = x$. Therefore, $\log_3(3^5) = 5$

Try It 4.5.2

Evaluate using properties of logarithms:

a. $5^{\log_5(15)}$ b. $\log_7(7^4)$

Answer

a. **15** b. **4**

c. $2^{\log_2(8)}$ d. $\log_2(2^{15})$

Answer

c. **8** d. **15**

e. $\log(100)$ f. $e^{\ln(7)}$

Answer

e. **2** f. **7**

Using the Product Rule for Logarithms

Recall that we use the *product rule of exponents* to combine the product of exponents by adding: $x^a x^b = x^{a+b}$. We have a similar property for logarithms, called the **product rule for logarithms**, which says that **the logarithm of a product is equal to a sum of logarithms**. Because logs are exponents, and we multiply like bases, we can add the exponents. We will use the inverse property to derive the product rule below.

Given any real number x and positive real numbers M, N, and b, where $b \neq 1$, we will show

$$\log_b(MN) = \log_b(M) + \log_b(N).$$

Let $m = \log_b M$ and $n = \log_b N$. In exponential form, these equations are $b^m = M$ and $b^n = N$. It follows that

$$\begin{aligned} \log_b(MN) &= \log_b(b^m b^n) && \text{Substitute for M and N} \\ &= \log_b(b^{m+n}) && \text{Apply the product rule for exponents} \\ &= m+n && \text{Apply the inverse property of logs} \\ &= \log_b(M)+\log_b(N) && \text{Substitute for m and n} \end{aligned}$$

Note that repeated applications of the product rule for logarithms allow us to simplify the logarithm of the product of any number of factors. For example, consider $\log_b(wxyz)$. Using the product rule for logarithms, we can rewrite this logarithm of a product as the sum of logarithms of its factors:

$$\log_b(wxyz) = \log_b(w) + \log_b(x) + \log_b(y) + \log_b(z)$$

PRODUCT RULE: The Log of a Product is the Sum of Logs

$$\log_b(MN) = \log_b(M) + \log_b(N) \text{ for } b > 0$$

Example 4.5.3

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible:

a. $\log_3(7x)$
b. $\log_4(64xy)$
c. $\log_3(30x(3x+4))$

Solution:

a. Use the Product Property, $\log_b(M \cdot N) = \log_b M + \log_b N$.

$$\begin{aligned} \log_3(7x) &= \log_3(7 \cdot x) \\ &= \log_3(7) + \log_3(x) \end{aligned}$$

b. Use the Product Property, $\log_b(M \cdot N) = \log_b M + \log_b N$.

$$\begin{aligned} \log_4(64xy) &= \log_4(64 \cdot x \cdot y) \\ &= \log_4(64) + \log_4(x) + \log_4(y) \end{aligned}$$

The argument (64) is a power of the base, so it must be simplified. Rewrite and use the Inverse Property $\log_b(b^x) = x$.

$$\begin{aligned} \log_4(64xy) &= \log_4(4^3) + \log_4(x) + \log_4(y) \\ &= 3 + \log_4(x) + \log_4(y) \end{aligned}$$

c. Use the Product Property, $\log_b(M \cdot N) = \log_b M + \log_b N$.

$$\begin{aligned} \log_3(30x(3x+4)) &= \log_3((30x) \cdot (3x+4)) \\ &= \log_3(30x) + \log_3(3x+4) \end{aligned}$$

The argument $(30x)$ is not a single factor, so it must be simplified. Use the Product Property again.

$$\begin{aligned} \log_3(30x(3x+4)) &= \log_3(30 \cdot x) + \log_3(3x+4) \\ &= \log_3(30) + \log_3(x) + \log_3(3x+4) \end{aligned}$$

However, the expression is still not completely simplified because the argument (30) is a power of the base (3). So write 30 as a product involving 3 as one of its factors. Then use the Product Property once more, followed by the property of One: $\log_b(b) = 1$.

$$\begin{aligned} \log_3(30x(3x+4)) &= \log_3(3 \cdot 10) + \log_3(x) + \log_3(3x+4) \\ &= \log_3(3) + \log_3(10) + \log_3(x) + \log_3(3x+4) \\ &= 1 + \log_3(10) + \log_3(x) + \log_3(3x+4) \end{aligned}$$

We will see in future that there are log properties for log arguments that are products, for log arguments that are quotients, and for log arguments that are powers. But, there is NO LOG PROPERTY for log arguments that are sums or differences. Therefore, $\log_3(3x+4)$ CANNOT be simplified!!

Try It 4.5.3

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible:

a. $\log_3(3x)$ b. $\log_2(8xy)$	a. $\log_9(81x)$ b. $\log_3(27xy)$	a. $\log(20(x))$ b. $\ln(4e)$
Answer a. $1+\log_3(x)$ b. $3+\log_2(x)+\log_2(y)$	**Answer** a. $2+\log_9(x)$ b. $3+\log_3(x)+\log_3(y)$	**Answer** a. $1+\log(2)+\log(x)$ b. $1+\ln(4)$

Using the Quotient Rule for Logarithms

For quotients, we have a similar rule for logarithms. Recall that we use the *quotient rule of exponents* to combine the quotient of exponents by subtracting: $x^{\frac{a}{b}} = x^{a-b}$. The **quotient rule for logarithms** says that **the logarithm of a quotient is equal to a difference of logarithms**.

QUOTIENT RULE: The Log of a Quotient is the Difference of Logs

$$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$$

Just as with the product rule, we can use the inverse property to derive the quotient rule.

Given any real number x and positive real numbers M, N, and b, where $b \neq 1$, we will show

$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N).$$

Let $m = \log_b M$ and $n = \log_b N$. In exponential form, these equations are $b^m = M$ and $b^n = N$. It follows that

$$\begin{aligned}
\log_b\left(\frac{M}{N}\right) &= \log_b\left(\frac{b^m}{b^n}\right) && \text{Substitute for M and N}\\
&= \log_b(b^{m-n}) && \text{Apply the quotient rule for exponents}\\
&= m-n && \text{Apply the inverse property of logs}\\
&= \log_b(M) - \log_b(N) && \text{Substitute for m and n}
\end{aligned}$$

Example 4.5.4

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

a. $\log_5(\frac{5}{7})$
b. $\log(\frac{x}{100})$

Solution:

a. Use the Quotient Property, $\log_b \frac{M}{N} = \log_b M - \log_b N$.

$$\begin{aligned}
\log_5(\tfrac{5}{7}) &= \log_5(5) - \log_5(7)\\
&= 1 - \log_5 7 && \text{Simplify with Property of One: } \log_b(b) = 1
\end{aligned}$$

b. Use the Quotient Property, $\log_b \frac{M}{N} = \log_b M - \log_b N$.

$$\begin{aligned}
\log(\tfrac{x}{100}) &= \log(x) - \log(100)\\
&= \log(x) - \log_{10}(10^2) && \text{Use Inverse Property: } \log_b(b^x) = x\\
&= \log(x) - 2
\end{aligned}$$

Try It 4.5.4

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

a. $\log_4(\frac{3}{4})$
b. $\log(\frac{x}{1000})$

Answer

a. $\log_4 3 - 1$
b. $\log x - 3$

c. $\log_2(\frac{5}{4})$
d. $\log(\frac{10}{y})$

Answer

c. $\log_2 5 - 2$
d. $1 - \log y$

Using the Power Rule for Logarithms

We've explored the product rule and the quotient rule, but how can we take the logarithm of a power, such as x^2? One method is as follows:

$$\begin{aligned}\log_b(x^2) &= \log_b(x \cdot x)\\ &= \log_b x + \log_b x\\ &= 2\log_b x\end{aligned}$$

Notice that we used the **product rule for logarithms** to find a solution for the example above. By doing so, we have derived the **power rule for logarithms**, which says that **the log of a power equals the product of the power and the log**. Keep in mind that, although the input to a logarithm may not be written as a power, we may be able to change it to a power. For example,

$$100 = 10^2 \qquad \sqrt{3} = 3^{\frac{1}{2}} \qquad \frac{1}{e} = e^{-1}$$

POWER RULE: The Log of a Power is the Product of the Power and the Log

$$\log_b(M^p) = p\log_b(M)$$

Example 4.5.5

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

a. $\log_5(4^3)$
b. $\log(x^{10})$
c. $\log_3(25)$

Solution:

a. Use the Power Property, $\log_b M^p = p\log_b M \longrightarrow \log_5(4^3) = 3\log_5 4$

b. Use the Power Property, $\log_b M^p = p\log_b M \longrightarrow \log(x^{10}) = 10\log(x)$.

c. Use the Power Property, $\log_b M^p = p\log_b M \longrightarrow \log_3(25) = \log_3(5^2) = 2\log_3(5)$.

Try It 4.5.5

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

a. $\log_7(5^4)$
b. $\log(x^{100})$

Answer

a. $4\log_7(5)$
b. $100\cdot\log(x)$

c. $\log_2(3^7)$
d. $\log(x^{20})$

Answer

c. $7\log_2(3)$
d. $20\cdot\log(x)$

e. $\ln(x^2)$.
f. $\ln\left(\frac{1}{x^2}\right)$

Answer

e. $2\ln(x)$
f. $-2\ln(x)$

Expanding Logarithmic Expressions

Taken together, the product rule, quotient rule, and power rule are often called "laws of logs." Sometimes we apply more than one rule in order to simplify an expression. For example:

$$\begin{aligned}\log_b\left(\frac{6x}{y}\right) &= \log_b(6x) - \log_b y \\ &= \log_b 6 + \log_b x - \log_b y\end{aligned}$$

We can use the power rule to expand logarithmic expressions involving negative and fractional exponents. Here is an alternate proof of the quotient rule for logarithms using the fact that a reciprocal is a negative power:

$$\begin{aligned}\log_b\left(\frac{A}{C}\right) &= \log_b(AC^{-1}) \\ &= \log_b(A) + \log_b(C^{-1}) \\ &= \log_b A + (-1)\log_b C \\ &= \log_b A - \log_b C\end{aligned}$$

We can also apply the product rule to express a sum or difference of logarithms as the logarithm of a product.

With practice, we can look at a logarithmic expression and expand it mentally, writing the final answer. Remember, however, that we can only do this with arguments that are products, quotients, powers, and roots—never with addition or subtraction inside the argument of the logarithm.

We generally apply the Product and Quotient Properties before we apply the Power Property.

Example 4.5.6: Logs of Products

Use the Properties of Logarithms to expand the following logarithms. Simplify, if possible.

a. $\log_4(2x^3y^2)$
b. $log_2(8x^4)$

Solution:

a.

$$\begin{aligned}\log_4(2x^3y^2) &= \log_4(2) + \log_4(x^3) + \log_4(y^2) && \text{Product Property, } \log_b M\cdot N = \log_b M + \log_b N. \\ &= \log_4(2) + 3\log_4(x) + 2\log_4(y) && \text{Power Property, } \log_b(M^p) = p\log_b(M).\end{aligned}$$

b. In this problem notice that it is incorrect to use the power rule first! $\log_2(8x^4) \neq 4\log_2(8x)$

$$\begin{aligned}\log_2(8x^4) &= \log_2(8) + \log_2(x^4) && \text{Product Property, } \log_b(M\cdot N) = \log_b M + \log_b N. \\ &= \log_2(2^3) + 4\log_2(x) && \text{Power Property, } \log_b(M^p) = p\log_b(M). \\ &= 3 + 4\log_2(x) && \text{Inverse Property, } \log_b(b^x) = x.\end{aligned}$$

Try It 4.5.6:

Use the Properties of Logarithms to expand the logarithms below. Simplify, if possible.

a. $\log_2(10x^4y^2)$
b. $\log_3(81x^5y^3)$

Answer

a. $1+\log_2 5+4\log_2(x)+2\log_2(y)$
b. $4+5\log_3 x+3\log_3 y$

Example 4.5.7: Logs of Quotients

Use the Properties of Logarithms to expand the following logarithms. Simplify, if possible.

a. $\ln\left(\frac{x^4y}{7}\right)$
b. $\log_2\left(\frac{x^3}{4y}\right)$
c. $\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right)$
d. $\log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right)$.
e. $\log\left(\frac{2x^2+6x}{3x+9}\right)$

a. Solution

$$\begin{aligned}\ln\left(\frac{x^4y}{7}\right) &= \ln(x^4y)-\ln(7) && \text{Quotient Rule } \log_b\tfrac{M}{N}=\log_b M-\log_b N\\ &= \ln(x^4\cdot y)-\ln(7)\\ &= \ln(x^4)+\ln(y)-\ln(7) && \text{Product Rule } \log_b(M\cdot N)=\log_b M+\log_b N\\ &= 4\ln(x)+\ln(y)-\ln(7) && \text{Power Rule } \log_b(M^p)=p\log_b(M)\end{aligned}$$

b. Solution

$$\begin{aligned}\log_2\left(\frac{x^3}{4y}\right) &= \log_2(x^3)-\log_2(4y) && \text{Quotient Rule } \log_b\tfrac{M}{N}=\log_b M-\log_b N\\ &= \log_2(x^3)-\log_2(4\cdot y)\\ &= \log_2(x^3)-\left(\log_2(4)+\log_2(y)\right) && \text{Product Rule } \log_b(M\cdot N)=\log_b M+\log_b N\\ &= 3\log_2(x)-2-\log_2(y) && \text{Power Rule } \log_b(M^p)=p\log_b(M);\ \text{distribute};\ \log_2(4)=2\end{aligned}$$

c. Solution

$$\begin{aligned}\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right) &= \log_6(64)+\log_6(x^3)+\log_6(4x+1)-\log_6(2x-1) && \text{Product and Quotient Rules}\\ &= \log_6(2^6)+\log_6(x^3)+\log_6(4x+1)-\log_6(2x-1)\\ &= 6\log_6(2)+3\log_6(x)+\log_6(4x+1)-\log_6(2x-1) && \text{Power Rule}\end{aligned}$$

d. Solution

$$\begin{aligned}\log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right) &= \log_2(15x(x-1))-\log_2((3x+4)(2-x)) && \text{Quotient Rule}\\ &= [\log_2(15)+\log_2(x)+\log_2(x-1)]-[\log_2(3x+4)+\log_2(2-x)] && \text{Product Rule}\\ &= \log_2(15)+\log_2(x)+\log_2(x-1)-\log_2(3x+4)-\log_2(2-x) && \text{Distribute}\end{aligned}$$

Analysis. There are exceptions to consider in this and later examples. First, because denominators must never be zero, this expression is not defined for $x=-\frac{4}{3}$ and $x=2$. Also, since the argument of a logarithm must be positive, the expanded logarithm requires that $x>0$, $x>1$, $x>-\frac{4}{3}$, and $x<2$. Combining these conditions yields a domain of $1<x<2$. In contrast the domain of the original log expression also includes the interval $-\frac{4}{3}<x<0$. However, this analysis is beyond the scope of this section, and we will not consider these considerations here or in subsequent exercises.

e. Factoring and canceling we get,

$$\begin{aligned}
\log\left(\frac{2x^2+6x}{3x+9}\right) &= \log(2x^2+6x)-\log(3x+9) && \text{Quotient Rule}\\
&= \log(2x(x+3))-\log(3(x+3)) && \text{Factor!!}\\
&= \log(2)+\log(x)+\log(x+3)-[\log(3)+\log(x+3)] && \text{Product Rule}\\
&= \log(2)+\log(x)+\log(x+3)-\log(3)-\log(x+3) && \text{Distribute negative}\\
&= \log(2)+\log(x)-\log(3) && \text{simplify}
\end{aligned}$$

Try It 4.5.7

Use the Properties of Logarithms to expand the following logarithms. Simplify, if possible.

a. $\log\left(\frac{x^2y^3}{z^4}\right)$

b. $\log_3\left(\frac{7x^2+21x}{7x(x-1)(x-2)}\right)$

c. $\ln\left(\frac{\sqrt{(x-1)(2x+1)^2}}{(x^2-9)}\right)$

a. Answer

$2\log x+3\log y-4\log z$

b. Answer

$\log_3(x+3)-\log_3(x-1)-\log_3(x-2)$

c. Answer

$\frac{1}{2}\ln(x-1)+\ln(2x+1)-\ln(x+3)-\ln(x-3)$

When we have a radical in the logarithmic expression, it is helpful to first write its radicand as a rational exponent.

Example 4.5.8: Logs of Radicals

Use the Properties of Logarithms to expand the following logarithms. Simplify, if possible.

a. $\ln(\sqrt[3]{7x^2})$

b. $\log_2\sqrt[4]{\frac{x^3}{3y^2z}}$

a. Solution

$$\begin{aligned}
\ln(\sqrt[3]{7x^2}) &= \ln(7x^2)^{\frac{1}{3}} && \text{Rewrite radical in exponential form}\\
&= \ln\left((7)^{\frac{1}{3}}\cdot(x^2)^{\frac{1}{3}}\right) && \text{Use Power Rule for Exponents}\\
&= \ln(7)^{\frac{1}{3}}+\ln(x^{\frac{2}{3}}) && \text{Product Rule } \log_b(M\cdot N)=\log_b M+\log_b N\\
&= \tfrac{1}{3}\ln(7)+\tfrac{2}{3}\ln(x) && \text{Power Rule } \log_b(M^p)=p\log_b(M)
\end{aligned}$$

b. Solution

Misplaced '&'

Try It 4.5.8

Use the Properties of Logarithms to expand the following logarithms. Simplify, if possible.

a. $\log_4\sqrt[5]{\frac{x^4}{2y^3z^2}}$.

a. Answer

$\frac{1}{5}\left(4\log_4 x-\frac{1}{2}-3\log_4 y-2\log_4 z\right)$

b. $\log_3\sqrt[3]{\frac{x^2}{5yz}}$.

b. Answer

$\frac{1}{3}\left(2\log_3 x-\log_3 5-\log_3 y-\log_3 z\right)$

Can we expand $\ln(x^2+y^2)$?

No. There is no way to expand the logarithm of a sum or difference inside the argument of the logarithm.

Example 4.5.8x

Given that $\log_2 x = a, \log_2 y = b$, and that $\log_2 z = c$, write the following in terms of a, b, and c:

a. $\log_2 (8x^2y)$

b. $\log_2 \left(\frac{2x^4}{\sqrt{z}}\right)$

Solution

a. Begin by expanding using sums and coefficients and then replace a and b with the appropriate logarithm.

$$\begin{aligned} \log_2 (8x^2y) &= \log_2 8 + \log_2 x^2 + \log_2 y \\ &= \log_2 8 + 2\log_2 x + \log_2 y \\ &= 3 + 2a + b \end{aligned} \tag{4.5.1}$$

b. Expand and then replace a, b, and c where appropriate.

$$\begin{aligned} \log_2 \left(\frac{2x^4}{\sqrt{z}}\right) &= \log_2 (2x^4) - \log_2 z^{1/2} \\ &= \log_2 2 + \log_2 x^4 - \log_2 z^{1/2} \\ &= \log_2 2 + 4\log_2 x - \frac{1}{2}\log_2 z \\ &= 1 + 4a - \frac{1}{2}b \end{aligned} \tag{4.5.2}$$

Condensing Logarithmic Expressions

We can use the rules of logarithms we just learned to condense sums, differences, and products with the same base as a single logarithm. It is important to remember that the logarithms must have the same base to be combined. We will learn later how to change the base of any logarithm before condensing.

How to: Condense a logarithmic expression into a single logarithm

Start with a sum and difference of logs all with the same base.

1. Apply the reverse power property first. Identify terms that are products of factors and a logarithm, and rewrite each as the logarithm of a power.
2. Next apply the reverse product property. Gather all the positive logs together and rewrite the sum of positive logarithms as the positive logarithm of a product. Gather all the negative logs together and rewrite the sum of negative logarithms as the negative logarithm of a product.
3. Apply the reverse quotient property last. Rewrite the difference of logarithms as the logarithm of a quotient.

The opposite of expanding a logarithm is to condense a sum or difference of logarithms that have the same base into a single logarithm. We again use the properties of logarithms to help us, but in reverse.

To condense logarithmic expressions with the same base into one logarithm, we start by using the Power Property to get the coefficients of the log terms to be one and then the Product and Quotient Properties as needed.

>Example 4.5.9: Using the Log Properties in Reverse

Use the Properties of Logarithms to condense the logarithms below. Simplify, if possible.

a. $4\ln(x)$

b. $2\log_3 x + 4\log_3(x+1)$

c. $\log_3(5)+\log_3(8)-\log_3(2)$

d. $2\log x-4\log(x+5)+\dfrac{1}{x}\log(3x+5)$

e. $5\left(\log_2(x^2)+\dfrac{1}{2}\log_2(x-1)-3\log_2\left((x+3)^2\right)\right)$

f. $\frac{1}{2}\ln x-3\ln y-\ln z$

g. $-\ln(x)-\frac{1}{2}$

a. Solution:

$$4\ln(x)=\ln(x^4) \qquad \text{Reverse Power Rule } p\log_b(M)=\log_b(M^p)$$

b. Solution:

$$\begin{aligned} 2\log_3 x+4\log_3(x+1) &= \log_3(x^2)+\log_3((x+1)^4) && \text{Reverse Power Rule } p\log_b(M)=\log_b(M^p)\\ &= \log_3((x^2)(x+1)^4) && \text{Reverse Product Property, } \log_b(M)+\log_b(N)=\log_b(M\cdot N).\end{aligned}$$

c. Solution

$$\begin{aligned} \log_3(5)+\log_3(8)-\log_3(2) &= \log_3(5\cdot 8)-\log_3(2) && \text{Reverse Product Rule}\\ &= \log_3\left(\frac{40}{2}\right) && \text{Reverse Quotient Rule}\\ &= \log_3(20)\end{aligned}$$

d. Solution

$$\begin{aligned} 2\log x-4\log(x+5)+\frac{1}{x}\log(3x+5) &= \log(x^2)-\log(x+5)^4+\log((3x+5)^{x^{-1}}) && \text{Reverse power rule}\\ &= \log(x^2)+\log((3x+5)^{x^{-1}}-\log(x+5)^4 && \text{Rearrange}\\ &= \log(x^2(3x+5)^{x^{-1}})-\log(x+5)^4 && \text{Reverse Product Rule}\\ &= \log\frac{x^2(3x+5)^{1/x}}{(x+5)^4} && \text{Reverse Quotient Rule}\end{aligned}$$

e. Solution

$$\begin{aligned} 5\left(\log_2(x^2)+\frac{1}{2}\log_2(x-1)-3\log_2(x+3)^2\right) &= 5\log_2(x^2)+\frac{5}{2}\log_2(x-1)-15\log_2(x+3) && \text{Create a sum of terms}\\ &= \log_2(x^2)^5+\log_2(x-1)^{\frac{5}{2}}-\log_2((x+3)^2)^{15} && \text{Reverse Power rule}\\ &= \log_2(x^{10})+\log_2(x-1)^{\frac{5}{2}}-\log_2(x+3)^{30}\\ &= \log_2(x^{10}(x-1)^{\frac{5}{2}})-\log_2(x+3)^{30} && \text{Reverse Product Rule}\\ &= \log_2\frac{x^2(x-1)^{\frac{5}{2}}}{(x+3)^{30}} && \text{Reverse Quotient Rule}\end{aligned}$$

f. Solution

$$\begin{aligned} \tfrac{1}{2}\ln x-3\ln y-\ln z &= \ln x^{1/2}-\ln y^3-\ln z && \text{Reverse Power Rule}\\ &= \ln x^{1/2}-(\ln y^3+\ln z) && \text{Combine negative terms}\\ &= \ln x^{1/2}-\ln(y^3z) && \text{Reverse Product Rule}\\ &= \ln\left(\frac{\sqrt{x}}{y^3z}\right) && \text{Reverse Quotient Rule}\end{aligned}$$

g. Solution

$$
\begin{aligned}
-\ln(x)-\tfrac{1}{2} &= (-1)\ln(x)-\tfrac{1}{2} \\
&= \ln\left(x^{-1}\right)-\tfrac{1}{2} && \text{Power Rule} \\
&= \ln\left(x^{-1}\right)-\ln\left(e^{1/2}\right) && \text{Since } \tfrac{1}{2}=\ln\left(e^{1/2}\right) \\
&= \ln\left(x^{-1}\right)-\ln\left(\sqrt{e}\right) \\
&= \ln\left(\frac{x^{-1}}{\sqrt{e}}\right) && \text{Quotient Rule} \\
&= \ln\left(\frac{1}{x\sqrt{e}}\right)
\end{aligned}
$$

Try It 4.5.11*a*

Use the Properties of Logarithms to condense the logarithms below. Simplify, if possible.

a. $2\log_3 4$
b. $3\log_2 x+2\log_2(x-1)$
c. $2\log x+2\log(x+1)$
d. $\log_2 5+\log_2 x-\log_2 y$
e. $\log_3 6-\log_3 x-\log_3 y$
f. $\log 3-\log 4+\log 5-\log 6$
g. $\log(5)+0.5\log(x)-\log(7x-1)+3\log(x-1)$
h. $4(3\log(x)+\log(x+5)-\log(2x+3))$.

a. Answer

$\log_3 16$

b. Answer

$\log_2 x^3(x-1)^2$

c. Answer

$\log x^2(x+1)^2$

d. Answer

$\log_2 \frac{5x}{y}$

e. Answer

$\log_3 \frac{6}{xy}$

f. Answer

$\log\left(\frac{5}{8}\right)$

g. Answer

$\log \frac{5(x-1)^3\sqrt{x}}{(7x-1)}$

h. Answer

$\log \frac{x^{12}(x+5)^4}{(2x+3)^4}$

Using the Change-of-Base Formula for Logarithms

Most calculators can evaluate only common and natural logs. In order to evaluate logarithms with a base other than **10** or, e, we use the **change-of-base formula** to rewrite the logarithm as the quotient of common or natural logs.

To evaluate a logarithm with any other base, we can use the **Change-of-Base Formula**. We will show how this is derived.

Suppose we want to evaluate $\log_a M$	$\log_a M$
Let $y=\log_a M$.	$y=\log_a M$
Rewrite the epression in exponential form.	$a^y=M$
Take the $\log_b$ of each side.	$\log_b a^y=\log_b M$
Use the Power Property.	$y\log_b a=\log_b M$
Solve for y.	$y=\frac{\log_b M}{\log_b a}$
Substitute $y=\log_a M$.	$\log_a M=\frac{\log_b M}{\log_b a}$

The Change-of-Base Formula introduces a new base b. This can be any base b we want where $b>0, b\neq 1$. Because our calculators have keys for logarithms base **10** and base e, the base used with the Change-of-Base Formula when using a calculator is **10** or e.

For example, to evaluate $\log_5 36$ using a calculator, we must first rewrite the expression as a quotient of common or natural logs. We will use the common log.

$$
\begin{aligned}
\log_5 36 &= \frac{\log(36)}{\log(5)} && \text{Apply the change of base formula using base 10} \\
&\approx 2.2266 && \text{Use a calculator to evaluate to 4 decimal places}
\end{aligned}
$$

THE CHANGE-OF-BASE FORMULA

The change-of-base formula can be used to evaluate a logarithm with any base.

For any positive real numbers M, b, and n, where $n \neq 1$ and $b \neq 1$,

$$\log_b M = \frac{\log_n M}{\log_n b}$$

It follows that the change-of-base formula can be used to rewrite a logarithm with any base as the quotient of common or natural logs.

$$\log_b M = \frac{\ln M}{\ln b} \qquad \text{and} \qquad \log_b M = \frac{\log M}{\log b}$$

How to: Rewrite $\log_b M$ as a quotient of logs with any positive base n, where $n \neq 1$

1. Determine the new base n, remembering that the common log, $\log(x)$, has base 10, and the natural log, $\ln(x)$, has base e.
2. Rewrite the log as a quotient using the change-of-base formula
   - The numerator of the quotient will be a logarithm with base n and argument M.
   - The denominator of the quotient will be a logarithm with base n and argument b.

Example 4.5.12: Changing Logarithmic Expressions to Expressions Involving Only Natural Logs

Change $\log_5 3$ to a quotient of natural logarithms.

Solution

Because we will be expressing $\log_5 3$ as a quotient of natural logarithms, the new base, $n = e$.

We rewrite the log as a quotient using the change-of-base formula. The numerator of the quotient will be the natural log with argument 3. The denominator of the quotient will be the natural log with argument 5.

$$\log_b M = \frac{\ln M}{\ln b}$$
$$\log_5 3 = \frac{\ln 3}{\ln 5}$$

Try It 4.5.12

Change $\log 0.58$ to a quotient of natural logarithms.

Answer

$$\frac{\ln(.58)}{\ln(10)}$$

Can we change common logarithms to natural logarithms?

Yes. Remember that $\log 9$ *means* $\log_{10} 9$. *So,* $\log 9 = \frac{\ln 9}{\ln 10}$.

Example 4.5.13: Using the Change-of-Base Formula with a Calculator

Evaluate $\log_2(10)$ using the change-of-base formula with a calculator.

Solution

According to the change-of-base formula, we can rewrite the log base 2 as a logarithm of any other base. Since our calculators can evaluate the natural log, we might choose to use the natural logarithm, which is the log base e.

$$\log_2 10 = \frac{\ln 10}{\ln 2} \quad \text{Apply the change of base formula using base } e$$
$$\approx 3.3219 \quad \text{Use a calculator to evaluate to 4 decimal places}$$

Try It 4.5.13

Evaluate $\log_5(100)$ using the change-of-base formula.

Answer

$$\frac{\ln 100}{\ln 5} \approx \frac{4.6051}{1.6094} = 2.861$$

Applications of Logs

Example 4.5.14: Application of the Laws of Logs

Recall that, in chemistry, $pH = -\log[H+]$. If the concentration of hydrogen ions in a liquid is doubled, what is the effect on pH?

Solution

Suppose C is the original concentration of hydrogen ions, and P is the original pH of the liquid. Then $P = -\log(C)$. If the concentration is doubled, the new concentration is $2C$. Then the pH of the new liquid is

$$pH = -\log(2C)$$

Using the product rule of logs

$$pH = -\log(2C) = -(\log(2) + \log(C)) = -\log(2) - \log(C)$$

Since $P = -\log(C)$, the new pH is

$$pH = P - \log(2) \approx P - 0.301$$

Try It 4.5.14

When the concentration of hydrogen ions is doubled, the pH decreases by about 0.301.

How does the pH change when the concentration of positive hydrogen ions is decreased by half?

Answer

The pH increases by about 0.301.

Key Equations

The Product Rule for Logarithms	$\log_b(MN) = \log_b(M) + \log_b(N)$
The Quotient Rule for Logarithms	$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$
The Power Rule for Logarithms	$\log_b(M^n) = n\log_b M$
The Change-of-Base Formula	$\log_b M = \frac{\log_n M}{\log_n b}$ $n > 0,\ n \neq 1,\ b \neq 1$

Key Concepts

- We can use the product rule of logarithms to rewrite the log of a product as a sum of logarithms. See Example $4.5.1$.
- We can use the quotient rule of logarithms to rewrite the log of a quotient as a difference of logarithms. See Example $4.5.2$.

- We can use the power rule for logarithms to rewrite the log of a power as the product of the exponent and the log of its base. See Example **4.5.3,** Example **4.5.4**, and Example **4.5.5**.
- We can use the product rule, the quotient rule, and the power rule together to combine or expand a logarithm with a complex input. See Example **4.5.6,** Example **4.5.7,** and Example **4.5.8**.
- The rules of logarithms can also be used to condense sums, differences, and products with the same base as a single logarithm. See Example **4.5.9,** Example **4.5.10**, Example **4.5.11**, and Example **4.5.12**.
- We can convert a logarithm with any base to a quotient of logarithms with any other base using the change-of-base formula. See Example **4.5.13**.
- The change-of-base formula is often used to rewrite a logarithm with a base other than 10 and e as the quotient of natural or common logs. That way a calculator can be used to evaluate. See Example **4.5.14**.

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4.5e: Exercises - Properties of Logarithms

A: Concepts

Exercise 4.5e.A

1. How does the power rule for logarithms help when solving logarithms with the form $\log_b(\sqrt[n]{x})$
2. Write the Product Property in your own words. Does it apply to each of the following? $\log_a 5x, \log_a(5+x)$. Why or why not?
3. Write the Power Property in your own words. Does it apply to each of the following? $\log_a x^p, (\log_a x)^r$. Why or why not?
4. Use an example to show that $\log(a+b) \neq \log a + \log b$?
5. Explain how to find the value of $\log_7 15$ using your calculator.
6. What does the change-of-base formula do? Why is it useful when using a calculator?

Answers to odd exercises:

1. Any root expression can be rewritten as an expression with a rational exponent so that the power rule can be applied, making the logarithm easier to calculate. Thus, $\log_b\left(x^{\frac{1}{n}}\right) = \frac{1}{n}\log_b(x)$.

3. Answers may vary

5. Answers may vary

B: Basic simplification of logs

Exercise 4.5e.B

★ For the following exercises, use properties of logarithms to evaluate without using a calculator.

7. $\log_7 1$	15. $\log_{1/2}\left(\frac{1}{2}\right)$	23. $e^{\ln x^2}$	32. a. $10^{\log\sqrt{5}}$, b. $\log 10^{-2}$
8. $\log_{1/2} 2$	16. $\log_{1/5} 5$	24. $e^{\ln e^x}$	33. a. $10^{\log\sqrt{3}}$, b. $\log 10^{-1}$
9. $\log 10^{14}$	17. $\log_{3/4}\left(\frac{4}{3}\right)$	25. a. $\log_{12} 1$, b. $\ln e$	34. a. $e^{\ln 4}$, b. $\ln e^2$
10. $\log 10^{-23}$	18. $\log_{2/3} 1$	26. a. $\log_4 1$, b. $\log_8 8$	35. a. $e^{\ln 3}$, b. $\ln e^7$
11. $\log_3 3^{10}$	19. $2^{\log_2 100}$	27. a. $5^{\log_5 10}$, b. $\log_4 4^{10}$	36. $\log_3\left(\frac{1}{9}\right) - 3\log_3(3)$
12. $\log_6 6$	20. $3^{\log_3 1}$	28. a. $3^{\log_3 6}$, b. $\log_2 2^7$	37. $6\log_8(2) + \frac{\log_8(64)}{3\log_8(4)}$
13. $\ln e^7$	21. $10^{\log 18}$	29. a. $6^{\log_6 15}$, b. $\log_8 8^{-4}$	38. $2\log_9(3) - 4\log_9(3)$
14. $\ln\left(\frac{1}{e}\right)$	22. $e^{\ln 23}$	30. a. $8^{\log_8 7}$, b. $\log_6 6^{-2}$	$+ \log_9\left(\frac{1}{729}\right)$

★ Find a:

39. $\ln a = 1$	41. $\log_9 a = -1$	43. $\log_2 a = 5$	45. $2^a = 7$	47. $\log_a 4^5 = 5$
40. $\log a = -1$	42. $\log_{12} a = 1$	44. $\log a = 13$	46. $e^a = 23$	48. $\log_a 10 = 1$

Answers to odd exercises:

7. 0	15. 1	23. x^2	33. $\sqrt{3}, -1$	41. $\frac{1}{9}$
9. 14	17. -1	25. $0, 1$	35. $3, 7$	43. $2^5 = 32$
11. 10	19. 100	27. $10, 10$	37. 3	45. $\log_2 7$
13. 7	21. 18	29. $15, -4$	39. e	47. 4

C: Expand logarithms

Exercise 4.5e.C

★ In the following exercises, use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify if possible.

49. $\log_5 8y$	50. $\log_4 6x$	51. $\log_3 81xy$	52. $\log_2 32xy$	53. $\log 1000y$	54. $\log 100x$

★ In the following exercises, use the Quotient Property of Logarithms to write each logarithm as a sum of logarithms. Simplify if possible.

55. $\log_6 \frac{5}{6}$	57. $\log_5 \frac{125}{x}$	59. $\log \frac{10,000}{y}$	61. $\ln \frac{e^4}{16}$
56. $\log_3 \frac{3}{8}$	58. $\log_4 \frac{16}{y}$	60. $\log \frac{x}{10}$	62. $\ln \frac{e^3}{3}$

★ In the following exercises, use the Power Property of Logarithms to expand each. Simplify if possible.

63. $\log_2 x^5$	65. $\log x^{-3}$	67. $\log_5 \sqrt[3]{x}$	69. $\ln x^{\sqrt[3]{4}}$
64. $\log_3 x^2$	66. $\log x^{-2}$	68. $\log_4 \sqrt{x}$	70. $\ln x^{\sqrt{3}}$

Answers to odd exercises:

49. $\log_5 8 + \log_5 y$ 51. $4 + \log_3 x + \log_3 y$	53. $3 + \log y$ 55. $\log_6 5 - 1$	57. $3 - \log_5 x$ 59. $4 - \log y$	61. $4 - \ln 16$ 63. $5 \log_2 x$	65. $-3 \log x$ 67. $\frac{1}{3}\log_5 x$	69. $\sqrt[3]{4} \ln x$

D: Expand logarithms

Exercise 4.5e.D

★ In the following exercises, use the Properties of Logarithms to expand the logarithm. Simplify if possible.

71. $\log_2 (3x^5y^3)$	77. $\log_3 (9x^2)$	83. $\log_6 (36(x+y)^4)$	89. $\log \left(\sqrt{x^3y^{-4}}\right)$
72. $\log_5 (4x^6y^4)$	78. $\log_2 (32x^7)$	84. $\ln [(e^4(x-y)^3)$	90. $\log_2(y^x)$
73. $\log_5 \left(\sqrt[4]{21}y^3\right)$	79. $\ln (3y^2)$	85. $\log_7(2\sqrt{xy})$	91. $\log \left(x^2y^3\sqrt[3]{x^2y^5}\right)$
74. $\log_3 \left(\sqrt{2}x^2\right)$	80. $\log (100x^2)$	86. $\ln(2x\sqrt{y})$	92. $\log(x^4y)$
75.$\log_4(xy)$	81. $\log (10x^2y^3)$	87. $\log_b(7x \cdot 2y)$	93. $\log_5 \sqrt{125xy^3}$
76. $\log(6x)$	82. $\log_2 (2x^4y^5)$	88. $\ln(3ab \cdot 5c)$	94. $\ln(a\sqrt[3]{b})$

Answers to odd exercises:

71. $\log_2 3 + 5\log_2 x + 3\log_2 y$ 73. $\frac{1}{4}\log_5 21 + 3\log_5 y$ 75. $\log_4 x + \log_4 y$ 77. $2 + 2\log_3 x$	79. $\ln 3 + 2\ln y$ 81. $1 + 2\log x + 3\log y$ 83. $2 + 4\log_6(x+y)$ 85. $\log_7 2 + \frac{1}{2}\log_7 x + \frac{1}{2}\log_7 y$	87. $\log_b 14 + \log_b(x) + \log_b(y)$ 89. $\frac{3}{2}\log(x) - 2\log(y)$ 91. $\frac{8}{3}\log(x) + \frac{14}{3}\log(y)$ 93: $\frac{3}{2} + \frac{1}{2}log_5 x + \frac{3}{2}log_5 y$

★ **In the following exercises, use the Properties of Logarithms to expand the logarithm. Simplify if possible.**

95. $\log_5 \frac{4ab^3c^4}{d^2}$

96. $\log_3 \frac{xy^2}{z^2}$

97. $\log_2 \left(\frac{x}{y^2}\right)$

98. $\log_5 \left(\frac{25}{x}\right)$

99. $\log_3 \left(\frac{x^3}{yz^2}\right)$

100. $\log \left(\frac{x}{y^3z^2}\right)$

101. $\log_5 \left(\frac{1}{x^2yz}\right)$

102. $\log_4 \left(\frac{1}{16x^2z^3}\right)$

103. $\log \left(\frac{100x^3}{(y+10)^3}\right)$

104. $\log \left(\frac{2(x+y)^3}{z^2}\right)$

105. $\log_b \left(\frac{13}{17}\right)$

106. $\log_4 \left(\frac{\frac{x}{z}}{w}\right)$

107. $\ln \left(\frac{1}{4e^k}\right)$

108. $ln \frac{9e^5}{4b}$

109. $\log \left(\frac{x^{15}y^{13}}{z^{19}}\right)$

110. $\ln \left(\frac{a^{-2}}{b^{-4}c^5}\right)$

Answers to odd exercises:

95. $\log_5 4 + \log_5 a + 3\log_5 b + 4\log_5 c - 2\log_5 d$

97. $\log_2 x - 2\log_2 y$

99. $3\log_3 x - \log_3 y - 2\log_3 z$

101. $-2\log_5 x - \log_5 y - \log_5 z$

103. $2 + 3\log x - 3\log(y+10)$

105. $\log_b(13) - \log_b(17)$

107. $-\ln(4) - k$

109. $15\log(x) + 13\log(y) - 19\log(z)$

★ **In the following exercises, use the Properties of Logarithms to expand the logarithm. Simplify if possible.**

111. $\log_3 \frac{\sqrt[3]{x^2}}{27y^4}$

112. $\log_4 \frac{\sqrt{x}}{16y^4}$

113. $\log_3 \frac{\sqrt{3x+2y^2}}{5z^2}$

114. $\log_2 \frac{\sqrt{2x+y^2}}{8z^2}$

115. $\log_5 \sqrt[3]{\frac{3x^2}{4y^3z}}$

116. $\log_2 \sqrt[4]{\frac{5x^3}{2y^2z^4}}$

117. $\log_3 \left(\frac{x^2\sqrt[3]{y}}{z}\right)$

118. $\log_7 \left(\frac{x}{\sqrt[5]{(y+z)^3}}\right)$

119. $\log_5 \left(\frac{x^3}{\sqrt[3]{yz^2}}\right)$

120. $\log \left(\frac{x^2}{\sqrt[5]{y^3z^2}}\right)$

122. $\ln \left(y\sqrt{\frac{y}{1-y}}\right)$

123. $\ln(\frac{6}{\sqrt{e^3}})$

124. $\log_4 \frac{\sqrt[3]{xy}}{64}$

Answers to odd exercises:

111. $\frac{2}{3}\log_3 x - 3 - 4\log_3 y$

113. $\frac{1}{2}\log_3(3x+2y^2) - \log_3 5 - 2\log_3 z$

115. $\frac{1}{3}(\log_5 3 + 2\log_5 x - \log_5 4) - 3\log_5 y - \log_5 z)$

117. $2\log_3 x + \frac{1}{3}\log_3 y - \log_3 z$

119. $3\log_5 x - \frac{1}{3}\log_5 y - \frac{2}{3}\log_5 z$

123: $-\frac{3}{2} + \ln 6$

E: Apply Log Properties

Exercise 4.5e.E

★ **Given $\log_3 x = a, \log_3 y = b$, and $\log_3 z = c$, write the following logarithms in terms of a, b, and and c.**

125. $\log_3(27x^2y^3z)$

126. $\log_3(xy^3\sqrt{z})$

127. $\log_3 \left(\frac{9x^2y}{z^3}\right)$

128. $\log_3 \left(\frac{\sqrt[3]{x}}{yz^2}\right)$

★ **Given $\log_b 2 = 0.43, \log_b 3 = 0.68$, and $\log_b 7 = 1.21$, calculate the following. (Hint: Expand using sums, differences, and quotients of the factors 2, 3, and 7.)**

129. $\log_b 42$

130. $\log_b(36)$

131. $\log_b \left(\frac{28}{9}\right)$

132. $\log_b \sqrt{21}$

★ **Expand using the properties of the logarithm and then approximate using a calculator to the nearest tenth.**

133. $\log\left(3.10\times10^{25}\right)$
134. $\log\left(1.40\times10^{-33}\right)$
135. $\ln\left(6.2e^{-15}\right)$
136. $\ln\left(1.4e^{22}\right)$

Answers to odd exercises:

125. $3+2a+3b+c$ 127. $2+2a+b-3c$ 129. 2.32 131. 0.71 133. $\log(3.1)+25\approx25.5$ 135. $\ln(6.2)-15\approx-13.2$

F: Condense Logarithms

Exercise 4.5e.F

★ **For the following exercises, condense each expression to a single logarithm with a coefficient 1 using the properties of logarithms.**

137. $\log x+\log y$
138. $\log_3 x-\log_3 y$
139. $\log_b(28)-\log_b(7)$
140. $\ln(a)-\ln(d)-\ln(c)$
141. $\ln(7)+\ln(x)+\ln(y)$
142. $\log_3 2+\log_3 a+\log_3 11+\log_3 b$
143. $\ln\left(6x^9\right)-\ln\left(3x^2\right)$
144. $\log\left(2x^4\right)+\log\left(3x^5\right)$
145. $\log 4+\log 25$
146. $\log_6 4+\log_6 9$
147. $\log_3 36-\log_3 4$
148. $\log_2 80-\log_2 5$
149. $\log_2 5-\log_2(x-1)$
150. $\log_3 4+\log_3(x+1)$
151. $\log_5 2-\log_5 x-\log_5 y$
152. $\log_7 3+\log_7 x-\log_7 y$
153. $\log(x+1)+\log(x-1)$
154. $\log_2(x+2)+\log_2(x+1)$
155. $\ln\left(x^2+2x+1\right)-\ln(x+1)$
156. $\ln\left(x^2-9\right)-\ln(x+3)$
157. $\log_5\left(x^3-8\right)-\log_5(x-2)$
158. $\log_3\left(x^3+1\right)-\log_3(x+1)$
159. $\log x+\log(x+5)-\log\left(x^2-25\right)$
160. $\log(2x+1)+\log(x-3)-\log\left(2x^2-5x-3\right)$

Answers to odd exercises:

137. $\log(xy)$
139. $\log_b(4)$
141. $\ln(7xy)$
143. $\ln\left(2x^7\right)$
145. 2
147. 2
149. $\log_2\frac{5}{x-1}$
151. $\log_5\frac{2}{xy}$
153. $\log\left(x^2-1\right)$
155. $\ln(x+1)$
157. $\log_5\left(x^2+2x+4\right)$
159. $\log\left(\frac{x}{x-5}\right)$

★ **For the following exercises, condense each expression to a single logarithm with a coefficient 1 using the properties of logarithms.**

161. $-\log_b\left(\frac{1}{7}\right)$
162. $\frac{1}{3}\ln(8)$
163. $6\log_3 x+9\log_3 y$
164. $4\log_2 x-6\log_2 y$
165. $\frac{1}{3}\log_2 x+\frac{2}{3}\log_2 y$
166. $-2\log_5 x+\frac{3}{5}\log_5 y$
167. $\log 5+3\log(x+y)$
168. $4\log_5(x+5)+\log_5 y$
169. $2\log(2x+3)+\frac{1}{2}\log(x+1)$
170. $\frac{1}{3}\log x-3\log(x+1)$
171. $\frac{2}{3}\log_3 x-\frac{1}{2}\left(\log_3 y+\log_3 z\right)$
172. $\log_3\left(x^2-1\right)-2\log_3(x-1)$
173. $\log\left(x^2+2x+1\right)-2\log(x+1)$
178. $\frac{1}{5}\left(\log_7 x+2\log_7 y\right)-2\log_7(z+1)$
174) $2\log(x)+3\log(x+1)$
175. $\frac{1}{3}(\ln x+2\ln y)-(3\ln 2+\ln z)$

Answers to odd exercises:

161. $\log_b(7)$
163. $\log_3 x^6y^9$
165. $\log_2\left(\sqrt[3]{xy^2}\right)$
167. $\log\left(5(x+y)^3\right)$
169. $\log\left(2x+3\right)^2\cdot\sqrt{x+1}$
171. $\log_3\left(\frac{\sqrt[3]{x^2}}{\sqrt{yz}}\right)$
173. 0
175. $\ln\left(\frac{\sqrt[3]{xy^2}}{8z}\right)$

★ **For the following exercises, condense each expression to a single logarithm with a coefficient 1 using the properties of logarithms.**

176. $4\log_7(c)+\frac{\log_7(a)}{3}+\frac{\log_7(b)}{3}$
177. $3\ln x+4\ln y-2\ln z$
178. $4\log x-2\log y-3\log z$
179. $\log_2 5+2\log_2 x+\log_2 y$
180. $\log_3 4+3\log_3 x+\frac{1}{2}\log_3 y$
181. $3\log_2 x-2\log_2 y+\frac{1}{2}\log_2 z$
182. $4\log x-\log y-\log 2$
183. $\ln x-6\ln y+\ln z$
184. $\log_3 x-2\log_3 y+5\log_3 z$
185. $7\log x-\log y-2\log z$
186. $2\ln x-3\ln y-\ln z$
187. $1+\log_2 x-\frac{1}{2}\log_2 y$
188. $2-3\log_3 x+\frac{1}{3}\log_3 y$
189. $-\ln 2+2\ln(x+y)-\ln z$
190. $-3\ln(x-y)-\ln z+\ln 5$
191. $\log(x)-\frac{1}{2}\log(y)+3\log(z)$
192. $4\log 2+\frac{2}{3}\log x-4\log(y+z)$
193. $\log_2 3-2\log_2 x+\frac{1}{2}\log_2 y-4\log_2 z$
194. $2\log_5 4-\log_5 x-3\log_5 y+\frac{2}{3}\log_5 z$

Answers to odd exercises:

177. $\ln\frac{x^3y^4}{z^2}$
179. $\log_2(5x^2y)$
181. $\log_2\left(\frac{x^3\sqrt{z}}{y^2}\right)$
183. $\ln\left(\frac{xz}{y^6}\right)$
185. $\log\left(\frac{x^7}{yz^2}\right)$
187. $\log_2\left(\frac{2x}{\sqrt{y}}\right)$
189. $\ln\left(\frac{(x+y)^2}{2z}\right)$
191. $\log\left(\frac{xz^3}{\sqrt{y}}\right)$
193. $\log_2\left(\frac{3\sqrt{y}}{x^2z^4}\right)$

G: Use the Change of Base Formula

Exercise 4.5e.G

★ **For the following exercises, rewrite each expression as an equivalent ratio of logs using the indicated base.**

197. $\log_7(15)$ to base e
198. $\log_{14}(55.875)$ to base 10

★ **For the following exercises, use the change-of-base formula and either base 10 or base e to evaluate the given expressions. Answer in exact form and in approximate form, rounding to four decimal places.**

199. $\log_3(22)$
200. $\log_8(65)$
201. $\log_6(5.38)$
202. $\log_4\left(\frac{15}{2}\right)$
203. $\log_{\frac{1}{2}}(4.7)$
204. $\log_3 42$
205. $\log_5 46$
206. $\log_{12} 87$
207. $\log_{15} 93$
208. $\log_{\sqrt{2}} 17$
209. $\log_{\sqrt{3}} 21$
210. $\log_5 47$
211. $\log_7 82$
212. $\log_6 103$
213. $\log_{0.5} 211$
214. $\log_2 \pi$
215. $\log_{0.2} 0.452$

★ **For the following exercises, suppose $\log_5(6)=a$ and $\log_5(11)=b$. Use the change-of-base formula along with properties of logarithms to rewrite each expression in terms of a and b. Show the steps for solving.**

216. $\log_{11}(5)$
217. $\log_6(55)$
218. $\log_{11}\left(\frac{6}{11}\right)$

Answers to odd exercises:

197. $\frac{\ln(15)}{\ln(7)}\approx 1.3917$
199. $\frac{\ln(22)}{\ln(3)}\approx 2.8136$
201. $\frac{\ln(5.38)}{\ln(6)}\approx 0.9391$
203. $\frac{\ln(4.7)}{\ln(.5)}\approx -2.2327$
205. $\frac{\ln(46)}{\ln(5)}\approx 2.3789$
207. $\frac{\ln(93)}{\ln(15)}\approx 1.6738$
209. $\frac{\ln(21)}{\ln(\sqrt{3}}\approx 5.5425$
211. $\frac{\log 82}{\log 7}\approx 2.2646$
213. $\frac{\log 211}{\log 0.5}\approx -7.7211$
215. $\frac{\log 0.452}{\log 0.2}\approx 0.4934$
217. $\frac{\log_5(5\cdot 11)}{\log_5(6)}=\frac{1+b}{a}$

H: "Extensions"

Exercise $4.5e.H$

220. Use the product rule for logarithms to find all x values such that $\log_{12}(2x+6)+\log_{12}(x+2)=2$.

221. Use the quotient rule for logarithms to find all x values such that $\log_6(x+2)-\log_6(x-3)=1$.

222. Can the power property of logarithms be derived from the power property of exponents using the equation $b^x=m$? If not, explain why. If so, show the derivation.

223. Prove that $\log_b(n)=\frac{1}{\log_n(b)}$ for any positive integers $b>1$ and $n>1$.

224. Does $\log_{81}(2401)=\log_3(7)$? Verify the claim algebraically.

Answers to odd exercises:

221. Rewriting as an exponential equation and solving for x: $6^1=\frac{x+2}{x-3}\rightarrow 0=\frac{x+2}{x-3}-6=\frac{x+2}{x-3}-\frac{6(x-3)}{(x-3)}=\frac{x+2-6x+18}{x-3}=\frac{x-4}{x-3}$ $\rightarrow x=4$. Checking, we find that $\log_6(4+2)-\log_6(4-3)=\log_6(6)-\log_6(1)$ is defined, so $x=4$

223. Let b and n be positive integers >1 . By the change-of-base formula, $\log_b(n)=\frac{\log_n(n)}{\log_n(b)}=\frac{1}{\log_n(b)}$

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4.6: Exponential and Logarithmic Equations

Learning Objectives

- Solve exponential equations by rewriting with a common base, or rewriting in logarithmic form.
- Solve logarithmic equations by rewriting in exponential form or using the one-to-one property of logarithms.

In 1859, an Australian landowner named Thomas Austin released **24** rabbits into the wild for hunting. Because Australia had few predators and ample food, the rabbit population exploded. In fewer than ten years, the rabbit population numbered in the millions. Uncontrolled population growth, as in the wild rabbits in Australia, can be modeled with exponential functions. Equations resulting from those exponential functions can be solved to analyze and make predictions about exponential growth. In this section, we will learn techniques for solving exponential functions.

Solve Exponential Equations

Use the One-to-One Property of Exponential Functions

The first technique involves two functions with like bases. Recall that the one-to-one property of exponential functions tells us that, for any real numbers b, S, and T, where $b > 0$, $b \neq 1$, $b^S = b^T$ if and only if $S = T$.

In other words, when an **exponential equation** has the same base on each side, the exponents must be equal. This also applies when the exponents are algebraic expressions. Therefore, we can solve many exponential equations by using the rules of exponents to rewrite each side as a power with the same base. Then, we use the fact that exponential functions are one-to-one to set the exponents equal to one another, and solve for the unknown.

For example, consider the equation $3^{4x-7} = \frac{3^{2x}}{3}$. To solve for x, we use the division property of exponents to rewrite the right side so that both sides have the common base, **3**. Then we apply the one-to-one property of exponents by setting the exponents equal to one another and solving for x:

$$\begin{aligned} 3^{4x-7} &= \frac{3^{2x}}{3} && \\ 3^{4x-7} &= \frac{3^{2x}}{3^1} && \text{Rewrite 3 as } 3^1 \\ 3^{4x-7} &= 3^{2x-1} && \text{Use the division property of exponents} \\ 4x-7 &= 2x-1 && \text{Apply the one-to-one property of exponents} \\ 2x &= 6 && \text{Subtract 2x and add 7 to both sides} \\ x &= 3 && \text{Divide by 3} \end{aligned}$$

THE 1-1 PROPERTY OF EXPONENTIAL FUNCTIONS

For any algebraic expressions S and T, and any positive real number $b \neq 1$,

$$b^S = b^T \text{ if and only if } S = T \tag{4.6.1}$$

How to: Solve an exponential equation of the form $b^S = b^T$, where S and T are algebraic expressions.

1. Use the rules of exponents to simplify, if necessary, so that the resulting equation has the form $b^S = b^T$.
2. Use the one-to-one property to set the exponents equal.
3. Solve the resulting equation, $S = T$, for the unknown.

Example 4.6.1: Solve an Exponential Equation with a Common Base

Solve $2^{x-1} = 2^{2x-4}$.

Solution

$2^{x-1}=2^{2x-4}$	The common base is 2
$x-1=2x-4$	By the one-to-one property the exponents must be equal
$x=3$	Solve for x

Try It 4.6.1

Solve $5^{2x}=5^{3x+2}$.

Answer

$x=-2$

Common Base Method

Sometimes the *common base* for an exponential equation is not explicitly shown. In these cases, we simply rewrite the terms in the equation as powers with a common base, and solve using the one-to-one property.

For example, consider the equation $256=4^{x-5}$. We can rewrite both sides of this equation as a power of 2. Then we apply the rules of exponents, along with the one-to-one property, to solve for x:

$256=4^{x-5}$	
$2^8=(2^2)^{x-5}$	Rewrite each side as a power with base 2
$2^8=2^{2x-10}$	Use the one-to-one property of exponents
$8=2x-10$	Apply the one-to-one property of exponents
$18=2x$	Add 10 to both sides
$x=9$	Divide by 2

How to: Given an exponential equation with unlike bases, use the one-to-one property to solve it.

1. Rewrite each side in the equation as a power with a common base.
2. Use the rules of exponents to simplify, if necessary, so that the resulting equation has the form $b^S=b^T$.
3. Use the one-to-one property to set the exponents equal.
4. Solve the resulting equation, $S=T$, for the unknown.

Example 4.6.2: Solve Equations by Rewriting Them to Have a Common Base

Solve $8^{x+2}=16^{x+1}$.

Solution

$8^{x+2}=16^{x+1}$	
$(2^3)^{x+2}=(2^4)^{x+1}$	Write 8 and 16 as powers of 2
$2^{3x+6}=2^{4x+4}$	To take a power of a power, multiply exponents
$3x+6=4x+4$	Use the one-to-one property to set the exponents equal
$x=2$	Solve for x

Try It 4.6.2

Solve $5^{2x}=25^{3x+2}$.

Answer

$x=-1$

4.6.3

Solve $2^{5x} = \sqrt{2}$.

Solution

$$2^{5x} = 2^{\frac{1}{2}} \quad \text{Write the square root of 2 as a power of 2}$$
$$5x = \frac{1}{2} \quad \text{Use the one-to-one property}$$
$$x = \frac{1}{10} \quad \text{Solve for } x$$

Try It 4.6.3

Solve $5^x = \sqrt{5}$.

Answer

$$x = \frac{1}{2}$$

Do all exponential equations have a solution? How can we tell if there is not a solution?

No. Recall that the range of an exponential function is always positive. In the process of solving an exponential equation, if the equation obtained is an exponential expression that is not equal to a positive number, there is no solution for that equation.

Example 4.6.4: Exponential Equation with no solution

Solve $3^{x+1} = -2$.

Solution

This equation has no solution. There is no real value of x that will make the equation a true statement because any power of a positive number is positive. The figure below shows that the two graphs do not cross so the left side of the equation is never equal to the right side. Thus the equation has no solution.

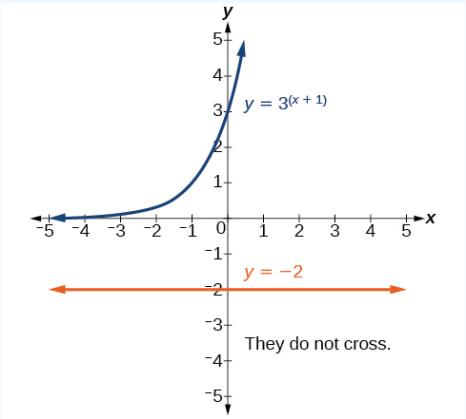

Figure 4.6.4

Try It 4.6.4

Solve $2^x = -100$.

Answer

The equation has no solution.

Rewrite in Logarithmic Form

Sometimes the terms of an exponential equation cannot be rewritten with a common base. In these cases, we solve by taking the logarithm of each side. Recall, since $\log(a) = \log(b)$ is equivalent to $a = b$, we may apply logarithms with the same base on both sides of an exponential equation.

How to: Solve an exponential equation in which a common base cannot be found

1. Apply the logarithm to both sides of the equation.
   - If one of the terms in the equation has base 10, use the common logarithm.
   - If one of the terms in the equation has base e, use the natural logarithm.
2. Use the rules of logarithms to solve for the unknown.

Example 4.6.5: Solve an Equation Containing Powers of Different Bases

Solve $5^{x+2} = 4^x$.

Solution

$$\begin{aligned}
5^{x+2} &= 4^x && \text{There is no easy way to get the powers to have the same base} \\
\ln 5^{x+2} &= \ln 4^x && \text{Take ln of both sides} \\
(x+2)\ln 5 &= x \ln 4 && \text{Power Rule for Logarithms} \\
x\ln 5 + 2\ln 5 &= x\ln 4 && \text{Use the distributive law} \\
x\ln 5 - x\ln 4 &= -2\ln 5 && \text{Get terms containing x on one side, terms without x on the other} \\
x(\ln 5 - \ln 4) &= -2\ln 5 && \text{On the left hand side, factor out an x} \\
x\ln\left(\frac{5}{4}\right) &= \ln\left(\frac{1}{25}\right) && \text{Power and Quotient Rules for Logarithms} \\
x &= \frac{\ln\left(\frac{1}{25}\right)}{\ln\left(\frac{5}{4}\right)} && \text{Divide by the coefficient of x}
\end{aligned}$$

Try It 4.6.5

Solve $2^x = 3^{x+1}$.

Answer

$$x = \frac{\ln 3}{\ln\left(\frac{2}{3}\right)}$$

Is there any way to solve $2^x = 3^x$?

Yes. The solution is $x = 0$. (Take the ln of both sides, use the power rule, and solve for x).

Equations Containing e

One common type of exponential equations are those with base e. This constant occurs again and again in nature, in mathematics, in science, in engineering, and in finance. When we have an equation with a base e on either side, we can use the natural logarithm to solve it.

How to: Given an equation of the form $y = Ae^{kt}$, solve for t.

1. Divide both sides of the equation by A.
2. Apply the natural logarithm of both sides of the equation.
3. Divide both sides of the equation by k.

Example 4.6.6: Solve an Equation of the Form $y = Ae^{kt}$

Solve $100 = 20e^{2t}$.

Solution

$$\begin{aligned} 100 &= 20e^{2t} && \\ 5 &= e^{2t} && \text{Divide by the coefficient of the power} \\ \ln 5 &= 2t && \text{Take ln of both sides. Use the fact that } ln(x) \text{ and } e^x \text{ are inverse functions} \\ t &= \frac{\ln 5}{2} && \text{Divide by the coefficient of t} \end{aligned}$$

Analysis

Using logarithm rules, this answer can be rewritten in the form $t = \ln\sqrt{5}$. A calculator can be used to obtain a decimal approximation of the answer, $t \approx 0.8047$.

Try It 4.6.6

Solve $3e^{0.5t} = 11$.

Answer

$t = 2\ln\left(\frac{11}{3}\right)$ or $\ln\left(\frac{11}{3}\right)^2$

Does every equation of the form $y = Ae^{kt}$ have a solution?

No. When $k \neq 0$, there is a solution when y and A are either both 0, or when neither is 0 and they have the same sign. An example of an equation with this form that does not have a solution is $2 = -3e^t$, which would mean e^t is negative, which is impossible.

Example 4.6.7: Solve an Equation That Can Be Simplified to the Form $y = Ae^{kt}$

Solve $4e^{2x} + 5 = 12$.

Solution

$$\begin{aligned} 4e^{2x} + 5 &= 12 && \\ 4e^{2x} &= 7 && \text{Combine like terms} \\ e^{2x} &= \frac{7}{4} && \text{Divide by the coefficient of the power} \\ 2x &= \ln\left(\frac{7}{4}\right) && \text{Take ln of both sides and use } \ln e^u = u \\ x &= \frac{1}{2}\ln\left(\frac{7}{4}\right) && \text{Solve for x} \end{aligned}$$

Try It 4.6.7

Solve $3 + e^{2t} = 7e^{2t}$.

Answer

$$t = \ln\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}\ln(2)$$

Use properties of exponents

Being able to solve equations of the form $y = Ae^{kt}$ suggests a final way of solving exponential equations that can be rewritten in the form $a = b^{p(x)}$. We will redo example 5 using this alternate method. The method used in example 5 is good practice using log properties. This alternative approach uses exponent properties instead.

How to: Solve an exponential equation in which a common base cannot be found

1. If possible, use Rules of Exponents to write the equation in the form $a = b^{p(x)}$.
2. Rewrite the exponential equation in its logarithmic form: $p(x) = \log_b(a)$, and solve for x.

Example 4.6.8: Simplify using Exponent Rules before writing in Logarithmic form

Solve $5^{x+2} = 4^x$.

Solution

$5^{x+2} = 4^x$	Use the Exponent Product Rule in reverse
$5^x \cdot 5^2 = 4^x$	
$5^x \cdot 25 = 4^x$	
$25 = \frac{4^x}{5^x}$	Like Powers Rule
$25 = \left(\frac{4}{5}\right)^x$	Rewrite as a log equation
$x = \log_{\frac{4}{5}}(25)$	Now use the change of base rule
$x = \frac{\ln 25}{\ln \frac{4}{5}} \approx -14.4251$	

Exponential Equations that are quadratic in form

Extraneous Solutions

Sometimes the methods used to solve an equation introduce an **extraneous solution**, which is a solution that is correct algebraically but does not satisfy the conditions of the original equation. One such situation arises in solving when the logarithm is taken on both sides of the equation. In such cases, remember that the argument of the logarithm must be positive. If the value of the argument of a logarithm is negative, there is no output.

Example 4.6.9: Solve Exponential Functions that are Quadratic in Form

Solve $e^{2x} - e^x = 56$.

Solution:

$e^{2x} - e^x = 56$	
$e^{2x} - e^x - 56 = 0$	Get one side of the equation equal to zero
$(e^x + 7)(e^x - 8) = 0$	Factor by the FOIL method
$e^x + 7 = 0$	Use the Zero Factor Property to solve for x
$e^x = -7$	Rewrite in log form
$x =$ undefined	Reject the equation in which the power equals a negative number

Analysis

When we plan to use factoring to solve a problem, we always get zero on one side of the equation, because zero has the unique property that when a product is zero, one or both of the factors must be zero. We reject the equation $e^x = -7$ because a positive

number never equals a negative number. The solution $\ln(-7)$ is not a real number, and in the real number system this solution is rejected as an extraneous solution.

Try It 4.6.9

Solve $e^{2x} = e^x + 2$.

Answer

$x = \ln 2$

Solving Logarithmic Equations

Does every logarithmic equation have a solution?

No. Keep in mind that we can only apply the logarithm to a positive number. Always check for extraneous solutions.

Rewrite in Exponential Form

We have already seen that every logarithmic equation $\log_b(x) = y$ is equivalent to the exponential equation $b^y = x$. We can use this fact, along with the rules of logarithms, to solve logarithmic equations where the argument is an algebraic expression.

For example, consider the equation $\log_2(2) + \log_2(3x - 5) = 3$. To solve this equation, we can use rules of logarithms to rewrite the left side in compact form and then rewrite the logarithmic equation in exponential form to solve for x:

$$\begin{aligned} \log_2(2) + \log_2(3x-5) &= 3 && \\ \log_2(2(3x-5)) &= 3 && \text{Apply the product rule of logarithms} \\ \log_2(6x-10) &= 3 && \text{Distribute} \\ 2^3 &= 6x-10 && \text{Apply the definition of a logarithm} \\ 8 &= 6x-10 && \text{Calculate } 2^3 \\ 18 &= 6x && \text{Add 10 to both sides} \\ x &= 3 && \text{Divide by 6, then check the solution!} \end{aligned}$$

Whenever solving an equation with logarithms, it is always necessary to check that the solution is in the domain of the original equation. If it is not, it must be rejected as a solution.

USE THE DEFINITION OF A LOGARITHM TO SOLVE LOGARITHMIC EQUATIONS

For any algebraic expression S and real numbers b and c, where $b > 0$, $b \neq 1$,

$$\log_b(S) = c \text{ if and only if } b^c = S \tag{4.6.2}$$

Example 4.6.10: Rewrite a Logarithmic Equation in Exponential Form

Solve $2\ln x + 3 = 7$.

Solution

$$\begin{aligned} 2\ln x + 3 &= 7 && \\ 2\ln x &= 4 && \text{Subtract 3} \\ \ln x &= 2 && \text{Divide by 2} \\ x &= e^2 && \text{Rewrite in exponential form} \end{aligned}$$

Try It 4.6.10

Solve $6 + \ln x = 10$.

Answer

$x = e^4$

Example 4.6.11

Solve $2\ln(6x) = 7$.

Solution:

$$\begin{aligned} 2\ln(6x) &= 7 \\ \ln(6x) &= \frac{7}{2} && \text{Divide by 2} \\ 6x &= e^{\left(\frac{7}{2}\right)} && \text{Use the definition of ln} \\ x &= \tfrac{1}{6}e^{\left(\frac{7}{2}\right)} && \text{Divide by 6} \end{aligned}$$

Try It 4.6.11

Solve $2\ln(x+1) = 10$.

Answer

$x = e^5 - 1$

Use the One-to-One Property of Logarithms

As with exponential equations, we can use the one-to-one property to solve logarithmic equations. The one-to-one property of logarithmic functions tells us that, for any real numbers $S > 0$, $T > 0$ and any positive real number b, where $b \neq 1$,

$$\log_b S = \log_b T \text{ if and only if } S = T.$$

For example,

$$\text{If } \log_2(x-1) = \log_2(8), \text{ then } x - 1 = 8.$$

So, if $x - 1 = 8$, then we can solve for x, and we get $x = 9$. To check, we can substitute $x = 9$ into the original equation: $\log_2(9-1) = \log_2(8) = 3$. In other words, when a logarithmic equation has the same base on each side, the arguments must be equal. This also applies when the arguments are algebraic expressions. Therefore, when given an equation with logs of the same base on each side, we can use rules of logarithms to rewrite each side as a single logarithm. Then we use the fact that logarithmic functions are one-to-one to set the arguments equal to one another and solve for the unknown.

For example, consider the equation $\log(3x-2) - \log(2) = \log(x+4)$. To solve this equation, we can use the rules of logarithms to rewrite the left side as a single logarithm, and then apply the one-to-one property to solve for x:

$$\begin{aligned} \log(3x-2) - \log(2) &= \log(x+4) \\ \log\left(\frac{3x-2}{2}\right) &= \log(x+4) && \text{Apply the quotient rule of logarithms} \\ \frac{3x-2}{2} &= x+4 && \text{Apply the one to one property of a logarithm} \\ 3x-2 &= 2x+8 && \text{Multiply both sides of the equation by 2} \\ x &= 10 && \text{Subtract 2x and add 2} \end{aligned}$$

To check the result, substitute $x = 10$ into $\log(3x-2) - \log(2) = \log(x+4)$.

$$\begin{aligned} \log(3(10)-2) - \log(2) &= \log((10)+4) \\ \log(28) - \log(2) &= \log(14) \\ \log\left(\frac{28}{2}\right) &= \log(14) && \text{The solution checks} \end{aligned}$$

USE THE ONE-TO-ONE PROPERTY OF LOGARITHMS TO SOLVE LOGARITHMIC EQUATIONS

For any algebraic expressions S and T and any positive real number b, where $b \neq 1$,

$$\log_b S = \log_b T \quad \text{if and only if} \quad S = T$$

Note, when solving an equation involving logarithms, always check to see if the answer is correct or if it is an extraneous solution.

How to: Given an equation containing logarithms, solve it using the one-to-one property

1. Use the rules of logarithms to combine like terms, if necessary, so that the resulting equation has the form $\log_b S = \log_b T$.
2. Use the one-to-one property to set the arguments equal.
3. Solve the resulting equation, $S = T$, for the unknown.
4. Confirm that each solution is correct. If a solution when substituted in the original equation makes one of the log arguments zero or negative, that solution must be rejected.

Example 4.6.12: Use the One-to-One Property of Logarithms to Solve an Equation

Solve $\ln(x^2) = \ln(2x + 3)$.

Solution

$$\begin{aligned} \ln(x^2) &= \ln(2x+3) \\ x^2 &= 2x+3 && \text{Use the one-to-one property of the logarithm} \\ x^2 - 2x - 3 &= 0 && \text{Get zero on one side before factoring} \\ (x-3)(x+1) &= 0 && \text{Factor using FOIL} \\ x-3=0 \quad &\text{or} \quad x+1=0 && \text{If a product is zero, one of the factors must be zero} \\ x &= -1 && \text{Solve for x} \end{aligned}$$

$3^2 > 0$ and $2(3)+3 > 0$ ✓

$(-1)^2 > 0$ and $2(-1)+3 > 0$ ✓ — Check the solution when substituted in the arguments is > 0

Analysis

There are two solutions: 3 or -1. The solution -1 is negative, but it checks when substituted into the original equation because the argument of the logarithm functions is still positive.

Try It 4.6.12

Solve $\ln(x^2) = \ln 1$.

Answer

$x = 1$ or $x = -1$

Key Equations

One-to-one property for exponential functions	For any algebraic expressions S and T and any positive real number b, where $b^S = b^T$ if and only if $S = T$.
Definition of a logarithm	For any algebraic expression S and positive real numbers b and c, where $b \neq 1$, $\log_b(S) = c$ if and only if $b^c = S$.
One-to-one property for logarithmic functions	For any algebraic expressions S and T and any positive real number b, where $b \neq 1$, $\log_b S = \log_b T$ if and only if $S = T$.

Key Concepts

- We can solve many exponential equations by using the rules of exponents to rewrite each side as a power with the same base. Then we use the fact that exponential functions are one-to-one to set the exponents equal to one another and solve for the unknown.
- When we are given an exponential equation where the bases are explicitly shown as being equal, set the exponents equal to one another and solve for the unknown.
- When we are given an exponential equation where the bases are *not* explicitly shown as being equal, rewrite each side of the equation as powers of the same base, then set the exponents equal to one another and solve for the unknown.
- When an exponential equation cannot be rewritten with a common base, solve by taking the logarithm of each side.
- We can solve exponential equations with base e, by applying the natural logarithm of both sides and then using the fact that $\ln(e^U) = U$.
- After solving an exponential equation, check each solution in the original equation to find and eliminate any extraneous solutions (Specifically, b^p is ALWAYS positive).
- When given an equation of the form $\log_b(S) = c$, where S is an algebraic expression, we can use the definition of a logarithm to rewrite the equation as the equivalent exponential equation $b^c = S$, and solve for the unknown.
- When given an equation of the form $\log_b S = \log_b T$, where S and T are algebraic expressions, we can use the one-to-one property of logarithms to solve the equation $S = T$ for the unknown. Check that the answers do not make any original log arguments zero or negative.

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4.6e: Exercises - Exponential and Logarithmic Equations

A: Concepts

Exercise 4.6e.1

1) How can an exponential equation be solved?

2) When does an extraneous solution occur? How can an extraneous solution be recognized?

3) When can the one-to-one property of logarithms be used to solve an equation? When can it not be used?

Answer 1

Determine first if the equation can be rewritten so that each side uses the same base. If so, the exponents can be set equal to each other. If the equation cannot be rewritten so that each side uses the same base, then apply the logarithm to each side and use properties of logarithms to solve.

Answer 3

The one-to-one property can be used if both sides of the equation can be rewritten as a single logarithm with the same base. If so, the arguments can be set equal to each other, and the resulting equation can be solved algebraically. The one-to-one property cannot be used when each side of the equation cannot be rewritten as a single logarithm with the same base.

B: Solve Exponential Equations Using the 1-1 Property (like Bases)

Exercise 4.6e.2

★ For the following exercises, use like bases to solve the exponential equation.

4. $2^{-x}=16$	10. $64^{3x-2}=\sqrt{2}$	16. $4^{x(2x+5)}=64$	22. $3^{2x+1}\cdot 3^{x}=243$
5. $3^{x}=81$	11. $81^{2x+1}=\sqrt{3}$	17. $9^{x(x+1)}=81$	23. $2^{-3n}\cdot\frac{1}{4}=2^{n+2}$
6. $3^{x+4}=27$	12. $8^{1-5x}-32=0$	18. $e^{3(3x^2-1)}-e=0$	24. $625\cdot 5^{3x+3}=125$
7. $5^{x-1}=25$	13. $9^{2-3x}-27=0$	19. $100^{x^2}-10^{7x-3}=0$	25. $\frac{36^{3b}}{36^{2b}}=216^{2-b}$
8. $2^{3x+7}=8$	14. $4^{x^2-1}-64=0$	20. $4^{-3v-2}=4^{-v}$	26. $\left(\frac{1}{64}\right)^{3n}\cdot 8=2^{6}$
9. $2^{5x-2}=16$	15. $16^{x^2}-2=0$	21. $64\cdot 4^{3x}=16$	

Answers to odd exercises:

5. 4	11. $-\frac{7}{16}$	17. $-2,1$	23. $n=-1$
7. 3	13. $\frac{1}{6}$	19. $\frac{1}{2},3$	25. $b=\frac{6}{5}$
9. $\frac{6}{5}$	15. $\pm\frac{1}{2}$	21. $x=-\frac{1}{3}$	

C: Solve exponential equations using logarithms

Exercise 4.6e.3

★ For the following exercises, use logarithms to solve.

27. $9^{x-10}=1$	32. $7e^{3n-5}+5=-89$	37. $e^{2x}-e^{x}-132=0$	42. $3^{2x+1}=7^{x-2}$
28. $2e^{6x}=13$	33. $e^{-3k}+6=44$	38. $7e^{8x+8}-5=-95$	43. $e^{2x}-e^{x}-6=0$
29. $e^{r+10}-10=-42$	34. $-5e^{9x-8}-8=-62$	39. $10e^{8x+3}+2=8$	44. $3e^{3-3x}+6=-31$
30. $2\cdot 10^{9a}=29$	35. $-6e^{9x+8}+2=-74$	40. $4e^{3x+3}-7=53$	44.1 $4^{2x+3}=5^{x-2}$
31. $-8\cdot 10^{p+7}-7=-24$	36. $2^{x+1}=5^{2x-1}$	41. $8e^{-5x-2}-4=-90$	44.2 $9^{x}\cdot 3^{x}=4^{x-1}$

Answers to odd exercises:

27. $x = 10$
29. No solution
31. $p = \log\left(\frac{17}{8}\right) - 7$
33. $k = -\frac{\ln(38)}{\frac{3}{5}}$
35. $x = \frac{\ln\left(\frac{38}{3}\right) - 8}{9}$
37. $x = \ln 12$
39. $x = \frac{\ln\left(\frac{3}{5}\right) - 3}{8}$
41. No solution
43. $x = \ln 3$

★ Solve. Give the exact answer and the approximate answer rounded to the nearest thousandth.

45. $3^x = 5$
46. $7^x = 2$
47. $4^x = 9$
48. $2^x = 10$
49. $5^{x-3} = 13$
50. $3^{x+5} = 17$
51. $7^{2x+5} = 2$
52. $3^{5x-9} = 11$
53. $5^{4x+3} + 6 = 4$
54. $10^{7x-1} - 2 = 1$
55. $e^{2x-3} - 5 = 0$
56. $e^{5x+1} - 10 = 0$
57. $6^{3x+1} - 3 = 7$
58. $8 - 10^{9x+2} = 9$
59. $15 - e^{3x} = 2$
60. $7 + e^{4x+1} = 10$
61. $7 - 9e^{-x} = 4$
62. $3 - 6e^{-x} = 0$
63. $5^{x^2} = 2$
64. $3^{2x^2-x} = 1$
65. $100e^{27x} = 50$
66. $6e^{12x} = 2$
67. $\frac{3}{1+e^{-x}} = 1$
68. $\frac{2}{1+3e^{-x}} = 1$

Answers to odd exercises:

45. $\frac{\log 5}{\log 3} \approx 1.465$
47. $\frac{\log 3}{\log 2} \approx 1.585$
49. $\frac{3\log 5 + \log 13}{\log 5} \approx 4.594$
51. $\frac{\log 2 - 5\log 7}{2\log 7} \approx -2.322$
53. $\varnothing$
55. $\frac{3+\ln 5}{2} \approx 2.305$
57. $\frac{1-\log 6}{3\log 6} \approx 0.095$
59. $\frac{\ln 13}{3} \approx 0.855$
61. $\ln 3 \approx 1.099$
63. $\pm\sqrt{\frac{\log 2}{\log 5}} \approx \pm 0.656$
65. $-\frac{\ln 2}{27} \approx -0.026$
67. $-\ln 2 \approx -0.693$

★ Find the x- and y-intercepts of the given function.

69. $f(x) = 3^{x+1} - 4$
70. $f(x) = 2^{3x-1} - 1$
71. $f(x) = 10^{x+1} + 2$
72. $f(x) = 10^{4x} - 5$
73. $f(x) = e^{x-2} + 1$
74. $f(x) = e^{x+4} - 4$

★ Use a u-substitution to solve the following.

75. $3^{2x} - 3^x - 6 = 0$ Hint: Let $u = 3^x$
76. $4^x + 2^x - 20 = 0$
77. $100^x + 10^x - 12 = 0$
78. $10^{2x} - 10^x - 30 = 0$
79. $e^{2x} - 3e^x + 2 = 0$
80. $e^{2x} - 8e^x + 15 = 0$

Answers to odd exercises:

69. x-intercept: $\left(\frac{2\log 2 - \log 3}{\log 3}, 0\right)$; y-intercept: $(0, -1)$
71. No x-int.; y-int.: $(0, 12)$
73. No x-int.; y-int.: $\left(0, \frac{1+e^2}{e^2}\right)$
75. 1
77. $\log 3$
79. $0, \ln 2$

D: Mixed exponential equations

Exercise 4.6e.4

★ Solve each exponential equation. Find the exact answer and then approximate it to three decimal places.

81. $2^x = 74$
82. $5^x = 110$
83. $4^x = 112$
84. $3^x = 89$
85. $e^x = 8$
86. $e^x = 16$
87. $\left(\frac{1}{3}\right)^x = 8$
88. $\left(\frac{1}{2}\right)^x = 6$
89. $3e^{x+2} = 9$
90. $6e^{2x} = 24$
91. $2e^{3x} = 32$
92. $4e^{x+1} = 16$
93. $\frac{1}{3}e^x = 2$
94. $\frac{1}{4}e^x = 3$
95. $e^{x-1} + 4 = 12$
96. $e^{x+1} + 2 = 16$
97. $6^{4x-17} = 216$
98. $3^{3x+1} = 81$
99. $\frac{e^{x^2}}{e^x} = e^{20}$
100. $\frac{e^{x^2}}{e^{14}} = e^{5x}$
101. $\left(\frac{1}{2}\right)^x = 10$
102. $6^x = 91$
103. $8e^{x+5} = 56$
104. $7e^{x-3} = 35$

Answers to odd exercises:

81. $x = \frac{\log 74}{\log 2} \approx 6.209$
83. $x = \frac{\log 112}{\log 4} \approx 3.404$
85. $x = \ln 8 \approx 2.079$
87. $x = \frac{\log 8}{\log \frac{1}{3}} \approx -1.893$
89. $x = \ln 3 - 2 \approx -0.901$
91. $x = \frac{\ln 16}{3} \approx 0.924$
93. $x = \ln 6 \approx 1.792$
95. $x = \ln 8 + 1 \approx 3.079$
97. $x = 5$
99. $x = -4, x = 5$
101. $x = \frac{\log 10}{\log \frac{1}{2}} \approx -3.322$
103. $x = \ln 7 - 5 \approx -3.054$

★ **For the following exercises, solve the exponential equation exactly.**

105. $e^{3x} - 15 = 0$
106. $5^x = 125$
107. $4^{x+1} - 32 = 0$
108. $8^x = 4$
109. $10^x = 7.21$
110. $3^{x/14} = \frac{1}{10}$
111. $7^{3x-2} = 11$
112. $4 \cdot 2^{3x} - 20 = 0$

★ **For the following exercises, solve each equation. Write the exact solution, and then approximate the answer to 3 decimal places.**

113. $e^{5x} = 17$
114. $1000(1.03)^t = 5000$
115. $3^{4x-5} = 38$
116. $3(1.04)^{3t} = 8$
117. $7e^{3x-5} + 7.9 = 47$
118. $50e^{-0.12t} = 10$
119. $\log(-0.7x - 9) = 1 + 5\log(5)$
120. $\ln(3) + \ln(4.4x + 6.8) = 2$

Answers to odd exercises:

105. $\frac{ln15}{3}$
107. $\frac{3}{2}$
109. $\log 7.21$
111. $\frac{2}{3} + \frac{log11}{3log7}$
113. $\frac{\ln(17)}{5} \approx 0.567$
115. $x = \frac{\log(38 + 5\log(3))}{4\log(3)} \approx 2.078$
117. $x = \frac{1}{3}(5 + \ln(39.1/7)) \approx 2.240$
119. $x = -31340/.7 \approx -44655.714$

E: Solve log equations by rewriting in exponential form

Exercise 4.6e.5

★ **For the following exercises, use the definition of a logarithm to rewrite the equation as an exponential equation.**

121. $\log\left(\frac{1}{100}\right) = -2$
122. $\log_{324}(18) = \frac{1}{2}$

★ **For the following exercises, use the definition of a logarithm to solve the equation.**

123. $5\log_7 n = 10$
124. $-8\log_9 x = 16$
125. $4 + \log_2(9k) = 2$
126. $2\log(8n + 4) + 6 = 10$
127. $10 - 4\ln(9 - 8x) = 6$

★ **Find the x- and y-intercepts of the given function.**

128. $f(x) = \log(x - 2) + 1$
129. $f(x) = \log(x + 3) - 1$
130. $f(x) = \log_3(x + 4) - 3$
131. $f(x) = \log_2(3x) - 4$
132. $f(x) = \ln(x + 1) + 2$
133. $f(x) = \ln(2x + 5) - 6$

Answers to odd exercises:

121. $10^{-2} = \frac{1}{100}$
123. $n = 49$
125. $k = \frac{1}{36}$
127. $x = \frac{9 - e}{8}$
129. x-intercept: $(7,\ 0)$; y-intercept: $(0,\ \log 3 - 1)$
131. x-intercept: $(\frac{16}{3},\ 0)$; y-intercept: None
133. x-intercept: $\left(\frac{e^6 - 5}{2},\ 0\right)$; y-intercept: $(0,\ \ln 5 - 6)$

★ **Solve.**

134. $\log_3(2x+1)=2$
135. $\log_2(3x-7)=5$
136. $\log_4(3x+5)=\frac{1}{2}$
137. $\log(2x+20)=1$
138. $\log\left(x^2+3x+10\right)=1$
139. $\log_3 x^2=2$
140. $\log_5\left(x^2+20\right)-2=0$
141. $\ln\left(x^2-1\right)=0$
142. $\log_2(x+5)+\log_2(x+1)=5$
143. $\log_2(x-5)+\log_2(x-9)=5$
144. $\log_6 x+\log_6(2x-1)=2$
145. $\log_4 x+\log_4(x-6)=2$
146. $\log_2(x+1)-\log_2(x-2)=4$
147. $\log_3(2x+5)-\log_3(x-1)=2$
148. $\ln(2x+1)-\ln x=2$
149. $\ln x-\ln(x-1)=1$
150. $2\log_2 x=3+\log_2(x-2)$
151. $2\log_3 x=2+\log_3(2x-9)$
152. $\log_2(x+3)+\log_2(x+1)-1=0$
153. $\log_2(x-2)=2-\log_2 x$
154. $\log_2(x+2)+\log_2(1-x)=1+\log_2(x+1)$
155. $\log x-\log(x+1)=1$

Answers to odd exercises:

135. 13
137. -5
139. ± 3
141. $\pm\sqrt{2}$
143. 13
145. 8
147. 2
149. $\frac{e}{e-1}$
151. 9
153. $1\pm\sqrt{5}$
155. $\varnothing$

F: Solve log equations using the 1-1 property

Exercise 4.6e.6

★ **For the following exercises, use the one-to-one property of logarithms to solve.**

156. $\ln(10-3x)=\ln(-4x)$
157. $\log_{13}(5n-2)=\log_{13}(8-5n)$
158. $\log(x+3)-\log(x)=\log(74)$
159. $\ln(-3x)=\ln(x^2-6x)$
160. $\log_4(6-m)=\log_4(3m)$
161. $\ln(x-2)-\ln(x)=\ln(54)$
162. $\log_9(2n^2-14n)=\log_9(-45+n^2)$
163. $\ln(x^2-10)+\ln(9)=\ln(10)$
164. $\log(x+12)=\log(x)+\log(12)$
165. $\ln(x)+\ln(x-3)=\ln(7x)$
166. $\log_2(7x+6)=3$
167. $\ln(7)+\ln(2-4x^2)=\ln(14)$
168. $\log_8(x+6)-\log_8(x)=\log_8(58)$
169. $\ln(3)-\ln(3-3x)=\ln(4)$
170. $\log_3(3x)-\log_3(6)=\log_3(77)$
171. $\log_5(2x+4)=\log_5(3x-6)$
172. $\log_4(7x)=\log_4(5x+14)$
173. $\log_2(x-2)-\log_2(6x-5)=0$
174. $\ln(2x-1)=\ln(3x)$
175. $\log(x+5)-\log(2x+7)=0$
176. $\ln\left(x^2+4x\right)=2\ln(x+1)$
177. $\log_3 2+2\log_3 x=\log_3(7x-3)$
178. $2\log x-\log 36=0$
179. $\ln(x+3)+\ln(x+1)=\ln 8$
180. $\log_5(x-2)+\log_5(x-5)=\log_5 10$

Answers to odd exercises:

157. $n=1$
159. No solution
161. No solution
163. $x=\pm\frac{10}{3}$
165. $x=10$
167. $x=0$
169. $x=\frac{3}{4}$
171. 10
173. No solution
175. -2
177. $\frac{1}{2}$, 3
179. 1

G: Mixed log equations

Exercise 4.6e.7

★ **Solve for x. Give exact answer (not a decimal approximation).**

182. $\log_4 64 = 2\log_4 x$
183. $\log 49 = 2\log x$
184. $3\log_3 x = \log_3 27$
185. $3\log_6 x = \log_6 64$
186. $\log_5(4x-2) = \log_5 10$
187. $\log_3(x^2+3) = \log_3 4x$
188. $\log_3 x + \log_3 x = 2$
189. $\log_4 x + \log_4 x = 3$
190. $\log_2 x + \log_2(x-3) = 2$
191. $\log_3 x + \log_3(x+6) = 3$
192. $\log x + \log(x+3) = 1$
193. $\log x + \log(x-15) = 2$
194. $\log(x+4) - \log(5x+12) = -\log x$
195. $\log(x-1) - \log(x+3) = \log\frac{1}{x}$
196. $\log_5(x+3) + \log_5(x-6) = \log_5 10$
197. $\log_5(x+1) + \log_5(x-5) = \log_5 7$
198. $\log_3(2x-1) = \log_3(x+3) + \log_3 3$
199. $\log(5x+1) = \log(x+3) + \log 2$

Answers to odd exercises:

183. $x = 7$
185. $x = 4$
187. $x = 1, x = 3$
189. $x = 8$
191. $x = 3$
193. $x = 20$
195. $x = 3$
197. $x = 6$
199. $x = \frac{5}{3}$

★ **Solve for x.**

200. $\log_a 64 = 2$
201. $\log_a 81 = 4$
202. $\ln x = -8$
203. $\ln x = 9$
204. $\log_5(3x-8) = 2$
205. $\log_4(7x+15) = 3$
206. $\ln e^{5x} = 30$
207. $\ln e^{6x} = 18$
208. $3\log x = \log 125$
209. $7\log_3 x = \log_3 128$
210. $\log_3 x = 0$
211. $\log_5 x = -2$
212. $\log_4(x+5) = 0$
213. $\log(2x-7) = 0$
214. $\log_3(x) + 3 = 2$
215. $\log_9(x) - 5 = -4$
216. $\ln(x-5) = 1$
217. $\ln(3x) = 2$
218. $\ln\sqrt{x+3} = 2$
219. $\ln(4x-10) - 6 = -5$

Answers to odd exercises:

201. $a = 3$
203. $x = e^9$
205. $x = 7$
207. $x = 3$
209. $x = 2$
211: $x = \frac{1}{25}$
213: $x = 4$
215. $x = 9$
217. $x = \frac{e^2}{3} \approx 2.5$
219. $x = \frac{e+10}{4} \approx 3.2$

★ **Solve for x.**

220. $\log_6 x + \log_6(x-5) = \log_6 24$
221. $\log_9 x + \log_9(x-4) = \log_9 12$
222. $\log_4(x+2) - \log_4(x-1) = 0$
223. $\log_6(x+9) + \log_6 x = 2$
224. $-7 + \log_3(4-x) = -6$
225. $\ln x + \ln(x-2) = \ln 4$
226. $\log(4-2x) = \log(-4x)$
227. $\log(4) + \log(-5x) = 2$
228. $\ln(2x+9) = \ln(-5x)$
229. $\log_{11}(-2x^2-7x) = \log_{11}(x-2)$
230. $\log(x^2+13) = \log(7x+3)$
231. $\log_9(3-x) = \log_9(4x-8)$
232. $\ln(x) - \ln(x+3) = \ln(6)$
233. $\dfrac{3}{\log_2(10)} - \log(x-9) = \log(44)$
234. $\log_2(x+2) - \log_2(2x+9) = -\log_2 x$
235. $\log_6(x+1) - \log_6(4x+10) = \log_6\frac{1}{x}$

Answers to odd exercises:

221. $x = 6$
223. $x = 3$
225. $1+\sqrt{5}$
227. $x = -5$
229. No solution
231. $x = \frac{11}{5} \approx 2.2$
233. $x = \frac{101}{11} \approx 9.2$
235. $x = 5$

H: Inverses of Log and Exponent Functions

Exercise 4.6e.8

★ Find the inverse of the following functions.

237. $f(x) = \log_2(x+5)$
238. $f(x) = 4 + \log_3 x$
239. $f(x) = \log(x+2) - 3$
240. $f(x) = \ln(x-4) + 1$
241. $f(x) = \ln(9x-2) + 5$
242. $f(x) = \log_6(2x+7) - 1$
243. $g(x) = e^{3x}$
244. $g(x) = 10^{-2x}$
245. $g(x) = 2^{x+3}$
246. $g(x) = 3^{2x} + 5$
247. $g(x) = 10^{x+4} - 3$
248. $g(x) = e^{2x-1} + 1$

Answers to odd exercises:

237. $f^{-1}(x) = 2^x - 5$
239. $f^{-1}(x) = 10^{x+3} - 2$
241. $f^{-1}(x) = \frac{e^{x-5}+2}{9}$
243. $g^{-1}(x) = \frac{\ln x}{3}$
245. $g^{-1}(x) = \log_2 x - 3$
247. $g^{-1}(x) = \log(x+3) - 4$

I: Mixed log and exponential equations

Exercise 4.6e.9

★ Solve.

249. $\log(9x+5) = 1 + \log(x-5)$
250. $2 + \log_2(x^2+1) = \log_2 13$
251. $e^{5x-2} - e^{3x} = 0$
252. $3^{x^2} - 11 = 70$
253. $2^{3x} - 5 = 0$
254. $\log_7(x+1) + \log_7(x-1) = 1$
255. $\ln(4x-1) - 1 = \ln x$
256. $\log(20x+1) = \log x + 2$
257. $\frac{3}{1+e^{2x}} = 2$
258. $2e^{-3x} = 4$
259. $2e^{3x} = e^{4x+1}$
260. $2\log x + \log x - 1 = 0$
261. $3\log x = \log(x-2) + 2\log x$
262. $2\ln 3 + \ln x^2 = \ln(x^2+1)$

Answers to odd exercises:

249. 55
251. 1
253. $\frac{\log_2 5}{3}$
255. $\frac{1}{4-e}$
257. $\frac{\ln(1/2)}{2}$
259. $\ln 2 - 1$
261. $\emptyset$

J: Applications

Exercise 4.6e.10

263. In chemistry, pH is a measure of acidity and is given by the formula $\mathrm{pH} = -\log(H^+)$, where H^+ is the hydrogen ion concentration (measured in moles of hydrogen per liter of solution.) Determine the hydrogen ion concentration if the pH of a solution is 4.

264. The volume of sound, L in decibels (dB), is given by the formula $L = 10\log\left(I/10^{-12}\right)$ where I represents the intensity of the sound in watts per square meter. Determine the intensity of an alarm that emits 120 dB of sound.

265. An account with an initial deposit of $\$6,500$ earns 7.25% annual interest, compounded continuously. How much will the account be worth after 20 years?

266. The formula for measuring sound intensity in decibels D is defined by the equation $D = 10\log\left(\frac{I}{I_0}\right)$,where I is the intensity of the sound in watts per square meter and $I_0 = 10^{-12}$ is the lowest level of sound that the average person can hear. How many decibels are emitted from a jet plane with a sound intensity of $8 \cdot 3 \cdot 10^2$ watts per square meter?

267. The population of a small town is modeled by the equation $P = 1650e^{0.5t}$ where t is measured in years. In approximately how many years will the town's population reach $20,000$?

268. Atmospheric pressure P in pounds per square inch is represented by the formula $P = 14.7e^{-0.21x}$, where x is the number of miles above sea level. To the nearest foot, how high is the peak of a mountain with an atmospheric pressure of 8.369 pounds per square inch? (Hint: there are 5280 feet in a mile)

269. The magnitude M of an earthquake is represented by the equation $M = \frac{2}{3}\log\left(\frac{E}{E_0}\right)$ where E is the amount of energy released by the earthquake in joules $E_0 = 10^{4.4}$ and is the assigned minimal measure released by an earthquake. To the nearest hundredth, what would the magnitude be of an earthquake releasing $1.4 \cdot 10^{13}$ joules of energy?

270. Use the definition of a logarithm along with the one-to-one property of logarithms to prove that $b^{\log_b x} = x$.

271. Recall the formula for continually compounding interest, $y = Ae^{kt}$. Use the definition of a logarithm along with properties of logarithms to solve the formula for time t such that t is equal to a single logarithm.

272. Recall the compound interest formula $A = a(1 + \frac{r}{k})^{kt}$. Use the definition of a logarithm along with properties of logarithms to solve the formula for time t.

273. Newton's Law of Cooling states that the temperature T of an object at any time t can be described by the equation $T = T_s + (T_0 - T_s)e^{-kt}$. where T_s is the temperature of the surrounding environment, T_0 is the initial temperature of the object, and k is the cooling rate. Use the definition of a logarithm along with properties of logarithms to solve the formula for time t such that t is equal to a single logarithm.

Answers to odd exercises:

263. 10^{-4} moles per liter. 265. about $\$27,710.24$	267. about 5 years 269. about 5.83	271. $t = \ln\left(\left(\frac{y}{A}\right)^{\frac{1}{k}}\right)$	273. $t = \ln\left(\left(\frac{T-T_s}{T_0-T_s}\right)^{-\frac{1}{k}}\right)$

⋆

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4.7: Exponential and Logarithmic Models

Learning Objectives

- Compound and Continuous Interest applications
- Exponential growth and decay models.
- Newton's Law of Heating and Cooling.
- Logistic-growth models.

In previous sections we were able to solve some applications that were modeled with exponential equations. Now that we have so many more techniques to solve these equations, we are able to solve more applications.

Compound and Continuous Interest Formulas

We will again use the Compound Interest Formulas and so we list them here for reference.

Definition 4.7.1: Compound Interest

For a principal, P, invested at an interest rate, r, for t years, the new balance, A is:

$$\begin{aligned} A &= P\left(1+\tfrac{r}{n}\right)^{nt} && \text{when compounded } n \text{ times a year.} \\ A &= Pe^{rt} && \text{when compounded continuously.} \end{aligned}$$

Recall that compound interest occurs when interest accumulated for one period is added to the principal investment before calculating interest for the next period. The amount A accrued in this manner over time t is modeled by the compound interest formula $A(t) = P\left(1+\tfrac{r}{n}\right)^{nt}$ where the initial principal P is accumulating compound interest at an annual rate r where the value n represents the number of times the interest is compounded in a year.

Example 4.7.1r: Compounded interest - Find time and total amount

Susan invested \$500 in an account earning $4\frac{1}{2}\%$ annual interest that is compounded monthly.

a. How much will be in the account after 3 years? b. How long will it take for the amount to grow to \$750?

Solution

State variable values and formula. In this example, the principal $P = \$500$, the interest rate $r = 4\frac{1}{2}\% = 0.045$, and because the interest is compounded monthly, $n = 12$. The investment can be modeled by the following formula:

$$\begin{aligned} A(t) &= P\left(1+\frac{r}{n}\right)^{nt} && \text{Formula} \\ A(t) &= 500\left(1+\frac{0.045}{12}\right)^{12t} \\ A(t) &= 500(1.00375)^{12t} && \text{Model equation} \end{aligned} \tag{4.7.1}$$

a. Use this model equation (4.7.1) to calculate the amount in the account after $t = 3$ years.

$$\begin{aligned} A(3) &= 500(1.00375)^{12(3)} \\ &= 500(1.00375)^{36} \\ &\approx 572.12 \end{aligned}$$

Rounded off to the nearest cent, after 3 years, the amount accumulated will be \$572.12.

b. To calculate the time it takes to accumulate \$750, set $A(t) = 750$ and solve for t using Equation (4.7.1).

$$\begin{aligned} A(t) &= 500(1.00375)^{12t} \\ 750 &= 500(1.00375)^{12t} \end{aligned}$$

This results in an exponential equation that can be solved by first isolating the exponential expression.

$$750 = 500(1.00375)^{12t}$$
$$\frac{750}{500} = (1.00375)^{12t}$$
$$1.5 = (1.00375)^{12t}$$

At this point take the common logarithm of both sides, apply the power rule for logarithms, and then solve for t.

$$\log(1.5) = \log(1.00375)^{12t}$$
$$\log(1.5) = 12t\log(1.00375)$$
$$\frac{\log(1.5)}{12\log(1.00375)} = \frac{\cancel{12}\,t\,\cancel{\log(1.00375)}}{\cancel{12}\,\cancel{\log(1.00375)}}$$
$$\frac{\log(1.5)}{12\,\log(1.00375)} = t$$

Using a calculator we can approximate the time it takes.

$$t = \log(1.5)/(12 * \log(1.00375)) \approx 9 \text{ years}$$

Answers:

a. $\$572.12$ b. Approximately 9 years

Example 4.7.2r: Compounded interest - Find time

Mario invested $\$1000$ in an account earning 6.3% annual interest, that is compounded semi-annually. How long will it take the investment to double?

Solution

State variable values and formula. Here the principal $P = \$1,000$, the interest rate $r = 6.3\% = 0.063$, and because the interest is compounded semi-annually $n = 2$. This investment can be modeled as follows:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} \qquad \text{Formula}$$
$$A(t) = 1,000\left(1 + \frac{0.063}{2}\right)^{2t}$$
$$A(t) = 1,000(1.0315)^{2t} \qquad \text{Model equation}$$

Since we are looking for the time it takes to double $\$1,000$, substitute $\$2,000$ for the resulting amount $A(t)$ and then solve for t.

$$2,000 = 1,000(1.0315)^{2t}$$
$$\frac{2,000}{1,000} = (1.0315)^{2t}$$
$$2 = (1.0315)^{2t}$$

At this point we take the common logarithm of both sides.

$$2 = (1.0315)^{2t}$$
$$\log 2 = \log(1.0315)^{2t}$$
$$\log 2 = 2t\log(1.0315)$$
$$\frac{\log 2}{2\log(1.0315)} = t$$

Using a calculator we can approximate the time it takes: $t = \log(2)/(2 * \log(1.0315)) \approx 11.17$.
So the **answer** is it takes approximately 11.17 years to double at 6.3%.

If the investment in the previous example was one million dollars, how long would it take to double? To answer this we would use $P = \$1,000,000$ and $A(t) = \$2,000,000$:

$$A(t) = 1,000(1.0315)^{2t}$$
$$2,000,000 = 1,000,000(1.0315)^{2t}$$

Dividing both sides by $1,000,000$ we obtain the same exponential function as before.

$$2 = (1.0315)^{2t}$$

Hence, the result will be the same, about 11.17 years. In fact, doubling time is independent of the initial investment P.

Interest is typically compounded semi-annually $(n = 2)$, quarterly $(n = 4)$, monthly $(n = 12)$, or daily $(n = 365)$. However if interest is compounded every instant we obtain a formula for continuously compounding interest: $A(t) = Pe^{rt}$ where P represents the initial principal amount invested, r represents the annual interest rate, and t represents the time in years the investment is allowed to accrue continuously compounded interest.

Example 4.7.3r: Continuous compounding - Find time

Mary invested $\$200$ in an account earning $5\frac{3}{4}\%$ annual interest that is compounded continuously. How long will it take the investment to grow to $\$350$?

Solution

State variable values and formula. Here the principal $P = \$200$ and the interest rate $r = 5\frac{3}{4}\% = 5.75\% = 0.0575$. Since the interest is compounded continuously, use the formula $A(t) = Pe^{rt}$. Hence, the investment can be modeled by the following,

$$\begin{aligned} A(t) &= Pe^{rt} && \text{Formula} \\ A(t) &= 200e^{0.0575t} && \text{Model equation} \end{aligned}$$

To calculate the time it takes to accumulate to $\$350$, set $A(t) = 350$ and solve for t.

$$\begin{aligned} A(t) &= 200e^{0.0575t} \\ 350 &= 200e^{0.0575t} \end{aligned}$$

Begin by isolating the exponential expression.

$$\begin{aligned} \frac{350}{200} &= e^{0.0575t} \\ \frac{7}{4} &= e^{0.0575t} \\ 1.75 &= e^{0.0575t} \end{aligned}$$

Because this exponential has base e, we choose to take the natural logarithm of both sides and then solve for t.

$$\begin{aligned} \ln(1.75) &= \ln e^{0.0575t} && \textit{Apply the power rule for logarithms.} \\ \ln(1.75) &= 0.0575t \ln e && \textit{Recall that } \ln e = 1. \\ \ln(1.75) &= 0.0575t \cdot 1 \\ \tfrac{\ln(1.75)}{0.0575} &= t \end{aligned}$$

Using a calculator we can approximate the time it takes:

$$t = \ln(1.75)/0.0575 \approx 9.73 \text{ years}$$

Answer:

It will approximately 9.73 years.

When solving applications involving compound interest, look for the keyword “continuous,” or the keywords that indicate the number of annual compoundings. It is these keywords that determine which formula to choose.

Example 4.7.4m: Continuously compounded - find interest rate

Jermael’s parents put $\$10,000$ in investments for his college expenses on his first birthday. They hope the investments will be worth $\$50,000$ when he turns 18. If the interest compounds continuously, approximately what rate of growth will they need to achieve their goal?

Solution:

Identify the variables and the formula. $A = \$50,000$, $P = \$10,000$, $r = ?$, $t = 17$ years.

$$\begin{aligned} A(t) &= Pe^{rt} && \text{Formula} \\ 50,000 &= 10,000e^{r \cdot 17} && \text{Model equation with substitutions made} \end{aligned}$$

Solve for r.

$A(t) = Pe^{rt}$	Formula
$50,000 = 10,000e^{r \cdot 17}$	Model equation with substitutions made
	Now solve for r
$5 = e^{17r}$	Divide each side by $10,000$
$\ln 5 = \ln e^{17r}$	Take the natural log of each side.
$\ln 5 = 17r \ln e$	Use the Power Property.
$\ln 5 = 17r$	Simplify.
$\frac{\ln 5}{17} = r$	Divide each side by 17.
$r \approx 0.095$	Approximate the answer.

They need the rate of growth (or interest rate) to be approximately 9.5%.

Try It 4.7.4rm

a. Mario invested $\$1,000$ in an account earning 6.3% annual interest that is compounded continuously. How long will it take the investment to double?

b. Hector invests $\$10,000$ at age 21. He hopes the investments will be worth $\$150,000$ when he turns 50. If the interest compounds continuously, approximately what rate of growth will he need to achieve his goal?

c. Rachel invests $\$15,000$ at age 25. She hopes the investments will be worth $\$90,000$ when she turns 40. If the interest compounds continuously, approximately what rate of growth will she need to achieve her goal?

a. Answer

Approximately 11 years.

b. Answer

$r \approx 9.3\%$

c. Answer

$r \approx 11.9\%$

Exponential Growth and Decay

In real-world applications, we need to model the behavior of a function. Two common examples are instances of rapid growth, where the **exponential growth** function could be used, or situations where a quantity is falling rapidly toward zero, where the **exponential decay** model may be a good choice.

EXPONENTIAL Growth and Decay Model, $y = a_0e^{kt}$

The function used to model exponential growth or decay is

$$A(t) = A_0e^{kt}$$

where

- A is the amount present at time t
- t is the amount of time that has elapsed since time $t = 0$
- A_0 is the initial amount present at time $t = 0$
- k is the **rate** of growth or decay per unit time
  - if $k > 0$ the function is increasing (Exponential Growth)
  - if $k < 0$ the function is decreasing (Exponential Decay)
- The decay **factor** is $\frac{A(t+1)}{A(t)}$ or e^k
- e is Euler's constant

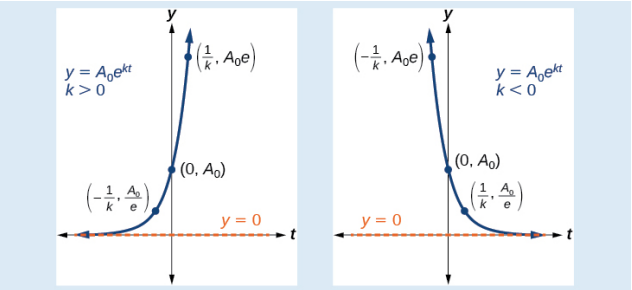

Figure 4.7.5: Exponential Functions model exponential growth ($k > 0$) and exponential decay ($k < 0$)

Exponential Growth

Example 4.7.5: Find growth rate and model equation

A population of bacteria doubles every hour. If the culture started with 10 bacteria, construct an equation that models the population as a function of time.

Solution

When an amount grows at a fixed percent per unit time, the growth is exponential. To find A_0 we use the fact that A_0 is the amount at time zero, so $A_0 = 10$. To find k, use the fact that after one hour $(t = 1)$ the population doubles from 10 to 20.The formula is derived as follows

$y = A_0e^{kt}$	**Formula**
$20 = 10e^{k \cdot 1}$	**Substitute given values**
$2 = e^k$	**Divide by 10**
$\ln 2 = k$	**Take the natural logarithm**

The growth rate is $k = \ln(2)$. $\ln(2) \approx 0.6931$ so the equation could be written $y = 10e^{0.6931t}$. However, a more exact version of the equation can be obtained by writing the equation in terms of e^k, the growth factor. Using some log properties we get $y = 10e^{(\ln 2)t} = 10(e^{\ln 2})^t = 10 \cdot 2^t$

Model Equations: using growth rate k approximation: $y = 10e^{0.6931t}$ or more exactly: $y = 10 \cdot 2^t$

Analysis. The population of bacteria after ten hours using $y = 10 \cdot 2^t$ is $10,240$. We could describe this amount is being of the order of magnitude 10^4. The population of bacteria after twenty hours is $10,485,760$ which is of the order of magnitude 10^7, so we could say that the population has increased by three orders of magnitude in ten hours.

A cautionary word about rounding. If the rounded approximation $k = 0.6931$ had been used, the result would be $10,235$. If the full precision of your calculator is used to represent k by using the [2nd][ANS] feature or [STO] and [MEMVAR] functions, for example, the result of $10,240$ would still be obtained. This illustrates that *using rounded intermediate results produces rounding errors*, so avoid getting answers that sometimes are significantly "off." Always use exact values or the full precision of your calculator in calculations to reduce or eliminate rounding errors.

Example 4.7.6r: Find model equation, total amount and time, given a growth rate

It is estimated that the population of a certain small town is $93,000$ people with an annual growth rate of 2.6%. If the population continues to increase exponentially at this rate:

a. Estimate the population in 7 years' time.
b. Estimate the time it will take for the population to reach 120,000 people.

Solution

We begin by constructing a mathematical model based on the given information. Here the initial population $P_0 = 93,000$ people and the growth rate $r = 2.6\% = 0.026$. The following model gives population in terms of time measured in years:

$$\begin{aligned} A(t) &= A_0 e^{kt} && \text{Formula} \\ P(t) &= 93,000 e^{0.026t} && \text{Model equation} \end{aligned} \tag{4.7.2}$$

a. Use this model equation (4.7.2) to estimate the population in $t = 7$ years.

$$\begin{aligned} P(t) &= 93,000 e^{0006(7)} \\ &= 93,000 e^{0.182} \\ &\approx 111,564 \quad \textit{people} \end{aligned}$$

b. Use this model equation (4.7.2) to determine the time it takes to reach $P(t) = 120,000$ people.

$$\begin{aligned} P(t) &= 93,000 e^{0.026t} \\ 120,000 &= 93,000 e^{0.026t} \\ \frac{120,000}{93,000} &= e^{0.026t} \\ \frac{40}{31} &= e^{0.026t} \end{aligned}$$

Take the natural logarithm of both sides and then solve for t.

$$\begin{aligned} \ln\left(\tfrac{40}{31}\right) &= \ln e^{0.026t} \\ \ln\left(\tfrac{40}{31}\right) &= 0.026t \ln e \\ \ln\left(\tfrac{40}{31}\right) &= 0.026t \cdot 1 \\ \tfrac{\ln\left(\frac{40}{31}\right)}{0.026} &= t \end{aligned}$$

Using a calculator,

$$t = \ln(40/31)/0.026 \approx 9.8 \quad \textit{years}$$

Answer:

a. $111,564$ people
b. 9.8 years

Often the growth rate k is not given. In this case, we look for some other information so that we can determine it and then construct a mathematical model. The general steps are outlined in the following example.

Example 4.7.7: Find growth rate, model equation, and total amount

Under optimal conditions *Escherichia coli (E. coli)* bacteria will grow exponentially with a doubling time of 20 minutes. If $1,000$ *E. coli* cells are placed in a Petri dish and maintained under optimal conditions, how many *E. coli* cells will be present in 2 hours?

Solution

The goal is to use the given information to construct a mathematical model based on the formula $A(t) = A_0 e^{kt}$.

Step 1: Find the growth rate k. Use the fact that the initial amount, $A_0 = 1,000$ cells, doubles in 20 minutes. That is, $A(t) = 2,000$ cells when $t = 20$ minutes.

$$\begin{aligned} A(t) &= A_0 e^{kt} \\ 2,000 &= 1,000 e^{k20} \end{aligned}$$

Solve for the only variable k.

$$\begin{aligned} 2,000 &= 1,000e^{k20} \\ \frac{2,000}{1,000} &= e^{k20} \\ 2 &= e^{k20} \\ \ln(2) &= \ln e^{k20} \\ \ln(2) &= k20 \ln e \\ \ln(2) &= k20 \cdot 1 \\ \frac{\ln(2)}{20} &= k \end{aligned}$$

Step 2: Write a mathematical model based on the given information for the number of *E. coli* cells in terms of time in minutes. Here $k \approx 0.0347$, which is about 3.5% growth rate per minute. So we could use the equation $A(t) = 1,000e^{0.0347t}$ for our model. However, we will use the exact value for k in our model. This will allow us to avoid round-off error in the final result. Use $A_0 = 1,000$ and $k = \ln(2)/20$:

$$\begin{aligned} A(t) &= 1,000e^{(\ln(2)/20)t} \\ &= 1,000e^{\frac{\ln(2)}{20}t} \\ &= 1,000e^{\ln(2)\frac{t}{20}} \\ &= 1,000(e^{\ln(2)})^{t/20} \\ A(t) &= 1,000(2)^{t/20} \qquad \text{Alternate form of Model equation} \end{aligned}$$

This equation models the number of *E. coli* cells in terms of time in minutes.

Step 3: Use the function to answer the questions. In this case, we are asked to find the number of cells present in 2 hours. Because time is measured in minutes, use $t = 120$ minutes to calculate the number of *E. coli* cells.

$$\begin{aligned} A(120) &= 1,000 \cdot (2)^{120/20} \\ &= 1,000 \cdot 2^6 \\ &= 64,000 \text{ cells} \end{aligned}$$

Answer:

In two hours $64,000$ cells will be present. (The answer obtained when $k = 0.0347$ is $64,328$ cells.)

We can now solve applications that give us enough information to determine the rate of growth. We can then use that rate of growth to predict other situations.

Example 4.7.8m: Find growth rate, model equation, and total amount

Researchers recorded that a certain bacteria population grew from 100 to 300 in 3 hours. At this rate of growth, how many bacteria will there be 24 hours from the start of the experiment?

Solution:

This problem requires two main steps. First we must find the unknown rate, k. Then we use that value of k to help us find the unknown number of bacteria. Use the formula $A(t) = A_0e^{kt}$.

Step 1: Find the growth rate k. Identify values for the variables in the formula. $A = 300$, $A_0 = 100$, $k = ?$, $t = 3$ hours.

$$\begin{aligned} A(t) &= A_0e^{kt} && \text{formula} \\ 300 &= 100e^{k \cdot 3} && \text{substitute. only one variable left - solve for k} \\ 3 &= e^{3k} \\ \ln 3 &= \ln e^{3k} && \text{Take the natural log of each side.} \\ \ln 3 &= 3k \ln e && \text{Use the Power Property.} \\ \ln 3 &= 3k \ln e \\ \frac{\ln 3}{3} &= k && \ln e = 1 \\ k &\approx 0.366 \end{aligned}$$

Step 2: Write the model equation using $A_0 = 100$ and $k = \frac{\ln 3}{3}$. We use this revised equation to predict the number of bacteria $A(t)$ there will be in t hours.

$$A(t) = 100e^{\frac{\ln 3}{3}t} \quad \text{or} \quad A(t) = 100e^{0.366t} \quad \text{Model equations}$$

Step 3: Use the function to answer the questions. How many bacteria will there be t hours from the start of the experiment? Substitute in the value $t = 24$ and evaluate.

$$A(24) = 100e^{\frac{\ln 3}{3} \cdot 24}$$
$$A(24) = 656,100$$

At this rate of growth, they can expect $656,100$ bacteria. (The answer obtained when $k = 0.366$ is $652,894$ bacteria.)

Example 4.7.9: Find model if not given initial amount

According to Moore's Law, the doubling time for the number of transistors that can be put on a computer chip is approximately two years. Give a function that describes this behavior.

Solution

Start with the formula Use the formula $A(t) = A_0 e^{kt}$.

Step 1. Find the growth rate k. Identify values for the variables in the formula. $A = 2A_0$, $k = ?$, $t = 2$ years.

$$\begin{aligned} A(t) &= A_0 e^{kt} && \text{formula} \\ 2A_0 &= A_0 e^{k \cdot 2} && \\ 2 &= e^{2k} && \text{DIvide both sides by } A_0 \\ \ln 2 &= \ln e^{2k} && \text{Take the natural log of each side.} \\ \ln 2 &= 2k \ln e && \text{Use the Power Property.} \\ \frac{\ln 2}{2} &= k && \ln e = 1 \end{aligned}$$

The function is $A(t) = A_0 e^{\frac{\ln 2}{2}t}$ or $A(t) = A_0(2)^{\frac{t}{2}}$.

Try It 4.7.9ma

a. Researchers recorded that a certain bacteria population grew from 100 to 500 in 6 hours. At this rate of growth, how many bacteria will there be 24 hours from the start of the experiment?

b. Recent data suggests that, as of 2013, the rate of growth predicted by Moore's Law no longer holds. Growth has slowed to a doubling time of approximately three years. Find the new function that takes that longer doubling time into account.

a. Answer

There will be $62,500$ bacteria.

b. Answer

$f(t) = A_0 e^{\frac{\ln 2}{3}t}$ or $A(t) = A_0(2)^{t/3}$

Exponential Decay

When the growth rate is negative the function models **exponential decay**. With exponential decay, quantities are declining at a rapid rate. The rate of decay k, is directly proportional to the size of the quantity and is negative. The main example of exponential decay is radioactive decay. Radioactive elements and isotopes spontaneously emit subatomic particles, and this process gradually changes the substance into a different isotope. For example, the radioactive isotope Uranium-238 eventually decays into the stable isotope Lead-206. This is a random process for individual atoms, but overall the mass of the substance decreases according to the exponential decay formula. One of the common terms associated with exponential decay is **half-life**, the length of time it takes an exponentially decaying quantity to decrease to half its original amount. Every radioactive isotope has a half-life, and the process describing the exponential decay of an isotope is called radioactive decay.

Example 4.7.10n: Exponential decay - find the equation

Suppose that a certain population of bacteria has an annual decay rate of 10%. Starting with 20,000 bacteria, how many bacteria will be left in 3 years?

Solution

Start with the formula $A(t) = A_0 e^{kt}$.

Identify values for the variables in the formula. $A = ?$, $A_0 = 20,000$, $k = 10\%$ decay $\longrightarrow k = -0.10$, $t = 3$ years.

$$\begin{aligned} A(t) &= A_0 e^{kt} && \text{formula} \\ A(3) &= 20000 e^{-0.10 \cdot 3} \\ A(3) &\approx 148.1636 && \text{grams} \end{aligned}$$

Example 4.7.11c: Exponential decay - find the equation and original amount

The loudness of a sound dies down at an exponential rate of 6.91. If it takes .2 seconds for a loud clap of thunder to diminish to the sound level of a whisper (of 30 decibels), how loud was the clap of thunder originally?

Solution

Start with the formula $A(t) = A_0 e^{kt}$.

Identify values for the variables in the formula. $A = 30$, $A_0 = ?$, $|k| = 6.91$ decay $\longrightarrow k = -6.91$, $t = .2$ seconds.

$$\begin{aligned} A(t) &= A_0 e^{kt} && \text{formula} \\ 30 &= A_0 e^{-6.91 \cdot .2} \\ 30 e^{6.91 \cdot .2} &= A_0 \\ &= A_0 \approx 119.5 && \text{decibels} \end{aligned}$$

Example 4.7.12c: Exponential decay - find the equation and elapsed time

A lawmaker once introduced legislation that would force Americans to cut back on their consumption of salt. It never passed the House, but it proposed that each year restaurants would be mandated to decrease sodium levels by two and a half percent annually. If all restaurants started out using a collective total of 5,000 kilograms of salt initially annually, and they were asked to reduce their consumption by two and a half percent each year, how long would it take to reduce their collective consumption to 3000 kg per year?

Start with the formula $A(t) = A_0 e^{kt}$.

Identify values for the variables in the formula. $A = 3000$, $A_0 = 5000$, $k = 2.5\%$ decrease $\longrightarrow k = -.025$, $t = ?$ seconds.

$$\begin{aligned} A(t) &= A_0 e^{kt} && \text{Formula} \\ 3000 &= 5000 e^{-.025t} && \text{Substitute values. Now solve for t} \\ \frac{3000}{5000} &= e^{-.025t} \\ \ln(.6) &= \ln(e^{-.025t}) && \text{Take ln of both sides} \\ \ln(.6) &= -.025t \ln(e) && \text{Power rule} \\ \ln(.6) &= -.025t && \ln(e) = 1 \\ t &= \frac{\ln(.6)}{-.025} \\ t &\approx 20.4 && \text{years} \end{aligned}$$

Half-Life

Radioactive substances decay or decompose according to the exponential decay formula. The amount of time it takes for the substance to decay to half of its original amount is called the **half-life** of the substance. As with exponential growth processes that have a fixed doubling time. Similarly, exponential decay processes have a fixed half-life, the time in which one-half the original amount decays.

Example 4.7.13: Radioactive Decay (half life) - find decay rate and equation

The half-life of carbon-14 is $5,730$ years. Express the amount of carbon-14 remaining as a function of time, t.

Solution

Start with the formula $A(t) = A_0 e^{kt}$.

Identify values for the variables in the formula. $A = \frac{1}{2}A_0$, $A_0 = ?$, $k = ?$, $t = 5730$ years.

$$\begin{aligned} A &= A_0 e^{kt} && \text{The continuous growth formula} \\ 0.5A_0 &= A_0 e^{k \cdot 5730} && \text{Substitute the half-life for t and } 0.5A_0 \text{ for } f(t) \\ 0.5 &= e^{5730k} && \text{Divide by } A_0 \\ \ln(0.5) &= 5730k && \text{Take the natural log of both sides} \\ k &= \frac{\ln(0.5)}{5730} && \text{Divide by the coefficient of k} \\ k &\approx -1.2097 \times 10^{-4} \end{aligned}$$

The function that describes this continuous decay is $A(t) = A_0 e^{\left(\frac{\ln(0.5)}{5730}\right)t}$. We observe that k is negative, as expected in the case of exponential decay. Other forms of this function include $A(t) = A_0 e^{-.00012t}$ and $A(t) = A_0\left(\frac{1}{2}\right)^{t/5730}$

Example 4.7.14m: Exponential Decay - find decay rate, equation, amount remaining

The half-life of radium-226 is $1,590$ years. How much of a 100 mg sample will be left in 500 years?

Solution:

Start with the formula $A(t) = A_0 e^{kt}$. Identify values for the variables in the formula.

Step 1. Identify values of the variables needed to find the decay constant k: At time t=1590 years, $A(1590) = \frac{1}{2}A_0$, $A_0 = ?$, $k = ?$, $t = 1590$ years. Since the next part of the question asks how much of a 100 mg sample will be left in 500 years, we may also choose to use $A(1590) = 50$ mg, and $A_0 = 100$ mg instead.

$$\begin{aligned} A(t) &= A_0 e^{kt} && \text{The continuous growth formula} \\ 50 &= 100 e^{k \cdot 1590} && \text{Substitute values. Now solve for k} \\ 0.5 &= e^{1590k} \\ \ln 0.5 &= \ln e^{1590k} && \text{Take the natural log of each side.} \\ \ln 0.5 &= 1590k \ln e && \text{Use the Power Property.} \\ \ln 0.5 &= 1590k && \ln(e) = 1 \\ k &= \frac{\ln 0.5}{1590} && \text{exact answer} \\ k &\approx -.00044 && \text{approximate answer} \end{aligned}$$

Step 2: Write the model equation that predicts the amount of radium-226 $A(t)$ there will be left in t years. The decay factor is Extra close brace or missing open brace. Also $e^{kt} = (e^k)^t$

$$A(t) = 100(.5)^{t/1590} \quad \text{or} \quad A(t) \approx 100\, e^{-.00044t} \quad \text{Model Equations}$$

Step 3: Use the model equation to answer the question. How much of a 100 mg sample will be left in 500 years? Here the values of the variables are $A(500) = ?$ mg and $t = 500$ years.

$$\begin{aligned} A(500) &= 100(.5)^{500/1590} \\ &\approx 80.41 \end{aligned}$$

In 500 years there would be approximately 80.4 mg remaining. (The answer obtained when $k = -.00044$ is 80.3 mg.)

Example 4.7.15r: Exponential Decay - find decay rate, model equation, and elapsed time

Due to radioactive decay, caesium-137 has a half-life of 30 years. How long will it take a 50-milligram sample to decay to 10 milligrams?

Solution:

Start with the formula $A(t) = A_0 e^{kt}$. Identify values for the variables in the formula.

Step 1: Find the growth rate k. Identify values for the variables in the formula: $A(30) = 25$ mg, $A_0 = 50$ mg, $k = ?$, $t = 30$ years.

$$\begin{aligned} A(t) &= A_0 e^{kt} \\ 25 &= 50e^{k\cdot 30} \\ \frac{25}{50} &= e^{30k} \\ \ln\left(\frac{1}{2}\right) &= \ln e^{30k} \\ \ln(.5) &= 30k \ln e \qquad \text{Recall that } \ln e = 1. \\ k &= \frac{\ln(.5)}{30} \\ k &\approx -0.0231049 \end{aligned}$$

Step 2: Write the model equation to predict the amount of caesium-137 that will be left in a 50-milligram sample after t years. Note that $k \approx -0.0231$ is negative. However, we will use the exact value to construct a model that gives the amount of cesium-137 with respect to time in years.

$$\begin{aligned} A(t) &= 50e^{(ln(.5)/30)t} \\ &= 50e^{ln(.5)t/30} \\ &= 50\left(e^{(ln(.5)}\right)^{t/30} \qquad \text{Use the Inverse Property } e^{\ln M} = M. \\ A(t) &= 50(.5)^{t/30} \qquad \text{or alternatively } A(t) = 50e^{-0.0231t} \text{ Model equations} \end{aligned}$$

Step 3: Use the model to find t when $A(t) = 10$ milligrams.

$$\begin{aligned} 10 &= 50(.5)^{t/30} \\ \frac{10}{50} &= (.5)^{t/30} \\ \ln(.2) &= \ln(.5)^{t/30} \\ \ln(.2) &= t/30 \ln(.5) \\ \ln(.2) &= \frac{\ln(.5)}{30} t \\ t &= \ln(.2) \cdot \frac{30}{\ln(.5)} \\ t &= \frac{30\ln(.2)}{\ln .5} \\ t &\approx 69.66 \end{aligned}$$

Answer: It will take $t \approx 69.66$ years to decay to 10 milligrams. (The answer obtained when $k = -0.0231$ is $t \approx 69.67$.)

Radiocarbon Dating

The formula for radioactive decay is important in **radiocarbon dating**, which is used to calculate the approximate date a plant or animal died. Radiocarbon dating was discovered in 1949 by Willard Libby, who won a Nobel Prize for his discovery. It compares the difference between the ratio of two isotopes of carbon in an organic artifact or fossil to the ratio of those two isotopes in the air. It is believed to be accurate to within about 1% error for plants or animals that died within the last 60,000 years. As long as a plant or animal is alive, the ratio of the two isotopes of carbon in its body is close to the ratio in the atmosphere. When it dies, the carbon-14 in its body decays and is not replaced. By comparing the ratio of carbon-14 to carbon-12 in a decaying sample to the known ratio in the atmosphere, the date the plant or animal died can be approximated. While the carbon-14 technique only works on plants and animals, there are other similar dating techniques, using other radioactive isotopes, that are used to date rocks and other inorganic matter.

Example 4.7.16r: Carbon dating - find decay rate, model equation, and elapsed time

An ancient bone tool is found to contain 25% of the carbon-14 normally found in bone. Given that carbon-14 has a half-life of 5,730 years, estimate the age of the tool.

Solution

Start with the formula $A(t) = A_0 e^{kt}$. Identify values for the variables in the formula.

Step 1: Find the growth rate k. Identify values for the variables in the formula: $A(5,730) = \frac{1}{2}A_0$ mg, $A_0 =?$ mg, $k =?$, $t = 5,730$ years. Notice below that when both sides are divided by A_0, the remaining exponential equation is only in terms of k. This shows that half-life is independent of the initial amount.

$$\begin{aligned} A(t) &= A_0 e^{kt} \\ \frac{1}{2}A_0 &= A_0 e^{k5,730} \\ .5 &= e^{k5,730} \\ \ln(.5) &= \ln e^{k5,730} \\ \ln(.5) &= 5,730k \ln e \\ k &= \frac{\ln(.5)}{5,730} \\ k &\approx -.00012 \end{aligned}$$

Step 2: Write the model equation that predicts the amount of carbon-14 remaining in the bone t years after death of the object.

$$A(t) = A_0 e^{-.00012t} \quad \text{or} \quad A(t) = A_0 e^{\frac{\ln(.5)}{5,730}t} = A_0 \left(e^{\ln(.5)}\right)^{\frac{t}{5,730}} = A_0 \, (.5)^{t/5730} \quad \text{Model equations}$$

Step 3: Use the model to find the time t it takes the carbon-14 to decay to 25% of the initial amount, or $A(t) = 0.25A_0$

$$\begin{aligned} A(t) &= A_0 \, (.5)^{t/5730} \\ 0.25A_0 &= A_0 (.5)^{t/5730} \\ 0.25 &= (.5)^{t/5730} \\ \ln(0.25) &= \ln \, (.5)^{t/5730} \\ \ln(0.25) &= t/5730 \, (\ln(.5)) \\ t &= \frac{5,730 \ln(0.25)}{\ln(.5)} \\ t &\approx 11,460 \end{aligned}$$

Answer: The tool is approximately $11,460$ years old. (The answer obtained when $k = -.00012$ is $11,552$ years old).

Try It 4.7.16*a*A

a. Researchers recorded that a certain bacteria population declined from $700,000$ to $400,000$ in 5 hours after the administration of medication. At this rate of decay, how many bacteria will there be 24 hours from the start of the experiment?

b. The half-life of plutonium-244 is $80,000,000$ years. Find function gives the amount of carbon-14 remaining as a function of time, measured in years.

a. Answer

There will be $47,699$ bacteria.
($k = \frac{1}{5}\ln(\frac{4}{7}) \approx -.11192$)

b. Answer

$f(t) = A_0 e^{-0.0000000087t}$
($k = \frac{1}{8}\ln(.5) \times 10^{-7}$)

Try It 4.7.16*b*m

a. The half-life of strontium-90 is about 28 years. How long will it take a 36 milligram sample of strontium-90 to decay to 30 milligrams?

b. Cesium-137 has a half-life of about 30 years. If we begin with 200 mg of cesium-137, how long will it take until only 1 milligram remains?

c. The half-life of magnesium-27 is 9.45 minutes. How much of a 10-mg sample will be left in 6 minutes?

d. The half-life of radioactive iodine is **60** days. How much of a **50**-mg sample will be left in **40** days?

a .Answer	**b .Answer**	**c .Answer**	**d .Answer**
7.4 years	**229.3** years	There will be **6.43** mg left.	There will be **31.5** mg left.

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Newton's Law of Cooling

Exponential decay can also be applied to temperature. When a hot object is put into a place that is at a lower temperature, the object's temperature will decrease exponentially, leveling off as it approaches the surrounding air temperature.

NEWTON'S LAW OF COOLING

The temperature of an object, T, in surrounding air with temperature S will behave according to the formula

$$T(t) = Ae^{kt} + S \qquad \begin{cases} t & \text{is time} \\ A & \text{is the difference between the initial temperature of the object and the surroundings} \\ k & \text{is a constant, the continuous rate of cooling of the object} \end{cases}$$

Example 4.7.17: Using Newton's Law of Cooling

A cheesecake is taken out of the oven with an ideal internal temperature of $165°F$, and is placed into a $35°F$ refrigerator. After 10 minutes, the cheesecake has cooled to $150°F$. If we must wait until the cheesecake has cooled to $70°F$ before we eat it, how long will we have to wait?

Solution

Because the surrounding air temperature in the refrigerator is 35 degrees, the cheesecake's temperature will decay exponentially toward 35, following the equation

$$T(t) = Ae^{kt} + 35$$

Step 1. Find A. We know the initial temperature was 165, so $T(0) = 165$.

$$\begin{aligned} 165 &= Ae^{k0} + 35 && \text{Substitute } (0, 165) \\ 165 &= A + 35 \\ A &= 130 && \text{Solve for A} \end{aligned}$$

At the conclusion of this step, we now have a modifed formula, $T(t) = 130e^{kt} + 35$

Step 2. Find k. To do this, use our modified formula, $T(t) = 130e^{kt} + 35$, and the fact that $T(10) = 150$.

$$\begin{aligned} T(t) &= 130e^{kt} + 35 \\ 150 &= 130e^{k10} + 35 && \text{Substitute } (10, 150) \\ 115 &= 130e^{k10} && \text{Subtract 35} \\ \frac{115}{130} &= e^{10k} && \text{Divide by 130} \\ \ln\left(\frac{115}{130}\right) &= 10k && \text{Take the natural log of both sides} \\ k &= \frac{\ln\left(\frac{115}{130}\right)}{10} \\ &= -0.0123 && \text{Divide by the coefficient of k} \end{aligned}$$

At the end of this step we now have the equation for the cooling of the cheesecake: $T(t) = 130e^{-0.0123t} + 35$.

Another form this equation can take is $T(t) = 130e^{\frac{\ln\left(\frac{115}{130}\right)}{10}} + 35$ which simplifies to $T(t) = 130\left(\frac{115}{130}\right)^{\frac{t}{10}} + 35$.

Step 3. Now we can solve for the time it will take for the temperature to cool to 70 degrees.

$$\begin{aligned} 70 &= 130e^{-0.0123t} + 35 && \text{Substitute in 70 for } T(t) \\ 35 &= 130e^{-0.0123t} && \text{Subtract 35} \\ \frac{35}{130} &= e^{-0.0123t} && \text{Divide by 130} \\ \ln\left(\frac{35}{130}\right) &= -0.0123t && \text{Take the natural log of both sides} \\ t &= \frac{\ln\left(\frac{35}{130}\right)}{-0.0123} \\ &\approx 106.68 && \text{Divide by the coefficient of t} \end{aligned}$$

It will take about 107 minutes, or one hour and 47 minutes, for the cheesecake to cool to $70°F$. (The answer, rounded to the nearest minute, obtained when \(k=\frac{\ln \left (\tfrac{115}{130} \right) \) is the same).

Try It 4.7.17

A pitcher of water at 40 degrees Fahrenheit is placed into a 70 degree room. One hour later, the temperature has risen to 45 degrees. How long will it take for the temperature to rise to 60 degrees?

Answer

6.026 hours

Logistic Growth Models

Exponential growth cannot continue forever. Exponential models, while they may be useful in the short term, tend to fall apart the longer they continue. Consider an aspiring writer who writes a single line on day one and plans to double the number of lines she writes each day for a month. By the end of the month, she must write over 17 billion lines, or one-half-billion pages. It is impractical, if not impossible, for anyone to write that much in such a short period of time. Eventually, an exponential model must begin to approach some limiting value, and then the growth is forced to slow. For this reason, it is often better to use a model with an upper bound instead of an exponential growth model, though the **exponential growth** model is still useful over a short term, before approaching the limiting value.

The **logistic growth model** is approximately exponential at first, but it has a reduced rate of growth as the output approaches the model's upper bound, called the **carrying capacity**. For constants a, b, and c, the logistic growth of a population over time x is represented by the model

$$f(x) = \frac{c}{1 + ae^{-bx}}$$

The graph in Figure 4.7.6 shows how the growth rate changes over time. The graph increases from left to right, but the growth rate only increases until it reaches its point of maximum growth rate, at which point the rate of increase decreases.

LOGISTIC GROWTH

The logistic growth model is

$$f(x) = \frac{c}{1 + ae^{-bx}} \qquad (4.7.3)$$

where

- $\frac{c}{1+a}$ is the initial value, $f(0)$
- c is the *carrying capacity*, or *limiting value*, $f(\infty)$
- b is a constant determined by the rate of growth.

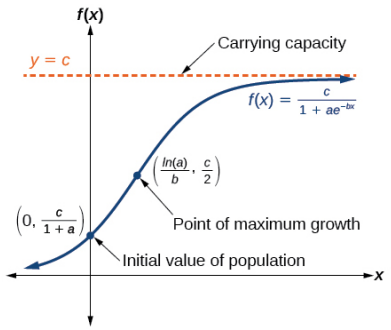

Figure 4.7.6

Example 4.7.18: Using the Logistic-Growth Model

An influenza epidemic spreads through a population rapidly, at a rate that depends on two factors: The more people who have the flu, the more rapidly it spreads, and also the more uninfected people there are, the more rapidly it spreads. These two factors make the logistic model a good one to study the spread of communicable diseases. And, clearly, there is a maximum value for the number of people infected: the entire population.

