MATH 333: INTRODUCTION TO COLLEGE ALGEBRA



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Math 333 Introduction to College Algebra

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Licensing

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CHAPTER OVERVIEW

1: Prerequisites

In this chapter, we will review sets of numbers and properties of operations used to manipulate numbers. This understanding will serve as prerequisite knowledge throughout our study of algebra and trigonometry.

1.0: Prelude to Prerequisites
1.1: Real Numbers - Algebra Essentials
1.2: Exponents and Scientific Notation
1.3: Radicals and Rational Expressions
1.4: Polynomials
1.5: Factoring Polynomials
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1.0: Prelude to Prerequisites

It’s a cold day in Antarctica. In fact, it’s always a cold day in Antarctica. Earth’s southernmost continent, Antarctica experiences the coldest, driest, and windiest conditions known. The coldest temperature ever recorded, over one hundred degrees below zero on the Celsius scale, was recorded by remote satellite. It is no surprise then, that no native human population can survive the harsh conditions. Only explorers and scientists brave the environment for any length of time.



Figure **1.0.1**: Credit - Andreas Kambanls

Measuring and recording the characteristics of weather conditions in in Antarctica requires a use of different kinds of numbers. Calculating with them and using them to make predictions requires an understanding of relationships among numbers. In this chapter, we will review sets of numbers and properties of operations used to manipulate numbers. This understanding will serve as prerequisite knowledge throughout our study of algebra and trigonometry.

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1.1: Real Numbers - Algebra Essentials

Learning Objectives

- Classify a real number as a natural, whole, integer, rational, or irrational number.
- Perform calculations using order of operations.
- Use the following properties of real numbers: commutative, associative, distributive, inverse, and identity.
- Evaluate algebraic expressions.
- Simplify algebraic expressions.

It is often said that mathematics is the language of science. If this is true, then the language of mathematics is numbers. The earliest use of numbers occurred 100 centuries ago in the Middle East to count, or enumerate items. Farmers, cattlemen, and tradesmen used tokens, stones, or markers to signify a single quantity—a sheaf of grain, a head of livestock, or a fixed length of cloth, for example. Doing so made commerce possible, leading to improved communications and the spread of civilization.

Three to four thousand years ago, Egyptians introduced fractions. They first used them to show reciprocals. Later, they used them to represent the amount when a quantity was divided into equal parts.

But what if there were no cattle to trade or an entire crop of grain was lost in a flood? How could someone indicate the existence of nothing? From earliest times, people had thought of a “base state” while counting and used various symbols to represent this null condition. However, it was not until about the fifth century A.D. in India that zero was added to the number system and used as a numeral in calculations.

Clearly, there was also a need for numbers to represent loss or debt. In India, in the seventh century A.D., negative numbers were used as solutions to mathematical equations and commercial debts. The opposites of the counting numbers expanded the number system even further.

Because of the evolution of the number system, we can now perform complex calculations using these and other categories of real numbers. In this section, we will explore sets of numbers, calculations with different kinds of numbers, and the use of numbers in expressions.

Classifying a Real Number

The numbers we use for counting, or enumerating items, are the natural numbers: $1, 2, 3, 4, 5$ and so on. We describe them in set notation as $\{1, 2, 3, ...\}$ where the ellipsis $(\cdots)$ indicates that the numbers continue to infinity. The natural numbers are, of course, also called the counting numbers. Any time we enumerate the members of a team, count the coins in a collection, or tally the trees in a grove, we are using the set of natural numbers. The set of whole numbers is the set of natural numbers plus zero: $\{0, 1, 2, 3, ...\}$.

The set of integers adds the opposites of the natural numbers to the set of whole numbers: $\{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$. It is useful to note that the set of integers is made up of three distinct subsets: negative integers, zero, and positive integers. In this sense, the positive integers are just the natural numbers. Another way to think about it is that the natural numbers are a subset of the integers.

$$\overbrace{\cdots, -3, -2, -1}^{\text{negative integers}}, \underbrace{0}_{\text{zero}}, \overbrace{1, 2, 3, \cdots}^{\text{positive integers}}$$

The set of rational numbers is written as $\{\frac{m}{n} | \text{m and n are integers and } n \neq 0\}$.Notice from the definition that rational numbers are fractions (or quotients) containing integers in both the numerator and the denominator, and the denominator is never 0. We can also see that every natural number, whole number, and integer is a rational number with a denominator of 1.

Because they are fractions, any rational number can also be expressed in decimal form. Any rational number can be represented as either:

1. a terminating decimal: $\frac{15}{8} = 1.875$, or
2. a repeating decimal: $\frac{4}{11} = 0.36363636\cdots = 0.\overline{36}$

We use a line drawn over the repeating block of numbers instead of writing the group multiple times.

Example 1.1.1: Writing Integers as Rational Numbers

Write each of the following as a rational number. Write a fraction with the integer in the numerator and 1 in the denominator.

a. 7
b. 0
c. -8

Solution

a. $7 = \frac{7}{1}$

b. $0 = \frac{0}{1}$

c. $-8 = \frac{-8}{1}$

Exercise 1.1.1

Write each of the following as a rational number.

a. 11
b. 3
c. -4

Answer

a. $\frac{11}{1}$
b. $\frac{3}{1}$
c. $-\frac{4}{1}$

Example 1.1.2: Identifying Rational Numbers

Write each of the following rational numbers as either a terminating or repeating decimal.

a. $-\frac{5}{7}$
b. $\frac{15}{5}$
c. $\frac{13}{25}$

Solution

a. a repeating decimal

b. $\frac{15}{5} = 3$(or 3.0), a terminating decimal

c. $\frac{13}{25} = 0.52$, a terminating decimal

Exercise 1.1.2

Write each of the following rational numbers as either a terminating or repeating decimal.

a. $\frac{68}{17}$
b. $\frac{8}{13}$
c. $-\frac{13}{25}$

Answer

a. 4 (or 4.0), terminating
b. $0.\overline{615384}$, repeating
c. -0.85, terminating

Irrational Numbers

At some point in the ancient past, someone discovered that not all numbers are rational numbers. A builder, for instance, may have found that the diagonal of a square with unit sides was not **2** or even **32**, but was something else. Or a garment maker might have observed that the ratio of the circumference to the diameter of a roll of cloth was a little bit more than **3**, but still not a rational number. Such numbers are said to be irrational because they cannot be written as fractions. These numbers make up the set of irrational numbers. Irrational numbers cannot be expressed as a fraction of two integers. It is impossible to describe this set of numbers by a single rule except to say that a number is irrational if it is not rational. So we write this as shown.

$$\{h \mid h \text{ is not a rational number}\} \tag{1.1.1}$$

Example 1.1.3: Differentiating Rational and Irrational Numbers

Determine whether each of the following numbers is rational or irrational. If it is rational, determine whether it is a terminating or repeating decimal.

a. $\sqrt{25}$
b. $\frac{33}{9}$
c. $\sqrt{11}$
d. $\frac{17}{34}$
e. $0.3033033303333\ldots$

Solution

a. $\sqrt{25}$: This can be simplified as $\sqrt{25} = 5$. Therefore,$\sqrt{25}$is rational.
b. $\frac{33}{9}$: Because it is a fraction,$\frac{33}{9}$ is a rational number. Next, simplify and divide.

$$\frac{33}{9} = \frac{33^{\diagup}}{9_{\diagup}}$$

So, $\frac{33}{9}$ is rational and a repeating decimal.
c. $\sqrt{11}$: This cannot be simplified any further. Therefore, $\sqrt{11}$ is an irrational number.
d. $\frac{17}{34}$: Because it is a fraction, $\frac{17}{34}$ is a rational number. Simplify and divide.

$$\frac{17}{34} = 0.5$$

So, $\frac{17}{34}$ is rational and a terminating decimal.
e. $0.3033033303333\ldots$ is not a terminating decimal. Also note that there is no repeating pattern because the group of $3s$ increases each time. Therefore it is neither a terminating nor a repeating decimal and, hence, not a rational number. It is an irrational number.

Exercise 1.1.3

Determine whether each of the following numbers is rational or irrational. If it is rational, determine whether it is a terminating or repeating decimal.

a. $\frac{7}{77}$
b. $\sqrt{81}$
c. $4.27027002700027\ldots$
d. $\frac{91}{13}$
e. $\sqrt{39}$

Answer

a. rational and repeating;
b. rational and terminating;
c. irrational;

d. rational and terminating;
e. irrational

Real Numbers

Given any number n, we know that n is either rational or irrational. It cannot be both. The sets of rational and irrational numbers together make up the set of real numbers. As we saw with integers, the real numbers can be divided into three subsets: negative real numbers, zero, and positive real numbers. Each subset includes fractions, decimals, and irrational numbers according to their algebraic sign (+ or –). Zero is considered neither positive nor negative.

The real numbers can be visualized on a horizontal number line with an arbitrary point chosen as 0, with negative numbers to the left of 0 and positive numbers to the right of 0. A fixed unit distance is then used to mark off each integer (or other basic value) on either side of 0. Any real number corresponds to a unique position on the number line.The converse is also true: Each location on the number line corresponds to exactly one real number. This is known as a one-to-one correspondence. We refer to this as the real number line as shown in Figure (1.1.1.

Figure 1.1.1: The real number line.

Example 1.1.4: Classifying Real Numbers

Classify each number as either positive or negative and as either rational or irrational. Does the number lie to the left or the right of 0 on the number line?

a. $-\frac{10}{3}$
b. $\sqrt{5}$
c. $-\sqrt{289}$
d. -6π
e. $0.615384615384\ldots$

Solution

a. $-\frac{10}{3}$ is negative and rational. It lies to the left of 0 on the number line.
b. $\sqrt{5}$ is positive and irrational. It lies to the right of 0.
c. $-\sqrt{289} = -\sqrt{17^2} = -17$ is negative and rational. It lies to the left of 0.
d. -6π is negative and irrational. It lies to the left of 0.
e. $0.615384615384\ldots$ is a repeating decimal so it is rational and positive. It lies to the right of 0.

Exercise 1.1.4

Classify each number as either positive or negative and as either rational or irrational. Does the number lie to the left or the right of 0 on the number line?

a. $\sqrt{73}$
b. $-11.411411411\ldots$
c. $\frac{47}{19}$
d. $-\frac{\sqrt{5}}{2}$
e. 6.210735

Answer

a. positive, irrational; right
b. negative, rational; left
c. positive, rational; right
d. negative, irrational; left
e. positive, rational; right

Sets of Numbers as Subsets

Beginning with the natural numbers, we have expanded each set to form a larger set, meaning that there is a subset relationship between the sets of numbers we have encountered so far. These relationships become more obvious when seen as a diagram, such as Figure(**1.1.2**).

Figure **1.1.2**: Sets of numbers. **N**: the set of natural numbers. **W**: the set of whole numbers. **I**: the set of integers. **Q**: the set of rational numbers. **Q´**: the set of irrational numbers

SETS OF NUMBERS

The set of natural numbers includes the numbers used for counting: $\{1, 2, 3, ...\}$.

The set of whole numbers is the set of natural numbers plus zero: $\{0, 1, 2, 3, ...\}$.

The set of integers adds the negative natural numbers to the set of whole numbers: $\{..., -3, -2, -1, 0, 1, 2, 3, ...\}$.

The set of rational numbers includes fractions written as $\{\frac{m}{n} | \text{m and n are integers and } n \neq 0\}$.

The set of irrational numbers is the set of numbers that are not rational, are nonrepeating, and are nonterminating: $\{h \parallel \text{h is not a rational number}\}$.

Example 1.1.5: Differentiating the Sets of Numbers

Classify each number as being a natural number (N), whole number (W), integer (I), rational number (Q), and/or irrational number (Q').

a. $\sqrt{36}$
b. $\frac{8}{3}$
c. $\sqrt{73}$
d. -6
e. $3.2121121112...$

Solution

	N	W	I	Q	Q'
a. $\sqrt{36} = 6$	X	X	X	X	
b. $\frac{8}{3} = 2.\overline{6}$				X	
c. $\sqrt{73}$					X
d. -6			X	X	
e. $3.2121121112...$					X

Exercise 1.1.5

Classify each number as being a natural number (N), whole number (W), integer (I), rational number (Q), and/or irrational number (Q').

a. $-\frac{35}{7}$
b. 0
c. $\sqrt{169}$
d. $\sqrt{24}$
e. $4.763763763...$

Answer

	N	W	I	Q	Q'
a. $-\frac{35}{7}$			X	X	
b. 0		X	X	X	
c. $\sqrt{169}$	X	X	X	X	
d. $\sqrt{24}$					X
e. $4.763763763...$				X	

Performing Calculations Using the Order of Operations

When we multiply a number by itself, we square it or raise it to a power of 2. For example, $4^2 = 4 \times 4 = 16$. We can raise any number to any power. In general, the exponential notation an means that the number or variable a is used as a factor n times.

$$a^n = a \cdot a \cdot a \cdots a \qquad \text{n factors}$$

In this notation, a^n is read as the nth power of a, where a is called the base and n is called the exponent. A term in exponential notation may be part of a mathematical expression, which is a combination of numbers and operations. For example, $24 + 6 \times \frac{2}{3} - 4^2$ is a mathematical expression.

To evaluate a mathematical expression, we perform the various operations. However, we do not perform them in any random order. We use the order of operations. This is a sequence of rules for evaluating such expressions.

Recall that in mathematics we use parentheses ( ), brackets [ ], and braces { } to group numbers and expressions so that anything appearing within the symbols is treated as a unit. Additionally, fraction bars, radicals, and absolute value bars are treated as grouping symbols. When evaluating a mathematical expression, begin by simplifying expressions within grouping symbols.

The next step is to address any exponents or radicals. Afterward, perform multiplication and division from left to right and finally addition and subtraction from left to right.

Let's take a look at the expression provided.

$$24 + 6 \times \frac{2}{3} - 4^2$$

There are no grouping symbols, so we move on to exponents or radicals. The number 4 is raised to a power of 2, so simplify 4^2 as 16.

$$24 + 6 \times \frac{2}{3} - 4^2$$

$$24 + 6 \times \frac{2}{3} - 16$$

Next, perform multiplication or division, left to right.

$$24 + 6 \times \frac{2}{3} - 16$$

$$24 + 4 - 16$$

Lastly, perform addition or subtraction, left to right.

$$24 + 4 - 16$$

$$28 - 16$$

$$12$$

Therefore,

$$24 + 6 \times \frac{2}{3} - 4^2 = 12$$

For some complicated expressions, several passes through the order of operations will be needed. For instance, there may be a radical expression inside parentheses that must be simplified before the parentheses are evaluated. Following the order of operations ensures that anyone simplifying the same mathematical expression will get the same result.

ORDER OF OPERATIONS

Operations in mathematical expressions must be evaluated in a systematic order, which can be simplified using the acronym **PEMDAS**:

- **P**(arentheses)
- **E**(xponents)
- **M(**ultiplication) and **D**(ivision)
- **A**(ddition) and **S**(ubtraction)

HOW TO: Given a mathematical expression, simplify it using the order of operations.

1. Simplify any expressions within grouping symbols.
2. Simplify any expressions containing exponents or radicals.
3. Perform any multiplication and division in order, from left to right.
4. Perform any addition and subtraction in order, from left to right.

Example 1.1.6: Using the Order of Operations

Use the order of operations to evaluate each of the following expressions.

a. $(3 \times 2)^2 - 4 \times (6 + 2)$

b. $\dfrac{5^2 - 4}{7} - \sqrt{11 - 2}$

c. $6 - |\, 5 - 8 \,| + 3 \times (4 - 1)$

d. $\dfrac{14 - 3 \times 2}{2 \times 5 - 3^2}$

e. $7 \times (5 \times 3) - 2 \times [(6 - 3) - 4^2] + 1$

Solution

a.

$$\begin{aligned}(3 \times 2)^2 - 4 \times (6 + 2) &= (6)^2 - 4 \times (8) && \text{Simplify parentheses}\\ &= 36 - 4 \times 8 && \text{Simplify exponent}\\ &= 36 - 32 && \text{Simplify multiplication}\\ &= 4 && \text{Simplify subtraction}\end{aligned}$$

b.

$$\begin{aligned}\frac{5^2 - 4}{7} - \sqrt{11 - 2} &= \frac{5^2 - 4}{7} - \sqrt{9} && \text{Simplify grouping symbols (radical)}\\ &= \frac{5^2 - 4}{7} - 3 && \text{Simplify radical}\\ &= \frac{25 - 4}{7} - 3 && \text{Simplify exponent}\\ &= \frac{21}{7} - 3 && \text{Simplify subtraction in numerator}\\ &= 3 - 3 && \text{Simplify division}\\ &= 0 && \text{Simplify subtraction}\end{aligned}$$

Note that in the first step, the radical is treated as a grouping symbol, like parentheses. Also, in the third step, the fraction bar is considered a grouping symbol so the numerator is considered to be grouped.

c.

$$\begin{aligned} 6-|5-8|+3\times(4-1) &= 6-|-3|+3\times 3 && \text{Simplify inside grouping symbols}\\ &= 6-3+3\times 3 && \text{Simplify absolute value}\\ &= 6-3+9 && \text{Simplify multiplication}\\ &= 3+9 && \text{Simplify subtraction}\\ &= 12 && \text{Simplify addition}\end{aligned}$$

d.

$$\begin{aligned} \frac{14-3\times 2}{2\times 5-3^2} &= \frac{14-3\times 2}{2\times 5-9} && \text{Simplify exponent}\\ &= \frac{14-6}{10-9} && \text{Simplify products}\\ &= \frac{8}{1} && \text{Simplify differences}\\ &= 8 && \text{Simplify quotient}\end{aligned}$$

In this example, the fraction bar separates the numerator and denominator, which we simplify separately until the last step.

e.

$$\begin{aligned} 7\times(5\times 3)-2\times[(6-3)-4^2]+1 &= 7\times(15)-2\times[(3)-4^2]+1 && \text{Simplify inside parentheses}\\ &= 7\times(15)-2\times(3-16)+1 && \text{Simplify exponent}\\ &= 7\times(15)-2\times(-13)+1 && \text{Subtract}\\ &= 105+26+1 && \text{Multiply}\\ &= 132 && \text{Add}\end{aligned}$$

? Exercise 1.1.6

Use the order of operations to evaluate each of the following expressions.

a. $\sqrt{5^2-4^2}+7\times(5-4)^2$

b. $1+\dfrac{7\times 5-8\times 4}{9-6}$

c. $|1.8-4.3|+0.4\times\sqrt{15+10}$

d. $\dfrac{1}{2}\times[5\times 3^2-7^2]+\dfrac{1}{3}\times 9^2$

e. $[(3-8^2)-4]-(3-8)$

Answer

a. 10

b. 2

c. 4.5

d. 25

e. −60

Using Properties of Real Numbers

For some activities we perform, the order of certain operations does not matter, but the order of other operations does. For example, it does not make a difference if we put on the right shoe before the left or vice-versa. However, it does matter whether we put on shoes or socks first. The same thing is true for operations in mathematics.

Commutative Properties

The commutative property of addition states that numbers may be added in any order without affecting the sum.

$$a+b=b+a \tag{1.1.2}$$

We can better see this relationship when using real numbers.

$$(-2)+7=5 \text{ and } 7+(-2)=5$$

Similarly, the **commutative property of multiplication** states that numbers may be multiplied in any order without affecting the product.

$$a \times b = b \times a \tag{1.1.3}$$

Again, consider an example with real numbers.

$$(-11) \times (-4) = 44 \text{ and } (-4) \times (-11) = 44$$

It is important to note that neither subtraction nor division is commutative. For example, $17 - 5$ is not the same as $5 - 17$. Similarly, $20 \div 5 \neq 5 \div 20$.

Associative Properties

The associative property of multiplication tells us that it does not matter how we group numbers when multiplying. We can move the grouping symbols to make the calculation easier, and the product remains the same.

$$a(bc) = (ab)c \tag{1.1.4}$$

Consider this example.

$$(3 \times 4) \times 5 = 60 \text{ and } 3 \times (4 \times 5) = 60$$

The associative property of addition tells us that numbers may be grouped differently without affecting the sum.

$$a + (b + c) = (a + b) + c \tag{1.1.5}$$

This property can be especially helpful when dealing with negative integers. Consider this example.

$$[15 + (-9)] + 23 = 29 \text{ and } 15 + [(-9) + 23] = 29$$

Are subtraction and division associative? Review these examples.

$$\begin{aligned} 8 - (3 - 15) &\stackrel{?}{=} (8 - 3) - 15 \\ 8 - (-12) &\stackrel{?}{=} 5 - 15 \\ 20 &\neq 10 \end{aligned}$$

$$\begin{aligned} 64 \div (8 \div 4) &\stackrel{?}{=} (64 \div 8) \div 4 \\ 64 \div 2 &\stackrel{?}{=} 8 \div 4 \\ 32 &\neq 2 \end{aligned}$$

As we can see, neither subtraction nor division is associative.

Distributive Property

The distributive property states that the product of a factor times a sum is the sum of the factor times each term in the sum.

$$a \times (b + c) = a \times b + a \times c \tag{1.1.6}$$

This property combines both addition and multiplication (and is the only property to do so). Let us consider an example.

Note that 4 is outside the grouping symbols, so we distribute the 4 by multiplying it by 12, multiplying it by -7, and adding the products.

Example 1.1.7

To be more precise when describing this property, we say that multiplication distributes over addition. The reverse is not true, as we can see in this example.

$$\begin{aligned} 6 + (3 \times 5) &\stackrel{?}{=} (6 + 3) \times (6 \times 5) \\ 6 + (15) &\stackrel{?}{=} (9) \times (11) \\ 21 &\neq 99 \end{aligned}$$

Multiplication does not distribute over subtraction, and division distributes over neither addition nor subtraction.

A special case of the distributive property occurs when a sum of terms is subtracted.

$$a - b = a + (-b) \tag{1.1.7}$$

For example, consider the difference $12 - (5 + 3)$. We can rewrite the difference of the two terms 12 and $(5 + 3)$ by turning the subtraction expression into addition of the opposite. So instead of subtracting $(5 + 3)$, we add the opposite.

$$12 + (-1) \times (5 + 3)$$

Now, distribute -1 and simplify the result.

$$\begin{aligned} 12 - (5 + 3) &= 12 + (-1) \times (5 + 3) \\ &= 12 + [(-1) \times 5 + (-1) \times 3] \\ &= 12 + (-8) \\ &= 4 \end{aligned}$$

This seems like a lot of trouble for a simple sum, but it illustrates a powerful result that will be useful once we introduce algebraic terms. To subtract a sum of terms, change the sign of each term and add the results. With this in mind, we can rewrite the last example.

$$\begin{aligned} 12 - (5 + 3) &= 12 + (-5 - 3) \\ &= 12 - 8 \\ &= 4 \end{aligned}$$

Identity Properties

The **identity property of addition** states that there is a unique number, called the additive identity (0) that, when added to a number, results in the original number.

$$a + 0 = a \tag{1.1.8}$$

The **identity property of multiplication** states that there is a unique number, called the multiplicative identity (1) that, when multiplied by a number, results in the original number.

$$a \times 1 = a \tag{1.1.9}$$

For example, we have $(-6) + 0 = -6$ and $23 \times 1 = 23$. There are no exceptions for these properties; they work for every real number, including 0 and 1.

Inverse Properties

The inverse property of addition states that, for every real number a, there is a unique number, called the additive inverse (or opposite), denoted $-a$, that, when added to the original number, results in the additive identity, 0.

$$a + (-a) = 0 \tag{1.1.10}$$

For example, if $a = -8$, the additive inverse is 8, since $(-8) + 8 = 0$.

The inverse property of multiplication holds for all real numbers except 0 because the reciprocal of 0 is not defined. The property states that, for every real number a, there is a unique number, called the multiplicative inverse (or reciprocal), denoted $1a$, that, when multiplied by the original number, results in the multiplicative identity, 1.

$$a \times \frac{1}{a} = 1 \tag{1.1.11}$$

For example, if $a = -\frac{2}{3}$, the reciprocal, denoted $\frac{1}{a}$, is $-\frac{3}{2}$ because

$$a \cdot \frac{1}{a} = \left(-\frac{2}{3}\right) \times \left(-\frac{3}{2}\right) = 1$$

PROPERTIES OF REAL NUMBERS

The following properties hold for real numbers a, b, and c.

Table 1.1.1

	Addition	Multiplication
Commutative Property	$a+b=b+a$	$a\times b=b\times a$
Associative Property	$a+(b+c)=(a+b)+c$	$a(bc)=(ab)c$
Distributive Property	$a\times(b+c)=a\times b+a\times c$	
Identity Property	There exists a unique real number called the additive identity, 0, such that, for any real number a $a+0=a$	

There exists a unique real number called the multiplicative identity, 1, such that, for any real number a

$$a\times 1=a$$

Inverse Property

Every real number a has an additive inverse, or opposite, denoted –a, such that

$$a+(-a)=0$$

Every nonzero real number a has a multiplicative inverse, or reciprocal, denoted 1a , such that

$$a\times\left(\frac{1}{a}\right)=1$$

Example 1.1.8: Using Properties of Real Numbers

Use the properties of real numbers to rewrite and simplify each expression. State which properties apply.

a. $3\times 6+3\times 4$
b. $(5+8)+(-8)$
c. $6-(15+9)$
d. $\frac{4}{7}\times\left(\frac{2}{3}\times\frac{7}{4}\right)$
e. $100\times[0.75+(-2.38)]$

Solution

a.

$$\begin{aligned} 3\times 6+3\times 4 &= 3\times(6+4) && \text{Distributive property}\\ &= 3\times 10 && \text{Simplify}\\ &= 30 && \text{Simplify}\end{aligned}$$

b.

$$\begin{aligned} (5+8)+(-8) &= 5+[8+(-8)] && \text{Associative property of addition}\\ &= 5+0 && \text{Inverse property of addition}\\ &= 5 && \text{Identity property of addition}\end{aligned}$$

c.

$$\begin{aligned} 6-(15+9) &= 6+[(-15)+(-9)] && \text{Distributive property}\\ &= 6+(-24) && \text{Simplify}\\ &= -18 && \text{Simplify}\end{aligned}$$

d.

$$\begin{aligned} \frac{4}{7}\times\left(\frac{2}{3}\times\frac{7}{4}\right) &= \frac{4}{7}\times\left(\frac{7}{4}\times\frac{2}{3}\right) && \text{Commutative property of multiplication}\\ &= \left(\frac{4}{7}\times\frac{7}{4}\right)\times\frac{2}{3} && \text{Associative property of multiplication}\\ &= 1\times\frac{2}{3} && \text{Inverse property of multiplication}\\ &= \frac{2}{3} && \text{Identity property of multiplication}\end{aligned}$$

e.

$$\begin{aligned} 100\times[0.75+(-2.38)] &= 100\times 0.75+100\times(-2.38) && \text{Distributive property}\\ &= 75+(-238) && \text{Simplify}\\ &= -163 && \text{Simplify}\end{aligned}$$

? Exercise 1.1.7

Use the properties of real numbers to rewrite and simplify each expression. State which properties apply.

a. $\left(-\frac{23}{5}\right) \times \left[11 \times \left(-\frac{5}{23}\right)\right]$
b. $5 \times (6.2 + 0.4)$
c. $18 - (7 - 15)$
d. $\frac{17}{18} + \left[\frac{4}{9} + \left(-\frac{17}{18}\right)\right]$
e. $6 \times (-3) + 6 \times 3$

Answer

1. 11, commutative property of multiplication
2. 33, distributive property
3. 26, distributive property
4. $\frac{4}{9}$, commutative property of addition, associative property of addition, inverse property of addition, identity property of addition
5. 0, distributive property, inverse property of addition, identity property of addition

Evaluating Algebraic Expressions

So far, the mathematical expressions we have seen have involved real numbers only. In mathematics, we may see expressions such as $x + 5$, $\frac{4}{3}\pi r^3$, or $\sqrt{2m^3n^2}$. In the expression $x + 5$, 5 is called a **constant** because it does not vary and x is called a **variable** because it does. (In naming the variable, ignore any exponents or radicals containing the variable.) An **algebraic expression** is a collection of constants and variables joined together by the algebraic operations of addition, subtraction, multiplication, and division.

We have already seen some real number examples of exponential notation, a shorthand method of writing products of the same factor. When variables are used, the constants and variables are treated the same way.

$$(-3)^5 = (-3) \times (-3) \times (-3) \times (-3) \times (-3) \Rightarrow x^5 = x \times x \times x \times x \times x$$
$$(2 \times 7)^3 = (2 \times 7) \times (2 \times 7) \times (2 \times 7) \quad \Rightarrow (yz)^3 = (yz) \times (yz) \times (yz)$$

In each case, the exponent tells us how many factors of the base to use, whether the base consists of constants or variables.

Any variable in an algebraic expression may take on or be assigned different values. When that happens, the value of the algebraic expression changes. To evaluate an algebraic expression means to determine the value of the expression for a given value of each variable in the expression. Replace each variable in the expression with the given value, then simplify the resulting expression using the order of operations. If the algebraic expression contains more than one variable, replace each variable with its assigned value and simplify the expression as before.

✓ Example 1.1.9: Describing Algebraic Expressions

List the constants and variables for each algebraic expression.

a. $x + 5$
b. $\frac{4}{3}\pi r^3$
c. $\sqrt{2m^3n^2}$

Solution

	Constants	Variables
a. $x+5$	5	x
b. $\frac{4}{3}\pi r^3$	$\frac{4}{3}, \pi$	r
c. $\sqrt{2m^3n^2}$	2	m,n

? Exercise 1.1.8

List the constants and variables for each algebraic expression.

a. $2(L+W)$

b. $4y^3+y$

Answer

	Constants	Variables
a. $2\pi r(r+h)$	$2,\pi$	r,h
b. $2(L+W)$	2	L, W
c. $4y^3+y$	4	y

✓ Example 1.1.10: Evaluating an Algebraic Expression at Different Values

Evaluate the expression $2x-7$ for each value for x.

a. $x=0$

b. $x=1$

c. $x=12$

d. $x=-4$

Solution

a. Substitute 0 for x.

$$\begin{aligned} 2x-7 &= 2(0)-7 \\ &= 0-7 \\ &= -7 \end{aligned}$$

b. Substitute 1 for x.

$$\begin{aligned} 2x-7 &= 2(1)-7 \\ &= 2-7 \\ &= -5 \end{aligned}$$

c. Substitute $\frac{1}{2}$ for x.

$$\begin{aligned} 2x-7 &= 2\left(\frac{1}{2}\right)-7 \\ &= 1-7 \\ &= -6 \end{aligned}$$

d. Substitute -4 for x.

$$\begin{aligned} 2x-7 &= 2(-4)-7 \\ &= -8-7 \\ &= -15 \end{aligned}$$

Exercise 1.1.9

Evaluate the expression $11-3y$ for each value for y.

a. $y=2$
b. $y=0$
c. $y=\frac{2}{3}$
d. $y=-5$

Answer

1. 11
2. 26

Example 1.1.11: Evaluating Algebraic Expressions

Evaluate each expression for the given values.

a. $x+5$ for $x=-5$
b. $\frac{t}{2t-1}$ for $t=10$
c. $\frac{4}{3}\pi r^3$ for $r=5$
d. $a+ab+b$ for $a=11$, $b=-8$
e. $\sqrt{2m^3n^2}$ for $m=2$, $n=3$

Solution

a. Substitute -5 for x.

$$\begin{aligned} x+5 &= (-5)+5 \\ &= 0 \end{aligned}$$

b. Substitute 10 for t.

$$\begin{aligned} \frac{t}{2t-1} &= \frac{(10)}{2(10)-1} \\ &= \frac{10}{20-1} \\ &= \frac{10}{19} \end{aligned}$$

c. Substitute 5 for r.

$$\begin{aligned}\frac{4}{3}\pi r^3 &= \frac{4}{3}\pi(5)^3\\ &= \frac{4}{3}\pi(125)\\ &= \frac{500}{3}\pi\end{aligned}$$

d. Substitute 11 for a and -8 for b.

$$\begin{aligned}a+ab+b &= (11)+(11)(-8)+(-8)\\ &= 11-88-8\\ &= -85\end{aligned}$$

e. Substitute 2 for m and 3 for n.

$$\begin{aligned}\sqrt{2m^3n^2} &= \sqrt{2(2)^3(3)^2}\\ &= \sqrt{2(8)(9)}\\ &= \sqrt{144}\\ &= 12\end{aligned}$$

? Exercise 1.1.10

Evaluate each expression for the given values.

a. $\dfrac{y+3}{y-3}$ for $y=5$

b. $7-2t$ for $t=-2$

c. $\dfrac{1}{3}\pi r^2$ for $r=11$

d. $(p^2q)^3$ for $p=-2$, $q=3$

e. $4(m-n)-5(n-m)$ for $m=\dfrac{2}{3}$ $n=\dfrac{1}{3}$

Answer

a. 4

b. 11

c. $\dfrac{121}{3}\pi$

d. 1728

e. 3

Formulas

An equation is a mathematical statement indicating that two expressions are equal. The expressions can be numerical or algebraic. The equation is not inherently true or false, but only a proposition. The values that make the equation true, the solutions, are found using the properties of real numbers and other results. For example, the equation $2x+1=7$ has the unique solution of 3 because when we substitute 3 for x in the equation, we obtain the true statement $2(3)+1=7$.

A formula is an equation expressing a relationship between constant and variable quantities. Very often, the equation is a means of finding the value of one quantity (often a single variable) in terms of another or other quantities. One of the most common

examples is the formula for finding the area A of a circle in terms of the radius r of the circle: $A = \pi r^2$. For any value of r, the area A can be found by evaluating the expression πr^2.

Example 1.1.12: Using a Formula

A right circular cylinder with radius r and height h has the surface area S (in square units) given by the formula $S = 2\pi r(r + h)$. See Figure 1.1.3. Find the surface area of a cylinder with radius 6 in. and height 9 in. Leave the answer in terms of π.

Figure 1.1.3

Evaluate the expression $2\pi r(r + h)$ for $r = 6$ and $h = 9$.

Solution

$$\begin{aligned} S &= 2\pi r(r + h) \\ &= 2\pi(6)[(6) + (9)] \\ &= 2\pi(6)(15) \\ &= 180\pi \end{aligned}$$

The surface area is 180π square inches.

Exercise 1.1.11

A photograph with length L and width W is placed in a matte of width 8 centimeters (cm). The area of the matte (in square centimeters, or cm^2 is found to be $A = (L + 16)(W + 16) - L \cdot W$.See Figure 1.1.4. Find the area of a matte for a photograph with length 32cm and width 24cm.

Figure 1.1.4

Answer

$1152cm^2$

Simplifying Algebraic Expressions

Sometimes we can simplify an algebraic expression to make it easier to evaluate or to use in some other way. To do so, we use the properties of real numbers. We can use the same properties in formulas because they contain algebraic expressions.

Example 1.1.13: Simplifying Algebraic Expressions

Simplify each algebraic expression.

a. $3x - 2y + x - 3y - 7$

b. $2r - 5(3 - r) + 4$

c. $\left(4t - \frac{5}{4}s\right) - \left(\frac{2}{3}t + 2s\right)$

d. $2mn - 5m + 3mn + n$

Solution

a.

$$\begin{aligned} 3x - 2y + x - 3y - 7 &= 3x + x - 2y - 3y - 7 && \text{Commutative property of addition} \\ &= 4x - 5y - 7 && \text{Simplify} \end{aligned}$$

b.

$$\begin{aligned} 2r-5(3-r)+4 &= 2r-15+5r+4 && \text{Distributive property} \\ &= 2r+5y-15+4 && \text{Commutative property of addition} \\ &= 7r-11 && \text{Simplify} \end{aligned}$$

c.

$$\begin{aligned} \left(4t-\frac{5}{4}s\right)-\left(\frac{2}{3}t+2s\right) &= 4t-\frac{5}{4}s-\frac{2}{3}t-2s && \text{Distributive property} \\ &= 4t-\frac{2}{3}t-\frac{5}{4}s-2s && \text{Commutative property of addition} \\ &= \frac{10}{3}t-\frac{13}{4}s && \text{Simplify} \end{aligned}$$

d.

$$\begin{aligned} 2mn-5m+3mn+n &= 2mn+3mn-5m+n && \text{Commutative property of addition} \\ &= 5mn-5m+n && \text{Simplify} \end{aligned}$$

? Exercise 1.1.12

Simplify each algebraic expression.

a. $\frac{2}{3}y-2\left(\frac{4}{3}y+z\right)$

b. $\frac{5}{t}-2-\frac{3}{t}+1$

c. $4p(q-1)+q(1-p)$

d. $9r-(s+2r)+(6-s)$

Answer

a. $-2y-2z$ or $-2(y+z)$

b. $\frac{2}{t}-1$

c. $3pq-4p+q$

d. $7r-2s+6$

✓ Example 1.1.14: Simplifying a Formula

A rectangle with length L and width W has a perimeter P given by $P=L+W+L+W$. Simplify this expression.

Solution

$$\begin{aligned} P &= L+W+L+W \\ P &= L+L+W+W && \text{Commutative property of addition} \\ P &= 2L+2W && \text{Simplify} \\ P &= 2(L+W) && \text{Distributive property} \end{aligned}$$

? Exercise 1.1.13

If the amount P is deposited into an account paying simple interest r for time t, the total value of the deposit A is given by $A=P+Prt$. Simplify the expression. (This formula will be explored in more detail later in the course.)

Answer

$$A=P(1+rt)$$

Access these online resources for additional instruction and practice with real numbers.

- Simplify an Expression
- Evaluate an Expression1
- Evaluate an Expression2

Key Concepts

- Rational numbers may be written as fractions or terminating or repeating decimals. See Example and Example.
- Determine whether a number is rational or irrational by writing it as a decimal. See Example.
- The rational numbers and irrational numbers make up the set of real numbers. See Example. A number can be classified as natural, whole, integer, rational, or irrational. See Example.
- The order of operations is used to evaluate expressions. See Example.
- The real numbers under the operations of addition and multiplication obey basic rules, known as the properties of real numbers. These are the commutative properties, the associative properties, the distributive property, the identity properties, and the inverse properties. See Example.
- Algebraic expressions are composed of constants and variables that are combined using addition, subtraction, multiplication, and division. See Example. They take on a numerical value when evaluated by replacing variables with constants. See Example,Example, and Example
- Formulas are equations in which one quantity is represented in terms of other quantities. They may be simplified or evaluated as any mathematical expression. See Example and Example.

Glossary

algebraic expression

constants and variables combined using addition, subtraction, multiplication, and division

associative property of addition

the sum of three numbers may be grouped differently without affecting the result; in symbols,$a+(b+c)=(a+b)+c$

associative property of multiplication

the product of three numbers may be grouped differently without affecting the result; in symbols,$a\cdot(b\cdot c)=(a\cdot b)\cdot c$

base

in exponential notation, the expression that is being multiplied

commutative property of addition

two numbers may be added in either order without affecting the result; in symbols,$a+b=b+a$

commutative property of multiplication

two numbers may be multiplied in any order without affecting the result; in symbols,$a\cdot b=b\cdot a$

constant

a quantity that does not change value

distributive property

the product of a factor times a sum is the sum of the factor times each term in the sum; in symbols,$a\cdot(b+c)=a\cdot b+a\cdot c$

equation

a mathematical statement indicating that two expressions are equal

exponent

in exponential notation, the raised number or variable that indicates how many times the base is being multiplied

exponential notation

a shorthand method of writing products of the same factor

formula

an equation expressing a relationship between constant and variable quantities

identity property of addition

there is a unique number, called the additive identity, 0, which, when added to a number, results in the original number; in symbols, $a+0=a$

identity property of multiplication

there is a unique number, called the multiplicative identity, 1, which, when multiplied by a number, results in the original number; in symbols, $a \cdot 1=a$

integers

the set consisting of the natural numbers, their opposites, and 0: $\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$

inverse property of addition

for every real number a, there is a unique number, called the additive inverse (or opposite), denoted $-a$, which, when added to the original number, results in the additive identity, 0; in symbols, $a+(-a)=0$

inverse property of multiplication

for every non-zero real number a, there is a unique number, called the multiplicative inverse (or reciprocal), denoted $\frac{1}{a}$, which, when multiplied by the original number, results in the multiplicative identity, 1; in symbols, $a \cdot \frac{1}{a}=1$

irrational numbers

the set of all numbers that are not rational; they cannot be written as either a terminating or repeating decimal; they cannot be expressed as a fraction of two integers

natural numbers

the set of counting numbers: $\{1,2,3,\ldots\}$

order of operations

a set of rules governing how mathematical expressions are to be evaluated, assigning priorities to operations

rational numbers

the set of all numbers of the form $\frac{m}{n}$, where m and n are integers and $n \neq 0$. Any rational number may be written as a fraction or a terminating or repeating decimal.

real number line

a horizontal line used to represent the real numbers. An arbitrary fixed point is chosen to represent 0; positive numbers lie to the right of 0 and negative numbers to the left.

real numbers

the sets of rational numbers and irrational numbers taken together

variable

a quantity that may change value

whole numbers

the set consisting of 0 plus the natural numbers: $\{0,1,2,3,\ldots\}$

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1.2: Exponents and Scientific Notation

Learning Objectives

- Various rules of Exponents
- Scientific Notation

Mathematicians, scientists, and economists commonly encounter very large and very small numbers. But it may not be obvious how common such figures are in everyday life. For instance, a pixel is the smallest unit of light that can be perceived and recorded by a digital camera. A particular camera might record an image that is $2,048$ pixels by $1,536$ pixels, which is a very high resolution picture. It can also perceive a color depth (gradations in colors) of up to 48 bits per frame, and can shoot the equivalent of 24 frames per second. The maximum possible number of bits of information used to film a one-hour ($3,600$-second) digital film is then an extremely large number.

Using a calculator, we enter $2,048 \times 1,536 \times 48 \times 24 \times 3,600$ and press ENTER. The calculator displays $1.304596316E13$. What does this mean? The "$E13$" portion of the result represents the exponent 13 of ten, so there are a maximum of approximately 1.3×10^{13} bits of data in that one-hour film. In this section, we review rules of exponents first and then apply them to calculations involving very large or small numbers.

Using the Product Rule of Exponents

Consider the product $x^3 \times x^4$. Both terms have the same base, x, but they are raised to different exponents. Expand each expression, and then rewrite the resulting expression.

$$\begin{aligned} x^3 \times x^4 &= \overbrace{x \times x \times x}^{3 \text{ factors}} \times \overbrace{x \times x \times x \times x}^{4 \text{ factors}} \\ &= \overbrace{x \times x \times x \times x \times x \times x \times x}^{7 \text{ factors}} \\ &= x^7 \end{aligned}$$

The result is that $x^3 \times x^4 = x^{3+4} = x^7$.

Notice that the exponent of the product is the sum of the exponents of the terms. In other words, when multiplying exponential expressions with the same base, we write the result with the common base and add the exponents. This is the product rule of exponents.

$$a^m \times a^n = a^{m+n} \tag{1.2.1}$$

Now consider an example with real numbers.

$$2^3 \times 2^4 = 2^{3+4} = 2^7$$

We can always check that this is true by simplifying each exponential expression. We find that 2^3 is 8, 2^4 is 16, and 2^7 is 128. The product 8×16 equals 128, so the relationship is true. We can use the product rule of exponents to simplify expressions that are a product of two numbers or expressions with the same base but different exponents.