For example, at time $t = 0$ there is one person in a community of 1,000 people who has the flu. So, in that community, at most 1,000 people can have the flu. Researchers find that for this particular strain of the flu, the logistic growth constant is b

$= 0.6030$. Estimate the number of people in this community who will have had this flu after ten days. Predict how many people in this community will have had this flu after a long period of time has passed.

Solution

We substitute the given data into the logistic growth model $f(x) = \frac{c}{1 + ae^{-bx}}$

Step 1. Find one of the constants by evaluating f when $x \to \infty$. This results in e^{-bx} approaching zero, so $f(x \to \infty) = c$. Because at most the entire population of the community, $1,000$ people, can eventually get the flu, we know the limiting value is $c = 1000$.

Step 2. Find one of the constants by evaluating f when $x = 0$. We are given $f(0) = 1$. Therefore $\frac{c}{1 + a} = 1$, from which it follows that $a = 999$.

Step 3. Given that the logistic growth constant is $b = 0.6030$, we obtain the model equation $f(x) = \frac{1000}{1 + 999e^{-0.6030x}}$.

Step 4. Use the model. This model predicts that, after ten days, the number of people who have had the flu is $f(10) = \frac{1000}{1 + 999e^{-0.6030 \cdot 10}} \approx 293.8$. Because the actual number must be a whole number (a person has either had the flu or not) we round to 294. In the long term, the number of people who will contract the flu is the limiting value, $c = 1000$.

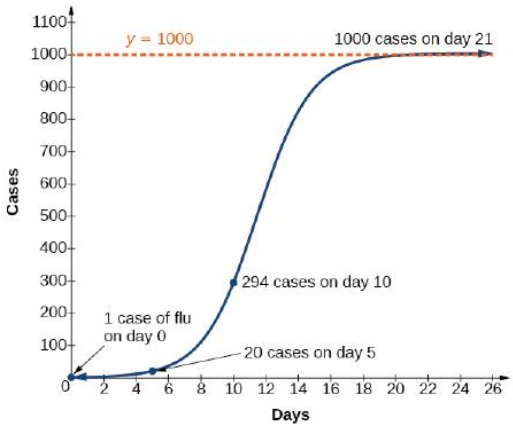

Figure 4.7.7: The graph of $f(x) = \frac{1000}{1 + 999e^{-0.6030x}}$ After a long time has passed, the entire community of 1000 people will have had this flu.

Analysis. Remember that, because we are dealing with a virus, we cannot predict with certainty the number of people infected. The model only approximates the number of people infected and will not give us exact or actual values. The graph in Figure 4.7.7 gives a good picture of how this model fits the data.

Try It 4.7.18

Using the model in the Example above, estimate the number of cases of flu on day 15.

Answer

895 cases on day 15

https://math.libretexts.org/@go/page/34908

Key Concepts

- The basic exponential function is $f(x) = ab^x$. If $b > 1$, we have exponential growth; if $0 < b < 1$, we have exponential decay.
- We can also write this formula in terms of continuous growth as $A = A_0e^{kx}$, where A_0 is the starting value. If A_0 is positive, then we have exponential growth when $k > 0$ and exponential decay when $k < 0$.
- In general, we solve problems involving exponential growth or decay in two steps. First, we set up a model and use the model to find the parameters. Then we use the formula with these parameters to predict growth and decay.
- We can find the age of an organic artifact by measuring the amount of carbon-14 (which is subject to exponential decay) remaining in the artifact.
- Given a substance's doubling time or half-time, we can find a function that represents its exponential growth or decay.
- We can use Newton's Law of Cooling to find how long it will take for a cooling object to reach a desired temperature, or to find what temperature an object will be after a given time.
- We can use logistic growth functions to model real-world situations where the rate of growth changes over time, such as population growth, spread of disease, and spread of rumors.
- Any exponential function with the form $y = A_0b^x$ can be rewritten as an equivalent exponential function with the form $y = A_0e^{kx}$ where $k = \ln b$.

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4.7e: Exercises - Exponential Applications

A: Concepts

Exercise 4.7e.A

1) With what kind of exponential model would *half-life* be associated? What role does half-life play in these models?

2) What is carbon dating? Why does it work? Give an example in which carbon dating would be useful.

3) With what kind of exponential model would *doubling time* be associated? What role does doubling time play in these models?

4) Define Newton's Law of Cooling. Then name at least three real-world situations where Newton's Law of Cooling would be applied.

5) What is an order of magnitude? Why are orders of magnitude useful? Give an example to explain.

7) A doctor and injects a patient with 13 milligrams of radioactive dye that decays exponentially. After 12 minutes, there are 4.75 milligrams of dye remaining in the patient's system. Which is an appropriate model for this situation?

a. $f(t) = 13(0.0805)^t$ b. $f(t) = 13e^{0.9195t}$ c. $f(t) = 13e^{(-0.0839t)}$ d. $f(t) = \frac{4.75}{1+13e^{-0.83925t}}$

9) What situations are best modeled by a logistic equation? Give an example, and state a case for why the example is a good fit.

10) What is a carrying capacity? What kind of model has a carrying capacity built into its formula? Why does this make sense?

Answers to odd exercises:

1. Half-life is a measure of decay and is thus associated with exponential decay models. The half-life of a substance or quantity is the amount of time it takes for half of the initial amount of that substance or quantity to decay.

3. Doubling time is a measure of growth and is thus associated with exponential growth models. The doubling time of a substance or quantity is the amount of time it takes for the initial amount of that substance or quantity to double in size.

5. An order of magnitude is the nearest power of ten by which a quantity exponentially grows. It is also an approximate position on a logarithmic scale; Sample response: Orders of magnitude are useful when making comparisons between numbers that differ by a great amount. For example, the mass of Saturn is 95 times greater than the mass of Earth. This is the same as saying that the mass of Saturn is about 10^2 times, or 2 orders of magnitude greater, than the mass of Earth.

7. C

9. Logistic models are best used for situations that have limited values. For example, populations cannot grow indefinitely since resources such as food, water, and space are limited, so a logistic model best describes populations.

B: Compound Interest

Exercise 4.7e.B

Find current amount

11. Billy's grandfather invested in a savings bond that earned 5.5% annual interest that was compounded annually. Currently, 30 years later, the savings bond is valued at $\$10,000$. Determine what the initial investment was.
12. Given a yearly interest rate of 3.5% and an initial principle of $100,000, find the amount A accumulated in 5 years for interest that is compounded a. daily, b., monthly, c. quarterly, and d. yearly.
13. Starting with $100,000 invested at an annual interest rate of 5.5% compounded continuously, find the amount accumulated after 5 years.
14. In 1935 Frank opened an account earning 3.8% annual interest that was compounded quarterly. He rediscovered this account while cleaning out his garage in 2005. If the account is now worth $\$11,294.30$, how much was his initial deposit in 1935?

Find principal

15. Given that the bank is offering 4.2% annual interest compounded monthly, what principal is needed to earn $\$25,000$ in interest for one year?
16. Given that the bank is offering 3.5% annual interest compounded continuously, what principal is needed to earn $\$12,000$ in interest for one year?

Find interest rate

17. Find the annual interest rate at which an account earning continuously compounding interest has a doubling time of 9 years.
18. Find the annual interest rate at which an account earning interest that is compounded monthly has a doubling time of 10 years.
19. Alice invests $\$15,000$ at age 30 from the signing bonus of her new job. She hopes the investments will be worth $\$30,000$ when she turns 40. If the interest compounds continuously, approximately what rate of growth will she need to achieve her goal?
20. Sung Lee invests $\$5,000$ at age 18. He hopes the investments will be worth $\$10,000$ when he turns 25. If the interest compounds continuously, approximately what rate of growth will he need to achieve his goal? Is that a reasonable expectation?

Find time

21. Jill invested $\$1,450$ in an account earning $4\frac{5}{8}\%$ annual interest that is compounded monthly.
    a. How much will be in the account after 6 years?
    b. How long will it take the account to grow to $\$2,200$?
22. James invested $\$825$ in an account earning $5\frac{2}{5}\%$ annual interest that is compounded monthly.
    a. How much will be in the account after 4 years?
    b. How long will it take the account to grow to $\$1,500$?
23. Bill wants to grow his $\$75,000$ inheritance to $\$100,000$ before spending any of it. How long will this take if the bank is offering 5.2% annual interest compounded quarterly?
24. Mary needs $\$25,000$ for a down payment on a new home. If she invests her savings of $\$21,350$ in an account earning 4.6% annual interest that is compounded semi-annually, how long will it take to grow to the amount that she needs?
25. Joe invested his $\$8,700$ savings in an account earning $6\frac{3}{4}\%$ annual interest that is compounded continuously. How long will it take to earn $\$300$ in interest?
26. Miriam invested $\$12,800$ in an account earning $5\frac{1}{4}\%$ annual interest that is compounded monthly. How long will it take to earn $\$1,200$ in interest?
27. Raul invested $\$8,500$ in an online money market fund earning 4.8% annual interest that is compounded continuously.
    a. How much will be in the account after 2 years?
    b. How long will it take the account to grow to $\$10,000$?
28. Ian deposited $\$500$ in an account earning 3.9% annual interest that is compounded continuously.
    a. How much will be in the account after 3 years?
    b. How long will it take the account to grow to $\$1,500$?

Find doubling time

29. Jose invested his $\$3,500$ bonus in an account earning $5\frac{1}{2}\%$ annual interest that is compounded quarterly. How long will it take to double his investment?
30. Maria invested her $\$4,200$ savings in an account earning $6\frac{3}{4}\%$ annual interest that is compounded semi-annually. How long will it take to double her savings?
31. Calculate the doubling time of an investment made at 7% annual interest that is compounded: (a) monthly (b) continuously
32. Coralee invests $\$5,000$ in an account that compounds interest monthly and earns 7%. How long will it take for her money to double?
33. If money is invested in an account earning 3.85% annual interest that is compounded continuously, how long will it take the amount to double?

34. Calculate the doubling time of an investment that is earning continuously compounding interest at an annual interest rate of: (a) 4% (b) 6%

Find Tripling time

36. Jill invested $\$1,450$ in an account earning $4\frac{5}{8}\%$ annual interest that is compounded monthly.
    a. Find the amount accumulated after 5 years and 10 years.
    b. Determine how long it takes for the original investment to triple.
37. Alice invested her savings of $\$7,000$ in an account earning 4.5% annual interest that is compounded monthly. How long will it take the account to triple in value?
38. Mary invested her $\$42,000$ bonus in an account earning 7.2% annual interest that is compounded continuously. How long will it take the account to triple in value?

Answers to odd exercises:

11. $\$2,006.44$ 13. Approximately \$131,653 15. $\$583,867$ 17. 7.7%	19. 6.9% 21. (a) $\$1,912.73$ (b) 9 years 23. 5.6 years 25. $\frac{1}{2}$ year	27. (a) $\$9,356.45$ (b) 3.4 years 29. 12.7 years	31. (a) 9.93 years (b) 9.90 years 33. 18 years 37. 24.5 years

C: Exponential Growth

Exercise $4.7e.C$

Find current amount

41. The number of bacteria N in a culture after t days can be modeled by the function $N(t) = 1300 \cdot (2)^{t/4}$. Find the number of bacteria present after 15 days.

42. A research student is working with a culture of bacteria that doubles in size every twenty minutes. The initial population count was 1350 bacteria. Rounding to five significant digits, write an exponential equation representing this situation. To the nearest whole number, what is the population size after 3 hours?

43. Suppose that the population of a certain town grows at an annual rate of 6%. If the population is currently 5, 000, what will it be in 7 years? Round your answer to the nearest integer.

44. Suppose that the population of a certain town grows at an annual rate of 4%. If the population is currently 1, 000, what will it be in 3 years? Round your answer to the nearest integer.

Find initial amount

46. A biologist recorded a count of 360 bacteria present in a culture after 5 minutes and 1000 bacteria present after 20 minutes.

    (a) To the nearest whole number, what was the initial population in the culture?

    (b) Rounding to six significant digits, write an exponential equation representing this situation. To the nearest minute, how long did it take the population to double?

47. Suppose that the population of a certain town grows at an annual rate of 9%. If the population grows to 7, 000 in 5 years, what was the original population? Round your answer to the nearest integer.

48. Suppose that the population of a certain town grows at an annual rate of 2%. If the population grows to 9, 000 in 4 years, what was the original population? Round your answer to the nearest integer.

Find growth rate

49. A bacterial colony grown in a lab is known to double in number in 12 hours. Suppose, initially, there are 1000 bacteria present.

a. Use the exponential function $Q = Q_0 e^{kt}$ to determine k, the **growth rate** of the bacteria. Round to four decimal places.

b. Determine approximately how long it takes for 200,000 bacteria to grow.

50. The rabbit population on a game reserve doubles every 6 months. Suppose there were 120 rabbits initially.

a. Use the exponential function $P = P_0 a^t$ to determine the **growth factor** a. Round to four decimal places.

b. Use the function in part a. to determine approximately how long it takes for the rabbit population to reach 3500.

Find time

51. Suppose that the population of a certain town grows at an annual rate of 8%. If the population is currently 2,000, how many years will it take for it to reach 7,000? Round your answer to the nearest hundredth.

52. Suppose that the population of a certain town grows at an annual rate of 4%. If the population is currently 7,000, how many years will it take for it to reach 17,000? Round your answer to the nearest hundredth.

Find doubling time

53. According to the World Bank, at the end of 2013 $(t = 0)$ the U.S. population was 316 million and was increasing according to the following model below where P is measured in millions of people and t is measured in years after 2013.

$$P(t) = 316e^{0.0074t},$$

a. Based on this model, what will be the population of the United States in 2020?

b. Determine when the U.S. population will be twice what it is in 2013.

54. Suppose that the population of a certain town grows at an annual rate of 8%. If the population is currently 4,000, how many years will it take for it to double? Round your answer to the nearest hundredth.

55. Suppose that the population of a certain town grows at an annual rate of 2%. If the population is currently 7,000, how many years will it take for it to double? Round your answer to the nearest hundredth.

Answers to odd exercises:

11. Solution: $17,491$ 43. 7610 people 47. 4,463 people	49. a. $k \approx 0.0578$, b. ≈ 92 hours 51. 15.66 yrs	53. Solution: a. 333 million *quad* b. 94 years from 2013, or in 2107 55. 34.66 yrs

D: Exponential Decay

Exercise 4.7e.D

Find current amount

61. Suppose that a certain radioactive isotope has an annual decay rate of 17.3%. Starting with a 214 gram sample, how many grams will be left after 5 years? Round your answer to the nearest hundredth.

62. Suppose that a certain radioactive isotope has an annual decay rate of 12.5%. Starting with a 127 gram sample, how many grams will be left after 6 years? Round your answer to the nearest hundredth.

Find original amount

63. Suppose that a certain radioactive isotope has an annual decay rate of 2.3%. If a particular sample decays to 25 grams after 8 years, how big (in grams) was the original sample? Round your answer to the nearest hundredth.

64. Suppose that a certain radioactive isotope has an annual decay rate of 9.8%. If a particular sample decays to 11 grams after 6 years, how big (in grams) was the original sample? Round your answer to the nearest hundredth.

Find decay rate

65. The half-life of Erbium-165 is 10.4 hours. What is the hourly decay rate? Express the decimal result to four significant digits and the percentage to two significant digits.

66. The half-life of Radium-226 is 1590 years. What is the annual decay rate? Express the decimal result to four significant digits and the percentage to two significant digits.

Find time (half life)

68. A doctor prescribes 125 milligrams of a therapeutic drug that has an hourly decay rate of about 30%.

(a) Write an exponential model, $f(t)$, representing the amount of the drug remaining in the patient's system after t hours. Then use the formula to find the amount of the drug that would remain in the patient's system after 3 hours. Round to the nearest tenth of a milligram.

(b) To the nearest tenth of an hour, what is the half-life of the drug?

(c) Using the model found in part (a), find $f(10)$ and interpret the result. Round to the nearest hundredth.

69. A scientist begins with 250 grams of a radioactive substance. After 225 minutes, the sample has decayed to 32 grams. Rounding to five significant digits, write an exponential equation representing this situation. To the nearest minute, what is the half-life of this substance?

70. A substance has a half-life of 2.045 minutes. If the initial amount of the substance was 132.8 grams, how many half-lives will have passed before the substance decays to 8.3 grams? What is the total time of decay?

71. Suppose that a certain radioactive isotope has an annual decay rate of 17.4%. What is the half-life (in years) of the isotope? Round your answer to the nearest hundredth.

72. Suppose that a certain radioactive isotope has an annual decay rate of 5.2%. What is the half-life (in years) of the isotope? Round your answer to the nearest hundredth.

Find time

73. Suppose that a certain radioactive isotope has an annual decay rate of 7.2%. How many years will it take for a 227 gram sample to decay to 93 grams? Round your answer to the nearest hundredth.

74. A tumor is injected with 0.5 grams of Iodine-125, which has a decay rate of 1.15% per day.

(a) Write an exponential model representing the amount of Iodine-125 remaining in the tumor after t days. Then use the formula to find the amount of Iodine-125 that would remain in the tumor after 30 days. Round to the nearest hundredth of a gram.

(b) To the nearest day, how long will it take for half of the Iodine-125 to decay?

75. Suppose that a certain radioactive isotope has an annual decay rate of 18.7%. How many years will it take for a 324 gram sample to decay to 163 grams? Round your answer to the nearest hundredth.

76. Iodine-131 is a radioactive substance that decays according to the function $Q(t) = Q_0 \cdot e^{-0.08664t}$, where Q_0 is the initial quantity of a sample of the substance and t is in days. Determine how long it takes (to the nearest day) for 95% of a quantity to decay.

77. Suppose that 90.1% of the normal amount of carbon-14 remains in a piece of wood. How old is the wood? The decay rate of carbon-14 is 0.012%. Round your answer to the nearest year.

78. A wooden artifact from an archeological dig contains 60 percent of the carbon-14 that is present in living trees. To the nearest year, about how many years old is the artifact? (The half-life of carbon-14 is 5730 years.)

79. Suppose that only 6.2% of the normal amount of carbon-14 remains in a fragment of bone. How old is the bone? The decay rate of carbon-14 is 0.012%. Round your answer to the nearest year.

80. Suppose that 96.7% of the normal amount of carbon-14 remains in a piece of cloth. How old is the cloth? The decay rate of carbon-14 is 0.012%. Round your answer to the nearest year.

Answers to odd exercises:

61. 90.11g 63. 30.05g 65. $r \approx -0.0667$, So hourly decay rate is $\approx 6.67\%$	69. $f(t) = 250e^{(-0.00914t)}$; half-life: about 76 minutes 71. 3.98 yrs 73. 12.39 yrs	75. 3.67 yrs 77. 869 years 79. 23172 years

E: Exercise 4.7e.E

Find current amount

81. The population of a small town of $24,000$ people is expected grow exponentially at a rate of 1.6% per year. Construct an exponential growth model and use it to:

a. Estimate the population in 3 years' time.

b. Estimate the time it will take for the population to reach $30,000$ people.

82. During the exponential growth phase, certain bacteria can grow at a rate of 4.1% per hour. If $10,000$ cells are initially present in a sample, construct an exponential growth model and use it to:

a. Estimate the population in 5 hours.

b. Estimate the time it will take for the population to reach $25,000$ cells.

83. In 2000, the world population was estimated to be 6.115 billion people and in 2010 the estimate was 6.909 billion people. If the world population continues to grow exponentially, estimate the total world population in 2020.

84. The population of the downtown area of a certain city decreased from $12,500$ people to $10,200$ people in two years. If the population continues to decrease exponentially at this rate, what would we expect the population to be in two more years?

85. Researchers recorded that a certain bacteria population declined from $800,000$ to $500,000$ in 6 hours after the administration of medication. At this rate of decay, how many bacteria will there be in 24 hours?

86. Researchers recorded that a certain bacteria population declined from $100,000$ to 100 in 24 hours. At this rate of decay, how many bacteria will there be in 16 hours?

87. An automobile was purchased new for $\$42,500$ and 2 years later it was valued at $\$33,400$. Estimate the value of the automobile in 5 years if it continues to decrease exponentially.

88. A new MP3 player was purchased for $\$320$ and in 1 year it was selling used online for $\$210$. If the value continues to decrease exponentially at this rate, determine the value of the MP3 player 3 years after it was purchased.

89. A bacteria doubles its original population in 24 hours. How big will its population be in 72 hours?

90. A new PC was purchased for $\$1,200$ and in 1.5 years it was worth $\$520$. Assume the value is decreasing exponentially and estimate the value of the PC four years after it is purchased.

91. Radioactive technetium-99m is often used in diagnostic medicine as it has a relatively short half-life but lasts long enough to get the needed testing done on the patient. If its half-life is 6 hours, how much of the radioactive material form a 0.5 ml injection will be in the body in 24 hours?

92. Carbon-14 is used for archeological carbon dating. Its half-life is $5,730$ years. How much of a 100-gram sample of Carbon-14 will be left in 1000 years?

93. The half-life of caesium-137 is about 30 years. What percent of an initial sample will remain in 100 years?

94. The half-life of radioactive iodine-131 is about 8 days. What percent of an initial sample will remain in 30 days?

95. If a bone is 100 years old, what percent of its original amount of carbon-14 do we expect to find in it? (The half-life of carbon-14 is 5730 years.)

96. The half-life of plutonium-239 is about $24,000$ years. What percent of an initial sample will remain in $1,000$ years?

Find time

97. The half-life of radium-226 is about $1,600$ years. How long will a 5-milligram sample of radium-226 take to decay to 1 milligram?

98. The half-life of plutonium-239 is about $24,000$ years. How long will a 5-milligram sample of plutonium-239 take to decay to 1 milligram? 121. Find the amount of time it will take for 10% of an initial sample of plutonium-239 to decay. (Hint: If 10% decays, then 90% will remain.)

99. The half-life of radioactive iodine-131 is about 8 days. How long will it take a 28-gram initial sample of iodine-131 to decay to 12 grams?

100. The half-life of caesium-137 is about 30 years. How long will it take a 15-milligram sample of caesium-137 to decay to 5 milligrams? 116. How long will it take a sample of caesium-137 to decay to 25% of the original amount?

101. The Rhind Mathematical Papyrus is considered to be the best example of Egyptian mathematics found to date. This ancient papyrus was found to contain 64% of the carbon-14 normally found in papyrus. Given that carbon-14 has a half-life of $5,730$ years, estimate the age of the papyrus.

102. A wooden bowl artifact carved from oak was found to contain 55% of the carbon-14 normally found in oak. Given that carbon-14 has a half-life of $5,730$ years, estimate the age of the bowl.

103. The half-life of radioactive iodine-131 is about 8 days. How long will it take a sample of iodine-131 to decay to 10% of the original amount?

104. A virus takes 6 days to double its original population. How long will it take to triple its population?

Answers to odd exercises:

81. (a) About $25,180$ people (b) About 14 years 83. About 7.806 billion people 85. $122,070$ bacteria	87. About $\$23,269.27$ 89. 8 times as large as the original population 91. 0.03 mL	93. 9.9% 95. 98.8% 97. $3,715$ years	99. 9.8 days 101. About $3,689$ years old 103. 26.6 days

F: Newton' s Law of Heating and Cooling

Exercise $4.7e.F$

111. The temperature of an object in degrees Fahrenheit after t minutes is represented by the equation $T(t) = 68e^{-0.0174t} + 72$. To the nearest degree, what is the temperature of the object after one and a half hours?

112. A pot of boiling soup with an internal temperature of $100°$ Fahrenheit was taken off the stove to cool in a $69°$ F room. After fifteen minutes, the internal temperature of the soup was $95°$ F.

a) Use Newton's Law of Cooling to write a formula that models this situation.

b) To the nearest minute, how long will it take the soup to cool to $80°$ F?

c) To the nearest degree, what will the temperature be after 2 and a half hours?

113. A turkey is taken out of the oven with an internal temperature of $165°$ F and is allowed to cool in a $75°$ F room. After half an hour, the internal temperature of the turkey is $145°$ F.

(a) Write a formula that models this situation.

(b) To the nearest degree, what will the temperature be after 50 minutes?

(c) To the nearest minute, how long will it take the turkey to cool to $110°$ F?

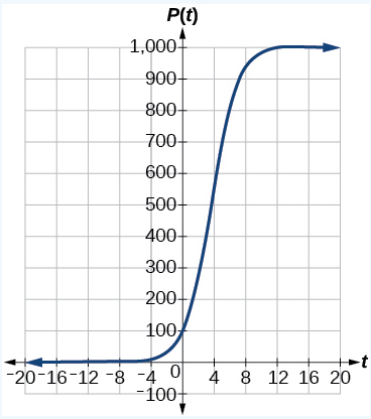
Answers to odd exercises:

111. about $86.2°$ F 113. (a) $T(t) = 90e^{(-0.008377t)} + 75$, where t is in minutes. (b) $134°$ F (c) about 113 minutes

G: Logistic Growth

Exercise 4.7e.G

121. The population of a fish farm in t years is modeled by the equation $P(t) = \frac{1000}{1+9e^{-0.6t}}$. The graph of the function is at the right.



a) What is the initial population of fish?

b) To the nearest tenth, what is the doubling time for the fish population?

c) To the nearest whole number, what will the fish population be after 2 years?

d) To the nearest tenth, how long will it take for the population to reach 900?

e) What is the carrying capacity for the fish population? Justify your answer using the graph of P.

122. The equation $N(t) = \frac{500}{1+49e^{-0.7t}}$ models the number of people in a town who have heard a rumor after t days.

a) How many people started the rumor?

b) To the nearest whole number, how many people will have heard the rumor after 3 days?

c) As t increases without bound, what value does $N(t)$ approach? Interpret your answer.

123. What is the y-intercept of the logistic growth model $y = \frac{c}{1+ae^{-rx}}$? Show the steps for calculation. What does this point tell us about the population?

124. A logistic model is given by the equation $P(t) = \frac{90}{1+5e^{-0.42t}}$. To the nearest hundredth, for what value of t does $P(t) = 45$

125. To the nearest whole number, what is the initial value of a population modeled by the logistic equation $P(t) = \frac{175}{1+6.995e^{-0.68t}}$? What is the carrying capacity?151) What is the y-intercept on the graph of the logistic model given in the previous exercise?

126. The number of cells in a certain bacteria sample is approximated by the logistic growth model $N(t) = \frac{1.2\times10^5}{1+9e^{-0.32t}}$, where t represents time in hours. Determine the time it takes the sample to grow to $24,000$ cells.

127. Given the logistic growth model $f(x) = \frac{150}{1+8e^{-2x}}$

a) Find and interpret $f(0)$. Round to the nearest tenth.

b) Find and interpret $f(4)$. Round to the nearest tenth.

c) Find the carrying capacity.

128. The population P of an endangered species habitat for wolves is modeled by the function $P(x) = \frac{558}{1+54.8e^{-0.462x}}$ where x is given in years.

a) What was the initial population of wolves transported to the habitat?

b) How many wolves will the habitat have after 3 years?

c) How many years will it take before there are 100 wolves in the habitat?

Answers to odd exercises:

121. (a) 100 fish, (b) about 1.4 years, (c) 269 fish, (d) about 7.3 years, (e) 1000 fish - the upper horizontal asymptote bound

123. $\frac{c}{1+a}$; initial population size 125. $P(0) = 22$; 175

127. (a) $f(0) \approx 16.7$; The amount initially present is about 16.7 units. (b) The quantity after 4 hours is 149.6 units (c) 150

*

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CHAPTER OVERVIEW

5: Trigonometric Functions and Graphs

The trigonometric functions are functions of an angle. and relate the angles of a triangle to the lengths of its sides. They are important in the study of triangles and modeling periodic phenomena, among many other applications.

5.1: Angles

5.1e: Exercises - Angles

5.2: Right Triangle Trigonometry

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5.6e: Exercises - Graphs of Other Trigonometric Functions

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5.1: Angles

Learning Objectives

- Draw angles in standard position.
- Convert between degrees and radians.
- Find coterminal angles.

Angles

In this section, we will examine properties of angles.

Properly defining an angle first requires that we define a *ray*. A ray consists of one point on a line and all points extending in one direction from that point. The first point is called the endpoint of the ray. We can refer to a specific ray by stating its *endpoint* and any other point on it. The ray illustrated in the figure to the right can be named as ray EF, or in symbolic form $\overrightarrow{EF}$.

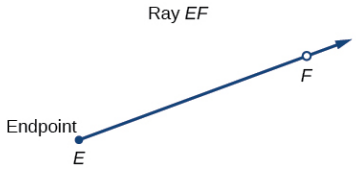


Figure 5.1.1: Ray $\overrightarrow{EF}$

An *angle* is the union of two rays having a common endpoint as shown in Figure 5.1.2. The endpoint is called the **vertex** of the angle, and the two rays are the sides of the angle. The angle illustrated in the figure on the right is formed from rays $\overrightarrow{ED}$ and $\overrightarrow{EF}$. Angles can be named using a point on each ray and the vertex, such as angle *DEF*, or in symbolic form $\angle DEF$.

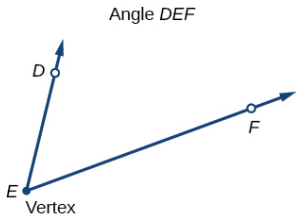


Figure 5.1.2: Angle $\angle DEF$. -- An angle is the union of two rays with a common endpoint

Greek letters are often used as variables for the measure of an angle. In the table below is a list of Greek letters commonly used to represent angles.

θ	φ or ϕ	α	β	γ
theta	phi	alpha	beta	gamma

Angle creation is a dynamic process. We start with two rays lying on top of one another. We leave one fixed in place, and rotate the other. The fixed ray is the **initial side,** and the rotated ray is the **terminal side**. In order to identify the different sides, we indicate the rotation with a small arc and arrow close to the vertex as in Figure 5.1.3.

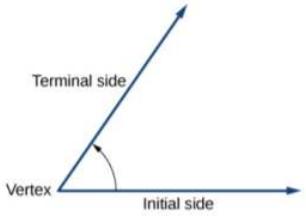


Figure 5.1.3 The **measure of an angle** indicates the amount rotation that separates the two rays of an angle

When applications use angles, we must be able to measure them.

Definition: Measurement of an angle

The **measure of an angle** indicates the amount of rotation from the initial side to the terminal side.

The measure of an angle is **positive** when the angle is measured in a *counterclockwise direction* from the initial side to the terminal side.

The measure of an angle is **negative** when the angle is measured in a *clockwise direction* from the initial side to the terminal side.

Degrees

Probably the most familiar unit of angle measurement is the degree. One **degree** is $\frac{1}{360}$ of a circular rotation, so a complete circular rotation contains 360 degrees. An angle measured in degrees should always include the unit “degrees” after the number, or include the degree symbol °. For example, 90 degrees = 90°.

To formalize our work, we will begin by drawing angles on an *x*-*y* coordinate plane. Angles can occur in any position on the coordinate plane, but for the purpose of comparison, the convention is to illustrate them in the same position whenever possible.

Definition: Standard Position

An angle is in **standard position** if its vertex is located at the origin, and its initial side extends along the positive *x*-axis. See Figure 5.1.5.

Angles in standard position are classified according to where their terminal side lies. For instance, an angle in standard position whose terminal side lies in Quadrant I is called a `Quadrant I angle' or 'an angle in Quadrant I'. If the terminal side of an angle lies on one of the coordinate axes, it is called a **quadrantal angle**. Two angles in standard position are called **coterminal** if they share the same terminal side.

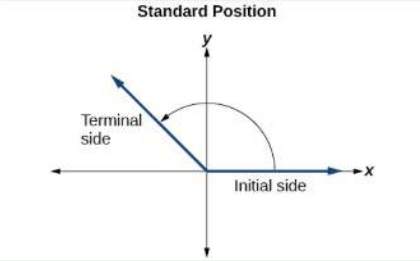


Figure 5.1.5

Drawing an angle in standard position always starts the same way—draw the initial side along the positive *x*-axis.

To place the terminal side of the angle, we must calculate the fraction of a full rotation the angle represents. We do that by dividing the angle measure in degrees by 360°. For example, to draw a 90° angle, we calculate that $\frac{90°}{360°} = \frac{1}{4}$. So, the terminal side will be one-fourth of the way around the circle, moving counterclockwise from the positive *x*-axis. To draw a 360° angle, we calculate that $\frac{360°}{360°} = 1$. So the terminal side will be 1 complete rotation around the circle, moving counterclockwise from the positive *x*-axis. In this case, the initial side and the terminal side overlap. See Figure 5.1.6 .

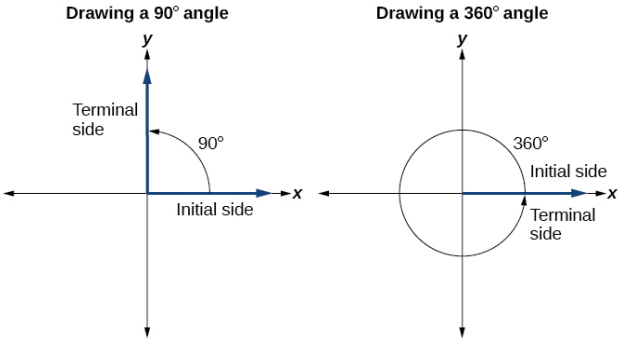


Figure 5.1.6

Since we define an angle in **standard position** by its initial side, we have a special type of angle whose terminal side lies on an axis.

Definition: QUADRANTAL ANGLES

Quadrantal angles are angles in standard position whose terminal side lies on an axis.

Examples of quadrantal angles include 0°, 90°, 180°, 270°, or 360°.

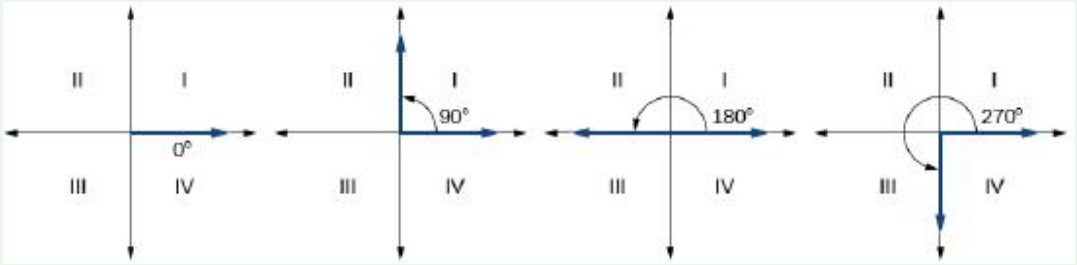


Figure 5.1.7: Examples of **Quadrantal** angles .

Howto: Draw an angle in standard position, given an angle measure in degrees.

1. Express the angle measure as a fraction of 360°.
2. Reduce the fraction to simplest form.
3. Draw an angle that contains that same fraction of the circle, beginning on the positive *x*-axis and moving counterclockwise for positive angles and clockwise for negative angles.

Example 5.1.1: Drawing an Angle in Standard Position Measured in Degrees

1. Sketch an angle of 30° in standard position.
2. Sketch an angle of −135° in standard position.

Solution

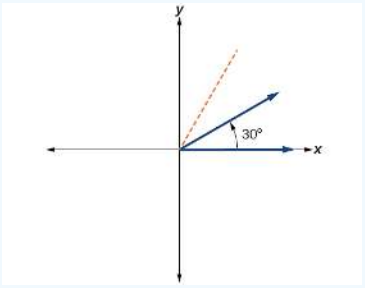
1. Divide the angle measure by 360°.

$$\frac{30°}{360°} = \frac{1}{12}$$

To rewrite the fraction in a more familiar fraction, we can recognize that

$$\frac{1}{12} = \frac{1}{3}\left(\frac{1}{4}\right)$$

One-twelfth equals one-third of a quarter, so by dividing a quarter rotation into thirds, we can sketch a line at 30° as shown in the figure to the right.



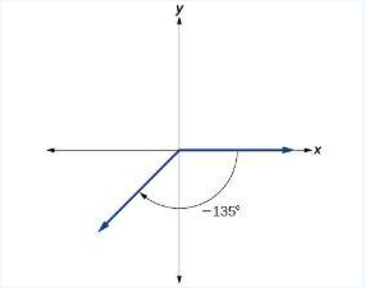
2. Divide the angle measure by 360°.

$$\frac{-135°}{360°} = -\frac{3}{8}$$

In this case, we can recognize that

$$-\frac{3}{8} = -\frac{3}{2}\left(\frac{1}{4}\right)$$

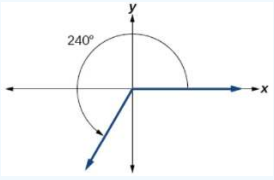
Negative three-eighths is one and one-half times a quarter, so we place a line by moving clockwise one full quarter and one-half of another quarter, as in the figure to the right.



Try It 5.1.1

Show an angle of 240° on a circle in standard position.

Answer



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Radians

Dividing a circle into 360 parts is an arbitrary choice, although it creates the familiar degree measurement. We may choose other ways to divide a circle. To find another unit, think of the process of drawing a circle. Imagine that you stop before the circle is completed. The portion that you drew is referred to as an arc. An **arc** may be a portion of a full circle, a full circle, or more than a full circle, represented by more than one full rotation. The length of the arc around an entire circle is called the **circumference** of that circle.

In the previous example we illustrated a method of drawing an angle by determining the fraction of the measure of an angle to the measure of a full rotation. This is the same as the fraction of the arclength of an angle to $2\pi r$, the arclength of a complete rotation (i.e. the *circumference*) of a circle. Building on this idea, we define a new way to measure an angle, as the fraction of the arclength of an angle to the *radius* of a circle, which is just 2π times the fraction used when drawing an angle.

Definition: Radian Measure

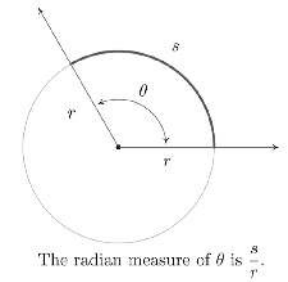
The **radian measure** of an angle is the ratio of an arc measuring s units to the radius r of the circle containing this arc.

The measure of the angle in radians is the ratio $\frac{s}{r}$.

The radian measure of an angle θ is the number of 'radius lengths' needed to sweep out along the circle to subtend the angle θ.

One complete revolution measures 2π radians.

Note that when an angle is described without a specific unit, it refers to radian measure.

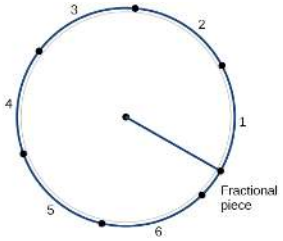


The radian measure of θ is $\frac{s}{r}$.

Radian measure is dimensionless, since it is the quotient of a length (portion of a circumference) divided by a length (radius) and the length units cancel out. Therefore, it is not necessary to write the label “radians” after a radian measure, and if we see an angle that is not labeled with “degrees” or the degree symbol, we can assume that it is a radian measure. In contrast, if the angle is measured in degrees, the units must be specified.

Radian measure is the number of radii in the length of an arc of a circle

One full rotation has an arc length equal to the circumference of a circle. The circumference of a circle is $C = 2\pi r$. The angle measure of one complete rotation in radian measure is $\frac{2\pi r}{r}$, which is always $2\pi \approx 6.28$, regardless of the length of the radius. So the circumference of any circle is $2\pi \approx 6.28$ times the length of the radius. That means that if we took a string as long as the radius and used it to measure consecutive lengths around the circumference, there would be room for six full string-lengths and a little more than a quarter of a seventh, as shown in the figure on the right.



This brings us to our new angle measure.

One **radian** is the measure of a central angle of a circle that intercepts an arc equal in length to the radius of that circle. A central angle is an angle formed at the center of a circle by two radii. Because the total circumference equals 2π times the radius, a full circular rotation is 2π radians. So

$$\begin{aligned} 2\pi \text{ radians} &= 360^\circ && \text{one complete rotation} \\ \pi \text{ radians} &= \frac{360^\circ}{2} = 180^\circ && \text{one-half of a complete rotation} \\ 1 \text{ radian} &= \frac{180^\circ}{\pi} \approx 57.3^\circ \end{aligned}$$

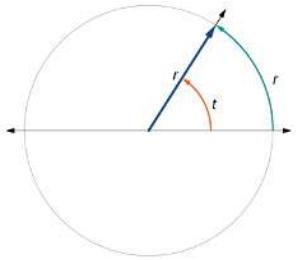


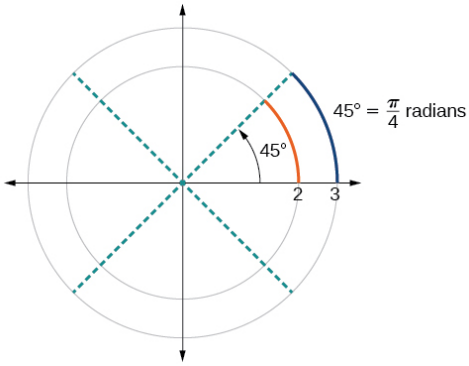
Fig. 5.1.12. A radian: arcLength=radius

The measure of one radian is depicted in Figure 5.1.12. The angle t sweeps out a measure of one radian. Notice that the length of the intercepted arc is the same as the length of the radius of the circle.

Radian measure is independent of the radius of the circle containing the arc

The ratio of the length of the curve along the arc of a circle to the radius of the circle containing that arc is the **radian measure** of the angle. This ratio is the same regardless of the radius of the circle—it depends only on the angle. This property allows us to define a measure of any angle as the ratio of the arc length s to the radius r.

To elaborate on this idea, consider two circles, one with radius 2 and the other with radius 3. Recall the circumference of a circle is $C = 2\pi r$, where r is the radius. The smaller circle then has circumference $2\pi(2) = 4\pi$ and the larger has circumference $2\pi(3) = 6\pi$. Now we draw a 45° angle on the two circles, as in the figure on the right. The length of the arc subtended by $45°$ is one-eighth of the circumference because $45°$ is one-eighth of a complete rotation. So the length of the arc subtended by $45°$ for the circle of radius 2 is $\frac{1}{8}4\pi = \frac{1}{2}\pi$ and the length of the arc subtended by $45°$ for the circle of radius 3 is $\frac{1}{8}6\pi = \frac{3}{4}\pi$.



Notice what happens if we find the ratio of the arc length divided by the radius of the circle.

$$\text{Smaller circle: } \frac{\frac{1}{2}\pi}{2} = \frac{1}{4}\pi \qquad \text{Larger circle: } \frac{\frac{3}{4}\pi}{3} = \frac{1}{4}\pi$$

Since both ratios are $\frac{1}{4}\pi$, the angle measures of both circles are the same, even though the arc lengths and radii differ.

Example 5.1.2: Finding a Radian Measure

Find the radian measure of one-third of a full rotation.

Solution

For any circle, the arc length along such a rotation would be one-third of the circumference. The circumference is $2\pi r$ so the arc length s is

$$s = \frac{1}{3}(2\pi r) = \frac{2\pi r}{3}$$

The radian measure would be the arc length divided by the radius.

$$\text{radian measure} = \frac{\frac{2\pi r}{3}}{r} = \frac{2\pi r}{3r} = \frac{2\pi}{3}$$

Try It 5.1.2

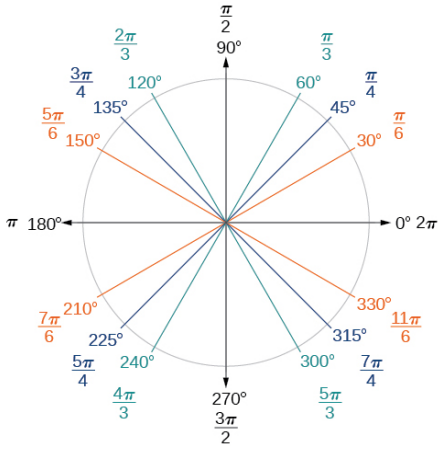
Find the radian measure of three-fourths of a full rotation.

Answer

$\frac{3\pi}{2}$

Special Angles

There are some frequently encountered angles in one revolution of a circle with which we should be familiar. The figure on the right illustrates these commonly encountered angles measured in radians and degrees. These angles include *basic angles* that have their terminal side in Quadrant I or along the positive axes, and multiples of these basic angles that have their terminal side in the other quadrants. Notice that angles drawn in the same color are multiples of the same *basic angle.*



The *basic angles* are $0°$, $30°$, $45°$, $60°$, and $90°$ and their radian measure equivalents, 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$.

The other angles can be obtained from the *basic angle* θ by performing simple addition or subtraction.

Angles in Quadrant II:	$180° - \theta$ (in degrees)	$= \pi - \theta$ (in radians)
Angles in Quadrant III:	$180° + \theta$ (in degrees)	$= \pi + \theta$ (in radians)
Angles in Quadrant IV:	$360° - \theta$ (in degrees)	$= 2\pi - \theta$ (in radians)

The basic angles in Quadrant I are examples of **reference angles**. We will see shortly that reference angles play an important role in the evaluation of trigonometric ratios.

Converting Between Radians and Degrees

Because degrees and radians both measure angles, we need to be able to convert between them. However, while converting back and forth between degrees and radians is certainly a good skill to have, it is best that you learn to `think in radians' as well as you can `think in degrees'.

Since one revolution counter-clockwise measures $360°$ and the same angle measures 2π radians, we can use the proportion $\frac{2\pi \text{ radians}}{360°}$, or its reduced equivalent, $\frac{\pi \text{ radians}}{180°}$, as the conversion factor between the two systems. A proportion equation can be set up to do the conversion, or cancellation of units can be done.

We summarize these conversions below.

How To: Howto: Convert between Degrees and Radians.

1. To convert degree measure to radian measure, multiply by $\frac{\pi \text{ radians}}{180°}$
2. To convert radian measure to degree measure, multiply by $\frac{180°}{\pi \text{ radians}}$

~or~ Solve the proportion equation $\frac{\theta \text{ (in degrees)}}{180°} = \frac{\theta \text{ (in radians)}}{\pi \text{ radians}}$

Example 5.1.3: Converting Radians to Degrees

Convert each radian measure to degrees. 1) 3 2) $\frac{\pi}{6}$

Solution

1. Cancellation of Units Method. Use the appropriate equivalency ratio, $\frac{180°}{\pi \text{ radians}}$.

$$\theta = 3 \text{ radians} \cdot \frac{180°}{\pi \text{ radians}} = 3 \cancel{\text{ radians}} \cdot \frac{180°}{\pi \cancel{\text{ radians}}} = \frac{3(180)°}{\pi} \approx 172°$$

2. Proportion Method. Use the proportion equation $\frac{\theta \text{ (in degrees)}}{180°} = \frac{\theta \text{ (in radians)}}{\pi \text{ radians}}$ then substitute.

$$\frac{\theta}{180^\circ} = \frac{\frac{\pi}{6}}{\pi}$$
$$\theta = 180\left(\frac{\pi}{6}\cdot\frac{1}{\pi}\right) = \frac{180^\circ}{6} = 30^\circ$$

Try It: 5.1.3

Convert $-\frac{3\pi}{4}$ radians to degrees.

Solution

−135°

Example 5.1.4: Converting Degrees to Radians

1. Convert 127° to radians. Write the answer to the nearest tenth of a radian.
2. Convert 15 degrees to radians. Write the answer in units of π

Solution

1. Cancellation of Units Method. Use the appropriate equivalency ratio, $\frac{\pi \text{ radians}}{180^\circ}$

$$\theta = 127^\circ \cdot \frac{\pi \text{ radians}}{180^\circ}$$
$$= 127^{\not\circ} \cdot \frac{\pi \text{ radians}}{180^{\not\circ}}$$
$$= \frac{127(\pi) \text{ radians}}{180} \approx 2.2 \text{ radians}$$

2. Proportion Method. In this example, we start with degrees and want radians, so we again set up a proportion and solve it, but we substitute the given information into a different part of the proportion.

$$\frac{\theta \text{ (in degrees)}}{180^\circ} = \frac{\theta \text{ (in radians)}}{\pi \text{ radians}}$$
$$\frac{15^\circ}{180^\circ} = \frac{\theta}{\pi \text{ radians}}$$
$$\pi\left(\frac{15^\circ}{180^\circ}\right) = \pi\left(\frac{\theta}{\pi \text{ radians}}\right)$$
$$\frac{15\pi \text{ radians}}{180} = \theta$$
$$\frac{\pi}{12} \text{ radians} = \theta$$

Try It: 5.1.4

Convert 126° to radians written as a multiple of π.

Solution

$\frac{7\pi}{10}$

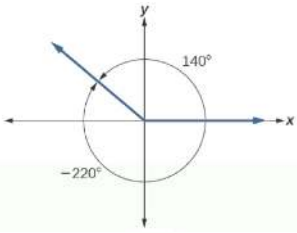
Coterminal Angles

Negative angles and angles greater than a full revolution are often more awkward to work with than those in the range of 0° to 360°, or 0 to 2π. It would be convenient to replace those out-of-range angles with a corresponding angle within the range of a single revolution.

Definition: Coterminal Angles

Coterminal angles are angles in standard position that have the same terminal side.

It is possible for more than one angle to have the same terminal side. Look at the figure on the right for an example. The angle of 140° is a **positive angle**, measured counterclockwise. The angle of –220° is a **negative angle**, measured clockwise. Because both angles have the same terminal side, they are **coterminal** angles. Every angle greater than 360° or less than 0° is coterminal with an angle between 0° and 360°, and it is often more convenient to find the coterminal angle within the range of 0° to 360° than to work with an angle that is outside that range.



Any angle has infinitely many **coterminal angles** because each time we add 360° to that angle—or subtract 360° from it—the resulting value has a terminal side in the same location. For example, 100° and 460° are coterminal for this reason, as is −260°. Recognizing that any angle has infinitely many coterminal angles explains the repetitive shape in the graphs of trigonometric functions.

How to: Find a Coterminal Angle.

Given an angle θ measured in degrees, any angle calculated by increasing or decreasing θ by a multiple of $360°$ is **coterminal** to θ.

Given an angle θ measured in radians, any angle calculated by increasing or decreasing θ by a multiple of 2π radians is **coterminal** to θ.

Example 5.1.5: Find an Angle Coterminal with an Angle Measured in Degrees

1. Find the least positive angle θ that is coterminal with an angle measuring 800°, where $0° \le \theta < 360°$.
2. Show the angle with measure −45° on a circle and find a positive coterminal angle α such that $0° \le \alpha < 360°$.

1. Solution

An angle with measure 800° is coterminal with an angle with measure 800° − 360° = 440°, but 440° is still greater than 360°, so we subtract 360° again to find another coterminal angle: 440° − 360° = 80°.

The angle $\theta = 80°$ is coterminal with 800°. To put it another way, 800° equals 80° plus two full rotations, as shown in Figure 5.1.19.

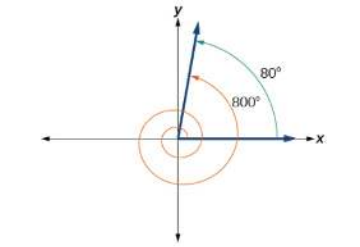


Figure 5.1.19

2. Solution

Since 45° is half of 90°, we can start at the positive horizontal axis and measure clockwise half of a 90° angle.

Because we can find coterminal angles by adding or subtracting a full rotation of 360°, we can find a positive coterminal angle here by adding 360°:

$-45° + 360° = 315°$ so the angle $315°$ is coterminal with -45°.

These two coterminal angles are drawn below in Figure 5.1.20.

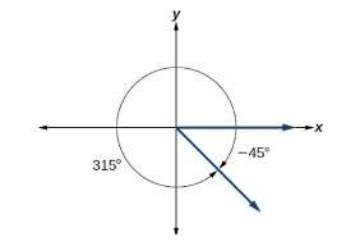


Figure 5.1.20

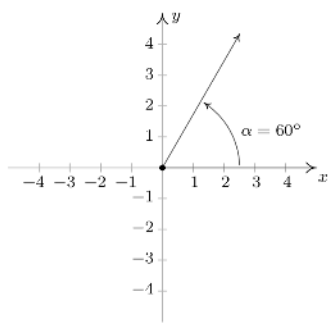
Example 5.1.5b: Coterminal angles in degrees

Graph each of the (oriented) angles below in standard position and classify them according to where their terminal side lies. Find three coterminal angles, at least one of which is positive and one of which is negative.

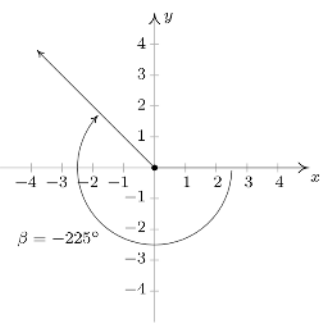
1. $\alpha = 60°$ 2. $\beta = -225°$ 3. $\gamma = 540°$ 4. $\phi = -750°$

Solution

1. To graph $\alpha = 60°$, we draw an angle with its initial side on the positive x-axis and rotate counter-clockwise $\frac{60°}{360°} = \frac{1}{6}$ of a revolution. We see that α is a Quadrant I angle. To find angles which are coterminal, we look for angles θ of the form $\theta = \alpha + 360° \cdot k$, for some integer k. When $k = 1$, we get $\theta = 60° + 360° = 420°$. Substituting $k = -1$ gives $\theta = 60° - 360° = -300°$. Finally, if we let $k = 2$, we get $\theta = 60° + 720° = 780°$.
2. Since $\beta = -225°$ is negative, we start at the positive x-axis and clockwise $\frac{225°}{360°} = \frac{5}{8}$ of a revolution. We see that β is a Quadrant II angle. To find coterminal angles, we proceed as before and compute $\theta = -225° + 360° \cdot k$ for integer values of k. We find $135°$, $-585°$ and $495°$ are all coterminal with $-225°$.

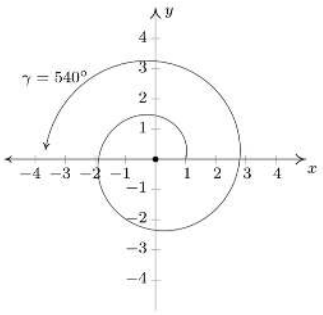


$\alpha = 60°$ in standard position.

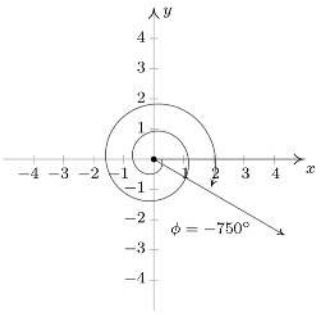


$\beta = -225°$ in standard position.

3. Since $\gamma = 540°$ is positive, we rotate counter-clockwise from the positive x-axis. One full revolution accounts for $360°$, with $180°$, or $\frac{1}{2}$ of a revolution remaining. Since the terminal side of γ lies on the negative x-axis, γ is a quadrantal angle. All angles coterminal with γ are of the form $\theta = 540° + 360° \cdot k$, where k is an integer. Working through the arithmetic, we find three such angles: $180°$, $-180°$ and $900°$.
4. The Greek letter ϕ is pronounced `fee' or `fie' and since ϕ is negative, we begin our rotation clockwise from the positive x-axis. Two full revolutions account for $720°$, with just $30°$ or $\frac{1}{12}$ of a revolution to go. We find that ϕ is a Quadrant IV angle. To find coterminal angles, we compute $\theta = -750° + 360° \cdot k$ for a few integers k and obtain $-390°$, $-30°$ and $330°$.



$\gamma = 540°$ in standard position.



$\phi = -750°$ in standard position.

Try It: 5.1.5

1. Find an angle α that is coterminal with an angle measuring 870°, where $0° \le \alpha < 360°$.
2. Find an angle β that is coterminal with an angle measuring −300° such that $0° \le \beta < 360°$.

Solution

1. $\alpha = 150°$ 2. $\beta = 60°$

We can find **coterminal** angles measured in radians in much the same way as we have found them using degrees. In both cases, we find coterminal angles by adding or subtracting one or more full rotations.

Example 5.1.6: Coterminal Angles Using Radians

Find an angle β that is coterminal with $\frac{19\pi}{4}$, where $0 \le \beta < 2\pi$.

Solution

When working in degrees, we found coterminal angles by adding or subtracting 360 degrees, a full rotation. Likewise, in radians, we can find coterminal angles by adding or subtracting full rotations of 2π radians:

$$\frac{19\pi}{4} - 2\pi = \frac{19\pi}{4} - \frac{8\pi}{4} = \frac{11\pi}{4}$$

The angle $\frac{11\pi}{4}$ is coterminal, but not less than 2π, so we subtract another rotation:

$$\frac{11\pi}{4} - 2\pi = \frac{11\pi}{4} - \frac{8\pi}{4} = \frac{3\pi}{4}$$

The angle $\frac{3\pi}{4}$ is coterminal with $\frac{19\pi}{4}$, as shown in Figure 5.1.21.

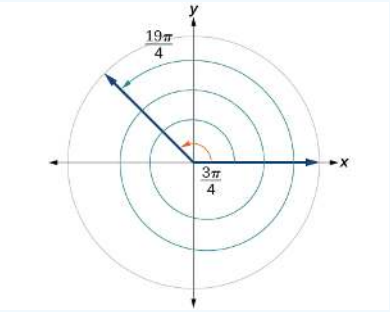


Figure 5.1.21

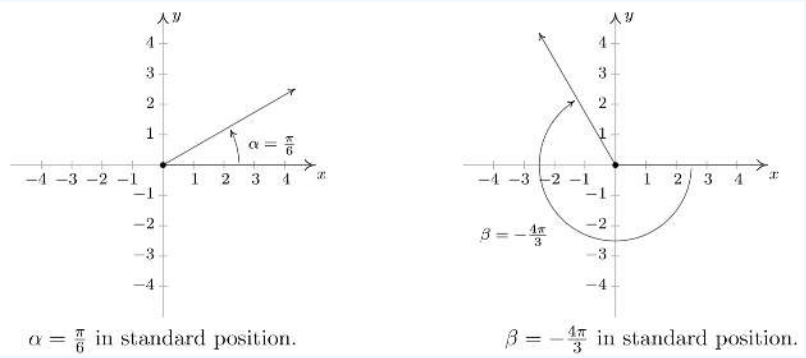
Example 5.1.7: Coterminal Angles in Radians

Graph each of the angles below in standard position and classify them according to where their terminal side lies. Find three coterminal angles, at least one of which is positive and one of which is negative.

1. $\alpha = \frac{\pi}{6}$ 2. $\beta = -\frac{4\pi}{3}$ 3. $\gamma = \frac{9\pi}{4}$ 4. $\phi = -\frac{5\pi}{2}$

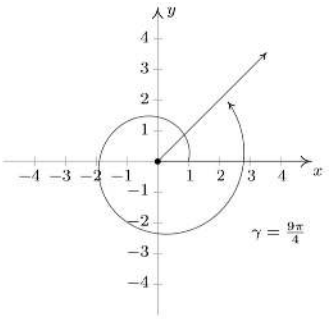
Solution

1. The angle $\alpha = \frac{\pi}{6}$ is positive, so we draw an angle with its initial side on the positive x-axis and rotate counter-clockwise $\frac{(\pi/6)}{2\pi} = \frac{1}{12}$ of a revolution. Thus α is a Quadrant I angle. Coterminal angles θ are of the form $\theta = \alpha + 2\pi \cdot k$, for some integer k. To make the arithmetic a bit easier, we note that $2\pi = \frac{12\pi}{6}$, thus when $k = 1$, we get $\theta = \frac{\pi}{6} + \frac{12\pi}{6} = \frac{13\pi}{6}$. Substituting $k = -1$ gives $\theta = \frac{\pi}{6} - \frac{12\pi}{6} = -\frac{11\pi}{6}$ and when we let $k = 2$, we get $\theta = \frac{\pi}{6} + \frac{24\pi}{6} = \frac{25\pi}{6}$.
2. Since $\beta = -\frac{4\pi}{3}$ is negative, we start at the positive x-axis and rotate clockwise $\frac{(4\pi/3)}{2\pi} = \frac{2}{3}$ of a revolution. We find β to be a Quadrant II angle. To find coterminal angles, we proceed as before using $2\pi = \frac{6\pi}{3}$, and compute $\theta = -\frac{4\pi}{3} + \frac{6\pi}{3} \cdot k$ for integer values of k. We obtain $\frac{2\pi}{3}$, $-\frac{10\pi}{3}$ and $\frac{8\pi}{3}$ as coterminal angles.

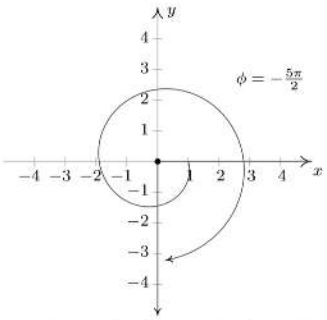


$\alpha = \frac{\pi}{6}$ in standard position. $\beta = -\frac{4\pi}{3}$ in standard position.

3. $\gamma = \frac{9\pi}{4}$ is positive, we rotate counter-clockwise from the positive x-axis. One full revolution accounts for $2\pi = \frac{8\pi}{4}$ of the radian measure with $\frac{\pi}{4}$ or $\frac{1}{8}$ of a revolution remaining. We have γ as a Quadrant I angle. All angles coterminal with γ are of the form $\theta = \frac{9\pi}{4} + \frac{8\pi}{4} \cdot k$, where k is an integer. Working through the arithmetic, we find: $\frac{\pi}{4}$, $-\frac{7\pi}{4}$ and $\frac{17\pi}{4}$.
4. To graph $\phi = -\frac{5\pi}{2}$, we begin our rotation clockwise from the positive x-axis. As $2\pi = \frac{4\pi}{2}$, after one full revolution clockwise, we have $\frac{\pi}{2}$ or $\frac{1}{4}$ of a revolution remaining. Since the terminal side of ϕ lies on the negative y-axis, ϕ is a quadrantal angle. To find coterminal angles, we compute $\theta = -\frac{5\pi}{2} + \frac{4\pi}{2} \cdot k$ for a few integers k and obtain $-\frac{\pi}{2}$, $\frac{3\pi}{2}$ and $\frac{7\pi}{2}$.



$\gamma = \frac{9\pi}{4}$ in standard position.



$\phi = -\frac{5\pi}{2}$ in standard position.

Try It: 5.1.7

Find an angle of measure θ that is coterminal with an angle of measure $-\frac{17\pi}{6}$ where $0 \le \theta < 2\pi$.

Solution

$\frac{7\pi}{6}$

Key Concepts

- An angle is formed from the union of two rays, by keeping the initial side fixed and rotating the terminal side. The amount of rotation determines the measure of the angle.
- An angle is in standard position if its vertex is at the origin and its initial side lies along the positive *x*-axis. A positive angle is measured counterclockwise from the initial side and a negative angle is measured clockwise.
- To draw an angle in standard position, draw the initial side along the positive *x*-axis and then place the terminal side according to the fraction of a full rotation the angle represents.
- In addition to degrees, the measure of an angle can be described in radians.
- To convert between degrees and radians, use the proportion $\frac{\theta}{180} = \frac{\theta^R}{\pi}$.
- Two angles that have the same terminal side are called coterminal angles.
- We can find coterminal angles by adding or subtracting 360° or 2π.

Glossary

angle
the union of two rays having a common endpoint

coterminal angles
description of positive and negative angles in standard position sharing the same terminal side

degree
a unit of measure describing the size of an angle as one-360th of a full revolution of a circle

initial side
the side of an angle from which rotation begins

measure of an angle
the amount of rotation from the initial side to the terminal side

negative angle
description of an angle measured clockwise from the positive *x*-axis

positive angle

description of an angle measured counterclockwise from the positive x-axis

quadrantal angle

an angle whose terminal side lies on an axis

radian measure

the ratio of the arc length formed by an angle divided by the radius of the circle

radian

the measure of a central angle of a circle that intercepts an arc equal in length to the radius of that circle

ray

one point on a line and all points extending in one direction from that point; one side of an angle

reference angle

the measure of the acute angle formed by the terminal side of the angle and the horizontal axis

standard position

the position of an angle having the vertex at the origin and the initial side along the positive x-axis

terminal side

the side of an angle at which rotation ends

vertex

the common endpoint of two rays that form an angle

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5.1e: Exercises - Angles

A: Concepts

Exercise 5.1e.A

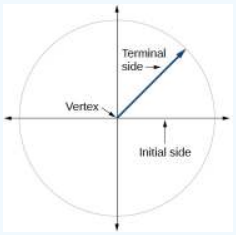
1) Draw an angle in standard position. Label the vertex, initial side, and terminal side.

2) Explain why there are an infinite number of angles that are coterminal to a certain angle.

3) State what a positive or negative angle signifies, and explain how to draw each.

4) How does radian measure of an angle compare to the degree measure? Include an explanation of 1 radian in your paragraph.

Answers to odd exercises:

1. 

3. Whether the angle is positive or negative determines the direction. A positive angle is drawn in the counterclockwise direction, and a negative angle is drawn in the clockwise direction.

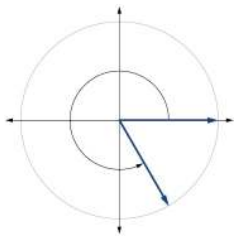
B: Draw Angles in Standard Position

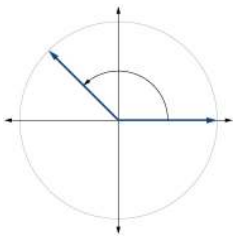
Exercise 5.1e.B

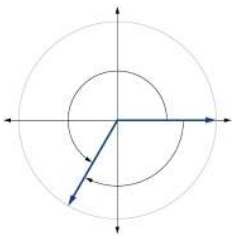
★ Draw an angle in standard position with the given measure. If the angle measure is not between 0 and 2π or between 0° and 360°, also state the coterminal angle that is within that interval.

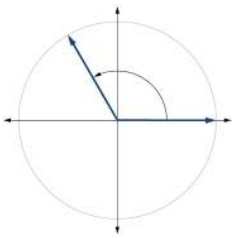
5. $300°$
6. $415°$
7. $135°$
8. $-30°$
9. $-120°$
10. $-315°$
11. $\frac{2\pi}{3}$
12. $\frac{5\pi}{3}$
13. $-\frac{4\pi}{3}$
14. $-\frac{5\pi}{3}$
15. $\frac{5\pi}{6}$
16. $\frac{11\pi}{6}$
17. $-\frac{\pi}{6}$
18. $\frac{7\pi}{6}$
19. $\frac{7\pi}{4}$
20. $\frac{5\pi}{4}$
21. $-\frac{3\pi}{4}$
22. $\frac{11\pi}{4}$
23. $\frac{22\pi}{3}$
24. $\frac{17\pi}{6}$
25. $-\frac{\pi}{10}$
26. $\frac{23\pi}{5}$
27. $\frac{\pi}{2}$
28. $-\frac{3\pi}{2}$
29. $\frac{7\pi}{2}$
30. -5π

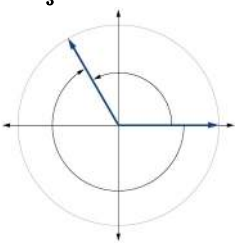
Answers to odd exercises:

5. 

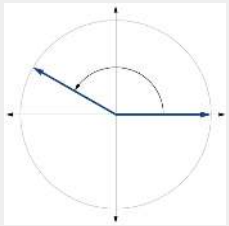
7. 

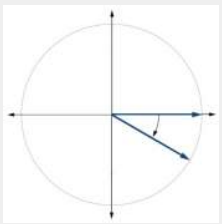
9. $240°$ 

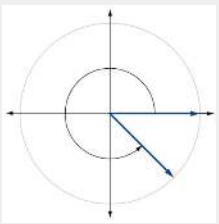
11. 

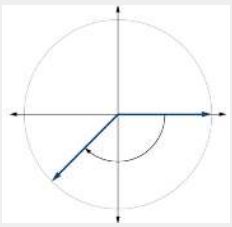
13. $\frac{2\pi}{3}$ 

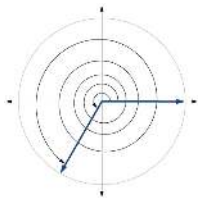
x

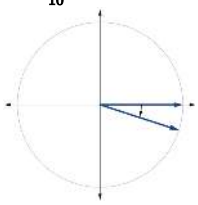
15. 

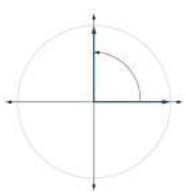
17. 

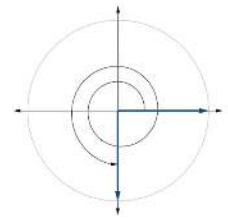
19. 

21. 

23. $\frac{4\pi}{3}$ 

25. $\frac{19\pi}{10}$ 

27. 

29 . $\frac{3\pi}{2}$ 

C: Convert between radians and degrees

Exercise $5.1e.C$

★ Convert angles in radians to degrees.

36. $\frac{3\pi}{4}$ radians	38. $-\frac{5\pi}{4}$ radians	40. $-\frac{7\pi}{3}$ radians	42. $\frac{11\pi}{6}$ radians
37. $\frac{\pi}{9}$ radians	39. $\frac{\pi}{3}$ radians	41. $-\frac{5\pi}{12}$ radians	

★ Convert angles in degrees to radians. Write the answer both as a multiple of pi and to the nearest hundredth of a radian.

43. $90°$	44. $100°$	45. $-540°$	46. $-120°$	47. $180°$	48. $-315°$	49. $150°$

Answers to odd exercises:

37. $20°$ 39. $60°$ 41. $-75°$ 43. $\frac{\pi}{2} \approx 1.57$ radians 45. $-3\pi \approx 9.42$ radians
47. $\pi \approx 3.14$ radians 49. $\frac{5\pi}{6} \approx 2.62$ radians

D: Coterminal Angles

Exercise 5.1e.D

★ **Find the angle between $0°$ and $360°$ that is coterminal to the given angle.**

50. $-40°$ 51. $-110°$ 52. $700°$ 53. $1400°$

★ **Find the angle between 0 and 2π in radians that is coterminal to the given angle.**

54. $-\frac{\pi}{9}$ 55. $\frac{10\pi}{3}$ 56. $\frac{13\pi}{6}$ 57. $\frac{44\pi}{9}$

Answers to odd exercises:

51. $250°$ 53. $320°$ 55. $\frac{4\pi}{3}$ 57. $\frac{8\pi}{9}$

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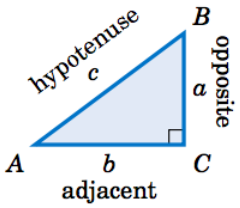
5.2: Right Triangle Trigonometry

Trigonometric Functions: Ratios of Sides of a Right Triangle

In this section we introduce right triangle trigonometry and explore right triangle applications of trigonometric functions. In later sections we will examine trigonometric functions as ratios of the sides of a triangle drawn inside a circle, and discuss the role of those functions in finding points on the circle.

Trigonometric functions are ratios of two of the three sides of a right triangle. The longest side of the right triangle is the hypotenuse. The other two sides are defined in terms of their relationship to one of the acute angles in the right triangle -- the side opposite the chosen angle, and the side adjacent (next to) that angle.

Angles in a right triangle are frequently designated with upper case letters like A, B, and C and the sides opposite each of these angles are labeled with the corresponding lower case letter. C is often the $90°$ angle, so side c would be the hypotenuse. Thus the side opposite angle A is side a, and the side adjacent to angle A is side b. These relationships and the naming convention is illustrated in the diagram on the right.



The triangle can be oriented any where in space, but when the triangle is placed in standard position with the chosen acute angle θ at the origin, these sides can be expressed equivalently in terms of y for the opposite side, x for the adjacent side, and r for the hypotenuse. The point (x, y) is a point on the terminal side of the angle.

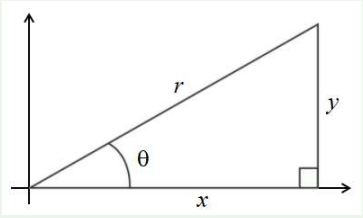
RIGHT TRIANGLE RELATIONSHIPS

Given a right triangle with an angle of θ

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{r}{y}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{r}{x}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{x}{y}$$



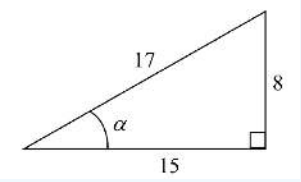
A common mnemonic for remembering these relationships is SohCahToa, formed from the first letters of "Sine is opposite over hypotenuse, Cosine is adjacent over hypotenuse, Tangent is opposite over adjacent."

Example 5.2.1: Find trigonometric ratios given 3 sides of a right triangle

Given the triangle shown, find the value for $\cos(\alpha)$.

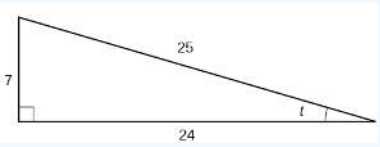
The side adjacent to the angle is 15, and the hypotenuse of the triangle is 17, so

$$\cos(\alpha) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{15}{17}$$



Try It 5.2.1

Given the triangle shown, find the value of $\sin t$.



Answer

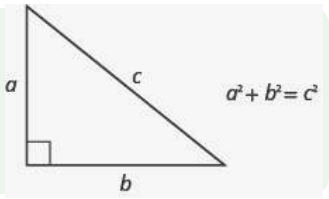
$$\sin t = \frac{7}{25}$$

Given Two Sides, Find the 3rd Side and All Trigonometric Ratios

It is not necessary that the lengths of all three sides of the right triangle be known initially. The **Theorem of Pythagoras** can be used to find the third side if we are given the lengths of only two sides of a right triangle.

THE PYTHAGOREAN THEOREM

In any right triangle, where a and b are the lengths of the legs, and c is the length of the hypotenuse, the sum of the squares of the lengths of the two legs equals the square of the length of the hypotenuse.

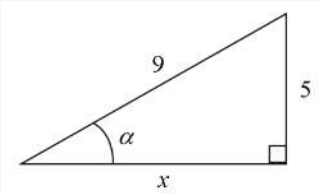


We will use the Pythagorean Theorem in the next example.

Example 5.2.2: Find trigonometric ratios given 2 sides of a right triangle

Given a triangle with a hypotenuse of 9 and side opposite to angle α of 5, find the sine, cosine and tangent for angle α.

Solution.



The triangle with the given information is illustrated on the right. The third side, which in this case is the "adjacent" side, can be found by using the Theorem of Pythagoras $a^2 + b^2 = c^2$. Always remember that in the formula, c is the length of the hypotenuse. From $x^2 + 5^2 = 9^2$ we obtain $x^2 = 81 - 25 = 56$. So $x = \sqrt{56} = 2\sqrt{14}$.

Now that all three sides have been found, values for the trigonometric functions can be calculated. Always remember to rationalize denominators!

$$\sin\alpha = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{5}{9} \qquad \cos\alpha = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{2\sqrt{14}}{9} \qquad \tan\alpha = \frac{\text{opposite}}{\text{adjacent}} = \frac{5}{2\sqrt{14}} = \frac{5\sqrt{14}}{28}$$

Try It 5.2.2

Given a triangle with the side adjacent to angle θ being $x = 2$ and the side opposite angle θ being $x = 5$, find the secant, cosecant, and cotangent for angle θ.

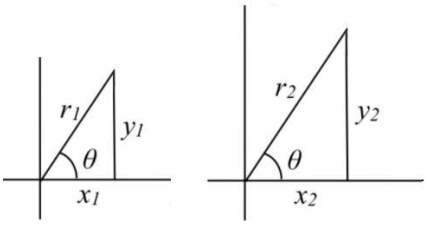
Answer

From $x^2 + y^2 = r^2$ we obtain $r = \sqrt{x^2 + y^2}$ and so $r = \sqrt{29}$.

Therefore, $\csc\theta = \frac{\sqrt{29}}{5}$, $\sec\theta = \frac{\sqrt{29}}{2}$, $\cot\theta = \frac{5}{2}$,

Given One Trigonometric Ratio, find all others

One important point to make here is that it is only the angle that determines the values of these trigonometric ratios, not the size of the triangle being used. This is because triangles constructed with the same angles but different length sides are similar triangles, so their ratios will be the same.

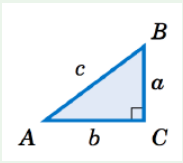


$$\text{Thus, } \frac{y_1}{r_1} = \frac{y_2}{r_2} \qquad \frac{x_1}{r_1} = \frac{x_2}{r_2}$$

A trigonometric ratio is the ratio of two sides of a right triangle. Since right triangles that are similar can be used to establish values for trigonometric functions, the numerator of the trigonometric ratio can serve as the length of one of the sides; the denominator can serve as the length of another side. Whether these sides are the hypotenuse of the triangle or the side opposite or adjacent to the given angle depends on which trigonometric ratio is being used.

How to: Find three sides of a right triangle, given one trigonometric ratio.

Given a trigonometric ratio, $\sin A = \frac{a}{c}$ or $\cos A = \frac{b}{c}$ or $\tan A = \frac{a}{b}$



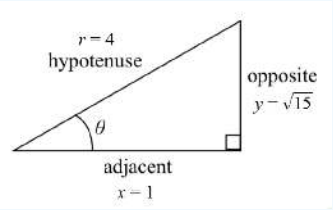
1. Draw a right triangle with angle A and the two known sides labeled according to the diagram on the right. If the trigonometric ratio has no denominator, use 1 for the value of the denominator.
2. Use the Pythagorean Theorem $a^2 + b^2 = c^2$ to find the value of the missing 3rd side of the right triangle.
3. Use the definitions to find the values of the other trigonometric ratios.

Example 5.2.3: Find all trigonometric ratios given one ratio

Given $\sec\theta = 4$, find the other five trigonometric ratios for angle θ.

Solution

Given $\sec\theta = 4$, we can construct the ratio $\sec\theta = \frac{4}{1} = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{r}{x}$.



Draw a triangle to help visualize the relationships represented by this trigonometric ratio.

Using the Theorem of Pythagoras $x^2 + y^2 = r^2$ and knowing $r = 4$, $x = 1$, we obtain $1^2 + y^2 = 4^2$. Thus, $y^2 = 16 - 1 = 15$ and so $y = \sqrt{15}$.

Now that all three sides of the right triangle are known, the definitions can be used to find the other five trigonometric ratios.

$$\sin\theta = \frac{y}{r} = \frac{\sqrt{15}}{4} \qquad \cos\theta = \frac{x}{r} = \frac{1}{4} \qquad \tan\theta = \frac{y}{x} = \sqrt{15}$$

$$\csc\theta = \frac{r}{y} = \frac{4}{\sqrt{15}} = \frac{4\sqrt{15}}{15} \qquad \cot\theta = \frac{x}{y} = \frac{1}{\sqrt{15}} = \frac{\sqrt{15}}{15}$$

Try It 5.2.3

Given $\tan\theta = \frac{\sqrt{3}}{2}$, find the other five trigonometric ratios for angle θ.

Answer

$$\sin\theta = \frac{\sqrt{21}}{7},\ \cos\theta = \frac{2\sqrt{7}}{7},\ \csc\theta = \frac{\sqrt{21}}{3},\ \sec\theta = \frac{\sqrt{7}}{2},\ \cot\theta = \frac{2\sqrt{3}}{3}$$

Given One Side and an Angle, Find all Other Sides and Angles

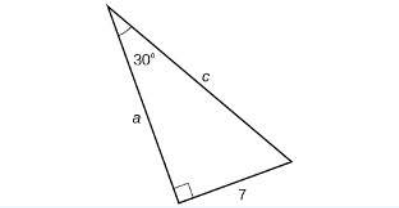
In previous examples, we evaluated the sine and cosine in triangles where we knew all three sides. But the real power of right-triangle trigonometry emerges when we look at triangles in which we know an angle but do not know all the sides.

How to: Given the side and one acute angle of a right triangle, find the remaining sides.

1. For each side, select the trigonometric function with the unknown side as either the numerator or the denominator and the known side as the corresponding denominator or the numerator.
2. Write an equation setting the function value of the known angle equal to the ratio of the corresponding sides.
3. Using the value of the trigonometric function and the known side length, solve for the missing side length.

Example 5.2.4: Finding Missing Side Lengths Using Trigonometric Ratios

Find the unknown sides of the triangle below.



Solution

We are given the angle $30°$ and the opposite side (OPP).

To find the adjacent side (ADJ) choose the ratio that uses both OPP (given) and ADJ (unknown): $\tan A = \frac{OPP}{ADJ}$.

Write the equation with the given and unknown values: $\tan(30°) = \frac{7}{a}$

Rearrange to solve for a. $a = \frac{7}{\tan(30°)} = 12.1$

To find the hypotenuse (HYP), choose the ratio that uses both OPP (given) and HYP (unknown): $\sin A = \frac{OPP}{HYP}$.

Substitute: $\sin(30°) = \frac{7}{c}$

Rearrange to solve for c. $c = \frac{7}{\sin(30°)} = 14$

Notice that if we know at least one of the non-right angles of a right triangle and one side, we can find the rest of the sides and angles.

Try It 5.2.4

A right triangle has one angle of $\frac{\pi}{3}$ (radians) and a hypotenuse of 20. Find the unknown sides and angles of the triangle.

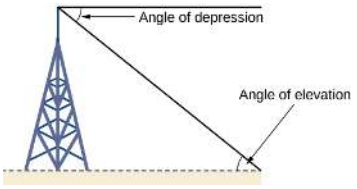
Answer

$$\cos\left(\frac{\pi}{3}\right) = \frac{\text{adjacent}}{\text{hypoteuse}} = \frac{\text{Adj}}{20} \quad \rightarrow \quad \text{adjacent} = 20\cos\left(\frac{\pi}{3}\right) = 20\left(\frac{1}{2}\right) = 10$$

$$\sin\left(\frac{\pi}{3}\right) = \frac{\text{Opposite}}{\text{hypotenuse}} = \frac{\text{Opp}}{20} \quad \rightarrow \quad \text{opposite} = 20\sin\left(\frac{\pi}{3}\right) = 20\left(\frac{\sqrt{3}}{2}\right) = 10\sqrt{3}$$

Missing angle = $\frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$.

Right Triangle Trigonometry Used to Solve Applied Problems



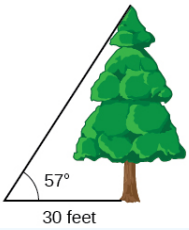
Right-triangle trigonometry has many practical applications. For example, the ability to compute the lengths of sides of a triangle makes it possible to find the height of a tall object without climbing to the top or having to extend a tape measure along its height. We do so by measuring a distance from the base of the object to a point on the ground some distance away, where we can look up to the top of the tall object at an angle. The **angle of elevation** of an object above an observer is the angle between the horizontal and the line from the observer's eye to the object. The right triangle created in this way has sides that represent the unknown height, the measured distance from the base, and the angled line of sight from the ground to the top of the object. Knowing the measured distance to the base of the object and the angle of the line of sight, we can use trigonometric functions to calculate the unknown height. Similarly, we can form a triangle from the top of a tall object by looking downward. The **angle of depression** of an object below an observer is the angle between the horizontal and the line from the observer's eye to the object.

Example 5.2.5

To find the height of a tree, a person walks to a point 30 feet from the base of the tree, and measures the angle from the ground to the top of the tree to be 57 degrees. Find the height of the tree.

Solution

We can introduce a variable, h, to represent the height of the tree. The two sides of the triangle that are most important to us are the side opposite the angle, the height of the tree we are looking for, and the adjacent side, which is 30 feet long.



The trigonometric function which relates the side opposite of the angle and the side adjacent to the angle is the tangent.

$$\tan(57°) = \frac{\text{opposite}}{\text{adjacent}} = \frac{h}{30}$$

Solving for h, $h = 30\tan(57°)$

Using a calculator, we can approximate a value for the height, $h = 30\tan(57°) \approx 46.2$ feet

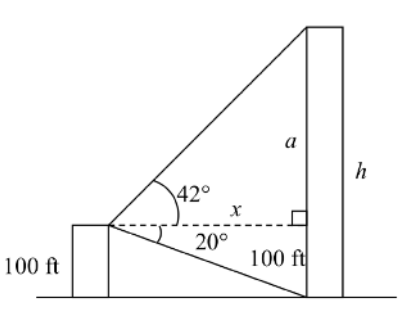
The tree is approximately 46 feet tall.

Example 5.2.6

A person standing on the roof of a 100 foot tall building is looking towards a skyscraper a few blocks away, wondering how tall it is. The angle of declination from the roof of the building to the base of the skyscraper is observed to be 20 degrees and the angle of inclination to the top of the skyscraper is 42 degrees.

Solution

To approach this problem, it would be good to start with a picture. Although we are interested in the height, h, of the skyscraper, it can be helpful to also label other unknown quantities in the picture – in this case the horizontal distance x between the buildings and a, the height of the skyscraper above the person.



To start solving this problem, notice we have two right triangles. In the top triangle, we know one angle is 42 degrees, but we don't know any of the sides of the triangle, so we don't yet know enough to work with this triangle.

In the lower right triangle, we know one angle is 20 degrees, and we know the vertical height measurement of 100 ft. Since we know these two pieces of information, we can solve for the unknown distance x.

$$\tan(20°) = \frac{\text{opposite}}{\text{adjacent}} = \frac{100}{x}$$

Solving for x

$$x\tan(20°) = 100$$
$$x = \frac{100}{\tan(20°)}$$

Now that we have found the distance x, we know enough information to solve the top right triangle.

$$\tan(42°) = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{x} = \frac{a}{100/\tan(20°)}$$
$$\tan(42°) = \frac{a\tan(20°)}{100} \quad \rightarrow \quad 100\tan(42°) = a\tan(20°) \quad \rightarrow \quad \frac{100\tan(42°)}{\tan(20°)} = a$$

Approximating, $a = \dfrac{100\tan(42°)}{\tan(20°)} \approx 247.4$ feet

Adding the height of the first building, we determine that the skyscraper is about 347 feet tall.

Find an angle given two sides of a right triangle.

Sometimes an angle that has a particular trigonometric ratio is desired. In this situation we use the inverse trigonometric functions $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ on a calculator to find an angle with a given trigonometric ratio.

Example 5.2.7

Use a calculator to find the angle in degrees to the nearest tenth with the following trigonometric ratios.

a. $\sin A = .3877$ b. $\cos A = .6691$ $\tan A = 4.6543$

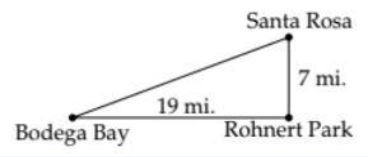
Solution

Use the inverse trigonometric function keys on a calculator. Be sure you are in degree mode in order to get an angle measurement in degrees!

a. $A = \sin^{-1}(.3877) \approx 22.8°$ b. $A = \cos^{-1}(.6691) \approx 48.0°$ c. $A = \tan^{-1}(4.6543) \approx 77.9°$

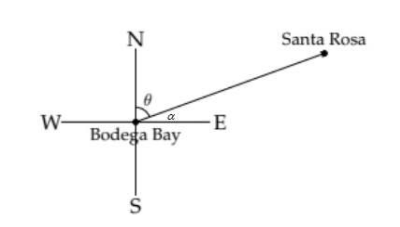
Example 5.2.8

Santa Rosa, California is 7 miles due north of Rohnert Park. Bodega Bay is 19 miles due west of Rohnert Park (as the crow flies). What is the bearing of Santa Rosa from Bodega Bay?



Solution

To answer the question we'll need another diagram:



If we knew the angle θ, then we could conclude that the bearing of Santa Rosa from Bodega Bay is θ degrees East of North. From the previous diagram we can see that we can't find θ directly, but we can find α, the complement of θ

$$\tan\alpha = \tfrac{7}{19} \quad \text{so} \quad \alpha = \tan^{-1}\tfrac{7}{19} \approx 20.2°$$

$$\text{Therefore, } \theta \approx 90° - 20.2° \approx 69.8°$$

Thus the bearing of Santa Rosa from Bodega Bay is $N69.8°E$, or $69.8°$ East of North.

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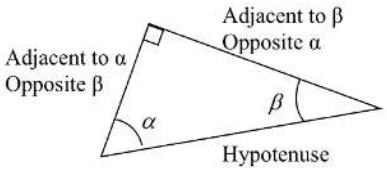
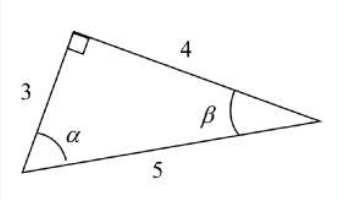
Cofunction Identities

When working with general right triangles, the same rules apply regardless of the orientation of the triangle. In fact, we can evaluate the sine and cosine of either of the two acute angles in the triangle.



Example 5.2.9

Using the triangle shown, evaluate $\sin\alpha, \cos\alpha, \tan\alpha, \sec\alpha, \csc\alpha,$ and $\cot\alpha$. Also evaluate $\sin(\beta)$ and $\cos(\beta)$.



Solution

$$\sin\alpha = \frac{\text{opposite } \alpha}{\text{hypotenuse}} = \frac{4}{5} \qquad \cos\alpha = \frac{\text{adjacent to } \alpha}{\text{hypotenuse}} = \frac{3}{5} \qquad \tan\alpha = \frac{\text{opposite } \alpha}{\text{adjacent to } \alpha} = \frac{4}{3}$$

$$\csc\alpha = \frac{\text{hypotenuse}}{\text{opposite } \alpha} = \frac{5}{4} \qquad \sec\alpha = \frac{\text{hypotenuse}}{\text{adjacent to } \alpha} = \frac{5}{3} \qquad \cot\alpha = \frac{\text{adjacent to } \alpha}{\text{opposite } \alpha} = \frac{3}{4}$$

$$\sin(\beta) = \frac{\text{opposite } \beta}{\text{hypotenuse}} = \frac{3}{5} \qquad \cos(\beta) = \frac{\text{adjacent to } \beta}{\text{hypotenuse}} = \frac{4}{5}$$

Try It 5.2.9

A right triangle is drawn with angle α opposite a side with length 33, angle β opposite a side with length 56, and hypotenuse 65. Evaluate $\sin\alpha, \cos\alpha, \tan\alpha, \sec\alpha, \csc\alpha,$ and $\cot\alpha$. Also, find the sine and cosine of β

Answer

$\sin\alpha = \frac{33}{65}$, $\cos\alpha = \frac{56}{65}$, $\tan\alpha = \frac{33}{56}$, $\sec\alpha = \frac{65}{56}$, $\csc\alpha = \frac{65}{33}$, $\cot\alpha = \frac{56}{33}$ $\quad$ $\sin(\beta) = \frac{56}{65}$, $\cos(\beta) = \frac{33}{65}$

You may have noticed in the examples above that $\cos(\alpha) = \sin(\beta)$ and $\cos(\beta) = \sin(\alpha)$.

This makes sense since the side opposite α is the same side that is adjacent to β. Since the three angles in a right triangle need to add up to $180°$ (or π if angles are measured in radians), then the two acute angles in a right triangle must add to $90°$ (or $\frac{\pi}{2}$ (radians). Therefore, $\beta = 90° - \alpha$, and $\alpha = 90° - \beta$. Since $\cos(\alpha) = \sin(\beta)$, then $\cos(\alpha) = \sin(90° - \alpha)$.

So we may state a *cofunction identity*: If any two angles are complementary, the sine of one is the cosine of the other, and vice versa. This identity is illustrated in Figure 5.2.4.

Using this identity, we can state without calculating, for instance, that $\sin(15°) = \cos(90° - 15°) = \cos(75°)$, and that $\sin(75°) = \cos(90° - 75°) = \cos(15°)$.

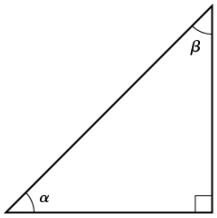


Figure 5.2.4: Cofunction identity illustration

In a similar manner it can be shown that $\sec(\alpha) = \csc(\beta)$ and $\sec(\beta) = \csc(\alpha)$. It can also be shown that $\cot(\alpha) = \tan(\beta)$ and $\cot(\beta) = \tan(\alpha)$. These relationships are summarized below.

COFUNCTION IDENTITIES

Cofunctions of **complementary angles** are equal.

The angles $d°$ and $(90° - d°)$ are **complementary angles**. Angles r and $\left(\frac{\pi}{2} - r\right)$ are **complementary angles**.
Sine and **co**sine are **cofunctions**. Secant and **co**secant are **cofunctions**. Tangent and **co**tangent are **cofunctions**.

The **cofunction identities** for an angle t in degrees are listed below. (If t is in radians, then use $\frac{\pi}{2}$ in place of $90°$).

$\sin t = \cos(90° - t)$	$\sec t = \csc(90° - t)$	$\tan t = \cot(90° - t)$
$\cos t = \sin(90° - t)$	$\csc t = \sec(90° - t)$	$\cot t = \tan(90° - t)$

How to: Given the sine and cosine of an angle, find the sine or cosine of its complement.

1. To find the sine of the complementary angle, find the cosine of the original angle.
2. To find the cosine of the complementary angle, find the sine of the original angle.

Example 5.2.10: Using Cofunction Identities

If $\sin t = \frac{5}{12}$, find $\cos(90° - t)$.

Solution

According to the cofunction identities for sine and cosine, $\sin t = \cos(90° - t)$. So $\cos(90° - t) = \frac{5}{12}$.

Try It 5.2.10

If $\csc(30°) = 2$, find $\sec(60°)$.

Answer

2

Key Concepts

- We can define trigonometric functions as ratios of the side lengths of a right triangle.
- The same side lengths can be used to evaluate the trigonometric functions of either acute angle in a right triangle.
- Any two complementary angles could be the two acute angles of a right triangle.
- If two angles are complementary, the cofunction identities state that the sine of one equals the cosine of the other and vice versa.
- We can use trigonometric functions of an angle to find unknown side lengths. Select the trigonometric function representing the ratio of the unknown side to the known side.
- Right-triangle trigonometry permits the measurement of inaccessible heights and distances. The unknown height or distance can be found by creating a right triangle in which the unknown height or distance is one of the sides, and another side and angle are known.

Glossary

adjacent side

in a right triangle, the side between a given angle and the right angle

angle of depression

the angle between the horizontal and the line made by an observer's eye looking at an object positioned lower than the observer

angle of elevation

the angle between the horizontal and the line made by an observer's eye looking at an object positioned higher than the observer

opposite side

in a right triangle, the side most distant from a given angle

hypotenuse

the side of a right triangle opposite the right angle

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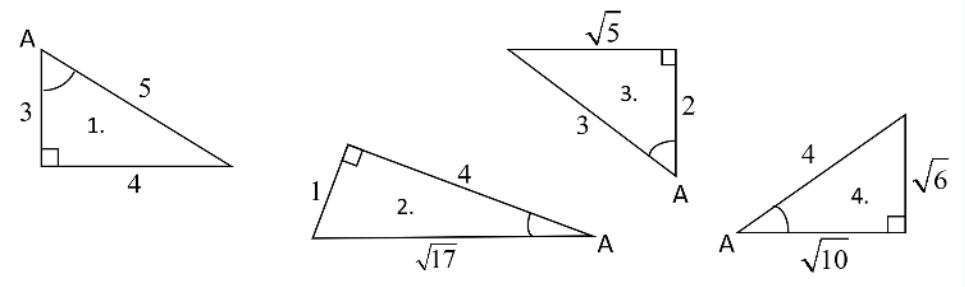
5.2e: Exercises - Right Angle Trigonometry

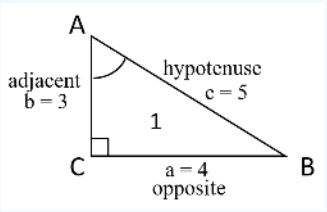
A: Given three sides of a right triangle, find all six trigonometric ratios

Exercise 5.2e.A

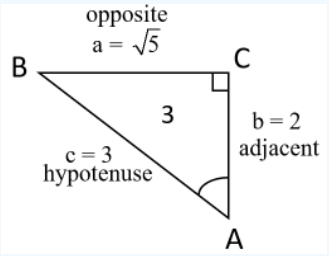
★ Given right triangle ABC where the right angle is angle C in each figure below,
(a) Label the remaining sides and angles
(b) Designate the hypotenuse, adjacent side or opposite side to angle A.
Determine the trigonometric ratios for (c) $\sin A$, (d) $\cos A$, (e) $\tan A$, (f) $\sec A$, (g) $\csc A$, (h) $\cot A$.
Give simplified exact answers - reduce fractions, rationalize all denominators!



Answers to odd exercises.



1. (c) $\sin A = \frac{4}{5}$, (d) $\cos A = \frac{3}{5}$, (e) $\tan A = \frac{4}{3}$,
(f) $\sec A = \frac{5}{3}$, (g) $\csc A = \frac{5}{4}$, (h) $\cot A = \frac{3}{4}$

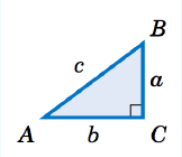


3. (c) $\sin A = \frac{\sqrt{5}}{3}$, (d) $\cos A = \frac{2}{3}$, (e) $\tan A = \frac{\sqrt{5}}{2}$,
(f) $\sec A = \frac{3}{2}$, (g) $\csc A = \frac{3\sqrt{5}}{5}$, (h) $\cot A = \frac{2\sqrt{5}}{5}$

B: Given two sides of right triangle, find all trigonometric ratios of the acute angles

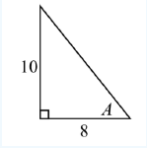
Exercise 5.2e.B

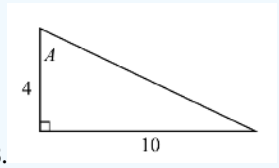
★ Find the exact values of all six trigonometric functions of angles A and B in the right triangle $\triangle ABC$ illustrated below. Simplify!



5. $b=12,\ c=13$	8. $a=20,\ c=29$	11. $a=1,\ b=3$	14. $a=2,\ c=\sqrt{6}$
6. $b=15,\ c=17$	9. $a=9,\ b=40$	12. $a=2,\ b=5$	15. $b=7,\ c=8$
7. $a=7,\ c=25$	10. $a=1,\ b=2$	13. $a=5,\ c=6$	16. $b=3,\ c=\sqrt{15}$

★ In each of the triangles below, find $\sin(A)$, $\cos(A)$, $\tan(A)$, $\sec(A)$, $\csc(A)$, $\cot(A)$.

17. 

18. 

Answers to odd exercises.

For all triangles #5-16, the trigonometric ratios can be determined using the following formulas:

$$\sin A=\cos B=\tfrac{a}{c},\ \cos A=\sin B=\tfrac{b}{c},\ \tan A=\cot B=\tfrac{a}{b},$$
$$\csc A=\sec B=\tfrac{c}{a},\ \sec A=\csc B=\tfrac{c}{b},\ \cot A=\tan B=\tfrac{b}{a},$$

The missing side needed to calculate these ratios is given below:

5. $a=5$ 7. $b=24$ 9. $c=41$ 11. $c=\sqrt{10}$ 13. $b=\sqrt{11}$ 15. a = $\sqrt{15}$

17. $\sin(A)=\frac{5\sqrt{41}}{41}$, $\cos(A)=\frac{4\sqrt{41}}{41}$, $\tan(A)=\frac{5}{4}$, $\sec(A)=\frac{\sqrt{41}}{4}$, $\csc(A)=\frac{\sqrt{41}}{5}$, $\cot(A)=\frac{4}{5}$

C: Given one trigonometric ratio of an acute angle, find all the others

Exercise 5.2e.C

★ Find the exact values of the other five trigonometric functions of the acute angle A given the indicated value of one of the functions. Simplify!

21. $\sin A=\frac{1}{2}$	24. $\cos A=\frac{2}{3}$	27. $\csc A=\frac{7}{3}$	30. $\sec A=\frac{5\sqrt{6}}{3}$
22. $\sin A=\frac{5}{12}$	25. $\tan A=\frac{3}{4}$	28. $\csc A=\frac{\sqrt{3}}{1}$	31. $\cot A=10$
23. $\cos A=\frac{1}{5}$	26. $\tan A=\frac{9}{5}$	29. $\sec A=3$	32. $\cot A=4$

Answers to odd exercises.

21. $\sin A=\frac{1}{2}$, $\cos A=\frac{\sqrt{3}}{2}$, $\tan A=\frac{\sqrt{3}}{3}$, $\csc A=2$, $\sec A=\frac{2\sqrt{3}}{3}$, $\cot A=\sqrt{3}$
23. $\sin A=\frac{2\sqrt{6}}{5}$, $\cos A=\frac{1}{5}$, $\tan A=2\sqrt{6}$, $\csc A=\frac{5\sqrt{6}}{12}$, $\sec A=5$, $\cot A=\frac{\sqrt{6}}{12}$
25. $\sin A=\frac{3}{5}$, $\cos A=\frac{4}{5}$, $\tan A=\frac{3}{4}$, $\csc A=\frac{5}{3}$, $\sec A=\frac{5}{4}$, $\cot A=\frac{4}{3}$
27. $\sin A=\frac{3}{7}$, $\cos A=\frac{2\sqrt{10}}{7}$, $\tan A=\frac{3\sqrt{10}}{20}$, $\csc A=\frac{7}{3}$, $\sec A=\frac{7\sqrt{10}}{20}$, $\cot A=\frac{2\sqrt{10}}{3}$
29. $\sin A=\frac{2\sqrt{2}}{3}$, $\cos A=\frac{1}{3}$, $\tan A=2\sqrt{2}$, $\csc A=\frac{3\sqrt{2}}{4}$, $\sec A=3$, $\cot\theta=\frac{\sqrt{2}}{4}$
31. $\sin A=\frac{\sqrt{101}}{101}$, $\cos A=\frac{10\sqrt{101}}{101}$, $\tan A=\frac{1}{10}$, $\csc A=\sqrt{101}$, $\sec A=\frac{\sqrt{101}}{10}$, $\cot A=10$

D: Cofunctions

Exercise 5.2e.D

Use cofunctions of complementary angles to write an equivalent expression.

36. $\cos(34°) = \sin(\_°)$	37. $\cos(\frac{\pi}{3}) = \sin(\_)$	38. $\csc(21°) = \sec(\_°)$	39. $\tan(\frac{\pi}{4}) = \cot(\_)$

Answers to odd exercises.

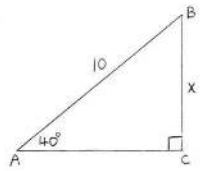
37. $\frac{\pi}{6}$ 39. $\frac{\pi}{4}$

E: Given one side and an acute angle of a right triangle, find the other sides and angles

Exercise 5.2e.E

★ In each of the following triangles, solve for x to the nearest tenth.

41.



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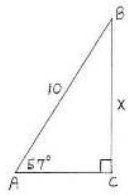

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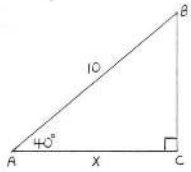

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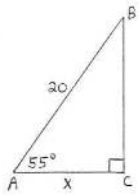

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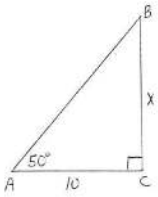

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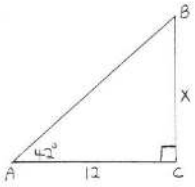

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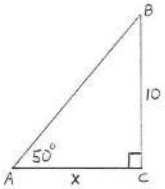

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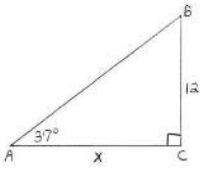

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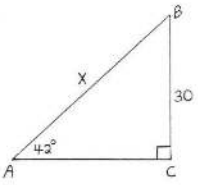

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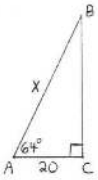

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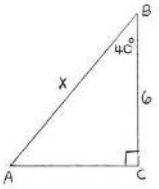

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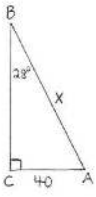
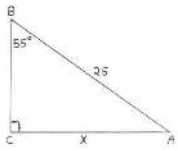


Answers to odd exercises.

41. **6.4** 43. **7.7** 45. **11.9** 47. **8.4** 49. **44.8** 51. **7.8**

★ In each of the following triangles, solve for x to the nearest tenth.

53.



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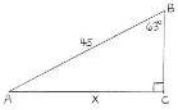

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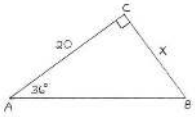

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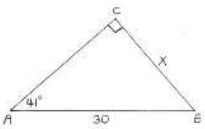

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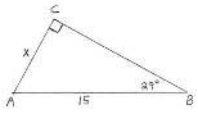

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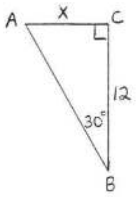

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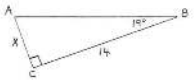

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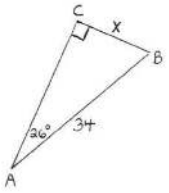
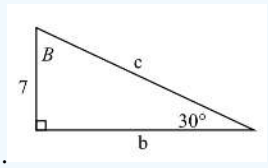


Answers to odd exercises.

53. 20.5 55. 14.5 57. 7.3 59. 4.8

★ In each of the following triangles, solve for the unknown sides and angles. Give answers to 4 decimal digits.

61.



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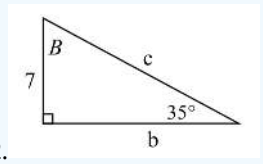
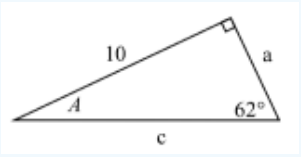


63.



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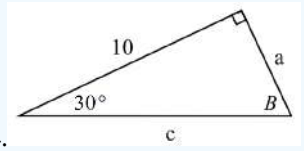

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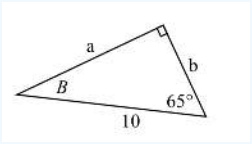

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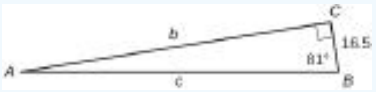

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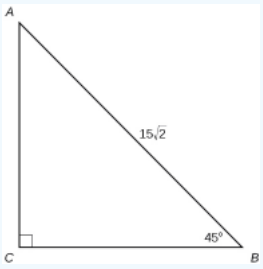
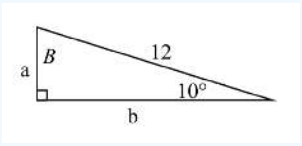


68.

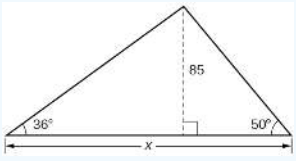


69. $a = 5$, angle opposite side a is $\measuredangle A = 60°$

70. Hypotenuse $c = 12$, and one acute angle is $\measuredangle A = 45°$

★ Find x. Give answers to 4 decimal digits.

71.



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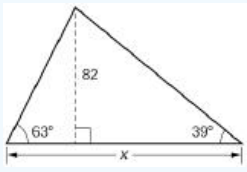

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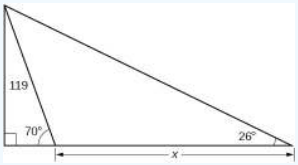

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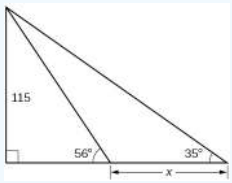


Answers to odd exercises.

61. $b \approx 12.1244$, $c = 14$, $B = 60°$ 63. $a \approx 5.3171$, $c \approx 11.3257$, $A = 28°$ 65. $a \approx 9.0631$, $b \approx 4.2262$, $B = 25°$
67. $a = 15$, $b = 15$, $A = 45°$ 69. $b \approx 2.8868$, $c \approx 5.7735$, $B = 30°$ 71. 188.3159 73. 200.6737

Exercise 5.2e.F

Solve. Round answers to the nearest 10th unless otherwise specified.

81. A 23-ft ladder leans against a building so that the angle between the ground and the ladder is 80°. How high does the ladder reach up the side of the building?

82. A 33-ft ladder leans against a building so that the angle between the ground and the ladder is 80°. How high does the ladder reach up the side of the building?

83. A 20 foot ladder is leaning against a wall, It makes an angle of 70° with the ground. How high is the top of the ladder from the ground (nearest tenth of a foot)?

84. A 275 foot guy wire is attached to the top of a communication tower. If the wire makes an angle of 53° with the ground, how tall is the tower?

ANGLE OF ELEVATION

85. At a point 50 feet from a tree the angle of elevation of the top of the tree is 43°. Find the height of the tree to the nearest tenth of a foot.

86. At a point 60 feet from a tree the angle of elevation of the top of the tree is 40°. Find the height of the tree to the nearest tenth of a foot.

87. The angle of elevation to the top of a mountain from a point 20 miles away from the base of the mountain is 6°. How high is the mountain to the nearest foot? One mile is 5280 feet.

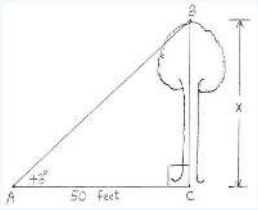


Figure 5.2e.#

89. At a point 100 feet from a tall building the angle of elevation of the top of the building is 65°. Find the height of the building to the nearest foot.

90. The angle of elevation to the top of a building in New York is found to be 9 degrees from the ground at a distance of 1 mile from the base of the building. Using this information, find the height of the building in feet. One mile is 5280 feet.

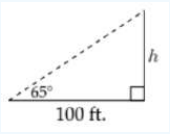


Figure 5.2e.#

91. The angle of elevation to the top of a building in Seattle is found to be 2 degrees from the ground at a distance of 2 miles from the base of the building. Using this information, find the height of the building in feet. One mile is 5280 feet.

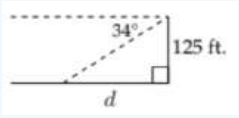
93. Assuming that a 370-foot tall giant redwood grows vertically, if I walk a certain distance from the tree and measure the angle of elevation to the top of the tree to be 60°, how far from the base of the tree am I?

94. The CN Tower in Toronto is 1815 feet tall. When the angle of elevation to the top of the tower is observed to be 40°, how far away from the Tower is that location?

ANGLE OF DEPRESSION

95. From the top of a building 125 feet tall, the angle of depression of an intersection is 34°. How far from the base of the building is the intersection?

Notice that the angle of elevation will be equal to the corresponding angle of depression.



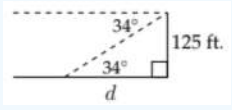


Figure 5.2e.#

96. A helicopter that is 700 feet in the air measures the angle of depression to a landing pad as 24°. How far is the landing pad from the point directly beneath the helicopter's current position?

97. From the top of a 100 foot lighthouse the angle of depression of a boat is $15°$. How far is the boat from the bottom of the lighthouse (nearest foot)?

98. From the top of a lighthouse 180 feet above sea level, the angle of depression to a ship in the ocean is $28°$. How far is the ship from the base of the lighthouse?

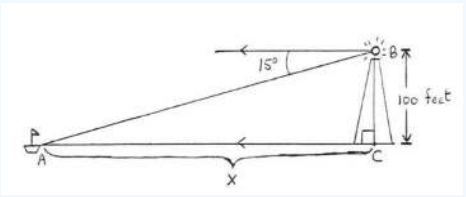


Figure 5.2e.#

99. From an airplane 5000 feet above the ground the angle of depression of an airport is $5°$. How far away is the airport to the nearest hundred feet?

100. From a helicopter 1000 feet above the ground the angle of depression of a heliport is $10°$. How far away is the heliport to the nearest foot?

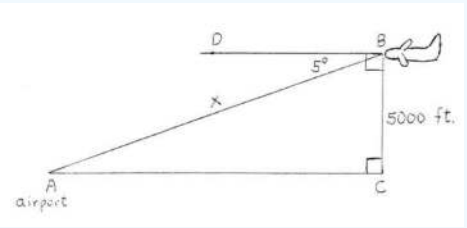


Figure 5.2e.#

TWO ANGLES of DEPRESSION and/or ELEVATION

101. A radio tower is located 325 feet from a building. From a window in the building, a person determines that the angle of elevation to the top of the tower is $43°$, and that the angle of depression to the bottom of the tower is $31°$. How tall is the tower?

102. A radio tower is located 400 feet from a building. From a window in the building, a person determines that the angle of elevation to the top of the tower is $36°$, and that the angle of depression to the bottom of the tower is $23°$. How tall is the tower?

103. A 400-foot tall monument is located in the distance. From a window in a building, a person determines that the angle of elevation to the top of the monument is $18°$, and that the angle of depression to the bottom of the monument is $3°$. How far is the person from the monument?

104. A 200-foot tall monument is located in the distance. From a window in a building, a person determines that the angle of elevation to the top of the monument is $15°$, and that the angle of depression to the bottom of the tower is $2°$. How far is the person from the monument?

105. There is lightning rod on the top of a building. From a location 500 feet from the base of the building, the angle of elevation to the top of the building is measured to be $36°$. From the same location, the angle of elevation to the top of the lightning rod is measured to be $38°$. Assuming the lightning rod is situated at the edge of the building facing the observer, find the height of the lightning rod.

106. There is an antenna on the top of a building. From a location 300 feet from the base of the building, the angle of elevation to the top of the building is measured to be $40°$. From the same location, the angle of elevation to the top of the antenna is measured to be $43°$. Find the height of the antenna.

Answers to odd exercises.

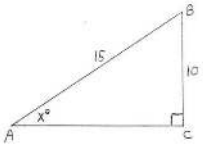
81. 22.7 ft 83. 18.8 ft 85. 46.6 feet. 87. $11,099$ feet. 89. 214 feet. 91. 368.8 ft 93. 213.6 ft
95. $d \approx 185.32$ ft 97. 373.2 ft 99. $57,300$ feet 101. 498.3 ft 103. 1060.1 ft 105. 27.4 ft

G: Find an angle given 2 sides of a right triangle

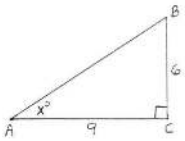
Exercise 5.2e.G

★ **Find the angle x to the nearest tenth of a degree:**

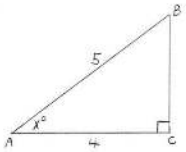
111.



112.



113.



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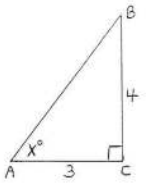

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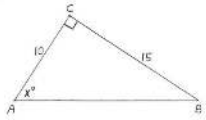

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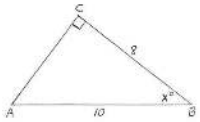

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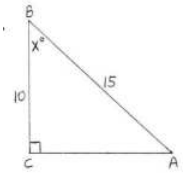
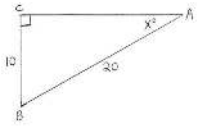


118.



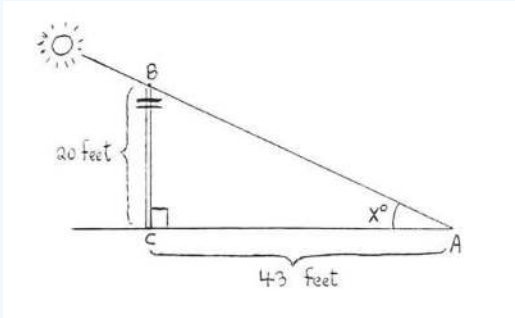
Answers to odd exercises.

111. **41.8°** 113. **36.9°** 115. **56.3°** 117. **48.2°**

H: Applications Finding an Angle

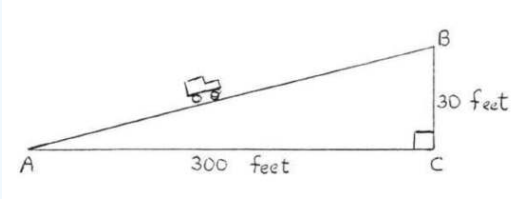
Exercise 5.2e.H

121. If a 20 foot telephone pole casts a shadow of 43 feet , what is the angle of elevation of the sun to the nearest tenth?



122. An 88 foot tree casts a shadow that is 135 feet long. What is the angle of elevation of the sun to the nearest tenth?

123. A road rises 30 feet in a horizontal distance of 300 feet. Find the angle the road makes with the horizontal to the nearest tenth of a degree.



124. A road rises 10 feet in a horizontal distance of 400 feet. Find the angle the road makes with the horizontal to the nearest tenth of a degree.

Answers to odd exercises.

121. **24.9°** 123. **5.7°**

I: Calculator Practice

Exercise 5.2e.I

★ Use a calculator to find the value to four decimal digits. Be sure your calculator is in the appropriate mode!

131. $\sin 10^\circ$ 132. $\sin 30^\circ$ 133. $\cos 80^\circ$ 134. $\cos 60^\circ$ 135. $\tan 45^\circ$

136. $\tan 60^\circ$ 137. $\sin 18^\circ$ 138. $\cos 72^\circ$ 139. $\tan 50^\circ$ 140. $\tan 80^\circ$

141. $\sin\left(\frac{\pi}{12}\right)$ 142. $\sin\left(\frac{5\pi}{12}\right)$ 143. $\cos\left(\frac{3\pi}{8}\right)$ 144. $\cos\left(\frac{3\pi}{8}\right)$

145. $\tan\left(\frac{\pi}{3}\right)$ 146. $\tan(0.8)$ 147. $\sin(1.5)$ 148. $\cos(1.0)$

Answers to odd exercises.

131. 0.1736 133. 0.1736 135. 1 137. 0.3090 139. 1.1918
141. 0.2588 143. 0.3827 145. 1.7321 147. 0.9975

★ Use a calculator to find the value (a) in degrees and (b) in radians to four decimal digits.

151. $\sin^{-1}(.1)$ 152. $\sin^{-1}(.1)$ 153. $\cos^{-1}(.3)$ 154. $\cos^{-1}(.9)$

155. $\tan^{-1}(.5)$ 156. $\tan^{-1}(4)$ 157. $\sin^{-1}(3)$ 158. $\cos^{-1}(2)$

Answers to odd exercises.

151. (a) 6.7392° (b) $.1002$ (radians) 153. (a) 72.5425° (b) 1.2661 (radians)
155. (a) 26.5651° (b) 1.3258 (radians) 157. Undefined ("Domain Error")

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5.3: Sine and Cosine Functions

Learning Objectives

- Define the sine and cosine functions
- Find function values for the sine and cosine of 30° or $(\frac{\pi}{6})$, 45° or $(\frac{\pi}{4})$, and 60° or $(\frac{\pi}{3})$.
- Find reference angles.
- Use reference angles to evaluate trigonometric functions.

While it is convenient to describe the location of a point on a circle using the radius of the circle and the measure of a central angle in the circle, relating this information to x and y coordinates is an important application of trigonometry.

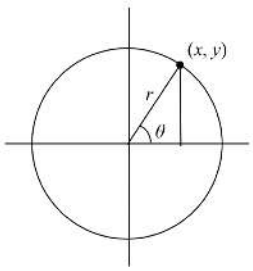
An example of a problem that can be solved using the techniques of this section follows: A distress signal is sent from a sailboat during a storm, but the transmission is unclear and the rescue boat sitting at the marina cannot determine the sailboat's location. Using high powered radar, they determine the distress signal is coming from a distance of 20 miles at an angle of 225 degrees from the marina. How many miles east/west and north/south of the rescue boat is the stranded sailboat?

Reference Triangles

We begin by drawing a circle centered at the origin with radius r, and marking the point on the circle indicated by some angle θ. This point has coordinates (x, y).

Definition: reference triangle

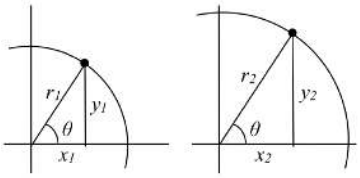
A **reference triangle** is the right triangle inside a circle formed by dropping a perpendicular line segment to the x-axis from a point (x, y) on the terminal side of an angle in standard position that has its vertex in the center of a circle of radius r. The positive acute angle between the x-axis and the terminal side of the angle θ is the **reference angle**.



Some important points about this are made here.

- No matter which quadrant our angle θ puts us in, we can draw a triangle by dropping a perpendicular line segment to the x axis. The values of x and y may be positive or negative, depending on the quadrant the terminal side is in. The value of r is always positive.
- If the angle θ puts us on an axis, we simply measure the radius as the magnitude of an x or y value on the axis, with the other value being 0, again ensuring we have appropriate signs on the coordinates based on the axis the point is on.
- Triangles obtained from different radii will all be similar triangles, meaning corresponding sides scale proportionally. While the lengths of the sides may change, the *ratios* of the side lengths will always remain constant for any given angle. Thus,

$$\frac{y_1}{r_1} = \frac{y_2}{r_2} \qquad \frac{x_1}{r_1} = \frac{x_2}{r_2}$$



To be able to refer to these ratios more easily, we will give them names. Since the ratios depend on the angle, we will write them as functions of the angle θ.

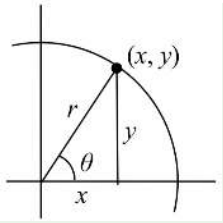
Sine and Cosine Function Values

Definition: sine and cosine

For the point (x, y) on a circle of radius r at an angle of θ in standard position, we can define two important functions as the ratios of the sides of the corresponding triangle:

The **sine** function: $\sin(\theta) = \frac{y}{r}$

The **cosine** function: $\cos(\theta) = \frac{x}{r}$



Note that the definition of the sine and cosine functions depends on the values of a point (x, y) on the terminal side of the angle. Consequently

- The point chosen can be ANY point on the terminal side of an angle in standard position.
- The sign of the ratio obtained depends on the signs of the point (x, y), which in turn is determined by the quadrant the angle is in.

The value of the radius r of a circle is ALWAYS positive. Therefore, the sign of a sine or cosine function is entirely determined by the signs of x and y for a point on the terminal side of the angle.	$\sin\theta = \frac{y}{r}$ QII: + | QI: + Sine is positive when the terminal side of the angle is in quadrants I and II	$\cos\theta = \frac{x}{r}$ QI: + QIV: + Cosine is positive when the terminal side of the angle is in quadrants I and IV

In this chapter, we will explore these functions using both circles and right triangles.

Using a Point on the Terminal Side of an Angle

The definition of the sine and cosine functions depends on the coordinates of a point on the terminal side of an angle in standard position. In order to find the sine and cosine function values, this point can be given to us. Alternatively, if we are given an angle with a terminal side on one of the axes, the point can be chosen by us.

Example 5.3.1: Find sine and cosine given a point on the terminal side of an angle

The point (3, 4) is on the circle of radius 5 at some angle θ. Find $\cos(\theta)$and $\sin(\theta)$.

Solution

Knowing the radius of the circle and coordinates of the point, we can evaluate the cosine and sine functions as the ratio of the sides.

$$\cos(\theta) = \frac{x}{r} = \frac{3}{5} \qquad \sin(\theta) = \frac{y}{r} = \frac{4}{5}$$

There are a few cosine and sine values which we can determine fairly easily because the corresponding point on the terminal side falls on the x or y axis.

Example 5.3.2: Find sine and cosine of a quadrantal angle

Find $\cos(90°)$ and $\sin(90°)$.

Solution

The terminal side of a 90 degree angle in standard position points straight up, so the coordinates of the corresponding point on the circle would be (0, r). Using our definitions of cosine and sine,

$$\cos(90°) = \frac{x}{r} = \frac{0}{r} = 0 \qquad \sin(90°) = \frac{y}{r} = \frac{r}{r} = 1$$

Try It 5.3.2

Find cosine and sine of the angle π.

Answer

$\cos(\pi) = -1 \quad \sin(\pi) = 0$

Given a Trigonometric Ratio and the Quadrant the Angle is in

Use a Reference Triangle

Given one trigonometric ratio and the quadrant where the terminal side lies, the **Pythagorean Theorem** can be used to find a corresponding third side to the triangle, and then other trigonometric ratios can be determined.

How to: Use a Reference Triangle to find sine or cosine given the other ratio and the quadrant the angle is in.

Method #1: Reference Triangle Approach. Given a trigonometric ratio and the quadrant the terminal side of the angle is in.

1. Use the definition of the given trigonometric function and the quadrant it is in to determine two of the three sides of a reference triangle (x, y, and r). Depending on the quadrant the angle is in, x or y can be negative.
2. Use the Pythagorean Theorem to find the third side
3. Choose the solution with the sign that is appropriate to the Quadrant the angle is in. Now that all three sides of a reference triangle for the angle are known, use the definitions to determine the other trigonometric ratio.

Example 5.3.3

Given $\sin(\theta) = -\frac{4}{11}$ and θ is in Quadrant III, find $\cos(\theta)$.

Solution (Method #1: Reference Triangle Approach)

1. Since $\sin(\theta) = -\frac{4}{11} = \frac{y}{r}$ and r is always positive, we can conclude $y = -4$ and $r = 11$
2. By the Theorem of Pythagoras, $x^2 + y^2 = r^2$. Substituting the known values from step one we have

$$\begin{aligned} x^2 + (-4)^2 &= (11)^2 \\ x^2 + 16 &= 121 \\ x^2 &= 105 \\ x &= \pm\sqrt{105} \end{aligned}$$

3. Because θ is in Quadrant III where x coordinates are always negative, $x = -\sqrt{105}$.

   The definition for cosine is $\cos(\theta) = \frac{x}{r}$. Substituting the values we have for x and r we have $\cos(\theta) = \frac{-\sqrt{105}}{11}$.

Use The Pythagorean Identity

The basic equation for a circle centered at the origin is $x^2 + y^2 = r^2$. Combine this with the definition for sine and cosine above, to establish a new identity.

$$\begin{aligned} x^2 + y^2 &= r^2 && \text{Start with equation for a circle} \\ \frac{x^2 + y^2}{r^2} &= \frac{r^2}{r^2} && \text{Divide through by } r^2 \\ \frac{x^2}{r^2} + \frac{y^2}{r^2} &= 1 && \text{Rewrite} \\ \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 &= 1 && \text{Rewrite} \\ (\cos(\theta))^2 + (\sin(\theta))^2 &= 1 && \text{Substitute the definitions above} \\ \cos^2(\theta) + \sin^2(\theta) &= 1 && \text{Rewrite} \end{aligned}$$

Here $\cos^2(\theta)$ is a commonly used shorthand notation for $(\cos(\theta))^2$. Be aware that many calculators and computers do not understand the shorthand notation.

Definition: Pythagorean Identity

The **Pythagorean Identity**. For any angle θ,

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

One use of this identity is that it helps us to find a cosine value of an angle if we know the sine value of that angle or vice versa. However, since the equation will yield two possible values, we will need to utilize additional knowledge of the Quadrant the angle is in to help us find the desired value.

How to: Use Identities to find sine or cosine given the other ratio and the quadrant the angle is in.

Method #2: Identities Approach. Given a trigonometric ratio and the quadrant the terminal side of the angle is in.

1. Use the Pythagorean Identity to determine the sine or cosine ratio of an angle from the other ratio that was given.
2. Choose the solution with the sign that is appropriate to the Quadrant the angle is in.

Example 5.3.4

If $\sin(\theta) = \frac{3}{7}$ and θ is in the second quadrant, find $\cos(\theta)$.

Solution (Method #2: Identities Approach)

Substitute the known value for sine into the Pythagorean identity,

$$\begin{aligned} \cos^2(\theta) + \sin^2(\theta) &= 1 && \text{Pythagorean Identity} \\ \cos^2(\theta) + \frac{9}{49} &= 1 && \text{Substitute given values} \\ \cos^2(\theta) &= 1 - \frac{9}{49} \\ \cos^2(\theta) &= \frac{40}{49} \\ \cos(\theta) &= \pm\sqrt{\frac{40}{49}} = \pm\frac{\sqrt{40}}{7} \\ &= \pm\frac{2\sqrt{10}}{7} \end{aligned}$$

Since we are given that θ is in the second quadrant, x coordinates of points in Quadrant II are negative, and $\cos(\theta) = \frac{x}{r}$, we can conclude the cosine value is also be negative. Therefore, $\cos(\theta) = -\frac{2\sqrt{10}}{7}$

Try It 5.3.4

If $\cos(\theta) = \frac{24}{25}$ and θ is in the fourth quadrant, find $\sin(\theta)$.

Answer

$$\sin(\theta) = -\frac{7}{25}$$

Sines and Cosines of Special Angles

Using the **Pythagorean Identity** and some knowledge about triangles, values for the sine and cosine of some special angles can be determined.

45° or π/4 Angles

First, consider a point on a circle at an angle of 45°, or $\frac{\pi}{4}$. At this angle, the x and y coordinates of the corresponding point on the circle will be equal because 45° divides the first quadrant in half. Since the x and y values will be the same, the sine and cosine values will also be equal. Utilizing the Pythagorean Identity,

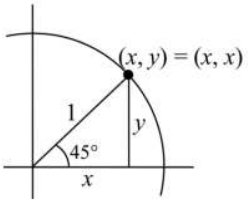


Figure 5.3.1

$$\begin{aligned}
\cos^2\left(\frac{\pi}{4}\right)+\sin^2\left(\frac{\pi}{4}\right) &= 1 && \text{since the sine and cosine are equal}\\
\cos^2\left(\frac{\pi}{4}\right)+\cos^2\left(\frac{\pi}{4}\right) &= 1 && \text{add like terms}\\
2\cos^2\left(\frac{\pi}{4}\right) &= 1 && \text{divide}\\
\cos^2\left(\frac{\pi}{4}\right) &= \frac{1}{2} && \text{since the } x \text{ value is positive}\\
\cos\left(\frac{\pi}{4}\right) &= \sqrt{\frac{1}{2}} && \text{rationalize the denominator}\\
\cos\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2} && \text{Because } = \sqrt{\frac{1}{2}}\sqrt{\frac{2}{2}} = \sqrt{\frac{2}{4}}
\end{aligned}$$

Since the sine and cosine are equal, $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ as well. The (x, y) coordinates for a point on a circle of radius 1 at an angle of 45° are $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

30° or π/6 Angles

Next, we will find the cosine and sine at an angle of 30°, or $\frac{\pi}{6}$. To do this, we will first draw a triangle inside a circle with one side at an angle of 30°, and another at an angle of -30°. If the resulting two right triangles are combined into one large triangle, notice that all three angles of this larger triangle will be 60°.

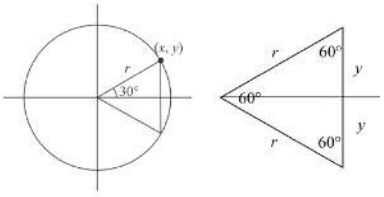


Figure 5.3.2

Since all the angles are equal, the sides will all be equal as well. The vertical line has length $2y$, and since the sides are all equal we can conclude that $2y = r$, or $y = \frac{r}{2}$. Using this, we can find the sine value:

$$\sin\left(\frac{\pi}{6}\right) = \frac{y}{r} = \frac{r/2}{r} = \frac{r}{2}\cdot\frac{1}{r} = \frac{1}{2}$$

Using the Pythagorean Identity, we can find the cosine value:

$$\cos^2\left(\frac{\pi}{6}\right)+\sin^2\left(\frac{\pi}{6}\right) = 1$$
$$\cos^2\left(\frac{\pi}{6}\right)+\left(\frac{1}{2}\right)^2 = 1$$
$$\cos^2\left(\frac{\pi}{6}\right) = \frac{3}{4}$$

since the x value is positive, we'll keep the positive root

$$\cos\left(\frac{\pi}{6}\right) = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

The (x, y) coordinates for the point on a circle of radius 1 at an angle of 30 degrees are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

60° or π/3 Angles

Consider an equilateral triangle ABC inscribed in a circle of radius 1 below. All angles in this triangle are 60° so Angle A has measure 60°. Construct BD that bisects angle B. Therefore BD is the perpendicular bisector of AC, so it cuts AC in half, which

means that AD is $\frac{1}{2}$ the radius, or $x = \frac{1}{2}$.

$$
\begin{aligned}
x^2 + y^2 &= 1 && \text{From the Pythagorean Theorem} \\
\left(\frac{1}{2}\right)^2 + y^2 &= 1 && \text{Because } x \text{ is } \tfrac{1}{2} \text{ the radius is 1} \\
y^2 &= 1 - \frac{1}{4} && \text{Solve for } y \\
y^2 &= \frac{3}{4} \\
y &= \pm\frac{\sqrt{3}}{2} \\
y &= \frac{\sqrt{3}}{2} && \text{in QI where } x \text{ and } y \text{ are positive}
\end{aligned}
$$

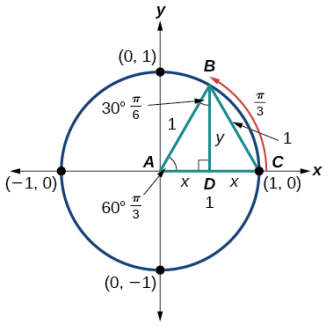


Figure 5.3.3

Thus, $\sin\left(\frac{\pi}{3}\right) = \frac{y}{r} = \frac{\sqrt{3}}{2}$ and $\cos\left(\frac{\pi}{3}\right) = \frac{x}{r} = \frac{1}{2}$

The (x, y) coordinates for the point on a circle of radius 1 at an angle of $60°$ are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

Summary

We have now found the cosine and sine values for all the commonly encountered angles that have their terminal side in the first quadrant. This information is summarized below in both a table format and in a diagram of the first quadrant of the unit circle. These values for sine and cosine should be thoroughly memorized.

Table 5.3.4 of sine and cosine values for "special" angles

Angle	0 or 0°	$\frac{\pi}{6}$ or 30°	$\frac{\pi}{4}$ or 45°	$\frac{\pi}{3}$ or 60°	$\frac{\pi}{2}$ or 90°
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

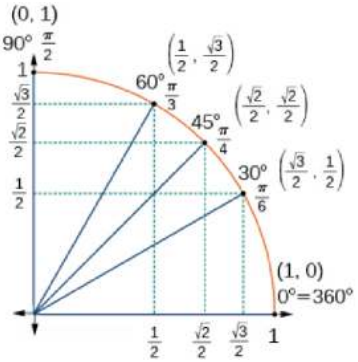


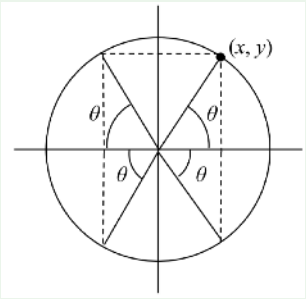
Figure 5.3.4: Angles with points $(x = \cos\theta,\ y = \sin\theta)$ on the Unit Circle in QI

Reference Angles

Definition: Reference Angles

An angle's **reference angle** is the measure of the smallest, positive, acute angle θ formed by the terminal side of the angle θ and the horizontal axis. It is the acute angle located at the origin of a reference triangle.

The diagram to the right illustrates the reference angle θ for angles in standard position that have a terminal side in one of the four quadrants. Notice the reference angle is NOT in standard position (except in Quadrant I)



A reference angle is always an angle between 0° and 90°, or 0 and $\frac{\pi}{2}$ radians.

Any reference angle in standard position is always in Quadrant I. Consequently, all trigonometric ratios for a reference angle are never negative.

Reference angles are always between 0 and $\frac{\pi}{2}$, so the sine and cosine values for reference angles will never be negative. Angles with the same reference angles all have the same cosine and sine values, except possibly for their sign (positive or negative). The sign of the sine or cosine ratio is determined by which quadrant the angle is in.

Angles with the Same Sine or Cosine

Any angle in the first quadrant is a reference angle. Any point (x, y) on the terminal side of this angle has x and y coordinates that are both positive.

For any given angle in the first quadrant, there is an angle in the second quadrant with the same sine value because the sine value is the ratio $\frac{y}{r}$, and y coordinates are positive in Quadrant I and Quadrant II. However, this angle in Quadrant II with the same positive y-value, will have the opposite (i.e. negative) value for x. Thus although the sine is the same, the cosine for the angle in Quadrant II will be negative.

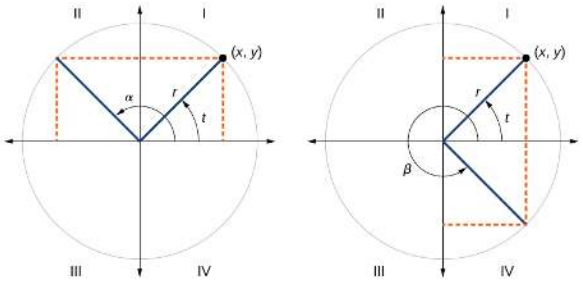


Figure 5.3.5 In QI, $\sin(t) = \sin(\alpha)$ and $\cos(t) = -\cos(\alpha)$; In QIV, $\sin(t) = -\sin(\beta)$ and $\cos(t) = \cos(\beta)$

Likewise, there will be an angle in the fourth quadrant with the same cosine as the original (reference) angle. The angle with the same cosine will share the same x-value but will have the opposite y-value. Therefore, its sine value will be the opposite of the original angle's sine value. This is illustrated in Figure 5.3.5.

Finding a Reference Angle

Recall that an angle's reference angle is the acute angle, α, formed by the terminal side of the angle θ and the horizontal axis. A reference angle is always an angle between 0 and 90°, or 0 and $\frac{\pi}{2}$ radians and it is not in standard position (except in QI). As we can see from Figure 5.3.6, for any angle in quadrants II, III, or IV, there is a reference angle in quadrant I.

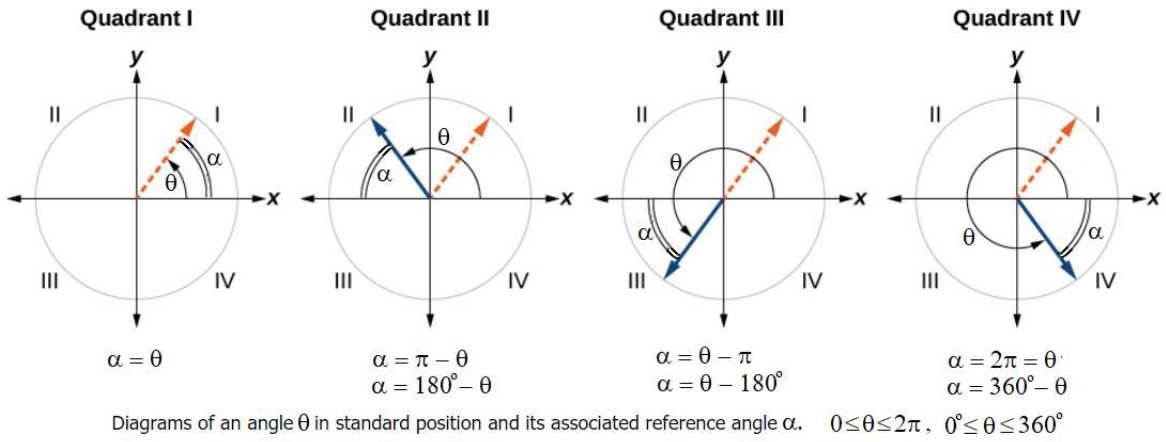


Figure 5.3.6 Diagrams of an angle θ in standard position and its associated reference angle α.

Howto: Find a reference angle.

1. Given an angle θ, if the angle is in degrees and not in the interval $[0°, 360°)$, or if the angle is in radians and not in the interval $[0, 2\pi)$, find a coterminal angle that is in these intervals (by adding or subtracting $360°$ or 2π).
2. An angle in the first quadrant is its own reference angle.
3. For an angle in the second or third quadrant, the reference angle is $|\pi - \theta|$ or $|180° - \theta|$.
4. For an angle in the fourth quadrant, the reference angle is $2\pi - \theta$ or $360° - \theta$.