THE PRODUCT RULE OF EXPONENTS

For any real number a and natural numbers m and n, the product rule of exponents states that

$$a^m \times a^n = a^{m+n} \tag{1.2.2}$$

Example 1.2.1: Using the Product Rule

Write each of the following products with a single base. Do not simplify further.

a. $t^5 \times t^3$

b. $(-3)^5 \times (-3)$

c. $x^2 \times x^5 \times x^3$

Solution

Use the product rule (Equation 1.2.2) to simplify each expression.

a. $t^5 \times t^3 = t^{5+3} = t^8$
b. $(-3)^5 \times (-3) = (-3)^5 \times (-3)^1 = (-3)^{5+1} = (-3)^6$
c. $x^2 \times x^5 \times x^3$

At first, it may appear that we cannot simplify a product of three factors. However, using the associative property of multiplication, begin by simplifying the first two.

$$x^2 \times x^5 \times x^3 = (x^2 \times x^5) \times x^3 = (x^{2+5}) \times x^3 = x^7 \times x^3 = x^{7+3} = x^{10}$$

Notice we get the same result by adding the three exponents in one step.

$$x^2 \times x^5 \times x^3 = x^{2+5+3} = x^{10}$$

? Exercise 1.2.1

Write each of the following products with a single base. Do not simplify further.

a. $k^6 \times k^9$
b. $\left(\frac{2}{y}\right)^4 \times \left(\frac{2}{y}\right)$
c. $t^3 \times t^6 \times t^5$

Answer a

k^{15}

Answer b

$\left(\frac{2}{y}\right)^5$

Answer c

t^{14}

Using the Quotient Rule of Exponents

The quotient rule of exponents allows us to simplify an expression that divides two numbers with the same base but different exponents. In a similar way to the product rule, we can simplify an expression such as $\frac{y^m}{y^n}$, where $m > n$. Consider the example $\frac{y^9}{y^5}$. Perform the division by canceling common factors.

$$\begin{aligned} \frac{y^9}{y^5} &= \frac{y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y}{y \cdot y \cdot y \cdot y \cdot y} \\ &= \frac{y \cdot y \cdot y \cdot y}{1} \\ &= y^4 \end{aligned}$$

Notice that the exponent of the quotient is the difference between the exponents of the divisor and dividend.

$$\frac{a^m}{a^n} = a^{m-n} \tag{1.2.3}$$

In other words, when dividing exponential expressions with the same base, we write the result with the common base and subtract the exponents.

$$\frac{y^9}{y^5} = y^{9-5} = y^4$$

For the time being, we must be aware of the condition $m > n$. Otherwise, the difference $m - n$ could be zero or negative. Those possibilities will be explored shortly. Also, instead of qualifying variables as nonzero each time, we will simplify matters and assume from here on that all variables represent nonzero real numbers.

THE QUOTIENT RULE OF EXPONENTS

For any real number a and natural numbers m and n, such that $m > n$, the quotient rule of exponents states that

$$\frac{a^m}{a^n} = a^{m-n} \tag{1.2.4}$$

Example 1.2.2: Using the Quotient Rule

Write each of the following products with a single base. Do not simplify further.

a. $\frac{(-2)^{14}}{(-2)^9}$
b. $\frac{t^{23}}{t^{15}}$
c. $\frac{(z\sqrt{2})^5}{z\sqrt{2}}$

Solution

Use the quotient rule (Equation 1.2.4) to simplify each expression.

a. $\frac{(-2)^{14}}{(-2)^9} = (-2)^{14-9} = (-2)^5$
b. $\frac{t^{23}}{t^{15}}$=t^{23−15}=t^8\)
c. $\frac{(z\sqrt{2})^5}{z\sqrt{2}} = (z\sqrt{2})^{5-1} = (z\sqrt{2})^4$

Exercise 1.2.2

Write each of the following products with a single base. Do not simplify further.

a. $\frac{s^{75}}{s^{68}}$
b. $\frac{(-3)^6}{-3}$
c. $\frac{(ef^2)^5}{(ef^2)^3}$

Answer a

s^7

Answer b

$(-3)^5$

Answer c

$(ef^2)^2$

Using the Power Rule of Exponents

Suppose an exponential expression is raised to some power. Can we simplify the result? Yes. To do this, we use the power rule of exponents. Consider the expression $(x^2)^3$. The expression inside the parentheses is multiplied twice because it has an exponent of 2. Then the result is multiplied three times because the entire expression has an exponent of 3.

$$\begin{aligned}(x^2)^3 &= (x^2) \times (x^2) \times (x^2) \\ &= x \times x \times x \times x \times x \times x \\ &= x^6\end{aligned}$$

The exponent of the answer is the product of the exponents: $(x^2)^3 = x^{2 \cdot 3} = x^6$. In other words, when raising an exponential expression to a power, we write the result with the common base and the product of the exponents.

$$(a^m)^n = a^{m \cdot n} \tag{1.2.5}$$

Be careful to distinguish between uses of the product rule and the power rule. When using the product rule, different terms with the same bases are raised to exponents. In this case, you add the exponents. When using the power rule, a term in exponential notation is raised to a power. In this case, you multiply the exponents.

Table 1.2.1

Product Rule	Power Rule
$5^3 \times 5^4 = 5^{3+4} = 5^7$	$(5^3)^4 = 5^{3 \times 4} = 5^{12}$
$x^5 \times x^2 = x^{5+2} = x^7$	$(x^5)^2 = x^{5 \times 2} = x^{10}$
$(3a)^7 \times (3a)^{10} = (3a)^{7+10} = (3a)^{17}$	$((3a)^7)^{10} = (3a)^{7 \times 10} = (3a)^{70}$

THE POWER RULE OF EXPONENTS

For any real number a and positive integers m and n, the power rule of exponents states that

$$(a^m)^n = a^{m \cdot n} \tag{1.2.6}$$

Example 1.2.3: Using the Power Rule

Write each of the following products with a single base. Do not simplify further.

a. $(x^2)^7$
b. $((2t)^5)^3$
c. $((-3)^5)^{11}$

Solution

Use the power rule (Equation 1.2.6) to simplify each expression.

a. $(x^2)^7 = x^{2 \cdot 7} = x^{14}$
b. $((2t)^5)^3 = (2t)^{5 \cdot 3} = (2t)^{15}$
c. $((-3)^5)^{11} = (-3)^{5 \cdot 11} = (-3)^{55}$

Exercise 1.2.3

Write each of the following products with a single base. Do not simplify further.

a. $((3y)^8)^3$
b. $(t^5)^7$
c. $((-g)^4)^4$

Answer a

$(3y)^{24}$

Answer b

t^{35}

Answer c

$(-g)^{16}$

Using the Zero Exponent Rule of Exponents

Return to the quotient rule. We made the condition that $m > n$ so that the difference $m - n$ would never be zero or negative. What would happen if $m = n$? In this case, we would use the zero exponent rule of exponents to simplify the expression to 1. To see how this is done, let us begin with an example.

$$\frac{t^8}{t^8} = 1$$

If we were to simplify the original expression using the quotient rule, we would have

$$\frac{t^8}{t^8} = t^{8-8} = t^0$$

If we equate the two answers, the result is $t^0 = 1$. This is true for any nonzero real number, or any variable representing a real number.

$$a^0 = 1$$

The sole exception is the expression 0^0. This appears later in more advanced courses, but for now, we will consider the value to be undefined.

THE ZERO EXPONENT RULE OF EXPONENTS

For any nonzero real number a, the zero exponent rule of exponents states that

$$a^0 = 1 \tag{1.2.7}$$

Example 1.2.4: Using the Zero Exponent Rule

Simplify each expression using the zero exponent rule of exponents.

a. $\frac{c^3}{c^3}$
b. $\frac{-3x^5}{x^5}$
c. $\frac{(j^2k)^4}{(j^2k) \times (j^2k)^3}$
d. $\frac{5(rs^2)^2}{(rs^2)^2}$

Solution

Use the zero exponent and other rules to simplify each expression.

a.

$$\begin{aligned}\frac{c^3}{c^3} &= c^{3-3}\\ &= c^0\\ &= 1\end{aligned}$$

b.

$$\begin{aligned}\frac{-3x^5}{x^5} &= -3 \times \frac{x^5}{x^5}\\ &= -3 \times x^{5-5}\\ &= -3 \times x^0\\ &= -3 \times 1\\ &= -3\end{aligned}$$

c.

$$\begin{aligned}\frac{(j^2k)^4}{(j^2k) \times (j^2k)^3} &= \frac{(j^2k)^4}{(j^2k)^{1+3}} && \text{Use the product rule in the denominator}\\ &= \frac{(j^2k)^4}{(j^2k)^4} && \text{Simplify}\\ &= (j^2k)^{4-4} && \text{Use the quotient rule}\\ &= (j^2k)^0 && \text{Simplify}\\ &= 1\end{aligned}$$

d.

$$\begin{aligned}\frac{5(rs^2)^2}{(rs^2)^2} &= 5(rs^2)^{2-2} && \text{Use the quotient rule}\\ &= 5(rs^2)^0 && \text{Simplify}\\ &= 5 \times 1 && \text{Use the zero exponent rule}\\ &= 5 && \text{Simplify}\end{aligned}$$

? Exercise 1.2.4

Simplify each expression using the zero exponent rule of exponents.

a. $\frac{t^7}{t^7}$
b. $\frac{(de^2)^{11}}{2(de^2)^{11}}$
c. $\frac{w^4 \times w^2}{w^6}$
d. $\frac{t^3 \times t^4}{t^2 \times t^5}$

Answer a

1

Answer b

$\frac{1}{2}$

Answer c

1

Answer d

1

Using the Negative Rule of Exponents

Another useful result occurs if we relax the condition that $m > n$ in the quotient rule even further. For example, can we simplify $\frac{t^3}{t^5}$? When $m < n$ —that is, where the difference $m - n$ is negative—we can use the negative rule of exponents to simplify the expression to its reciprocal.

Divide one exponential expression by another with a larger exponent. Use our example, $\frac{t^3}{t^5}$.

$$\begin{aligned}\frac{t^3}{t^5} &= \frac{t \times t \times t}{t \times t \times t \times t \times t} \\ &= \frac{1}{t \times t} \\ &= \frac{1}{h^2}\end{aligned}$$

If we were to simplify the original expression using the quotient rule, we would have

$$\begin{aligned}\frac{t^3}{t^5} &= h^{3-5} \\ &= h^{-2}\end{aligned}$$

Putting the answers together, we have $h^{-2} = \frac{1}{h^2}$. This is true for any nonzero real number, or any variable representing a nonzero real number.

A factor with a negative exponent becomes the same factor with a positive exponent if it is moved across the fraction bar—from numerator to denominator or vice versa.

$$a^{-n} = \frac{1}{a^n} \text{ and } a^n = \frac{1}{a^{-n}}$$

We have shown that the exponential expression an is defined when n is a natural number, 0, or the negative of a natural number. That means that an is defined for any integer n. Also, the product and quotient rules and all of the rules we will look at soon hold for any integer n.

THE NEGATIVE RULE OF EXPONENTS

For any nonzero real number a and natural number n, the negative rule of exponents states that

$$a^{-n} = \frac{1}{a^n} \tag{1.2.8}$$

Example 1.2.5: Using the Negative Exponent Rule

Write each of the following quotients with a single base. Do not simplify further. Write answers with positive exponents.

a. $\frac{\theta^3}{\theta^{10}}$
b. $\frac{z^2 \times z}{z^4}$
c. $\frac{(-5t^3)^4}{(-5t^3)^8}$

Solution

a. $\frac{\theta^3}{\theta^{10}} = \theta^{3-10} = \theta^{-7} = \frac{1}{\theta^7}$
b. $\frac{z^2 \times z}{z^4} = \frac{z^{2+1}}{z^4} = \frac{z^3}{z^4} = z^{3-4} = z^{-1} = \frac{1}{z}$
c. $\frac{(-5t^3)^4}{(-5t^3)^8} = (-5t^3)^{4-8} = (-5t^3)^{-4} = \frac{1}{(-5t^3)^4}$

? Exercise 1.2.5

Write each of the following quotients with a single base. Do not simplify further. Write answers with positive exponents.

a. $\frac{(-3t)^2}{(-3t)^8}$

b. $\frac{f^{47}}{f^{49} \times f}$

c. $\frac{2k^4}{5k^7}$

Answer a

$$\frac{1}{(-3t)^6}$$

Answer b

$$\frac{1}{f^3}$$

Answer c

$$\frac{2}{5k^3}$$

✓ Example 1.2.6: Using the Product and Quotient Rules

Write each of the following products with a single base. Do not simplify further. Write answers with positive exponents.

a. $b^2 \times b^{-8}$

b. $(-x)^5 \times (-x)^{-5}$

c. $\frac{-7z}{(-7z)^5}$

Solution

a. $b^2 \times b^{-8} = b^{2-8} = b^{-6} = \frac{1}{b^6}$

b. $(-x)^5 \times (-x)^{-5} = (-x)^{5-5} = (-x)^0 = 1$

c. $\frac{-7z}{(-7z)^5} = \frac{(-7z)^1}{(-7z)^5} = (-7z)^{1-5} = (-7z)^{-4} = \frac{1}{(-7z)^4}$

? Exercise 1.2.6

Write each of the following products with a single base. Do not simplify further. Write answers with positive exponents.

a. $t^{-11} \times t^6$

b. $\frac{25^{12}}{25^{13}}$

Answer a

$$t^{-5} = \frac{1}{t^5}$$

Answer b

$$\frac{1}{25}$$

Finding the Power of a Product

To simplify the power of a product of two exponential expressions, we can use the power of a product rule of exponents, which breaks up the power of a product of factors into the product of the powers of the factors. For instance, consider $(pq)^3$. We begin by using the associative and commutative properties of multiplication to regroup the factors.

$$\begin{aligned}(pq)^3 &= (pq) \times (pq) \times (pq) \\ &= p \times q \times p \times q \times p \times q \\ &= p^3 \times q^3\end{aligned}$$

In other words, $(pq)^3 = p^3 \times q^3$.

THE POWER OF A PRODUCT RULE OF EXPONENTS

For any real numbers a and b and any integer n, the power of a product rule of exponents states that

$$(ab)^n = a^n b^n \tag{1.2.9}$$

Example 1.2.7: Using the Power of a Product Rule

Simplify each of the following products as much as possible using the power of a product rule. Write answers with positive exponents.

a. $(ab^2)^3$
b. $(2t)^{15}$
c. $(-2w^3)^3$
d. $\dfrac{1}{(-7z)^4}$
e. $(e^{-2}f^2)^7$

Solution

Use the product and quotient rules and the new definitions to simplify each expression.

a. $(ab^2)^3 = (a)^3 \times (b^2)^3 = a^{1\times 3} \times b^{2\times 3} = a^3b^6$
b. $(2t)^{15} = (2)^{15} \times (t)^{15} = 2^{15}t^{15} = 32,768t^{15}$
c. $(-2w^3)^3 = (-2)^3 \times (w^3)^3 = -8 \times w^{3\times 3} = -8w^9$
d. $\dfrac{1}{(-7z)^4} = \dfrac{1}{(-7)^4 \times (z)^4} = \dfrac{1}{2401z^4}$
e. $(e^{-2}f^2)^7 = (e^{-2})^7 \times (f^2)^7 = e^{-2\times 7} \times f^{2\times 7} = e^{-14}f^{14} = \dfrac{f^{14}}{e^{14}}$

Exercise 1.2.7

Simplify each of the following products as much as possible using the power of a product rule. Write answers with positive exponents.

a. $(g^2h^3)^5$
b. $(5t)^3$
c. $(-3y^5)^3$
d. $\dfrac{1}{(a^6b^7)^3}$
e. $(r^3s^{-2})^4$

Answer a

$g^{10}h^{15}$

Answer b

$125t^3$

Answer c

$-27y^{15}$

Answer d

$\frac{1}{a^{18}b^{21}}$

Answer e

$\frac{r^{12}}{s^8}$

Finding the Power of a Quotient

To simplify the power of a quotient of two expressions, we can use the power of a quotient rule, which states that the power of a quotient of factors is the quotient of the powers of the factors. For example, let's look at the following example.

$$(e^{-2}f^2)^7 = \frac{f^{14}}{e^{14}} \tag{1.2.10}$$

Let's rewrite the original problem differently and look at the result.

$$\begin{aligned}(e^{-2}f^2)^7 &= \left(\frac{f^2}{e^2}\right)^7 \\ &= \frac{f^{14}}{e^{14}}\end{aligned}$$

It appears from the last two steps that we can use the power of a product rule as a power of a quotient rule.

$$\begin{aligned}(e^{-2}f^2)^7 &= \left(\frac{f^2}{e^2}\right)^7 \\ &= \frac{(f^2)^7}{(e^2)^7} \\ &= \frac{f^{2\times 7}}{e^{2\times 7}} \\ &= \frac{f^{14}}{e^{14}}\end{aligned}$$

THE POWER OF A QUOTIENT RULE OF EXPONENTS

For any real numbers a and b and any integer n, the power of a quotient rule of exponents states that

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \tag{1.2.11}$$

Example 1.2.8: Using the Power of a Quotient Rule

Simplify each of the following quotients as much as possible using the power of a quotient rule. Write answers with positive exponents.

a. $\left(\frac{4}{z^{11}}\right)^3$
b. $\left(\frac{p}{q^3}\right)^6$
c. $\left(\frac{-1}{t^2}\right)^{27}$
d. $(j^3k^{-2})^4$
e. $(m^{-2}n^{-2})^3$

Solution

a. $\left(\frac{4}{z^{11}}\right)^3 = \frac{(4)^3}{(z^{11})^3} = \frac{64}{z^{11\times 3}} = \frac{64}{z^{33}}$
b. $\left(\frac{p}{q^3}\right)^6 = \frac{(p)^6}{(q^3)^6} = \frac{p^{1\times 6}}{q^{3\times 6}} = \frac{p^6}{q^{18}}$
c. $\left(\frac{-1}{t^2}\right)^{27} = \frac{(-1)^{27}}{(t^2)^{27}} = \frac{-1}{t^{2\times 27}} = \frac{-1}{t^{54}} = -\frac{1}{t^{54}}$

d. $(j^3k^{-2})^4 = \left(\frac{j^3}{k^2}\right)^4 = \frac{(j^3)^4}{(k^2)^4} = \frac{j^{3\times4}}{k^{2\times4}} = \frac{j^{12}}{k^8}$

e. $(m^{-2}n^{-2})^3 = \left(\frac{1}{m^2n^2}\right)^3 = \frac{(1)^3}{(m^2n^2)^3} = \frac{1}{(m^2)^3(n^2)^3} = \frac{1}{m^{2\times3}n^{2\times3}} = \frac{1}{m^6n^6}$

? Exercise 1.2.8

Simplify each of the following quotients as much as possible using the power of a quotient rule. Write answers with positive exponents.

a. $\left(\frac{b^5}{c}\right)^3$

b. $\left(\frac{5}{u^8}\right)^4$

c. $\left(\frac{-1}{w^3}\right)^{35}$

d. $(p^{-4}q^3)^8$

e. $(c^{-5}d^{-3})^4$

Answer a

$\frac{b^{15}}{c^3}$

Answer b

$\frac{625}{u^{32}}$

Answer c

$\frac{-1}{w^{105}}$

Answer d

$\frac{q^{24}}{p^{32}}$

Answer e

$\frac{1}{c^{20}d^{12}}$

Simplifying Exponential Expressions

Recall that to simplify an expression means to rewrite it by combing terms or exponents; in other words, to write the expression more simply with fewer terms. The rules for exponents may be combined to simplify expressions.

✓ Example 1.2.9: Simplifying Exponential Expressions

Simplify each expression and write the answer with positive exponents only.

a. $(6m^2n^{-1})^3$

b. $17^5 \times 17^{-4} \times 17^{-3}$

c. $\left(\frac{u^{-1}v}{v^{-1}}\right)^2$

d. $(-2a^3b^{-1})(5a^{-2}b^2)$

e. $(x^2\sqrt{2})^4(x^2\sqrt{2})^{-4}$

f. $\frac{(3w^2)^5}{(6w^{-2})^2}$

Solution

a.

$$\begin{aligned}(6m^2n^{-1})^3 &= (6)^3(m^2)^3(n^{-1})^3 && \text{The power of a product rule}\\ &= 6^3m^{2\times3}n^{-1\times3} && \text{The power rule}\\ &= 216m^6n^{-3} && \text{The power rule}\\ &= \frac{216m^6}{n^3} && \text{The negative exponent rule}\end{aligned}$$

b.

$$\begin{aligned}17^5\times17^{-4}\times17^{-3} &= 17^{5-4-3} && \text{The product rule}\\ &= 17^{-2} && \text{Simplify}\\ &= \frac{1}{17^2} \text{ or } \frac{1}{289} && \text{The negative exponent rule}\end{aligned}$$

c.

$$\begin{aligned}\left(\frac{u^{-1}v}{v^{-1}}\right)^2 &= \frac{(u^{-1}v)^2}{(v^{-1})^2} && \text{The power of a quotient rule}\\ &= \frac{u^{-2}v^2}{v^{-2}} && \text{The power of a product rule}\\ &= u^{-2}v^{2-(-2)} && \text{The quotient rule}\\ &= u^{-2}v^4 && \text{Simplify}\\ &= \frac{v^4}{u^2} && \text{The negative exponent rule}\end{aligned}$$

d.

$$\begin{aligned}\left(-2a^3b^{-1}\right)\left(5a^{-2}b^2\right) &= \left(x^2\sqrt{2}\right)^{4-4} && \text{Commutative and associative laws of multiplication}\\ &= -10\times a^{3-2}\times b^{-1+2} && \text{The product rule}\\ &= -10ab && \text{Simplify}\end{aligned}$$

e.

$$\begin{aligned}\left(x^2\sqrt{2}\right)^4\left(x^2\sqrt{2}\right)^{-4} &= \left(x^2\sqrt{2}\right)^{4-4} && \text{The product rule}\\ &= \left(x^2\sqrt{2}\right)^0 && \text{Simplify}\\ &= 1 && \text{The zero exponent rule}\end{aligned}$$

f.

$$\begin{aligned}\frac{(3w^2)^5}{(6w^{-2})^2} &= \frac{(3)^5\times(w^2)^5}{(6)^2\times(w^{-2})^2} && \text{The power of a product rule}\\ &= \frac{3^5w^{2\times5}}{6^2w^{-2\times2}} && \text{The power rule}\\ &= \frac{243w^{10}}{36w^{-4}} && \text{Simplify}\\ &= \frac{27w^{10-(-4)}}{4} && \text{The quotient rule and reduce fraction}\\ &= \frac{27w^{14}}{4} && \text{Simplify}\end{aligned}$$

Using Scientific Notation

Recall at the beginning of the section that we found the number 1.3×10^{13} when describing bits of information in digital images. Other extreme numbers include the width of a human hair, which is about $0.00005\ m$, and the radius of an electron, which is about $0.00000000000047\ m$. How can we effectively work read, compare, and calculate with numbers such as these?

A shorthand method of writing very small and very large numbers is called scientific notation, in which we express numbers in terms of exponents of 10. To write a number in scientific notation, move the decimal point to the right of the first digit in the number. Write the digits as a decimal number between 1 and 10. Count the number of places n that you moved the decimal point. Multiply the decimal number by 10 raised to a power of n. If you moved the decimal left as in a very large number, n is positive. If you moved the decimal right as in a small large number, n is negative.

For example, consider the number $2,780,418$. Move the decimal left until it is to the right of the first nonzero digit, which is 2.

We obtain 2.780418 by moving the decimal point 6 places to the left. Therefore, the exponent of 10 is 6, and it is positive because we moved the decimal point to the left. This is what we should expect for a large number.

$$2.780418 \times 10^{6}$$

Working with small numbers is similar. Take, for example, the radius of an electron, $0.00000000000047\ m$. Perform the same series of steps as above, except move the decimal point to the right.

Be careful not to include the leading 0 in your count. We move the decimal point 13 places to the right, so the exponent of 10 is 13. The exponent is negative because we moved the decimal point to the right. This is what we should expect for a small number.

$$4.7 \times 10^{-13}$$

SCIENTIFIC NOTATION

A number is written in scientific notation if it is written in the form $a \times 10^{n}$, where $1 \leq |a| < 10$ and n is an integer.

Example 1.2.10: Converting Standard Notation to Scientific Notation

Write each number in scientific notation.

a. Distance to Andromeda Galaxy from Earth: $24,000,000,000,000,000,000,000\ m$
b. Diameter of Andromeda Galaxy: $1,300,000,000,000,000,000,000\ m$
c. Number of stars in Andromeda Galaxy: $1,000,000,000,000$
d. Diameter of electron: $0.00000000000094\ m$
e. Probability of being struck by lightning in any single year: 0.00000143

Solution

a. $24,000,000,000,000,000,000,000\ m$ 22 places

$2.4 \times 10^{22}\ m$

b. $1,300,000,000,000,000,000,000\ m$ 21 places

$1.3 \times 10^{21}\ m$

c. $1,000,000,000,000$ 12 places

1×10^{12}

d. $0.00000000000094\ m$ 13 places

$9.4 \times 10^{-13}\ m$

e. 0.00000143 6 places

1.43×10^{6}

Analysis

Observe that, if the given number is greater than 1, as in examples a–c, the exponent of 10 is positive; and if the number is less than 1, as in examples d–e, the exponent is negative.

Exercise 1.2.10

Write each number in scientific notation.

a. U.S. national debt per taxpayer (April 2014): $\$152,000$
b. World population (April 2014): $7,158,000,000$
c. World gross national income (April 2014): $\$85,500,000,000,000$
d. Time for light to travel $1\ m$: $0.00000000334\ s$
e. Probability of winning lottery (match 6 of 49 possible numbers): 0.0000000715

Answer a

$\$1.52 \times 10^5$

Answer b

7.158×10^9

Answer c

$\$8.55 \times 10^{13}$

Answer d

3.34×10^{-9}

Answer e

7.15×10^{-8}

Converting from Scientific to Standard Notation

To convert a number in scientific notation to standard notation, simply reverse the process. Move the decimal n places to the right if n is positive or n places to the left if n is negative and add zeros as needed. Remember, if n is positive, the value of the number is greater than 1, and if n is negative, the value of the number is less than one.

Example 1.2.11: Converting Scientific Notation to Standard Notation

Convert each number in scientific notation to standard notation.

a. 3.547×10^{14}
b. -2×10^6
c. 7.91×10^{-7}
d. -8.05×10^{-12}

Solution

a. 3.547×10^{14}

3.54700000000000

$\rightarrow 14$ places

$354,700,000,000,000$

b. -2×10^6

-2.000000

$\rightarrow 6$ places

$-2,000,000$

c. 7.91×10^{-7}

0000007.91

$\rightarrow 7$ places

0.000000791

d. -8.05×10^{-12}

-000000000008.05

$\rightarrow 12$ places

-0.00000000000805

? Exercise 1.2.11

Convert each number in scientific notation to standard notation.

a. 7.03×10^{5}
b. -8.16×10^{11}
c. -3.9×10^{-13}
d. 8×10^{-6}

Answer a

$$703,000$$

Answer b

$$-816,000,000,000$$

Answer c

$$-0.00000000000039$$

Answer d

$$0.000008$$

Using Scientific Notation in Applications

Scientific notation, used with the rules of exponents, makes calculating with large or small numbers much easier than doing so using standard notation. For example, suppose we are asked to calculate the number of atoms in $1\ L$ of water. Each water molecule contains 3 atoms (2 hydrogen and 1 oxygen). The average drop of water contains around 1.32×1021 molecules of water and $1\ L$ of water holds about 1.22×10^{4} average drops. Therefore, there are approximately $3 \cdot (1.32 \times 10^{21}) \cdot (1.22 \times 10^{4}) \approx 4.83 \times 10^{25}$ atoms in $1\ L$ of water. We simply multiply the decimal terms and add the exponents. Imagine having to perform the calculation without using scientific notation!

When performing calculations with scientific notation, be sure to write the answer in proper scientific notation. For example, consider the product $(7 \times 10^{4}) \cdot (5 \times 10^{6}) = 35 \times 10^{10}$. The answer is not in proper scientific notation because 35 is greater than 10. Consider 35 as 3.5×10. That adds a ten to the exponent of the answer.

$$(35) \times 10^{10} = (3.5 \times 10) \times 10^{10} = 3.5 \times (10 \times 10^{10}) = 3.5 \times 10^{11}$$

✓ Example 1.2.12: Using Scientific Notation

Perform the operations and write the answer in scientific notation.

a. $(8.14 \times 10^{-7})(6.5 \times 10^{10})$

b. $(4 \times 10^5) \div (-1.52 \times 10^9)$
c. $(2.7 \times 10^5)(6.04 \times 10^{13})$
d. $(1.2 \times 10^8) \div (9.6 \times 10^5)$
e. $(3.33 \times 10^4)(-1.05 \times 10^7)(5.62 \times 10^5)$

Solutions

a.
$$\begin{aligned}(8.14 \times 10^{-7})(6.5 \times 10^{10}) &= (8.14 \times 6.5)(10^{-7} \times 10^{10}) \text{ Commutative and associative properties of multiplication} \\ &= (52.91)(10^3) \text{ Product rule of exponents} \\ &= 5.291 \times 10^4 \text{ Scientific notation}\end{aligned}$$

b.
$$\begin{aligned}(4 \times 10^5) \div (-1.52 \times 10^9) &= \left(\frac{4}{-1.52}\right)\left(\frac{10^5}{10^9}\right) \text{ Commutative and associative properties of multiplication} \\ &\approx (-2.63)(10^{-4}) \text{ Quotient rule of exponents} \\ &= -2.63 \times 10^{-4} \text{ Scientific notation}\end{aligned}$$

c.
$$\begin{aligned}(2.7 \times 10^5)(6.04 \times 10^{13}) &= (2.7 \times 6.04)(10^5 \times 10^{13}) \text{ Commutative and associative properties of multiplication} \\ &= (16.308)(10^{18}) \text{ Product rule of exponents} \\ &= 1.6308 \times 10^{19} \text{ Scientific notation}\end{aligned}$$

d.
$$\begin{aligned}(1.2 \times 10^8) \div (9.6 \times 10^5) &= \left(\frac{1.2}{9.6}\right)\left(\frac{10^8}{10^5}\right) \text{ Commutative and associative properties of multiplication} \\ &= (0.125)(10^3) \text{ Quotient rule of exponents} \\ &= 1.25 \times 10^2 \text{ Scientific notation}\end{aligned}$$

e.
$$\begin{aligned}(3.33 \times 10^4)(-1.05 \times 10^7)(5.62 \times 10^5) &= [3.33 \times (-1.05) \times 5.62](10^4 \times 10^7 \times 10^5) \\ &\approx (-19.65)(10^{16}) \\ &= -1.965 \times 10^{17}\end{aligned}$$

? Exercise 1.2.12

Perform the operations and write the answer in scientific notation.

a. $(-7.5 \times 10^8)(1.13 \times 10^{-2})$
b. $(1.24 \times 10^{11}) \div (1.55 \times 10^{18})$
c. $(3.72 \times 10^9)(8 \times 10^3)$
d. $(9.933 \times 10^{23}) \div (-2.31 \times 10^{17})$
e. $(-6.04 \times 10^9)(7.3 \times 10^2)(-2.81 \times 10^2)$

Answer a

$$-8.475 \times 10^6$$

Answer b

$$8 \times 10^{-8}$$

Answer c

$$2.976 \times 10^{13}$$

Answer d

-4.3×10^6

Answer e

$\approx 1.24 \times 10^{15}$

Example 1.2.13: Applying Scientific Notation to Solve Problems

In April 2014, the population of the United States was about $308,000,000$ people. The national debt was about $\$17,547,000,000,000$. Write each number in scientific notation, rounding figures to two decimal places, and find the amount of the debt per U.S. citizen. Write the answer in both scientific and standard notations.

Solution

The population was $308,000,000 = 3.08 \times 10^8$.

The national debt was $\$17,547,000,000,000 \approx \$1.75 \times 10^{13}$.

To find the amount of debt per citizen, divide the national debt by the number of citizens.

$$\begin{aligned}(1.75 \times 10^{13}) \div (3.08 \times 10^8) &= \left(\frac{1.75}{3.08}\right)(10^5) \\ &\approx 0.57 \times 10^5 \\ &= 5.7 \times 10^4\end{aligned}$$

The debt per citizen at the time was about $\$5.7 \times 10^4$, or $\$57,000$.

Exercise 1.2.13

An average human body contains around $30,000,000,000,000$ red blood cells. Each cell measures approximately $0.000008\ m$ long. Write each number in scientific notation and find the total length if the cells were laid end-to-end. Write the answer in both scientific and standard notations.

Answer

Number of cells: 3×10^{13}; length of a cell: $8 \times 10^{-6}\ m$; total length: $2.4 \times 10^8\ m$ or $240,000,000\ m$.

Access these online resources for additional instruction and practice with exponents and scientific notation.

Exponential Notation

Properties of Exponents

Zero Exponent

Simplify Exponent Expressions

Quotient Rule for Exponents

Scientific Notation

Converting to Decimal Notation

Key Equations

Rules of Exponents For nonzero real numbers a and b and integers m and n

Product rule	$a^m \cdot a^n = a^{m+n}$
Quotient rule	$\frac{a^m}{a^n} = a^{m-n}$
Power rule	$(a^m)^n = a^{m \cdot n}$
Zero exponent rule	$a^0 = 1$
Negative rule	$a^{-n} = \frac{1}{a^n}$
Power of a product rule	$(a \cdot b)^n = a^n \cdot b^n$
Power of a quotient rule	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Key Concepts

- Products of exponential expressions with the same base can be simplified by adding exponents. See Example.
- Quotients of exponential expressions with the same base can be simplified by subtracting exponents. See Example.
- Powers of exponential expressions with the same base can be simplified by multiplying exponents. See Example.
- An expression with exponent zero is defined as 1. See Example.
- An expression with a negative exponent is defined as a reciprocal. See Example and Example.
- The power of a product of factors is the same as the product of the powers of the same factors. See Example.
- The power of a quotient of factors is the same as the quotient of the powers of the same factors. See Example.
- The rules for exponential expressions can be combined to simplify more complicated expressions. See Example.
- Scientific notation uses powers of 10 to simplify very large or very small numbers. See Example and Example.
- Scientific notation may be used to simplify calculations with very large or very small numbers. See Example and Example.

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1.3: Radicals and Rational Expressions

Learning Objectives

- Evaluate square roots.
- Use the product rule to simplify square roots.
- Use the quotient rule to simplify square roots.
- Add and subtract square roots.
- Rationalize denominators.
- Use rational roots.

A hardware store sells 16-ft ladders and 24-ft ladders. A window is located 12 feet above the ground. A ladder needs to be purchased that will reach the window from a point on the ground 5 feet from the building. To find out the length of ladder needed, we can draw a right triangle as shown in Figure $1.3.1$, and use the Pythagorean Theorem.

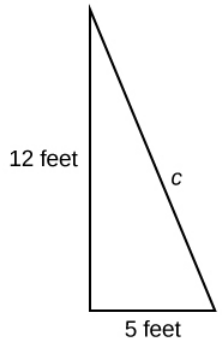


Figure $1.3.1$: A right triangle

Multiple \label

Now, we need to find out the length that, when squared, is 169, to determine which ladder to choose. In other words, we need to find a square root. In this section, we will investigate methods of finding solutions to problems such as this one.

Evaluating Square Roots

When the square root of a number is squared, the result is the original number. Since $4^2 = 16$, the square root of 16 is 4.The square root function is the inverse of the squaring function just as subtraction is the inverse of addition. To undo squaring, we take the square root.

In general terms, if a is a positive real number, then the square root of a is a number that, when multiplied by itself, gives a.The square root could be positive or negative because multiplying two negative numbers gives a positive number. The principal square root is the nonnegative number that when multiplied by itself equals a. The square root obtained using a calculator is the principal square root.

The principal square root of a is written as $\sqrt{a}$. The symbol is called a radical, the term under the symbol is called the radicand, and the entire expression is called a **radical expression**.



Example 1.3.1

Does $\sqrt{25} = \pm 5$?

Solution

No. Although both 5^2 and $(-5)^2$ are 25, the radical symbol implies only a *nonnegative* root, the principal square root. The principal square root of 25 is $\sqrt{25} = 5$.

Note

The principal square root of a is the nonnegative number that, when multiplied by itself, equals a. It is written as a radical expression, with a symbol called a *radical* over the term called the *radicand*: $\sqrt{a}$.

Example 1.3.2: Evaluating Square Roots

Evaluate each expression.

a. $\sqrt{\sqrt{16}}$
b. $\sqrt{49}$-$\sqrt{81}$

Solution

a. $\sqrt{\sqrt{16}} = \sqrt{4} = 2$ because $4^2 = 16$ and $2^2 = 4$
b. $\sqrt{49} - \sqrt{81} = 7 - 9 = -2$ because $7^2 = 49$ and $9^2 = 81$

Example 1.3.3:

For $\sqrt{25 + 144}$,can we find the square roots before adding?

Solution

No. $\sqrt{25} + \sqrt{144} = 5 + 12 = 17$. This is not equivalent to $\sqrt{25 + 144} = 13$. The order of operations requires us to add the terms in the radicand before finding the square root.

Exercise 1.3.1

Evaluate each expression.

a. $\sqrt{\sqrt{81}}$
b. $\sqrt{36} + \sqrt{121}$

Answer a

3

Answer b

17

Using the Product Rule to Simplify Square Roots

To simplify a square root, we rewrite it such that there are no perfect squares in the radicand. There are several properties of square roots that allow us to simplify complicated radical expressions. The first rule we will look at is the product rule for simplifying square roots, which allows us to separate the square root of a product of two numbers into the product of two separate rational expressions. For instance, we can rewrite $\sqrt{15}$ as $\sqrt{3} \times \sqrt{5}$. We can also use the product rule to express the product of multiple radical expressions as a single radical expression.

The Product Rule For Simplifying Square Roots

If a and b are nonnegative, the square root of the product ab is equal to the product of the square roots of a and b

$$\sqrt{ab} = \sqrt{a} \times \sqrt{b} \tag{1.3.1}$$

HOWTO: Given a square root radical expression, use the product rule to simplify it.

1. Factor any perfect squares from the radicand.
2. Write the radical expression as a product of radical expressions.
3. Simplify.

Example 1.3.4: Using the Product Rule to Simplify Square Roots

Simplify the radical expression.

a. $\sqrt{300}$
b. $\sqrt{162a^5b^4}$

Solution

a. $\sqrt{100 \times 3}$ Factor perfect square from radicand.

$\sqrt{100} \times \sqrt{3}$ Write radical expression as product of radical expressions.

$10\sqrt{3}$ Simplify

b. $\sqrt{81a^4b^4 \times 2a}$ Factor perfect square from radicand

$\sqrt{81a^4b^4} \times \sqrt{2a}$ Write radical expression as product of radical expressions

$9a^2b^2\sqrt{2a}$ Simplify

Exercise 1.3.2

Simplify $\sqrt{50x^2y^3z}$

Answer

$$5|x||y|\sqrt{2yz}$$

Notice the absolute value signs around x and y? That's because their value must be positive!

Howto: Given the product of multiple radical expressions, use the product rule to combine them into one radical expression

1. Express the product of multiple radical expressions as a single radical expression.
2. Simplify.

Example 1.3.5: Using the Product Rule to Simplify the Product of Multiple Square Roots

Simplify the radical expression.

$\sqrt{12} \times \sqrt{3}$

Solution

$\sqrt{12 \times 3}$ Express the product as a single radical expression
$\sqrt{36}$ Simplify
6

Exercise 1.3.3

Simplify $\sqrt{50x} \times \sqrt{2x}$ assuming $x > 0$.

Answer

$10|x|$

Using the Quotient Rule to Simplify Square Roots

Just as we can rewrite the square root of a product as a product of square roots, so too can we rewrite the square root of a quotient as a quotient of square roots, using the quotient rule for simplifying square roots. It can be helpful to separate the numerator and denominator of a fraction under a radical so that we can take their square roots separately. We can rewrite

$$\sqrt{\frac{5}{2}} = \frac{\sqrt{5}}{\sqrt{2}}.$$

THE QUOTIENT RULE FOR SIMPLIFYING SQUARE ROOTS

The square root of the quotient $\frac{a}{b}$ is equal to the quotient of the square roots of a and b, where $b \neq 0$.

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \tag{1.3.2}$$

Howto: Given a radical expression, use the quotient rule to simplify it

1. Write the radical expression as the quotient of two radical expressions.
2. Simplify the numerator and denominator.

Example 1.3.6: Using the Quotient Rule to Simplify Square Roots

Simplify the radical expression.

$\sqrt{\frac{5}{36}}$

Solution

$\frac{\sqrt{5}}{\sqrt{36}}$ Write as quotient of two radical expressions
$\frac{\sqrt{5}}{6}$ Simplify denominator

Exercise 1.3.4

Simplify $\sqrt{\frac{2x^2}{9y^4}}$

Answer

$\frac{x\sqrt{2}}{3y^2}$

We do not need the absolute value signs for y^2 because that term will always be nonnegative.

Example 1.3.7: Using the Quotient Rule to Simplify an Expression with Two Square Roots

Simplify the radical expression.

$\frac{\sqrt{234x^{11}y}}{\sqrt{26x^7y}}$

Solution

$\sqrt{\frac{234x^{11}y}{26x^7y}}$ Combine numerator and denominator into one radical expression

$\sqrt{9x^4}$ Simplify fraction

$3x^2$ Simplify square root

Exercise 1.3.5

Simplify $\frac{\sqrt{9a^5b^{14}}}{\sqrt{3a^4b^5}}$

Answer

$b^4\sqrt{3ab}$

Adding and Subtracting Square Roots

We can add or subtract radical expressions only when they have the same radicand and when they have the same radical type such as square roots. For example, the sum of $\sqrt{2}$ and $3\sqrt{2}$ is $4\sqrt{2}$. However, it is often possible to simplify radical expressions, and that may change the radicand. The radical expression $\sqrt{18}$ can be written with a 2 in the radicand, as $3\sqrt{2}$, so $\sqrt{2}+\sqrt{18}=\sqrt{2}+3\sqrt{2}=4\sqrt{2}$

Howto: Given a radical expression requiring addition or subtraction of square roots, solve

1. Simplify each radical expression.
2. Add or subtract expressions with equal radicands.

Example 1.3.8: Adding Square Roots

Add $5\sqrt{12}+2\sqrt{3}$.

Solution

We can rewrite $5\sqrt{12}$ as $5\sqrt{4\times3}$. According the product rule, this becomes $5\sqrt{4}\sqrt{3}$. The square root of $\sqrt{4}$ is 2, so the expression becomes $5\times2\sqrt{3}$, which is $10\sqrt{3}$. Now we can the terms have the same radicand so we can add.

$$10\sqrt{3}+2\sqrt{3}=12\sqrt{3}$$

Exercise 1.3.6

Add $\sqrt{5}+6\sqrt{20}$

Answer

$13\sqrt{5}$

Example 1.3.9: Subtracting Square Roots

Subtract $20\sqrt{72a^3b^4c}-14\sqrt{8a^3b^4c}$

Solution

Rewrite each term so they have equal radicands.

$$\begin{aligned}20\sqrt{72a^3b^4c} &= 20\sqrt{9}\sqrt{4}\sqrt{2}\sqrt{a}\sqrt{a^2}\sqrt{(b^2)^2}\sqrt{c}\\ &= 20(3)(2)|a|b^2\sqrt{2ac}\\ &= 120|a|b^2\sqrt{2ac}\end{aligned}$$

$$\begin{aligned}14\sqrt{8a^3b^4c} &= 14\sqrt{2}\sqrt{4}\sqrt{a}\sqrt{a^2}\sqrt{(b^2)^2}\sqrt{c}\\ &= 14(2)|a|b^2\sqrt{2ac}\\ &= 28|a|b^2\sqrt{2ac}\end{aligned}$$

Now the terms have the same radicand so we can subtract.

$$120|a|b^2\sqrt{2ac}-28|a|b^2\sqrt{2ac}=92|a|b^2\sqrt{2ac} \tag{1.3.3}$$

Exercise 1.3.7

Subtract $3\sqrt{80x}-4\sqrt{45x}$

Answer

0

Rationalizing Denominators

When an expression involving square root radicals is written in simplest form, it will not contain a radical in the denominator. We can remove radicals from the denominators of fractions using a process called rationalizing the denominator.

We know that multiplying by 1 does not change the value of an expression. We use this property of multiplication to change expressions that contain radicals in the denominator. To remove radicals from the denominators of fractions, multiply by the form of 1 that will eliminate the radical.

For a denominator containing a single term, multiply by the radical in the denominator over itself. In other words, if the denominator is $b\sqrt{c}$, multiply by $\frac{\sqrt{c}}{\sqrt{c}}$.