Example 5.3.6: Find a Reference Angle

Find the reference angle for $225°$ as shown in Figure 5.3.7.

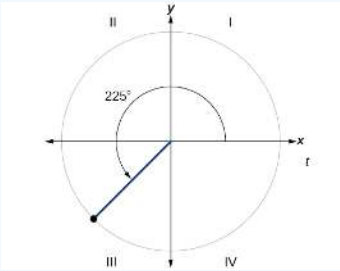


Figure 5.3.7

Solution

Because $225°$ is in the third quadrant, the reference angle is $|(180° - 225°)| = |-45°| = 45°$

Try It 5.3.6

Find the reference angle for $\frac{5\pi}{3}$.

Answer

$\frac{\pi}{3}$

The reader may have noticed that when expressed in radian measure, the reference angle for a non-quadrantal angle is easy to spot. Reduced fraction multiples of π with a denominator of 6 have $\frac{\pi}{6}$ as a reference angle, those with a denominator of 4 have $\frac{\pi}{4}$ as their reference angle, and those with a denominator of 3 have $\frac{\pi}{3}$ as their reference angle.

Use Reference Angles to Evaluate Trigonometric Functions

We can find the cosine and sine of any angle in any quadrant if we know the cosine or sine of its reference angle. The absolute values of the cosine and sine of an angle are the same as those of the reference angle. The sign depends on the quadrant of the original angle. The cosine will be positive or negative depending on the sign of the x-values in that quadrant. The sine will be positive or negative depending on the sign of the y-values in that quadrant.

Reference Angle Theorem

Angles have cosines and sines with the same absolute value as cosines and sines of their reference angles. The sign (positive or negative) can be determined from the quadrant of the angle.

Suppose α is the reference angle for θ. Then $\cos(\theta) = \pm\cos(\alpha)$ and $\sin(\theta) = \pm\sin(\alpha)$, where the choice of the $(\pm)$ depends on the quadrant in which the terminal side of θ lies.

Howto: Use reference angles to find the sine or cosine of an angle.

1. Measure the angle between the terminal side of the given angle and the horizontal axis. That is the reference angle.
2. Determine the values of the cosine and sine of the reference angle.
3. Give the cosine the same sign as the x-values in the quadrant of the original (given) angle.
4. Give the sine the same sign as the y-values in the quadrant of the original (given) angle.

Example 5.3.7: Use Reference Angles to Find the Sine and Cosine for Multiples of Special Angles

a. Use a reference angle to find the exact value of $\cos(150°)$ and $\sin(150°)$.

b. Use a reference angle to find $\cos\ \frac{5\pi}{4}$ and $\sin\ \frac{5\pi}{4}$.

a. Solution

1. The reference angle for 150° is 30°
2. $\cos 30° = \frac{\sqrt{3}}{2}$ and $\sin 30° = \frac{\sqrt{3}}{2}$
3. Since $150°$ is in the second quadrant, the x-coordinate of the point on the circle is negative, so the cosine value is negative. Thus, $\cos 150° = -\frac{\sqrt{3}}{2}$
4. Since $150°$ is in the second quadrant, the y-coordinate of the point on the circle is positive, so the sine value is positive. Thus, $\sin 150° = \frac{1}{2}$.

b. Solution

1. The reference angle for $\frac{5\pi}{4}$ is $\frac{\pi}{4}$
2. $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$
3. Since $\frac{5\pi}{4}$ is in the third quadrant, the x-coordinate of the point on the circle is negative, so the cosine value is negative. Thus, $\cos\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$
4. Since $\frac{5\pi}{4}$ is in the third quadrant, the y-coordinate of the point on the circle is negative, so the sine value is negative. Thus, $\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$.

Try It 5.3.7

1. Use the reference angle for $315°$ to find $\cos(315°)$ and $\sin(315°)$.
2. Use the reference angle for $-\frac{\pi}{6}$ to find $\cos\left(-\frac{\pi}{6}\right)$ and $\sin\left(-\frac{\pi}{6}\right)$

Answer

1. $\cos(315°) = \frac{\sqrt{2}}{2}, \quad \sin(315°) = \frac{-\sqrt{2}}{2}$
2. $\cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, \quad \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$

Because the cosine and sine values for special angles in the first quadrant are known, we can use symmetry and reference angles to fill in cosine and sine values for the rest of the special angles on the unit circle. The complete unit circle appears at the end of this section in Figure 5.3.19. This ENTIRE circle does NOT need to be memorized. Just learning the information that is in Quadrant I as illustrated in Table 5.3.4 or Figure 5.3.4 is all that is needed. Take time to learn the sine and cosine values for all the major angles in the first quadrant! From this information, the trigonometric ratio of any angle can be determined by determining the sign of that ratio based on the quadrant the terminal side lies.

Use a Calculator to Find Sine and Cosine

The special values of sine and cosine in the first quadrant are very useful to know, since knowing them allows you to quickly evaluate the sine and cosine of very common angles without needing to look at a reference or use your calculator. However, scenarios do come up where we need to know the sine and cosine of other angles.

To find the cosine and sine of any other angle, we turn to a computer or calculator. **Be aware**: most calculators can be set into "degree" or "radian" mode, which tells the calculator the units for the input value. When you evaluate "cos(30)" on your calculator, it will evaluate it as the cosine of 30 degrees if the calculator is in degree mode, or the cosine of 30 radians if the calculator is in radian mode. Most computer software with cosine and sine functions only operate in radian mode.

Howto: Use a calculator to find the sine or cosine of an angle.

1. If the angle is in radians, set the calculator to radian mode; if the angle is in degrees, set the calculator to degree mode.
2. Press the COS key or SIN key.
3. Enter the value of the angle, press the close-parentheses key ")" and press ENTER.

Example 5.3.8: Use a Calculator to Find Sine and Cosine

1. Use a calculator to evaluate $\cos\left(\frac{5\pi}{3}\right)$.
2. Use a calculator to evaluate the cosine of 20°.

Solution

1. Make sure you are in radian mode because the angle is in radians. Then enter the following keystrokes: COS(5 × π ÷ 3) ENTER.
The result obtained is 0.5

2. Make sure you are in degree mode because the angle is in degrees. Then enter the following keystrokes: COS(20) ENTER
The result is approximately 0.939693.

If the calculator does not have degree mode, the angle must be converted to radians: COS(20 × 180 ÷ π) ENTER

Try It 5.3.8

Evaluate $\sin\left(\frac{\pi}{3}\right)$.

Answer

approximately 0.866025

Identify the Domain and Range of Sine and Cosine Functions

Now that we can find the sine and cosine of an angle, we need to discuss their domains and ranges.

What are the domains of the sine and cosine functions? The arguments to the sine and cosine function can be any angle, positive or negative. Therefore the input to the sine and cosine functions can be any real number. Thus the domain is $(-\infty, \infty)$.

What are the ranges of the sine and cosine functions? What are the least and greatest possible values for their output? Using the Pythagorean Identity $\cos^2(\theta) + \sin^2(\theta) = 1$, we can see that each term at a minimum can be 0, which makes the square of the other term at most 1. Thus $-1 \leq \sin\theta \leq 1$ and $-1 \leq \cos\theta \leq 1$. In other words, the range of both the sine and cosine functions is $[-1, 1]$.

Find Coordinates on a Circle

Notice that the definitions for sine and cosine can also be stated in the following form.

How to: Find coordinates of a point given r and an angle .

Given a circle of radius r at an angle θ, then the **coordinates of the point (x, y)** on that circle where it intersects the terminal side of the standard position angle θ are:

$$x = r\cos(\theta)$$

$$y = r\sin(\theta)$$

Example 5.3.9a: Find coordinates of a point on a circle given r and an angle.

Find the coordinates of the point on a circle of radius 6 at an angle of $\frac{\pi}{4}$.

Solution

Using our knowledge that $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, along with our relationships that stated $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we can find the coordinates of the point desired:

$$x = 6\cos\left(\frac{\pi}{4}\right) = 6\left(\frac{\sqrt{2}}{2}\right) = 3\sqrt{2}$$

$$y = 6\sin\left(\frac{\pi}{4}\right) = 6\left(\frac{\sqrt{2}}{2}\right) = 3\sqrt{2}$$

Try It 5.3.9a

Find the coordinates of the point on a circle of radius 3 at an angle of $90°$.

Answer

$$x = 3\cos\left(\frac{\pi}{2}\right) = 3 \cdot 0 = 0$$
$$y = 3\sin\left(\frac{\pi}{2}\right) = 3 \cdot 1 = 3$$

Example 5.3.9b: Find coordinates of a point on a circle given r and an angle .

1. Find the coordinates of the point on a circle of radius 12 at an angle of $\frac{7\pi}{6}$.
2. Find the coordinates of the point on the unit circle at an angle of $\frac{7\pi}{6}$.

1. Solution

Note that this angle is in the third quadrant, where both *x* and *y* are negative. Keeping this in mind can help you check your signs of the sine and cosine function.

$$x = 12\cos\left(\frac{7\pi}{6}\right) = 12\left(\frac{-\sqrt{3}}{2}\right) = -6\sqrt{3}$$

$$y = 12\sin\left(\frac{7\pi}{6}\right) = 12\left(\frac{-1}{2}\right) = -6$$

The coordinates of the point are $(-6\sqrt{3}, -6)$.

2. Solution

We know that the angle $\frac{7\pi}{6}$ is in the third quadrant. Because this angle is a reduced multiple of $\frac{\pi}{6}$, its reference angle is simply $\frac{\pi}{6}$

Next, we will find the cosine and sine of the reference angle: $\cos\left(\frac{\pi}{6}\right) = \frac{3}{2}$ $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

We must determine the appropriate signs for x and y in the given quadrant. Because our original angle is in the third quadrant, where both x and y are negative, both cosine and sine are negative.

$$\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$
$$\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$$

Now we can calculate the (x, y) coordinates on the unit circle, where $r = 1$, are $x = \cos\theta$ and $y = \sin\theta$.

The coordinates of the point on the unit circle are $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

Try It 5.3.9b

Find the coordinates of the point on a circle of radius 5 at an angle of $\frac{5\pi}{3}$.

Answer

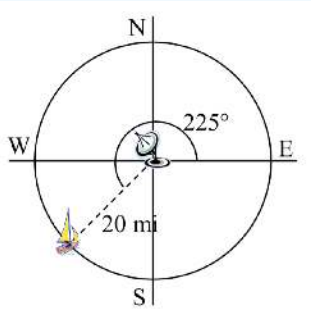
$$\left(5\cos\left(\frac{5\pi}{3}\right), 5\sin\left(\frac{5\pi}{3}\right)\right) = \left(\frac{5}{2}, \frac{-5\sqrt{3}}{2}\right)$$

Example 5.3.10

We now have the tools to return to the sailboat question posed at the beginning of this section.

Solution

A distress signal is sent from a sailboat during a storm, but the transmission is unclear and the rescue boat sitting at the marina cannot determine the sailboat's location. Using high powered radar, they determine the distress signal is coming from a point 20 miles away at an angle of 225 degrees from the marina. How many miles east/west and north/south of the rescue boat is the stranded sailboat?



We can now answer the question by finding the coordinates of the point on a circle with a radius of 20 miles at an angle of 225 degrees.

$$x = 20\cos(225°) = 20\left(\frac{-\sqrt{2}}{2}\right) \approx -14.142 \text{ miles}$$

$$y = 20\sin(225°) = 20\left(\frac{-\sqrt{2}}{2}\right) \approx -14.142 \text{ miles}$$

The sailboat is located 14.142 miles west and 14.142 miles south of the marina.

Key Equations

Cosine	$\cos\, t = x$
Sine	$\sin\, t = y$
Pythagorean Identity	$\cos^2 t + \sin^2 t = 1$

Key Concepts

- Finding the function values for the sine and cosine begins with drawing a unit circle, which is centered at the origin and has a radius of 1 unit.
- Using the unit circle, the sine of an angle θ equals the y-value of the endpoint on the unit circle of an arc of length θ whereas the cosine of an angle θ equals the x-value of the endpoint.
- The sine and cosine values are most directly determined when the corresponding point on the unit circle falls on an axis.
- When the sine or cosine is known, we can use the Pythagorean Identity to find the other. The Pythagorean Identity is also useful for determining the sines and cosines of special angles.
- Calculators and graphing software are helpful for finding sines and cosines if the proper procedure for entering information is known.
- The domain of the sine and cosine functions is all real numbers.
- The range of both the sine and cosine functions is $[-1, 1]$.
- The sine and cosine of an angle have the same absolute value as the sine and cosine of its reference angle.
- The signs of the sine and cosine are determined from the x- and y-values in the quadrant of the original angle.
- An angle's reference angle is the size angle, α, formed by the terminal side of the angle θ and the horizontal axis.
- Reference angles can be used to find the sine and cosine of the original angle.
- Reference angles can also be used to find the coordinates of a point on a circle.

Glossary

cosine function

the x-value of the point on a unit circle corresponding to a given angle

Pythagorean Identity

a corollary of the Pythagorean Theorem stating that the square of the cosine of a given angle plus the square of the sine of that angle equals 1

sine function

the y-value of the point on a unit circle corresponding to a given angle

unit circle

a circle with a center at $(0, 0)$ and radius 1.

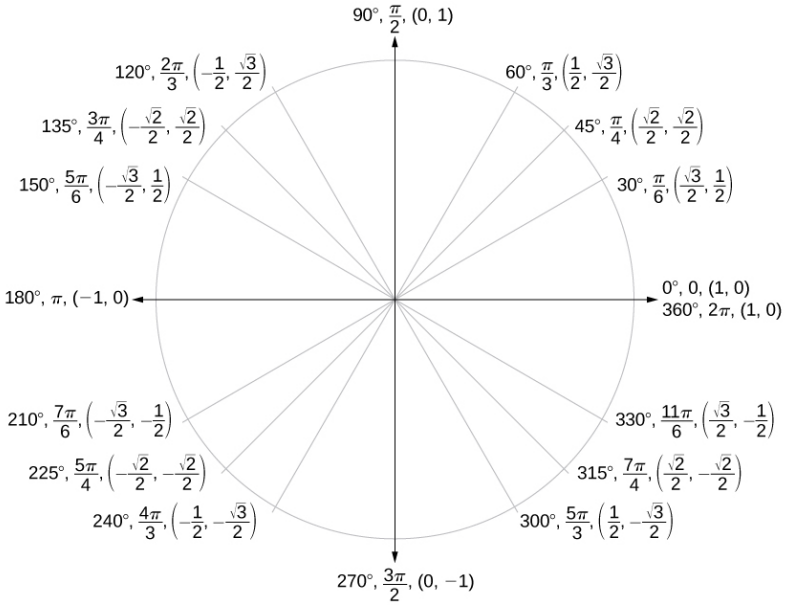


Figure **5.3.19**: The Unit Circle. Special angles and their coordinates $(\cos\theta, \sin\theta)$ on the unit circle

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5.3e: Exercises - Sine and Cosine Functions

A: Concepts

Exercise 5.3e.A

1) Describe the unit circle.

2) What do the x- and y-coordinates of the points on the unit circle represent?

3) Discuss the difference between a coterminal angle and a reference angle.

4) Explain how the cosine of an angle in the second quadrant differs from the cosine of its reference angle in the unit circle.

5) Explain how the sine of an angle in the second quadrant differs from the sine of its reference angle in the unit circle.

6) State the domain of the sine and cosine functions.

7) State the range of the sine and cosine functions.

Answers to Odd Problems:

1. The unit circle is a circle of radius 1 centered at the origin.
3. Coterminal angles are angles that share the same terminal side. A reference angle is the size of the smallest acute angle, t, formed by the terminal side of the angle t and the horizontal axis.
5. The sine values are equal.
7. $[-1, 1]$

B. Find sine or cosine values given a point on the terminal side of an angle or given a quadrantal angle

Exercise 5.3e.B

★ Find the sine and cosine of an angle θ that has the following point on its terminal side. Give exact answers.

11. $(2, 3)$ 12. $(11, 6)$ 13. $(-4, 1)$ 14. $(-2, 8)$ 15. $(-3, -5)$ 16. $(-10, -6)$ 17. $(6, -17)$ 18. $(9, -5)$

★ Find the sine and cosine of the following angles.

21. $0°$ 22. $900°$ 23. $-90°$ 24. $-630°$ 25. 7π 26. 4π 27. $\frac{-7\pi}{2}$ 28. $\frac{-13\pi}{2}$

Answers to Odd Problems:

11. $\sin\theta = \frac{3\sqrt{13}}{13}$, $\cos\theta = \frac{2\sqrt{13}}{13}$
13. $\sin\theta = \frac{\sqrt{17}}{17}$, $\cos\theta = \frac{-4\sqrt{17}}{17}$
15. $\sin\theta = \frac{-5\sqrt{34}}{34}$, $\cos\theta = \frac{-3\sqrt{34}}{34}$
17. $\sin\theta = \frac{-17\sqrt{13}}{65}$, $\cos\theta = \frac{6\sqrt{13}}{65}$

21. $\sin\theta = 0$, $\cos\theta = 1$ 23. $\sin\theta = -1$, $\cos\theta = 0$ 25. $\sin\theta = 0$, $\cos\theta = -1$ 27. $\sin\theta = 1$, $\cos\theta = 0$

C. Find the quadrant an angle is in from the signs of a sine and cosine function

Exercise 5.3e.C

★ Use the given sign of the sine and cosine functions to find the quadrant in which the terminal point determined by t lies.

31. $\sin(t) > 0$ and $\cos(t) > 0$
32. $\sin(t) < 0$ and $\cos(t) < 0$
33. $\sin(t) < 0$ and $\cos(t) > 0$
34. $\sin(t) > 0$ and $\cos(t) < 0$

Answers to Odd Problems:

31. I 33. IV

D. Find sine or cosine values given another trig ratio and the quadrant the angle is in

Exercise 5.3e.D

★ **Find $\cos\theta$ given the following information.**

41. $\sin(\theta) = \frac{3}{8}$ and θ is in the 2nd quadrant.
42. $\sin(\theta) = -\frac{1}{4}$ and θ is in the 3rd quadrant.
43. $\sin\theta = -\frac{1}{3}$ and $\frac{3\pi}{2} < \theta < 2\pi$
44. $\sin\theta = -\frac{3}{4}$ and $\pi < \theta < \frac{3\pi}{2}$
45. $\sin\theta = \frac{5}{7}$ and $\cos\theta < 0$
46. $\sin\theta = \frac{1}{3}$ and $\cos\theta > 0$

★ **Find $\sin\theta$ given the following information.**

51. $\cos(\theta) = \frac{1}{7}$ and θ is in the 4th quadrant.
52. $\cos(\theta) = \frac{2}{9}$ and θ is in the 1st quadrant.
53. $\cos\theta = -\frac{3}{5}$ and $\frac{\pi}{2} < \theta < \pi$
54. $\cos\theta = -\frac{2}{3}$ and $\pi < \theta < \frac{3\pi}{2}$
55. $\cos\theta = -\frac{1}{5}$ and $\sin\theta < 0$
56. $\cos\theta = \frac{5}{6}$ and $\sin\theta < 0$

Answers to Odd Problems:

41. $-\frac{\sqrt{55}}{8}$ 43. $\frac{2\sqrt{2}}{3}$ 45. $-\frac{2\sqrt{6}}{7}$ 51. $-\frac{4\sqrt{3}}{7}$ 53. $\frac{4}{5}$ 55. $-\frac{2\sqrt{6}}{5}$

E. Reference angles

Exercise 5.3e.E

★ **State the reference angle for the given angle.**

71. 135°
72. 240°
73. 100°
74. −315°
75. −170°
76. −47°
77. $\frac{2\pi}{3}$
78. $\frac{5\pi}{4}$
79. $\frac{11\pi}{6}$
80. $-\frac{11\pi}{3}$
81. $\frac{29\pi}{6}$
82. $\frac{10\pi}{7}$
83. $\frac{-3\pi}{8}$
84. $\frac{7\pi}{5}$
85. 1.33
86. 1.98
87. 4.19
88. −1.07
89. 15.20
90. −19.50

Answers to Odd Problems:

71. 45° 73. 80° 75. 10° 77. $\frac{\pi}{3}$ 79. $\frac{\pi}{6}$ 81. $\frac{\pi}{6}$ 83. $\frac{3\pi}{8}$ 85. 1.33 87. 1.05 89. 0.51

★ **Find an angle θ with $0° < \theta < 360°$ or $0 < \theta < 2\pi$ that has the same sine value as the following angles.**

91. $\frac{\pi}{3}$
92. $\frac{\pi}{4}$
93. $\frac{4\pi}{3}$
94. $\frac{7\pi}{6}$
95. 80°
96. 15°
97. 140°
98. 160°
99. 305°
100. 340°

★ **Find an angle θ with $0° < \theta < 360°$ or $0 < \theta < 2\pi$ that has the same cosine value as the following angles.**

101. $\frac{\pi}{3}$
102. $\frac{\pi}{4}$
103. $\frac{4\pi}{3}$
104. $\frac{7\pi}{6}$
105. 80°
106. 15°
107. 140°
108. 160°
109. 305°
110. 340°

Answers to Odd Problems:

91. $\frac{2\pi}{3}$ 93. $\frac{5\pi}{3}$ 95. 100° 97. 40° 99. 235°
101. $\frac{5\pi}{3}$ 103. $\frac{2\pi}{3}$ 105. 280° 107. 220° 109. 55°

★ **Find the reference angle, the quadrant of the terminal side, and the sine and cosine of each angle.**

111. $300°$	113. $135°$	115. $120°$	117. $150°$	119. $\frac{7\pi}{6}$	121. $\frac{3\pi}{4}$	123. $\frac{2\pi}{3}$	125. $\frac{7\pi}{4}$
112. $320°$	114. $225°$	116. $250°$	118. $210°$	120. $\frac{5\pi}{3}$	122. $\frac{5\pi}{4}$	124. $\frac{5\pi}{6}$	126. $\frac{4\pi}{3}$

Answers to Odd Problems:

111. $60°$, QIV, $\sin(300°) = -\frac{\sqrt{3}}{2}$, $\cos(300°) = \frac{1}{2}$
113. $45°$, QII, $\sin(135°) = \frac{\sqrt{2}}{2}$, $\cos(135°) = -\frac{\sqrt{2}}{2}$
115. $60°$, QII, $\sin(120°) = \frac{\sqrt{3}}{2}$, $\cos(120°) = -\frac{1}{2}$
117. $30°$, QII, $\sin(150°) = \frac{1}{2}$, $\cos(150°) = -\frac{\sqrt{3}}{2}$
119. $\frac{\pi}{6}$, QIII, $\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$, $\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$
121. $\frac{\pi}{4}$, QII, $\sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$, $\cos\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{2}}{2}$
123. $\frac{\pi}{3}$, QII, $\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$, $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$
125. $\frac{\pi}{4}$, QIV, $\sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2}$, $\cos\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2}$

F. Find sine or cosine for special angles

Exercise 5.3e.F

★ **Find the exact value of each trigonometric function.**

131. $\sin\frac{\pi}{3}$ 132. $\sin\frac{\pi}{4}$	133. $\sin\frac{\pi}{6}$ 134. $\sin\frac{\pi}{2}$	135. $\sin\pi$ 136. $\sin\frac{3\pi}{2}$	137. $\cos\frac{\pi}{3}$ 138. $\cos\frac{\pi}{6}$	139. $\cos\frac{\pi}{4}$ 140. $\cos 0$	141. $\cos\pi$ 142. $\cos\frac{\pi}{2}$

Answers to Odd Problems:

131. $\frac{\sqrt{3}}{2}$ 133. $\frac{1}{2}$ 135. 0 137. $\frac{1}{2}$ 139. $\frac{\sqrt{2}}{2}$ 141. -1

★ **Evaluate. Give an exact answer. Do not use a calculator.**

143. $\sin\left(\frac{11\pi}{3}\right)$ 144. $\sin\left(-\frac{4\pi}{3}\right)$ 145. $\sin\left(\frac{-11\pi}{4}\right)$	146, $\sin\left(\frac{-5\pi}{4}\right)$ 147. $\sin\left(\frac{\pi}{6}\right)$ 148. $\sin\left(\frac{11\pi}{6}\right)$	149. $\cos\left(\frac{3\pi}{4}\right)$ 150. $\cos\left(\frac{-\pi}{4}\right)$ 151. $\cos\left(\frac{-2\pi}{3}\right)$	152. $\cos\left(\frac{-4\pi}{3}\right)$ 153. $\cos\left(\frac{-5\pi}{6}\right)$ 154. $\cos\left(\frac{-\pi}{6}\right)$	155. $\cos\left(\frac{-11\pi}{2}\right)$ 156. $\sin 7\pi$

★ **Give exact values for $\sin(\theta)$ and $\cos(\theta)$ for each of these angles.**

157. $-\frac{3\pi}{4}$	158. $\frac{23\pi}{6}$	159. $-\frac{\pi}{2}$	150. 5π	161. $315°$

Answers to Odd Problems:

143. $-\frac{\sqrt{3}}{2}$ 145. $-\frac{\sqrt{2}}{2}$ 147. $\frac{1}{2}$ 149. $-\frac{\sqrt{2}}{2}$ 151. $-\frac{1}{2}$ 153. $-\frac{\sqrt{3}}{2}$ 155. 0

157. $\sin(-\frac{3\pi}{4}) = -\frac{\sqrt{2}}{2}$, $\cos(-\frac{3\pi}{4}) = -\frac{\sqrt{2}}{2}$ 159. $\sin(-\frac{\pi}{2}) = -1$, $\cos(-\frac{\pi}{2}) = 0$

161. $\sin(315°) = -\frac{\sqrt{2}}{2}$, $\cos(315°) = \frac{\sqrt{2}}{2}$

G. Find sine or cosine values using a calculator

Exercise 5.3e.G

★ **Use a calculator to evaluate.**

171. $\cos\frac{5\pi}{9}$	172. $\sin\frac{5\pi}{9}$	173. $\sin\frac{\pi}{10}$	174. $\cos\frac{\pi}{10}$	175. $\cos\frac{3\pi}{4}$	176. $\sin\frac{3\pi}{4}$
177. $\sin 310°$	178. $\cos 310°$	179. $\cos 98°$	180. $\sin 98°$		

Answers to Odd Problems:

171. -0.1736 173. 0.3090 175. -0.7071 177. -0.7660 179. -0.1392

H. Find points on a circle intersected by given angle

Exercise 5.3e.H

★ **Find exact values if possible, otherwise give answers to three decimal places.**

182. Find the coordinates of the point on a circle with radius 15 corresponding to an angle of $220°$.

183. Find the coordinates of the point on a circle with radius 20 corresponding to an angle of $120°$.

184. Find the coordinates of the point on a circle with radius 20 corresponding to an angle of 280°.

185. Find the coordinates of the point on a circle with radius 16 corresponding to an angle of $\frac{5\pi}{9}$.

186. Find the coordinates of the point on a circle with radius 8 corresponding to an angle of $\frac{7\pi}{4}$.

187. The point P is on the unit circle. If the y-coordinate of P is $\frac{3}{5}$, and P is in quadrant II, find the x coordinate.

188. The point P is on the unit circle. If the x-coordinate of P is $\frac{1}{5}$, and P is in quadrant IV, find the y coordinate.

Answers to Odd Problems:

183. $(-10, 10\sqrt{3})$ 185. $(-2.778, 15.757)$ 187. $-\frac{4}{5}$

I: Applications

Exercise 5.3e.I

★ **For the exercises below, use this scenario: A child enters a carousel that takes one minute to revolve once around. The child enters at the point $(0,1)$, that is, on the due north position. Assume the carousel revolves counter clockwise.**

190. What are the coordinates of the child after 45 seconds?

191. What are the coordinates of the child after 90 seconds?

192. What is the coordinates of the child after 125 seconds?

193. When will the child have coordinates $(0.707, -0.707)$ if the ride lasts 6 minutes? (There are multiple answers.)

194. When will the child have coordinates $(-0.866, -0.5)$ if the ride last 6 minutes?

196. Marla is running clockwise around a circular track. She runs at a constant speed of 3 meters per second. She takes 46 seconds to complete one lap of the track. From her starting point, it takes her 12 seconds to reach the northernmost point of the track. Impose a coordinate system with the center of the track at the origin, and the northernmost point on the positive y-axis.

a. Give Marla's coordinates at her starting point.
b. Give Marla's coordinates when she has been running for 10 seconds.
c. Give Marla's coordinates when she has been running for 901.3 seconds.

Answers to Odd Problems:

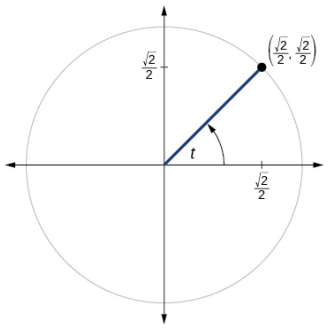
191. $(0, -1)$ 193. 37.5 seconds, 97.5 seconds, 157.5 seconds, 217.5 seconds, 277.5 seconds, 337.5 seconds

J: Using the Unit Circle

Exercise $5.3e.J$

★ **For the exercises below, use the given point on the unit circle to find the value of the sine and cosine of t.**

200)



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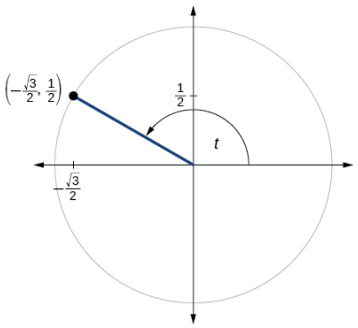

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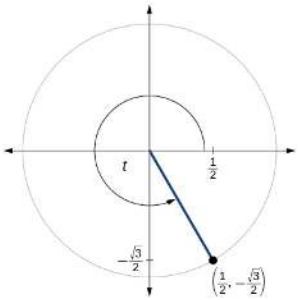

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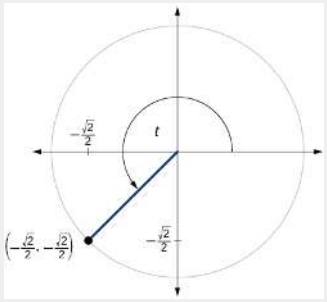

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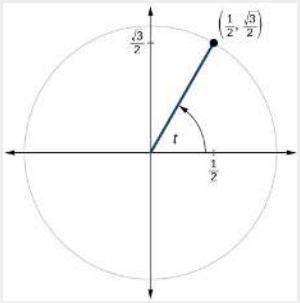

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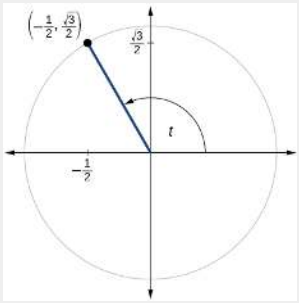

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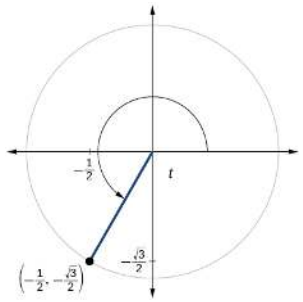

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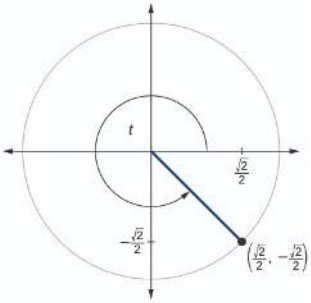

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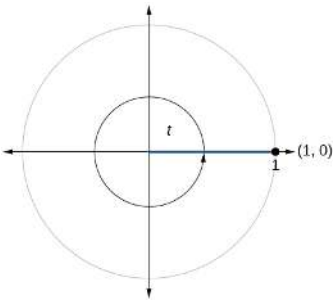

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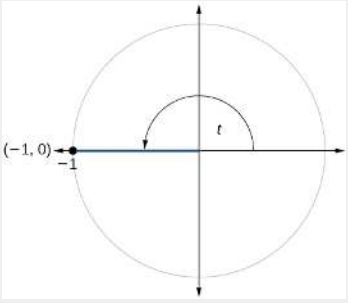

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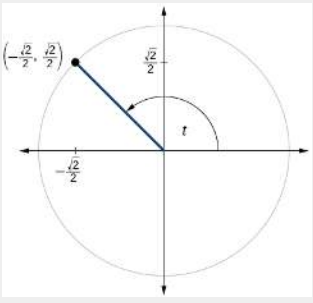

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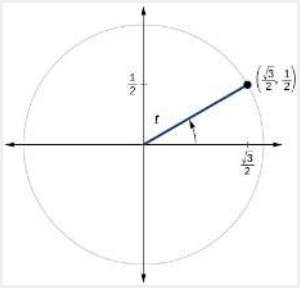

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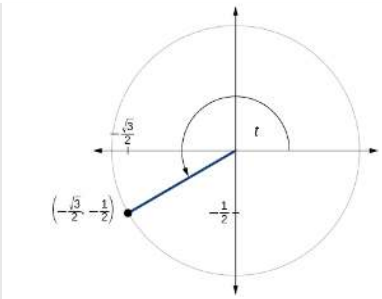

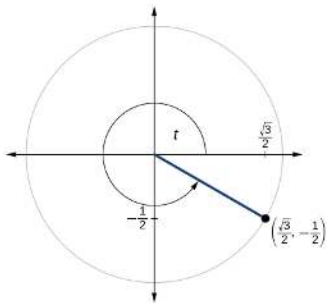

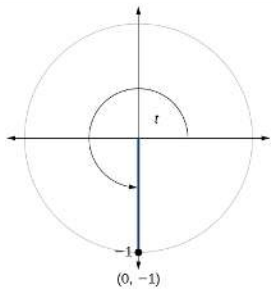

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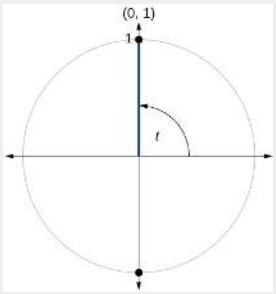

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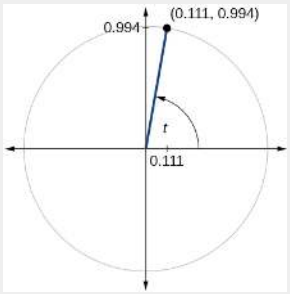

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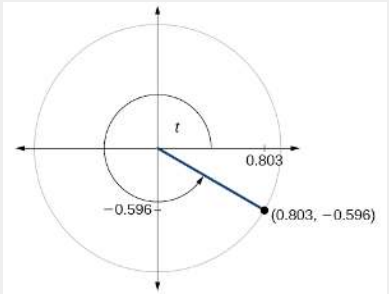

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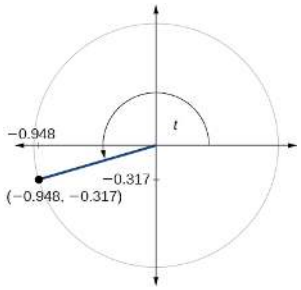

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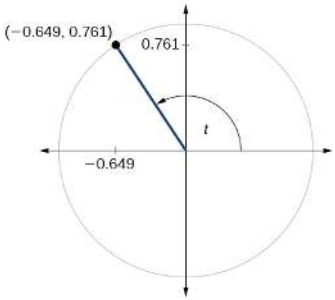

Answers to Odd Problems:

201. $\sin t = \frac{1}{2}, \cos t = -\frac{\sqrt{3}}{2}$
203. $\sin t = -\frac{\sqrt{2}}{2}, \cos t = -\frac{\sqrt{2}}{2}$
205. $\sin t = \frac{\sqrt{3}}{2}, \cos t = -\frac{1}{2}$
207. $\sin t = -\frac{\sqrt{2}}{2}, \cos t = \frac{\sqrt{2}}{2}$
209. $\sin t = 0, \cos t = -1$
211. $\sin t = \frac{1}{2}, \cos t = \frac{\sqrt{3}}{2}$
213. $\sin t = -\frac{1}{2}, \cos t = \frac{\sqrt{3}}{2}$
215. $\sin t = 1, \cos t = 0$
217. $\sin t = -0.596, \cos t = 0.803$
219. $\sin t = 0.761, \cos t = -0.649$

★

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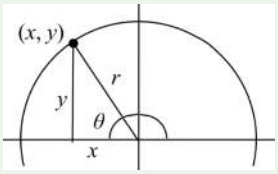
5.4: The Other Trigonometric Functions

In the previous section, we defined the sine and cosine functions as ratios of sides of a right triangle formed by the point of intersection (x, y) between a circle of radius r and the terminal side of an angle in standard position. Since a triangle has 3 sides, there are 6 possible combinations of ratios. While the sine and cosine are the two prominent ratios that can be formed, there are four others, and together they define the 6 trigonometric functions.

Definition: Sine, Cosine, Tangent, Secant, Cosecant, and Cotangent Functions

For the point (x, y) on a circle of radius r at an angle of θ, six trigonometric functions are defined as the ratios of the sides of the corresponding triangle. They are given below.



The **sine** function: $\sin(\theta) = \frac{y}{r}$ The **cosecant** function: $\csc(\theta) = \frac{r}{y}$

The **cosine** function: $\cos(\theta) = \frac{x}{r}$ The **secant** function: $\sec(\theta) = \frac{r}{x}$

The **tangent** function: $\tan(\theta) = \frac{y}{x}$ The **cotangent** function: $\cot(\theta) = \frac{x}{y}$

The signs that x and y have depend on the quadrant the terminal side of the angle is in. The sign of r is always positive and $r = \sqrt{x^2 + y^2}$. Geometrically, notice that the definition of tangent corresponds with the slope of the line segment between the origin (0, 0) and the point (x, y). This relationship can be very helpful in thinking about tangent values.

Use Definitions to Obtain Trigonometric Function Values

If the coordinates of a point on the terminal side of an angle are known, the values of the trigonometric function can be found using the definitions.

Given a point on the terminal side of an angle.

Example 5.4.1a: Find trigonometric ratios given a point on the terminal side of an angle

The point (-1, 7) is on the terminal side of some angle θ. Evaluate the six trigonometric functions for this angle.

Solution

We know a triangle is formed in the 2nd Quadrant, with sides $x = -1$ and $y = 7$.
The Pythagorean Theorem can be used to find the third side: $r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + 7^2} = \sqrt{50} = 5\sqrt{2}$.

Now the definitions of the six trigonometric functions can be used to obtain the desired function values.

$\sin\theta = \frac{y}{r} = \frac{7}{5\sqrt{2}} = \frac{7}{5\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{7\sqrt{2}}{10}$	$\cos\theta = \frac{x}{r} = \frac{-1}{5\sqrt{2}} = \frac{-1}{5\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = -\frac{\sqrt{2}}{10}$	$\tan\theta = \frac{y}{x} = \frac{7}{-1} = -7$
$\csc\theta = \frac{r}{y} = \frac{5\sqrt{2}}{7}$	$\sec\theta = \frac{r}{x} = \frac{5\sqrt{2}}{-1} = -5\sqrt{2}$	$\cot\theta = \frac{x}{y} = -\frac{1}{7}$

Example 5.4.1b

The point $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ is on the unit circle, as shown in Figure 5.4.1b. Find $\sin t, \cos t, \tan t, \sec t, \csc t$, and $\cot t$.

Solution

Because this is a point on the unit circle, $r = 1$, and furthermore, $x = \frac{\sqrt{2}}{2}$ and $y = -\frac{\sqrt{2}}{2}$. Using these values for x, y, and r and the definitions, the values of the six trigonometric

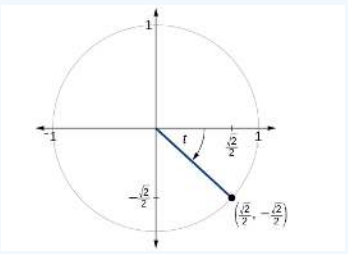


Figure 5.4.1b

ratios are obtained. (Because this is a point on the unit circle, we could have directly stated the values of $\sin t = y$ and $\cos t = x$, but the definitions were used below.)

$\sin t = \frac{y}{r} = \frac{-\frac{\sqrt{2}}{2}}{1} = -\frac{\sqrt{2}}{2}$	$\cos t = \frac{x}{r} = \frac{\frac{\sqrt{2}}{2}}{1} = \frac{\sqrt{2}}{2}$	$\tan t = \frac{y}{x} = \frac{-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = -1$
$\csc t = \frac{r}{y} = \frac{1}{-\frac{\sqrt{2}}{2}} = -\frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = -\sqrt{2}$	$\sec t = \frac{r}{y} = \frac{1}{\frac{\sqrt{2}}{2}} = \sqrt{2}$	$\cot t = \frac{x}{y} = \frac{\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = -1$

Try It 5.4.1

1. Suppose the terminal side of θ, when plotted in standard position, contains the point $Q(3, -4)$. Evaluate the six trigonometric functions for this angle.
2. The point $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ is on the unit circle, as shown in Figure 5.4.1c. Find $\sin t, \cos t, \tan t, \sec t, \csc t$, and $\cot t$.

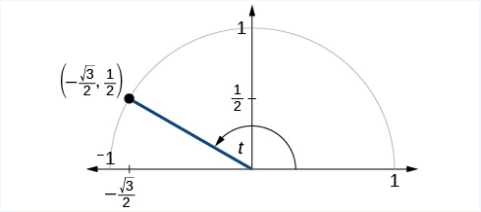


Figure 5.4.1c

Answers

1. Since $x = 3$ and $y = -4$,
   then $r = \sqrt{x^2 + y^2} = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5$.
   $$\sin(\theta) = -\frac{4}{5}, \quad \cos(\theta) = \frac{3}{5}, \quad \tan(\theta) = -\frac{4}{3}, \quad \csc(\theta) = -\frac{5}{4}, \quad \sec(\theta) = \frac{5}{3}, \quad \cot(\theta) = -\frac{3}{4}.$$
2. We have $x = -\frac{\sqrt{3}}{2}$ and $y = \frac{1}{2}$. Because this is a point on the unit circle, $r = 1$.
   $$\sin(\theta) = \frac{1}{2}, \quad \cos(\theta) = -\frac{\sqrt{3}}{2}, \quad \tan(\theta) = -\frac{\sqrt{3}}{3}, \quad \csc(\theta) = 2, \quad \sec(\theta) = -\frac{2\sqrt{3}}{3}, \quad \cot(\theta) = -\sqrt{3}.$$

Given one trigonometric ratio, find other trigonometric ratios

When given a trigonometric ratio, it is possible to narrow down the possible quadrants the angle is in to two, based on the definition of the trigonometric ratio and the signs that x and y take on in the different quadrants.

$\sin\theta = \frac{y}{r}$, $\csc\theta = \frac{r}{y}$ QII: + | QI: +	$\cos\theta = \frac{x}{r}$, $\sec\theta = \frac{r}{x}$ QI: + QIV: +	$\tan\theta = \frac{y}{x}$, $\cot\theta = \frac{x}{y}$ QI: + QIII: +

Figure 5.4.2. Quadrants in which trigonometric functions are positive

If given a trigonometric ratio, the definitions can be used 'in reverse' to determine the values of two of the three legs (x, y, and r) of a right triangle inscribed in the circle. Remember r is ALWAYS positive, so the sign of the ratio depends entirely on the signs of x and y.

How to: Find trig ratios given one trig ratio.

Reference Triangle or Pythagorean Theorem Approach. Given a trigonometric ratio and perhaps some other information.

1. Determine the quadrant the angle is in, and the signs of x and y for any point on the terminal side of the angle.
2. Use the definition of the given trigonometric function and the quadrant it is in to determine two of the three sides of a reference triangle and their appropriate signs.
3. Use the Pythagorean Theorem to find the third side, and its appropriate sign. Now all three sides of the reference triangle are known.
4. Use the definitions to determine the other trigonometric ratios.

Given a quadrant the angle is in

Example 5.4.2 Find Trigonometric Functions given one Trig Ratio and the quadrant the angle is in

Given $\sec(\theta) = \frac{2\sqrt{6}}{3}$ with θ in Quadrant IV. Find the value of $\sin\theta$ and $\cot\theta$.

Solution

Step 1. Angle θ is in QIV where $x > 0$ and $y < 0$.

Step 2. $\sec(\theta) = \frac{2\sqrt{6}}{3} = \frac{r}{x}$, therefore we can choose $r = 2\sqrt{6}$ and $x = 3$

Step 3. Find y with the Pythagorean Theorem: $x^2 + y^2 = r^2 \Rightarrow 3^2 + y^2 = (2\sqrt{6})^2 \Rightarrow y^2 = 4(6) - 9 = 15$. Since y is known to be negative, $y = -\sqrt{15}$.

Step 4. Using the definitions,

$$\sin\theta = \frac{y}{r} = \frac{-\sqrt{15}}{2\sqrt{6}} = \frac{-\sqrt{15}}{2\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{-\sqrt{90}}{2(6)} = \frac{-3\sqrt{10}}{12} = -\frac{\sqrt{10}}{4} \text{ and}$$

$$\cot\theta = \frac{x}{y} = \frac{3}{-\sqrt{15}} = \frac{3}{-\sqrt{15}} \cdot \frac{\sqrt{15}}{\sqrt{15}} = -\frac{3\sqrt{15}}{15} = -\frac{\sqrt{15}}{5}$$

Try It 5.4.2

Suppose θ is a Quadrant IV angle with $\cot(\theta) = -4$. Find the values of the five remaining circular functions of θ.

Answers

We need to find a point $Q(x, y)$ which lies on the terminal side of θ, when θ is plotted in standard position. Since θ is a Quadrant IV angle, we also know $x > 0$ and $y < 0$.

We have that $\cot(\theta) = -4 = \frac{x}{y}$. We may choose any values x and y so long as $x > 0$, $y < 0$ and $\frac{x}{y} = -4$. (For example, we could choose $x = 8$ and $y = -2$ or we could choose $x = 4$ and $y = -1$.

Using $x = 4$ and $y = -1$ we obtain $r = \sqrt{x^2 + y^2} = \sqrt{(4)^2 + (-1)^2} = \sqrt{17}$.

Thus $\sin(\theta) = -\frac{1}{\sqrt{17}} = -\frac{\sqrt{17}}{17}$, $\cos(\theta) = \frac{4}{\sqrt{17}} = \frac{4\sqrt{17}}{17}$, $\tan(\theta) = -\frac{1}{4}$, $\sec(\theta) = \frac{\sqrt{17}}{4}$, $\csc(\theta) = -\sqrt{17}$.

Given a range of angle values

If a range of values an angle can have is given, that information can be translated to the quadrant the angle is in, and therefore the signs that x and y can take on. This follows from the definition of an angle in standard position.

QII: $\frac{\pi}{2} < \theta < \pi$ $90° < \theta < 180°$	QI: $0 < \theta < \frac{\pi}{2}$ $0° < \theta < 90°$
QIII: $\pi < \theta < \frac{3\pi}{2}$ $180° < \theta < 270°$	QIV: $\frac{3\pi}{2} < \theta < 2\pi$ $270° < \theta < 360°$

Figure 5.4.3. Values of Angles in Standard Position in the Different Quadrants

Example 5.4.3: Use Definitions to Evaluate Trigonometric Functions

1. If $\sin(\phi) = -\frac{3}{7}$ and $\frac{3\pi}{2} < \phi < 2\pi$, find $\cos(\phi)$ and $\tan(\phi)$.
2. If $\tan(\theta) = 3$ and $\pi < \theta < \frac{3\pi}{2}$, find $\sin(\theta)$.

Solution

1. Angle ϕ is in QIV where $x > 0$ and $y < 0$.

$\sin(\phi) = -\frac{3}{7} = \frac{y}{r}$, therefore we can choose $r = 7$ and $y = -3$

Find x with the Theorem of Pythagoras: $x^2 + y^2 = r^2 \Rightarrow x^2 + (-3)^2 = 7^2 \Rightarrow x^2 = 49 - 9 = 40$. Thus $x = \pm\sqrt{40}$. Because x is known to be positive, $x = 2\sqrt{10}$.

Using the definitions, $\cos(\phi) = \frac{x}{r} = \frac{2\sqrt{10}}{7}$ and $\tan(\phi) = \frac{y}{x} = \frac{-3}{2\sqrt{10}} = \frac{3\sqrt{10}}{20}$.

2. Angle θ is in QIII where $x < 0$ and $y < 0$.

$\tan(\theta) = 3 = \frac{3}{1} = \frac{y}{x}$, and because θ is in QIII, $x = -1$ and $y = -3$

Find r with the formula $r = \sqrt{x^2 + y^2} \Rightarrow r = \sqrt{(-1)^2 + (-3)^2} \Rightarrow r = \sqrt{10}$.

Using the definitions, $\sin(\theta) = \frac{y}{r} = -\frac{3}{\sqrt{10}} = -\frac{3\sqrt{10}}{10}$.

Try It 5.4.3

If $\sec(t) = -\frac{17}{8}$ and $0 < t < \pi$, find the values of the other five trigonometric functions.

Answers

t is in QII where $x < 0$ and $y > 0$. $x = -8$, $y = 15$, and $r = 17$.

$\sin t = \frac{15}{17}$, $\cos t = -\frac{8}{17}$, $\tan t = -\frac{15}{8}$, $\csc t = \frac{17}{15}$, $\cot t = -\frac{8}{15}$.

Given the sign of another trigonometric ratio

Example 5.4.4: Find other trig ratios given one trig ratio and the sign of another ratio

1. If $\csc(\theta) = -\frac{10}{7}$ and $\tan(\theta) > 0$, determine the exact values of $\cos(\theta)$ and $\tan(\theta)$.
2. If $\cot(t) = -6$ and $\cos(t) < 0$, determine the exact values of $\sin(t)$ and $\sec(t)$.

1. Solution

Step 1. Angle θ is either in QIII or QIV because $\csc\theta < 0$. Angle θ is in either QI or QIII because $\tan\theta > 0$. Therefore θ must be in QIII where $x < 0$ and $y < 0$.

Step 2. $\csc(\theta) = -\frac{10}{7} = \frac{r}{y}$, therefore we can choose $r = 10$ and $y = -7$

Step 3. Find x with the Theorem of Pythagoras: $x^2 + y^2 = r^2 \Rightarrow x^2 + (-7)^2 = 10^2 \Rightarrow x^2 = 100 - 49 = 51$. Thus $x = \pm\sqrt{51}$. Because x is known to be negative, $x = -\sqrt{51}$.

Step 4. Using the definitions, $\cos(\theta) = \frac{x}{r} = -\frac{\sqrt{51}}{10}$ and $\tan(\theta) = \frac{y}{x} = \frac{-7}{-\sqrt{51}} = \frac{7\sqrt{51}}{51}$

2. Solution

Step 1. Angle t is either in QII or QIV because $\cot t < 0$. Angle t is in either QII or QIII because $\cos t > 0$. Therefore t must be in QII where $x < 0$ and $y > 0$.

Step 2. $\cot(t) = -6 = -\frac{6}{1} = \frac{x}{y}$, and because t is in QII, $x = -6$ and $y = 1$

Step 3. Find r with the formula $r = \sqrt{x^2 + y^2} \Rightarrow r = \sqrt{(-6)^2 + 1^2} \Rightarrow r = \sqrt{37}$.

Step 4. Using the definitions, $\sin(t) = \frac{y}{r} = \frac{1}{\sqrt{37}} = \frac{\sqrt{37}}{37}$ and $\sec(t) = \frac{r}{x} = -\frac{\sqrt{37}}{6}$

Try It 5.4.4

If $\sec(\theta) = 3$ and $\sin(\theta) < 0$, find the values of the other five trigonometric functions.

Answers

θ is in QIV where $x > 0$ and $y < 0$. $x = 1$, $y = -2\sqrt{2}$, and $r = 3$.

$$\sin(\theta) = -\frac{2\sqrt{2}}{3}, \qquad \cos(\theta) = -\frac{1}{3}, \qquad \tan(\theta) = -2\sqrt{2}, \qquad \csc(\theta) = -\frac{3\sqrt{2}}{4}, \qquad \cot(\theta) = -\frac{\sqrt{2}}{4}.$$

Given only one trigonometric ratio

Notice in the problems above we were given both a trigonometric ratio and the quadrant the angle was in. When the quadrant is not given, there are two possible solutions, as the example below illustrates.

Example 5.4.5

Given $\cos\theta = -\frac{4}{5}$, draw possible angles for θ in standard position and determine the value of $\sin\theta$ and $\tan\theta$.

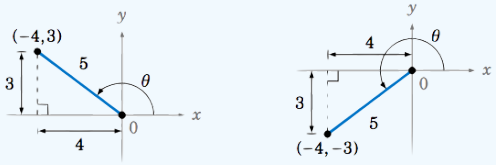
Solution

Given $\cos\theta = \frac{x}{r} = -\frac{4}{5}$ we can conclude a point on the terminal side of θ has coordinate $(-4, y)$ on a circle of radius $r = 5$.

Using the Pythagorean Theorem to determine y we find $x^2 + y^2 = r^2$ $\Rightarrow (-4)^2 + y^2 = 5^2 \Rightarrow y^2 = 9 \Rightarrow y = \pm 3$.

Thus there are actually two possible angles for θ.

If the point $(-4, 3)$ is on the terminal side of θ, then θ is in QII, and so we have $x = -4$, $y = 3$, and $r = 5$. Thus, $\boxed{\sin\theta = \frac{y}{r} = \frac{3}{5}}$ and $\boxed{\tan\theta = \frac{y}{x} = \frac{3}{-4}}$.



(a) θ in QII (b) θ in QIII

Figure 5.4.5: $\cos\theta = -\frac{4}{5}$

If the point $(-4, -3)$ is on the terminal side of θ, then θ is in QIII, and so we have $x = -4$, $y = -3$, and $r = 5$. Thus, $\boxed{\sin\theta = \frac{y}{r} = \frac{-3}{5}}$ and $\boxed{\tan\theta = \frac{y}{x} = \frac{-3}{-4} = \frac{3}{4}}$.

Trigonometric Function Values From Identities

Whenever a trigonometric ratio is given, and other function values are desired, there are two different approaches that can be used. We have been illustrating the use of the definitions and the Pythagorean Theorem. Another approach which will now be discussed uses identities.

Reciprocal and Quotient Identities

You may have noticed that the ratios defining the secant, cosecant, and cotangent are the reciprocals of the ratios defining the cosine, sine, and tangent functions, respectively. Additionally, notice that

$$\tan(\theta) = \frac{y}{x} = \frac{r\sin(\theta)}{r\cos(\theta)} = \frac{\sin(\theta)}{\cos(\theta)}$$

Applying this concept to the other trigonometric functions we can state the Reciprocal and Quotient Identities.

Identities

Reciprocal Identities	$\csc(\theta)=\frac{1}{\sin(\theta)}$	$\sec(\theta)=\frac{1}{\cos(\theta)}$	$\cot(\theta)=\frac{1}{\tan(\theta)}$
	$\sin(\theta)=\frac{1}{\csc(\theta)}$	$\cos(\theta)=\frac{1}{\sec(\theta)}$	$\tan(\theta)=\frac{1}{\cot(\theta)}$
Quotient Identities	$\tan(\theta)=\frac{\sin(\theta)}{\cos(\theta)}$	$\cot(\theta)=\frac{\cos(\theta)}{\sin(\theta)}$	

These relationships are called **identities**. Identities are statements that are true for all values of the input on which they are defined. Identities are usually something that can be derived from definitions and relationships we already know.

Trigonometric Functions of Special Angles

Example 5.4.6*a*: Using Identities to Evaluate Trigonometric Functions

a. Given $\sin(45°)=\frac{\sqrt{2}}{2}$ and $\cos(45°)=\frac{\sqrt{2}}{2}$, use identities to evaluate $\tan(45°)$.

b. Given $\sin\left(\frac{5\pi}{6}\right)=\frac{1}{2}$ and $\cos\left(\frac{5\pi}{6}\right)=-\frac{\sqrt{3}}{2}$, use identities to evaluate $\sec\left(\frac{5\pi}{6}\right)$.

Solution

Because we know the sine and cosine values for these angles, we can use identities to evaluate the other functions.

a. Use the quotient identity for tangent. $\tan(45°)=\frac{\sin(45°)}{\cos(45°)}=\frac{\sqrt{2}/2}{\sqrt{2}/2}=1$

Notice this result is consistent with our interpretation of the tangent value as the slope of the line passing through the origin at the given angle: a line at 45 degrees would indeed have a slope of 1.

b. Use the reciprocal identity for secant. $\sec\left(\frac{5\pi}{6}\right)=\frac{1}{\cos\left(\frac{5\pi}{6}\right)}=\frac{1}{-\frac{\sqrt{3}}{2}}=\frac{-2}{\sqrt{3}}=\frac{-2\sqrt{3}}{3}$

We know the values of sine and cosine for "special" angles 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$. Therefore, the values of the other four trigonometric functions for these "special" angles can be found by using the reciprocal and quotient identities.

Example 5.4.6*b*: Finding the Trigonometric Functions of an Angle

Find $\sin t, \cos t, \tan t, \sec t, \csc t,$ and $\cot t$ when $t=\frac{\pi}{6}$.

Solution

We have previously used the properties of equilateral triangles to demonstrate that $\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$ and $\cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$. We can use these values and the definitions of tangent, secant, cosecant, and cotangent as functions of sine and cosine to find the remaining function values.

$$\tan\left(\frac{\pi}{6}\right)=\frac{\sin\left(\frac{\pi}{6}\right)}{\cos\left(\frac{\pi}{6}\right)}=\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3} \qquad \cot\left(\frac{\pi}{6}\right)=\frac{\cos\left(\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{6}\right)}=\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}=\sqrt{3}$$

$$\sec\left(\frac{\pi}{6}\right)=\frac{1}{\cos\left(\frac{\pi}{6}\right)}=\frac{1}{\frac{\sqrt{3}}{2}}=\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3} \qquad \csc\left(\frac{\pi}{6}\right)=\frac{1}{\sin\left(\frac{\pi}{6}\right)}=\frac{1}{\frac{1}{2}}=2$$

A common mistake is that although $\tan(\theta)=\frac{\sqrt{3}}{3}=\frac{\sin(\theta)}{\cos(\theta)}=3$, it does NOT mean we can take $\sin(\theta)=\sqrt{3}$ and $\cos(\theta)=3$. Instead, from $\frac{\sin(\theta)}{\cos(\theta)}=\frac{\sqrt{3}}{3}$ we get: $3\sin(\theta)=\sqrt{3}\cos(\theta)$.

Try It 5.4.6

1. Find $\sec(60°)$
2. Find $\sin t,\ \cos t,\ \tan t,\ \sec t,\ \csc t,$ and $\cot t$ when $t = \frac{\pi}{3}$.

Answer

1. $\sec(60°) = \frac{1}{\cos(60°)}$. Hence, $\sec(60°) = \frac{1}{(1/2)} = 2$.
2. $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2},\ \cos\left(\frac{\pi}{3}\right) = \frac{1}{2},\ \tan\left(\frac{\pi}{3}\right) = \sqrt{3},\ \sec\left(\frac{\pi}{3}\right) = 2,\ \csc\left(\frac{\pi}{3}\right) = \frac{2\sqrt{3}}{3},\ \cot\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{3}$

Because we know the sine and cosine values for the common first-quadrant angles, we can find the other four function values for those angles by rewriting the definitions of tangent, secant, cosecant, and cotangent in terms of sine and cosine and evaluating the resulting expressions. The results are shown in Table **5.4.1**. It is worth taking the time to memorize the function values for at least the sine, cosine and tangent of the common 'special angles' listed in the table below.

Table **5.4.6** Values of Trigonometric functions of "Special Angles"

Angle	**0**	$\frac{\pi}{6}$ **or 30°**	$\frac{\pi}{4}$ **or 45°**	$\frac{\pi}{3}$ **or 60°**	$\frac{\pi}{2}$ **or 90°**
Sine	0	$d\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined
Cosecant	Undefined	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
Secant	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	Undefined
Cotangent	Undefined	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

Pythagorean Identities

The Pythagorean Identity we learned earlier was derived from the Pythagorean Theorem and the definitions of sine and cosine. We can build new identities from previously established identities. For example, if we divide both sides of the Pythagorean Identity by cosine squared, we can create a new identity:

$$\frac{\cos^2(\theta) + \sin^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$$

Splitting the fraction on the left,

$$\frac{\cos^2(\theta)}{\cos^2(\theta)} + \frac{\sin^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$$

Simplifying and using the definitions of tangent and secant

$$1 + \tan^2(\theta) = \sec^2(\theta).$$

Try It 5.4.7

Use a similar approach to establish that $\cot^2(\theta) + 1 = \csc^2(\theta)$.

Answer

$$\frac{\cos^2(\theta)+\sin^2(\theta)}{\sin^2\theta}=1 \rightarrow \frac{\cos^2(\theta)}{\sin^2(\theta)}+\frac{\sin^2(\theta)}{\sin^2(\theta)}=\frac{1}{\sin^2(\theta)} \rightarrow \cot^2(\theta)+1=\csc^2(\theta)$$

Identities: Alternate forms of the Pythagorean Identity

$$\sin^2(\theta)+\cos^2(\theta)=1 \qquad 1+\tan^2(\theta)=\sec^2(\theta) \qquad \cot^2(\theta)+1=\csc^2(\theta)$$

How to: Given one trigonometric ratio, find other ratios.

Identities Approach. Given a trigonometric ratio and other information.

1. Determine the quadrant the angle must be in, and thus the signs of the various trig ratios in that quadrant.
2. Use identities to find the other trig ratios and their appropriate signs. Typically, use an identity to find the value of sine or cosine or their reciprocal functions, and then use other identities to find the rest of the trigonometric ratios.

Given a Trigonometric Ratio and a Quadrant

Example 5.4.8a

If $\tan(\theta)=\frac{2}{7}$ and θ is in the 3rd quadrant, find $\cos(\theta)$.

Solution

There are two approaches to this problem, both of which work equally well.

Approach 1 (Use the Pythagorean Theorem)

Angle θ is in QIII where $x<0$ and $y<0$.

$\tan(\theta)=\frac{2}{7}=\frac{-2}{-7}=\frac{y}{x}$, and because θ is in QIII, $x=-2$ and $y=-7$

Find r with the formula $r=\sqrt{x^2+y^2} \Rightarrow r=\sqrt{(-2)^2+(-7)^2} \Rightarrow r=\sqrt{53}$.

Using the definitions, $\cos(\theta)=\frac{x}{r}=\frac{-7}{\sqrt{53}}=-\frac{7\sqrt{53}}{53}$.

Approach 2 (Use Identities)

Use the $1+\tan^2(\theta)=\sec^2(\theta)$ form of the Pythagorean Identity because we are given a tangent value,

$$\begin{aligned} 1+\tan^2(\theta)&=\sec^2(\theta) && \text{Use a Pythagorean Identity}\\ 1+\left(\frac{2}{7}\right)^2&=\sec^2(\theta) && \text{Substitute the known tangent value}\\ \frac{53}{49}&=\sec^2(\theta)\\ \sec(\theta)&=\pm\sqrt{\frac{53}{49}}=\pm\frac{\sqrt{53}}{7} \end{aligned}$$

Since the angle is in the third quadrant, the cosine value will be negative so the secant value will also be negative. Keeping the negative result, and using definition of secant,

$$\begin{aligned} \sec(\theta)&=-\frac{\sqrt{53}}{7}\\ \cos(\theta)&=-\frac{1}{\sec(\theta)} && \text{Use the Reciprocal Identity for Secant}\\ \cos(\theta)&=-\frac{1}{\frac{\sqrt{53}}{7}}=-\frac{7}{\sqrt{53}}=-\frac{7\sqrt{53}}{53} \end{aligned}$$

Example 5.4.8b: Using Identities to Relate Trigonometric Functions

If $\cos(t) = \frac{12}{13}$ and t is in quadrant IV, as shown in Figure 5.4.8b, find the values of the other five trigonometric functions.

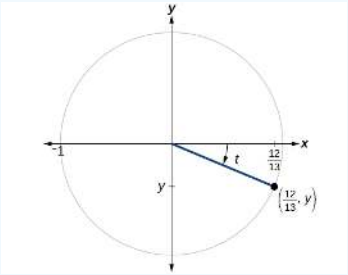


Figure 5.4.8b

Solution

We are given the value of cosine so we can find the sine using the Pythagorean Identity, $\cos^2 t + \sin^2 t = 1$. Since the angle t is in the fourth quadrant, the sine ratio will be negative.

$$\left(\frac{12}{13}\right)^2 + \sin^2 t = 1$$
$$\sin^2 t = 1 - \left(\frac{12}{13}\right)^2 = 1 - \frac{144}{169}$$
$$\sin^2 t = \frac{25}{169}$$
$$\sin t = \pm\sqrt{\frac{25}{169}} = \pm\frac{\sqrt{25}}{\sqrt{169}}$$
$$\sin t = \pm\frac{5}{13} = -\frac{5}{13}$$

The remaining functions can be calculated using identities relating them to sine and cosine.

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{5}{13}}{\frac{12}{13}} = -\frac{5}{12} \quad \sec t = \frac{1}{\cos t} = \frac{1}{\frac{12}{13}} = \frac{13}{12}$$
$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{5}{13}} = -\frac{13}{5} \quad \cot t = \frac{1}{\tan t} = \frac{1}{-\frac{5}{12}} = -\frac{12}{5}$$

Try It 5.4.8

Evaluate $\cos(\theta)$, where $\csc(\theta) = -\sqrt{5}$ and θ is a Quadrant IV angle.

Answer

Given $\csc(\theta) = -\sqrt{5}$, use the Reciprocal Identity $\sin(\theta) = -\frac{1}{\csc(\theta)}$ to obtain $\sin(\theta) = -\frac{1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$. Use the Pythagorean Identity, $\cos^2(\theta) + \sin^2(\theta) = 1$, to find $\cos(\theta)$. Substituting, we get $\cos^2(\theta) + \left(-\frac{\sqrt{5}}{5}\right)^2 = 1$, which gives $\cos^2(\theta) + \frac{5}{25}$. Therefore, $\cos^2(\theta) = \frac{4}{5}$, or $\cos(\theta) = \pm\frac{2\sqrt{5}}{5}$. Since θ is a Quadrant IV angle, $\cos(\theta) > 0$, so $\cos(\theta) = \frac{2\sqrt{5}}{5}$

Trigonometric Function Values from Reference Angles

We can evaluate trigonometric functions of angles outside the first quadrant using reference angles as we have already done with the sine and cosine functions. The procedure is the same: Find the **reference angle** formed by the terminal side of the given angle with the horizontal axis. The trigonometric function values for the original angle will be the same as those for the reference angle, except for the positive or negative sign, which is determined by x- and y-values in the quadrant where the terminal side is located.

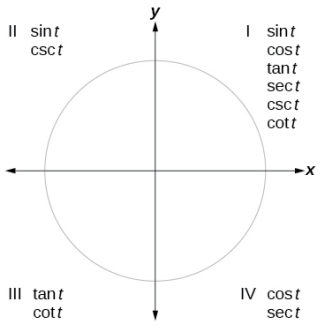


Figure 5.4.9 Quadrants where trigonometric functions are positive

Figure 5.4.9 shows which functions are positive in which quadrant.

To help us remember which of the six trigonometric functions are positive in each quadrant, we can use the mnemonic phrase "All Students Take Classes." Each of the four words in the phrase corresponds to one of the four quadrants, starting with quadrant I and rotating counterclockwise. In quadrant I, which is "**A**," **a**ll of the six trigonometric functions are positive. In quadrant II, "**S**tudents," only **s**ine and its reciprocal function, cosecant, are positive. In quadrant III, "**T**ake," only **t**angent and its reciprocal function, cotangent, are positive. Finally, in quadrant IV, "**C**lasses," only **c**osine and its reciprocal function, secant, are positive.

Example 5.4.9: Coterminal Angles and Reference Angles

Let $\theta = 928°$.

a. Which angle between $0°$ and $360°$ has the the same trigonometric function values as θ?
b. What is the reference angle for θ?
c. Draw these angles in standard position.

Solution

(a) Since $928° = 2 \times 360° + 208°$, θ has the same terminal side as $208°$, and thus the angles $928°$ and $208°$ are **coterminal**. This is illustrated in Figure 5.4.9*a*. Therefore, $208°$ has the same trigonometric function values as $\theta = 928°$.

(b) $928°$ and $208°$ have the same terminal side in QIII, so the reference angle for $\theta = 928°$ is $208° - 180° = 28°$.

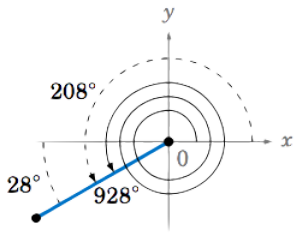


Figure 5.4.9*a*

Generalized Reference Angle Theorem

The absolute value of trigonometric functions of an angle θ, are the same as the corresponding trigonometric functions of its reference angle α. More specifically, if α is the reference angle for an θ, then:

$\sin(\theta) = \pm\sin(\alpha)$	$\cos(\theta) = \pm\cos(\alpha)$	$\tan(\theta) = \pm\tan(\alpha)$
$\csc(\theta) = \pm\csc(\alpha)$	$\sec(\theta) = \pm\sec(\alpha)$	$\cot(\theta) = \pm\cot(\alpha)$

The choice of the $(\pm)$ depends on the quadrant in which the terminal side of θ lies.

How to: Find Trigonometric Functions using Reference Angles.

1. Measure the angle formed by the terminal side of the given angle and the horizontal axis. This is the reference angle. The measure of a reference angle is ALWAYS a positive number between 0 and $\frac{\pi}{2}$
2. Observe the quadrant where the terminal side of the original angle is located. Based on the quadrant, determine whether the output is positive or negative. Rewrite the function in terms of the reference angle and the appropriate sign.
3. Evaluate the function at the reference angle.

Example 5.4.10: Using Reference Angles to Find Trigonometric Functions

Use reference angles to find all six trigonometric functions of $-\frac{5\pi}{6}$.

Solution

The angle between this angle's terminal side and the x-axis is $\frac{\pi}{6}$, so that is the reference angle. Since $-\frac{5\pi}{6}$ is in the third quadrant, where both x and y are negative, then sine, cosine, secant, and cosecant will be negative, while tangent and cotangent will be positive.

$\sin\left(-\frac{5\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$	$\cos\left(-\frac{5\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$	$\tan\left(-\frac{5\pi}{6}\right) = +\tan\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$
$\csc\left(-\frac{5\pi}{6}\right) = -\csc\left(\frac{\pi}{6}\right) = -2$	$\sec\left(-\frac{5\pi}{6}\right) = -\sec\left(\frac{\pi}{6}\right) = -\frac{2\sqrt{3}}{3}$	$\cot\left(-\frac{5\pi}{6}\right) = +\cot\left(\frac{\pi}{6}\right) = \sqrt{3}$

Try It 5.4.10

Use reference angles to find all six trigonometric functions of $-\frac{7\pi}{4}$.

Answer

Angle $-\frac{7\pi}{4}$ is in QI where all trigonometric functions are positive.

$\sin\left(-\frac{7\pi}{4}\right)=\frac{\sqrt{2}}{2},\ \cos\left(-\frac{7\pi}{4}\right)=\frac{\sqrt{2}}{2},\ \tan\left(-\frac{7\pi}{4}\right)=1,\ \csc\left(-\frac{7\pi}{4}\right)=\sqrt{2},\ \sec\left(-\frac{7\pi}{4}\right)=\sqrt{2},\ \cot\left(-\frac{7\pi}{4}\right)=1$

Example 5.4.11

Use Identities to determine the exact value of each of the following function values.

1. $\sec\left(\frac{7\pi}{4}\right)$ 2. $\csc\left(\frac{7\pi}{6}\right)$ 3. $\tan(\theta)$, where θ is any angle coterminal with $\frac{3\pi}{2}$.

Solution

1. $\sec\left(\frac{7\pi}{4}\right) = \frac{1}{\cos\left(\frac{7\pi}{4}\right)} = \frac{1}{+\cos\left(\frac{\pi}{4}\right)} = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$ in QIV so x and cos positive

2. $\csc\left(\frac{7\pi}{6}\right) = \frac{1}{\sin\left(\frac{7\pi}{6}\right)} = \frac{1}{-\sin\left(\frac{\pi}{6}\right)} = \frac{1}{-1/2} = -2$ in QIII so y and sine negative

3. $\tan(\theta) = \tan\left(\frac{3\pi}{2}\right) = \frac{\sin\left(\frac{3\pi}{2}\right)}{\cos\left(\frac{3\pi}{2}\right)} = \frac{-1}{0} =$ Undefined

Or $= \tan\left(\frac{3\pi}{2}\right) = \pm\tan\left(\frac{\pi}{2}\right) =$ Undefined $\pm$ undefined is undefined

Try It 5.4.11

Use Identities to determine the exact value of each of the following function values.

1. $\csc\left(\frac{-\pi}{4}\right)$ 2. $\cot\left(\frac{4\pi}{3}\right)$

Answer

Misplaced '&'

Trigonometric Function Values from a Calculator

Values for the six trigonometric functions of "special" first-quadrant angles, and multiples of these angles that lie in other quadrants, can be evaluated using the value of the trig function for its reference angle and modifying the sign when appropriate.

For other angles, a calculator can be used to evaluate trigonometric functions. If the calculator has degree mode and radian mode, confirm the correct mode is chosen before making a calculation. There are calculator keys for SIN, COS, and TAN, but typically not for CSC, SEC, or COT. To evaluate these last three functions, the function must be evaluated as the reciprocal of a sine, cosine, or tangent function.

Example 5.4.12: Evaluating the Cosecant Using Technology

1. Use a calculator to find the cosecant of $\frac{5\pi}{7}$. Use a calculator to find the secant of $30°$.

1. Solution

- The angle is in radians so be sure the calculator is in radian mode.
- There is not a cosecant button on the calculator, so the reciprocal identity $\csc(\theta)=\frac{1}{\sin(\theta)}$ must be used.

- Enter the following keystrokes: $1 \div (\text{SIN}(5 \times \pi \div 7))$ **ENTER**.
- The result obtained is $1.2790480\ldots$

2. Solution

- The angle is in degrees so be sure the calculator is in degree mode.
- Enter the following keystrokes: $1 \div \text{COS}(30)$ **ENTER**.
- The result is $1.15470053\ldots$
- If the calculator does not have a degree mode, the angle must also be converted to radians. When the calculator only has a radian mode, enter the following keystrokes: $1 \div \text{COS}(30 \times \pi \div 180)$ **ENTER**.

Try It 5.4.12

Use a calculator to evaluate the following. Round to four decimal digits.

1. $\sec\left(\frac{7\pi}{8}\right)$ 2. $\csc\left(-\frac{\pi}{8}\right)$ 3. $\cot(3)$

Answer

$$\sec\left(\frac{7\pi}{8}\right) = \frac{1}{\cos\left(\frac{7\pi}{8}\right)} \approx -1.0824 \qquad \csc\left(-\frac{\pi}{8}\right) = \frac{1}{\sin\left(-\frac{\pi}{8}\right)} \approx -2.6131 \qquad \cot(3) = \frac{1}{\tan(3)} \approx -7.015$$

Even and Odd Trigonometric Functions

To be able to use our six trigonometric functions freely with both positive and negative angle inputs, we should examine how each function treats a negative input. As it turns out, there is an important difference among the functions in this regard. Recall in Section 2.2: Attributes of Functions, even and odd functions were defined. This definition is repeated here.

Definition: Odd and Even functions

- An **even function** is one in which $f(-x) = f(x)$. An even function has y-axis symmetry.
- An **odd function** is one in which $f(-x) = -f(x)$. An odd function has origin axis symmetry.

We can test whether a trigonometric function is even or odd by drawing a **unit circle** with a positive and a negative angle, as illustrated in Figure 5.4.13.

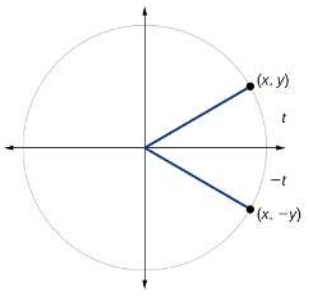


Figure 5.4.13

- The sine of the positive angle is $\sin(t) = \frac{y}{r}$. The sine of the negative angle is $\sin(-t) = \frac{-y}{r} = -\frac{y}{r}$. Thus, $\sin(-t) = -\sin(t)$ and the **sine function** is an odd function.
- The cosine of the positive angle is $\cos(t) = \frac{x}{r}$. The cosine of the negative angle is $\cos(-t) = \frac{x}{r}$. Thus, $\cos(-t) = \cos(t)$ and the **cosine function** is an even function.
- The tangent of the positive angle is $\tan(t) = \frac{y}{x}$. The tangent of the negative angle is $\tan(-t) = \frac{-y}{x} = -\frac{y}{x}$. Thus, $\tan(-t) = -\tan(t)$ and the **tangent function** is an odd function.

Similar arguments can be made for the secant, cosecant and cotangent functions.

Even and Odd Trigonometric Functions; Even/Odd Identities

Cosine and secant are even functions:

$$\cos(-t) = \cos t$$
$$\sec(-t) = \sec t$$

Sine, tangent, cosecant, and cotangent are odd functions:

$$\sin(-t) = -\sin t \qquad \tan(-t) = -\tan t$$
$$\csc(-t) = -\csc t \qquad \cot(-t) = -\cot t$$

Example 5.4.13: Using Even and Odd Properties of Trigonometric Functions

If $\sec t = 2$, what is $\sec(-t)$?

Solution

Secant is an even function. The secant of an angle is the same as the secant of its opposite. So if the secant of angle t is 2, the secant of $-t$ is also 2.

Try It 5.4.13

If the cotangent of angle t is $\sqrt{3}$, what is the cotangent of $-t$?

Answer

$-\sqrt{3}$

Cofunction Identities

If we look more closely at the relationship between the sine and cosine of the special angles relative to the unit circle, we will notice a pattern. In a right triangle with angles of $\frac{\pi}{6}$ and $\frac{\pi}{3}$, we see that the sine of $\frac{\pi}{3}$, namely $\frac{\sqrt{3}}{2}$, is also the cosine of $\frac{\pi}{6}$, while the sine of $\frac{\pi}{6}$, namely $\frac{1}{2}$, is also the cosine of $\frac{\pi}{3}$ (Figure 5.4.14*a*).

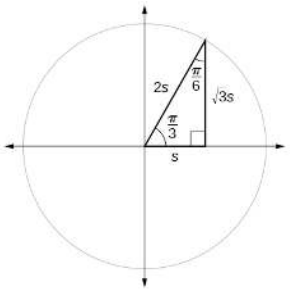


Figure 5.4.14*a*: The sine of $\frac{\pi}{3}$ equals the cosine of $\frac{\pi}{6}$ and vice versa.

$$\sin\frac{\pi}{3} = \cos\frac{\pi}{6} = \frac{\sqrt{3}s}{2s} = \frac{\sqrt{3}}{2}$$
$$\sin\frac{\pi}{6} = \cos\frac{\pi}{3} = \frac{s}{2s} = \frac{1}{2}$$

This result should not be surprising because, as we see from Figure 5.4.14*a*, the side opposite the angle of $\frac{\pi}{3}$ is also the side adjacent to $\frac{\pi}{6}$, so $\sin(\frac{\pi}{3})$ and $\cos(\frac{\pi}{6})$ are exactly the same ratio of the same two sides, $\sqrt{3}s$ and $2s$. Similarly, $\cos(\frac{\pi}{3})$ and $\sin(\frac{\pi}{6})$ are also the same ratio using the same two sides, s and $2s$.

The interrelationship between the sines and cosines of $\frac{\pi}{6}$ and $\frac{\pi}{3}$ also holds for the two acute angles in any right triangle, since in every case, the ratio of the same two sides would constitute the sine of one angle and the cosine of the other. Since the three angles of a triangle add to π, and the right angle is $\frac{\pi}{2}$, the remaining two angles must also add up to $\frac{\pi}{2}$. That means that a right triangle can be formed with any two angles that add to $\frac{\pi}{2}$—in other words, any two complementary angles. So we may state a *cofunction identity*: If any two angles are complementary, the sine of one is the cosine of the other, and vice versa. This identity is illustrated in Figure 5.4.14*b*.

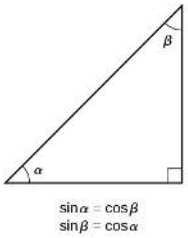


Figure 5.4.14*b*: Cofunction identity

Using this identity, we can state without calculating, for instance, that the sine of $\frac{\pi}{12}$ equals the cosine of $\frac{5\pi}{12}$, and that the sine of $\frac{5\pi}{12}$ equals the cosine of $\frac{\pi}{12}$. We can also state that if, for a certain angle t, $\cos t = \frac{5}{13}$, then $\sin(\frac{\pi}{2} - t) = \frac{5}{13}$ as well.

Cofunction Identities

The **cofunction identities** in terms of radians are listed below.

$\sin(t) = \cos\left(\frac{\pi}{2} - t\right)$	$\cos(t) = \sin\left(\frac{\pi}{2} - t\right)$	$\tan(t) = \cot\left(\frac{\pi}{2} - t\right)$
$\csc(t) = \sec\left(\frac{\pi}{2} - t\right)$	$\sec(t) = \csc\left(\frac{\pi}{2} - t\right)$	$\sec(t) = \csc\left(\frac{\pi}{2} - t\right)$

Example 5.4.14: Using Cofunction Identities

If $\sin(t) = \frac{5}{12}$, find $\cos\left(\frac{\pi}{2} - t\right)$.

Solution

According to the cofunction identities for sine and cosine, $\sin(t) = \cos\left(\frac{\pi}{2} - t\right)$. So $\cos\left(\frac{\pi}{2} - t\right) = \frac{5}{12}$.

Try It 5.4.14

If $\csc\left(\frac{\pi}{6}\right) = 2$, find $\sec\left(\frac{\pi}{3}\right)$.

Answer

2

Key Equations

Tangent function	$\tan t = \frac{\sin t}{\cos t}$
Secant function	$\sec t = \frac{1}{\cos t}$
Cosecant function	$\csc t = \frac{1}{\sin t}$
Cotangent function	$\cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t}$

Reciprocal Identities	$\csc(\theta) = \frac{1}{\sin(\theta)}$ $\sin(\theta) = \frac{1}{\csc(\theta)}$	$\sec(\theta) = \frac{1}{\cos(\theta)}$ $\cos(\theta) = \frac{1}{\sec(\theta)}$	$\cot(\theta) = \frac{1}{\tan(\theta)}$ $\tan(\theta) = \frac{1}{\cot(\theta)}$
Quotient Identities	$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$	$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$	
Pythagorean Identities	$\sin^2(\theta) + \cos^2(\theta) = 1$	$1 + \tan^2(\theta) = \sec^2(\theta)$	$\cot^2(\theta) + 1 = \csc^2(\theta)$
Even/Odd Identities	$\sin(-t) = -\sin t$ $\csc(-t) = -\csc t$	$\cos(-t) = \cos t$ $\sec(-t) = \sec t$	$\tan(-t) = -\tan t$ $\cot(-t) = -\cot t$
Cofunction Identities	$\sin t = \cos(\frac{\pi}{2} - t)$ $\cos t = \sin(\frac{\pi}{2} - t)$	$\sec t = \csc(\frac{\pi}{2} - t)$ $\csc t = \sec(\frac{\pi}{2} - t)$	$\tan t = \cot(\frac{\pi}{2} - t)$ $\cot t = \tan(\frac{\pi}{2} - t)$

Key Concepts

- The tangent of an angle is the ratio of the *y*-value to the *x*-value of the corresponding point on the unit circle.
- The secant, cotangent, and cosecant are all reciprocals of other functions. The secant is the reciprocal of the cosine function, the cotangent is the reciprocal of the tangent function, and the cosecant is the reciprocal of the sine function.
- The six trigonometric functions can be found from a point on the unit circle.
- Trigonometric functions can also be found from an angle.
- Trigonometric functions of angles outside the first quadrant can be determined using reference angles.
- The Pythagorean Identity makes it possible to find a cosine from a sine or a sine from a cosine.

- Identities can be used to evaluate trigonometric functions.
- Fundamental identities such as the Pythagorean Identity can be manipulated algebraically to produce new identities.
- The values of trigonometric functions of special angles can be found by mathematical analysis.
- To evaluate trigonometric functions of other angles, we can use a calculator or computer software.
- A function is said to be even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$.
- Cosine and secant are even; sine, tangent, cosecant, and cotangent are odd.
- Even and odd properties can be used to evaluate trigonometric functions.

Glossary

cosecant

the reciprocal of the sine function: on the unit circle, $\csc t = \frac{1}{y}, y \neq 0$, or more generally $\csc t = \frac{r}{y}, y \neq 0$

cotangent

the reciprocal of the tangent function: on the unit circle, $\cot t = \frac{x}{y}, y \neq 0$

identities

statements that are true for all values of the input on which they are defined

secant

the reciprocal of the cosine function: on the unit circle, $\sec t = \frac{1}{x}, x \neq 0$, or more generally $\sec t = \frac{r}{x}, x \neq 0$

tangent

the quotient of the sine and cosine: on the unit circle, $\tan t = \frac{y}{x}, x \neq 0$

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5.4e: Exercises - Other Trigonometric Functions

A: Concepts

Exercise 5.4e.A

1) On an interval of $[0, 2\pi)$, can the sine and cosine values of a radian measure ever be equal? If so, where?

2) What would you estimate the cosine of π degrees to be? Explain your reasoning.

3) For any angle in quadrant II, if you knew the sine of the angle, how could you determine the cosine of the angle?

4) Describe the secant function.

Answers to Odd Problems

1. Yes, when the reference angle is $\frac{\pi}{4}$ and the terminal side of the angle is in quadrants I and III. Thus, at $x = \frac{\pi}{4}, \frac{5\pi}{4}$, the sine and cosine values are equal.
3. Substitute the sine of the angle in for y in the Pythagorean Theorem $x^2 + y^2 = 1$. Solve for x and take the negative solution.

B: Given a point

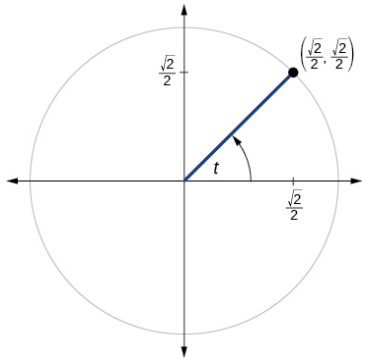
Exercise 5.4e.B

★ Given the following point on the terminal side of an angle θ, find the value of each of the six trigonometric functions.

5. $(\sqrt{5}, 2)$	6. $(-4, \sqrt{2})$	7. $(-1, -\sqrt{7})$	8. $(2\sqrt{3}, -5)$

★ For the below, use the angle in the unit circle to find the value of each of the six trigonometric functions.

9.



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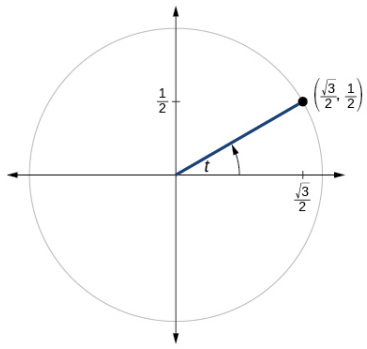

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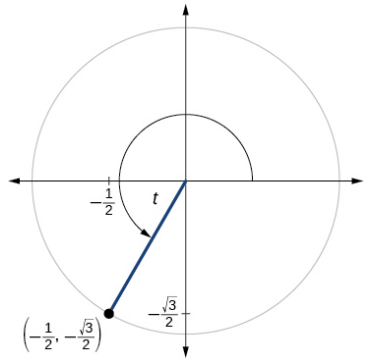

Answers to Odd Problems

5. $\sin\theta = \frac{2}{3}$, $\cos\theta = \frac{\sqrt{5}}{3}$, $\tan\theta = \frac{2\sqrt{5}}{5}$, $\csc\theta = \frac{3}{2}$, $\sec\theta = \frac{3\sqrt{5}}{5}$, $\cot\theta = \frac{\sqrt{5}}{2}$
7. $\sin\theta = -\frac{\sqrt{14}}{4}$, $\cos\theta = -\frac{\sqrt{2}}{4}$, $\tan\theta = \sqrt{7}$, $\csc\theta = -\frac{2\sqrt{14}}{7}$, $\sec\theta = -2\sqrt{2}$, $\cot\theta = \frac{\sqrt{7}}{7}$
9. $\sin t = \frac{\sqrt{2}}{2}$, $\cos t = \frac{\sqrt{2}}{2}$, $\tan t = 1$, $\cot t = 1$, $\sec t = \sqrt{2}$, $\csc t = \sqrt{2}$
11. $\sin t = -\frac{\sqrt{3}}{2}$, $\cos t = -\frac{1}{2}$, $\tan t = \sqrt{3}$, $\cot t = \frac{\sqrt{3}}{3}$, $\sec t = -2$, $\csc t = -\frac{2\sqrt{3}}{3}$

C: Get information about an angle from the sign of its trigonometric ratio

Exercise 5.4e.C

★ **State the quadrant or quadrants for which the following conditions are true.**

15. Sine and cosine have the same sign.	**18.** Sine and tangent have the opposite sign.
16. Sine and cosine have the opposite sign.	**19.** Cosine and tangent have the same sign.
17. Sine and tangent have the same sign.	**20.** Cosine and tangent have the opposite sign.

Answers to Odd Problems

15. QI or QIII 17. QI or QIV 19. QI or QII

D: Given one ratio and information to find which quadrant the angle is in

Exercise 5.4e.D

Given one trigonometric ratio and information about the angle, find the other five trigonometric ratios.

21. $\sin(\theta) = \frac{3}{4}$, θ in quadrant II	25. $\tan(\theta) = \frac{12}{5}$, $\pi \le \theta < \frac{3\pi}{2}$	29. $\cot(\theta) = -\frac{4}{3}$, $\csc\theta < 0$
22. $\csc(\theta) = -\frac{7}{2}$, θ in quadrant IV	26. $\cot(\theta) = -4$, $\frac{\pi}{2} \le \theta < \pi$	30. $\tan(\theta) = -7$, $\sec\theta > 0$
23. $\cos(\theta) = -\frac{1}{3}$, θ in quadrant III	27. $\sec(\theta) = -7$, $\frac{\pi}{2} < \theta < \pi$	31. $\csc(\theta) = 4$, $\cot\theta < 0$
24. $\sec(\theta) = -5$, θ in quadrant II	28. $\csc(\theta) = -\frac{7}{3}$, $\pi \le \theta < \frac{3\pi}{2}$	32. $\sec(\theta) = \frac{5}{2}$, $\sin\theta < 0$

Answers to Odd Problems

21. $\cos(\theta) = -\frac{\sqrt{7}}{4}$, $\tan(\theta) = -\frac{3\sqrt{7}}{7}$, $\csc(\theta) = \frac{4}{3}$, $\sec(\theta) = -\frac{4\sqrt{7}}{7}$, $\cot(\theta) = -\frac{\sqrt{7}}{3}$

23. $\sin(\theta) = -\frac{2\sqrt{2}}{3}$, $\tan(\theta) = 2\sqrt{2}$, $\csc(\theta) = -\frac{3\sqrt{2}}{4}$, $\sec(\theta) = -3$, $\cot(\theta) = \frac{\sqrt{2}}{4}$

25. $\sin(\theta) = -\frac{12}{13}$, $\cos(\theta) = -\frac{5}{13}$, $\csc(\theta) = -\frac{13}{12}$, $\sec(\theta) = -\frac{13}{5}$, $\cot(\theta) = \frac{5}{12}$

27. $\sin(\theta) = \frac{4\sqrt{3}}{7}$, $\cos(\theta) = -\frac{1}{7}$, $\tan(\theta) = -4\sqrt{3}$, $\csc(\theta) = \frac{7\sqrt{3}}{12}$, $\sec(\theta) = -7$, $\cot(\theta) = -\frac{\sqrt{3}}{12}$

29. $\sin(\theta) = -\frac{3}{5}$, $\cos(\theta) = \frac{4}{5}$, $\tan(\theta) = -\frac{3}{4}$, $\csc(\theta) = -\frac{5}{3}$, $\sec(\theta) = \frac{5}{4}$, $\cot(\theta) = -\frac{4}{3}$

31. $\sin(\theta) = \frac{1}{4}$, $\cos(\theta) = -\frac{\sqrt{15}}{4}$, $\tan(\theta) = -\frac{\sqrt{15}}{15}$, $\csc(\theta) = 4$, $\sec(\theta) = -\frac{4\sqrt{15}}{15}$, $\cot(\theta) = -\sqrt{15}$

E: Find quadrants or axes where a ratio is the same; find other ratios for each location

Exercise 5.4e.E

★ **Find the quadrants or axes and exact values of $\sin\theta$ and $\tan\theta$ when $\cos\theta$ has the indicated value**

41. $\cos\theta = \frac{1}{2}$	**42.** $\cos\theta = -\frac{1}{2}$	**43.** $\cos\theta = 0$	**44.** $\cos\theta = \frac{2}{5}$	**45.** $\cos\theta = 1$

★ **Find the quadrants or axes and the exact values of $\cos\theta$ and $\tan\theta$ when $\sin\theta$ has the indicated value.**

46. $\sin\theta = \frac{1}{2}$	**47.** $\sin\theta = -\frac{1}{2}$	**48.** $\sin\theta = 0$	**49.** $\sin\theta = -\frac{2}{3}$	**50.** $\sin\theta = 1$

★ **Find the quadrants or axes and the exact values of $\sin\theta$ and $\cos\theta$ when $\tan\theta$ has the indicated value.**

51. $\tan\theta = \frac{1}{2}$	**52.** $\tan\theta = -\frac{1}{2}$	**53.** $\tan\theta = 0$	**54.** $\tan\theta = \frac{5}{12}$	**55.** $\tan\theta = 1$

Answers to Odd Problems:

41. QI: $\sin\theta = \frac{\sqrt{3}}{2}, \tan\theta = \sqrt{3}$ or QIV: $\sin\theta = -\frac{\sqrt{3}}{2}, \tan\theta = -\sqrt{3}$
43. positive y-axis: $\sin\theta = 1$, $\tan\theta$ = undefined or negative y-axis: $\sin\theta = -1, \tan\theta$ = undefined
45. positive x-axis: $\sin\theta = 0, \tan\theta = 0$ or negative x-axis: $\sin\theta = 0, \tan\theta = 0$
47. QIII: $\cos\theta = -\frac{\sqrt{3}}{2}, \tan\theta = \frac{\sqrt{3}}{3}$ or QIV: $\cos\theta = \frac{\sqrt{3}}{2}, \tan\theta = -\frac{\sqrt{3}}{3}$
49. QIII: $\cos\theta = -\frac{\sqrt{5}}{3}, \tan\theta = \frac{2\sqrt{5}}{5}$ or QIV: $\cos\theta = \frac{\sqrt{5}}{3}, \tan\theta = -\frac{2\sqrt{5}}{5}$
51. QI: $\sin\theta = \frac{\sqrt{5}}{5}, \cos\theta = \frac{2\sqrt{5}}{5}$ or QIII: $\sin\theta = -\frac{\sqrt{5}}{5}, \cos\theta = -\frac{2\sqrt{5}}{5}$
53. positive x-axis: $\sin\theta = 0, \cos\theta = 1$ or negative x-axis: $\sin\theta = 0, \cos\theta = -1$
55. QI: $\sin\theta = \frac{\sqrt{2}}{2}, \cos\theta = \frac{\sqrt{2}}{2}$ or QIII: $\sin\theta = -\frac{\sqrt{2}}{2}, \cos\theta = -\frac{\sqrt{2}}{2}$

F: Use identities to find other trig ratios from two given ratios

Exercise 5.4e.F

61. If $\sin t = \frac{\sqrt{3}}{2}$ and $\cos t = \frac{1}{2}$, find $\sec t, \csc t, \tan t$, and $\cot t$.

62. If $\sin 40° \approx 0.643$ $\cos 40° \approx 0.766$ $\sec 40°, \csc 40°, \tan 40°,$ and $\cot 40°$.

63. If $\tan(t) \approx 1.3$, and $\cos(t) \approx 0.61$, find $\sin(t)$.

64. If $\tan(t) \approx 2.7$, and $\sin(t) \approx 0.94$, find $\cos(t)$.

65. If $\cot(t) \approx 0.58$, and $\cos(t) \approx 0.5$, find $\csc(t)$.

66. If $\csc(t) \approx 3.2$, and $\cos(t) \approx 0.95$, find $\tan(t)$.

67. Use identities to simplify $\frac{\sec t}{\csc t}$

68. Use identities to simplify $\csc t \tan t$

70. Use the given information and identities to find the value of the other five trigonometric functions

a. $\sin(\theta) = -\frac{1}{3}$, θ in quadrant IV	c. $\cos(\theta) = \frac{1}{4}$, θ in quadrant IV	e. $\tan(\theta) = 5$, θ in quadrant III
b. $\csc(\theta) = \frac{5}{3}$, θ in quadrant I	d. $\sec(\theta) = -4$, θ in quadrant II	f. $\cot(\theta) = -\frac{1}{6}$, θ in quadrant II

Answers to Odd Problems:

61. $\sec t = 2, \csc t = \frac{2\sqrt{3}}{3}, \tan t = \sqrt{3}, \cot t = \frac{\sqrt{3}}{3}$ 63. $\sin(t) \approx 0.79$ 65. $\csc(t) \approx 1.16$ 67. $\tan t$.

G: Quadrants and Reference angles

Exercise 5.4e.G

★ State in which quadrant or on which axis the given angle lies.

71. $127°$	**73.** $313°$	**75.** $-90°$	**77.** $230°$	**79.** $1079°$
72. $-127°$	**74.** $-313°$	**76.** $621°$	**78.** $2009°$	**80** $-514°$

★ Find the reference angle for the given angle.

81. $317°$	**82.** $63°$	**83.** $-126°$	**84.** $696°$	**85.** $275°$

Answers to Odd Problems:

71. QII 73. QIV 75. on negative y-axis 77. QIII 79. QIV 81. $43°$ 83. $54°$ 85. $85°$

H: Evaluate trig ratios for Multiples of "Special" Angles

Exercise 5.4e.H

★ **Find the exact value of each expression.**

89. $\cos\frac{\pi}{6}$	91. $\cot\frac{\pi}{6}$	93. $\csc\frac{\pi}{4}$	95. $\tan\frac{\pi}{4}$	97. $\sec\frac{\pi}{3}$	99. $\sin\frac{\pi}{3}$
90. $\sin\frac{\pi}{6}$	92. $\tan\frac{\pi}{6}$	94. $\sec\frac{\pi}{4}$	96. $\cos\frac{\pi}{4}$	98. $\csc\frac{\pi}{3}$	100. $\cot\frac{\pi}{3}$

Answers to Odd Problems

89. $\frac{\sqrt{3}}{2}$ 91. $\sqrt{3}$ 93. $\sqrt{2}$ 95. 1 97. 2 99. $\frac{\sqrt{3}}{2}$

★ **Use reference angles to evaluate the expression.**

101. $\sec\frac{7\pi}{6}$	103. $\cot\frac{13\pi}{6}$	105. $\sec\frac{3\pi}{4}$	107. $\cot\frac{19\pi}{4}$	109. $\csc\frac{10\pi}{3}$	111. $\tan\frac{11\pi}{3}$
102. $\tan\frac{5\pi}{6}$	104. $\csc\frac{11\pi}{6}$	106. $\tan\frac{7\pi}{4}$	108. $\csc\frac{11\pi}{4}$	110. $\cot\frac{10\pi}{3}$	112. $\sec\frac{5\pi}{3}$

113. $\sec 300°$	115. $\cot 240°$	117. $\sec 120°$	119. $\cot 315°$
114. $\tan 225°$	116. $\csc 150°$	118. $\tan 330°$	120. $\csc 210°$

★ **For each of the following angles, find exact values for $\sec(\theta)$, $\csc(\theta)$, $\tan(\theta)$, $\cot(\theta)$.**

121. $\theta = \frac{5\pi}{6}$	122. $\theta = \frac{7\pi}{4}$	123. $\theta = \frac{2\pi}{3}$	124. $\theta = \frac{4\pi}{3}$

★ **Evaluate the following expressions exactly.**

125. a. $\sec(135°)$ b. $\csc(210°)$ c. $\tan(60°)$ d. $\cot(225°)$	126. a. $\sec(30°)$ b. $\csc(315°)$ c. $\tan(135°)$ d. $\cot(150°)$

Answers to Odd Problems

101. $-\frac{2\sqrt{3}}{3}$ 103. $\sqrt{3}$ 105. $-\sqrt{2}$ 107. -1 109. $-\frac{2\sqrt{3}}{3}$ 111. $-\sqrt{3}$ 113. 2 115. $\frac{\sqrt{3}}{3}$ 117. -2 119. -1.

121. $\sec(\theta) = -\frac{2\sqrt{3}}{3}$, $\csc(\theta) = 2$, $\tan(\theta) = -\frac{\sqrt{3}}{3}$, $\cot(\theta) = -\sqrt{3}$

123. $\sec(\theta) = -2$, $\csc(\theta) = \frac{2\sqrt{3}}{3}$, $\tan(\theta) = -\sqrt{3}$, $\cot(\theta) = -\frac{\sqrt{3}}{3}$

125. a. $\sec(135°) = -\sqrt{2}$ b. $\csc(210°) = -2$ c. $\tan(60°) = \sqrt{3}$ d. $\cot(225°) = 1$

I: Use a calculator

Exercise 5.4e.I

★ **Use a calculator to evaluate to three decimal digits.**

131. $\cot\frac{4\pi}{7}$	133. $\tan\frac{5\pi}{8}$	135. $\csc\frac{\pi}{4}$	137. $\cot 33°$	139. $\sec 310°$
132. $\csc\frac{5\pi}{9}$	134. $\sec\frac{\pi}{10}$	136. $\sec\frac{3\pi}{4}$	138. $\tan 98°$	140. $\cot 140°$

★ **Use a calculator to find secant, cosecant, and cotangent of the following angles to four decimal digits:**

141. 0.15	143. 4	145. 70°	147. 283°
142. 0.5	144. 5.2	146. 10°	148. 195°

Answers to Odd Problems

131. –0.228 133. –2.414 135. 1.414 137. 1.540 139. 1.556

141. $\csc(0.15) = 6.6917$ $\sec(0.15) = 1.0114$ $\cot(0.15) = 6.6166$
143. $\csc(4) = -1.3213$ $\sec(4) = -1.5299$ $\cot(4) = 0.8637$
145. $\csc(70°) = 1.0642$ $\sec(70°) = 2.9238$ $\cot(70°) = 0.3640$
147. $\csc(283°) = -1.0263$ $\sec(283°) = 4.4454$ $\cot(283°) = -0.2309$

J: Symmetry, Even/Odd Identities, Cofunction Identities

Exercise 5.4e.J

151. If $\sin(t) = \frac{\sqrt{2}}{2}$, what is the $\sin(-t)$?
152. If $\cos(t) = \frac{1}{2}$, what is the $\cos(-t)$?
153. If $\sec(t) = 3.1$, what is the $\sec(-t)$?
154. If $\csc(t) = 0.34$, what is the $\csc(-t)$?
155. If $\tan(t) = -1.4$, what is the $\tan(-t)$?
156. If $\cot(t) = 9.23$, what is the $\cot(-t)$?

★ **Determine whether the function is even, odd, or neither.**

157. $f(x) = 3\sin^2 x \cos x + \sec x$
158. $f(x) = 2\sin x \cos x$
159. $f(x) = \csc^2 x + \sec x$
160. $f(x) = \sin x - 2\cos^2 x$

Answers to Odd Problems

151. $-\frac{\sqrt{2}}{2}$ 153. 3.1 155. 1.4 157. even 159. even .

★ **Use cofunctions of complementary angles.**

161. $\cos\left(\frac{2\pi}{7}\right) = \_\_(\_\_)$
162. $\sin(76°) = \_\_(\_°)$
163. $\csc(38°) = \_\_(\_°)$
164. $\tan\left(\frac{3\pi}{5}\right) = \_\_(\_)$

★ **Use reciprocal, quotient, and Pythagorean Identities to determine the following.**

171. Given $\sin\theta = 0.4$, find (a) $\cos\theta$ (b) $\tan\theta$ (c) $\csc\theta$
172. Given $\cos\theta = 0.7$, find (a) $\sin\theta$ (b) $\sec\theta$ (c) $\cot\theta$
173. Given $\tan\theta = 2$, find (a) $\cot\theta$ (b) $\sec\theta$ (c) $\cos\theta$
174. Given $\csc\theta = 3$, find (a) $\sin\theta$ (b) $\cot\theta$ (c) $\tan\theta$
175. Given $\sec\theta = 4$, find (a) $\cos\theta$ (b) $\sin\theta$ (c) $\tan\theta$
176. Given $\cot\theta = 5$, find (a) $\tan\theta$ (b) $\csc\theta$ (c) $\sec\theta$

Answers to Odd Problems

161. $\sin\frac{3\pi}{14}$ 163. $\sec(52°)$. 171. (a) $\pm .2\sqrt{21} \approx \pm 0.9165$, (b) $\pm\frac{2}{\sqrt{21}} \approx \pm .4364$, (c) 2.5
173. (a) 0.5, (b) $\pm\sqrt{5} \approx \pm 2.236$, (c) $\pm\frac{\sqrt{5}}{5} \approx \pm .447$ 175. (a) 0.25, (b) $\pm\frac{\sqrt{15}}{4} \approx \pm .968$, (c) $\pm\sqrt{15} \approx \pm 3.873$

★

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5.5: Graphs of the Sine and Cosine Functions

Learning Objectives

- Graph variations of sinusoidal functions $y = \sin(x)$ and $y = \cos(x)$.
- Identify properties of a sinusoidal graph: amplitude, period, vertical shift, and phase shift.
- Construct a sinusoidal equation from a graph or a description



The London Eye (photo by Lippman&Rasmussen, 2010, CC-BY) is a huge Ferris wheel 135 meters (394 feet) tall in London, England, which completes one rotation every 30 minutes. When we look at the behavior of this Ferris wheel it is clear that it completes 1 cycle, or 1 revolution, and then repeats this revolution over and over again.

This is an example of a periodic function, because the Ferris wheel repeats its revolution or one cycle every 30 minutes, and so we say it has a period of 30 minutes.

In this section, we will work to sketch a graph of a rider's height above the ground over time and express this height as a function of time.

Definition: PERIODIC FUNCTIONS

A **periodic function** is a function for which a specific **horizontal shift**, $P > 0$, results in a function equal to the original function: $f(x + P) = f(x)$ for all values of x in the domain of f.

The **period** of a periodic function is the *smallest positive* number P such that $f(x + P) = f(x)$ for all values of x.

Graphing Sine and Cosine Functions

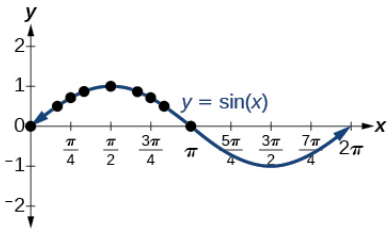
The graph of $y = \sin(x)$

One way to graph the function $y = \sin(x)$ is to construct a table of x and y values for $y = \sin(x)$. Table 5.5.1 lists some of the values for the sine function.

Table 5.5.1 Values for the sine function

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$y = \sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Plotting the points from the table and continuing along the x-axis gives the shape of the sine function, which is illustrated in the figure below.



Notice that this graph starts to repeat at $x = 2\pi$, because trigonometric functions of coterminal angles (angles that differ from each other by multiples of 2π radians or $360°$, have the same value. Thus, trigonometric functions are **periodic functions**.

Another way to create the graph of the function $y = \sin(x)$ uses a geometrical approach and the *unit circle*. The unit circle is the circle of radius 1 in the xy-plane consisting of all points (x, y) which satisfy the equation $x^2 + y^2 = 1$.

We see in Figure 5.5.3 that any point on the unit circle has coordinates $(x, y) = (\cos\ \theta, \sin\ \theta)$, where θ is the angle that the line segment from the origin to (x, y) makes with the positive x-axis. So as the point (x, y) goes around the circle, its y-coordinate is $\sin$

θ.

We thus get a correspondence between the y-coordinates of points on the unit circle and the values $f(\theta) = \sin\ \theta$, as shown by the horizontal lines from the unit circle to the graph of $f(\theta) = \sin\ \theta$ in Figure 5.5.4 for the angles $\theta = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$. These correspond to the values of the sine function in quadrant I.

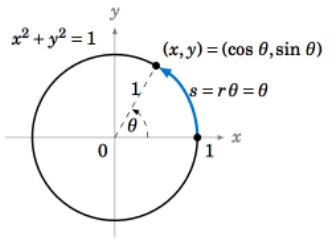


Figure 5.5.3: The Unit Circle

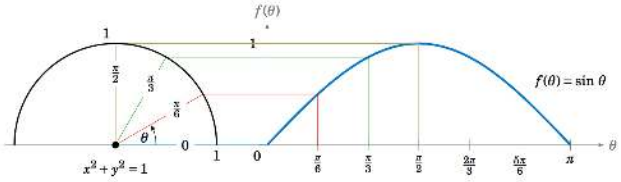


Figure 5.5.4: Graph of $y = \sin x$

We can extend the above picture to include angles from 0 to 2π radians. Notice how the sine values are positive between 0 and π, which correspond to the values of the sine function in quadrants I and II on the unit circle, and the sine values are negative between π and 2π, which correspond to the values of the sine function in quadrants III and IV on the unit circle. Figure 5.5.5 below illustrates what is sometimes called the *unit circle definition of the sine function.*

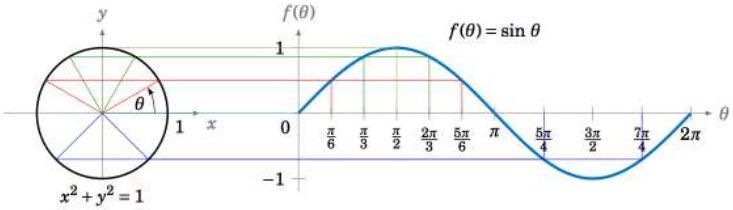
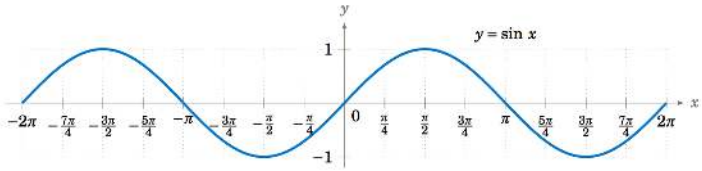


Figure 5.5.5 Unit circle definition of the sine function

Since the trigonometric functions repeat every 2π radians (360°), we get, for example, the following graph of the function $y = \sin\ x$ for x in the interval $[-2\pi, 2\pi]$:



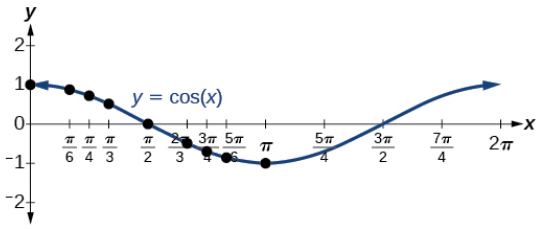
The graph of $y = \cos(x)$

Now let's take a similar look at the **cosine function**. To graph the cosine function, we could again use the unit circle idea (using the x-coordinate of a point that moves around the circle), or we can create a table of values and use them to sketch a graph. Table 5.5.2 below lists some of the values for the cosine function.

Table 5.5.2 Values for the cosine function

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$y = \cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

As with the sine function, we can plots points to create a graph of the cosine function as in the figure below.



Once again, because trigonometric functions of coterminal angles have exactly the same values, the cosine function, like the sine function, is periodic, and has a period of 2π.

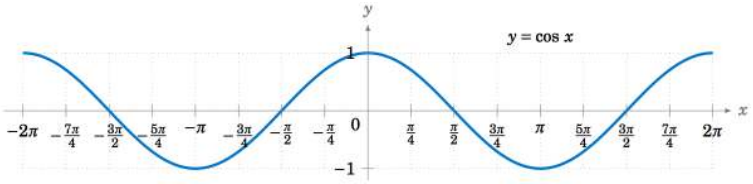


Figure 5.5.8: Graph of $y = \cos x$

Characteristics of Sinusoidal Functions

A function that has the same general shape as a sine or cosine function is known as a **sinusoidal function**.

In both graphs, the shape of the graph begins repeating after 2π, which means the functions are periodic with a period of 2π. Indeed, since any coterminal angles will have the same sine and cosine values, we could conclude that $\sin(\theta + 2\pi) = \sin(\theta)$ and $\cos(\theta + 2\pi) = \cos(\theta)$

In other words, if you were to shift either graph horizontally by 2π, the resulting shape would be identical to the original function.

PERIOD OF SINE AND COSINE

The periods of the sine function $y = \sin(x)$, and cosine function $y = \cos(x)$, are both 2π.

The diagrams below illustrate one period of the sine and cosine functions. A cycle is one complete pattern of the graph. Notice that the sine function starts in the vertical middle of a cycle whereas the cosine function starts at the vertical top of a cycle.

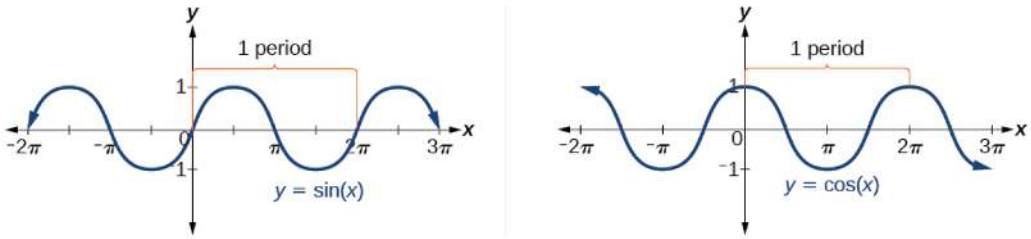


Figure 5.5.9: Illustration of one period of the sine and cosine functions

Both of these functions are defined for all real numbers, since we can evaluate the sine and cosine of any angle. It is also clear by looking at the graphs that the range of both functions must be the interval [-1, 1].

DOMAIN AND RANGE OF SINE AND COSINE

The domain of sine and cosine is all real numbers, $(-\infty, \infty)$.

The range of sine and cosine is the interval [-1, 1].

Looking again at the sine and cosine functions on a domain centered at the y-axis helps reveal symmetries.

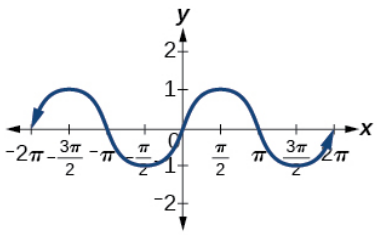


Figure 5.5.10: Odd symmetry of the sine function

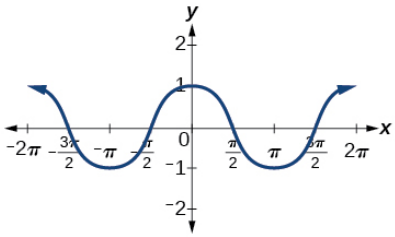


Figure 5.5.11: Even symmetry of the cosine function

As we can see in Figure 5.5.10, the sine function is symmetric about the origin, the same symmetry the cubic function has, making it an odd function. Figure 5.5.11 shows that the cosine function is symmetric about the y-axis, the same symmetry as the quadratic function, making it an even function.

NEGATIVE ANGLE IDENTITIES

The sine is an odd function, symmetric about the *origin*, so

$$\sin(-\theta) = -\sin(\theta)$$

The cosine is an even function, symmetric about the *y*-axis, so

$$\cos(-\theta) = \cos(\theta)$$

The sine and cosine graphs are horizontal transformations of each other. We can prove this by using the Cofunction Identity and the Negative Angle Identity for cosine.

$$\begin{aligned} \sin(\theta) &= \cos\left(\frac{\pi}{2} - \theta\right) && \text{Cofunction Identity} \\ &= \cos\left(-\theta + \frac{\pi}{2}\right) = \cos\left(-\left(\theta - \frac{\pi}{2}\right)\right) \\ &= \cos\left(\theta - \frac{\pi}{2}\right) && \text{Negative Angle Identity} \end{aligned}$$

Now we can clearly see that if we horizontally shift the cosine function to the right by $\pi/2$ we get the sine function.

CHARACTERISTICS OF SINE AND COSINE FUNCTIONS

The sine and cosine functions have several distinct characteristics:

- They are periodic functions with a period of 2π.
- The domain of each function is $(-\infty, \infty)$ and the range is $[-1, 1]$.
- The graph of $y = \sin\ x$ is symmetric about the origin, because it is an odd function.
- The graph of $y = \cos\ x$ is symmetric about they- y-axis, because it is an even function.

Investigating Sinusoidal Functions

As we can see, sine and cosine functions have a regular period and range. If we watch ocean waves or ripples on a pond, we will see that they resemble the sine or cosine functions. However, they are not necessarily identical. Some are taller or longer than others.

Sinusoidal Functions

A function that has the same general shape as a sine or cosine function is known as a **sinusoidal function**. The general forms of sinusoidal functions are

$$\boxed{y = A\sin(Bx - C) + D} \quad \text{and} \quad \boxed{y = A\sin(Bx - C) + D} \tag{5.5.1}$$

The variables A, B, C, and D are sometimes called **parameters**, and their values affect what the graph of the function looks like. We often refer to the expression inside the parentheses as the **argument** of the function.

Amplitude of Sinusoidal Functions

Starting with the parameter A, we will analyze how it is related to the **amplitude**, or greatest distance from rest.

AMPLITUDE OF SINUSOIDAL FUNCTIONS and vertical stretching

The **amplitude** of the sinusoidal functions $y = A\sin(Bx - C) + D$ and $y = A\cos(Bx - C) + D$ is $|A|$, the absolute value of the parameter A.

The **amplitude** can be directly observed on a graph of a sinusoidal function. It is half the height between the top and the bottom of the curve and can be calculated using the formula:

$$|A| = \text{amplitude} = \frac{1}{2}\,|\,\text{maximum} - \text{minimum}|$$

Amplitude can also be determined by using the fact that it is the vertical height from either the top or bottom of the curve to the horizontal line (called the midline) that divides the curve horizontally in half.

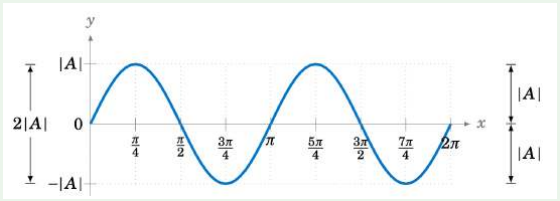


Figure 5.5.12: Amplitude = \dfrac{|A|−(−|A|)}{ 2} = |A|\)

Equation parameter A represents the **vertical stretch factor.**

- If $|A| > 1$, the function is vertically stretched.
- If $|A| < 1$, the function is vertically compressed.
- Reflection over the x-axis occurs when A is negative*.

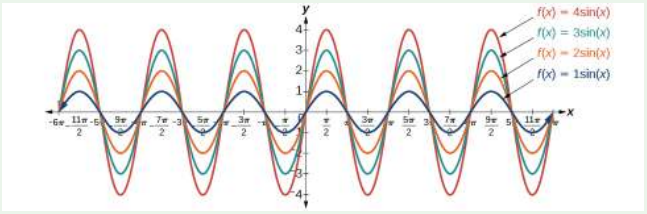


Figure 5.5.12: Amplitude = \dfrac{|A|−(−|A|)}{ 2} = |A|\)

Before preparing to graph the function, if B is negative, rewrite the equation using the Negative Angle Identities so the equation under analysis has $B > 0$. (The argument $Bx - C$ becomes $-Bx + C$ and if the sine function is being used, the sign of A changes.)

- When parameter B is positive, then if parameter A is **negative**, the result is a **reflection** across the x-axis.

Example 5.5.1: Identify the Amplitude of a Sine or Cosine Function

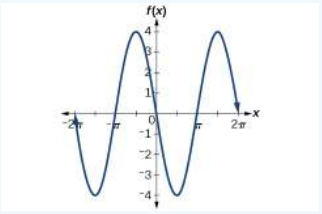
What is the amplitude of the sinusoidal function $f(x) = -4\sin(x)$? Is the function stretched or compressed vertically?

Solution

Let's begin by comparing the function to the simplified form $y = A\sin(Bx)$.

In the given function, $A = -4$, so the amplitude is $|A| = |-4| = 4$. The function is stretched vertically because $|A| > 1$.

The negative value of A results in a reflection across the x-axis of the sine function, as shown in the figure to the right.



Reflection of $\sin x$ across the x-axis

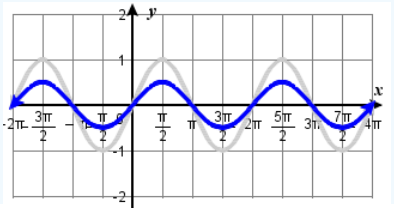
Try It 5.5.1a

What is the amplitude of the sinusoidal function $f(x) = \frac{1}{2}\sin(x)$? Is the function stretched or compressed vertically?

Answer

The amplitude is $\frac{1}{2}$. The graph is vertically compressed.

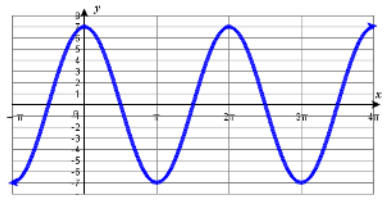
$f(x)$ is in dark blue in the graph below. The basic sine function is in light grey.



Try It 5.5.1b

What is the amplitude of the function $f(x) = 7\cos(x)$? Sketch a graph of this function.

Answer



A graph of $f(x) = 7\cos(x)$. Amplitude is 7

Vertical shift of Sinusoidal Functions

Definition: MIDLINE

The **midline** of a sinusoidal function is the center value of the function -- the y-value that the function oscillates above and below.

The midline of the basic sine and cosine functions, $y = \sin(x)$ and $y = \cos(x)$, is the line $y = 0$.

From our work with transformations, we know that adding a constant to a function shifts the entire graph vertically. So when using the general form of our sinusoidal equations (Equation 5.5.1), the parameter D indicates the vertical shift of the graph up or down from $y = 0$. For example, in Figure 5.5.14, the function $y = \sin(x) + D$ has its midline at $y = D$ and the entire basic graph $y = \sin(x)$ has been shifted up D units.

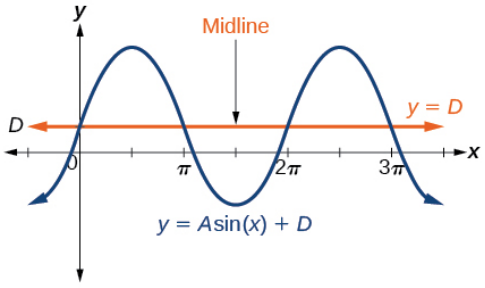


Figure 5.5.14

VERTICAL SHIFT OF SINUSOIDAL FUNCTIONS

The **vertical shift** of the sinusoidal functions $y = A\sin(Bx - C) + D$ and $y = A\cos(Bx - C) + D$ is indicated by the value of the parameter D.

The equation for the **midline** of a sinusoidal function is $y = D$

- If $D > 0$ the graph has been shifted **up** D units
- If $D < 0$ the graph has been shifted **down** D units

The **vertical shift** can be directly observed on a graph of a sinusoidal function. It is the vertical distance the midline has moved up or down from the x-axis.

- **Local maxima** are a distance $|A|$ above the vertical **midline** of the graph, and have y coordinates of $y = D + |A|$.
- **Local minima** will be the same distance below the midline, and have y coordinates of $y = D - |A|$.
- The **range** of sinusoidal functions in interval notation is $[D - |A|, D + |A|]$

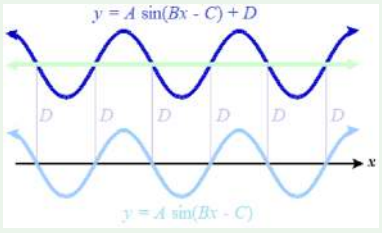


Figure 5.5.15: Graphs of $y = A\sin(Bx - C) + D)$ and $y = A\sin(Bx - C)$.

Any value of D other than zero shifts the graph up or down. Figure 5.5.16 compares $f(x) = \sin x$ with $f(x) = \sin x + 2$, which is shifted 2 units up on a graph.

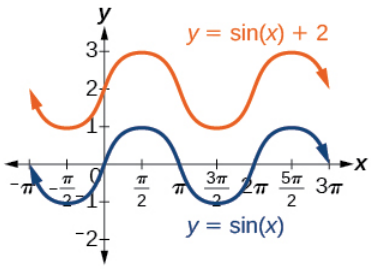


Figure 5.5.16

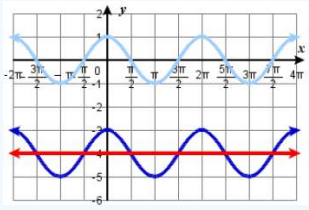
Example 5.5.2: Identify the Vertical Shift of a Sinusoidal Function

Given the function $f(\theta) = \cos(\theta) - 4$,

1. What is the midline of the function $f(\theta) = \cos(\theta) - 4$?
2. Determine the direction and magnitude of the transformation needed on the basic cosine function to obtain $f(\theta)$.
3. Sketch a graph of the function.

Solution

(1). Because the parameter $D = -4$, the midline is $y = -4$.
(2). Shift $y = \cos(x)$ **down 4 units** to obtain $f(\theta)$.
(3). The graph below shows the basic function $y = \cos(x)$, the midline for f, and $f(\theta)$.

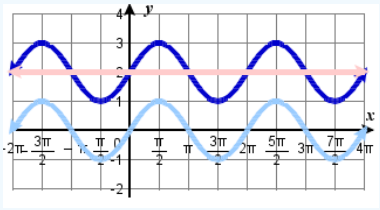


Try It 5.5.2

Determine the direction and magnitude of the vertical shift for $f(x) = \sin(x) + 2$. State the equation for the midline and sketch the graph.

Answer

The midline is $y = 2$. Shift $y = \sin(x)$ **up 2 units** to obtain $f(x)$.
The graph below shows $f(x)$, the midline $y = 2$, and the basic function $y = \sin(x)$.

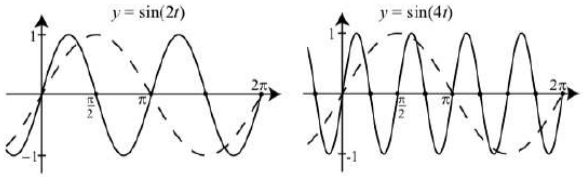


Period of Sinusoidal Functions

Looking at the forms of sinusoidal functions, we can see that they are transformations of the sine and cosine functions. We can use what we know about transformations to determine the period. Some examples of sinusoidal equations and their graphs are below. Look at each graph and observe what the length of one complete cycle (period) of the function is and then relate the observed period to the equation itself. The graph of the basic function $y = \sin(t)$ is also shown (dashes).

The graph of $y = \sin(2t)$ has one complete cycle over the interval $[0, \pi]$ and so its period is $P = \pi = \frac{2\pi}{2}$.

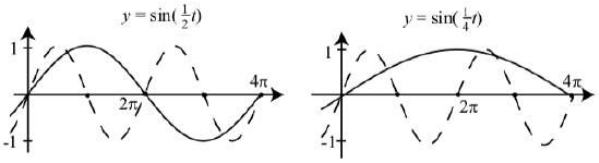
The graph of $y = \sin(4t)$ has one complete cycle over the interval $[0, \frac{\pi}{2}]$ and so its period is $P = \frac{\pi}{2} = \frac{2\pi}{4}$.



Graphs of $y = \sin(2t)$ and $y = \sin(4t)$

The graph of $y = \sin\left(\frac{1}{2}t\right)$ has one complete cycle over the interval $[0, 4\pi]$ and so its period is $P = 4\pi = \frac{2\pi}{\frac{1}{2}}$.

The graph of $y = \sin\left(\frac{1}{4}t\right)$ has one-half of a complete cycle over the interval $[0, 4\pi]$ and so its period is $P = 8\pi = \frac{2\pi}{\frac{1}{4}}$.



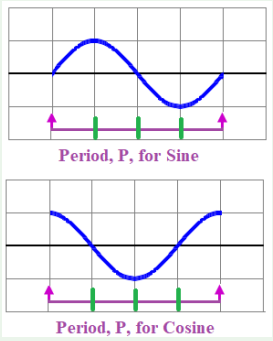
Graphs of $y = \sin(\frac{1}{2}t)$ and $y = \sin(\frac{1}{4}t)$

The conclusion we reach is that the period for equations of the sine and cosine functions $y = A\sin(Bx)$ and $y = A\cos(Bx)$ is $P = \frac{2\pi}{B}$. In fact this is also true for the more general forms of the sinusoidal functions.

PERIOD OF SINUSOIDAL FUNCTIONS and horizontal stretching

The **period** P of the sinusoidal functions

$y = A\sin(Bx - C) + D$ and $y = A\cos(Bx - C) + D$ is $P = \frac{2\pi}{|B|}$.



The **period** P can be directly observed on a graph of a sinusoidal function. It is the **horizontal length** of one complete cycle. P is always positive.

Notice that the parameter B is not directly visible from a graph, but it can be calculated once the period P IS obtained by looking at the graph. The period is inversely proportional to the parameter $|B|$.

- If $|B| > 1$, then the period is less than 2π and the function undergoes a horizontal compression
- if $|B| < 1$, then the period is greater than 2π and the function undergoes a horizontal stretch
- If B is negative, *before graphing the function*, rewrite the equation using the Negative Angle Identities (also known as the Even/Odd Properties) so the equation under analysis has $B > 0$ and reflection will be done properly.

When graphing a sinusoidal function, the period is broken down into quarters. When $C = 0$, these **quarter points** are at the beginning of a cycle $x = 0$, $x = \frac{1}{4}P$, $x = \frac{1}{2}P$, $x = \frac{3}{4}P$, and at the end of a cycle $x = P$. These quarter points, using the shape of the parent function as a guide, are the locations, in one cycle, of the local minimum, local maximum, and intersection with the midline.

A good question now is, "Why are the periods of sinusoidal functions always equal to $\frac{2\pi}{B}$"? From our work with transformations, we know that the coefficient of x affects the amount of horizontal stretch or compression of a graph. When the argument of the general form of a sinusoidal function is zero, the graph of the function will start a cycle. This occurs when $Bx_0 - C = 0$. When the argument of that same sinusoidal function is 2π , the graph of that function will end the cycle that was started. This occurs when

$Bx_1 - C = 2\pi$. The length of a cycle is the difference between x-coordinates at the end and start of a cycle. By subtracting these two equations $(Bx_1 - C) - (Bx_0 - C) = 2\pi - 0$ we obtain $B(x_1 - x_0) = 2\pi$ or $x_1 - x_0 = \frac{2\pi}{B}$.

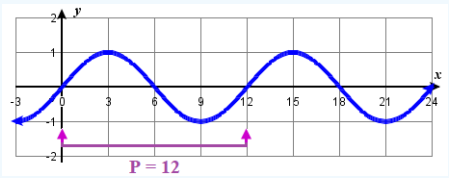
Example 5.5.3: Identify the Period of a Sine or Cosine Function

Determine the period of the function $f(x) = \sin\left(\frac{\pi}{6}x\right)$. Then sketch the function.

Solution

Let's begin by comparing the equation to the general form $y = A\sin(Bx)$.

In the given equation, $B = \frac{\pi}{6}$, so the period will be $P = \frac{2\pi}{|B|} = \frac{2\pi}{\frac{\pi}{6}} = 2\pi \cdot \frac{6}{\pi} = 12$



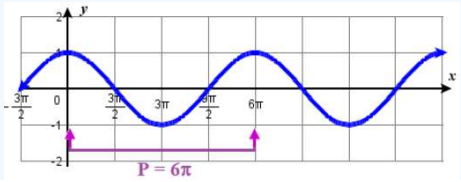
To graph the function, the sketch will take on the general shape of a sine function, starting a cycle at $x = 0$ with quarter points at $x = 3$, $x = 6$, $x = 9$, and $x = 12$ to complete one cycle.

Try It 5.5.3

Determine the period of the function $g(x) = \cos\left(\frac{x}{3}\right)$. Then sketch the function.

Answer

6π. To graph the function, the sketch will take on the general shape of a cosine function, starting a cycle at $x = 0$ with quarter points at $x = 1.5\pi$, $x = 3\pi$, $x = 4.5\pi$, and $x = 6\pi$ to complete one cycle.



Summary: Period, Amplitude, and Vertical Shift

TRANSFORMATIONS OF SINE AND COSINE

Given an equation in the form

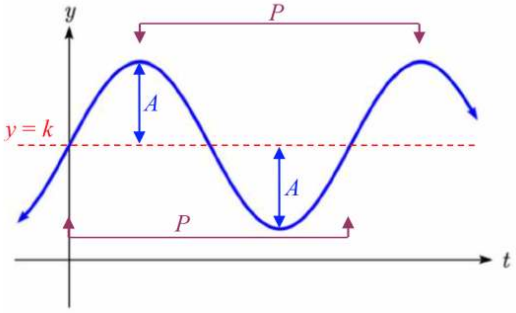
$$y = A\sin(Bx) + D \quad \text{or} \quad y = A\cos(Bx) + D$$

Rewrite the equation if necessary so that $B > 0$.

A is the **vertical stretch**, and $|A|$ is the **amplitude** of the function. When A is negative, the graph has been reflected over the x-axis.

B is inversely proportional to the amount of horizontal stretch/compression, and is related to the **period,** P, by the equation $P = \frac{2\pi}{B}$.

D is the **vertical shift** and determines the equation, $y = D$, for the **midline** of the function.



Graph for $y = A\sin\left(\frac{2\pi}{|P|}t\right) + k$

Sketch a Graph From an Equation

Instead of focusing on the general form equations $y = A\sin(Bx - C) + D$ and $y = A\cos(Bx - C) + D$ we will let $C = 0$ and work with a simplified form of the equations in the following examples.

Howto: Sketch the graph of a sinusoidal function that does not have a phase shift.

Before doing anything else, if B, the coefficient of x in the argument is negative, use the Even/Odd Properties to rewrite the sinusoidal equation so that B is positive (This *could* change the sign of A).

Given sinusoidal functions of the form

$$y = A\sin(Bx) + D \quad \text{or} \quad y = A\cos(Bx) + D$$

1. Identify the **amplitude**, $|A|$. If B is negative, rewrite the sinusoidal equation so that B is positive (This rewrite *could* change the sign of A).
2. Use the equation and its sign for A to determine the **shape** of the graph. The four options are shown in the graphs to the right.
3. Identify the **vertical shift** from the parameter D. The midline is the line $y = D$. The graph will oscillate $|A|$ units above and below this line, so the maximum is $y = D + |A|$, and the minimum is $y = D - |A|$.
4. Identify the **period**, $P = \dfrac{2\pi}{|B|}$.
5. Divide the period into quarters, These **quarter points** are at the beginning of a cycle ($x = 0$), $x = \frac{1}{4}P$, $x = \frac{1}{2}P$, $x = \frac{3}{4}P$, and at the end of a cycle ($x = P$). These quarter points, using the shape of the parent function as a guide, are the locations, in one cycle, of the local minimum, local maximum, and intersection with the midline. (See accompanying graphs)

Quarter points correspond to intervals of equal length each representing $1/4$ of the period. They indicate the horizontal location of the maximum and minimum values and where the midline is crossed for one period of the function. The green dots in the figures to the right show where the quarter points are for the four basic sinusoidal functions.

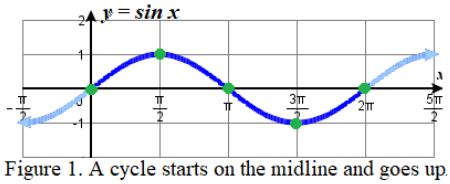


Figure 1. A cycle starts on the midline and goes up.
A sine function with positive parameter A

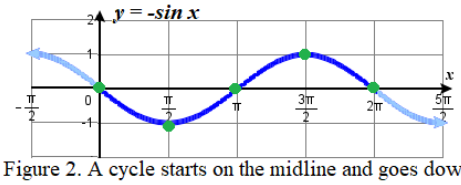


Figure 2. A cycle starts on the midline and goes down.
A sine function with negative parameter A

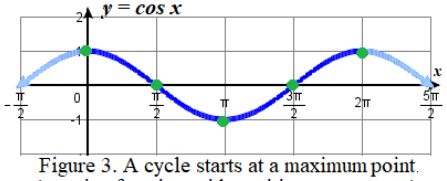


Figure 3. A cycle starts at a maximum point.
A cosine function with positive parameter A

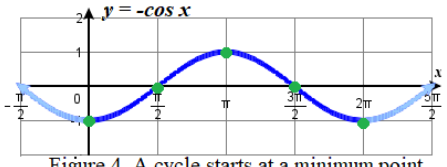


Figure 4. A cycle starts at a minimum point.
A cosine function with negative parameter A

Example 5.5.4a: Graphing a Function and Identifying the Amplitude and Period

Sketch the graph of

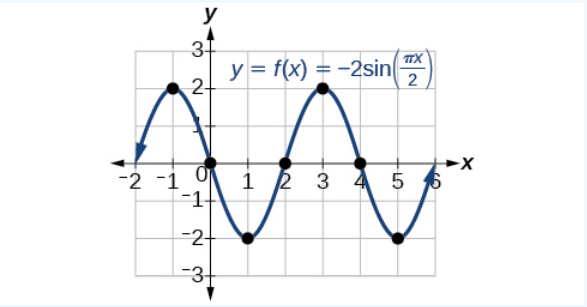
a. $f(x) = -2\sin\left(\dfrac{\pi x}{2}\right)$.
b. $g(x) = 2\sin\left(\dfrac{-\pi x}{2}\right)$.

Solution

a. Let's begin by comparing the equation to the form $y = A\sin(Bx) + D$.

- *Step* 1. We can see from the equation that $A = -2$, so the **amplitude** is 2. $\quad |A| = 2$
- *Step* 2. Because A is negative, the graph is a **reflection** over the x-axis of the basic sine function. The **shape** of the graph is like the graph for $y = -\sin(x)$, starting on the y-axis at the midline and going down.
- *Step* 3. The parameter D is zero so the **midline** is along $y = D \Rightarrow y = 0$ or the x-axis, and, since the amplitude is 2, the graph oscillates between 2 and -2.
- *Step* 4. The equation shows that $B = \dfrac{\pi}{2}$, so the period is $\quad P = \dfrac{2\pi}{|B|} = \dfrac{2\pi}{\frac{\pi}{2}} = 2\pi \cdot \dfrac{2}{\pi} = 4$

- *Step* 5. The **quarter points** are at $x = 0, 1, 2, 3, 4$. One cycle starts at the midline and goes down. The graph intersects the midline at $x = 0$, $x = 2$, and $x = 4$. A local minimum will occur at 2 units below the midline, at $x = 1$. A local maximum will occur at 2 units above the midline, at $x = 3$. The graph of the function is below.



The graph of $f(x) = -2\sin\left(\frac{\pi x}{2}\right)$

b.

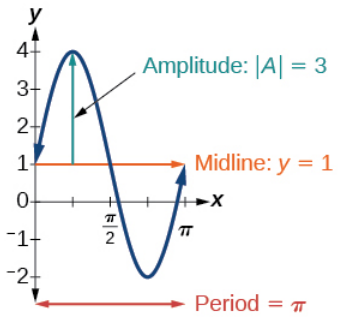
- Preparation. Because B is negative, the equation needs to be rewritten using the Negative Angle Identity. Since $\sin(-x) = -\sin(x)$, then $g(x) = -2\sin\left(\frac{\pi x}{2}\right)$, which is the same as $f(x)$ in part (a).
- Step 1. We can see from the equation that $|A| = 2$, so the **amplitude** is 2.
- Step 2. Using the modified version of g, A is negative, so the **shape** of the graph for g is like the shape of $y = -\sin(x)$. [*Notice that if the Negative Angle Identity is not used, the shape chosen would be that of* $y = \sin(x)$ *, which would result in a graph that would not be correct*].
- Step 3. The **midline** is $y = 0$.
- Step 4. The **period** is $P = 4$.
- Step 5. The **quarter points** are at $x = 0, 1, 2, 3, 4$, and the graph is the same as that in the graph above.

Example 5.5.4b

Determine the midline, amplitude, and period of the function $y = 3\sin(2x) + 1$. Then sketch the function.

Solution

Step 1. $A = 3$, so the **amplitude** is $|A| = 3$.
Step 2. The graph has the same **shape** as the graph for $y = \sin(x)$, starting on the y-axis at the midline and going up.
Step 3. $D = 1$, so the **midline** is $y = 1$, and the graph will oscillate between $D - |A|$ and $|D + |A|$ or between $y = -2$ and $y = 4$.
Step 4. $B = 2$, so the **period** is $P = \frac{2\pi}{|B|} = \frac{2\pi}{2} = \pi$.
Step 5. The **quarter points** are at $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3}{4}, \pi$. One cycle starts at the midline and goes up.
One cycle of the function appears in the graph on the right.



Graph of $y = 3\sin(2x) + 1$

Try It 5.5.4

Sketch a graph of $g(x) = -0.8\cos(2x)$. Determine the midline, amplitude, and period.

Answer

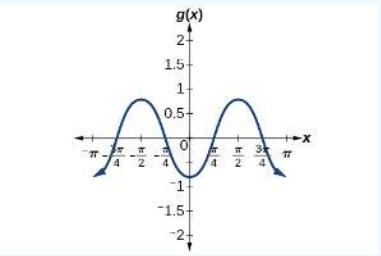
amplitude: $|A| = 0.8$;

shape: $y = -\cos(x)$;

midline: $y = 0$, minimum $y = -.8$, maximum: $y = .8$;

period: $P = \frac{2\pi}{|B|} = \pi$;

quarter points: $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$, and π.



$g(x) = -0.8\cos(2x)$

Construct an Equation From a Graph

Howto: Construct an equation from the graph of a sinusoidal function.

Given a sinusoidal graph, construct one of the sinusoidal functions: $y = A\sin(Bx) + D$ and $y = A\cos(Bx) + D$

1. Identify the **amplitude**, $|A|$, half the distance between the highest and lowest values of y on the graph.
2. Identify the **midline**, the line halfway between the highest and lowest values of y on the graph. This is **parameter** D.
3. Look at the **shape** of the graph at the point where it intersects the y-axis.
   - If the point is on the midline and the graph is going up, use the **sine function** and make parameter \(**A**\) positive. The shape is the same as $y = \sin x$.
   - If the point is on the midline and the graph is going down, use the **sine function** and make parameter \(**A**\) negative. The shape is the same as $y = -\sin x$.
   - If the point is a maximum, use the **cosine function** and make parameter \(**A**\) positive. The shape is the same as $y = \cos x$.
   - If the point is a minimum, use the **cosine function** and make parameter \(**A**\) negative. The shape is the same as $y = -\cos x$.
4. Identify the **period** P, the horizontal distance between peaks on the graph. Calculate **parameter** B from the equation $B = \frac{2\pi}{P}$.
5. Use the values of the parameters obtained above to construct a possible sinusoidal equation for the graph.

Example 5.5.5a: Identifying the Equation for a Sinusoidal Function from a Graph

Determine the formula for the function in Figure 5.5.27.

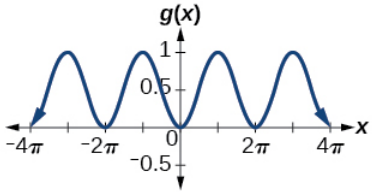


Figure 5.5.27

Solution

To determine the equation, we need to identify each parameter value in the general form of a sinusoidal function.

$$y = A\sin(Bx) + D \qquad y = A\cos(Bx) + D$$

The **amplitude** is half the difference between the height of local maxima and minima. This difference is 1, so $|A| = \frac{1}{2} = 0.5$. Another way to obtain the amplitude is to identify the midline and observe the greatest distance above and below the midline, which is the amplitude.

In general, the **shape** of the graph indicates a sinusoidal function. When $x = 0$, the graph has an extreme point, $(0, 0)$. Since the reflected cosine function has an extreme (minimum) point for $x = 0$, let us write our equation in terms of a reflected cosine function. Therefore, the parameter A must be negative, and $A = -0.5$.

We can see that the graph rises and falls an equal distance above and below $y = 0.5$. This value, which is the **midline**, is D in the equation, so $D = 0.5$.

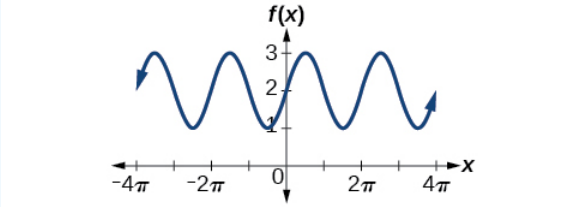
The **period** is $P = 2\pi$, so the graph is not horizontally stretched or compressed, so $B = \frac{2\pi}{P} = 1$.

Putting this all together,

$$g(x) = -0.5\cos(x) + 0.5$$

Try It 5.5.5a

Determine the formula for the function in the graph below



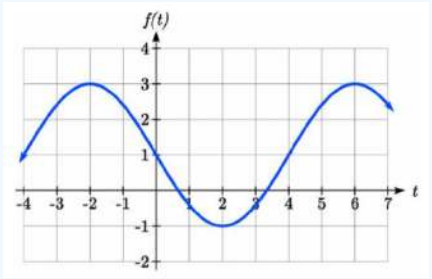
Graph of a Sinusoidal Function

Answer

amplitude is half the difference between maximum and minimum so $|A| = 1$; **midline** is $y = 2$ so $D = 2$. When $x = 0$, the graph is on the midline and is going up, so the **shape** is the same as for the function $y = sin(x)$, so write the equation in terms of a sine function, and A is positive, so $A = 1$. The period is $P = 2\pi$, so there has been no horizontal stretching, so $B = \frac{2\pi}{P} = 1$. Putting it all together, the equation is: $f(x) = \sin(x) + 2$

Example 5.5.5b

Find a formula for the sinusoidal function graphed here.



Solution

The **amplitude** is half the distance between the highest value and lowest value of the graph. So the amplitude is $\frac{3-(-1)}{2} = 2$.

The graph oscillates from a low of -1 to a high of 3, putting the **midline** at $y = 1$, halfway between.

At $t = 0$, the graph is at the midline value, which tells us the graph can most easily be represented as a sine function. Since the graph then decreases, this must be a vertical reflection of the sine function. Therefore the **shape** is the same as that of $y = -\sin(t)$ so A is negative, and thus $A = -2$.

The **period** of the graph is 8. We can measure this from the first peak at $x = -2$ to the second at $x = 6$. Since the period is 8, the stretch/compression factor we will use will be

$$B = \frac{2\pi}{P} = \frac{2\pi}{8} = \frac{\pi}{4}$$

Putting this all together,

$$f(t) = -2\sin\left(\frac{\pi}{4}t\right) + 1$$

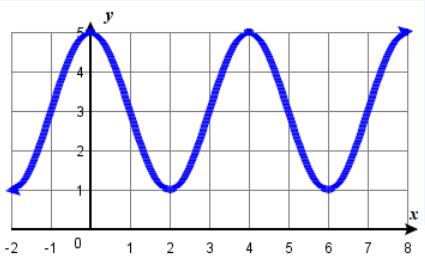
Try It 5.5.5b

If a sinusoidal function has a maximum value at point (0,5), has an amplitude of 2, and a period of 4, write a formula for the function. Sketch the graph.

Answer

We are given $|A| = 2$ and there is a maximum is on the y-axis. Therefore the **shape** is the same as that of $y = cos(x)$, so A is positive, and thus $A = 2$. The midline is at $y = 3$, so $D = 3$. $B = \frac{2\pi}{4} = \frac{\pi}{2}$.

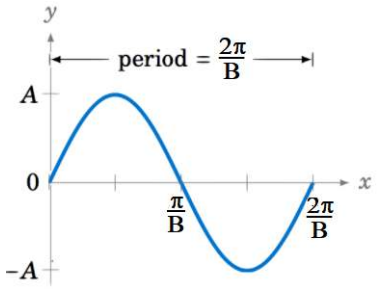
Putting it all together, a possible equation would be $f(x) = 2\cos\left(\frac{\pi}{2}x\right) + 3$. Its graph is below.



Phase Shift of Sinusoidal Functions

Now that we understand how equation parameters A, B, and D relate to the graph of sine and cosine functions, we will explore the parameter C. We have seen how adding a constant to a function shifts the entire graph vertically. We will now see how to shift the entire graph of a periodic curve horizontally.

Consider a function of the form $y = A\ \sin(Bx)$, where A and B are nonzero constants. For simplicity we will assume that $A > 0$ and $B > 0$ (in general either one could be negative). Then the amplitude is A and the period is $\frac{2\pi}{B}$. The graph is shown on the right.



$y = A\sin(Bx)$ which has a cycle starting at $x = 0$ and ending at $\frac{2\pi}{B}$.

Now consider the function $y = A\ \sin\ (Bx - C)$, where C is some constant. The amplitude is still A, and the period is still $\frac{2\pi}{B}$, since $Bx - C$ is a linear function of x. Also, we know that the sine function goes through an entire cycle when its angle goes from 0 to 2π. Here, we are taking the sine of the angle $Bx - C$. So as $Bx - C$ goes from 0 to 2π, an entire cycle of the function $y = A\ \sin\ (Bx - C)$ will be traced out.

$$\text{The cycle starts when} \quad Bx - C = 0 \quad \Rightarrow \quad x = \frac{C}{B}$$

$$\text{and ends when} \quad Bx - C = 2\pi \quad \Rightarrow \quad x = \frac{2\pi}{B} + \frac{C}{B}\ .$$

Thus, the graph of $y = A\ \sin\ (Bx - C)$ is just the graph of $y = A\ \sin(Bx)$ shifted horizontally by $\frac{C}{B}$, as in Figure 5.2.11. The graph is shifted to the right when $C > 0$, and to the left when $C < 0$. The amount $\frac{C}{B}$ of the shift is called the **phase shift** of the graph.

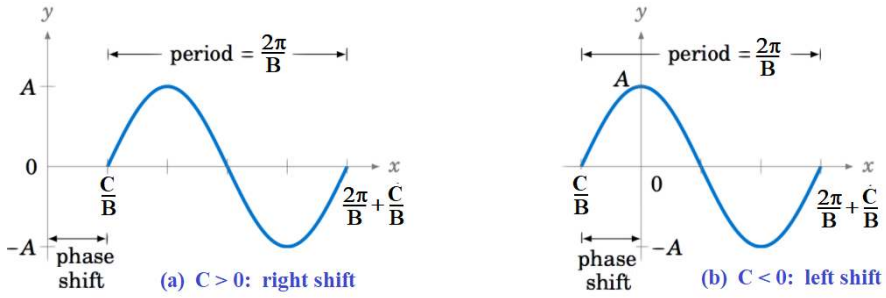


Figure 5.2.11 Phase shift for $y = A\sin(Bx - C)$

The phase shift is defined similarly for the cosine function.

PHASE SHIFT OF SINUSOIDAL FUNCTIONS (horizontal shift)

The **phase shift** is the horizontal displacement of a sine or cosine function.

- If the graph starts to the left of the y axis, the phase shift is negative.
- If the graph starts to the right of the y axis, the phase shift is positive.

The formula for the **phase shift** given one of the sinusoidal functions $y = A\sin(Bx - C) + D$ and $y = A\cos(Bx - C) + D$ is

$$\text{Phase Shift: } H = \frac{C}{B}.$$

Be careful!! When determining the value for C, notice the negative sign in front of C in the sinusoidal equations!

The **phase shift** $H = \frac{C}{B}$ can be directly observed on a graph of a sinusoidal function. It is the horizontal distance between the start of a sinusoidal function and the y-axis.

Notice that the parameter C is not necessarily visible when looking at a graph.

- If the graph starts to the left of the y axis, the phase shift H is negative, and therefore C will be negative (but the operation in the argument will be +).
- If the graph starts to the right of the y axis, the phase shift H is positive, and therefore C will be positive (but the operation in the argument will be -).

If we rewrite the above equations by factoring B out of the argument expression, the argument $Bx - C$ becomes $B\left(x - \frac{C}{B}\right)$ or $B(x - H)$. Then the general equations can be written in the following form, which makes the phase shift H readily visible in the equation.

$$y = A\sin(B(x - H)) + D \qquad \text{and} \qquad y = A\cos(B(x - H)) + D$$

How To: Howto: Identify the midline, amplitude, period, and phase shift from a sinusoidal equation .

1. Determine the **amplitude** as $|A|$.
2. Determine the **period** as $P = \frac{2\pi}{|B|}$.
3. Determine the **phase shift** as $\frac{C}{B}$. Shift is to the *left* if negative and to the *right* if positive.
4. Determine the **midline** as $y = D$.

Example 5.5.6: Identify phase shift in a sinusoidal equation

Determine the direction and magnitude of the phase shift for $f(x) = \sin\left(x + \frac{\pi}{6}\right) - 2$.

Solution

Let's begin by comparing the equation to the general form $y = A\sin(Bx - C) + D$.

In the given equation, notice that $B = 1$ and $C = -\frac{\pi}{6}$. So the phase shift is $\frac{C}{B} = -\frac{\frac{\pi}{6}}{1} = -\frac{\pi}{6}$

or $\frac{\pi}{6}$ units to the left.

Analysis

We must pay attention to the sign in the equation for the general form of a sinusoidal function. The equation shows a minus sign before C. Therefore $f(x) = \sin(x + \frac{\pi}{6}) - 2$ can be rewritten as $f(x) = \sin\left(x - \left(-\frac{\pi}{6}\right)\right) - 2$. If the value of C is negative, the shift is to the left.

Try It 5.5.6a

Determine the direction and magnitude of the phase shift for $f(x) = 3\cos\left(x - \frac{\pi}{2}\right)$.

Answer

$\frac{\pi}{2}$; right

Try It 5.5.6b

Determine the midline, amplitude, period, and phase shift of the function $y = \frac{1}{2}\cos\left(\frac{x}{3} - \frac{\pi}{3}\right)$.

Answer

midline: $y = 0$; amplitude: $|A| = \frac{1}{2}$; period: $P = \frac{2\pi}{|B|} = 6\pi$; phase shift: $\frac{C}{B} = \pi$ or π units to the right.

Draw a graph from an equation

Throughout this section, we have learned about types of variations of sine and cosine functions and used that information to write equations from graphs. Now we can use the same information to create graphs from equations.

How To: Howto: Given a sinusoidal function with a phase shift and a vertical shift, sketch its graph

1. Express the function in the general form $y = A\sin(Bx - C) + D$ or $y = A\cos(Bx - C) + D$. Rewrite the equation so B is positive and determine the **basic** shape of the graph.
2. Identify the **amplitude**, $|A|$ and the **midline** $y = D$, and maximum and minimum y values.
3. Identify the **period**, $P = \frac{2\pi}{|B|}$ and the value of x at the **quarter points**.
4. Identify the **phase shift**, $\frac{C}{B}$ and adjust the quarter points accordingly.
5. Draw the graph of $f(x) = A\sin(Bx)$ or $f(x) = A\cos(Bx)$ shifted to the right or left by $\frac{C}{B}$ units and up or down by D units.

Example 5.5.7a: Graph a Sinusoidal Equation

Sketch a graph of $f(x) = 3\sin\left(\frac{\pi}{4}x - \frac{\pi}{4}\right)$.

Solution

- *Step 1.* The function is already written in general form: $f(x) = 3\sin\left(\frac{\pi}{4}x - \frac{\pi}{4}\right)$.This graph will have the shape of a sine function, starting at the midline and increasing to the right.
- *Step 2.* $|A| = |3| = 3$. The amplitude is 3. $D = 0$ so the midline is $y = 0$ and the graph has not been shifted vertically. The maximum and minimun values for y are 3 and -3.
- *Step 3.* Since $|B| = |\frac{\pi}{4}| = \frac{\pi}{4}$, we determine the period as follows.

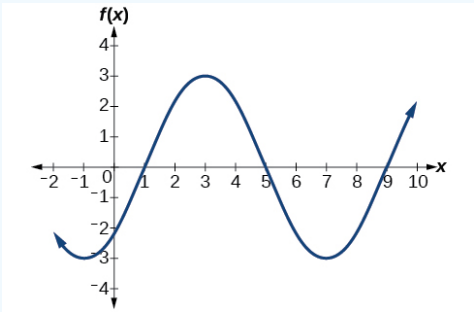
$$P = \frac{2\pi}{|B|} = \frac{2\pi}{\frac{\pi}{4}} = 2\pi \cdot \frac{4}{\pi} = 8$$

  The period is 8. The quarter points are $x = 0,\ 2,\ 4,\ 6,$ and 8.

- *Step 4.* Since $C = \frac{\pi}{4}$, the phase shift is $\frac{C}{B} = \frac{\frac{\pi}{4}}{\frac{\pi}{4}} = 1$.

  The phase shift is 1 unit. This means 1 is added to each of the quarter points making them $x = 1,\ 3,\ 5,\ 7,$ and 9.

- *Step 5.* The graph of the function is below.



Graph of $f(x) = 3\sin\left(\frac{\pi}{4}x - \frac{\pi}{4}\right)$

Example 5.5.7b

Given $y = -2cos\left(\frac{\pi}{2}x + \pi\right) + 3$, determine the amplitude, period, phase shift, and horizontal shift. Then graph the function.

Solution

Begin by comparing the equation to the general form.

$$y = A\cos(Bx - C) + D$$

- *Step 1.* The function is already written in general form. It will have the shape of a reflected cosine function, starting at a minimum and going up to the right.
- *Step 2.* Since $A = -2$, the amplitude is $|A| = 2$. $D = 3$ so the midline is $y = 3$. The maximum and minimum values for y are $y = 3 + 2 = 5$ and $y = 3 - 2 = 1$.
- *Step 3.* $|B| = \frac{\pi}{2}$, so the period is $P = \frac{2\pi}{|B|} = \frac{2\pi}{\frac{pi}{2}} = 2\pi \cdot \frac{2}{\pi} = 4$.

  The period is 4. The quarter points are $x = 0,\ 1,\ 2,\ 3,\ \text{and } 4$.
- *Step 4.* $C = -\pi$, so we calculate the phase shift as $\frac{C}{B} = -\frac{\pi}{\frac{\pi}{2}} = -\pi \cdot \frac{2}{\pi} = -2$.

  The phase shift is -2 or two units to the left. This changes the quarter points to $x = -2,\ -1,\ 0,\ 1,\ \text{and } 2$.

One complete cycle starts at $x = -2$ and since A is negative, the graph will start a a local minimum ($y = 1$) and go up (like the shape of the function $y = -\cos(x)$. Another cycle will start at $x = 2$ and this is the cycle indicated in the graph of the function illustrated in Figure 5.5.23.

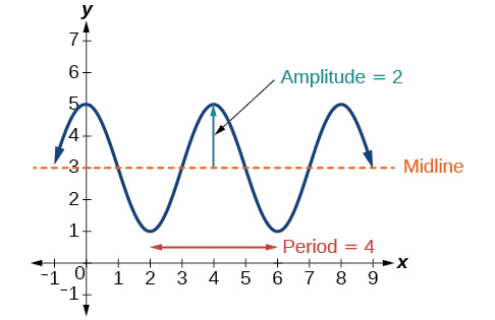


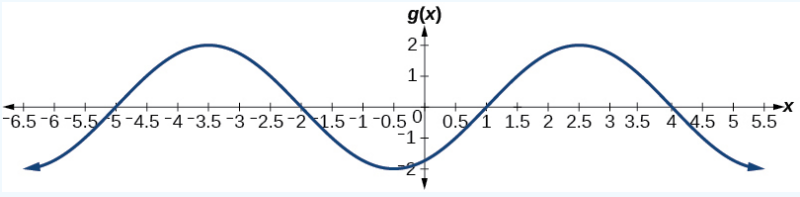
Figure 5.5.23

Try It 5.5.7

Draw a graph of $g(x) = -2\cos\left(\frac{\pi}{3}x + \frac{\pi}{6}\right)$. Determine the midline, amplitude, period, and phase shift.

Answer

midline: $y = 0$; amplitude: $|A| = 2$; period: $P = \frac{2\pi}{|B|} = 6$; phase shift: $\frac{C}{B} = -\frac{1}{2}$ or $\frac{1}{2}$ unit to the left.



Graph of $g(x) = -2\cos\left(\frac{\pi}{3}x + \frac{\pi}{6}\right)$

Construct an Equation From a Graph

Howto: Construct an equation from the graph of a sinusoidal function.

From a sinusoidal graph, construct one of the sinusoidal functions: $y = A\sin(Bx - C) + D$ and $y = A\cos(Bx - C) + D$

Determine the values of the parameters used in the

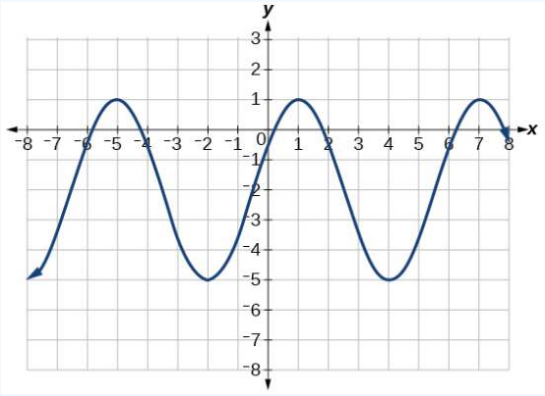
1. Identify the **amplitude**, $|A|$, half the distance between the highest and lowest values of y on the graph.
2. Identify the **midline**, the line halfway between the highest and lowest values of y on the graph. This is **parameter** D.
3. Identify the **period** P, the horizontal distance between peaks on the graph. Calculate **parameter** B from the equation $B = \frac{2\pi}{P}$.
4. Look for a highest point, a lowest point, or a point on the midline that is on (ideally - then there is no phase shift) or close to the y axis and determine the horizontal distance, H, this point is from the y-axis. This is the **phase shift** and marks the beginning of one cycle of the function. Calculate **parameter** C from the equation $C = B \cdot H$.
5. Identify the **function to be used** and the **parameter** A from the shape of the graph beginning at the starting point for one cycle, determined in the previous step.

Starting point	Graph direction	Function	parameter A
a Maximum	Down	cos	$\|A\|$
on Midline	Up	sin	$\|A\|$
on Midline	Down	sin	$-\|A\|$
a Minimum	Up	cos	$-\|A\|$

Use the values of the parameters obtained above to construct a possible sinusoidal equation for the graph.

Example 5.5.8 Identifying the Equation for a Sinusoidal Function from a Graph

Determine the equation for the sinusoidal function below.



A sinusoidal function

Solution

Step 1. Half the distance between the highest or lowest values gives an amplitude of $|A| = 3$.

Step 2. With the highest value at 1 and the lowest value at -5, the midline will be halfway between at -2. So $D = -2$.

Step 3. The period of the graph is 6, which can be measured from the peak at $x = 1$ to the next peak at $x = 7$, or from the distance between adjacent lowest points. Therefore, $P = \frac{2\pi}{|B|} = 6$. Using the positive value for B, we find that

$$B = \frac{2\pi}{P} = \frac{2\pi}{6} = \frac{\pi}{3}$$

So far, our equation is either $y = 3\sin\left(\frac{\pi}{3}x - C\right) - 2$ or $y = 3\cos\left(\frac{\pi}{3}x - C\right) - 2$.

Steps 4 and 5. For the shape and horizontal shift, we have more than one option. We could write this as any one of the following:

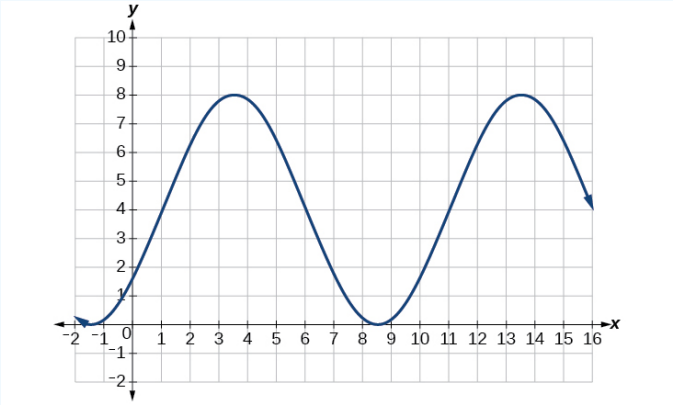
- a cosine shifted to the right Starting at $x = 1 = H = C/B$
- a negative cosine shifted to the left Starting at $x = -2$
- a sine shifted to the left Starting at $x \approx 1/2$
- a negative sine shifted to the right Starting at $x \approx 2\,1/2$

While any of these would be correct, the cosine shifts are easier to work with than the sine shifts in this case because they involve integer values. So our function becomes either $y = 3\cos\left(\frac{\pi}{3}x - C\right) - 2$ or $y = -3\cos\left(\frac{\pi}{3}x - C\right) - 2$. The horizontal displacement when $A = 3$ to the highest point on the graph is one unit to the right, so the equation used to solve for C is $\frac{C}{B} = 1 \Rightarrow C = B = \frac{\pi}{3}$. Alternatively, we could have chosen to use the equation in which $A = -3$ (indicating *reflection* over the x-axis). In this case, the horizontal displacement to the *lowest* point on the graph is two units to the left, and the equation used to solve for C is $\frac{C}{B} = -2 \Rightarrow C = -2B = -\frac{2\pi}{3}$.

So, two possibilities for our equation are $y = 3\cos\left(\frac{\pi}{3}x - \frac{\pi}{3}\right) - 2$ and $y = -3\cos\left(\frac{\pi}{3}x + \frac{2\pi}{3}\right) - 2$. Again, these functions are equivalent, so both yield the same graph.

Try It 5.5.8

Write a formula for the function graphed below.



A Sinusoidal Graph

Answer

Two possible equations are: $y = 4\sin\left(\frac{\pi}{5}x - \frac{\pi}{5}\right) + 4$ or $y = -4\sin\left(\frac{\pi}{5}x + \frac{4\pi}{5}\right) + 4$

Construct Functions for Applications

We can use the transformations of sine and cosine functions in numerous applications. As mentioned at the beginning of the chapter, **circular motion** can be modeled using either the sine or cosine function.

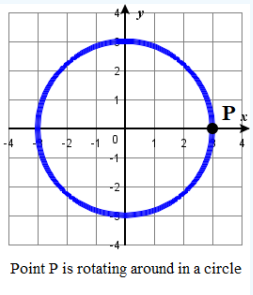
Circular Motion

Example 5.5.9: Finding the Vertical Component of Circular Motion

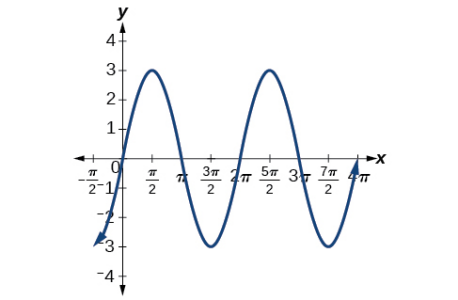
A point rotates around a circle of radius 3 centered at the origin. Sketch a graph of the y-coordinate of the point as a function of the angle of rotation.

Solution

Recall that, for a point on a circle of radius r, the y-coordinate of the point is $y = r\sin(x)$, so in this case, we get the equation $y(x) = 3\sin(x)$. The constant 3 causes a vertical stretch of the y-values of the function by a factor of 3, which we can see in the graph in Figure 5.5.24.



Point P is rotating around in a circle



y-coordinate of P as a function of the angle of rotation

Analysis

Notice that the period of the function is still 2π; as we travel around the circle, we return to the point $(3, 0)$ for $x = 2\pi, 4\pi, 6\pi$,....Because the outputs of the graph will now oscillate between -3 and 3, the amplitude of the sine wave is 3.

Example 5.5.10: Finding the Vertical Component of Circular Motion

A circle with radius 3 ft is mounted with its center 4 ft off the ground. The point closest to the ground is labeled P, as shown in Figure 5.5.26. Sketch a graph of the height above the ground of the point P as the circle is rotated; then find a function that gives the height in terms of the angle of rotation.

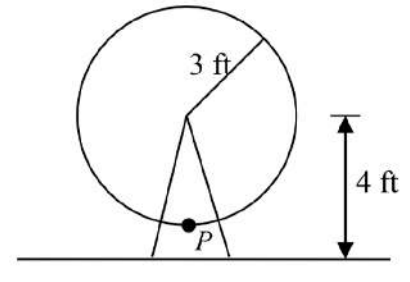


Figure 5.5.26

Solution

For a point rotating on a circle of radius r, the height is the y-coordinate of the point defined by the equation $y = r\sin(\theta)$ so $y = 3\sin(\theta)$. The amplitude is $|A| = 3$.

Sketching the height, we note that it will start 1 ft above the ground, then increase up to 7 ft above the ground, and continue to oscillate 3 ft above and below the center value of 4 ft, as shown in Figure 5.5.27.

Although we could use a transformation of either the sine or cosine function, we start by looking for characteristics that would make one function easier to use than the other. Let's use a cosine function because it starts at the highest or lowest value, while a sine function starts at the middle value. A standard cosine starts at the highest value, and this graph starts at the lowest value, so we need to incorporate a vertical reflection.

Second, we see that the graph oscillates 3 above and below the center, while a basic cosine has an amplitude of 1, so this graph has been vertically stretched by 3, as in the last example.

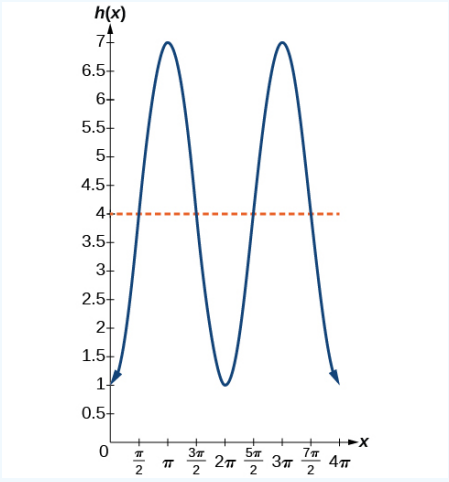


Figure 5.5.27

Finally, to move the center of the circle up to a height of 4, the graph has been vertically shifted up by 4. Putting these transformations together, we find that

$$y = -3\cos(x) + 4$$

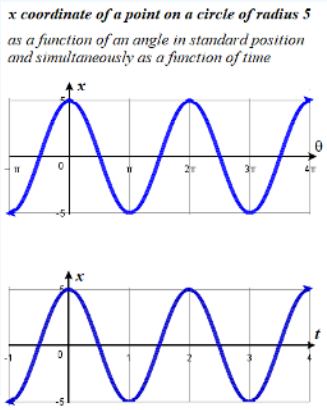
To answer the Ferris wheel problem at the beginning of the section, we need to be able to express our sine and cosine functions at inputs of time. The basic sinusoidal functions are functions of an angle instead of time. One approach would be to use composition. For example, write one sinusoidal equation for vertical height, h, as a function of an angle and another equation for an angle, a as a function of time. The composition $h(a(t))$ is a sinusoidal function of time. Here, instead, we observe the equivalency of two graphs that are identical except for the units on the x axis.

Example 5.5.11

A point completes 1 revolution every 2 minutes around a circle of radius 5. Find the x coordinate of the point as a function of time, if it starts at (5, 0).

Solution

For a point moving on a rotating circle of radius r, the x-coordinate of the point is $x = r\cos(\theta)$ so $x = 5\cos(\theta)$. The amplitude is $|A| = 5$.



We are told the starting point is at a maximum point, so this is a cosine function.

Simultaneously, from a different perspective of time rather than rotational angle, the same motion completes one revolution every $t = 2$ minutes. A cosine function with the same amplitude but a period $P = 2$ has equation parameter B defined by the formula $P = \frac{2\pi}{B}$. So $B = \pi$ and the equation in terms of time is $x = 5\cos(\pi t)$.

Example 5.5.12: Determining a Rider's Height on a Ferris Wheel

The London Eye is a huge Ferris wheel with a diameter of 135 meters (443 feet). It completes one rotation every 30 minutes. Riders board from a platform 2 meters above the ground. Express a rider's height above ground as a function of time in minutes.

Solution

With a diameter of 135 m, the wheel has a radius of 67.5 m. The height will oscillate with amplitude 67.5 m above and below the center.

Passengers board 2 m above ground level, so the center of the wheel must be located $67.5 + 2 = 69.5$ m above ground level. The midline of the oscillation will be at 69.5 m.

The wheel takes 30 minutes to complete 1 revolution, so the height will oscillate with a period of 30 minutes.

Lastly, because the rider boards at the lowest point, the height will start at the smallest value and increase, following the shape of a vertically reflected cosine curve.

- Amplitude: 67.5, so $A = 67.5$
- Midline: 69.5, so $D = 69.5$
- Period: 30, so $B = \frac{2\pi}{30} = \frac{\pi}{15}$
- Shape: $-\cos(t)$

An equation for the rider's height would be

$$y = -67.5\cos\left(\frac{\pi}{15}t\right) + 69.5$$

where t is in minutes and y is measured in meters.

Try It 5.5.12

The Ferris wheel at the Puyallup Fair (Photo by photogirl7.1, www.flickr.com/photos/kitkaph...86205/sizes/z/, CC-BY) has a diameter of about 70 feet and takes 3 minutes to complete a full rotation. Passengers board from a platform 10 feet above the ground. Write an equation for a rider's height above ground over time.



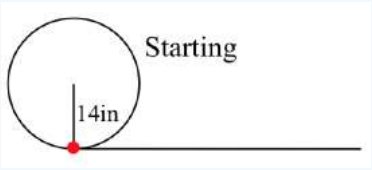
Answer

$$h(t) = -35\cos\left(\frac{2\pi}{3}t\right) + 45$$

While it is common to compose sine or cosine with functions involving time, the composition can be done so that the input represents any reasonable quantity.

Example 5.5.13

A bicycle wheel with radius 14 inches has the bottom-most point on the wheel marked in red. The wheel then begins rolling down the street. Write a formula for the height above ground of the red point after the bicycle has travelled x inches.



Solution

For a point moving on a rotating circle of radius r, the height of the point above the ground is the y-coordinate of the point, $y = r\sin(\theta)$. Therefore $y = 14\cos(\theta)$, so the **amplitude is** $|A| = 14$. The height oscillates between 0 inches and 28 inches above ground, so the **vertical displacement is** $D = 14$.

We are told the starting point is at a minimum point, so this is a **reflected cosine function**.

Simultaneously, from a different perspective of distance d travelled rather than rotational angle, the same motion completes one revolution every $d = 28\pi$ inches. A cosine function with the same amplitude and vertical displacement, but a **period** $P = 28\pi$ has equation parameter B defined by the formula $P = \frac{2\pi}{B}$. So $B = \frac{\pi}{14}$ and the equation for the height of the point in terms of distance travelled is

$$y = -14\cos\left(\frac{\pi}{14}d\right) + 14).$$

Periodic Behaviour

We will now apply these ideas to problems involving periodic behavior.

Example 5.5.14: Modeling an Equation and Sketching a Sinusoidal Graph to Fit Criteria

The average monthly temperatures for a small town in Oregon are given in the table below. Find a sinusoidal function of the form $y = A\sin(Bt - C) + D$ that fits the data (round to the nearest tenth) and sketch the graph.

Table 5.5.2

Month	Temperature,^oF			Month	Temperature,^oF
January	42.5	May	58	September	64.5
February	44.5	June	63	October	55.5
March	48.5	July	68.5	November	46.5
April	52.5	August	69	December	43.5

Solution

Recall that amplitude is found using the formula $A = \frac{\text{largest value} - \text{smallest value}}{2}$. Thus, the amplitude is

$$|A| = \frac{69 - 42.5}{2} = 13.25$$

The data covers a period of 12 months, so $\frac{2\pi}{B} = 12$ which gives

$$B = \frac{2\pi}{12} = \frac{\pi}{6}.$$

The vertical shift is found using the equation $D = \frac{\text{highest value} + \text{lowest value}}{2}$. Thus, the vertical shift is

$$D = \frac{69 + 42.5}{2} = 55.8$$

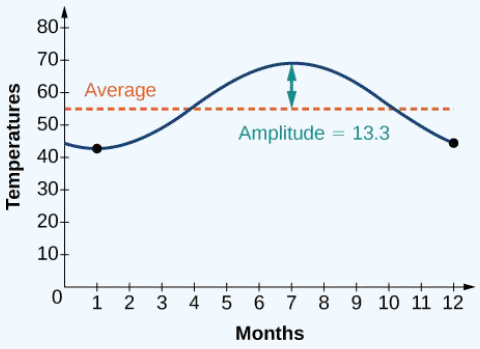


Figure 5.5.14

So far, we have the equation $y = 13.3\sin(\frac{\pi}{6}x - C) + 55.8$. To find the horizontal shift, we input the x and y values for the first month and solve for C.

$$\begin{aligned} 42.5 &= 13.3\sin(\tfrac{\pi}{6}(1) - C) + 55.8 \\ -13.3 &= 13.3\sin(\tfrac{\pi}{6} - C) \\ -1 &= \sin(\tfrac{\pi}{6} - C) \qquad \sin\theta = -1 \to \theta = -\tfrac{\pi}{2} \\ -\tfrac{\pi}{2} &= \tfrac{\pi}{6} - C \qquad C = \tfrac{\pi}{6} + \tfrac{\pi}{2} \to C = \tfrac{2\pi}{3} \end{aligned}$$

We have the equation $y = 13.3\sin(\frac{\pi}{6}x - \frac{2\pi}{3}) + 55.8$. See the graph in Figure 5.5.14.

Try It 5.5.15

A weight is attached to a spring that is then hung from a board, as shown in Figure 5.5.28. As the spring oscillates up and down, the position y of the weight relative to the board ranges from –1 in. (at time $t = 0$) to –7 in. (at time $t = \pi$) below the board. Assume the position of y is given as a sinusoidal function of time t. Sketch a graph of the function, and then find a function that gives the position y in terms of t.

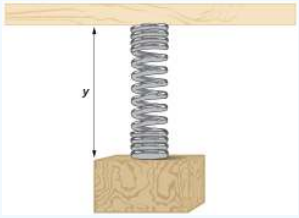
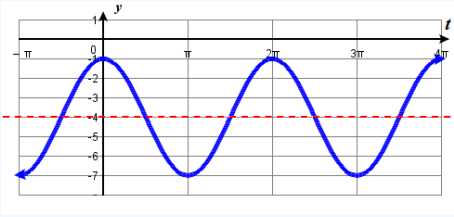


Figure 5.5.15

Answer



$y = 3\cos(x) - 4$

Key Equations

Sinusoidal functions	$f(x) = A\sin(Bx - C) + D$ $f(x) = A\cos(Bx - C) + D$

Key Concepts

- Periodic functions repeat after a given value. The smallest such value is the period. The basic sine and cosine functions have a period of 2π.
- The function $\sin x$ is odd, so its graph is symmetric about the origin. The function $\cos x$ is even, so its graph is symmetric about the y-axis.
- The graph of a sinusoidal function has the same general shape as a sine or cosine function.
- In the general formula for a sinusoidal function, the period is $P = \frac{2\pi}{|B|}$.

- In the general formula for a sinusoidal function, $|A|$ represents amplitude. If $|A| > 1$, the function is stretched, whereas if $|A| < 1$, the function is compressed.
- The value $\frac{C}{B}$ in the general formula for a sinusoidal function indicates the phase shift.
- The value D in the general formula for a sinusoidal function indicates the vertical shift from the midline.
- An equation for a sinusoidal function can be determined from a graph.
- A function can be graphed by identifying its amplitude, period, phase shift, and horizontal shift.
- Sinusoidal functions can be used to solve real-world problems.

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5.5e: Exercises - Graphs of Sine and Cosine Functions

A: Concepts

Exercise 5.5e.A

1. Why are the sine and cosine functions called periodic functions?

2. How does the graph of $y = \sin x$ compare with the graph of $y = \cos x$? Explain how you could horizontally translate the graph of $y = \sin x$ to obtain $y = \cos x$.

3. For the equation $A\cos(Bx + C) + D$, what constants affect the range of the function and how do they affect the range?

4. How does the range of a translated sine function relate to the equation $y = A\sin(Bx + C) + D$?

5. How can the unit circle be used to construct the graph of $f(t) = \sin t$?

Answers to odd-numbered problems:

1. The sine and cosine functions have the property that $f(x + P) = f(x)$ for a certain P. This means that the function values repeat for every P units on the x-axis.
3. The absolute value of the constant A (amplitude) increases the total range and the constant D (vertical shift) shifts the graph vertically.
5. At the point where the terminal side of t intersects the unit circle, you can determine that the $\sin t$ equals the y-coordinate of the point.

B1: Identify Attributes and Graph non-phase-shifted sine equations

Exercise 5.5e.B1

★ For the following exercises, (a) State the amplitude and midline, (b) the period, and (c) the maximum and minimum y-values and their corresponding x-values on one period and (d) Graph two full periods of each function. Give exact answers (fractions and multiples of pi, rather than decimal approximations).

11. $f(x) = \sin(5x) + 2$
12. $f(x) = 4\sin(2x) - 1$
13. $f(x) = 4\sin x$
14. $f(x) = 3\sin x + 2$
15. $f(x) = 2\sin(6\pi x) + 3$
16. $f(x) = 4\sin(3\pi x) + 1$
17. $f(x) = -3\sin x$
18. $f(x) = -2\sin(4x) - 2$

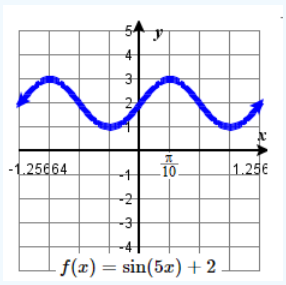
18.1 $f(x) = 2\sin(-4x) - 2$

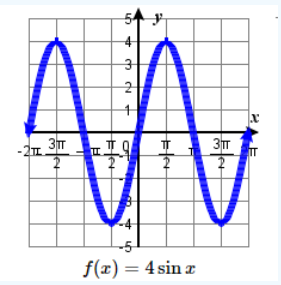
19. $f(x) = 3\sin\left(\frac{5}{4}x\right) + 1$
20. $f(x) = \sin\left(\frac{4}{5}x\right)$
21. $f(x) = 2\sin\left(\frac{1}{2}x\right)$
22. $f(x) = \sin\left(\frac{2}{5}x\right) - 4$
23. $f(x) = \sin\left(\frac{1}{6}x\right) + 3$
24. $f(x) = 4\sin\left(\frac{5\pi}{2}x\right)$
25. $f(x) = \sin\left(\frac{4\pi}{3}x\right) - 3$
26. $f(x) = 3\sin\left(\frac{3\pi}{4}x\right) - 1$
27. $f(x) = 3\sin\left(\frac{\pi}{2}x\right)$
28. $f(x) = 2\sin\left(\frac{\pi}{5}x\right) - 1$

Answers to odd numbered problems

11. amplitude: 1; period: $\frac{2\pi}{5}$ midline: $y = 2$;
maximum: $y = 3$ occurs at $x = \frac{1\pi}{10}$;
minimum: $y = 1$ occurs at $x = \frac{3\pi}{10}$;
one period starts at 0 and ends at $\frac{2\pi}{5}$.

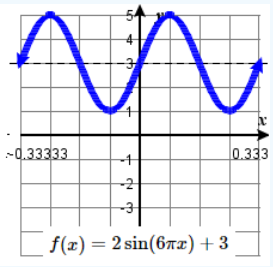
13. amplitude: 4; period: 2π midline: $y = 0$;
maximum: $y = 4$ occurs at $x = \frac{1\pi}{2}$;



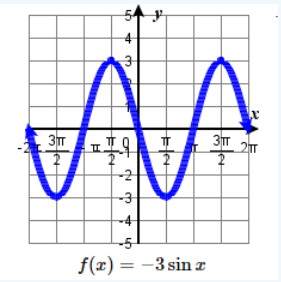


minimum: $y = -4$ occurs at $x = \frac{3\pi}{2}$;
one period starts at 0 and ends at 2π.

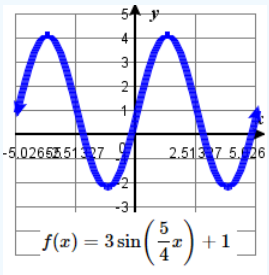
15. amplitude: 2; period: $\frac{1}{3}$ midline: $y = 3$;
maximum: $y = 5$ occurs at $x = \frac{1}{12}$;
minimum: $y = 1$ occurs at $x = \frac{1}{4}$;
one period starts at 0 and ends at $\frac{1}{3}$.



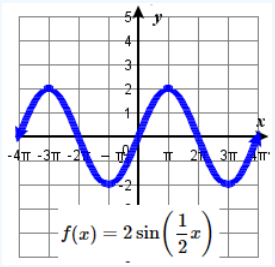
17. amplitude: 3; period: 2π midline: $y = 0$;
maximum: $y = 3$ occurs at $x = \frac{3\pi}{2}$;
minimum: $y = -3$ occurs at $x = \frac{\pi}{2}$;
one period starts at 0 and ends at 2π.



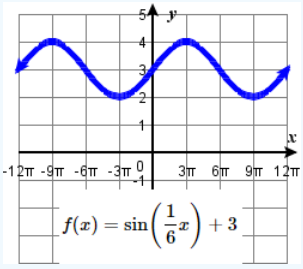
19. amplitude: 3; period: $\frac{8\pi}{5}$ midline: $y = 1$;
maximum: $y = 4$ occurs at $x = \frac{2\pi}{5}$;
minimum: $y = -2$ occurs at $x = \frac{6\pi}{5}$;
one period starts at 0 and ends at $\frac{8\pi}{5}$.



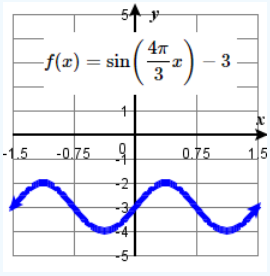
21. amplitude: 2; period: 4π midline: $y = 0$;
maximum: $y = 2$ occurs at $x = \pi$;
minimum: $y = -2$ occurs at $x = 3\pi$;
one period starts at 0 and ends at 4π.



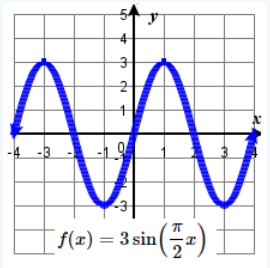
23. amplitude: 1; period: 12π midline: $y = 3$;
maximum: $y = 4$ occurs at $x = 3\pi$;
minimum: $y = 2$ occurs at $x = 9\pi$;
one period starts at 0 and ends at 12π.



25. amplitude: 1; period: $\frac{3}{2}$ midline: $y = -3$;
maximum: $y = -2$ occurs at $x = \frac{3}{8}$;
minimum: $y = -4$ occurs at $x = \frac{9}{8}$;
one period starts at 0 and ends at $\frac{3}{2}$.



27. amplitude: 3; period: 4 midline: $y = 0$;
maximum: $y = 3$ occurs at $x = 1$;
minimum: $y = -3$ occurs at $x = 3$;
one period starts at 0 and ends at 4.



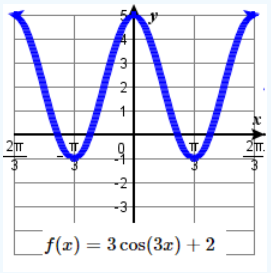
Exercise $5.5e.B2$

★ **For the following exercises, (a) State the amplitude and midline, (b) the period, and (c) the maximum and minimum y-values and their corresponding x-values on one period and (d) Graph two full periods of each function. Give exact answers (fractions and multiples of pi, rather than decimal approximations).**

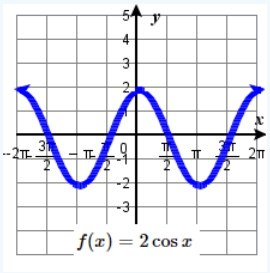
31. $f(x) = 3\cos(3x) + 2$
32. $f(x) = \cos(2x)$
33. $f(x) = 2\cos x$
34. $f(x) = -3\cos x + 1$
35. $f(x) = 2\cos(\pi x) - 3$
36. $f(x) = \cos(4\pi x)$
37. $f(x) = -\cos(2\pi x) - 3$
38. $f(x) = -\cos(6x) + 3$
38.1 $f(x) = \cos(-6x) + 3$
39. $f(x) = 3\cos\left(\frac{6}{5}x\right)$
40. $f(x) = \cos\left(\frac{3}{2}x\right) - 2$
41. $f(x) = \cos\left(\frac{5}{6}x\right) + 1$
42. $f(x) = 2\cos\left(\frac{3}{5}x\right)$
43. $f(x) = 4\cos\left(\frac{1}{3}x\right) + 1$
44. $f(x) = 2\cos\left(\frac{1}{4}x\right) - 2$
45. $f(x) = 5\cos\left(\frac{5\pi}{3}x\right)$
46. $f(x) = 2\cos\left(\frac{6\pi}{5}x\right) + 2$
47. $f(x) = 2\cos\left(\frac{4\pi}{5}x\right) + 3$
48. $f(x) = \cos\left(\frac{3\pi}{4}x\right) + 2$
49. $f(x) = \cos\left(\frac{2\pi}{3}x\right) - 1$

Answers to odd numbered problems:

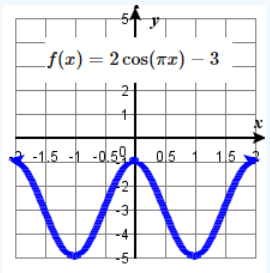
31. amplitude: 3; period: $\frac{2\pi}{3}$ midline: $y = 2$;
maximum: $y = 5$ occurs at $x = 0$;
minimum: $y = -1$ occurs at $x = \frac{1\pi}{3}$;
one period starts at 0 and ends at $\frac{2\pi}{3}$.



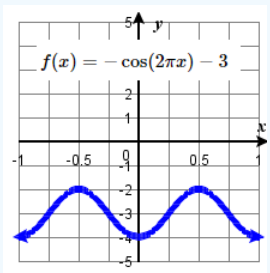
33. amplitude: 2; period: 2π midline: $y = 0$;
maximum: $y = 2$ occurs at $x = 0$;
minimum: $y = -2$ occurs at $x = \pi$;
one period starts at 0 and ends at 2π.



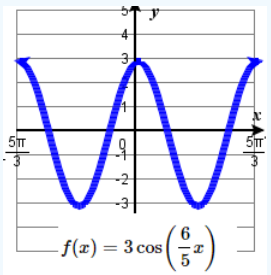
35. amplitude: 2; period: 2 midline: $y = -3$;
maximum: $y = -1$ occurs at $x = 0$;
minimum: $y = -5$ occurs at $x = 1$;
one period starts at 0 and ends at 2.



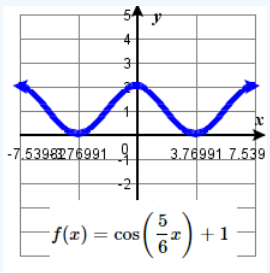
37. amplitude: 1; period: 1 midline: $y = -3$;
maximum: $y = -2$ occurs at $x = \frac{1}{2}$;
minimum: $y = -4$ occurs at $x = 0$;
one period starts at 0 and ends at 1.



39. amplitude: 3; period: $\frac{5\pi}{3}$ midline: $y = 0$;
maximum: $y = 3$ occurs at $x = 0$;
minimum: $y = -3$ occurs at $x = \frac{5\pi}{6}$;
one period starts at 0 and ends at $\frac{5\pi}{3}$.

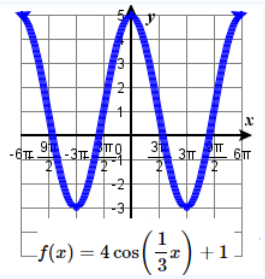


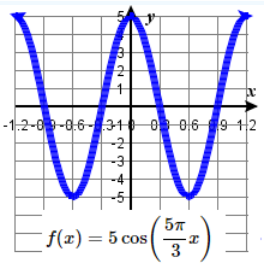
41. amplitude: 1; period: $\frac{12\pi}{5}$ midline: $y = 1$;
maximum: $y = 2$ occurs at $x = 0$;
minimum: $y = 0$ occurs at $x = \frac{6\pi}{5}$;
one period starts at 0 and ends at $\frac{12\pi}{5}$.



43. amplitude: 4; period: 6π midline: $y = 1$;
maximum: $y = 5$ occurs at $x = 0$;
minimum: $y = -3$ occurs at $x = 3\pi$;
one period starts at 0 and ends at 6π.

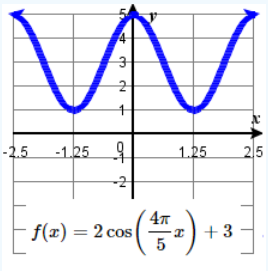
45. amplitude: 5; period: $\frac{6}{5}$ midline: $y = 0$;
maximum: $y = 5$ occurs at $x = 0$;
minimum: $y = -5$ occurs at $x = \frac{3}{5}$;
one period starts at 0 and ends at $\frac{6}{5}$.

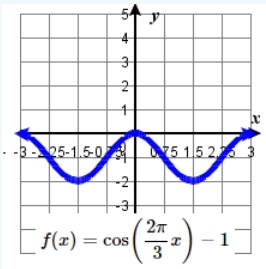




47. amplitude: 2; period: $\frac{5}{2}$ midline: $y = 3$;
maximum: $y = 5$ occurs at $x = 0$;
minimum: $y = 1$ occurs at $x = \frac{5}{4}$;
one period starts at 0 and ends at $\frac{5}{2}$.

49. amplitude: 1; period: 3 midline: $y = -1$;
maximum: $y = 0$ occurs at $x = 0$;
minimum: $y = -2$ occurs at $x = \frac{3}{2}$;
one period starts at 0 and ends at 3.





★

C: Find attributes of sinusoidal functions

Exercise 5.5e.C

★ For each of the following equations, find the amplitude, period, horizontal shift, and midline.

51. $y = 3\sin(8(x+4)) + 5$	53. $y = 2\sin(3x - 21) + 4$	55. $y = \sin\left(\frac{\pi}{6}x + \pi\right) - 3$
52. $y = 4\sin\left(\frac{\pi}{2}(x-3)\right) + 7$	54. $y = 5\sin(5x + 20) - 2$	56. $y = 8\sin\left(\frac{7\pi}{6}x + \frac{7\pi}{2}\right) + 6$

Answer

51. Amp: 3, Period = $\frac{\pi}{4}$, Shift: 4 left, Midline: $y = 5$
53. Amp: 2, Period = $\frac{2\pi}{3}$, Shift: 7 right, Midline: $y = 4$
55. Amp: 1, Period = 12, Shift: 6 left, Midline: $y = -3$

D: Identify Attributes and Graph Phase-Shifted Sinusoidal Functions

Exercise 5.5e.D

★ For the following exercises, state (a) the amplitude, vertical translation and midline, (b) the period, (c) the phase shift, and (d) the maximum and minimum y-values and their corresponding x-values on one period. Give exact answers (fractions and multiples of pi, rather than decimal approximations). (e) Graph two full periods of each function.

61) $y = 3\sin\left(\frac{\pi}{4}x + \frac{\pi}{8}\right) + 2$

62) $y = 2\sin\left(\frac{\pi}{3}x + \frac{\pi}{6}\right) - 2$

63) $y = -2\sin\left(\frac{\pi}{2}x + \frac{\pi}{4}\right) + 1$

64) $y = -3\sin\left(\pi x + \frac{\pi}{4}\right) + 1$

65) $y = 4\sin\left(\pi x - \frac{\pi}{3}\right) - 1$

66) $y = 2\sin\left(\frac{2\pi}{3}x - \frac{\pi}{3}\right) - 3$

67) $y = 3\sin\left(\frac{5}{3}x - \frac{\pi}{6}\right) - 2$

68) $y = \sin\left(\frac{1}{6}x - \frac{\pi}{6}\right) - 2$

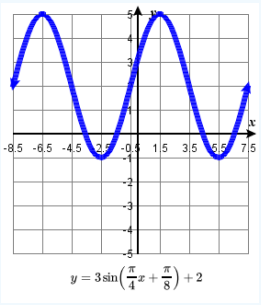
69) $y = 2\sin\left(-\frac{4}{3}x + \frac{\pi}{3}\right) + 1$

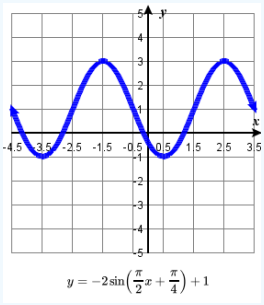
70) $y = 4\sin\left(-\frac{5}{3}x - \frac{\pi}{3}\right) - 1$

71) $y = 4\sin\left(\frac{5}{4}x - \frac{\pi}{2}\right) + 1$

Answers to Odd Problems

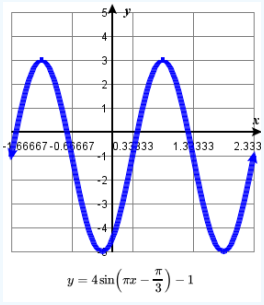
61. amplitude: 3; period: 8; midline: $y = 2$;
vertical translation: up 2; horizontal shift: left $\frac{1}{2}$;
one period occurs from $x = -\frac{1}{2}$ to $x = \frac{15}{2}$
maximum: $y = 5$ occurs at $x = \frac{3}{2}$;
minimum: $y = -1$ occurs at $x = \frac{11}{2}$;

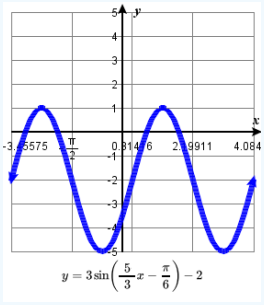




63. amplitude: 2; period: 4; midline: $y = 1$;
vertical translation: up 1; horizontal shift: left $\frac{1}{2}$;
one period occurs from $x = -\frac{1}{2}$ to $x = \frac{7}{2}$
maximum: $y = 3$ occurs at $x = \frac{5}{2}$;
minimum: $y = -1$ occurs at $x = \frac{1}{2}$;

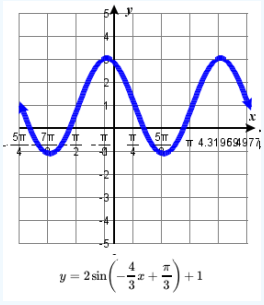
65. amplitude: 4; period: \(2 \); midline: $y = -1$;
vertical translation: down 1; horizontal shift: right $\frac{1}{3}$;
one period occurs from $x = \frac{1}{3}$ to $x = \frac{7}{3}$
maximum: $y = 3$ occurs at $x = \frac{5}{6}$;
minimum: $y = -5$ occurs at $x = \frac{11}{6}$;

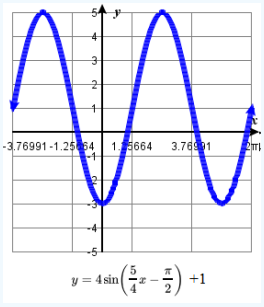




67. amplitude: 3; period: $\frac{6\pi}{5}$; midline: $y = -2$;
vertical translation: down 2; horizontal shift: right $\frac{\pi}{10}$;
one period occurs from $x = \frac{\pi}{10}$ to $x = \frac{13\pi}{10}$
maximum: $y = 1$ occurs at $x = \frac{2\pi}{5}$;
minimum: $y = -5$ occurs at $x = \pi$;

69. amplitude: 2; period: $\frac{3\pi}{2}$; midline: $y = 1$;
vertical translation: up 1; horizontal shift: right $\frac{\pi}{4}$;
one period occurs from $x = \frac{\pi}{4}$ to $x = \frac{7\pi}{4}$
maximum: $y = 3$ occurs at $x = \frac{11\pi}{8}$;
minimum: $y = -1$ occurs at $x = \frac{5\pi}{8}$;





71. amplitude: 4; period: $\frac{8\pi}{5}$; midline: $y = 1$;
vertical translation: none; horizontal shift: right $\frac{2\pi}{5}$;
one period occurs from $x = \frac{2\pi}{5}$ to $x = 2\pi$
maximum: $y = 5$ occurs at $x = \frac{4\pi}{5}$;
minimum: $y = -3$ occurs at $x = \frac{8\pi}{5}$;

★ **For the following exercises, state (a) the amplitude, vertical translation and midline, (b) the period, (c) the phase shift, and (d) the maximum and minimum y-values and their corresponding x-values on one period. Give exact answers (fractions and multiples of pi, rather than decimal approximations). (e) Graph two full periods of each function.**

75) $y=\cos\left(\frac{\pi}{4}x-\frac{\pi}{4}\right)+3$

76) $y=\cos\left(\frac{\pi}{6}x-\frac{\pi}{3}\right)-1$

77) $y=-\cos\left(\frac{5\pi}{2}x-\frac{\pi}{4}\right)+2$

78) $y=-3\cos\left(\frac{5\pi}{6}x-\frac{\pi}{6}\right)-1$

79) $y=4\cos\left(\frac{3\pi}{2}x+\frac{\pi}{4}\right)+1$

80) $y=2\cos\left(\frac{\pi}{2}x+\frac{\pi}{3}\right)+3$

81) $y=2\cos\left(\frac{1}{3}x+\frac{\pi}{3}\right)-1$

82) $y=2\cos\left(\frac{1}{2}x+\frac{\pi}{6}\right)+2$

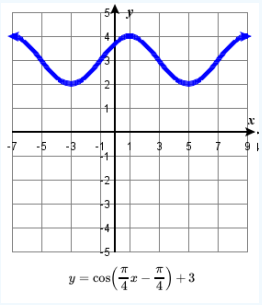
83) 97) $y=\cos\left(-2x+\frac{\pi}{4}\right)-3$

84) $y=3\cos\left(-\frac{5}{6}x-\frac{\pi}{3}\right)$

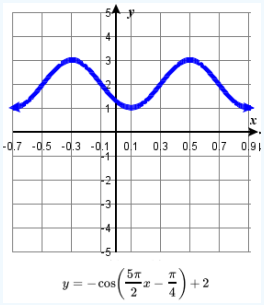
85) $y=\cos\left(\frac{2}{3}x-\frac{\pi}{6}\right)+4$

Answers to odd-numbered problems:

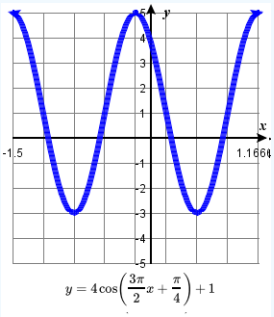
75. amplitude: 1; period: 8; midline: $y=3$;
vertical translation: up 3; horizontal shift: right 1;
one period occurs from $x=1$ to $x=9$
maximum: $y=4$ occurs at $x=1$;
minimum: $y=2$ occurs at $x=5$;



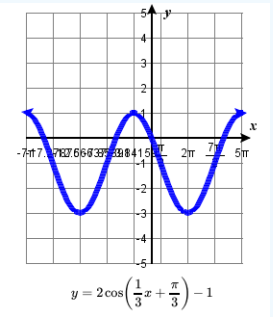
77. amplitude: 1; period: $\frac{4}{5}$; midline: $y=2$;
vertical translation: up 2; horizontal shift: right $\frac{1}{10}$;
one period occurs from $x=\frac{1}{10}$ to $x=\frac{9}{10}$
maximum: $y=3$ occurs at $x=\frac{1}{2}$;
minimum: $y=1$ occurs at $x=\frac{1}{10}$;



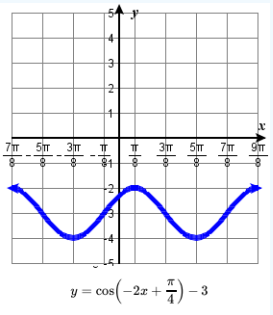
79. amplitude: 4; period: $\frac{4}{3}$; midline: $y=1$;
vertical translation: up 1; horizontal shift: left $\frac{1}{6}$;
one period occurs from $x=-\frac{1}{6}$ to $x=\frac{7}{6}$
maximum: $y=5$ occurs at $x=-\frac{1}{6}$;
minimum: $y=-3$ occurs at $x=\frac{1}{2}$;



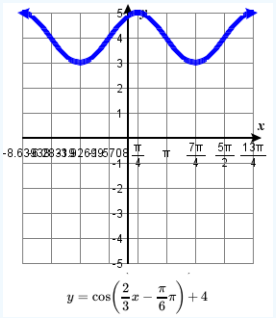
81. amplitude: 2; period: 6π; midline: $y=-1$;
vertical translation: down 1 ; horizontal shift: left π;
one period occurs from $x=-\pi$ to $x=5\pi$
maximum: $y=1$ occurs at $x=-\pi$;
minimum: $y=-3$ occurs at $x=2\pi$;



83. amplitude: 1; period: π; midline: $y=-3$;
vertical translation: down 3; horizontal shift: right $\frac{\pi}{8}$;
one period occurs from $x=\frac{\pi}{8}$ to $x=\frac{9\pi}{8}$
maximum: $y=-2$ occurs at $x=\frac{\pi}{8}$;
minimum: $y=-4$ occurs at $x=\frac{5\pi}{8}$;



85. amplitude: 1; period: 3π; midline: $y=4$;
vertical translation: up 4; horizontal shift: right $\frac{\pi}{4}$;
one period occurs from $x=\frac{\pi}{4}$ to $x=\frac{13\pi}{4}$
maximum: $y=5$ occurs at $x=\frac{\pi}{4}$;
minimum: $y=3$ occurs at $x=\frac{7\pi}{4}$;



★ For the following exercises, state (a) the amplitude, vertical translation and midline, (b) the period, (c) the phase shift, and (d) the maximum and minimum y-values and their corresponding x-values on one period. Give exact answers (fractions and multiples of pi, rather than decimal approximations). (e) Graph two full periods of each function.

90. $f(t) = 2\sin\left(t - \frac{5\pi}{6}\right)$

91. $f(t) = -\sin\left(\frac{1}{2}t + \frac{5\pi}{3}\right)$

92. $f(x) = 4\sin\left(\frac{\pi}{2}(x-3)\right) + 7$

95. $f(t) = -\cos\left(t + \frac{\pi}{3}\right) + 1$

96. $f(t) = 4\cos\left(2\left(t + \frac{\pi}{4}\right)\right) - 3$

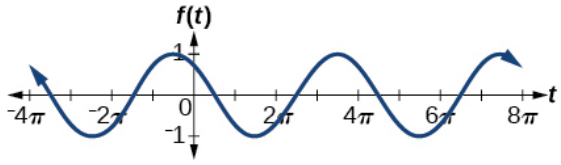
97. $y = 3\sin(8(x+4)) + 5$

98. $y = 2\sin(3x - 21) + 4$

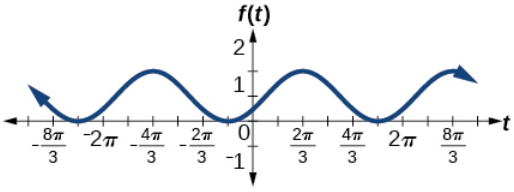
99. $y = 5\sin(5x + 20) - 2$

Answers to odd-numbered problems:

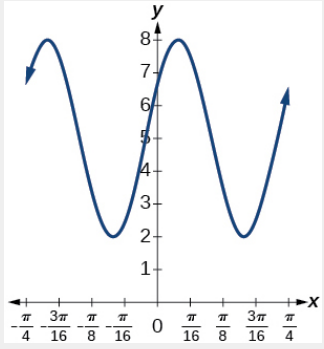
91.
amplitude: 1; period: 4π; midline: $y = 0$;
maximum: $y = 1$ occurs at $t = \frac{11\pi}{3} \approx 11.52$;
minimum: $y = -1$ occurs at $t = \frac{5\pi}{3} \approx 5.24$;
phase shift: $-\frac{10\pi}{3}$; vertical shift: 0;
one full period is from $t = \frac{2\pi}{3} \approx 2.094$ to $t = \frac{14\pi}{3} \approx 14.661$



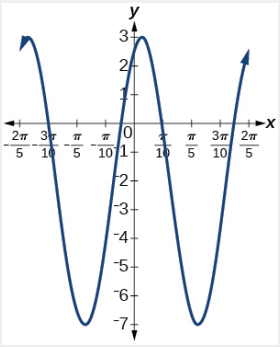
95.
amplitude: 1; period: 2π; midline: $y = 1$;
maximum: $y = 2$ occurs at $t = \frac{2\pi}{3} \approx 2.094$;
minimum: $y = 0$ occurs at $t = \frac{5\pi}{3} \approx 5.24$;
phase shift: $-\frac{\pi}{3}$; vertical translation: 1;
one full period is from $t = \frac{2\pi}{3} \approx 2.094$ to $t = \frac{8\pi}{3} \approx 8.378$



97.
amplitude: 3; period: $\frac{\pi}{4}$; midline: $y = 5$;
maximum: $y = 8$ occurs at $x = -4 + \frac{21\pi}{16} \approx 0.123$;
minimum: $y = 2$ occurs at $x = -4 + \frac{23\pi}{16} \approx 0.516$;
horizontal shift: -4; vertical translation 5;
one period occurs from $x = -4 + \frac{22\pi}{16} \approx 0.320$
to $x = -4 + \frac{26\pi}{16} \approx 1.105$



99.
amplitude: 5; period: $\frac{2\pi}{5}$; midline: $y = -2$;
maximum: $y = 3$ occurs at $x = -4 + \frac{13\pi}{10} \approx 0.084$;
minimum: $y = -7$ occurs at $x = -4 + \frac{15\pi}{10} \approx 0.712$;
phase shift: -4; vertical translation: -2;
one full period can be graphed on $x = -4 + \frac{7\pi}{5} \approx 0.398$
to $x = -4 + \frac{9\pi}{5} \approx 1.655$



E: Construct an equation for a non-phase-shifted sinusoidal graph

Exercise 5.5e.E

★ **For each non-phase shifted sine or cosine graph, determine the amplitude, midline, and period, then find a formula for the function.**

101.

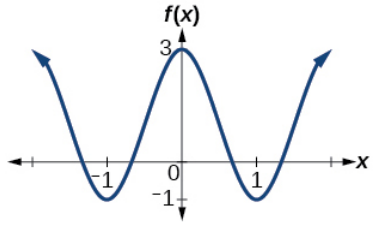


Figure 5.5e.101

102.

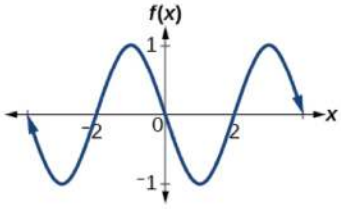


Figure 5.5e.102

103.

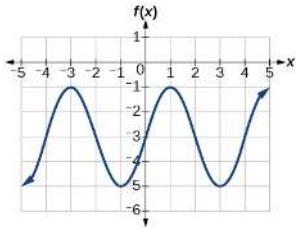


Figure 5.5e.103

104.

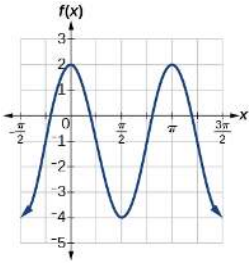


Figure 5.5e.104

105.

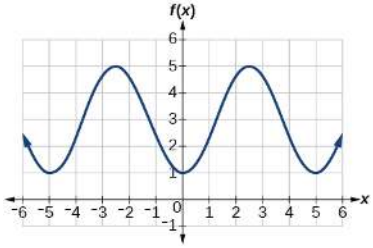


Figure 5.5e.105

106.

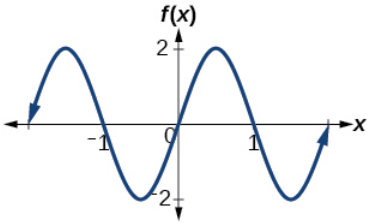
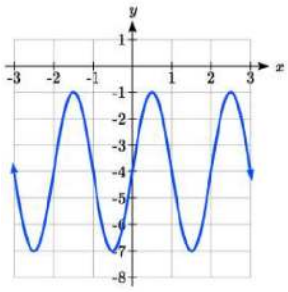
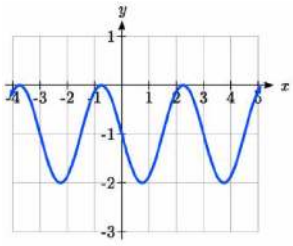


Figure 5.5e.106

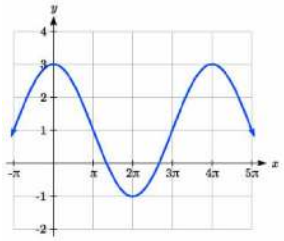
107.



108.



109.



Answers to odd-numbered problems:

101. amplitude: 2; period: 2; midline $y = 1$ equation: $f(x) = 2\cos(\pi x) + 1$

103. amplitude: 2; midline: $y = -3$ period: 4; equation: $f(x) = 2\sin\left(\frac{\pi}{2}x\right) - 3$

105. amplitude: 2; period: 5; midline: $y = 3$ equation: $f(x) = -2\cos\left(\frac{2\pi}{5}x\right) + 3$

107. Amp: 3. Period = 2. Midline: $y = -4$. $f(t) = 3\sin(\pi t) - 4$

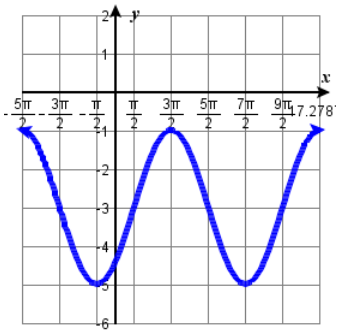
109. Amp: 2. Period = 4π. Midline: $y = 1$. $f(t) = 2\cos(\frac{1}{2}t) + 1$

F: Construct an equation for a phase-shifted sinusoidal graph

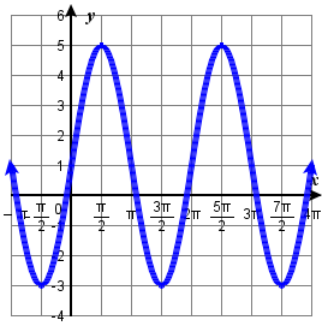
Exercise 5.5e.F

★ **Find a formula for each of the functions graphed below.**

111.



112.



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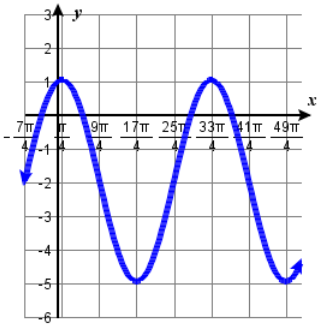

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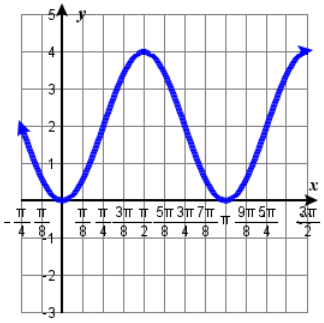

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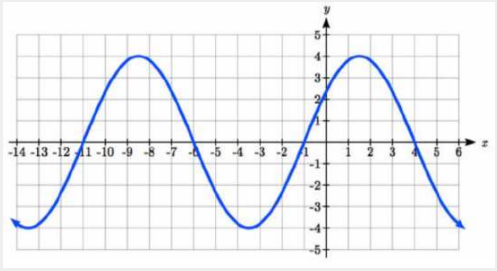

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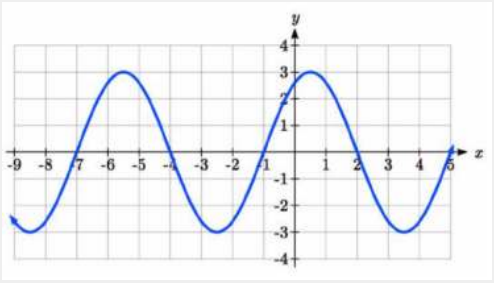

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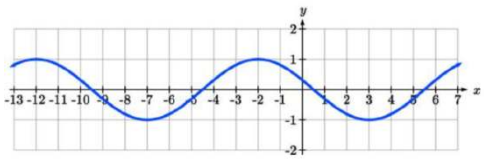

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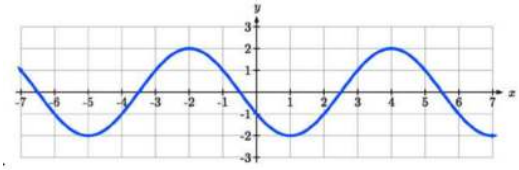

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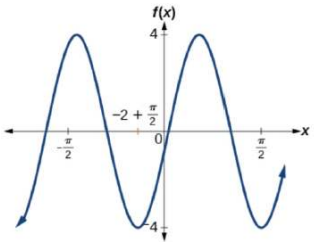

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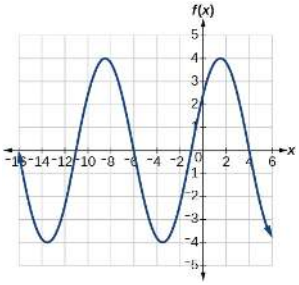


Answers to odd-numbered problems:

111. $f(x) = 2\sin(\frac{1}{2}x + \frac{\pi}{4})) - 3$ 113. $f(x) = 3\cos(\frac{1}{4}x - \frac{\pi}{16})) - 2$
115. $f(x) = 4\sin(\frac{\pi}{5}(x+1))$ 117. $f(x) = \cos(\frac{\pi}{5}(x+2))$
119. amplitude: 4; period: 2; midline: $y = 0$; equation: $f(x) = -4\cos\left(\pi\left(x - \frac{\pi}{2}\right)\right)$

G: Find an equation for an application using a phase-shifted sinusoidal graph

Exercise $5.5e.G$

Note: In addition to constructing a sinusoidal equation, some of these problems also have an additional part that requires solving an equation. Solving sinusoidal equations is covered later in the next chapter.

Temperature

121. Outside temperatures over the course of a day can be modeled as a sinusoidal function. Suppose the high temperature of $84\,°F$ occurs at 6PM and the average temperature for the day is $70\,°F$. (a) Construct an equation that models the daily temperature. (b) Find the temperature, to the nearest degree, at 7AM.

122. Outside temperatures over the course of a day can be modeled as a sinusoidal function. Suppose the high temperature of $105\,°F$ occurs at 5PM and the average temperature for the day is $85\,°F$. (a) Construct an equation that models the daily temperature. (b) Find the temperature, to the nearest degree, at 9AM.

123. Outside temperatures over the course of a day can be modeled as a sinusoidal function. Suppose the temperature varies between $64°F$ and $86°F$ during the day and the average daily temperature first occurs at 12 AM. (a) Construct an equation that models the daily temperature. (b) How many hours after midnight does the temperature first reach $70°F$?

124. Outside temperatures over the course of a day can be modeled as a sinusoidal function. Suppose the temperature varies between $47°F$ and $63°F$ during the day and the average daily temperature first occurs at 10 AM. (a) Construct an equation that models the daily temperature. (b) How many hours after midnight does the temperature first reach $51°F$?

125. Outside temperature over the course of a day can be modeled as a sinusoidal function. Suppose you know the temperature is 50 degrees at midnight and the high and low temperature during the day are 57 and 43 degrees, respectively. Assuming t is the number of hours since midnight, find a function for the temperature, D, in terms of t.

126. Outside temperature over the course of a day can be modeled as a sinusoidal function. Suppose you know the temperature is 68 degrees at midnight and the high and low temperature during the day are 80 and 56 degrees, respectively. Assuming t is the number of hours since midnight, find a function for the temperature, D, in terms of t.

Ferris Wheel

127. A Ferris wheel is 45 meters in diameter and boarded from a platform that is 1 meter above the ground. The six o'clock position on the Ferris wheel is level with the loading platform. The wheel completes 1 full revolution in 10 minutes. (a) Construct an equation that models the height above ground t minutes after the wheel begins to turn. (b) How many minutes of the ride are spent higher than 27 meters above the ground? Round to the nearest second.

128. A Ferris wheel is 20 meters in diameter and boarded from a platform that is 2 meters above the ground. The six o'clock position on the Ferris wheel is level with the loading platform. The wheel completes 1 full revolution in 6 minutes. (a) Construct an equation that models the height above ground t minutes after the wheel begins to turn. (b) How much of the ride, in minutes and seconds, is spent higher than 13 meters above the ground?

129. A Ferris wheel is 25 meters in diameter and boarded from a platform that is 1 meters above the ground. The six o'clock position on the Ferris wheel is level with the loading platform. The wheel completes 1 full revolution in 10 minutes. The function $h(t)$ gives your height in meters above the ground t minutes after the wheel begins to turn.

a. Find the amplitude, midline, and period of $h(t)$.
b. Find a formula for the height function $h(t)$.
c. How high are you off the ground after 5 minutes?

130. A Ferris wheel is 35 meters in diameter and boarded from a platform that is 3 meters above the ground. The six o'clock position on the Ferris wheel is level with the loading platform. The wheel completes 1 full revolution in 8 minutes. The function $h(t)$ gives your height in meters above the ground t minutes after the wheel begins to turn.

a. Find the amplitude, midline, and period of $h(t)$.
b. Find a formula for the height function $h(t)$.
c. How high are you off the ground after 4 minutes?

Ice melt

131. The sea ice area around the South Pole fluctuates sinusoidally between about 18 million square kilometers in September to 3 million square kilometers in March. (a) Construct an equation that models the sea ice area. (b) When are there more than 15 million square kilometers of sea ice? Give your answer as a range of dates, to the nearest day.

132. The sea ice area around the North Pole fluctuates sinusoidally between about 6 million square kilometers on September 1 to 14 million square kilometers on March 1. (a) Construct an equation that models the sea ice area. (b) When are there less than 9 million square kilometers of sea ice? Give your answer as a range of dates, to the nearest day.

Rainfall

133. During a 90-day monsoon season, daily rainfall can be modeled by sinusoidal functions. A low of 4 inches of rainfall was recorded on day 30, and overall the average daily rainfall was 8 inches. (a) Construct an equation that models the daily rainfall during a monsoon season. (b) During what period was daily rainfall less than 5 inches?

134. During a 90-day monsoon season, daily rainfall can be modeled by sinusoidal functions. The rainfall fluctuates between a low of 2 inches on day 10 and 12 inches on day 55. (a) Construct an equation that models the daily rainfall during a monsoon season. (b) During what period is daily rainfall more than 10 inches?

135. In a certain region, monthly precipitation peaks at 24 inches in September and falls to a low of 4 inches in March. (a) Construct an equation that models the monthly precipitation. (b) Identify the periods when the region is under flood conditions (greater than 22 inches) and drought conditions (less than 5 inches). Give your answer to the nearest day.

136. In a certain region, monthly precipitation peaks at 8 inches on June 1 and falls to a low of 1 inch on December 1. (a) Construct an equation that models the monthly precipitation. (b) Identify the periods when the region is under flood conditions (greater than 7 inches) and drought conditions (less than 2 inches). Give your answer to the nearest day.

More

137. The hour hand of the large clock on the wall in Union Station measures 24 inches in length. At noon, the tip of the hour hand is 30 inches from the ceiling. At 3 PM, the tip is 54 inches from the ceiling, and at 6 PM, 78 inches. At 9 PM, it is again 54 inches from the ceiling, and at midnight, the tip of the hour hand returns to its original position 30 inches from the ceiling. Let y equal the distance from the tip of the hour hand to the ceiling x hours after noon. Find the equation that models the motion of the clock and sketch the graph.

138. The height of the tide in a small beach town is measured along a seawall. Water levels oscillate between 7 feet at low tide and 15 feet at high tide. On a particular day, low tide occurred at 6 AM and high tide occurred at noon. Approximately every 12 hours, the cycle repeats. Find an equation to model the water levels.

139. The daily temperature in the month of March in a certain city varies from a low of $24°F$ at 6 AM to a high of $40°F$. (a) Let $t = 0$ correspond to noon. Find a sinusoidal function to model daily temperature and sketch the graph. (b) Approximate the time when the temperature reaches the freezing point $32°F$.

140. Blood pressure of $\frac{120}{80}$ is considered to be normal. The top number is the maximum or systolic reading, which measures the pressure in the arteries when the heart contracts. The bottom number is the minimum or diastolic reading, which measures the pressure in the arteries as the heart relaxes between beats, refilling with blood. Thus, normal blood pressure can be modeled by a periodic function with a maximum of 120 and a minimum of 80. In a blood pressure function, frequency represents the number of heart beats per minute. Frequency is the reciprocal of period and is 80 beats per minute for a normal heart

rate. Write a function that models the average person's blood pressure, where $f(t)$ represents the blood pressure at time t, measured in minutes.

Answers to odd exercises.

121. (a) $y = -14\sin(\frac{2\pi}{24}t) + 70$ (b) $56°F$
123. (a) $y = -11\sin(\frac{2\pi}{24}t) + 75$ (b) 1.8024 hours
125. $D(t) = 50 - 7\sin(\frac{\pi}{12}t)$
127. (a) $y = 22.5\sin(\frac{2\pi}{10}t + \frac{3\pi}{2}) + 23.5$ (b) $4:30$
129. (a). Amp: 12.5. Midline: $y = 13.5$. Period: 10 (b). $h(t) = -12.5\cos(\frac{\pi}{5}t) + 13.5$ or $h(t) = 12.5\sin\left(\frac{\pi}{5}(t-2.5)\right) + 13.5$; (c). $h(t) = 26$ meters
131. (a) $y = -7.5\sin(\frac{2\pi}{12}t) + 10.5$ (b) From July 8 to October 23
133. (a) $y = -5\cos(\frac{2\pi}{12}t) + 10.5$ (b) From day 19 through day 40
135. (a) $y = -10\sin(\frac{2\pi}{10}t) + 14$ (b) Floods: July 24 through October 7. Droughts: February 4 through March 27
137. $y = -24\cos(\frac{\pi}{6}x) + 54$ 138. $y = 4\cos(\frac{\pi}{6}t) + 11$
139. (a) $y = 8\sin(\frac{2\pi}{24}t) + 32$ (b) The temperature reaches freezing at noon and at midnight.
140. $f(t) = 20\sin(160\pi t) + 100$

Extension - cyclical behaviour with a linear or exponential growth rate

140.1 A rabbit population oscillates 15 above and below average during the year, reaching the lowest value in January. The average population starts at 650 rabbits and increases by 110 each year. Find a function that models the population, P, in terms of months since January, t.

Answer 140.1

$$P(t) = -15\cos\left(\frac{\pi}{6}t\right) + 650 + \frac{55}{6}t$$

140.2 A deer population oscillates 19 above and below average during the year, reaching the lowest value in January. The average population starts at 800 deer and increases by 160 each year. Find a function that models the population, P, in terms of months since January, t.

140.3 A fish population oscillates 40 above and below average during the year, reaching the lowest value in January. The average population starts at 800 fish and increases by 4% each month. Find a function that models the population, P, in terms of months since January, t.

Answer 140.3

$$P(t) = -40\cos\left(\frac{\pi}{6}t\right) + 800(1.04)^t$$

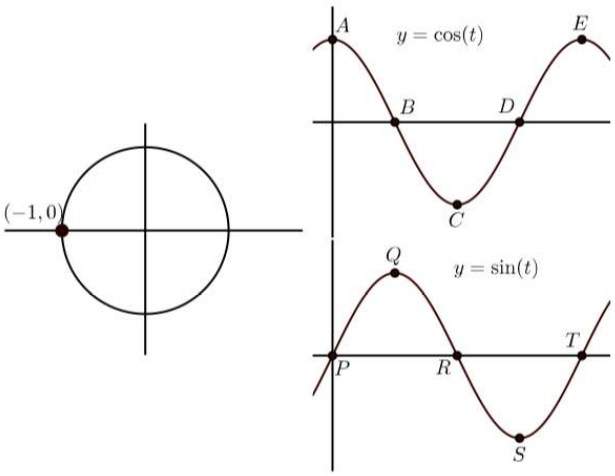
140.4 A muskrat population oscillates 33 above and below average during the year, reaching the lowest value in January. The average population starts at 900 muskrats and increases by 7% each month. Find a function that models the population, P, in terms of months since January, t.

H: Unit Circle concepts

Exercise 5.5e.H

★ In each of the following, the graph on the left shows the terminal point of an arc t (with $0 \le t \le 2\pi$) on the unit circle. The graphs on the right show the graphs of $y = \cos(t)$ and $y = \sin(t)$ with some points on the graph labeled. Match the point on the graphs of $y = \cos(t)$ and $y = \sin(t)$ that correspond to the point on the unit circle. In addition, state the coordinates of the points on $y = \cos(t)$ and $y = \sin(t)$.

141.



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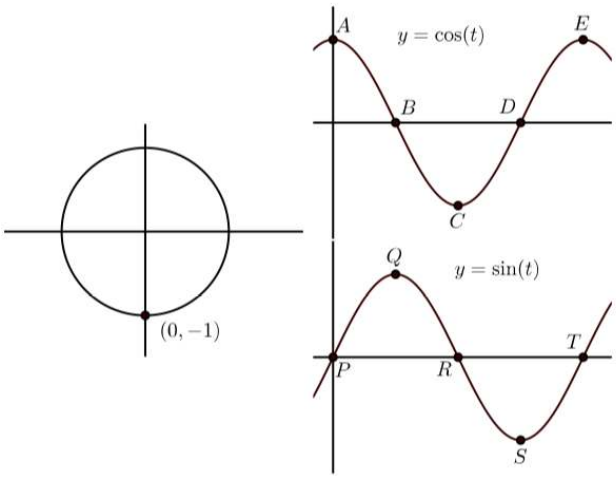

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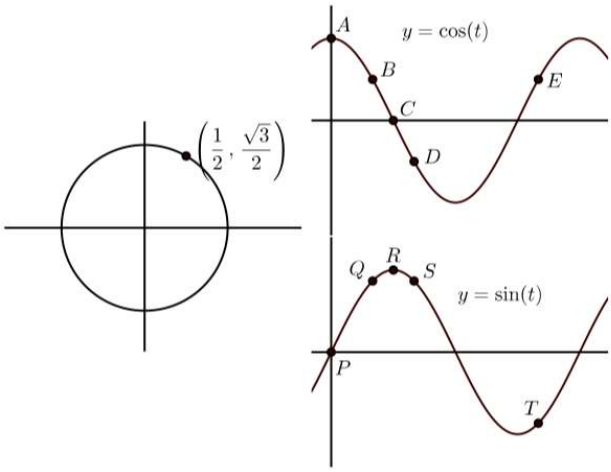

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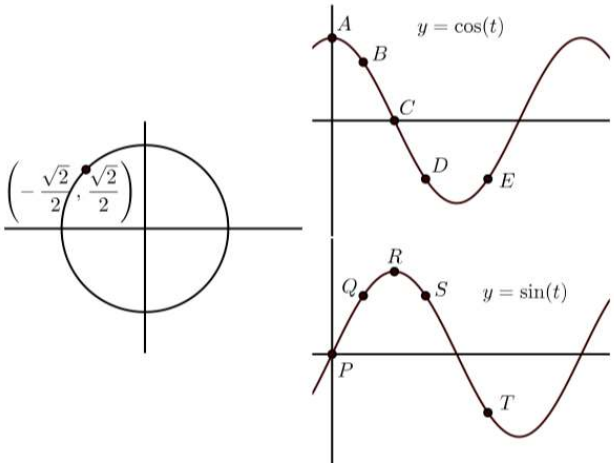

Answers to odd exercises.

141. $C(\pi, -1)$ $R(\pi, 0)$ 143. $B(\frac{\pi}{3}, \frac{1}{2})$ $Q(\frac{\pi}{3}, \frac{\sqrt{3}}{2})$

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5.6: Graphs of the Other Trigonometric Functions

Learning Objectives

- Graph $y = \tan x$, $y = \cot x$, $y = \sec x$, and $y = \csc x$, and transformations of these functions.
- Construct an equation from a graph of a transformation of $y = \tan x$.

We know the tangent function can be used to find distances, such as the height of a building, mountain, or flagpole. But what if we want to measure repeated occurrences of distance? Imagine, for example, a police car parked next to a warehouse. The rotating light from the police car would travel across the wall of the warehouse in regular intervals. If the input is time, the output would be the distance the beam of light travels. The beam of light would repeat the distance at regular intervals. The tangent function can be used to approximate this distance. Asymptotes would be needed to illustrate the repeated cycles when the beam runs parallel to the wall because, seemingly, the beam of light could appear to extend forever. The graph of the tangent function would clearly illustrate the repeated intervals. In this section, we will explore the graphs of the tangent and other trigonometric functions.

Analyzing the Graph of $y = \tan x$

We can determine whether tangent is an odd or even function by using the definition of tangent and the Negative Angle Properties for sine and cosine.

$$\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin x}{\cos x} = -\frac{\sin x}{\cos x} = -\tan x$$

Therefore, tangent is an odd function. We can further analyze the graphical behavior of the tangent function by looking at values for some of the special angles, as listed in Table 5.6.1.

Table 5.6.1

x	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\tan x$	undefined	-1	0	1	undefined	-1	0	1	undefined	-1	0

These points will help us draw our graph, but we need to determine how the graph behaves where it is undefined. If we look more closely at values when when $-\frac{\pi}{2} < x < -\frac{\pi}{4}$, and when $\frac{\pi}{4} < x < \frac{\pi}{2}$, we can use a table to look for a trend. Because $\frac{\pi}{4} \approx 0.79$ and $\frac{\pi}{2} \approx 1.57$, we choose values of x in Table 5.6.2 below, for which $-1.57 < x < -0.79$ and $0.79 < x < 1.57$.

Table 5.6.2

−1.56	−1.55	−1.5	−1.3	-1	x	1	1.3	1.5	1.55	1.56
−92.6	−48.1	−14.1	−3.6	-1.6	$\tan x$	1.6	3.6	14.1	48.1	92.6

We can see that, as x approaches $\frac{\pi}{2}$ or $-\frac{\pi}{2}$, the magnitude of the outputs of the function grow without bound, so the graph of the tangent function has vertical asymptotes at these values. This behaviour is a result of the fact that $\tan(x) = \frac{\sin(x)}{\cos(x)}$, and the cosine function can be zero. At every value of x for which $\cos(x) = 0$, the tangent function is undefined, and the graph of the tangent function is discontinuous.

The period of the tangent function is π because the graph repeats itself on intervals of $k\pi$ where k is an integer. Examining the graph of the tangent function on $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, we can see the behavior of the graph completes one cycle. If we look at any larger interval, we will see that the characteristics of the graph repeat.

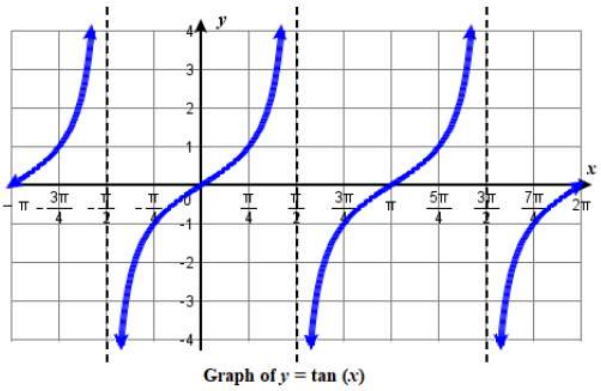


Figure 5.6.1: Graph of the tangent function

Graph a Stretched or Compressed Tangent Function

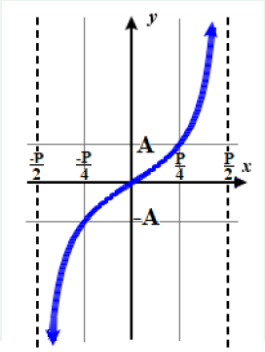
As with the sine and cosine functions, the *tangent function* can be described by a general equation: $\boxed{y = A\tan(Bx)}$

We can identify horizontal and vertical stretches and compressions using values of A and B. The horizontal stretch can typically be determined from the period of the graph. With tangent graphs, it is often necessary to determine a vertical stretch using a point on the graph.

Because there are no maximum or minimum values of a tangent function, the term *amplitude* cannot be interpreted as it is for the sine and cosine functions. Instead, we will use the phrase *stretching/compressing factor* when referring to the constant A.

FEATURES OF THE GRAPH OF $y = A\tan(Bx)$

- The **vertical stretching factor** is $|A|$.
- The **period** is $P = \frac{\pi}{|B|}$.
- The **asymptotes** occur at $x = \frac{P}{2} + P \cdot k$ where k is any integer and P is the period.
- The **domain** is all real numbers x, where $\{x | x \neq \frac{P}{2} + P \cdot k\}$ where k is any integer and P is the period.
- The **range** is $(-\infty, \infty)$.
- $y = A\tan(Bx)$ is **an odd function**.



One cycle of $y = A\tan(Bx)$
Period, $P = \frac{\pi}{B}$

Figure 5.6.2a: Generic Graph for $y = A\tan(Bx)$, with A and B both positive (or both negative).

These results can be confirmed by examining the start of a cycle of $f(x) = A\tan(Bx)$ and relating it to the behaviour of the parent function $y = \tan(x)$. A cycle for f starts when its argument $Bx = -\frac{\pi}{2}$ and ends when $Bx = \frac{\pi}{2}$. Solving these equations we obtain the cycle for f starts when $x = \frac{-\pi}{2B}$, ends when $x = \frac{\pi}{2B}$, and consequently has a length (known as the period) of $\frac{\pi}{B}$. The vertical stretch factor is A by what we know about transformations. When the argument of f, Bx, is $\frac{\pi}{4}$, the value of f is $A\tan\frac{\pi}{4} = A$, whereas the value of the parent function with that same argument $\frac{\pi}{4}$, has a value of just 1.

We can use what we know about the properties of the **tangent function** to quickly sketch a graph of any stretched and/or compressed tangent function of the form $f(x) = A\tan(Bx)$. We focus on a single **period** of the function including the origin, because the periodic property enables us to extend the graph to the rest of the function's domain if we wish. Our limited domain is then the interval $\left(-\frac{P}{2}, \frac{P}{2}\right)$ and the graph has vertical asymptotes at $\pm\frac{P}{2}$ where $P = \frac{\pi}{B}$. On the interval $\left(-\frac{P}{2}, \frac{P}{2}\right)$, the graph will come up from the left asymptote at $x = -\frac{P}{2}$, cross through the origin, and continue to increase as it approaches the right asymptote at $x = \frac{P}{2}$. To make the function approach the asymptotes at the correct rate, we also need to set the vertical scale by actually evaluating the function for three other **quarter points** that the graph will pass through. For example, we can use

$$f\left(\frac{P}{4}\right) = A\tan\left(B \cdot \frac{P}{4}\right) = A\tan\left(B \cdot \frac{\pi}{4B}\right) = A\tan\left(\frac{\pi}{4}\right) = A \cdot 1 = 1$$

How To: Howto: Given the function $f(x) = A\tan(Bx)$, graph one period.

Rewrite the equation so that B is positive. Express the equation in the form $f(x) = A\tan(Bx)$.

1. Identify the **verticalstretching factor**, $|A|$. If A is negative, the graph will be a reflection over the x-axis and will be decreasing (rather than increasing) within a cycle.
2. Identify B and determine the **period**, $P = \frac{\pi}{|B|}$.
3. Evaluate the x values for the **five quarter points**, starting at $-\frac{P}{2}$ and spaced $\frac{P}{4}$ units apart.
4. Draw **vertical asymptotes** at the first and last quarter points, $x = -\frac{P}{2}$ and $x = \frac{P}{2}$.
5. Evaluate the function at the middle three **quarter points** and draw the graph through these points.

In summary, quarter point values of x and corresponding values for $f(x) = A\tan(|B|x)$ for one cycle are shown in the table below.

$x = -\frac{P}{2}$	$x = -\frac{P}{4}$	$x = 0$	$x = \frac{P}{4}$	$x = \frac{P}{2}$
asymptote	$y = -A$	$y = 0$	$y = A$	asymptote

Example 5.6.1: Sketching a Compressed Tangent

Sketch a graph of one period of the function $y = 0.5\tan\left(\frac{\pi}{2}x\right)$. State the domain and range.

Solution

First, we identify A and B.

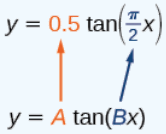


Figure 5.6.2b

The **period** is $P = \frac{\pi}{B} = \frac{\pi}{\frac{\pi}{2}} = 2$. The asymptotes are at $x = \pm\frac{P}{2} = \pm\frac{2}{2}$. Therefore, the asymptotes are at $x = -1$ and $x = 1$.

The horizontal distance between **quarter points** is $\frac{P}{4} = 0.5$, so quarter points from the start of this cycle (at $x = -1$, the asymptote on the left, occur at $x = -1,\ -.5,\ 0,\ .5,$ and 1

$$f\left(-\frac{P}{4}\right) = f(-0.5) \quad = 0.5\tan\left(\frac{-0.5\pi}{2}\right) \quad = 0.5\tan\left(\frac{-\pi}{4}\right) \quad = -0.5 = -A$$

$$f(0) = f(0) \quad = 0.5\tan\left(\frac{0\cdot\pi}{2}\right) \quad = 0.5\cdot 0 \quad = 0$$

$$f\left(\frac{P}{4}\right) = f(0.5) \quad = 0.5\tan\left(\frac{0.5\pi}{2}\right) \quad = 0.5\tan\left(\frac{\pi}{4}\right) \quad = 0.5 = A$$

This means the curve must pass through the points $(-0.5, -0.5)$, $(0, 0)$, and $(0.5, 0.5)$.

Domain is $\{x | x \neq 1 + 2k\}$, k is any integer. Range: $(-\infty, \infty)$.

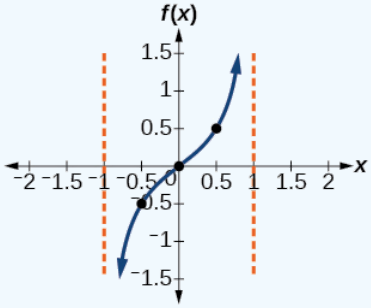


Figure 5.6.2c One period of the function $y = 0.5\tan\left(\frac{\pi}{2}x\right)$.

Try It 5.6.2

Sketch a graph of $f(x) = 3\tan\left(\frac{\pi}{6}x\right)$. State the domain and range.

Answer

Vertical stretch is 3; Period (Horizontal stretch) is $P = \frac{\pi}{\frac{\pi}{6}} = 6$; quarter point x values, starting at -3 and 1.5 units apart, are $x = -3,\ -1.5,\ 0,\ 1.5,\ 3$; Asymptotes at $x = -3$ and $x = 3$; middle quarter point coordinates: $(-1.5, -3)$, $(0, 0)$, and $(1.5,\ 3)$. Domain: $\{x | x \neq 3 + 6k\}$, k is any integer. Range: $(-\infty, \infty)$.

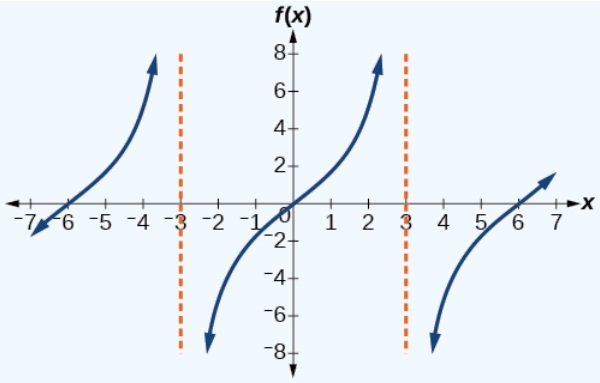


Figure 5.6.2d Graph of $f(x) = 3\tan\left(\frac{\pi}{6}x\right)$.

Graph a Vertically or Horizontally Shifted Tangent Function

Now that we can graph a **tangent function** that is stretched or compressed, we will add a vertical shift and/or a phase shift (horizontal shift). In this case, we add C and D to the general form of the tangent function, $f(x) = A\tan(Bx - C) + D$.

FEATURES OF THE GRAPH OF $g(x) = A\tan(Bx - C) + D$

- The **vertical stretching factor** is $|A|$. There is no amplitude.
- The **period** is $P = \frac{\pi}{|B|}$.
- The **phase shift** is $H = \frac{C}{B}$ units (right when H is positive, left when H is negative).
- The **vertical shift** is D units (up when D is positive, down when D is negative)
- The **vertical asymptotes** occur at $x = H + \frac{P}{2} + P \cdot k$, where k is any integer.
- The **domain** is $\{x | x \neq H + \frac{P}{2} + P \cdot k\}$, where k is any integer.
- The **range** is $(-\infty, \infty)$.

The graph of this transformed tangent function is fundamentally the generic tangent function $y = A\tan(Bx)$, that has been shifted H units right or left and D units up or down'.

How To: Given the function $y = A\tan(Bx - C) + D$, sketch the graph of one period.

Rewrite the equation so that B is positive. Express the function in the form $y = A\tan(Bx - C) + D$.

1. Identify the **vertical stretching** factor, $|A|$. (If A is negative, the graph will be a reflection over the x-axis and will be decreasing (rather than increasing) within a cycle.)
2. Identify B and determine the **period**, $P = \frac{\pi}{|B|}$.
3. Identify C and determine the **phase shift**, $H = \frac{C}{B}$.
4. Evaluate the x values for the **five quarter points**, starting at $x = -\frac{P}{2}$ and spaced $\frac{P}{4}$ units apart.
5. Draw **vertical asymptotes** at the first and last quarter points, $x = -\frac{P}{2} + H$ and $x = \frac{P}{2} + H$.
6. Identify D, which is the **vertical shift**.
7. Evaluate the function at the middle three **quarter points** and draw the graph through these points. Notice that each of these points are the middle three points from the graph of $y = A\tan(Bx)$, that have had H added to their x coordinates, and D added to their y coordinates.

   In summary, quarter point values of x and corresponding values for $g(x) = A\tan(|B|x - C) + D$ for one cycle are shown in the table below.

$x=-\frac{P}{2}+H$	$x=-\frac{P}{4}+H$	$x=H$	$x=\frac{P}{4}+H$	$x=\frac{P}{2}+H$
asymptote	$y=-A+D$	$y=D$	$y=A+D$	asymptote

Example 5.6.3: Graphing One Period of a Shifted Tangent Function

Graph one period of the function $y=-2\tan(\pi x+\pi)-1$. State the domain and range.

Solution

- The function is already written in the form $y=A\tan(Bx-C)+D$, with $B>0$.
- Step 1. $A=-2$, so the **vertical stretching factor** is $|A|=2$. Because A is negative the graph is decreasing over one cycle.
- Step 2. $B=\pi$, so the **period** is $P=\frac{\pi}{|B|}=\frac{\pi}{\pi}$. So $P=1$.
- Step 3. $C=-\pi$, so the **phase shift** is $H=\frac{C}{B}=\frac{-\pi}{\pi}$. So $H=-1$, corresponding to a **1 unit shift to the left**.
- Step 4. The five quarter points start at $x=H-\frac{P}{2}=-1-\frac{1}{2}=-1.5$ and are spaced $\frac{P}{4}=\frac{1}{4}=.25$ units apart. They are $x=-1.5,\ -1.25,\ -1\ ,-.75,\ -.5$.
- Step 4. Asymptotes, at the first and last quarter points, are $x=-1.5$ and $x=-0.5$
- Step 5. $D=-1$, so the **vertical shift** moves everything **down 1 unit**.
- Step 6. Coordinates of the middle three quarter points are $(-1.25,\ 1)$, $(-1,\ -1)$, and $(-0.75,\ -3)$. The graph is shown in Figure 5.6.2.
- The domain is $\{x\mid x\neq -\frac{1}{2}+k\}$, where k is any integer. The range is $(-\infty,\infty)$.

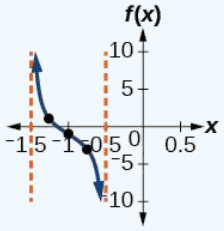


Figure 5.6.3 One period of the function $y=-2\tan(\pi x+\pi)-1$.

Try It 5.6.3

How would the graph in Example 5.6.2 look different if we made $A=2$ instead of -2?

Answer

It would be reflected across the line $y=-1$, becoming an increasing function.

Construct an Equation for a Stretched or Compressed Tangent Function

Howto: Given the graph of a tangent function, identify horizontal and vertical stretches.

1. Find the period P from the spacing between successive vertical asymptotes or x-intercepts.
2. Write $f(x)=A\tan\left(\frac{\pi}{P}x\right)$.
3. Determine a convenient point $(x,f(x))$ on the given graph and use it to determine A.

Example 5.6.4: Identifying the Graph of a Stretched Tangent

Find a formula for the function graphed in Figure 5.6.4a. The graph goes through the point $(2,\ 2)$.

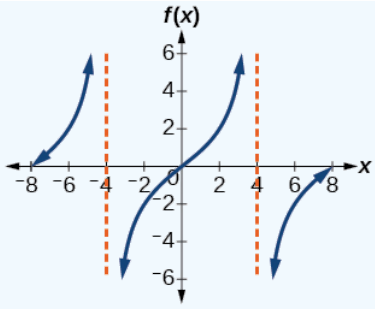


Figure 5.6.4*a*: A stretched tangent function

Solution

The graph has the shape of a tangent function.

- *Step 1*. One cycle extends from –4 to 4, so the period is $P = 8$. Since $P = \frac{\pi}{|B|}$, we have $B = \frac{\pi}{P} = \frac{\pi}{8}$.
- *Step 2*. The equation must have the form $f(x) = A\tan\left(\frac{\pi}{8}x\right)$.
- *Step 3*. To find the vertical stretch A, we can use the point $(2, 2)$: $2 = A\tan\left(\frac{\pi}{8}\cdot 2\right) = A\tan\left(\frac{\pi}{4}\right) = A$

This function would have a formula $f(x) = 2\tan\left(\frac{\pi}{8}x\right)$.

Try It 5.6.4

Find a formula for the function in Figure 5.6.4*b*. (The point $\left(\frac{\pi}{8}, 4\right)$ is on the graph.)

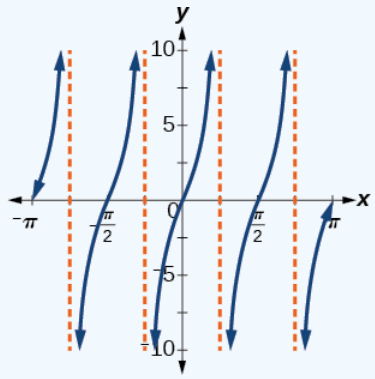


Figure 5.6.4*b*

Answer

$g(x) = 4\tan(2x)$

Analyzing the Graph of $y = \cot x$

The next trigonometric function we need to explore is cotangent. The cotangent is defined by the reciprocal identity $\cot x = \frac{1}{\tan x}$. Notice that the function is undefined when the tangent function is 0, leading to a vertical asymptote in the graph at 0, π, etc. Since the output of the tangent function is all real numbers, the output of the cotangent function is also all real numbers. Because the two functions are reciprocals of each other, when one graph is increasing, the other is decreasing.

The cotangent graph has vertical asymptotes at each value of x where $\tan x = 0$; we show these in the graph below with dashed lines. Since the cotangent is the reciprocal of the tangent, $\cot x$ has vertical asymptotes at all values of x where $\tan x = 0$, and $\cot x = 0$ at all values of x where $\tan x$ has its vertical asymptotes.

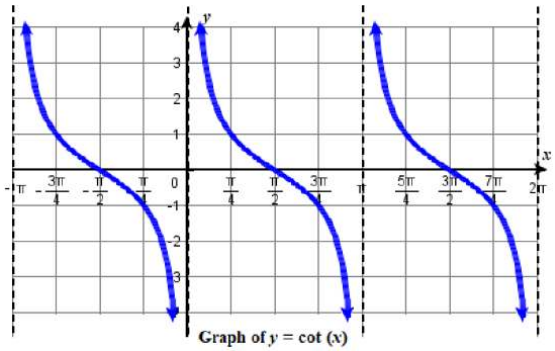


Figure 5.6.5: The cotangent function

Graph a Stretched or Compressed Cotangent Function

FEATURES OF THE GRAPH OF $y = A\cot(Bx)$

- The **vertical stretching factor** $|A|$. There is no amplitude.
- The **period** is $P = \frac{\pi}{|B|}$.
- The **vertical asymptotes** occur at $x = P \cdot k$, where k is any integer and P is the period.
- The **domain** is all real numbers x, where $\{x|x \neq P \cdot k\}$ where k is any integer and P is the period.
- The **range** is $(-\infty, \infty)$.
- $y = A\cot(Bx)$ is an odd function.

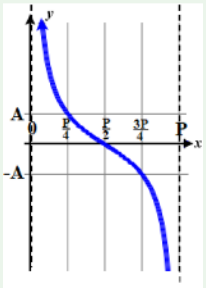


Figure 5.6.6a: Generic Graph for $y = A\cot(Bx)$, with A and B both positive (or both negative)

As with the tangent function, these results can be confirmed by examining the start of a cycle of \(f(x) = =A \tan(Bx)\), and relating it to the the behaviour of the parent function $y = cot(x)$. A cycle for f starts when its argument $Bx = 0$ and ends when $Bx = \pi$. Solving these equations we obtain the cycle for f starts when $x = 0$, ends when $x = \frac{\pi}{B}$, and consequently has a length (known as the period) of $\frac{\pi}{B}$.

How To: Howto: Given a modified cotangent function of the form $f(x) = A\cot(Bx)$, graph one period.

Rewrite the equation so that B is positive. Express the equation in the form $f(x) = A\cot(Bx)$.

1. Identify the **vertical stretching factor**, $|A|$. If A is negative, the graph will be a reflection over the x-axis and will increase over one cycle (rather than decrease).
2. Identify B and determine the **period**, $P = \frac{\pi}{|B|}$.
3. Evaluate the x values for the **five quarter points**, starting at $x = 0$ and spaced $\frac{P}{4}$ units apart.
4. Draw **vertical asymptotes** at the first and last quarter points, $x = 0$ and $x = P$.
5. Evaluate the function at the middle three **quarter points** and draw the graph through these points.

In summary, quarter point values of x and corresponding values for $f(x) = A\cot(|B|x)$ for one cycle are shown in the table below.

$x = 0$	$x = \frac{P}{4}$	$x = \frac{P}{2}$	$x = \frac{3P}{4}$	$x = P$
asymptote	$y = A$	$y = 0$	$y = -A$	asymptote

Example 5.6.6: Graphing Variations of the Cotangent Function

Determine the vertical stretching factor, period, and phase shift of $y = 3\cot(4x)$, and then sketch a graph. State the domain and range.

Solution

- The function is in the form $f(x) = A\cot(Bx)$ gives $f(x) = 3\cot(4x)$.
- *Step 1.* $A = 3$. The vertical stretching factor is $|A| = 3$.
- *Step 2.* $B = 4$. The period is $P = \frac{\pi}{|B|} = \frac{\pi}{4}$.
- *Step 3.* The five quarter points start at $x = 0$ and are spaced $\frac{P}{4} = \frac{\frac{\pi}{4}}{4} = \frac{\pi}{16}$ units apart. They are $x = 0, \frac{\pi}{16}, \frac{\pi}{8}, \frac{3\pi}{16}, \frac{\pi}{4}$.
- *Step 4.* Sketch the asymptotes at the first and last quarter points, $x = 0, x = \frac{\pi}{4}$.
- *Step 5.* The coordinates of the three middle quarter points are $\left(\frac{\pi}{16}, 3\right)$, $\left(\frac{\pi}{8}, 0\right)$, and $\left(\frac{3\pi}{16}, -3\right)$
- The domain is $\{x | x \neq \frac{\pi}{4}k\}$ where k is any integer; the range is $(-\infty, \infty)$.

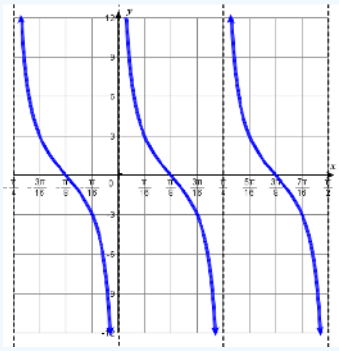


Figure **5.6.6** Graph of three periods of $y = 3\cot(4x)$

Graph a Vertically or Horizontally Shifted Cotangent Function

We can transform the graph of the cotangent in much the same way as we did for the tangent. The equation becomes $y = A\cot(Bx - C) + D$.

PROPERTIES OF THE GRAPH OF $y = A\cot(Bx - C) + D$

- The **vertical stretching factor** is $|A|$. There is no amplitude.
- The **period** is $P = \frac{\pi}{|B|}$
- The **phase shift** is $H = \frac{C}{B}$ units (right when H is positive, left when H is negative).
- The **vertical shift** is D units (up when D is positive, down when D is negative).
- The **vertical asymptotes** occur at $x = H + P \cdot k$, where k is any integer and P is the period.
- The **domain** is $x \neq H + P \cdot k$, where k is an integer, P is the period and H is the phase shift..
- The **range** is $(-\infty, \infty)$.

The graph of this transformed cotangent function is fundamentally the generic cotangent function $y = A\cot(Bx)$, that has been shifted H units 'right' and D units 'up'.

Howto: Given a modified cotangent function of the form $f(x) = A\cot(Bx - C) + D$, graph one period.

Rewrite the equation so that B is positive. Express the function in the form $f(x) = A\cot(Bx - C) + D$.

1. Identify the **vertical stretching factor**, $|A|$. If A is negative, the graph will be a reflection over the x-axis and will increase over one cycle (rather than decrease).
2. Identify B and determine the **period**, $P = \frac{\pi}{|B|}$.
3. Identify C and determine the **phase shift**, $H = \frac{C}{B}$.
4. Evaluate the x values for the **five quarter points**, starting at $x = H$ and spaced $\frac{P}{4}$ units apart.
5. Draw **vertical asymptotes** at at the first and last quarter points, $x = H$ and $x = P + H$.
6. Identify D, which is the **vertical shift**.

7. Evaluate the function at the middle three **quarter points** and draw the graph through these points. Notice that each of these points are the middle three points from the graph of $y = A\cot(Bx)$, that have had H added to their x coordinates, and D added to their y coordinates.

   In summary, quarter point values of x and corresponding values for $g(x) = A\cot(|B|x - C) + D$ for one cycle are shown in the table below.

$x = H$	$x = \frac{P}{4} + H$	$x = \frac{P}{2} + H$	$x = \frac{3P}{4} + H$	$x = P + H$
asymptote	$y = A + D$	$y = D$	$y = -A + D$	asymptote

Example 5.6.7: Graphing a Modified Cotangent

Sketch a graph of one period of the function $f(x) = 4\cot\left(\frac{\pi}{8}x - \frac{\pi}{2}\right) - 2$.

Solution

- The function is already written in the general form $f(x) = A\cot(Bx - C) + D$.
- Step 1. $A = 4$, so the vertical stretching factor is 4.
- Step 2. $B = \frac{\pi}{8}$, so the period is $P = \frac{\pi}{|B|} = \frac{\pi}{\frac{\pi}{8}} = 8$.
- Step 3. $C = \frac{\pi}{2}$, so the phase shift is $H = \frac{C}{B} = \frac{\frac{\pi}{2}}{\frac{\pi}{8}} = 4$.
- Step 4. The five quarter points start at $x = 4$ and are spaced $\frac{P}{4} = \frac{8}{4} = 2$ units apart. They are $x = 4,\ 6,\ 8\ ,10,\ 12$.
- Step 4. Draw vertical asymptotes $x = 4$ and $x = 12$.
- Step 5. $D = -2$, so the graph is shifted down two units.
- Step 6. The coordinates of the three middle quarter points are $(6, 2)$, $(8, -2)$, and $(10, -6)$.
- The domain is $\{x \mid x \neq 4 + 8k\}$, where k is any integer. The range is $(-\infty, \infty)$.

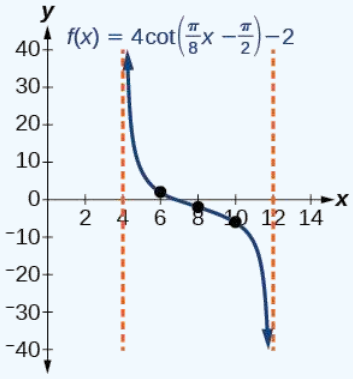


Figure 5.6.7: One period of a modified cotangent function

The graph is shown in Figure 5.6.7.

Analyzing the Graph of $y = \sec x$

For shifted, compressed, and/or stretched versions of the secant and cosecant functions, we can follow similar methods to those we used for tangent and cotangent. That is, we locate the vertical asymptotes and also evaluate the functions for a few points (specifically the local extrema).

The secant was defined by the reciprocal identity $\sec x = \dfrac{1}{\cos x}$. Notice that the function is undefined when the cosine is 0, leading to vertical asymptotes at $\dfrac{\pi}{2}$, $\dfrac{3\pi}{2}$ etc. Because the cosine is never more than 1 in absolute value, the secant, being the reciprocal, will never be less than 1 in absolute value.

We can graph $y = \sec x$ by observing the graph of the cosine function because these two functions are reciprocals of one another. See Figure 5.6.8. The graph of the cosine is shown as a grey curve so we can see the relationship. Where the graph of the cosine function decreases, the graph of the secant function increases. Where the graph of the cosine function increases, the graph of the secant function decreases. When the cosine function is zero, the secant is undefined.

The secant graph has vertical asymptotes at each value of x where the cosine graph crosses the x-axis; we show these in the graph below with dashed vertical lines.

Note that, because cosine is an even function, secant is also an even function. That is, $\sec(-x) = \sec x$.

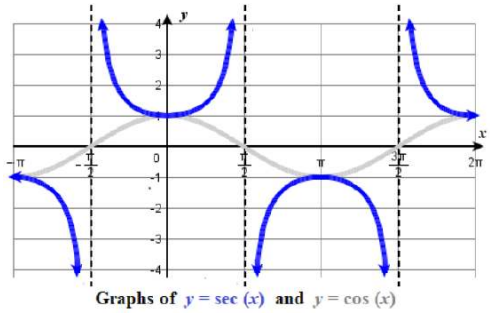


Figure 5.6.8: Graph of the secant function, $f(x) = \sec x = \dfrac{1}{\cos x}$

Graph a Stretched or Compressed Cotangent Function

As we did for the tangent function, we will again refer to the constant $|A|$ as the stretching factor, not the amplitude.

FEATURES OF THE GRAPH OF $y = A\sec(Bx)$

- The **vertical stretching factor** is $|A|$. There is no amplitude.
- The **period** is $P = \dfrac{2\pi}{|B|}$.
- The **vertical asymptotes** occur at $x = \dfrac{P}{4} + \dfrac{P}{2} \cdot k$ where k is any integer and P is the period.
- The **domain** is $\{x \mid x \neq \dfrac{P}{4} + \dfrac{P}{2} \cdot k\}$, where k is any integer and P is the period.
- The **range** is $(-\infty, -|A|] \cup [|A|, \infty)$.
- $y = A\sec(Bx)$ is an even function because cosine is an even function.

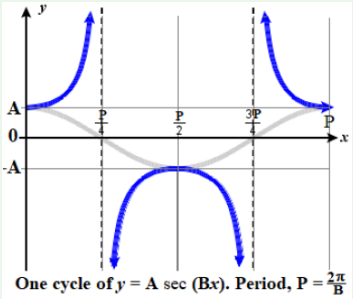


Figure 5.6.9a: Generic Graph for $y = A\sec(Bx)$, with A positive

These results can be confirmed by examining the start of a cycle of $f(x) = A\sec(Bx)$, and relating it to the the behaviour of the parent function $y = \sec(x)$. A cycle for f starts when its argument $Bx = 0$ and ends when $Bx = 2\pi$. Solving these equations we obtain the cycle for f starts when $x = 0$, ends when $x = \dfrac{2\pi}{B}$, and consequently has a length (known as the period) of $\dfrac{2\pi}{B}$.

Howto: Given a function of the form $y = A\sec(Bx)$, graph one period

Rewrite the equation so that B is positive. Express the function given in the form $y = A\sec(Bx)$.

1. Identify the **vertical stretching** factor, $|A|$. If A is negative, the graph will be a reflection over the x-axis.
2. Identify B and determine the **period**, $P = \frac{2\pi}{|B|}$.
3. Evaluate the x values for the **five quarter points**, starting at $x = 0$ and spaced $\frac{P}{4}$ units apart.
4. Draw **vertical asymptotes** for the secant function, $x = \frac{P}{4}$ and $x = \frac{3P}{4}$. These asymptotes are half a period apart at the second and fourth quarter points. These places are where the cosine function is zero.
5. Determine the location of **local extrema** which are also half a period apart - at the first, third and fifth quarter points when $x = 0$, $x = \frac{P}{2}$ and $x = P$. These are places where the cosine function is a maximum at the same time the secant is a minimum, and vice versa. Sketch the graph of $y = A\sec(Bx)$.

   In summary, quarter point values of x and corresponding values for $f(x) = A\sec(|B|x)$ for one cycle are shown in the table below.

$x = 0$	$x = \frac{P}{4}$	$x = \frac{P}{2}$	$x = \frac{3P}{4}$	$x = P$
$y = A$	asymptote	$y = -A$	asymptote	$y = A$

Example 5.6.9: Graphing a Variation of the Secant Function

Graph one period of $f(x) = 3\sec(0.4x)$. State the domain and range of f.

Solution

- The given function is already written in the general form, $y = A\sec(Bx)$.
- *Step 1.* $A = 3$ so the **vertical stretching factor** is 3.
- *Step 2.* $B = 0.4$ so $P = \frac{2\pi}{0.4} = 5\pi$. The **period** is 5π units.
- *Step 3.* The **five quarter points** start at $x = 0$ and are $\frac{P}{4} = \frac{5\pi}{4}$ units apart. They are $x = 0,\ 1.25\pi,\ 2.5\pi\ ,3.75\pi,\ 5pi$.
- *Step 4.* **Vertical asymptotes** for the secant function (or places where the cosine function is zero), occur half a period apart at the second and fourth quarter points, $x = 1.25\pi$ and $x = 3.75\pi$.
- *Step 5.* **Extrema** occur half a period apart at the first, third and fifth quarter points. The coordinates of these points are at $(0,\ 3)$, $(2.5\pi,\ -3)$, and $(5\pi,\ 3)$. Sketch the graph of the function $f(x) = 3\sec(0.4x)$ and if desired, its reciprocal function, $g(x) = 3\cos(0.4x)$.
- The **domain** is $x \neq \frac{5\pi}{4} + \frac{5\pi}{2} \cdot k$, where k is any integer. The **range** is $(-\infty, -3] \cup [3, \infty)$.

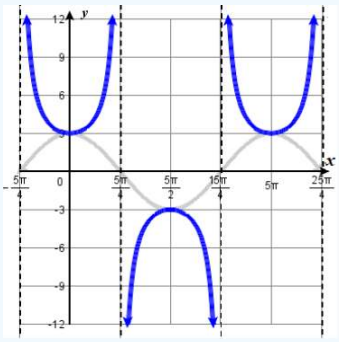


Figure 5.6.9a $f(x) = 3\sec(0.4x)$ **and** $g(x) = 3\cos(0.4x)$

Try It 5.6.9

Graph one period of $f(x) = -2.5\sec(0.4x)$.

Answer

This is a vertical reflection of the preceding graph because A is negative.

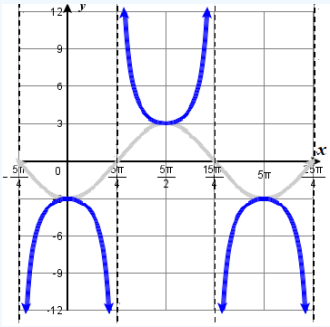


Figure 5.6.9b $f(x) = -3\sec(0.4x)$ **and** $g(x) = -3\cos(0.4x)$

Graph a Vertically or Horizontally Shifted Secant Function

Vertical shifts and (horizontal) phase shifts may be applied to the secant function in the same way as for the other functions. The general equation for secant is $y = A\sec(Bx - C) + D$.

FEATURES OF THE GRAPH OF $Y = A\sec(Bx - C) + D$

- The **vertical stretching factor** is $|A|$. There is no amplitude.
- The **period** is $P = \dfrac{2\pi}{|B|}$.
- The **phase shift** is $H = \dfrac{C}{B}$ units (right when H is positive, left when H is negative).
- The **vertical shift** is D units (up when D is positive, down when D is negative).
- The **vertical asymptotes** occur half a period apart at $x = H + \dfrac{P}{4} + \dfrac{P}{2} \cdot k$, where k is any integer and P is the period.
- The **domain** is $\{x \mid x \neq H + \dfrac{P}{4} + \dfrac{P}{2} \cdot k\}$, where k is any integer and P is the period.
- The **range** is $(-\infty, D - |A|] \cup [D + |A|, \infty)$.

The graph of this transformed secant function is fundamentally the generic secant function $y = A\sec(Bx)$, that has been shifted H units 'right' and D units 'up'.

Do the vertical shift and stretch/compression affect the secant's range?

Yes. The range of $f(x) = A\sec(Bx - C) + D$ is $(-\infty, -|A| + D] \cup [|A| + D, \infty)$.

Howto: Given a function of the form $f(x) = A\sec(Bx - C) + D$, graph one period.

Rewrite the equation so that B is positive. Express the function in the form $f(x) = A\sec(Bx - C) + D$.

1. Identify the **vertical stretching factor**, $|A|$. If A is negative, the graph will be a reflection over the x-axis.
2. Identify B and determine the **period**, $P = \dfrac{2\pi}{|B|}$.
3. Identify C and determine the **phase shift**, $H = \dfrac{C}{B}$.
4. Evaluate the x values for the **five quarter points**, starting at $x = H$ and spaced $\dfrac{P}{4}$ units apart.
5. Vertical asymptotes are at the second and fourth quarter points which are $x = H + \dfrac{P}{4}$ and $x = H + \dfrac{3P}{4}$
6. Identify D, which is the **vertical shift**.

7. Evaluate the function where local extrema exist, which occur at the first, third and fifth quarter points, and draw the graph through these points. Notice that this is the same graph that would be obtained by graphing $y = A\sec(Bx)$, but shifting it to the right/left by H units and up/down by D units.

   Local extrema are at $(H,\ D+A)$, $(H+\frac{P}{2},\ D-A)$, and $(H+P,\ D+A)$,

   In summary, quarter point values of x and corresponding values for $g(x) = A\sec(|B|x - C) + D$ for one cycle are shown in the table below.

$x = H$	$x = \frac{P}{4} + H$	$x = \frac{P}{2} + H$	$x = \frac{3P}{4} + H$	$x = P + H$
$y = A + D$	asymptote	$y = -A + D$	asymptote	$y = A + D$

Try It 5.6.10

Given the graph of $f(x) = 2\cos\left(\frac{\pi}{2}x\right) + 1$ shown in Figure 5.6.10*a*, sketch the graph of $g(x) = 2\sec\left(\frac{\pi}{2}x\right) + 1$ on the same axes.

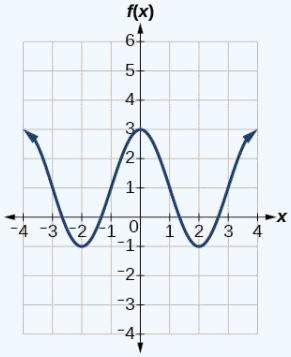


Figure 5.6.10*a*

Answer

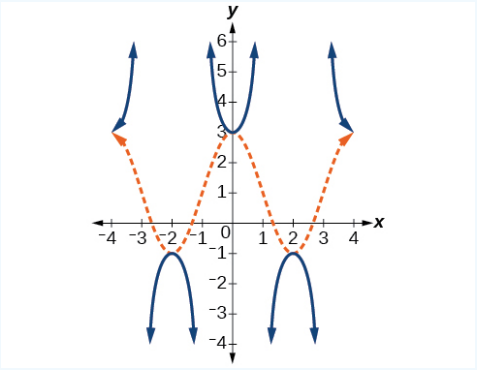


Figure 5.6.10*b*

Example 5.6.11: Graphing a Variation of the Secant Function

Graph one period of $f(x) = 5\sec\left(\frac{x}{6} - \frac{\pi}{3}\right) + 11$. State the domain and range.

Solution

- The function parameters are $A = 5$, $B = \frac{1}{6}$, $C = \frac{\pi}{3}$, and $D = 11$.
- *Step 1.* The **vertical stretching factor** is $|A| = 5$.
- *Step 2.* The **period** is $P = \frac{2\pi}{|B|} = \frac{2\pi}{\frac{1}{6}} = 2\pi \cdot \frac{6}{1} = 12\pi$.

- *Step 3.* The **phase shift** is $H = \dfrac{C}{B} = \dfrac{\frac{\pi}{3}}{\frac{1}{6}} = \dfrac{\pi}{3} \cdot \dfrac{6}{1} = 2\pi$.
- *Step 4.* The **quarter points** starting at $H = 2\pi$ and $\dfrac{P}{4} = \dfrac{12\pi}{4} = 3\pi$ units apart are $x = 2\pi,\ 5\pi,\ 8\pi\ , 11\pi,\ 14\pi$.
- *Step 5.* **Vertical asymptotes**, at the second and fourth quarter points, are $x = 5\pi$ and $x = 11\pi$.
- *Step 6.* The **vertical shift** is $D = 11$ or up one unit.
- *Step 7.* Coordinates of the **extrema**, at the first, third and fifth quarter points, are $(2\pi, f(2\pi) = A + D = 16)$, $(8\pi,\ 6)$, and $(14\pi,\ 16)$. Figure 5.6.12 shows a graph.
- The **domain** is $\{x \mid x \neq 5\pi + 6\pi \cdot k\}$, where k is any integer. The **range** is $(-\infty, 6] \cup [16, \infty)$.

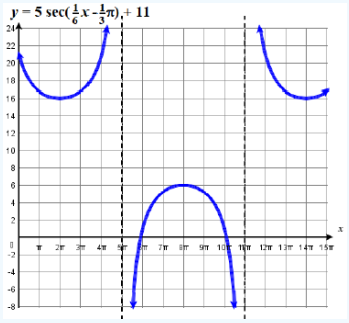


Figure 5.6.11 Graph of $y = 5\sec\left(\dfrac{x}{6} - \dfrac{\pi}{3}\right) + 11$

Try It 5.6.11

Graph one period of $f(x) = 4\sec\left(-\dfrac{\pi}{8}x - \dfrac{3\pi}{8}\right) - 8$. State the domain and range.

Answer

$(f(x) = 4\sec\left(\dfrac{\pi}{8}x + \dfrac{3\pi}{8}\right) - 8$. (Rewritten because B initially was negative). Parameters are $A = 4$, $B = \dfrac{\pi}{8}$, $C = -\dfrac{3\pi}{8}$, and $D = -8$. Period $P = 16$, Phase shift $H = -3$. The domain is $\{x \mid x \neq 1 + 8k\}$, where k is any integer. The range is $(-\infty, -12] \cup [-4, \infty)$

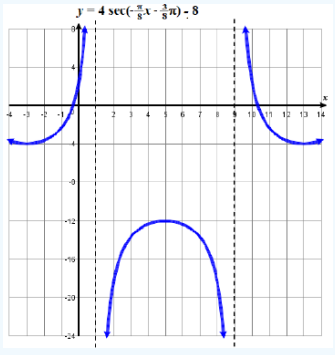


Figure 5.6.11 Graph of $f(x) = 4\sec\left(-\dfrac{\pi}{8}x - \dfrac{3\pi}{8}\right) - 8$

Analyzing the Graph of $y = \csc x$

Similar to the secant, the **cosecant** is defined by the reciprocal identity $\csc x = \dfrac{1}{\sin x}$. Notice that the function is undefined when the sine is 0, leading to a vertical asymptote in the graph at 0, π, etc. Since the sine is never more than 1 in absolute value, the cosecant, being the reciprocal, will never be less than 1 in absolute value.

We can graph $y = \csc x$ by observing the graph of the sine function because these two functions are reciprocals of one another. See Figure 5.6.7. The graph of sine is shown as a grey curve so we can see the relationship. Where the graph of the sine function decreases, the graph of the cosecant function increases. Where the graph of the sine function increases, the graph of the **cosecant function** decreases.

The cosecant graph has vertical asymptotes at each value of x where the sine graph crosses the x-axis; we show these in the graph below with dashed vertical lines.

Note that, since sine is an odd function, the cosecant function is also an odd function. That is, $\csc(-x) = -\csc x$.

The graph of cosecant, which is shown in Figure 5.6.9, is similar to the graph of secant.

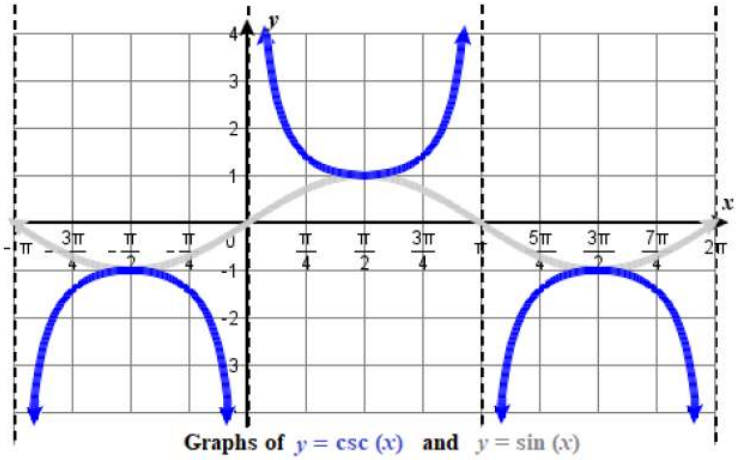


Figure 5.6.12: The graph of the cosecant function, $f(x) = \csc x$, and its reciprocal function $g(x) = sin(x)$.

Graph a Stretched or Compressed Cosecant Function

FEATURES OF THE GRAPH OF $y = A\csc(Bx)$

- The **vertical stretching factor** is $|A|$. There is no amplitude.
- The **period** is $\frac{2\pi}{|B|}$.
- The **vertical asymptotes** occur at $x = P \cdot k$, where k is any integer and P is the period.
- The **domain** is $\{x \mid x \neq P \cdot k$, where k is any integer and P is the period.
- The **range** is $(-\infty, -|A|] \cup [|A|, \infty)$.
- $y = A\csc(Bx)$ is an odd function because sine is an odd function.