For a denominator containing the sum or difference of a rational and an irrational term, multiply the numerator and denominator by the conjugate of the denominator, which is found by changing the sign of the radical portion of the denominator. If the denominator is $a+b\sqrt{c}$, then the conjugate is $a-b\sqrt{c}$.

HowTo: Given an expression with a single square root radical term in the denominator, rationalize the denominator

1. Multiply the numerator and denominator by the radical in the denominator.
2. Simplify.

Example 1.3.10: Rationalizing a Denominator Containing a Single Term

Write $\frac{2\sqrt{3}}{3\sqrt{10}}$ in simplest form.

Solution

The radical in the denominator is $\sqrt{10}$. So multiply the fraction by $\frac{\sqrt{10}}{\sqrt{10}}$. Then simplify.

$$\frac{2\sqrt{3}}{3\sqrt{10}} \times \frac{\sqrt{10}}{\sqrt{10}}$$
$$\frac{2\sqrt{30}}{30}$$
$$\frac{\sqrt{30}}{15}$$

Exercise 1.3.8

Write $\frac{12\sqrt{3}}{\sqrt{2}}$ in simplest form.

Answer

$6\sqrt{6}$

How to: Given an expression with a radical term and a constant in the denominator, rationalize the denominator

1. Find the conjugate of the denominator.
2. Multiply the numerator and denominator by the conjugate.
3. Use the distributive property.
4. Simplify.

Example 1.3.11: Rationalizing a Denominator Containing Two Terms

Write $\frac{4}{1+\sqrt{5}}$ in simplest form.

Solution

Begin by finding the conjugate of the denominator by writing the denominator and changing the sign. So the conjugate of $1+\sqrt{5}$ is $1-\sqrt{5}$. Then multiply the fraction by $\frac{1-\sqrt{5}}{1-\sqrt{5}}$.

$$\frac{4}{1+\sqrt{5}} \times \frac{1-\sqrt{5}}{1-\sqrt{5}}$$
$$\frac{4-4\sqrt{5}}{-4} \quad \text{Use the distributive property}$$
$$\sqrt{5}-1 \quad \text{Simplify}$$

Exercise 1.3.9

Write $\frac{7}{2+\sqrt{3}}$ in simplest form.

Answer

$14-7\sqrt{3}$

Using Rational Roots

Although square roots are the most common rational roots, we can also find cube roots, 4th roots, 5th roots, and more. Just as the square root function is the inverse of the squaring function, these roots are the inverse of their respective power functions. These functions can be useful when we need to determine the number that, when raised to a certain power, gives a certain number.

Understanding n^{th} Roots

Suppose we know that $a^3 = 8$. We want to find what number raised to the 3rd power is equal to 8. Since $2^3 = 8$, we say that 2 is the cube root of 8.

The nth root of a is a number that, when raised to the nth power, gives a. For example, -3 is the 5th root of -243 because $(-3)^5 = -243$. If a is a real number with at least one n^{th} root, then the principal n^{th} root of a is the number with the same sign as a that, when raised to the nth power, equals a.

The principal n^{th} root of a is written as $\sqrt[n]{a}$, where n is a positive integer greater than or equal to 2. In the radical expression, n is called the index of the radical.

Note: Principal n^{th} Root

If a is a real number with at least one n^{th} root, then the **principal n^{th} root** of a, written as $\sqrt[n]{a}$, is the number with the same sign as a that, when raised to the nth power, equals a. The **index** of the radical is n.

Example 1.3.12: Simplifying n^{th} Roots

Simplify each of the following:

a. $\sqrt[5]{-32}$
b. $\sqrt[4]{4} \times \sqrt[4]{10234}$
c. $-\sqrt[3]{\frac{8x^6}{125}}$
d. $8\sqrt[4]{3} - \sqrt[4]{48}$

Solution

a. $\sqrt[5]{-32} = -2$ because $(-2)^5 = -32$

b. First, express the product as a single radical expression. $\sqrt[4]{4096} = 8$ because $8^4 = 4096$

c.

$$\frac{-\sqrt[3]{8x^6}}{\sqrt[3]{125}} \quad \text{Write as quotient of two radical expressions}$$
$$\frac{-2x^2}{5} \quad \text{Simplify}$$

d.

$$8\sqrt[4]{3} - 2\sqrt[4]{3} \quad \text{Simplify to get equal radicands}$$
$$6\sqrt[4]{3} \quad \text{Add}$$

Exercise 1.3.10

Simplify

a. $\sqrt[3]{-216}$
b. $\frac{3\sqrt[4]{80}}{\sqrt[4]{5}}$
c. $6\sqrt[3]{9000} + 7\sqrt[3]{576}$

Answer a

-6

Answer b

6

Answer c

$88\sqrt[3]{9}$

Using Rational Exponents

Radical expressions can also be written without using the radical symbol. We can use rational (fractional) exponents. The index must be a positive integer. If the index n is even, then a cannot be negative.

$$a^{\frac{1}{n}} = \sqrt[n]{a} \tag{1.3.4}$$

We can also have rational exponents with numerators other than 1. In these cases, the exponent must be a fraction in lowest terms. We raise the base to a power and take an nth root. The numerator tells us the power and the denominator tells us the root.

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m} \tag{1.3.5}$$

All of the properties of exponents that we learned for integer exponents also hold for rational exponents.

Note: Rational Exponents

Rational exponents are another way to express principal n^{th} roots. The general form for converting between a radical expression with a radical symbol and one with a rational exponent is

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m} \tag{1.3.6}$$

Howto: Given an expression with a rational exponent, write the expression as a radical

1. Determine the power by looking at the numerator of the exponent.
2. Determine the root by looking at the denominator of the exponent.
3. Using the base as the radicand, raise the radicand to the power and use the root as the index.

Example 1.3.13: Writing Rational Exponents as Radicals

Write $343^{\frac{2}{3}}$ as a radical. Simplify.

Solution

The 2 tells us the power and the 3 tells us the root.

$343^{\frac{2}{3}} = (\sqrt[3]{343})^2 = \sqrt[3]{343^2}$

We know that $\sqrt[3]{343} = 7$ because $7^3 = 343$. Because the cube root is easy to find, it is easiest to find the cube root before squaring for this problem. In general, it is easier to find the root first and then raise it to a power.

$$343^{\frac{2}{3}} = (\sqrt[3]{343})^2 = 7^2 = 49 \tag{1.3.7}$$

Exercise 1.3.11

Write $9^{\frac{5}{2}}$ as a radical. Simplify.

Answer

$(\sqrt{9})^5 = 3^5 = 243$

Example 1.3.14: Writing Radicals as Rational Exponents

Write $\frac{4}{\sqrt[7]{a^2}}$ using a rational exponent.

Solution

The power is 2 and the root is 7, so the rational exponent will be $\frac{2}{7}$. We get $\frac{4}{a^{\frac{2}{7}}}$. Using properties of exponents, we get $\frac{4}{\sqrt[7]{a^2}} = 4a^{\frac{-2}{7}}$

Exercise 1.3.12

Write $x\sqrt{(5y)^9}$ using a rational exponent.

Answer

$x(5y)^{\frac{9}{2}}$

Example 1.3.15: Simplifying Rational Exponents

Simplify:

a. $5(2x^{\frac{3}{4}})(3x^{\frac{1}{5}})$

b. $\left(\frac{16}{9}\right)^{-\frac{1}{2}}$

Solution

a.

$$\begin{aligned} &30x^{\frac{3}{4}}x^{\frac{1}{5}} && \text{Multiply the coefficients} \\ &30x^{\frac{3}{4}+\frac{1}{5}} && \text{Use properties of exponents} \\ &30x^{\frac{19}{20}} && \text{Simplify} \end{aligned}$$

b.

$$\begin{aligned} &\left(\frac{9}{16}\right)^{\frac{1}{2}} && \text{Use definition of negative exponents} \\ &\sqrt{\frac{9}{16}} && \text{Rewrite as a radical} \\ &\frac{\sqrt{9}}{\sqrt{16}} && \text{Use the quotient rule} \\ &\frac{3}{4} && \text{Simplify} \end{aligned}$$

Exercise 1.3.13

Simplify $(8x)^{\frac{1}{3}}\left(14x^{\frac{6}{5}}\right)$

Answer

$28x^{\frac{23}{15}}$

Media

Access these online resources for additional instruction and practice with radicals and rational exponents.

Radicals

Rational Exponents

Simplify Radicals

Rationalize Denominator

Key Concepts

- The principal square root of a number a is the nonnegative number that when multiplied by itself equals a. See Example.
- If a and b are nonnegative, the square root of the product ab is equal to the product of the square roots of a and b See Example and Example.
- If a and b are nonnegative, the square root of the quotient $\frac{a}{b}$ is equal to the quotient of the square roots of a and b See Example and Example.
- We can add and subtract radical expressions if they have the same radicand and the same index. See Example and Example.
- Radical expressions written in simplest form do not contain a radical in the denominator. To eliminate the square root radical from the denominator, multiply both the numerator and the denominator by the conjugate of the denominator. See Example and Example.
- The principal n^{th} root of a is the number with the same sign as a that when raised to the nth power equals a. These roots have the same properties as square roots. See Example.
- Radicals can be rewritten as rational exponents and rational exponents can be rewritten as radicals. See Example and Example.
- The properties of exponents apply to rational exponents. See Example.

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1.4: Polynomials

Learning Objectives

In this section students will:

- Identify the degree and leading coefficient of polynomials.
- Add and subtract polynomials.
- Multiply polynomials.
- Use FOIL to multiply binomials.
- Perform operations with polynomia
- ls of several variables.

Earl is building a doghouse, whose front is in the shape of a square topped with a triangle. There will be a rectangular door through which the dog can enter and exit the house. Earl wants to find the area of the front of the doghouse so that he can purchase the correct amount of paint. Using the measurements of the front of the house, shown in Figure **1.4.1**, we can create an expression that combines several variable terms, allowing us to solve this problem and others like it.

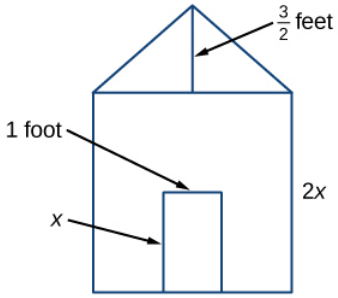


Figure **1.4.1**

- First find the area of the square in square feet.

$$\begin{aligned} A &= s^2 \\ &= (2x)^2 \\ &= 4x^2 \end{aligned}$$

- Then find the area of the triangle in square feet.

$$\begin{aligned} A &= \frac{1}{2}bh \\ &= \frac{1}{2}(2x)\left(\frac{3}{2}\right) \\ &= \frac{3}{2}x \end{aligned}$$

- Next find the area of the rectangular door in square feet.

$$\begin{aligned} A &= lw \\ &= x \times 1 \\ &= x \end{aligned}$$

The area of the front of the doghouse can be found by adding the areas of the square and the triangle, and then subtracting the area of the rectangle. When we do this, we get

$$4x^2 + \frac{3}{2}x - x \ ft^2$$

or

$$4x^2 + \frac{1}{2}x \ ft^2$$

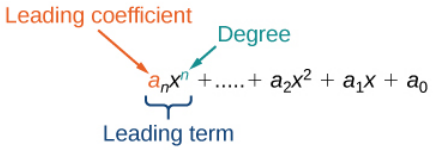
In this section, we will examine expressions such as this one, which combine several variable terms.

Identifying the Degree and Leading Coefficient of Polynomials

The formula just found is an example of a **polynomial**, which is a sum of or difference of terms, each consisting of a variable raised to a nonnegative integer power. A number multiplied by a variable raised to an exponent, such as 384π, is known as a **coefficient**. Coefficients can be positive, negative, or zero, and can be whole numbers, decimals, or fractions. Each product $a_i x^i$, such as $384\pi w$, is a **term of a polynomial**. If a term does not contain a variable, it is called a constant.

A polynomial containing only one term, such as $5x^4$, is called a **monomial**. A polynomial containing two terms, such as $2x - 9$, is called a **binomial**. A polynomial containing three terms, such as $-3x^2 + 8x - 7$, is called a **trinomial**.

We can find the **degree** of a polynomial by identifying the highest power of the variable that occurs in the polynomial. The term with the highest degree is called the **leading term** because it is usually written first. The coefficient of the leading term is called the **leading coefficient**. When a polynomial is written so that the powers are descending, we say that it is in standard form.



Polynomials

A polynomial is an expression that can be written in the form

$$a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0 \tag{1.4.1}$$

Each real number ai is called a **coefficient**. The number a_0 that is not multiplied by a variable is called aconstant. Each product $a_i x^i$ is a **term of a polynomial**. The highest power of the variable that occurs in the polynomial is called the **degree** of a polynomial. The **leading term** is the term with the highest power, and its coefficient is called the **leading coefficient.**

How to: Given a polynomial expression, identify the degree and leading coefficient.

1. Find the highest power of x to determine the degree.
2. Identify the term containing the highest power of x to find the leading term.
3. Identify the coefficient of the leading term.

Example 1.4.1: Identifying the Degree and Leading Coefficient of a Polynomial

For the following polynomials, identify the degree, the leading term, and the leading coefficient.

a. $3 + 2x^2 - 4x^3$
b. $5t^5 - 2t^3 + 7t$
c. $6p - p^3 - 2$

Solution

a. The highest power of x is 3, so the degree is 3. The leading term is the term containing that degree, $-4x^3$. The leading coefficient is the coefficient of that term, -4.
b. The highest power of t is 5, so the degree is 5. The leading term is the term containing that degree, $5t^5$. The leading coefficient is the coefficient of that term, 5.
c. The highest power of p is 3, so the degree is 3. The leading term is the term containing that degree, $-p^3$, The leading coefficient is the coefficient of that term, -1.

Exercise 1.4.1

Identify the degree, leading term, and leading coefficient of the polynomial $4x^2 - x^6 + 2x - 6$.

Answer

The degree is 6, the leading term is $-x^6$, and the leading coefficient is -1.

Adding and Subtracting Polynomials

We can add and subtract polynomials by combining like terms, which are terms that contain the same variables raised to the same exponents. For example, $5x^2$ and $-2x^2$ are like terms, and can be added to get $3x^2$, but $3x$ and $3x^2$ are not like terms, and therefore cannot be added.

How to: Given multiple polynomials, add or subtract them to simplify the expressions.

1. Combine like terms.
2. Simplify and write in standard form.

Example 1.4.2: Adding Polynomials

Find the sum.

$$(12x^2 + 9x - 21) + (4x^3 + 8x^2 - 5x + 20)$$

Solution

$$\begin{aligned} &4x^3 + (12x^2 + 8x^2) + (9x - 5x) + (-21 + 20) && \text{Combine like terms} \\ &4x^3 + 20x^2 + 4x - 1 && \text{Simplify} \end{aligned}$$

Analysis

We can check our answers to these types of problems using a graphing calculator. To check, graph the problem as given along with the simplified answer. The two graphs should be equivalent. Be sure to use the same window to compare the graphs. Using different windows can make the expressions seem equivalent when they are not.

Exercise 1.4.2

Find the sum.

$$(2x^3 + 5x^2 - x + 1) + (2x^2 - 3x - 4)$$

Answer

$2x^3 + 7x^2 - 4x - 3$

Example 1.4.3: Subtracting Polynomials

Find the difference.

$$(7x^4 - x^2 + 6x + 1) - (5x^3 - 2x^2 + 3x + 2)$$

Solution

$7x^4 - 5x^3 + (-x^2 + 2x^2) + (6x - 3x) + (1 - 2)$ Combine like terms

$7x^4 - 5x^3 + x^2 + 3x - 1$ Simplify

Analysis

Note that finding the difference between two polynomials is the same as adding the opposite of the second polynomial to the first.

Exercise 1.4.3

Find the difference.

$$(-7x^3 - 7x^2 + 6x - 2) - (4x^3 - 6x^2 - x + 7)$$

Answer

$-11x^3 - x^2 + 7x - 9$

Multiplying Polynomials

Multiplying polynomials is a bit more challenging than adding and subtracting polynomials. We must use the distributive property to multiply each term in the first polynomial by each term in the second polynomial. We then combine like terms. We can also use a shortcut called the FOIL method when multiplying binomials. Certain special products follow patterns that we can memorize and use instead of multiplying the polynomials by hand each time. We will look at a variety of ways to multiply polynomials.

Multiplying Polynomials Using the Distributive Property

To multiply a number by a polynomial, we use the distributive property. The number must be distributed to each term of the polynomial. We can distribute the 2 in $2(x + 7)$ to obtain the equivalent expression $2x + 14$. When multiplying polynomials, the distributive property allows us to multiply each term of the first polynomial by each term of the second. We then add the products together and combine like terms to simplify.

How to: Given the multiplication of two polynomials, use the distributive property to simplify the expression.

1. Multiply each term of the first polynomial by each term of the second.
2. Combine like terms.
3. Simplify.

Example 1.4.4: Multiplying Polynomials Using the Distributive Property

Find the product.

$$(2x + 1)(3x^2 - x + 4)$$

Solution

$$\begin{aligned} &2x(3x^2 - x + 4) + 1(3x^2 - x + 4) && \text{Use the distributive property} \\ &(6x^3 - 2x^2 + 8x) + (3x^2 - x + 4) && \text{Multiply} \\ &6x^3 + (-2x^2 + 3x^2) + (8x - x) + 4 && \text{Combine like terms} \\ &6x^3 + x^2 + 7x + 4 && \text{Simplify} \end{aligned}$$

Analysis

We can use a table to keep track of our work, as shown in Table 1.4.1. Write one polynomial across the top and the other down the side. For each box in the table, multiply the term for that row by the term for that column. Then add all of the terms together, combine like terms, and simplify.

Table 1.4.1

	$3x^2$	$-x$	$+4$
$2x$	$6x^3$	$-2x^2$	$8x$
$+1$	$3x^2$	$-x$	4

Exercise 1.4.4

Find the product.

$(3x+2)(x^3-4x^2+7)$

Answer

$3x^4-10x^3-8x^2+21x+14$

Using FOIL to Multiply Binomials

A shortcut called FOIL is sometimes used to find the product of two binomials. It is called FOIL because we multiply the first terms, the outer terms, the inner terms, and then the last terms of each binomial.

The FOIL method arises out of the distributive property. We are simply multiplying each term of the first binomial by each term of the second binomial, and then combining like terms.

FOIL to simplify expression

Given two binomials, use FOIL to simplify the expression.

1. Multiply the outer terms of the binomials.
2. Multiply the last terms of each binomial.
3. Combine like terms and simplify.

Example 1.4.5: Using FOIL to Multiply Binomials

Use FOIL to find the product.

$$(2x-10)(3x+3)$$

Solution

Find the product of the first terms.

Find the product of the outer terms.

Find the product of the inner terms.

Find the product of the last terms.

$$\begin{aligned} &6x^2+6x-54x-54 && \text{Add the products}\\ &6x^2+(6x-54x)-54 && \text{Combine like terms}\\ &6x^2-48x-54 && \text{Simplify}\end{aligned}$$

Exercise 1.4.5

Use FOIL to find the product.

$$(x+7)(3x-5)$$

Answer

$3x^2 + 16x - 35$

Perfect Square Trinomials

Certain binomial products have special forms. When a binomial is squared, the result is called a perfect square trinomial. We can find the square by multiplying the binomial by itself. However, there is a special form that each of these perfect square trinomials takes, and memorizing the form makes squaring binomials much easier and faster. Let's look at a few perfect square trinomials to familiarize ourselves with the form.

$$(x+5)^2 = x^2 + 10x + 25$$

$$(x-3)^2 = x^2 - 6x + 9$$

$$(4x-1)^2 = 16x^2 - 8x + 1$$

Notice that the first term of each trinomial is the square of the first term of the binomial and, similarly, the last term of each trinomial is the square of the last term of the binomial. The middle term is double the product of the two terms. Lastly, we see that the first sign of the trinomial is the same as the sign of the binomial.

Perfect Square Trinomials

When a binomial is squared, the result is the first term squared added to double the product of both terms and the last term squared.

$$(x+a)^2 = (x+a)(x+a) = x^2 + 2ax + a^2 \tag{1.4.2}$$

How to: Given a binomial, square it using the formula for perfect square trinomials.

1. Square the first term of the binomial.
2. Square the last term of the binomial.
3. For the middle term of the trinomial, double the product of the two terms.
4. Add and simplify.

Example 1.4.6: Expanding Perfect Squares

Expand $(3x-8)^2$.

Solution

Begin by squaring the first term and the last term. For the middle term of the trinomial, double the product of the two terms.

$$(3x)^2 - 2(3x)(8) + (-8)^2$$
$$9x^2 - 48x + 64 \qquad \text{Simplify}$$

Exercise 1.4.6

Expand $(4x-1)^2$.

Answer

$16x^2 - 8x + 1$

Difference of Squares

Another special product is called the difference of squares, which occurs when we multiply a binomial by another binomial with the same terms but the opposite sign. Let's see what happens when we multiply $(x+1)(x-1)$ using the FOIL method.

$$\begin{aligned}(x+1)(x-1) &= x^2 - x + x - 1 \\ &= x^2 - 1\end{aligned}$$

The middle term drops out, resulting in a difference of squares. Just as we did with the perfect squares, let's look at a few examples.

$$(x+5)(x-5) = x^2 - 25$$

$$(x+11)(x-11) = x^2 - 121$$

$$(2x+3)(2x-3) = 4x^2 - 9$$

Because the sign changes in the second binomial, the outer and inner terms cancel each other out, and we are left only with the square of the first term minus the square of the last term.

Q&A

Is there a special form for the sum of squares?

No. The difference of squares occurs because the opposite signs of the binomials cause the middle terms to disappear. There are no two binomials that multiply to equal a sum of squares.

Difference of Squares

When a binomial is multiplied by a binomial with the same terms separated by the opposite sign, the result is the square of the first term minus the square of the last term.

$$(a+b)(a-b) = a^2 - b^2 \tag{1.4.3}$$

Howto: Given a binomial multiplied by a binomial with the same terms but the opposite sign, find the difference of squares.

1. Square the first term of the binomials.
2. Square the last term of the binomials.
3. Subtract the square of the last term from the square of the first term.

Example 1.4.7: Multiplying Binomials Resulting in a Difference of Squares

Multiply $(9x+4)(9x-4)$.

Solution

Square the first term to get $(9x)^2 = 81x^2$. Square the last term to get $4^2 = 16$. Subtract the square of the last term from the square of the first term to find the product of $81x^2 - 16$.

Exercise 1.4.7

Multiply $(2x+7)(2x-7)$.

Answer

$4x^2 - 49$

Performing Operations with Polynomials of Several Variables

We have looked at polynomials containing only one variable. However, a polynomial can contain several variables. All of the same rules apply when working with polynomials containing several variables. Consider an example:

$(a+2b)(4a-b-c)a(4a-b-c)+2b(4a-b-c)$	Use the distributive property
$4a^2 - ab - ac + 8ab - 2b^2 - 2bc$	Multiply
$4a^2 + (-ab+8ab) - ac - 2b^2 - 2bc$	Combine like terms
$4a^2 + 7ab - ac - 2bc - 2b^2$	Simplify

Example 1.4.8: Multiplying Polynomials Containing Several Variables

Multiply $(x+4)(3x-2y+5)$.

Solution

$$\begin{aligned} &x(3x-2y+5)+4(3x-2y+5) && \text{Use the distributive property} \\ &3x^2-2xy+5x+12x-8y+20 && \text{Multiply} \\ &3x^2-2xy+(5x+12x)-8y+20 && \text{Combine like terms} \\ &3x^2-2xy+17x-8y+20 && \text{Simplify} \end{aligned}$$

Exercise 1.4.8

Multiply $(3x-1)(2x+7y-9)$.

Answer

$6x^2+21xy-29x-7y+9$

Media

Access these online resources for additional instruction and practice with polynomials.

1. Adding and Subtracting Polynomials
2. Multiplying Polynomials
3. Special Products of Polynomials

Key Equations

perfect square trinomial	$(x+a)^2=(x+a)(x+a)=x^2+2ax+a^2$
difference of squares	$(a+b)(a-b)=a^2-b^2$

Key Concepts

- A polynomial is a sum of terms each consisting of a variable raised to a non-negative integer power. The degree is the highest power of the variable that occurs in the polynomial. The leading term is the term containing the highest degree, and the leading coefficient is the coefficient of that term. See Example.
- We can add and subtract polynomials by combining like terms. See Example and Example.
- To multiply polynomials, use the distributive property to multiply each term in the first polynomial by each term in the second. Then add the products. See Example.
- FOIL (First, Outer, Inner, Last) is a shortcut that can be used to multiply binomials. See Example.
- Perfect square trinomials and difference of squares are special products. See Example and Example.
- Follow the same rules to work with polynomials containing several variables. See Example.

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1.5: Factoring Polynomials

Learning Objectives

In this section students will:

- Factor the greatest common factor of a polynomial.
- Factor a trinomial.
- Factor by grouping.
- Factor a perfect square trinomial.
- Factor a difference of squares.
- Factor the sum and difference of cubes.

Imagine that we are trying to find the area of a lawn so that we can determine how much grass seed to purchase. The lawn is the green portion in Figure 1.5.1.

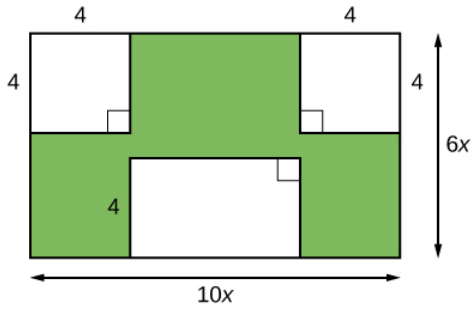


Figure 1.5.1

The area of the entire region can be found using the formula for the area of a rectangle.

$$\begin{aligned} A &= lw \\ &= 10x \times 6x \\ &= 60x^2 \ units^2 \end{aligned}$$

The areas of the portions that do not require grass seed need to be subtracted from the area of the entire region. The two square regions each have an area of $A = s^2 = 4^2 = 16 \ units^2$. The other rectangular region has one side of length $10x - 8$ and one side of length 4, giving an area of

$$A = lw = 4(10x - 8) = 40x - 32 \text{ units}^2.$$

So the region that must be subtracted has an area of

$$2(16) + 40x - 32 = 40x \text{ units}^2.$$

The area of the region that requires grass seed is found by subtracting $60x^2 - 40x \text{ units}^2$. This area can also be expressed in factored form as $20x(3x - 2) \text{ units}^2$. We can confirm that this is an equivalent expression by multiplying.

Many polynomial expressions can be written in simpler forms by factoring. In this section, we will look at a variety of methods that can be used to factor polynomial expressions.

Factoring the Greatest Common Factor of a Polynomial

When we study fractions, we learn that the greatest common factor (GCF) of two numbers is the largest number that divides evenly into both numbers. For instance, 4 is the GCF of 16 and 20 because it is the largest number that divides evenly into both 16 and 20 The GCF of polynomials works the same way: $4x$ is the GCF of $16x$ and $20x^2$ because it is the largest polynomial that divides evenly into both $16x$ and $20x^2$.

When factoring a polynomial expression, our first step should be to check for a GCF. Look for the GCF of the coefficients, and then look for the GCF of the variables.

Definition: Greatest Common Factor

The *greatest common factor* (GCF) of polynomials is the largest polynomial that divides evenly into the polynomials.

Howto: Given a polynomial expression, factor out the greatest common factor

1. Identify the GCF of the coefficients.
2. Identify the GCF of the variables.
3. Combine to find the GCF of the expression.
4. Determine what the GCF needs to be multiplied by to obtain each term in the expression.
5. Write the factored expression as the product of the GCF and the sum of the terms we need to multiply by.

Example 1.5.1: Factoring the Greatest Common Factor

Factor $6x^3y^3 + 45x^2y^2 + 21xy$.

Solution

First, find the GCF of the expression. The GCF of 6, 45, and 21 is 3. The GCF of x^3,x^2, and x is x. (Note that the GCF of a set of expressions in the form x^n will always be the exponent of lowest degree.) And the GCF of y^3,y^2, and y is y. Combine these to find the GCF of the polynomial, $3xy$.

Next, determine what the GCF needs to be multiplied by to obtain each term of the polynomial. We find that

- $3xy(2x^2y^2) = 6x^3y^3$,
- $3xy(15xy) = 45x^2y^2$, and
- $3xy(7) = 21xy$.

Finally, write the factored expression as the product of the GCF and the sum of the terms we needed to multiply by.

$$(3xy)(2x^2y^2 + 15xy + 7)$$

Analysis

After factoring, we can check our work by multiplying. Use the distributive property to confirm that

$$(3xy)(2x^2y^2 + 15xy + 7) = 6x^3y^3 + 45x^2y^2 + 21xy$$

Exercise 1.5.1

Factor $x(b^2 - a) + 6(b^2 - a)$ by pulling out the GCF.

Answer

$(b^2 - a)(x + 6)$

Factoring a Trinomial with Leading Coefficient 1

Although we should always begin by looking for a GCF, pulling out the GCF is not the only way that polynomial expressions can be factored. The polynomial $x^2 + 5x + 6$ has a GCF of 1, but it can be written as the product of the factors $(x + 2)$ and $(x + 3)$.

Trinomials of the form $x^2 + bx + c$ can be factored by finding two numbers with a product of c and a sum of b. The trinomial $x^2 + 10x + 16$, for example, can be factored using the numbers 2 and 8 because the product of those numbers is 16 and their sum is 10. The trinomial can be rewritten as the product of $(x + 2)$ and $(x + 8)$.

FACTORING A TRINOMIAL WITH LEADING COEFFICIENT 1

A trinomial of the form $x^2 + bx + c$ can be written in factored form as $(x + p)(x + q)$ where $pq = c$ and $p + q = b$.

Q&A: Can every trinomial be factored as a product of binomials?

No. Some polynomials cannot be factored. These polynomials are said to be *prime*.

Howto: Given a trinomial in the form $x^2 + bx + c$, factor it

1. List factors of c.
2. Find p and q, a pair of factors of c with a sum of b.
3. Write the factored expression $(x + p)(x + q)$.

Example 1.5.2: Factoring a Trinomial with Leading Coefficient 1

Factor $x^2 + 2x - 15$.

Solution

We have a trinomial with leading coefficient 1, $b = 2$, and $c = -15$. We need to find two numbers with a product of -15 and a sum of 2. In Table 1.5.1, we list factors until we find a pair with the desired sum.

Table 1.5.1

Factors of −15	Sum of Factors
1,−15	−14
−1,15	14
3,−5	−2
−3,5	

Now that we have identified p and q as -3 and 5, write the factored form as $(x - 3)(x + 5)$.

Analysis

We can check our work by multiplying. Use FOIL to confirm that $(x - 3)(x + 5) = x^2 + 2x - 15$.

Q&A: Does the order of the factors matter?

No. Multiplication is commutative, so the order of the factors does not matter.

Exercise 1.5.2

Factor $x^2 - 7x + 6$.

Answer

$(x - 6)(x - 1)$

Factoring by Grouping

Trinomials with leading coefficients other than 1 are slightly more complicated to factor. For these trinomials, we can **factor by grouping** by dividing the x term into the sum of two terms, factoring each portion of the expression separately, and then factoring out the GCF of the entire expression. The trinomial $2x^2 + 5x + 3$ can be rewritten as $(2x + 3)(x + 1)$ using this process. We begin by

rewriting the original expression as $2x^2+2x+3x+3$ and then factor each portion of the expression to obtain $2x(x+1)+3(x+1)$. We then pull out the GCF of $(x+1)$ to find the factored expression.

Factor by Grouping

To factor a trinomial in the form ax^2+bx+c by grouping, we find two numbers with a product of ac and a sum of b. We use these numbers to divide the x term into the sum of two terms and factor each portion of the expression separately, then factor out the GCF of the entire expression.

Howto: Given a trinomial in the form ax^2+bx+c, factor by grouping.

1. List factors of ac.
2. Find p and q, a pair of factors of ac with a sum of b.
3. Rewrite the original expression as $ax^2+px+qx+c$.
4. Pull out the GCF of ax^2+px.
5. Pull out the GCF of $qx+c$.
6. Factor out the GCF of the expression.

Example 1.5.3: Factoring a Trinomial by Grouping

Factor $5x^2+7x-6$ by grouping.

Solution

We have a trinomial with $a=5, b=7$, and $c=-6$. First, determine $ac=-30$. We need to find two numbers with a product of -30 and a sum of 7. In the table below, we list factors until we find a pair with the desired sum.

Table 1.5.2

Factors of −30	Sum of Factors
1,−30	−29
−1,30	29
2,−15	−13
−2,15	13
3,−10	−7
−3,10	7

So $p=-3$ and $q=10$.

$5x^2-3x+10x-6$ Rewrite the original expression as $ax^2+px+qx+c$.

$x(5x-3)+2(5x-3)$ Factor out the GCF of each part

$(5x-3)(x+2)$ Factor out the GCF of the expression.

Analysis

We can check our work by multiplying. Use FOIL to confirm that $(5x-3)(x+2)=5x^2+7x-6$.

Exercise 1.5.3

Factor:

a. $2x^2+9x+9$
b. $6x^2+x-1$

Answer a

$(2x+3)(x+3)$

Answer b

$(3x-1)(2x+1)$

Factoring a Perfect Square Trinomial

A perfect square trinomial is a trinomial that can be written as the square of a binomial. Recall that when a binomial is squared, the result is the square of the first term added to twice the product of the two terms and the square of the last term.

$$a^2+2ab+b^2=(a+b)^2 \tag{1.5.1}$$

and

$$a^2-2ab+b^2=(a-b)^2 \tag{1.5.2}$$

We can use this equation to factor any perfect square trinomial.

Perfect Square Trinomials

A perfect square trinomial can be written as the square of a binomial:

$$a^2+2ab+b^2=(a+b)^2 \tag{1.5.3}$$

Howto: Given a perfect square trinomial, factor it into the square of a binomial

1. Confirm that the first and last term are perfect squares.
2. Confirm that the middle term is twice the product of ab.
3. Write the factored form as $(a+b)^2$.

Example 1.5.4: Factoring a Perfect Square Trinomial

Factor $25x^2+20x+4$.

Solution

Notice that $25x^2$ and 4 are perfect squares because $25x^2=(5x)^2$ and $4=2^2$. Then check to see if the middle term is twice the product of $5x$ and 2. The middle term is, indeed, twice the product: $2(5x)(2)=20x$. Therefore, the trinomial is a perfect square trinomial and can be written as $(5x+2)^2$.

Exercise 1.5.4

Factor $49x^2-14x+1$.

Answer

$(7x-1)^2$

Factoring a Difference of Squares

A difference of squares is a perfect square subtracted from a perfect square. Recall that a difference of squares can be rewritten as factors containing the same terms but opposite signs because the middle terms cancel each other out when the two factors are multiplied.

$$a^2-b^2=(a+b)(a-b) \tag{1.5.4}$$

We can use this equation to factor any differences of squares.

Differences of Squares

A difference of squares can be rewritten as two factors containing the same terms but opposite signs.

$$a^2 - b^2 = (a + b)(a - b) \tag{1.5.5}$$

Howto: Given a difference of squares, factor it into binomials

1. Confirm that the first and last term are perfect squares.
2. Write the factored form as $(a + b)(a - b)$.

Example 1.5.5: Factoring a Difference of Squares

Factor $9x^2 - 25$.

Solution

Notice that $9x^2$ and 25 are perfect squares because $9x^2 = (3x)^2$ and $25 = 5^2$. The polynomial represents a difference of squares and can be rewritten as $(3x + 5)(3x - 5)$.

Exercise 1.5.5

Factor $81y^2 - 100$.

Answer

$(9y + 10)(9y - 10)$

Q&A: Is there a formula to factor the sum of squares?

No. A sum of squares cannot be factored.

Factoring the Sum and Difference of Cubes

Now, we will look at two new special products: the sum and difference of cubes. Although the sum of squares cannot be factored, the sum of cubes can be factored into a binomial and a trinomial.

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2) \tag{1.5.6}$$

Similarly, the sum of cubes can be factored into a binomial and a trinomial, but with different signs.

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2) \tag{1.5.7}$$

We can use the acronym SOAP to remember the signs when factoring the sum or difference of cubes. The first letter of each word relates to the signs: Same Opposite Always Positive. For example, consider the following example.

$$x^3 - 2^3 = (x - 2)(x^2 + 2x + 4) \tag{1.5.8}$$

The sign of the first 2 is the same as the sign between $x^3 - 2^3$. The sign of the $2x$ term is opposite the sign between $x^3 - 2^3$. And the sign of the last term, 4, is always positive.

Sum and Difference of Cubes

We can factor the sum of two cubes as

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2) \tag{1.5.9}$$

We can factor the difference of two cubes as

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2) \tag{1.5.10}$$

Howto: Given a sum of cubes or difference of cubes, factor it

1. Confirm that the first and last term are cubes, $a^3 + b^3$ or $a^3 - b^3$.
2. For a sum of cubes, write the factored form as $(a + b)(a^2 - ab + b^2)$. For a difference of cubes, write the factored form as $(a - b)(a^2 + ab + b^2)$.

Example 1.5.6: Factoring a Sum of Cubes

Factor $x^3 + 512$.

Solution

Notice that x^3 and 512 are cubes because $8^3 = 512$. Rewrite the sum of cubes as $(x + 8)(x^2 - 8x + 64)$.

Analysis

After writing the sum of cubes this way, we might think we should check to see if the trinomial portion can be factored further. However, the trinomial portion cannot be factored, so we do not need to check.

Exercise 1.5.6

Factor the sum of cubes: $216a^3 + b^3$.

Answer

$(6a + b)(36a^2 - 6ab + b^2)$

Example 1.5.7: Factoring a Difference of Cubes

Factor $8x^3 - 125$.

Solution

Notice that $8x^3$ and 125 are cubes because $8x^3 = (2x)^3$ and $125 = 5^3$. Write the difference of cubes as $(2x - 5)(4x^2 + 10x + 25)$.

Analysis

Just as with the sum of cubes, we will not be able to further factor the trinomial portion.

Exercise 1.5.7

Factor the difference of cubes: $1000x^3 - 1$

Answer

$(10x - 1)(100x^2 + 10x + 1)$

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1.6: Rational Expressions

Learning Objectives

In this section students will:

- Simplify rational expressions.
- Multiply rational expressions.
- Divide rational expressions.
- Add and subtract rational expressions.
- Simplify complex rational expressions.

A pastry shop has fixed costs of $280 per week and variable costs of $9 per box of pastries. The shop's costs per week in terms of x, the number of boxes made, is $280 + 9x$. We can divide the costs per week by the number of boxes made to determine the cost per box of pastries.

$$\frac{280 + 9x}{x}$$

Notice that the result is a polynomial expression divided by a second polynomial expression. In this section, we will explore quotients of polynomial expressions.

Simplifying Rational Expressions

The quotient of two polynomial expressions is called a rational expression. We can apply the properties of fractions to rational expressions, such as simplifying the expressions by canceling common factors from the numerator and the denominator. To do this, we first need to factor both the numerator and denominator. Let's start with the rational expression shown.

$$\frac{x^2 + 8x + 16}{x^2 + 11x + 28}$$

We can factor the numerator and denominator to rewrite the expression.

$$\frac{(x+4)^2}{(x+4)(x+7)}$$

Then we can simplify that expression by canceling the common factor $(x + 4)$.

$$\frac{x+4}{x+7}$$

Howto: Given a rational expression, simplify it

1. Factor the numerator and denominator.
2. Cancel any common factors.

Example 1.6.1: Simplifying Rational Expressions

Simplify $\frac{x^2 - 9}{x^2 + 4x + 3}$

Solution

$$\frac{(x+3)(x-3)}{(x+3)(x+1)} \qquad \text{Factor the numerator and the denominator}$$

$$\frac{x-3}{x+1} \qquad \text{Cancel common factor } (x+3)$$

Analysis

We can cancel the common factor because any expression divided by itself is equal to 1.

Q&A

Can the x^2 term be cancelled in the last example?

No. A factor is an expression that is multiplied by another expression. The x^2 term is not a factor of the numerator or the denominator.

Exercise 1.6.1

Simplify $\frac{x-6}{x^2-36}$

Answer

$$\frac{1}{x+6}$$

Multiplying Rational Expressions

Multiplication of rational expressions works the same way as multiplication of any other fractions. We multiply the numerators to find the numerator of the product, and then multiply the denominators to find the denominator of the product. Before multiplying, it is helpful to factor the numerators and denominators just as we did when simplifying rational expressions. We are often able to simplify the product of rational expressions.

Howto: Given two rational expressions, multiply them

1. Factor the numerator and denominator.
2. Multiply the numerators.
3. Multiply the denominators.
4. Simplify.

Example 1.6.2: Multiplying Rational Expressions

Multiply the rational expressions and show the product in simplest form:

$$\frac{(x+5)(x-1)}{3(x+6)} \times \frac{(2x-1)}{(x+5)}$$

Solution

$$\frac{(x+5)(x-1)}{3(x+6)} \times \frac{(2x-1)}{(x+5)}$$ Factor the numerator and denominator.

$$\frac{(x+5)(x-1)(2x-1)}{3(x+6)(x+5)}$$ Multiply numerators and denominators

$$\frac{(x-1)(2x-1)}{3(x+6)}$$ Cancel common factors to simplify

Exercise 1.6.2

Multiply the rational expressions and show the product in simplest form:

$$\frac{x^2+11x+30}{x^2+5x+6} \times \frac{x^2+7x+12}{x^2+8x+16}$$

Answer

$$\frac{(x+5)(x+6)}{(x+2)(x+4)}$$

Dividing Rational Expressions

Division of rational expressions works the same way as division of other fractions. To divide a rational expression by another rational expression, multiply the first expression by the reciprocal of the second. Using this approach, we would rewrite $\frac{1}{x} \div \frac{x^2}{3}$ as the product $\frac{1}{x} \cdot \frac{3}{x^2}$. Once the division expression has been rewritten as a multiplication expression, we can multiply as we did before.

$$\frac{1}{x} \cdot \frac{3}{x^2} = \frac{3}{x^3}$$

Howto: Given two rational expressions, divide them

1. Rewrite as the first rational expression multiplied by the reciprocal of the second.
2. Factor the numerators and denominators.
3. Multiply the numerators.
4. Multiply the denominators.
5. Simplify.

Example 1.6.3: Dividing Rational Expressions

Divide the rational expressions and express the quotient in simplest form:

$$\frac{2x^2+x-6}{x^2-1} \div \frac{x^2-4}{x^2+2x+1}$$

Solution

$$\frac{2x^2+x-6}{x^2-1} \div \frac{x^2-4}{x^2+2x+1}$$

$$\frac{2x^2+x-6}{x^2-1} \times \frac{x^2+2x+1}{x^2-4} \qquad \text{Rewrite as a multiplication problem}$$

$$\frac{(2x-3)(x+2)}{(x-1)(x+1)} \times \frac{(x+1)(x+1)}{(x-2)(x+2)} \qquad \text{Factor the numerator and denominator.}$$

$$\frac{(2x-3)(x+2)(x+1)(x+1)}{(x-1)(x+1)(x-2)(x+2)} \qquad \text{Multiply numerators and denominators}$$

$$\frac{(2x-3)(x+1)}{(x-1)(x-2)} \qquad \text{Cancel common factors to simplify}$$

Exercise 1.6.3

Divide the rational expressions and express the quotient in simplest form:

$$\frac{9x^2-16}{3x^2+17x-28} \div \frac{3x^2-2x-8}{x^2+5x-14}$$

Answer

1

Adding and Subtracting Rational Expressions

Adding and subtracting rational expressions works just like adding and subtracting numerical fractions. To add fractions, we need to find a common denominator. Let's look at an example of fraction addition.

$$\begin{aligned}\frac{5}{24}+\frac{1}{40}&=\frac{25}{120}+\frac{3}{120}\\&=\frac{28}{120}\\&=\frac{7}{30}\end{aligned}$$

We have to rewrite the fractions so they share a common denominator before we are able to add. We must do the same thing when adding or subtracting rational expressions.