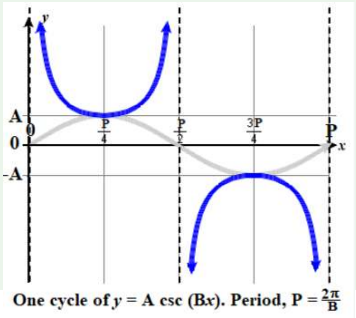


Figure 5.6.13a: Generic Graph for $y = A\csc(Bx)$, with A and B both positive

These results can be confirmed by examining the start of a cycle of $f(x) = A\csc(Bx)$, and relating it to the the behaviour of the parent function $y = \csc(x)$. A cycle for f starts when its argument $Bx = 0$ and ends when $Bx = 2\pi$. Solving these equations we obtain the cycle for f starts when $x = 0$, ends when $x = \frac{2\pi}{B}$, and consequently has a length (known as the period) of $\frac{2\pi}{B}$.

How To: Howto: Given a function of the form $y = A\csc(Bx)$, graph one period.

Rewrite the equation so that B is positive. Express the function given in the form $y = A\csc(Bx)$.

1. Identify the **vertical stretching** factor, $|A|$. If A is negative, the graph will be a reflection over the x-axis.
2. Identify B and determine the **period**, $P = \frac{2\pi}{|B|}$.
3. Evaluate the x values for the **five quarter points**, starting at $x = 0$ and spaced $\frac{P}{4}$ units apart.

4. Draw **vertical asymptotes** for one period of the cosecant function, $x = 0$, $x = \frac{P}{2}$ and $x = P$. The asymptotes are half a period apart, at the first, third and fifth quarter points. These places are where the sine function is zero.
5. Determine the location of **local extrema** which are also half a period apart - at the second and fourth quarter points when $x = \frac{P}{4}$ and $x = \frac{3P}{4}$. These are places where the sine function is a maximum at the same time the cosecant is a minimum, and vice versa. Sketch the graph of $y = A\csc(Bx)$.

   In summary, quarter point values of x and corresponding values for $f(x) = A\csc(|B|x)$ for one cycle are shown in the table below.

$x = 0$	$x = \frac{P}{4}$	$x = \frac{P}{2}$	$x = \frac{3P}{4}$	$x = P$
asymptote	$y = A$	asymptote	$y = -A$	asymptote

Example 5.6.13: Graphing a Variation of the Cosecant Function

Graph one period of $f(x) = -3\csc(4x)$. State the domain and range.

Solution

- Parameters are $A = -3$, $B = 4$.
- *Step 1.* $|A| = |-3| = 3$, so the vertical stretching factor is 3.
- *Step 2.* $B = 4$, so $P = \frac{2\pi}{4} = \frac{\pi}{2}$. The period is $P = \frac{\pi}{2}$ units.
- *Step 3.* The five quarter points, starting at $x = 0$ and spaced $\frac{\pi}{8}$ units apart, are $x = 0$, $\frac{\pi}{8}$, $\frac{\pi}{4}$, $\frac{3\pi}{8}$, $\frac{\pi}{2}$.
- *Step 4.* Sketch three asymptotes at $x = 0$, $x = \frac{\pi}{4}$, and $x = \frac{\pi}{2}$.
- *Step 5.* Coordinates of the extrema are $\left(\frac{\pi}{8}, -3\right)$ and $\left(\frac{3\pi}{8}, 3\right)$.
- Figure 5.6.14 shows the graph. Domain is $\{x \mid x \neq \frac{\pi}{4} \cdot k$, where k is any integer. The **range** is $(-\infty, -3] \cup [3, \infty)$.

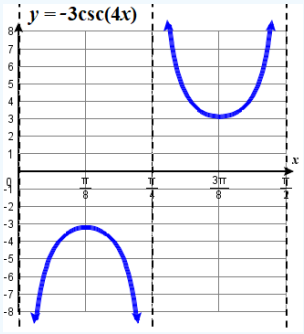


Figure 5.6.13*b* Graph of one period of $f(x) = -3\csc(4x)$.

Try It 5.6.13

Graph one period of $f(x) = 0.5\csc(2\pi x)$.

Answer

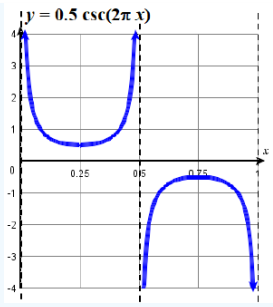


Figure 5.6.13c. Graph of one period of $f(x) = 0.5\csc(2\pi x)$.

Graph a Vertically or Horizontally Shifted Cosecant Function

Vertical and phase shifts may be applied to the cosecant function in the same way as for the secant and other functions.The equation for cosecant becomes $y = A\csc(Bx - C) + D$

FEATURES OF THE GRAPH OF $Y = A\csc(Bx - C) + D$

- The **vertical stretching factor** is $|A|$. There is no amplitude.
- The **period** is $P = \frac{2\pi}{|B|}$.
- The **phase shift** is $H = \frac{C}{B}$ units (right when H is positive, left when H is negative).
- The **vertical shift** is D units (up when D is positive, down when D is negative).
- The **vertical asymptotes** occur half a period apart at $x = H + \frac{P}{2} \cdot k$, where k is any integer and P is the period.
- The **domain** is $\{x \mid x \neq H + \frac{P}{2} \cdot k\}$, where k is any integer, P is the period, and H is the phase shift.
- The **range** is \ $(-\infty, D - |A|] \cup [D + |A|, \infty)$.

The graph of this transformed secant function is fundamentally the generic secant function $y = A\sec(Bx)$, that has been shifted H units 'right' and D units 'up'.

Q A The domain of $\csc\ x$ was given to be all x such that $x \neq k\pi$ for any integer k. Would the domain of $y = A\csc(Bx - C) + D$ be $x \neq \frac{C + k\pi}{B}$?

Yes. The excluded points of the domain follow the vertical asymptotes. Their locations show the horizontal shift and compression or expansion implied by the transformation to the original function's input.

How To: Howto: Given a function of the form $f(x) = A\csc(Bx - C) + D$, graph one period

Rewrite the equation so that B is positive. Express the function given in the form $y = A\csc(Bx - C) + D$.

1. Identify the **vertical stretching factor**, $|A|$. If A is negative, the graph will be a reflection over the x-axis.
2. Identify B and determine the **period**, $P = \frac{2\pi}{|B|}$.
3. Identify C and determine the **phase shift**, $H = \frac{C}{B}$. (Shift right when H is positive, left when H is negative).
4. Evaluate the x values for the **five quarter points**, starting at $x = H$ and spaced $\frac{P}{4}$ units apart.
5. Draw **vertical asymptotes** $x = H$, $x = \frac{P}{2} + H$ and $x = P + H$. The asymptotes are half a period apart, at the first, third and fifth quarter points. These places are where the sine function is zero.
6. Identify D, which is the **vertical shift**. (Shift up when D is positive, down when D is negative).
7. Evaluate the coordinates for the **local extrema** located at the second and fourth quarter points. These are places where the sine function is a maximum at the same time the cosecant is a minimum, and vice versa. Sketch the graph of $y = A\csc(Bx - C) + D$.

In summary, quarter point values of x and corresponding values for $f(x) = A\csc(|B|x - C) + D$ for one cycle are shown in the table below.

$x = H$	$x = \frac{P}{4} + H$	$x = \frac{P}{2} + H$	$x = \frac{3P}{4} + H$	$x = P + H$
asymptote	$y = A + D$	asymptote	$y = -A + D$	asymptote

Example 5.6.14: Graphing a Vertically Stretched, Horizontally Compressed, and Vertically Shifted Cosecant

Sketch a graph of $y = 2\csc\left(\frac{\pi}{2} - \frac{\pi}{4}\right) + 1$. State the domain and range.

Solution

- Parameters are $A = 2$, $B = \frac{\pi}{2}$, $C = \frac{\pi}{4}$, and $D = 1$.
- *Step 1.* Vertical stretching factor is $|A| = 2$.
- *Step 2.* The period is $\frac{2\pi}{|B|} = \frac{2\pi}{\frac{\pi}{2}} = 2\pi \cdot \frac{2}{\pi}$. So $P = 4$
- *Step 3.* The phase shift is $\frac{\frac{\pi}{4}}{\frac{\pi}{2}} = \frac{1}{2}$. So $H = 0.5$
- *Step 4.* quarter points, starting at $x = .5$ and spaced 1 unit apart, are $x = 0.5,\ 1.5,\ 2.5\ ,3.5,\ 4.5$.
- *Step 5.* Vertical asymptotes are $x = 0.5$, $x = 2.5$, $x = 4.5$.
- *Step 6.* Draw the graph of $y = A\csc(Bx)$ but shift it up $D = 1$.
- *Step 7.* Coordinates of extrema are $(1.5,\ 3)$ and $(3.5,\ -3)$.
- Domain is $\{x \mid x \neq 0.5 + 2k\}$, where k is any integer, domain is $(-\infty, -1] \cup [3, \infty)$.

The graph for this function is shown in Figure 5.6.14*a*.

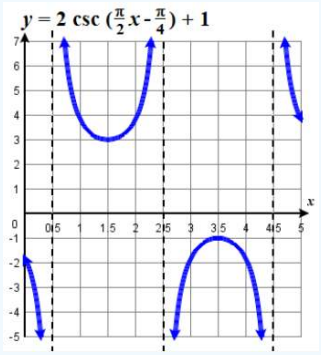


Figure 5.6.14*a*: Graph of $y = 2\csc\left(\frac{\pi}{2} - \frac{\pi}{4}\right) + 1$

Try It 5.6.14

Graph the function $y = -3\csc\left(-\frac{x}{3} + \frac{\pi}{6}\right) - 5$. State the domain and range.

Answer

Function rewrite necessary: $y = 3\csc\left(\frac{x}{3} - \frac{\pi}{6}\right) - 5$. Parameters are $A = 3$, $B = \frac{1}{3}$, $C = \frac{\pi}{6}$, and $D = -5$. $P = 6\pi$. $H = \frac{\pi}{2}$. Domain is $\{x \mid x \neq \frac{\pi}{2} + 3\pi k\}$, where k is any integer, domain is $(-\infty, -8] \cup [-2, \infty)$.

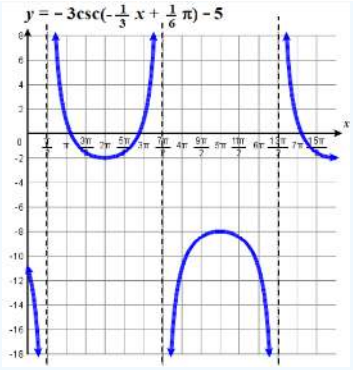


Figure 5.6.14b: Graph of the function $y = -3\csc\left(-\frac{x}{3} + \frac{\pi}{6}\right) - 5$.

Using the Graphs of Trigonometric Functions to Solve Real-World Problems

Many real-world scenarios represent periodic functions and may be modeled by trigonometric functions. As an example, let's return to the scenario from the section opener. Have you ever observed the beam formed by the rotating light on a police car and wondered about the movement of the light beam itself across the wall? The periodic behavior of the distance the light shines as a function of time is obvious, but how do we determine the distance? We can use the **tangent function**.

Example 5.6.15: Using Trigonometric Functions to Solve Real-World Scenarios

Suppose the function $y = 5\tan(\frac{\pi}{4}t)$ marks the distance in the movement of a light beam from the top of a police car across a wall where t is the time in seconds and y is the distance in feet from a point on the wall directly across from the police car.

a. Find and interpret the vertical stretching factor and period.
b. Graph on the interval $[0, 5]$.
c. Evaluate $f(1)$ and discuss the function's value at that input.

Solution

a. We know from the general form of $y = A\tan(Bt)$ that $|A|$ is the vertical stretching factor and $\frac{\pi}{B}$ is the period.

$$y = 5\tan\left(\frac{\pi}{4}t\right)$$
A B

Figure 5.6.15a

We see that the vertical stretching factor is 5. This means that the beam of light will have moved 5 ft after 1 second.

The period is $\frac{\pi}{\frac{\pi}{4}} = \frac{\pi}{1} \cdot \frac{4}{\pi} = 4$. This means that every 4 seconds, the beam of light sweeps the wall. The distance from the spot across from the police car grows larger as the police car approaches.

b. To graph the function, we draw an asymptote at $t = 2$ and use the vertical stretching factor and period. See Figure 5.6.23

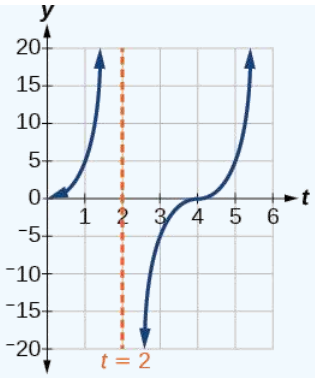


Figure **5.6.15***b*

c. period: $f(1) = 5\tan(\frac{\pi}{4}(1)) = 5(1) = 5$; after 1 second, the beam of has moved 5 ft from the spot across from the police car.

Media

Access these online resources for additional instruction and practice with graphs of other trigonometric functions.

- Graphing the Tangent
- Graphing Cosecant and Secant
- Graphing the Cotangent

Key Equations

Shifted, compressed, and/or stretched tangent function	$y = A\tan(Bx - C) + D$
Shifted, compressed, and/or stretched secant function	$y = A\sec(Bx - C) + D$
Shifted, compressed, and/or stretched cosecant function	$y = A\csc(Bx - C) + D$
Shifted, compressed, and/or stretched cotangent function	$y = A\cot(Bx - C) + D$

Key Concepts

- The tangent function has period π.
- $f(x) = A\tan(Bx - C) + D$ is a tangent with vertical and/or horizontal stretch/compression and shift.
- The secant and cosecant are both periodic functions with a period of 2π. $f(x) = A\sec(Bx - C) + D$ gives a shifted, compressed, and/or stretched secant function graph.
- $f(x) = A\csc(Bx - C) + D$ gives a shifted, compressed, and/or stretched cosecant function graph.
- The cotangent function has period π and vertical asymptotes at $0, \pm\pi, \pm 2\pi, \ldots$.
- The range of cotangent is $(-\infty, \infty)$, and the function is decreasing at each point in its range.
- The cotangent is zero at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$.
- $f(x) = A\cot(Bx - C) + D$ is a cotangent with vertical and/or horizontal stretch/compression and shift.
- Real-world scenarios can be solved using graphs of trigonometric functions.

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5.6e: Exercises - Graphs of Other Trigonometric Functions

A: Concepts

Exercise 5.6e.A

1) Explain how the graph of the sine function can be used to graph $y = \csc x$.

2) How can the graph of $y = \cos x$ be used to construct the graph of $y = \sec x$?

3) Explain why the period of $y = \tan x$ is equal to π.

4) Why are there no intercepts on the graph of $y = \csc x$?

5) How does the period of $y = \csc x$ compare with the period of $y = \sin x$?

Answer

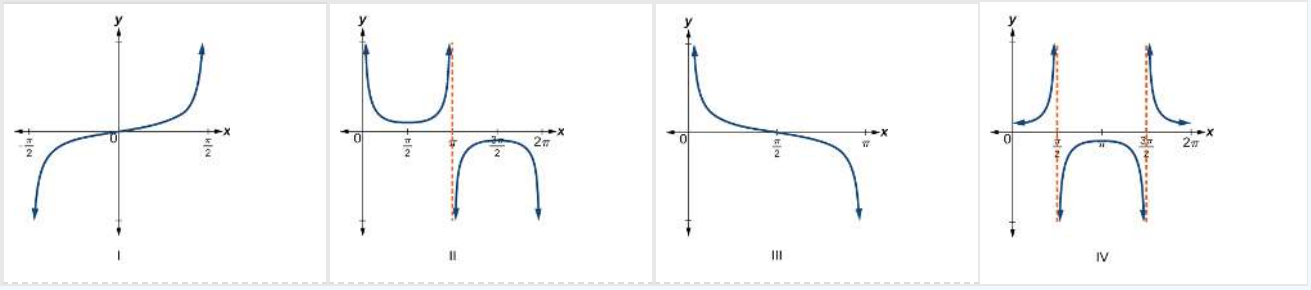
1. Since $y = \csc x$ is the reciprocal function of $y = \sin x$, you can plot the reciprocal of the coordinates on the graph of $y = \sin x$ obtain the y-coordinates of $y = \csc x$. The x-intercepts of the graph $y = \sin x$ are the vertical asymptotes for the graph of $y = \csc x$.
3. Answers will vary. Using the unit circle, one can show that $y = \tan(x + \pi) = \tan x$.
5. The period is the same: 2πadd texts here. Do not delete this text first.

B: Graph Equations without Phase Shifts

Exercise 5.6e.B

★ Match each trigonometric function with one of the following graphs.

6. $f(x) = \tan x$	7. $f(x) = \sec x$	8. $f(x) = \csc x$	9. $f(x) = \cot x$



★ For each function, identify attributes of the graph: (a) Stretching factor and vertical shift, (b) period, and (c) asymptotes, (d) domain and range. (e) Sketch two periods of the graph.

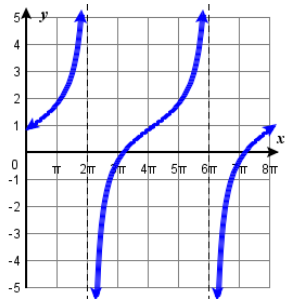
11. $f(x) = 4\tan\left(\frac{1}{4}x\right) + 1$
12. $f(x) = 2\tan(x) - 4$
13. $f(x) = \tan(6x) + 3$
14. $f(x) = 2\tan\left(\frac{\pi}{2}x\right) + 2$
15. $f(x) = -3\tan\left(\frac{3\pi}{2}x\right) - 1$
16. $f(x) = \tan(\pi x)$
17. $f(x) = \cot\left(\frac{2}{5}x\right) - 2$
18. $f(x) = 2\cot(2x) + 2$
19. $f(x) = \cot(x)$
20. $f(x) = 2\cot\left(\frac{\pi}{4}x\right) - 1$
21. $f(x) = -\cot\left(\frac{2\pi}{3}x\right) + 1$
22. $f(x) = 3\cot(4\pi x) - 3$
23. $f(x) = \sec\left(\frac{1}{3}x\right) - 2$
24. $f(x) = 3\sec(x) + 2$
25. $f(x) = 4\sec(3x)$
26. $f(x) = 2\sec\left(\frac{2\pi}{5}x\right) + 1$
27. $f(x) = -2\sec\left(\frac{3\pi}{4}x\right) + 3$
28. $f(x) = 3\sec(\pi x)$
29. $f(x) = \frac{1}{2}\csc(x) + 4$
30. $f(x) = 3\csc\left(\frac{2}{3}x\right) - 2$
31. $f(x) = 2\csc(x)$
32. $f(x) = 2\csc\left(\frac{\pi}{3}x\right) + 1$
33. $f(x) = -3\csc(\pi x) + 2$
34. $f(x) = \csc(3\pi x) - 1$

Answers to Odd Exercises

k used below represents any integer

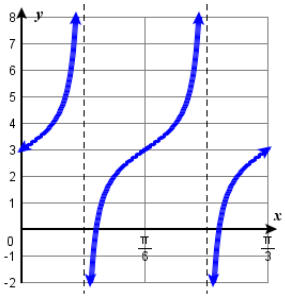
7. IV 9. III

11. V stretch factor: 4, up 1.
period: 4π,
VA: $x = 2\pi + 4\pi k$
D: $\{x \mid x \neq 2\pi + 4\pi + k\}$
R: $(-\infty, \infty)$



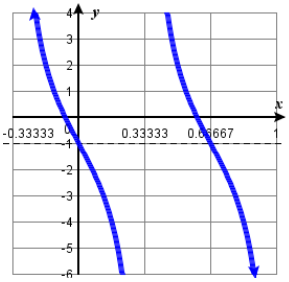
$$f(x) = 4\tan\left(\frac{1}{4}x\right) + 1$$

13. no vertical stretch, up 3;
period: $\pi/6$,
VA: $x = \frac{\pi}{12} + \frac{\pi}{6}k$
D: $\{x \mid x \neq \frac{\pi}{12} + \frac{\pi}{6}k\}$
R: $(-\infty, \infty)$



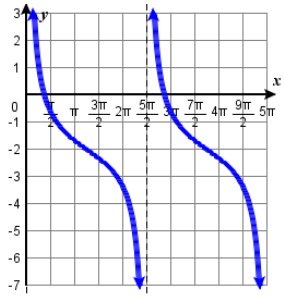
$$f(x) = \tan(6x) + 3$$

15. $y \to -3y$, down 1;
period: $\frac{2}{3}$,
VA: $x = \frac{1}{3} + \frac{2}{3}k$
D: $\{x \mid x \neq \frac{1}{3} + \frac{2}{3}k\}$
R: $(-\infty, \infty)$



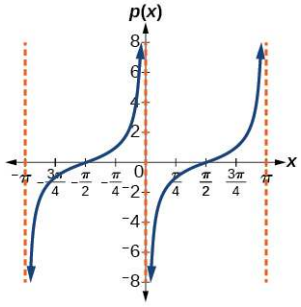
$$f(x) = -3\tan\left(\frac{3\pi}{2}x\right) - 1$$

17. no vertical stretch, down 2; period: $\frac{5\pi}{2}$,
VA: $x = \frac{5\pi}{2}k$
D: $\{x \mid x \neq \frac{5\pi}{2}k\}$
R: $(-\infty, \infty)$

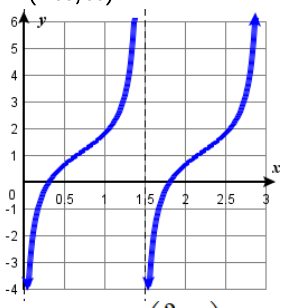


$$f(x) = \cot\left(\frac{2}{5}x\right) - 2$$

19. no V stretch, no V shift;
period: π ;
VA: $x = \pi k$
D: $\{x \mid x \neq \pi k\}$
R: $(-\infty, \infty)$

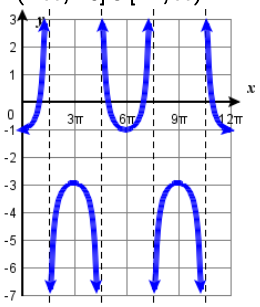


21. reflection over x-axis, up 1;
period: 1.5;
VA: $x = 1.5k$
D: $\{x \mid x \neq 1.5k\}$
R: $(-\infty, \infty)$



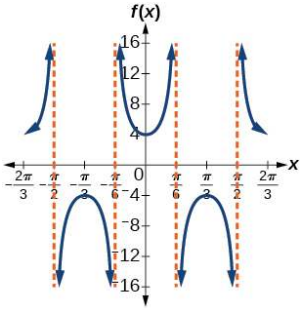
$$f(x) = -\cot\left(\frac{2\pi}{3}x\right) + 1$$

23. no vertical stretch, down 2;
period: 6π,
VA: $x = 1.5\pi + 3\pi k$
D: $\{x \mid x \neq 1.5\pi + 3\pi k\}$
R: $(-\infty, -3] \cup [-1, \infty)$

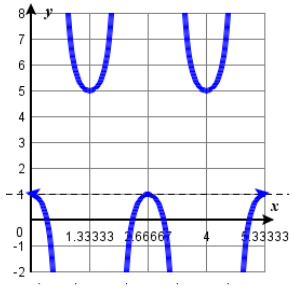


$$f(x) = \sec\left(\frac{1}{3}x\right) - 2$$

25. V stretching factor: 4;
period: $\frac{2\pi}{3}$;
VA: $x = \frac{\pi}{6} + \frac{\pi}{3}k$
D: $\{x \mid x \neq \frac{\pi}{6} + \frac{\pi}{3}k\}$
R: $(-\infty, -4] \cup [4, \infty)$

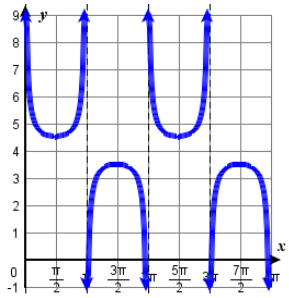


27. $y \to -2y$, up 3;
period: $\frac{8}{3}$,
VA: $x = \frac{2}{3} + \frac{4}{3}k$
D: $\{x \mid x \neq \frac{2}{3} + \frac{4}{3}k\}$
R: $(-\infty, 1] \cup [5, \infty)$



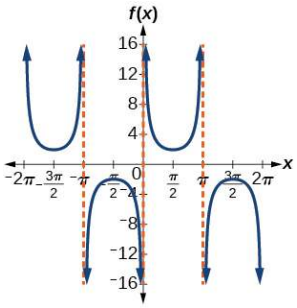
$f(x) = -2\sec\left(\frac{3\pi}{4}x\right) + 3$

29. $y \to \frac{1}{2}y$, up 4;
VA: 2π,
VA: $x = \pi k$
D: $\{x \mid x \neq \pi k\}$
R: $(-\infty, 3.5] \cup [4.5, \infty)$

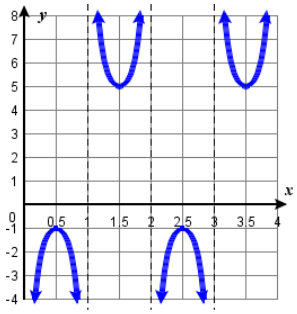


$f(x) = \frac{1}{2}\csc(x) + 4$

31. V stretch factor: 2;
period: 2π ;
VA: $x = \pi k$
D: $\{x \mid x \neq \pi k\}$
R: $(-\infty, -2] \cup [2, \infty)$



33. $y \to -3y$, up 2;
period: 2,
VA: $x = k$
D: $\{x \mid x \neq k\}$
R: $(-\infty, -1] \cup [5, \infty)$



$f(x) = -3\csc(\pi x) + 2$

C: Graph Equations with Phase Shifts

Exercise 5.6e.C

★ For each function, identify attributes of the graph: (a) Stretching factor and vertical shift, (b) period, (c) phase shift, and (d) asymptotes, (e) domain and range. (f) Sketch two periods of the graph.

41. $f(x) = \tan\left(x + \frac{\pi}{4}\right)$
42. $f(x) = 4\tan(\pi x - \pi) - 5$
43. $f(x) = 2\tan(4x - 32)$
44. $g(x) = 3\tan(6x + 42)$
45. $y = \cot\left(3x - \frac{\pi}{4}\right) + 2$
46. $f(x) = 5\cot\left(x + \frac{\pi}{2}\right) - 3$
49. $h(x) = 2\sec\left(\frac{\pi}{4}(x+1)\right)$
50. $f(x) = 3\sec\left(\frac{\pi}{2}x - \frac{\pi}{3}\right) - 2$
51. $f(x) = 2\sec\left(x - \frac{\pi}{4}\right) - 1$
52. $k(x) = 3\sec\left(2\left(x + \frac{\pi}{2}\right)\right)$
53. $m(x) = 6\csc\left(\frac{\pi}{3}x + \pi\right)$
54. $n(x) = 4\csc\left(\frac{5\pi}{3}x - \frac{20\pi}{3}\right) + 1$
55. $f(x) = \frac{7}{5}\csc\left(x - \frac{\pi}{4}\right)$
56. $n(x) = 4\csc\left(\frac{\pi}{4}x + \frac{\pi}{20}\right) + 3$

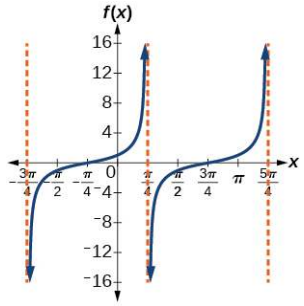
57. Construct an equation and graph a tangent curve, $A = 1$period of $\frac{\pi}{3}$and phase shift $(h, k) = \left(\frac{\pi}{4}, 2\right)$

58. Construct an equation and graph a tangent curve, $A = -2$, period of $\frac{\pi}{4}$; and phase shift $(h, k) = \left(-\frac{\pi}{4}, -2\right)$

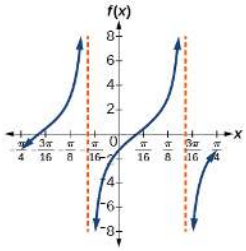
Answers to odd exercises:

k used below represents any integer

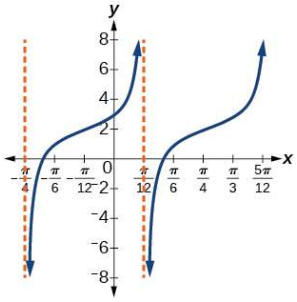
41. No V stretch, no V shift;
period: π; shift left $\frac{\pi}{4}$ units;
VA: $x = \frac{\pi}{4} + \pi k$
D: $\{x \mid x \neq \frac{\pi}{4} + \pi k\}$
R: $(-\infty, \infty)$



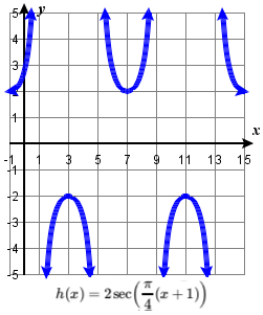
43. $y \to 2y$, no V shift;
period: $\frac{\pi}{4}$; shift right 8 units (the point (8, 0) is on the graph);
VA: $x = \frac{1}{4}\left(\frac{\pi}{2} + \pi k\right) + 8$
D: $\{x \mid x \neq \frac{1}{4}\left(\frac{\pi}{2} + \pi k\right) + 8\}$
R: $(-\infty, \infty)$



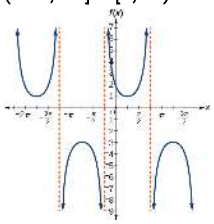
45. No vertical stretch, up 2;
period: $\pi/3$; shift right $\frac{\pi}{12}$;
VA: $x = \frac{\pi}{12} + \frac{\pi}{3}k$
D: $\{x \mid x \neq \frac{\pi}{12} + \frac{\pi}{3}k\}$
R: $(-\infty, \infty)$



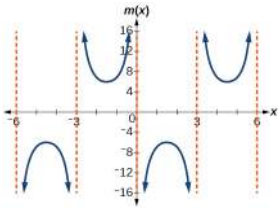
49. $y \to 2y$, no vertical shift;
period: 8; shift left 1 unit;
VA: $x = 1 + 4k$
D: $\{x \mid x \neq 1 + 4k\}$
R: $(-\infty, -2] \cup [2, \infty)$



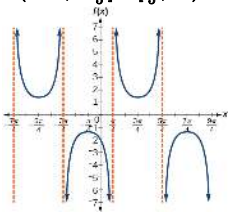
51. V stretch factor: 2; down 1
period: 2π; shift right $\frac{\pi}{4}$ units;
VA: $x = -\frac{\pi}{4} + \pi k$
D: $\{x \mid x \neq -\frac{\pi}{4} + \pi k\}$
R: $(-\infty, -3] \cup [1, \infty)$



53. V stretch factor: 6;
period: 6; shift left 3 units;
VA: $x = 3k$
D: $\{x \mid x \neq 3k\}$
R: $(-\infty, -6] \cup [6, \infty)$



55. V stretch factor: $\frac{7}{5}$;
VA: 2π; shift right $\frac{\pi}{4}$ units;
VA: $x = \frac{\pi}{4} + \pi k$
D: $\{x \mid x \neq \frac{\pi}{4} + \pi k\}$
R: $(-\infty, -\frac{7}{5}] \cup [\frac{7}{5}, \infty)$



D: Practice Negative Angle Identities

Exercise 5.6e.D

61. If $\tan x = -1.5$, find $\tan(-x)$.
62. If $\sec x = 2$, find $\sec(-x)$.
63. If $\csc x = -5$, find $\csc(-x)$.
64. If $x \sin x = 2$, find $(-x)\sin(-x)$.

65. If $\tan x = 3$, find $\tan(-x)$.
66. If $\sec x = -4$, find $\sec(-x)$.
67. If $\csc x = 2$, find $\csc(-x)$.

Rewrite each expression such that the argument x is positive.
69. $\cot(-x)\cos(-x) + \sin(-x)$
70. $\cos(-x) + \tan(-x)\sin(-x)$

Answers to odd exercises:

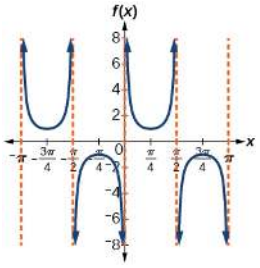
61. 1.5 63. 5 65. -3 67. -2 69. $-\cot x \cos x - \sin x$

E: Given a graph, construct an equation

Exercise 5.6e.E

★ Find an equation for the graph of each function.

71.



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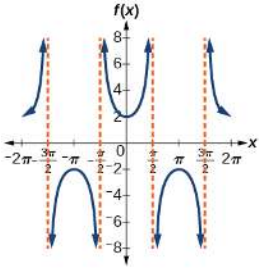

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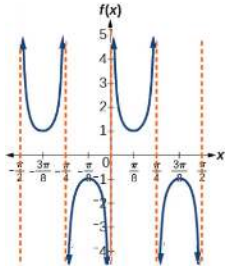

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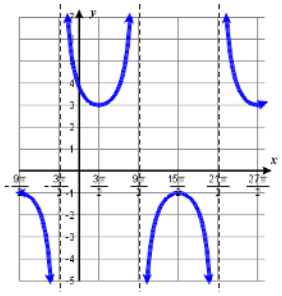

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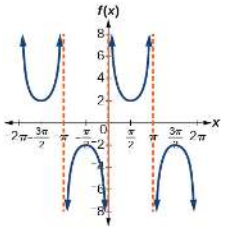

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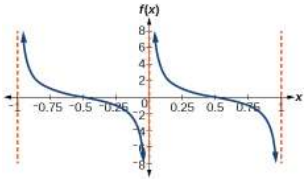

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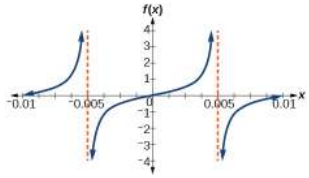

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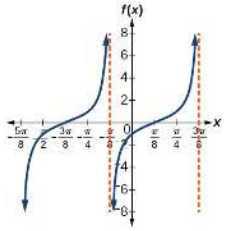

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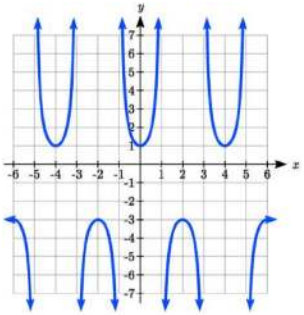

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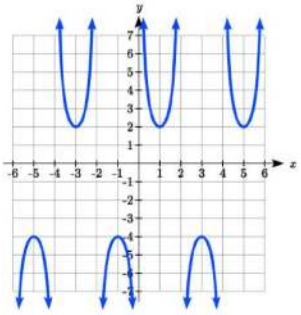

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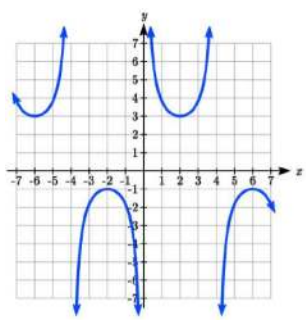
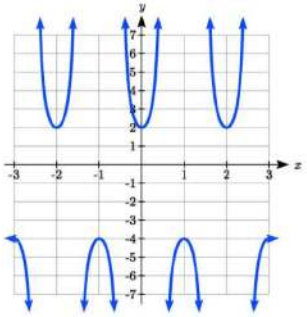


82.



Answers to odd exercises:

71. $f(x) = \csc(2x)$ 73. $f(x) = \csc(4x)$ 75. $f(x) = 2\csc x$ 77. $f(x) = \frac{1}{2}\tan(100\pi x)$
79. $f(x) = 2\sec(\frac{\pi}{2}x) - 1$ 81. $f(x) = 2\csc(\frac{\pi}{4}x) + 1$

F: Applications

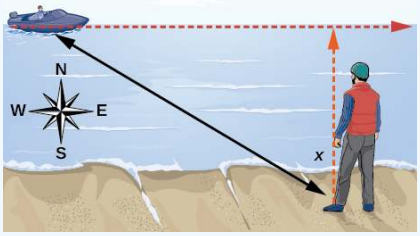
Exercise 5.6e.F

90. The function $f(x) = 20\tan\left(\frac{\pi}{10}x\right)$ marks the distance in the movement of a light beam from a police car across a wall for time x, in seconds, and distance $f(x)$, in feet.

a. Graph on the interval $[0, 5]$
b. Find and interpret the stretching factor, period, and asymptote.
c. Evaluate $f(10)$ and $f(2.5)$ and discuss the function's values at those inputs.

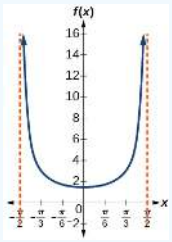
91. Standing on the shore of a lake, a fisherman sights a boat far in the distance to his left. Let x measured in radians, be the angle formed by the line of sight to the ship and a line due north from his position. Assume due north is 0 and x is measured negative to the left and positive to the right. (See Figure below.) The boat travels from due west to due east and, ignoring the curvature of the Earth, the distance $d(x)$ in kilometers, from the fisherman to the boat is given by the function $d(x) = 1.5\sec(x)$

a. What is a reasonable domain for $d(x)$?
b. Graph $d(x)$ on this domain.
c. Find and discuss the meaning of any vertical asymptotes on the graph of $d(x)$.
d. Calculate and interpret $d\left(-\frac{\pi}{3}\right)$. Round to the second decimal place.
e. Calculate and interpret $d\left(\frac{\pi}{6}\right)$. Round to the second decimal place.
f. What is the minimum distance between the fisherman and the boat? When does this occur?



91. Answer

a. $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
b. see diagram on the right -->
c. $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$ the distance grows without bound as $|x|$ approaches $\frac{\pi}{2}$—i.e., at right angles to the line representing due north, the boat would be so far away, the fisherman could not see it;
d. 3; when $x = -\frac{\pi}{3}$ the boat is 3 km away;
e. 1.73; when $x = \frac{\pi}{6}$ the boat is about 1.73 km away;
f. 1.5 km; when $x = 0$



92. A laser rangefinder is locked on a comet approaching Earth. The distance $g(x)$, in kilometers, of the comet after x days, for x in the interval 0 to 30 days, is given by $g(x) = 250,000 \csc\left(\frac{\pi}{30}x\right)$.

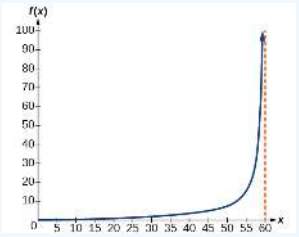
a. Graph $g(x)$ on the interval $[0, 35]$.
b. Evaluate $g(5)$ and interpret the information.
c. What is the minimum distance between the comet and Earth? When does this occur? To which constant in the equation does this correspond?
d. Find and discuss the meaning of any vertical asymptotes.

93. A video camera is focused on a rocket on a launching pad 2 miles from the camera. The angle of elevation from the ground to the rocket after x seconds is $\frac{\pi}{120}x$.

a. Write a function expressing the altitude $h(x)$, in miles, of the rocket above the ground after x seconds. Ignore the curvature of the Earth.
b. Graph $h(x)$ on the interval $(0, 60)$.
c. Evaluate and interpret the values $h(0)$ and $h(30)$.
d. What happens to the values of $h(x)$ as x approaches 60 seconds? Interpret the meaning of this in terms of the problem.

93. Answer

a. $h(x) = 2\tan\left(\frac{\pi}{120}x\right)$
b. $h(0) = 0$after 0 seconds, the rocket is 0 mi above the ground; $h(30) = 2$after 30 seconds, the rockets is 2 mi high;
c. As x approaches 60 seconds, the values of $h(x)$ grow increasingly large. The distance to the rocket is growing so large that the camera can no longer track it.



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CHAPTER OVERVIEW

6: Analytic Trigonometry

In this chapter, we discuss how to manipulate trigonometric equations algebraically by applying various formulas and trigonometric identities. We will also investigate some of the ways that trigonometric equations are used to model real-life phenomena.

6.1: Inverse Trigonometric Functions

6.1e: Exercises - Inverse Trigonometric Functions

6.2: Trigonometric Equations

6.2e: Exercises - Trig Equations

6.3: Verifying Trigonometric Identities

6.3e: Verifying Trigonometric Identities

6.4: Sum and Difference Identities

6.4e: Exercises - Sum and Difference Identities

6.5: Double-Angle, Half-Angle, and Reduction Formulas

6.5e: Exercises: Double Angle, Half Angle and Reductions Formulas

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6.1: Inverse Trigonometric Functions

Learning Objectives

- Understand and use the inverse sine, cosine, and tangent functions.
- Find the exact value of expressions involving the inverse sine, cosine, and tangent functions.
- Use a calculator to evaluate inverse trigonometric functions.
- Find exact values of composite functions with inverse trigonometric functions.

For any right triangle, given one other angle and the length of one side, we can figure out what the other angles and sides are. But what if we are given only two sides of a right triangle? We need a procedure that leads us from a ratio of sides to an angle. This is where the notion of an inverse to a trigonometric function comes into play. In this section, we will explore the inverse trigonometric functions.

Understanding and Using the Inverse Sine, Cosine, and Tangent Functions

In order to use inverse trigonometric functions, we need to understand that an inverse trigonometric function "undoes" what the original trigonometric function "does," as is the case with any other function and its inverse. In other words, the domain of the inverse function is the range of the original function, and vice versa, as summarized in Figure **6.1.1***a*.

Trig Functions	Inverse Trig Functions
Domain: Measure of an angle	Domain: Ratio
Range: Ratio	Range: Measure of an angle

Figure **6.1.1***a*

For example, if $f(x) = \sin\ x$, then we would write $f^{-1}(x) = \sin^{-1}x$. Be aware that $\sin^{-1}x$ does not mean $\frac{1}{\sin\ x}$. The following examples illustrate the inverse trigonometric functions:

- Since $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, then $\frac{\pi}{6} = \sin^{-1}\left(\frac{1}{2}\right)$.
- Since $\cos(\pi) = -1$, then $\pi = \cos^{-1}(-1)$.
- Since $\tan\left(\frac{\pi}{4}\right) = 1$, then $\frac{\pi}{4} = \tan^{-1}(1)$.

In previous sections, we evaluated the trigonometric functions at various angles, but at times we need to know what angle would yield a specific sine, cosine, or tangent value. For this, we need inverse functions. Recall that, for a one-to-one function, if $f(a) = b$, then an inverse function would satisfy $f^{-1}(b) = a$.

Bear in mind that the sine, cosine, and tangent functions are not one-to-one functions. The graph of each function would fail the horizontal line test. In fact, no periodic function can be one-to-one because each output in its range corresponds to at least one input in every period, and there are an infinite number of periods. As with other functions that are not one-to-one, we will need to restrict the **domain** of each function to yield a new function that is one-to-one. We choose a domain for each function that includes the number 0. Figure **6.1.1***b* shows how the domains are restricted to obtain one-to-one functions.

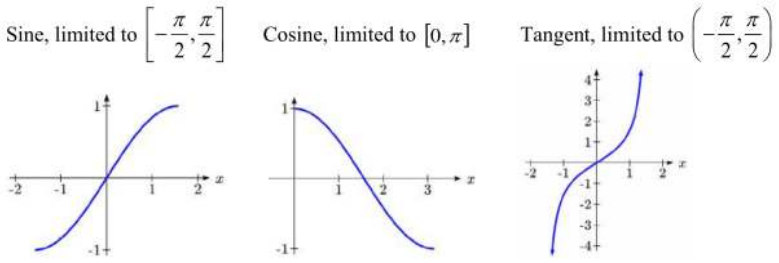

Figure **6.1.1***b*: Domain restrictions applied to sine, cosine and tangent in order to define inverse functions for these trigonometric functions.

These conventional choices for the restricted domain are somewhat arbitrary, but they have important, helpful characteristics. Each domain includes the origin and some positive values, and most importantly, each results in a one-to-one function that is invertible.

The conventional choice for the restricted domain of the tangent function also has the useful property that it extends from one **vertical asymptote** to the next instead of being divided into two parts by an asymptote.

On these restricted domains, we can define the inverse trigonometric functions.

- The inverse sine function $y = \sin^{-1} x$ means $x = \sin\ y$. The inverse sine function is sometimes called the **arcsine** function, and notated $\arcsin\ x$.

$$y = \sin^{-1} x \text{ has domain } [-1, 1] \text{ and range } [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]$$

- The inverse cosine function $y = \cos^{-1} x$ means $x = \cos\ y$. The inverse cosine function is sometimes called the **arccosine** function, and notated $\arccos\ x$.

$$y = \cos^{-1} x \text{ has domain } [-1, 1] \text{ and range } [0, \pi]$$

- The inverse tangent function $y = \tan^{-1} x$ means $x = \tan\ y$. The inverse tangent function is sometimes called the **arctangent** function, and notated $\arctan\ x$.

$$y = \tan^{-1} x \text{ has domain } (-\infty, \infty) \text{ and range } (-\tfrac{\pi}{2}, \tfrac{\pi}{2})$$

The graphs of the inverse functions are shown in Figure **6.1.1c**. Notice that the output of each of these inverse functions is an *angle* in radian measure. We see that $\sin^{-1} x$ has domain $[-1, 1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\cos^{-1} x$ has domain $[-1, 1]$ and range $[0, \pi]$, and $\tan^{-1} x$ has domain of all real numbers and range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. To find the domain and range of inverse trigonometric functions, switch the domain and range of the original domain restricted functions.

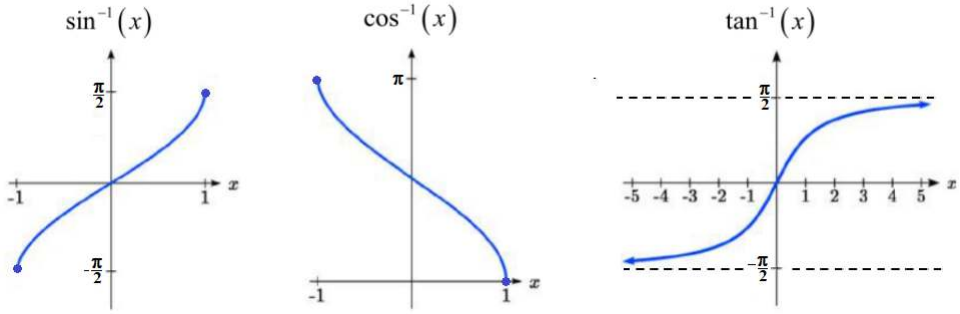

Figure **6.1.1c**: The inverse sine , inverse cosine, and inverse tangent

Recall we previously defined a reference angle, often denoted α, to be an angle between 0 and $\frac{\pi}{2}$. Any defined inverse trigonometric function with a positive argument will return an angle that can be regarded as a reference angle in Quadrant I. If the argument is negative, the result will be an angle in standard position either in Quadrant IV ($\sin^{-1}$ or $\tan^{-1}$) or in Quadrant II ($\cos^{-1}$). A summary of these definitions is below.

INVERSE SINE, COSINE, AND TANGENT FUNCTIONS

Given that the reference angle of y is α,

$$\sin^{-1} x = y \quad \leftrightarrow \quad \sin y = x, \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2} \qquad \text{So } \sin^{-1} x = \begin{cases} \text{undefined} & \text{if } |x| > 1 \quad \text{because } -1 \le \sin y \le 1 \\ \alpha & \text{if } x \ge 0, \quad \alpha \text{ is in QI} \\ -\alpha & \text{if } x < 0, \quad -\alpha \text{ is in QIV} \end{cases}$$

$$\cos^{-1} x = y \quad \leftrightarrow \quad \cos y = x, \quad 0 \le y \le \pi \qquad \text{So } \cos^{-1} x = \begin{cases} \text{undefined} & \text{if } |x| > 1 \quad \text{because } -1 \le \cos y \le 1 \\ \alpha & \text{if } x \ge 0, \quad \alpha \text{ is in QI} \\ \pi - \alpha & \text{if } x < 0, \quad \pi - \alpha \text{ is in QII} \end{cases}$$

$$\tan^{-1} x = y \quad \leftrightarrow \quad \tan y = x, \quad -\frac{\pi}{2} < y < \frac{\pi}{2} \qquad \text{So } \tan^{-1} x = \begin{cases} \alpha & \text{if } x \ge 0, \quad \alpha \text{ is in QI} \\ -\alpha & \text{if } x < 0, \quad -\alpha \text{ is in QIV} \end{cases}$$

Example 6.1.1: Write a Trig Function in its Inverse Form

Given $\sin\left(\frac{5\pi}{12}\right) \approx 0.96593$, write a relation involving the inverse sine.

Solution

Use the relation for the inverse sine. If $\sin y = x$, then $\sin^{-1} x = y$.

In this problem, $x = 0.96593$, and $y = \frac{5\pi}{12}$.

$\sin^{-1}(0.96593) \approx \frac{5\pi}{12}$

Try It 6.1.1

Given $\cos(0.5) \approx 0.8776$, write a relation involving the inverse cosine.

Answer

$\arccos(0.8776) \approx 0.5$

Find Exact Special Angle Values for Inverse Sine, Cosine, and Tangent Functions

Now that we can identify inverse functions, we will learn to evaluate them. For most values in their domains, we must evaluate the inverse trigonometric functions by using a calculator, interpolating from a table, or using some other numerical technique. Just as we did with the original trigonometric functions, we can give exact values for the inverse functions when we are using the special angles, specifically $\frac{\pi}{6}$ (or 30°), $\frac{\pi}{4}$ (or 45°), and $\frac{\pi}{3}$ (or 60°), and their reflections into other quadrants.

Given a "special" input value, evaluate an inverse trigonometric function.

1. Find angle x for which the original trigonometric function has an output equal to the given input for the inverse trigonometric function.
2. If x is not in the defined range of the inverse, find another angle y that is in the defined range and has the same sine, cosine, or tangent as x, depending on which corresponds to the given inverse function.

Example 6.1.2: Evaluating Inverse Trigonometric Functions for Special Input Values

Evaluate each of the following.

a. $\sin^{-1}\left(\frac{1}{2}\right)$	b. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$	c. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$	d. $\tan^{-1}(1)$

Solution

a. Evaluating $\sin^{-1}\left(\frac{1}{2}\right)$ is the same as determining the angle that would have a sine value of $\frac{1}{2}$. In other words, what angle x would satisfy $\sin(x) = \frac{1}{2}$? There are multiple values that would satisfy this relationship, such as $\frac{\pi}{6}$ and $\frac{5\pi}{6}$, but we know we need the angle in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so the answer will be $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$. Remember that the inverse is a function, so for each input, we will get exactly one output.

b. To evaluate $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$, we know that $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$ both have a sine value of $-\frac{\sqrt{2}}{2}$, but neither is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. For that, we need the negative angle coterminal with $\frac{7\pi}{4}$: $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$.

c. To evaluate $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$, we are looking for an angle in the interval $[0, \pi]$ with a cosine value of $-\frac{\sqrt{3}}{2}$. The angle that satisfies this is $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$.

d. Evaluating $\tan^{-1}(1)$, we are looking for an angle in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with a tangent value of 1. The correct angle is $\tan^{-1}(1) = \frac{\pi}{4}$.

Try It 6.1.2

Evaluate each of the following.

a. $\sin^{-1}(-1)$	b. $\tan^{-1}(-1)$	c. $\cos^{-1}(-1)$	d. $\cos^{-1}\left(\frac{1}{2}\right)$
Answer a $-\frac{\pi}{2}$	**Answer b** $-\frac{\pi}{4}$	**Answer c** π	**Answer d** $\frac{\pi}{3}$

Use a Calculator to Evaluate Inverse Trigonometric Functions

To evaluate **inverse trigonometric functions** that do not involve the special angles discussed previously, we will need to use a calculator or other type of technology. Most scientific calculators and calculator-emulating applications have specific keys or buttons for the inverse sine, cosine, and tangent functions. These may be labeled, for example, SIN^{-1}, ARCSIN, or ASIN.

In the previous chapter, we worked with trigonometry on a right triangle to solve for the sides of a triangle given one side and an additional angle. Using the inverse trigonometric functions, we can solve for the angles of a right triangle given two sides, and we can use a calculator to find the values to several decimal places.

In these examples and exercises, the answers will be interpreted as angles and we will use θ as the independent variable. The value displayed on the calculator may be in degrees or radians, so be sure to set the mode appropriate to the application.

Example 6.1.3: Evaluate Inverse Sine on a Calculator

Evaluate $\sin^{-1}(0.97)$ using a calculator.

Solution

Because the output of the inverse function is an angle, the calculator will give us a degree value if in degree mode and a radian value if in radian mode. Calculators also use the same domain restrictions on the angles as we are using.

In radian mode, $\sin^{-1}(0.97) \approx 1.3252$. In degree mode, $\sin^{-1}(0.97) \approx 75.93°$. Note that in calculus and beyond we will use radians in almost all cases.

Try It 6.1.3

Evaluate $\cos^{-1}(-0.4)$ using a calculator.

Answer

1.9823 or $113.578°$

Given two sides of a right triangle, find an angle.

h is the length of the **hypotenuse**

a is the length of the side **adjacent** to the desired angle

p is the length of the side **opposite** to the desired angle

θ is the "desired" angle with unknown measure between the hypotenuse and the adjacent side

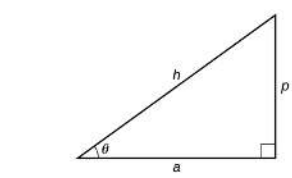

Figure 6.1.3

1. Given h and a, find the angle using $\theta = \cos^{-1}\left(\frac{a}{h}\right)$ and the other side using $p = \sqrt{h^2 - a^2}$.
2. Given h and p, find the angle using $\theta = \sin^{-1}\left(\frac{p}{h}\right)$ and the other side using $a = \sqrt{h^2 - p^2}$.
3. Given p and a find the angle using $\theta = \tan^{-1}\left(\frac{p}{a}\right)$ and the other side using $h = \sqrt{p^2 + a^2}$.

Example 6.1.4: Applying the Inverse Cosine to a Right Triangle

Solve the triangle in Figure 6.1.4 for the angle θ.

Solution

Because we know the hypotenuse and the side adjacent to the angle, it makes sense for us to use the cosine function.

$$\cos\theta = \frac{9}{12}$$
$$\theta = \cos^{-1}\left(\frac{9}{12}\right) \quad \text{Apply definition of the inverse}$$
$$\theta \approx 0.7227 \quad \text{or about } 41.4096°$$

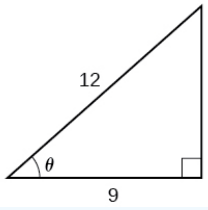

Figure 6.1.4

Try It 6.1.4

Solve the triangle in Figure 6.1.4t for the angle θ.

Answer

$\sin^{-1}(0.6) = 36.87° = 0.6435$ radians

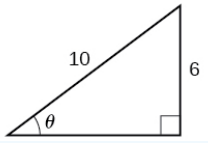

Figure 6.1.4t

Finding Exact Values of Composite Functions with Inverse Trigonometric Functions

There are times when we need to compose a trigonometric function with an inverse trigonometric function. In these cases, we can usually find exact values for the resulting expressions without resorting to a calculator. Even when the input to the composite function is a variable or an expression, we can often find an expression for the output. To help sort out different cases, let $f(x)$ and $g(x)$ be two different trigonometric functions belonging to the set{ $\sin(x)$, $\cos(x)$, $\tan(x)$ } and let $f^{-1}(y)$ and $g^{-1}(y)$ be their inverses.

Evaluating Compositions of the Form $f(f^{-1}(y))$ and $f^{-1}(f(x))$

For any trigonometric function, $f(f^{-1}(y)) = y$ for all y in the proper domain for the given function. This follows from the definition of the inverse and from the fact that the range of f was defined to be identical to the domain of f^{-1}. However, we have to be a little more careful with expressions of the form $f^{-1}(f(x))$.

COMPOSITIONS OF A TRIGONOMETRIC FUNCTION AND ITS INVERSE

For all x below, α is the reference angle for x

$\sin(\sin^{-1}(x)) = \begin{cases} x & \text{if } -1 \le x \le 1 \\ \text{Undefined} & \text{if } \lvert x\rvert > 1 \end{cases}$	$\sin^{-1}(\sin(x)) = \begin{cases} x & \text{if } -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ \alpha & \text{if } x \ge 0 \\ -\alpha & \text{if } x < 0 \end{cases}$
$\cos(\cos^{-1}(x)) = \begin{cases} x & \text{if } -1 \le x \le 1 \\ \text{Undefined} & \text{if } \lvert x\rvert > 1 \end{cases}$	$\cos^{-1}(\cos(x)) = \begin{cases} x & \text{if } \lvert 0 \le x \le \pi \\ \alpha & \text{if } x \ge 0 \\ \pi - \alpha & \text{if } x < 0 \end{cases}$
$\tan(\tan^{-1}(x)) = \quad x \qquad$ for all real numbers x	$\tan^{-1}(\tan(x)) = \begin{cases} x & \text{if } -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ \alpha & \text{if } x \ge 0 \\ -\alpha & \text{if } x < 0 \end{cases}$

Is it correct that $\sin^{-1}(\sin x) = x$?

No. This equation is correct only if x belongs to the restricted domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, but sine is defined for all real input values, and for x outside the restricted interval, the equation is not correct because its inverse always returns a value in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The situation is similar for cosine and tangent and their inverses. For example, $\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right) = \frac{\pi}{4}$.

Given an expression of the form $f^{-1}(f(\theta))$ where $f(\theta) = \sin\theta$, $\cos\theta$, or $\tan\theta$, evaluate.

1. If θ is in the restricted domain of f, then $f^{-1}(f(\theta)) = \theta$.
2. If not, then find an angle ϕ within the restricted domain of f such that $f(\phi) = f(\theta)$. Then $f^{-1}(f(\theta)) = \phi$.

Example 6.1.5: Using Inverse Trigonometric Functions

Evaluate the following:

a. $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$	b. $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$	c. $\cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right)$	d. $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right)$

Solution

1. $\frac{\pi}{3}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$.
2. $\frac{2\pi}{3}$ is not in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, but $\sin\left(\frac{2\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right)$, so $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \frac{\pi}{3}$.
3. $\frac{2\pi}{3}$ is in $[0, \pi]$, so $\cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right) = \frac{2\pi}{3}$.
4. $-\frac{\pi}{3}$ is not in $[0, \pi]$, but $\cos\left(-\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right)$ because cosine is an even function. $\frac{\pi}{3}$ is in $[0, \pi]$, so $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$.

Try It 6.1.5

Evaluate $\tan^{-1}\left(\tan\left(\frac{\pi}{8}\right)\right)$ and $\tan^{-1}\left(\tan\left(\frac{11\pi}{9}\right)\right)$.

Answer

$\frac{\pi}{8}; \frac{2\pi}{9}$

Evaluating Compositions of the Form $f^{-1}(g(x))$

Now that we can compose a trigonometric function with its inverse, we can explore how to evaluate a composition of a trigonometric function and the inverse of another trigonometric function. We will begin with compositions of the form $f^{-1}(g(x))$. For special values of x, we can exactly evaluate the inner function and then the outer, inverse function.

Example 6.1.6: Evaluating the Composition of an Inverse Sine with a Cosine

Evaluate $\sin^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right)$

Solution

We can directly evaluate the inside of the composition. $\cos\left(\frac{13\pi}{6}\right)=\cos\left(\frac{\pi}{6}+2\pi\right)=\cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$

Now, we can evaluate the inverse function as we did earlier. $\sin^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right)=\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{3}$

Try It 6.1.6

Evaluate $\cos^{-1}\left(\sin\left(-\frac{11\pi}{4}\right)\right)$.

Answer

$\frac{3\pi}{4}$

Evaluating Compositions of the Form $f(g^{-1}(x))$

To evaluate compositions of the form $f(g^{-1}(x))$, where f and g are any two of the functions sine, cosine, or tangent and x is any input in the domain of g^{-1}, sometimes Identities can be used, like the Pythagorean identity, $\sin^2 x+\cos^2 x=1$. Another approach uses the fact that the inner function is an inverse trigonometric function, and therefore tells us the quadrant the angle is in and the value of a trigonometric ratio for that angle. From this information, the other side of the triangle, and any other trigonometric ratio can be obtained.

Example 6.1.7: Evaluating the Composition of a Sine with an Inverse Cosine

Find an exact value for $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right)$.

Solution

A restatement of the problem is: Find $\sin\theta$, given that $\theta=\cos^{-1}\left(\frac{4}{5}\right)$.

Method I. Using Definitions

The inverse cosine equation gives us two pieces of information.

- The cosine ratio ($4/5$) is positive, so the angle θ is in Quadrant I, where x and y are both positive.
- $\cos\theta=\frac{4}{5}=\frac{x}{r}$ so $\boxed{x=4}$ and $\boxed{r=5}$

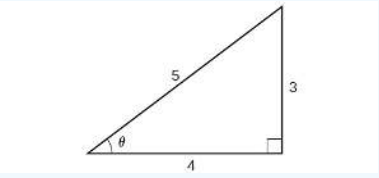

Figure 6.1.7: Right triangle illustrating that if $\cos\theta=\frac{4}{5}$, then $\sin\theta=\frac{3}{5}$

Use the Pythagorean Theorem to find the third side.
y will be positive because θ is in Quadrant I.
$x^2+y^2=r^2\rightarrow 4^2+y^2=5^2\rightarrow y^2=25-16$, so $y=\pm 3$ but since θ is in Quadrant I, $\boxed{y=3}$.

Use the definition of the sine ratio: $\sin\theta=\frac{y}{r}=\frac{3}{5}$.

Method II. Using Identities.

We can say that $\theta=\cos^{-1}\left(\frac{4}{5}\right)$ means $\cos\theta=\frac{4}{5}$.

We are looking for $\sin\theta$. We can use the Pythagorean identity to do this.

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 && \text{Use our known value for cosine} \\ \sin^2\theta + \left(\frac{4}{5}\right)^2 &= 1 && \text{Solve for sine} \\ \sin^2\theta &= 1 - \frac{16}{25} \\ \sin\theta &= \pm\frac{9}{25} \\ &= \pm\frac{3}{5} \end{aligned}$$

We know that the inverse cosine always gives an angle on the interval $[0, \pi]$, so we know that the sine of that angle must be positive. Since $\theta = \cos^{-1}\left(\frac{4}{5}\right)$ is in quadrant I, $\sin\theta$ must be positive, so the solution is \(\dfrac{3}{5}\

Try It 6.1.7

Evaluate $\cos\left(\tan^{-1}\left(\frac{5}{12}\right)\right)$.

Answer

$\frac{12}{13}$

Example 6.1.8: Evaluating the Composition of a Sine with an Inverse Tangent

Find an exact value for $\sin\left(\tan^{-1}\left(-\frac{7}{4}\right)\right)$.

Solution

Restating the problem: Find $\sin\theta$, given that $\theta = \tan^{-1}\left(-\frac{7}{4}\right)$.

From $\theta = \tan^{-1}\left(-\frac{7}{4}\right)$ we can obtain 2 pieces of information.

- Inverse tangent always gives an angle in QI or QIV, but since the tangent ratio ($-7/4$) is negative, the angle is in Quadrant IV where x is positive and y is negative.
- Rewriting the inverse tangent expression, and applying the definition of tangent, we can obtain $\tan\theta = -\frac{7}{4} = \frac{y}{x=4}$ so $\boxed{x=4}$ and $\boxed{y=-7}$

Use the Pythagorean Theorem $r^2 = x^2 + y^2$ to find r.
$r = \sqrt{(4)^2 + (-7)^2} = \sqrt{16+49}$.
r is always positive so $\boxed{r = \sqrt{65}}$.

Using the definition of the sine function, and the diagram of angle θ in standard position, we find $\sin\theta = \frac{y}{r} = \frac{-7}{\sqrt{65}} = -\frac{7\sqrt{65}}{65}$.

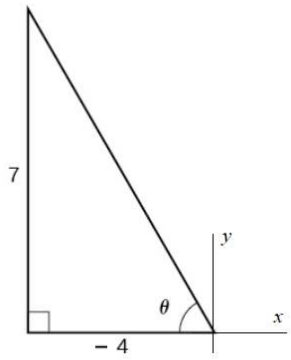

Figure 6.1.8: A right triangle with two sides known

Try It 6.1.8

Evaluate $\cos\left(\sin^{-1}\left(-\frac{7}{9}\right)\right)$.

Answer

$\frac{4\sqrt{2}}{9}$

Example 6.1.9: Finding the Cosine of the Inverse Sine of an Algebraic Expression

Find a simplified expression for $\cos\left(\sin^{-1}\left(\frac{u}{3}\right)\right)$. State any restrictions to u.

Solution

Restating the problem: Find an expression for $\cos\theta$ given $\theta = \sin^{-1}\left(\frac{u}{3}\right)$

Method using Definitions

The inverse sine function gives us two pieces of information:

1. θ will be in either the first or fourth quadrants, where we know x is only positive.
2. $\sin\theta = \left(\frac{u}{3}\right) = \left(\frac{y}{r}\right)$, so therefore, $\boxed{y = u}$, and $\boxed{r = 3}$.

Use the Pythagorean Theorem to find x:

$x^2 + y^2 = r^2 \rightarrow x^2 + u^2 = (3)^2 \rightarrow x^2 = 9 - u^2$.

Therefore, $\boxed{x = \sqrt{9 - u^2}}$

Finally, using the definition for cosine, we obtain $\cos\theta = \frac{x}{r} = \frac{\sqrt{9-u^2}}{3}$

Method Using Identities

We know there is an angle θ such that $\sin\theta = \frac{u}{3}$.

$\sin^2\theta + \cos^2\theta = 1$ Use the Pythagorean Theorem

$\left(\frac{u}{3}\right)^2 + \cos^2\theta = 1$ Solve for cosine

$\cos^2\theta = 1 - \frac{u^2}{9}$

$\cos\theta = \pm\sqrt{\frac{9-u^2}{9}}$

$= \pm\frac{\sqrt{9-u^2}}{3}$

Because we know that the inverse sine must give an angle on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, we can deduce that the cosine of that angle must be positive.

$\cos\left(\sin^{-1}\left(\frac{u}{3}\right)\right) = \frac{\sqrt{9-u^2}}{3}$

Given $\sin^{-1}\left(\frac{u}{3}\right)$ then $\sin\theta = \frac{u}{3}$. Since $-1 \le \sin\theta \le 1$ then $-1 \le \frac{u}{3} \le 1$ and thus $-3 \le u \le 3$.

Try It 6.1.9

Find a simplified expression for $\sin(\tan^{-1}(4u))$. State any restrictions to u.

Answer

$\frac{4u}{\sqrt{16^2+1}}$. No restrictions to u because $\tan\theta$ can be any real number.

Key Concepts

- An inverse function is one that "undoes" another function. The domain of an inverse function is the range of the original function and the range of an inverse function is the domain of the original function.
- Because the trigonometric functions are not one-to-one on their natural domains, their domains are restricted so that inverse trigonometric functions can be defined for them.
- For any trigonometric function $f(x)$, if $x = f^{-1}(y)$, then $f(x) = y$. However, $f(x) = y$ only implies $x = f^{-1}(y)$ if x is in the restricted domain of f.
- Special angles are the outputs of inverse trigonometric functions for special input values; for example, $\frac{\pi}{4} = \tan^{-1}(1)$ and $\frac{\pi}{6} = \sin^{-1}(\frac{1}{2})$.
- A calculator will return an angle within the restricted domain of the original trigonometric function.
- Inverse functions allow us to find an angle when given two sides of a right triangle.
- In function composition, if the inside function is an inverse trigonometric function, then the result of the composition is an exact expression; for example,$\sin(\cos^{-1}(x)) = \sqrt{1-x^2}$.

- When evaluating the composition of a trigonometric function with an inverse trigonometric function, draw a reference triangle to assist in determining the ratio of sides that represents the output of the trigonometric function.
- When evaluating the composition of a trigonometric function with an inverse trigonometric function, you may use trig identities to assist in determining the ratio of sides.

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6.1e: Exercises - Inverse Trigonometric Functions

A: Concepts.

Exercise 6.1e.A

1. Why do the functions $f(x) = \sin^{-1} x$ and $g(x) = \cos^{-1} x$ have different ranges?
2. Since the functions $y = \cos x$ and $y = \cos^{-1} x$ are inverse functions, why is $\cos^{-1}\left(\cos\left(-\frac{\pi}{6}\right)\right)$ not equal to $-\frac{\pi}{6}$?
3. Explain the meaning of $\frac{\pi}{6} = \arcsin(0.5)$.
4. Most calculators do not have a key to evaluate $\sec^{-1}(2)$. Explain how this can be done using the cosine function or the inverse cosine function.
5. Why must the domain of the sine function, $\sin x$, be restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for the inverse sine function to exist?
6. Discuss why this statement is incorrect: $\arccos(\cos x) = x$ for all x.
7. Determine whether the following statement is true or false and explain your answer: $\arccos(-x) = \pi - \arccos x$

Answers to odd exercises.

1. The function $y = \sin x$ is one-to-one on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$; thus, this interval is the range of the inverse function of $y = \sin x$, $f(x) = \sin^{-1} x$ The function $y = \cos x$ is one-to-one on $[0, \pi]$; thus, this interval is the range of the inverse function of $y = \cos x$, $f(x) = \cos^{-1} x$

3. $\frac{\pi}{6}$ is the radian measure of an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is 0.5.

5. In order for any function to have an inverse, the function must be one-to-one and must pass the horizontal line test. The regular sine function is not one-to-one unless its domain is restricted in some way. Mathematicians have agreed to restrict the sine function to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so that it is one-to-one and possesses an inverse.

7. True . The angle, θ_1 that equals $\arccos(-x)$, $x > 0$, will be a second quadrant angle with reference angle, θ_2, where θ_2 equals $\arccos x$, $x > 0$. Since θ_2 is the reference angle for θ_1, $\theta_2 = \pi - \theta_1$ and $\arccos(-x) = \pi - \arccos x$—add texts here. Do not delete this text first.

B: Evaluate Inverse Trigonometric Functions for "Special Angles"

Exercise 6.1e.B

★ Evaluate the expressions.

9. $\sin^{-1}(0)$
10. $\cos^{-1}(0)$
11. $\tan^{-1}(0)$
12. $\sin^{-1}(1)$
13. $\cos^{-1}(1)$
14. $\tan^{-1}(1)$
15. $\sin^{-1}\left(\frac{1}{2}\right)$
16. $\cos^{-1}\left(\frac{1}{2}\right)$
17. $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$
18. $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$
19. $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$
20. $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$
21. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$
22. $\tan^{-1}(\sqrt{3})$
23. $\sin^{-1}(-1)$
24. $\cos^{-1}(-1)$
25. $\tan^{-1}(-1)$
26. $\sin^{-1}\left(-\frac{1}{2}\right)$
27. $\cos^{-1}\left(-\frac{1}{2}\right)$
28. $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$
29. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$
30. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$
31. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
32. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
33. $\tan^{-1}(-\sqrt{3})$
34. $\csc^{-1}(-2)$
35. $\cot^{-1}(-\sqrt{3})$
36. $\sec^{1}\left(\frac{-2\sqrt{3}}{3}\right)$

Answers to odd exercises.

9. 0 11. 0 13. 0 15. $\frac{\pi}{6}$ 17. $\frac{\pi}{6}$ 19. $\frac{\pi}{4}$ 21. $\frac{\pi}{6}$
23. $-\frac{\pi}{2}$ 25. $-\frac{\pi}{4}$ 27. $\frac{2\pi}{3}$ 29. $\frac{3\pi}{4}$ 31. $-\frac{\pi}{3}$ 33. $-\frac{\pi}{3}$ 35. $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

C: Evaluate Inverse Trigonometric Functions with a Calculator

Exercise 6.1e.C

★ **Use a calculator to evaluate each expression. Express answers to the nearest hundredth.**

37. $\cos^{-1}(-0.4)$	38. $\arcsin(0.23)$	39. $\tan^{-1}(6)$	40 $\sec^{-1}(2.5)$	41. $\csc^{-1}(1.5)$	42. $\cot^{-1}(5)$

Answers to odd exercises.

37. 1.98 39. 1.41 41. 0.73

D: Evaluate $f^{-1}(f(\theta))$ Compositions

Exercise 6.1e.D

★ **Evaluate without a calculator.**

45. $\sin^{-1}\left(\sin\left(\frac{\pi}{10}\right)\right)$
46. $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$
47. $\sin^{-1}\left(\sin\left(\frac{5\pi}{4}\right)\right)$
48. $\sin^{-1}\left(\sin\left(-\frac{9\pi}{8}\right)\right)$
49. $\sin^{-1}\left(\sin\left(-\frac{3\pi}{2}\right)\right)$
50. $\sin^{-1}\left(\sin\left(-\frac{\pi}{7}\right)\right)$
51. $\cos^{-1}\left(\cos\left(\frac{\pi}{10}\right)\right)$
52. $\cos^{-1}\left(\cos\left(\frac{17\pi}{9}\right)\right)$
53. $\cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right)$
54. $\cos^{-1}\left(\cos\left(-\frac{11\pi}{6}\right)\right)$
55. $\cos^{-1}\left(\cos\left(-\frac{4\pi}{5}\right)\right)$
56. $\cos^{-1}\left(\cos\left(-\frac{\pi}{7}\right)\right)$
57. $\tan^{-1}\left(\tan\left(\frac{2\pi}{3}\right)\right)$
58. $\tan^{-1}\left(\tan\left(\frac{3\pi}{2}\right)\right)$
59. $\tan^{-1}\left(\tan\left(\frac{17\pi}{9}\right)\right)$
60. $\tan^{-1}\left(\tan\left(-\frac{11\pi}{6}\right)\right)$
61. $\tan^{-1}\left(\tan\left(-\frac{9\pi}{8}\right)\right)$
62. $\tan^{-1}\left(\tan\left(-\frac{4\pi}{5}\right)\right)$

Answers to odd exercises.

45. $\frac{\pi}{10}$ 47. $-\frac{\pi}{4}$ 49. $\frac{\pi}{2}$ 51. $\frac{\pi}{10}$ 53. $\frac{2\pi}{3}$ 55. $\frac{4\pi}{5}$ 57. $-\frac{\pi}{3}$ 59. $-\frac{\pi}{9}$ 61. $-\frac{\pi}{8}$

E: Evaluate $f(f^{-1}(\frac{a}{b}))$ Compositions

Exercise 6.1e.E

★ **Evaluate without a calculator.**

65. $\sin\left(\sin^{-1}\left(\frac{1}{5}\right)\right)$
66. $\cos\left(\cos^{-1}\left(\frac{2}{3}\right)\right)$
67. $\tan\left(\tan^{-1}(4)\right)$
68. $\sin\left(\sin^{-1}(-5)\right)$
69. $\cos\left(\cos^{-1}\left(-\frac{3}{2}\right)\right)$
70. $\tan\left(\tan^{-1}\left(-\frac{1}{4}\right)\right)$

Answers to odd exercises.

65. $\frac{1}{5}$ 67. 4 69. undefined

F: Evaluate $f(g^{-1}(\frac{a}{b}))$ Compositions

Exercise 6.1e.F

★ **Evaluate without a calculator.**

71. $\sin\left(\cos^{-1}\left(-\frac{2}{\sqrt{5}}\right)\right)$
72. $\sin\left(\cos^{-1}\left(\frac{3}{4}\right)\right)$
73. $\sin\left(\cos^{-1}\left(\frac{3}{5}\right)\right)$
74. $\sin\left(\tan^{-1}\left(\frac{4}{3}\right)\right)$
75. $\sin\left(\tan^{-1}\left(-\frac{2}{\sqrt{5}}\right)\right)$
76. $\sin\left(\tan^{-1}(1)\right)$
77. $\cos\left(\sin^{-1}\left(\frac{1}{2}\right)\right)$
78. $\cos\left(\sin^{-1}\left(\frac{4}{5}\right)\right)$
79. $\cos\left(\tan^{-1}\left(\frac{12}{5}\right)\right)$
80. $\cos\left(\tan^{-1}\left(-\frac{1}{2}\right)\right)$
81. $\tan\left(\sin^{-1}\left(-\frac{2}{\sqrt{5}}\right)\right)$
82. $\tan\left(\sin^{-1}\left(-\frac{\sqrt{2}}{3}\right)\right)$
83. $\tan\left(\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$
84. $\cot\left(\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$
85. $\cot\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$
86. $\sec\left(\tan^{-1}\left(-\frac{3}{4}\right)\right)$
87. $\sec\left(\sin^{-1}\left(-\frac{3}{4}\right)\right)$
88. $\csc\left(\tan^{-1}\left(\frac{\sqrt{2}}{3}\right)\right)$
89. $\csc\left(\cos^{-1}\left(\frac{\sqrt{2}}{3}\right)\right)$

90. Find the exact value of $\dfrac{\sin^{-1}\left(\frac{1}{2}\right)-\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)+\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)-\cos^{-1}(1)}{\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)-\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)+\cos^{-1}\left(\frac{1}{2}\right)-\sin^{-1}(0)}$

Answers to odd exercises.

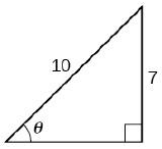
71. $\frac{\sqrt{5}}{5}$ 73. $\frac{4}{5}$ 75. $-\frac{2}{3}$ 77. $\frac{\sqrt{3}}{2}$ 79. $\frac{5}{13}$
81. -2 83. $-\frac{\sqrt{3}}{3}$ 85. $\frac{\sqrt{3}}{3}$ 87. $\frac{4\sqrt{7}}{7}$ 89. $\frac{3\sqrt{7}}{7}$

G: Evaluate $f^{-1}(g(\theta))$ Compositions

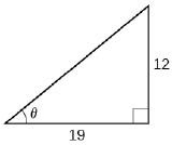
Exercise 6.1e.G

★ **Find the angle θ in the given right triangle. Round answers to the nearest hundredth.**

102.



103.



★ **Find the exact value, if possible, without a calculator, or round to the nearest hundredth.**

104. $\sin^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right)$
105. $\sin^{-1}\left(\cos\left(\frac{-\pi}{2}\right)\right)$
106. $\sin^{-1}(\cos(\pi))$
107. $\sin^{-1}\left(\tan\left(-\frac{4\pi}{3}\right)\right)$
108. $\cos^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$
109. $\cos^{-1}\left(\sin\left(\frac{7\pi}{4}\right)\right)$
110. $\cos^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)$
111. $\cos^{-1}\left(\cot\left(-\frac{3\pi}{4}\right)\right)$
112. $\tan^{-1}\left(\sin\left(\frac{4\pi}{3}\right)\right)$
113. $\tan^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$
114. $\tan^{-1}(\sin(\pi))$
115. $\tan^{-1}\left(\sin\left(-\frac{5\pi}{2}\right)\right)$
116. $\tan^{-1}\left(\csc\left(\frac{7\pi}{6}\right)\right)$
117. $\tan^{-1}\left(\sec\left(-\frac{\pi}{6}\right)\right)$

Answers to odd exercises.

103. **0.56** radians 105. **0** 107. undefined 109. $\frac{3\pi}{4}$
111. **0** 113. **0.71** 115. $-\frac{\pi}{4}$ 117. **0.86**

H: Evaluate $f(g^{-1}(h(u)))$ Compositions

Exercise 6.1e.H

For the exercises below, (a) Find the exact value of the expression in terms of u. (b) State any restrictions to u.

121. $\cos\left(\sin^{-1}(u)\right)$
122. $\tan\left(\sin^{-1}(u)\right)$
123. $\sin\left(\tan^{-1}(u)\right)$
124. $\cos\left(\tan^{-1}(u)\right)$
125. $\tan\left(\cos^{-1}(u)\right)$
126. $\sin\left(\cos^{-1}(u)\right)$
127. $\tan\left(\sin^{-1}(u-1)\right)$
128. $\cos\left(\sin^{-1}(1-u)\right)$
129. $\cos\left(\sin^{-1}\left(\frac{1}{u}\right)\right)$
130. $\tan\left(\sin^{-1}\left(\frac{u}{u+1}\right)\right)$
131. $\sin\left(\tan^{-1}\left(u+\frac{1}{2}\right)\right)$
132. $\cos\left(\tan^{-1}(3u-1)\right)$
133. $\sin\left(\tan^{-1}\left(\frac{u}{\sqrt{2u+1}}\right)\right)$

Answers to odd exercises.

121. $\sqrt{1-u^2}$, $-1 \le u \le 1$
123. $\frac{u}{\sqrt{1+u^2}}$, no restrictions
125. $\frac{\sqrt{1-u^2}}{u}$, $-1 \le u \le 1$
127. $\frac{u-1}{\sqrt{-u^2+2u}}$, $0 \le u \le 2$
129. $\frac{\sqrt{u^2-1}}{u}$, $u \ge 1$ or $u \le -1$
131. $\frac{u+\frac{1}{2}}{\sqrt{u^2+u+\frac{5}{4}}}$, no restrictions
133. $\frac{u}{u+1}$, $u > -\frac{1}{2}$

I: Graphs of inverses

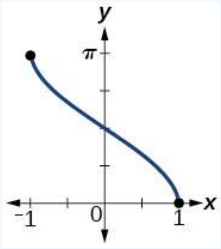
Exercise 6.1e.I

138. Graph $y = \sin^{-1} x$ and state the domain and range of the function.

139. Graph $y = \arccos x$ and state the domain and range of the function.

140. Graph one cycle of $y = \tan^{-1} x$ and state the domain and range of the function.

Answers to odd exercises.

139. 

domain $[-1, 1]$ range $[0, \pi]$

J: Applications

Exercise 6.1e.J

143. Suppose a 13-foot ladder is leaning against a building, reaching to the bottom of a second-floor window 12 feet above the ground. What angle, in radians, does the ladder make with the building?

144. Suppose you drive 0.6 miles on a road so that the vertical distance changes from 0 to 150 feet. What is the angle of elevation of the road?

145. An isosceles triangle has two congruent sides of length 9 inches. The remaining side has a length of 8 inches. Find the angle that a side of 9 inches makes with the 8-inch side.

146. Without using a calculator, approximate the value of $\arctan(10,000)$. Explain why your answer is reasonable.

147. A truss for the roof of a house is constructed from two identical right triangles. Each has a base of 12 feet and height of 4 feet. Find the measure of the acute angle adjacent to the 4-foot side.

148. The line $y = \frac{3}{5}x$ passes through the origin in the x,y-plane. What is the measure of the angle that the line makes with the positive x-axis?

149. The line $y = \frac{-3}{7}x$ passes through the origin in the x, y-plane. What is the measure of the angle that the line makes with the negative x-axis?

150. What percentage grade should a road have if the angle of elevation of the road is 4 degrees? (The percentage grade is defined as the change in the altitude of the road over a 100-foot horizontal distance. For example a 5% grade means that the road rises 5 feet for every 100 feet of horizontal distance.)

151. A 20-foot ladder leans up against the side of a building so that the foot of the ladder is 10 feet from the base of the building. If specifications call for the ladder's angle of elevation to be between 35 and 45 degrees, does the placement of this ladder satisfy safety specifications?

152. Suppose a 15-foot ladder leans against the side of a house so that the angle of elevation of the ladder is 42 degrees. How far is the foot of the ladder from the side of the house?

Answers to odd exercises.

143. 0.395 radians 145. 1.11 radians 147. 1.25 radians 149. 0.405 radians
151. No. The angle the ladder makes with the horizontal is 60 degrees..

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6.2: Trigonometric Equations

Linear Trigonometric Equations

In Section 5.5, we determined the height of a rider on the London Eye Ferris wheel could be determined by the equation

$$h(t) = -65\cos\left(\frac{\pi}{15}t\right) + 70.$$

If we wanted to know the length of time during which the rider is more than 100 meters above ground, we would need to solve equations involving trigonometric functions.

Trigonometric equations are, as the name implies, equations that involve trigonometric functions. Although similar in many ways to solving polynomial equations or rational equations, only specific values of the variable will be solutions, if there are solutions at all. Often we will solve a trigonometric equation over a specified interval. However, just as often, we will be asked to find all possible solutions, and as trigonometric functions are periodic, solutions are repeated within each period. In other words, trigonometric equations may have an infinite number of solutions. Additionally, like rational equations, the domain of the function must be considered before we assume that any solution is valid. The **period** of both the sine function and the cosine function is 2π. In other words, every 2π units, the *y*-values repeat. If we need to find all possible solutions, then we must add $2\pi k$, where k is an integer, to the initial solution. Recall the rule that gives the format for stating all possible solutions for a function where the period is 2π:

$$\sin\theta = \sin(\theta \pm 2k\pi)$$

There are similar rules for indicating all possible solutions for the other trigonometric functions.

How to: Solve Linear Trigonometric Equations having a Simple Argument

1. Isolate the trigonometric function (sine, cosine, or tangent) on one side of the equation. The final result for this step should look like $\sin(x) = y$ or $\cos(x) = y$ or $\tan(x) = y$, where y is a real number or real number expression. If the function is not sine, cosine, or tangent, use a Reciprocal Identity to change it:
$$\sec(t) = \frac{1}{\cos(t)} \qquad \csc(t) = \frac{1}{\sin(t)} \qquad \cot(t) = \frac{1}{\tan(t)}$$
2. Use inverse trigonometric functions to find one solution
3. Find a second solution on one cycle, when a second solution exists. Sine and cosine have 2 solutions per cycle (except when the sine or cosine ratio is 1 or -1) because the period of one cycle is $P = 2\pi$ and they oscillate between 1 and -1 and back again. A tangent function just has one solution, because the period $P = \pi$ for tangent is half the length of that for sine or cosine, and the graph only increases or only decreases within that cycle.
4. Find additional solutions if needed, by writing a general solution that involves multiples of the period (use k to represent any integer). Given a solution s, these general solutions look like
$$x = s + 2\pi k \text{ for sine or cosine} \qquad \text{or} \qquad x = s + \pi k \text{ for tangent}$$

Solutions that are Special Angles

Simple Arguments

In the last chapter, we learned sine and cosine values at commonly encountered ("special") angles. We can use these to solve sine and cosine equations involving these common angles.

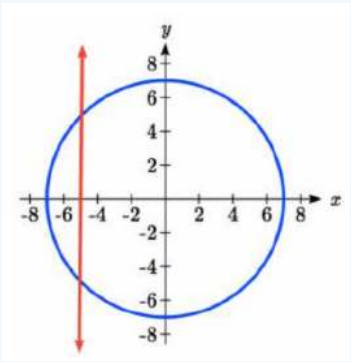
Often the question will ask just for solutions in a given interval.

Example 6.2.1

A circle of radius $5\sqrt{2}$ intersects the line $x = -5$ at two points. Find the angles θ on the interval $0 \le \theta < 2\pi$, where the circle and line intersect.

Solution

The x coordinate of a point on a circle can be found as $x = r\cos(\theta)$, so the x coordinate of points on this circle would be $x = 5\sqrt{2}\cos(\theta)$. To find where the line $x = -5$ intersects the circle, we can solve for where the x value on the circle would be -5.



$$\begin{aligned} 5\sqrt{2}\cos(\theta) &= -5 && \text{Isolate the cosine} \\ \cos(\theta) &= \frac{-5}{5\sqrt{2}} = \frac{-1}{\sqrt{2}} = \frac{-1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \\ \cos(\theta) &= -\frac{\sqrt{2}}{2} \end{aligned}$$

We recognize this as one of our special cosine values. The reference angle is $\frac{\pi}{4}$ because $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, but the desired angles are in Quadrants II and III where the cosine function is negative, as the unit circle diagram suggests. Our solutions are angles

$$\theta = \frac{3\pi}{4} \text{ and } \theta = \frac{5\pi}{4}$$

Example 6.2.2: Solve a Linear Equation Involving Cosine

Solve the equation exactly: $2\cos\theta - 3 = -5$, $0 \le \theta < 2\pi$.

Solution

Use algebraic techniques to solve the equation.

$$\begin{aligned} 2\cos\theta - 3 &= -5 \\ 2\cos\theta &= -2 \\ \cos\theta &= -1 \\ \theta &= \pi \end{aligned}$$

Other times, an expression that gives all possible solutions is desired. Two of the solutions are unique; all the others are coterminal with those two solutions.

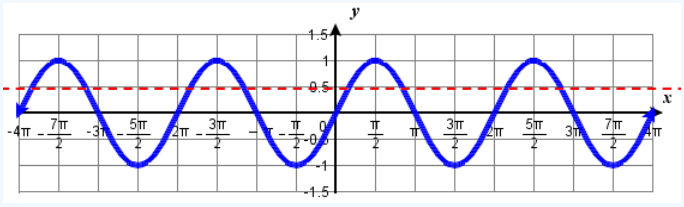
Example 6.2.3: Find the General Solution to a Linear Equation Involving Sine

Solve $\sin(t) = \frac{1}{2}$ for all possible values of t.

Solution

Notice this is asking us to identify all angles, t, that have a sine value of $\frac{1}{2}$. While evaluating a function always produces one result, solving for an input can yield multiple solutions. Two solutions should immediately jump to mind from the last chapter: $t = \frac{\pi}{6}$ and $t = \frac{5\pi}{6}$ because they have the same reference angle and are in quadrants where y (and thus the sine function) are positive, and because they are the common angles on the unit circle with a sin of $\frac{1}{2}$.

Looking at a graph confirms that there are more than these two solutions. While eight are seen on this graph, there are an infinite number of solutions!



Remember that any coterminal angle will also have the same sine value, so any angle coterminal with these our first two solutions is also a solution. Coterminal angles can be found by adding or subtracting full rotations of 2π, so we can write the full set of solutions:

$t = \frac{\pi}{6} + 2\pi k$ where k is any integer, and $t = \frac{5\pi}{6} + 2\pi k$ where k is any integer.

Example 6.2.4: Find the General Solution to a Linear Equation Involving Cosine

Find all possible exact solutions for the equation $\cos\theta = \frac{1}{2}$.

Solution

The cosine ratio is positive, so one of the angles is in Quadrant I. We know $\cos\frac{\pi}{3} = \frac{1}{2}$, so therefore one of the solutions is $\frac{\pi}{3}$. The other solution is in Quadrant IV, with the same reference angle. Thus the other solution is $\frac{5\pi}{3}$. The same result could be obtained from analysis using the unit circle

These are the solutions in the interval $[0, 2\pi]$. All possible solutions are given by

$$\theta = \frac{\pi}{3} + 2k\pi \quad \text{and} \quad \theta = \frac{5\pi}{3} + 2k\pi$$

where k is any integer.

When we are given equations that involve only one of the six trigonometric functions, their solutions involve using algebraic techniques and solving inverse trigonometric equations. We need to make several considerations when the equation involves trigonometric functions other than sine and cosine. Problems that involve reciprocals of the primary trigonometric functions must first be rewritten in terms of the primary function and then solved.

Example 6.2.5: Solve a Linear Equation Involving Cosecant

Solve the following equation exactly: $\csc\theta = -2,\ 0 \le \theta < 4\pi$.

Solution

We want all values of θ for which $\csc\theta = -2$ over the interval $0 \le \theta < 4\pi$.

$$\begin{aligned} \csc\theta &= -2 && \text{Rewrite as the reciprocal function} \\ \frac{1}{\sin\theta} &= -2 \\ \sin\theta &= -\frac{1}{2} && \text{The reference angle is } \frac{\pi}{6} \text{ and solutions are in QIII and QIV} \\ \theta &= \frac{7\pi}{6},\ \frac{11\pi}{6} && \text{Solutions in one cycle} \end{aligned}$$

Because we are told to find solutions up to 4π, we must find other solutions that are coterminal to these. This is done by adding 2π to each of these answers. Our final result is $\boxed{\frac{7\pi}{6},\ \frac{11\pi}{6},\ \frac{19\pi}{6},\ \frac{23\pi}{6}}$.

An equation involving the tangent function is slightly different from one containing a sine or cosine function. First, as we know, the period of tangent is π, not 2π. Further, the domain of tangent is all real numbers with the exception of odd integer multiples of $\frac{\pi}{2}$, unless, of course, a problem places its own restrictions on the domain.

Example 6.2.6: Find the General Solution to a Linear Equation Involving Tangent

Identify all exact solutions to the equation $2(\tan x + 3) = 5 + \tan x,\ 0 \le x < 2\pi$.

Solution

Isolate the expression $\tan x$ on the left side of the equals sign. $2\tan x + 6 = 5 + \tan x \rightarrow \tan x = -1$.

From $\tan\frac{\pi}{4} = 1$ we know the reference angle for the solutions is $\frac{\pi}{4}$. Because the tangent ratio (-1) is negative, the solution angles have the same reference angle but are in quadrants QII and QIV.

Therefore the two angles (which could also be derived from using the unit circle) are: $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$.

Try It 6.2.7

Solve

a. $2\sin x + 1 = 0$ for all exact solutions on the interval $[0, 2\pi)$.
b. $\sec x = -2$ for all possible values of x.
c. $\tan(t) = 1$ for all possible values of t.

Answer

a. $x = \frac{7\pi}{6}, \frac{11\pi}{6}$

b. $x = \frac{2\pi}{3} + 2\pi k, \frac{4\pi}{3} + 2\pi k$

c. From our special angles, we know one answer is $t = \frac{\pi}{4}$. Recall that one cycle for the tangent function is π, not 2π like for sine and cosine. Tangent equations only have one unique solution per cycle or period, so additional solutions can be found by adding multiples of a full period, π.

$$t = \frac{\pi}{4} + \pi k$$

Arguments that are Expressions

When the argument is an expression, rather than just a simple variable, a substitution approach is helpful.

How to: Solve Linear Trigonometric Equations having an Argument that is an Expression

1. Rewrite the equation so it just contains one trigonometric function - sine, cosine, or tangent. Isolate the trigonometric function on one side of the equation.
2. If the argument is an expression, substitute the expression with the variable u.
3. Use inverse trigonometric functions to find one solution for u.
4. Find a second solution for u on one cycle, when a second solution exists. Sine and cosine have 2 solutions (or one if the ratio is ± 1); tangent just has one solution.
5. Find additional solutions for u if needed by writing a general solution that involves multiples of the period (use k to represent any integer).
6. Undo the substitution, if one was made in step 2. Replace u in the solution formula with the expression that was replaced with u in step 2, and then solve for the original variable.

Example 6.2.8: Solving a Multiple Angle Trigonometric Equation

Solve exactly: $\cos(2x) = \frac{1}{2}$ on $[0, 2\pi)$.

Solution

We can see that this equation is the standard equation with a multiple of an angle. (Replace the argument expression, $2x$, with u and find solutions for u in one cycle of the cosine function.

$$\cos(u) = \frac{1}{2} \quad \text{Substitution made: } u = 2x$$
$$u = \cos^{-1}\left(\frac{1}{2}\right) \quad \text{Solutions in QI and QIV with ref. angle } \frac{\pi}{3}$$
$$u = \frac{\pi}{3} \quad \text{and} \quad u = \frac{5\pi}{3}$$

We saw the solutions to this equation were

$u = \frac{\pi}{3}$	$u = \frac{5\pi}{3}$	
$u = \frac{\pi}{3} + 2\pi k$	$u = \frac{5\pi}{3} + 2\pi k$	General solutions, where k is any integer
$2x = \frac{\pi}{3} + 2\pi k$	$2x = \frac{5\pi}{3} + 2\pi k$	Replace the u in the solutions with $2x$
$x = \frac{\pi}{6} + \pi k$	$x = \frac{5\pi}{6} + \pi k$	Solve for x.
$x = \frac{\pi}{6}$	$x = \frac{5\pi}{6}$	solutions when $k = 0$
$x = \frac{7\pi}{6}$	$x = \frac{11\pi}{6}$	solutions when $k = 1$
$x = \frac{13\pi}{6}$	$x = \frac{17\pi}{6}$	$k = 2$ solutions are too big!

Our solutions are $x = \frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{7\pi}{6}$, and $\frac{11\pi}{6}$. Note that whenever we solve a problem in the form of $\sin(nx) = c$, we will get n times as many solutions as exist in one cycle of the function.

Example 6.2.9: Multiple Angle Application

The depth of water at a dock rises and falls with the tide, following the equation $f(t) = 4\sin\left(\frac{\pi}{12}t\right) + 7$, where t is measured in hours after midnight. A boat requires a depth of 9 feet to tie up at the dock. Between what times will the depth be 9 feet or more?

Solution

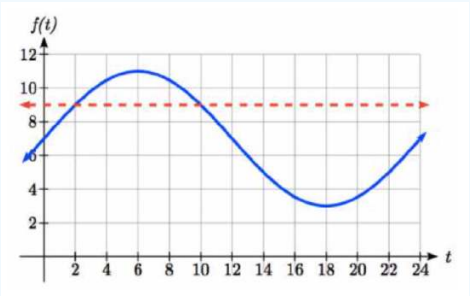
To find when the depth is 9 feet, we need to solve $f(t) = 9$.

$$\begin{aligned} 4\sin\left(\frac{\pi}{12}t\right) + 7 &= 9 && \text{Isolate the sine} \\ 4\sin\left(\frac{\pi}{12}t\right) &= 2 && \text{Divide by 4} \\ \sin\left(\frac{\pi}{12}t\right) &= \frac{1}{2} \end{aligned}$$

Define a new temporary variable u to be $u = \frac{\pi}{12}t$, so our equation $\sin\left(\frac{\pi}{12}t\right) = \frac{1}{2}$ becomes

$$\sin(u) = \frac{1}{2}.$$

$u = \frac{\pi}{6}$	$u = \frac{5\pi}{6}$	
$u = \frac{\pi}{6} + 2\pi k$	$u = \frac{5\pi}{6} + 2\pi k$	General solutions
$\frac{\pi}{12}t = \frac{\pi}{6} + 2\pi k$	$\frac{\pi}{12}t = \frac{5\pi}{6} + 2\pi k$	"Undo" sub - replace u
$t = \frac{\frac{\pi}{6}}{\frac{\pi}{12}} + \frac{2\pi k}{\frac{\pi}{12}}$	$t = \frac{\frac{5\pi}{6}}{\frac{\pi}{12}} + \frac{2\pi k}{\frac{\pi}{12}}$	Solve for t.
$t = 2 + 24k$	$t = 10 + 24k$	



The depth will be 9 feet and the boat will be able to approach the dock between 2 am and 10 am.

Notice how in both scenarios, the 24k shows how every 24 hours the cycle will be repeated.

In the previous example, looking back at the original simplified equation $\sin\left(\frac{\pi}{12}t\right) = \frac{1}{2}$, we can use the ratio of the "normal period" to the stretch factor to find the period: $\frac{2\pi}{\left(\frac{\pi}{12}\right)} = 2\pi\left(\frac{12}{\pi}\right) = 24$. Notice that the sine function has a period of 24, which is reflected in the solutions: there were two unique solutions on one full cycle of the sine function, and additional solutions were found by adding multiples of a full period.

Example 6.2.10: Multiple Angle Equation Involving Tangent

Solve the equation exactly: $\tan\left(\theta-\frac{\pi}{2}\right)=1,\ 0\le\theta<2\pi$.

Solution

The initial substitution is $u=\theta-\frac{\pi}{2}$. Solving the equation $\tan u=1$ produces one solution, $u=\frac{\pi}{4}$.

Recall that the tangent function has a period of π. So there is only one solution in one cycle of the tangent function. All others can be obtained from the general form of the solution set, $u=\frac{\pi}{4}+\pi k$, where k is any integer.

Now we need to solve for our original variable, θ: $\theta-\frac{\pi}{2}=\frac{\pi}{4}+\pi k\ \rightarrow\theta=\frac{3\pi}{4}+\pi k$

When $k=0$ we obtain the solution $\theta=\frac{3\pi}{4}$. When $k=1$ we get $\theta=\frac{7\pi}{4}$. When $k=2$ the answer we get is $\frac{11\pi}{4}$, which is larger than 2π, so it must be rejected for this problem.

Over the interval $[0,2\pi)$, we have two solutions: $\theta=\frac{3\pi}{4}$ and $\theta=\frac{7\pi}{4}$

Try It 6.2.11

Solve $4\sin(5t)-1=1$ for all possible values of t.

Answer

$4\sin(5t)-1=1$	Isolate the cosine	$u=\frac{\pi}{6}+2\pi k$	$u=\frac{5\pi}{6}+2\pi k$	General solutions
$\sin(5t)=\frac{1}{2}$	Let $u=5t$	$5t=\frac{\pi}{6}+2\pi k$	$5t=\frac{5\pi}{6}+2\pi k$	Solve for t
$\sin(u)=\frac{1}{2}$		$t=\frac{\pi}{30}+\frac{2\pi}{5}k$	$t=\frac{\pi}{6}+\frac{2\pi}{5}k$	

Solutions that Require a Calculator

Not all equations involve the “special” values of the trigonometric functions to we have learned. To find solutions to these kind of equations, we need to use an inverse trig function operation on a calculator.

Simple Arguments that need a calculator

Example 6.2.12

Use the inverse sine function to find one solution to $\sin(\theta)=0.8$.

Solution

Since this is not a known value, a calculator must be used to find the angle $\theta=\sin^{-1}(0.8)$. The result is an approximate value for this angle. If your calculator is in degree mode, your calculator will give you an angle in degrees as the output. If your calculator is in radian mode, your calculator will give you an angle in radians. In radians, $\theta=\sin^{-1}(0.8)\approx0.927$, or in degrees, $\theta=\sin^{-1}(0.8)\approx53.130°$.

Always make sure you are using the appropriate mode on your calculator -- degrees or radians. In calculus, we will almost always want to work with radians since they are unit-less.

Notice that the inverse trigonometric functions do exactly what you would expect of any function – for each input they give exactly one output. While this is necessary for these to be a function, it means that to find *all* the solutions to an equation like $\sin(\theta)=0.8$, we need to do more than just evaluate the inverse function.

To find additional solutions, it is good to remember four things:

- The sine is the y-value of a point on the unit circle
- The cosine is the x-value of a point on the unit circle

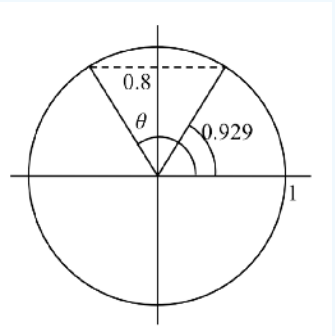
- The tangent is the slope of a line at a given angle
- Other angles with the same sin/cos/tan will have the same reference angle

Example 6.2.13: Find all solutions

Find all solutions to $\sin(\theta) = 0.8$.

Solution

We would expect two unique angles on one cycle to have this sine value. In the previous example, we found one solution to be $\theta = \sin^{-1}(0.8) \approx 0.927$. To find the other, we need to answer the question "what other angle has the same sine value as an angle of 0.927?"



We can think of this as finding all the angles where the y-value on the unit circle is 0.8. Drawing a picture of the circle helps to show in which quadrant another angle with the same y coordinate would be positioned.

The second angle would have the same reference angle and reside in the second quadrant (where y coordinates are also positive). This second angle would be located at $\theta = \pi - \sin^{-1}(0.8)$, or approximately $\theta \approx \pi - 0.927 = 2.214$.

To find more solutions we recall that angles coterminal with these two would have the same sine value, so we can add full cycles of 2π.

$$\theta = \sin^{-1}(0.8) + 2\pi k \quad \text{and} \quad \theta = \pi - \sin^{-1}(0.8) + 2\pi k$$
$$\theta = 0.927 + +2\pi k \quad \text{and} \quad \theta = 2.214 + 2\pi k$$

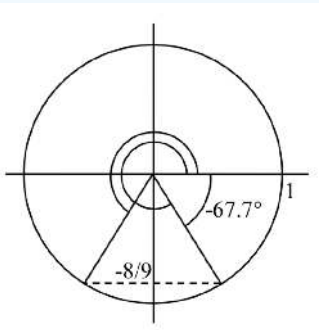
Approximate solutions, where k is an integer.

Example 6.2.14

Find all solutions to $\sin(x) = -\frac{8}{9}$ on the interval $0° \le x < 360°$.

Solution

We are looking for the angles with a y-value of -8/9 on the unit circle. Immediately we can see the solutions will be in the third and fourth quadrants (where the y coordinates are negative).



First, we will turn our calculator to degree mode. Using the inverse, we can find one solution $x = \sin^{-1}\left(-\frac{8}{9}\right) \approx -62.734°$. While this angle satisfies the equation, it does not lie in the domain we are looking for. To find the angles in the desired domain, we start looking for additional solutions.

First, an angle coterminal with $-62.734°$ will have the same sine. By adding a full rotation, we can find an angle in the desired domain with the same sine.

$$x = -62.734° + 360° = 297.266°$$

There is a second angle in the desired domain that lies in the third quadrant. Notice that $62.734°$ is the reference angle for all solutions, so this second solution would be $62.734°$ past $180°$

$$x = 62.734° + 180° = 242.734°$$

The two solutions on $0° \le x < 360°$ are x = 297.266° and x = 242.734°

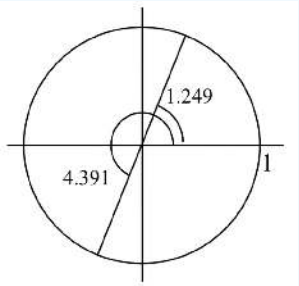
Example 6.2.15

Find all solutions to $\tan(x) = 3$ on $0 \le x < 2\pi$.

Solution

Using the inverse tangent function, we can find one solution $x = \tan^{-1}(3) \approx 1.249$. Unlike the sine and cosine, the tangent function only attains any output value once per cycle, so there is no second solution in any one cycle.

By adding π, a full period of tangent function, we can find a second angle with the same tangent value. Notice this gives another angle where the line (along the terminal side of the angle) has the same slope.



If additional solutions were desired, we could continue to add multiples of π, so all solutions would take on the form $x = 1.249 + k\pi$, however we are only interested in $0 \le x < 2\pi$.

$$x = 1.249 + \pi = 4.391$$

The two solutions on $0 \le x < 2\pi$ are $x = 1.249$ and $x = 4.391$.

Try It 6.2.16

Find all solutions to $\tan(x) = 0.7$ on $0° \le x < 360°$.

Answer

The first solution is $x = \tan^{-1}(0.7) \approx 34.992°$.

For a standard tangent, the second solution can be found by adding a full period, $180°$, giving

$$x = 180° + 34.99° = 214.992°$$

Example 6.2.17

Solve $3\cos(t) + 4 = 2$ for all solutions on one cycle, $0 \le t < 2\pi$

Solution

$$\begin{aligned} 3\cos(t) + 4 &= 2 && \text{Isolate the cosine} \\ 3\cos(t) &= -2 \\ \cos(t) &= -\frac{2}{3} && \text{Use the inverse, to find one solution} \\ t &= \cos^{-1}\left(-\frac{2}{3}\right) \\ t &\approx 2.301 \end{aligned}$$

We're looking for two angles where the x-coordinate on a unit circle is -2/3. A second angle with the same cosine would be located in the third quadrant. Notice that the location of this angle could be represented as $t = -2.301$. To represent this as a positive angle we could find a coterminal angle by adding a full rotation.

$$t = -2.301 + 2\pi = 3.982$$

The equation has two solutions between 0 and 2π, at $t = 2.301$ and $t = 3.982$.

Example 6.2.18: Using a Calculator to Solve a Trigonometric Equation Involving Secant

Use a calculator to solve the equation $\sec\theta = -4$, giving your answer in radians.

Solution

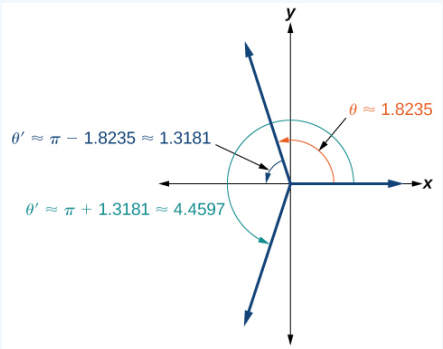
We can begin with some algebra.

$$\begin{aligned} \sec\theta &= -4 \\ \frac{1}{\cos\theta} &= -4 \\ \cos\theta &= -\frac{1}{4} \end{aligned}$$

Check that the MODE is in radians. Now use the inverse cosine function

$$\cos^{-1}\left(-\frac{1}{4}\right) \approx 1.8235$$
$$\theta \approx 1.8235 + 2\pi k$$

Since $\frac{\pi}{2} \approx 1.57$ and $\pi \approx 3.14$, and we know 1.8235 is between these two numbers, so$\theta \approx 1.8235$ is in quadrant II. Cosine is also negative in quadrant III. Note that a calculator will only return an angle in quadrants I or II for the inverse cosine function, since that is the range of the inverse cosine. See the Figure on the right.



So, we also need to find the measure of the angle in quadrant III. In quadrant III, the reference angle is $\alpha \approx \pi - 1.8235 \approx 1.3181$. The other solution in quadrant III is $\pi + \alpha \approx \pi + 1.3181 \approx 4.4597$.

The solutions are $\theta \approx 1.8235 + 2\pi k$ and $\theta \approx 4.4597 + 2\pi k$.

Try It 6.2.19

Solve $\cos\theta = -0.2$.

Answer

$\theta \approx 1.7722 + 2\pi k$ and $\theta \approx 4.5110 + 2\pi k$

Arguments that are Expressions that need a calculator

Example 6.2.20

Solve $\cos(3t) = 0.2$ for all solutions on two cycles, $0 \le t < \frac{4\pi}{3}$.

Solution

As before, with a horizontal compression it can be helpful to make a substitution, $u = 3t$ Making this substitution simplifies the equation to a form we have already solved.

$$\cos(u) = 0.2$$
$$u = \cos^{-1}(0.2) \approx 1.369$$

A second solution on one cycle would be located in the fourth quadrant with the same reference angle.

$$u \approx 2\pi - 1.369 = 4.914$$

In this case, we need all solutions on two cycles, so we need to find the solutions on the second cycle. We can do this by adding a full rotation to the previous two solutions.

$u = 1.369 + 2\pi k$	$u = 4.914 + 2\pi k$	General solutions, k is any integer
$3t = 7.653 + 2\pi k$	$3t = 11.197 + 2\pi k$	Replace u with $3t$
$t = 0.456 + \frac{2\pi}{3}k$	$t = 1.638 + \frac{2\pi}{3}k$	General solutions for t
$t = 0.456$	$t = 1.638$	Solutions obtained when $k = 0$
$t = 0.456... + \frac{2\pi}{3}$	$t = 1.638.... + \frac{2\pi}{3}$	Solutions obtained when $k = 1$
$t = 2.551$	$t = 3.732$	

The question asks for all solutions less than $\frac{4\pi}{3} \approx 4.189$, so the Solution Set is $\{0.456,\ 1.638,\ 2.551,\ 3.732\}$.

Example 6.2.21

Solve $3\sin(\pi t) = -2$ for all solutions.

Solution

$$\begin{aligned} 3\sin(\pi t) &= -2 && \text{Isolate the sine} \\ \sin(\pi t) &= -\frac{2}{3} && \text{Make the substitution } u = \pi t \\ \sin(u) &= -\frac{2}{3} && \text{Use the inverse} \\ u &= \sin^{-1}\left(-\frac{2}{3}\right) && \text{Find one solution} \\ u &\approx -0.730 \end{aligned}$$

This angle is in the fourth quadrant. A second angle with the same (negative) sine ratio would be in the third quadrant with 0.730 as a reference angle: $u = \pi + 0.730 = 3.871$. We can write all solutions to the equation $\sin(u) = -\frac{2}{3}$ as

$$\begin{array}{rl|rl} u = -0.730 + 2\pi k & & u = 3.871 + 2\pi k & \text{General solutions, } k \text{ is any integer} \\ \pi t = -0.730 + 2\pi k & & \pi t = 3.871 + 2\pi k & \text{Replace } u \text{ with } \pi t \\ t = -0.232 + 2k & & t = 1.232 + 2k & \text{Divide by } \pi \text{ to get all (approximate) solutions} \end{array}$$

The solutions are $t = -0.232 + 2k$ and $t = 1.232 + 2k$, where k represents any integer.

Try It 6.2.22

Solve $5\sin\left(\frac{\pi}{2}t\right) + 3 = 0$ for all solutions on one cycle, $0 \le t < 4$.

Answer

$$\sin\left(\frac{\pi}{2}t\right) = -\frac{3}{5}$$

Let $u = \frac{\pi}{2}t$, so this becomes

$$\sin(u) = -\frac{3}{5}$$

Using the inverse, $u = \sin^{-1}\left(-\frac{3}{5}\right) \approx -0.6435$. Since we want positive solutions, we can find the coterminal solution by adding a full rotation: $u = -0.6435 + 2\pi = 5.6397$. Another angle with the same sin would be in the third quadrant with the reference angle 0.6435. $u = \pi + 0.6435 = 3.7851$.

$$\begin{array}{rl|rl} u = 5.6397 & & u = 3.7851 & \text{Solutions for } u \\ \frac{\pi}{2}t = 5.6397 & & \frac{\pi}{2}t = 3.7851 & \text{Replace } u \text{ with } \frac{\pi}{2}t \\ t = 5.6397\left(\frac{2}{\pi}\right) & & t = 3.7851\left(\frac{2}{\pi}\right) & \text{Solve for } t \\ t = 2.4097 & & t = 3.5903 & \end{array}$$

The solutions on one cycle are $t = 2.4097$ and $t = 3.5903$.

Applications

We now can return to the question we began the section with.

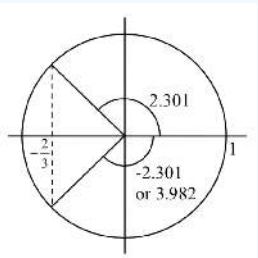
Example 6.2.23

The height of a rider on the London Eye Ferris wheel can be determined by the equation $h(t) = -65\cos\left(\frac{\pi}{15}t\right) + 70$. How long is the rider more than 100 meters above ground?

Solution

To find how long the rider is above 100 meters, we first find the times at which the rider is at a height of 100 meters by solving h(t) = 100.

$$
\begin{aligned}
-65\cos\left(\frac{\pi}{15}t\right)+70 &= 100 && \text{Isolate the cosine}\\
-65\cos\left(\frac{\pi}{15}t\right) &= 30\\
\cos\left(\frac{\pi}{15}t\right) &= \frac{30}{-65} && \text{Make the substitution } u=\frac{\pi}{15}t\\
\cos(u) &= \frac{30}{-65} && \text{Use the inverse}\\
u &= \cos^{-1}\left(\frac{30}{-65}\right) && \text{Find one solution}\\
u &\approx 2.051
\end{aligned}
$$



This angle is in the second quadrant. A second angle with the same cosine would be symmetric in the third quadrant. This angle could be represented as u = -2.051, but we need a coterminal positive angle, so we add 2π: $u = 2\pi - 2.051 \approx 4.230$.

Now we can undo the substitution to solve for t

$$
\begin{aligned}
u = 2.051 \quad &\text{and} \quad u = 4.230 && \text{Solutions for } u\\
\frac{\pi}{15}t = 2.051 \quad &\text{and} \quad \frac{\pi}{15}t = 4.230 && \text{Replace } u \text{ with } \frac{\pi}{15}\\
t = \frac{2.051}{\frac{\pi}{15}} \quad &\text{and} \quad t = \frac{4.230}{\frac{\pi}{15}} && \text{Solve for } t\\
t = 9.793 \quad &\text{and} \quad t = 20.197
\end{aligned}
$$

A rider will be at 100 meters after 9.793 minutes, and again after 20.197 minutes. From the behavior of the height graph, we know the rider will be above 100 meters between these times. A rider will be above 100 meters for 20.197 - 9.793 = 10.404 minutes of the ride.

Quadratic Trigonometric Equations

At the beginning of this section, we solved linear trigonometric equations. In this section, we explore the techniques needed to solve trigonometric equations that are quadratic in form. Techniques used to solve quadratic trigonometric equations include factoring out a GCF, factoring a difference of squares, using the square root property, factoring a trinomial, and using the Quadratic Formula.

Factor out a GCF

Consider the function $f(x) = 2x^2 + x$. If you were asked to solve $f(x) = 0$, it requires simple algebra:

$$
\begin{aligned}
x^2 + x &= 0 && \text{Factor}\\
x(2x+1) &= 0 && \text{Obtain solutions}\\
x = 0 \;\text{ or }\; x &= -\frac{1}{2}
\end{aligned}
$$

We use a similar process on trigonometric equations.

Example 6.2.24: Use GCF Factoring

Solve exactly: $2\sin^2\theta + \sin\theta = 0;\ 0 \le \theta < 2\pi$

Solution

This problem should appear familiar as it is similar to a quadratic. Let $\sin\theta = s$. The equation becomes $2s^2 + s = 0$. We begin by factoring:

$$
\begin{aligned}
2s^2 + s &= 0\\
s(2s+1) &= 0 && \text{Set each factor equal to zero.}\\
\boxed{s = 0} \quad &2s + 1 = 0\\
&\boxed{s = -\frac{1}{2}}
\end{aligned}
$$

Then, substitute the original expression $\sin\theta$ back for s in each of these solutions. Thus,

$$\begin{array}{l|l} s=0 & s=-\frac{1}{2} \\ \sin\theta=0 & \sin\theta=-\frac{1}{2} \\ \theta=0,\ \theta=\pi & \theta=\frac{7\pi}{6},\ \theta=\frac{11\pi}{6} \end{array}$$

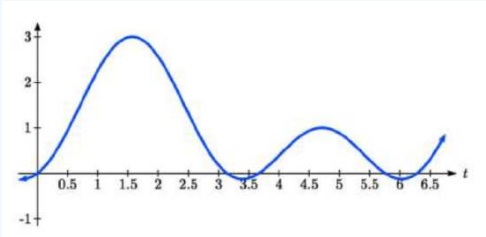
The solutions within the domain $0\le\theta<2\pi$ are $\theta=0,\pi,\frac{7\pi}{6},\frac{11\pi}{6}$.

As one gets more familiar with factoring trig expressions, you may prefer not to substitute, in which case we can solve the equation more directly:

$$\begin{gathered} \sin(t)\,(2\sin(t)+1)=0 \\ \sin(t)=0 \quad \text{or} \quad 2\sin(t)+1=0 \end{gathered}$$

Analysis

We could check these answers are reasonable by graphing the function and comparing the zeros. We can see the solutions on the graph in the figure to the right. On the interval $0\le\theta<2\pi$, the graph crosses the x-axis four times, at the solutions noted. Notice that trigonometric equations that are in quadratic form can yield up to four solutions instead of the expected two that are found with quadratic equations. In this example, each solution (angle) corresponding to a positive sine value will yield two angles that would result in that value.



Try It 6.2.24

Solve the quadratic equation $2\cos^2\theta+\cos\theta=0$.

Answer

$\frac{\pi}{2},\frac{2\pi}{3},\frac{4\pi}{3},\frac{3\pi}{2}$

Factor a Difference of Squares

When confronted with two terms that are both perfect squares and one is subtracted from the other, we have what is called a difference of squares. In general, the difference of squares formula is $a^2-b^2=(a-b)(a+b)$.

Example 6.2.25: Use the Difference of Squares

Solve: $4cos^2\theta-1=0,\ 0\le\theta<2\pi$.

Solution

Notice that both the coefficient and the trigonometric expression in the first term are squared, and the square of the number 1 is 1. This is a difference of squares.

$$\begin{gathered} 4\cos^2\theta-1=0 \\ (2\cos\theta)^2-1=0 \\ (2\cos\theta-1)(2\cos\theta+1)=0 \end{gathered}$$

$$\begin{array}{l|l} 2\cos\theta=1 & 2\cos\theta=-1 \\ \cos\theta=\frac{1}{2} & \cos\theta=-\frac{1}{2} \\ \theta=\frac{\pi}{3},\ \theta=\frac{5\pi}{3} & \theta=\frac{2\pi}{3},\ \theta=\frac{4\pi}{3} \end{array}$$

Solution Set: $\left\{\frac{\pi}{3}, \frac{5\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}\right\}$

Try It 6.2.25

Solve: $25 - 9\sin^2\theta = 0, 0 \le \theta < 2\pi$.

Answer

$25 - 9\sin^2\theta = (5 - 3\sin\theta)(5 + 3\sin\theta) = 0$. Therefore, $\sin\theta = \pm\frac{5}{3}$. However, because it is always true that $-1 \le \sin\theta \le 1$, this equation has no solution.

Example 6.2.26: Use the square root property

Solve the problem exactly: $2\sin^2\theta - 1 = 0, 0 \le \theta < 2\pi$.

Solution

As this problem is not easily factored, we will solve using the square root property. First, we use algebra to isolate $\sin\theta$. Then we will find the angles.

$$\begin{aligned} 2\sin^2\theta - 1 &= 0 \\ 2\sin^2\theta &= 1 \\ \sin^2\theta &= \frac{1}{2} \\ \sqrt{\sin^2\theta} &= \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}} \\ \sin\theta &= \pm\frac{\sqrt{2}}{2} \end{aligned}$$

Solution Set: $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Factor a Trinomial

Example 6.2.27: Factor a trinomial

Solve: $2\cos^2\theta + \cos\theta - 1, 0 \le \theta < 2\pi$.

Solution

Letting $\cos\theta = c$, we can rewrite a simplified version of the expression as follows:

$$2c^2 + c - 1 = 0$$

This quadratic expression can be factored as $(2c + 1)(c - 1)$. Replacing c with $\cos\theta$ in this factored form yields

$$(2\cos\theta + 1)(\cos\theta - 1) = 0$$

$$\begin{array}{l|l} 2\cos\theta = -1 & \cos\theta = 1 \\ \cos\theta = -\frac{1}{2} & \theta = 0,\ \theta = \pi \\ \theta = \frac{2\pi}{3},\ \theta = \frac{4\pi}{3} & \end{array}$$

Solution Set: $\left\{0, \pi, \frac{2\pi}{3}, \frac{4\pi}{3}\right\}$

Example 6.2.28

Solve the quadratic in form equation exactly: $2\sin^2\theta - 3\sin\theta + 1 = 0,\ 0 \le \theta < 2\pi$.

Solution

We can factor using the AC grouping method or the Reverse Foil method. Initially replacing $\sin\theta$ with s may also be helpful. Solution values of θ can be found on the unit circle.

$$(2\sin\theta - 1)(\sin\theta - 1) = 0$$

$2\sin\theta - 1 = 0$	$\sin\theta - 1 = 0$
$\sin\theta = \frac{1}{2}$	$\sin\theta = 1$
$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$	$\theta = \frac{\pi}{2}$

Solution Set: $\left\{\frac{\pi}{6},\ \frac{5\pi}{6},\ \frac{\pi}{2}\right\}$

Example 6.2.29

Solve the equation exactly: $2\sin^2\theta - 5\sin\theta + 3 = 0,\ 0 \le \theta \le 2\pi$.

Solution

Using grouping, this quadratic can be factored. Either make the real substitution, $\sin\theta = u$,or imagine it, as we factor:

$$2\sin^2\theta - 5\sin\theta + 3 = 0$$
$$(2\sin\theta - 3)(\sin\theta - 1) = 0$$

$2\sin\theta - 3 = 0$	$\sin\theta - 1 = 0$
$\sin\theta = \frac{3}{2}$ Impossible	$\sin\theta = 1$

Solution Set: $\left\{\frac{\pi}{2}\right\}$

Always make sure to check all solutions on the given domain as some factors have no solution.

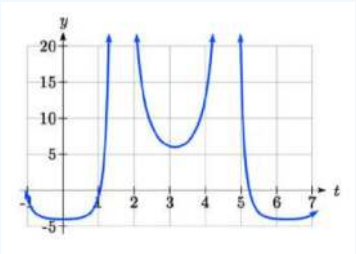
Example 6.2.30

Solve $3\sec^2(t) - 5\sec(t) - 2 = 0$ for all solutions with $0 \le t < 2\pi$.

Solution

If it is easier to for you to consider factoring without the trig function present, consider using a substitution $u = \sec(t)$, resulting in $3u^2 - 5u - 2 = 0$,. When factored, this becomes $(3u + 1)(u - 2) = 0$. Then,

$3u + 1 = 0$	$u - 2 = 0$	Zero factor property
$u = -\frac{1}{3}$	$u = 2$	
$\sec(t) = -\frac{1}{3}$	$\sec(t) = 2$	Substitute back
$\cos(t) = -3$	$\cos(t) = \frac{1}{2}$	Reciprocal Identity
Impossible	$t = \frac{\pi}{3},\ t = \frac{5\pi}{3}$	



By utilizing technology to graph $f(t) = 3\sec^2(t) - 5\sec(t) - 2$, a look at a graph confirms there are only two zeros for this function on the interval $[0, 2\pi)$, which assures us that we didn't miss anything.

Try It 6.2.31

Solve $2\sin^2(t)+3\sin(t)+1=0$ for all solutions with $0 \le t < 2\pi$.

Answer

$$2\sin^2(t)+3\sin(t)+1=0$$
$$(2\sin(t)+1)(\sin(t)+1)=0$$

$2\sin(t)+1$	$=0$	$\sin(t)+1$	$=0$	Zero factor property
$\sin(t)$	$=-\frac{1}{2}$	$\sin(t)$	$=-1$	
$t=\frac{7\pi}{6},$	$\frac{11\pi}{6}$	t	$=\frac{3\pi}{2}$	

Solution Set: $\left\{\frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}\right\}$

Quadratic Formula

Example 6.2.32: Use the Quadratic Formula

Solve: $\cos^2\theta+3\cos\theta-1=0,\ 0 \le \theta < 2\pi$.

Solution

We begin by using substitution and replacing $\cos\theta$ with x. It is not necessary to use substitution, but it may make the problem easier to solve visually. Let $\cos\theta = x$. We have

$$x^2+3x-1=0$$

The equation cannot be factored, so we will use the **quadratic formula:** $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

$$x=\frac{-3\pm\sqrt{(-3)^2-4(1)(-1)}}{2}=\frac{-3\pm\sqrt{13}}{2}$$

Replace x with $\cos\theta$ and solve.

$$\cos\theta=\frac{-3\pm\sqrt{13}}{2}$$
$$\theta=\cos^{-1}\left(\frac{-3+\sqrt{13}}{2}\right)$$

Note that only the + sign is used. This is because we get an error when we solve $\theta=\cos^{-1}\left(\frac{-3-\sqrt{13}}{2}\right)$ on a calculator, since the domain of the inverse cosine function is $[-1,1]$. However, there is a second solution:

$$\theta=\cos^{-1}\left(\frac{-3+\sqrt{13}}{2}\right)\approx 1.26$$

This terminal side of the angle lies in quadrant I. Since cosine is also positive in quadrant IV, the second solution is

$$\theta=2\pi-\cos^{-1}\left(\frac{-3+\sqrt{13}}{2}\right)\approx 5.02$$

Use Identities to pre-process an Equation

Before a trigonometric equation can be solved, the equation must usually contain just one trigonometric function and one angle. In cases where there are two different trig functions in the equation, and one is a square of a trigonometric function, it is sometimes helpful apply a Pythagorean Identity to create an equation that has just one trig function in it. The Pythagorean Identities, as well as some other identities we have learned so far, are listed below.

Definitions: IDENTITIES

Pythagorean Identities

$$\cos^2(t)+\sin^2(t)=1 \qquad 1+\cot^2(t)=\csc^2(t) \qquad 1+\tan^2(t)=\sec^2(t)$$

Reciprocal and Quotient Identities

$$\sec(t)=\frac{1}{\cos(t)} \qquad \csc(t)=\frac{1}{\sin(t)} \qquad \tan(t)=\frac{\sin(t)}{\cos(t)} \qquad \cot(t)=\frac{1}{\tan(t)}$$

Example 6.2.33: Use an Identity

Solve the equation exactly using an identity: $3\cos\theta+3=2\sin^2\theta$, $0\le\theta<2\pi$.

Solution

If we use a Pythagorean Identity to replace $\sin^2\theta$ with $1-\cos^2\theta$, we can obtain an equation that only has the cosine function in it:

$$\begin{aligned}
3\cos\theta+3&=2\sin^2\theta\\
3\cos\theta+3&=2(1-\cos^2\theta)\\
3\cos\theta+3&=2-2\cos^2\theta\\
2\cos^2\theta+3\cos\theta+1&=0\\
(2\cos\theta+1)(\cos\theta+1)&=0
\end{aligned}$$

$$\begin{array}{l|l}
2\cos\theta+1=0 & \cos\theta+1=0\\
\cos(\theta)=-\frac{1}{2} & \cos\theta=-1\\
\theta=\frac{2\pi}{3},\ \frac{4\pi}{3} & \theta=\pi
\end{array}$$

Our solutions are $\theta=\frac{2\pi}{3},\ \frac{4\pi}{3},\ \pi$.

Try It 6.2.34

Solve $\sin^2\theta=2\cos\theta+2$, $0\le\theta\le2\pi$. [Hint: Make a substitution to express the equation only in terms of cosine.]

Answer

$\cos\theta=-1,\ \theta=\pi$

The Pythagorean Identities become very useful whenever an equation involves a combination of sine and cosine functions.

Example 6.2.35

Solve $2\sin^2(t)-\cos(t)=1$ for all solutions with $0\le t<2\pi$.

Solution

Since this equation has a mix of sine and cosine functions, it becomes more complicated to solve. It is usually easier to work with an equation involving only one trig function. This is where we can use the Pythagorean Identity.

$$\begin{aligned}
2\sin^2(t)-\cos(t)&=1 && \text{Use the Pythagorean Identity: } \sin^2(\theta)=1-\cos^2(\theta)\\
2\left(1-\cos^2(t)\right)-\cos(t)&=1 && \text{Distribute the 2}\\
2-2\cos^2(t)-\cos(t)&=1
\end{aligned}$$

Since this is now quadratic in cosine, we rearrange the equation so one side is zero and factor.

$$\begin{aligned}
-2\cos^2(t)-\cos(t)+1&=0 && \text{Multiply by -1 to simplify the factoring}\\
2\cos^2(t)+\cos(t)-1&=0 && \text{Factor}\\
(2\cos(t)-1)(\cos(t)+1)&=0
\end{aligned}$$

This product will be zero if either factor is zero, so we can break this into two separate cases and solve each independently.

$$\begin{array}{l|l} 2\cos(t)-1=0 & \cos(t)+1=0 \\ \cos(t)=\frac{1}{2} & \cos(t)=-1 \\ t=\frac{\pi}{3} \text{ or } t=\frac{5\pi}{3} & t=\pi \end{array}$$

The solutions are $t=\frac{\pi}{3},\ \frac{5\pi}{3},\ \pi$.

Try It 6.2.36

Solve $2\sin^2(t)=3\cos(t)$ for all solutions with $0\le t<2\pi$.

Answer

$$\begin{aligned} 2\left(1-\cos^2(t)\right)&=3\cos(t)\\ 2\cos^2(t)+3\cos(t)-2&=0\\ (2\cos(t)-1)(\cos(t)+2)&=0 \end{aligned}$$

$2\cos(t)-1=0$ has two solutions, $t=\frac{\pi}{3},\frac{5\pi}{3}$. $\cos(t)+2=0$ has no solutions.

In addition to the Pythagorean Identity, it is often necessary to rewrite the tangent, secant, cosecant, and cotangent as part of solving an equation.

Example 6.2.37

Solve $\tan(x)=3\sin(x)$ for all solutions with $0\le x<2\pi$.

Solution

With a combination of tangent and sine, we might try rewriting tangent

$$\begin{aligned} \tan(x)&=3\sin(x)\\ \frac{\sin(x)}{\cos(x)}&=3\sin(x) && \text{Multiply both sides by cosine}\\ \sin(x)&=3\sin(x)\cos(x) \end{aligned}$$

At this point, you may be tempted to divide both sides of the equation by $\sin(x)$.

Resist the urge. When we divide both sides of an equation by a quantity, we are assuming the quantity is never zero. In this case, when $\sin(x) = 0$ the equation is satisfied, so we'd lose those solutions if we divided by the sine.

To avoid this problem, we can rearrange the equation so that one side is zero (You technically *can* divide by $\sin(x)$, as long as you separately consider the case where $\sin(x) = 0$. Since it is easy to forget this step, the factoring approach used in the example is recommended.).

$$\begin{aligned} \sin(x)-3\sin(x)\cos(x)&=0 && \text{Factor out } \sin(x) \text{ from both parts}\\ \sin(x)(1-3\cos(x))&=0 \end{aligned}$$

From here, we can see we get solutions when $\sin(x)=0$ or $1-3\cos(x)=0$.

$$\begin{array}{l|ll} \sin(x)=0 & 1-3\cos(x)=0 & \\ x=0 \text{ or } x=\pi & \cos(x)=\frac{1}{3} & \\ & x=\cos^{-1}\left(\frac{1}{3}\right)\approx 1.231 & \text{First solution (in QI)}\\ & x=2\pi-1.231=5.052 & \text{Second solution in QIV} \end{array}$$

We have four solutions on $0\le x<2\pi$: $x=0,\ 1.231,\ \pi,\ 5.052$

Example 6.2.38

Solve $\sec(\theta) = 2\cos(\theta)$ to find the first four positive solutions.

Answer

$$\frac{1}{\cos(\theta)} = 2\cos(\theta)$$
$$\frac{1}{2} = \cos^2(\theta)$$
$$\cos(\theta) = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$$

Solutions are: $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Example 6.2.39

Solve $\frac{4}{\sec^2(\theta)} + 3\cos(\theta) = 2\cot(\theta)\tan(\theta)$ for all solutions with $0 \le \theta < 2\pi$.

Solution

$\frac{4}{\sec^2(\theta)} + 3\cos(\theta) = 2\cot(\theta)\tan(\theta)$	Use the reciprocal identities
$4\cos^2(\theta) + 3\cos(\theta) = 2\frac{1}{\tan(\theta)}\tan(\theta)$	Simplify
$4\cos^2(\theta) + 3\cos(\theta) = 2$	Subtract 2 from each side
$4\cos^2(\theta) + 3\cos(\theta) - 2 = 0$	

This does not appear to factor nicely so we use the quadratic formula, remembering that we are solving for $\cos(\theta)$.

$$\cos(\theta) = \frac{-3 \pm \sqrt{3^2 - 4(4)(-2)}}{2(4)} = \frac{-3 \pm \sqrt{41}}{8}$$

$\cos(\theta) = \frac{-3-\sqrt{41}}{8}$	$\cos(\theta) = \frac{-3+\sqrt{41}}{8}$	
$\cos(\theta) = -1.175$	$\cos(\theta) = 0.425$	
No solution	$\theta = \cos^{-1}(0.425) = 1.131$	First solution (in QI)
	$\theta = 2\pi - 1.131 = 5.152$	Second solution in QIV

Solutions are: $\theta = 1.131,\ 5.152$

Applications

We will now apply these ideas to problems involving periodic behavior.

Example 6.2.40: Modeling an Equation and Sketching a Sinusoidal Graph to Fit Criteria

The average monthly temperatures for a small town in Oregon are given in Table 6.2.1. Find a sinusoidal function of the form $y = A\sin(Bt - C) + D$ that fits the data (round to the nearest tenth) and sketch the graph.

Table 6.2.2

Month	Temperature, ^oF
January	42.5
February	44.5
March	48.5
April	52.5
May	58
June	63
July	68.5
August	69
September	64.5
October	55.5
November	46.5
December	43.5

Solution

Recall that amplitude is found using the formula

$$A = \frac{\text{largest value} - \text{smallest value}}{2}$$

Thus, the amplitude is

$$|A| = \frac{69 - 42.5}{2} = 13.25$$

The data covers a period of 12 months, so $\frac{2\pi}{B} = 12$ which gives $B = \frac{2\pi}{12} = \frac{\pi}{6}$.

The vertical shift is found using the following equation.

$$D = \frac{\text{highest value} + \text{lowest value}}{2}$$

Thus, the vertical shift is

$$D = \frac{69 + 42.5}{2} \qquad = 55.8$$

So far, we have the equation $y = 13.3\sin(\frac{\pi}{6}x - C) + 55.8$.

To find the horizontal shift, we input the x and y values for the first month and solve for C.

$$\begin{aligned} 42.5 &= 13.3\sin(\frac{\pi}{6}(1) - C) + 55.8 \\ -13.3 &= 13.3\sin(\frac{\pi}{6} - C) \\ -1 &= \sin(\frac{\pi}{6} - C) \qquad \sin\theta = -1 \rightarrow \theta = -\frac{\pi}{2} \\ \frac{\pi}{6} - C &= -\frac{\pi}{2} \\ \frac{\pi}{6} + \frac{\pi}{2} &= C \\ C &= \frac{2\pi}{3} \end{aligned}$$

We have the equation $y = 13.3\sin(\frac{\pi}{6}x - \frac{2\pi}{3}) + 55.8$. See the graph in Figure 6.2.9.

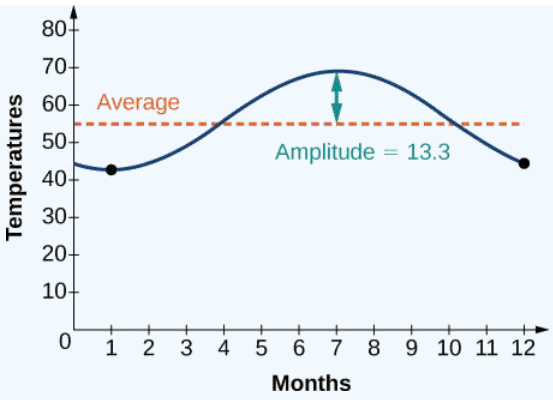


Figure **6.2.9**

Important Topics of This Section

- Solving trig equations using known values
- Using substitution to solve equations
- Finding answers in one cycle or period vs. finding all possible solutions
- Methods for solving trig equations

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6.2e: Exercises - Trig Equations

Give all answers in radians unless otherwise indicated.

A: Concepts

Exercise 6.2e.A

1. Will there always be solutions to trigonometric function equations? If not, describe an equation that would not have a solution. Explain why or why not.

2. When solving a trigonometric equation involving more than one trig function, do we always want to try to rewrite the equation so it is expressed in terms of one trigonometric function? Why or why not?

3. When solving linear trig equations in terms of only sine or cosine, how do we know whether there will be solutions?

Answers to odd exercises.

1. There will not always be solutions to trigonometric function equations. For a basic example, $\cos(x) = -5$.

3. Rewrite the equation in the form $\sin(u) = c$ or $\cos(u) = c$. If $|c| \leq 1$, then the equation will have a solution; otherwise it will not have a solution.

B: Linear Equations - Special Angle Solutions

Exercise 6.2e.B

★ **Find all solutions on the interval** $0 \leq \theta < 2\pi$.

5. $2\sin(\theta) = -\sqrt{2}$	9. $2\cos(\theta) = 1$	13. $\tan x = 1$	17. $\sqrt{3}\csc x + 2 = 0$
6. $2\sin(\theta) = \sqrt{3}$	10. $2\cos(\theta) = -\sqrt{2}$	14. $\tan\theta = -1$	18. $\sqrt{3}\sec\theta + 2 = 0$
7. $\sin(\theta) = 1$	11. $\cos(\theta) = 0$	15. $\sqrt{3}\tan\theta = 1$	19. $\cot x + 1 = 0$
8. $\sin(\theta) = 0$	12. $\cos(\theta) = 1$	16. $\sqrt{3} + \tan\theta = 0$	

★ **State the general solution and the first four non-negative solutions.**

21. $2\sin(\theta) = -1$

22. $2\sin(\theta) = 1$

23. $2\sin(3\theta) = \sqrt{2}$

24. $2\sin(3\pi\theta) = -\sqrt{2}$

25. $2\sin\left(\frac{\theta}{2}\right) = \sqrt{3}$

26. $\sin\left(\frac{2\pi}{3}\theta\right) = -1$

27. $\sin\left(\frac{3\pi}{4}\theta + \frac{\pi}{2}\right) = 0$

28. $2\sin\left(4\pi\theta - \frac{\pi}{3}\right) = -\sqrt{3}$

29. $\csc(2\pi x) = -2$

30. $\csc\left(\frac{x}{3}\right) = \sqrt{2}$

31. $2\cos(\theta) = \sqrt{2}$

32. $2\cos(\theta) = -1$

33. $2\cos(2\pi\theta) = 1$

34. $2\cos(3\theta) = -\sqrt{2}$

35. $\cos\left(\frac{\pi}{4}\theta\right) = -1$

36. $2\cos\left(\frac{\theta}{3}\right) = \sqrt{3}$

37. $2\cos\left(3\pi\theta - \frac{\pi}{4}\right) = -\sqrt{3}$

38. $\cos\left(\frac{\theta}{2} + \frac{\pi}{6}\right) = 0$

39. $\sec(3x) = -2$

40. $\sec\left(\frac{\pi}{5}x\right) = \sqrt{2}$

41. $\sqrt{3}\tan(3x) = -1$

42. $\tan(\pi x) = \sqrt{3}$

43. $\tan\left(\frac{\theta}{3}\right) = 1$

44. $\tan\left(\frac{2\pi}{3}\theta\right) = -1$

45. $\tan\left(2x + \frac{\pi}{2}\right) = -\sqrt{3}$

46. $\tan\left(\frac{\pi}{6}\theta - \frac{\pi}{9}\right) = 0$

47. $\cot(2\pi x) = -1$

48. $\cot\left(\frac{\pi}{6}x\right) = \sqrt{3}$

Answers to odd exercises.

5. $\frac{5\pi}{4}, \frac{7\pi}{4}$ 7. $\frac{\pi}{2}$ 9. $\frac{\pi}{3}, \frac{5\pi}{3}$ 11. $\frac{\pi}{2}, \frac{3\pi}{2}$ 13. $\frac{\pi}{4}, \frac{5\pi}{4}$ 15. $\frac{\pi}{6}, \frac{7\pi}{6}$ 17. $\frac{4\pi}{3}, \frac{5\pi}{3}$ 19. $\frac{3\pi}{4}, \frac{7\pi}{4}$

k below represents any integer

21. $\frac{7\pi}{6} + 2\pi k, \frac{11\pi}{6} + 2\pi k;$ $\frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$

23. $\frac{\pi}{12} + \frac{2\pi}{3}k, \frac{\pi}{4} + \frac{2\pi}{3}k;$ $\frac{\pi}{12}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{11\pi}{12}$

25. $\frac{2\pi}{3} + 4\pi k, \frac{4\pi}{3} + 4\pi k;$ $\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{14\pi}{3}, \frac{16\pi}{3}$

27. $-\frac{2}{3} + \frac{4}{3}k;$ $\frac{2}{3}, 2, \frac{10}{3}, \frac{14}{3}$

29. $\frac{7}{12} + k, \frac{11}{12} + k;$ $\frac{7}{12}, \frac{11}{12}, \frac{19}{12}, \frac{23}{12}$

31. $\frac{\pi}{4} + 2\pi k, \frac{7\pi}{4} + 2\pi k;$ $\frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}$

33. $\frac{1}{6} + k, \frac{5}{6} + k;$ $\frac{1}{6}, \frac{5}{6}, \frac{7}{6}, \frac{11}{6}$

35. $4 + 8k;$ $4, 12, 20, 28$

37. $\frac{13}{36} + \frac{2}{3}k, \frac{17}{36} + \frac{2}{3}k;$ $\frac{13}{36}, \frac{17}{36}, \frac{37}{36}, \frac{41}{36}$

39. $\frac{2\pi}{9} + \frac{2\pi}{3}k, \frac{4\pi}{9} + \frac{2\pi}{3}k;$ $\frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}$

41. $-\frac{\pi}{18} + \frac{\pi}{3}k;$ $\frac{5\pi}{18}, \frac{11\pi}{18}, \frac{17\pi}{18}, \frac{23\pi}{18}$

43. $\frac{3\pi}{4} + 3\pi k;$ $\frac{3\pi}{4}, \frac{15\pi}{4}, \frac{27\pi}{4}, \frac{39\pi}{4}$

45. $\frac{\pi}{12} + \frac{\pi}{2}k;$ $\frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, 19\frac{\pi}{12}$

47. $\frac{3}{8} + \frac{k}{2};$ $\frac{3}{8}, \frac{7}{8}, \frac{11}{8}, \frac{15}{8}$

C: Quadratic Equations - Special Angle Solutions

Exercise 6.2e.C

★ **Find all solutions on the interval $[0, 2\pi)$. Give exact answers.**

51. $\sin^2 x = \frac{1}{4}$
52. $4\sin^2 x - 2 = 0$
53. $\sin^3 t = \sin t$
54. $\sin^2 x + \sin x - 2 = 0$
55. $2\sin^2 w + 3\sin w + 1 = 0$
56. $2\sin^2 x + 3\sin x - 2 = 0$
57. $\csc^2 x - 4 = 0$
59. $\cos^2 \theta = \frac{1}{2}$
60. $4\cos^2 x - 3 = 0$
61. $\cos^3 t = \cos t$
62. $2\cos^2 t + \cos(t) = 1$
63. $\cos^2 x - 2\cos x - 3 = 0$
64. $6\cos^2(\theta) = 4 - 5\cos(\theta)$
65. $\sec^2 x = 1$
67 $\tan^2 \theta = 3$
68. $3\tan^2 \theta = 1$
69. $\tan^3(x) = 3\tan(x)$
70. $\tan^5(x) = \tan(x)$
71. $9\tan^5(x) - \tan(x) = 0$
72. $\tan^2 x - \sqrt{3}\tan x = 0$
73. $\cot^2 x = -\cot x$

Answers to odd exercises

51. $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$
53. $0, \pi, \frac{\pi}{2}, \frac{3\pi}{2}$
55. $\frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$
57. $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$
59. $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
61. $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$
63. π
65. $0, \pi$
67. $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$
69. $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$
71. $0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}$
73. $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{3\pi}{4}, \frac{7\pi}{4}$

★ **Find all solutions on the interval $[0, 2\pi)$. Give exact answers. Use fundamental identities as needed.**

79. $\sin^2 x(1 - \sin^2 x) + \cos^2 x(1 - \sin^2 x) = 0$
80. $\sin^2 x - \cos^2 x - \sin x = 0$
81. $\sin^2 x - \cos^2 x - 1 = 0$
82. $\sin^2 x - \cos^2 x - \cos x = 1$
83. $2\cos^2 x + 3\sin x - 3 = 0$
84. $\cos^2 x - 2\sin x - 2 = 0$
85. $2\sin^2 x - \cos x - 1 = 0$
86. $2\sin^2 x - \cos x - 2 = 0$
87. $4\sin(x)\cos(x) + 2\sin(x) - 2\cos(x) - 1 = 0$
88. $2\sin(x)\cos(x) - \sin(x) + 2\cos(x) - 1 = 0$
89. $\sec(x)\sin(x) - 2\sin(x) = 0$
90. $\frac{1}{\sec^2 x} + 2 + \sin^2 x + 4\cos^2 x = 4$
91. $\cos(x) = \sin(-x)$
92. $\tan(x)\sin(x) = 3\cos(x)$
93. $\tan(x) = \cot(x)$
94. $\tan^2(x) = -1 + 2\tan(-x)$
95. $\tan(x)\sin(x) - \sin(x) = 0$
96. $\tan(x) - 2\sin(x)\tan(x) = 0$
97. $2\tan^2(t) = 3\sec(t)$
98. $\sec x = \tan x - 1$
99. $\csc x = 1 - \cot x$

Answers to odd exercises

79. $\frac{\pi}{2}, \frac{3\pi}{2}$
81. $\frac{\pi}{2}, \frac{3\pi}{2}$
83. $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}$
85. $\frac{\pi}{3}, \pi, \frac{5\pi}{3}$
87. $\frac{\pi}{6}, \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}$
89. $0, \pi, \frac{\pi}{3}, \frac{5\pi}{3}$
91. $\frac{3\pi}{4}, \frac{7\pi}{4}$
93. $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
95. $0, \pi, \frac{\pi}{4}, \frac{5\pi}{4}$
97. $\frac{\pi}{3}, \frac{5\pi}{3}$
99. $\frac{\pi}{2}$

D: Use a calculator to solve linear equations

Exercise 6.2e.D

★ **Find all solutions on the interval $0 \le x < 2\pi$.**

101. $\sin(x) = 0.27$
102. $\sin(x) = -0.48$
103. $\sin(x) = -0.58$
104. $\cos(x) = -0.34$
105. $\cos(x) = -0.55$
106. $\cos(x) = 0.28$
107. $\tan(x) = 0.71$
108. $\tan(x) = -4.73$

★ **Find the first two positive solutions**

111. $\csc(2x) - 9 = 0$
112. $\sec(2\theta) = 3$
115. $7\sin(6x) = 2$
116. $7\sin(5x) = 6$
117. $3\sin\left(\frac{\pi}{4}x\right) = 2$
118. $7\sin\left(\frac{\pi}{5}x\right) = 6$
119. $5\cos(3x) = -3$
120. $3\cos(4x) = 2$
121. $5\cos\left(\frac{\pi}{3}x\right) = 1$
122. $3\cos\left(\frac{\pi}{2}x\right) = -2$

★ **Use a calculator to find all solutions to four decimal places.**

131. $\tan x = -0.34$
132. $\sin x = -0.55$
133. $3\cos\left(\frac{\pi}{5}x\right) = 2$
134. $8\cos\left(\frac{\pi}{2}x\right) = 6$
135. $7\sin(3t) = -2$
136. $4\sin(4t) = 1$

Answers to odd exercises.

101. $0.2734,\ 2.8682$ 103. $3.7603,\ 5.6645$ 105. $2.1532,\ 4.1300$ 107. $0.6174,\ 3.7590$
111. $0.056,\ 1.515,\ 3.197,\ 4.657$ 115. $0.04829,\ 0.47531$
117. $0.9291,\ 3.0709$ 119. $0.7381,\ 1.3563$ 121. $1.3077,\ 4.6923$
131. $\pi k - 0.3277$ 133. $0.1339 + 10k$ and $8.6614 + 10k$, where k is an integer
135. $1.1438 + \frac{2\pi}{3}k$ and $1.9978 + \frac{2\pi}{3}k$, where k is an integer

E: Use calculator to Solve Quadratic Equations

Exercise $6.2e.E$

★ **Find all solutions on the interval $[0, 2\pi)$. Use the quadratic formula if the equations do not factor.**

147. $10\sin(x)\cos(x) = 6\cos(x)$
148. $-3\sin(t) = 15\cos(t)\sin(t)$
149. $\sec^2 x = 7$
150. $\csc^2 t = 3$
151. $4\cos^2(x) - 4 = 15\cos(x)$
152. $8\sin^2 x + 6\sin(x) + 1 = 0$
153. $\sin^2 x - 2\sin x - 4 = 0$
154. $9\sin(w) - 2 = 4\sin^2(w)$
155. $\tan^2 x + 3\tan x - 3 = 0$
156. $6\sin^2 x - 5\sin x + 1 = 0$
157. $2\tan^2 x + 9\tan x - 6 = 0$
158. $6\tan^2 x + 13\tan x = -6$
159. $-\tan^2 x - \tan x - 2 = 0$
160. $5\cos^2 x + 3\cos x - 1 = 0$
161. $2\cos^2 x - \cos x + 15 = 0$
162. $5\sin^2 x + 2\sin x - 1 = 0$
163. $3\cos^2 x - 3\cos x - 2 = 0$
164. $100\tan^2 x + 20\tan x - 3 = 0$
165. $\tan^2 x + 5\tan x - 1 = 0$
166. $20\sin^2 x - 27\sin x + 7 = 0$
167. $8\cos^2 x - 2\cos x - 1 = 0$
168. $130\tan^2 x + 69\tan x - 130 = 0$
169. $2\tan^2 x + 7\tan x + 6 = 0$

Answers to odd exercises.

147. $\frac{\pi}{2}, \frac{3\pi}{2}, 0.644, 2.498$ 149. $1.183,\ 1.958,\ 4.325,\ 5.100$
151. $1.823,\ 4.460$ 153. There are no solutions. 155. $0.6694,\ 1.8287,\ 3.8110,\ 4.9703$
157. $0.5326,\ 1.7648,\ 3.6742,\ 4.9064$ 159. There are no solutions. 161. There are no solutions.
163. $\cos^{-1}\left(\frac{3-\sqrt{33}}{3}\right) \approx 2.0459,\ 2\pi - \cos^{-1}\left(\frac{3-\sqrt{33}}{3}\right) \approx 4.2373$
165. $\tan^{-1}\left(\frac{\sqrt{29}-5}{2}\right), \pi + \tan^{-1}\left(\frac{-\sqrt{29}-5}{2}\right),\ \pi + \tan^{-1}\left(\frac{\sqrt{29}-5}{2}\right), 2\pi + \tan^{-1}\left(\frac{-\sqrt{29}-5}{2}\right)$
167. $\frac{\pi}{3}, \cos^{-1}\left(-\frac{1}{4}\right), 2\pi - \cos^{-1}\left(-\frac{1}{4}\right), \frac{5\pi}{3}$
169. $\pi + \tan^{-1}(-2), \pi + \tan^{-1}\left(-\frac{3}{2}\right), 2\pi + \tan^{-1}(-2), 2\pi + \tan^{-1}\left(-\frac{3}{2}\right)$

★ Find all solutions on the interval $[0, 2\pi)$. Use identities. Use the quadratic formula if the equations do not factor.

171. $12\sin^2(t) + \cos(t) - 6 = 0$
172. $6\cos^2(x) + 7\sin(x) - 8 = 0$
173. $\cos^2\phi = -6\sin\phi$
174. $\sin^2 t = \cos t$
175. $\tan(x) - 3\sin(x) = 0$
176. $3\cos(x) = \cot(x)$
177. $\sin^2 x + \cos^2 x = 0$
178. $\dfrac{2\tan x}{2 - \sec^2 x} - \sin^2 x = \cos^2 x$
179. $\tan^2 x - \sec x = 1$
180. $\sin^2 x - 2\cos^2 x = 0$
181. $12\sin^2 t + \cos t - 6 = 0$
182. $1 - 2\tan(w) = \tan^2(w)$
183. $\csc^2 x - 3\csc x - 4 = 0$
184. $3\sec^2 x + 2 + \sin^2 x + \cos^2 x = \tan^2 x$
185. $2\cos^2 x - \sin^2 x - \cos x - 5 = 0$
186. $\tan^2 x - 1 - \sec^3 x \cos x = 0$

Answers to odd exercises.

171. $2.301,\ 3.983,\ 0.723,\ 5.560$ 173. $3.305,\ 6.120$ 175. $0,\ \pi,\ 1.231,\ 5.052$ 177. There are no solutions.
179. $1.0472, 3.1416, 5.2360$ 181. $\cos^{-1}\left(\frac{3}{4}\right), \cos^{-1}\left(-\frac{2}{3}\right), 2\pi - \cos^{-1}\left(-\frac{2}{3}\right), 2\pi - \cos^{-1}\left(\frac{3}{4}\right)$
183. $\sin^{-1}\left(\frac{1}{4}\right), \pi - \sin^{-1}\left(\frac{1}{4}\right), \frac{3\pi}{2}$ 185. There are no solutions.

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6.3: Verifying Trigonometric Identities

Learning Objectives.

- Verify the fundamental trigonometric identities.
- Simplify trigonometric expressions using algebra and the identities.

In this section, we will begin an examination of the fundamental trigonometric identities, including how we can verify them and how we can use them to simplify trigonometric expressions.

Trigonometric Identities

Identities enable us to simplify complicated expressions. We can use algebraic techniques to simplify trigonometric expressions. Basic properties and formulas of algebra, such as the difference of squares formula and the perfect squares formula, will simplify the work involved with trigonometric expressions and equations. Consequently, any trigonometric identity can be written in many ways.

To verify the trigonometric identities, we usually start with the more complicated side of the equation and essentially rewrite the expression until it has been transformed into the same expression as the other side of the equation. Sometimes we have to factor expressions, expand expressions, find common denominators, or use other algebraic strategies to obtain the desired result.

We will begin with reviewing the fundamental identities already introduced in a previous section: the **Pythagorean Identities**, the **Even-Odd** (or **Negative Angle**) **Identities**, the **Reciprocal Identities**, and the **Quotient Identities**.

Definitions: Basic TRIGONOMETRIC IDENTITIES

The P**ythagorean Identities** are based on the properties of a right triangle.

$\sin^2\theta+\cos^2\theta=1$	$1+\cot^2\theta=\csc^2\theta$	$1+\tan^2\theta=\sec^2\theta$

The **Even-Odd** (or **Negative Angle**) **Identities** relate the value of a trigonometric function at a given angle to the value of the function at the opposite angle of a right triangle.

$\tan(-\theta)=-\tan\theta$	$\sin(-\theta)=-\sin\theta$	$\cos(-\theta)=\cos\theta$
$\cot(-\theta)=-\cot\theta$	$\csc(-\theta)=-\csc\theta$	$\sec(-\theta)=\sec\theta$

The **Reciprocal Identities** define reciprocals of the trigonometric functions.

$\sin\theta=\dfrac{1}{\csc\theta}$	$\cos\theta=\dfrac{1}{\sec\theta}$	$\tan\theta=\dfrac{1}{\cot\theta}$
$\csc\theta=\dfrac{1}{\sin\theta}$	$\sec\theta=\dfrac{1}{\cos\theta}$	$\cot\theta=\dfrac{1}{\tan\theta}$

The **Quotient Identities** define relationships among the trigonometric functions.

$\tan\theta=\dfrac{\sin\theta}{\cos\theta}$	$\cot\theta=\dfrac{\cos\theta}{\sin\theta}$

6.3.1 Summary of Basic Trigonometric Identities

These relationships are called **identities**. Identities are statements that are true for all values of the input on which they are defined. For example, $2x+6=2(x+3)$ is an example of an identity. Identities are usually something that can be derived from definitions and relationships we already know. The Pythagorean Identity we learned earlier was derived from the Pythagorean Theorem and the definitions of sine and cosine. We will discuss the role of identities more after an example.

Simplifying a Trigonometric Expression

Just as we often need to **simplify** algebraic expressions, it is often also necessary or helpful to simplify trigonometric expressions. To do so, we utilize the definitions and identities we have established.

Example 6.3.2: Simplify

Simplify $\frac{\sec(\theta)}{\tan(\theta)}$.

Solution

$$\begin{aligned}
\frac{\sec(\theta)}{\tan(\theta)} &= \frac{\frac{1}{\cos(\theta)}}{\frac{\sin(\theta)}{\cos(\theta)}} && \text{Simplify: rewrite both functions in terms of sine and cosine} \\
&= \frac{1}{\cos(\theta)} \cdot \frac{\cos(\theta)}{\sin(\theta)} && \text{Divide fractions: invert and multiply} \\
&= \frac{1}{\sin(\theta)} && \text{Cancel the cosines} \\
&= \csc(\theta) && \text{Simplify: use Reciprocal Identity}
\end{aligned}$$

By showing that $\frac{\sec(\theta)}{\tan(\theta)}$ can be simplified to $\csc(\theta)$, we have, in fact, established a new identity: $\frac{\sec(\theta)}{\tan(\theta)} = \csc(\theta)$.

Try It 6.3.2

Simplify the expression by rewriting and using identities: $\csc^2\theta - \cot^2\theta$

Answer

$$\begin{aligned}
\csc^2\theta - \cot^2\theta &= (1 + \cot^2\theta) - \cot^2\theta && \text{Use the Pythagorean Identity: } 1 + \cot^2\theta = \csc^2 \\
&= 1 + \cot^2\theta - \cot^2\theta \\
&= 1
\end{aligned}$$

Verifying A Trigonometric Identity

Occasionally a question may ask you to "prove the identity" or "establish the identity." This is the same idea as when an algebra book asks a question like "show that $(x-1)^2 = x^2 - 2x + 1$." In this type of question, we must show the algebraic manipulations that demonstrate that the left and right side of the equation are in fact equal. You can think of a "prove the identity" problem as a simplification problem where you *know the answer*: you know what the end goal of the simplification should be, and just need to show the steps to get there.

To prove an identity, in most cases you will start with the expression on one side of the identity and manipulate it using algebra and trigonometric identities until you have simplified it to the expression on the other side of the equation. **Do not** treat the identity like an equation to solve – it is NOT an equation! The proof is establishing the two expressions are equal, so we must take care to work with one side at a time rather than applying an operation simultaneously to both sides of the equation.

Example 6.3.3: Verify

Prove the identity $\frac{1+\cot(\alpha)}{\csc(\alpha)} = \sin(\alpha) + \cos(\alpha)$.

Solution

Since the left side seems a bit more complicated, we will start there and simplify the expression until we obtain the right side. We can use the right side as a guide for what might be good steps to make. In this case, the left side involves a fraction while the right side doesn't, which suggests we should look to see if the fraction can be reduced.

Additionally, since the right side involves sine and cosine and the left does not, it suggests that rewriting the cotangent and cosecant using sine and cosine might be a good idea.

$\dfrac{1+\cot(\alpha)}{\csc(\alpha)}$	$=\sin(\alpha)+\cos(\alpha)$	Rewrite the cotangent and cosecant
$=\dfrac{1+\dfrac{\cos(\alpha)}{\sin(\alpha)}}{\dfrac{1}{\sin(\alpha)}}$		Divide fractions: invert and multiply
$=\left(1+\dfrac{\cos(\alpha)}{\sin(\alpha)}\right)\cdot\dfrac{\sin(\alpha)}{1}$		Distribute
$=1\cdot\dfrac{\sin(\alpha)}{1}+\dfrac{\cos(\alpha)}{\sin(\alpha)}\cdot\dfrac{\sin(\alpha)}{1}$		Simplify fractions
$=\sin(\alpha)+\cos(\alpha)$ ✓		Establish the identity

Notice that in the second step, we could have combined the 1 and $\dfrac{\cos(\alpha)}{\sin(\alpha)}$ before inverting and multiplying. It is very common when proving or simplifying identities for there to be more than one way to obtain the same result.

We can also utilize identities we have previously learned, like the Pythagorean Identity, while simplifying or proving identities.

Graphically Confirming a Trigonometric Identity

One way to quickly confirm whether or not an identity is valid, is to graph the expression on each side of the equal sign. If the resulting gtaphs are identical, then the equation is an identity.

Example 6.3.4: Confirm graphically

Graph both sides of the identity $\cot\theta=\dfrac{1}{\tan\theta}$. In other words, on the graphing calculator, graph $y=\cot\theta$ and $y=\dfrac{1}{\tan\theta}$.

Solution

See Figure 6.3.4.

Analysis

We see only one graph because both expressions generate the same image. One is on top of the other. This is a good way to prove any identity. If both expressions give the same graph, then they must be identities.

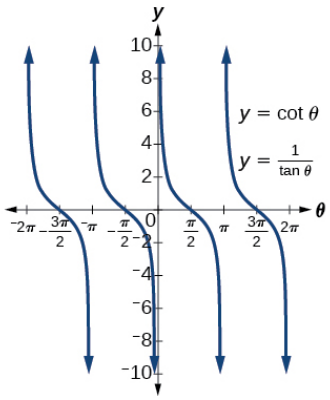

Figure 6.3.4

Verifying Trig Identities

How to: Given a trigonometric identity, verify that it is true.

1. Work on one side of the equation. Do NOT - absolutely NOT EVER - use Properties of Equality like adding/subtracting/multiplying/or dividing the same expression to both sides of the equal sign. Work ONLY on one side of the equal sign.
2. It is usually better to start with the more complex side, as it is easier to simplify than to build.
3. Look for opportunities to
   - Multiply expressions out and combine like terms
     or factor expressions and cancel,
   - Split apart fractions
     or rewrite fractions with a common denominator and combine fractions into a single fraction,

- Simplify two term denominators by using a Pythagorean substitution
  or simplify two term denominators by multiplying numerator and denominator by the conjugate of the binomial denominator.

3. Observe which functions are in the final expression, and look for opportunities to use identities and/or employ substitutions that would make both sides of both sides of the equal sign have
   - the number of terms
   - the same functions in the numerators and/or denominators.
4. If these steps do not yield the desired result, try converting all terms to sines and cosines.

Example 6.3.5: Creating and Verifying an Identity

Create an identity for the expression $2\tan\theta\sec\theta$ by rewriting strictly in terms of sine.

Solution

There are a number of ways to begin, but here we will use the quotient and reciprocal identities to rewrite the expression:

$$\begin{aligned} 2\tan\theta\sec\theta &= 2\left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{1}{\cos\theta}\right) \\ &= \frac{2\sin\theta}{\cos^2\theta} \\ &= \frac{2\sin\theta}{1-\sin^2\theta} \qquad \text{Substitute } 1-\sin^2\theta \text{ for } \cos^2\theta \end{aligned}$$

Thus, $2\tan\theta\sec\theta = \dfrac{2\sin\theta}{1-\sin^2\theta}$

Example 6.3.6: Verify a Trigonometric Identity - Cancel

Verify $\tan\theta\cos\theta = \sin\theta$.

Solution

We will start on the left side, as it is the more complicated side:

$\tan\theta\cos\theta$	$= \sin\theta$	Use the Quotient Identity: $\frac{\sin\theta}{\cos\theta}$
$= \left(\frac{\sin\theta}{\cos\theta}\right)\cos\theta$		Rewrite as product of fractions
$= \frac{\sin\theta}{\cos\theta}\cdot\frac{\cos\theta}{1}$		Cancel
$= \sin(\theta)$ ✓		Establish the identity

Analysis

This identity was fairly simple to verify, as it only required writing $\tan\theta$ in terms of $\sin\theta$ and $\cos\theta$.

Try It 6.3.6

Verify the identity $\csc\theta\cos\theta\tan\theta = 1$.

Answer

$$\csc\theta\cos\theta\tan\theta = \left(\frac{1}{\sin\theta}\right)\cos\theta\left(\frac{\sin\theta}{\cos\theta}\right) = \frac{1}{\sin\theta}\cdot\frac{\cos\theta}{1}\cdot\frac{\sin\theta}{\cos\theta} = \frac{\cos\theta\sin\theta}{\sin\theta\cos\theta} = 1 \ \checkmark$$

Example 6.3.7: Verify a Trigonometric Identity - Multiply Out

Verify the following equivalency:

$$(1+\sin x)(1+\sin(-x)) = \cos^2 x$$

Solution

Working on the left side of the equation, we have

$(1+\sin x)(1+\sin(-x))$	$=\cos^2 x$	Use the Even/Odd Identity: $\sin(-x) = -\sin x$
$=(1+\sin x)(1-\sin x)$		Multiply out
$=1-\sin x+\sin x-\sin^2 x$		Combine like terms
$=1-\sin^2 x$		Use the Pythagorean Identity: $\cos^2 x+\sin^2 x=1$
$=\cos^2 x$ ✓		Establish the identity

Example 6.3.8: Verify a Trigonometric Identity - Factor

Establish the identity $\dfrac{\cos^2(\theta)}{1+\sin(\theta)} = 1-\sin(\theta)$.

Solution

Since the left side of the identity is more complicated, it makes sense to start there. To simplify this, we will have to reduce the fraction, which would require the numerator to have a factor in common with the denominator. Additionally, we notice that the right side only involves sine. Both of these suggest that we need to convert the cosine into something involving sine.

Recall the Pythagorean Identity told us $\cos^2(\theta)+\sin^2(\theta)=1$. By moving one of the trig functions to the other side, we can establish:

$$\sin^2(\theta) = 1-\cos^2(\theta) \quad \text{and} \quad \cos^2(\theta) = 1-\sin^2(\theta)$$

Utilizing this, we now can establish the identity.

$\dfrac{\cos^2(\theta)}{1+\sin(\theta)}$	$=1-\sin(\theta)$	Use the Pythagorean Identity: $\cos^2(\theta)+\sin^2(\theta)=1$
$=\dfrac{1-\sin^2(\theta)}{1+\sin(\theta)}$		Factor the numerator
$=\dfrac{(1-\sin(\theta))(1+\sin(\theta))}{1+\sin(\theta)}$		Cancel like factors
$=1-\sin(\theta)$ ✓		Establish the identity

Example 6.3.9: Verify a Trigonometric Identity - Factor

Verify the identity:

$$\frac{\sin^2(-\theta)-\cos^2(-\theta)}{\sin(-\theta)-\cos(-\theta)} = \cos\theta-\sin\theta$$

Solution

Let's start with the left side and simplify:

$\dfrac{\sin^2(-\theta)-\cos^2(-\theta)}{\sin(-\theta)-\cos(-\theta)}$	$=\cos\theta-\sin\theta$	
$=\dfrac{[\sin(-\theta)]^2-[\cos(-\theta)]^2}{\sin(-\theta)-\cos(-\theta)}$		Use Even\Odd Identities: $\sin(-x)=-\sin\ x$ and $\cos(-x)=\cos\ x$
$=\dfrac{(-\sin\theta)^2-(\cos\theta)^2}{-\sin\theta-\cos\theta}$		Factor a difference of squares; Factor out a common negative.
$=\dfrac{(\sin\theta-\cos\theta)(\sin\theta+\cos\theta)}{-(\sin\theta+\cos\theta)}$		Cancel
$=\dfrac{(\sin\theta-\cos\theta)}{-1}$		
$=\cos\theta-\sin\theta$ ✓		Establish the identity

Try It 6.3.10: Factor

Verify the identity $\dfrac{\sin^2\theta-1}{\tan\theta\sin\theta-\tan\theta}=\dfrac{\sin\theta+1}{\tan\theta}$.

Answer

$$\frac{\sin^2\theta-1}{\tan\theta\sin\theta-\tan\theta}=\frac{(\sin\theta+1)(\sin\theta-1)}{\tan\theta(\sin\theta-1)}=\frac{\sin\theta+1}{\tan\theta}$$

Example 6.3.11: Verify a Trigonometric Identity - Add fractions

Verify the identity: $(1-\cos^2x)(1+\cot^2x)=1$.

Solution

$(1-\cos^2x)(1+\cot^2x)$	$=1$	Use Quotient Identity: $\cot x=\dfrac{\cos x}{\sin x}$
$=(1-\cos^2x)\left(1+\dfrac{\cos^2x}{\sin^2x}\right)$		Find the common denominator
$=(1-\cos^2x)\left(\dfrac{\sin^2x}{\sin^2x}+\dfrac{\cos^2x}{\sin^2x}\right)$		
$=(1-\cos^2x)\left(\dfrac{\sin^2x+\cos^2x}{\sin^2x}\right)$		Use the Pythagorean Identity: $\cos^2x)+\sin^2x)=1$
$=(\sin^2x)\left(\dfrac{1}{\sin^2x}\right)$		Cancel
$=1$		

Example 6.3.12: Verify a Trigonometric Identity - Split or Shear Fractions

Verify the identity $\dfrac{\sec^2\theta-1}{\sec^2\theta}=\sin^2\theta$

Solution

As the left side is more complicated, let's begin there.

$\dfrac{\sec^2\theta - 1}{\sec^2\theta}$	Use the Pythagorean Identity: $\tan^2(\theta) + 1 = \sec^2(\theta)$
$= \dfrac{(\tan^2(\theta) + 1) - 1}{\sec^2\theta}$	Simplify
$= \dfrac{\tan^2\theta}{\sec^2\theta}$	"Shear" the fraction
$= \dfrac{\tan^2\theta}{1} \cdot \dfrac{1}{\sec^2\theta}$	Use Reciprocal Identity: $\dfrac{1}{\sec\theta} = \cos\theta$
$= \tan^2\theta \cdot \cos^2\theta$	Use Quotient Identity: $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
$= \dfrac{\sin^2\theta}{\cos^2\theta} \cdot \cos^2\theta$	Cancel
$= \sin^2\theta$ ✓	Establish the identity

There is more than one way to verify an identity. Here is another possibility. Again, we can start with the left side.

$\dfrac{\sec^2\theta - 1}{\sec^2\theta}$	Split the fraction apart
$= \dfrac{\sec^2\theta}{\sec^2\theta} - \dfrac{1}{\sec^2\theta}$	Simplify. Use Reciprocal Identity: $\dfrac{1}{\sec\theta} = \cos\theta$
$= 1 - \cos^2\theta$	Use the Pythagorean Identity: $\cos^2(\theta) + \sin^2(\theta) = 1$
$= \sin^2\theta$ ✓	Establish the identity

Analysis

In the first method, we used the identity $\sec^2\theta = \tan^2\theta + 1$ and continued to simplify. In the second method, we split the fraction, putting both terms in the numerator over the common denominator. This problem illustrates that there are multiple ways we can verify an identity. Employing some creativity can sometimes simplify a procedure. As long as the substitutions are correct, the answer will be the same.

Try It 6.3.13: Complex Fraction

Show that $\dfrac{\cot\theta}{\csc\theta} = \cos\theta$.

Answer

$$\frac{\cot\theta}{\csc\theta} = \frac{\dfrac{\cos\theta}{\sin\theta}}{\dfrac{1}{\sin\theta}} = \frac{\cos\theta}{\sin\theta} \cdot \frac{\sin\theta}{1} = \cos\theta$$

Example 6.3.14: Verify a Trigonometric Identity - 2 term denominator

Use algebraic techniques to verify the identity: $\dfrac{\cos\theta}{1 + \sin\theta} = \dfrac{1 - \sin\theta}{\cos\theta}$.

(Hint: Multiply the numerator and denominator on the left side by $1 - \sin\theta$, the conjugate of the denominator.)

Solution

$\dfrac{\cos\theta}{1 + \sin\theta}$	$= \dfrac{1 - \sin\theta}{\cos\theta}$ Multiply by a "1" composed of the conjugate of the denominator
$= \dfrac{\cos\theta}{1 + \sin\theta}\left(\dfrac{1 - \sin\theta}{1 - \sin\theta}\right)$	
$= \dfrac{\cos\theta(1 - \sin\theta)}{1 - \sin^2\theta}$	Use the Pythagorean Identity: $\cos^2\theta + \sin^2\theta = 1$
$= \dfrac{\cos\theta(1 - \sin\theta)}{\cos^2\theta}$	
$= \dfrac{1 - \sin\theta}{\cos\theta}$ ✓	Establish the identity

Use Identities to Solve Trigonometric Equations

When solving some trigonometric equations, it becomes necessary to first rewrite the equation using trigonometric identities. One of the most common is the Pythagorean Identity, $\sin^2(\theta)+\cos^2(\theta)=1$ which allows you to rewrite $\sin^2(\theta)$ in terms of $\cos^2(\theta)$ or vice versa. These identities become very useful whenever an equation involves a combination of sine and cosine functions.

Example 6.3.15: Use Pythagorean Identities

Solve $2\sin^2(t)-\cos(t)=1$ for all solutions with $0\le t<2\pi$.

Solution

Since this equation has a mix of sine and cosine functions, it becomes more complicated to solve. It is usually easier to work with an equation involving only one trig function. This is where we can use the Pythagorean Identity.

$$2\sin^2(t)-\cos(t)=1$$

Using $\sin^2(\theta)=1-\cos^2(\theta)$

$$2\left(1-\cos^2(t)\right)-\cos(t)=1$$

Distributing the 2

$$2-2\cos^2(t)-\cos(t)=1$$

Since this is now quadratic in cosine, we rearrange the equation so one side is zero and factor.

$$-2\cos^2(t)-\cos(t)+1=0$$

Multiply by -1 to simplify the factoring

$$2\cos^2(t)+\cos(t)-1=0$$

Factor

$$(2\cos(t)-1)(\cos(t)+1)=0$$

This product will be zero if either factor is zero, so we can break this into two separate cases and solve each independently.

$$2\cos(t)-1=0 \text{ or } \cos(t)+1=0$$
$$\cos(t)=\frac{1}{2} \text{ or } \cos(t)=-1$$
$$t=\frac{\pi}{3} \text{ or } t=\frac{5\pi}{3} \text{ or } t=\pi$$

Try It 6.3.15

Solve $2\sin^2(t)=3\cos(t)$ for all solutions with $0\le t<2\pi$.

Answer

$$2\left(1-\cos^2(t)\right)=3\cos(t)$$
$$2\cos^2(t)+3\cos(t)-2=0$$
$$(2\cos(t)-1)(\cos(t)+2)=0$$
$\cos(t)+2=0$ has no solutions; $\quad 2\cos(t)-1=0$ at $t=\frac{\pi}{3},\frac{5\pi}{3}$

In addition to the Pythagorean Identity, it is often necessary to rewrite the tangent, secant, cosecant, and cotangent as part of solving an equation.

Example 6.3.16: Use a Quotient Identity

Solve $\tan(x) = 3\sin(x)$ for all solutions with $0 \le x < 2\pi$.

Solution

With a combination of tangent and sine, we might try rewriting tangent

$$\tan(x) = 3\sin(x)$$

$$\frac{\sin(x)}{\cos(x)} = 3\sin(x)$$

Multiplying both sides by cosine

$$\sin(x) = 3\sin(x)\cos(x)$$

At this point, you may be tempted to divide both sides of the equation by $\sin(x)$.

Resist the urge. When we divide both sides of an equation by a quantity, we are assuming the quantity is never zero. In this case, when $\sin(x) = 0$ the equation is satisfied, so we'd lose those solutions if we divided by the sine.

To avoid this problem, we can rearrange the equation so that one side is zero (You technically *can* divide by $\sin(x)$, as long as you separately consider the case where $\sin(x) = 0$. Since it is easy to forget this step, the factoring approach used in the example is recommended.).

$$\sin(x) - 3\sin(x)\cos(x) = 0$$

Factoring out $\sin(x)$ from both parts

$$\sin(x)\left(1 - 3\cos(x)\right) = 0$$

From here, we can see we get solutions when $\sin(x) = 0$ or $1 - 3\cos(x) = 0$.

Using our knowledge of the special angles of the unit circle,

$$\sin(x) = 0 \text{ when } x = 0 \text{ or } x = \pi$$

For the second equation, we will need the inverse cosine.

$$1 - 3\cos(x) = 0$$

$$\cos(x) = \frac{1}{3}$$

Using our calculator or technology

$$x = \cos^{-1}\left(\frac{1}{3}\right) \approx 1.231$$

Using symmetry to find a second solution

$$x = 2\pi - 1.231 = 5.052$$

We have four solutions on $0 \le x < 2\pi$:

$$x = 0, 1.231, \quad \pi, 5.052$$

Example 6.3.17: Use Reciprocal Identities

Solve $\sec(\theta) = 2\cos(\theta)$ to find the first four positive solutions.

Answer

$$\frac{1}{\cos(\theta)} = 2\cos(\theta)$$
$$\frac{1}{2} = \cos^2(\theta)$$

$$\cos(\theta) = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$$
$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

Example 6.3.18: Use Reciprocal Identities

Solve $\frac{4}{\sec^2(\theta)} + 3\cos(\theta) = 2\cot(\theta)\tan(\theta)$ for all solutions with $0 \le \theta < 2\pi$.

Solution

$$\frac{4}{\sec^2(\theta)} + 3\cos(\theta) = 2\cot(\theta)\tan(\theta)$$

Using the reciprocal identities

$$4\cos^2(\theta) + 3\cos(\theta) = 2\frac{1}{\tan(\theta)}\tan(\theta)$$

Simplifying

$$4\cos^2(\theta) + 3\cos(\theta) = 2$$

Subtracting 2 from each side

$$4\cos^2(\theta) + 3\cos(\theta) - 2 = 0$$

This does not appear to factor nicely so we use the quadratic formula, remembering that we are solving for cos(θ).

$$\cos(\theta) = \frac{-3 \pm \sqrt{3^2 - 4(4)(-2)}}{2(4)} = \frac{-3 \pm \sqrt{41}}{8}$$

Using the negative square root first,

$$\cos(\theta) = \frac{-3 - \sqrt{41}}{8} = -1.175$$

This has no solutions, since the cosine can't be less than -1.

Using the positive square root,

$$\cos(\theta) = \frac{-3 + \sqrt{41}}{8} = 0.425$$

$$\theta = \cos^{-1}(0.425) = 1.131$$

By symmetry, a second solution can be found

$$\theta = 2\pi - 1.131 = 5.152$$

Key Concepts

- There are multiple ways to represent a trigonometric expression. Verifying the identities illustrates how expressions can be rewritten to simplify a problem.
- Graphing both sides of an identity will verify it.
- Simplifying one side of the equation to equal the other side is another method for verifying an identity.
- The approach to verifying an identity depends on the nature of the identity. It is often useful to begin on the more complex side of the equation.
- We can create an identity and then verify it.
- Verifying an identity may involve algebra with the fundamental identities.

- Algebraic techniques can be used to simplify trigonometric expressions. We use algebraic techniques throughout this text, as they consist of the fundamental rules of mathematics.

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6.3e: Verifying Trigonometric Identities

A: Simplify

Exercise 6.3e.A

★

Simplify each of the following to an expression involving a single trig function with no fractions.

1. $\csc(t)\tan(t)$
2. $\cos(t)\csc(t)$
3. $\sin x \cos x \sec x$
4. $\tan x \sin x + \sec x \cos^2 x$
5. $\sin(-x)\cos(-x)\csc(-x)$
6. $\csc x + \cos x \cot(-x)$
7. $-\tan(-x)\cot(-x)$
8. $3\sin^3 t \csc t + \cos^2 t + 2\cos(-t)\cos t$
9. $\dfrac{\sec(t)}{\csc(t)}$
10. $\dfrac{\cot(t)}{\csc(t)}$
11. $\dfrac{\sec(t)-\cos(t)}{\sin(t)}$
12. $\dfrac{\cot t + \tan t}{\sec(-t)}$
13. $\dfrac{-\sin(-x)\cos x \sec x \csc x \tan x}{\cot x}$
14. $\dfrac{\sin^2(t)+\cos^2(t)}{\cos^2(t)}$
15. $\dfrac{1-\sin^2(t)}{\sin^2(t)}$
16. $\dfrac{1-\cos^2 x}{\tan^2 x} + 2\sin^2 x$
17. $\dfrac{1+\tan^2\theta}{\csc^2\theta} + \sin^2\theta + \dfrac{1}{\sec^2\theta}$
18. $\dfrac{\tan(t)}{\sec(t)-\cos(t)}$
19. $\dfrac{1+\cot(t)}{1+\tan(t)}$
20. $\dfrac{1+\sin(t)}{1+\csc(t)}$
21. $\left(\dfrac{\tan x}{\csc^2 x} + \dfrac{\tan x}{\sec^2 x}\right)\left(\dfrac{1+\tan x}{1+\cot x}\right) - \dfrac{1}{\cos^2 x}$

Answers to odd exercises.

1. $\sec(t)$ 3. $\sin x$ 5. $\cos x$ 7. -1 9. $\tan(t)$ 11. $\tan(t)$
13. $\tan^2(x)$ 15. $\cot^2(t)$ 17. $\sec^2(x)$ 19. $\cot(t)$ 21. -1

★ **Simplify the first trigonometric expression into an expression that is exclusively a function of the second expression.**

24. $\tan x;\ f(\sec x)$

25. $\sec x;\ f(\cot x)$

26. $\sec x;\ f(\sin x)$

27. $\cot x;\ f(\sin x)$

28. $\cot x;\ f(\csc x)$

29. $(\sec x+\csc x)(\sin x+\cos x)-2-\cot x;\ f(\tan x)$

30. $\dfrac{\tan x+\cot x}{\csc x};\ f(\cos x)$

31. $\dfrac{\sec x+\csc x}{1+\tan x};\ f(\sin x)$

32. $\dfrac{1}{\csc x-\sin x};\ f(\sec x,\ \tan x)$

33. $\dfrac{1}{\sin x\cos x}-\cot x;\ f(\cot x)$

34. $\dfrac{\cos x}{1+\sin x}+\tan x;\ f(\cos x)$

35. $\dfrac{1-\sin x}{1+\sin x}-\dfrac{1+\sin x}{1-\sin x};\ f(\sec x,\ \tan x)$

36. $\dfrac{1}{1-\cos x}-\dfrac{\cos x}{1+\cos x};\ f(\csc x)$

Answers to odd exercises.

25. $\pm\sec x$ 27. $\pm\cot x$ 29. $\tan x$ 31. $\dfrac{1}{\sin x}$ 33. $\dfrac{1}{\cot x}$ 35. $-4\sec x\tan x$

B: Verify (no fractions)

Exercise $6.3e.B$

★ **Verify the following identities.**

41. $\cos\theta\tan\theta=\ \sin\ \theta$

42. $\sin\ \theta\ \cot\ \theta\ =\ \cos\ \theta$

43. $\sec(a)-\cos(a)=\sin(a)\tan(a)$

44. $\cos x-\cos^3 x=\cos x\sin^2 x$

45. $\tan^2\theta\csc^2\theta-\tan^2\theta=1$

46. $\sin^2\theta\cot^2\theta+\sin^2\theta=1$

47. $3\sin^2\theta+4\cos^2\theta=3+\cos^2\theta$

48. $\sin^4\ \theta\ -\ \cos^4\ \theta\ =\ \sin^2\ \theta\ -\ \cos^2\ \theta$

49. $\cos^2 x-\tan^2 x=2-\sin^2 x-\sec^2 x$

50. $\cos^4\ \theta\ -\ \sin^4\ \theta\ =\ 1\ -\ 2\ \sin^2\ \theta$

51. $\sec^4\theta-\sec^2\theta=\tan^4\theta+\tan^2\theta$

52. $\csc^4\theta-\csc^2\theta=\cot^4\theta+\cot^2\theta$

53. $\cos\theta(\sec\theta-\cos\theta)=\sin^2\theta$

54. $\tan\theta(\cot\theta+\tan\theta)=\sec^2\theta$

55. $\tan\theta(\csc\theta+\cot\theta)-\sec\theta=1$

56. $\cot\theta(\sec\theta+\tan\theta)-\csc\theta=1$

57. $\cos\theta(\tan\theta+\cot\theta)=\csc\theta$

58. $\sin\theta(\cot\theta+\tan\theta)=\sec\theta$

59. $1+\cot(x)=\cos(x)(\sec(x)+\csc(x))$

60. $\cos x(\tan x-\sec(-x))=\sin x-1$

61. $\csc^2 x(1-\sin^2 x)=\cot^2 x$

62. $(\sin\theta+\cos\theta)^2+(\sin\theta-\cos\theta)^2=2$

63. $(\sin x+\cos x)^2=1+2\sin x\cos x$

Answers to odd exercises.

41. $\cos\theta\tan\theta = \dfrac{\cos\theta}{1}\cdot\dfrac{\sin\theta}{\cos\theta} = \sin\theta$

43. $\cos x - \cos^3 x = \cos x(1-\cos^2 x) = \cos x \sin^x$

45. $\tan^2\theta\csc^2\theta - \tan^2\theta = \dfrac{\sin^2\theta}{\cos^2\theta}\cdot\dfrac{1}{\sin^2\theta} - \dfrac{\sin^2\theta}{\cos^2\theta} = \dfrac{1-\sin^2\theta}{\cos^2\theta} = \dfrac{\cos^2\theta}{\cos^2\theta} = 1$

47. $3\sin^2\theta + 4\cos^2\theta = 3\sin^2\theta + 3\cos^2\theta + \cos^2\theta = 3\left(\sin^2\theta + \cos^2\theta\right) + \cos^2\theta = 3 + \cos^2\theta$

49. $\cos^2 x - \tan^2 x = 1 - \sin^2 x - \left(\sec^2 x - 1\right) = 1 - \sin^2 x - \sec^2 x + 1 = 2 - \sin^2 x - \sec^2 x$

51. $\sec^4\theta - \sec^2\theta = \sec^2\theta(\sec^2\theta - 1) = (1+\tan^2\theta)(1+\tan^2\theta - 1) = (1+\tan^2\theta)\tan^2\theta = \tan^4\theta + \tan^2\theta$

53. $\cos\theta(\sec\theta - \cos\theta) = \cos\theta\sec\theta - \cos^2\theta = 1 - \cos^2\theta = \sin^2\theta$

55. $\tan\theta(\csc\theta + \cot\theta) - \sec\theta = \dfrac{\sin\theta}{\cos\theta}\cdot\dfrac{1}{\sin\theta} + \dfrac{\sin\theta}{\cos\theta}\cdot\dfrac{\cos\theta}{\sin\theta} - \dfrac{1}{\cos\theta} = \dfrac{1}{\cos\theta} + 1 - \dfrac{1}{\cos\theta} = 1$

57. $\cos\theta(\tan\theta + \cot\theta) = \dfrac{\cos\theta}{1}\cdot\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{1}\cdot\dfrac{\cos\theta}{\sin\theta} = \dfrac{\sin^2\theta}{\sin\theta} + \dfrac{\cos^2\theta}{\sin\theta} = \dfrac{\sin^2\theta + \cos^2\theta}{\sin\theta} = \dfrac{1}{\sin\theta} = \csc\theta$

59. $\cos(x)(\sec(x) + \csc(x)) = \dfrac{cos\theta}{1}\cdot\dfrac{1}{cos\theta} + \dfrac{cos\theta}{1}\cdot\dfrac{1}{\sin\theta} = 1 + \cot\theta$

61. $\csc^2 x(1 - \sin^2 x) = \csc^2 x - 1 = \cot^2 x + 1 - 1 = \cot^2 x$

63. $(\sin x + \cos x)^2 = \sin^2 + 2\sin x\cos x + \cos^2 x = 1 + 2\sin x\cos x$

C: Verify (fractions)

Exercise 6.3e.C

★ Verify the following identities.

71. $\frac{\tan\ \theta}{\cot\ \theta}\ =\ \tan^2\ \theta$

72. $\frac{\csc\ \theta}{\sin\ \theta}\ =\ \csc^2\ \theta$

73. $\frac{\tan x}{\sec x}\sin(-x) = \cos^2 x - 1$

74. $\frac{1\ -\ 2\ \cos^2\ \theta}{\sin\ \theta \cos\ \theta}\ =\ \tan\ \theta\ -\ \cot\ \theta$

75. $\frac{1+\sin^2 x}{\cos^2 x} = 1 + 2\tan^2 x$

76. $\frac{1+\tan^2(b)}{\tan^2(b)} = \csc^2(b)$

77. $\frac{\cos\theta}{\tan\theta} - \csc\theta = -\sin\theta$

78. $\frac{\sin\theta}{\cot\theta} - \sec\theta = -\cos\theta$

79. $\tan^2(t) = \frac{1}{\cos^2(t)} - 1$

80. $\frac{(\sin\theta+\cos\theta)^2}{\cos\theta} - \sec\theta = 2\sin\theta$

81. $\frac{(1+\cos A)(1-\cos A)}{\sin A} = \sin A$

Answers to odd exercises.

71. $\frac{\tan\theta}{\cot\theta} = \tan\theta \cdot \frac{1}{\cot\theta} = \tan\theta \cdot \tan\theta = \tan^2\theta$

73. $\frac{\tan x}{\sec x}\sin(-x) = \frac{\sin x}{\cos x} \cdot \frac{\cos x}{1} \cdot \sin x = -sin^2 x = -(1-\cos^2 x) = \cos^2 x - 1$

75. $\frac{1+\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \sec^2 x + \tan^2 x = \tan^2 x + 1 + \tan^2 x = 1 + 2\tan^2 x$

77. $\frac{\cos\theta}{\tan\theta} - \csc\theta = \frac{\cos\theta}{1} \cdot \frac{\cos\theta}{\sin\theta} - \frac{1}{\sin\theta} = \frac{\cos^2\theta - 1}{\sin\theta} = \frac{\sin^2\theta}{\sin\theta} = -\sin\theta$

79. $\tan^2(t) = \frac{1}{\cos^2(t)} - 1 = \frac{1}{\cos^2 t} - \frac{\cos^2 t}{\cos^2 t} = \frac{\cos^2 t - 1}{\cos^2 t} = \frac{-(1-\cos^2 t)}{\cos^2 t} = \frac{\sin^2 t}{\cos^2 t} = \tan^2 t$

81. $\frac{(1+\cos(A))(1-\cos(A))}{\sin(A)} = \frac{1-\cos^2(A))}{\sin(A)} = \frac{\sin^2(A))}{\sin(A)} = \sin(A)$

★ Verify the following identities.

82. $\frac{\csc\theta}{\cos\theta} - \frac{\cos\theta}{\csc\theta} = \frac{\cot^2\theta + \sin^2\theta}{\cot\theta}$

83. $\frac{\sec\theta}{\csc\theta} + \frac{\sin\theta}{\cos\theta} = 2\tan\theta$

84. $\frac{\csc^2(\alpha)-1}{\csc^2(\alpha)-\csc(\alpha)} = 1 + \sin(\alpha)$

85. $\frac{\sin\theta\tan\theta + \sin\theta}{\tan\theta + \tan^2\theta} = \cos\theta$

86. $\frac{\cos\theta\cot\theta + \cos\theta}{\cot\theta + \cot^2\theta} = \sin\theta$

87. $\frac{\sin^2\ \theta}{1\ -\ \sin^2\ \theta}\ =\ \tan^2\ \theta$

88. $\frac{1\ -\ \tan^2\ \theta}{1\ -\ \cot^2\ \theta}\ =\ 1\ -\ \sec^2\ \theta$

89. $\frac{\sec^2\theta}{1+\cot^2\theta} = \tan^2\theta$

90. $\frac{\cos^2\theta - \sin^2\theta}{1-\tan^2\theta} = 1 - \sin^2\theta$

91. $\frac{\csc^2\theta}{1+\tan^2\theta} = \cot^2\theta$

92. $\sin\ \theta\ =\ \pm\ \frac{\tan\ \theta}{\sqrt{1\ +\ \tan^2\ \theta}}$

93. $\frac{\sec\theta + \tan\theta}{\sin\theta(\cot\theta + \cos\theta)} = \sec^2\theta$

Answers to odd exercises.

83. $\dfrac{\sec\theta}{\csc\theta}+\dfrac{\sin\theta}{\cos\theta}=\dfrac{\sec\theta}{1}\cdot\dfrac{1}{\csc\theta}+\dfrac{\sin\theta}{\cos\theta}=\dfrac{1}{\cos\theta}\cdot\dfrac{\sin\theta}{1}+\dfrac{\sin\theta}{\cos\theta}=2\dfrac{\sin\theta}{\cos\theta}=2\tan\theta$

85. $\dfrac{\sin\theta\tan\theta+\sin\theta}{\tan\theta+\tan^2\theta}=\dfrac{\sin\theta(\tan\theta+1)}{\tan\theta(1+\tan\theta)}=\sin\theta\cdot\cot\theta=\sin\theta\cdot\dfrac{\cos\theta}{\sin\theta}=\cos\theta$

87. $\dfrac{\sin^2\theta}{1-\sin^2\theta}=\dfrac{\sin^2\theta}{\cos^2\theta}=\left(\dfrac{\sin\theta}{\cos\theta}\right)^2=\tan^2\theta$

89. $\dfrac{\sec^2\theta}{1+\cot^2\theta}=\dfrac{\sec^2\theta}{\csc^2\theta}=\dfrac{1}{\cos^2\theta}\cdot\dfrac{\sin^2\theta}{1}=\tan^2\theta$

91. $\dfrac{\csc^2\theta}{1+\tan^2\theta}=\dfrac{\csc^2\theta}{\sec^2\theta}=\dfrac{1}{\sin^2\theta}\cdot\dfrac{\cos^2\theta}{1}=\cot^2\theta$

93. $\dfrac{\sec\theta+\tan\theta}{\sin\theta(\cot\theta+\cos\theta)}=\dfrac{\dfrac{1}{\cos\theta}+\dfrac{\sin\theta}{\cos\theta}}{\sin\theta\left(\dfrac{\cos\theta}{\sin\theta}+\cos\theta\right)}=\dfrac{\dfrac{1}{\cos\theta}(1+\sin\theta)}{\cos\theta\sin\theta\left(\dfrac{1}{\sin\theta}+1\right)}=\dfrac{\dfrac{1}{\cos\theta}(1+\sin\theta)}{\cos\theta(1+\sin\theta)}=\dfrac{1}{\cos^2\theta}=\sec^2\theta$

D: Verify (more fractions)

Exercise 6.3e.D

★ Verify the following identities.

101. $\dfrac{1+\sin x}{\cos x}=\dfrac{\cos x}{1+\sin(-x)}$

102. $\dfrac{\cos^2\theta}{1+\sin\theta}=1-\sin\theta$

103. $\dfrac{1+\cos(u)}{\sin(u)}=\dfrac{\sin(u)}{1-\cos(u)}$

104. $\dfrac{\sin^2(\theta)}{1+\cos(\theta)}=1-\cos(\theta)$

105. $\dfrac{1-\tan\theta}{1+\tan\theta}=\dfrac{\cot\theta-1}{\cot\theta+1}$

106. $\dfrac{\sin^4(\gamma)-\cos^4(\gamma)}{\sin(\gamma)-\cos(\gamma)}=\sin(\gamma)+\cos(\gamma)$

107. $\dfrac{\csc^2(x)-\sin^2(x)}{\csc(x)+\sin(x)}=\cos(x)\cot(x)$

108. $\dfrac{\sec\theta-\cos\theta}{\sec\theta+\cos\theta}=\dfrac{\sin^2\theta}{1+\cos^2\theta}$

109. $\dfrac{\sec\theta+\tan\theta}{\cot\theta+\cos\theta}=\tan\theta\sec\theta$

110. $\dfrac{\tan\theta+\tan\phi}{\cot\theta+\cot\phi}=\tan\theta\tan\phi$

111. $\dfrac{\sec(-x)}{\tan x+\cot x}=-\sin(-x)$

112. $\dfrac{\sec\theta+\csc\theta}{\tan\theta+\cot\theta}=\sin\theta+\cos\theta$

113. $\dfrac{\tan\theta-\cot\theta}{\tan\theta+\cot\theta}=\sin^2\theta-\cos^2\theta$

114. $\dfrac{\sin(\theta)-\cos(\theta)}{\sec(\theta)-\csc(\theta)}=\sin(\theta)\cos(\theta)$

Answers to odd exercises.

101. $= \dfrac{\cos x}{1-\sin(x)} \times \dfrac{1+\sin x}{1+\sin x} = \dfrac{\cos x(1+\sin x)}{1-\sin^2 x} = \dfrac{\cos x(1+\sin x)}{\cos^2 x} = \dfrac{1+\sin x}{\cos x}$

103. $\dfrac{1+\cos(u)}{\sin(u)} = \dfrac{1+\cos(u)}{\sin(u)} \cdot \dfrac{\sin u}{\sin u} = \dfrac{\sin u(1+\cos u)}{\sin^2 u} = \dfrac{\sin u(1+\cos u)}{1-\cos^2 u} = \dfrac{\sin u(1+\cos u)}{(1-\cos u)(1+\cos u)}$

$= \dfrac{\sin(u)}{1-\cos(u)}$

105. $\dfrac{1 - \tan\theta}{1 + \tan\theta} = \dfrac{(1 - \tan\theta)}{(1 + \tan\theta)} \cdot \dfrac{\cot\theta}{\cot\theta} = \dfrac{\cot\theta - 1}{\cot\theta + 1}$

107. $\dfrac{\csc^2(x) - \sin^2(x)}{\csc(x) + \sin(x)} = \dfrac{(\csc x - \sin x)(\csc x + \sin x)}{(\csc x + \sin x)} = \csc x - \sin x = \dfrac{1}{\sin x} - \dfrac{\sin^2 x}{\sin x} = \dfrac{1-\sin^2 x}{\sin x}$

$= \dfrac{\cos^2 x}{\sin x} = \cos x \cdot \dfrac{\cos x}{\sin x} = \cos(x)\cot(x)$

109. $\dfrac{\sec\theta + \tan\theta}{\cot\theta + \cos\theta} = \dfrac{\dfrac{1}{\cos\theta} + \dfrac{\sin\theta}{\cos\theta}}{\dfrac{\cos\theta}{\sin\theta} + \dfrac{\cos\theta\sin\theta}{\sin\theta}} = \dfrac{\dfrac{1+\sin\theta}{\cos\theta}}{\dfrac{\cos\theta(1+\sin\theta)}{\sin\theta}} = \dfrac{1+\sin\theta}{\cos\theta} \cdot \dfrac{\sin\theta}{\cos\theta(1+\sin\theta)} = \dfrac{\sin\theta}{\cos^2\theta}$

$= \dfrac{\sin\theta}{\cos\theta} \cdot \dfrac{1}{\cos\theta} = \tan\theta\sec\theta$

111. $\dfrac{\sec(-x)}{\tan x + \cot x} = \dfrac{\sec(x)}{\dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x}} = \dfrac{\dfrac{1}{\cos x}}{\dfrac{\sin^2 x + \cos^2 x}{\cos x \sin x}} = \dfrac{\dfrac{1}{\cos x}}{\dfrac{1}{\cos x \sin x}} = \dfrac{1}{\cos x} \cdot \dfrac{\cos x \sin x}{1} = \sin x = -\sin(-x)$

113. $\dfrac{\tan\theta - \cot\theta}{\tan\theta + \cot\theta} = \dfrac{\dfrac{\sin\theta}{\cos\theta} - \dfrac{\cos\theta}{\sin\theta}}{\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta}} = \dfrac{\dfrac{\sin^2\theta - \cos^2\theta}{\sin\theta\cos\theta}}{\dfrac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}} = \dfrac{\dfrac{\sin^2\theta - \cos^2\theta}{\sin\theta\cos\theta}}{\dfrac{1}{\sin\theta\cos\theta}} = \dfrac{\sin^2\theta - \cos^2\theta}{\sin\theta\cos\theta} \cdot \dfrac{\sin\theta\cos\theta}{1}$

$= \sin^2\theta - \cos^2\theta$

★ **Verify the following identities.**

115. $\dfrac{\sin\theta}{1+\sin\theta} - \dfrac{\sin\theta}{1-\sin\theta} = -2\tan^2\theta$

116. $\dfrac{\cos\theta}{1+\cos\theta} - \dfrac{\cos\theta}{1-\cos\theta} = -2\cot^2\theta$

117. $\dfrac{1}{1+\cos x} - \dfrac{1}{1-\cos(-x)} = -2\cot x \csc x$

118. $\dfrac{\cot\theta}{1-\csc\theta} - \dfrac{\cot\theta}{1+\csc\theta} = 2\sec\theta$

119. $\dfrac{\tan\theta}{1+\sec\theta} - \dfrac{\tan\theta}{1-\sec\theta} = 2\csc\theta$

120. $\dfrac{1-\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{1-\sin\theta} = 2\sec\theta$

121. $\dfrac{\cos\theta}{1+\sin\theta} + \dfrac{1+\sin\theta}{\cos\theta} = 2\sec\theta$

122. $1 - \dfrac{\cos^2\theta}{1+\sin\theta} = \sin\theta$

123. $1 - \dfrac{\sin^2\theta}{1+\cos\theta} = \cos\theta$

124. $2\sec^2(t) = \dfrac{1-\sin(t)}{\cos^2(t)} + \dfrac{1}{1-\sin(t)}$

125. $\left(\dfrac{\sec^2(-x) - \tan^2 x}{\tan x}\right)\left(\dfrac{2+2\tan x}{2+2\cot x}\right) - 2\sin^2 x$
$= \cos^2 x - \sin^2 x$

Answers to odd exercises.

115. $\dfrac{\sin\theta}{1+\sin\theta}-\dfrac{\sin\theta}{1-\sin\theta}=\dfrac{\sin\theta}{(1+\sin\theta)}\cdot\dfrac{(1-\sin\theta)}{(1-\sin\theta)}-\dfrac{\sin\theta}{(1-\sin\theta)}\cdot\dfrac{(1+\sin\theta)}{(1+\sin\theta)}$

$=\dfrac{(\sin\theta-\sin^2\theta)-(\sin\theta+\sin^2\theta)}{1-\sin^2\theta}=\dfrac{-2\sin^2\theta}{\cos^2\theta}=-2\tan^2\theta$

117. $\dfrac{1}{1+\cos x}-\dfrac{1}{1-\cos(-x)}=\dfrac{1}{(1+\cos x)}\dfrac{(1-\cos x)}{(1-\cos x)}-\dfrac{1}{(1-\cos x)}\dfrac{(1+\cos x)}{(1+\cos x)}=\dfrac{(1-\cos x)-(1+\cos x)}{1-\cos^2 x}$

$=\dfrac{-2\cos x}{\sin^2 x}=\dfrac{-2\cos x}{\sin x}\cdot\dfrac{1}{\sin x}=-2\cot x\csc x$

119. $\dfrac{\tan\theta}{1+\sec\theta}-\dfrac{\tan\theta}{1-\sec\theta}=\dfrac{\tan\theta}{(1+\sec\theta)}\dfrac{(1-\sec\theta)}{(1-\sec\theta)}-\dfrac{\tan\theta}{(1-\sec\theta)}\dfrac{(1+\sec\theta)}{(1+\sec\theta)}=\dfrac{-2\tan\theta\sec\theta}{1-\sec^2\theta}=\dfrac{-2\tan\theta\sec\theta}{-\tan^2\theta}$

$=\dfrac{2\sec\theta}{\tan\theta}=\dfrac{2\sec\theta}{1}\cdot\dfrac{\cos\theta}{\sin\theta}=\dfrac{2}{\sin\theta}=2\csc\theta$

121. $\dfrac{\cos\theta}{1+\sin\theta}+\dfrac{1+\sin\theta}{\cos\theta}=\dfrac{\cos\theta}{(1+\sin\theta)}\dfrac{\cos\theta}{\cos\theta}+\dfrac{(1+\sin\theta)}{\cos\theta}\dfrac{(1+\sin\theta)}{(1+\sin\theta)}=\dfrac{\cos^2\theta+1+2\sin\theta+\sin^2\theta}{\cos\theta(1+\sin\theta)}$

$=\dfrac{2+2\sin\theta}{\cos\theta(1+\sin\theta)}=\dfrac{2(1+\sin\theta)}{\cos\theta(1+\sin\theta)}=\dfrac{2}{\cos\theta}=2\sec\theta$

123. $1-\dfrac{\sin^2\theta}{1+\cos\theta}=1-\dfrac{1-\cos^2\theta}{1+\cos\theta}=1-\dfrac{(1-\cos\theta)(1+\cos\theta)}{1+\cos\theta}=1-(1-\cos\theta)=\cos\theta$

125. $\left(\dfrac{\sec^2(-x)-\tan^2 x}{\tan x}\right)\left(\dfrac{2+2\tan x}{2+2\cot x}\right)-2\sin^2 x=\left(\dfrac{1+\tan^2 x-\tan^2 x}{\tan x}\right)\dfrac{2\left(1+\dfrac{\sin x}{\cos x}\right)}{2\left(1+\dfrac{\cos x}{\sin x}\right)}-2\sin^2 x$

$=\dfrac{1}{\tan x}\cdot\dfrac{\dfrac{\cos x+\sin x}{\cos x}}{\dfrac{\sin x+\cos x}{\sin x}}-\sin^2 x=\dfrac{\cos x}{\sin x}\cdot\dfrac{\cos x+\sin x}{\cos x}\cdot\dfrac{\sin x}{\sin x+\cos x}-2\sin^2 x=1-2\sin^2 x$

$=\cos^2 x+\sin^2 x-2\sin^2 x=\cos^2 x-\sin^2 x$

★

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6.4: Sum and Difference Identities

Learning Objectives

- Use sum and difference formulas for sine, cosine, and tangent.
- Use sum and difference formulas to solve equations.
- Use sum and difference formulas to verify identities.

The trigonometric identities we will examine in this section can be traced to a Persian astronomer who lived around 950 AD, but the ancient Greeks discovered these same formulas much earlier and stated them in terms of chords. These are special equations or postulates, true for all values input to the equations, and with innumerable applications.

In this section, we will learn techniques that will enable us to solve problems such as the ones presented above. The formulas that follow will simplify many trigonometric expressions and equations. Keep in mind that, throughout this section, the term *formula* is used synonymously with the word *identity*.

Sum and Difference Formulas for Cosine

Finding the exact value of the sine, cosine, or tangent of an angle is often easier if we can rewrite the given angle in terms of two angles that have known trigonometric values. The sum and difference formulas for cosine are given below. We can use special angles and directly obtain the cosine of an angle that is the combination of known "special" angles. Alternatively, we may be a given trigonometric ratio and the quadrant the angle is in, and from that information, indirectly deduce the cosine of an angle described in this way.

Cosine: Sum and Difference Formulas

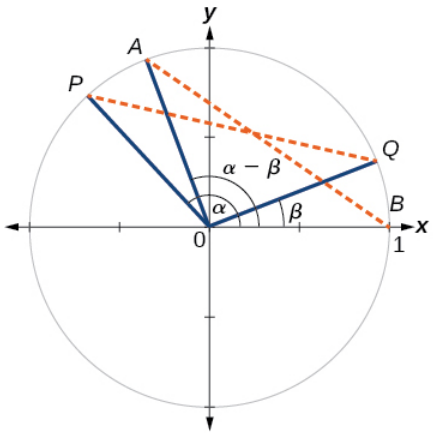
$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

Proof of difference formula for cosines

First, we will prove the difference formula for cosines. Let's consider two points on the unit circle illustrated in the figure to the right. Point P is at an angle α from the positive x-axis with coordinates $(\cos\alpha,\sin\alpha)$ and point Q is at an angle of β from the positive x-axis with coordinates $(\cos\beta,\sin\beta)$. Note the measure of angle POQ is $\alpha-\beta$.

Label two more points: A at an angle of $(\alpha-\beta)$ from the positive x-axis with coordinates $(\cos(\alpha-\beta),\sin(\alpha-\beta))$; and point B with coordinates $(1,0)$. Triangle POQ is a rotation of triangle AOB and thus the distance from P to Q is the same as the distance from A to B.We can find the distance from P to Q using the distance formula.



$$
\begin{aligned}
d_{PQ} &= \sqrt{(\cos\alpha - \cos\beta)^2 + (sin\alpha - sin\beta)^2} \\
&= \sqrt{\cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta} \\
&= \sqrt{(\cos^2\alpha + \sin^2\alpha) + (\cos^2\beta + \sin^2\beta) - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta} \\
&= \sqrt{1 + 1 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta} \\
&= \sqrt{2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta}
\end{aligned}
$$

Similarly, using the distance formula we can find the distance from A to B.

$$
\begin{aligned}
d_{AB} &= \sqrt{(\cos(\alpha - \beta) - 1)^2 + (\sin(\alpha - \beta) - 0)^2} \\
&= \sqrt{\cos^2(\alpha - \beta) - 2\cos(\alpha - \beta) + 1 + \sin^2(\alpha - \beta)} \\
&= \sqrt{(\cos^2(\alpha - \beta) + \sin^2(\alpha - \beta)) - 2\cos(\alpha - \beta) + 1} \\
&= \sqrt{1 - 2\cos(\alpha - \beta) + 1} \\
&= \sqrt{2 - 2\cos(\alpha - \beta)}
\end{aligned}
$$

Equating these two distances, we get $\sqrt{2 - 2\cos(\alpha - \beta)} = \sqrt{2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta}$. If we then square both sides, subtract 2 from both sides and divide both sides by -2, the result is the difference formula for cosine: $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

Proof of sum formula for cosines

To derive the cosine of the sum of two angles, start with writing the sum of two angles as a difference, and use the difference formula we just derived.

$$
\begin{aligned}
\cos(\alpha + \beta) &= \cos(\alpha - (-\beta)) \\
&= \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta) && \text{Use the Even/Odd Identities to remove the negative angle} \\
&= \cos\alpha\cos(\beta) - \sin\alpha\sin(-\beta) && \text{This is the sum formula for cosine}
\end{aligned}
$$

Example 6.4.1: Find the Exact Value for the Cosine of the Difference of Two Angles

Using the formula for the cosine of the difference of two angles, find the exact value of $\cos\left(\frac{5\pi}{4} - \frac{\pi}{6}\right)$.

Solution

Begin by writing the formula for the cosine of the difference of two angles. Then substitute the given values.

$$
\begin{aligned}
\cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\
\cos\left(\frac{5\pi}{4} - \frac{\pi}{6}\right) &= \cos\left(\frac{5\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{5\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) \\
&= \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) \\
&= -\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \\
&= \frac{-\sqrt{6} - \sqrt{2}}{4}
\end{aligned}
$$

Keep in mind that we can always check the answer using a calculator in radian mode.

Try It 6.4.1:

Find the exact value of $\cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$.

Answer

$\frac{\sqrt{2}+\sqrt{6}}{4}$

Example 6.4.2: Find the Exact Value for the Cosine of the Sum of Two Angles

Find the exact value of $\cos(75°)$.

Solution

As $75° = 45° + 30°$,we can evaluate $\cos(75°)$ as $\cos(45° + 30°)$.

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(45°+30°) = \cos(45°)\cos(30°) - \sin(45°)\sin(30°)$$
$$= \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)$$
$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$
$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$

Keep in mind that we can always check the answer using a calculator in degree mode.

Analysis

Note that we could have also solved this problem using the fact that $75° = 135° - 60°$.

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\cos(135°-60°) = \cos(135°)\cos(60°) + \sin(135°)\sin(60°)$$
$$= \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$
$$= -\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}$$
$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$

Try It 6.4.2

Find the exact value of $\cos(105°)$.

Answer

$\frac{\sqrt{2}-\sqrt{6}}{4}$

Sum and Difference Formulas for Sine

The sum and difference formulas for sine can be derived in the same manner as those for cosine, and they resemble the cosine formulas.

Sine: Sum and Difference Formulas

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \quad (6.4.1)$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \quad (6.4.2)$$

Proof of the sum and difference formulas for sine

The derivation for the sine of a sum of two angles can be shown using the Cofunction Identities and the formula for the cosine of the difference of two angles.

$$\begin{aligned}\sin(\alpha+\beta) &= \cos\left(\frac{\pi}{2}-(\alpha+\beta)\right) && \text{Cofunction Identity: } \sin A=\cos(\pi/2-A)\\ &= \cos\left(\left(\frac{\pi}{2}-\alpha\right)-\beta\right)\\ &= \cos\left(\frac{\pi}{2}-\alpha\right)\cdot\cos\beta+\sin\left(\frac{\pi}{2}-\alpha\right)\cdot\sin\beta && \text{Cosine of a difference of angles formula}\\ &= \sin\alpha\cos\beta+\cos\alpha\sin\beta && \text{Cofunction Identity: } \cos(\pi/2-A)=\sin A\end{aligned}$$

The derivation for the sine of a difference of two angles comes from using the formula for the sine of the sum of two angles.

$$\begin{aligned}\sin(\alpha-\beta) &= \sin(\alpha+(-\beta))\\ &= \sin\alpha\cos(-\beta)+\cos\alpha\sin(-\beta)\\ &= \sin\alpha\cos\beta-\cos\alpha\sin\beta && \text{Even/Odd Properties}\end{aligned}$$

Example 6.4.3: Using Sum and Difference Identities to Evaluate the Difference of Angles

Use the sum and difference identities to evaluate the difference of the angles and show that part *a* equals part *b*.

a. $\sin(45°-30°)$
b. $\sin(135°-120°)$

Solution

a. Let's begin by writing the formula and substitute the given angles.

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$
$$\sin(45°-30°)=\sin(45°)\cos(30°)-\cos(45°)\sin(30°)$$

Next, we need to find the values of the trigonometric expressions.

$$\sin(45°)=\frac{\sqrt{2}}{2},\qquad \cos(30°)=\frac{\sqrt{3}}{2},\qquad \cos(45°)=\frac{\sqrt{2}}{2},\qquad \sin(30°)=\frac{1}{2}$$

Now we can substitute these values into the equation and simplify.

$$\sin(45°-30°)=\frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right)-\frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)=\frac{\sqrt{6}-\sqrt{2}}{4}$$

b. Again, we write the formula and substitute the given angles.

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$
$$\sin(135°-120°)=\sin(135°)\cos(120°)-\cos(135°)\sin(120°)$$

Next, we find the values of the trigonometric expressions.

$$\sin(135°)=\frac{\sqrt{2}}{2},\qquad \cos(120°)=-\frac{1}{2},\qquad \cos(135°)=\frac{\sqrt{2}}{2},\qquad \sin(120°)=\frac{\sqrt{3}}{2}$$

Now we can substitute these values into the equation and simplify.

$$\sin(135°-120°)=\frac{\sqrt{2}}{2}\left(-\frac{1}{2}\right)-\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)=\frac{\sqrt{6}-\sqrt{2}}{4}$$

Example 6.4.4: Finding the Exact Value of an Expression Involving an Inverse Trigonometric Function

Find the exact value of $\sin\left(\cos^{-1}\frac{1}{2}+\sin^{-1}\frac{3}{5}\right)$. Then check the answer with a graphing calculator.

Solution

The pattern displayed in this problem is $\sin(\alpha+\beta)$, where $\alpha=\cos^{-1}\frac{1}{2}$ and $\beta=\sin^{-1}\frac{3}{5}$. Then we can write

$$\cos\alpha = \frac{1}{2}, \quad 0 \le \alpha \le \pi$$
$$\sin\beta = \frac{3}{5}, \quad -\frac{\pi}{2} \le \beta \le \frac{\pi}{2}$$

We will use the Pythagorean identities to find $\sin\alpha$ and $\cos\beta$

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-\frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$
$$\cos\beta = \sqrt{1-\sin^2\beta} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

Using the sum formula for sine,

$$\begin{aligned}\sin\left(\cos^{-1}\frac{1}{2}+\sin^{-1}\frac{3}{5}\right) &= \sin(\alpha+\beta)\\ &= \sin\alpha\cos\beta+\cos\alpha\sin\beta\\ &= \frac{\sqrt{3}}{2}\cdot\frac{4}{5}+\frac{1}{2}\cdot\frac{3}{5}\\ &= \frac{4\sqrt{3}+3}{10}\end{aligned}$$

Sum and Difference Formulas for Tangent

Finding the sum of two angles formula for tangent involves taking quotient of the sum formulas for sine and cosine and simplifying. Recall, $\tan x = \frac{\sin x}{\cos x}$, when $\cos x \neq 0$.

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}\\ &= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}\\ &= \frac{\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \frac{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}\\ &= \frac{\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}}{1-\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}\\ &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\end{aligned}$$

We can derive the difference formula for tangent by using the sum formula above, plus the Odd/Even Identity for tangent.

Tangent: Sum and Difference Formulas

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \tag{6.4.3}$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta} \tag{6.4.4}$$

Example 6.4.5: Finding the Exact Value of an Expression Involving Tangent

Find the exact value of $\tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right)$.

Solution

Let's first write the sum formula for tangent and then substitute the given angles into the formula.

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right)=\frac{\tan\left(\frac{\pi}{6}\right)+\tan\left(\frac{\pi}{4}\right)}{1-\left(\tan\left(\frac{\pi}{6}\right)\right)\left(\tan\left(\frac{\pi}{4}\right)\right)}$$

Next, we determine the individual function values within the formula:

$$\tan\left(\frac{\pi}{6}\right)=\frac{1}{\sqrt{3}},\quad \text{and}\quad \tan\left(\frac{\pi}{4}\right)=1$$

So we have,

$$\begin{aligned}
\tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right)&=\frac{\frac{1}{\sqrt{3}}+1}{1-\left(\frac{1}{\sqrt{3}}\right)(1)} \\
&=\frac{\frac{1+\sqrt{3}}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}} && \text{Simplify the complex fraction} \\
&=\frac{1+\sqrt{3}}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}-1} \\
&=\frac{\sqrt{3}+1}{\sqrt{3}-1} && \text{Rationalize the denominator} \\
&=\frac{\sqrt{3}+1}{\sqrt{3}-1}\cdot\frac{\sqrt{3}+1}{\sqrt{3}+1} && \text{Multiply out} \\
&=\frac{3+\sqrt{3}+\sqrt{3}+1}{3+\sqrt{3}-\sqrt{3}-1} && \text{Simplify} \\
&=\frac{4+2\sqrt{3}}{2}=2+\sqrt{3} && \text{Simplify}
\end{aligned}$$

Try It 6.4.5:

Find the exact value of $\tan\left(\frac{2\pi}{3}+\frac{\pi}{4}\right)$.

Answer

$$\frac{1-\sqrt{3}}{1+\sqrt{3}}$$

Example 6.4.6: Finding Multiple Sums and Differences of Angles

Given $\sin\alpha=\frac{3}{5},\quad 0<\alpha<\frac{\pi}{2}$, and $\cos\beta=-\frac{5}{13},\quad \pi<\beta<\frac{3\pi}{2}$, find

a. $\sin(\alpha+\beta)$ b. $\cos(\alpha+\beta)$ c. $\tan(\alpha+\beta)$ d. $\tan(\alpha-\beta)$

Solution

We can use the sum and difference formulas to identify the sum or difference of angles when the ratio of sine, cosine, or tangent is provided for each of the individual angles. To do so, we construct what is called a reference triangle to help find each

component of the sum and difference formulas. Since these formulas will need the sine, cosine or tangent of angle α and β, we will determine them here and then use the results below.

Given $\sin\alpha = \frac{3}{5}, \quad 0 < \alpha < \frac{\pi}{2}$,

- Angle α is in QI, where x and y are both positive.
- $\sin\alpha = \frac{3}{5} = \frac{y}{r}$, so we can use $y = 3$ and $r = 5$.
- Use the Pythagorean Theorem to find x: $x^2 + y^2 = r^2$ so $x^2 + 3^2 = 5^2$. Solving for x we obtain $x^2 = 25 - 9 = 16$ so in QI, $x = 4$.

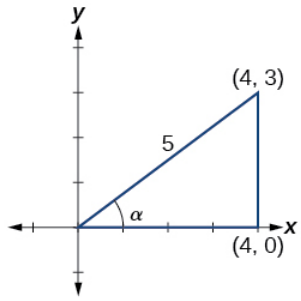


Figure 6.4.4 Angle α in standard position.

$$\sin\alpha = \frac{3}{5}, \cos\alpha = \frac{4}{5}, \tan\alpha = \frac{3}{4}$$

Given $\cos\beta = -\frac{5}{13}, \quad \pi < \beta < \frac{3\pi}{2}$

- Angle β is in QIII, where x and y are both negative.
- $\cos\beta = -\frac{5}{13} = \frac{x}{r}$ so we can use $x = -5$ and $r = 13$. (r is always positive.
- Use the Pythagorean Theorem to find y: $x^2 + y^2 = r^2$ so $(-5)^2 + y^2 = 13^2$. Solving for y we obtain $y^2 = 169 - 25 = 144$ so in QIII, $y = -12$.

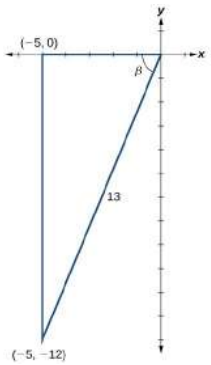


Figure 6.4.5 Angle β in standard position.

$$\sin\beta = \frac{-12}{13}, \cos\beta = \frac{-5}{13}, \tan\beta = \frac{12}{5}$$

Now we are ready to use the formulas and perform the appropriate substitutions.

a. To find $\sin(\alpha + \beta)$,

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ &= \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right) + \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right) \quad = -\frac{15}{65} - \frac{48}{65} = -\frac{63}{65}\end{aligned}$$

b. Find $\cos(\alpha + \beta)$

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right) - \left(\frac{3}{5}\right)\left(-\frac{12}{13}\right) \quad = -\frac{20}{65} + \frac{36}{65} = \frac{16}{65}\end{aligned}$$

c. For $\tan(\alpha + \beta)$

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \\ &= \frac{\frac{3}{4} + \frac{12}{5}}{1 - \frac{3}{4}\left(\frac{12}{5}\right)} \quad = \frac{\frac{63}{20}}{-\frac{16}{20}} = -\frac{63}{16}\end{aligned}$$

d. To find $\tan(\alpha - \beta)$

$$\begin{aligned}\tan(\alpha-\beta) &= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} \\ &= \frac{\frac{3}{4} - \frac{12}{5}}{1 + \frac{3}{4}\left(\frac{12}{5}\right)} \quad = \frac{-\frac{33}{20}}{\frac{56}{20}} = -\frac{33}{56}\end{aligned}$$

Use Sum and Difference Formulas to Solve Equations

These identities can also be used to solve equations.

Example 6.4.7

Solve $\sin(x)\sin(2x)+\cos(x)\cos(2x)=\frac{\sqrt{3}}{2}$.

Solution

By recognizing the left side of the equation as the result of the difference of angles identity for cosine, we can simplify the equation

$$\begin{aligned}\cos(x)\cos(2x)+\sin(x)\sin(2x)&=\frac{\sqrt{3}}{2} && \text{Apply the difference of angles identity}\\ \cos(x-2x)&=\frac{\sqrt{3}}{2}\\ \cos(-x)&=\frac{\sqrt{3}}{2} && \text{Use the negative angle identity}\\ \cos(x)&=\frac{\sqrt{3}}{2}\end{aligned}$$

Since this is a special cosine value, so we can quickly write the answers: $x=\frac{\pi}{6}+2\pi k$ and $x=\frac{11\pi}{6}+2\pi k$, where k is any integer

Use Sum and Difference Formulas to Verify Identities

Verifying an identity means demonstrating that the equation holds for all values of the variable. It helps to be very familiar with the identities or to have a list of them accessible while working the problems.

Example 6.4.8*A*: Verifying an Identity Involving Sine

Verify the identity $\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta$.

Solution

We see that the left side of the equation includes the sines of the sum and the difference of angles.

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

We can rewrite each using the sum and difference formulas.

$$\begin{aligned}\sin(\alpha+\beta)+\sin(\alpha-\beta)&=\sin\alpha\cos\beta+\cos\alpha\sin\beta+\sin\alpha\cos\beta-\cos\alpha\sin\beta\\ &=2\sin\alpha\cos\beta\end{aligned}$$

We see that the identity is verified.

Example 6.4.8*B*: Verifying an Identity Involving Tangent

Verify the following identity.

$$\frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}=\tan\alpha-\tan\beta$$

Solution

We can begin by rewriting the numerator on the left side of the equation.

$$\begin{aligned}\frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta} &= \frac{\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\cos\alpha\cos\beta} && \text{Split fraction apart}\\ &= \frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}-\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta} && \text{Cancel}\\ &= \frac{\sin\alpha}{\cos\alpha}-\frac{\sin\beta}{\cos\beta} && \text{Rewrite in terms of tangent}\\ &= \tan\alpha-\tan\beta\end{aligned}$$

Try It 6.4.8:

Verify the identity: $\tan(\pi-\theta)=-\tan\theta$.

Answer

$$\begin{aligned}\tan(\pi-\theta) &= \frac{\tan(\pi)-\tan\theta}{1+\tan(\pi)\tan\theta}\\ &= \frac{0-\tan\theta}{1+0\cdot\tan\theta}\\ &= -\tan\theta\end{aligned}$$

Applications of Sum and Difference Formulas

Example 6.4.9A: Using Sum and Difference Formulas to Solve an Application Problem

Let L_1 and L_2 denote two non-vertical intersecting lines, and let θ denote the acute angle between L_1 and L_2. See Figure 6.4.7. Show that

$$\tan\theta=\frac{m_2-m_1}{1+m_1m_2}$$

where m_1 and m_2 are the slopes of L_1 and L_2 respectively. (**Hint:** Use the fact that $\tan\theta_1=m_1$ and $\tan\theta_2=m_2$.)

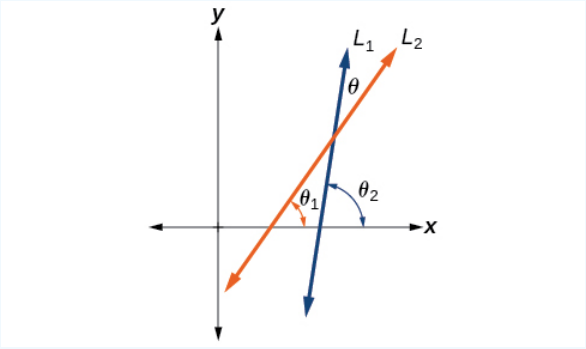


Figure 6.4.7

Solution

Observe that $\theta_1+(180^\circ-\theta_2)+\theta=180^\circ$. Thus $\theta=\theta_2-\theta_1$. To find θ, use the difference formula for tangent, plus the knowledge that the tangent of an angle is the slope of the terminal side of that angle when it is placed in standard position.

$$\begin{aligned}\tan\theta &= \tan(\theta_2-\theta_1)\\ &= \frac{\tan\theta_2-\tan\theta_1}{1+\tan\theta_1\tan\theta_2}\\ &= \frac{m_2-m_1}{1+m_1m_2}\end{aligned}$$

Example 6.4.9B: Investigating a Guy-wire Problem

For a climbing wall, a guy-wire R is attached 47 feet high on a vertical pole. Added support is provided by another guy-wire S attached 40 feet above ground on the same pole. If the wires are attached to the ground 50 feet from the pole, find the angle α between the wires. See Figure 6.4.8.

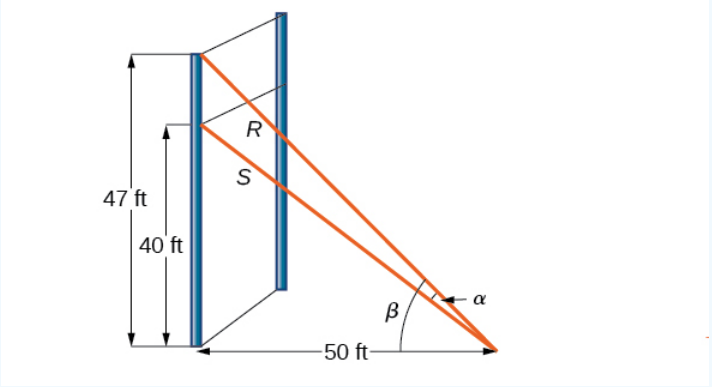


Figure 6.4.8

Solution

Let's first summarize the information we can gather from the diagram. As only the sides adjacent to the right angle are known, we can use the tangent function. Notice that $\tan\beta = \frac{47}{50}$, and $\tan(\beta - \alpha) = \frac{40}{50} = \frac{4}{5}$. We can then use difference formula for tangent.

$$\tan(\beta - \alpha) = \frac{\tan\beta - \tan\alpha}{1 + \tan\beta\tan\alpha}$$

Now, substituting the values we know into the formula, we have,

$$\frac{4}{5} = \frac{\frac{47}{50} - \tan\alpha}{1 + \frac{47}{50}\tan\alpha}$$

$$4\left(1 + \frac{47}{50}\tan\alpha\right) = 5\left(\frac{47}{50} - \tan\alpha\right)$$

Use the distributive property, and then simplify the functions.

$$4(1) + 4\left(\frac{47}{50}\right)\tan\alpha = 5\left(\frac{47}{50}\right) - 5\tan\alpha$$

$$4 + 3.76\tan\alpha = 4.7 - 5\tan\alpha$$

$$5\tan\alpha + 3.76\tan\alpha = 0.7$$

$$8.76\tan\alpha = 0.7$$

$$\tan\alpha \approx 0.07991$$

$$\tan^{-1}(0.07991) \approx .079741 \text{ radians or } 4.57^\circ$$

Key Equations

Sum and Difference Formulas for Cosine	$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$	$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$
Sum and Difference Formulas for Sine	$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$	$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$
Sum and Difference Formulas for Tangent	$\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$	$\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$
Cofunction identities	$\sin\theta=\cos\left(\dfrac{\pi}{2}-\theta\right)$ $\cos\theta=\sin\left(\dfrac{\pi}{2}-\theta\right)$ $\tan\theta=\cot\left(\dfrac{\pi}{2}-\theta\right)$	$\cot\theta=\tan\left(\dfrac{\pi}{2}-\theta\right)$ $\sec\theta=\csc\left(\dfrac{\pi}{2}-\theta\right)$ $\csc\theta=\sec\left(\dfrac{\pi}{2}-\theta\right)$

Key Concepts

- The sum formula for cosines states that the cosine of the sum of two angles equals the product of the cosines of the angles minus the product of the sines of the angles. The difference formula for cosines states that the cosine of the difference of two angles equals the product of the cosines of the angles plus the product of the sines of the angles.
- The sum and difference formulas can be used to find the exact values of the sine, cosine, or tangent of an angle.
- The sum formula for sines states that the sine of the sum of two angles equals the product of the sine of the first angle and cosine of the second angle plus the product of the cosine of the first angle and the sine of the second angle. The difference formula for sines states that the sine of the difference of two angles equals the product of the sine of the first angle and cosine of the second angle minus the product of the cosine of the first angle and the sine of the second angle.
- The sum formula for tangent states that the tangent of the sum of two angles equals the sum of the tangents of the angles divided by 1 minus the product of the tangents of the angles. The difference formula for tangent states that the tangent of the difference of two angles equals the difference of the tangents of the angles divided by 1 plus the product of the tangents of the angles.
- The Pythagorean Theorem along with the sum and difference formulas can be used to find multiple sums and differences of angles.
- The cofunction identities apply to complementary angles and pairs of reciprocal functions.
- Sum and difference formulas are useful in verifying identities.
- Application problems are often easier to solve by using sum and difference formulas.

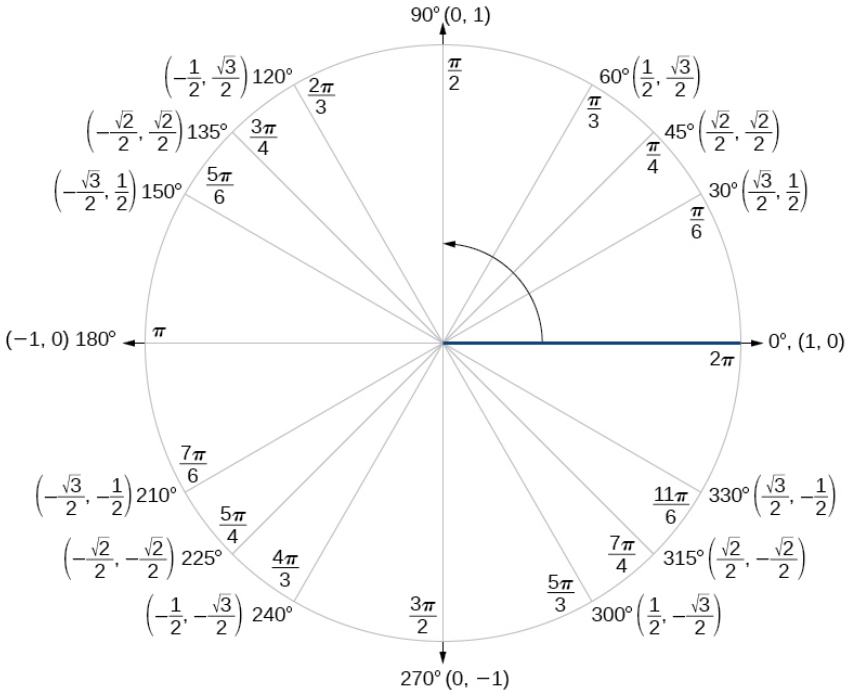


Figure **6.4.2**: The Unit Circle

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6.4e: Exercises - Sum and Difference Identities

A: Evaluate sum and difference formulas from a given angle.

Exercise 6.4e.A

★ Find the exact value.

1. $\cos\left(\frac{7\pi}{4}-\frac{5\pi}{3}\right)$
2. $\cos\left(\frac{11\pi}{6}-\frac{3\pi}{4}\right)$
3. $\cos\left(\frac{\pi}{4}+\frac{2\pi}{3}\right)$
4. $\cos\left(\frac{5\pi}{4}+\frac{\pi}{3}\right)$
5. $\sin\left(\frac{5\pi}{4}-\frac{7\pi}{6}\right)$
6. $\sin\left(\frac{5\pi}{3}-\frac{3\pi}{4}\right)$
7. $\sin\left(\frac{\pi}{4}+\frac{5\pi}{6}\right)$
8. $\sin\left(\frac{7\pi}{4}+\frac{\pi}{6}\right)$
9. $\tan\left(\frac{11\pi}{6}-\frac{5\pi}{4}\right)$
10. $\tan\left(\frac{7\pi}{4}-\frac{2\pi}{3}\right)$
11. $\tan\left(\frac{\pi}{4}+\frac{\pi}{6}\right)$
12. $\tan\left(\frac{3\pi}{4}+\frac{7\pi}{6}\right)$

★ Find the exact value.

13. $\cos\left(\frac{\pi}{12}\right)=\cos\left(\frac{4\pi}{3}-\frac{5\pi}{4}\right)$
14. $\cos\left(\frac{7\pi}{12}\right)=\cos\left(\frac{5\pi}{6}-\frac{\pi}{4}\right)$
15. $\sin\left(\frac{11\pi}{12}\right)=\sin\left(\frac{3\pi}{4}+\frac{\pi}{6}\right)$
16. $\sin\left(\frac{5\pi}{12}\right)=\sin\left(\frac{7\pi}{4}-\frac{4\pi}{3}\right)$
17. $\tan\left(\frac{19\pi}{12}\right)=\tan\left(\frac{3\pi}{4}+\frac{5\pi}{6}\right)$
18. $\tan\left(-\frac{\pi}{12}\right)=\tan\left(\frac{7\pi}{6}-\frac{5\pi}{4}\right)$
19. $\sin(195°)$
20. $\sin(75°)$
21. $\cos(345°)$
22. $\cos(165°)$
23. $\tan(-15°)$
24. $\cos(105°)$

★ Simplify

25. $\sin\left(x-\frac{3\pi}{4}\right)$
26. $\sin\left(x+\frac{11\pi}{6}\right)$
27. $\cos\left(x+\frac{2\pi}{3}\right)$
28. $\cos\left(x-\frac{5\pi}{6}\right)$
29. $\sec\left(\frac{\pi}{2}-\theta\right)$
30. $\csc\left(\frac{\pi}{2}-t\right)$
31. $\cot\left(\frac{\pi}{2}-x\right)$
32. $\tan\left(\frac{\pi}{4}-x\right)$

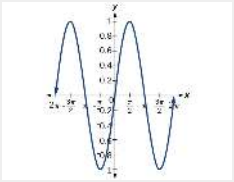
★ (a) Simplify and then (b) graph

33. $\cos\left(\frac{\pi}{2}-x\right)$
34. $\sin(\pi-x)$
35. $\tan\left(\frac{\pi}{3}+x\right)$
36. $\sin\left(\frac{\pi}{3}+x\right)$
37. $\tan\left(\frac{\pi}{4}-x\right)$
38. $\cos\left(\frac{7\pi}{6}+x\right)$
39. $\sin\left(\frac{\pi}{4}+x\right)$
40. $\cos\left(\frac{5\pi}{4}+x\right)$

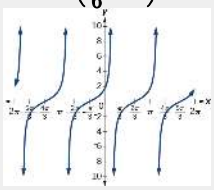
Answers to odd exercises.

1. $\frac{\sqrt{2}+\sqrt{6}}{4}$ 3. $\frac{-\sqrt{2}-\sqrt{6}}{4}$ 5. $\frac{\sqrt{6}-\sqrt{2}}{4}$ 7. $\frac{\sqrt{2}-\sqrt{6}}{4}$ 9. $-2-\sqrt{3}$ 11. $2+\sqrt{3}$
13. $\frac{\sqrt{2}+\sqrt{6}}{4}$ 15. $\frac{\sqrt{6}-\sqrt{2}}{4}$ 17. $-2-\sqrt{3}$ 19. $-\frac{\sqrt{3}-1}{2\sqrt{2}}$ 21. $\frac{1+\sqrt{3}}{2\sqrt{2}}$ 23. $-2+\sqrt{3}$
25. $-\frac{\sqrt{2}}{2}\sin x-\frac{\sqrt{2}}{2}\cos x$ 27. $-\frac{1}{2}\cos x-\frac{\sqrt{3}}{2}\sin x$ 29. $\csc\theta$ 31. $\tan x$

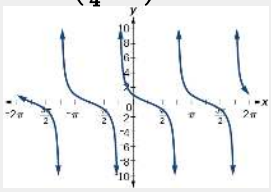
33. $\sin x$



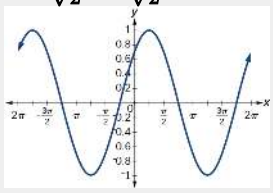
35. $\cot\left(\frac{\pi}{6}-x\right)$



37. $\cot\left(\frac{\pi}{4}+x\right)$



39. $\frac{\sin x}{\sqrt{2}}+\frac{\cos x}{\sqrt{2}}$



B: Evaluate sum and difference formulas given trig ratios of angles

Exercise 6.4e.B

41. Given that $\sin a=\frac{4}{5}$ and $\cos b=\frac{1}{3}$, with a and b both in the interval $\left[0,\frac{\pi}{2}\right)$

(a) Find $\sin(a-b)$ (b) Find $\cos(a+b)$.

42. Given that $\sin a=\frac{2}{3}$ and $\cos b=-\frac{1}{4}$, with a and b both in the interval $\left[\frac{\pi}{2},\pi\right)$.

(a) Find $\sin(a+b)$ (b) Find $\cos(a-b)$.

43. Angles A and B are in standard position and $\sin(A)=\frac{1}{2}, \cos(A)>0)$, $\cos(B)=\frac{3}{4}$, and $\sin(B)<0$.
Draw a picture of the angles A and B in the plane and then find each of the following.

(a) $\cos(A+B)$ (b) $\cos(A-B)$ (c) $\sin(A+B)$ (d) $\sin(A-B)$ (e) $\tan(A+B)$ (f) $\tan(A-B)$

★ Given the information about angles A and B in the exercises below, find the exact value for each of the following.

(a) $\cos(A+B)$ (b) $\cos(A-B)$ (c) $\sin(A+B)$ (d) $\sin(A-B)$ (e) $\tan(A+B)$ (f) $\tan(A-B)$

45. $\sin A=-\frac{3}{5}$ with A in Quadrant III, and $\cos B=\frac{1}{2}$ with B in Quadrant IV.

46. $\sin A=\frac{4}{5}$ with A in Quadrant I, and $\tan B=-\frac{\sqrt{5}}{2}$ with B in Quadrant II.

47. $\cos A=\frac{1}{2}$ with $0\le A\le\frac{\pi}{2}$, and $\tan B=2\sqrt{2}$ with $\pi\le B\le\frac{3\pi}{2}$.

48. $\cos A=-\frac{\sqrt{3}}{3}$ with $\pi\le A\le\frac{3\pi}{2}$, and $\sin B=\frac{\sqrt{3}}{3}$ with $\frac{\pi}{2}\le B\le\pi$.

49. $A=\tan^{-1}\left(\sqrt{5}\right)$ and $B=\sin^{-1}\left(-\frac{\sqrt{5}}{3}\right)$

50. $A=\tan^{-1}(-2)$ and $B=\cos^{-1}\left(-\frac{\sqrt{6}}{6}\right)$

★ Find the exact value of each expression.

52. $\sin\left(\cos^{-1}(0)-\cos^{-1}\left(\frac{1}{2}\right)\right)$	53. $\cos\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)+\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$	54. $\tan\left(\sin^{-1}\left(\frac{1}{2}\right)-\cos^{-1}\left(\frac{1}{2}\right)\right)$
58. $\tan(\sin^{-1}u+\cos^{-1}u)$	59. $\cos(\sin^{-1}u+\tan^{-1}u)$	60. $\sin(\cos^{-1}u+tan^{-1}u)$

Answers to odd exercises.

41a. $\sin(a-b)=\left(\frac{4}{5}\right)\left(\frac{1}{3}\right)-\left(\frac{3}{5}\right)\left(\frac{2\sqrt{2}}{3}\right)=\frac{4-6\sqrt{2}}{15}$

41b. $\cos(a+b)=\left(\frac{3}{5}\right)\left(\frac{1}{3}\right)-\left(\frac{4}{5}\right)\left(\frac{2\sqrt{2}}{3}\right)=\frac{3-8\sqrt{2}}{15}$

43. a. $\frac{\sqrt{3}}{2}\cdot\frac{3}{4}-\frac{1}{2}\cdot\frac{-\sqrt{7}}{4}=\frac{3\sqrt{3}+\sqrt{7}}{8}$, b. $\frac{3\sqrt{3}-\sqrt{7}}{8}$, c. $\frac{3-\sqrt{21}}{8}$, d. $\frac{3+\sqrt{21}}{8}$, e. $\frac{4\sqrt{3}-3\sqrt{7}}{5}$, f. $\frac{4\sqrt{3}+3\sqrt{7}}{5}$

45. $\sin(A+B)=\frac{-3}{5}\cdot\frac{1}{2}+\frac{-4}{5}\cdot\frac{-\sqrt{3}}{2}=\frac{-3+4\sqrt{3}}{10}$, $\cos(A+B)=\frac{-4-3\sqrt{3}}{10}$, $\tan(A+B)=\frac{48-25\sqrt{3}}{-11}$

$\sin(A-B)=\frac{-3-4\sqrt{3}}{10}$, $\cos(A-B)=\frac{-4+3\sqrt{3}}{10}$, $\tan(A-B)=\frac{48+25\sqrt{3}}{-11}$

47. $\sin(A+B)=\frac{\sqrt{3}}{2}\cdot\frac{-1}{3}+\frac{1}{2}\cdot\frac{-2\sqrt{2}}{3}=\frac{-\sqrt{3}-2\sqrt{2}}{6}$, $\cos(A+B)=\frac{-1+2\sqrt{6}}{6}$, $\tan(A+B)=\frac{-9\sqrt{3}-8\sqrt{2}}{23}$

$\sin(A-B)=\frac{-\sqrt{3}+2\sqrt{2}}{6}$, $\cos(A-B)=\frac{-1-2\sqrt{6}}{6}$, $\tan(A-B)=\frac{-9\sqrt{3}+8\sqrt{2}}{23}$

49. $\sin(A+B)=\frac{\sqrt{3}}{6}\cdot\frac{2}{3}+\frac{\sqrt{6}}{6}\cdot\frac{-\sqrt{5}}{3}=\frac{\sqrt{30}}{18}$, $\cos(A+B)=\frac{7\sqrt{6}}{18}$, $\tan(A+B)=\frac{\sqrt{5}}{7}$

$\sin(A-B)=\frac{\sqrt{30}}{6}$, $\cos(A-B)=\frac{-\sqrt{6}}{6}$, $\tan(A-B)=-\sqrt{5}$

53. $\frac{\sqrt{2}-\sqrt{6}}{4}$ 59. $\frac{\sqrt{1-u^4}-u^2\sqrt{1+u^2}}{1+u^2}$

C: Solve Equations

Exercise 6.4e.D

★ Solve each equation for all solutions.

65. $\sin(3x)\cos(6x)-\cos(3x)\sin(6x)=-0.9$
66. $\sin(6x)\cos(11x)-\cos(6x)\sin(11x)=-0.1$
67. $\cos(2x)\cos(x)+\sin(2x)\sin(x)=1$
68. $\cos(5x)\cos(3x)-\sin(5x)\sin(3x)=\frac{\sqrt{3}}{2}$

Answers to odd exercises.

65. $0.373+\frac{2\pi}{3}k$ and $0.674+\frac{2\pi}{3}k$, where k is an integer 67. $2\pi k$, where k is an integer

D: Verify Identities

Exercise 6.4e.F

★ Simplify.

71. $\dfrac{\tan\left(\frac{3}{2}x\right)-\tan\left(\frac{7}{5}x\right)}{1+\tan\left(\frac{3}{2}x\right)\tan\left(\frac{7}{5}x\right)}$
72. $\sin(2x)\cos(5x)-\sin(5x)\cos(2x)$

★ Verify the Identity.

73. $\dfrac{\sin(x)+\sin(y)}{\cos(x)+\cos(y)}=\tan\left(\frac{1}{2}(x+y)\right)$
74. $\dfrac{\sin(A-B)}{\sin(A+B)}=\dfrac{\cot B-\cot A}{\cot B+\cot A}$
75. $\cot A+\cot B=\dfrac{\sin(A+B)}{\sin A\sin B}$
76. $\dfrac{\sin(r+s)}{\cos(r)\cos(s)}=\tan(r)+\tan(s)$
77. $\dfrac{\sin(r-s)}{\cos(r)\cos(s)}=\tan(r)-\tan(s)$
78. $\dfrac{\cos(A+B)}{\sin A\cos B}=\cot A-\tan B$
79. $\dfrac{\cos(a+b)}{\cos a\cos b}=1-\tan a\tan b$
80. $\dfrac{\tan(a+b)}{\tan(a-b)}=\dfrac{\sin a\cos a+\sin b\cos b}{\sin a\cos a-\sin b\cos b}$
81. $\tan\left(x+\frac{\pi}{4}\right)=\dfrac{\tan x+1}{1-\tan x}$
82. $\tan\left(\frac{\pi}{4}-t\right)=\dfrac{1-\tan(t)}{1+\tan(t)}$
83. $\dfrac{\tan(x+y)}{1+\tan x\tan y}=\dfrac{\tan x+\tan y}{1-\tan^2 x\tan^2 y}$
84. $\cot(A+B)=\dfrac{\cot A\cot B-1}{\cot A+\cot B}$
85. $\cot(A-B)=\dfrac{\cot A\cot B+1}{\cot B-\cot A}$
87. $\dfrac{\cos(x+h)-\cos(x)}{h}=\cos x\dfrac{\cos(h)-1}{h}-\sin x\dfrac{\sin(h)}{h}$

Answers to odd exercises

71. $\tan\left(\frac{x}{10}\right)$

75. $\cot\ A\ +\ \cot\ B = \dfrac{\cos(A)}{\sin(A)} + \dfrac{\cos(B)}{\sin(B)}$
$= \dfrac{\cos(A)}{\sin(A)} \cdot \dfrac{\sin(B)}{\sin(B)} + \dfrac{\cos(B)}{\sin(B)} \cdot \dfrac{\sin(A)}{\sin(A)}$

$= \dfrac{\cos(A)\sin(B) + \cos(B)\sin(A)}{\sin(A)\sin(B)} = \dfrac{\sin\ (A+B)}{\sin\ A\ \sin\ B}$

77. $\dfrac{\sin(r-s)}{\cos(r)\cos(s)} = \dfrac{\sin(r)\cos(s) - \cos(r)\sin(s)}{\cos(r)\cos(s)}$

$= \dfrac{\sin(r)\cos(s)}{\cos(r)\cos(s)} - \dfrac{\cos(r)\sin(s)}{\cos(r)\cos(s)} = \dfrac{\sin(r)}{\cos(r)} - \dfrac{\sin(s)}{\cos(s)} = \tan(r) - \tan(s)$

79. $\dfrac{\cos(a+b)}{\cos a \cos b} = \dfrac{\cos a \cos b}{\cos a \cos b} - \dfrac{\sin a \sin b}{\cos a \cos b} = 1 - \tan a \tan b$

81. $\tan\left(x + \dfrac{\pi}{4}\right) = \dfrac{\tan x + \tan\left(\frac{\pi}{4}\right)}{1 - \tan x \tan\left(\frac{\pi}{4}\right)} = \dfrac{\tan x + 1}{1 - \tan x(1)} = \dfrac{\tan x + 1}{1 - \tan x}$

83. $\dfrac{\tan(x+y)}{1+\tan x \tan y} = \dfrac{\tan x + \tan y}{1 - \tan x \tan y} \cdot \dfrac{1}{1 + \tan x \tan y} = \dfrac{\tan x + \tan y}{1 - \tan^2 x \tan^2 y}$

85. $\cot\ (A-B) = \dfrac{1}{\tan(A-B)} = \dfrac{1 + \tan A \tan B}{\tan A + \tan B} = \dfrac{1 + \tan A \tan B}{\tan A + \tan B} \cdot \dfrac{\cot A \cot B}{\cot A \cot B} = \dfrac{\cot\ A\ \cot\ B\ +\ 1}{\cot\ B\ -\ \cot\ A}$

87. $\dfrac{\cos(x+h) - \cos(x)}{h} = \dfrac{\cos x \cos(h) - \sin x \sin(h) - \cos x}{h} = \dfrac{\cos x(\cos(h) - 1) - \sin x(\sin(h))}{h}$

$= \cos x \cdot \dfrac{\cos(h) - 1}{h} - \sin x \cdot \dfrac{\sin(h)}{h}$

★ Verify the Identity.

90. $2\sin\ (a+b)\sin\ (a-b) = \cos\ (2b) - \cos(2a)$
91. $\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y$
92. $\cos\ (a+b) + \cos\ (a-b) = 2\cos\ (a)\cos\ (b)$
93. $\sin(4x) - \sin(3x)\cos x = \sin x \cos(3x)$
94. $\cos(4x) + \sin x \sin(3x) = \cos x \cos(3x)$
95. $\sin(3x)\cos(6x) = \sin(9x) - \cos(3x)\sin(6x)$
96. $\sin(4x) = \sin(5x)\cos x - \cos(5x)\sin x$
97. $\sin(3x) = 3\sin x \cos^2 x - \sin^3 x$
98. $\cos(3x) = \cos^3 x - 3\sin^2 x \cos x$
99. $\sin(2x) = 2\sin x \cos x$
100. $\cos(2\theta) = \cos^2\theta - \sin^2\theta$
101. $\tan(2\theta) = \dfrac{2\tan\theta}{1 - \tan^2\theta}$
102. $\tan(-x) = \dfrac{\tan x - \tan(2x)}{1 + \tan x \tan(2x)}$

Answers to odd exercises.

91. $\cos(x+y)\cos(x-y) = (\cos x \cos y - \sin x \sin y)(\cos x \cos y + \sin x \sin y) = \cos^2 x \cos^2 y - \sin^2 x \sin^2 y$
$= (1 - \sin^2 x)(1 - \sin^2 y) - \sin^2 x \sin^2 y = 1 - \sin^2 x - \sin^2 y + \sin^2 x \sin^2 y - \sin^2 x \sin^2 y$
$= 1 - \sin^2 x - \sin^2 y = \cos^2 x - \sin^2 y$

93. $\sin(4x) - \sin(3x)\cos x = \sin(x+3x) - \sin(3x)\cos x$
$= \sin x \cos(3x) + \cos x \sin(3x) - \sin(3x)\cos x = \sin x \cos(3x)$

95. $\sin(3x)\cos(6x) = \sin(3x)\cos(6x) + \cos(3x)\sin(6x) - \cos(3x)\sin(6x) = \sin(9x) - \cos(3x)\sin(6x)$

97. $\sin(x+2x) = \sin x \cos(2x) + \sin(2x)\cos x = \sin x(\cos^2 x - \sin^2 x) + 2\sin x \cos x \cos x$
$= \sin x \cos^2 x - \sin^3 x + 2\sin x \cos^2 x = 3\sin x \cos^2 x - \sin^3 x$

99. $\sin(2x) = \sin x \cos x + \cos x \sin x = 2\sin x \cos x$

101. $\tan(2\theta) = \dfrac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta} = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

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6.5: Double-Angle, Half-Angle, and Reduction Formulas

Learning Objectives.

- Use double-angle formulas to find exact values
- Use double-angle formulas to solve equations
- Use double-angle formulas to verify identities
- Use reduction formulas to simplify an expression
- Use half-angle formulas to find exact values

Use Double-Angle Formulas to Find Exact Values

In the previous section, we used addition and subtraction formulas for trigonometric functions. Now, we take another look at those same formulas. The **double-angle formulas** are a special case of the sum formulas, where $\alpha = \beta$. Deriving the double-angle formula for sine begins with the sum formula,

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

If we let $\alpha = \beta = \theta$, then we have

$$sin(2\theta) = \sin(\theta + \theta) = \sin\theta\cos\theta + \cos\theta\sin\theta \ = 2\sin\theta\cos\theta$$

Deriving the double-angle for cosine gives us three options. First, starting from the sum formula, $\cos(\alpha + \beta) = \cos\ \alpha\ \cos\ \beta - \sin\ \alpha\ \sin\ \beta$, and letting $\alpha = \beta = \theta$, we have

$$cos(2\theta) = \cos(\theta + \theta) = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta$$

Using the Pythagorean properties, we can expand this double-angle formula for cosine and get two more variations. The first variation is:

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = (1 - \sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta$$

The second variation is:

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = \cos^2\theta - (1 - \cos^2\theta) = 2\cos^2\theta - 1$$

Similarly, to derive the double-angle formula for tangent, replacing $\alpha = \beta = \theta$ in the sum formula gives

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan(\theta + \theta) = \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta}$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$

DOUBLE-ANGLE FORMULAS

$$\sin(2\theta) = 2\sin\theta\cos\theta \tag{6.5.1}$$

$$\cos(2\theta) = \begin{cases} \cos^2\theta - \sin^2\theta \\ 1 - 2\sin^2\theta \\ 2\cos^2\theta - 1 \end{cases} \tag{6.5.2}$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta} \tag{6.5.3}$$

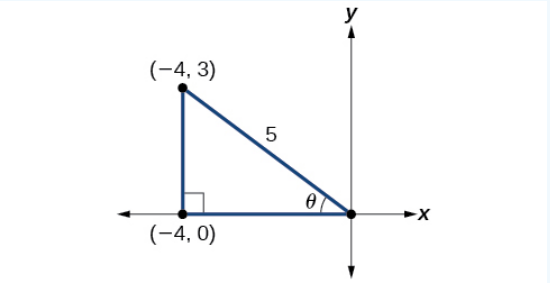
Example 6.5.1: Use Double-Angle Formulas to Evaluate a Trigonometric Expression

Given that $\tan\theta = -\frac{3}{4}$ and θ is in quadrant II, find the following:

a. $\sin(2\theta)$ b. $\cos(2\theta)$ c. $\tan(2\theta)$

Solution

If we draw a triangle to reflect the information given, we can find the values needed to solve the problems on the image. We are given $\tan\theta = -\frac{3}{4}$, such that θ is in quadrant II. The tangent of an angle is equal to the opposite side over the adjacent side, and because θ is in the second quadrant, the adjacent side is on the x-axis and is negative. Use the **Pythagorean Theorem** to find the length of the hypotenuse:



$$c^2 = (-4)^2 + (3)^2 = 16 + 9 = 25 \rightarrow c = 5 \text{ Therefore, } \boxed{\begin{array}{rcl} x &=& -4 \\ y &=& 3 \\ r &=& 5 \end{array}}.$$

Using these values we obtain: $\boxed{\begin{array}{rcl} \cos\theta &=& -\frac{4}{5} \\ \sin\theta &=& \frac{3}{5} \\ \tan\theta &=& -\frac{3}{4} \end{array}}$.

a. Write the double-angle formula for sine. Then substitute the values found above and simplify:

$$\begin{aligned} \sin(2\theta) &= 2\sin\theta\cos\theta \\ &= 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right) \quad = -\frac{24}{25} \end{aligned}$$

b. Write the double-angle formula for cosine. Then substitute the values found above and simplify:

$$\begin{aligned} \cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= \left(-\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 \quad = \frac{16}{25} - \frac{9}{25} = \frac{7}{25} \end{aligned}$$

c. Write the double-angle formula for tangent. Then substitute the values found above and simplify:

$$\begin{aligned} \tan(2\theta) &== \frac{2\tan\theta}{1-\tan^2\theta} \\ &= \frac{2\left(-\frac{3}{4}\right)}{1-\left(-\frac{3}{4}\right)^2} \quad = \frac{-\frac{3}{2}}{1-\frac{9}{16}} = -\frac{3}{2}\left(\frac{16}{7}\right) = -\frac{24}{7} \end{aligned}$$

Try It 6.5.1

Given $\sin\alpha = \frac{5}{8}$, with θ in quadrant I, find $\cos(2\alpha)$.

Answer

$$\cos(2\alpha) = \frac{7}{32}$$

Example 6.5.2: Use Double-Angle Formulas to Rewrite Trigonometric Expressions

Use the double-angle formula for cosine to write $\cos(6x)$ in terms of $\cos(3x)$.

Solution

$$\begin{aligned} \cos(6x) &= \cos(3x + 3x) \\ &= \cos 3x \cos 3x - \sin 3x \sin 3x \\ &= \cos^2 3x - \sin^2 3x \end{aligned}$$

Analysis

This example illustrates that we can use the double-angle formula without having exact values. It emphasizes that the pattern is what we need to remember and that identities are true for all values in the domain of the trigonometric function.

Use Double-Angle Formulas to Solve Equations

When solving equations, fundamental identities are used to make solving equations simpler. Then basic rules of algebra are used to solve the equation.

Example 6.5.3a: Use Double-Angle Formulas to Solve an Equation

Solve the equation exactly using a double-angle formula: $\cos(2\theta) = \cos\theta$.

Solution

We have three choices of expressions to substitute for the double-angle of cosine. As it is simpler to solve for one trigonometric function at a time, we will choose the double-angle identity involving only cosine:

$$\begin{aligned} \cos(2\theta) &= \cos\theta \\ 2\cos^2\theta - 1 &= \cos\theta \\ 2\cos^2\theta - \cos\theta - 1 &= 0 \\ (2\cos\theta + 1)(\cos\theta - 1) &= 0 \\ 2\cos\theta + 1 &= 0 \\ \cos\theta &= -\frac{1}{2} \\ \cos\theta - 1 &= 0 \\ \cos\theta &= 1 \end{aligned}$$

So, if $\cos\theta = -\frac{1}{2}$,then $\theta = \frac{2\pi}{3} \pm 2\pi k$ and $\theta = \frac{4\pi}{3} \pm 2\pi k$; if $\cos\theta = 1$,then $\theta = 0 \pm 2\pi k$.

Example 6.5.3b

Solve the equation exactly using an identity: $3\cos\theta + 3 = 2\sin^2\theta$, $0 \le \theta < 2\pi$.

Solution

If we rewrite the right side, we can write the equation in terms of cosine:

$$
\begin{aligned}
3\cos\theta + 3 &= 2\sin^2\theta \\
3\cos\theta + 3 &= 2(1-\cos^2\theta) \\
3\cos\theta + 3 &= 2 - 2\cos^2\theta \\
2\cos^2\theta + 3\cos\theta + 1 &= 0 \\
(2\cos\theta + 1)(\cos\theta + 1) &= 0 \\
2\cos\theta + 1 &= 0 \\
\cos\theta &= -\frac{1}{2} \\
\theta &= \frac{2\pi}{3},\ \frac{4\pi}{3} \\
\cos\theta + 1 &= 0 \\
\cos\theta &= -1 \\
\theta &= \pi
\end{aligned}
$$

Our solutions are $\theta = \frac{2\pi}{3},\ \frac{4\pi}{3},\ \pi$.

Use Double-Angle Formulas to Verify Identities

The techniques used for solving equations are not the same as those for verifying identities. To verify an identity, one side of the identity is rewritten to match the other side.

Simplify

A first step in verifying identities or solving equations is recognizing patterns.

Example 6.5.4 Simplify by Recognizing Identities

Simplify the expressions

a) $2\cos^2(12°) - 1$

b) $8\sin(3x)\cos(3x)$

Solution

a) Notice that the expression is in the same form as one version of the double angle identity for cosine: $\cos(2\theta) = 2\cos^2(\theta) - 1$. Using this,

$$2\cos^2(12°) - 1 = \cos(2 \cdot 12°) = \cos(24°)$$

b) This expression looks similar to the result of the double angle identity for sine.

$$
\begin{aligned}
8\sin(3x)\cos(3x) &\quad \text{Factor a 4 out of the original expression} \\
4 \cdot 2\sin(3x)\cos(3x) &\quad \text{Applying the double angle identity} \\
4\sin(6x)
\end{aligned}
$$

Try It 6.5.4: Recognize and simplify

Use an identity to find the exact value of $\cos^2(75°) - \sin^2(75°)$.

Answer

$$\cos^2(75°) - \sin^2(75°) = \cos(2 \cdot 75°) = \cos(150°) = \frac{-\sqrt{3}}{2}$$

Double angle formulas can be used to simplify expressions.

Example 6.5.5: Simplify

Simplify $\dfrac{\cos(2t)}{\cos(t)-\sin(t)}$.

Solution

With three choices for how to rewrite the double angle, we need to consider which will be the most useful. To simplify this expression, it would be great if the denominator would cancel with something in the numerator, which would require a factor of $\cos(t)-\sin(t)$ in the numerator, which is most likely to occur if we rewrite the numerator with a mix of sine and cosine.

$$\begin{aligned} &\frac{\cos(2t)}{\cos(t)-\sin(t)} && \text{Apply the double angle identity}\\ &= \frac{\cos^2(t)-\sin^2(t)}{\cos(t)-\sin(t)} && \text{Factor the numerator}\\ &= \frac{(\cos(t)-\sin(t))\,(\cos(t)+\sin(t))}{\cos(t)-\sin(t)} && \text{Cancel the common factor}\\ &\cos(t)+\sin(t) && \text{Result is in the most simplified form}\end{aligned}$$

Verify

Establishing identities using the double-angle formulas is performed using the same steps as have been used previously to verify identities. Choose the more complicated side of the equation and rewrite it until it matches the other side.

Example 6.5.6: Use Double-Angle Formulas to Verify an Identity

Verify the following identity using double-angle formulas:

$$1+\sin(2\theta)=(\sin\theta+\cos\theta)^2$$

Solution

We will work on the right side of the equal sign and rewrite the expression until it matches the left side.

$$\begin{aligned}(\sin\theta+\cos\theta)^2 &= \sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta\\ &= (\sin^2\theta+\cos^2\theta)+2\sin\theta\cos\theta\\ &= 1+2\sin\theta\cos\theta\\ &= 1+\sin(2\theta)\end{aligned}$$

Analysis

This process is not complicated, as long as we recall the perfect square formula from algebra: $(a\pm b)^2=a^2\pm 2ab+b^2$, where here, $a=\sin\theta$ and $b=\cos\theta$. Part of being successful in mathematics is the ability to recognize patterns. While the terms or symbols may change, the algebra remains consistent.

Try It 6.5.6

Verify the identity: $\cos^4\theta - \sin^4\theta = \cos(2\theta)$.

Answer

$$\cos^4\theta - \sin^4\theta = (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta) = \cos(2\theta)$$

Example 6.5.7

Verify the identity: $\tan(2\theta) = 2\cot\theta - \tan\theta$

Solution

In this case, we will work with the left side of the equation and simplify or rewrite until it equals the right side of the equation.

$$\begin{aligned}
\tan(2\theta) &= \frac{2\tan\theta}{1-\tan^2\theta} && \text{Double-angle formula} \\
&= \frac{2\tan\theta\left(\frac{1}{\tan\theta}\right)}{(1-\tan^2\theta)\left(\frac{1}{\tan\theta}\right)} && \text{Multiply by a term that results in desired numerator} \\
&= \frac{2}{\frac{1}{\tan\theta} - \frac{\tan^2\theta}{\tan\theta}} \\
&= \frac{2}{\cot\theta - \tan\theta} && \text{Use reciprocal identity for } \frac{1}{\tan\theta}
\end{aligned} \tag{6.5.4}$$

Analysis

Here is a case where the more complicated side of the initial equation appeared on the right, but we chose to work the left side. However, if we had chosen the left side to rewrite, we would have been working backwards to arrive at the equivalency. For example, suppose that we wanted to show

$$\begin{aligned}
\frac{2\tan\theta}{1-\tan^2\theta} &= \frac{2}{\cot\theta - \tan\theta} \\
&= \frac{\frac{2}{\frac{1}{\tan\theta} - \tan\theta}\left(\frac{\tan\theta}{\tan\theta}\right)}{} && \text{Multiply by a term that results in desired numerator} \\
&= \frac{2\tan\theta}{\frac{1}{\tan\theta}(\tan\theta) - \tan\theta(\tan\theta)} \\
&= \frac{2\tan\theta}{1-\tan^2\theta}
\end{aligned} \tag{6.5.5}$$

When using the identities to simplify a trigonometric expression or solve a trigonometric equation, there are usually several paths to a desired result. There is no set rule as to what side should be manipulated. However, we should begin with the guidelines set forth earlier.

Try It 6.5.7

Verify the identity: $\cos(2\theta)\cos\theta = \cos^3\theta - \cos\theta\sin^2\theta$.

Answer

$$\cos(2\theta)\cos\ \theta = (\cos^2\theta - \sin^2\theta)\cos\ \theta = \cos^3\theta - \cos\ \theta\sin^2\theta$$

Example 6.5.8

Prove $\sec(2\alpha) = \dfrac{\sec^2(\alpha)}{2-\sec^2(\alpha)}$.

Solution

Since the right side seems a bit more complicated than the left side, we begin there.

$$\begin{aligned}
\sec(2\alpha) &= \frac{\sec^2(\alpha)}{2-\sec^2(\alpha)} && \text{Rewrite the secants in terms of cosine}\\
&= \frac{\dfrac{1}{\cos^2\alpha}}{2-\dfrac{1}{\cos^2\alpha}} && \text{Simplify complex fraction* by multiplying by a convenient 1}\\
&= \frac{\dfrac{1}{\cos^2(\alpha)}}{\left(2-\dfrac{1}{\cos^2(\alpha)}\right)}\cdot\frac{\cos^2(\alpha)}{\cos^2(\alpha)} && \text{Multiply numerator; multiply denominator}\\
&= \frac{\dfrac{\cos^2(\alpha)}{\cos^2(\alpha)}}{2\cos^2(\alpha)-\dfrac{\cos^2(\alpha)}{\cos^2(\alpha)}}. && \text{Simplify}\\
&= \frac{1}{2\cos^2(\alpha)-1} && \text{Rewrite the denominator as a double angle}\\
&= \frac{1}{\cos(2\alpha)} && \text{Rewrite as a secant}\\
&= \sec(2\alpha) && \text{Identity is verified}
\end{aligned}$$

*At this point, we could have rewritten the bottom with common denominators, subtracted the terms, inverted and multiplied, then simplified. Instead, we chose to clear the fractions in numerator and denominator by multiplying both the top and bottom by $\cos^2(\alpha)$, the common denominator.

Power Reduction Formulas

The double-angle formulas can be used to derive the **reduction formulas,** which are formulas we can use to reduce the power of a given expression involving even powers of sine or cosine. They allow us to rewrite the even powers of sine or cosine in terms of the first power of cosine. These formulas are especially important in higher-level math courses, calculus in particular. Also called the power-reducing formulas, three identities are included and are easily derived from the double-angle formulas.

We can use two of the three double-angle formulas for cosine to derive the reduction formulas for sine and cosine. Let's begin with $\cos(2\theta) = 1 - 2\sin^2\theta$. Solve for $\sin^2\theta$:

$$\cos(2\theta) = 1 - 2\sin^2\theta$$
$$2\sin^2\theta = 1 - \cos(2\theta)$$
$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

Next, we use the formula $\cos(2\theta) = 2\cos^2\theta - 1$. Solve for $\cos^2\theta$:

$$\cos(2\theta) = 2\cos^2\theta - 1$$
$$1 + \cos(2\theta) = 2\cos^2\theta$$
$$\frac{1+\cos(2\theta)}{2} = \cos^2\theta$$

The last reduction formula is derived by writing tangent in terms of sine and cosine:

$$\begin{aligned}
\tan^2\theta &= \frac{\sin^2\theta}{\cos^2\theta} \\
&= \frac{\frac{1-\cos(2\theta)}{2}}{\frac{1+\cos(2\theta)}{2}} && \text{Substitute the reduction formulas} \\
&= \left(\frac{1-\cos(2\theta)}{2}\right)\left(\frac{2}{1+\cos(2\theta)}\right) \\
&= \frac{1-\cos(2\theta)}{1+\cos(2\theta)}
\end{aligned}$$

POWER REDUCTION FORMULAS

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2} \tag{6.5.6}$$
$$\cos^2\theta = \frac{1+\cos(2\theta)}{2} \tag{6.5.7}$$
$$\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)} \tag{6.5.8}$$

Example 6.5.9: Use Power Reduction Formulas

Write an equivalent expression for $\cos^4 x$ that does not involve any powers of sine or cosine greater than 1.

Solution

We will apply the reduction formula for cosine twice.

$$\begin{aligned}\cos^4 x &= (\cos^2 x)^2 \\ &= \left(\frac{1+\cos(2x)}{2}\right)^2 \qquad \text{Substitute reduction formula} \\ &= \frac{1}{4}(1+2\cos(2x)+\cos^2(2x)) \\ &= \frac{1}{4}+\frac{1}{2}\cos(2x)+\frac{1}{4}\left(\frac{1+\cos^2(2x)}{2}\right) \qquad \text{Substitute reduction formula for } \cos^2 x \\ &= \frac{1}{4}+\frac{1}{2}\cos(2x)+\frac{1}{8}+\frac{1}{8}\cos(4x) \\ &= \frac{3}{8}+\frac{1}{2}\cos(2x)+\frac{1}{8}\cos(4x)\end{aligned}$$

Analysis

The solution is found by using the reduction formula twice, as noted, and the perfect square formula from algebra.

Example 6.5.10: Use Power-Reducing Formulas to Prove an Identity

Use the power-reducing formulas to prove $\sin^3(2x) = \left[\frac{1}{2}\sin(2x)\right][1-\cos(4x)]$

Solution

We will work on simplifying the left side of the equation:

$$\begin{aligned}\sin^3(2x) &= [\sin(2x)][\sin^2(2x)] \\ &= \sin(2x)\left[\frac{1-\cos(4x)}{2}\right] \qquad \text{Substitute the power-reduction formula.} \\ &= \sin(2x)\left(\frac{1}{2}\right)[1-\cos(4x)] \\ &= \frac{1}{2}[\sin(2x)][1-\cos(4x)]\end{aligned}$$

Analysis

Note that in this example, we substituted $\frac{1-\cos(4x)}{2}$ for $\sin^2(2x)$. The formula states $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$. We let $\theta = 2x$, so $2\theta = 4x$.

Try It 6.5.11

Use the power-reducing formulas to prove that $10\cos^4 x = \frac{15}{4} + 5\cos(2x) + \frac{5}{4}\cos(4x)$.

Answer

$$
\begin{aligned}
10\cos^4 x &= 10(\cos^2 x)^2 \\
&= 10\left[\frac{1+\cos(2x)}{2}\right]^2 \qquad \text{Substitute reduction formula for } \cos^2 x \\
&= \frac{10}{4}[1+2\cos(2x)+\cos^2(2x)] \\
&= \frac{10}{4}+\frac{10}{2}\cos(2x)+\frac{10}{4}\left(\frac{1+\cos^2(2x)}{2}\right) \qquad \text{Substitute reduction formula for } \cos^2 x \\
&= \frac{10}{4}+\frac{10}{2}\cos(2x)+\frac{10}{8}+\frac{10}{8}\cos(4x) \\
&= \frac{30}{8}+5\cos(2x)+\frac{10}{8}\cos(4x) \\
&= \frac{15}{4}+5\cos(2x)+\frac{5}{4}\cos(4x)
\end{aligned}
$$

Half-Angle Formulas

The next set of identities is the set of **half-angle formulas**, which can be derived from the reduction formulas and we can use when we have an angle that is half the size of a special angle. If we replace θ with $\frac{\alpha}{2}$, the half-angle formula for sine is found by simplifying the equation and solving for $\sin\left(\frac{\alpha}{2}\right)$. Note that the half-angle formulas are preceded by a $\pm$ sign. This does not mean that both the positive and negative expressions are valid. Rather, it depends on the quadrant in which $\frac{\alpha}{2}$ terminates.

The half-angle formula for sine is derived as follows:

$$
\begin{aligned}
\sin^2\theta &= \frac{1-\cos(2\theta)}{2} \\
\sin^2\left(\frac{\alpha}{2}\right) &= \frac{1-\left(\cos 2\cdot\frac{\alpha}{2}\right)}{2} = \frac{1-\cos\alpha}{2} \\
\sin\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1-\cos\alpha}{2}}
\end{aligned}
$$

To derive the half-angle formula for cosine, we have

$$
\begin{aligned}
\cos^2\theta &= \frac{1+\cos(2\theta)}{2} \\
\cos^2\left(\frac{\alpha}{2}\right) &= \frac{1+\cos\left(2\cdot\frac{\alpha}{2}\right)}{2} = \frac{1+\cos\alpha}{2} \\
\cos\left(\frac{\pi}{2}\right) &= \pm\sqrt{\frac{1+\cos\alpha}{2}}
\end{aligned}
$$

For the tangent identity, we have

$$\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$$

$$\tan^2\left(\frac{\alpha}{2}\right) = \frac{1-\cos\left(2\cdot\frac{\alpha}{2}\right)}{1+\cos\left(2\cdot\frac{\alpha}{2}\right)}$$

$$\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

HALF-ANGLE FORMULAS

$$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}} \tag{6.5.9}$$

$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}} \tag{6.5.10}$$

$$\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{\sin\alpha}{1+\cos\alpha} = \frac{1-\cos\alpha}{\sin\alpha} \tag{6.5.11}$$

Example 6.5.12: Use Half Angle Formulas to Evaluate a Trigonometric Expression

Using a Half-Angle Formula to Find the Exact Value of a Sine Function. Find $\sin(15°)$ using a half-angle formula.

Solution

Since $15° = \frac{30°}{2}$,we use the half-angle formula for sine (Equation 6.5.9):

$$\sin 15° = \sin\frac{30°}{2} = \sqrt{\frac{1-\cos 30°}{2}}$$

$$= \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{2}$$

Remember that we can check the answer with a graphing calculator.

Analysis

Notice that we used only the positive root because $\sin(15°)$ is positive.

Example 6.5.13

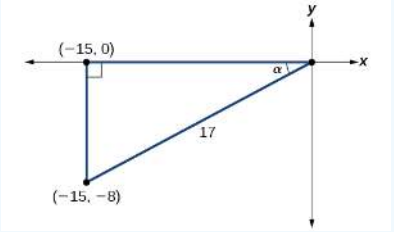
Given that $\tan\alpha = \frac{8}{15}$ and α lies in quadrant III, find the exact value of the following:

a. $\sin\left(\frac{\alpha}{2}\right)$ b. $\cos\left(\frac{\alpha}{2}\right)$ c. $\tan\left(\frac{\alpha}{2}\right)$

Solution

Using the given information, we can draw the triangle shown to the right. Using the Pythagorean Theorem, we find the hypotenuse to be 17. Therefore, we can calculate $\sin\alpha = -\frac{8}{17}$ and $\cos\alpha = -\frac{15}{17}$. in summary

then, $\begin{aligned} \cos\theta &= -\frac{15}{17} \\ \sin\theta &= -\frac{8}{17} \\ \tan\theta &= \frac{8}{15} \end{aligned}$.