The easiest common denominator to use will be the **least common denominator**, or LCD. The LCD is the smallest multiple that the denominators have in common. To find the LCD of two rational expressions, we factor the expressions and multiply all of the distinct factors. For instance, if the factored denominators were $(x+3)(x+4)$ and $(x+4)(x+5)$, then the LCD would be $(x+3)(x+4)(x+5)$.

Once we find the LCD, we need to multiply each expression by the form of 1 that will change the denominator to the LCD. We would need to multiply the expression with a denominator of $(x+3)(x+4)$ by $\frac{x+5}{x+5}$ and the expression with a denominator of $(x+4)(x+5)$ by $\frac{x+3}{x+3}$.

Howto: Given two rational expressions, add or subtract them

1. Factor the numerator and denominator.
2. Find the LCD of the expressions.
3. Multiply the expressions by a form of 1 that changes the denominators to the LCD.
4. Add or subtract the numerators.
5. Simplify.

Example 1.6.4: Adding Rational Expressions

Add the rational expressions:

$$\frac{5}{x}+\frac{6}{y}$$

Solution

First, we have to find the LCD. In this case, the LCD will be xy. We then multiply each expression by the appropriate form of 1 to obtain xy as the denominator for each fraction.

$$\begin{aligned}&\frac{5}{x}\times\frac{y}{y}+\frac{6}{y}\times\frac{x}{x}\\&\frac{5y}{xy}+\frac{6x}{xy}\end{aligned}$$

Now that the expressions have the same denominator, we simply add the numerators to find the sum.

$$\frac{6x+5y}{xy}$$

Analysis

Multiplying by $\frac{y}{y}$ or $\frac{x}{x}$ does not change the value of the original expression because any number divided by itself is 1, and multiplying an expression by 1 gives the original expression.

Example 1.6.5: Subtracting Rational Expressions

Subtract the rational expressions:

$$\frac{6}{x^2+4x+4}-\frac{2}{x^2-4} \tag{1.6.1}$$

Solution

$\frac{6}{(x+2)^2}-\frac{2}{(x+2)(x-2)}$	Factor
$\frac{6}{(x+2)^2}\times\frac{x-2}{x-2}-\frac{2}{(x+2)(x-2)}\times\frac{x+2}{x+2}$	Multiply each fraction to get LCD as denominator
$\frac{6(x-2)}{(x+2)^2(x-2)}-\frac{2(x+2)}{(x+2)^2(x-2)}$	Multiply
$\frac{6x-12-(2x+4)}{(x+2)^2(x-2)}$	Apply distributive property
$\frac{4x-16}{(x+2)^2(x-2)}$	Subtract
$\frac{4(x-4)}{(x+2)^2(x-2)}$	Simplify

Q&A

Do we have to use the LCD to add or subtract rational expressions?

No. Any common denominator will work, but it is easiest to use the LCD.

Exercise 1.6.4

Subtract the rational expressions: $\frac{3}{x+5}-\frac{1}{x-3}$

Answer

$$\frac{2(x-7)}{(x+5)(x-3)}$$

Simplifying Complex Rational Expressions

A complex rational expression is a rational expression that contains additional rational expressions in the numerator, the denominator, or both. We can simplify complex rational expressions by rewriting the numerator and denominator as single rational expressions and dividing. The complex rational expression $\frac{a}{\frac{1}{b}+c}$ can be simplified by rewriting the numerator as the fraction $\frac{a}{1}$ and combining the expressions in the denominator as $\frac{1+bc}{b}$. We can then rewrite the expression as a multiplication problem using the reciprocal of the denominator. We get $\frac{a}{1}\cdot\frac{b}{1+bc}$, which is equal to $\frac{ab}{1+bc}$.

Howto: Given a complex rational expression, simplify it

1. Combine the expressions in the numerator into a single rational expression by adding or subtracting.
2. Combine the expressions in the denominator into a single rational expression by adding or subtracting.
3. Rewrite as the numerator divided by the denominator.
4. Rewrite as multiplication.
5. Multiply.
6. Simplify.

Example 1.6.6: Simplifying Complex Rational Expressions

Simplify: $\frac{y+\frac{1}{x}}{\frac{x}{y}}$

Solution

Begin by combining the expressions in the numerator into one expression.

$$y \times \frac{x}{x} + \frac{1}{x} \qquad \text{Multiply by } \frac{x}{x} \text{ to get LCD as denominator}$$

$$\frac{xy}{x} + \frac{1}{x}$$

$$\frac{xy+1}{x} \qquad \text{Add numerators}$$

Now the numerator is a single rational expression and the denominator is a single rational expression.

$$\frac{\frac{xy+1}{x}}{\frac{x}{y}}$$

We can rewrite this as division, and then multiplication.

$$\frac{xy+1}{x} \div \frac{x}{y}$$

$$\frac{xy+1}{x} \times \frac{y}{x} \qquad \text{Rewrite as multiplication}$$

$$\frac{y(xy+1)}{x^2} \qquad \text{Multiply}$$

? Exercise 1.6.5

Simplify: $\dfrac{\frac{x}{y} - \frac{y}{x}}{y}$

Answer

$$\frac{x^2 - y^2}{xy^2}$$

Q&A

Can a complex rational expression always be simplified?

Yes. We can always rewrite a complex rational expression as a simplified rational expression.

Media

Access these online resources for additional instruction and practice with rational expressions.

1. Simplify Rational Expressions
2. Multiply and Divide Rational Expressions
3. Add and Subtract Rational Expressions
4. Simplify a Complex Fraction

Key Concepts

- Rational expressions can be simplified by cancelling common factors in the numerator and denominator. See Example.
- We can multiply rational expressions by multiplying the numerators and multiplying the denominators. See Example.
- To divide rational expressions, multiply by the reciprocal of the second expression. See Example.
- Adding or subtracting rational expressions requires finding a common denominator. See Example and Example.
- Complex rational expressions have fractions in the numerator or the denominator. These expressions can be simplified. See Example.

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CHAPTER OVERVIEW

2: Equations and Inequalities

Recall that a function is a relation that assigns to every element in the domain exactly one element in the range. Linear functions are a specific type of function that can be used to model many real-world applications, such as plant growth over time. In this chapter, we will explore linear functions, their graphs, and how to relate them to data.

2.0: Prelude to Equations and Inequalities
2.1: The Rectangular Coordinate Systems and Graphs
2.2: Linear Equations in One Variable
2.3: Models and Applications
2.4: Complex Numbers
2.5: Quadratic Equations
2.6: Other Types of Equations
2.7: Solving with rational exponents
2.8: Linear Inequalities and Absolute Value Inequalities

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2.0: Prelude to Equations and Inequalities

For most people, the term territorial possession indicates restrictions, usually dealing with trespassing or rite of passage and takes place in some foreign location. What most Americans do not realize is that from September through December, territorial possession dominates our lifestyles while watching the NFL. In this area, territorial possession is governed by the referees who make their decisions based on what the chains reveal. If the ball is at point $A(x_1, y_1)$, then it is up to the quarterback to decide which route to point $B(x_2, y_2)$, the end zone, is most feasible.

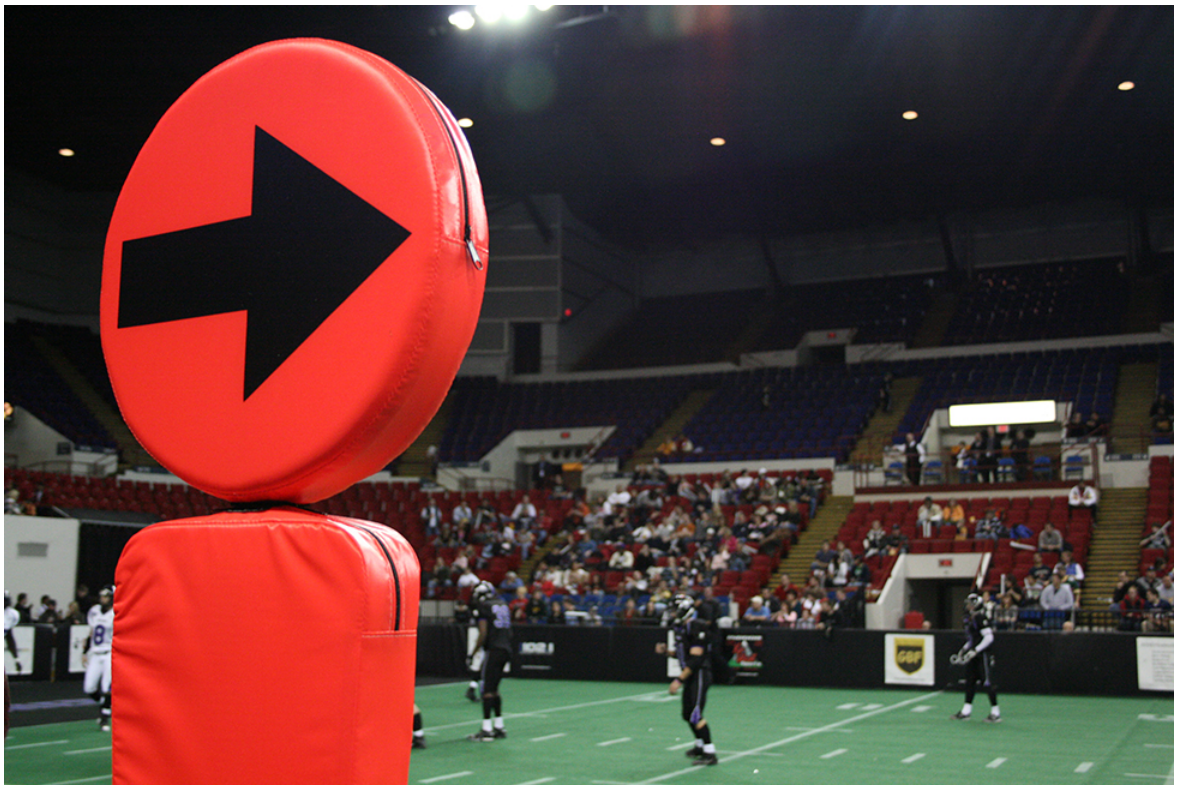


Figure **2.0.1**

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2.1: The Rectangular Coordinate Systems and Graphs

Learning Objectives

- Plot ordered pairs in a Cartesian coordinate system.
- Graph equations by plotting points.
- Graph equations with a graphing utility.
- Find x-intercepts and y-intercepts.

Tracie set out from Elmhurst, IL, to go to Franklin Park. On the way, she made a few stops to do errands. Each stop is indicated by a red dot in Figure **2.1.1**. Laying a rectangular coordinate grid over the map, we can see that each stop aligns with an intersection of grid lines. In this section, we will learn how to use grid lines to describe locations and changes in locations.

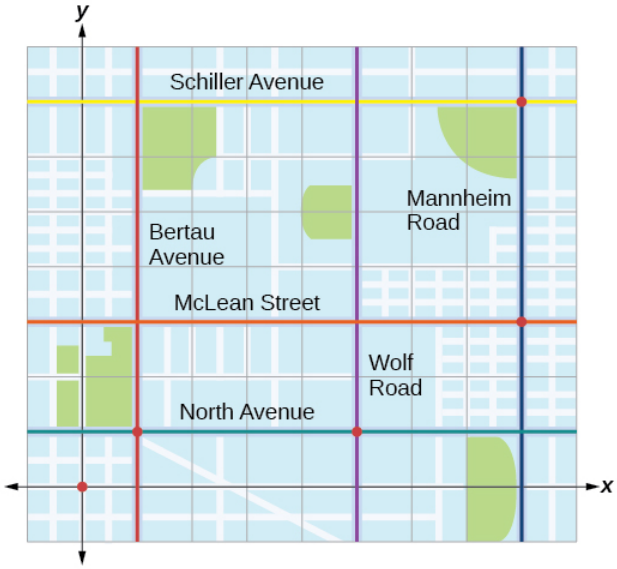


Figure **2.1.1**

Plotting Ordered Pairs in the Cartesian Coordinate System

An old story describes how seventeenth-century philosopher/mathematician René Descartes invented the system that has become the foundation of algebra while sick in bed. According to the story, Descartes was staring at a fly crawling on the ceiling when he realized that he could describe the fly's location in relation to the perpendicular lines formed by the adjacent walls of his room. He viewed the perpendicular lines as horizontal and vertical axes. Further, by dividing each axis into equal unit lengths, Descartes saw that it was possible to locate any object in a two-dimensional plane using just two numbers—the displacement from the horizontal axis and the displacement from the vertical axis.

While there is evidence that ideas similar to Descartes' grid system existed centuries earlier, it was Descartes who introduced the components that comprise the Cartesian coordinate system, a grid system having perpendicular axes. Descartes named the horizontal axis the x-axis and the vertical axis the y-axis.

The Cartesian coordinate system, also called the *rectangular coordinate system*, is based on a two-dimensional plane consisting of the x-axis and the y-axis. Perpendicular to each other, the axes divide the plane into four sections. Each section is called a quadrant; the quadrants are numbered counterclockwise as shown in Figure **2.1.2**.

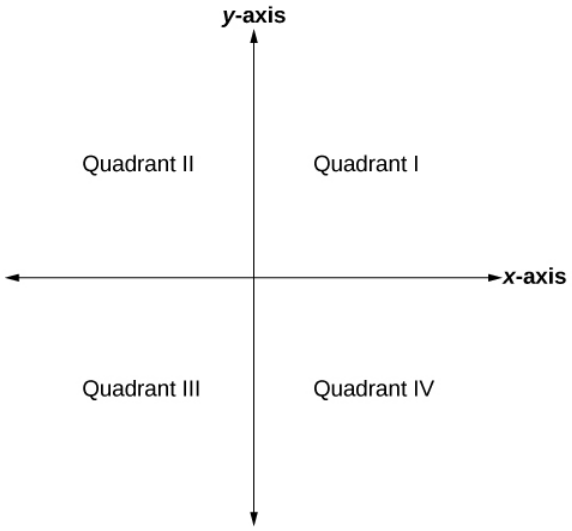


Figure 2.1.2

The center of the plane is the point at which the two axes cross. It is known as the origin, or point $(0, 0)$. From the origin, each axis is further divided into equal units: increasing, positive numbers to the right on the x-axis and up the y-axis; decreasing, negative numbers to the left on the x-axis and down the y-axis. The axes extend to positive and negative infinity as shown by the arrowheads in Figure 2.1.3.

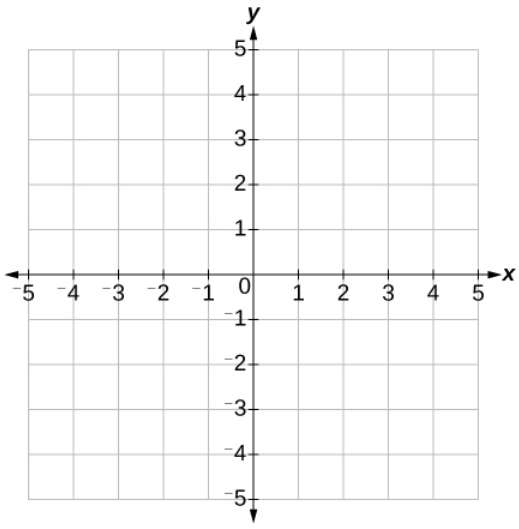


Figure 2.1.3

Each point in the plane is identified by its x-coordinate, or horizontal displacement from the origin, and its y-coordinate, or vertical displacement from the origin. Together, we write them as an ordered pair indicating the combined distance from the origin in the form (x, y). An ordered pair is also known as a coordinate pair because it consists of x- and y-coordinates. For example, we can represent the point $(3, -1)$ in the plane by moving three units to the right of the origin in the horizontal direction, and one unit down in the vertical direction. See Figure 2.1.4.

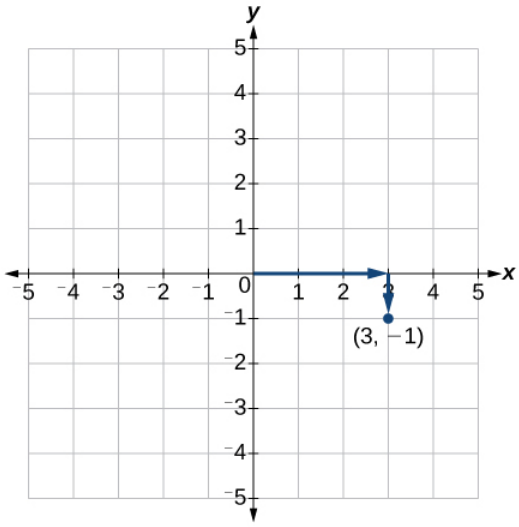


Figure 2.1.4

When dividing the axes into equally spaced increments, note that the x-axis may be considered separately from the y-axis. In other words, while the x-axis may be divided and labeled according to consecutive integers, the y-axis may be divided and labeled by increments of 2, or 10, or 100. In fact, the axes may represent other units, such as years against the balance in a savings account, or quantity against cost, and so on. Consider the rectangular coordinate system primarily as a method for showing the relationship between two quantities.

Cartesian Coordinate System

A two-dimensional plane where the

- x-axis is the horizontal axis
- y-axis is the vertical axis

A point in the plane is defined as an ordered pair, (x, y), such that x is determined by its horizontal distance from the origin and y is determined by its vertical distance from the origin.

Example 2.1.1: Plotting Points in a Rectangular Coordinate System

Plot the points $(-2, 4)$, $(3, 3)$, and $(0, -3)$ in the plane.

Solution

To plot the point $(-2, 4)$, begin at the origin. The x-coordinate is -2, so move two units to the left. The y-coordinate is 4, so then move four units up in the positive y direction.

To plot the point $(3, 3)$, begin again at the origin. The x-coordinate is 3, so move three units to the right. The y-coordinate is also 3, so move three units up in the positive y direction.

To plot the point $(0, -3)$, begin again at the origin. The x-coordinate is 0. This tells us not to move in either direction along the x-axis. The y-coordinate is -3, so move three units down in the negative y direction. See the graph in Figure 2.1.5.

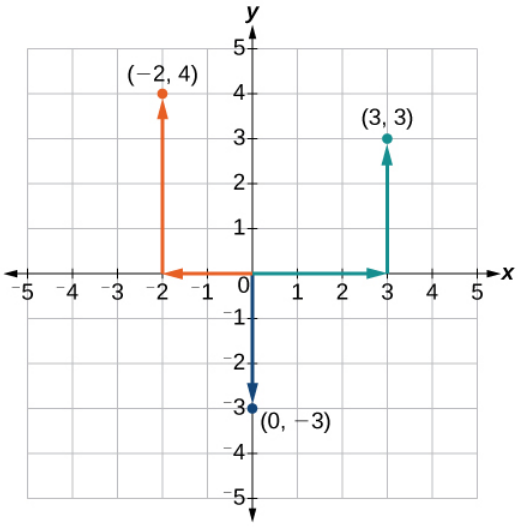


Figure 2.1.5

Analysis

Note that when either coordinate is zero, the point must be on an axis. If the x-coordinate is zero, the point is on the y-axis. If the y-coordinate is zero, the point is on the x-axis.

Graphing Equations by Plotting Points

We can plot a set of points to represent an equation. When such an equation contains both an x variable and a y variable, it is called an **equation in two variables**. Its graph is called a **graph in two variables**. Any graph on a two-dimensional plane is a graph in two variables.

Suppose we want to graph the equation $y = 2x - 1$. We can begin by substituting a value for x into the equation and determining the resulting value of y. Each pair of x- and y-values is an ordered pair that can be plotted. Table 2.1.1 lists values of x from –3 to 3 and the resulting values for y.

Table 2.1.1

x	$y = 2x - 1$	(x, y)
-3	$y = 2(-3) - 1 = -7$	$(-3, -7)$
-2	$y = 2(-2) - 1 = -5$	$(-2, -5)$
-1	$y = 2(-1) - 1 = -3$	$(-1, -3)$
0	$y = 2(0) - 1 = -1$	$(0, -1)$
1	$y = 2(1) - 1 = 1$	$(1, 1)$
2	$y = 2(2) - 1 = 3$	$(2, 3)$
3	$y = 2(3) - 1 = 5$	$(3, 5)$

We can plot the points in the table. The points for this particular equation form a line, so we can connect them (Figure 2.1.6). This is not true for all equations.

Figure 2.1.6

Note that the x-values chosen are arbitrary, regardless of the type of equation we are graphing. Of course, some situations may require particular values of x to be plotted in order to see a particular result. Otherwise, it is logical to choose values that can be calculated easily, and it is always a good idea to choose values that are both negative and positive. There is no rule dictating how many points to plot, although we need at least two to graph a line. Keep in mind, however, that the more points we plot, the more accurately we can sketch the graph.

Howto: Given an equation, graph by plotting points

1. Make a table with one column labeled x, a second column labeled with the equation, and a third column listing the resulting ordered pairs.
2. Enter x-values down the first column using positive and negative values. Selecting the x-values in numerical order will make the graphing simpler.
3. Select x-values that will yield y-values with little effort, preferably ones that can be calculated mentally.
4. Plot the ordered pairs.
5. Connect the points if they form a line.

Example 2.1.2: Graphing an Equation in Two Variables by Plotting Points

Graph the equation $y = -x + 2$ by plotting points.

Solution

First, we construct a table similar to Table 2.1.2. Choose x values and calculate y.

Table 2.1.2

x	$y=-x+2$	(x,y)
-5	$y=-(-5)+2=7$	$(-5,7)$
-3	$y=-(-3)+2=5$	$(-3,5)$
-1	$y=-(-1)+2=3$	$(-1,3)$
0	$y=-(0)+2=2$	$(0,2)$
1	$y=-(1)+2=1$	$(1,1)$
3	$y=-(3)+2=-1$	$(3,-1)$
5	$y=-(5)+2=-3$	$(5,-3)$

Now, plot the points. Connect them if they form a line. See Figure 2.1.7.

Figure 2.1.7

? Exercise 2.1.1

Construct a table and graph the equation by plotting points: $y=\frac{1}{2}x+2$.

Answer

Please see Table 2.1.3 and graph below.

Table 2.1.3

x	$y=12x+2$	(x,y)
-2	$y=12(-2)+2=1$	$(-2,1)$
-1	$y=12(-1)+2=32$	$(-1,32)$
0	$y=12(0)+2=2$	$(0,2)$
1	$y=12(1)+2=52$	$(1,52)$
2	$y=12(2)+2=3$	$(2,3)$

Figure 2.1.8

Graphing Equations with a Graphing Utility

Most graphing calculators require similar techniques to graph an equation. The equations sometimes have to be manipulated so they are written in the style $y=$_____ . The TI-84 Plus, and many other calculator makes and models, have a mode function, which allows the window (the screen for viewing the graph) to be altered so the pertinent parts of a graph can be seen.

For example, the equation $y=2x-20$ has been entered in the TI-84 Plus shown in Figure 2.1.9*a*. In Figure 2.1.9*b*, the resulting graph is shown. Notice that we cannot see on the screen where the graph crosses the axes. The standard window screen on the TI-84 Plus shows $-10\le x\le 10$, and $-10\le y\le 10$. See Figure \(\PageIndex{9c}\).

Figure 2.1.9: a. Enter the equation. b. This is the graph in the original window. c. These are the original settings.

By changing the window to show more of the positive x-axis and more of the negative y-axis, we have a much better view of the graph and the x- and y-intercepts. See Figure 2.1.10*a* and Figure 2.1.10*b*.

Figure 2.1.10: a. This screen shows the new window settings. b. We can clearly view the intercepts in the new window.

Example 2.1.3: Using a Graphing Utility to Graph an Equation

Use a graphing utility to graph the equation: $y = -\frac{2}{3}x - \frac{4}{3}$.

Solution

Enter the equation in the $y =$ function of the calculator. Set the window settings so that both the x- and y- intercepts are showing in the window. See Figure 2.1.11.

Figure 2.1.11

Finding x-intercepts and y-intercepts

The **intercepts** of a graph are points at which the graph crosses the axes. The x-intercept is the point at which the graph crosses the \(x\)-axis. At this point, the y-coordinate is zero. The y-intercept is the point at which the graph crosses the y-axis. At this point, the x-coordinate is zero.

To determine the x-intercept, we set y equal to zero and solve for x. Similarly, to determine the y-intercept, we set x equal to zero and solve for y. For example, lets find the intercepts of the equation $y = 3x - 1$.

To find the x-intercept, set $y = 0$.

$$\begin{aligned} y &= 3x - 1 \\ 0 &= 3x - 1 \\ 1 &= 3x \\ \frac{1}{3} &= x \end{aligned}$$

x–intercept: $\left(\frac{1}{3}, 0\right)$

To find the y-intercept, set $x = 0$.

$$\begin{aligned} y &= 3x - 1 \\ y &= 3(0) - 1 \\ y &= -1 \end{aligned}$$

y–intercept: $(0, -1)$

We can confirm that our results make sense by observing a graph of the equation as in Figure 2.1.12. Notice that the graph crosses the axes where we predicted it would.

Figure 2.1.12

Howto: GIVEN AN EQUATION, FIND THE INTERCEPTS

1. Find the x-intercept by setting $y = 0$ and solving for x.
2. Find the y-intercept by setting $x = 0$ and solving for y.

Example 2.1.4: Finding the Intercepts of the Given Equation

Find the intercepts of the equation $y = -3x - 4$. Then sketch the graph using only the intercepts.

Solution

Set $y = 0$ to find the x-intercept.

$$\begin{aligned} y &= -3x - 4 \\ 0 &= -3x - 4 \\ 4 &= -3x \\ \frac{4}{3} &= x \end{aligned}$$

x–intercept: $\left(-\frac{4}{3}, 0\right)$

Set $x = 0$ to find the y-intercept.

$$y = -3x - 4$$
$$y = -3(0) - 4$$
$$y = -4$$

y–intercept: $(0, -4)$

Plot both points, and draw a line passing through them as in Figure 2.1.13.

Figure 2.1.13

? Exercise 2.1.2

Find the intercepts of the equation and sketch the graph: $y = -\frac{3}{4}x + 3$.

Answer

x-intercept is $(4, 0)$; y-intercept is $(0, 3)$

Figure 2.1.14

Media

Access these online resources for additional instruction and practice with the Cartesian coordinate system.

1. Plotting points on the coordinate plane
2. Find x and y intercepts based on the graph of a line

Key Concepts

- We can locate, or plot, points in the Cartesian coordinate system using ordered pairs, which are defined as displacement from the x-axis and displacement from the y-axis. See Example.
- An equation can be graphed in the plane by creating a table of values and plotting points. See Example.
- Using a graphing calculator or a computer program makes graphing equations faster and more accurate. Equations usually have to be entered in the form $y =$_____. See Example.
- Finding the x- and y-intercepts can define the graph of a line. These are the points where the graph crosses the axes. See Example.

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2.2: Linear Equations in One Variable

Learning Objectives

- Solve equations in one variable algebraically.
- Solve a rational equation.
- Find a linear equation.
- Given the equations of two lines, determine whether their graphs are parallel or perpendicular.
- Write the equation of a line parallel or perpendicular to a given line.

Caroline is a full-time college student planning a spring break vacation. To earn enough money for the trip, she has taken a part-time job at the local bank that pays $\$15.00/hr$, and she opened a savings account with an initial deposit of $\$400$ on January 15. She arranged for direct deposit of her payroll checks. If spring break begins March 20 and the trip will cost approximately $\$2,500$, how many hours will she have to work to earn enough to pay for her vacation? If she can only work 4 hours per day, how many days per week will she have to work? How many weeks will it take? In this section, we will investigate problems like this and others, which generate graphs like the line in Figure 2.2.1.

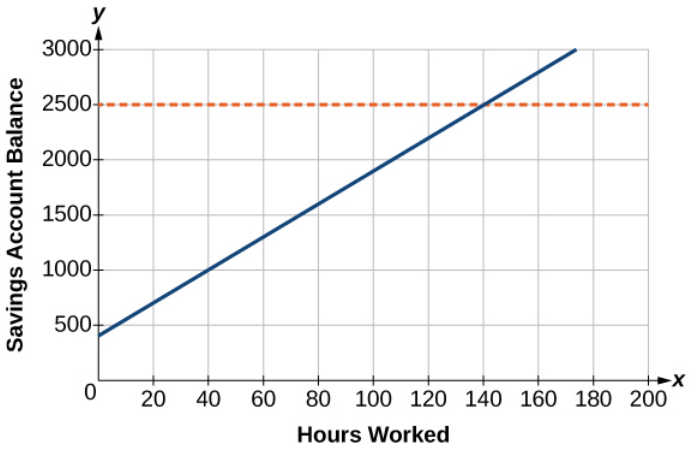


Figure 2.2.1

Solving Linear Equations in One Variable

A *linear equation* is an equation of a straight line, written in one variable. The only power of the variable is 1. Linear equations in one variable may take the form $ax + b = 0$ and are solved using basic algebraic operations. We begin by classifying linear equations in one variable as one of three types: identity, conditional, or inconsistent.

- An **identity equation** is true for all values of the variable. Here is an example of an identity equation:

$$3x = 2x + x$$

  The *solution set* consists of all values that make the equation true. For this equation, the solution set is all real numbers because any real number substituted for x will make the equation true.
- A **conditional equation** is true for only some values of the variable. For example, if we are to solve the equation $5x + 2 = 3x - 6$, we have the following:

$$\begin{aligned} 5x + 2 &= 3x - 6 \\ 2x &= -8 \\ x &= -4 \end{aligned}$$

  The solution set consists of one number: -4. It is the only solution and, therefore, we have solved a conditional equation.
- An **inconsistent equation** results in a false statement. For example, if we are to solve $5x - 15 = 5(x - 4)$, we have the following:

$$5x - 15 = 5x - 20$$
$$5x - 15 - 5x = 5x - 20 - 5x$$
$$-15 \neq -20$$

Indeed, $-15 \neq -20$. There is no solution because this is an inconsistent equation.

Solving linear equations in one variable involves the fundamental properties of equality and basic algebraic operations. A brief review of those operations follows.

LINEAR EQUATION IN ONE VARIABLE

A linear equation in one variable can be written in the form

$$ax + b = 0 \tag{2.2.1}$$

where a and b are real numbers, $a \neq 0$.

Howto: Given a linear equation in one variable, use algebra to solve it

The following steps are used to manipulate an equation and isolate the unknown variable, so that the last line reads $x =$ _____, if x is the unknown. There is no set order, as the steps used depend on what is given:

1. We may add, subtract, multiply, or divide an equation by a number or an expression as long as we do the same thing to both sides of the equal sign. Note that we cannot divide by zero.
2. Apply the distributive property as needed: $a(b + c) = ab + ac$.
3. Isolate the variable on one side of the equation.
4. When the variable is multiplied by a coefficient in the final stage, multiply both sides of the equation by the reciprocal of the coefficient.

Example 2.2.1: Solving an Equation in One Variable

Solve the following equation: $2x + 7 = 19$.

Solution

This equation can be written in the form $ax + b = 0$ by subtracting 19 from both sides. However, we may proceed to solve the equation in its original form by performing algebraic operations.

$$\begin{aligned} 2x + 7 &= 19 \\ 2x &= 12 \quad \text{Subtract 7 from both sides} \\ x &= 6 \quad \text{Multiply both sides by } \frac{1}{2} \text{ or divide by 2} \end{aligned}$$

The solution is 6.

Exercise 2.2.1

Solve the linear equation in one variable: $2x + 1 = -9$.

Answer

$x = -5$

Example 2.2.2: Solving When the Variable Appears on Both Sides

Solve the following equation: $4(x - 3) + 12 = 15 - 5(x + 6)$.

Solution

Apply standard algebraic properties.

$$
\begin{aligned}
4(x-3)+12 &= 15-5(x+6) \\
4x-12+12 &= 15-5x-30 \quad \text{Apply the distributive property} \\
4x &= -15-5x \quad \text{Combine like terms} \\
9x &= -15 \quad \text{Place x terms on one side and simplify} \\
x &= -\frac{15}{9} \quad \text{Multiply both sides by } \frac{1}{9} \text{, the reciprocal of 9} \\
x &= -\frac{5}{3}
\end{aligned}
$$

Analysis

This problem requires the distributive property to be applied twice, and then the properties of algebra are used to reach the final line, $x=-\frac{5}{3}$.

? Exercise 2.2.2

Solve the equation in one variable: $-2(3x-1)+x=14-x$.

Answer

$x=-3$

Solving a Rational Equation

In this section, we look at rational equations that, after some manipulation, result in a linear equation. If an equation contains at least one rational expression, it is a considered a **rational equation**. Recall that a **rational number** is the ratio of two numbers, such as $\frac{2}{3}$ or $\frac{7}{2}$. A rational expression is the ratio, or quotient, of two polynomials. Here are three examples.

$$\frac{x+1}{x^2-4}$$

$$\frac{1}{x-3}$$

or

$$\frac{4}{x^2+x-2}$$

Rational equations have a variable in the denominator in at least one of the terms. Our goal is to perform algebraic operations so that the variables appear in the numerator. In fact, we will eliminate all denominators by multiplying both sides of the equation by the **least common denominator** (LCD). Finding the LCD is identifying an expression that contains the highest power of all of the factors in all of the denominators. We do this because when the equation is multiplied by the LCD, the common factors in the LCD and in each denominator will equal one and will cancel out.

✓ Example 2.2.3: Solving a Rational Equation

Solve the rational equation:

$$\frac{7}{2x}-\frac{5}{3x}=\frac{22}{3}$$

Solution

We have three denominators; $2x$,$3x$, and 3. The LCD must contain $2x$,$3x$, and 3. An LCD of $6x$ contains all three denominators. In other words, each denominator can be divided evenly into the LCD. Next, multiply both sides of the equation by the LCD $6x$.

$$(6x)\left[\frac{7}{2x}-\frac{5}{3x}\right]=\left[\frac{22}{3}\right](6x)$$

$$(6x)\left(\frac{7}{2x}\right)-(6x)\left(\frac{5}{3x}\right)=\left(\frac{22}{3}\right)(6x) \quad \text{Use the distributive property.}$$

Cancel out the common factors

$$3(7)-2(5)=22(2x) \quad \text{Multiply remaining factors by each numerator.}$$

$$21-10=44x$$

$$11=44x$$

$$\frac{11}{44}=x$$

$$\frac{1}{4}=x$$

A common mistake made when solving rational equations involves finding the LCD when one of the denominators is a binomial—two terms added or subtracted—such as $(x+1)$. Always consider a binomial as an individual factor—the terms cannot be separated. For example, suppose a problem has three terms and the denominators are x, $x-1$, and $3x-3$. First, factor all denominators. We then have x, $(x-1)$, and $3(x-1)$ as the denominators. (Note the parentheses placed around the second denominator.) Only the last two denominators have a common factor of $(x-1)$. The x in the first denominator is separate from the x in the $(x-1)$ denominators. An effective way to remember this is to write factored and binomial denominators in parentheses, and consider each parentheses as a separate unit or a separate factor. The LCD in this instance is found by multiplying together the x, one factor of $(x-1)$, and the 3. Thus, the LCD is the following:

$$x(x-1)3=3x(x-1)$$

So, both sides of the equation would be multiplied by $3x(x-1)$. Leave the LCD in factored form, as this makes it easier to see how each denominator in the problem cancels out.

Another example is a problem with two denominators, such as x and x^2+2x. Once the second denominator is factored as $x^2+2x=x(x+2)$, there is a common factor of x in both denominators and the LCD is $x(x+2)$.

Sometimes we have a rational equation in the form of a proportion; that is, when one fraction equals another fraction and there are no other terms in the equation.

$$\frac{a}{b}=\frac{c}{d} \tag{2.2.2}$$

We can use another method of solving the equation without finding the LCD: cross-multiplication. We multiply terms by crossing over the equal sign.

$$\text{If } \frac{a}{b}=\frac{c}{d}, \text{ then } \frac{a}{b}\times\frac{c}{d}.$$

Multiply a(d) and b(c), which results in $ad=bc$.

Any solution that makes a denominator in the original expression equal zero must be excluded from the possibilities.

RATIONAL EQUATIONS

A r*ational equation* contains at least one rational expression where the variable appears in at least one of the denominators.

Howto: Given a rational equation, solve it.

1. Factor all denominators in the equation.
2. Find and exclude values that set each denominator equal to zero.
3. Find the LCD.
4. Multiply the whole equation by the LCD. If the LCD is correct, there will be no denominators left.
5. Solve the remaining equation.
6. Make sure to check solutions back in the original equations to avoid a solution producing zero in a denominator

Example 2.2.4: Solving a Rational Equation without Factoring

Solve the following rational equation:

$$\frac{2}{x}-\frac{3}{2}=\frac{7}{2x}$$

Solution

We have three denominators: x, 2, and $2x$. No factoring is required. The product of the first two denominators is equal to the third denominator, so, the LCD is $2x$. Only one value is excluded from a solution set, 0. Next, multiply the whole equation (both sides of the equal sign) by $2x$.

$$\begin{aligned} 2x\left[\frac{2}{x}-\frac{3}{2}\right] &= \left[\frac{7}{2x}\right](2x) \\ 2x\left(\frac{2}{x}\right)-2x\left(\frac{3}{2}\right) &= \left(\frac{7}{2x}\right)(2x) \quad \text{Distribute } 2x \\ 2(2)-3x &= 7 \quad \text{Denominators cancel out.} \\ 4-3x &= 7 \\ -3x &= 3 \\ x &= -1 \text{ or } \{-1\} \end{aligned}$$

The proposed solution is -1, which is not an excluded value, so the solution set contains one number, $x=-1$, or $\{-1\}$ written in set notation.

Exercise 2.2.4

Solve the rational equation:

$$\frac{2}{3x}=\frac{1}{4}-\frac{1}{6x}$$

Answer

$$x=\frac{10}{3}$$

Example 2.2.5: Solving a Rational Equation by Factoring the Denominator

Solve the following rational equation:

$$\frac{1}{x}=\frac{1}{10}-\frac{3}{4x}$$

Solution

First find the common denominator. The three denominators in factored form are $x, 10=2\cdot 5$, and $4x=2\cdot 2\cdot x$. The smallest expression that is divisible by each one of the denominators is $20x$. Only $x=0$ is an excluded value. Multiply the whole equation by $20x$.

$$\begin{aligned} 20x\left(\frac{1}{x}\right) &= \left(\frac{1}{10}-\frac{3}{4x}\right)20x \\ 20 &= 2x-15 \\ 35 &= 2x \\ \frac{35}{2} &= x \end{aligned}$$

The solution is $\frac{35}{2}$.

Exercise 2.2.5

Solve the rational equation:

$$-\frac{5}{2x}+\frac{3}{4x}=-\frac{7}{4}$$

Answer

$x = 1$

Example 2.2.6: Solving Rational Equations with a Binomial in the Denominator

Solve the following rational equations and state the excluded values:

a. $\dfrac{3}{x-6} = \dfrac{5}{x}$

b. $\dfrac{x}{x-3} = \dfrac{5}{x-3} - \dfrac{1}{2}$

c. $\dfrac{x}{x-2} = \dfrac{5}{x-2} - \dfrac{1}{2}$

Solution

a.

The denominators x and $x-6$ have nothing in common. Therefore, the LCD is the product $x(x-6)$. However, for this problem, we can cross-multiply.

$$\begin{aligned} \frac{3}{x-6} &= \frac{5}{x} \\ 3x &= 5(x-6) \qquad \text{Distribute.} \\ 3x &= 5x - 30 \\ -2x &= -30 \\ x &= 15 \end{aligned}$$

The solution is 15. The excluded values are 6 and 0.

b.

The LCD is $2(x-3)$. Multiply both sides of the equation by $2(x-3)$.

$$\begin{aligned} 2(x-3)\left[\frac{x}{x-3}\right] &= \left[\frac{5}{x-3} - \frac{1}{2}\right] 2(x-3) \\ \frac{2(x-3)x}{x-3} &= \frac{2(x-3)5}{x-3} - \frac{2(x-3)}{2} \\ 2x &= 10 - (x-3) \\ 2x &= 13 - x \\ 3x &= 13 \\ x &= \frac{13}{3} \end{aligned}$$

The solution is $\dfrac{13}{3}$. The excluded value is 3.

c.

The least common denominator is $2(x-2)$. Multiply both sides of the equation by $x(x-2)$.

$$\begin{aligned} 2(x-2)\left[\frac{x}{x-2}\right] &= \left[\frac{5}{x-2} - \frac{1}{2}\right] 2(x-2) \\ 2x &= 10 - (x-2) \\ 2x &= 12 - x \\ 3x &= 12 \\ x &= 4 \end{aligned}$$

The solution is 4. The excluded value is 2.

Exercise 2.2.6

Solve $\dfrac{-3}{2x+1} = \dfrac{4}{3x+1}$. State the excluded values.

Answer

$x = -\frac{7}{17}$. Excluded values are $x = -12$ and $x = -13$.

Example 2.2.7: Solving a Rational Equation with Factored Denominators and Stating Excluded Values

Solve the rational equation after factoring the denominators: $\frac{2}{x+1} - \frac{1}{x-1} = \frac{2x}{x^2-1}$. State the excluded values.

Solution

We must factor the denominator $x^2 - 1$. We recognize this as the difference of squares, and factor it as $(x-1)(x+1)$. Thus, the LCD that contains each denominator is $(x-1)(x+1)$. Multiply the whole equation by the LCD, cancel out the denominators, and solve the remaining equation.

$$\begin{aligned}(x+1)(x-1)\left[\frac{2}{x+1} - \frac{1}{x-1}\right] &= \left[\frac{2x}{x^2-1}\right](x+1)(x-1)\\ 2(x-1) - (x+1) &= 2x\\ 2x - 2 - x - 1 &= 2x \text{ Distribute the negative sign}\\ -3 - x &= 0\\ x &= -3\end{aligned}$$

The solution is -3. The excluded values are 1 and -1.

Exercise 2.2.7

Solve the rational equation:

$$\frac{2}{x-2} + \frac{1}{x+1} = \frac{1}{x^2-x-2}$$

Answer

$x = \frac{1}{3}$

Finding a Linear Equation

Perhaps the most familiar form of a linear equation is the *slope-intercept form*, written as

$$y = mx + b \tag{2.2.3}$$

where $m =$ **slope** and $b =$ **y—intercept**. Let us begin with the slope.

The **slope** of a line refers to the ratio of the vertical change in y over the horizontal change in x between any two points on a line. It indicates the direction in which a line slants as well as its steepness. Slope is sometimes described as rise over run.

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{2.2.4}$$

If the slope is positive, the line slants to the right. If the slope is negative, the line slants to the left. As the slope increases, the line becomes steeper. Some examples are shown in Figure 2.2.2. The lines indicate the following slopes: $m = -3$, $m = 2$, and $m = \frac{1}{3}$.

Figure 2.2.2

THE SLOPE OF A LINE

The slope of a line, m, represents the change in y over the change in x. Given two points, (x_1, y_1) and (x_2, y_2), the following formula determines the slope of a line containing these points:

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{2.2.5}$$

Example 2.2.8: Finding the Slope of a Line Given Two Points

Find the slope of a line that passes through the points $(2,-1)$ and $(-5,3)$.

Solution

We substitute the y-values and the x-values into the formula.

$$\begin{aligned} m &= \frac{3-(-1)}{-5-2} \\ &= \frac{4}{-7} \\ &= -\frac{4}{7} \end{aligned}$$

The slope is $-\frac{4}{7}$

Analysis

It does not matter which point is called (x_1, y_1) or (x_2, y_2). As long as we are consistent with the order of the y terms and the order of the x terms in the numerator and denominator, the calculation will yield the same result.

Exercise 2.2.8

Find the slope of the line that passes through the points $(-2,6)$ and $(1,4)$.

Answer

$$m = -\frac{2}{3}$$

Example 2.2.9: Identifying the Slope and y-intercept of a Line Given an Equation

Identify the slope and y-intercept, given the equation $y = -\frac{3}{4}x - 4$.

Solution

As the line is in $y = mx + b$ form, the given line has a slope of $m = -\frac{3}{4}$. The y-intercept is $b = -4$.

Analysis

The y-intercept is the point at which the line crosses the y-axis. On the y-axis, $x = 0$. We can always identify the y-intercept when the line is in slope-intercept form, as it will always equal b. Or, just substitute $x = 0$ and solve for y.

The Point-Slope Formula

Given the slope and one point on a line, we can find the equation of the line using the point-slope formula.

$$y - y_1 = m(x - x_1) \tag{2.2.6}$$

This is an important formula, as it will be used in other areas of college algebra and often in calculus to find the equation of a tangent line. We need only one point and the slope of the line to use the formula. After substituting the slope and the coordinates of one point into the formula, we simplify it and write it in slope-intercept form.

THE POINT-SLOPE FORMULA

Given one point and the slope, the point-slope formula will lead to the equation of a line:

$$y - y_1 = m(x - x_1) \tag{2.2.7}$$

Example 2.2.10: Finding the Equation of a Line Given the Slope and One Point

Write the equation of the line with slope $m = -3$ and passing through the point $(4, 8)$. Write the final equation in slope-intercept form.

Solution

Using the point-slope formula, substitute -3 for m and the point $(4, 8)$ for (x_1, y_1).

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 8 &= -3(x - 4) \\ y - 8 &= -3x + 12 \\ y &= -3x + 20 \end{aligned}$$

Analysis

Note that any point on the line can be used to find the equation. If done correctly, the same final equation will be obtained.

Exercise 2.2.10

Given $m = 4$, find the equation of the line in slope-intercept form passing through the point $(2, 5)$.

Answer

$y = 4x - 3$

Example 2.2.11: Finding the Equation of a Line Passing Through Two Given Points

Find the equation of the line passing through the points $(3, 4)$ and $(0, -3)$. Write the final equation in slope-intercept form.

Solution

First, we calculate the slope using the slope formula and two points.

$$\begin{aligned} m &= \frac{-3 - 4}{0 - 3} \\ m &= \frac{-7}{-3} \\ m &= \frac{7}{3} \end{aligned}$$

Next, we use the point-slope formula with the slope of $\frac{7}{3}$, and either point. Let's pick the point $(3, 4)$ for (x_1, y_1).

$$\begin{aligned} y - 4 &= \frac{7}{3}(x - 3) \\ y - 4 &= \frac{7}{3}x - 7 \\ y &= \frac{7}{3}x - 3 \end{aligned}$$

In slope-intercept form, the equation is written as $y = \frac{7}{3}x - 3$

Analysis

To prove that either point can be used, let us use the second point $(0, -3)$ and see if we get the same equation.

$$\begin{aligned} y - (-3) &= \frac{7}{3}(x - 0) \\ y + 3 &= \frac{7}{3}x \\ y &= \frac{7}{3}x - 3 \end{aligned}$$

We see that the same line will be obtained using either point. This makes sense because we used both points to calculate the slope.

Standard Form of a Line

Another way that we can represent the equation of a line is in **standard form**. Standard form is given as

$$Ax + By = C \tag{2.2.8}$$

where A, B, and C are integers. The x- and y-terms are on one side of the equal sign and the constant term is on the other side.

Example 2.2.12: Finding the Equation of a Line and Writing It in Standard Form

Find the equation of the line with $m = -6$ and passing through the point $\left(\frac{1}{4}, -2\right)$. Write the equation in standard form.

Solution

We begin using the point-slope formula.

$$\begin{aligned} y - (-2) &= -6\left(x - \frac{1}{4}\right) \\ y + 2 &= -6x + \frac{3}{2} \end{aligned}$$

From here, we multiply through by 2, as no fractions are permitted in standard form, and then move both variables to the left aside of the equal sign and move the constants to the right.

$$\begin{aligned} 2(y + 2) &= \left(-6x + \frac{3}{2}\right)2 \\ 2y + 4 &= -12x + 3 \\ 12x + 2y &= -1 \end{aligned}$$

This equation is now written in standard form.

Exercise 2.2.12

Find the equation of the line in standard form with slope $m = -\frac{1}{3}$ and passing through the point $(1, 13)$.

Answer

$x + 3y = 2$

Vertical and Horizontal Lines

The equations of vertical and horizontal lines do not require any of the preceding formulas, although we can use the formulas to prove that the equations are correct. The equation of a **vertical line** is given as

$$x = c \tag{2.2.9}$$

where c is a constant. The slope of a vertical line is undefined, and regardless of the y-value of any point on the line, the x-coordinate of the point will be c.

Suppose that we want to find the equation of a line containing the following points: $(-3, -5)$,$(-3, 1)$,$(-3, 3)$, and $(-3, 5)$. First, we will find the slope.

$$m = \frac{5 - 3}{-3 - (-3)} = \frac{2}{0}$$

Zero in the denominator means that the slope is undefined and, therefore, we cannot use the point-slope formula. However, we can plot the points. Notice that all of the x-coordinates are the same and we find a vertical line through $x = -3$. See Figure 2.2.3.

The equation of a **horizontal line** is given as

$$y = c \tag{2.2.10}$$

where c is a constant. The slope of a horizontal line is zero, and for any x-value of a point on the line, the y-coordinate will be c.

Suppose we want to find the equation of a line that contains the following set of points: $(-2,-2)$,$(0,-2)$,$(3,-2)$, and $(5,-2)$. We can use the point-slope formula. First, we find the slope using any two points on the line.

$$\begin{aligned} m &= \frac{-2-(-2)}{0-(-2)} \\ &= \frac{0}{2} \\ &= 0 \end{aligned}$$

Use any point for (x_1, y_1) in the formula, or use the y-intercept.

$$\begin{aligned} y-(-2) &= 0(x-3) \\ y+2 &= 0 \\ y &= -2 \end{aligned}$$

The graph is a horizontal line through $y=-2$. Notice that all of the y-coordinates are the same. See Figure 2.2.3.

Figure 2.2.3: The line x = −3 is a vertical line. The line y = −2 is a horizontal line

Example 2.2.13: Finding the Equation of a Line Passing Through the Given Points

Find the equation of the line passing through the given points: $(1,-3)$ and $(1,4)$.

Solution

The x-coordinate of both points is 1. Therefore, we have a vertical line, $x=1$.

Exercise 2.2.13

Find the equation of the line passing through $(-5,2)$ and $(2,2)$.

Answer

Horizontal line: $y=2$

Determining Whether Graphs of Lines are Parallel or Perpendicular

Parallel lines have the same slope and different y-intercepts. Lines that are parallel to each other will never intersect. For example, Figure 2.2.4 shows the graphs of various lines with the same slope, $m=2$.

Figure 2.2.4: Parallel lines

All of the lines shown in the graph are parallel because they have the same slope and different y-intercepts.

Lines that are **perpendicular** intersect to form a $90°$ -angle. The slope of one line is the negative **reciprocal** of the other. We can show that two lines are perpendicular if the product of the two slopes is -1 : $m_1 \cdot m_2 = -1$. For example, Figure 2.2.5 shows the graph of two perpendicular lines. One line has a slope of 3; the other line has a slope of $-\frac{1}{3}$.

$$\begin{aligned} m_1 \cdot m_2 &= -1 \\ 3 \cdot \left(-\frac{1}{3}\right) &= -1 \end{aligned}$$

Figure 2.2.5: Perpendicular lines

Example 2.2.14: Graphing Two Equations, and Determining Whether the Lines are Parallel, Perpendicular, or Neither

Graph the equations of the given lines, and state whether they are parallel, perpendicular, or neither: $3y=-4x+3$ and $3x-4y=8$.

Solution

The first thing we want to do is rewrite the equations so that both equations are in slope-intercept form.

First equation:

$$3y = -4x + 3$$
$$y = -\frac{4}{3}x + 1$$

Second equation:

$$3x - 4y = 8$$
$$-4y = -3x + 8$$
$$y = \frac{3}{4}x - 2$$

See the graph of both lines in Figure **2.2.6**.

Figure **2.2.6**

From the graph, we can see that the lines appear perpendicular, but we must compare the slopes.

$$m_1 = -\frac{4}{3}$$
$$m_2 = \frac{3}{4}$$
$$m_1 \cdot m_2 = \left(-\frac{4}{3}\right)\left(\frac{3}{4}\right)$$
$$= -1$$

The slopes are negative reciprocals of each other, confirming that the lines are perpendicular.

? Exercise 2.2.14

Graph the two lines and determine whether they are parallel, perpendicular, or neither: $2y - x = 10$ and $2y = x + 4$.

Answer

Parallel lines: equations are written in slope-intercept form.

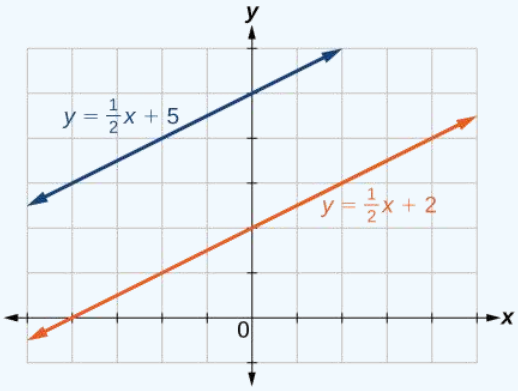


Figure **2.2.7**

Writing the Equations of Lines Parallel or Perpendicular to a Given Line

As we have learned, determining whether two lines are parallel or perpendicular is a matter of finding the slopes. To write the equation of a line parallel or perpendicular to another line, we follow the same principles as we do for finding the equation of any line. After finding the slope, use the **point-slope formula** to write the equation of the new line.

Given an equation for a line, write the equation of a line parallel or perpendicular to it.

1. Find the slope of the given line. The easiest way to do this is to write the equation in slope-intercept form.
2. Use the slope and the given point with the point-slope formula.
3. Simplify the line to slope-intercept form and compare the equation to the given line.

Example 2.2.15: Writing the Equation of a Line Parallel to a Given Line Passing Through a Given Point

Write the equation of line parallel to a $5x + 3y = 1$ and passing through the point $(3, 5)$.

Solution

First, we will write the equation in slope-intercept form to find the slope.

$$\begin{aligned} 5x + 3y &= 1 \\ 3y &= -5x + 1 \\ y &= -\frac{5}{3}x + \frac{1}{3} \end{aligned}$$

The slope is $m = -\frac{5}{3}$. The y-intercept is 1, but that really does not enter into our problem, as the only thing we need for two lines to be parallel is the same slope. The one exception is that if the y-intercepts are the same, then the two lines are the same line. The next step is to use this slope and the given point with the point-slope formula.

$$\begin{aligned} y - 5 &= -\frac{5}{3}(x - 3) \\ y - 5 &= -\frac{5}{3}x + 5 \\ y &= -\frac{5}{3}x + 10 \end{aligned}$$

The equation of the line is $y = -\frac{5}{3}x + 10$. See Figure 2.2.8.

Figure 2.2.8

Exercise 2.2.15

Find the equation of the line parallel to $5x = 7 + y$ and passing through the point $(-1, -2)$.

Answer

$y = 5x + 3$

Example 2.2.16: Finding the Equation of a Line Perpendicular to a Given Line Passing Through a Given Point

Find the equation of the line perpendicular to $5x - 3y + 4 = 0$ $(-4, 1)$.

Solution

The first step is to write the equation in slope-intercept form.

$$\begin{aligned} 5x - 3y + 4 &= 0 \\ -3y &= -5x - 4 \\ y &= \frac{5}{3}x + \frac{4}{3} \end{aligned}$$

We see that the slope is $m = \frac{5}{3}$. This means that the slope of the line perpendicular to the given line is the negative reciprocal, or $-\frac{3}{5}$. Next, we use the point-slope formula with this new slope and the given point.

$$y-1=-\frac{3}{5}(x-(-4))$$
$$y-1=-\frac{3}{5}x-\frac{12}{5}$$
$$y=-\frac{3}{5}x-\frac{12}{5}+\frac{5}{5}$$
$$y=-\frac{3}{5}x-\frac{7}{5}$$

Media

Access these online resources for additional instruction and practice with linear equations.

1. Solving rational equations
2. Equation of a line given two points
3. Finding the equation of a line perpendicular to another line through a given point
4. Finding the equation of a line parallel to another line through a given point

Key Concepts

- We can solve linear equations in one variable in the form $ax+b=0$ using standard algebraic properties. See Example and Example.
- A rational expression is a quotient of two polynomials. We use the LCD to clear the fractions from an equation. See Example and Example.
- All solutions to a rational equation should be verified within the original equation to avoid an undefined term, or zero in the denominator. See Example and Example.
- Given two points, we can find the slope of a line using the slope formula. See Example.
- We can identify the slope and y-intercept of an equation in slope-intercept form. See Example.
- We can find the equation of a line given the slope and a point. See Example.
- We can also find the equation of a line given two points. Find the slope and use the point-slope formula. See Example.
- The standard form of a line has no fractions. See Example.
- Horizontal lines have a slope of zero and are defined as $y=c$, where c is a constant.
- Vertical lines have an undefined slope (zero in the denominator), and are defined as $x=c$, where c is a constant. See Example.
- Parallel lines have the same slope and different y-intercepts. See Example.
- Perpendicular lines have slopes that are negative reciprocals of each other unless one is horizontal and the other is vertical. See Example.

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2.3: Models and Applications

Learning Objectives

- Set up a linear equation to solve a real-world application.
- Use a formula to solve a real-world application.

Josh is hoping to get an A in his college algebra class. He has scores of 75, 82, 95, 91, and 94 on his first five tests. Only the final exam remains, and the maximum of points that can be earned is 100. Is it possible for Josh to end the course with an A? A simple linear equation will give Josh his answer.



Figure 2.4.1 Credit: Kevin Dooley

Many real-world applications can be modeled by linear equations. For example, a cell phone package may include a monthly service fee plus a charge per minute of talk-time; it costs a widget manufacturer a certain amount to produce x widgets per month plus monthly operating charges; a car rental company charges a daily fee plus an amount per mile driven. These are examples of applications we come across every day that are modeled by linear equations. In this section, we will set up and use linear equations to solve such problems.

Setting up a Linear Equation to Solve a Real-World Application

To set up or model a linear equation to fit a real-world application, we must first determine the known quantities and define the unknown quantity as a variable. Then, we begin to interpret the words as mathematical expressions using mathematical symbols. Let us use the car rental example above. In this case, a known cost, such as $\$0.10/mi$, is multiplied by an unknown quantity, the number of miles driven. Therefore, we can write $0.10x$. This expression represents a variable cost because it changes according to the number of miles driven.

If a quantity is independent of a variable, we usually just add or subtract it, according to the problem. As these amounts do not change, we call them fixed costs. Consider a car rental agency that charges $\$0.10/mi$ plus a daily fee of $\$50$. We can use these quantities to model an equation that can be used to find the daily car rental cost C.

$$C = 0.10x + 50$$

When dealing with real-world applications, there are certain expressions that we can translate directly into math. Table $2.3.1$ lists some common verbal expressions and their equivalent mathematical expressions.

Table $2.3.1$: Verbal to math conversion

Verbal	Translation to Math Operations
One number exceeds another by a	$x, x+a$
Twice a number	$2x$
One number is a more than another number	$x, x+a$
One number is a less than twice another number	$x, 2x-a$
The product of a number and a, decreased by b	$ax-b$
The quotient of a number and the number plus a is three times the number	$\frac{x}{x+a}=3x$
The product of three times a number and the number decreased by b is c	$3x(x-b)=c$

How to: Given a real-world problem, model a linear equation to fit it

1. Identify known quantities.
2. Assign a variable to represent the unknown quantity.
3. If there is more than one unknown quantity, find a way to write the second unknown in terms of the first.
4. Write an equation interpreting the words as mathematical operations.
5. Solve the equation. Be sure the solution can be explained in words, including the units of measure.

Example 2.3.1

Find a linear equation to solve for the following unknown quantities: One number exceeds another number by 17 and their sum is 31. Find the two numbers.

Solution

Let x equal the first number. Then, as the second number exceeds the first by 17, we can write the second number as $x+17$. The sum of the two numbers is 31. We usually interpret the word is as an equal sign.

$$\begin{aligned} x+(x+17) &= 31 \\ 2x+17 &= 31 \\ 2x &= 14 \\ x &= 7 \end{aligned}$$

$$\begin{aligned} x+17 &= 7+17 \\ &= 24 \end{aligned}$$

The two numbers are 7 and 24.

Exercise 2.3.1

Find a linear equation to solve for the following unknown quantities: One number is three more than twice another number. If the sum of the two numbers is 36, find the numbers.

Answer

11 and 25

Example 2.3.2: Setting Up a Equation to Solve a Real-World Application

There are two cell phone companies that offer different packages. Company A charges a monthly service fee of $\$34$ plus $\$.05/min$ talk-time. Company B charges a monthly service fee of $\$40$ plus $\$.04/min$ talk-time.

a. Write a linear equation that models the packages offered by both companies.
b. If the average number of minutes used each month is $1,160$, which company offers the better plan?
c. If the average number of minutes used each month is 420, which company offers the better plan?
d. How many minutes of talk-time would yield equal monthly statements from both companies?

Solution

a.

The model for Company A can be written as $A = 0.05x + 34$. This includes the variable cost of $0.05x$ plus the monthly service charge of $\$34$. Company B's package charges a higher monthly fee of $\$40$, but a lower variable cost of $0.04x$. Company B's model can be written as $B = 0.04x + \$40$.

b.

If the average number of minutes used each month is $1,160$, we have the following:

$$\begin{aligned} \text{Company A} &= 0.05(1.160) + 34 \\ &= 58 + 34 \\ &= 92 \end{aligned}$$

$$\begin{aligned} \text{Company B} &= 0.04(1,1600) + 40 \\ &= 46.4 + 40 \\ &= 86.4 \end{aligned}$$

So, Company B offers the lower monthly cost of $\$86.40$ as compared with the $\$92$ monthly cost offered by Company A when the average number of minutes used each month is $1,160$.

c.

If the average number of minutes used each month is 420, we have the following:

$$\begin{aligned} \text{Company A} &= 0.05(420) + 34 \\ &= 21 + 34 \\ &= 55 \end{aligned}$$

$$\begin{aligned} \text{Company B} &= 0.04(420) + 40 \\ &= 16.8 + 40 \\ &= 56.8 \end{aligned}$$

If the average number of minutes used each month is 420, then Company A offers a lower monthly cost of $\$55$ compared to Company B's monthly cost of $\$56.80$.

d.

To answer the question of how many talk-time minutes would yield the same bill from both companies, we should think about the problem in terms of (x, y) coordinates: At what point are both the x-value and the y-value equal? We can find this point by setting the equations equal to each other and solving for x.

$$\begin{aligned} 0.05x + 34 &= 0.04x + 40 \\ 0.01x &= 6 \\ x &= 600 \end{aligned}$$

Check the x-value in each equation.

$$0.05(600) + 34 = 64$$

$$0.04(600) + 40 = 64$$

Therefore, a monthly average of 600 talk-time minutes renders the plans equal. See Figure $2.3.2$.

Figure 2.3.2

Exercise 2.3.2

Find a linear equation to model this real-world application: It costs ABC electronics company $2.50 per unit to produce a part used in a popular brand of desktop computers. The company has monthly operating expenses of $350 for utilities and $3,300 for salaries. What are the company's monthly expenses?

Answer

$C = 2.5x + 3,650$

Using a Formula to Solve a Real-World Application

Many applications are solved using known formulas. The problem is stated, a formula is identified, the known quantities are substituted into the formula, the equation is solved for the unknown, and the problem's question is answered. Typically, these problems involve two equations representing two trips, two investments, two areas, and so on. Examples of formulas include the **area** of a rectangular region,

$$A = LW \tag{2.4.2}$$

the **perimeter** of a rectangle,

$$P = 2L + 2W \tag{2.4.3}$$

and the **volume** of a rectangular solid,

$$V = LWH. \tag{2.4.4}$$

When there are two unknowns, we find a way to write one in terms of the other because we can solve for only one variable at a time.

Example 2.3.3: Solving an Application Using a Formula

It takes Andrew $30\ min$ to drive to work in the morning. He drives home using the same route, but it takes \(10\; min\) longer, and he averages $10\ mi/h$ less than in the morning. How far does Andrew drive to work?

Solution

This is a distance problem, so we can use the formula $d = rt$, where distance equals rate multiplied by time. Note that when rate is given in mi/h, time must be expressed in hours. Consistent units of measurement are key to obtaining a correct solution.

First, we identify the known and unknown quantities. Andrew's morning drive to work takes $30\ min$, or $12\ h$ at rate r. His drive home takes $40\ min$, or $23\ h$, and his speed averages $10\ mi/h$ less than the morning drive. Both trips cover distance d. A table, such as Table 2.3.2, is often helpful for keeping track of information in these types of problems.

Table 2.3.2

	d	r	t
To Work	d	r	12
To Home	d	$r - 10$	23

Write two equations, one for each trip.

$$d = r\left(\frac{1}{2}\right) \quad \text{To work}$$

$$d = (r - 10)\left(\frac{2}{3}\right) \quad \text{To home}$$

As both equations equal the same distance, we set them equal to each other and solve for r.

$$\begin{aligned} r\left(\frac{1}{2}\right) &= (r-10)\left(\frac{2}{3}\right) \\ \frac{1}{2r} &= \frac{2}{3}r - \frac{20}{3} \\ \frac{1}{2}r - \frac{2}{3}r &= -\frac{20}{3} \\ -\frac{1}{6}r &= -\frac{20}{3} \\ r &= -\frac{20}{3}(-6) \\ r &= 40 \end{aligned}$$

We have solved for the rate of speed to work, $40\ mph$. Substituting 40 into the rate on the return trip yields $30 mi/h$. Now we can answer the question. Substitute the rate back into either equation and solve for d.

$$\begin{aligned} d &= 40\left(\frac{1}{2}\right) \\ &= 20 \end{aligned}$$

The distance between home and work is $20\ mi$.

Analysis

Note that we could have cleared the fractions in the equation by multiplying both sides of the equation by the LCD to solve for r.

$$\begin{aligned} r\left(\frac{1}{2}\right) &= (r-10)\left(\frac{2}{3}\right) \\ 6 \times r\left(\frac{1}{2}\right) &= 6 \times (r-10)\left(\frac{2}{3}\right) \\ 3r &= 4(r-10) \\ 3r &= 4r - 40 \\ r &= 40 \end{aligned}$$

? Exercise 2.3.3

On Saturday morning, it took Jennifer $3.6\ h$ to drive to her mother's house for the weekend. On Sunday evening, due to heavy traffic, it took Jennifer $4\ h$ to return home. Her speed was $5\ mi/h$ slower on Sunday than on Saturday. What was her speed on Sunday?

Answer

$45\ mi/h$

✓ Example 2.3.4: Solving a Perimeter Problem

The perimeter of a rectangular outdoor patio is $54\ ft$. The length is $3\ ft$ greater than the width. What are the dimensions of the patio?

Solution

The perimeter formula is standard: $P = 2L + 2W$. We have two unknown quantities, length and width. However, we can write the length in terms of the width as $L = W + 3$. Substitute the perimeter value and the expression for length into the formula. It is often helpful to make a sketch and label the sides as in Figure 2.3.3.

Figure 2.3.3

Now we can solve for the width and then calculate the length.

$$P = 2L + 2W$$
$$54 = 2(W + 3) + 2W$$
$$54 = 2W + 6 + 2W$$
$$54 = 4W + 6$$
$$48 = 4W$$
$$W = 12$$

$$L = 12 + 3$$
$$L = 15$$

The dimensions are $L = 15\ ft$ and $W = 12\ ft$.

Exercise 2.3.4

Find the dimensions of a rectangle given that the perimeter is $110\ cm$ and the length is $1\ cm$ more than twice the width.

Answer

$L = 37\ cm$, $W = 18\ cm$

Example 2.3.5: Solving an Area Problem

The perimeter of a tablet of graph paper is $48\ in.^2$. The length is $6\ in.$ more than the width. Find the area of the graph paper.

Solution

The standard formula for area is $A = LW$; however, we will solve the problem using the perimeter formula. The reason we use the perimeter formula is because we know enough information about the perimeter that the formula will allow us to solve for one of the unknowns. As both perimeter and area use length and width as dimensions, they are often used together to solve a problem such as this one.

We know that the length is $6\ in.$ more than the width, so we can write length as $L = W + 6$. Substitute the value of the perimeter and the expression for length into the perimeter formula and find the length.

$$P = 2L + 2W$$
$$48 = 2(W + 6) + 2W$$
$$48 = 2W + 12 + 2W$$
$$48 = 4W + 12$$
$$36 = 4W$$
$$W = 9$$

$$L = 9 + 6$$
$$L = 15$$

Now, we find the area given the dimensions of $L = 15\ in.$ and $W = 9\ in.$

$$A = LW$$
$$A = 15(9)$$
$$A = 135\ in.^2$$

The area is $135\ in.^2$.

Exercise 2.3.5

A game room has a perimeter of $70\ ft$. The length is five more than twice the width. How many ft^2 of new carpeting should be ordered?

Answer

$250\ ft^2$

Example 2.3.6: Solving a Volume Problem

Find the dimensions of a shipping box given that the length is twice the width, the height is 8 in, and the volume is $1,600\ in.^3$.

Solution

The formula for the volume of a box is given as $V = LWH$, the product of length, width, and height. We are given that $L = 2W$, and $H = 8$. The volume is $1,600$ cubic inches.

$$\begin{aligned} V &= LWH \\ 1600 &= (2W)W(8) \\ 1600 &= 16W^2 \\ 100 &= W^2 \\ 10 &= W \end{aligned}$$

The dimensions are $L = 20\ in$, $W = 10\ in$, and $H = 8\ in$.

Analysis

Note that the square root of W^2 would result in a positive and a negative value. However, because we are describing width, we can use only the positive result.

Media

Access these online resources for additional instruction and practice with models and applications of linear equations.

1. Problem solving using linear equations
2. Problem solving using equations
3. Finding the dimensions of area given the perimeter
4. Find the distance between the cities using the distance = rate * time formula
5. Linear equation application (Write a cost equation)

Key Concepts

- A linear equation can be used to solve for an unknown in a number problem. See Example.
- Applications can be written as mathematical problems by identifying known quantities and assigning a variable to unknown quantities. See Example.
- There are many known formulas that can be used to solve applications. Distance problems, for example, are solved using the $d = rt$ formula. See Example.
- Many geometry problems are solved using the perimeter formula $P = 2L + 2W$, the area formula $A = LW$, or the volume formula $V = LWH$. See Example, Example, and Example.

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2.4: Complex Numbers

Learning Objectives

- Definition of i, imaginary and complex numbers.
- Simplifying square roots with complex numbers

Discovered by Benoit Mandelbrot around 1980, the Mandelbrot Set is one of the most recognizable fractal images. The image is built on the theory of self-similarity and the operation of iteration. Zooming in on a fractal image brings many surprises, particularly in the high level of repetition of detail that appears as magnification increases. The equation that generates this image turns out to be rather simple.

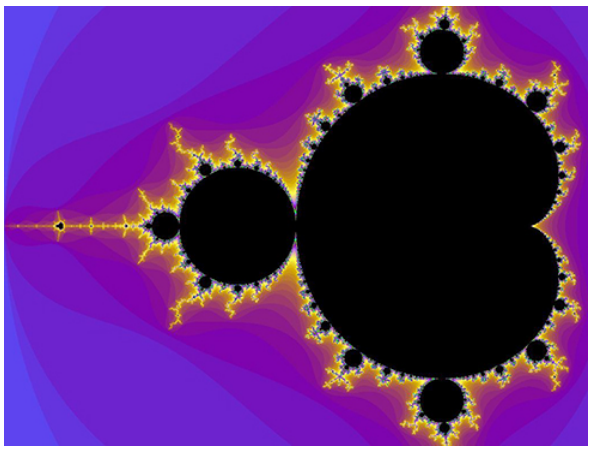


Figure 2.4.1: The Mandelbrot set exhibits similarity, which is best shown in an animation.

In order to better understand it, we need to become familiar with a new set of numbers. Keep in mind that the study of mathematics continuously builds upon itself. Negative integers, for example, fill a void left by the set of positive integers. The set of rational numbers, in turn, fills a void left by the set of integers. The set of real numbers fills a void left by the set of rational numbers. Not surprisingly, the set of real numbers has voids as well. In this section, we will explore a set of numbers that fills voids in the set of real numbers and find out how to work within it.

Expressing Square Roots of Negative Numbers as Multiples of i

We know how to find the square root of any positive real number. In a similar way, we can find the square root of any negative number. The difference is that the root is not real. If the value in the radicand is negative, the root is said to be an imaginary number.The imaginary number i is defined as the square root of -1.

$$\sqrt{-1} = i \tag{2.4.1}$$

So, using properties of radicals,

$$i^2 = (\sqrt{-1})^2 = -1 \tag{2.4.2}$$

We can write the square root of any negative number as a multiple of i. Consider the square root of -49.

$$\begin{aligned} \sqrt{-49} &= \sqrt{49 \times (-1)} \\ &= \sqrt{49}\sqrt{-1} \\ &= 7i \end{aligned}$$

We use $7i$ and not $-7i$ because the principal root of 49 is the positive root.

A complex number is the sum of a real number and an imaginary number. A complex number is expressed in standard form when written $a + bi$ where a is the real part and b is the imaginary part. For example, $5 + 2i$ is a complex number. So, too, is $3 + 4i\sqrt{3}$.



Imaginary numbers differ from real numbers in that a squared imaginary number produces a negative real number. Recall that when a positive real number is squared, the result is a positive real number and when a negative real number is squared, the result is also a positive real number. Complex numbers consist of real and imaginary numbers.

Definition: IMAGINARY AND COMPLEX NUMBERS

A *complex number* is a number of the form $a+bi$ where

1. a is the real part of the complex number.
2. b is the imaginary part of the complex number.

If $b=0$, then $a+bi$ is a real number. If $a=0$ and b is not equal to 0, the complex number is called a pure imaginary number. An **imaginary number** is an even root of a negative number.

Howto: Given an imaginary number, express it in the standard form of a complex number

1. Write $\sqrt{-a}$ as $\sqrt{a}\sqrt{-1}$.
2. Express $\sqrt{-1}$ as i.
3. Write $\sqrt{a}\times i$ in simplest form.

Example 2.4.1: Expressing an Imaginary Number in Standard Form

Express $\sqrt{-9}$ in standard form.

Solution

$$\begin{aligned}\sqrt{-9} &= \sqrt{9}\sqrt{-1)}\\ &= 3i\end{aligned}$$

In standard form, this is $0+3i$.

Exercise 2.4.1

Express $\sqrt{-24}$ in standard form.

Answer

$\sqrt{-24}=0+2i\sqrt{6}$

Key Concepts

- The square root of any negative number can be written as a multiple of i. See Example.

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2.5: Quadratic Equations

Learning Objectives

- Solve quadratic equations by factoring.
- Solve quadratic equations by the square root property.
- Solve quadratic equations by completing the square.
- Solve quadratic equations by using the quadratic formula.

The computer monitor on the left in Figure 2.5.1 is a 23.6-inch model and the one on the right is a 27-inch model. Proportionally, the monitors appear very similar. If there is a limited amount of space and we desire the largest monitor possible, how do we decide which one to choose? In this section, we will learn how to solve problems such as this using four different methods.



Figure 2.5.1

Solving Quadratic Equations by Factoring

An equation containing a second-degree polynomial is called a **quadratic equation**. For example, equations such as $2x^2+3x-1=0$ and $x^2-4=0$ are quadratic equations. They are used in countless ways in the fields of engineering, architecture, finance, biological science, and, of course, mathematics.

Often the easiest method of solving a quadratic equation is **factoring**. Factoring means finding expressions that can be multiplied together to give the expression on one side of the equation.

If a quadratic equation can be factored, it is written as a product of linear terms. Solving by factoring depends on the zero-product property, which states that if $a \cdot b = 0$, then $a = 0$ or $b = 0$, where a and b are real numbers or algebraic expressions. In other words, if the product of two numbers or two expressions equals zero, then one of the numbers or one of the expressions must equal zero because zero multiplied by anything equals zero.

Multiplying the factors expands the equation to a string of terms separated by plus or minus signs. So, in that sense, the operation of multiplication undoes the operation of factoring. For example, expand the factored expression $(x-2)(x+3)$ by multiplying the two factors together.

$$\begin{aligned}(x-2)(x+3) &= x^2+3x-2x-6\\ &= x^2+x-6\end{aligned}$$

The product is a quadratic expression. Set equal to zero, $x^2+x-6=0$ is a quadratic equation. If we were to factor the equation, we would get back the factors we multiplied.

The process of factoring a quadratic equation depends on the leading coefficient, whether it is 1 or another integer. We will look at both situations; but first, we want to confirm that the equation is written in standard form, $ax^2+bx+c=0$, where a, b, and c are real numbers, and $a \neq 0$. The equation $x^2+x-6=0$ is in standard form.

We can use the zero-product property to solve quadratic equations in which we first have to factor out the greatest common factor(GCF), and for equations that have special factoring formulas as well, such as the difference of squares, both of which we will see later in this section.

ZERO-PRODUCT PROPERTY AND QUADRATIC EQUATIONS

The **zero-product property** states

$$\text{If } a \cdot b = 0, \text{ then } a = 0 \text{ or } b = 0,$$

where a and b are real numbers or algebraic expressions.

A quadratic equation is an equation containing a second-degree polynomial; for example

$$ax^2 + bx + c = 0 \tag{2.5.1}$$

where a, b, and c are real numbers, and if $a \neq 0$, it is in standard form.

Solving Quadratics with a Leading Coefficient of 1

In the quadratic equation $x^2 + x - 6 = 0$, the leading coefficient, or the coefficient of x^2, is 1. We have one method of factoring quadratic equations in this form.

How to: Factor a quadratic equation with the leading coefficient of 1

1. Find two numbers whose product equals c and whose sum equals b.
2. Use those numbers to write two factors of the form $(x + k)$ or $(x - k)$, where k is one of the numbers found in step 1. Use the numbers exactly as they are. In other words, if the two numbers are 1 and -2, the factors are $(x + 1)(x - 2)$.
3. Solve using the zero-product property by setting each factor equal to zero and solving for the variable.

Example 2.5.1: Solving a Quadratic with Leading Coefficient of 1

Factor and solve the equation: $x^2 + x - 6 = 0$.

Solution

To factor $x^2 + x - 6 = 0$, we look for two numbers whose product equals -6 and whose sum equals 1. Begin by looking at the possible factors of -6.

$$1 \cdot (-6)$$

$$(-6) \cdot 1$$

$$2 \cdot (-3)$$

$$3 \cdot (-2)$$

The last pair, $3 \cdot (-2)$ sums to 1, so these are the numbers. Note that only one pair of numbers will work. Then, write the factors.

$$(x - 2)(x + 3) = 0$$

To solve this equation, we use the zero-product property. Set each factor equal to zero and solve.

$$\begin{aligned} (x - 2)(x + 3) &= 0 \\ (x - 2) &= 0 \\ x &= 2 \\ (x + 3) &= 0 \\ x &= -3 \end{aligned}$$

The two solutions are 2 and -3. We can see how the solutions relate to the graph in Figure 2.5.2. The solutions are the x-intercepts of $x^2 + x - 6 = 0$.

Figure 2.5.2

? Exercise 2.5.1

Factor and solve the quadratic equation: $x^2 - 5x - 6 = 0$.

Answer

$(x-6)(x+1) = 0$, $x = 6$, $x = -1$

✓ Example 2.5.2: Solve the Quadratic Equation by Factoring

Solve the quadratic equation by factoring: $x^2 + 8x + 15 = 0$.

Solution

Find two numbers whose product equals 15 and whose sum equals 8. List the factors of 15.

$$1 \cdot 15$$

$$3 \cdot 5$$

$$(-1) \cdot (-15)$$

$$(-3) \cdot (-5)$$

The numbers that add to 8 are 3 and 5. Then, write the factors, set each factor equal to zero, and solve.

$$\begin{aligned} (x+3)(x+5) &= 0 \\ (x+3) &= 0 \\ x &= -3 \\ (x+5) &= 0 \\ x &= -5 \end{aligned}$$

The solutions are -3 and -5.

? Exercise 2.5.2

Solve the quadratic equation by factoring: $x^2 - 4x - 21 = 0$.

Answer

$(x-7)(x+3) = 0$, $x = 7$, $x = -3$

✓ Example 2.5.3: Using Zero-Product Property to Solve a Quadratic Equation

Solve the difference of squares equation using the zero-product property: $x^2 - 9 = 0$.

Solution

Recognizing that the equation represents the difference of squares, we can write the two factors by taking the square root of each term, using a minus sign as the operator in one factor and a plus sign as the operator in the other. Solve using the zero-factor property.

$$\begin{aligned} x^2 - 9 &= 0 \\ (x-3)(x+3) &= 0 \\ x - 3 &= 0 \\ x &= 3 \\ (x+3) &= 0 \\ x &= -3 \end{aligned}$$

The solutions are 3 and -3.

Exercise 2.5.3

Solve by factoring: $x^2 - 25 = 0$.

Answer

$(x + 5)(x - 5) = 0, x = -5, x = 5$

Factoring and Solving a Quadratic Equation of Higher Order

When the leading coefficient is not 1, we factor a quadratic equation using the method called *grouping*, which requires four terms.

Grouping: Steps for factoring quadratic equations

With the equation in standard form, let's review the grouping procedures

1. With the quadratic in standard form, $ax^2 + bx + c = 0$, multiply $a \cdot c$.
2. Find two numbers whose product equals ac and whose sum equals b.
3. Rewrite the equation replacing the bx term with two terms using the numbers found in step 1 as coefficients of x.
4. Factor the first two terms and then factor the last two terms. The expressions in parentheses must be exactly the same to use grouping.
5. Factor out the expression in parentheses.
6. Set the expressions equal to zero and solve for the variable.

Example 2.5.4: Solving a Quadratic Equation Using Grouping

Use grouping to factor and solve the quadratic equation: $4x^2 + 15x + 9 = 0$.

Solution

First, multiply $ac: 4(9) = 36$. Then list the factors of 36.

$$1 \cdot 36$$

$$2 \cdot 18$$

$$3 \cdot 12$$

$$4 \cdot 9$$

$$6 \cdot 6$$

The only pair of factors that sums to 15 is $3 + 12$. Rewrite the equation replacing the b term, $15x$, with two terms using 3 and 12 as coefficients of x. Factor the first two terms, and then factor the last two terms.

$$\begin{aligned} 4x^2 + 3x + 12x + 9 &= 0 \\ x(4x + 3) + 3(4x + 3) &= 0 \\ (4x + 3)(x + 3) &= 0 \quad \text{Solve using the zero-product property} \\ (4x + 3) &= 3 \\ x &= -\frac{3}{4} \\ (x + 3) &= 0 \\ x &= -3 \end{aligned}$$

The solutions are $-\frac{3}{4}$, and -3. See Figure 2.5.3.

Figure 2.5.3

? Exercise 2.5.4

Solve using factoring by grouping: $12x^2 + 11x + 2 = 0$.

Answer

$(3x + 2)(4x + 1) = 0, x = -\frac{2}{3}, x = -\frac{1}{4}$

✓ Example 2.5.5: Solving a Higher Degree Quadratic Equation by Factoring

Solve the equation by factoring: $-3x^3 - 5x^2 - 2x = 0$.

Solution

This equation does not look like a quadratic, as the highest power is 3, not 2. Recall that the first thing we want to do when solving any equation is to factor out the GCF, if one exists. And it does here. We can factor out $-x$ from all of the terms and then proceed with grouping.

$$\begin{aligned} -3x^3 - 5x^2 - 2x &= 0 \\ -x(3x^2 + 5x + 2) &= 0 \\ -x(3x^2 + 3x + 2x + 2) &= 0 \quad \text{Use grouping on the expression in parentheses} \\ -x[3x(x + 1) + 2(x + 1)] &= 0 \\ -x(3x + 2)(x + 1) &= 0 \end{aligned}$$

Now, we use the zero-product property. Notice that we have three factors.

$$\begin{aligned} -x &= 0 \\ x &= 0 \\ 3x + 2 &= 0 \\ x &= -\frac{2}{3} \\ x + 1 &= 0 \\ x &= -1 \end{aligned}$$

The solutions are 0, $-\frac{2}{3}$, and -1.

? Exercise 2.5.5

Solve by factoring: $x^3 + 11x^2 + 10x = 0$.

Answer

$x = 0, x = -10, x = -1$

Using the Square Root Property

When there is no linear term in the equation, another method of solving a quadratic equation is by using the **square root property**, in which we isolate the x^2 term and take the square root of the number on the other side of the equals sign. Keep in mind that sometimes we may have to manipulate the equation to isolate the x^2 term so that the square root property can be used.

THE SQUARE ROOT PROPERTY

With the x^2 term isolated, the square root property states that:

$$\text{if } x^2 = k, \text{ then } x = \pm\sqrt{k}$$

where k is a nonzero real number.

Howto: Given a quadratic equation with an x^2 term but no x term, use the square root property to solve it

1. Isolate the x^2 term on one side of the equal sign.
2. Take the square root of both sides of the equation, putting a $\pm$ sign before the expression on the side opposite the squared term.
3. Simplify the numbers on the side with the $\pm$ sign.

Example 2.5.6: Solving a Simple Quadratic Equation Using the Square Root Property

Solve the quadratic using the square root property: $x^2 = 8$.

Solution

Take the square root of both sides, and then simplify the radical. Remember to use a $\pm$ sign before the radical symbol.

$$\begin{aligned} x^2 &= 8 \\ x &= \pm\sqrt{8} \\ &= \pm 2\sqrt{2} \end{aligned}$$

The solutions are $2\sqrt{2}, -2\sqrt{2}$

Example 2.5.7: Solving a Quadratic Equation Using the Square Root Property

Solve the quadratic equation: $4x^2 + 1 = 7$.

Solution

First, isolate the x^2 term. Then take the square root of both sides.

$$\begin{aligned} 4x^2 + 1 &= 7 \\ 4x^2 &= 6 \\ x^2 &= \frac{6}{4} \\ x &= \pm\frac{\sqrt{6}}{2} \end{aligned}$$

The solutions are $\frac{\sqrt{6}}{2}$, and $-\frac{\sqrt{6}}{2}$.

Exercise 2.5.6

Solve the quadratic equation using the square root property: $3(x-4)^2 = 15$.

Answer

$x = 4 \pm \sqrt{5}$

Completing the Square

Not all quadratic equations can be factored or can be solved in their original form using the square root property. In these cases, we may use a method for solving a **quadratic equation** known as **completing the square**. Using this method, we add or subtract terms to both sides of the equation until we have a perfect square trinomial on one side of the equal sign. We then apply the square root property. To complete the square, the leading coefficient, a, must equal 1. If it does not, then divide the entire equation by a. Then, we can use the following procedures to solve a quadratic equation by completing the square.

We will use the example $x^2 + 4x + 1 = 0$ to illustrate each step.

Given a quadratic equation that cannot be factored, and with $a = 1$, first add or subtract the constant term to the right sign of the equal sign.

$$x^2 + 4x + 1 = 0$$
$$x^2 + 4x = -1 \quad \text{Multiply the b term by } \frac{1}{2} \text{ and square it.}$$
$$\frac{1}{2}(4) = 2$$
$$2^2 = 4 \quad \text{Add } \left(\frac{1}{2}\right)^2 \text{ to both sides of the equal sign and simplify the right side. We have}$$
$$x^2 + 4x + 4 = -1 + 4$$
$$x^2 + 4x + 4 = 3 \quad \text{The left side of the equation can now be factored as a perfect square.}$$
$$(x+2)^2 = 3$$
$$\sqrt{(x+2)^2} = \pm\sqrt{3} \quad \text{Use the square root property and solve.}$$
$$\sqrt{(x+2)^2} = \pm\sqrt{3}$$
$$x + 2 = \pm\sqrt{3}$$
$$x = -2 \pm \sqrt{3}$$

The solutions are $-2 + \sqrt{3}$, and $-2 - \sqrt{3}$.

Example 2.5.8: Solving a Quadratic by Completing the Square

Solve the quadratic equation by completing the square: $x^2 - 3x - 5 = 0$.