Before we start, we must remember that if α is in quadrant III, then $180° < \alpha < 270°$, so $\frac{180°}{2} < \frac{\alpha}{2} < \frac{270°}{2}$. This means that the terminal side of $\frac{\alpha}{2}$ is in quadrant II, since $90° < \frac{\alpha}{2} < 135°$.

a. To find $\sin\frac{\alpha}{2}$, we begin by writing the half-angle formula for sine. Then we substitute the value of the cosine we found from the triangle above and simplify.

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

$$= \pm\sqrt{\frac{1-\left(-\frac{15}{17}\right)}{2}} = \pm\sqrt{\frac{\frac{32}{17}}{2}} = \pm\sqrt{\frac{32}{17}\cdot\frac{1}{2}} = \pm\sqrt{\frac{16}{17}} = \pm\frac{4}{\sqrt{17}} = \frac{4\sqrt{17}}{17}$$

We choose the positive value of $\sin\frac{\alpha}{2}$ because the angle $\frac{\alpha}{2}$ terminates in quadrant II and sine is positive in quadrant II.

b. To find $\cos\frac{\alpha}{2}$, we will write the half-angle formula for cosine, substitute the value of the cosine we found from the triangle above, and simplify.

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$$= \pm\sqrt{\frac{1+\left(-\frac{15}{17}\right)}{2}} = \pm\sqrt{\frac{\frac{2}{17}}{2}} = \pm\sqrt{\frac{2}{17}\cdot\frac{1}{2}} = \pm\sqrt{\frac{1}{17}} = -\frac{\sqrt{17}}{17}$$

We choose the negative value of $\cos\frac{\alpha}{2}$ because the angle $\frac{\alpha}{2}$ is in quadrant II because cosine is negative in quadrant II.

c. To find $\tan\frac{\alpha}{2}$, we write the half-angle formula for tangent. Again, we substitute the value of the cosine we found from the triangle above and simplify.

$$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

$$= \pm\sqrt{\frac{1-\left(-\frac{15}{17}\right)}{1+\left(-\frac{15}{17}\right)}} = \pm\sqrt{\frac{\frac{32}{17}}{\frac{2}{17}}} = \pm\sqrt{\frac{32}{2}} = -\sqrt{16} = -4$$

We choose the negative value of $\tan\frac{\alpha}{2}$ because $\frac{\alpha}{2}$ lies in quadrant II, and tangent is negative in quadrant II.

Try It 6.5.14

Given that $\sin\ \alpha = -\frac{4}{5}$ and α lies in quadrant IV, find the exact value of $\cos\ \left(\frac{\alpha}{2}\right)$.

Answer

$$-\frac{2}{\sqrt{5}}$$

Key Equations

Double-angle formulas		
$\sin(2\theta) = 2\sin\ \theta\ \cos\ \theta$	$\cos(2\theta) = \cos^2\theta - \sin^2\theta$ $= 1 - 2\sin^2\theta$ $= 2\cos^2\theta - 1$	$\tan(2\theta) = \frac{2\tan\ \theta}{1-\tan^2\theta}$
Reduction formulas		
$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$	$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$	$\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$
Half-angle formulas		
$\sin\ \frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\ \alpha}{2}}$	$\cos\ \frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\ \alpha}{2}}$	$\tan\ \frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\ \alpha}{1+\cos\ \alpha}}$ $= \frac{\sin\ \alpha}{1+\cos\ \alpha}$ $= \frac{1-\cos\ \alpha}{\sin\ \alpha}$

Key Concepts

- Double-angle identities are derived from the sum formulas of the fundamental trigonometric functions: sine, cosine, and tangent.
- Reduction formulas are especially useful in calculus, as they allow us to reduce the power of the trigonometric term.
- Half-angle formulas allow us to find the value of trigonometric functions involving half-angles, whether the original angle is known or not.

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6.5e: Exercises: Double Angle, Half Angle and Reductions Formulas

A: Concepts.

Exercise 6.5e.A

1) Explain how to determine the reduction identities from the double-angle identity $\cos(2x) = \cos^2 x - \sin^2 x$

2) Explain how to determine the double-angle formula for $\tan(2x)$ using the double-angle formulas for $\cos(2x)$ and $\sin(2x)$.

3) We can determine the half-angle formula for $\tan\left(\frac{x}{2}\right) = \frac{\sqrt{1-\cos x}}{\sqrt{1+\cos x}}$ by dividing the formula for $\sin\left(\frac{x}{2}\right)$ by $\cos\left(\frac{x}{2}\right)$. Explain how to determine two formulas for $\tan\left(\frac{x}{2}\right)$ that do not involve any square roots.

4) For the half-angle formula given in the previous exercise for $\tan\left(\frac{x}{2}\right)$, explain why dividing by 0 is not a concern. (Hint: examine the values of $\cos x$ necessary for the denominator to be 0.)

Answers to odd exercises.

1. Use the Pythagorean identities and isolate the squared term.
3. $\frac{1-\cos x}{\sin x}$, $\frac{\sin x}{1+\cos x}$, multiplying the top and bottom by $\sqrt{1-\cos x}$ and $\sqrt{1+\cos x}$, respectively.

B: Evaluate Double Angle Trigonometric Expressions

Exercise 6.5e.B

★ Find the exact values of (a) $\sin(2x)$, (b) $\cos(2x)$, and (c) $\tan(2x)$.

5. If $\sin x = \frac{1}{8}$, and x is in quadrant I
.
6. If $\sin x = \frac{2}{3}$, and x is in quadrant II
.
7. If $\cos x = -\frac{1}{2}$, and x is in quadrant III.

8. If $\cos x = \frac{5}{12}$, and x is in quadrant I.
9. If $\tan x = \frac{12}{5}$, and x is in quadrant I
.
10. If $\tan x = -8$, and x is in quadrant IV
.

★ Find the values of the six trigonometric functions for angle θ if the conditions provided hold.

11. $\cos(2\theta) = \frac{3}{5}$ and $\frac{\pi}{2} \le \theta \le \pi$

12. $\cos(2\theta) = \frac{1}{\sqrt{2}}$ and $\pi \le \theta \le \frac{3\pi}{2}$

Answers to odd exercises.

5a. $\frac{3\sqrt{7}}{32}$ 5b. $\frac{31}{32}$ 5c. $\frac{3\sqrt{7}}{31}$ 7a. $\frac{\sqrt{3}}{2}$ 7b. $-\frac{1}{2}$ 7c. $-\sqrt{3}$ 9a. $\frac{120}{169}$, 9b. $-\frac{119}{169}$, 9c. $-\frac{120}{119}$

11. $\cos\theta = -\frac{2\sqrt{5}}{5}, \sin\theta = \frac{\sqrt{5}}{5}, \tan\theta = -\frac{1}{2}, \csc\theta = \sqrt{5}, \sec\theta = -\frac{\sqrt{5}}{2}, \cot\theta = -2$

C: Use Double Angle Formulas to Solve Equations

Exercise 6.5e.C

★ **Use double angle formulas to solve exactly (where possible) on the interval $[0, 2\pi)$.**

20. $\sin(2t) = \cos t$

21. $\sin(2t) + \sin t = 0$

22. $\sin(2x) - \sin x = 0$

23. $\sin(2t) + 3\cos(t) = 0$

24. $\sin(4x) - \sin(2x) = 0$

25. $\dfrac{\sin(2x)}{\sec^2 x} = 0$

26. $\dfrac{\sin(2x)}{2\csc^2 x} = 0$

27. $\cos(2t) = \sin t$

28. $\cos(6x) - \cos(3x) = 0$

29. $9\cos(2\theta) = 9\cos^2\theta - 4$

30. $3\cos(2\alpha) = 2\cos^2(\alpha)$

31. $\cos(2x) - \cos x = 0$

32. $4\sin^2 x + \sin(2x)\sec x - 3 = 0$

33. $\sin^2 x - 1 + 2\cos(2x) - \cos^2 x = 1$

34. $8\cos(2\alpha) = 8\cos^2(\alpha) - 1$

35. $6\sin(2t) + 9\sin(t) = 0$

Answers to odd exercises.

21. $0, \dfrac{2\pi}{3}, \pi, \dfrac{4\pi}{3}$ 23. $\dfrac{\pi}{2}, \dfrac{3\pi}{2}$ 25. $0, \pi$ 27. $\dfrac{3\pi}{2}, \dfrac{\pi}{6}, \dfrac{5\pi}{6}$
29. $0.7297,\ 2.4119,\ 3.8713,\ 5.5535$ 31. $0, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$ 33. No solution. 35. $0,\ \pi,\ 2.4189,\ 3.8643$

D: Recognize patterns

Exercise 6.5e.D

★ **Simplify to one trigonometric expression. Do not evaluate.**

41. $2\sin\left(\dfrac{\pi}{4}\right)2\cos\left(\dfrac{\pi}{4}\right)$

42. $4\sin\left(\dfrac{\pi}{8}\right)\cos\left(\dfrac{\pi}{8}\right)$

43. $6\sin(5x)\cos(5x)$

44. $4\sin(8x)\cos(8x)$

45. $\cos^2(6x) - \sin^2(6x)$

46. $\cos^2(28°) - \sin^2(28°)$

47. $2\cos^2(37°) - 1$

48. $1 - 2\sin^2(17°)$

49. $\cos^2(9x) - \sin^2(9x)$

50. $\cos^2(37°) - \sin^2(37°)$

Answers to odd exercises.

41. $2\sin\left(\dfrac{\pi}{2}\right)$ 43. $3\sin(10x)$ 45. $\cos(12x)$ 47. $\cos(74°)$ 49. $\cos(18x)$

E: Verify identities

Exercise 6.5e.E

★ **Prove the identity.**

51. $(\sin^2 x - 1)^2 = \cos(2x) + \sin^4 x$

52. $(\sin t - \cos t)^2 = 1 - \sin(2t)$

53. $\sin(2x) = -2\sin(-x)\cos(-x)$

54. $\cot x - \tan x = 2\cot(2x)$

55. $\dfrac{1 + \cos(2\theta)}{\sin(2\theta)}\tan^2\theta = \tan\theta$

56. $\dfrac{1 + \cos(2t)}{\sin(2t) - \cos t} = \dfrac{2\cos t}{2\sin t - 1}$

57. $\sin(2x) = \dfrac{2\tan x}{1 + \tan^2 x}$

58. $\cos(2\alpha) = \dfrac{1 - \tan^2\alpha}{1 + \tan^2\alpha}$

59. $\tan(2x) = \dfrac{2\sin x\cos x}{2\cos^2 x - 1}$

60. $\dfrac{\sin(2\theta)}{1 + \cos(2\theta)} = \tan(\theta)$

61. $\cos(16x) = \left(\cos^2(4x) - \sin^2(4x) - \sin(8x)\right) \cdot \left(\cos^2(4x) - \sin^2(4x) + \sin(8x)\right)$

62. $\sin(16x) = 16\sin x\cos x\cos(2x)\cos(4x)\cos(8x)$

Answers to odd exercises.

51. $(\sin^2 x - 1)^2 = \sin^4 x - 2\sin^2 x + 1 = \cos(2x) + \sin^4 x$

53. $-2\sin(-x)\cos(-x) = -2(-\sin(x)\cos(x)) = \sin(2x)$

55. $\dfrac{\sin(2\theta)}{1+\cos(2\theta)}\tan^2\theta = \dfrac{2\sin(\theta)\cos(\theta)}{1+\cos^2\theta-\sin^2\theta}\tan^2\theta = \dfrac{2\sin(\theta)\cos(\theta)}{2\cos^2\theta}\tan^2\theta = \dfrac{\sin(\theta)}{\cos(\theta)}\tan^2\theta$
$= \cot(\theta)\tan^2\theta = \tan\theta$

57. $\dfrac{2\tan x}{1+\tan^2 x} = \dfrac{\frac{2\sin x}{\cos x}}{1+\frac{\sin^2 x}{\cos^2 x}} = \dfrac{\frac{2\sin x}{\cos x}}{\frac{\cos^2 x+\sin^2 x}{\cos^2 x}} = \dfrac{2\sin x}{\cos x}\cdot\dfrac{\cos^2 x}{1} = 2\sin x\cos x = \sin(2x)$

59. $\dfrac{2\sin x\cos x}{2\cos^2 x - 1} = \dfrac{\sin(2x)}{\cos(2x)} = \tan(2x)$

61. $(\cos^2(4x) - \sin^2(4x) - \sin(8x))(\cos^2(4x) - \sin^2(4x) + \sin(8x))$
$= (\cos(8x) - \sin(8x))(\cos(8x) + \sin(8x)) = \cos^2(8x) - \sin^2(8x) = \cos(16x)$

F: Use Power Reduction Formulas

Exercise 6.5e.F

★ **Rewrite the expression with an exponent no higher than 1.**

70. $\cos^2(5x)$	74. $\sin^2(2x)$	78. $\cos^2 x\sin(2x)$	82. $\tan^2 x\sin x$
71. $\cos^2(6x)$	75. $\sin^2 x\cos^2 x$	79. $\tan^4 x\cos^2 x$	83. $\cos^4 x\sin^2 x$
72. $\sin^4(8x)$	76. $\tan^2 x\sin^2 x$	80. $\tan^2\left(\frac{x}{2}\right)\sin x$	84. $\cos^2 x\sin^4 x$
73. $\sin^4(3x)$	77. $\tan^4 x$	81. $\cos^2(2x)\sin x$	

★ **Algebraically find an equivalent function, in terms of only $\sin x$ and/or $\cos x$**

85. $\sin(4x)$	86. $\cos(4x)$

Answers to odd exercises.

71. $\dfrac{1+\cos(12x)}{2}$ 73. $\dfrac{3+\cos(12x)-4\cos(6x)}{8}$ 75. $\dfrac{1-\cos(4x)}{8}$
77. $\dfrac{3+\cos(4x)-4\cos(2x)}{3+\cos(4x)+4\cos(2x)}$ 79. $\dfrac{3+\cos(4x)-4\cos(2x)}{4(\cos(2x)+1)}$ 81. $\dfrac{(1+\cos(4x))\sin x}{2}$
83. $\dfrac{2+\cos(2x)-2\cos(4x)-\cos(6x)}{32}$ 85. $4\sin x\cos x(\cos^2 x - \sin^2 x)$

G: Use Half Angle Formulas

Exercise 6.5e.G

★ **Find the exact value using half-angle formulas.**

91. $\sin\left(\frac{\pi}{8}\right)$	93. $\sin\left(\frac{11\pi}{12}\right)$	95. $\tan\left(\frac{5\pi}{12}\right)$	97. $\tan\left(-\frac{3\pi}{8}\right)$
92. $\cos\left(-\frac{11\pi}{12}\right)$	94. $\cos\left(\frac{7\pi}{8}\right)$	96. $\tan\left(-\frac{3\pi}{12}\right)$	

★ **Find the exact values of a) $\sin\left(\frac{x}{2}\right)$ b) $\cos\left(\frac{x}{2}\right)$, and c) $\tan\left(\frac{x}{2}\right)$, when $0 \le \theta \le 2\pi$**

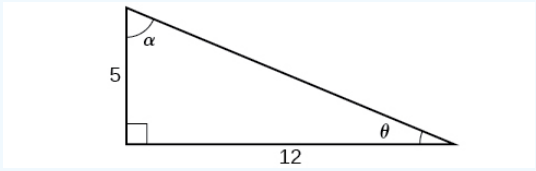
101. If $\sin x = -\frac{12}{13}$, and x is in quadrant III
.

102. If $\tan x = -\frac{4}{3}$, and x is in quadrant IV.

103. If $\sec x = -4$, and x is in quadrant II
.

104. If $\csc x = 7$, and x is in quadrant II.

★ **Use the figure below to find the requested half angle trigonometric expressions.**



107. Find $\sin\left(\frac{\alpha}{2}\right)$, $\cos\left(\frac{\alpha}{2}\right)$, and $\tan\left(\frac{\alpha}{2}\right)$.

108. Find $\sin\left(\frac{\theta}{2}\right)$, $\cos\left(\frac{\theta}{2}\right)$, and $\tan\left(\frac{\theta}{2}\right)$
.

Answers to odd exercises.

91. $\frac{\sqrt{2-\sqrt{2}}}{2}$ 93. $\frac{\sqrt{2-\sqrt{3}}}{2}$ 95. $2+\sqrt{3}$ 97. $-1-\sqrt{2}$

101a. $\frac{3\sqrt{13}}{13}$ 101b. $-\frac{2\sqrt{13}}{13}$ 101c. $-\frac{3}{2}$ 103a. $\frac{\sqrt{10}}{4}$ 103b. $\frac{\sqrt{6}}{4}$ 103c. $\frac{\sqrt{15}}{3}$

107. $\frac{2\sqrt{13}}{13}, \frac{3\sqrt{13}}{13}, \frac{2}{3}$

★

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CHAPTER OVERVIEW

7: Further Applications of Trigonometry

In this chapter, we will explore applications of trigonometry that will enable us to solve many different kinds of problems, including finding the height of a tree. We extend topics we introduced in Trigonometric Functions and investigate applications more deeply and meaningfully.

7.1: Non-right Triangles - Law of Sines

7.1e: Exercises - Law of Sines

7.2: Non-right Triangles - Law of Cosines

7.2e: Exercises - Law of Cosines

7.3: Vectors in 2D

7.3e: Exercises - Vectors in 2D

7.4: Vectors in Three Dimensions

7.4e: Vectors in 3D

7.5: The Dot Product

7.5e: The Dot Product

7.6: The Cross Product

7.6e: Exercises - Cross Product

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7.1: Non-right Triangles - Law of Sines

Learning Objectives.

- Use the Law of Sines to solve oblique triangles and applied problems

In any triangle, we can draw an **altitude**, a perpendicular line from one vertex to the opposite side, forming two right triangles. It would be preferable, however, to have methods that we can apply directly to non-right triangles without first having to create right triangles.

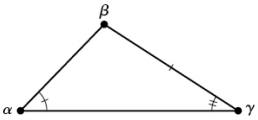
Any triangle that is not a right triangle is an **oblique triangle**. It could be an **acute triangle** (all three angles of the triangle are less than right angles) or it could be an **obtuse triangle** (one of the three angles is greater than a right angle). Solving an oblique triangle means finding the measurements of all three angles and all three sides. To do so, we need to start with at least three of these values, including at least one of the sides. We will investigate three possible oblique triangle problem situations:

ASA (angle-side-angle)



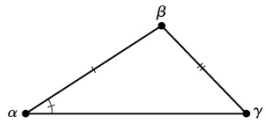
The measurements of two angles and the included side are known.

AAS (angle-angle-side)



The measurements of two angles and a side opposite one of those angles is known.

SSA (side-side-angle)



The measurements of two sides and an angle opposite one of those sides is known.

Knowing how to approach each of these situations enables oblique triangles to be solved without having to drop a perpendicular to form two right triangles. Instead, the fact that the ratio of the measurement of one of the angles to the length of its opposite side will be equal to the other two ratios of angle measure to opposite side can be used. This statement is derived by considering the triangle in Figure **7.1.1**.

Using right triangle relationships, equations can be found for $\sin\alpha$ and $\sin\beta$. Solving both equations for h gives two different expressions for h, $h = b\sin\alpha$ and $h = a\sin\beta$.

$$b\sin\alpha = a\sin\beta \qquad \text{Equate expressions for } h$$
$$\frac{(b\sin\alpha)}{ab} = \frac{(a\sin\beta)}{ab} \qquad \text{Divide both sides by } ab$$
$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \qquad \text{Equivalent side/angle ratios}$$

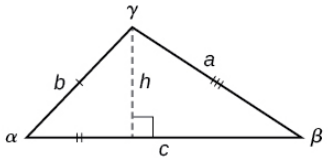


Figure **7.1.1**: Triangle used to derive Law of Sines: $\sin\alpha = \frac{h}{b}$ and $\sin\beta = \frac{h}{a}$

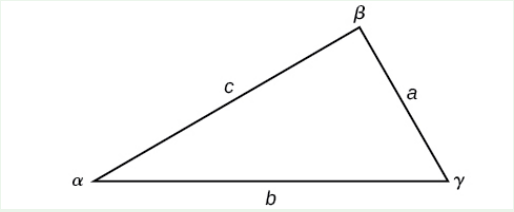
Similarly, ratios between other angle/side pairs can be obtained. Together, these relationships are called the **Law of Sines**.

LAW OF SINES

Given a triangle with angles and opposite sides labeled as in the figure to the right, the ratio of the measurement of an angle to the length of its opposite side will be equal to the other two ratios of angle measure to opposite side. All proportions will be equal.

The **Law of Sines** is based on proportions and is presented symbolically two ways.

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$
$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$



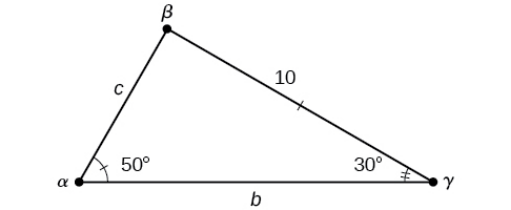
To solve an oblique triangle, use any pair of applicable ratios.

Note **the standard way of labeling triangles**: angle α (alpha) is opposite side a; angle β (beta) is opposite side b; and angle γ (gamma) is opposite side c. Also, when calculating angles and sides, be sure to carry the exact values through to the final answer. Generally, final answers are rounded to the nearest tenth, unless otherwise specified.

Use the Law of Sines to Solve AAS and ASA Triangles (Two Angles and One Side Known)

Example 7.1.1: Solve an AAS Triangle

Solve the triangle illustrated below to the nearest tenth.



Solution

The three angles must add up to 180 degrees. From this, we can determine that

$\beta = 180° - 50° - 30° = 100°$

To find an unknown side, we need to know the corresponding angle and a known ratio. We know angle $\alpha = 50°$ and its corresponding side $a = 10$. The following proportion from the Law of Sines can be used to find the length of c.

$$\frac{\sin(50°)}{10} = \frac{\sin(30°)}{c}$$

$$c \cdot \frac{\sin(50°)}{10} = \sin(30°) \qquad \text{Multiply both sides by } c$$

$$c = \sin(30°)\frac{10}{\sin(50°)} \approx 6.5 \qquad \text{Multiply by the reciprocal to isolate } c$$

Similarly, to solve for b, we set up another proportion.

$$\frac{\sin(50°)}{10} = \frac{\sin(100°)}{b}$$

$$b\sin(50°) = 10\sin(100°) \qquad \text{Multiply both sides by } b$$

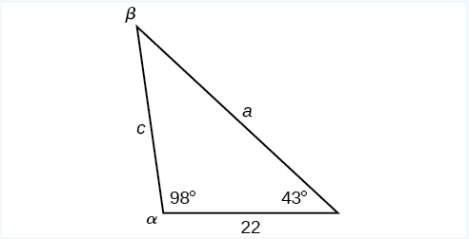
$$b = \frac{10\sin(100°)}{\sin(50°)} \approx 12.9 \qquad \text{Multiply by the reciprocal to isolate } b$$

Therefore, the complete set of angles and sides is:

$$\alpha = 50° \quad a = 10$$
$$\beta = 100° \quad b \approx 12.9$$
$$\gamma = 30° \quad c \approx 6.5$$

Try It 7.1.1: Solve an ASA triangle

Solve the triangle shown below to the nearest tenth.



Answer

https://math.libretexts.org/@go/page/34939

$\alpha = 98^\circ \quad a \approx 34.6$
$\beta = 39^\circ \quad b = 22$
$\gamma = 43^\circ \quad c \approx 23.8$

Use the Law of Sines to Solve SSA Triangles (Two Sides and One Angle Known)

We can use the Law of Sines to solve any oblique triangle, but some solutions may not be straightforward. In some cases, more than one triangle may satisfy the given criteria, which we describe as an ambiguous case. Triangles classified as SSA, those in which we know the lengths of two sides and the measurement of the angle opposite one of the given sides, may result in one or two solutions, or even no solution.

POSSIBLE OUTCOMES FOR SSA TRIANGLES

Oblique triangles in the category SSA may have four different outcomes. Figure **7.1.2** illustrates the solutions with the known sides a and b and known angle α. When angle α is obtuse, there are only two outcomes: no triangle when $a \le b$ and one triangle when $a > b$.

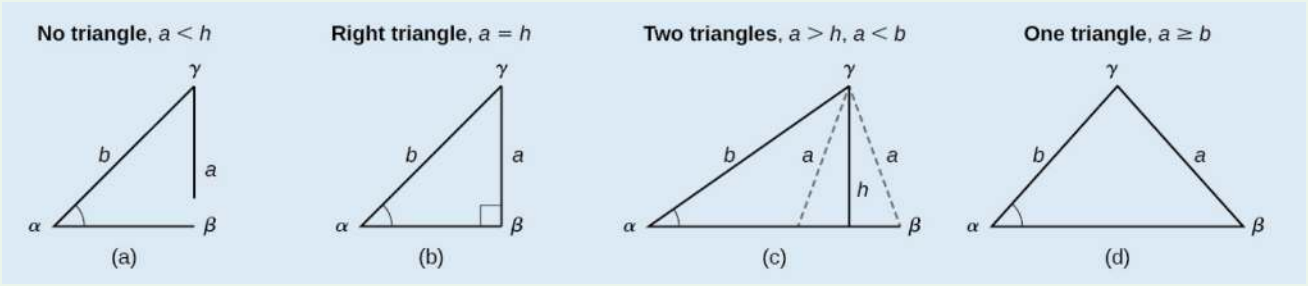
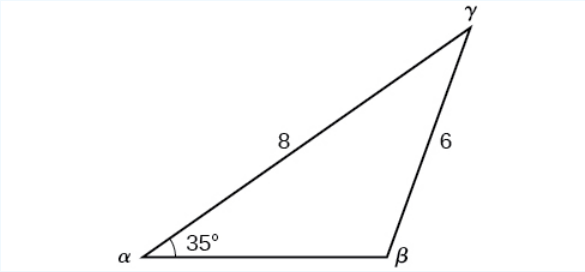


***Figure* 7.1.2** Sides a and b and angle α are given. The height of the triangle is h.

The angle determined, β, (a) Does not exist, (b) is equal to 90°, (c) is greater than α, or (d) is less than α.

Example 7.1.2: Solve an Oblique SSA Triangle

Solve the triangle in the diagram below for the missing side and find the missing angle measures to the nearest tenth.



Solution

Use the Law of Sines to find angle β and angle γ, and then side c. Solving for β, we have the proportion

$$\begin{aligned} \frac{\sin\alpha}{a} &= \frac{\sin\beta}{b} \\ \frac{\sin(35^\circ)}{6} &= \frac{\sin\beta}{8} \\ \frac{8\sin(35^\circ)}{6} &= \sin\beta \\ 0.7648 &\approx \sin\beta \\ \sin^{-1}(0.7648) &\approx 49.9^\circ \\ \beta &\approx 49.9^\circ \end{aligned}$$

However, in the diagram, angle β appears to be an obtuse angle and may be greater than $90°$. How did we get an acute angle, and how do we find the measurement of β? Remember that the sine function is positive in both the first and second quadrants and thus finding an angle using the $\sin^{-1}$ function will only produce an angle between $0°$ and $90°$!! Dropping a perpendicular from γ and viewing the triangle from a right angle perspective, we have Figure 7.1.2*a*. It appears that there may be a second triangle that will fit the given criteria.

Our calculations have found the angle measure $\beta' \approx 49.9°$ in the acute triangle. The measure of this angle β in the oblique triangle, is supplementary to β', which means that $\beta = 180° - \beta'$ so $\beta = 180° - 49.9° = 130.1°$.

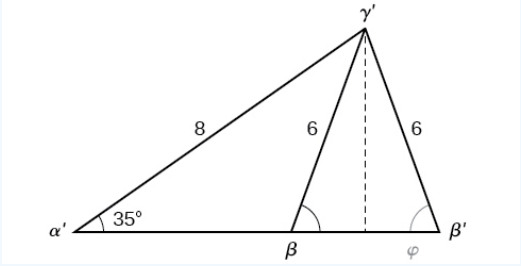


Figure 7.1.2*a*

Solving for γ in the oblique triangle, we have

$$\gamma = 180° - 35° - 130.1° \approx 14.9°$$

Solving for γ' in the acute triangle, we have

$$\gamma' = 180° - 35° - 49.5° \approx 95.1°$$

Now we need to find c and c'.

$$\frac{c}{\sin(14.9°)} = \frac{6}{\sin(35°)} \quad \rightarrow \quad c = \frac{6\sin(14.9°)}{\sin(35°)} \approx 2.7$$

and

$$\frac{c'}{\sin(95.1°)} = \frac{6}{\sin(35°)} \quad \rightarrow \quad c' = \frac{6\sin(95.1°)}{\sin(35°)} \approx 10.4$$

To summarize, there are two triangles with an angle of $35°$, an adjacent side of 8, and an opposite side of 6, as shown in Figure 7.1.2*b*.

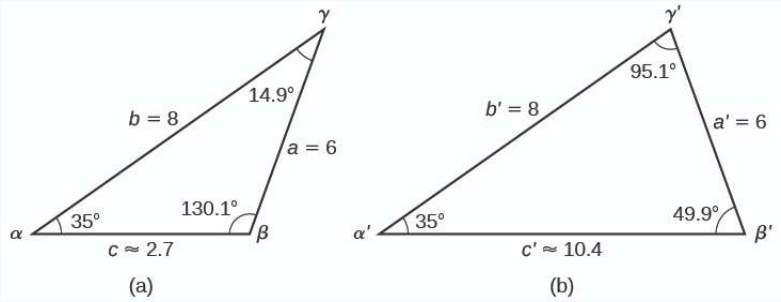


Figure* 7.1.2*b

However, we were looking for the values for the triangle with an obtuse angle β. We can see them in the first triangle (a) in Figure 7.1.2*b*.

Try It 7.1.2

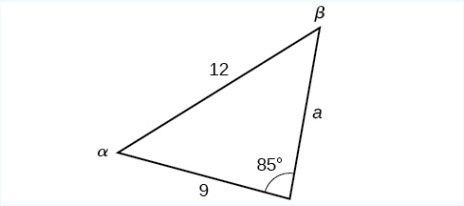
Given $\alpha = 80°$, $a = 120$, and $b = 121$, find the missing side and angles. If there is more than one possible solution, show both.

Answer

Solution 1		Solution 2	
$\alpha = 80°$	$a = 120$	$\alpha' = 80°$	$a' = 120$
$\beta \approx 83.2°$	$b = 121$	$\beta' \approx 96.8°$	$b' = 121$
$\gamma \approx 16.8°$	$c \approx 35.2$	$\gamma' \approx 3.2°$	$c' \approx 6.8$

Example 7.1.3: Solve an Acute SSA Triangle

In the triangle shown below, solve for the unknown side and angles. Round your answers to the nearest tenth.



Solution

In choosing the pair of ratios from the Law of Sines to use, look at the information given. In this case, we know the angle, $\gamma = 85°$, and its corresponding side $c = 12$, and we know side $b = 9$. We will use this proportion to solve for β.

$$\frac{\sin(85°)}{12} = \frac{\sin\beta}{9} \qquad \text{Isolate the unknown.}$$
$$\frac{9\sin(85°)}{12} = \sin\beta$$

To find β, apply the inverse sine function. The inverse sine will produce a single result, but keep in mind that there may be two values for β. It is important to verify the result, as there may be two viable solutions, only one solution (the usual case), or no solutions.

$$\beta = \sin^{-1}\left(\frac{9\sin(85°)}{12}\right) \approx \sin^{-1}(0.7471) \approx 48.3°$$

Because one solution has been found, and this is an SSA triangle, there may be a second possible solution. The other possivle angle is found by subtracting β from $180°$, so $\beta = 180° - 48.3° \approx 131.7°$. To check if this is also a solution, subtract both angles, the given angle $\gamma = 85°$ and the calculated angle $\beta = 131.7°$, from $180°$. This gives

$$\alpha = 180° - 85° - 131.7° \approx -36.7°$$

which is impossible, and so there is only one possible solution, $\beta \approx 48.3°$.

To find the remaining missing values, we calculate $\alpha = 180° - 85° - 48.3° \approx 46.7°$. Now, only side a is needed. Use the Law of Sines to solve for a by one of the proportions.

$$\frac{\sin(85°)}{12} = \frac{\sin(46.7°)}{a}$$
$$a \cdot \frac{\sin(85°)}{12} = \sin(46.7°)$$
$$a = \frac{12\sin(46.7°)}{\sin(85°)} \approx 8.8$$

The complete set of solutions for the given triangle is:

$\alpha \approx 46.7°$	$a \approx 8.8$
$\beta \approx 48.3°$	$b = 9$
$\gamma = 85°$	$c = 12$

Try It 7.1.3

Given $\alpha = 80°$, $a = 100$, $b = 10$, find the missing side and angles. If there is more than one possible solution, show both. Round your answers to the nearest tenth.

Answer

$\beta \approx 5.7°$, $\gamma \approx 94.3°$, $c \approx 101.3$

Example 7.1.4: Solve a Triangle That Does Not Meet the Given Criteria

Find all possible triangles if one side has length 4 opposite an angle of $50°$, and a second side has length 10.

Solution

Using the given information, we can solve for the angle opposite the side of length 10. See Figure 7.1.4.

$$\frac{\sin\alpha}{10} = \frac{\sin(50°)}{4}$$
$$\sin\alpha = \frac{10\sin(50°)}{4}$$
$$\sin\alpha \approx 1.915$$

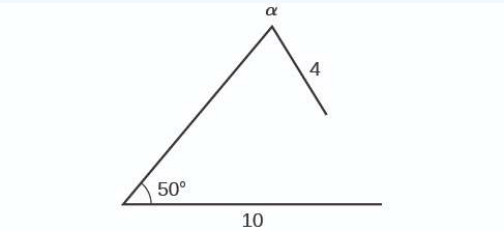


Figure 7.1.4: Impossible Triangle

We can stop here without finding the value of α. Because the range of the sine function is $[-1, 1]$, it is impossible for the sine value to be 1.915. In fact, inputting $\sin^{-1}(1.915)$ in a graphing calculator generates an ERROR DOMAIN. Therefore, no triangles can be drawn with the provided dimensions.

Try It 7.1.4

Determine the number of triangles possible given $a = 31$, $b = 26$, $\beta = 48°$.

Answer

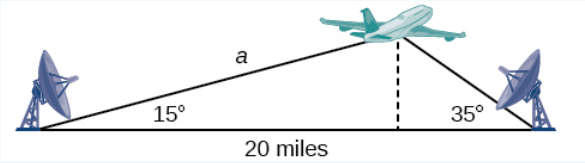
two

Applied Problems Using the Law of Sines

The more we study trigonometric applications, the more we discover that the applications are countless. Some are flat, diagram-type situations, but many applications in calculus, engineering, and physics involve three dimensions and motion.

Example 7.1.6: Find Altitude

Suppose two radar stations located 20 miles apart each detect an aircraft between them. The angle of elevation measured by the first station is 35 degrees, whereas the **angle of elevation** measured by the second station is 15 degrees, shown here. Find the altitude of the aircraft. Round the altitude to the nearest tenth of a mile.



Solution

To find the elevation of the aircraft, we first find the distance from one station to the aircraft, such as the side a, and then use right triangle relationships to find the height of the aircraft, h.

Because the angles in the triangle add up to 180 degrees, the unknown angle must be $180° - 15° - 35° = 130°$. This angle is opposite the side of length 20, allowing us to set up a Law of Sines relationship.

$$\frac{\sin(130°)}{20} = \frac{\sin(35°)}{a}$$
$$a\sin(130°) = 20\sin(35°)$$
$$a = \frac{20\sin(35°)}{\sin(130°)} \approx 14.98$$

The distance from one station to the aircraft is about 14.98 miles.

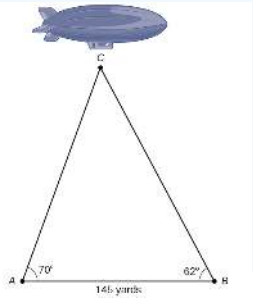
Now that we know a, we can use right triangle relationships to solve for h.

$$\sin(15^\circ) = \frac{opposite}{hypotenuse}$$
$$\sin(15^\circ) = \frac{h}{a}$$
$$\sin(15^\circ) = \frac{h}{14.98}$$
$$h = 14.98\sin(15^\circ) \approx 3.88$$

The aircraft is at an altitude of approximately 3.9 miles.

Try It 7.1.6

The accompanying diagram represents the height of a blimp flying over a football stadium. Find the height of the blimp if the angle of elevation at the southern end zone, point A, is $70°$, the angle of elevation from the northern end zone, point B, is $62°$, and the distance between the viewing points of the two end zones is 145 yards.



Answer

161.9 yd.

Key Equations

Law of Sines	$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$	$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$
Area for oblique triangles	$Area = \frac{1}{2}bc\sin\alpha$	$= \frac{1}{2}ac\sin\beta$ $= \frac{1}{2}ab\sin\gamma$

Key Concepts

- The Law of Sines can be used to solve oblique triangles, which are non-right triangles.
- According to the Law of Sines, the ratio of the measurement of one of the angles to the length of its opposite side equals the other two ratios of angle measure to opposite side.
- There are three possible cases: ASA, AAS, SSA. Depending on the information given, we can choose the appropriate equation to find the requested solution.
- The ambiguous case arises when an oblique triangle can have different outcomes.
- There are three possible cases that arise from SSA arrangement—a single solution, two possible solutions, and no solution.
- The Law of Sines can be used to solve triangles with given criteria.
- The general area formula for triangles translates to oblique triangles by first finding the appropriate height value.
- There are many trigonometric applications. They can often be solved by first drawing a diagram of the given information and then using the appropriate equation.

Contributors

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7.1e: Exercises - Law of Sines

For all exercises below, assume α or A is the angle opposite side a, β or B is the angle opposite side b, and γ or C is the angle opposite side c.

A: Concepts.

Exercise 7.1e.A

1) Describe the altitude of a triangle.

2) Compare right triangles and oblique triangles.

3) When can you use the Law of Sines to find a missing angle?

4) In the Law of Sines, what is the relationship between the angle in the numerator and the side in the denominator?

5) What type of triangle results in an ambiguous case?

Answers to Odd Exercises:

1. The altitude extends from any vertex to the opposite side or to the line containing the opposite side at a $90°$ angle.
3. When the known values are the side opposite the missing angle and another side and its opposite angle.
5. A triangle with two given sides and a non-included angle.

B: Find a side or angle

Exercise 7.1e.B

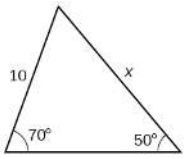
★ Find the length of the specified side or the length of side x. Round each answer to the nearest tenth.

11) $A = 37°, B = 49°, c = 5$
Find side b.

12) $A = 132°, C = 23°, b = 10$
Find side a.

13) Find side c when $B = 37°, C = 21°, b = 23$

15)



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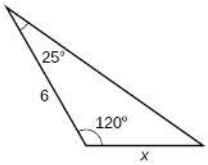

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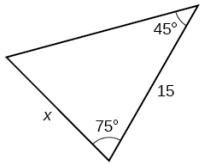

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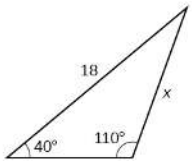

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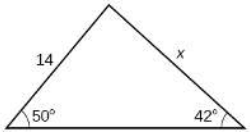

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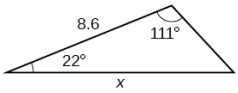


Answers to Odd Exercises:

11. $b \approx 3.78$
13. $c \approx 13.70$
15. $x \approx 12.3$
17. $x \approx 12.2$
19. $x \approx 16.0$.

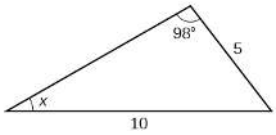
★ State the number of possible triangles that can be formed from the given information. Find all possible values for angle x f. Round each answer to the nearest tenth.

22) $a = 24, b = 5, B = 22°$
Find angle A.

23) $a = 13, b = 6, B = 20°$
Find angle A.

24) $A = 12°, a = 2, b = 9$
Find angle B.

25)



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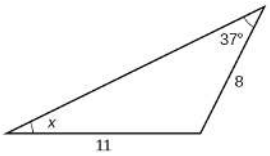

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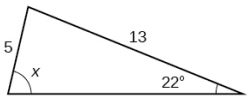

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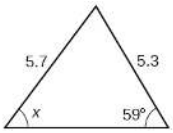

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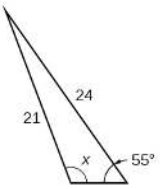
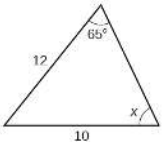


30)



Answers to Odd Exercises:

23. Two triangles. $A \approx 47.8°$ or $A' \approx 132.2°$
25. One triangle. $29.7°$
27. Two triangles. $x = 76.9°$ or $x = 103.1°$
29. Two triangles. $69.4°$ or $110.6°$

C: Solve a Triangle

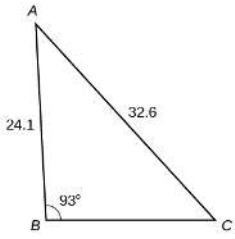
Exercise 7.1e.C

★ Determine whether there is no triangle, one triangle, or two triangles. Then solve each triangle, if possible. Round each answer to the nearest tenth.

32) $\alpha = 43°, \gamma = 69°, a = 20$
33) $\alpha = 35°, \gamma = 73°, c = 20$
34) $\alpha = 60°, \beta = 20°, c = 60$
35) $a = 4, \alpha = 60°, \beta = 100°$
36) $b = 10, \beta = 95°, \gamma = 30°$
37) $\gamma = 113°, b = 10, c = 32$
38) $\alpha = 119°, a = 14, b = 26$

39) $a = 12, c = 17, \alpha = 35°$
40) $b = 3.5, c = 5.3, \gamma = 80°$
41) $a = 7, c = 9, \alpha = 43°$
42) $a = 20.5, b = 35.0, \beta = 25°$
43) $b = 13, c = 5, \gamma = 10°$
44) $a = 7, b = 3, \beta = 24°$
45) $\beta = 119°, b = 8.2, a = 11.3$
46) $a = 2.3, c = 1.8, \gamma = 28°$

47) Solve the triangle



Answers to Odd Exercises:

33. one triangle, $\beta = 72°, a \approx 12.0, b \approx 19.9$
35. one triangle, $\gamma = 20°, b \approx 4.5, c \approx 1.6$
37. one triangle, $\alpha \approx 50.3°, \beta \approx 16.7°, a \approx 26.7$
39. two triangles, $\gamma \approx 54.3°, \beta \approx 90.7°, b \approx 20.9$ or $\gamma' \approx 125.7°, \beta' \approx 19.3°, b' \approx 6.9$
41. two triangles, $\beta \approx 75.7°, \gamma \approx 61.3°, b \approx 9.9$ or $\beta' \approx 18.3°, \gamma' \approx 118.7°, b' \approx 3.2$
43. two triangles, $\alpha \approx 143.2°, \beta \approx 26.8°, a \approx 17.3$ or $\alpha' \approx 16.8°, \beta' \approx 153.2°, a' \approx 8.3$

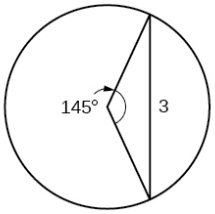
45. no triangle possible

47. one triangle, $A \approx 39.4^\circ, C \approx 47.6^\circ, BC \approx 20.7$.

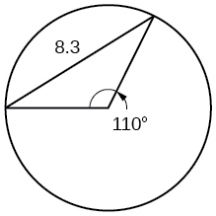
D: Extensions

Exercise 7.1e.D

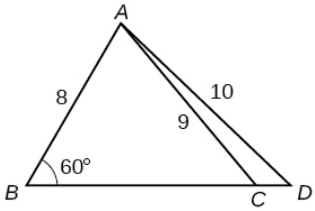
50) Find the radius of the circle in the Figure below. Round to the nearest tenth.



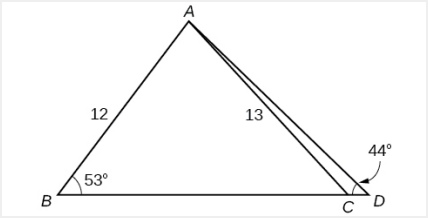
51) Find the diameter of the circle in the Figure below. Round to the nearest tenth.



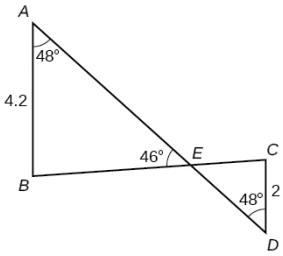
52) Find $m\angle ADC$ in the Figure below. Round to the nearest tenth.



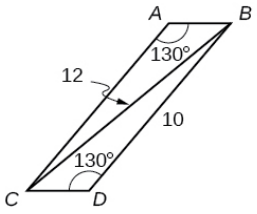
53) Find side AD in the Figure below. Round to the nearest tenth.



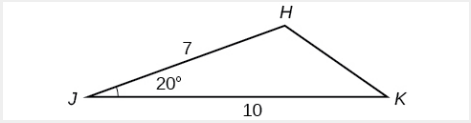
54) Solve both triangles in the Figure below. Round each answer to the nearest tenth.



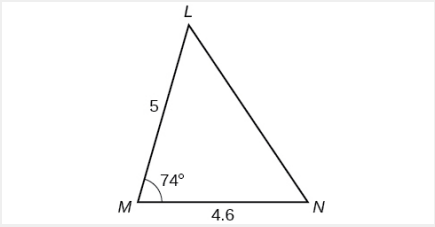
55) Find side AB in the parallelogram shown below. Round to the nearest tenth.



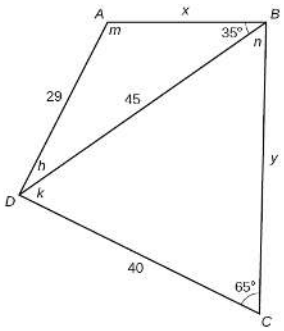
56) Solve the triangle in the Figure below. (**Hint:** Draw a perpendicular from H to JK. Round each answer to the nearest tenth.



57) Solve the triangle in the Figure below. (Hint: Draw a perpendicular from N to LM. Round each answer to the nearest tenth.



58) In the Figure to the right, $ABCD$ is not a parallelogram. $\angle m$ is obtuse. Solve both triangles. Round each answer to the nearest tenth.



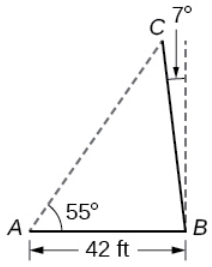
Answers to Odd Exercises:

51. 10.1 53. $AD \approx 13.8$ 55. $AB \approx 2.8$ 57. $L \approx 49.7°,\ N \approx 56.3°,\ LN \approx 5.8.$

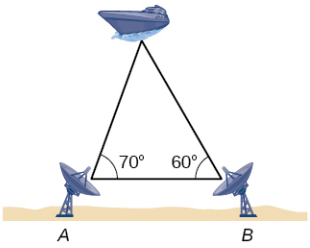
E: Real-World Applications

Exercise 7.1e.E

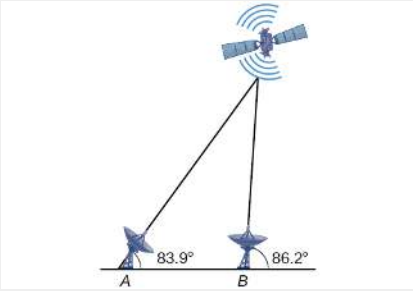
59) A pole leans away from the sun at an angle of $7°$ to the vertical, as shown in the Figure below. When the elevation of the sun is $55°$, the pole casts a shadow 42 feet long on the level ground. How long is the pole? Round the answer to the nearest tenth.



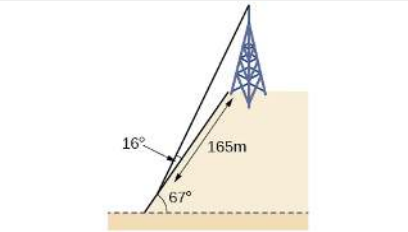
60) To determine how far a boat is from shore, two radar stations 500 feet apart find the angles out to the boat, as shown in the Figure below. Determine the distance of the boat from station A and the distance of the boat from shore. Round your answers to the nearest whole foot.



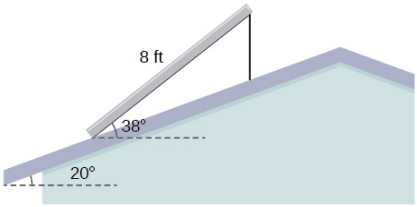
61) The Figure below shows a satellite orbiting Earth. The satellite passes directly over two tracking stations A and B, which are 69 miles apart. When the satellite is on one side of the two stations, the angles of elevation at A and B are measured to be $86.2°$ and $83.9°$ respectively. How far is the satellite from station A and how high is the satellite above the ground? Round answers to the nearest whole mile.



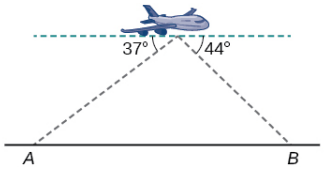
62) A communications tower is located at the top of a steep hill, as shown in the Figure below. The angle of inclination of the hill is $67°$. A guy wire is to be attached to the top of the tower and to the ground, 165 meters downhill from the base of the tower. The angle formed by the guy wire and the hill is $16°$. Find the length of the cable required for the guy wire to the nearest whole meter.



63) The roof of a house is at a $20°$ angle. An 8-foot solar panel is to be mounted on the roof and should be angled $38°$ relative to the horizontal for optimal results. (See the Figure below). How long does the vertical support holding up the back of the panel need to be? Round to the nearest tenth.

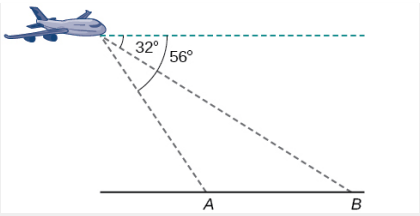


64) Similar to an angle of elevation, an *angle of depression* is the acute angle formed by a horizontal line and an observer's line of sight to an object below the horizontal. A pilot is flying over a straight highway. He determines the angles of depression to two mileposts, 6.6 km apart, to be $37°$ and $44°$ as shown in the Figure below. Find the distance of the plane from point A to the nearest tenth of a kilometer.



65) A pilot is flying over a straight highway. He determines the angles of depression to two mileposts, 4.3 km apart, to be $32°$ and $56°$, as shown in the Figure below. Find the distance of the plane from point A to the nearest tenth of a kilometer.

66) In order to estimate the height of a building, two students stand at a certain distance from the building at street level. From this point, they find the angle of elevation from the street to the top of the building to be $39°$. They then move 300 feet closer to the building and find the angle of elevation to be $50°$. Assuming that the street is level, estimate the height of the building to the nearest foot.



67) In order to estimate the height of a building, two students stand at a certain distance from the building at street level. From this point, they find the angle of elevation from the street to the top of the building to be $35°$. They then move 250 feet closer to the building and find the angle of elevation to be $53°$. Assuming that the street is level, estimate the height of the building to the nearest foot.

68) Points A and B are on opposite sides of a lake. Point C is 97 meters from A. The measure of $\angle BAC$ is determined to be $101°$, and the measure of $\angle ACB$ is determined to be $53°$. What is the distance from A to B, rounded to the nearest whole meter?

69) A man and a woman standing 3.5 miles apart spot a hot air balloon between them at the same time. If the angle of elevation from the man to the balloon is $27°$, and the angle of elevation from the woman to the balloon is $41°$, find the altitude of the balloon to the nearest tenth of a mile.

70) Two search teams spot a stranded climber between them on a mountain. The first search team is 0.5 miles from the second search team, and both teams are at an altitude of 1 mile. The angle of elevation from the first search team to the stranded climber is $15°$. The angle of elevation from the second search team to the climber is $22°$. What is the altitude of the climber? Round to the nearest hundredth of a mile.

71) A street light is mounted on a pole. A 6-foot-tall man is standing on the street a short distance from the pole, casting a shadow. The angle of elevation from the tip of the man's shadow to the top of his head of $28°$. A 6-foot-tall woman is standing on the same street on the opposite side of the pole from the man. The angle of elevation from the tip of her shadow to the top of her head is $28°$. If the man and woman are 20 feet apart, how far is the street light from the tip of the shadow of each person? Round the distance to the nearest tenth of a foot.

72) Three cities, A, B, and C, are located so that city A is due east of city B. If city C is located $35°$ west of north from city B and is 100 miles from city A and 70 miles from city B, how far is city A from city B? Round the distance to the nearest tenth of a mile.

Answers to Odd Exercises:

59. 51.4 ft

61. The distance from the satellite to station A is approximately 1716 miles. The satellite is approximately 1706 miles above the ground.

63. 2.6 ft 65. 5.6 km 67. 371 ft 69. 1.1 miles 71. 21.3 ft.

★

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7.2: Non-right Triangles - Law of Cosines

Learning Objectives

- Use the Law of Cosines to solve oblique triangles and applied problems.

Unfortunately, while the Law of Sines enables us to address many non-right triangle cases, it does not help us with triangles where the known angle is between two known sides, a SAS (side-angle-side) triangle, or when all three sides are known, but no angles are known, a SSS (side-side-side) triangle. In this section, we will investigate another tool for solving oblique triangles described by these last two cases.

The tool we need to solve these types of problems is the *Law of Cosines*, which defines the relationship among angle measurements and side lengths in oblique triangles. Three formulas make up the Law of Cosines. At first glance, the formulas may appear complicated because they include many variables. However, once the pattern is understood, the Law of Cosines is easier to work with than most formulas at this mathematical level.

Understanding how the Law of Cosines is derived will be helpful in using the formulas. The derivation begins with the Generalized Pythagorean Theorem, which is an extension of the Pythagorean Theorem to non-right triangles. Here is how it works: An arbitrary non-right triangle ABC is placed in the coordinate plane with vertex A at the origin, side c drawn along the x-axis, and vertex C located at some point (x, y) in the plane, as illustrated in Figure 7.2.2. Generally, triangles exist anywhere in the plane, but for this explanation we will place the triangle as noted.

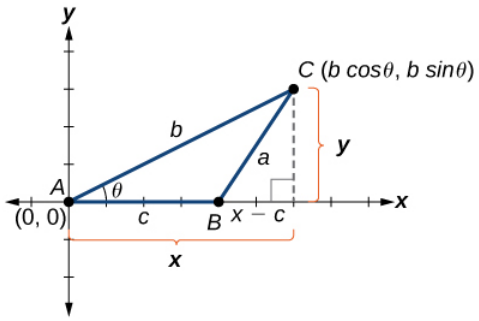


Figure 7.2.1: Triangle used to derive Law of Sines:

We can drop a perpendicular from C to the x-axis (this is the altitude or height). Recalling the basic trigonometric identities, we know that

$$\cos\theta = \frac{x\ (\text{adjacent})}{b\ (\text{hypotenuse})} \text{ and } \sin\theta = \frac{y\ (\text{opposite})}{b\ (\text{hypotenuse})}$$

In terms of θ, $x = b\cos\theta$ and $y = b\sin\theta$. The (x, y) point located at C has coordinates $(b\cos\theta, b\sin\theta)$. Using the side $(x - c)$ as one leg of a right triangle and y as the second leg, we can find the length of hypotenuse a using the Pythagorean Theorem. Thus,

$$\begin{aligned}
a^2 &= (x-c)^2 + y^2 \\
&= (b\cos\theta - c)^2 + (b\sin\theta)^2 && \text{Substitute } (b\cos\theta) \text{ for } x \text{ and } (b\sin\theta) \text{ for } y \\
&= (b^2\cos^2\theta - 2bc\cos\theta + c^2) + b^2\sin^2\theta && \text{Expand the perfect square.} \\
&= b^2\cos^2\theta + b^2\sin^2\theta + c^2 - 2bc\cos\theta && \text{Group terms noting that } \cos^2\theta + \sin^2\theta = 1 \\
&= b^2(\cos^2\theta + \sin^2\theta) + c^2 - 2bc\cos\theta && \text{Factor out } b^2 \\
a^2 &= b^2 + c^2 - 2bc\cos\theta
\end{aligned}$$

The formula derived is one of the three equations of the Law of Cosines. The other equations are found in a similar fashion.

Keep in mind that it is always helpful to sketch the triangle when solving for angles or sides. In a real-world scenario, try to draw a diagram of the situation. As more information emerges, the diagram may have to be altered. Make those alterations to the diagram and, in the end, the problem will be easier to solve.

The LAW OF COSINES

The Law of Cosines states that the square of any side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of the other two sides and the cosine of the included angle.

For triangles labeled as in the figure on the right, the Law of Cosines is given as three equations.

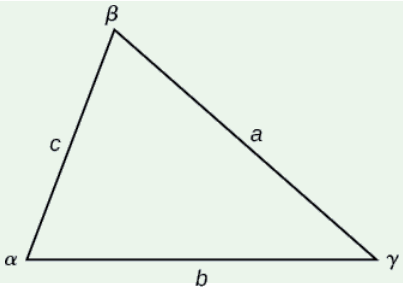
$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$
$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

To solve for a missing side measurement, the corresponding opposite angle measure is needed.

When solving for an angle, the corresponding opposite side measure is needed. We can use another version of the Law of Cosines to solve for an angle.

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc} \qquad \cos\beta = \frac{a^2 + c^2 - b^2}{2ac} \qquad \cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$$



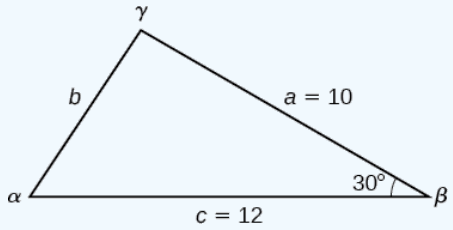
Use Law of Cosines to Solve SAS Triangles

How to: Given two sides and the angle between them (SAS), find the measures of the remaining side and angles of a triangle

1. Sketch the triangle. Identify the measures of the known sides and angles. Use variables to represent the measures of the unknown sides and angles.
2. Apply the Law of Cosines to find the length of the unknown side or angle.
3. Apply the Law of Sines or Cosines to find the measure of a second angle.
4. Compute the measure of the remaining angle.

Example 7.2.1: Solve an SAS Triangle

Find the unknown side and angles of the triangle below.



Solution

First, make note of what is given: two sides and the angle between them. This arrangement is classified as SAS and supplies the data needed to apply the Law of Cosines.

Each one of the three laws of cosines begins with the square of an unknown side opposite a known angle. For this example, the first side to solve for is side b, as we know the measurement of the opposite angle β.

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

$$b^2 = 10^2 + 12^2 - 2(10)(12)\cos(30°) \quad \text{Substitute the measurements for the known quantities.}$$

$$b^2 = 100 + 144 - 240\left(\frac{\sqrt{3}}{2}\right) \quad \text{Evaluate the cosine and begin to simplify.}$$

$$b = \sqrt{244 - 120\sqrt{3}} \approx 6.013 \quad \text{Use the square root property.}$$

Because we are solving for a length, we use only the positive square root. Now that we know the length b, we can use the Law of Sines to fill in the remaining angles of the triangle. Solving for angle α, we have

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b}$$

$$\frac{\sin\alpha}{10} = \frac{\sin(30°)}{6.013}$$

$$\sin\alpha = \frac{10\sin(30°)}{6.013} \quad \text{Multiply both sides of the equation by 10.}$$

$$\alpha = \sin^{-1}\left(\frac{10\sin(30°)}{6.013}\right) \approx 56.3° \quad \text{Find the inverse sine of } \frac{10\sin(30°)}{6.013}.$$

The other possibility for α would be $\alpha = 180° - 56.3° \approx 123.7°$. In the original diagram,α is adjacent to the longest side, so α is an acute angle and, therefore, $123.7°$ does not make sense. Notice that if we choose to apply the Law of Cosines, we arrive at a unique answer. We do not have to consider the other possibilities, as cosine is unique for angles between $0°$ and $180°$. Proceeding with $\alpha \approx 56.3°$, we can then find the third angle of the triangle.

$$\gamma = 180° - 30° - 56.3° \approx 93.7°$$

The complete set of angles and sides is:

$$\begin{array}{ll} \alpha = 56.3° & a = 10 \\ \beta = 30° & b \approx 6.013 \\ \gamma = 93.7° & c = 12 \end{array}$$

Try It 7.2.1

Find the missing side and angles of the given triangle: $\alpha = 30°$, $b = 12$, $c = 24$.

Answer

$a \approx 14.9$, $\beta \approx 23.8°$, $\gamma \approx 126.2°$.

Use Law of Cosines to Solve SSS Triangles

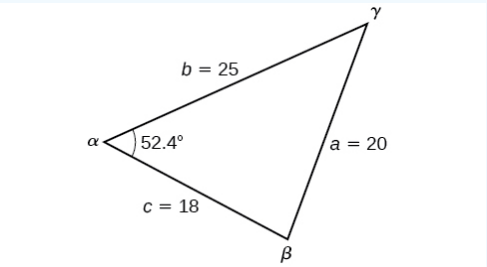
Example 7.2.2: Solve an SSS Triangle

Find the angle α for the given triangle if side $a = 20$, side $b = 25$, and side $c = 18$.

Solution

For this example, we have no angles. We can solve for any angle using the Law of Cosines. To solve for angle α, we have

$$\begin{array}{ll} a^2 = b^2 + c^2 - 2bc\cos\alpha & \\ 20^2 = 25^2 + 18^2 - 2(25)(18)\cos\alpha & \text{Substitute.} \\ 400 = 625 + 324 - 900\cos\alpha & \text{Simplify .} \\ 400 = 949 - 900\cos\alpha & \\ -549 = -900\cos\alpha & \text{Isolate } \cos\alpha. \\ 0.61 \approx \cos\alpha & \text{Find angle.} \\ \alpha \approx 52.4° & \end{array}$$



Analysis. Because the inverse cosine can return any angle between 0 and 180 degrees, there will not be any ambiguous cases using this method.

Try It 7.2.2

Given $a = 5$, $b = 7$, and $c = 10$, find the missing angles.

Answer

$\alpha \approx 27.7°$, $\beta \approx 40.5°$, $\gamma \approx 111.8°$

Solve Applied Problems Using the Law of Cosines

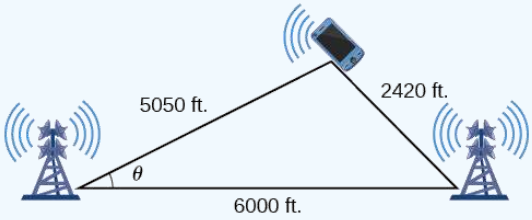
Just as the Law of Sines provided the appropriate equations to solve a number of applications, the Law of Cosines is applicable to situations in which the given data fits the cosine models. We may see these in the fields of navigation, surveying, astronomy, and geometry, just to name a few.

Example 7.2.3A: Using the Law of Cosines to Solve a Communication Problem

On many cell phones with GPS, an approximate location can be given before the GPS signal is received. This is accomplished through a process called triangulation, which works by using the distances from two known points. Suppose there are two cell phone towers within range of a cell phone. The two towers are located 6000 feet apart along a straight highway, running east to west, and the cell phone is north of the highway. Based on the signal delay, it can be determined that the signal is 5050 feet from the first tower and 2420 feet from the second tower. Determine the position of the cell phone north and east of the first tower, and determine how far it is from the highway.

Solution

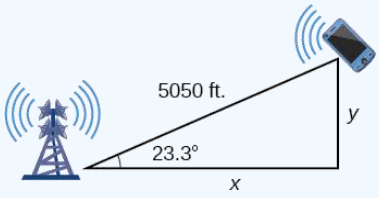
For simplicity, we start by drawing a diagram similar to the figure on the right and labeling our given information.



Using the Law of Cosines, we can solve for the angle θ. Remember that the Law of Cosines uses the square of one side to find the cosine of the opposite angle. For this example, let $a = 2420$, $b = 5050$, and $c = 6000$. Thus, θ corresponds to the opposite side $a = 2420$.

$$a^2 = b^2 + c^2 - 2bc\cos\theta$$
$$(2420)^2 = (5050)^2 + (6000)^2 - 2(5050)(6000)\cos\theta$$
$$\cos\theta \approx 0.9183$$
$$\theta \approx \cos^{-1}(0.9183) \approx 23.3°$$

To answer the questions about the phone's position north and east of the tower, and the distance to the highway, drop a perpendicular from the position of the cell phone, illustrated on the right. This forms two right triangles, although we only need the right triangle that includes the first tower for this problem.



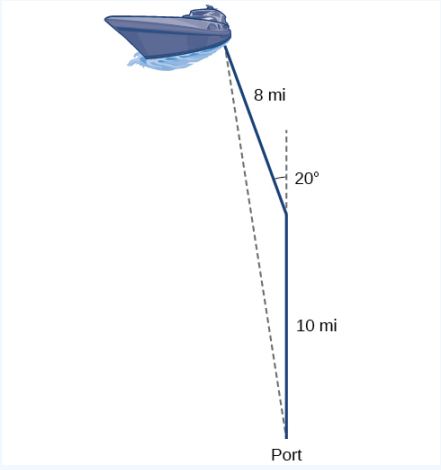
Using the angle $\theta = 23.3°$ and the basic trigonometric identities, we can find the solutions. Thus

$$\cos(23.3°) = \frac{x}{5050}$$
$$x = 5050\cos(23.3°) \approx 4638.15 \text{ feet}$$
$$\sin(23.3°) = \frac{y}{5050}$$
$$y = 5050\sin(23.3°) \approx 1997.5 \text{ feet}$$

The cell phone is approximately 4638 feet east and 1998 feet north of the first tower, and 1998 feet from the highway.

Example 7.2.3B: Calculating Distance Traveled Using a SAS Triangle

Suppose a boat leaves port, travels 10 miles, turns 20 degrees, and travels another 8 miles. How far from port is the boat? The diagram is given at the right.



Solution

The boat turned 20 degrees, so the obtuse angle of the non-right triangle is the supplemental angle, $180° - 20° = 160°$. With this, we can utilize the Law of Cosines to find the missing side of the obtuse triangle—the distance of the boat to the port.

$$x^2 = 8^2 + 10^2 - 2(8)(10)\cos(160°)$$
$$x^2 = 314.35$$
$$x = \sqrt{314.35} \approx 17.7 \text{ miles}$$

The boat is about **17.7** miles from port.

Key Equations

Law of Cosines	$a^2 = b^2 + c^2 - 2bc\cos\alpha$	$b^2 = a^2 + c^2 - 2ac\cos\beta$	$c^2 = a^2 + b^2 - 2ab\cos\gamma$
Heron's formula	$Area = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{(a+b+c)}{2}$		

Key Concepts

- The Law of Cosines defines the relationship among angle measurements and lengths of sides in oblique triangles.
- The Generalized Pythagorean Theorem is the Law of Cosines for two cases of oblique triangles: SAS and SSS.
- The Law of Cosines is useful for many types of applied problems.
- Heron's formula allows the calculation of area in oblique triangles if all three sides are known.

Contributors

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7.2e: Exercises - Law of Cosines

For all exercises below, assume α or A is the angle opposite side a, β or B is the angle opposite side b, and γ or C is the angle opposite side c.

A: Concepts

Exercise 7.2e.A

1) If you are looking for a missing side of a triangle, what do you need to know when using the Law of Cosines?

2) If you are looking for a missing angle of a triangle, what do you need to know when using the Law of Cosines?

3) Explain what s represents in Heron's formula.

4) Explain the relationship between the Pythagorean Theorem and the Law of Cosines.

5) When must you use the Law of Cosines instead of the Pythagorean Theorem?

Answers to odd exercises.

1. two sides and the angle opposite the missing side.
3. s is the semi-perimeter, which is half the perimeter of the triangle.
5. The Law of Cosines must be used for any oblique (non-right) triangle.

B: Find a side or an angle

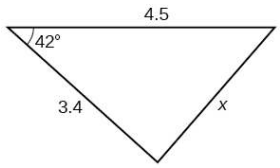
Exercise 7.2e.A

★ **If possible, solve each triangle for the unknown side. Round to the nearest tenth.**

7) $\alpha = 120°, b = 6, c = 7$
8) $\alpha = 115°, b = 18, c = 23$
9) $\gamma = 41.2°, a = 2.49, b = 3.13$
10) $\beta = 58.7°, a = 10.6, c = 15.7$
11) $\gamma = 113°, b = 10, c = 32$

12) $\alpha = 119°, a = 26, b = 14$
13) $\alpha = 43.1°, a = 184.2, b = 242.8$
14) $\alpha = 36°, a = 22, b = 26$
15) $\beta = 50°, a = 105, b = 45$
16) $\beta = 67°, a = 49, b = 38$

17)



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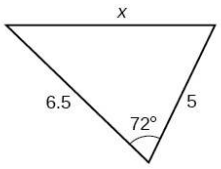

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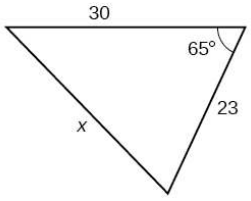

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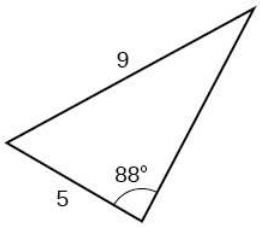

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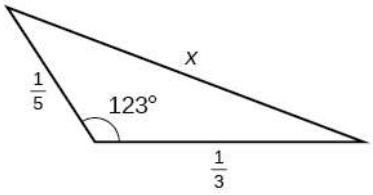

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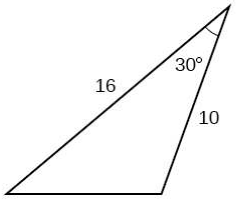

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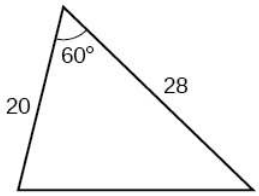

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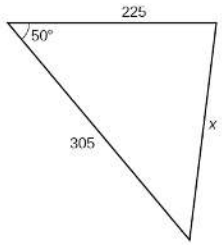

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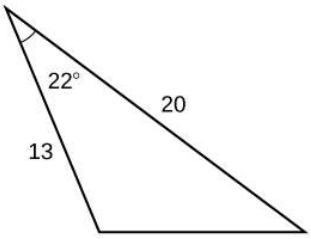

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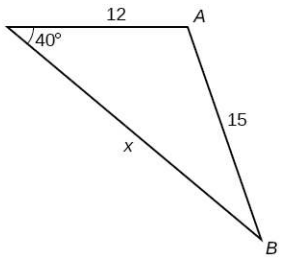


Answers to odd exercises.

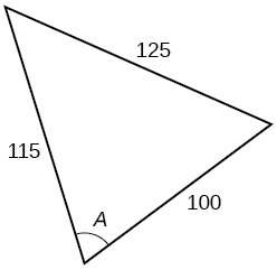
7. **11.3** 9. **2.1** 11. **26.7** 13. **257.3** or **97.2** 15. not possible.
17. **3.0** 19. **29.1** 21. **0.5** 23. **25.0** 25. **9.3**

★ Use the Law of Cosines to solve for the missing angle of the oblique triangle. Round to the nearest tenth.

32) $a = 42, b = 19, c = 30$; find angle A.
33) $a = 14, b = 13, c = 20$; find angle C.
34) $a = 16, b = 31, c = 20$; find angle B.

35) $a = 13, b = 22, c = 28$; find angle A.
36) $a = 108, b = 132, c = 160$; find angle C.

37)



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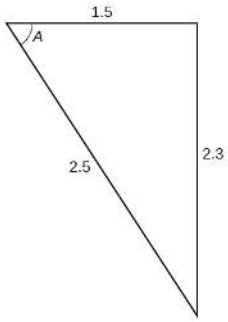

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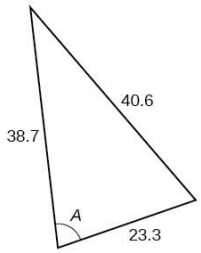

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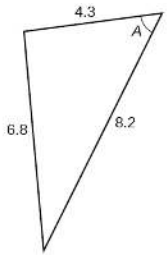


Answers to odd exercises.

33. $95.5°$ 35. $26.9°$ 37. $70.7°$ 39. $77.4°$

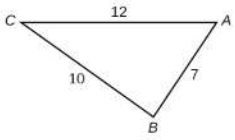
C: Solve a Triangle

Exercise 7.2e.B

★ **Solve the triangle. Round to the nearest tenth.**

41. $B = 88°, a = 4.4, c = 5.2$
42. $C = 121°, a = 21, b = 37$
43. $A = 35°, b = 8, c = 11$
44. $B = 43°, a = 48, c = 32$

47.



48) $a = 13, b = 11, c = 15$
49) $a = 3.1, b = 3.5, c = 5$
50) $a = 25, b = 51, c = 29$

Answers to odd exercises.

41. $A \approx 41.1°, C \approx 50.9°, b \approx 6.7$
43. $B \approx 45.9°, C \approx 99.1°, a \approx 6.4$
47. $A \approx 56.4°, B \approx 88.0°, C \approx 35.7$
49. $A \approx 37.8°, B \approx 43.8°, C \approx 98.4$

D: Extensions

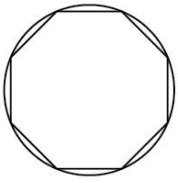
Exercise 7.2e.D

52) A parallelogram has sides of length 16 units and 10 units. The shorter diagonal is 12 units. Find the measure of the longer diagonal.

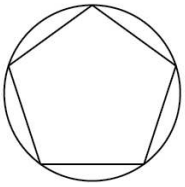
53) The sides of a parallelogram are 11 feet and 17 feet. The longer diagonal is 22 feet. Find the length of the shorter diagonal.

54) The sides of a parallelogram are 28 centimeters and 40 centimeters. The measure of the larger angle is $100°$. Find the length of the shorter diagonal.

55) A regular octagon is inscribed in a circle with a radius of 8 inches. (See Figure below.) Find the perimeter of the octagon.



56) A regular pentagon is inscribed in a circle of radius 12 cm. (See Figure below.) Find the perimeter of the pentagon. Round to the nearest tenth of a centimeter.



★ **Suppose that $x^2 = 25 + 36 - 60\cos(52)$ represents the relationship of three sides of a triangle and the cosine of an angle.**

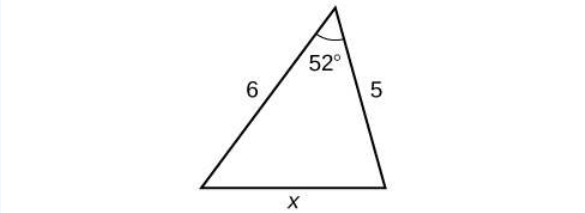
57) Draw the triangle.

58) Find the length of the third side

Answers to odd exercises.

53. **18.3** 55. **48.98**

57.



E: Real-World Applications

Exercise 7.2e.E

62) A surveyor has taken the measurements shown in the Figure below. Find the distance across the lake. Round answers to the nearest tenth.

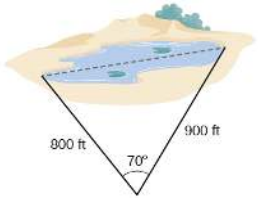
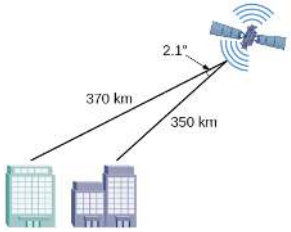


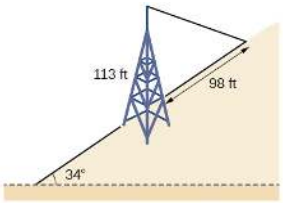
Figure 62.

63) A satellite calculates the distances and angle shown in the Figure below (not to scale). Find the distance between the two cities. Round answers to the nearest tenth.



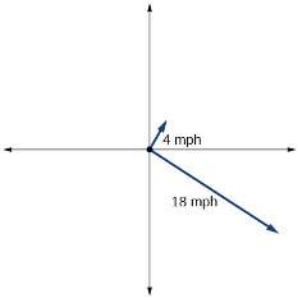
64) An airplane flies **220** miles with a heading of **40°**, and then flies **180** miles with a heading of **170°**. How far is the plane from its starting point, and at what heading? Round answers to the nearest tenth.

65) A **113**-foot tower is located on a hill that is inclined **34°** to the horizontal, as shown in the Figure below. A guy-wire is to be attached to the top of the tower and anchored at a point **98** feet uphill from the base of the tower. Find the length of wire needed.



66) Two ships left a port at the same time. One ship traveled at a speed of **18** miles per hour at a heading of **320°**. The other ship traveled at a speed of **22** miles per hour at a heading of **194°**. Find the distance between the two ships after **10** hours of travel.

67) The graph in the Figure below represents two boats departing at the same time from the same dock. The first boat is traveling at **18** miles per hour at a heading of **327°** and the second boat is traveling at **4** miles per hour at a heading of **60°**. Find the distance between the two boats after **2** hours.



68) A triangular swimming pool measures **40** feet on one side and **65** feet on another side. These sides form an angle that measures **50°**. How long is the third side (to the nearest tenth)?

69) A pilot flies in a straight path for 1 hour 30 min. She then makes a course correction, heading $10°$ to the right of her original course, and flies 2 hours in the new direction. If she maintains a constant speed of 680 miles per hour, how far is she from her starting position?

70) Los Angeles is $1,744$ miles from Chicago, Chicago is 714 miles from New York, and New York is $2,451$ miles from Los Angeles. Draw a triangle connecting these three cities, and find the angles in the triangle.

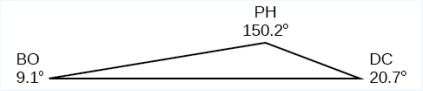
71) Philadelphia is 140 miles from Washington, D.C., Washington, D.C. is 442 miles from Boston, and Boston is 315 miles from Philadelphia. Draw a triangle connecting these three cities and find the angles in the triangle.

72) Two planes leave the same airport at the same time. One flies at $20°$ east of north at 500 miles per hour. The second flies at $30°$ east of south at 600 miles per hour. How far apart are the planes after 2 hours?

73) Two airplanes take off in different directions. One travels 300 mph due west and the other travels $25°$ west of north at 420 mph. After 90 minutes, how far apart are they, assuming they are flying at the same altitude?

74) A parallelogram has sides of length 15.4 units and 9.8 units. Its area is 72.9 square units. Find the measure of the longer diagonal.

Answers to odd exercises.

63. 24.0 km 65. 99.9 ft 67. 37.3 miles 69. 2371 miles 71. 

73. 599.8 miles .

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7.3: Vectors in 2D

Learning Objectives

- Perform basic vector operations (scalar multiplication, addition, subtraction).
- Express a vector in component form.
- Evaluate the magnitude and direction angle of a vector.
- Express a vector in terms of unit vectors.

When describing the movement of an airplane in flight, it is important to communicate two pieces of information: the direction in which the plane is traveling and the plane's speed. When measuring a force, such as the thrust of the plane's engines, it is important to describe not only the strength of that force, but also the direction in which it is applied. Some quantities, such as velocity or force, are defined in terms of both size (also called magnitude) and direction. A quantity that has magnitude and direction is called a **vector**. In this text, we denote vectors by boldface letters, such as $\mathbf{v}$. Many times, we will also include an arrow or harpoon above the boldface letter, giving us $\vec{v}$ or $\vec{\mathbf{v}}$. This is how we will write vectors by hand, since it's hard to write in boldface.

Definition: Vector

A **vector** is a quantity that has both magnitude and direction.

Vector Representation

Geometric Perspective

A vector in a plane is represented by a directed line segment (an arrow). The endpoints of the segment are called the **initial point** and the **terminal point** of the vector. An arrow from the initial point to the terminal point indicates the direction of the vector. The length of the line segment represents its **magnitude**. We use the notation $\|\vec{\mathbf{v}}\|$ to denote the magnitude of the vector $\vec{\mathbf{v}}$.

Definition: Zero vector

A vector with an initial point and terminal point that are the same is called the **zero vector**, denoted $\vec{\mathbf{0}}$. The zero vector is the only vector without a direction, and by convention can be considered to have any direction convenient to the problem at hand.

Vectors with the same magnitude and direction are called equivalent vectors. We treat equivalent vectors as equal, even if they have different initial points. Thus, if $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ are equivalent, we write

$$\vec{\mathbf{v}} = \vec{\mathbf{w}}.$$

Definition I: equivalent vectors

Vectors are said to be ***equivalent vectors*** if they have the same magnitude and direction.

The vectors in Figure **7.3.0** are equivalent. Each vector has the same length and direction. A closely related concept is the idea of **parallel vectors**. Two vectors are said to be parallel if they have the same or opposite directions. We explore this idea in more detail later in the chapter. A vector is defined by its magnitude and direction, regardless of where its initial point is located.

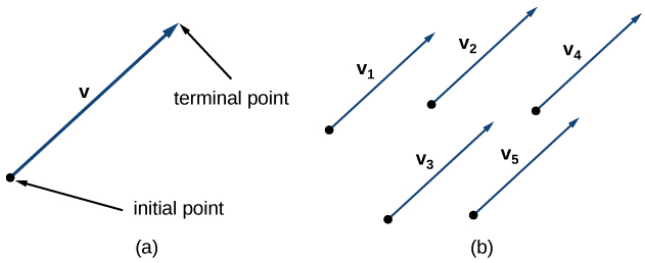


Figure **7.3.0**: (a) A vector is represented by a directed line segment from its initial point to its terminal point. (b) Vectors $\vec{\mathbf{v}}_1$ through $\vec{\mathbf{v}}_5$ *are equivalent.*

The use of boldface, lowercase letters to name vectors is a common representation in print, but there are alternative notations. When writing the name of a vector by hand, for example, it is easier to sketch an arrow over the variable to show it is a vector: $\vec{v}$. When a vector has initial point P and terminal point Q, the notation $\overrightarrow{PQ}$ is useful because it indicates the direction and location of the vector.

Example 7.3.1: Sketch a Vector

Sketch a vector in the plane from initial point $P(1,1)$ to terminal point $Q(8,5)$.

Solution

See Figure 7.3.1. Because the vector goes from point P to point Q, we name it $\overrightarrow{PQ}$.

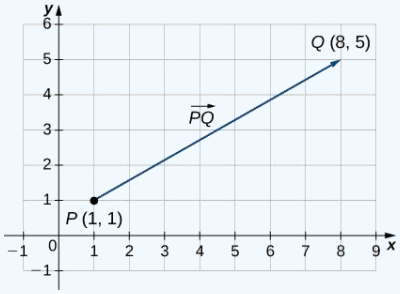


Figure 7.3.1: The vector with initial point $(1,1)$ and terminal point $(8,5)$ is named $\overrightarrow{PQ}$.

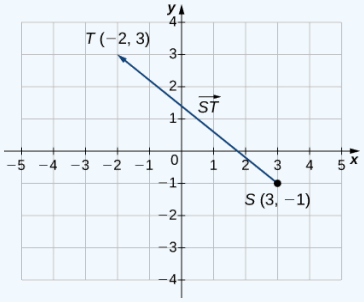
Try It 7.3.1

Sketch the vector $\overrightarrow{ST}$ where S is point $(3,-1)$ and T is point $(-2,3)$.

Hint

The first point listed in the name of the vector is the initial point of the vector.

Answer



Working with vectors in a plane is easier when we are working in a coordinate system. When the initial points and terminal points of vectors are given in Cartesian coordinates, computations become straightforward.

Example 7.3.2: Compare Vectors Graphically

Are $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ equivalent vectors?

a. $\vec{\mathbf{v}}$ has initial point $(3,2)$ and terminal point $(7,2)$, $\vec{\mathbf{w}}$ has initial point $(1,-4)$ and terminal point $(1,0)$.

b. $\vec{\mathbf{v}}$ has initial point $(0,0)$ and terminal point $(1,1)$, $\vec{\mathbf{w}}$ has initial point $(-2,2)$ and terminal point $(-1,3)$.

Solution

a. The vectors are each 4 units long, but they are oriented in different directions. So $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ are not equivalent (Figure 7.3.2a).

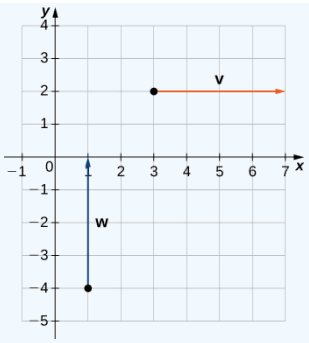


Figure **7.3.2***a*: These vectors are not equivalent.

b. Based on Figure **7.3.2***b*, and using a bit of geometry, it is clear these vectors have the same length and the same direction, so $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ are equivalent.

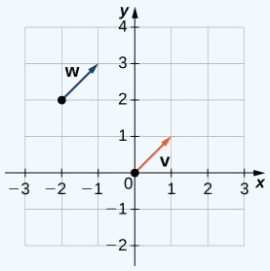
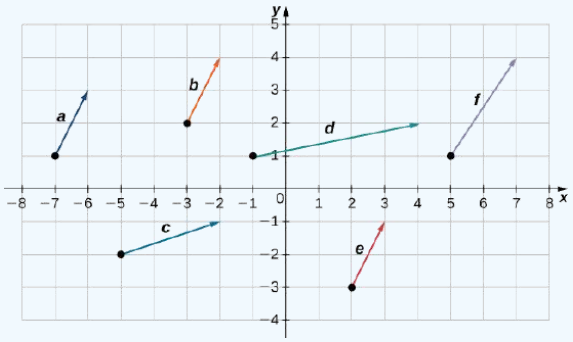


Figure **7.3.2***b*: These vectors are equivalent.

Try It 7.3.2

Which of the following vectors are equivalent?



Hint

Equivalent vectors have both the same magnitude and the same direction.

Answer

Vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{e}}$ are equivalent.

Algebraic Perspective

Vector Components

Since the location of the initial point of a vector is irrelevant when determining equivalent vectors, knowing the location of the initial and terminal point of a vector is unnecessary. Consequently, working with vectors in a plane can be further simplified by writing them in component form.

Definition: Component Form

Irregardless of the location of its initial point, a vector can be written in **component form** as

$$\vec{\mathbf{v}} = \langle x, y \rangle.$$

The scalars x and y are called the **components of the vector** $\vec{\mathbf{v}}$. These scalars are measures of the horizontal and vertical distances from the initial point to the terminal point of a vector. The horizontal distance is x and the vertical distance is y.

We have seen how to plot a vector when we are given an initial point and a terminal point. However, because a vector can be placed anywhere in a plane, it may be easier to perform calculations with a vector when its initial point coincides with the origin. We call a vector with its initial point at the origin a **standard-position vector.**

Definition: Standard Position Vector

The **standard-position vector** is the vector with its initial point at the origin. This vector can be written in component form as

$$\vec{\mathbf{v}} = \langle x, y \rangle.$$

Because its initial point is $(0, 0)$, the terminal point is of the standard position vector is (x, y) and the scalars x and y are the **components** of $\vec{\mathbf{v}}$.

When writing the component form of a vector, it is important to distinguish between $\langle x, y \rangle$ and (x, y). The first ordered pair uses angle brackets to describe a vector, whereas the second uses parentheses to describe a point in a plane.

When we have a vector not already in standard position, we can determine its component form algebraically using the coordinates of the initial point and the terminal point. To find it algebraically, we subtract the x-coordinate of the initial point from the x-coordinate of the terminal point to get the x-component, and we subtract the y-coordinate of the initial point from the y-coordinate of the terminal point to get the y-component.

Howto: Given an initial and terminal point, find component form.

Let $\vec{\mathbf{v}}$ be a vector with initial point $P(x_i, y_i)$ and terminal point $Q(x_t, y_t)$. Then we can express $\vec{\mathbf{v}}$ in component form as

$$\vec{\mathbf{v}} = \langle x_t - x_i, y_t - y_i \rangle.$$

Because the vector goes from point P to point Q, it is named $\overrightarrow{PQ}$.

When we have a vector not already in standard position, we can determine its component form in one of two ways. We can use a geometric approach, in which we sketch the vector in the coordinate plane and then sketch an equivalent standard-position vector, or, we can find it algebraically.

Example 7.3.3: Express a Vector in Component Form Given its Initial and Terminal Points

Express vector $\vec{\mathbf{v}}$ with initial point $(-3, 4)$ and terminal point $(1, 2)$ in component form.

Solution:

a. Geometric Approach

1. Sketch the vector in the coordinate plane (Figure 7.3.3*a*).
2. The terminal point is 4 units to the right and 2 units down from the initial point.
3. Find the point that is 4 units to the right and 2 units down from the origin.
4. In standard position, this vector has initial point $(0, 0)$ and terminal point $(4, -2)$. Therefore, its component form is $\vec{\mathbf{v}} = \langle 4, -2 \rangle$.

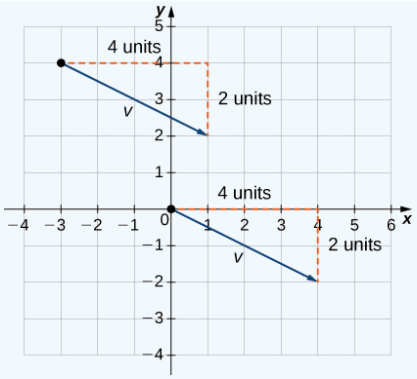


Figure **7.3.3a**: These vectors are equivalent.

b. Algebraic Approach

In the first solution, we used a sketch of the vector to see that the terminal point lies 4 units to the right. We can accomplish this algebraically by finding the difference of the x-coordinates:

$$x_t - x_i = 1 - (-3) = 4.$$

Similarly, the difference of the y-coordinates shows the vertical length of the vector.

$$y_t - y_i = 2 - 4 = -2.$$

So, in component form, $\vec{\mathbf{v}} = \langle x_t - x_i, y_t - y_i \rangle = \langle 1 - (-3), 2 - 4 \rangle = \langle 4, -2 \rangle$.

Try It 7.3.3

Vector $\vec{\mathbf{w}}$ has initial point $(-4, -5)$ and terminal point $(-1, 2)$. Express $\vec{\mathbf{w}}$ in component form.

Hint

You may use either geometric or algebraic method.

Answer

$\langle 3, 7 \rangle$

Earlier in this section we stated that two vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are considered equal if they have the same magnitude and the same direction. Another (and much easier) way to show that two vectors are equal is to show that both vectors have the same components.

Definition II: equivalent vectors

Vectors are **equal** if they have the same components.

Example 7.3.4: Show Two Vectors Are Equal

Given vector $\vec{\mathbf{v}}$ with initial point at $(5, -3)$ and terminal point at $(-1, 2)$ and vector $\vec{\mathbf{u}}$ with initial point at $(-1, -3)$ and terminal point at $(-7, 2)$,

a. Show the vectors are equal

b. Draw on the same grid: $\vec{\mathbf{v}}$ and $\vec{\mathbf{u}}$ and their position vector.

Solution

a. The vectors are equal if they have the same components.

$$\begin{aligned} \vec{\mathbf{v}} &= \langle -1 - 5, 2 - (-3) \rangle &&= \langle -6, 5 \rangle \\ \vec{\mathbf{u}} &= \langle -7 - (-1), 2 - (-3) \rangle &&= \langle -6, 5 \rangle \end{aligned}$$

Since the components of both vectors are the same, $\vec{v}$ and $\vec{u}$ are the same. Furthermore, their position vector is the same.

b. As shown in Figure 7.3.4, draw the vector $\vec{v}$ starting at initial $(5,-3)$ and terminal point $(-1,2)$. Draw the vector $\vec{u}$ with initial point $(-1,-3)$ and terminal point $(-7,2)$. The position vector $\langle -6,5\rangle$ is the same for both vectors. Furthermore, notice the coordinates of the terminal point of the position vector $(-6,\ 5)$, are the same as its components.

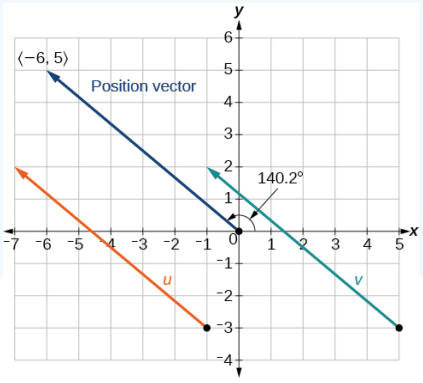


Figure 7.3.4

Combining Vectors

Vectors have many real-life applications, including situations involving force or velocity. For example, consider the forces acting on a boat crossing a river. The boat's motor generates a force in one direction, and the current of the river generates a force in another direction. Both forces are vectors. We must take both the magnitude and direction of each force into account if we want to know where the boat will go.

A second example that involves vectors is a quarterback throwing a football. The quarterback does not throw the ball parallel to the ground; instead, he aims up into the air. The velocity of his throw can be represented by a vector. If we know how hard he throws the ball (magnitude—in this case, speed), and the angle (direction), we can tell how far the ball will travel down the field.

Geometric Approach

Scalar Multiplication

A real number is often called a **scalar** in mathematics and physics. Unlike vectors, scalars are generally considered to have a magnitude only, but no direction. Multiplying a vector by a scalar changes the vector's magnitude. This is called scalar multiplication. Note that changing the magnitude of a vector does not indicate a change in its direction. For example, wind blowing from north to south might increase or decrease in speed while maintaining its direction from north to south.

Definition: Scalar Multiplication

The product $k\vec{v}$ of a vector $\vec{v}$ and a scalar k is a vector with a magnitude that is $|k|$ times the magnitude of $\vec{v}$, and with a direction that is the same as the direction of $\vec{v}$ if $k>0$, and opposite the direction of $\vec{v}$ if $k<0$. This is called *scalar multiplication*. If $k=0$ or $\vec{v}=\vec{0}$, then $k\vec{v}=\vec{0}$.

As you might expect, if $k=-1$, we denote the product $k\vec{v}$ as

$$k\vec{v}=(-1)\vec{v}=-\vec{v}.$$

Note that $-\vec{v}$ has the same magnitude as $\vec{v}$, but has the opposite direction (Figure 7.3.5.01).

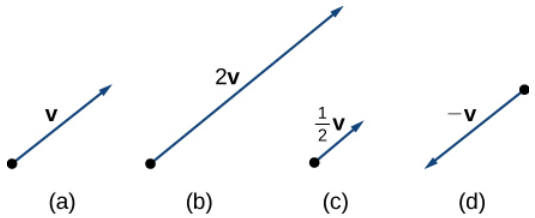


Figure 7.3.5.01: (a) The original vector $\vec{v}$ has length n units. (b) The length of $2\vec{v}$ equals $2n$ units. (c) The length of $\vec{v}/2$ is $n/2$ units. (d) The vectors $\vec{v}$ and $-\vec{v}$ have the same length but opposite directions.

Vector Addition

Another operation we can perform on vectors is to add them together in vector addition, but because each vector may have its own direction, the process is different from adding two numbers. The most common graphical method for adding two vectors is to place the initial point of the second vector at the terminal point of the first, as in Figure 7.3.5.02. To see why this makes sense, suppose, for example, that both vectors represent displacement. If an object moves first from the initial point to the terminal point of vector $\vec{v}$, then from the initial point to the terminal point of vector $\vec{w}$, the overall displacement is the same as if the object had made just

one movement from the initial point to the terminal point of the vector $\vec{v}+\vec{w}$. For obvious reasons, this approach is called the **triangle method**. Notice that if we had switched the order, so that $\vec{w}$ was our first vector and $\vec{v}$ was our second vector, we would have ended up in the same place. (Again, see Figure 7.3.5.02.) Thus,

$$\vec{v}+\vec{w}=\vec{w}+\vec{v}.$$

A second method for adding vectors is called the **parallelogram method**. With this method, we place the two vectors so they have the same initial point, and then we draw a parallelogram with the vectors as two adjacent sides, as in Figure 7.3.5.02. The length of the diagonal of the parallelogram is the sum. Comparing Figure 7.3.5.02 and Figure 7.3.5.02, we can see that we get the same answer using either method. The vector $\vec{v}+\vec{w}$ is called the **vector sum.**

Definition: Vector Addition

The sum of two vectors $\vec{v}$ and $\vec{w}$ can be constructed graphically by placing the initial point of $\vec{w}$ at the terminal point of $\vec{v}$. Then, the vector sum, $\vec{v}+\vec{w}$, is the vector with an initial point that coincides with the initial point of $\vec{v}$ and has a terminal point that coincides with the terminal point of $\vec{w}$. This operation is known as *vector addition*.

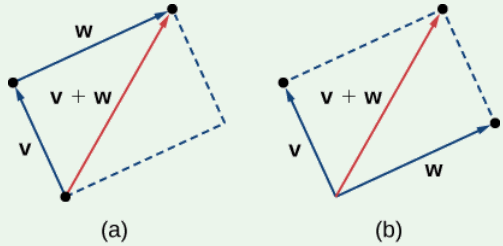


Figure 7.3.5.02: (a) When adding vectors by the triangle method, the initial point of $\vec{w}$ is the terminal point of $\vec{v}$. (b) When adding vectors by the parallelogram method, the vectors $\vec{v}$ and $\vec{w}$ have the same initial point.

Vector Subtraction

It is also appropriate here to discuss vector subtraction. We define $\vec{v}-\vec{w}$ as $\vec{v}+(-\vec{w})=\vec{v}+(-1)\vec{w}$. The vector $\vec{v}-\vec{w}$ is called the **vector difference**. Graphically, the vector $\vec{v}-\vec{w}$ is depicted by drawing a vector from the terminal point of $\vec{w}$ to the terminal point of $\vec{v}$ (Figure 7.3.5.03).

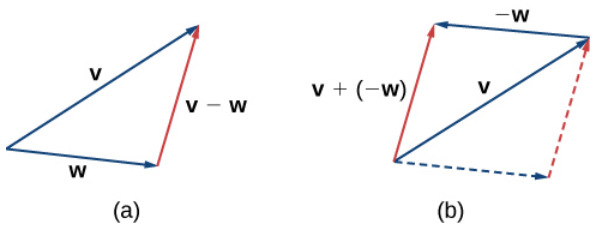


Figure 7.3.5.03: (a) The vector difference $\vec{v}-\vec{w}$ is depicted by drawing a vector from the terminal point of $\vec{w}$ to the terminal point of $\vec{v}$. (b) The vector $\vec{v}-\vec{w}$ is equivalent to the vector $\vec{v}+(-\vec{w})$.

Example 7.3.5: Perform Vector Operations Graphically

Given the vectors v and w shown in Figure 7.3.5, sketch the vectors

a. $3\vec{w}$
b. $\vec{v}+\vec{w}$
c. $2\vec{v}-\vec{w}$

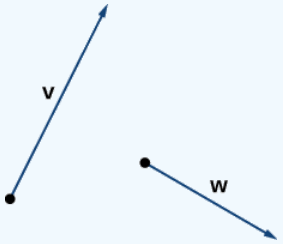
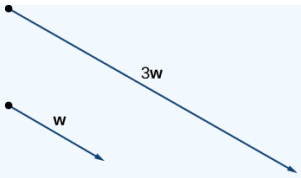


Figure 7.3.5: Vectors $\vec{v}$ *and* $\vec{w}$ *lie in the same plane.*

Solution

a. The vector $3\vec{w}$ has the same direction as $\vec{w}$;
it is three times as long as $\vec{w}$.



b. Use either addition method to find $\vec{v} + \vec{w}$.

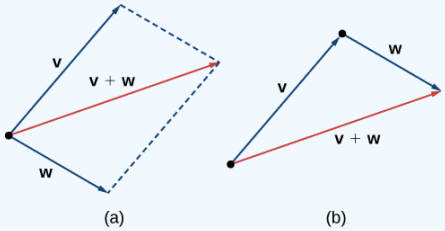


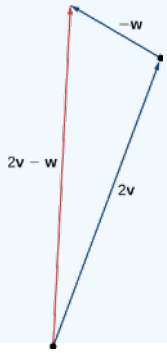
Figure 7.3.5*b*: To find $\vec{v} + \vec{w}$, align the vectors at their initial points or place the initial point of one vector at the terminal point of the other. (a) The vector$\vec{v} + \vec{w}$ is the diagonal of the parallelogram with sides $\vec{v}$ and $\vec{w}$. (b) The vector $\vec{v} + \vec{w}$ is the third side of a triangle formed with $\vec{w}$ placed at the terminal point of $\vec{v}$.

c. To find $2\vec{v} - \vec{w}$,

First rewrite the expression as $2\vec{v} + (-\vec{w})$.

Then draw the vector $-\vec{w}$,

Then add it to the vector $2\vec{v}$.



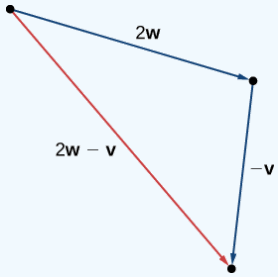
Try It 7.3.5

Using vectors $\vec{v}$ and $\vec{w}$ from Example 7.3.5, sketch the vector $2\vec{w} - \vec{v}$.

Hint

First sketch vectors $2\vec{w}$ and $-\vec{v}$.

Answer



Algebraic Approach

We have defined scalar multiplication and vector addition geometrically. Expressing vectors in component form allows us to perform these same operations algebraically.

Definition: Scalar multiplication and Vector addition

Let $\vec{v} = \langle x_1, y_1 \rangle$ and $\vec{w} = \langle x_2, y_2 \rangle$ be vectors, and let k be a scalar.

- *Scalar multiplication:* $k\vec{v} = \langle kx_1, ky_1 \rangle$
- *Vector addition:* $\vec{v} + \vec{w} = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle = \langle x_1 + x_2, y_1 + y_2 \rangle$

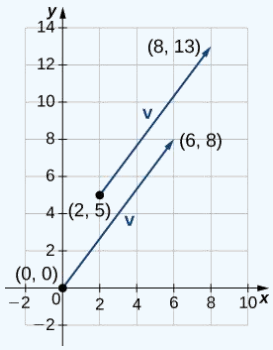
Example 7.3.6: Perform Vector Operations Algebraically

Let $\vec{v}$ be the vector with initial point $(2, 5)$ and terminal point $(8, 13)$, and let $\vec{w} = \langle -2, 4 \rangle$.

a. Express $\vec{v}$ in component form. Then, using algebra, find
b. $\vec{v} + \vec{w}$,
c. $3\vec{v}$, and
d. $\vec{v} - 2\vec{w}$.

Solution

a. To place the initial point of $\vec{v}$ at the origin, we must translate the vector 2 units to the left and 5 units down (see illustration ton the right). Using the algebraic method, we can express $\vec{v}$ as $\vec{v} = \langle 8 - 2, 13 - 5 \rangle = \langle 6, 8 \rangle$.



b. To find $\vec{v} + \vec{w}$, add the x-components and the y-components separately:

$$\vec{v} + \vec{w} = \langle 6, 8 \rangle + \langle -2, 4 \rangle = \langle 4, 12 \rangle.$$

c. To find $3\vec{v}$, multiply $\vec{v}$ by the scalar $k = 3$:

$$3\vec{v} = 3 \cdot \langle 6, 8 \rangle = \langle 3 \cdot 6, 3 \cdot 8 \rangle = \langle 18, 24 \rangle.$$

d. To find $\vec{v} - 2\vec{w}$, find $-2\vec{w}$ and add it to $\vec{v}$:

$$\vec{v} - 2\vec{w} = \langle 6, 8 \rangle - 2 \cdot \langle -2, 4 \rangle = \langle 6, 8 \rangle + \langle 4, -8 \rangle = \langle 10, 0 \rangle.$$

Try It 7.3.6

Let $\vec{a} = \langle 7, 1 \rangle$ and let $\vec{b}$ be the vector with initial point $(3, 2)$ and terminal point $(-1, -1)$.

a. Express $\vec{b}$ in component form.
b. Find $3\vec{a} - 4\vec{b}$.

Hint	**Answer a**	**Answer b**
To find $3\vec{a} - 4\vec{b}$, start by finding the scalar multiples $3\vec{a}$ and $-4\vec{b}$.	$\vec{b} = \langle -4, -3 \rangle$	$3\vec{a} - 4\vec{b} = \langle 37, 15 \rangle$

Vector Magnitude and Direction

To find the magnitude of a vector, we calculate the distance between its initial point and its terminal point. The **magnitude** of vector $\vec{v} = \langle x, y \rangle$ is denoted $\|\vec{v}\|$, or $|\vec{v}|$, and can be computed using the formula

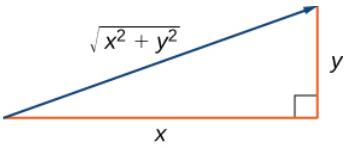
$$\|\vec{v}\| = \sqrt{x^2 + y^2}.$$

Note that because this vector is written in component form, it is equivalent to a vector in standard position, with its initial point at the origin and terminal point (x, y). Thus, it suffices to calculate the magnitude of the vector in standard position. Using the distance formula to calculate the distance between initial point $(0, 0)$ and terminal point (x, y), we have

$$\|\vec{v}\| = \sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{x^2 + y^2}.$$

Based on this formula, it is clear that for any vector $\vec{\mathbf{v}}$, $\|\vec{\mathbf{v}}\| \geq 0$, and $\|\vec{\mathbf{v}}\| = 0$ if and only if $\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

The magnitude of a vector can also be derived using the Pythagorean theorem, as in the following figure.



If you use the components of a vector to define a right triangle, the magnitude of the vector is the length of the triangle's hypotenuse.

Magnitude and Direction

How To: Howto: Given a vector in component form, find its magnitude and direction angle.

Given a vector $\vec{\mathbf{v}} = \langle a, b \rangle$, the ***magnitude*** *is found by* $\|\vec{\mathbf{v}}\| = \sqrt{a^2 + b^2}$.

The **direction angle** is the positive angle formed between with the positive x-axis and a vector in standard position. (In other words, place the initial point of the vector at the origin of a coordinate system where the positive x-axis is horizontal and extends to the right).

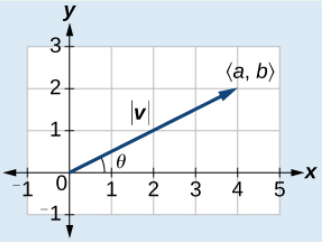


Figure 7.3.7.01 Illustration of a vector's Magnitude $\|\vec{\mathbf{v}}\|$ and Direction Angle θ

The ***direction angle*** *is found by* first finding $\tan\theta = \left(\frac{b}{a}\right) \Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right)$, as illustrated in Figure 7.3.7.01, and then adjusting the angle if the vector is not in Quadrant I. If the vector is in Quadrant II or Quadrant III, add $180°$. If the vector is in Quadrant IV, add $360°$.

Magnitude and Direction Angle

Example 7.3.7a: Find the Magnitude and Direction Angle of a Vector

Find the magnitude and direction of the vector with initial point $P(-8, 1)$ and terminal point $Q(-2, -5)$. Draw the vector.

Solution

First, find the position vector $\vec{\mathbf{u}}$ for $\overrightarrow{PQ}$.

$$\vec{\mathbf{u}} = \langle -2, -(-8), -5 - 1 \rangle = \langle 6, -6 \rangle$$

We use the Pythagorean Theorem to find the magnitude.

$$\|\vec{\mathbf{u}}\| = \sqrt{(6)^2 + (-6)^2} = \sqrt{72} = 6\sqrt{2}$$

The direction angle is given as

$$\tan\theta = \frac{-6}{6} = -1 \rightarrow \theta = \tan^{-1}(-1) = -45°$$

However, the angle terminates in the fourth quadrant, so we add $360°$ to obtain a positive angle. Thus, $-45° + 360° = 315°$. See Figure 7.3.7.

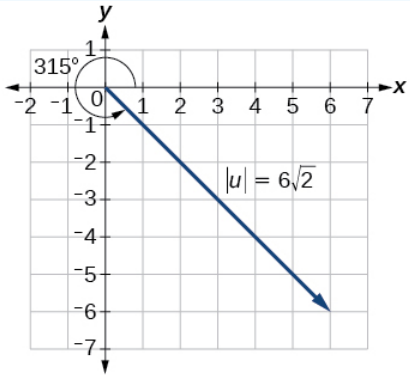


Figure 7.3.7

Example 7.3.7b

Find the magnitude and direction of vector $\vec{\mathbf{v}}$ with initial point at $(2, -7)$ and terminal point at $(-3, -4)$

Solution

First, find the position vector $\vec{\mathbf{v}}$.

$$\vec{\mathbf{v}} = \langle -3-2, -4-(-7)\rangle = \langle -5, 3\rangle$$

We use the Pythagorean Theorem to find the magnitude.

$$\|\vec{\mathbf{v}}\| = \sqrt{(-5)^2 + 3^2} = \sqrt{34}$$

The direction angle is given as

$$\tan\theta = \frac{3}{-5} = -0.6. \rightarrow \theta = \tan^{-1}(-0.6) \approx -31.0°$$

However, the angle terminates in the second quadrant, so we add $180°$ to obtain correct angle. Thus, the direction angle is $-31.0° + 180° = 149.0°$.

Example 7.3.7c

Let $\vec{\mathbf{v}}$ be the vector with initial point $(2, 5)$ and terminal point $(8, 13)$. Express $\vec{\mathbf{v}}$ in component form and find $\|\vec{\mathbf{v}}\|$ and its direction angle.

Solution

Using the algebraic method, we can express $\vec{\mathbf{v}}$ as $\vec{\mathbf{v}} = \langle 8-2, 13-5\rangle = \langle 6, 8\rangle$. The magnitude of $\vec{\mathbf{v}}$ is:

$$\|\vec{\mathbf{v}}\| = \sqrt{6^2 + 8^2} = \sqrt{36+64} = \sqrt{100} = 10.$$

The vector $\vec{\mathbf{v}}$ is in Quadrant I so the direction angle is

$$\tan\theta = \frac{8}{6} \Rightarrow \theta = \tan^{-1}\left(\frac{8}{6}\right) = 53.1°$$

Try It 7.3.7

Let $\vec{\mathbf{a}} = \langle -7, -1\rangle$. Find $\|\vec{\mathbf{a}}\|$ and the direction angle (to the nearest tenth of a degree) for $\vec{\mathbf{a}}$.

Hint

Use the Pythagorean Theorem to find $\|\vec{\mathbf{a}}\|$. The vector $\vec{\mathbf{v}}$ is in Quadrant III.

Answer

$\|\vec{\mathbf{a}}\| = 5\sqrt{2}$. The direction angle is $188.1°$

Find the component form of the vector

We have found the components of a vector given its initial and terminal points. In some cases, we may only have the magnitude and direction of a vector, not the points. For these vectors, we can identify the horizontal and vertical components using trigonometry (Figure 7.3.8.0).

Consider the angle θ formed by the vector $\vec{\mathbf{v}}$ and the positive x-axis. We can see from the triangle that the component form of vector $\vec{\mathbf{v}}$ is $\langle \|\vec{\mathbf{v}}\|\cos\theta, \|\vec{\mathbf{v}}\|\sin\theta\rangle$. Therefore, given an angle and the magnitude of a vector, we can use the cosine and sine of the angle to find the components of the vector.

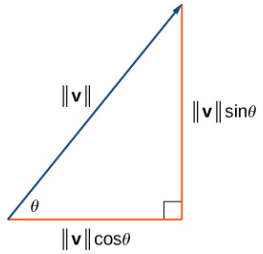


Figure 7.3.8.0: The components of a vector form the legs of a right triangle, with the vector as the hypotenuse.

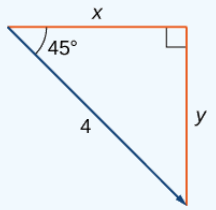
How to: Construct the component form of the vector given its magnitude and direction.

Given vector $\vec{\mathbf{v}}$ with magnitude $\|\vec{\mathbf{v}}\|$ and direction angle θ, the positive angle the vector makes with the positive x-axis. Then we can express $\vec{\mathbf{v}}$ in component form as

$$\vec{\mathbf{v}} = \langle \|\vec{\mathbf{v}}\|\cos\theta, \|\vec{\mathbf{v}}\|\sin\theta\rangle.$$

Example 7.3.8: Express a Vector in Component Form Given Its Magnitude and Direction

Find the component form of a vector with magnitude 4 that forms an angle of $-45°$ with the x-axis.



Solution

Let x and y represent the components of the vector (see the Figure on the right).
Then $x = 4\cos(-45°) = 2\sqrt{2}$ and $y = 4\sin(-45°) = -2\sqrt{2}$.
The component form of the vector is $\langle 2\sqrt{2}, -2\sqrt{2}\rangle$.

Try It 7.3.8

Find the component form of vector $\vec{v}$ with magnitude 10 that forms an angle of $120°$ with the positive x-axis.

Hint	Answer
$x = \|\vec{v}\|\cos\theta$ and $y = \|\vec{v}\|\sin\theta$	$v = \langle -5, 5\sqrt{3}\rangle$

Unit Vectors

Notice that $\|k\vec{v}\| = |k| \cdot \|\vec{v}\|$. This is because, if given any nonzero vector $\vec{v} = < a, b >$, then a scalar multiple of $\vec{v}$ can be written $k\vec{v}$ or $< ka, kb >$. The magnitude of $\vec{v}$ is $\sqrt{a^2 + b^2}$. The corresponding magnitude of the scalar multiple of $\vec{v}$ is $\sqrt{k^2a^2 + k^2b^2}$ which can be rewritten as $k\sqrt{a^2 + b^2}$ or $\|k\vec{v}\|$.

If we define a vector $\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$, it follows that $\|\vec{u}\| = \frac{1}{\|\vec{v}\|}(\|\vec{v}\|) = 1$ and we say that $\vec{u}$ is the ***unit vector in the direction of*** $\vec{v}$ (see Figure on the right).



Unit vector $\vec{u}$ in the direction of $\vec{v}$.

Definition: Unit Vector

The **unit vector** for any nonzero vector $\vec{v}$ is a vector that is in the same direction as $\vec{v}$ but with a magnitude of 1.

The unit vector for $\vec{v}$ is denoted $\hat{v}$ and is obtained by multiplying a vector by the reciprocal of its magnitude:

$$\hat{v} = \frac{1}{\|\vec{v}\|}\vec{v}.$$

The process of using scalar multiplication to find a unit vector with a given direction is called **normalization.**

Example 7.3.9: Find a Unit Vector

Let $\vec{v} = \langle 1, 2\rangle$.

a. Find a unit vector with the same direction as $\vec{v}$.
b. Find a vector $\vec{w}$ with the same direction as $\vec{v}$ such that $\|\vec{w}\| = 7$.

Solution:

a. First, find the magnitude of $\vec{v}$, then divide the components of $\vec{v}$ by the magnitude:

$$\|\vec{v}\| = \sqrt{1^2 + 2^2} = \sqrt{1 + 4} = \sqrt{5}$$

$$\vec{u} = \frac{1}{\|\vec{v}\|}v = \frac{1}{\sqrt{5}}\langle 1, 2\rangle = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle = \left\langle \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5} \right\rangle.$$

b. The vector $\vec{u}$ is in the same direction as $\vec{v}$ and $\|\vec{u}\| = 1$. Use scalar multiplication to increase the length of $\vec{u}$ without changing direction:

$$\vec{w} = 7\vec{u} = 7\left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle = \left\langle \frac{7}{\sqrt{5}}, \frac{14}{\sqrt{5}} \right\rangle = \left\langle \frac{7\sqrt{5}}{5}, \frac{14\sqrt{5}}{5} \right\rangle.$$

Try It 7.3.9

Let $\vec{v} = \langle 9, 2 \rangle$. Find a vector with magnitude 5 in the opposite direction as $\vec{v}$. Then negate it.

Hint

First, find a unit vector in the same direction as $\vec{v}$.

Answer

$$\langle -\frac{45}{\sqrt{85}}, -\frac{10}{\sqrt{85}} \rangle = \langle -\frac{9\sqrt{85}}{17}, -\frac{2\sqrt{85}}{17} \rangle$$

Standard Unit Vectors

In some applications involving vectors, it is helpful for us to be able to break a vector down into its components. Recall that vectors are comprised of two components: the horizontal component is the x direction, and the vertical component is the y direction. For example, we can see in the graph in Figure 7.3.13 that the position vector $\langle 2, 3 \rangle$ comes from adding the vectors \vecs{v_1} and $\vec{v_2}$. We have $\vec{v_2}$ with initial point $(0, 0)$ and terminal point $(2, 0)$.

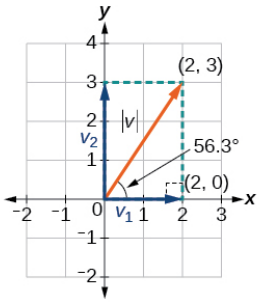


Figure 7.3.10.0: $v = v_1 + v_2$

$$\vec{v_1} = \langle 2 - 0, 0 - 0 \rangle = \langle 2, 0 \rangle$$

We also have $\vec{v_2}$ with initial point $(0, 0)$ and terminal point $(0, 3)$.

$$\vec{v_2} = \langle 0 - 0, 3 - 0 \rangle = \langle 0, 3 \rangle$$

Therefore, the position vector is $\vec{v} = \vec{v_1} + \vec{v_2} = \langle 2 + 0, 0 + 3 \rangle = \langle 2, 3 \rangle$

Example 7.3.10: Write a Vector as a Sum of Vectors

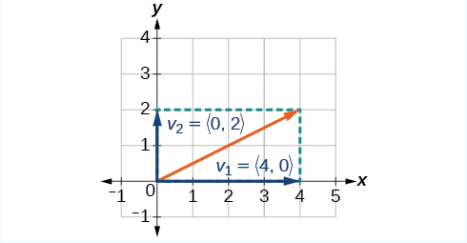
Write the vector $\vec{v}$ with initial point $(3, 2)$ and terminal point $(7, 4)$ as a sum of a horizontal and a vertical vector.

Solution

First find the standard position.

$$\begin{aligned} v &= \langle 7 - 3, 4 - 2 \rangle \\ &= \langle 4, 2 \rangle \end{aligned}$$

See the illustration on the right.



The horizontal component is $\vec{v_1} = \langle 4, 0 \rangle$ and the vertical component is $\vec{v_2} = \langle 0, 2 \rangle$.

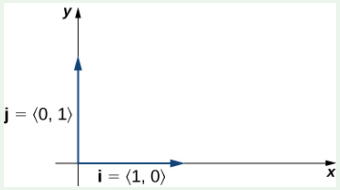
We have seen how convenient it can be to write a vector in component form. Sometimes, though, it is more convenient to write a vector as a sum of a horizontal vector and a vertical vector. To make this easier, let's look at standard unit vectors.

Definition: Standard Unit Vectors

The **standard unit vectors** are the vectors $\hat{\mathbf{i}} = \langle 1, 0 \rangle$ and $\hat{\mathbf{j}} = \langle 0, 1 \rangle$

Vector $\hat{\mathbf{i}}$ is the **horizontal unit vector** with a magnitude 1.
Vector $\hat{\mathbf{j}}$ is the **vertical unit vector** with a magnitude 1.

Any vector in component form can be expressed as a sum of the standard unit vectors (called **standard unit vector form**):



Standard unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$

$$\vec{\mathbf{v}} =< a, b >= a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$$

By applying the properties of vectors, it is possible to express any vector in terms of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ in what we call a *linear combination:*

$$\vec{\mathbf{v}} = \langle x, y\rangle = \langle x, 0\rangle + \langle 0, y\rangle = x\langle 1, 0\rangle + y\langle 0, 1\rangle = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}.$$

Thus, $\vec{\mathbf{v}}$ is the sum of a horizontal vector with magnitude x, and a vertical vector with magnitude y, as in Figure 7.3.11*b*.

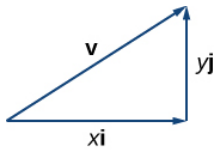


Figure 7.3.11*b*: The vector $\vec{\mathbf{v}}$ *is the sum of* $x\hat{\mathbf{i}}$ *and* $y\hat{\mathbf{j}}$.

Example 7.3.11: Write a Vector in Standard Unit Vector Form

Express the vector $\vec{\mathbf{w}} = \langle 3, -4\rangle$ as a sum of the standard unit vectors.

Solution:

$$\vec{\mathbf{w}} = \langle 3, -4\rangle = 3\hat{\mathbf{i}} - 4\hat{\mathbf{j}}.$$

Try It 7.3.11

Let $\vec{\mathbf{a}} = \langle 16, -11\rangle$ Express $\vec{\mathbf{a}}$ in standard unit vector form.

Answer

$\vec{\mathbf{a}} = 16\hat{\mathbf{i}} - 11\hat{\mathbf{j}}$

Example 7.3.12: Express a Vector in Standard Unit Vector Form Given its Initial and Terminal Points

Given a vector $\vec{\mathbf{v}}$ with initial point $P = (2, -6)$ and terminal point $Q = (-6, 6)$, write the vector in terms of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$.

Solution

$$\begin{aligned}\vec{\mathbf{v}} &= (x_Q - x_P)\hat{\mathbf{i}} + (y_Q - y_P)\hat{\mathbf{j}} \\ &= (-6 - 2)\hat{\mathbf{i}} + (6 - (-6))\hat{\mathbf{j}} = -8\hat{\mathbf{i}} + 12\hat{\mathbf{j}}\end{aligned}$$

Try It 7.3.12

Write the vector $\vec{\mathbf{u}}$ with initial point $P = (-1, 6)$ and terminal point $Q = (7, -5)$ in terms of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$.

Answer

$\vec{\mathbf{u}} = 8\hat{\mathbf{i}} - 11\hat{\mathbf{j}}$

Example 7.3.13: Perform Scalar Multiplication with Vectors written in Standard Unit Vector Form

Given vector $\vec{\mathbf{v}} = 3\hat{\mathbf{i}} - 1\hat{\mathbf{j}}$, find $3\vec{\mathbf{v}}$, $\frac{1}{2}\vec{\mathbf{v}}$, and $-\vec{\mathbf{v}}$.

Solution

$$\begin{aligned}3\vec{\mathbf{v}} &= 3 \cdot (3\hat{\mathbf{i}} - 1\hat{\mathbf{j}}) = 9\hat{\mathbf{i}} - 3\hat{\mathbf{j}} \\ \frac{1}{2}\vec{\mathbf{v}} &= \frac{1}{2} \cdot (3\hat{\mathbf{i}} - 1\hat{\mathbf{j}}) = \frac{3}{2}\hat{\mathbf{i}} - \frac{1}{2}\hat{\mathbf{j}} \\ -\vec{\mathbf{v}} &= -(3\hat{\mathbf{i}} - 1\hat{\mathbf{j}}) = -3\hat{\mathbf{i}} + 1\hat{\mathbf{j}}\end{aligned}$$

Try It 7.3.13

Find the scalar multiple $3\vec{u}$ given $\vec{u} = 5\hat{i} + 4\hat{j}$.

Answer

$3\vec{u} = 15\hat{i} + 12\hat{j}$

Example 7.3.14 Algebraic Operations using Vectors Written in Standard Unit Vector Form

Given $\vec{u} = 3\hat{i} - 2\hat{j}$ and $\vec{v} = -\hat{i} + 4\hat{j}$, find new vectors $\vec{w_1} = 3\vec{u} + 2\vec{v}$ and $\vec{w_2} = \vec{u} - \vec{v}$.

Solution

First, we must multiply each vector by the scalar.

$$3\vec{u} = 3(3\hat{i} - 2\hat{j}) = 9\hat{i} - 6\hat{j}$$
$$2\vec{v} = 2(-\hat{i} + 4\hat{j}) = -2\hat{i} + 8\hat{j}$$

Then, add the two together.

$$\begin{aligned}\vec{w_1} &= 3\vec{u} + 2\vec{v} \\ &= (9\hat{i} - 6\hat{j}) + (-2\hat{i} + 8\hat{j}) \\ &= (9-2)\hat{i} + (-6+8)\hat{j} = 7\hat{i} + 2\hat{j}\end{aligned}$$

To find the difference of two vectors, add the negative components of $\vec{v}$ to $\vec{u}$. Thus,

$$\begin{aligned}\vec{w_2} &= \vec{u} - \vec{v} \\ &= (3\hat{i} - 2\hat{j}) + (-(-\hat{i} + 4\hat{j})) \\ &= 3\hat{i} - 2\hat{j} + \hat{i} - 4\hat{j} \\ &= (9+1)\hat{i} + (-2-4)\hat{j} = 10\hat{i} - 6\hat{j}\end{aligned}$$

Try It 7.3.14

Given $\vec{u} = 3\hat{i} - 2\hat{j}$, $\vec{v} = -\hat{i} + 4\hat{j}$, $\vec{v_1} = 2\hat{i} - 3\hat{j}$ and $\vec{v_2} = 4\hat{i} + 5\hat{j}$, find vectors $\vec{u} + \vec{v}$, $\vec{u} - \vec{v}$ and $2\vec{v_1} - 3\vec{v_2}$.

Answer

$$\vec{u} + \vec{v} = 2\hat{i} + 2\hat{j}$$
$$\vec{u} - \vec{v} = 4\hat{i} - 6\hat{j}$$
$$2\vec{v_1} - 3\vec{v_2} = -8\hat{i} - 21\hat{j}$$

Example 7.3.15: Construct a Vector Given Its Magnitude and Direction

Write a vector in standard unit vector notation that has a length of 7 and makes an angle of $135°$ with the positive x-axis.

Solution

Using the conversion formulas $x = \|\vec{v}\|\cos\theta\hat{i}$ and $y = \|\vec{v}\|\sin\theta\hat{j}$, we find that

$$x = 7\cos(135°)\hat{i} = -\frac{7\sqrt{2}}{2}\hat{i}$$
$$y = 7\sin(135°)\hat{j} = \frac{7\sqrt{2}}{2}\hat{j}$$

This vector can be written as $\vec{v} = 7\cos(135°)\hat{i} + 7\sin(135°)\hat{j}$ which simplified becomes

$$\vec{\mathbf{v}} = -\frac{7\sqrt{2}}{2}\hat{\mathbf{i}} + \frac{7\sqrt{2}}{2}\hat{\mathbf{j}}$$

Try It 7.3.15

Let $\vec{\mathbf{b}}$ be a unit vector that forms an angle of $225°$ with the positive x-axis. Express $\vec{\mathbf{b}}$ in terms of the standard unit vectors.

Hint

Use sine and cosine to find the components of $\vec{\mathbf{b}}$.

Answer

$$\vec{\mathbf{b}} = -\frac{\sqrt{2}}{2}\hat{\mathbf{i}} - \frac{\sqrt{2}}{2}\hat{\mathbf{j}}$$

Example 7.3.16: Find the Unit Vector in the Direction of $\vec{\mathbf{v}}$

Find a unit vector in the same direction as $\vec{\mathbf{v}} = -5\hat{\mathbf{i}} + 12\hat{\mathbf{j}}$.

Solution

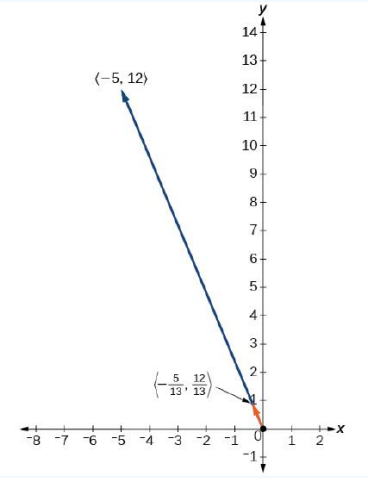
First, we will find the magnitude.

$$\|\vec{\mathbf{v}}\| = \sqrt{(-5)^2 + (12)^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

Then we divide each component by $|\vec{\mathbf{v}}|$, which gives a unit vector in the same direction as $\vec{v}$:

$$\frac{1}{\|\vec{\mathbf{v}}\|} \cdot \vec{\mathbf{v}} = -\frac{5}{13}\hat{\mathbf{i}} + \frac{12}{13}\hat{\mathbf{j}}$$

The figure on the right illustrates vector $\vec{\mathbf{v}}$ and its corresponding unit vector $\hat{\mathbf{v}}$



Verify that the magnitude of the unit vector equals 1. The magnitude of $-\frac{5}{13}\hat{\mathbf{i}} + \frac{12}{13}\hat{\mathbf{j}}$ is given as

$$\sqrt{\left(-\frac{5}{13}\right)^2 + \left(\frac{12}{13}\right)^2} = \sqrt{\frac{25}{169} + \frac{144}{169}}$$
$$= \sqrt{\frac{169}{169}} = 1$$

The vector $\hat{\mathbf{u}} = \frac{5}{13}\hat{\mathbf{i}} + \frac{12}{13}\hat{\mathbf{j}}$ is a unit vector because it has a magnitude of 1. It is in the same direction as $\vec{\mathbf{v}}$ because it is a positive scalar multiple of $\vec{\mathbf{v}}$.

Try It 7.3.16

A vector travels from the origin to the point $(-3, -5)$. Find the magnitude and direction angle of the vector (exactly and to the nearest hundredth). Write the vector in standard unit vector notation and find the unit vector that goes in the same direction.

Answer

Magnitude = $\sqrt{34} \approx 5.83$

Direction Angle: Vector is in QIII so $\theta = \tan^{-1}\left(\frac{-5}{-3}\right) + 180° \approx 239.04°$

$\vec{\mathbf{v}} = \sqrt{34}\cos(239.04°)\hat{\mathbf{i}} + \sqrt{34}\sin(239.04°)\hat{\mathbf{j}} = -3\hat{\mathbf{i}} - 5\hat{\mathbf{j}}$

Unit Vector: $\hat{\mathbf{u}} = -\frac{3\sqrt{34}}{34}\hat{\mathbf{i}} - \frac{5\sqrt{34}}{34}\hat{\mathbf{j}}$

Applications of Vectors

Because vectors have both direction and magnitude, they are valuable tools for solving problems involving such applications as motion and **force.** Recall the boat example and the quarterback example we described earlier. Here we look at two other examples in detail.

Example 7.3.17: Finding Resultant Force

Jane's car is stuck in the mud. Lisa and Jed come along in a truck to help pull her out. They attach one end of a tow strap to the front of the car and the other end to the truck's trailer hitch, and the truck starts to pull. Meanwhile, Jane and Jed get behind the car and push. The truck generates a horizontal force of **300** lb on the car. Jane and Jed are pushing at a slight upward angle and generate a force of **150** lb on the car. These forces can be represented by vectors, as shown in Figure **7.3.17**. The angle between these vectors is **15°**. Find the resultant force (the vector sum) and give its magnitude to the nearest tenth of a pound and its direction angle from the positive x-axis.



Figure **7.3.17**: Two forces acting on a car in different directions.

Solution

To find the effect of combining the two forces, add their representative vectors. First, express each vector in component form or in terms of the standard unit vectors. For this purpose, it is easiest if we align one of the vectors with the positive x-axis. The horizontal vector, then, has initial point $(0, 0)$ and terminal point $(300, 0)$. It can be expressed as $\langle 300, 0\rangle$ or $300\hat{\mathbf{i}}$.

The second vector has magnitude 150 and makes an angle of $15°$ with the first, so we can express it as $\langle 150\cos(15°), 150\sin(15°)\rangle$, or $150\cos(15°)\hat{\mathbf{i}} + 150\sin(15°)\hat{\mathbf{j}}$. Then, the sum of the vectors, or resultant vector, is $\vec{\mathbf{r}} = \langle 300, 0\rangle + \langle 150\cos(15°), 150\sin(15°)\rangle$, and we have

$$\|\vec{\mathbf{r}}\| = \sqrt{(300 + 150\cos(15°))^2 + (150\sin(15°))^2} \approx 446.6.$$

The angle θ made by $\vec{\mathbf{r}}$ and the positive x-axis has $\tan\theta = \dfrac{150\sin 15°}{(300 + 150\cos 15°)} \approx 0.09$, so $\theta \approx \tan^{-1}(0.09) \approx 5°$, which means the resultant force $\vec{\mathbf{r}}$ has an angle of $5°$ above the horizontal axis.

Example 7.3.18: Finding Resultant Velocity

An airplane flies due west at an airspeed of **425** mph. The wind is blowing from the northeast at **40** mph. What is the ground speed of the airplane? What is the bearing of the airplane?

Solution

Let's start by sketching the situation described (Figure **7.3.18**).

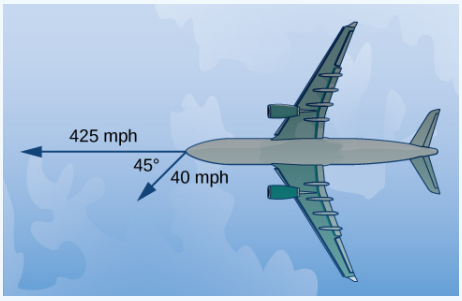


Figure **7.3.18**: Initially, the plane travels due west. The wind is from the northeast, so it is blowing to the southwest. The angle between the plane's course and the wind is **45°**. (Figure not drawn to scale.)

Set up a sketch so that the initial points of the vectors lie at the origin. Then, the plane's velocity vector is $\vec{\mathbf{p}} = -425\hat{\mathbf{i}}$. The vector describing the wind makes an angle of $225°$ with the positive x-axis:

$$\vec{\mathbf{w}} = \langle 40\cos(225°), 40\sin(225°)\rangle = \langle -\frac{40}{\sqrt{2}}, -\frac{40}{\sqrt{2}}\rangle = -\frac{40}{\sqrt{2}}\hat{\mathbf{i}} - \frac{40}{\sqrt{2}}\hat{\mathbf{j}}.$$

When the airspeed and the wind act together on the plane, we can add their vectors to find the resultant force:

$$\vec{\mathbf{p}} + \vec{\mathbf{w}} = -425\hat{\mathbf{i}} + (-\frac{40}{\sqrt{2}}\hat{\mathbf{i}} - \frac{40}{\sqrt{2}}\hat{\mathbf{j}}) = (-425 - \frac{40}{\sqrt{2}})\hat{\mathbf{i}} - \frac{40}{\sqrt{2}}\hat{\mathbf{j}}.$$

The magnitude of the resultant vector shows the effect of the wind on the ground speed of the airplane:

$$\|\vec{\mathbf{p}} + \vec{\mathbf{w}}\| = \sqrt{(-425 - \frac{40}{\sqrt{2}})^2 + (-\frac{40}{\sqrt{2}})^2} \approx 454.17\, mph$$

As a result of the wind, the plane is traveling at approximately 454 mph relative to the ground.

To determine the bearing of the airplane, we want to find the direction of the vector $\vec{\mathbf{p}} + \vec{\mathbf{w}}$:

$$\tan\theta = \frac{-\frac{40}{\sqrt{2}}}{(-425 - \frac{40}{\sqrt{2}})} \approx 0.06$$

$$\theta \approx 3.57°.$$

The overall direction of the plane is $3.57°$ south of west.

Try It 7.3.18

An airplane flies due north at an airspeed of 550 mph. The wind is blowing from the northwest at 50 mph. What is the ground speed of the airplane?

Hint

Sketch the vectors with the same initial point and find their sum.

Answer

Approximately 516 mph

Key Concepts

- Vectors are used to represent quantities that have both magnitude and direction.
- We can add vectors by using the parallelogram method or the triangle method to find the sum. We can multiply a vector by a scalar to change its length or give it the opposite direction.
- Subtraction of vectors is defined in terms of adding the negative of the vector.
- A vector is written in component form as $\vec{\mathbf{v}} = \langle x, y\rangle$.
- The magnitude of a vector is a scalar: $\|\vec{\mathbf{v}}\| = \sqrt{x^2 + y^2}$.
- A unit vector $\vec{\mathbf{u}}$ has magnitude 1 and can be found by dividing a vector by its magnitude: $\vec{\mathbf{u}} = \frac{1}{\|\vec{\mathbf{v}}\|}\vec{\mathbf{v}}$. The standard unit vectors are $\hat{\mathbf{i}} = \langle 1, 0\rangle$ and $\hat{\mathbf{j}} = \langle 0, 1\rangle$. A vector $\vec{\mathbf{v}} = \langle x, y\rangle$ can be expressed in terms of the standard unit vectors as $\vec{\mathbf{v}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$.
- Vectors are often used in physics and engineering to represent forces and velocities, among other quantities.

Glossary

component

a scalar that describes either the vertical or horizontal direction of a vector

equivalent vectors

vectors that have the same magnitude and the same direction

initial point
the starting point of a vector

magnitude
the length of a vector

parallelogram method

a method for finding the sum of two vectors; position the vectors so they share the same initial point; the vectors then form two adjacent sides of a parallelogram; the sum of the vectors is the diagonal of that parallelogram

scalar
a real number

scalar multiplication
a vector operation that defines the product of a scalar and a vector

standard-position vector
a vector with initial point $(0, 0)$

standard unit vectors
unit vectors along the coordinate axes: $\hat{\mathbf{i}} = \langle 1, 0 \rangle$, $\hat{\mathbf{j}} = \langle 0, 1 \rangle$

terminal point
the endpoint of a vector

triangle method

a method for finding the sum of two vectors; position the vectors so the terminal point of one vector is the initial point of the other; these vectors then form two sides of a triangle; the sum of the vectors is the vector that forms the third side; the initial point of the sum is the initial point of the first vector; the terminal point of the sum is the terminal point of the second vector

unit vector
a vector with magnitude 1

vector
a mathematical object that has both magnitude and direction

vector addition
a vector operation that defines the sum of two vectors

vector difference
the vector difference $\vec{\mathbf{v}} - \vec{\mathbf{w}}$ is defined as $\vec{\mathbf{v}} + (-\vec{\mathbf{w}}) = \vec{\mathbf{v}} + (-1)\vec{\mathbf{w}}$

vector sum
the sum of two vectors, $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$, can be constructed graphically by placing the initial point of $\vec{\mathbf{w}}$ at the terminal point of $\vec{\mathbf{v}}$; then the vector sum $\vec{\mathbf{v}} + \vec{\mathbf{w}}$ is the vector with an initial point that coincides with the initial point of $\vec{\mathbf{v}}$, and with a terminal point that coincides with the terminal point of $\vec{\mathbf{w}}$

zero vector
the vector with both initial point and terminal point $(0, 0)$

Contributors and Attributions

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7.3e: Exercises - Vectors in 2D

A: Vector Overview

Exercise $7.3e.A$: Vector Overview

1. What are the characteristics of the letters that are commonly used to represent vectors?

2. How is a vector more specific than a line segment?

3. What are $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, and what do they represent?

4. What is component form?

5. When a unit vector is expressed as $\langle a, b\rangle$ which letter is the coefficient of the $\hat{\mathbf{i}}$ and which the $\hat{\mathbf{j}}$?

Answers to odd exercises:

1. lowercase, bold letter, usually u, v, w
3. They are unit vectors. They are used to represent the horizontal and vertical components of a vector. They each have a magnitude of 1.
5. The first number always represents the coefficient of the $\hat{\mathbf{i}}$ and the second represents the $\hat{\mathbf{j}}$.