Solution

First, move the constant term to the right side of the equal sign.

$$x^2 - 3x = 5 \quad \text{Then, take } \frac{1}{2} \text{ of the b term and square it.}$$
$$\frac{1}{2}(-3) = -\frac{3}{2}$$
$$\left(-\frac{3}{2}\right)^2 = \frac{9}{4}$$
$$x^2 - 3x + \left(-\frac{3}{2}\right)^2 = 5 + \left(-\frac{3}{2}\right)^2 \quad \text{Add the result to both sides of the equal sign.}$$
$$x^2 - 3x + \frac{9}{4} = 5 + \frac{9}{4}$$

Factor the left side as a perfect square and simplify the right side.

$$\left(x - \frac{3}{2}\right)^2 = \frac{29}{4}$$
$$(x - \frac{3}{2}) = \pm\frac{\sqrt{29}}{2} \quad \text{Use the square root property and solve.}$$
$$x = \frac{3}{2} \pm \frac{\sqrt{29}}{2}$$

The solutions are $\frac{3}{2} + \frac{\sqrt{29}}{2}$, and $\frac{3}{2} - \frac{\sqrt{29}}{2}$

Exercise 2.5.7

Solve by completing the square: $x^2 - 6x = 13$.

Answer

$x = 3 \pm \sqrt{22}$

Using the Quadratic Formula

The fourth method of solving a **quadratic equation** is by using the **quadratic formula**, a formula that will solve all quadratic equations. Although the quadratic formula works on any quadratic equation in standard form, it is easy to make errors in substituting the values into the formula. Pay close attention when substituting, and use parentheses when inserting a negative number.

We can derive the quadratic formula by **completing the square**. We will assume that the leading coefficient is positive; if it is negative, we can multiply the equation by -1 and obtain a positive a. Given $ax^2 + bx + c = 0, a \neq 0$, we will complete the square as follows:

First, move the constant term to the right side of the equal sign:

$$ax^2 + bx = -c$$

As we want the leading coefficient to equal 1, divide through by a:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Then, find $\frac{1}{2}$ of the middle term, and add $(\frac{1}{2}\frac{b}{a})^2 = \frac{b^2}{4a^2}$ to both sides of the equal sign:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

Next, write the left side as a perfect square. Find the common denominator of the right side and write it as a single fraction:

$$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$$

Now, use the square root property, which gives

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

Finally, add $-\frac{b}{2a}$ to both sides of the equation and combine the terms on the right side. Thus,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

THE QUADRATIC FORMULA

Written in standard form, $ax^2 + bx + c = 0$, any quadratic equation can be solved using the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{2.5.2}$$

where a, b, and c are real numbers and $a \neq 0$.

How to

Given a quadratic equation, solve it using the quadratic formula

1. Make sure the equation is in standard form: $ax^2 + bx + c = 0$.
2. Make note of the values of the coefficients and constant term, a, b, and c.
3. Carefully substitute the values noted in step 2 into the equation. To avoid needless errors, use parentheses around each number input into the formula.

4. Calculate and solve.

Example 2.5.9: Solve the Quadratic Equation Using the Quadratic Formula

Solve the quadratic equation: $x^2+5x+1=0$.

Solution

Identify the coefficients: $a=1, b=5, c=1$. Then use the quadratic formula.

$$\begin{aligned} x &= \frac{-(5) \pm \sqrt{(5)^2-4(1)(1)}}{2(1)} \\ &= \frac{-5 \pm \sqrt{25-4}}{2} \\ &= \frac{-5 \pm \sqrt{21}}{2} \end{aligned}$$

Example 2.5.10: Solving a Quadratic Equation with the Quadratic Formula

Use the quadratic formula to solve $x^2+x+2=0$.

Solution

First, we identify the coefficients: $a=1$,$b=1$, and $c=2$.

Substitute these values into the quadratic formula.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\ &= \frac{-(1) \pm \sqrt{(1)^2-4(1)(2)}}{2(1)} \\ &= \frac{-1 \pm \sqrt{1-8}}{2} \\ &= \frac{-1 \pm \sqrt{-7}}{2} \\ &= \frac{-1 \pm i\sqrt{7}}{2} \end{aligned}$$

Example 2.5.11: Solving a Quadratic Equation with Complex Solutions

Solve: $3x^2+4x+8=2x^2+6x-5$

Solution

We first rewrite the equation so that zero is on one side of the equation. Then we can solve as usual.

$$
\begin{aligned}
3x^2+4x+8 &= 2x^2+6x-5 && \text{Rewrite where zero is on one side}\\
x^2-2x+13 &= 0 && \text{Apply the quadratic formula}\\
x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} && \text{Plug-n-chug } a=1,\ b=-2,\ c=13\\
x &= \frac{2\pm\sqrt{(-2)^2-4(1)(13)}}{2(1)} && \text{Simplify using order of operations}\\
x &= \frac{2\pm\sqrt{4-52}}{2}\\
x &= \frac{2\pm\sqrt{-48}}{2} && \text{Rewrite the radical using } i\\
x &= \frac{2\pm i\sqrt{48}}{2} && \text{Simplify the radical}\\
x &= \frac{2\pm 4i\sqrt{3}}{2} && \text{Factor a 2 from the numerator}\\
x &= \frac{2(1\pm 2i\sqrt{3})}{2} && \text{Reduce by a factor of 2}\\
x &= \frac{\cancel{2}(1\pm 2i\sqrt{3})}{\cancel{2}} && \text{Rewrite}\\
x &= 1\pm 2i\sqrt{3} && \text{Solutions}
\end{aligned}
$$

When there is a negative value as the radicand, we rewrite the radical using the imaginary unit and the solutions are non-real number.

? Exercise 2.5.8

Solve the quadratic equation using the quadratic formula: $9x^2+3x-2=0$.

Answer

$x=-\frac{2}{3}, x=\frac{1}{3}$

The Discriminant

The **quadratic formula** not only generates the solutions to a quadratic equation, it tells us about the nature of the solutions when we consider the **discriminant**, or the expression under the radical, b^2-4ac. The discriminant tells us whether the solutions are real numbers or complex numbers, and how many solutions of each type to expect. Table 2.5.1 relates the value of the discriminant to the solutions of a quadratic equation.

Table 2.5.1

Value of Discriminant	Results
$b^2-4ac=0$	One rational solution (double solution)
$b^2-4ac>0$, perfect square	Two rational solutions
$b^2-4ac>0$, not a perfect square	Two irrational solutions
$b^2-4ac<0$	Two complex solutions

THE DISCRIMINANT

For $ax^2+bx+c=0$, where a, b, and c are real numbers, the **discriminant** is the expression under the radical in the quadratic formula: b^2-4ac. It tells us whether the solutions are real numbers or complex numbers and how many solutions of each type to expect.

Example 2.5.12: Using the Discriminant to Find the Nature of the Solutions to a Quadratic Equation

Use the discriminant to find the nature of the solutions to the following quadratic equations:

a. $x^2+4x+4=0$
b. $8x^2+14x+3=0$
c. $3x^2-5x-2=0$
d. $3x^2-10x+15=0$

Solution

Calculate the discriminant b^2-4ac for each equation and state the expected type of solutions.

a.

$x^2+4x+4=0$

$b^2-4ac=(4)^2-4(1)(4)=0$ There will be one rational double solution.

b.

$8x^2+14x+3=0$

$b^2-4ac=(14)^2-4(8)(3)=100$ As 100 is a perfect square, there will be two rational solutions.

c.

$3x^2-5x-2=0$

$b^2-4ac=(-5)^2-4(3)(-2)=49$ As 49 is a perfect square, there will be two rational solutions.

d.

$3x^2-10x+15=0$

$b^2-4ac=(-10)^2-4(3)(15)=-80$ There will be two complex solutions.

Using the Pythagorean Theorem

One of the most famous formulas in mathematics is the **Pythagorean Theorem**. It is based on a right triangle, and states the relationship among the lengths of the sides as $a^2+b^2=c^2$, where a and b refer to the legs of a right triangle adjacent to the $90°$ angle, and c refers to the hypotenuse. It has immeasurable uses in architecture, engineering, the sciences, geometry, trigonometry, and algebra, and in everyday applications.

We use the Pythagorean Theorem to solve for the length of one side of a triangle when we have the lengths of the other two. Because each of the terms is squared in the theorem, when we are solving for a side of a triangle, we have a quadratic equation. We can use the methods for solving quadratic equations that we learned in this section to solve for the missing side.

The Pythagorean Theorem is given as

$$a^2+b^2=c^2 \tag{2.5.3}$$

where a and b refer to the legs of a right triangle adjacent to the $90°$ angle, and c refers to the hypotenuse, as shown in .

Figure 2.5.4

Example 2.5.13: Finding the Length of the Missing Side of a Right Triangle

Find the length of the missing side of the right triangle in Figure 2.5.5.

Figure 2.5.5

Solution

As we have measurements for side b and the hypotenuse, the missing side is a.

$$
\begin{aligned}
a^2+b^2&=c^2\\
a^2+(4)^2&=(12)^2\\
a^2+16&=144\\
a^2&=128\\
a&=\sqrt{128}\\
&=8\sqrt{2}
\end{aligned}
$$

? Exercise 2.5.9

Use the Pythagorean Theorem to solve the right triangle problem: Leg a measures 4 units, leg b measures 3 units. Find the length of the hypotenuse.

Answer

5 units

Media

Access these online resources for additional instruction and practice with quadratic equations.

1. Solving Quadratic Equations by Factoring
2. The Zero-Product Property
3. Completing the Square
4. Quadratic Formula with Two Rational Solutions
5. Length of a leg of a right triangle

Key Equations

quadratic formula	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Key Concepts

- Many quadratic equations can be solved by factoring when the equation has a leading coefficient of 1 or if the equation is a difference of squares. The zero-factor property is then used to find solutions. See Example, Example, and Example.
- Many quadratic equations with a leading coefficient other than 1 can be solved by factoring using the grouping method. See Example and Example.
- Another method for solving quadratics is the square root property. The variable is squared. We isolate the squared term and take the square root of both sides of the equation. The solution will yield a positive and negative solution. See Example and Example.
- Completing the square is a method of solving quadratic equations when the equation cannot be factored. See Example.
- A highly dependable method for solving quadratic equations is the quadratic formula, based on the coefficients and the constant term in the equation. See Example.
- The discriminant is used to indicate the nature of the roots that the quadratic equation will yield: real or complex, rational or irrational, and how many of each. See Example.
- The Pythagorean Theorem, among the most famous theorems in history, is used to solve right-triangle problems and has applications in numerous fields. Solving for the length of one side of a right triangle requires solving a quadratic equation. See Example.

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2.6: Other Types of Equations

Learning Objectives

- Solve equations involving rational exponents.
- Solve equations using factoring.
- Solve radical equations.
- Solve absolute value equations.
- Solve other types of equations.

We have solved linear equations, rational equations, and quadratic equations using several methods. However, there are many other types of equations, and we will investigate a few more types in this section. We will look at equations involving rational exponents, polynomial equations, radical equations, absolute value equations, equations in quadratic form, and some rational equations that can be transformed into quadratics. Solving any equation, however, employs the same basic algebraic rules. We will learn some new techniques as they apply to certain equations, but the algebra never changes.

Solving Equations Involving Rational Exponents

Rational exponents are exponents that are fractions, where the numerator is a power and the denominator is a root. For example, $16^{\frac{1}{2}}$ is another way of writing $\sqrt{16}$; $8^{\frac{1}{3}}$ is another way of writing $\sqrt[3]{8}$. The ability to work with rational exponents is a useful skill, as it is highly applicable in calculus.

We can solve equations in which a variable is raised to a rational exponent by raising both sides of the equation to the reciprocal of the exponent. The reason we raise the equation to the reciprocal of the exponent is because we want to eliminate the exponent on the variable term, and a number multiplied by its reciprocal equals 1. For example,

$$\frac{2}{3}\left(\frac{3}{2}\right) = 1$$

$$3\left(\frac{1}{3}\right) = 1,$$

and so on.

RATIONAL EXPONENTS

A *rational exponent* indicates a power in the numerator and a root in the denominator. There are multiple ways of writing an expression, a variable, or a number with a rational exponent:

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = a^{m\frac{1}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m \tag{2.6.1}$$

Example 2.6.1: Evaluating a Number Raised to a Rational Exponent

Evaluate $8^{\frac{2}{3}}$

Solution

Whether we take the root first or the power first depends on the number. It is easy to find the cube root of 8, so rewrite $8^{\frac{2}{3}}$ as $\left(8^{\frac{1}{3}}\right)^2$.

$$\begin{aligned}\left(8^{\frac{1}{3}}\right)^2 &= (2)^2 \\ &= 4\end{aligned}$$

? Exercise 2.6.1

Evaluate $64^{-\frac{1}{3}}$

Answer

$\frac{1}{4}$

✓ Example 2.6.2: Solve the Equation Including a Variable Raised to a Rational Exponent

Solve the equation in which a variable is raised to a rational exponent: $x^{\frac{5}{4}} = 32$.

Solution

The way to remove the exponent on x is by raising both sides of the equation to a power that is the reciprocal of $\frac{5}{4}$, which is $\frac{4}{5}$.

$$\begin{aligned} x^{\frac{5}{4}} &= 32 \\ \left(x^{\frac{5}{4}}\right)^{\frac{4}{5}} &= (32)^{\frac{4}{5}} \\ x &= (2)^4 \\ &= 16 \end{aligned}$$

? Exercise 2.6.2

Solve the equation $x^{\frac{3}{2}} = 125$.

Answer

25

? Exercise 2.6.3

Solve: $(x+5)^{\frac{3}{2}} = 8$.

Answer

-1

Solving Equations Using Factoring

We have used factoring to solve quadratic equations, but it is a technique that we can use with many types of polynomial equations, which are equations that contain a string of terms including numerical coefficients and variables. When we are faced with an equation containing polynomials of degree higher than 2, we can often solve them by factoring.

POLYNOMIAL EQUATIONS

A polynomial of degree n is an expression of the type

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \tag{2.6.2}$$

where n is a positive integer and $a_n, \ldots, a_0$ are real numbers and $a_n \neq 0$.

Setting the polynomial equal to zero gives a **polynomial equation**. The total number of solutions (real and complex) to a polynomial equation is equal to the highest exponent n.

Example 2.6.4: Solving a Polynomial by Factoring

Solve the polynomial by factoring: $5x^4 = 80x^2$.

Solution

First, set the equation equal to zero. Then factor out what is common to both terms, the GCF.

$$\begin{aligned} 5x^4 - 80x^2 &= 0 \\ 5x^2(x^2 - 16) &= 0 \end{aligned}$$

Notice that we have the difference of squares in the factor $x^2 - 16$, which we will continue to factor and obtain two solutions. The first term, $5x^2$, generates, technically, two solutions as the exponent is 2, but they are the same solution.

$$\begin{aligned} 5x^2 &= 0 \\ x &= 0 \\ x^2 - 16 &= 0 \\ (x+4)(x-4) &= 0 \\ x &= 4 \\ x &= -4 \end{aligned}$$

The solutions are 0 (double solution), 4, and -4.

Analysis

We can see the solutions on the graph in Figure 2.6.1. The x-coordinates of the points where the graph crosses the x-axis are the solutions—the x-intercepts. Notice on the graph that at the solution 0, the graph touches the x-axis and bounces back. It does not cross the x-axis. This is typical of double solutions.

Figure 2.6.1

Exercise 2.6.4

Solve by factoring: $12x^4 = 3x^2$.

Answer

$x = 0, x = 12, x = -12$

Example 2.6.5: Solve a Polynomial by Grouping

Solve a polynomial by grouping: $x^3 + x^2 - 9x - 9 = 0$.

Solution

This polynomial consists of 4 terms, which we can solve by grouping. Grouping procedures require factoring the first two terms and then factoring the last two terms. If the factors in the parentheses are identical, we can continue the process and solve, unless more factoring is suggested.

$$\begin{aligned} x^3 + x^2 - 9x - 9 &= 0 \\ x^2(x+1) - 9(x+1) &= 0 \\ (x^2 - 9)(x+1) &= 0 \end{aligned}$$

The grouping process ends here, as we can factor $x^2 - 9$ using the difference of squares formula.

$$\begin{aligned} (x^2 - 9)(x+1) &= 0 \\ (x-3)(x+3)(x+1) &= 0 \\ x &= 3 \\ x &= -3 \\ x &= -1 \end{aligned}$$

The solutions are 3, -3, and -1. Note that the highest exponent is 3 and we obtained 3 solutions. We can see the solutions, the x-intercepts, on the graph in Figure $2.6.2$.

Figure $2.6.2$

Analysis

We looked at solving quadratic equations by factoring when the leading coefficient is 1. When the leading coefficient is not 1, we solved by grouping. Grouping requires four terms, which we obtained by splitting the linear term of quadratic equations. We can also use grouping for some polynomials of degree higher than 2, as we saw here, since there were already four terms.

Solving Radical Equations

Radical equations are equations that contain variables in the **radicand** (the expression under a radical symbol), such as

$$\sqrt{3x+18}=x$$

$$\sqrt{x+3}=x-3$$

$$\sqrt{x+5}-\sqrt{x-3}=2$$

Radical equations may have one or more radical terms, and are solved by eliminating each radical, one at a time. We have to be careful when solving radical equations, as it is not unusual to find **extraneous solutions**, roots that are not, in fact, solutions to the equation. These solutions are not due to a mistake in the solving method, but result from the process of raising both sides of an equation to a power. However, checking each answer in the original equation will confirm the true solutions.

RADICAL EQUATIONS

An equation containing terms with a variable in the radicand is called a **radical equation.**

Howto: Given a radical equation, solve it

1. Isolate the radical expression on one side of the equal sign. Put all remaining terms on the other side.
2. If the radical is a square root, then square both sides of the equation. If it is a cube root, then raise both sides of the equation to the third power. In other words, for an nth root radical, raise both sides to the nth power. Doing so eliminates the radical symbol.
3. Solve the remaining equation.
4. If a radical term still remains, repeat steps 1–2.
5. Confirm solutions by substituting them into the original equation.

Example $2.6.6$: Solving an Equation with One Radical

Solve $\sqrt{15-2x}=x$.

Solution

The radical is already isolated on the left side of the equal side, so proceed to square both sides.

$$\begin{aligned}\sqrt{15-2x}&=x\\ \left(\sqrt{15-2x}\right)^2&=(x)^2\\ 15-2x&=x^2\end{aligned}$$

We see that the remaining equation is a quadratic. Set it equal to zero and solve.

$$\begin{aligned}0&=x^2+2x-15\\ 0&=(x+5)(x-3)\\ x&=-5\\ x&=3\end{aligned}$$

The proposed solutions are -5 and 3. Let us check each solution back in the original equation. First, check $x=-5$.

$$\begin{aligned}\sqrt{15-2x}&=x\\ \sqrt{15-2(-5)}&=-5\\ \sqrt{25}&=-5\\ 5&\neq -5\end{aligned}$$

This is an extraneous solution. While no mistake was made solving the equation, we found a solution that does not satisfy the original equation.

Check $x=3$.

$$\begin{aligned}\sqrt{15-2x}&=x\\ \sqrt{15-2(3)}&=3\\ \sqrt{9}&=3\\ 3&=3\end{aligned}$$

The solution is 3.

Exercise 2.6.5

Solve the radical equation: $\sqrt{x+3}=3x-1$

Answer

$x=1$, extraneous solution $x=-\frac{2}{9}$

Example 2.6.7: Solving a Radical Equation Containing Two Radicals

Solve $\sqrt{2x+3}+\sqrt{x-2}=4$

Solution

As this equation contains two radicals, we isolate one radical, eliminate it, and then isolate the second radical.

$$\sqrt{2x+3}+\sqrt{x-2}=4$$

$$\begin{aligned}\sqrt{2x+3}&=4-\sqrt{x-2} && \text{Subtract } \sqrt{x-2} \text{ from both sides}\\ \left(\sqrt{2x+3}\right)^2&=\left(4-\sqrt{x-2}\right)^2 && \text{Square both sides}\end{aligned}$$

Use the perfect square formula to expand the right side: $(a-b)^2=a^2-2ab+b^2$.

$$\begin{aligned}2x+3&=(4)^2-2(4)\sqrt{x-2}+\left(\sqrt{x-2}\right)^2\\ 2x+3&=16-8\sqrt{x-2}+(x-2)\\ 2x+3&=14+x-8\sqrt{x-2} && \text{Combine like terms}\\ x-11&=-8\sqrt{x-2} && \text{Isolate the second radical}\\ (x-11)^2&=\left(-8\sqrt{x-2}\right)^2 && \text{Square both sides}\\ x^2-22x+121&=64(x-2)\end{aligned}$$

Now that both radicals have been eliminated, set the quadratic equal to zero and solve.

$$\begin{aligned}x^2-22x+121&=64x-128\\ x^2-86x+249&=0\\ (x-3)(x-83)&=0\\ x&=3\\ x&=83\end{aligned}$$

The proposed solutions are 3 and 83. Check each solution in the original equation.

$$\begin{aligned}\sqrt{2x+3}+\sqrt{x-2}&=4\\ \sqrt{2x+3}&=4-\sqrt{x-2}\\ \sqrt{2(3)+3}&=4-\sqrt{(3)-2}\\ \sqrt{9}&=4-\sqrt{1}\\ 3&=3\end{aligned}$$

One solution is 3.

Check $x=83$.

$$\begin{aligned}\sqrt{2x+3}+\sqrt{x-2}&=4\\ \sqrt{2x+3}&=4-\sqrt{x-2}\\ \sqrt{2(83)+3}&=4-\sqrt{(83)-2}\\ \sqrt{169}&=4-\sqrt{81}\\ 13&\neq -5\end{aligned}$$

The only solution is 3. We see that $x=83$ is an extraneous solution.

? Exercise 2.6.6

Solve the equation with two radicals: $\sqrt{3x+7}+\sqrt{x+2}=1$

Answer

$x=-2$, extraneous solution $x=-1$

Solving an Absolute Value Equation

Next, we will learn how to solve an **absolute value equation**. To solve an equation such as $|2x-6|=8$, we notice that the absolute value will be equal to 8 if the quantity inside the absolute value bars is 8 or -8. This leads to two different equations we can solve independently.

$$\begin{aligned}2x-6&=8\\ 2x&=14\\ x&=7\end{aligned}$$

OR

$$\begin{aligned}2x-6&=-8\\ 2x&=-2\\ x&=-1\end{aligned}$$

Knowing how to solve problems involving absolute value functions is useful. For example, we may need to identify numbers or points on a line that are at a specified distance from a given reference point.

ABSOLUTE VALUE EQUATIONS

The absolute value of x is written as $|x|$. It has the following properties:

If $x\geq 0$, then $|x|=x$.If $x<0$, then $x=-x$.

For real numbers A and B, an equation of the form $|A|=B$, with $B\geq 0$, will have solutions when $A=B$ or $A=-B$. If $B<0$, the equation $|A|=B$ has no solution.

An **absolute value equation** in the form $|ax+b|=c$ has the following properties:

- If $c<0$,$|ax+b|=c$ has no solution.
- If $c=0$,$|ax+b|=c$has one solution.
- If $c>0$,$|ax+b|=c$ has two solutions.

How to

Given an absolute value equation, solve it.

1. Isolate the absolute value expression on one side of the equal sign.
2. If $c > 0$, write and solve two equations: $ax + b = c$ and $ax + b = -c$.

Example 2.6.8: Solving Absolute Value Equations

Solve the following absolute value equations:

a. $|6x + 4| = 8$
b. $|3x + 4| = -9$
c. $|3x - 5| - 4 = 6$
d. $|-5x + 10| = 0$

Solution

a. $|6x + 4| = 8$

Write two equations and solve each:

$$\begin{aligned} 6x + 4 &= 8 \\ 6x &= 4 \\ x &= \frac{2}{3} \end{aligned}$$

OR

$$\begin{aligned} 6x + 4 &= -8 \\ 6x &= -12 \\ x &= -2 \end{aligned}$$

The two solutions are $\frac{2}{3}$ and -2.

b. $|3x + 4| = -9$

There is no solution as an absolute value cannot be negative.

c. $|3x - 5| - 4 = 6$

Isolate the absolute value expression and then write two equations.

$$\begin{aligned} |3x - 5| - 4 &= 6 \\ |3x - 5| &= 10 \\ 3x - 5 &= 10 \\ 3x &= 15 \\ x &= 5 \end{aligned}$$

OR

$$\begin{aligned} 3x - 5 &= -10 \\ 3x &= -5 \\ x &= \frac{5}{3} \end{aligned}$$

There are two solutions: 5, and $-\frac{5}{3}$.

d. $|-5x + 10| = 0$

The equation is set equal to zero, so we have to write only one equation.

$$\begin{aligned} -5x + 10 &= 0 \\ -5x &= -10 \\ x &= 2 \end{aligned}$$

There is one solution: \(2\).

Exercise 2.6.7

Solve the absolute value equation: $|1-4x|+8=13$.

Answer

$x=-1, x=\frac{3}{2}$

Solving Rational Equations Resulting in a Quadratic

Earlier, we solved rational equations. Sometimes, solving a rational equation results in a quadratic. When this happens, we continue the solution by simplifying the quadratic equation by one of the methods we have seen. It may turn out that there is no solution.

Example 2.6.11: Solving a Rational Equation Leading to a Quadratic

Solve the following rational equation: $\frac{-4x}{x-1}+\frac{4}{x+1}=\frac{-8}{x^2-1}$

Solution

We want all denominators in factored form to find the LCD. Two of the denominators cannot be factored further. However, $x^2-1=(x+1)(x-1)$. Then, the LCD is $(x+1)(x-1)$. Next, we multiply the whole equation by the LCD.

$$\begin{aligned}(x+1)(x-1)\left(\frac{-4x}{x-1}+\frac{4}{x+1}\right)&=\left(\frac{-8}{x^2-1}\right)(x+1)(x-1)\\ -4x(x+1)+4(x-1)&=-8\\ -4x^2-4x+4x-4&=-8\\ -4x^2+4&=0\\ -4(x^2-1)&=0\\ -4(x+1)(x-1)&=0\\ x&=-1\\ x&=1\end{aligned}$$

In this case, either solution produces a zero in the denominator in the original equation. Thus, there is no solution.

Exercise 2.6.10

Solve $\frac{3x+2}{x-2}+\frac{1}{x}=\frac{-2}{x^2-2x}$

Answer

$x=-1, x=0$ is not a solution.

Media

Access these online resources for additional instruction and practice with different types of equations.

1. Rational Equation with no Solution
2. Solving equations with rational exponents using reciprocal powers
3. Solving radical equations part 1 of 2
4. Solving radical equations part 2 of 2

Key Concepts

- Rational exponents can be rewritten several ways depending on what is most convenient for the problem. To solve, both sides of the equation are raised to a power that will render the exponent on the variable equal to 1. See Example, Example, and

Example.

- Factoring extends to higher-order polynomials when it involves factoring out the GCF or factoring by grouping. See Example and Example.
- We can solve radical equations by isolating the radical and raising both sides of the equation to a power that matches the index. See Example and Example.
- To solve absolute value equations, we need to write two equations, one for the positive value and one for the negative value. See Example.
- Solving a rational equation may also lead to a quadratic equation or an equation in quadratic form. See Example.

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2.7: Solving with rational exponents

In this section, we take solving with radicals one step further and apply radicals and rational exponents to solve equations with exponents. Since radicals have some restrictions on the radicand, we will also have some restrictions here when applying a rational exponents in the solving process.

The Odd Root Property

Let's start with the case that we take an odd root of an equation.

The Odd Root Property

If $x^n = p$, where n is odd, then $\sqrt{x} = \sqrt[n]{p}$. Note, the radicand can be any real number, i.e., p is any number in $(-\infty, \infty)$.

Example 10.7.1

Solve: $x^5 = 32$

Solution

We can easily apply the odd root property to solve for x.

$$\begin{aligned} x^5 &= 32 && \text{Apply odd root property} \\ \sqrt[5]{x^5} &= \sqrt[5]{32} && \text{Simplify} \\ x &= 2 && \text{Solution} \end{aligned}$$

Example 10.7.2

Solve: $4r^3 - 2 = 106$

Solution

We can easily apply the odd root property to solve for r.

$$\begin{aligned} 4r^3 - 2 &= 106 && \text{Isolate the variable term} \\ 4r^3 &= 108 && \text{Isolate } r^3 \\ r^3 &= 27 && \text{Apply odd root property} \\ \sqrt[3]{r^3} &= \sqrt[3]{27} && \text{Simplify} \\ r &= 3 && \text{Solution} \end{aligned}$$

The Even Root Property

With even roots, we have the restriction on the radicand where the radicand is required to be non-negative here. We discussed this in the previous section, e.g., $\sqrt{-4}$ is not a real number. We continue this restriction when taking even roots of an equation.

The Even Root Property

If $x^n = p$, where n is even, then $x = \sqrt[n]{p}$ or$x = -\sqrt[n]{p}$ or we can write $x = \pm\sqrt[n]{p}$. Note, the radicand can be any real non-negative number, i.e., $p \geq 0$.

Example 10.7.3

Solve: $x^4 = 16$

Solution

We can easily apply the even root property to solve for x.

$$\begin{aligned} x^4 &= 16 && \text{Apply even root property} \\ \sqrt[4]{x^4} &= \sqrt[4]{16} && \text{Simplify} \\ |x| &= \pm 2 && \\ x &= \pm 2 && \text{Solution} \end{aligned}$$

Notice, it wasn't given that $x \geq 0$. Hence, we cannot assume it is, so we put absolute value around x. Once we verify the solution(s), then we can remove the absolute value around x.

Note

In 1545, French mathematician *Gerolamo Cardano* published his book *The Great Art, or the Rules of Algebra*, which included the solution to an equation with a fourth power, but it was considered absurd by many to take a quantity to the fourth power because there are only three dimensions!

Example 10.7.4

Solve: $(2x+4)^2 = 36$ *Find and verify all solutions that satisfy the equation.*

Solution

We can easily apply the even root property to solve for x.

$$\begin{aligned} (2x+4)^2 &= 36 && \text{Apply even root property} \\ \sqrt{(2x+4)^2} &= \pm\sqrt{36} && \text{Simplify } \sqrt{36} \\ 2x+4 &= \pm 6 && \text{Rewrite into two equations} \\ 2x+4 = 6 \quad \text{or} \quad & 2x+4 = -6 && \text{Isolate the variable term in each equation} \\ 2x = 2 \quad \text{or} \quad & 2x = -10 && \text{Solve each equation} \\ x = 1 \quad \text{or} \quad & x = -5 && \text{Solutions} \end{aligned}$$

We can always verify the solutions by substituting back in 1, -5 into the original equation:

$$\begin{aligned} (2x+4)^2 &= 36 && \text{Plug-n-chug } x = 1 \\ (2(1)+4)^2 &\stackrel{?}{=} 36 && \text{Simplify each side} \\ (2+4)^2 &\stackrel{?}{=} 36 && \\ 6^2 &\stackrel{?}{=} 36 && \\ 36 &= 36 && \checkmark \text{True} \end{aligned}$$

Let's try the next solution $x = -5$:

$$\begin{aligned} (2x+4)^2 &= 36 && \text{Plug-n-chug } x = -5 \\ (2(-5)+4)^2 &\stackrel{?}{=} 36 && \text{Simplify each side} \\ (-10+4)^2 &\stackrel{?}{=} 36 && \\ (-6)^2 &\stackrel{?}{=} 36 && \\ 36 &= 36 && \checkmark \text{True} \end{aligned}$$

Thus, 1, (-5) are, in fact, solutions to the original equation.

Example 10.7.5

Solve: $(6x-9)^2 = 45$ *Find and verify all solutions that satisfy the equation.*

Solution

$$
\begin{aligned}
(6x-9)^2 &= 45 && \text{Apply even root property}\\
\sqrt{(6x-9)^2} &= \pm\sqrt{45} && \text{Simplify } \sqrt{45}\\
6x-9 &= \pm 3\sqrt{5} && \text{Isolate the variable term}\\
6x &= 9 \pm 3\sqrt{5} && \text{Divide both sides by 6}\\
x &= \frac{9 \pm 3\sqrt{5}}{6} && \text{Factor a GCF from numerator}\\
x &= \frac{\cancel{3}(3 \pm \sqrt{5})}{\cancel{6}_{2}} && \text{Simplify}\\
x &= \frac{3 \pm \sqrt{5}}{2} && \text{Solution}
\end{aligned}
$$

Notice, we didn't split the equation into two different equations and solve. Since $\sqrt{45}$ is an irrational number, we can leave the $\pm$ and solve as usual. *We leave verifying the solutions to the student.*

Example 10.7.6

Solve: $256w^8 + 40 = 41$

Solution

We have to isolate the variable term first, then we can apply the even root property.

$$
\begin{aligned}
256w^8 + 40 &= 41 && \text{Isolate the variable term.}\\
256w^8 &= 1 && \text{Divide each side by 256}\\
w^8 &= \frac{1}{256} && \text{Apply even root property}\\
\sqrt[8]{w^8} &= \pm\sqrt[8]{\frac{1}{256}} && \text{Simplify the radicals}\\
|w| &= \pm\frac{1}{2}\\
w &= \pm\frac{1}{2} && \text{Solution}
\end{aligned}
$$

Notice, it wasn't given that $w \geq 0$. Hence, we cannot assume it is and we put absolute value around w. Once we verify the solution(s), then we can remove the absolute value around w.

Solving Equations with Rational Exponents

When exponents are fractions, we convert the rational exponent into a radical expression to solve. Recall, $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$. Then we clear the exponent by applying either the even or odd root property and solve as usual.

Steps for solving equations with rational exponents

Given an equation with rational exponents, we can follow the following steps to solve.

Step 1. Rewrite any rational exponents as radicals.

Step 2. Apply the odd or even root property. *Recall, even roots require the radicand to be positive unless otherwise noted.*

Step 3. Raise each side to the power of the root.

Step 4. Solve. *Verify the solutions, especially when there is an even root.*

Example 10.7.7

Solve: $(4x+1)^{\frac{2}{5}} = 9$. *Assume all variables are positive.*

Solution

We follow the steps in order to solve the equation with a rational exponent.

Step 1. Rewrite any rational exponents as radicals.

$$\begin{aligned}(4x+1)^{\frac{2}{5}} &= 9 \\ (\sqrt[5]{4x+1})^2 &= 9\end{aligned} \tag{2.7.1}$$

Step 2. Apply the odd or even root property. *Recall, even roots require the radicand to be positive unless otherwise noted.*
Since we are taking the square root, which is even, then we apply the even root property:

$$\begin{aligned}(\sqrt[5]{4x+1})^2 &= 9 \\ \sqrt[5]{4x+1} &= \pm\sqrt{9} \\ \sqrt[5]{4x+1} &= \pm 3\end{aligned} \tag{2.7.2}$$

Step 3. Raise each side to the power of the root.
Since the root is 5, then we can raise each side to the fifth power:

$$\begin{aligned}\sqrt[5]{4x+1} &= \pm 3 \\ (\sqrt[5]{4x+1})^5 &= (\pm 3)^5 \\ 4x+1 &= \pm 243\end{aligned} \tag{2.7.3}$$

Step 4. Solve. *Verify the solutions, especially when there is an even root.*

$$\begin{aligned} 4x+1 = 243 \quad &\text{or} \quad 4x+1 = -243 \\ 4x = 242 \quad &\text{or} \quad 4x = -244 \\ x = \frac{242}{4} \quad &\text{or} \quad x = -61 \\ x = \frac{121}{2} \quad &\text{or} \quad x = -61 \end{aligned}$$

We can verify all solutions. Let's start by verifying that $x = -61$ is a solution.

$$\begin{aligned}(\sqrt[5]{4x+1})^2 &= 9 \\ (\sqrt[5]{4(-61)+61})^2 &\stackrel{?}{=} 9 \\ (\sqrt[5]{-244+1})^2 &\stackrel{?}{=} 9 \\ (\sqrt[5]{-243})^2 &\stackrel{?}{=} 9 \\ (-3)^2 &\stackrel{?}{=} 9 \\ 9 = 9 &\checkmark \text{ True}\end{aligned} \tag{2.7.4}$$

We leave the verification of the second solution to the student.

Thus, the solutions to the equation are $\frac{121}{2}, -61$.

Example 10.7.8

Solve: $(3x-2)^{\frac{3}{4}} = 64$

Solution

We follow the steps in order to solve the equation with a rational exponent.

Step 1. Rewrite any rational exponents as radicals.

$$\begin{aligned}(3x-2)^{\frac{3}{4}} &= 64 \\ (\sqrt[4]{3x-2})^3 &= 64\end{aligned} \tag{2.7.5}$$

Step 2. Apply the odd or even root property. *Recall, even roots require the radicand to be positive unless otherwise noted.*
Since we are taking the cube root, which is odd, then we apply the odd root property:

$$\begin{aligned}(\sqrt[4]{3x-2})^3 &= 64\\ \sqrt[4]{3x-2} &= \sqrt[3]{64}\\ \sqrt[4]{3x-2} &= 4\end{aligned} \tag{2.7.6}$$

Step 3. Raise each side to the power of the root.
Since the root is **4**, then we can raise each side to the fourth power:

$$\begin{aligned}\sqrt[4]{3x-2} &= 4\\ (\sqrt[4]{3x-2})^4 &= 4^4\\ 3x-2 &= 256\end{aligned} \tag{2.7.7}$$

Step 4. Solve. *Verify the solutions, especially when there is an even root.*

$$\begin{aligned}3x-2 &= 256\\ 3x &= 258\\ x &= 86\end{aligned} \tag{2.7.8}$$

Since there is an even root in the original equation, we should verify the solution.

$$\begin{aligned}(\sqrt[4]{3x-2})^3 &= 64\\ (\sqrt[4]{3(86)-2})^3 &\stackrel{?}{=} 64\\ (\sqrt[4]{258-2})^3 &\stackrel{?}{=} 64\\ (\sqrt[4]{256})^3 &\stackrel{?}{=} 64\\ (4)^3 &\stackrel{?}{=} 64\\ 64 &= 64 \checkmark \text{ True}\end{aligned} \tag{2.7.9}$$

Thus, the solution is **86**.

When solving equations with rational exponents, it is very helpful to convert the equations into their radical form so we can see which property we need to use and to identify whether we need to verify the solutions due to an even root in the original equation.

Solving with Rational Exponents Homework

Solve.

? Exercise 10.7.1

$x^2 = 75$

? Exercise 10.7.2

$x^2 + 5 = 13$

? Exercise 10.7.3

$3x^2 + 1 = 73$

? Exercise 10.7.4

$(x+2)^5 = -243$

? Exercise 10.7.5

$(2x+5)^3-6=21$

Exercise 10.7.6

$(x-1)^{\frac{2}{3}}=4$

Exercise 10.7.7

$(2-x)^{\frac{3}{2}}=27$

Exercise 10.7.8

$(2x-3)^{\frac{2}{3}}=4$

Exercise 10.7.9

$(x+\frac{1}{2})^{-\frac{2}{3}}=4$

Exercise 10.7.10

$(x-1)^{-\frac{5}{2}}=32$

Exercise 10.7.11

$(3x-2)^{\frac{4}{5}}=16$

Exercise 10.7.12

$(4x+2)^{\frac{3}{5}}=-8$

Exercise 10.7.13

$x^3=-8$

Exercise 10.7.14

$4x^3-2=106$

Exercise 10.7.15

$(x-4)^2=49$

Exercise 10.7.16

$(5x+1)^4=16$

Exercise 10.7.17

$(2x+1)^2+3=21$

Exercise 10.7.18

$(x-1)^{\frac{3}{2}}=8$

Exercise 10.7.19

$(2x+3)^{\frac{4}{3}}=16$

Exercise 10.7.20

$(x+3)^{-\frac{1}{3}}=4$

Exercise 10.7.21

$(x-1)^{-\frac{5}{3}}=32$

Exercise 10.7.22

$(x+3)^{\frac{3}{2}}=-8$

Exercise 10.7.23

$(2x+3)^{\frac{3}{2}}=27$

Exercise 10.7.24

$(3-2x)^{\frac{4}{3}}=-81$

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2.8: Linear Inequalities and Absolute Value Inequalities

Learning Objectives

- Use interval notation.
- Use properties of inequalities.
- Solve inequalities in one variable algebraically.
- Solve absolute value inequalities.

It is not easy to make the honor role at most top universities. Suppose students were required to carry a course load of at least 12 credit hours and maintain a grade point average of 3.5 or above. How could these honor roll requirements be expressed mathematically? In this section, we will explore various ways to express different sets of numbers, inequalities, and absolute value inequalities.



Figure 2.8.1

Using Interval Notation

Indicating the solution to an inequality such as $x \geq 4$ can be achieved in several ways.

- We can use a number line as shown in Figure 2.8.2. The blue ray begins at $x = 4$ and, as indicated by the arrowhead, continues to infinity, which illustrates that the solution set includes all real numbers greater than or equal to 4.
- We can use **set-builder notation**: $\{x|x \geq 4\}$, which translates to "all real numbers x such that x is greater than or equal to 4." Notice that braces are used to indicate a set.
- The third method is **interval notation**, in which solution sets are indicated with parentheses or brackets. The solutions to $x \geq 4$ are represented as $[4, \infty)$. This is perhaps the most useful method, as it applies to concepts studied later in this course and to other higher-level math courses.

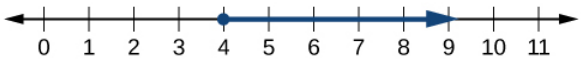


Figure 2.8.2

The main concept to remember is that parentheses represent solutions greater or less than the number, and brackets represent solutions that are greater than or equal to or less than or equal to the number. Use parentheses to represent infinity or negative infinity, since positive and negative infinity are not numbers in the usual sense of the word and, therefore, cannot be "equaled." A few examples of an interval, or a set of numbers in which a solution falls, are $[-2, 6)$, or all numbers between -2 and 6, including -2, but not including 6; $(-1, 0)$, all real numbers between, but not including -1 and 0; and $(-\infty, 1]$, all real numbers less than and including 1. Table 2.8.1 outlines the possibilities.

a}; (a , infinity). The entries in the fourth row are: All real numbers less than b, but not including b; {x| x < b}; (negative infinity, b). The entries in the fifth row are: All real numbers greater than a, including a; {x| x a}; [a, infinity). The entries in the sixth row are: All real numbers less than b, including b; {x| x b}; (negative infinity, b]. The entries in the seventh row are: All real numbers between a and b, including a; {x| a x < b}; [a, b). The entries in the eighth row are: All real numbers between a and b, including b; {x| a < x b}; (a, b]. The entries in the ninth row are: All real numbers between a and b, including a and b; {x| a x b}; [a, b]. The

entries in the tenth row are: all real numbers less than a and greater than b; {x| x < a and x > b}; (negative infinity, a) union (b, infinity). The entries in the eleventh row are: All real numbers; {x| x is all real numbers}; (negative infinity, infinity)."> Table **2.8.1**

Set Indicated	Set-Builder Notation	Interval Notation
All real numbers between a and b, but not including a or b	$\{x \mid a < x < b\}$	(a, b)
All real numbers greater than a, but not including a	$\{x \mid x > a\}$	(a, ∞)
All real numbers less than b, but not including b	$\{x \mid x < b\}$	$(-\infty, b)$
All real numbers greater than a, including a	$\{x \mid x \geq a\}$	$[a, \infty)$
All real numbers less than b, including b	$\{x \mid x \leq b\}$	$(-\infty, b]$
All real numbers between a and b, including a	$\{x \mid a \leq x < b\}$	$[a, b)$
All real numbers between a and b, including b	$\{x \mid a < x \leq b\}$	$(a, b]$
All real numbers between a and b, including a and b	$\{x \mid a \leq x \leq b\}$	$[a, b]$
All real numbers less than a or greater than b	$\{x \mid x < a \text{ and } x > b\}$	$(-\infty, a) \cup (b, \infty)$
All real numbers	$\{x \mid x \text{ is all real numbers}\}$	$(-\infty, \infty)$

✔ Example 2.8.1: Using Interval Notation to Express All Real Numbers Greater Than or Equal to a

Use interval notation to indicate all real numbers greater than or equal to -2.

Solution

Use a bracket on the left of -2 and parentheses after infinity: $[-2, \infty)$. The bracket indicates that -2 is included in the set with all real numbers greater than -2 to infinity.

? Try It! 2.8.1

Use interval notation to indicate all real numbers between and including -3 and 5.

Answer

$[-3, 5]$

✔ Example 2.8.2: Using Interval Notation to Express All Real Numbers Less Than or Equal to a or Greater Than or Equal to b

Write the interval expressing all real numbers less than or equal to -1 or greater than or equal to 1.

Solution

We have to write two intervals for this example. The first interval must indicate all real numbers less than or equal to 1. So, this interval begins at $-\infty$ and ends at -1, which is written as $(-\infty, -1]$.

The second interval must show all real numbers greater than or equal to 1, which is written as $[1, \infty)$. However, we want to combine these two sets. We accomplish this by inserting the union symbol, $\cup$, between the two intervals.

$$(-\infty, -1] \cup [1, \infty)$$

Try It! 2.8.2

Express all real numbers less than -2 or greater than or equal to 3 in interval notation.

Answer

$$(-\infty, -2) \cup [3, \infty)$$

Using the Properties of Inequalities

When we work with inequalities, we can usually treat them similarly to but not exactly as we treat equalities. We can use the **addition property** and the **multiplication property** to help us solve them. The one exception is when we multiply or divide by a negative number; doing so reverses the inequality symbol.

PROPERTIES OF INEQUALITIES

Addition Property

- If $a < b$, then $a + c < b + c$.

Multiplication Property

- If $a < b$ and $c > 0$, then $ac < bc$.
- If $a < b$ and $c < 0$, then $ac > bc$.

These properties also apply to $a \leq b$, $a > b$, and $a \geq b$.

Example 2.8.3: Demonstrating the Addition Property

Illustrate the addition property for inequalities by solving each of the following:

a. (a) $x - 15 < 4$
b. (b) $6 \geq x - 1$
c. (c) $x + 7 > 9$

Solution

The addition property for inequalities states that if an inequality exists, adding or subtracting the same number on both sides does not change the inequality.

a.

$$\begin{aligned} x - 15 &< 4 \\ x - 15 + 15 &< 4 + 15 \\ x &< 19 \end{aligned}$$

b.

$$\begin{aligned} 6 &\geq x - 1 \\ 6 + 1 &\geq x - 1 + 1 \\ 7 &\geq x \end{aligned}$$

c.

$$\begin{aligned} x + 7 &> 9 \\ x + 7 - 7 &> 9 - 7 \\ x &> 2 \end{aligned}$$

Try It! 2.8.3

Solve: $3x - 2 < 1$.

Answer

$x < 1$

Example 2.8.4: Demonstrating the Multiplication Property

Illustrate the multiplication property for inequalities by solving each of the following:

a. $3x < 6$
b. $-2x - 1 \geq 5$
c. $5 - x > 10$

Solution

a.

$$\begin{aligned} 3x &< 6 \\ \frac{1}{3}(3x) &< (6)\frac{1}{3} \\ x &< 2 \end{aligned}$$

b.

$$\begin{aligned} -2x - 1 &\geq 5 \\ -2x &\geq 6 \\ \left(-\frac{1}{2}\right)(-2) &\geq (6)\left(-\frac{1}{2}\right) && \text{Multiply by } \left(-\frac{1}{2}\right) \\ x &\leq -3 && \text{Reverse the inequality.} \end{aligned}$$

c.

$$\begin{aligned} 5 - x &> 10 \\ -x &> 5 \\ (-1)(-x) &> (5)(-1) && \text{Multiply by } -1 \\ x &< -5 && \text{Reverse the inequality.} \end{aligned}$$

Try It! 2.8.4

Solve: $4x + 7 \geq 2x - 3$.

Answer

$x \geq -5$

Solving Inequalities in One Variable Algebraically

As the examples have shown, we can perform the same operations on both sides of an inequality, just as we do with equations; we combine like terms and perform operations. To solve, we isolate the variable.

Example 2.8.5: Solving an Inequality Algebraically

Solve the inequality: $13 - 7x \geq 10x - 4$.

Solution

Solving this inequality is similar to solving an equation up until the last step.

$$\begin{aligned} 13 - 7x &\geq 10x - 4 \\ 13 - 17x &\geq -4 && \text{Move variable terms to one side of the inequality} \\ -17x &\geq -17 && \text{Isolate the variable term} \\ x &\leq 1 && \text{Dividing both sides by -17 reverses the inequality.} \end{aligned}$$

The solution set is given by the interval $(-\infty, 1]$, or all real numbers less than and including 1.

? Try It! 2.8.5

Solve the inequality and write the answer using interval notation: $-x+4<\frac{1}{2}x+1$.

Answer

$(2,\infty)$

✓ Example 2.8.6: Solving an Inequality with Fractions

Solve the following inequality and write the answer in interval notation: $-\frac{3}{4}x\geq-\frac{5}{8}+\frac{2}{3}x$.

Solution

We begin solving in the same way we do when solving an equation.

$$-\frac{3}{4}x\geq-\frac{5}{8}+\frac{2}{3}x$$

$$-\frac{3}{4}x-\frac{2}{3}x\geq-\frac{5}{8} \qquad \text{Put variable terms on one side.}$$

$$-\frac{9}{12}x-\frac{8}{12}x\geq-\frac{5}{8} \qquad \text{Write fractions with common denominator.}$$

$$-\frac{17}{12}x\geq-\frac{5}{8}$$

$$x\leq-\frac{5}{8}\left(-\frac{12}{17}\right) \qquad \text{Multiplying by a negative number reverses the inequality.}$$

$$x\leq\frac{15}{34}$$

The solution set is the interval $\left(-\infty,\frac{15}{34}\right]$.

? Try It! 2.8.6

Solve the inequality and write the answer in interval notation: $-\frac{5}{6}x\leq\frac{3}{4}+\frac{8}{3}x$.

Answer

$\left[-\frac{3}{14},\infty\right)$

Understanding Compound Inequalities

A **compound inequality** includes two inequalities in one statement. A statement such as $4<x\leq6$ means $4<x$ and $x\leq6$. There are two ways to solve compound inequalities: separating them into two separate inequalities or leaving the compound inequality intact and performing operations on all three parts at the same time. We will illustrate both methods.

✓ Example 2.8.7: Solving a Compound Inequality

Solve the compound inequality: $3\leq2x+2<6$

Solution

The first method is to write two separate inequalities: $3\leq2x+2$ and $2x+2<6$. We solve them independently.

$$\begin{array}{ll} 3\leq2x+2 \quad \text{and} & 2x+2<6 \\ 1\leq2x & 2x<4 \\ \frac{1}{2}\leq x & x<2 \end{array}$$

Then, we can rewrite the solution as a compound inequality, the same way the problem began.

$$\frac{1}{2} \leq x < 2$$

In interval notation, the solution is written as $[\frac{1}{2}, 2)$.

The second method is to leave the compound inequality intact, and perform solving procedures on the three parts at the same time.

$3 \leq 2x + 2 < 6$

$1 \leq 2x < 4$ **Isolate the variable term, and subtract 2 from all three parts.**

$\frac{1}{2} \leq x < 2$ **Divide through all three parts by 2.**

We get the same solution: $[\frac{1}{2}, 2)$.

Try It! 2.8.7

Solve the compound inequality: $4 < 2x - 8 \leq 10$.

Answer

$6 < x \leq 9$ or $(6, 9]$

▼

Example 2.8.8: Solving a Compound Inequality with the Variable in All Three Parts

Solve the compound inequality with variables in all three parts: $3 + x > 7x - 2 > 5x - 10$.

Solution

Let's try the first method. Write two inequalities:

$$\begin{array}{cc} 3 + x > 7x - 2 & \text{and} \quad 7x - 2 > 5x - 10 \\ 3 > 6x - 2 & 2x - 2 > -10 \\ 5 > 6x & 2x > -8 \\ \frac{5}{6} > x & -4 < x \end{array}$$

The solution set is $-4 < x < \frac{5}{6}$ or in interval notation $\left(-4, \frac{5}{6}\right)$. Notice that when we write the solution in interval notation, the smaller number comes first. We read intervals from left to right, as they appear on a number line. See Figure 2.8.3.

Figure 2.8.3

Try It! 2.8.8

Solve the compound inequality: $3y < 4 - 5y < 5 + 3y$.

Answer

$(-\frac{1}{8}, \frac{1}{2})$

▼

Solving Absolute Value Inequalities

As we know, the absolute value of a quantity is a positive number or zero. From the origin, a point located at $(-x, 0)$ has an absolute value of x, as it is x units away. Consider absolute value as the distance from one point to another point. Regardless of direction, positive or negative, the distance between the two points is represented as a positive number or zero.

An absolute value inequality is an equation of the form

$$|A| < B, \quad |A| \leq B, \quad |A| > B, \quad \text{or} \quad |A| \geq B,$$

Where A, and sometimes B, represents an algebraic expression dependent on a variable x. Solving the inequality means finding the set of all x-values that satisfy the problem. Usually this set will be an interval or the union of two intervals and will include a range of values.

There are two basic approaches to solving absolute value inequalities: graphical and algebraic. The advantage of the graphical approach is we can read the solution by interpreting the graphs of two equations. The advantage of the algebraic approach is that solutions are exact, as precise solutions are sometimes difficult to read from a graph.

Suppose we want to know all possible returns on an investment if we could earn some amount of money within \$200 of \$600. We can solve algebraically for the set of x-values such that the distance between$x$ and 600 is less than or equal to 200. We represent the distance between x and 600 as $|x-600|$, and therefore, $|x-600| \leq 200$ or

$$\begin{aligned} -200 \leq x-600 &\leq 200 \\ -200+600 \leq x-600+600 &\leq 200+600 \\ 400 \leq x &\leq 800 \end{aligned}$$

This means our returns would be between \$400 and \$800.

To solve absolute value inequalities, just as with absolute value equations, we write two inequalities and then solve them independently.

Definition: Absolute Value Inequalities

For an algebraic expression X and $k>0$, an **absolute value inequality** is an inequality of the form

$$\begin{aligned} |X|<k &\text{ is equivalent to } -k<X<k \\ |X|>k &\text{ is equivalent to } X<-k \text{ or } X>k \end{aligned}$$

These statements also apply to $|X| \leq k$ and $|X| \geq k$.

Example 2.8.9: Determining a Number within a Prescribed Distance

Describe all values x within a distance of 4 from the number 5.

Solution

We want the distance between x and 5 to be less than or equal to 4. We can draw a number line, such as in Figure $2.8.4$, to represent the condition to be satisfied.

Figure $2.8.4$

The distance from x to 5 can be represented using an absolute value symbol, $|x-5|$. Write the values of x that satisfy the condition as an absolute value inequality.

$$|x-5| \leq 4$$

We need to write two inequalities as there are always two solutions to an absolute value equation.

$$\begin{aligned} x-5 \leq 4 \quad &\text{and} \quad x-5 \geq -4 \\ x \leq 9 \quad &\qquad\quad x \geq 1 \end{aligned}$$

If the solution set is $x \leq 9$ and $x \geq 1$, then the solution set is an interval including all real numbers between and including 1 and 9.

So $|x-5| \leq 4$ is equivalent to $[1,9]$ in interval notation.

▼

Try It! 2.8.9

Describe all x-values within a distance of **3** from the number **2.**

Answer

$|x-2| \leq 3$

Example 2.8.10: Solving an Absolute Value Inequality

Solve $|x-1| \leq 3$.

Solution

$$|x-1| \leq 3$$
$$-3 \leq x-1 \leq 3$$
$$-2 \leq x \leq 4$$
$$[-2,4]$$

Example 2.8.11: Using a Graphical Approach to Solve Absolute Value Inequalities

Given the equation $y=-\frac{1}{2}|4x-5|+3$, determine the x-values for which the y-values are negative.

Solution

We are trying to determine where $y<0$, which is when $-\frac{1}{2}|4x-5|+3<0$. We begin by isolating the absolute value.

$$-\frac{1}{2}|4x-5|<-3 \qquad \text{Multiply both sides by } -2, \text{ and reverse the inequality.}$$
$$|4x-5|>6$$

Next, we solve for the equality $|4x-5|=6$.

$$\begin{aligned} 4x-5=6 \quad & \text{or} \quad 4x-5=-6 \\ 4x=11 \quad & \text{or} \quad 4x=-1 \\ x=\tfrac{11}{4} \quad & \text{or} \quad x=-\tfrac{1}{4} \end{aligned}$$

Now, we can examine the graph to observe where the y-values are negative. We observe where the branches are below the x-axis. Notice that it is not important exactly what the graph looks like, as long as we know that it crosses the horizontal axis at $x=-\frac{1}{4}$ and $x=\frac{11}{4}$, and that the graph opens downward. See Figure **2.8.5.**

Figure **2.8.5**

Try It! 2.8.10

Solve $-2|k-4| \leq -6$.

Answer

$k \leq 1$ or $k \geq 7$; in interval notation, this would be $(-\infty,1] \cup [7,\infty)$.

Figure **2.8.6**

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CHAPTER OVERVIEW

3: Functions

3.0: Prelude to Functions
3.1: Functions and Function Notation
3.2: Domain and Range
3.3: Composition of Functions
3.4: Transformation of Functions
3.5: Absolute Value Functions
3.6: Inverse Functions

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3.0: Prelude to Functions

Toward the end of the twentieth century, the values of stocks of internet and technology companies rose dramatically. As a result, the Standard and Poor's stock market average rose as well. The *Standard and Poor's Index* tracks the value of that initial investment of just under $100 over the 40 years. It shows that an investment that was worth less than $500 until about 1995 skyrocketed up to about $1100 by the beginning of 2000. That five-year period became known as the "dot-com bubble" because so many internet startups were formed. As bubbles tend to do, though, the dot-com bubble eventually burst. Many companies grew too fast and then suddenly went out of business. The result caused the sharp decline represented on the graph beginning at the end of 2000.

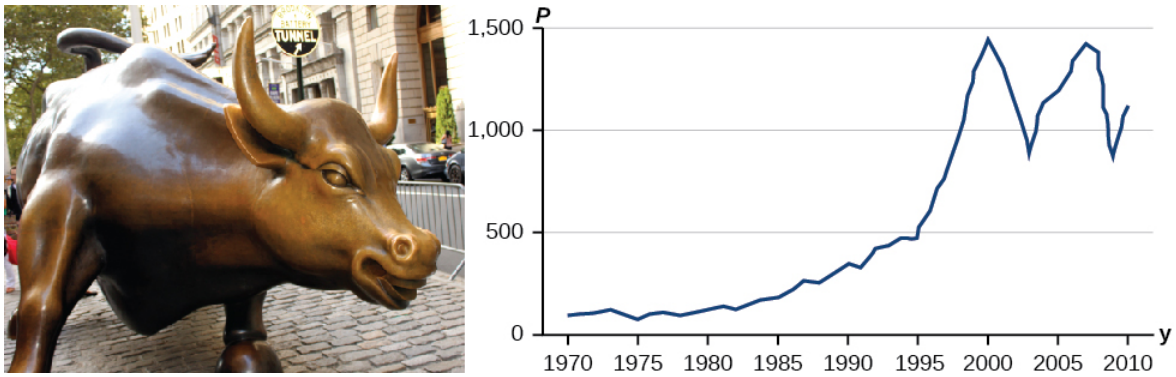


Figure **3.0.1**: Figure of a Bull and a graph of market prices. Standard and Poor's Index with dividends reinvested (credit "bull": modification of work by Prayitno Hadinata; credit "graph": modification of work by MeasuringWorth)

Notice, as we consider this example, that there is a definite relationship between the year and stock market average. For any year we choose, we can determine the corresponding value of the stock market average. In this chapter, we will explore these kinds of relationships and their properties.

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3.1: Functions and Function Notation

Learning Objectives

- Determine whether a relation represents a function.
- Find the value of a function.
- Determine whether a function is one-to-one.
- Use the vertical line test to identify functions.
- Graph the functions listed in the library of functions.

A jetliner changes altitude as its distance from the starting point of a flight increases. The weight of a growing child increases with time. In each case, one quantity depends on another. There is a relationship between the two quantities that we can describe, analyze, and use to make predictions. In this section, we will analyze such relationships.

Determining Whether a Relation Represents a Function

A relation is a set of ordered pairs. The set of the first components of each ordered pair is called the domain and the set of the second components of each ordered pair is called the range. Consider the following set of ordered pairs. The first numbers in each pair are the first five natural numbers. The second number in each pair is twice that of the first.

$$\{(1,2),(2,4),(3,6),(4,8),(5,10)\} \tag{1.1.1}$$

The domain is $\{1,2,3,4,5\}$. The range is $\{2,4,6,8,10\}$.

Note that each value in the domain is also known as an **input** value, or **independent variable**, and is often labeled with the lowercase letter x. Each value in the range is also known as an output value, or **dependent variable**, and is often labeled lowercase letter y.

A function f is a relation that assigns a single value in the range to each value in the domain. In other words, no x-values are repeated. For our example that relates the first five **natural numbers** to numbers double their values, this relation is a function because each element in the domain, {1, 2, 3, 4, 5}, is paired with exactly one element in the range, $\{2,4,6,8,10\}$.

Now let's consider the set of ordered pairs that relates the terms "even" and "odd" to the first five natural numbers. It would appear as

$$\{(\text{odd},1),(\text{even},2),(\text{odd},3),(\text{even},4),(\text{odd},5)\} \tag{1.1.2}$$

Notice that each element in the domain, {even, odd} is not paired with exactly one element in the range, $\{1,2,3,4,5\}$. For example, the term "odd" corresponds to three values from the range, $\{1,3,5\}$, and the term "even" corresponds to two values from the range, $\{2,4\}$. This violates the definition of a function, so this relation is not a function.

Figure 3.1.1 compares relations that are functions and not functions.

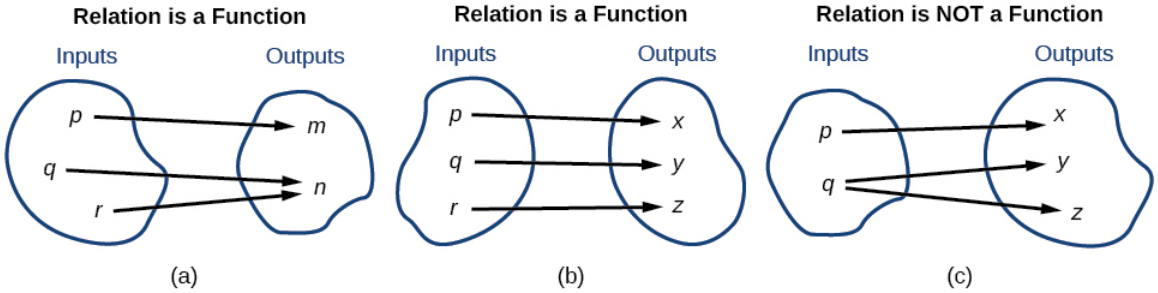


Figure 3.1.1: (a) This relationship is a function because each input is associated with a single output. Note that input q and r both give output n. (b) This relationship is also a function. In this case, each input is associated with a single output. (c) This relationship is not a function because input q is associated with two different outputs.

Function

A **function** is a relation in which each possible input value leads to exactly one output value. We say "the output is a function of the input."

The **input** values make up the **domain**, and the **output** values make up the **range**.

How To: Given a relationship between two quantities, determine whether the relationship is a function

1. Identify the input values.
2. Identify the output values.
3. If each input value leads to only one output value, classify the relationship as a function. If any input value leads to two or more outputs, do not classify the relationship as a function.

Example 3.1.1: Determining If Menu Price Lists Are Functions

The coffee shop menu, shown in Figure **3.1.2** consists of items and their prices.

a. Is price a function of the item?
b. Is the item a function of the price?

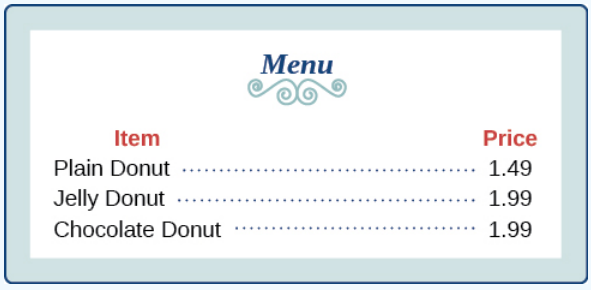


Figure **3.1.2**: A menu of donut prices from a coffee shop where a plain donut is $1.49 and a jelly donut and chocolate donut are $1.99.

Solution

a. Let's begin by considering the input as the items on the menu. The output values are then the prices. See Figure **3.1.3**.



Figure **3.1.3**: A menu of donut prices from a coffee shop where a plain donut is $1.49 and a jelly donut and chocolate donut are $1.99.

Each item on the menu has only one price, so the price is a function of the item.

a. Two items on the menu have the same price. If we consider the prices to be the input values and the items to be the output, then the same input value could have more than one output associated with it. See Figure **3.1.4**.

Figure **3.1.4**: Association of the prices to the donuts.

Therefore, the item is a not a function of price.

Example 3.1.2: Determining If Class Grade Rules Are Functions

In a particular math class, the overall percent grade corresponds to a grade point average. Is grade point average a function of the percent grade? Is the percent grade a function of the grade point average? Table **3.1.1** shows a possible rule for assigning grade points.

Table **3.1.1**: Class grade points.

Percent grade	0–56	57–61	62–66	67–71	72–77	78–86	87–91	92–100
Grade point average	0.0	1.0	1.5	2.0	2.5	3.0	3.5	4.0

Solution

For any percent grade earned, there is an associated grade point average, so the grade point average is a function of the percent grade. In other words, if we input the percent grade, the output is a specific grade point average.

In the grading system given, there is a range of percent grades that correspond to the same grade point average. For example, students who receive a grade point average of 3.0 could have a variety of percent grades ranging from 78 all the way to 86. Thus, percent grade is not a function of grade point average.

Exercise 3.1.2

Table **3.1.2** lists the five greatest baseball players of all time in order of rank.

Table **3.1.2**: Five greatest baseball players.

Player	**Rank**
Babe Ruth	1
Willie Mays	2
Ty Cobb	3
Walter Johnson	4
Hank Aaron	5

a. Is the rank a function of the player name?
b. Is the player name a function of the rank?

Answer a

Yes

Answer b

yes. (Note: If two players had been tied for, say, 4th place, then the name would not have been a function of rank.)

Using Function Notation

Once we determine that a relationship is a function, we need to display and define the functional relationships so that we can understand and use them, and sometimes also so that we can program them into computers. There are various ways of representing functions. A standard function notation is one representation that facilitates working with functions.

To represent "height is a function of age," we start by identifying the descriptive variables h for height and a for age. The letters f, g,and h are often used to represent functions just as we use x, y,and z to represent numbers and A, B, and C to represent sets.

$$\begin{aligned} &h \text{ is } f \text{ of } a && \text{We name the function } f\text{; height is a function of age.} \\ &h = f(a) && \text{We use parentheses to indicate the function input.} \\ &f(a) && \text{We name the function } f \text{ ; the expression is read as "} f \text{ of } a\text{."} \end{aligned} \tag{3.1.1}$$

Remember, we can use any letter to name the function; the notation $h(a)$ shows us that h depends on a. The value a must be put into the function h to get a result. The parentheses indicate that age is input into the function; they do not indicate multiplication.

We can also give an algebraic expression as the input to a function. For example $f(a+b)$ means "first add a and b, and the result is the input for the function f." The operations must be performed in this order to obtain the correct result.

Function Notation

The notation $y=f(x)$ defines a function named f. This is read as "y is a function of x." The letter x represents the input value, or independent variable. The letter y, or $f(x)$, represents the output value, or dependent variable.

Example 3.1.3: Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month.

Solution

Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month.

The number of days in a month is a function of the name of the month, so if we name the function f, we write $\text{days}=f(\text{month})$ or $d=f(m)$. The name of the month is the input to a "rule" that associates a specific number (the output) with each input.

Figure 3.1.5: The function $31=f(January)$ where 31 is the output, f is the rule, and January is the input.

For example, $f(\text{March})=31$, because March has 31 days. The notation $d=f(m)$ reminds us that the number of days, d (the output), is dependent on the name of the month, m (the input).

Analysis

Note that the inputs to a function do not have to be numbers; function inputs can be names of people, labels of geometric objects, or any other element that determines some kind of output. However, most of the functions we will work with in this book will have numbers as inputs and outputs.

Example 3.1.3B: Interpreting Function Notation

A function $N=f(y)$ gives the number of police officers, N, in a town in year y. What does $f(2005)=300$ represent?

Solution

When we read $f(2005)=300$, we see that the input year is 2005. The value for the output, the number of police officers (N), is 300. Remember, $N=f(y)$. The statement $f(2005)=300$ tells us that in the year 2005 there were 300 police officers in the town.

Exercise 3.1.3

Use function notation to express the weight of a pig in pounds as a function of its age in days d.

Answer

$w=f(d)$

Q&A

Instead of a notation such as $y = f(x)$, could we use the same symbol for the output as for the function, such as $y = y(x)$, meaning "y is a function of x?"

Yes, this is often done, especially in applied subjects that use higher math, such as physics and engineering. However, in exploring math itself we like to maintain a distinction between a function such as f, which is a rule or procedure, and the output y we get by applying f to a particular input x. This is why we usually use notation such as $y = f(x), P = W(d)$, and so on.

Representing Functions Using Tables

A common method of representing functions is in the form of a table. The table rows or columns display the corresponding input and output values. In some cases, these values represent all we know about the relationship; other times, the table provides a few select examples from a more complete relationship.

Table **3.1.3** lists the input number of each month ($\textbf{January} = 1$, $\textbf{February} = 2$, and so on) and the output value of the number of days in that month. This information represents all we know about the months and days for a given year (that is not a leap year). Note that, in this table, we define a days-in-a-month function f where $D = f(m)$ identifies months by an integer rather than by name.

Table **3.1.3**: Months and number of days per month.

Month number, m (input)	1	2	3	4	5	6	7	8	9	10	11	12
Days in month, D (output)	31	28	31	30	31	30	31	31	30	31	30	31

Table **3.1.4** defines a function $Q = g(n)$ Remember, this notation tells us that g is the name of the function that takes the input n and gives the output Q.

Table **3.1.4**: Function $Q = g(n)$

n	1	2	3	4	5
Q	8	6	7	6	8

Table **3.1.5** displays the age of children in years and their corresponding heights. This table displays just some of the data available for the heights and ages of children. We can see right away that this table does not represent a function because the same input value, 5 years, has two different output values, 40 in. and 42 in.

Table **3.1.5**: Age of children and their corresponding heights.

Age in years, a (input)	5	5	6	7	8	9	10
Height in inches, h (output)	40	42	44	47	50	52	54

How To: Given a table of input and output values, determine whether the table represents a function

1. Identify the input and output values.
2. Check to see if each input value is paired with only one output value. If so, the table represents a function.

Example **3.1.5**: Identifying Tables that Represent Functions

Which table, Table **3.1.6**, Table **3.1.7**, or Table **3.1.8**, represents a function (if any)?

Table **3.1.6**

Input	Output
2	1
5	3
8	6

Table 3.1.7

Input	Output
-3	5
0	1
4	5

Table 3.1.8

Input	Output
1	0
5	2
5	4

Solution

Table 3.1.6 and Table 3.1.7 define functions. In both, each input value corresponds to exactly one output value. Table 3.1.8 does not define a function because the input value of 5 corresponds to two different output values.

When a table represents a function, corresponding input and output values can also be specified using function notation.

The function represented by Table 3.1.6 can be represented by writing

$$f(2) = 1,\ f(5) = 3,\ \text{and}\ f(8) = 6$$

Similarly, the statements

$$g(-3) = 5,\ g(0) = 1,\ \text{and}\ g(4) = 5$$

represent the function in Table 3.1.7.

Table 3.1.8 cannot be expressed in a similar way because it does not represent a function.

Exercise 3.1.5

Does Table 3.1.9 represent a function?

Table 3.1.9

Input	Output
1	10
2	100
3	1000

Answer

yes

Finding Input and Output Values of a Function

When we know an input value and want to determine the corresponding output value for a function, we evaluate the function. Evaluating will always produce one result because each input value of a function corresponds to exactly one output value.

When we know an output value and want to determine the input values that would produce that output value, we set the output equal to the function's formula and solve for the input. Solving can produce more than one solution because different input values can produce the same output value.

Evaluation of Functions in Algebraic Forms

When we have a function in formula form, it is usually a simple matter to evaluate the function. For example, the function $f(x) = 5 - 3x^2$ can be evaluated by squaring the input value, multiplying by 3, and then subtracting the product from 5.

How To: Given the formula for a function, evaluate.

Given the formula for a function, evaluate.

1. Replace the input variable in the formula with the value provided.
2. Calculate the result.

Example 3.1.6A: Evaluating Functions at Specific Values

1. Evaluate $f(x) = x^2 + 3x - 4$ at
   a. 2
   b. a
   c. $a + h$
   d. Evaluate $\frac{f(a+h)-f(a)}{h}$

Solution

Replace the x in the function with each specified value.

a. Because the input value is a number, 2, we can use simple algebra to simplify.

$$\begin{aligned} f(2) &= 2^2 + 3(2) - 4 \\ &= 4 + 6 - 4 \\ &= 6 \end{aligned}$$

b. In this case, the input value is a letter so we cannot simplify the answer any further.

$$f(a) = a^2 + 3a - 4$$

c. With an input value of $a + h$, we must use the distributive property.

$$\begin{aligned} f(a+h) &= (a+h)^2 + 3(a+h) - 4 \\ &= a^2 + 2ah + h^2 + 3a + 3h - 4 \end{aligned}$$

d. In this case, we apply the input values to the function more than once, and then perform algebraic operations on the result. We already found that

$$f(a+h) = a^2 + 2ah + h^2 + 3a + 3h - 4$$

and we know that

$$f(a) = a^2 + 3a - 4$$

Now we combine the results and simplify.

$$\begin{aligned}\frac{f(a+h)-f(a)}{h} &= \frac{(a^2+2ah+h^2+3a+3h-4)-(a^2+3a-4)}{h} \\ &= \frac{(2ah+h^2+3h)}{h} \\ &= \frac{h(2a+h+3)}{h} && \text{Factor out h.} \\ &= 2a+h+3 && \text{Simplify.}\end{aligned}$$

Example 3.1.6B: Evaluating Functions

Given the function $h(p)=p^2+2p$, evaluate $h(4)$.

Solution

To evaluate $h(4)$, we substitute the value 4 for the input variable p in the given function.

$$\begin{aligned}h(p) &= p^2+2p \\ h(4) &= (4)^2+2(4) \\ &= 16+8 \\ &= 24\end{aligned}$$

Therefore, for an input of 4, we have an output of 24.

Exercise 3.1.6

Given the function $g(m)=\sqrt{m-4}$, evaluate $g(5)$.

Answer

$g(5)=1$

Example 3.1.7: Solving Functions

Given the function $h(p)=p^2+2p$, solve for $h(p)=3$.

Solution

$$\begin{aligned}h(p) &= 3 \\ p^2+2p &= 3 && \text{Substitute the original function} \\ p^2+2p-3 &= 0 && \text{Subtract 3 from each side.} \\ (p+3)(p-1) &= 0 && \text{Factor.}\end{aligned}$$

If $(p+3)(p-1)=0$, either $(p+3)=0$ or $(p-1)=0$ (or both of them equal 0). We will set each factor equal to 0 and solve for p in each case.

$$(p+3)=0,\ p=-3$$

$$(p-1)=0,\ p=1$$

This gives us two solutions. The output $h(p)=3$ when the input is either $p=1$ or $p=-3$. We can also verify by graphing as in Figure 3.1.6. The graph verifies that $h(1)=h(-3)=3$ and $h(4)=24$.

Figure 3.1.6: Graph of $h(p)=p^2+2p$

Exercise 3.1.7

Given the function $g(m)=\sqrt{m-4}$, solve $g(m)=2$.

Answer

$m=8$

Evaluating Functions Expressed in Formulas

Some functions are defined by mathematical rules or procedures expressed in **equation** form. If it is possible to express the function output with a formula involving the input quantity, then we can define a function in algebraic form. For example, the equation $2n + 6p = 12$ expresses a functional relationship between n and p. We can rewrite it to decide if p is a function of n.

How to: Given a function in equation form, write its algebraic formula.

1. Solve the equation to isolate the output variable on one side of the equal sign, with the other side as an expression that involves only the input variable.
2. Use all the usual algebraic methods for solving equations, such as adding or subtracting the same quantity to or from both sides, or multiplying or dividing both sides of the equation by the same quantity.

Example 3.1.8A: Finding an Equation of a Function

Express the relationship $2n + 6p = 12$ as a function $p = f(n)$, if possible.

Solution

To express the relationship in this form, we need to be able to write the relationship where p is a function of n, which means writing it as $p =$ [expression involving n].

$$\begin{aligned} 2n + 6p &= 12 \\ 6p &= 12 - 2n && \text{Subtract 2n from both sides.} \\ p &= \frac{12 - 2n}{6} && \text{Divide both sides by 6 and simplify.} \\ p &= \frac{12}{6} - \frac{2n}{6} \\ p &= 2 - \frac{1}{3}n \end{aligned}$$

Therefore, p as a function of n is written as

$$p = f(n) = 2 - \frac{1}{3}n$$

Analysis

It is important to note that not every relationship expressed by an equation can also be expressed as a function with a formula.

Example 3.1.8B: Expressing the Equation of a Circle as a Function

Does the equation $x^2 + y^2 = 1$ represent a function with x as input and y as output? If so, express the relationship as a function $y = f(x)$.

Solution

First we subtract x^2 from both sides.

$$y^2 = 1 - x^2$$

We now try to solve for y in this equation.

$$y = \pm\sqrt{1 - x^2}$$

$$\text{so, } y = \sqrt{1 - x^2} \text{ and } y = -\sqrt{1 - x^2}$$

We get two outputs corresponding to the same input, so this relationship cannot be represented as a single function $y = f(x)$.

Exercise 3.1.8

If $x - 8y^3 = 0$, express y as a function of x.

Answer

$$y = f(x) = \frac{\sqrt[3]{x}}{2}$$

Q & A

Are there relationships expressed by an equation that do represent a function but which still cannot be represented by an algebraic formula?

Yes, this can happen. For example, given the equation $x = y + 2^y$, if we want to express y as a function of x, there is no simple algebraic formula involving only x that equals y. However, each x does determine a unique value for y, and there are mathematical procedures by which y can be found to any desired accuracy. In this case, we say that the equation gives an implicit (implied) rule for y as a function of x, even though the formula cannot be written explicitly.

Evaluating a Function Given in Tabular Form

As we saw above, we can represent functions in tables. Conversely, we can use information in tables to write functions, and we can evaluate functions using the tables. For example, how well do our pets recall the fond memories we share with them? There is an urban legend that a goldfish has a memory of 3 seconds, but this is just a myth. Goldfish can remember up to 3 months, while the beta fish has a memory of up to 5 months. And while a puppy's memory span is no longer than 30 seconds, the adult dog can remember for 5 minutes. This is meager compared to a cat, whose memory span lasts for 16 hours.

The function that relates the type of pet to the duration of its memory span is more easily visualized with the use of a table (Table **3.1.10**).

Table **3.1.10**

Pet Memory	span in hours
Puppy	0.008
Adult Dog	0.083
Cat	3
Goldfish	2160
Beta Fish	3600

At times, evaluating a function in table form may be more useful than using equations. Here let us call the function P. The domain of the function is the type of pet and the range is a real number representing the number of hours the pet's memory span lasts. We can evaluate the function P at the input value of "goldfish." We would write $P(goldfish) = 2160$. Notice that, to evaluate the function in table form, we identify the input value and the corresponding output value from the pertinent row of the table. The tabular form for function P seems ideally suited to this function, more so than writing it in paragraph or function form.

How To: Given a function represented by a table, identify specific output and input values

1. Find the given input in the row (or column) of input values.
2. Identify the corresponding output value paired with that input value.
3. Find the given output values in the row (or column) of output values, noting every time that output value appears.
4. Identify the input value(s) corresponding to the given output value.

Example 3.1.9: Evaluating and Solving a Tabular Function

Using Table 3.1.11,

a. Evaluate $g(3)$.
b. Solve $g(n) = 6$.

Table 3.1.11

n	1	2	3	4	5
$g(n)$	8	6	7	6	8

Solution

a. Evaluating $g(3)$ means determining the output value of the function g for the input value of $n = 3$. The table output value corresponding to $n = 3$ is 7, so $g(3) = 7$.
b. Solving $g(n) = 6$ means identifying the input values, n,that produce an output value of 6. Table 3.1.12 shows two solutions: 2 and 4.

Table 3.1.12

n	1	2	3	4	5
$g(n)$	8	6	7	6	8

When we input 2 into the function g, our output is 6. When we input 4 into the function g, our output is also 6.

Exercise 3.1.1

Using Table 3.1.12, evaluate $g(1)$.

Answer

$g(1) = 8$

Finding Function Values from a Graph

Evaluating a function using a graph also requires finding the corresponding output value for a given input value, only in this case, we find the output value by looking at the graph. Solving a function equation using a graph requires finding all instances of the given output value on the graph and observing the corresponding input value(s).

Example 3.1.10: Reading Function Values from a Graph

Given the graph in Figure 3.1.7,

a. Evaluate $f(2)$.
b. Solve $f(x) = 4$.

Figure 3.1.7: Graph of a positive parabola centered at $(1, 0)$.

Solution

To evaluate $f(2)$, locate the point on the curve where $x = 2$, then read the y-coordinate of that point. The point has coordinates $(2, 1)$, so $f(2) = 1$. See Figure 3.1.8.

3.1.8: Graph of a positive parabola centered at $(1, 0)$ with the labeled point $(2, 1)$ where $f(2) = 1$.

To solve $f(x) = 4$, we find the output value 4 on the vertical axis. Moving horizontally along the line $y = 4$, we locate two points of the curve with output value 4: $(-1, 4)$ and $(3, 4)$. These points represent the two solutions to $f(x) = 4$: -1 or 3. This means $f(-1) = 4$ and $f(3) = 4$, or when the input is -1 or 3, the output is 4. See Figure 3.1.9.

Figure 3.1.9: Graph of an upward-facing parabola with a vertex at $(0, 1)$ and labeled points at $(-1, 4)$ and $(3, 4)$. A line at $y = 4$ intersects the parabola at the labeled points.

Exercise 3.1.10

Given the graph in Figure 3.1.7, solve $f(x) = 1$.

Answer

$x = 0$ or $x = 2$

Determining Whether a Function is One-to-One

Some functions have a given output value that corresponds to two or more input values. For example, in the stock chart shown in the Figure at the beginning of this chapter, the stock price was $1000 on five different dates, meaning that there were five different input values that all resulted in the same output value of $1000.

However, some functions have only one input value for each output value, as well as having only one output for each input. We call these functions one-to-one functions. As an example, consider a school that uses only letter grades and decimal equivalents, as listed in Table 3.1.13.

Table 3.1.13: Letter grades and decimal equivalents.

Letter Grade	Grade Point Average
A	4.0
B	3.0
C	2.0
D	1.0

This grading system represents a one-to-one function, because each letter input yields one particular grade point average output and each grade point average corresponds to one input letter.

To visualize this concept, let's look again at the two simple functions sketched in Figures 3.1.1a and 3.1.1b. The function in part (a) shows a relationship that is not a one-to-one function because inputs q and r both give output n. The function in part (b) shows a relationship that is a one-to-one function because each input is associated with a single output.

One-to-One Functions

A **one-to-one function** is a function in which each output value corresponds to exactly one input value.

Example 3.1.11: Determining Whether a Relationship Is a One-to-One Function

Is the area of a circle a function of its radius? If yes, is the function one-to-one?

Solution

A circle of radius r has a unique area measure given by $A = \pi r^2$, so for any input, r, there is only one output, A. The area is a function of radius$r$.

If the function is one-to-one, the output value, the area, must correspond to a unique input value, the radius. Any area measure A is given by the formula $A = \pi r^2$. Because areas and radii are positive numbers, there is exactly one solution: $\sqrt{\frac{A}{\pi}}$. So the area of a circle is a one-to-one function of the circle's radius.

Exercise 3.1.11A

a. Is a balance a function of the bank account number?
b. Is a bank account number a function of the balance?
c. Is a balance a one-to-one function of the bank account number?

Answer

a. yes, because each bank account has a single balance at any given time;

b. no, because several bank account numbers may have the same balance;

c. no, because the same output may correspond to more than one input.

Exercise 3.1.11B

Evaluate the following:

a. If each percent grade earned in a course translates to one letter grade, is the letter grade a function of the percent grade?
b. If so, is the function one-to-one?

Answer

a. Yes, letter grade is a function of percent grade;
b. No, it is not one-to-one. There are 100 different percent numbers we could get but only about five possible letter grades, so there cannot be only one percent number that corresponds to each letter grade.

Using the Vertical Line Test

As we have seen in some examples above, we can represent a function using a graph. Graphs display a great many input-output pairs in a small space. The visual information they provide often makes relationships easier to understand. By convention, graphs are typically constructed with the input values along the horizontal axis and the output values along the vertical axis.

The most common graphs name the input value x and the output y, and we say y is a function of x, or $y = f(x)$ when the function is named f. The graph of the function is the set of all points (x, y) in the plane that satisfies the equation $y = f(x)$. If the function is defined for only a few input values, then the graph of the function is only a few points, where the x-coordinate of each point is an input value and the y-coordinate of each point is the corresponding output value. For example, the black dots on the graph in Figure 3.1.10 tell us that $f(0) = 2$ and $f(6) = 1$. However, the set of all points (x, y) satisfying $y = f(x)$ is a curve. The curve shown includes $(0, 2)$ and $(6, 1)$ because the curve passes through those points

Figure 3.1.10: Graph of a polynomial.

The vertical line test can be used to determine whether a graph represents a function. If we can draw any vertical line that intersects a graph more than once, then the graph does not define a function because a function has only one output value for each input value. See Figure 3.1.11.

Figure 3.1.11: Three graphs visually showing what is and is not a function.

Howto: Given a graph, use the vertical line test to determine if the graph represents a function

1. Inspect the graph to see if any vertical line drawn would intersect the curve more than once.
2. If there is any such line, determine that the graph does not represent a function.

Example 3.1.12: Applying the Vertical Line Test

Which of the graphs in Figure 3.1.12 represent(s) a function $y = f(x)$?

Figure 3.1.12: Graph of a polynomial (a), a downward-sloping line (b), and a circle (c).

Solution

If any vertical line intersects a graph more than once, the relation represented by the graph is not a function. Notice that any vertical line would pass through only one point of the two graphs shown in parts (a) and (b) of Figure **3.1.12**. From this we can conclude that these two graphs represent functions. The third graph does not represent a function because, at most x-values, a vertical line would intersect the graph at more than one point, as shown in Figure **3.1.13**.

Figure **3.1.13**: Graph of a circle.

Exercise 3.1.12

Does the graph in Figure **3.1.14** represent a function?

Figure **3.1.14**: Graph of absolute value function.

Answer

yes

Using the Horizontal Line Test

Once we have determined that a graph defines a function, an easy way to determine if it is a one-to-one function is to use the **horizontal line test**. Draw horizontal lines through the graph. If any horizontal line intersects the graph more than once, then the graph does not represent a one-to-one function.

Howto: Given a graph of a function, use the horizontal line test to determine if the graph represents a one-to-one function

1. Inspect the graph to see if any horizontal line drawn would intersect the curve more than once.
2. If there is any such line, determine that the function is not one-to-one.

Example 3.1.13: Applying the Horizontal Line Test

Consider the functions shown in Figure **3.1.12*a*** and Figure **3.1.12*b***. Are either of the functions one-to-one?

Solution

The function in Figure **3.1.12*a*** is not one-to-one. The horizontal line shown in Figure **3.1.15** intersects the graph of the function at two points (and we can even find horizontal lines that intersect it at three points.)

Figure **3.1.15**: Graph of a polynomial with a horizontal line crossing through 2 points

The function in Figure **3.1.12*b*** is one-to-one. Any horizontal line will intersect a diagonal line at most once.

Exercise 3.1.13

Is the graph shown in Figure **3.1.13** one-to-one?

Answer

No, because it does not pass the horizontal line test.