D: Perform Vector Operations Geometrically

Exercise $7.3e.D$: Perform Vector Operations Graphically

★ **Given $\vec{\mathbf{v}}$ below, draw $\vec{\mathbf{v}}$, $3\vec{\mathbf{v}}$, and $\frac{1}{2}v$.**

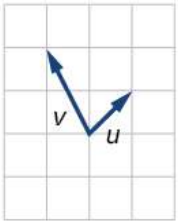
10. $\langle 2, -1\rangle$

11. $\langle -1, 4\rangle$

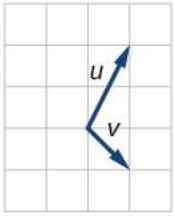
12. $\langle -3, -2\rangle$

★ **Use the vectors shown to sketch $\vec{\mathbf{u}} + \vec{\mathbf{v}}$, $\vec{\mathbf{u}} - \vec{\mathbf{v}}$, and $2\vec{\mathbf{u}}$.**

13.



14.



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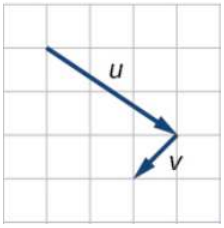
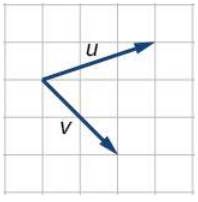
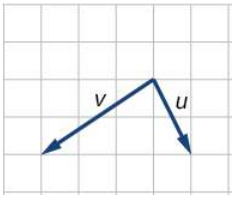


★ **Use the vectors shown to sketch $2\vec{\mathbf{u}} + \vec{\mathbf{v}}$.**

16.

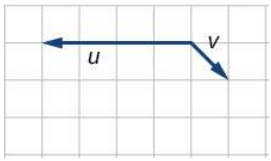


17.

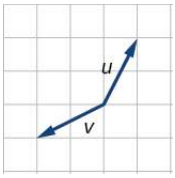


★ **Use the vectors shown to sketch $\vec{u} - 3\vec{v}$.**

18.

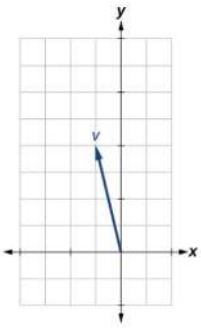


19.

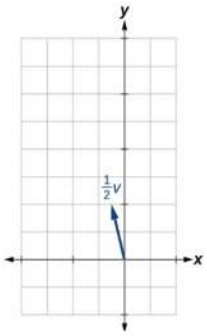


Answers to odd exercises:

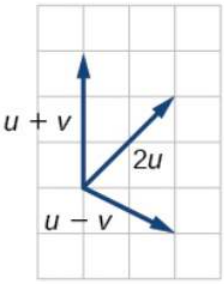
11.



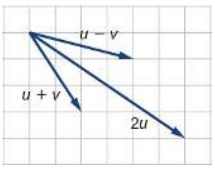




13.



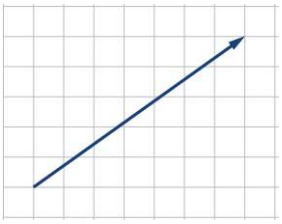
15.



17.



19.

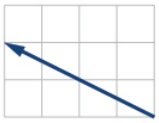


B: Construct a Vector Given a Graph or its Initial and Terminal Points

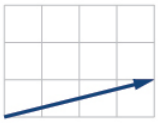
Exercise 7.3e.B: Construct Vectors

★ **Write the vector shown in component form.**

20.



21.



24. Given initial point $P_1 = (-3, 1)$ and terminal point $P_2 = (5, 2)$, write the vector $\vec{\mathbf{v}}$ in component form.

25. Given initial point $P_1 = (6, 0)$ and terminal point $P_2 = (-1, -3)$, write the vector $\vec{\mathbf{v}}$ in component form.

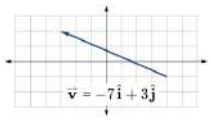
26. Given initial point $P_1 = (2, 1)$ and terminal point $P_2 = (-1, 2)$, write the vector $\vec{\mathbf{v}}$ in terms of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, then draw the vector on the graph.

27. Given initial point $P_1 = (4, -1)$ and terminal point $P_2 = (-3, 2)$, write the vector $\vec{\mathbf{v}}$ in terms of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. Draw the points and the vector on the graph.

28). Given initial point $P_1 = (3,3)$ and terminal point $P_2 = (-3,3)$, write the vector $\vec{v}$ in terms of $\hat{i}$ and $\hat{j}$. Draw the points and the vector on the graph.

Answers to odd exercises:

21. $\langle 4,1 \rangle$
25. $\langle -7,-3 \rangle$
27. $\vec{v} = -7\hat{i} + 3\hat{j}$



C: Equal Vectors

Exercise $7.3e.C$: Compare Vectors

★ **Determine whether the two vectors $\vec{u}$ and $\vec{v}$ are equal, where $\vec{u}$ has an initial point P_1 and a terminal point P_2 and $\vec{v}$ has an initial point P_3 and a terminal point P_4.**

31. $P_1 = (5,1), P_2 = (3,-2), P_3 = (-1,3), P4 = (9,-4)$
32. $P_1 = (2,-3), P_2 = (5,1), P_3 = (6,-1), P_4 = (9,3)$
33. $P_1 = (-1,-1), P_2 = (-4,5), P_3 = (-10,6), P_4 = (-13,12)$
34. $P_1 = (3,7), P_2 = (2,1), P_3 = (1,2), P_4 = (-1,-4)$
35. $P_1 = (8,3), P_2 = (6,5), P_3 = (11,8), P4 = (9,10)$

37. Given a vector with initial point $(5,2)$ and terminal point $(-1,-3)$, find an equivalent vector whose initial point is $(0,0)$. Write the vector in component form $\langle a,b \rangle$.

38. Given a vector with initial point $(7,-1)$ and terminal point $(-1,-7)$, find an equivalent vector whose initial point is $(0,0)$. Write the vector in component form $\langle a,b \rangle$.

39. Given a vector with initial point $(-4,2)$ and terminal point $(3,-3)$, find an equivalent vector whose initial point is $(0,0)$. Write the vector in component form $\langle a,b \rangle$.

Answers to odd exercises:

31. not equal 33. equal 35. equal 37. $\langle -6,-5 \rangle$ 39. $\langle 7,-5 \rangle$

E: Perform Vector Operations Algebraically

Exercise $7.3e.E$

★ **Use the given vectors to compute $\vec{u} + \vec{v}$, $\vec{u} - \vec{v}$, $2\vec{u} - 3\vec{v}$**

41. $\vec{u} = \langle -3,4 \rangle, \vec{v} = \langle -2,1 \rangle$
42. $\vec{u} = \langle 2,-3 \rangle, \vec{v} = \langle 1,5 \rangle$

★ **Use the vectors $u\vec{u} = \hat{i} + 5\hat{j}, \vec{v} = -2\hat{i} - 3\hat{j}, \vec{w} = 4\hat{i} - \hat{j}$ to evaluate the following vector operations.**

45. Find $4\vec{v} + 2\vec{u}$
46. Find $\vec{u} + (\vec{v} - \vec{w})$

Answers to odd exercises:

41. $\vec{u} + \vec{v} = \langle -5,5 \rangle, \vec{u} - \vec{v} = \langle -1,3 \rangle, 2\vec{u} - 3\vec{v} = \langle 0,5 \rangle$
45. $-6\hat{i} - 2\hat{j}$

F: Find the Magnitude and Direction Angle of a Vector

Exercise 7.3e.F

51. Let $\vec{\mathbf{a}}$ be a standard-position vector with terminal point $(-2,-4)$. Let $\vec{\mathbf{b}}$ be a vector with initial point $(1,2)$ and terminal point $(-1,4)$. Find the magnitude of vector $-3\vec{\mathbf{a}}+\vec{\mathbf{b}}-\langle 4,-15\rangle$.

52. Let $\vec{\mathbf{a}}$ be a standard-position vector with terminal point at $(2,5)$. Let $\vec{\mathbf{b}}$ be a vector with initial point $(-1,3)$ and terminal point $(1,0)$. Find the magnitude of vector $\vec{\mathbf{a}}-3\vec{\mathbf{b}}+14\hat{\mathbf{i}}-14\hat{\mathbf{j}}$.

★ Find the direction angle $\theta \in [0,2\pi)$ for vector $\vec{\mathbf{u}}$.

53. $\vec{\mathbf{u}}=5\sqrt{2}\hat{\mathbf{i}}-5\sqrt{2}\hat{\mathbf{j}}$

54. $\vec{\mathbf{u}}=-\sqrt{3}\hat{\mathbf{i}}-\hat{\mathbf{j}}$

★ Find the magnitude and direction angle for the vector.

55. $\langle 6,5\rangle$

56. $\langle 0,4\rangle$

57. $\langle -4,-6\rangle$

58. $\langle 2,-5\rangle$

Answers to odd exercises:

51. 29 53. $\theta=\frac{7\pi}{4}$ 55. $\|\vec{\mathbf{v}}\|=7.810, \theta=39.806°$ 57. $\|\vec{\mathbf{v}}\|=7.211, \theta=236.310°$

G: Find Vector components given its Magnitude and Direction

Exercise 7.3e.G

★ Find the component form of vector $\vec{\mathbf{u}}$, given its magnitude and the angle the vector makes with the positive x-axis. Give exact answers when possible.

61. $\|\vec{\mathbf{u}}\|=2, \theta=30°$
62. $\|\vec{\mathbf{u}}\|=6, \theta=60°$
63. $\|\vec{\mathbf{u}}\|=5, \theta=90°$
64. $\|\vec{\mathbf{u}}\|=8, \theta=\pi$
65. $\|\vec{\mathbf{u}}\|=10, \theta=\frac{5\pi}{6}$
66. $\|\vec{\mathbf{u}}\|=50, \theta=\frac{3\pi}{4}$

★ Write the vector in standard unit vector form, given its magnitude and direction angle θ.

67. $\|\vec{\mathbf{v}}\|=6, \theta=45°$
68. $\|\vec{\mathbf{v}}\|=8, \theta=220°$
69. $\|\vec{\mathbf{v}}\|=2, \theta=300°$
70. $\|\vec{\mathbf{v}}\|=5, \theta=135°$

★ Find the x and y components of a vector

71. Find the magnitude of the horizontal and vertical components of a vector with magnitude 8 pounds pointed in a direction of $27°$ above the horizontal. Round to the nearest hundredth.

72. Find the magnitude of the horizontal and vertical components of the vector with magnitude 4 pounds pointed in a direction of $127°$ above the horizontal. Round to the nearest hundredth.

73. Find the magnitude of the horizontal and vertical components of a vector with magnitude 5 pounds pointed in a direction of $55°$ above the horizontal. Round to the nearest hundredth.

74. Find the magnitude of the horizontal and vertical components of the vector with magnitude 1 pound pointed in a direction of $8°$ above the horizontal. Round to the nearest hundredth.

★ Find components parallel and perpendicular to another item

75. Find the components parallel and perpendicular to a $12°$ inclined ramp on which a 60-pound box is resting. Round to the nearest tenth.

76. Find the components parallel and perpendicular to $8°$ inclined ramp on which a 25-pound box is resting. Round to the nearest tenth.

Answers to odd exercises:

61. $\vec{\mathbf{u}} = \langle \sqrt{3}, 1 \rangle$
63. $\vec{\mathbf{u}} = \langle 0, 5 \rangle$
65. $\vec{\mathbf{u}} = \langle -5\sqrt{3}, 5 \rangle$
67. $3\sqrt{2}\hat{\mathbf{i}} + 3\sqrt{2}\hat{\mathbf{j}}$
69. $\hat{\mathbf{i}} - \sqrt{3}\hat{\mathbf{j}}$
71. $x = 7.13$ pounds, $y = 3.63$ pounds
73. $x = 2.87$ pounds, $y = 4.10$ pounds
75. perpendicular to ramp = $58.7 lbs$, parallel to ramp = $12.5 lbs$

H: Unit Vectors

Exercise $7.3e.H$: Unit Vectors

81. A vector $\vec{\mathbf{v}}$ has initial point $(-1, -3)$ and terminal point $(2, 1)$. Find the unit vector in the direction of $\vec{\mathbf{v}}$. Express the answer in component form.

82. A vector $\vec{\mathbf{v}}$ has initial point $(-2, 5)$ and terminal point $(3, -1)$. Find the unit vector in the direction of $\vec{\mathbf{v}}$. Express the answer in component form.

★ Find a unit vector in the same direction as the given vector.

83. $\vec{\mathbf{b}} = -2\hat{\mathbf{i}} + 5\hat{\mathbf{j}}$
84. $\vec{\mathbf{a}} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$
85. $\vec{\mathbf{d}} = -\frac{1}{3}\hat{\mathbf{i}} + \frac{5}{2}\hat{\mathbf{j}}$
86. $\vec{\mathbf{c}} = 10\hat{\mathbf{i}} - \hat{\mathbf{j}}$
87. $\vec{\mathbf{u}} = -14\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$
88. $\vec{\mathbf{u}} = 100\hat{\mathbf{i}} + 200\hat{\mathbf{j}}$

★ Find vector $\vec{\mathbf{v}}$ with the given magnitude and in the same direction as vector $\vec{\mathbf{u}}$.

89. $\|\vec{\mathbf{v}}\| = 7$, $\vec{\mathbf{u}} = \langle 3, 4 \rangle$
90. $\|\vec{\mathbf{v}}\| = 3$, $\vec{\mathbf{u}} = \langle -2, 5 \rangle$
91. $\|\vec{\mathbf{v}}\| = 7$, $\vec{\mathbf{u}} = \langle 3, -5 \rangle$
92. $\|\vec{\mathbf{v}}\| = 10$, $\vec{\mathbf{u}} = \langle 2, -1 \rangle$

93. Let $\vec{\mathbf{v}} = 5\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$. Find a vector that is twice the length and points in the opposite direction as $\vec{\mathbf{v}}$.

94). Let $\vec{\mathbf{v}} = -4\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$. Find a vector that is halfthe length and points in the same direction as $\vec{\mathbf{v}}$.

Answers to odd exercises:

81. $\langle \frac{3}{5}, \frac{4}{5} \rangle$
83. $-\frac{2\sqrt{29}}{29}\hat{\mathbf{i}} + \frac{5\sqrt{29}}{29}\hat{\mathbf{j}}$
85. $-\frac{2\sqrt{229}}{229}\hat{\mathbf{i}} + \frac{15\sqrt{229}}{229}\hat{\mathbf{j}}$
87. $-\frac{7\sqrt{2}}{10}\hat{\mathbf{i}} + \frac{\sqrt{2}}{10}\hat{\mathbf{j}}$
89. $\vec{\mathbf{v}} = \langle \frac{21}{5}, \frac{28}{5} \rangle$
91. $\vec{\mathbf{v}} = \langle \frac{21\sqrt{34}}{34}, -\frac{35\sqrt{34}}{34} \rangle$
93. $-10\hat{\mathbf{i}} - 4\hat{\mathbf{j}}$

I: Mixed practice

Exercise $7.3e.I$: Vector Mixed Practice

★ **Consider points $P(-1,3), Q(1,5)$, and $R(-3,7)$. Determine the requested vectors and express each of them**
a. in component form and
b. by using the standard unit vectors.

101. $\overrightarrow{PQ}$	106). $\overrightarrow{PQ}-\overrightarrow{PR}$
102. $\overrightarrow{PR}$	107. $2\overrightarrow{PQ}-2\overrightarrow{PR}$
103. $\overrightarrow{QP}$	108. $2\overrightarrow{PQ}+\frac{1}{2}\overrightarrow{PR}$
104. $\overrightarrow{RP}$	109. The unit vector in the direction of $\overrightarrow{PQ}$
105. $\overrightarrow{PQ}+\overrightarrow{PR}$	110. The unit vector in the direction of $\overrightarrow{PR}$

111. The vector $\vec{\mathbf{v}}$ has initial point $P(1,0)$ and terminal point Q that is on the *y-axis* and above the initial point. Find the coordinates of terminal point Q such that the magnitude of the vector $\vec{\mathbf{v}}$ is $\sqrt{5}$.

112. The vector $\vec{\mathbf{v}}$ has initial point $P(1,1)$ and terminal point Q that is on the *x-axis* and left of the initial point. Find the coordinates of terminal point Q such that the magnitude of the vector $\vec{\mathbf{v}}$ is $\sqrt{10}$.

For the following exercises, use the given vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.
a. Determine the vector sum $\vec{\mathbf{a}}+\vec{\mathbf{b}}$ and express it in both the component form and by using the standard unit vectors.
b. Find the vector difference $\vec{\mathbf{a}}-\vec{\mathbf{b}}$ and express it in both the component form and by using the standard unit vectors.
c. Determine the vectors $2\vec{\mathbf{a}}, -\vec{\mathbf{b}}$, and $2\mathbf{a}-\mathbf{b}$. Express the vectors in both the component form and by using standard unit vectors.

113. $\vec{\mathbf{a}}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}, \quad \vec{\mathbf{b}}=\hat{\mathbf{i}}+3\hat{\mathbf{j}}$

114. $\vec{\mathbf{a}}=2\hat{\mathbf{i}}, \quad \vec{\mathbf{b}}=-2\hat{\mathbf{i}}+2\hat{\mathbf{j}}$

Answers to odd exercises:

101. *a.* $\overrightarrow{PQ}=\langle 2,2\rangle$; *b.* $\overrightarrow{PQ}=2\hat{\mathbf{i}}+2\hat{\mathbf{j}}$
103. *a.* $\overrightarrow{QP}=\langle -2,-2\rangle$; *b.* $\overrightarrow{QP}=-2\hat{\mathbf{i}}-2\hat{\mathbf{j}}$
105. *a.* $\overrightarrow{PQ}+\overrightarrow{PR}=\langle 0,6\rangle$; *b.* $\overrightarrow{PQ}+\overrightarrow{PR}=6\hat{\mathbf{j}}$
107. *a.* $2\overrightarrow{PQ}-2\overrightarrow{PR}=\langle 8,-4\rangle$; *b.* $2\overrightarrow{PQ}-2\overrightarrow{PR}=8\hat{\mathbf{i}}-4\hat{\mathbf{j}}$
109. *a.* $\langle \frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\rangle$; *b.* $\frac{\sqrt{2}}{2}\hat{\mathbf{i}}+\frac{\sqrt{2}}{2}\hat{\mathbf{j}}$
111. $Q(0,2)$
113. *a.* $\vec{\mathbf{a}}+\vec{\mathbf{b}}=\langle 3,4\rangle=3\hat{\mathbf{i}}+4\hat{\mathbf{j}}$ *b.* $\vec{\mathbf{a}}-\vec{\mathbf{b}}=\langle 1,-2\rangle=\hat{\mathbf{i}}-2\hat{\mathbf{j}}$

c. $2\vec{\mathbf{a}}=\langle 4,2\rangle=4\hat{\mathbf{i}}+2\hat{\mathbf{j}}, \quad -\vec{\mathbf{b}}=\langle -1,-3\rangle=-\hat{\mathbf{i}}-3\hat{\mathbf{j}}, \quad 2\vec{\mathbf{a}}-\vec{\mathbf{b}}=\langle 3,-1\rangle=3\hat{\mathbf{i}}-\hat{\mathbf{j}}$

J: Find Resultant Vectors

Exercise 7.3e.J: Applications of Vectors

121. A woman leaves home and walks 3 miles west, then 2 miles southwest. How far from home is she, and in what direction must she walk to head directly home?

122. A boat leaves the marina and sails 6 miles north, then 2 miles northeast. How far from the marina is the boat, and in what direction must it sail to head directly back to the marina?

123. A man starts walking from home and walks 4 miles east, 2 miles southeast, 5 miles south, 4 miles southwest, and 2 miles east. How far has he walked? If he walked straight home, how far would he have to walk?

124. A woman starts walking from home and walks 4 miles east, 7 miles southeast, 6 miles south, 5 miles southwest, and 3 miles east. How far has she walked? If she walked straight home, how far would she have to walk?

125. A man starts walking from home and walks 3 miles at 20° north of west, then 5 miles at 10° west of south, then 4 miles at 15° north of east. If he walked straight home, how far would he have to the walk, and in what direction?

126. A woman starts walking from home and walks 6 miles at $40°$ north of east, then 2 miles at $15°$ east of south, then 5 miles at $30°$ south of west. If she walked straight home, how far would she have to walk, and in what direction?

127). Two children are throwing a ball back and forth straight across the back seat of a car. The ball is being thrown 10 mph relative to the car, and the car is traveling 25 mph down the road. If one child doesn't catch the ball, and it flies out the window, in what direction does the ball fly (ignoring wind resistance)?

128. Two children are throwing a ball back and forth straight across the back seat of a car. The ball is being thrown 8 mph relative to the car, and the car is traveling 45 mph down the road. If one child doesn't catch the ball, and it flies out the window, in what direction does the ball fly (ignoring wind resistance)?

129. An airplane is heading north at an airspeed of 600 km/hr, but there is a wind blowing from the southwest at 80 km/hr. How many degrees off course will the plane end up flying, and what is the plane's speed relative to the ground?

130. An airplane is heading north at an airspeed of 500 km/hr, but there is a wind blowing from the northwest at 50 km/hr. How many degrees off course will the plane end up flying, and what is the plane's speed relative to the ground?

131. An airplane needs to head due north, but there is a wind blowing from the southwest at 60 km/hr. The plane flies with an airspeed of 550 km/hr. To end up flying due north, how many degrees west of north will the pilot need to fly the plane?

132. An airplane needs to head due north, but there is a wind blowing from the northwest at 80 km/hr. The plane flies with an airspeed of 500 km/hr. To end up flying due north, how many degrees west of north will the pilot need to fly the plane?

133. Suppose a body has a force of 10 pounds acting on it to the right, 25 pounds acting on it $-135°$ from the horizontal, and 5 pounds acting on it directed $150°$ from the horizontal. What single force is the resultant force acting on the body?

134. Suppose a body has a force of 10 pounds acting on it to the right, 25 pounds acting on it upward, and 5 pounds acting on it $45°$ from the horizontal. What single force is the resultant force acting on the body?

135. Suppose a body has a force of 3 pounds acting on it to the left, 4 pounds acting on it upward, and 2 pounds acting on it $30°$ from the horizontal. What single force is needed to produce a state of equilibrium on the body? Draw the vector.

136. The condition of equilibrium is when the sum of the forces acting on a body is the zero vector. Suppose a body has a force of 2 pounds acting on it to the right, 5 pounds acting on it upward, and 3 pounds acting on it $45°$ from the horizontal. What single force is needed to produce a state of equilibrium on the body?

137. As part of a video game, the point $(5, 7)$ is rotated counterclockwise about the origin through an angle of $35°$. Find the new coordinates of this point.

138. As part of a video game, the point $(7, 3)$ is rotated counterclockwise about the origin through an angle of $40°$. Find the new coordinates of this point.

139. A 50-pound object rests on a ramp that is inclined $19°$. Find the magnitude of the components of the force parallel to and perpendicular to (normal) the ramp to the nearest tenth of a pound.

Answers to odd exercises:

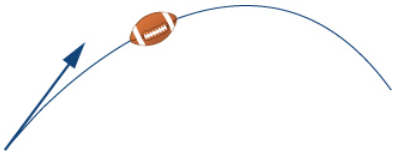
121. 4.635 miles, $17.764°$ N of E
123. 17 miles, 10.318 miles
125. Distance: 2.868, Direction: $86.474°$ North of West, or $3.526°$ West of North
127. $21.801°$, relative to the car's forward direction
129. $4.924°$, 659 km/hr
131. $4.424°$
133. 19.35 pounds, $231.54°$ from the horizontal
135. 5.1583 pounds, $-75.8°$ from the horizontal
137. $(0.081, 8.602)$
139. parallel: 16.28, perpendicular: 47.28 pounds

Real-World Applications II

Exercise $7.3e.A$: Applications of Vectors

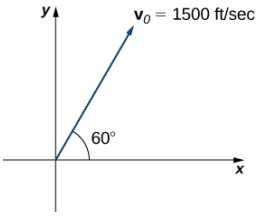
Solve the following vector applications.

145. The **speed** of an object is the magnitude of its related velocity vector. A football thrown by a quarterback has an initial speed of 70 mph and an angle of elevation of $30°$. Determine the velocity vector in mph and express it in component form. (Round to two decimal places.)



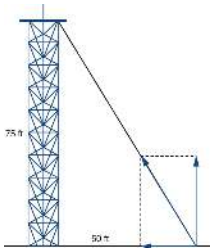
146. A baseball player throws a baseball at an angle of $30°$ with the horizontal. If the initial speed of the ball is 100 mph, find the horizontal and vertical components of the initial velocity vector of the baseball. (Round to two decimal places.)

147. A bullet is fired with an initial velocity of 1500 ft/sec at an angle of $60°$ with the horizontal. Find the horizontal and vertical components of the velocity vector of the bullet. (Round to two decimal places.)



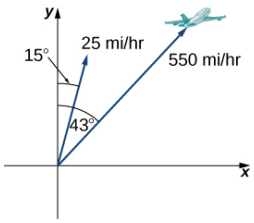
148. A 65-kg sprinter exerts a force of 798 N at a $19°$ angle with respect to the ground on the starting block at the instant a race begins. Find the horizontal component of the force. (Round to two decimal places.)

149. A guy-wire supports a pole that is 75 ft high. One end of the wire is attached to the top of the pole and the other end is anchored to the ground 50 ft from the base of the pole. Determine the horizontal and vertical components of the force of tension in the wire if its magnitude is 50 lb. (Round to the nearest integer.)

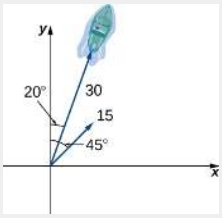


150. A telephone pole guy-wire has an angle of elevation of $35°$ with respect to the ground. The force of tension in the guy-wire is 120 lb. Find the horizontal and vertical components of the force of tension. (Round to the nearest integer.)

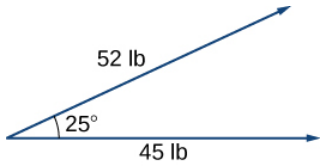
151. An airplane is flying in the direction of $43°$ east of north (also abbreviated as $N43E$) at a speed of 550 mph. A wind with speed 25 mph comes from the southwest at a bearing of $N15E$. What are the ground speed and new direction of the airplane?



152. A boat is traveling in the water at 30 mph in a direction of $N20E$ (that is, $20°$ east of north). A strong current is moving at 15 mph in a direction of $N45E$. What are the new speed and direction of the boat?



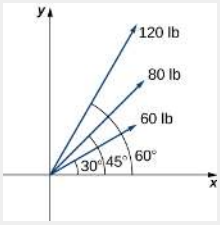
153. Two forces, a horizontal force of 45 lb and another of 52 lb, act on the same object. The angle between these forces is $25°$. Find the magnitude and direction angle from the positive x-axis of the resultant force that acts on the object. (Round to two decimal places.)



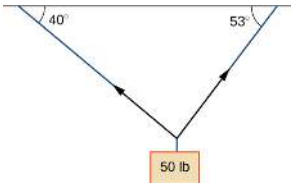
154. Two forces, a vertical force of 26 lb and another of 45 lb, act on the same object. The angle between these forces is $55°$. Find the magnitude and direction angle from the positive x-axis of the resultant force that acts on the object. (Round to two decimal places.)

155. Three forces act on object. Two of the forces have the magnitudes 58 N and 27 N, and make angles $53°$ and $152°$, respectively, with the positive *x*-axis. Find the magnitude and the direction angle from the positive *x*-axis of the third force such that the resultant force acting on the object is zero. (Round to two decimal places.)

156. Three forces with magnitudes 80 lb, 120 lb, and 60 lb act on an object at angles of $45°, 60°$ and $30°$, respectively, with the positive x-axis. Find the magnitude and direction angle from the positive x-axis of the resultant force. (Round to two decimal places.)

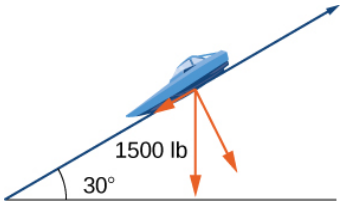


157. A 50-lb weight is hung by a cable so that the two portions of the cable make angles of $40°$ and $53°$, respectively, with the horizontal. Find the magnitudes of the forces of tension $\vec{\mathbf{T}}_1$ and $\vec{\mathbf{T}}_2$ in the cables if the resultant force acting on the object is zero. (Round to two decimal places.)



158. A 62-lb weight hangs from a rope that makes the angles of $29°$ and $61°$, respectively, with the horizontal. Find the magnitudes of the forces of tension $\vec{\mathbf{T}}_1$ and $\vec{\mathbf{T}}_2$ in the cables if the resultant force acting on the object is zero. (Round to two decimal places.)

159. A 1500-lb boat is parked on a ramp that makes an angle of $30°$ with the horizontal. The boat's weight vector points downward and is a sum of two vectors: a horizontal vector $\vec{\mathbf{v}}_1$ that is parallel to the ramp and a vertical vector $\vec{\mathbf{v}}_2$ that is perpendicular to the inclined surface. The magnitudes of vectors $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ are the horizontal and vertical component, respectively, of the boat's weight vector. Find the magnitudes of $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$. (Round to the nearest integer.)



160. An 85-lb box is at rest on a $26°$ incline. Determine the magnitude of the force parallel to the incline necessary to keep

the box from sliding. (Round to the nearest integer.)

Answers to odd exercises:

145. $\langle 60.62, 35 \rangle$
147. The horizontal and vertical components are 750 ft/sec and 1299.04 ft/sec, respectively.
149. The two horizontal and vertical components of the force of tension are 28 lb and 42 lb, respectively.
151. The new ground speed of the airplane is 572.19 mph; the new direction is $N41.82E$.
153. The magnitude of resultant force is 94.71 lb; the direction angle is $13.42°$.
155. The magnitude of the third vector is 60.03N; the direction angle is $259.38°$.
157. $\| \vec{\mathbf{T}}_1 \| = 30.13lb, \| \vec{\mathbf{T}}_2 \| = 38.35lb$
159. $\|\vec{v1}\| = 750$ lb, $\|\vec{v2}\| = 1299$ lb

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7.4: Vectors in Three Dimensions

Learning Objectives

- Locate points in space using coordinates.
- Write the distance formula in three dimensions.
- Perform vector operations in $\mathbb{R}^3$.

Vectors are useful tools for solving two-dimensional problems. Life, however, happens in three dimensions. To expand the use of vectors to more realistic applications, it is necessary to create a framework for describing three-dimensional space. For example, although a two-dimensional map is a useful tool for navigating from one place to another, in some cases the topography of the land is important. Does your planned route go through the mountains? Do you have to cross a river? To appreciate fully the impact of these geographic features, you must use three dimensions. This section presents a natural extension of the two-dimensional Cartesian coordinate plane into three dimensions.

Three-Dimensional Coordinate Systems

As we have learned, the two-dimensional rectangular coordinate system contains two perpendicular axes: the horizontal x-axis and the vertical y-axis. We can add a third dimension, the z-axis, which is perpendicular to both the x-axis and the y-axis. We call this system the three-dimensional rectangular coordinate system. It represents the three dimensions we encounter in real life.

Definition: Three-dimensional Rectangular Coordinate System

The three-dimensional rectangular coordinate system consists of three perpendicular axes: the x-axis, the y-axis, and the z-axis. Because each axis is a number line representing all real numbers in $\mathbb{R}$, the three-dimensional system is often denoted by $\mathbb{R}^3$.

In Figure 7.4.1*a*, the positive z-axis is shown above the plane containing the x- and y-axes. The positive x-axis appears to the left and the positive y-axis is to the right. A natural question to ask is: How was this arrangement determined? The system displayed follows the **right-hand rule**. If we take our right hand and align the fingers with the positive x-axis, then curl the fingers so they point in the direction of the positive y-axis, our thumb points in the direction of the positive z-axis (Figure 7.4.1*b*). In this text, we always work with coordinate systems set up in accordance with the right-hand rule. Some systems do follow a left-hand rule, but the right-hand rule is considered the standard representation.

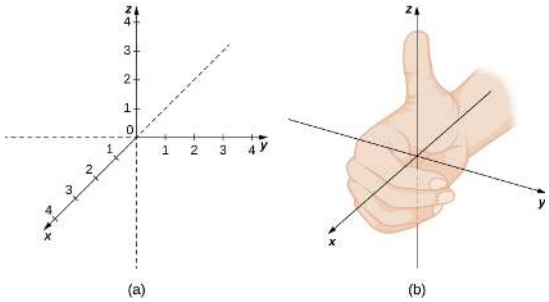


Figure 7.4.1: (a) We can extend the two-dimensional rectangular coordinate system by adding a third axis, the z-axis, that is perpendicular to both the x-axis and the y-axis. (b) The right-hand rule is used to determine the placement of the coordinate axes in the standard Cartesian plane.

In two dimensions, we describe a point in the plane with the coordinates (x, y). Each coordinate describes how the point aligns with the corresponding axis. In three dimensions, a new coordinate, z, is appended to indicate alignment with the z-axis: (x, y, z). A point in space is identified by all three coordinates (Figure 7.4.2). To plot the point (x, y, z), go x units along the x-axis, then y units in the direction of the y-axis, then z units in the direction of the z-axis.

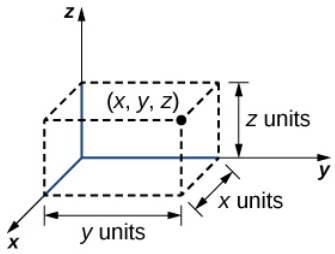


Figure 7.4.2: To plot the point (x, y, z) go x units along the x-axis, then y units in the direction of the y-axis, then z units in the direction of the z-axis.

Example 7.4.1: Locating Points in Space

Sketch the point $(1, -2, 3)$ in three-dimensional space.

Solution

To sketch a point, start by sketching three sides of a rectangular prism along the coordinate axes: one unit in the positive x direction, 2 units in the negative y direction, and 3 units in the positive z direction. Complete the prism to plot the point (Figure 3).

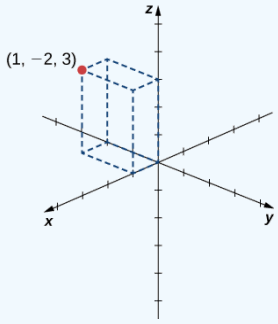


Figure 7.4.3: Sketching the point $(1, -2, 3)$.

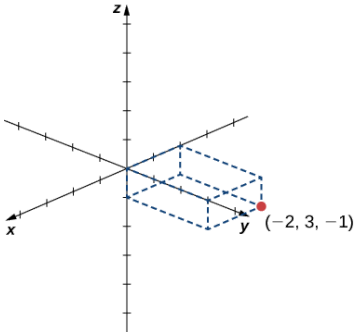
Try It 7.4.1

Sketch the point $(-2, 3, -1)$ in three-dimensional space.

Hint

Start by sketching the coordinate axes. e.g., Figure 7.4.3b. Then sketch a rectangular prism to help find the point in space.

Answer



Most work in three-dimensional space is a comfortable extension of the corresponding concepts in two dimensions. We begin by adapting the distance formula to three-dimensional space.

If two points lie in the same coordinate plane, then it is straightforward to calculate the distance between them. We that the distance d between two points (x_1, y_1) and (x_2, y_2) in the xy-coordinate plane is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

The formula for the distance between two points in space is a natural extension of this formula.

Definition: The Distance between Two Points in Space

The **distance d between points** (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

The proof of this theorem is left as an exercise. (Hint: First find the distance d_1 between the points (x_1, y_1, z_1) and (x_2, y_2, z_1) as shown in Figure 7.4.4.

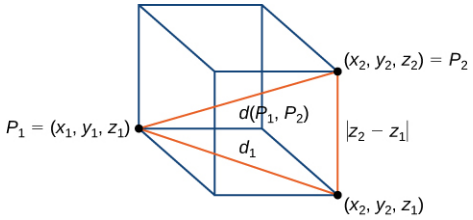


Figure 7.4.4: The distance between P_1 and P_2 is the length of the diagonal of the rectangular prism having P_1 and P_2 as opposite corners.

Example 7.4.2: Distance in Space

Find the distance between points $P_1 = (3, -1, 5)$ and $P_2 = (2, 1, -1)$.

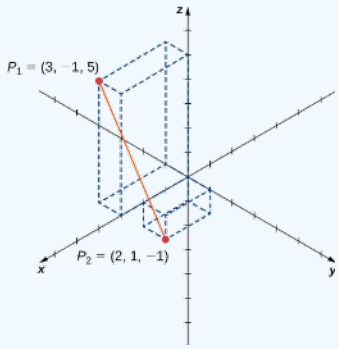


Figure 7.4.5: Find the distance between the two points.

Solution

Substitute values directly into the distance formula:

$$\begin{aligned} d(P_1, P_2) &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ &= \sqrt{(2-3)^2 + (1-(-1))^2 + (-1-5)^2} \\ &= \sqrt{(-1)^2 + 2^2 + (-6)^2} = \sqrt{41}. \end{aligned}$$

Try It 7.4.2

Find the distance between points $P_1 = (1, -5, 4)$ and $P_2 = (4, -1, -1)$.

Hint

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Answer

$5\sqrt{2}$

Working with Vectors in $\mathbb{R}^3$

Just like two-dimensional vectors, three-dimensional vectors are quantities with both magnitude and direction, and they are represented by directed line segments (arrows). With a three-dimensional vector, we use a three-dimensional arrow.

Three-dimensional vectors can also be represented in component form. The notation $\vec{\mathbf{v}} = \langle x, y, z \rangle$ is a natural extension of the two-dimensional case, representing a vector with the initial point at the origin, $(0, 0, 0)$, and terminal point (x, y, z). The zero vector is $\vec{\mathbf{0}} = \langle 0, 0, 0 \rangle$. So, for example, the three dimensional vector $\vec{\mathbf{v}} = \langle 2, 4, 1 \rangle$ is represented by a directed line segment from point $(0, 0, 0)$ to point $(2, 4, 1)$ (Figure 7.4.6).

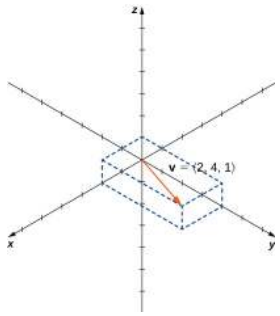


Figure 7.4.6: Vector $\vec{\mathbf{v}} = \langle 2, 4, 1 \rangle$ is represented by a directed line segment from point $(0, 0, 0)$ to point $(2, 4, 1)$.

Vector addition and scalar multiplication are defined analogously to the two-dimensional case. If $\vec{\mathbf{v}} = \langle x_1, y_1, z_1 \rangle$ and $\vec{\mathbf{w}} = \langle x_2, y_2, z_2 \rangle$ are vectors, and k is a scalar, then

$$\vec{\mathbf{v}} + \vec{\mathbf{w}} = \langle x_1 + x_2, y_1 \rangle + \langle y_2, z_1 + z_2 \rangle$$

and

$$k\vec{\mathbf{v}} = \langle kx_1, ky_1, kz_1 \rangle.$$

If $k = -1$, then $k\vec{\mathbf{v}} = (-1)\vec{\mathbf{v}}$ is written as $-\vec{\mathbf{v}}$, and vector subtraction is defined by $\vec{\mathbf{v}} - \vec{\mathbf{w}} = \vec{\mathbf{v}} + (-\vec{\mathbf{w}}) = \vec{\mathbf{v}} + (-1)\vec{\mathbf{w}}$.

The standard unit vectors extend easily into three dimensions as well, $\hat{\mathbf{i}} = \langle 1, 0, 0 \rangle$, $\hat{\mathbf{j}} = \langle 0, 1, 0 \rangle$, and $\hat{\mathbf{k}} = \langle 0, 0, 1 \rangle$, and we use them in the same way we used the standard unit vectors in two dimensions. Thus, we can represent a vector in $\mathbb{R}^3$ in the following ways:

$$\vec{\mathbf{v}} = \langle x, y, z \rangle = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}.$$

Example 7.4.3: Vector Representations

Let $\overrightarrow{PQ}$ be the vector with initial point $P = (3, 12, 6)$ and terminal point $Q = (-4, -3, 2)$ as shown in Figure 7.4.7. Express $\overrightarrow{PQ}$ in both component form and using standard unit vectors.

Solution

In component form,

$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$
$$= \langle -4 - 3, -3 - 12, 2 - 6 \rangle = \langle -7, -15, -4 \rangle.$$

In standard unit form,

$$\overrightarrow{PQ} = -7\hat{\mathbf{i}} - 15\hat{\mathbf{j}} - 4\hat{\mathbf{k}}.$$

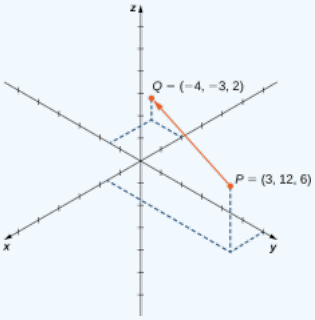


Figure 7.4.7: The vector with initial point $P = (3, 12, 6)$ and terminal point $Q = (-4, -3, 2)$.

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Try It 7.4.3

Let $S = (3, 8, 2)$ and $T = (2, -1, 3)$. Express $\vec{ST}$ in component form and in standard unit form.

Hint

Write $\overrightarrow{ST}$ in component form first. T is the terminal point of $\overrightarrow{ST}$.

Answer

$\overrightarrow{ST} = \langle -1, -9, 1 \rangle = -\hat{\mathbf{i}} - 9\hat{\mathbf{j}} + \hat{\mathbf{k}}$

As described earlier, vectors in three dimensions behave in the same way as vectors in a plane. The geometric interpretation of vector addition, for example, is the same in both two- and three-dimensional space (Figure 7.4.8).

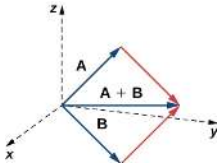

Figure 7.4.8: To add vectors in three dimensions, we follow the same procedures we learned for two dimensions.

We have already seen how some of the algebraic properties of vectors, such as vector addition and scalar multiplication, can be extended to three dimensions. Other properties can be extended in similar fashion. They are summarized here for our reference.

Properties of Vectors in Space

Let $\vec{\mathbf{v}} = \langle x_1, y_1, z_1 \rangle$ and $\vec{\mathbf{w}} = \langle x_2, y_2, z_2 \rangle$ be vectors, and let k be a scalar.

- **Scalar multiplication**: $k\vec{\mathbf{v}} = \langle kx_1, ky_1, kz_1 \rangle$
- **Vector addition**: $\vec{\mathbf{v}} + \vec{\mathbf{w}} = \langle x_1, y_1, z_1 \rangle + \langle x_2, y_2, z_2 \rangle = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$
- **Vector subtraction**: $\vec{\mathbf{v}} - \vec{\mathbf{w}} = \langle x_1, y_1, z_1 \rangle - \langle x_2, y_2, z_2 \rangle = \langle x_1 - x_2, y_1 - y_2, z_1 - z_2 \rangle$
- **Vector magnitude**: $\|\vec{\mathbf{v}}\| = \sqrt{x_1^2 + y_1^2 + z_1^2}$
- **Unit vector in the direction of $\vec{\mathbf{v}}$**: $\dfrac{1}{\|\vec{\mathbf{v}}\|}\vec{\mathbf{v}} = \dfrac{1}{\|\vec{\mathbf{v}}\|}\langle x_1, y_1, z_1 \rangle = \left\langle \dfrac{x_1}{\|\vec{\mathbf{v}}\|}, \dfrac{y_1}{\|\vec{\mathbf{v}}\|}, \dfrac{z_1}{\|\vec{\mathbf{v}}\|} \right\rangle, \quad \text{if } \vec{\mathbf{v}} \neq \vec{\mathbf{0}}$

Example 7.4.4: Vector Operations in Three Dimensions

Let $\vec{\mathbf{v}} = \langle -2, 9, 5 \rangle$ and $\vec{\mathbf{w}} = \langle 1, -1, 0 \rangle$. Find the following vectors.

a. $3\vec{\mathbf{v}} - 2\vec{\mathbf{w}}$
b. $5\|\vec{\mathbf{w}}\|$
c. $\|5\vec{\mathbf{w}}\|$
d. A unit vector in the direction of $\vec{\mathbf{v}}$

Solution

a. First, use scalar multiplication of each vector, then combine:

$$3\vec{\mathbf{v}} - 2\vec{\mathbf{w}} = 3\langle -2, 9, 5 \rangle - 2\langle 1, -1, 0 \rangle = \langle -6, 27, 15 \rangle + \langle -2, 2, 0 \rangle = \langle -6 - 2, 27 + 2, 15 + 0 \rangle = \langle -8, 29, 15 \rangle.$$

b. Write the equation for the magnitude of the vector, then use scalar multiplication:

$$5\|\vec{\mathbf{w}}\| = 5\sqrt{1^2 + (-1)^2 + 0^2} = 5\sqrt{2}.$$

c. First, use scalar multiplication, then find the magnitude of the new vector. Notice the result is the same as in part b.:

$$\|5\vec{\mathbf{w}}\| = \| \langle 5, -5, 0 \rangle \| = \sqrt{5^2 + (-5)^2 + 0^2} = \sqrt{50} = 5\sqrt{2}$$

d. Recall that to find a unit vector in two dimensions, we divide a vector by its magnitude. The procedure is the same in three dimensions:

$$\frac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|}=\frac{\langle -2,9,5\rangle}{\|\vec{\mathbf{v}}\|}=\frac{\langle -2,9,5\rangle}{\sqrt{(-2)^2+9^2+5^2}}=\frac{\langle -2,9,5\rangle}{\sqrt{110}}=\left\langle \frac{-2}{\sqrt{110}},\frac{9}{\sqrt{110}},\frac{5}{\sqrt{110}}\right\rangle=\left\langle -\frac{\sqrt{110}}{55},\frac{9\sqrt{110}}{110},\frac{\sqrt{110}}{22}\right\rangle.$$

Try It 7.4.4:

Let $\vec{\mathbf{v}}=\langle -1,-1,1\rangle$ and $\vec{\mathbf{w}}=\langle 2,0,1\rangle$. Find a unit vector in the direction of $5\vec{\mathbf{v}}+3\vec{\mathbf{w}}$.

Hint

Start by writing $5\vec{\mathbf{v}}+3\vec{\mathbf{w}}$ in component form.

Answer

$$\left\langle \frac{\sqrt{10}}{30},-\frac{\sqrt{10}}{6},\frac{4\sqrt{10}}{15}\right\rangle$$

Key Concepts

- The three-dimensional coordinate system is built around a set of three axes that intersect at right angles at a single point, the origin. Ordered triples (x,y,z) are used to describe the location of a point in space.
- The distance d between points (x_1,y_1,z_1) and (x_2,y_2,z_2) is given by the formula

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}.$$

- In three dimensions, as in two, vectors are commonly expressed in component form, $v=\langle x,y,z\rangle$, or in terms of the standard unit vectors, $xi+yj+zk$.
- Properties of vectors in space are a natural extension of the properties for vectors in a plane.

Glossary

Right-hand rule

a common way to define the orientation of the three-dimensional coordinate system; when the right hand is curved around the z-axis in such a way that the fingers curl from the positive x-axis to the positive y-axis, the thumb points in the direction of the positive z-axis

three-dimensional rectangular coordinate system

a coordinate system defined by three lines that intersect at right angles; every point in space is described by an ordered triple (x,y,z) that plots its location relative to the defining axes

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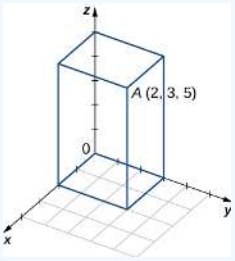
7.4e: Vectors in 3D

A: 3-D Vector Practice

Exercise $7.4e.A$: Vector Overview

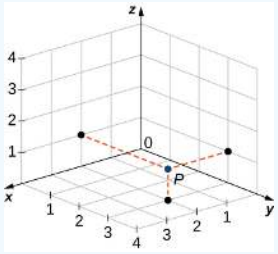
1) Consider a rectangular box with one of the vertices at the origin, as shown in the following figure. If point $A(2,3,5)$ is the opposite vertex to the origin, then find



a. the coordinates of the other six vertices of the box and

b. the length of the diagonal of the box determined by the vertices O and A.

2) Find the coordinates of point P and determine its distance to the origin.



Express vector $\overrightarrow{PQ}$ with the initial point at P and the terminal point at Q (a). in component form and (b). by using standard unit vectors.

3) $P(3,0,2)$ and $Q(-1,-1,4)$

4) $P(0,10,5)$ and $Q(1,1,-3)$

5) $P(-2,5,-8)$ and $M(1,-7,4)$, where M is the midpoint of the line segment $\overline{PQ}$

6) $Q(0,7,-6)$ and $M(-1,3,2)$, where M is the midpoint of the line segment $\overline{PQ}$ Q(0,7,−6)\) and $M(-1,3,2)$, where M is the midpoint of the line segment PQ

7) Find terminal point Q of vector $\vec{PQ} = \langle 7,-1,3\rangle$ with the initial point at $P(-2,3,5)$.

8) Find initial point P of vector $\vec{PQ} = \langle -9,1,2\rangle$ with the terminal point at $Q(10,0,-1)$.

Use the given vectors $\vec{a}$ and $\vec{b}$ to find and express the vectors $\vec{a}+\vec{b}$, $4\vec{a}$, and $-5\vec{a}+3\vec{b}$ in component form.

11) $\vec{a} = \langle -1,-2,4\rangle$, $\vec{b} = \langle -5,6,-7\rangle$

12) $\vec{a} = \langle 3,-2,4\rangle$, $\vec{b} = \langle -5,6,-9\rangle$

13) $\vec{a} = -\hat{k}$, $\vec{b} = -\hat{i}$

14) $\vec{a} = \hat{i}+\hat{j}+\hat{k}$, $\vec{b} = 2\hat{i}-3\hat{j}+2\hat{k}$

Given vectors $\vec{u}$ and $\vec{v}$ below, find the magnitudes of vectors $\vec{u}-\vec{v}$ and $-2\vec{u}$.

21) $\vec{u} = 2\hat{i}+3\hat{j}+4\hat{k}$, $\vec{v} = -\hat{i}+5\hat{j}-\hat{k}$

22) $\vec{u} = \hat{i}+\hat{j}$, $\vec{v} = \hat{j}-\hat{k}$

23) $\vec{u} = \langle 2\cos t, -2\sin t, 3\rangle$, $\vec{v} = \langle 0,0,3\rangle$, where t is a real number.

24) $\vec{u} = \langle 0,1,\sin t\rangle$, $\vec{v} = \langle 1,1,0\rangle$, where t is a real number.

Find the unit vector in the direction of the given vector a and express it using standard unit vectors.

31) $\vec{a} = 3\hat{i}-4\hat{j}$

32) $\vec{a} = \langle 4,-3,6\rangle$

33) $\vec{a} = \overrightarrow{PQ}$, where $P(-2,3,1)$ and $Q(0,-4,4)$

34) $\vec{\mathbf{a}} = \overrightarrow{OP}$, where $P(-1,-1,1)$

35) $\vec{\mathbf{a}} = \vec{\mathbf{u}} - \vec{\mathbf{v}} + \vec{\mathbf{w}}$, where $\vec{\mathbf{u}} = \hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}$, $\vec{\mathbf{v}} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$, and $\vec{\mathbf{w}} = -\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

36) $\vec{\mathbf{a}} = 2\vec{\mathbf{u}} + \vec{\mathbf{v}} - \vec{\mathbf{w}}$, where $\vec{\mathbf{u}} = \hat{\mathbf{i}} - \hat{\mathbf{k}}$, $\vec{\mathbf{v}} = 2\hat{\mathbf{j}}$, and $\vec{\mathbf{w}} = \hat{\mathbf{i}} - \hat{\mathbf{j}}$

37) Determine whether $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are equivalent vectors, where $A(1,1,1)$, $B(3,3,3)$, $P(1,4,5)$, and $Q(3,6,7)$.

38) Determine whether the vectors $\vec{AB}$ and $\overrightarrow{PQ}$ are equivalent, where $A(1,4,1), B(-2,2,0), P(2,5,7)$, and $Q(-3,2,1)$.

Find vector $\vec{\mathbf{u}}$ with a magnitude that is given and which satisfies the given conditions.

41) $\vec{\mathbf{v}} = \langle 7,-1,3\rangle$, $\|\vec{\mathbf{u}}\| = 10$, and $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ have the same direction

42) $\vec{\mathbf{v}} = \langle 2,4,1\rangle$, $\|\vec{\mathbf{u}}\| = 15$, and $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ have the same direction

43) $\vec{\mathbf{v}} = \langle 2\sin t,\ 2\cos t, 1\rangle, \|\vec{\mathbf{u}}\| = 2$, $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ have opposite directions for any t, where t is a real number

44) $\vec{\mathbf{v}} = \langle 3\sin t, 0, 3\rangle$, $\|\vec{\mathbf{u}}\| = 5$, and $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ have opposite directions for any t, where t is a real number

45) Determine a vector of magnitude 5 in the direction of vector $\vec{AB}$, where $A(2,1,5)$ and $B(3,4,-7)$.

46) Find a vector of magnitude 2 that points in the opposite direction than vector $\vec{AB}$, where $A(-1,-1,1)$ and $B(0,1,1)$. Express the answer in component form.

Answers to odd exercises:

1. *a.* $(2,0,5),(2,0,0),(2,3,0),(0,3,0),(0,3,5),(0,0,5)$; *b.* $\sqrt{38}$
3. *a.* $\overrightarrow{PQ} = \langle -4,-1,2\rangle$ *b.* $\overrightarrow{PQ} = -4\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$
5. *a.* $\overrightarrow{PQ} = \langle 6,-24,24\rangle$ *b.* $\overrightarrow{PQ} = 6\hat{\mathbf{i}} - 24\hat{\mathbf{j}} + 24\hat{\mathbf{k}}$
7. $Q(5,2,8)$
11. $\vec{\mathbf{a}} + \vec{\mathbf{b}} = \langle -6,4,-3\rangle, 4\vec{\mathbf{a}} = \langle -4,-8,16\rangle, -5\vec{\mathbf{a}} + 3\vec{\mathbf{b}} = \langle -10,28,-41\rangle$
13. $\vec{\mathbf{a}} + \vec{\mathbf{b}} = \langle -1,0,-1\rangle, 4\vec{\mathbf{a}} = \langle 0,0,-4\rangle, -5\vec{\mathbf{a}} + 3\vec{\mathbf{b}} = \langle -3,0,5\rangle$
21. $\|\vec{\mathbf{u}} - \vec{\mathbf{v}}\| = \sqrt{38}$, $\|-2\vec{\mathbf{u}}\| = 2\sqrt{29}$
23. $\|\vec{\mathbf{u}} - \vec{\mathbf{v}}\| = 2$, $\|-2\vec{\mathbf{u}}\| = 2\sqrt{13}$
31. $\frac{3}{5}\hat{\mathbf{i}} - \frac{4}{5}\hat{\mathbf{j}}$
33. $\frac{\sqrt{62}}{31}\hat{\mathbf{i}} - \frac{7\sqrt{62}}{62}\hat{\mathbf{j}} + \frac{3\sqrt{62}}{62}\hat{\mathbf{k}}$
35. $-\frac{\sqrt{6}}{3}\hat{\mathbf{i}} + \frac{\sqrt{6}}{6}\hat{\mathbf{j}} + \frac{\sqrt{6}}{6}\hat{\mathbf{k}}$
37. Equivalent vectors
41. $\vec{\mathbf{u}} = \langle \frac{70\sqrt{59}}{59}, -\frac{10\sqrt{59}}{59}, \frac{30\sqrt{59}}{59}\rangle$
43. $\vec{\mathbf{u}} = \langle -\frac{4\sqrt{5}}{5}\sin t, -\frac{4\sqrt{5}}{5}\cos t, -\frac{2\sqrt{5}}{5}\rangle$
45. $\langle \frac{5\sqrt{154}}{154}, \frac{15\sqrt{154}}{154}, -\frac{30\sqrt{154}}{77}\rangle$
47. $\alpha = -\sqrt{7}$, $\beta = -\sqrt{15}$

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7.5: The Dot Product

Learning Objectives

- Calculate the dot product of two given vectors.
- Determine whether two given vectors are perpendicular.
- Explain what is meant by the vector projection of one vector onto another vector, and describe how to compute it.
- Calculate the work done by a given force.

If we apply a force to an object so that the object moves, we say that work is done by the force. Previously, we looked at a constant force and we assumed the force was applied in the direction of motion of the object. Under those conditions, work can be expressed as the product of the force acting on an object and the distance the object moves. In this chapter, however, we have seen that both force and the motion of an object can be represented by vectors.

In this section, we develop an operation called the dot product, which allows us to calculate work in the case when the force vector and the motion vector have different directions. The dot product essentially tells us how much of the force vector is applied in the direction of the motion vector. The dot product can also help us measure the angle formed by a pair of vectors and the position of a vector relative to the coordinate axes. It even provides a simple test to determine whether two vectors meet at a right angle.

The Dot Product and Its Properties

We have already learned how to add and subtract vectors. In this chapter, we investigate two types of vector multiplication. The first type of vector multiplication is called the dot product, based on the notation we use for it, and it is defined as follows:

Definition: dot product

The **dot product** (also called the **scalar product**) of two vectors $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ is given by the sum of the products of the components

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

When vectors are written in standard unit vector form, $\vec{\mathbf{u}} = u_1\hat{\mathbf{i}} + u_2\hat{\mathbf{j}} + u_3\hat{\mathbf{k}}$ and $\vec{\mathbf{v}} = v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}} + v_3\hat{\mathbf{k}}$ the dot product is the same sum of products. Notice the standard unit vectors themselves are not included in the dot product.

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Note that if u and v are two-dimensional vectors, we calculate the dot product in a similar fashion. Thus, if $\vec{\mathbf{u}} = \langle u_1, u_2 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2 \rangle$, then

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2.$$

When two vectors are combined under addition or subtraction, the result is a vector. When two vectors are combined using the dot product, the result is a scalar. For this reason, the dot product is often called the ***scalar product***. It may also be called the ***inner product***.

Example 7.5.1: Calculate Dot Products

a. Find the dot product of $\vec{\mathbf{u}} = \langle 3, 5, 2 \rangle$ and $\vec{\mathbf{v}} = \langle -1, 3, 0 \rangle$.
b. Find the scalar product of $\vec{\mathbf{p}} = 10\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$ and $\vec{\mathbf{q}} = -2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 6\hat{\mathbf{k}}$.

Solution:

a. Substitute the vector components into the formula for the dot product:

$$\begin{aligned} \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} &= u_1 v_1 + u_2 v_2 + u_3 v_3 \\ &= 3(-1) + 5(3) + 2(0) \\ &= -3 + 15 + 0 = 12. \end{aligned}$$

b. The calculation is the same if the vectors are written using standard unit vectors. We still have three components for each vector to substitute into the formula for the dot product:

$$\begin{aligned}\vec{\mathbf{p}} \cdot \vec{\mathbf{q}} &= u_1v_1 + u_2v_2 + u_3v_3 \\ &= 10(-2) + (-4)(1) + (7)(6) \\ &= -20 - 4 + 42 = 18.\end{aligned}$$

Try It 7.5.1

a. Find $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$, where $\vec{\mathbf{u}} = \langle 2, 9, -1\rangle$ and $\vec{\mathbf{v}} = \langle -3, 1, -4\rangle$.
b. Find $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$, where $\vec{\mathbf{u}} = -8\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ and $\vec{\mathbf{v}} = 2\hat{\mathbf{i}} + 9\hat{\mathbf{j}}$.

Hint

Multiply corresponding components and then add their products.

Answer

a. 7 b. 11

Like vector addition and subtraction, the dot product has several algebraic properties.

Magnitude of a vector and the Dot Product

The dot product of a vector $\vec{\mathbf{v}}$ with itself is the square of its magnitude:

$$\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = \|\vec{\mathbf{v}}\|^2 \tag{7.5.1}$$

Example 7.5.2: Dot Product Expressions

Let $\vec{\mathbf{a}} = \langle 1, 2, -3\rangle$, $\vec{\mathbf{b}} = \langle 0, 2, 4\rangle$, and $\vec{\mathbf{c}} = \langle 5, -1, 3\rangle$.

Find each of the following products.

a. $(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})\vec{\mathbf{c}}$
b. $\vec{\mathbf{a}} \cdot (2\vec{\mathbf{c}})$
c. $\|\vec{\mathbf{b}}\|^2$

Solution

a. Note that this expression asks for the scalar multiple of $\vec{\mathbf{c}}$ by $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$:

$$\begin{aligned}(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})\vec{\mathbf{c}} &= (\langle 1, 2, -3\rangle \cdot \langle 0, 2, 4\rangle)\langle 5, -1, 3\rangle \\ &= (1(0) + 2(2) + (-3)(4))\langle 5, -1, 3\rangle \\ &= (-8)\langle 5, -1, 3\rangle = \langle -40, 8, -24\rangle.\end{aligned}$$

b. This expression is a dot product of vector $\vec{\mathbf{a}}$ and scalar multiple $2\vec{\mathbf{c}}$:

$$\begin{aligned}\vec{\mathbf{a}} \cdot (2\vec{\mathbf{c}}) &= \langle 1, 2, -3\rangle \cdot 2(\langle 5, -1, 3\rangle) \\ &= \langle 1, 2, -3\rangle \cdot \langle 10, -2, 6\rangle \\ &= (1)(10) + 2(-2) + (-3)(6)) \\ &= 10 - 4 - 18 = -12.\end{aligned}$$

c. Simplifying this expression is a straightforward application of the dot product:

$$\begin{aligned}\|\vec{b}\|^2 &= \vec{b}\cdot\vec{b}\\ &= \langle 0,2,4\rangle\cdot\langle 0,2,4\rangle\\ &= 0^2+2^2+4^2\\ &= 0+4+16=20.\end{aligned}$$

Try It 7.5.2

Find the following products for $\vec{p}=7\hat{i}+0\hat{j}+2\hat{k}$, $\vec{q}=-2\hat{i}+2\hat{j}-2\hat{k}$, and $\vec{r}=0\hat{i}+2\hat{j}-3\hat{k}$.

a. $(\vec{r}\cdot\vec{p})\vec{q}$
b. $\|\vec{p}\|^2$

Hint

$\vec{r}\cdot\vec{p}$ is a scalar.

Answer

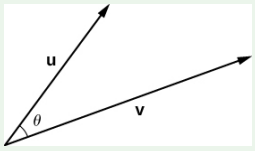
a. $(\vec{r}\cdot\vec{p})\vec{q}=12\hat{i}-12\hat{j}+12\hat{k}$; *b.* $\|\vec{p}\|^2=53$

Find the Angle between Two Vectors

When two nonzero vectors are placed in standard position, whether in two dimensions or three dimensions, they form an angle between them. The dot product provides a way to find the measure of this angle.

How to: Evaluate the dot product given the magnitude of 2 vectors and the angle between them

Given two non-zero vectors $\vec{u}$ and $\vec{v}$ and the angle between them, θ, such that $0\le\theta\le\pi$.

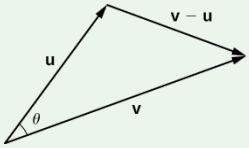


The dot product of the two vectors is the product of the magnitude of each vector and the cosine of the angle between them:

$$\vec{u}\cdot\vec{v}=\|\vec{u}\|\|\vec{v}\|\cos\theta.$$

Proof

Place vectors $\vec{u}$ and $\vec{v}$ in standard position and consider the vector $\vec{v}-\vec{u}$ (see the figure on the right). These three vectors form a triangle with side lengths $\|\vec{u}\|$, $\|\vec{v}\|$, and $\|\vec{v}-\vec{u}\|$.



Recall from trigonometry that the law of cosines describes the relationship among the side lengths of the triangle and the angle θ. Applying the law of cosines here gives

$$\|\vec{v}-\vec{u}\|^2=\|\vec{u}\|^2+\|\vec{v}\|^2-2\|\vec{u}\|\|\vec{v}\|\cos\theta.$$

The dot product provides a way to rewrite the left side of Equation ???:

$$\begin{aligned}\|\vec{v}-\vec{u}\|^2 &= (\vec{v}-\vec{u})\cdot(\vec{v}-\vec{u})\\ &= (\vec{v}-\vec{u})\cdot\vec{v}-(\vec{v}-\vec{u})\cdot\vec{u}\\ &= \vec{v}\cdot\vec{v}-\vec{u}\cdot\vec{v}-\vec{v}\cdot\vec{u}+\vec{u}\cdot\vec{u}\\ &= \vec{v}\cdot\vec{v}-\vec{u}\cdot\vec{v}-\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{u}\\ &= \|\vec{v}\|^2-2\vec{u}\cdot\vec{v}+\|\vec{u}\|^2.\end{aligned}$$

Substituting into the law of cosines yields

$$\|\vec{v}-\vec{u}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$$
$$\|\vec{v}\|^2 - 2\vec{u}\cdot\vec{v} + \|\vec{u}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$$
$$-2\vec{u}\cdot\vec{v} = -2\|\vec{u}\|\|\vec{v}\|\cos\theta$$
$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta. \quad \square$$

(Another proof uses the formula for the cosine of the sum of two angles instead of the Law of Cosines).

We can use the form of the dot product in Equation ??? to find the measure of the angle between two nonzero vectors by rearranging Equation ??? to solve for the cosine of the angle. Using this equation, we can find the cosine of the angle between two nonzero vectors. Since we are considering the smallest angle between the vectors, we assume $0° \le \theta \le 180°$ (or $0 \le \theta \le \pi$ if we are working in radians). The inverse cosine is unique over this range, so we are then able to determine the measure of the angle θ.

How to: Find the angle between two vectors

Given two vectors $\vec{u}$ and $\vec{v}$, then the smallest angle θ between them can be found using the equation:

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}.$$

Example 7.5.3: Find the Angle between Two Vectors

Find the measure of the angle between each pair of vectors.

a. $\hat{i}+\hat{j}+\hat{k}$ and $2\hat{i}-\hat{j}-3\hat{k}$
b. $\langle 2,5,6\rangle$ and $\langle -2,-4,4\rangle$

Solution

a. To find the cosine of the angle formed by the two vectors, substitute the components of the vectors into Equation ???:

$$\cos\theta = \frac{(\hat{i}+\hat{j}+\hat{k})\cdot(2\hat{i}-\hat{j}-3\hat{k})}{\|\,\hat{i}+\hat{j}+\hat{k}\,\|\cdot\|\,2\hat{i}-\hat{j}-3\hat{k}\,\|} = \frac{1(2)+(1)(-1)+(1)(-3)}{\sqrt{1^2+1^2+1^2}\sqrt{2^2+(-1)^2+(-3)^2}} = \frac{-2}{\sqrt{3}\sqrt{14}} = \frac{-2}{\sqrt{42}}.$$

Therefore, $\theta = \arccos\dfrac{-2}{\sqrt{42}} \approx 107.98°$ or ≈ 1.88 rad.

b. Start by finding the value of the cosine of the angle between the vectors:

$$\cos\theta = \frac{\langle 2,5,6\rangle\cdot\langle -2,-4,4\rangle}{\|\,\langle 2,5,6\rangle\,\|\cdot\|\,\langle -2,-4,4\rangle\,\|} = \frac{2(-2)+(5)(-4)+(6)(4)}{\sqrt{2^2+5^2+6^2}\sqrt{(-2)^2+(-4)^2+4^2}} = \frac{0}{\sqrt{65}\sqrt{36}} = 0.$$

Now, $\cos\theta = 0$ and $0 \le \theta \le \pi$, so $\theta = 90°$ or $\pi/2$ radians.

Try It 7.5.3a

Find the measure of the angle, in radians, formed by vectors $\vec{a} = \langle 1,2,0\rangle$ and $\vec{b} = \langle 2,4,1\rangle$. Round to the nearest hundredth.

Hint

Use the Equation ???.

Answer

$\theta \approx 12.61°$ or $\theta \approx 0.22$ rad

Try It 7.5.3b

Let $\vec{\mathbf{v}} = \langle 3, -5, 1 \rangle$. Find the measure of the angles formed by each pair of vectors.

a. $\vec{\mathbf{v}}$ and $\hat{\mathbf{i}}$
b. $\vec{\mathbf{v}}$ and $\hat{\mathbf{j}}$
c. $\vec{\mathbf{v}}$ and $\hat{\mathbf{k}}$

Hint

$\hat{\mathbf{i}} = \langle 1, 0, 0 \rangle, \hat{\mathbf{j}} = \langle 0, 1, 0 \rangle$, and $\hat{\mathbf{k}} = \langle 0, 0, 1 \rangle$

Answer

a. $\alpha \approx 1.04$ rad; b. $\beta \approx 2.58$ rad; c. $\gamma \approx 1.40$ rad

Possible angles between two vectors is illustrated in Figure 7.5.3. The angle between two vectors can be (a) acute $(0 < \cos\theta < 1)$, (b) obtuse $(-1 < \cos\theta < 0)$, or (c) straight $(\cos\theta = -1)$. If $\cos\theta = 1$, then (d) both vectors have the same direction. If $\cos\theta = 0$, then the vectors, when placed in standard position, form (e) a right angle. We can formalize this result into a theorem regarding orthogonal (perpendicular) vectors.

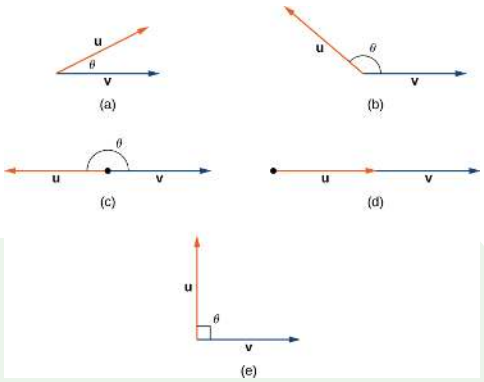


Figure 7.5.3 Possible angles between vectors.

Orthogonal Vectors

The nonzero vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are **orthogonal vectors** if and only if $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0$.

The terms **orthogonal, perpendicular,** and **normal** each indicate that mathematical objects are intersecting at right angles. The use of each term is determined mainly by its context. We say that vectors are **orthogonal** and lines are **perpendicular**. The term **normal** is used most often when measuring the angle made with a plane or other surface.

Example 7.5.4: Identifying Orthogonal Vectors

Determine whether $\vec{\mathbf{p}} = \langle 1, 0, 5 \rangle$ and $\vec{\mathbf{q}} = \langle 10, 3, -2 \rangle$ are orthogonal vectors.

Solution

Using the definition, we need only check the dot product of the vectors:

$$\vec{\mathbf{p}} \cdot \vec{\mathbf{q}} = 1(10) + (0)(3) + (5)(-2) = 10 + 0 - 10 = 0.$$

Because $\vec{\mathbf{p}} \cdot \vec{\mathbf{q}} = 0$, the vectors are orthogonal (Figure 7.5.4).

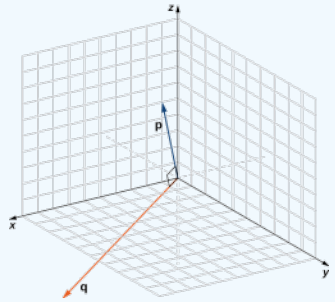


Figure 7.5.4: Vectors $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ form a right angle when their initial points are aligned.

Try It 7.5.4

For which value of x is $\vec{p} = \langle 2, 8, -1 \rangle$ orthogonal to $\vec{q} = \langle x, -1, 2 \rangle$?

Hint

Vectors $\vec{p}$ and $\vec{q}$ are orthogonal if and only if $\vec{p} \cdot \vec{q} = 0$.

Answer

$x = 5$

Projections

As we have seen, addition combines two vectors to create a resultant vector. But what if we are given a vector and we need to find its component parts? We use vector projections to perform the opposite process; they can break down a vector into its components. The magnitude of a vector projection is a scalar projection. For example, if a child is pulling the handle of a wagon at a 55° angle, we can use projections to determine how much of the force on the handle is actually moving the wagon forward (**7.5.6**). We return to this example and learn how to solve it after we see how to calculate projections.

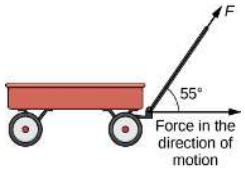


Figure **7.5.6**: When a child pulls a wagon, only the horizontal component of the force propels the wagon forward.

Definition: Vector Projection

The **vector projection** of $\vec{v}$ onto $\vec{u}$ is the portion of$\vec{v}$ that is in the direction of $\vec{u}$. This is the vector labeled $\text{proj}_{\vec{u}}\vec{v}$ in Figure **7.5.7**. It has the same initial point as $\vec{u}$ and $\vec{v}$ and the same direction as $\vec{u}$, and represents the component of $\vec{v}$ that acts in the direction of $\vec{u}$.

If θ represents the angle between $\vec{u}$ and $\vec{v}$, then, by properties of triangles, we know the length of $\text{proj}_{\vec{u}}\vec{v}$ is $\|\text{proj}_{\vec{u}}\vec{v}\| = \|\vec{v}\| \cos\theta$, which can also be expressed in terms of the dot product.

The magnitude of the projection of $\vec{v}$ onto $\vec{u}$:

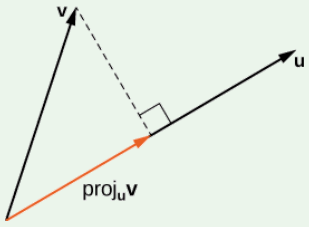


Figure **7.5.7**: Vector Projection.

$$\|\text{proj}_{\vec{u}}\vec{v}\| = \|\vec{v}\| \cos\theta = \|\vec{v}\| \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} \right) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|.} \tag{7.5.2}$$

We now multiply by a unit vector in the direction of $\vec{u}$ to get $\text{proj}_{\vec{u}}\vec{v}$, the vector that is the projection of $\vec{v}$ onto $\vec{u}$:

$$\text{proj}_{\vec{u}}\vec{v} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|} \left(\frac{1}{\|\vec{u}\|} \vec{u} \right) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \vec{u}. \tag{7.5.3}$$

Example 7.5.5: Finding Projections

Find the projection of $\vec{v}$ onto $\vec{u}$.

a. $\vec{v} = \langle 3, 5, 1 \rangle$ and $\vec{u} = \langle -1, 4, 3 \rangle$
b. $\vec{v} = 3\hat{i} - 2\hat{j}$ and $\vec{u} = \hat{i} + 6\hat{j}$

Solution

a. Substitute the components of $\vec{v}$ and $\vec{u}$ into the formula for the projection:

$$\begin{aligned}\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} &= \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2}\vec{\mathbf{u}} \\ &= \frac{\langle -1,4,3\rangle \cdot \langle 3,5,1\rangle}{\| \langle -1,4,3\rangle \|^2}\langle -1,4,3\rangle \\ &= \frac{-3+20+3}{(-1)^2+4^2+3^2}\langle -1,4,3\rangle \\ &= \frac{20}{26}\langle -1,4,3\rangle \\ &= \left\langle -\frac{10}{13},\frac{40}{13},\frac{30}{13}\right\rangle.\end{aligned}$$

b. To find the two-dimensional projection, simply adapt the formula to the two-dimensional case:

$$\begin{aligned}\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} &= \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2}\vec{\mathbf{u}} \\ &= \frac{(\hat{\mathbf{i}}+6\hat{\mathbf{j}})\cdot(3i-2j)}{\|\hat{\mathbf{i}}+6\hat{\mathbf{j}}\|^2}(\hat{\mathbf{i}}+6\hat{\mathbf{j}})j) \\ &= \frac{1(3)+6(-2)}{1^2+6^2}(\hat{\mathbf{i}}+6\hat{\mathbf{j}}) \\ &= -\frac{9}{37}(\hat{\mathbf{i}}+6\hat{\mathbf{j}}) \\ &= -\frac{9}{37}\hat{\mathbf{i}}-\frac{54}{37}\hat{\mathbf{j}}.\end{aligned}$$

Sometimes it is useful to decompose vectors—that is, to break a vector apart into a sum. This process is called the **resolution of a vector into components.** Projections allow us to identify two orthogonal vectors having a desired sum. For example, let $\vec{\mathbf{v}} = \langle 6,-4\rangle$ and let $\vec{\mathbf{u}} = \langle 3,1\rangle$. We want to decompose the vector $\vec{\mathbf{v}}$ into orthogonal components such that one of the component vectors has the same direction as $\vec{\mathbf{u}}$.

We first find the component that has the same direction as $\vec{\mathbf{u}}$ by projecting $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$. Let $\vec{\mathbf{p}} = \text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}$. Then, we have

$$\begin{aligned}\vec{\mathbf{p}} &= \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2}\vec{\mathbf{u}} = \frac{18-4}{9+1}\vec{\mathbf{u}} = \frac{7}{5}\vec{\mathbf{u}} \\ &= \frac{7}{5}\langle 3,1\rangle = \left\langle \frac{21}{5},\frac{7}{5}\right\rangle.\end{aligned}$$

Now consider the vector $\vec{\mathbf{q}} = \vec{\mathbf{v}} - \vec{\mathbf{p}}$. We have

$$\begin{aligned}\vec{\mathbf{q}} &= \vec{\mathbf{v}} - \vec{\mathbf{p}} \\ &= \left\langle 6,-4\right\rangle - \left\langle \frac{21}{5},\frac{7}{5}\right\rangle \\ &= \left\langle \frac{9}{5},-\frac{27}{5}\right\rangle.\end{aligned}$$

Clearly, by the way we defined $\vec{\mathbf{q}}$, we have $\vec{\mathbf{v}} = \vec{\mathbf{q}} + \vec{\mathbf{p}}$, and

$$\begin{aligned}\vec{\mathbf{q}}\cdot\vec{\mathbf{p}} &= \left\langle \frac{9}{5},-\frac{27}{5}\right\rangle \cdot \left\langle \frac{21}{5},\frac{7}{5}\right\rangle \\ &= \frac{9(21)}{25} + -\frac{27(7)}{25} \\ &= \frac{189}{25} - \frac{189}{25} = 0.\end{aligned}$$

Therefore, $\vec{\mathbf{q}}$ and $\vec{\mathbf{p}}$ are orthogonal.

Example 7.5.6: Resolving Vectors into Components

Express $\vec{\mathbf{v}} = \langle 8, -3, -3 \rangle$ as a sum of orthogonal vectors such that one of the vectors has the same direction as $\vec{\mathbf{u}} = \langle 2, 3, 2 \rangle$.

Solution

Let $\vec{\mathbf{p}}$ represent the projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$:

$$\begin{aligned} \vec{\mathbf{p}} = \text{proj}_{\vec{\mathbf{u}}} \vec{\mathbf{v}} &= \frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2} \vec{\mathbf{u}} \\ &= \frac{\langle 2, 3, 2 \rangle \cdot \langle 8, -3, -3 \rangle}{\| \langle 2, 3, 2 \rangle \|^2} \langle 2, 3, 2 \rangle \\ &= \frac{16 - 9 - 6}{2^2 + 3^2 + 2^2} \langle 2, 3, 2 \rangle = \frac{1}{17} \langle 2, 3, 2 \rangle = \left\langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17} \right\rangle. \end{aligned}$$

Then,

$$\begin{aligned} \vec{\mathbf{q}} = \vec{\mathbf{v}} - \vec{\mathbf{p}} &= \langle 8, -3, -3 \rangle - \left\langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17} \right\rangle \\ &= \left\langle \frac{134}{17}, -\frac{54}{17}, -\frac{53}{17} \right\rangle. \end{aligned}$$

To check our work, we can use the dot product to verify that $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ are orthogonal vectors:

$$\begin{aligned} \vec{\mathbf{p}} \cdot \vec{\mathbf{q}} &= \left\langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17} \right\rangle \cdot \left\langle \frac{134}{17}, -\frac{54}{17}, -\frac{53}{17} \right\rangle \\ &= \frac{268}{17} - \frac{162}{17} - \frac{106}{17} = 0. \end{aligned}$$

Then,

$$\vec{\mathbf{v}} = \vec{\mathbf{p}} + \vec{\mathbf{q}} = \left\langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17} \right\rangle + \left\langle \frac{134}{17}, -\frac{54}{17}, -\frac{53}{17} \right\rangle.$$

Try It 7.5.6

Express $\vec{\mathbf{v}} = 5\hat{\mathbf{i}} - \hat{\mathbf{j}}$ as a sum of orthogonal vectors such that one of the vectors has the same direction as $\vec{\mathbf{u}} = 4\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$.

Hint

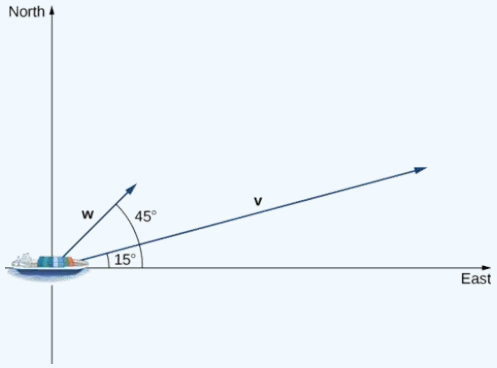
Start by finding the projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$.

Answer

$\vec{\mathbf{v}} = \vec{\mathbf{p}} + \vec{\mathbf{q}}$, where $\vec{\mathbf{p}} = \frac{18}{5}\hat{\mathbf{i}} + \frac{9}{5}\hat{\mathbf{j}}$ and $\vec{\mathbf{q}} = \frac{7}{5}\hat{\mathbf{i}} - \frac{14}{5}\hat{\mathbf{j}}$

Example 7.5.7: Scalar Projection of Velocity

A container ship leaves port traveling $15°$ north of east. Its engine generates a speed of 20 knots along that path (see the following figure). In addition, the ocean current moves the ship northeast at a speed of 2 knots. Considering both the engine and the current, how fast is the ship moving in the direction $15°$ north of east? Round the answer to two decimal places.



Solution

Let $\vec{\mathbf{v}}$ be the velocity vector generated by the engine, and let w be the velocity vector of the current. We already know $\|\vec{\mathbf{v}}\| = 20$ along the desired route. We just need to add in the scalar projection of $\vec{\mathbf{w}}$ onto $\vec{\mathbf{v}}$. We get

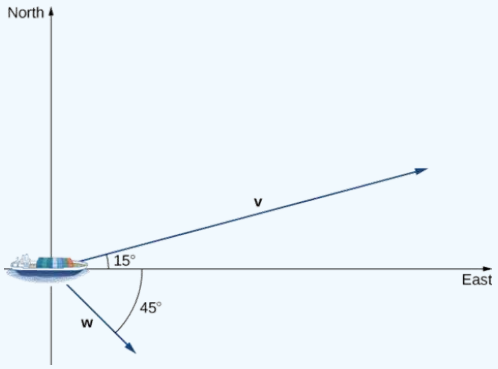
https://math.libretexts.org/@go/page/34953

$$\text{magnitude of projection} = \frac{\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}}{\|\vec{\mathbf{v}}\|} = \frac{\|\vec{\mathbf{v}}\|\|\vec{\mathbf{w}}\| \cos(30°)}{\|\vec{\mathbf{v}}\|} = \|\vec{\mathbf{w}}\| \cos(30°) = 2\frac{\sqrt{3}}{2} = \sqrt{3} \approx 1.76 \text{ knots}.$$ [4pt]

The ship is moving at 21.73 knots in the direction $15°$ north of east.

Try It 7.5.7

Repeat the previous example, but assume the ocean current is moving southeast instead of northeast, as shown in the following figure.



Hint

Compute the scalar projection of $\vec{\mathbf{w}}$ onto $\vec{\mathbf{v}}$.

Answer

21 knots

Work

Now that we understand dot products, we can see how to apply them to real-life situations. The most common application of the dot product of two vectors is in the calculation of work.

From physics, we know that work is done when an object is moved by a force. When the force is constant and applied in the same direction the object moves, then we define the work done as the product of the force and the distance the object travels: $W = Fd$. Now imagine the direction of the force is different from the direction of motion, as with the example of a child pulling a wagon. To find the work done, we need to multiply the component of the force that acts in the direction of the motion by the magnitude of the displacement. The dot product allows us to do just that. If we represent an applied force by a vector $\vec{\mathbf{F}}$ and the displacement of an object by a vector $\vec{\mathbf{s}}$, then the **work done by the force** is the dot product of $\vec{\mathbf{F}}$ and $\vec{\mathbf{s}}$.

Definition: Constant Force

When a constant force is applied to an object so the object moves in a straight line from point P to point Q, the work W done by the force $\vec{\mathbf{F}}$, acting at an angle θ from the line of motion, is given by

$$W = \vec{\mathbf{F}} \cdot \overrightarrow{PQ} = \| \vec{\mathbf{F}} \| \| \overrightarrow{PQ} \| \cos\theta.$$

Let's revisit the problem of the child's wagon introduced earlier. Suppose a child is pulling a wagon with a force having a magnitude of 8 lb on the handle at an angle of $55°$. If the child pulls the wagon 50 ft, find the work done by the force (Figure 7.5.8).

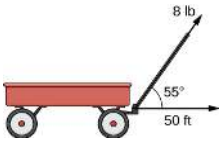


Figure 7.5.8: The horizontal component of the force is the projection of $\vec{\mathbf{F}}$ *onto the positive* x*-axis.*

We have

$$W = \| \vec{\mathbf{F}} \| \| \overrightarrow{PQ} \| \cos\theta = 8(50)(\cos(55°)) \approx 229 \text{ ft·lb}.$$

In U.S. standard units, we measure the magnitude of force $\| \vec{\mathbf{F}} \|$ in pounds. The magnitude of the displacement vector $\| \overrightarrow{PQ} \|$ tells us how far the object moved, and it is measured in feet. The customary unit of measure for work, then, is the foot-pound. One foot-

pound is the amount of work required to move an object weighing 1 lb a distance of 1 ft straight up. In the metric system, the unit of measure for force is the newton (N), and the unit of measure of magnitude for work is a newton-meter (N·m), or a joule (J).

Example 7.5.8: Calculating Work

A conveyor belt generates a force $\vec{\mathbf{F}} = 5\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$ that moves a suitcase from point $(1, 1, 1)$ to point $(9, 4, 7)$ along a straight line. Find the work done by the conveyor belt. The distance is measured in meters and the force is measured in newtons.

Solution

The displacement vector $\overrightarrow{PQ}$ has initial point $(1, 1, 1)$ and terminal point $(9, 4, 7)$:

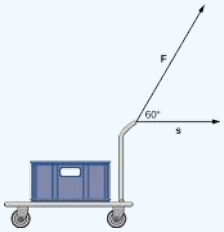
$$\overrightarrow{PQ} = \langle 9 - 1, 4 - 1, 7 - 1 \rangle = \langle 8, 3, 6 \rangle = 8\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}.$$

Work is the dot product of force and displacement:

$$\begin{aligned} W &= \vec{\mathbf{F}} \cdot \overrightarrow{PQ} \\ &= (5\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}) \cdot (8\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) \\ &= 5(8) + (-3)(3) + 1(6) \\ &= 37\,\text{N·m} = 37\,\text{J} \end{aligned}$$

Try It 7.5.8

A constant force of 30 lb is applied at an angle of 60° to pull a handcart 10 ft across the ground. What is the work done by this force?



Hint

Use the definition of work as the dot product of force and distance.

Answer

150 ft-lb

Key Concepts

- The **dot product**, or scalar product, of two vectors $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ is $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1v_1 + u_2v_2 + u_3v_3$.
- The **dot product** of two vectors can be expressed, alternatively, as $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\| \cos\theta$. This form of the dot product is useful for finding the measure of the angle formed by two vectors.
- The cosine of the angle formed by $\vec{\mathbf{u}}$ **and** $\vec{\mathbf{v}}$ is $\cos\theta = \dfrac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|}$
- The dot product satisfies the following properties:
  - $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \vec{\mathbf{v}} \cdot \vec{\mathbf{u}}$
  - $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} + \vec{\mathbf{u}} \cdot \vec{\mathbf{w}}$
  - $c(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}) = (c\vec{\mathbf{u}}) \cdot \vec{\mathbf{v}} = \vec{\mathbf{u}} \cdot (c\vec{\mathbf{v}})$
  - $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = \|\vec{\mathbf{v}}\|^2$
- Vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are **orthogonal** if $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0$.
- The **vector projection** of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$ is the vector $\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} = \dfrac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2}\vec{\mathbf{u}}$.

- The **magnitude of the vector projection** is known as the ***scalar projection*** of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$, given by $\text{comp}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} = \dfrac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|} = \|\vec{\mathbf{v}}\| \cos\theta$.

- **Work** is done when a force is applied to an object, causing displacement. When the force is represented by the vector $\vec{\mathbf{F}}$ and the displacement is represented by the vector $\vec{\mathbf{s}}$, then the work done W is given by the formula $W = \vec{\mathbf{F}} \cdot \vec{\mathbf{s}} = \| \vec{\mathbf{F}} \| \, \|\vec{\mathbf{s}}\| \cos\theta$.

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7.5e: The Dot Product

A: Evaluate the Dot Product

Exercise 7.5e.A

Given vectors $\vec{u}$ and $\vec{v}$, calculate the dot product $\vec{u} \cdot \vec{v}$

1) $\vec{u} = \langle 3, 0 \rangle, \quad \vec{v} = \langle 2, 2 \rangle$

2) $\vec{u} = \langle 3, -4 \rangle, \quad \vec{v} = \langle 4, 3 \rangle$

3) $\vec{u} = \langle 2, 2, -1 \rangle, \quad \vec{v} = \langle -1, 2, 2 \rangle$

4) $\vec{u} = \langle 4, 5, -6 \rangle, \quad \vec{v} = \langle 0, -2, -3 \rangle$

5) $\vec{u} = \langle -1, 6 \rangle, \quad \vec{v} = \langle 6, -1 \rangle$

6) $\vec{u} = \langle -2, 4 \rangle, \quad \vec{v} = \langle -3, 1 \rangle$

7) $\vec{u} = -\hat{i} - \hat{j}, \quad \vec{v} = \hat{i} + 5\hat{j}$

8) $\vec{u} = 3\hat{i} - 4\hat{j}, \quad \vec{v} = -2\hat{i} + 3\hat{j}$

Given vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, determine the vectors $(\vec{a} \cdot \vec{b})\vec{c}$ and $(\vec{a} \cdot \vec{c})\vec{b}$. Express the vectors in component form.

11) $\vec{a} = \langle 2, 0, -3 \rangle, \quad \vec{b} = \langle -4, -7, 1 \rangle, \quad \vec{c} = \langle 1, 1, -1 \rangle$

12) $\vec{a} = \langle 0, 1, 2 \rangle, \quad \vec{b} = \langle -1, 0, 1 \rangle, \quad \vec{c} = \langle 1, 0, -1 \rangle$

13) $\vec{a} = \hat{i} + \hat{j}, \quad \vec{b} = \hat{i} - \hat{k}, \quad \vec{c} = \hat{i} - 2\hat{k}$

14) $\vec{a} = \hat{i} - \hat{j} + \hat{k}, \quad \vec{b} = \hat{j} + 3\hat{k}, \quad \vec{c} = -\hat{i} + 2\hat{j} - 4\hat{k}$

Answers to odd exercises:

1. **6** 3. **0** 5. **-12** 7. **-6** 11. $(\vec{a} \cdot \vec{b})\vec{c} = \langle -11, -11, 11 \rangle$; $(\vec{a} \cdot \vec{c})b = \langle -20, -35, 5 \rangle$
13. $(\vec{a} \cdot \vec{b})\vec{c} = \langle 1, 0, -2 \rangle$; $(\vec{a} \cdot \vec{c})\vec{b} = \langle 1, 0, -1 \rangle$

B: Find the angle between 2 Vectors

Exercise 7.5e.B

Given two vectors,

a. Find the measure of the angle θ between a and b. Express the answer in degrees rounded to the nearest tenth, if it is not possible to express it exactly.

b. Is θ an acute angle?

21) $\vec{a} = \langle 3, -1 \rangle, \quad \vec{b} = \langle -4, 0 \rangle$

22) $\vec{a} = \langle 2, 1 \rangle, \quad \vec{b} = \langle -1, 3 \rangle$

23) $\vec{u} = 3\hat{i}, \quad \vec{v} = 4\hat{i} + 4\hat{j}$

24) $\vec{u} = 5\hat{i}, \quad \vec{v} = -6\hat{i} + 6\hat{j}$

Find the measure of the angle between the three-dimensional vectors $\vec{a}$ and $\vec{b}$. Express the answer in degrees rounded to the nearest tenth, if it is not possible to express it exactly.

25) $\vec{a} = \langle 3, -1, 2 \rangle, \quad \vec{b} = \langle 1, -1, -2 \rangle$

26) $\vec{a} = \langle 0, -1, -3 \rangle, \quad \vec{b} = \langle 2, 3, -1 \rangle$

27) $\vec{a} = \hat{i} + \hat{j}, \quad \vec{b} = \hat{j} - \hat{k}$

28) $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}, \quad \vec{b} = \hat{i} + \hat{j} - 2\hat{k}$

29) $\vec{a} = 3\hat{i} - \hat{j} - 2\hat{k}, \quad \vec{b} = \vec{v} + \vec{w}$, where $\vec{v} = -2\hat{i} - 3\hat{j} + 2\hat{k}$ and $\vec{w} = \hat{i} + 2\hat{k}$

30) $\vec{a} = 3\hat{i} - \hat{j} + 2\hat{k}, \quad \vec{b} = \vec{v} - \vec{w}$, where $\vec{v} = 2\hat{i} + \hat{j} + 4\hat{k}$ and $\vec{w} = 6\hat{i} + \hat{j} + 2\hat{k}$

33) Consider the points $P(4, 5)$ and $Q(5, -7)$, and note that O represents the origin.

a. Determine vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. Express the answer by using standard unit vectors.

b. Determine the measure of angle O in triangle OPQ. Express the answer in degrees rounded to two decimal places.

34) Consider points $A(1, 1), B(2, -7)$, and $C(6, 3)$.

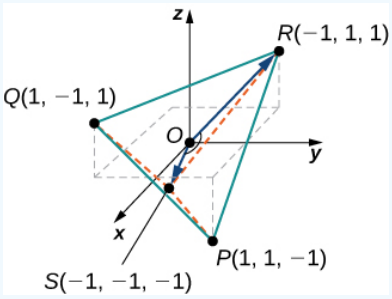
a. Determine vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$. Express the answer in component form.

b. Determine the measure of angle B in triangle ABC. Express the answer in degrees rounded to two decimal places.

35) Determine the measure of angle A in triangle ABC, where $A(1, 1, 8), B(4, -3, -4)$, and $C(-3, 1, 5)$. Express your answer in degrees rounded to two decimal places.

36) Consider points $P(3, 7, -2)$ and $Q(1, 1, -3)$, with O representing the origin. Determine the angle between vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. Express the answer in degrees rounded to two decimal places.

37. A methane molecule has a carbon atom situated at the origin and four hydrogen atoms located at points $P(1, 1, -1), Q(1, -1, 1), R(-1, 1, 1)$, and $S(-1, -1, -1)$ (see figure).



a. Find the distance between the hydrogen atoms located at P and R.

b. Find the angle between vectors $\vec{OS}$ and $\vec{OR}$ that connect the carbon atom with the hydrogen atoms located at S and R, which is also called the *bond angle*. Express the answer in degrees rounded to two decimal places.

38. Find the vectors that join the center of a clock to the hours 1:00, 2:00, and 3:00. Assume the clock is circular with a radius of 1 unit.

Answers to odd exercises:

21. $a.\ \theta = 161.6°$; $b.\ \theta$ is not acute. 23. $a.\ \theta = 45°$; $b.\ \theta$ is acute. 25. $\theta = 90°$ 27. $\theta = 60°$
29. $\theta = 114.6°$ 33. a. $\overrightarrow{OP} = 4\hat{i} + 5\hat{j}, \ \overrightarrow{OQ} = 5\hat{i} - 7\hat{j}$ b. $105.8°$ 35. $68.33°$ 37. $a.\ 2\sqrt{2}$; $b.\ 109.47°$

D: Orthogonality

Exercise 7.5e.D

Determine whether the given vectors are orthogonal.

41) $\vec{a} = \langle x, y \rangle, \quad \vec{b} = \langle -y, x \rangle$, where x and y are nonzero real numbers

42) $\vec{a} = \langle x, x \rangle, \quad \vec{b} = \langle -y, y \rangle$, where x and y are nonzero real numbers

43) $\vec{a} = 3\hat{i} - \hat{j} - 2\hat{k}, \quad \vec{b} = -2\hat{i} - 3\hat{j} + \hat{k}$

44) $\vec{a} = \hat{i} - \hat{j}, \quad \vec{b} = 7\hat{i} + 2\hat{j} - \hat{k}$

45) Determine the real number α such that vectors $\vec{a} = 2\hat{i} + 3\hat{j}$ and $\vec{b} = 9\hat{i} + \alpha\hat{j}$ are orthogonal.

46) Determine the real number α such that vectors $\vec{a} = -3\hat{i} + 2\hat{j}$ and $\vec{b} = 2\hat{i} + \alpha\hat{j}$ are orthogonal.

Determine which (if any) pairs of the following vectors are orthogonal.

47) $\vec{\mathbf{u}} = \langle 3, 7, -2 \rangle, \quad \vec{\mathbf{v}} = \langle 5, -3, -3 \rangle, \quad \vec{\mathbf{w}} = \langle 0, 1, -1 \rangle$

48) $\vec{\mathbf{u}} = \hat{\mathbf{i}} - \hat{\mathbf{k}}, \quad \vec{\mathbf{v}} = 5\hat{\mathbf{j}} - 5\hat{\mathbf{k}}, \quad \vec{\mathbf{w}} = 10\hat{\mathbf{j}}$

Answers to odd exercises:

41. Orthogonal 43. Not orthogonal 45. $\alpha = -6$

47. $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are orthogonal; $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ are orthogonal, but $\vec{\mathbf{u}}$ and $\vec{\mathbf{w}}$ are **not** orthogonal.

E: Projections

Exercise 7.5e.E

For the following exercises, the vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are given.

a. Find the vector projection $\text{Proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}$ of vector $\vec{\mathbf{v}}$ onto vector $\vec{\mathbf{u}}$ and the component of $\vec{\mathbf{v}}$ that is orthogonal to $\vec{\mathbf{u}}$, i.e., $\vec{\mathbf{v}}_{\text{perp}}$. Express your answers in component form.

b. Find the magnitude of the projection, $\|\text{Proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}\|$, of vector $\vec{\mathbf{v}}$ onto vector $\vec{\mathbf{u}}$.

c. Find the vector projection $\text{Proj}_{\vec{\mathbf{v}}}\vec{\mathbf{u}}$ of vector $\vec{\mathbf{u}}$ onto vector $\vec{\mathbf{v}}$ and the component of $\vec{\mathbf{u}}$ that is orthogonal to $\vec{\mathbf{v}}$, i.e., $\vec{\mathbf{u}}_{\text{perp}}$. Express your answers in unit vector form.

d. Find the magnitude of the projection, $\|\text{Proj}_{\vec{\mathbf{v}}}\vec{\mathbf{u}}\|$, of vector $\vec{\mathbf{u}}$ onto vector $\vec{\mathbf{v}}$.

51. $\vec{\mathbf{u}} = 5\hat{\mathbf{i}} + 2\hat{\mathbf{j}}, \quad \vec{\mathbf{v}} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$

52. $\vec{\mathbf{u}} = \langle 4, 4, 0 \rangle, \quad \vec{\mathbf{v}} = \langle 0, 4, 1 \rangle$

53. $\vec{\mathbf{u}} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{k}}, \quad \vec{\mathbf{v}} = 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$

55. Consider the vectors $\vec{\mathbf{u}} = 4\hat{\mathbf{i}} - 3\hat{\mathbf{j}}$ and $\vec{\mathbf{v}} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$.

a. Find the component form of vector $\text{Proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}$ that represents the projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$..

b. Write the decomposition $\vec{\mathbf{v}} = \vec{\mathbf{w}} + \vec{\mathbf{q}}$ of vector $\vec{\mathbf{v}}$ into the orthogonal components $\vec{\mathbf{w}}$ and $\vec{\mathbf{q}}$, where $\vec{\mathbf{w}}$ is the projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$ and $\vec{\mathbf{q}}$ is the vector component of $\vec{\mathbf{v}}$ orthogonal to the direction of $\vec{\mathbf{u}}$. That is, $\vec{\mathbf{q}} = \vec{\mathbf{v}}_{\text{perp}}$.

56. Consider vectors $\vec{\mathbf{u}} = 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$ and $\vec{\mathbf{v}} = 4\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$.

a. Find the component form of vector $\vec{\mathbf{w}} = \text{Proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}$ that represents the projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$.b. Write the decomposition $\vec{\mathbf{v}} = \vec{\mathbf{w}} + \vec{\mathbf{q}}$ of vector $\vec{\mathbf{v}}$ into the orthogonal components $\vec{\mathbf{w}}$ and $\vec{\mathbf{q}}$, where $\vec{\mathbf{w}}$ is the projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$ and $\vec{\mathbf{q}}$ is a vector orthogonal to the direction of $\vec{\mathbf{u}}$.

Answers to odd exercises:

51. a. $\text{Proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} = \langle \frac{80}{29}, \frac{32}{29} \rangle$ and $\vec{\mathbf{v}}_{\text{perp}} = < -\frac{22}{29}, \frac{55}{29} >$;
b. $\|\text{Proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}\| = \frac{16}{\sqrt{29}} = \frac{16\sqrt{29}}{29}$;
c. $\text{Proj}_{\vec{\mathbf{v}}}\vec{\mathbf{u}} = \frac{32}{13}\hat{\mathbf{i}} + \frac{48}{13}\hat{\mathbf{j}}$ and $\vec{\mathbf{u}}_{\text{perp}} = \frac{33}{13}\hat{\mathbf{i}} - \frac{22}{13}\hat{\mathbf{j}}$;
d. $\|\text{Proj}_{\vec{\mathbf{v}}}\vec{\mathbf{u}}\| = \frac{16}{\sqrt{13}} = \frac{16\sqrt{13}}{13}$

53. a. $\text{Proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} = \langle \frac{24}{13}, 0, \frac{16}{13} \rangle$ and $\vec{\mathbf{v}}_{\text{perp}} = < -\frac{24}{13}, \frac{26}{13}, \frac{36}{13} >$;
b. $\|\text{Proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}\| = \frac{8}{\sqrt{13}} = \frac{8\sqrt{13}}{13}$
c. $\text{Proj}_{\vec{\mathbf{v}}}\vec{\mathbf{u}} = \frac{4}{5}\hat{\mathbf{j}} + \frac{8}{5}\hat{\mathbf{k}}$ and $\vec{\mathbf{u}}_{\text{perp}} = 3\hat{\mathbf{i}} - \frac{4}{5}\hat{\mathbf{j}} + \frac{2}{5}\hat{\mathbf{k}}$;
d. $\|\text{Proj}_{\vec{\mathbf{v}}}\vec{\mathbf{u}}\| = \frac{\sqrt{80}}{5} = \frac{4\sqrt{5}}{5}$;

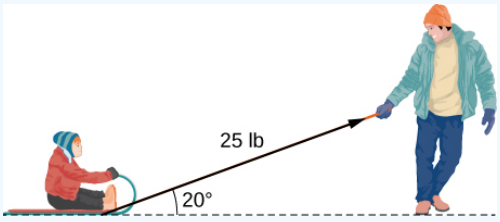
55. a. $\text{Proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} = \langle \frac{24}{25}, -\frac{18}{25} \rangle$;
b. $\vec{\mathbf{q}} = \vec{\mathbf{v}}_{\text{perp}} = \langle \frac{51}{25}, \frac{68}{25} \rangle$,
$\vec{\mathbf{v}} = \vec{\mathbf{w}} + \vec{\mathbf{q}} = \text{Proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} + \vec{\mathbf{v}}_{\text{perp}} = \langle \frac{24}{25}, -\frac{18}{25} \rangle + \langle \frac{51}{25}, \frac{68}{25} \rangle$
So we have that, $\vec{\mathbf{v}} = \langle \frac{24}{25}, -\frac{18}{25} \rangle + \langle \frac{51}{25}, \frac{68}{25} \rangle$.

Exercise 7.5e.F

61. Find the work done by force $\vec{\mathbf{F}} = \langle 5, 6, -2 \rangle$ (measured in Newtons) that moves a particle from point $P(3, -1, 0)$ to point $Q(2, 3, 1)$ along a straight line (the distance is measured in meters).

62. A sled is pulled by exerting a force of 100 N on a rope that makes an angle of $25°$ with the horizontal. Find the work done in pulling the sled 40 m. (Round the answer to one decimal place.)

63. A father is pulling his son on a sled at an angle of $20°$with the horizontal with a force of 25 lb (see the following image). He pulls the sled in a straight path of 50 ft. How much work was done by the man pulling the sled? (Round the answer to the nearest integer.)



64. A car is towed using a force of 1600 N. The rope used to pull the car makes an angle of 25° with the horizontal. Find the work done in towing the car 2 km. Express the answer in joules ($1J = 1N \cdot m$) rounded to the nearest integer.

65. A boat sails north aided by a wind blowing in a direction of $N30°E$ with a magnitude of 500 lb. How much work is performed by the wind as the boat moves 100 ft? (Round the answer to two decimal places.)

66. Vector $\vec{\mathbf{p}} = \langle 150, 225, 375 \rangle$ represents the price of certain models of bicycles sold by a bicycle shop. Vector $\vec{\mathbf{n}} = \langle 10, 7, 9 \rangle$ represents the number of bicycles sold of each model, respectively. Compute the dot product $\vec{\mathbf{p}} \cdot \vec{\mathbf{n}}$ and state its meaning.

67. Two forces $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ are represented by vectors with initial points that are at the origin. The first force has a magnitude of 20 lb and the terminal point of the vector is point $P(1, 1, 0)$. The second force has a magnitude of 40 lb and the terminal point of its vector is point $Q(0, 1, 1)$. Let $\vec{\mathbf{F}}$ be the resultant force of forces $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$.

a. Find the magnitude of $\vec{\mathbf{F}}$. (Round the answer to one decimal place.)

Answers to odd exercises:

61. $17N \cdot m$

63. 1175 ft·lb

65. $25000\sqrt{3}$ ft-lbs $\approx 43,301.27$ ft-lbs
Vector representing the wind: $\vec{\mathbf{w}} = 500 \cos 60° \hat{\mathbf{i}} + 500 \sin 60° \hat{\mathbf{j}}$
Vector representing the displacement to the north: $\vec{\mathbf{d}} = 100\hat{\mathbf{j}}$
Work done by the wind: $W = \vec{\mathbf{w}} \cdot \vec{\mathbf{d}} = 25000\sqrt{3}$ ft-lbs $\approx 43,301.27$ ft-lbs

67. $a. \| F_1 + F_2 \| = 52.9$ lb;

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7.6: The Cross Product

Learning Objectives

- Calculate the cross product of two given vectors.
- Use determinants to calculate a cross product.
- Find a vector orthogonal to two given vectors.
- Determine areas by using the cross product.

Imagine a mechanic turning a wrench to tighten a bolt. The mechanic applies a force at the end of the wrench. This creates rotation, or torque, which tightens the bolt. We can use vectors to represent the force applied by the mechanic, and the distance (radius) from the bolt to the end of the wrench. Then, we can represent torque by a vector oriented along the axis of rotation. Note that the torque vector is orthogonal to both the force vector and the radius vector.

In this section, we develop an operation called the **cross product**, which allows us to find a vector orthogonal to two given vectors. Calculating torque is an important application of cross products, and we examine torque in more detail later in the section.

The Cross Product and Its Properties

The dot product is a multiplication of two vectors that results in a scalar. In this section, we introduce a product of two vectors that generates a third vector orthogonal to the first two. Consider how we might find such a vector. Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ be nonzero vectors. We want to find a vector $\vec{\mathbf{w}} = \langle w_1, w_2, w_3 \rangle$ orthogonal to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$—that is, we want to find $\vec{\mathbf{w}}$ such that $\vec{\mathbf{u}} \cdot \vec{\mathbf{w}} = 0$ and $\vec{\mathbf{v}} \cdot \vec{\mathbf{w}} = 0$. Therefore, w_1, w_2, and w_3 must satisfy

$$u_1 w_1 + u_2 w_2 + u_3 w_3 = 0 \tag{7.6.1}$$

$$v_1 w_1 + v_2 w_2 + v_3 w_3 = 0. \tag{7.6.2}$$

If we multiply the top equation by v_3 and the bottom equation by u_3 and subtract, we can eliminate the variable w_3, which gives

$$(u_1 v_3 - v_1 u_3) w_1 + (u_2 v_3 - v_2 u_3) w_2 = 0.$$

If we select

$$w_1 = u_2 v_3 - u_3 v_2$$
$$w_2 = -(u_1 v_3 - u_3 v_1),$$

we get a possible solution vector. Substituting these values back into the original equations (Equations 7.6.1 and 7.6.2) gives

$$w_3 = u_1 v_2 - u_2 v_1.$$

That is, vector

$$\vec{\mathbf{w}} = \langle u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 \rangle \tag{7.6.3}$$

is orthogonal to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$. Unfortunately, this formula is complicated and difficult to remember, so determinant notation can be used to simplify the process.

Determinants and the Cross Product

Using the formula in Equation 7.6.3 to find the cross product is difficult to remember. Fortunately, we have an alternative. We can calculate the cross product of two vectors using determinant notation. Using determinants to evaluate a cross product is easier because there is fundamentally just a simple pattern to remember, rather than a complicated formula.

A 2×2 determinant is defined by

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - b_1 a_2. \tag{7.6.4}$$

For example,

$$\begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix} = 3(1) - 5(-2) = 3 + 10 = 13. \tag{7.6.5}$$

A 3×3 determinant is defined in terms of 2×2 determinants as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}. \tag{7.6.6}$$

Equation 7.6.6 is referred to as the expansion of the determinant along the first row. Notice that the multipliers of each of the 2×2 determinants on the right side of this expression are the entries in the first row of the 3×3 determinant. Furthermore, each of the 2×2 determinants contains the entries from the 3×3 determinant that would remain if you crossed out the row and column containing the multiplier.

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Thus, for the first term on the right, a_1 is the multiplier, and the 2×2 determinant contains the entries that remain if you cross out the first row and first column of the 3×3 determinant. Similarly, for the second term, the multiplier is a_2, and the 2×2 determinant contains the entries that remain if you cross out the first row and second column of the 3×3 determinant. Notice, however, that the coefficient of the second term is negative. The third term can be calculated in similar fashion.

Observe that the expansion of each 2×2 determinant corresponds to the components listed in Equation 7.6.3 when $\vec{\mathbf{u}} = \langle b_1, b_2, b_3 \rangle$ and $\vec{\mathbf{v}} = \langle c_1, c_2, c_3 \rangle$. If the entries along the first row of the determinant are made the standard unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$, we can obtain the determinant definition of the cross product.

Definition: Cross Product

The **cross product** $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is a vector that is perpendicular to two given vectors.

Given vectors $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$, the *cross product* is vector is defined by the formula

$$\begin{aligned} \vec{\mathbf{u}} \times \vec{\mathbf{v}} &= (u_2 v_3 - u_3 v_2)\hat{\mathbf{i}} - (u_1 v_3 - u_3 v_1)\hat{\mathbf{j}} + (u_1 v_2 - u_2 v_1)\hat{\mathbf{k}} \\ &= \langle u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 \rangle. \end{aligned} \tag{7.6.7}$$

The cross product is more easily calculated using determinant notation:

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \hat{\mathbf{i}} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \hat{\mathbf{j}} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \hat{\mathbf{k}} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_4 \end{vmatrix}. \tag{7.6.8}$$

The next few examples provide practice evaluating determinants in general.

Example 7.6.1: Using Expansion Along the First Row to Compute a 3×3 Determinant

Evaluate the determinant $\begin{vmatrix} 2 & 5 & -1 \\ -1 & 1 & 3 \\ -2 & 3 & 4 \end{vmatrix}$.

Solution

We have

$$\begin{aligned} \begin{vmatrix} 2 & 5 & -1 \\ -1 & 1 & 3 \\ -2 & 3 & 4 \end{vmatrix} &= 2 \begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} - 5 \begin{vmatrix} -1 & 3 \\ -2 & 4 \end{vmatrix} - 1 \begin{vmatrix} -1 & 1 \\ -2 & 3 \end{vmatrix} \\ &= 2(4 - 9) - 5(-4 + 6) - 1(-3 + 2) \\ &= 2(-5) - 5(2) - 1(-1) = -10 - 10 + 1 \\ &= -19 \end{aligned}$$

Try It 7.6.1

Evaluate the determinant $\begin{vmatrix} 1 & -2 & -1 \\ 3 & 2 & -3 \\ 1 & 5 & 4 \end{vmatrix}$.

Hint

Expand along the first row. Don't forget the second term is negative!

Answer

40

Cross Product Evaluation

Technically, determinants are defined only in terms of arrays of real numbers. However, the determinant notation provides a useful mnemonic device for the cross product formula. Now for some practice calculating cross products.

Example 7.6.2: Using Determinant Notation to find $\vec{p} \times \vec{q}$

Let $\vec{p} = \langle -1, 2, 5 \rangle$ and $\vec{q} = \langle 4, 0, -3 \rangle$. Find $\vec{p} \times \vec{q}$.

Solution

We set up our determinant by putting the standard unit vectors across the first row, the components of $\vec{u}$ in the second row, and the components of $\vec{v}$ in the third row. Then, we have

$$\begin{aligned} \vec{p} \times \vec{q} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 5 \\ 4 & 0 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 5 \\ 0 & -3 \end{vmatrix} \hat{i} - \begin{vmatrix} -1 & 5 \\ 4 & -3 \end{vmatrix} \hat{j} + \begin{vmatrix} -1 & 2 \\ 4 & 0 \end{vmatrix} \hat{k} \\ &= (-6 - 0)\hat{i} - (3 - 20)\hat{j} + (0 - 8)\hat{k} \\ &= -6\hat{i} + 17\hat{j} - 8\hat{k}. \end{aligned}$$

Notice that this answer confirms the calculation of the cross product in Example 7.6.1.

Try It 7.6.2

Use determinant notation to find $\vec{a} \times \vec{b}$, where $\vec{a} = \langle 8, 2, 3 \rangle$ and $\vec{b} = \langle -1, 0, 4 \rangle$.

Hint

Calculate the determinant $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 8 & 2 & 3 \\ -1 & 0 & 4 \end{vmatrix}$.

Answer

$\vec{a} \times \vec{b} = 8\hat{i} - 35\hat{j} + 2\hat{k}$

Properties of the Cross Product

The Cross Product is a vector orthogonal to two other vectors

From the way we have developed $\vec{u} \times \vec{v}$, it should be clear that the cross product is orthogonal to both $\vec{u}$ and $\vec{v}$. However, it never hurts to check. To show that $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$, we calculate the dot product of $\vec{u}$ and $\vec{u} \times \vec{v}$.

$$\begin{aligned}\vec{u}\cdot(\vec{u}\times\vec{v}) &= \langle u_1,u_2,u_3\rangle\cdot\langle u_2v_3-u_3v_2,-u_1v_3+u_3v_1,u_1v_2-u_2v_1\rangle\\ &= u_1(u_2v_3-u_3v_2)+u_2(-u_1v_3+u_3v_1)+u_3(u_1v_2-u_2v_1)\\ &= u_1u_2v_3-u_1u_3v_2-u_1u_2v_3+u_2u_3v_1+u_1u_3v_2-u_2u_3v_1\\ &= (u_1u_2v_3-u_1u_2v_3)+(-u_1u_3v_2+u_1u_3v_2)+(u_2u_3v_1-u_2u_3v_1)\\ &= 0\end{aligned}$$

In a similar manner, we can show that the cross product is also orthogonal to $\vec{v}$.

Although it may not be obvious from Equation 7.6.7, the direction of $\vec{u}\times\vec{v}$ is given by the right-hand rule. If we hold the right hand out with the fingers pointing in the direction of $\vec{u}$, then curl the fingers toward vector $\vec{v}$, the thumb points in the direction of the cross product, as shown in Figure 7.6.2.

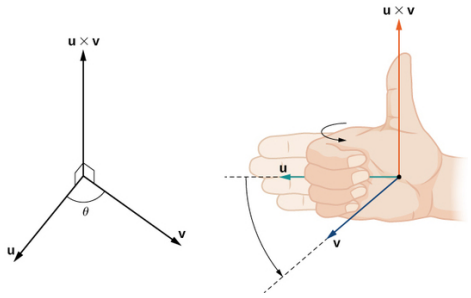

Figure 7.6.3: The direction of $\vec{u}\times\vec{v}$ *is determined by the right-hand rule.*

Notice what this means for the direction of $\vec{v}\times\vec{u}$. If we apply the right-hand rule to $\vec{v}\times\vec{u}$, we start with our fingers pointed in the direction of $\vec{v}$, then curl our fingers toward the vector $\vec{u}$. In this case, the thumb points in the opposite direction of $\vec{u}\times\vec{v}$. (Try it!)

Example 7.6.3: Anticommutativity of the Cross Product

Let $\vec{u}=\langle 0,2,1\rangle$ and $\vec{v}=\langle 3,-1,0\rangle$. Calculate $\vec{u}\times\vec{v}$ and $\vec{v}\times\vec{u}$ and graph them.

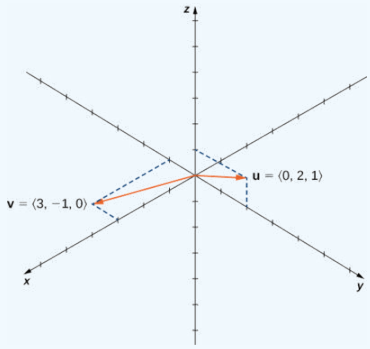

Figure 7.6.3: Are the cross products $\vec{u}\times\vec{v}$ and $\vec{v}\times\vec{u}$ in the same direction?

Solution

We have

$$\vec{u}\times\vec{v}=\langle(0+1),-(0-3),(0-6)\rangle=\langle 1,3,-6\rangle$$

$$\vec{v}\times\vec{u}=\langle(-1-0),-(3-0),(6-0)\rangle=\langle -1,-3,6\rangle.$$

We see that, in this case, $\vec{u}\times\vec{v}=-(\vec{v}\times\vec{u})$ (Figure 7.6.4). We prove this in general later in this section.

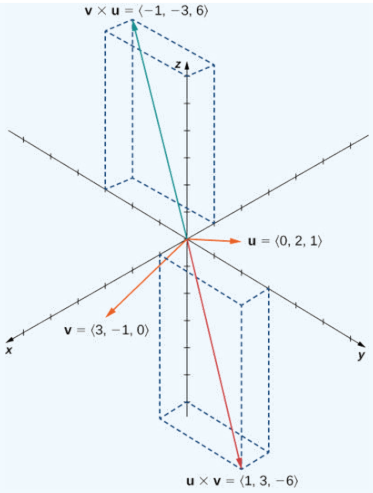

Figure 7.6.4: The cross products $\vec{u} \times \vec{v}$ *and* $\vec{v} \times \vec{u}$ *are both orthogonal to* $\vec{u}$ *and* $\vec{v}$*, but in opposite directions.*

Try It 7.6.3

Suppose vectors $\vec{u}$ and $\vec{v}$ lie in the xy-plane (the z-component of each vector is zero). Now suppose the x- and y-components of $\vec{u}$ and the y-component of $\vec{v}$ are all positive, whereas the x-component of $\vec{v}$ is negative. Assuming the coordinate axes are oriented in the usual positions, in which direction does $\vec{u} \times \vec{v}$ point?

Hint

Remember the right-hand rule (Figure 7.6.2).

Answer

Up (the positive z-direction)

Magnitude of the Cross Product

So far in this section, we have been concerned with the direction of the vector $\vec{u} \times \vec{v}$, but we have not discussed its magnitude. It turns out there is a simple expression for the magnitude of $\vec{u} \times \vec{v}$ involving the magnitudes of $\vec{u}$ and $\vec{v}$, and the sine of the angle between them.

Magnitude of the Cross Product

Let $\vec{u}$ and $\vec{v}$ be vectors, and let θ be the angle between them. Then,

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \sin\theta.$$

Proof

Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ be vectors, and let θ denote the angle between them. Then

$$\begin{aligned}
\|\vec{u} \times \vec{v}\|^2 &= (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2 \\
&= u_2^2v_3^2 - 2u_2u_3v_2v_3 + u_3^2v_2^2 + u_3^2v_1^2 - 2u_1u_3v_1v_3 + u_1^2v_3^2 + u_1^2v_2^2 - 2u_1u_2v_1v_2 + u_2^2v_1^2 \\
&= u_1^2v_1^2 + u_1^2v_2^2 + u_1^2v_3^2 + u_2^2v_1^2 + u_2^2v_2^2 + u_2^2v_3^2 + u_3^2v_1^2 + u_3^2v_2^2 + u_3^2v_3^2 \\
&\quad - (u_1^2v_1^2 + u_2^2v_2^2 + u_3^2v_3^2 + 2u_1u_2v_1v_2 + 2u_1u_3v_1v_3 + 2u_2u_3v_2v_3) \\
&= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2 \\
&= \|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2 \\
&= \|\vec{u}\|^2\|\vec{v}\|^2 - \|\vec{u}\|^2\|\vec{v}\|^2 \cos^2\theta \\
&= \|\vec{u}\|^2\|\vec{v}\|^2(1 - \cos^2\theta) \\
&= \|\vec{u}\|^2\|\vec{v}\|^2(\sin^2\theta).
\end{aligned}$$

Taking square roots and noting that $\sqrt{\sin^2\theta} = \sin\theta$ for $0 \le \theta \le 180°$, we have the desired result:

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin\theta. \tag{7.6.9}$$

This definition of the cross product allows us to visualize or interpret the product geometrically. It is clear, for example, that the cross product is defined only for vectors in three dimensions, not for vectors in two dimensions. In two dimensions, it is impossible to generate a vector simultaneously orthogonal to two nonparallel vectors.

Example 7.6.4: Calculating the Cross Product

Find the magnitude of the cross product of $\vec{u} = \langle 0, 4, 0\rangle$ and $\vec{v} = \langle 0, 0, -3\rangle$.

Solution

We have

$$\begin{aligned}\|\vec{u} \times \vec{v}\| &= \|\vec{u}\| \cdot \|\vec{v}\| \cdot \sin\theta \\ &= \sqrt{0^2 + 4^2 + 0^2} \cdot \sqrt{0^2 + 0^2 + (-3)^2} \cdot \sin\frac{\pi}{2} \\ &= 4(3)(1) = 12\end{aligned}$$

Try It 7.6.4

Find the magnitude of $\vec{u} \times \vec{v}$, where $\vec{u} = \langle -8, 0, 0\rangle$ and $\vec{v} = \langle 0, 2, 0\rangle$.

Hint

Vectors $\vec{u}$ and $\vec{v}$ are orthogonal.

Answer

16

The Unit Vector Orthogonal to Two Other Vectors

The cross product is very useful for several types of calculations, including finding a vector orthogonal to two given vectors, computing areas of triangles and parallelograms, and even determining the volume of the three-dimensional geometric shape made of parallelograms known as a *parallelepiped*. The following examples illustrate these calculations.

Example 7.6.5: Finding a Unit Vector Orthogonal to Two Given Vectors

Let $\vec{a} = \langle 5, 2, -1\rangle$ and $\vec{b} = \langle 0, -1, 4\rangle$. Find a unit vector orthogonal to both $\vec{a}$ and $\vec{b}$.

Solution

The cross product $\vec{a} \times \vec{b}$ is orthogonal to both vectors $\vec{a}$ and $\vec{b}$. We can calculate it with a determinant:

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 2 & -1 \\ 0 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix}\hat{i} - \begin{vmatrix} 5 & -1 \\ 0 & 4 \end{vmatrix}\hat{j} + \begin{vmatrix} 5 & 2 \\ 0 & -1 \end{vmatrix}\hat{k} \\ &= (8-1)\hat{i} - (20-0)\hat{j} + (-5-0)\hat{k} \\ &= 7\hat{i} - 20\hat{j} - 5\hat{k}.\end{aligned}$$

Normalize this vector to find a unit vector in the same direction:

$$\|\vec{a} \times \vec{b}\| = \sqrt{(7)^2 + (-20)^2 + (-5)^2} = \sqrt{474}.$$

Thus, $\left\langle \frac{7}{\sqrt{474}}, \frac{-20}{\sqrt{474}}, \frac{-5}{\sqrt{474}} \right\rangle$ is a unit vector orthogonal to $\vec{a}$ and $\vec{b}$.

Simplified, this vector becomes $\left\langle \frac{7\sqrt{474}}{474}, \frac{-10\sqrt{474}}{237}, \frac{-5\sqrt{474}}{474} \right\rangle$.

Try It 7.6.5

Find a unit vector orthogonal to both $\vec{a}$ and $\vec{b}$, where $\vec{a} = \langle 4, 0, 3 \rangle$ and $\vec{b} = \langle 1, 1, 4 \rangle$.

Hint

Normalize the cross product.

Answer

$\left\langle \frac{-3}{\sqrt{194}}, \frac{-13}{\sqrt{194}}, \frac{4}{\sqrt{194}} \right\rangle$ or, simplified as $\left\langle \frac{-3\sqrt{194}}{194}, \frac{-13\sqrt{194}}{194}, \frac{2\sqrt{194}}{97} \right\rangle$

Area of a Parallelogram

To use the cross product for calculating areas, we state and prove the following theorem.

Area of a Parallelogram

If we locate vectors $\vec{u}$ and $\vec{v}$ such that they form adjacent sides of a parallelogram, then the area of the parallelogram is given by $\|\vec{u} \times \vec{v}\|$ (Figure 7.6.5).

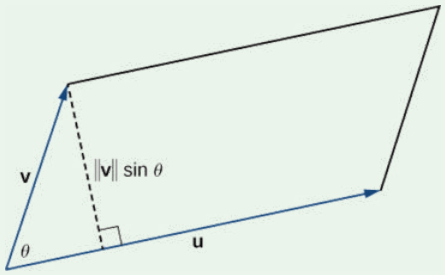

Figure 7.6.6: The parallelogram with adjacent sides $\vec{u}$ and $\vec{v}$ has base $\|\vec{u}\|$ and height $\|\vec{v}\| \sin\theta$.

Proof

We show that the magnitude of the cross product is equal to the base times height of the parallelogram.

$$\begin{aligned} \text{Area of a parallelogram} &= \text{base} \times \text{height} \\ &= \|\vec{u}\|(\|\vec{v}\| \sin\theta) \\ &= \|\vec{u} \times \vec{v}\| \end{aligned}$$

Example 7.6.6: Finding the Area of a Triangle

Let $P = (1, 0, 0), Q = (0, 1, 0)$, and $R = (0, 0, 1)$ be the vertices of a triangle (Figure 7.6.6*a*). Find its area.

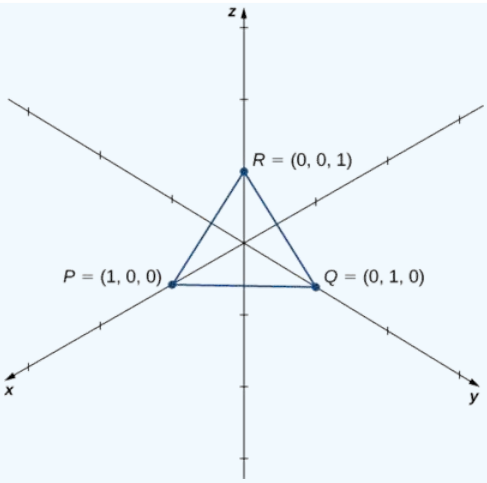

Figure **7.6.6a**: Finding the area of a triangle by using the cross product.

Solution

We have $\overrightarrow{PQ} = \langle 0-1, 1-0, 0-0\rangle = \langle -1, 1, 0\rangle$ and $\overrightarrow{PR} = \langle 0-1, 0-0, 1-0\rangle = \langle -1, 0, 1\rangle$. The area of the parallelogram with adjacent sides $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is given by $\| \overrightarrow{PQ} \times \overrightarrow{PR} \|$:

$$\begin{aligned} \overrightarrow{PQ} \times \overrightarrow{PR} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} \\ &= (1-0)\hat{\mathbf{i}} - (-1-0)\hat{\mathbf{j}} + (0-(-1))\hat{\mathbf{k}} \\ &= \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}} \end{aligned}$$

$$\begin{aligned} \| \overrightarrow{PQ} \times \overrightarrow{PR} \| &= \| \langle 1, 1, 1\rangle \| \\ &= \sqrt{1^2 + 1^2 + 1^2} \\ &= \sqrt{3}. \end{aligned}$$

The area of ΔPQR is half the area of the parallelogram or $\sqrt{3}/2$ units^2.

Try It 7.6.6

Find the area of the parallelogram $PQRS$ with vertices $P(1,1,0)$, $Q(7,1,0)$, $R(9,4,2)$, and $S(3,4,2)$.

Hint

Sketch the parallelogram and identify two vectors that form adjacent sides of the parallelogram.

Answer

$6\sqrt{13}$ units^2

Key Concepts

- The cross product $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ of two vectors $\vec{\mathbf{u}} = \langle u_1, u_2, u_3\rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3\rangle$ is a vector orthogonal to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$. Its length is given by $\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\| = \|\vec{\mathbf{u}}\| \cdot \|\vec{\mathbf{v}}\| \cdot \sin\theta$, where θ is the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$. Its direction is given by the right-hand rule.
- The algebraic formula for calculating the cross product of two vectors,

  $\vec{\mathbf{u}} = \langle u_1, u_2, u_3\rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3\rangle$, is

  $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (u_2v_3 - u_3v_2)\hat{\mathbf{i}} - (u_1v_3 - u_3v_1)\hat{\mathbf{j}} + (u_1v_2 - u_2v_1)\hat{\mathbf{k}}$.

- The cross product of vectors $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ is the determinant $\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
- If vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ form adjacent sides of a parallelogram, then the area of the parallelogram is given by $\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\|$.
- The cross product can be used to identify a vector orthogonal to two given vectors or to a plane.

Key Equations

- **The cross product of two vectors in terms of the unit vectors**

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (u_2v_3 - u_3v_2)\hat{\mathbf{i}} - (u_1v_3 - u_3v_1)\hat{\mathbf{j}} + (u_1v_2 - u_2v_1)\hat{\mathbf{k}} \tag{7.6.10}$$

Glossary

cross product

$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (u_2v_3 - u_3v_2)\hat{\mathbf{i}} - (u_1v_3 - u_3v_1)\hat{\mathbf{j}} + (u_1v_2 - u_2v_1)\hat{\mathbf{k}}$, where $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$

determinant

a real number associated with a square matrix

vector product

the cross product of two vectors

Contributors and Attributions

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7.6e: Exercises - Cross Product

A: Cross Products

Exercise 7.6e.A: Cross Products

1) Calculate the determinant $\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 7 \\ 2 & 0 & 3 \end{vmatrix}$.

2) Calculate the determinant $\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 3 & -4 \\ 1 & 6 & -1 \end{vmatrix}$.

Given the vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.

a. Find the cross product $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ of the vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$. Express the answer in component form.

b. Sketch the vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$.

3) $\vec{\mathbf{u}} = \langle 2, 0, 0 \rangle$, $\vec{\mathbf{v}} = \langle 2, 2, 0 \rangle$

4) $\vec{\mathbf{u}} = \langle 3, 2, -1 \rangle$, $\vec{\mathbf{v}} = \langle 1, 1, 0 \rangle$

5) $\vec{\mathbf{u}} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$, $\vec{\mathbf{v}} = \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

6) $\vec{\mathbf{u}} = 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, $\vec{\mathbf{v}} = 3\hat{\mathbf{i}} + \hat{\mathbf{k}}$

7) Find vector $(\vec{\mathbf{a}} - 2\vec{\mathbf{b}}) \times \vec{\mathbf{c}}$, where $\vec{\mathbf{a}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -1 & 5 \\ 0 & 1 & 8 \end{vmatrix}$, $\vec{\mathbf{b}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 1 & 1 \\ 2 & -1 & -2 \end{vmatrix}$, and $\vec{\mathbf{c}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$.

8) Find vector $\vec{\mathbf{c}} \times (\vec{\mathbf{a}} + 3\vec{\mathbf{b}})$, where $\vec{\mathbf{a}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 5 & 0 & 9 \\ 0 & 1 & 0 \end{vmatrix}$, $\vec{\mathbf{b}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & -1 & 1 \\ 7 & 1 & -1 \end{vmatrix}$, and $\vec{\mathbf{c}} = \hat{\mathbf{i}} - \hat{\mathbf{k}}$.

Find vector $\vec{\mathbf{w}}$ orthogonal to vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.

9) $\vec{\mathbf{u}} = \langle -1, 0, e^t \rangle$, $\vec{\mathbf{v}} = \langle 1, e^{-t}, 0 \rangle$, where t is a real number

10) $\vec{\mathbf{u}} = \langle 1, 0, x \rangle$, $\vec{\mathbf{v}} = \langle \frac{2}{x}, 1, 0 \rangle$, where x is a nonzero real number

Find unit vector $\vec{\mathbf{w}}$ in the direction of the cross product vector $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$. Express your answer using standard unit vectors.

11) $\vec{\mathbf{u}} = \langle 3, -1, 2 \rangle$, $\vec{\mathbf{v}} = \langle -2, 0, 1 \rangle$

12) $\vec{\mathbf{u}} = \langle 2, 6, 1 \rangle$, $\vec{\mathbf{v}} = \langle 3, 0, 1 \rangle$

13) $\vec{\mathbf{u}} = \overrightarrow{AB}$, $\vec{\mathbf{v}} = \overrightarrow{AC}$, where $A(1, 0, 1)$, $B(1, -1, 3)$, and $C(0, 0, 5)$

14) $\vec{\mathbf{u}} = \overrightarrow{OP}$, $\vec{\mathbf{v}} = \overrightarrow{PQ}$, where $P(-1, 1, 0)$ and $Q(0, 2, 1)$

15) Determine the real number α such that $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ and $\hat{\mathbf{i}}$ are orthogonal, where $\vec{\mathbf{u}} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ and $\vec{\mathbf{v}} = 4\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \alpha\hat{\mathbf{k}}$.

16) Show that $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ and $2\hat{\mathbf{i}} - 14\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ cannot be orthogonal for any α real number, where $\vec{\mathbf{u}} = \hat{\mathbf{i}} + 7\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\vec{\mathbf{v}} = \alpha\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + \hat{\mathbf{k}}$.

17) Show that $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is orthogonal to $\vec{\mathbf{u}} + \vec{\mathbf{v}}$ and $\vec{\mathbf{u}} - \vec{\mathbf{v}}$, where $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are nonzero vectors.

18) Show that $\vec{\mathbf{v}} \times \vec{\mathbf{u}}$ is orthogonal to $(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}})(\vec{\mathbf{u}} + \vec{\mathbf{v}}) + \vec{\mathbf{u}}$, where $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are nonzero vectors.

21) Find the area of the parallelogram with adjacent sides $\vec{\mathbf{u}} = \langle 3, 2, 0 \rangle$ and $\vec{\mathbf{v}} = \langle 0, 2, 1 \rangle$.

22) Find the area of the parallelogram with adjacent sides $\vec{\mathbf{u}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\vec{\mathbf{v}} = \hat{\mathbf{i}} + \hat{\mathbf{k}}$.

23) Consider points $A(3, -1, 2)$, $B(2, 1, 5)$, and $C(1, -2, -2)$.

a. Find the area of parallelogram $ABCD$ with adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

b. Find the area of triangle ABC.

c. Find the distance from point A to line BC.

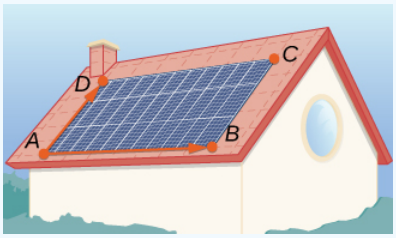
24) Consider points $A(2,-3,4)$, $B(0,1,2)$, and $C(-1,2,0)$.

a. Find the area of parallelogram $ABCD$ with adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

b. Find the area of triangle ABC.

c. Find the distance from point B to line AC.

26) A solar panel is mounted on the roof of a house. The panel may be regarded as positioned at the points of coordinates (in meters) $A(8,0,0), B(8,18,0), C(0,18,8)$, and $D(0,0,8)$ (see the following figure).



a. Find vector $\vec{\mathbf{n}} = \overrightarrow{AB} \times \overrightarrow{AD}$ perpendicular to the surface of the solar panels. Express the answer using standard unit vectors. Note that the magnitude of this vector should give us the area of rectangle $ABCD$.

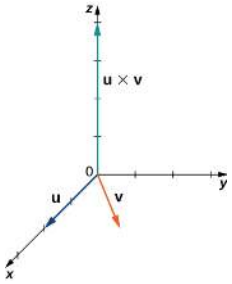
b. Assume unit vector $\vec{\mathbf{s}} = \frac{1}{\sqrt{3}}\hat{\mathbf{i}} + \frac{1}{\sqrt{3}}\hat{\mathbf{j}} + \frac{1}{\sqrt{3}}\hat{\mathbf{k}}$ points toward the Sun at a particular time of the day and the flow of solar energy is $\vec{\mathbf{F}} = 900\vec{\mathbf{s}}$ (in watts per square meter $[W/m^2]$). Find the predicted amount of electrical power the panel can produce, which is given by the dot product of vectors $\vec{\mathbf{F}}$ and $\vec{\mathbf{n}}$ (expressed in watts).

c. Determine the angle of elevation of the Sun above the solar panel. Express the answer in degrees rounded to the nearest whole number. (Hint: The angle between vectors $\vec{\mathbf{n}}$ and $\vec{\mathbf{s}}$ and the angle of elevation are complementary.)

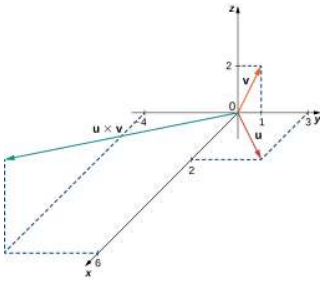
Answers 1-29:

1. $-3\hat{\mathbf{i}} + 11\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

3. a. $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \langle 0,0,4 \rangle$;
3. b.



5. a. $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \langle 6,-4,2 \rangle$;
5. b.



7. $-26\hat{\mathbf{i}} + 17\hat{\mathbf{j}} + 9\hat{\mathbf{k}}$ 9. $\vec{\mathbf{w}} = \langle -1, e^t, -e^{-t} \rangle$
11. $\vec{\mathbf{w}} = -\frac{\sqrt{6}}{18}\hat{\mathbf{i}} - \frac{7\sqrt{6}}{18}\hat{\mathbf{j}} - \frac{\sqrt{6}}{9}\hat{\mathbf{k}}$
13. $\vec{\mathbf{w}} = -\frac{4\sqrt{21}}{21}\hat{\mathbf{i}} - \frac{2\sqrt{21}}{21}\hat{\mathbf{j}} - \frac{\sqrt{21}}{21}\hat{\mathbf{k}}$ 15. $\alpha = 10$ 21. 7 23. a. $5\sqrt{6}$; b. $\frac{5\sqrt{6}}{2}$; c. $\frac{5\sqrt{6}}{\sqrt{59}} = \frac{5\sqrt{354}}{59}$

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