Identifying Basic Toolkit Functions

In this text, we will be exploring functions—the shapes of their graphs, their unique characteristics, their algebraic formulas, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of building-block elements. We call these our "toolkit functions," which form a set of basic named functions for which we know the graph, formula, and special properties. Some of these functions are programmed to individual buttons on many calculators. For these definitions we will use x as the input variable and $y = f(x)$ as the output variable.

We will see these toolkit functions, combinations of toolkit functions, their graphs, and their transformations frequently throughout this book. It will be very helpful if we can recognize these toolkit functions and their features quickly by name, formula, graph, and

basic table properties. The graphs and sample table values are included with each function shown in Table 3.1.14.

Table 3.1.14: Toolkit Functions

Name	Function	Graph
Constant	$f(x) = c$ where c is a constant	
Identity	$f(x) = x$	
Absolute Value	$f(x) = \|x\|$	
Quadratic	$f(x) = x^2$	
Cubic	$f(x) = x^3$	
reciprocal	$f(x) = \frac{1}{x}$	
Reciprocal squared	$f(x) = \frac{1}{x^2}$	
Square root	$f(x) = \sqrt{x}$	
Cube root	$f(x) = \sqrt[3]{x}$	

Key Equations

- Constant function $f(x) = c$, where c is a constant
- Identity function $f(x) = x$
- Absolute value function $f(x) = |x|$
- Quadratic function $f(x) = x^2$
- Cubic function $f(x) = x^3$
- Reciprocal function $f(x) = \frac{1}{x}$
- Reciprocal squared function $f(x) = \frac{1}{x^2}$
- Square root function $f(x) = \sqrt{x}$
- Cube root function $f(x) = 3\sqrt{x}$

Key Concepts

- A relation is a set of ordered pairs. A function is a specific type of relation in which each domain value, or input, leads to exactly one range value, or output.
- Function notation is a shorthand method for relating the input to the output in the form $y = f(x)$.
- In tabular form, a function can be represented by rows or columns that relate to input and output values.
- To evaluate a function, we determine an output value for a corresponding input value. Algebraic forms of a function can be evaluated by replacing the input variable with a given value.
- To solve for a specific function value, we determine the input values that yield the specific output value.
- An algebraic form of a function can be written from an equation.
- Input and output values of a function can be identified from a table.
- Relating input values to output values on a graph is another way to evaluate a function.
- A function is one-to-one if each output value corresponds to only one input value.
- A graph represents a function if any vertical line drawn on the graph intersects the graph at no more than one point.
- The graph of a one-to-one function passes the horizontal line test.

Footnotes

1 http://www.baseball-almanac.com/lege.../lisn100.shtml. Accessed 3/24/2014.
2 www.kgbanswers.com/how-long-i...y-span/4221590. Accessed 3/24/2014.

Glossary

dependent variable
an output variable

domain
the set of all possible input values for a relation

function
a relation in which each input value yields a unique output value

horizontal line test
a method of testing whether a function is one-to-one by determining whether any horizontal line intersects the graph more than once

independent variable
an input variable

input
each object or value in a domain that relates to another object or value by a relationship known as a function

one-to-one function
a function for which each value of the output is associated with a unique input value

output
each object or value in the range that is produced when an input value is entered into a function

range
the set of output values that result from the input values in a relation

relation
a set of ordered pairs

vertical line test
a method of testing whether a graph represents a function by determining whether a vertical line intersects the graph no more than once

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3.2: Domain and Range

Learning Objectives

- Find the domain of a function defined by an equation.
- Graph piecewise-defined functions.

If you're in the mood for a scary movie, you may want to check out one of the five most popular horror movies of all time—I am Legend, Hannibal, The Ring, The Grudge, and The Conjuring. Figure **3.2.1** shows the amount, in dollars, each of those movies grossed when they were released as well as the ticket sales for horror movies in general by year. Notice that we can use the data to create a function of the amount each movie earned or the total ticket sales for all horror movies by year. In creating various functions using the data, we can identify different independent and dependent variables, and we can analyze the data and the functions to determine the domain and range. In this section, we will investigate methods for determining the domain and range of functions such as these.

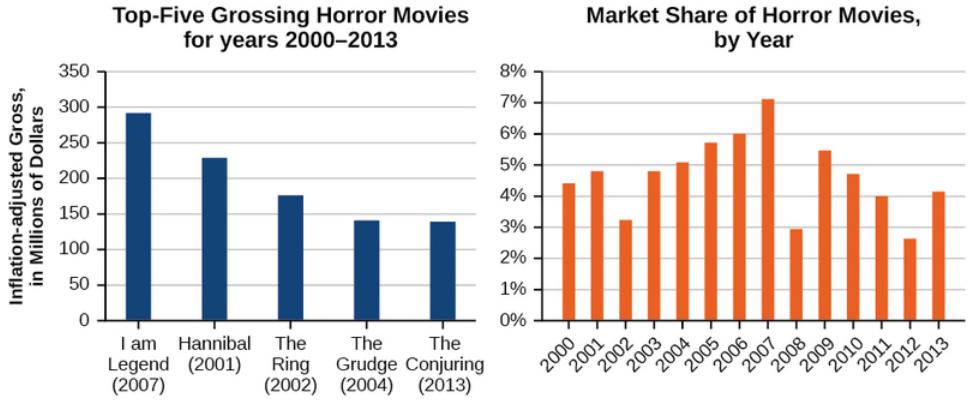


Figure **3.2.1**: Graph of the Top-Five Grossing Horror Movies for years 2000-2003, and a Graph of the Market Share of Horror Movies by Year. Based on data compiled by www.the-numbers.com.

Finding the Domain of a Function Defined by an Equation

In Functions and Function Notation, we were introduced to the concepts of **domain and range**. In this section, we will practice determining domains and ranges for specific functions. Keep in mind that, in determining domains and ranges, we need to consider what is physically possible or meaningful in real-world examples, such as tickets sales and year in the horror movie example above. We also need to consider what is mathematically permitted. For example, we cannot include any input value that leads us to take an even root of a negative number if the domain and range consist of real numbers. Or in a function expressed as a formula, we cannot include any input value in the domain that would lead us to divide by 0.

We can visualize the domain as a "holding area" that contains "raw materials" for a "function machine" and the range as another "holding area" for the machine's products (Figure **3.2.2**).

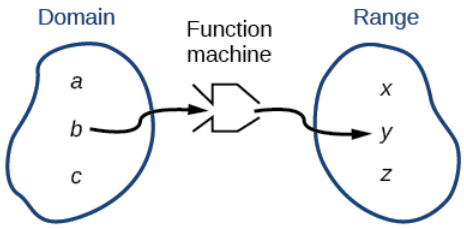


Figure **3.2.2**: Diagram of how a function relates two relations

We can write the **domain and range** in **interval notation**, which uses values within brackets to describe a set of numbers. In interval notation, we use a square bracket [when the set includes the endpoint and a parenthesis (to indicate that the endpoint is either not included or the interval is unbounded. For example, if a person has $100 to spend, he or she would need to express the interval that is more than 0 and less than or equal to 100 and write $(0, 100]$. We will discuss interval notation in greater detail later.

Let's turn our attention to finding the domain of a function whose equation is provided. Oftentimes, finding the domain of such functions involves remembering three different forms. First, if the function has no denominator or an even root, consider whether the domain could be all real numbers. Second, if there is a denominator in the function's equation, exclude values in the domain that force the denominator to be zero. Third, if there is an even root, consider excluding values that would make the radicand negative.

Before we begin, let us review the conventions of interval notation:

- The smallest term from the interval is written first.
- The largest term in the interval is written second, following a comma.
- Parentheses, (or), are used to signify that an endpoint is not included, called exclusive.
- Brackets, [or], are used to indicate that an endpoint is included, called inclusive.

See Figure **3.2.3** for a summary of interval notation.

Inequality	Interval Notation	Graph on Number Line	Description
$x > a$	(a, ∞)		x is greater than a
$x < a$	$(-\infty, a)$		x is less than a
$x \geq a$	$[a, \infty)$		x is greater than or equal to a
$x \leq a$	$(-\infty, a]$		x is less than or equal to a
$a < x < b$	(a, b)		x is strictly between a and b
$a \leq x < b$	$[a, b)$		x is between a and b, to include a
$a < x \leq b$	$(a, b]$		x is between a and b, to include b
$a \leq x \leq b$	$[a, b]$		x is between a and b, to include a and b

Figure **3.2.3**: Summary of interval notation.

Example 3.2.1: Finding the Domain of a Function as a Set of Ordered Pairs

Find the domain of the following function: $\{(2, 10), (3, 10), (4, 20), (5, 30), (6, 40)\}$.

Solution

First identify the input values. The input value is the first coordinate in an ordered pair. There are no restrictions, as the ordered pairs are simply listed. The domain is the set of the first coordinates of the ordered pairs.

$$\{2, 3, 4, 5, 6\}$$

Exercse 3.2.1

Find the domain of the function:

$$\{(-5, 4), (0, 0), (5, -4), (10, -8), (15, -12)\}$$

Answer

$\{-5, 0, 5, 10, 15\}$

How To: Given a function written in equation form, find the domain.

1. Identify the input values.
2. Identify any restrictions on the input and exclude those values from the domain.
3. Write the domain in interval form, if possible.

Example 3.2.2: Finding the Domain of a Function

Find the domain of the function $f(x) = x^2 - 1$.

Solution

The input value, shown by the variable x in the equation, is squared and then the result is lowered by one. Any real number may be squared and then be lowered by one, so there are no restrictions on the domain of this function. The domain is the set of real numbers.

In interval form, the domain of f is $(-\infty, \infty)$.

Exercse 3.2.2

Find the domain of the function:

$$f(x) = 5 - x + x^3$$

Answer

$(-\infty, \infty)$

Howto: Given a function written in an equation form that includes a fraction, find the domain

1. Identify the input values.
2. Identify any restrictions on the input. If there is a denominator in the function's formula, set the denominator equal to zero and solve for x . If the function's formula contains an even root, set the radicand greater than or equal to 0, and then solve.
3. Write the domain in interval form, making sure to exclude any restricted values from the domain.

Example 3.2.3: Finding the Domain of a Function Involving a Denominator

Find the domain of the function $f(x) = \dfrac{x+1}{2-x}$.

Solution

When there is a denominator, we want to include only values of the input that do not force the denominator to be zero. So, we will set the denominator equal to 0 and solve for x.

$$2 - x = 0$$
$$-x = -2$$
$$x = 2$$

Now, we will exclude 2 from the domain. The answers are all real numbers where $x < 2$ or $x > 2$. We can use a symbol known as the union, ∪,to combine the two sets. In interval notation, we write the solution:$(-\infty, 2) \cup (2, \infty)$.

In interval form, the domain of f is $(-\infty, 2) \cup (2, \infty)$.

Exercse 3.2.3

Find the domain of the function:

$$f(x) = \frac{1 + 4x}{2x - 1}$$

Answer

$$(-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$$

How To: Given a function written in equation form including an even root, find the domain.

1. Identify the input values.
2. Since there is an even root, exclude any real numbers that result in a negative number in the radicand. Set the radicand greater than or equal to zero and solve for x.
3. The solution(s) are the domain of the function. If possible, write the answer in interval form.

Example 3.2.4: Finding the Domain of a Function with an Even Root

Find the domain of the function:

$$f(x) = \sqrt{7 - x}.$$

Solution

When there is an even root in the formula, we exclude any real numbers that result in a negative number in the radicand.

Set the radicand greater than or equal to zero and solve for x.

$$7 - x \geq 0$$
$$-x \geq -7$$
$$x \leq 7$$

Now, we will exclude any number greater than 7 from the domain. The answers are all real numbers less than or equal to 7, or $(-\infty, 7]$.

Exercse 3.2.4

Find the domain of the function

$$f(x) = \sqrt{5 + 2x}.$$

Answer

$$[-2.5, \infty)$$

Q&A: Can there be functions in which the domain and range do not intersect at all?

Yes. For example, the function $f(x) = -\frac{1}{\sqrt{x}}$ *has the set of all positive real numbers as its domain but the set of all negative real numbers as its range. As a more extreme example, a function's inputs and outputs can be completely different categories (for example, names of weekdays as inputs and numbers as outputs, as on an attendance chart), in such cases the domain and range have no elements in common.*

Using Notations to Specify Domain and Range

In the previous examples, we used inequalities and lists to describe the domain of functions. We can also use inequalities, or other statements that might define sets of values or data, to describe the behavior of the variable in set-builder notation. For example, $\{x|10 \le x < 30\}$ describes the behavior of x in set-builder notation. The braces $\{\}$ are read as "the set of," and the vertical bar | is read as "such that," so we would read$\{x|10 \le x < 30\}$ as "the set of x-values such that 10 is less than or equal to x, and x is less than 30."

Figure **3.2.4** compares inequality notation, set-builder notation, and interval notation.

Figure **3.2.4**: Summary of notations for inequalities, set-builder, and interval.

To combine two intervals using inequality notation or set-builder notation, we use the word "or." As we saw in earlier examples, we use the union symbol, $\cup$,to combine two unconnected intervals. For example, the union of the sets$\{2, 3, 5\}$ and $\{4, 6\}$ is the set $\{2, 3, 4, 5, 6\}$. It is the set of all elements that belong to one or the other (or both) of the original two sets. For sets with a finite number of elements like these, the elements do not have to be listed in ascending order of numerical value. If the original two sets have some elements in common, those elements should be listed only once in the union set. For sets of real numbers on intervals, another example of a union is

$$\{x||x| \ge 3\} = (-\infty, -3] \cup [3, \infty) \tag{3.2.1}$$

Set-Builder Notation and Interval Notation

Set-builder notation is a method of specifying a set of elements that satisfy a certain condition. It takes the form $\{x|$ **statement about x**$\}$ which is read as, "the set of all x such that the statement about x is true." For example,

$$\{x|4 < x \le 12\}$$

Interval notation is a way of describing sets that include all real numbers between a lower limit that may or may not be included and an upper limit that may or may not be included. The endpoint values are listed between brackets or parentheses. A square bracket indicates inclusion in the set, and a parenthesis indicates exclusion from the set. For example,

$$(4, 12]$$

Given a line graph, describe the set of values using interval notation.

1. Identify the intervals to be included in the set by determining where the heavy line overlays the real line.
2. At the left end of each interval, use [with each end value to be included in the set (solid dot) or (for each excluded end value (open dot).
3. At the right end of each interval, use] with each end value to be included in the set (filled dot) or) for each excluded end value (open dot).
4. Use the union symbol $\cup$ to combine all intervals into one set.

Example 3.2.5: Describing Sets on the Real-Number Line

Describe the intervals of values shown in Figure **3.2.5** using inequality notation, set-builder notation, and interval notation.

Figure **3.2.5**: Line graph of $1 \le x \le 3$ or $x > 5$.

Solution

To describe the values, x, included in the intervals shown, we would say, "x is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5."

Inequality

$$1 \leq x \leq 3 \text{ or } x > 5$$

Set-builder Notation

$$\{x | 1 \leq x \leq 3 \text{ or } x > 5\}$$

Interval notation

$$[1, 3] \cup (5, \infty)$$

Remember that, when writing or reading interval notation, using a square bracket means the boundary is included in the set. Using a parenthesis means the boundary is not included in the set.

? Exercse 3.2.5

Given Figure **3.2.6**, specify the graphed set in

a. words
b. set-builder notation
c. interval notation

Figure **3.2.6**: Line graph of $-2 \leq x$, $-1 \leq x < 3$.leq}x\), $-1 \leq x < 3$.

Answer a

Values that are less than or equal to –2, or values that are greater than or equal to –1 and less than 3;

Answer b

$\{x | x \leq -2 or -1 \leq x < 3\}$

Answer c

$(-\infty, -2] \cup [-1, 3)$

Finding Domain and Range from Graphs

Another way to identify the domain and range of functions is by using graphs. Because the domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the x-axis. The range is the set of possible output values, which are shown on the y-axis. Keep in mind that if the graph continues beyond the portion of the graph we can see, the domain and range may be greater than the visible values. See Figure **3.2.7**.

Figure **3.2.7**: Graph of a polynomial that shows the x-axis is the domain and the y-axis is the range

We can observe that the graph extends horizontally from −5 to the right without bound, so the domain is $[-5, \infty)$. The vertical extent of the graph is all range values 5 and below, so the range is $(-\infty, 5]$. Note that the domain and range are always written from smaller to larger values, or from left to right for domain, and from the bottom of the graph to the top of the graph for range.

✓ Example 3.2.6A: Finding Domain and Range from a Graph

Find the domain and range of the function f whose graph is shown in Figure 1.2.8.

Figure **3.2.8**: Graph of a function from (-3, 1].

Solution

We can observe that the horizontal extent of the graph is –3 to 1, so the domain of f is $(-3, 1]$.

The vertical extent of the graph is 0 to –4, so the range is $[-4, 0)$. See Figure **3.2.9**.

Figure 3.2.9: Graph of the previous function shows the domain and range

Example 3.2.6B: Finding Domain and Range from a Graph of Oil Production

Find the domain and range of the function f whose graph is shown in Figure 3.2.10.

Figure 3.2.10: Graph of the Alaska Crude Oil Production where the y-axis is thousand barrels per day and the -axis is the years(credit: modification of work by the U.S. Energy Information Administration)

Solution

The input quantity along the horizontal axis is “years,” which we represent with the variable t for time. The output quantity is “thousands of barrels of oil per day,” which we represent with the variable b for barrels. The graph may continue to the left and right beyond what is viewed, but based on the portion of the graph that is visible, we can determine the domain as $1973 \le t \le 2008$ and the range as approximately $180 \le b \le 2010$.

In interval notation, the domain is $[1973, 2008]$, and the range is about $[180, 2010]$. For the domain and the range, we approximate the smallest and largest values since they do not fall exactly on the grid lines.

? Exercse 3.2.6

Given Figure 3.2.11, identify the domain and range using interval notation.

Figure 3.2.11: Graph of World Population Increase where the y-axis represents millions of people and the x-axis represents the year.

Answer

domain =$[1950, 2002]$

range = $[47,000,000, 89,000,000]$

Can a function’s domain and range be the same?

Yes. For example, the domain and range of the cube root function are both the set of all real numbers.

Finding Domains and Ranges of the Toolkit Functions

We will now return to our set of toolkit functions to determine the domain and range of each.

Figure 3.2.12: Constant function $f(x) = c$.

For the **constant function** $f(x) = c$, the domain consists of all real numbers; there are no restrictions on the input. The only output value is the constant c, so the range is the set $\{c\}$ that contains this single element. In interval notation, this is written as $[c, c]$, the interval that both begins and ends with c.

Figure* 3.2.13*: *Identity function f(x)=x.*

For the **identity function** $f(x) = x$, there is no restriction on x. Both the domain and range are the set of all real numbers.

Figure 3.2.14: Absolute function $f(x) = |x|$.

For the **absolute value function** $f(x) = |x|$, there is no restriction on x. However, because absolute value is defined as a distance from 0, the output can only be greater than or equal to 0.

Figure 3.2.15: Quadratic function $f(x) = x^2$.

For the **quadratic function** $f(x) = x^2$, the domain is all real numbers since the horizontal extent of the graph is the whole real number line. Because the graph does not include any negative values for the range, the range is only nonnegative real numbers.

Figure 3.2.16: Cubic function $f(x) - x^3$.

For the **cubic function** $f(x) = x^3$, the domain is all real numbers because the horizontal extent of the graph is the whole real number line. The same applies to the vertical extent of the graph, so the domain and range include all real numbers.

Figure 3.2.17: Reciprocal function $f(x) = \frac{1}{x}$.

For the **reciprocal function** $f(x) = \frac{1}{x}$, we cannot divide by 0, so we must exclude 0 from the domain. Further, 1 divided by any value can never be 0, so the range also will not include 0. In set-builder notation, we could also write $\{x|x \neq 0\}$,the set of all real numbers that are not zero.

Figure 3.2.18: Reciprocal squared function $f(x) = \frac{1}{x^2}$

For the **reciprocal squared function** $f(x) = \frac{1}{x^2}$,we cannot divide by 0, so we must exclude 0 from the domain. There is also no x that can give an output of 0, so 0 is excluded from the range as well. Note that the output of this function is always positive due to the square in the denominator, so the range includes only positive numbers.

Figure* 3.2.19*: *Square root function* $f(x) = \sqrt{(x)}$.

For the **square root function** $f(x) = \sqrt{x}$, we cannot take the square root of a negative real number, so the domain must be 0 or greater. The range also excludes negative numbers because the square root of a positive number x is defined to be positive, even though the square of the negative number $-\sqrt{x}$ also gives us x.

Figure 3.2.20: Cube root function $f(x) = \sqrt[3]{x}$.

For the **cube root function** $f(x) = \sqrt[3]{x}$, the domain and range include all real numbers. Note that there is no problem taking a cube root, or any odd-integer root, of a negative number, and the resulting output is negative (it is an odd function).

Given the formula for a function, determine the domain and range.

1. Exclude from the domain any input values that result in division by zero.
2. Exclude from the domain any input values that have nonreal (or undefined) number outputs.
3. Use the valid input values to determine the range of the output values.
4. Look at the function graph and table values to confirm the actual function behavior.

Finding the Domain and Range Using Toolkit Functions

Find the domain and range of $f(x) = 2x^3 - x$.

Solution

There are no restrictions on the domain, as any real number may be cubed and then subtracted from the result.

The domain is $(-\infty, \infty)$ and the range is also $(-\infty, \infty)$.

Example 3.2.7*B*: Finding the Domain and Range

Find the domain and range of $f(x) = \frac{2}{x+1}$.

Solution

We cannot evaluate the function at −1 because division by zero is undefined. The domain is $(-\infty, -1) \cup (-1, \infty)$. Because the function is never zero, we exclude 0 from the range. The range is $(-\infty, 0) \cup (0, \infty)$.

Example 3.2.7*C*: Finding the Domain and Range

Find the domain and range of $f(x) = 2\sqrt{x+4}$.

Solution

We cannot take the square root of a negative number, so the value inside the radical must be nonnegative.

$x + 4 \geq 0$ when $x \geq -4$

The domain of $f(x)$ is $[-4, \infty)$.

We then find the range. We know that $f(-4) = 0$, and the function value increases as x increases without any upper limit. We conclude that the range of f is $[0, \infty)$.

Analysis

Figure **3.2.19** represents the function f.

Figure **3.2.19**: Graph of a square root function at $(-4, 0)$.

? Exercise 3.2.7

Find the domain and range of

$f(x) = \sqrt{-2 - x}$.

Answer

domain: $(-\infty, -2]$

range: $[0, \infty)$

Key Concepts

- The domain of a function includes all real input values that would not cause us to attempt an undefined mathematical operation, such as dividing by zero or taking the square root of a negative number.
- The domain of a function can be determined by listing the input values of a set of ordered pairs.
- The domain of a function can also be determined by identifying the input values of a function written as an equation.
- Interval values represented on a number line can be described using inequality notation, set-builder notation, and interval notation.
- For many functions, the domain and range can be determined from a graph.
- An understanding of toolkit functions can be used to find the domain and range of related functions.

Footnotes

1 The Numbers: Where Data and the Movie Business Meet. “Box Office History for Horror Movies.” http://www.the-numbers.com/market/genre/Horror. Accessed 3/24/2014
2 www.eia.gov/dnav/pet/hist/Lea...s=MCRFPAK2&f=A.

Glossary

interval notation

a method of describing a set that includes all numbers between a lower limit and an upper limit; the lower and upper values are listed between brackets or parentheses, a square bracket indicating inclusion in the set, and a parenthesis indicating exclusion

set-builder notation

a method of describing a set by a rule that all of its members obey; it takes the form {x| statement about x}

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3.3: Composition of Functions

Learning Objectives

- Combine functions using algebraic operations.
- Create a new function by composition of functions.
- Evaluate composite functions.

Suppose we want to calculate how much it costs to heat a house on a particular day of the year. The cost to heat a house will depend on the average daily temperature, and in turn, the average daily temperature depends on the particular day of the year. Notice how we have just defined two relationships: The cost depends on the temperature, and the temperature depends on the day.

Using descriptive variables, we can notate these two functions. The function $C(T)$ gives the cost C of heating a house for a given average daily temperature in T degrees Celsius. The function $T(d)$ gives the average daily temperature on day d of the year. For any given day, $Cost = C(T(d))$ means that the cost depends on the temperature, which in turns depends on the day of the year. Thus, we can evaluate the cost function at the temperature $T(d)$. For example, we could evaluate $T(5)$ to determine the average daily temperature on the 5th day of the year. Then, we could evaluate the **cost function** at that temperature. We would write $C(T(5))$.

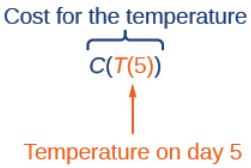


Figure 3.3.1: Explanation of $C(T(5))$, which is the cost for the temperature and $T(5)$ is the temperature on day 5.

By combining these two relationships into one function, we have performed function composition, which is the focus of this section.

Combining Functions Using Algebraic Operations

Function composition is only one way to combine existing functions. Another way is to carry out the usual algebraic operations on functions, such as addition, subtraction, multiplication and division. We do this by performing the operations with the function outputs, defining the result as the output of our new function.

Suppose we need to add two columns of numbers that represent a husband and wife's separate annual incomes over a period of years, with the result being their total household income. We want to do this for every year, adding only that year's incomes and then collecting all the data in a new column. If $w(y)$ is the wife's income and $h(y)$ is the husband's income in year y, and we want T to represent the total income, then we can define a new function.

$$T(y) = h(y) + w(y)$$

If this holds true for every year, then we can focus on the relation between the functions without reference to a year and write

$$T = h + w$$

Just as for this sum of two functions, we can define difference, product, and ratio functions for any pair of functions that have the same kinds of inputs (not necessarily numbers) and also the same kinds of outputs (which do have to be numbers so that the usual operations of algebra can apply to them, and which also must have the same units or no units when we add and subtract). In this way, we can think of adding, subtracting, multiplying, and dividing functions.

For two functions $f(x)$ and $g(x)$ with real number outputs, we define new functions $f + g$, $f - g$, fg, and $\frac{f}{g}$ by the relations.

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)g(x)$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

Example 3.3.1: Performing Algebraic Operations on Functions

Find and simplify the functions $(g-f)(x)$ and $\left(\frac{g}{f}\right)(x)$, given $f(x)=x-1$ and $g(x)=x^2-1$. Are they the same function?

Solution

Begin by writing the general form, and then substitute the given functions.

$$\begin{aligned}(g-f)(x) &= g(x)-f(x)\\ (g-f)(x) &= x^2-1-(x-1)\\ &= x^2-x\\ &= x(x-1)\end{aligned}$$

$$\begin{aligned}\left(\frac{g}{f}\right)(x) &= g(x)f(x)\\ \left(\frac{g}{f}\right)(x) &= \frac{x^2-1}{x-1}\\ &= \frac{(x+1)(x-1)}{x-1}\\ &= x+1\end{aligned}$$

No, the functions are not the same.

Note: For $\left(\frac{g}{f}\right)(x)$, the condition $x\neq 1$ is necessary because when $x=1$, the denominator is equal to 0, which makes the function undefined.

Exercise 3.3.1

Find and simplify the functions $(fg)(x)$ and $(f-g)(x)$.

$$f(x)=x-1$$

and

$$g(x)=x^2-1$$

Are they the same function?

Answer

$(fg)(x)=f(x)g(x)=(x-1)(x2-1)=x^3-x^2-x+1$
$(f-g)(x)=f(x)-g(x)=(x-1)-(x^2-1)=x-x^2$

No, the functions are not the same.

Create a Function by Composition of Functions

Performing algebraic operations on functions combines them into a new function, but we can also create functions by composing functions. When we wanted to compute a heating cost from a day of the year, we created a new function that takes a day as input and yields a cost as output. The process of **combining functions** so that the output of one function becomes the input of another is known as a **composition of functions**. The resulting function is known as a **composite function**. We represent this combination by the following notation:

$$f\circ g(x)=f(g(x)) \tag{3.3.1}$$

We read the left-hand side as"f composed with g at x," and the right-hand side as"f of g of x."The two sides of the equation have the same mathematical meaning and are equal. The open circle symbol $\circ$ is called the composition operator. We use this operator mainly when we wish to emphasize the relationship between the functions themselves without referring to any particular input

value. Composition is a binary operation that takes two functions and forms a new function, much as addition or multiplication takes two numbers and gives a new number. However, it is important not to confuse function composition with multiplication because, as we learned above, in most cases $f(g(x)) \neq f(x)g(x)$.

It is also important to understand the order of operations in evaluating a composite function. We follow the usual convention with parentheses by starting with the innermost parentheses first, and then working to the outside. In the equation above, the function g takes the input x first and yields an output $g(x)$. Then the function f takes $g(x)$ as an input and yields an output $f(g(x))$.

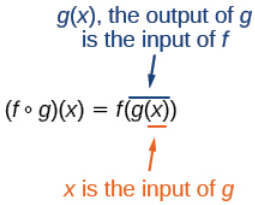


Figure **3.3.2**: Explanation of the composite function.

In general, $f \circ g$ and $g \circ f$ are different functions. In other words, in many cases $f(g(x)) \neq g(f(x))$ for all x. We will also see that sometimes two functions can be composed only in one specific order.

For example, if $f(x) = x^2$ and $g(x) = x + 2$, then

$$\begin{aligned} f(g(x)) &= f(x+2) \\ &= (x+2)^2 \\ &= x^2 + 4x + 4 \end{aligned}$$

but

$$\begin{aligned} g(f(x)) &= g(x^2) \\ &= x^2 + 2 \end{aligned}$$

These expressions are not equal for all values of x, so the two functions are not equal. It is irrelevant that the expressions happen to be equal for the single input value $x = -\frac{1}{2}$.

Note that the range of the inside function (the first function to be evaluated) needs to be within the domain of the outside function. Less formally, the composition has to make sense in terms of inputs and outputs.

Composition of Functions

When the output of one function is used as the input of another, we call the entire operation a composition of functions. For any input x and functions f and g, this action defines a **composite function**, which we write as $f \circ g$ such that

$$(f \circ g)(x) = f(g(x)) \tag{3.3.2}$$

The domain of the composite function $f \circ g$ is all x such that x is in the domain of g and $g(x)$ is in the domain of f.

It is important to realize that the product of functions fg is not the same as the function composition $f(g(x))$, because, in general, $f(x)g(x) \neq f(g(x))$.

Example 3.3.2: Determining whether Composition of Functions is Commutative

Using the functions provided, find $f(g(x))$ and $g(f(x))$. Determine whether the composition of the functions is **commutative**.

$$f(x) = 2x + 1 \quad g(x) = 3 - x$$

Solution

Let's begin by substituting $g(x)$ into $f(x)$.

$$\begin{aligned} f(g(x)) &= 2(3-x)+1 \\ &= 6-2x+1 \\ &= 7-2x \end{aligned}$$

Now we can substitute $f(x)$ into $g(x)$.

$$\begin{aligned} g(f(x)) &= 3-(2x+1) \\ &= 3-2x-1 \\ &= 2-2x \end{aligned}$$

We find that $g(f(x)) \neq f(g(x))$, so the operation of function composition is not commutative.

Example 3.3.3: Interpreting Composite Functions

The function $c(s)$ gives the number of calories burned completing s sit-ups, and $s(t)$ gives the number of sit-ups a person can complete in t minutes. Interpret $c(s(3))$.

Solution

The inside expression in the composition is $s(3)$. Because the input to the s-function is time, $t=3$ represents 3 minutes, and $s(3)$ is the number of sit-ups completed in 3 minutes.

Using $s(3)$ as the input to the function $c(s)$ gives us the number of calories burned during the number of sit-ups that can be completed in 3 minutes, or simply the number of calories burned in 3 minutes (by doing sit-ups).

Example 3.3.4: Investigating the Order of Function Composition

Suppose $f(x)$ gives miles that can be driven in x hours and $g(y)$ gives the gallons of gas used in driving y miles. Which of these expressions is meaningful: $f(g(y))$ or $g(f(x))$?

Solution

The function $y=f(x)$ is a function whose output is the number of miles driven corresponding to the number of hours driven.

$$\text{number of miles } = f(\text{number of hours})$$

The function $g(y)$ is a function whose output is the number of gallons used corresponding to the number of miles driven. This means:

$$\text{number of gallons } = g(\text{number of miles})$$

The expression $g(y)$ takes miles as the input and a number of gallons as the output. The function $f(x)$ requires a number of hours as the input. Trying to input a number of gallons does not make sense. The expression $f(g(y))$ is meaningless.

The expression $f(x)$ takes hours as input and a number of miles driven as the output. The function $g(y)$ requires a number of miles as the input. Using $f(x)$ (miles driven) as an input value for $g(y)$, where gallons of gas depends on miles driven, does make sense. The expression $g(f(x))$ makes sense, and will yield the number of gallons of gas used, g, driving a certain number of miles, $f(x)$, in x hours.

Question/Answer

Are there any situations where $f(g(y))$ and $g(f(x))$ would both be meaningful or useful expressions?

Yes. For many pure mathematical functions, both compositions make sense, even though they usually produce different new functions. In real-world problems, functions whose inputs and outputs have the same units also may give compositions that are meaningful in either order

Exercise 3.3.2

The gravitational force on a planet a distance r from the sun is given by the function $G(r)$. The acceleration of a planet subjected to any force F is given by the function $a(F)$. Form a meaningful composition of these two functions, and explain what it means.

Answer

A gravitational force is still a force, so $a(G(r))$ makes sense as the acceleration of a planet at a distance r from the Sun (due to gravity), but $G(a(F))$ does not make sense.

Evaluating Composite Functions

Once we compose a new function from two existing functions, we need to be able to evaluate it for any input in its domain. We will do this with specific numerical inputs for functions expressed as tables, graphs, and formulas and with variables as inputs to functions expressed as formulas. In each case, we evaluate the inner function using the starting input and then use the inner function's output as the input for the outer function.

Evaluating Composite Functions Using Tables

When working with functions given as tables, we read input and output values from the table entries and always work from the inside to the outside. We evaluate the inside function first and then use the output of the inside function as the input to the outside function.

Example 3.3.5: Using a Table to Evaluate a Composite Function

Using Table 3.3.1, evaluate $f(g(3))$ and $g(f(3))$.

Table 3.3.1

x	$f(x)$	$g(x)$
1	6	3
2	8	5
3	3	2
4	1	7

Solution

To evaluate $f(g(3))$, we start from the inside with the input value 3. We then evaluate the inside expression $g(3)$ using the table that defines the function g : $g(3) = 2$. We can then use that result as the input to the function f, so $g(3)$ is replaced by 2 and we get $f(2)$. Then, using the table that defines the function f, we find that $f(2) = 8$.

$$g(3) = 2$$

$$f(g(3)) = f(2) = 8$$

To evaluate $g(f(3))$, we first evaluate the inside expression $f(3)$ using the first table: $f(3) = 3$. Then, using the table for g, we can evaluate

$$g(f(3)) = g(3) = 2$$

Table 3.3.2 shows the composite functions $f \circ g$ and $g \circ f$ as tables.

Table 3.3.2

x	$g(x)$	$f(g(x))$	$f(x)$	$g(f(x))$
3	2	8	3	2

? Exercise 3.3.3

Using Table 3.3.1, evaluate $f(g(1))$ and $g(f(4))$.

Answer

$f(g(1)) = f(3) = 3$ and $g(f(4)) = g(1) = 3$

Evaluating Composite Functions Using Graphs

When we are given individual functions as graphs, the procedure for evaluating composite functions is similar to the process we use for evaluating tables. We read the input and output values, but this time, from the x- and y-axes of the graphs.

How To ...

Given a composite function and graphs of its individual functions, evaluate it using the information provided by the graphs.

1. Locate the given input to the inner function on the x-axis of its graph.
2. Read off the output of the inner function from the y-axis of its graph.
3. Locate the inner function output on the x-axis of the graph of the outer function.
4. Read the output of the outer function from the y-axis of its graph. This is the output of the composite function.

✓ Example 3.3.6: Using a Graph to Evaluate a Composite Function

Using Figure 3.3.3, evaluate $f(g(1))$.

Figure 3.3.3: Two graphs of a positive and negative parabola.

Solution

To evaluate $f(g(1))$, we start with the inside evaluation. See Figure 3.3.4.

Figure 3.3.4: Two graphs of a positive parabola $g(x)$ and a negative parabola $f(x)$. The following points are plotted: $g(1) = 3$ and $f(3) = 6$.

We evaluate $g(1)$ using the graph of $g(x)$, finding the input of 1 on the x-axis and finding the output value of the graph at that input. Here, $g(1) = 3$. We use this value as the input to the function f.

$$f(g(1)) = f(3)$$

We can then evaluate the composite function by looking to the graph of $f(x)$, finding the input of 3 on the x-axis and reading the output value of the graph at this input. Here, $f(3) = 6$, so $f(g(1)) = 6$.

Analysis

Figure 3.3.5 shows how we can mark the graphs with arrows to trace the path from the input value to the output value.

Figure 3.3.5: Two graphs of a positive and negative parabola.

? Exercise 3.3.4

Using Figure 3.3.3, evaluate $g(f(2))$.

Answer

$g(f(2)) = g(5) = 3$

Evaluating Composite Functions Using Formulas

When evaluating a composite function where we have either created or been given formulas, the rule of working from the inside out remains the same. The input value to the outer function will be the output of the inner function, which may be a numerical value, a variable name, or a more complicated expression.

While we can compose the functions for each individual input value, it is sometimes helpful to find a single formula that will calculate the result of a composition $f(g(x))$. To do this, we will extend our idea of function evaluation. Recall that, when we evaluate a function like $f(t) = t^2 - t$, we substitute the value inside the parentheses into the formula wherever we see the input variable.

How To...

Given a formula for a composite function, evaluate the function.

1. Evaluate the inside function using the input value or variable provided.
2. Use the resulting output as the input to the outside function.

Example 3.3.7: Evaluating a Composition of Functions Expressed as Formulas with a Numerical Input

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate $f(h(1))$.

Solution

Because the inside expression is $h(1)$, we start by evaluating $h(x)$ at 1.

$$h(1) = 3(1) + 2$$
$$h(1) = 5$$

Then $f(h(1)) = f(5)$, so we evaluate $f(t)$ at an input of 5.

$$f(h(1)) = f(5)$$
$$f(h(1)) = 5^2 - 5$$
$$f(h(1)) = 20$$

Analysis

It makes no difference what the input variables t and x were called in this problem because we evaluated for specific numerical values.

Exercise 3.3.5

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate

a. $h(f(2))$
b. $h(f(-2))$

Answer a

8

Answer b

20

Key Equation

- Composite function $(f \circ g)(x) = f(g(x))$

Key Concepts

- We can perform algebraic operations on functions. See Example.
- When functions are combined, the output of the first (inner) function becomes the input of the second (outer) function.
- The function produced by combining two functions is a composite function. See Example and Example.
- The order of function composition must be considered when interpreting the meaning of composite functions. See Example.
- A composite function can be evaluated by evaluating the inner function using the given input value and then evaluating the outer function taking as its input the output of the inner function.
- A composite function can be evaluated from a table. See Example.
- A composite function can be evaluated from a graph. See Example.
- A composite function can be evaluated from a formula. See Example.

Glossary

composite function

the new function formed by function composition, when the output of one function is used as the input of another

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3.4: Transformation of Functions

Learning Objectives

- Graph functions using vertical and horizontal shifts.
- Graph functions using reflections about the x-axis.
- Graph functions using compressions and stretches.
- Combine transformations.

We all know that a flat mirror enables us to see an accurate image of ourselves and whatever is behind us. When we tilt the mirror, the images we see may shift horizontally or vertically. But what happens when we bend a flexible mirror? Like a carnival funhouse mirror, it presents us with a distorted image of ourselves, stretched or compressed horizontally or vertically. In a similar way, we can distort or transform mathematical functions to better adapt them to describing objects or processes in the real world. In this section, we will take a look at several kinds of transformations.



Figure **3.4.1**: (credit: "Misko"/Flickr)

Often when given a problem, we try to model the scenario using mathematics in the form of words, tables, graphs, and equations. One method we can employ is to adapt the basic graphs of the toolkit functions to build new models for a given scenario. There are systematic ways to alter functions to construct appropriate models for the problems we are trying to solve.

Identifying Vertical Shifts

One simple kind of transformation involves shifting the entire graph of a function up, down, right, or left. The simplest shift is a vertical shift, moving the graph up or down, because this transformation involves adding a positive or negative constant to the function. In other words, we add the same constant to the output value of the function regardless of the input. For a function $g(x) = f(x) + k$, the function $f(x)$ is shifted vertically k units. See Figure **3.4.2** for an example.

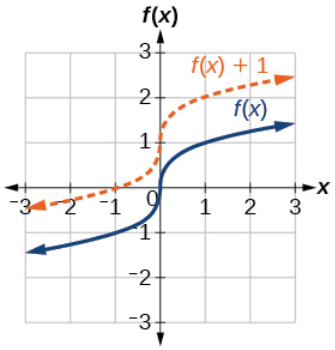


Figure **3.4.2**: Vertical shift by $k = 1$ of the cube root function $f(x) = \sqrt[3]{x}$.

To help you visualize the concept of a vertical shift, consider that $y = f(x)$. Therefore, $f(x) + k$ is equivalent to $y + k$. Every unit of y is replaced by $y + k$, so the y-value increases or decreases depending on the value of k. The result is a shift upward or downward.

Definition: Vertical Shift

Given a function $f(x)$, a new function $g(x) = f(x) + k$, where k is a constant, is a **vertical shift** of the function $f(x)$. All the output values change by k units. If k is positive, the graph will shift up. If k is negative, the graph will shift down.

Example 3.4.1: Adding a Constant to a Function

To regulate temperature in a green building, airflow vents near the roof open and close throughout the day. Figure 3.4.3 shows the area of open vents V (in square feet) throughout the day in hours after midnight, t. During the summer, the facilities manager decides to try to better regulate temperature by increasing the amount of open vents by 20 square feet throughout the day and night. Sketch a graph of this new function.

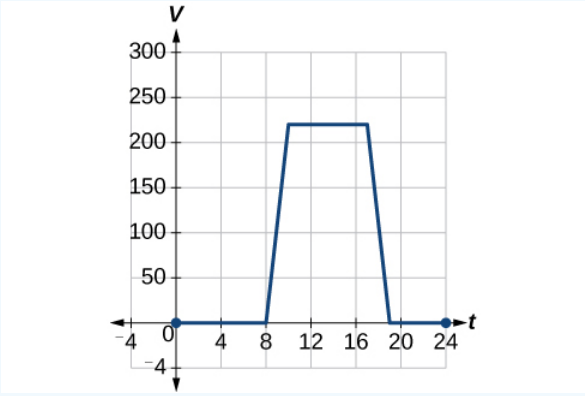


Figure 3.4.3

Solution

We can sketch a graph of this new function by adding 20 to each of the output values of the original function. This will have the effect of shifting the graph vertically up, as shown in Figure 3.4.4.

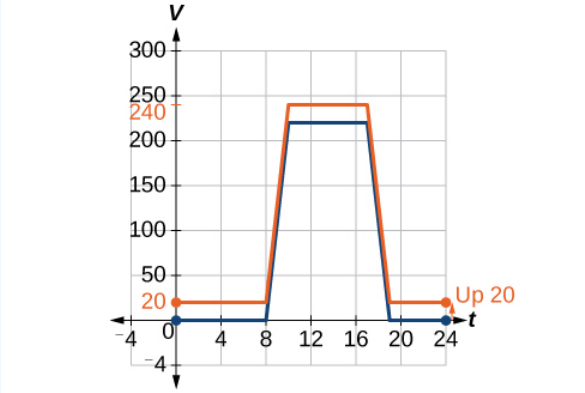


Figure 3.4.4

Notice that in Figure 3.4.4, for each input value, the output value has increased by 20, so if we call the new function $S(t)$,we could write

$$S(t) = V(t) + 20 \tag{3.4.1}$$

This notation tells us that, for any value of t,$S(t)$ can be found by evaluating the function V at the same input and then adding 20 to the result. This defines S as a transformation of the function V, in this case a vertical shift up 20 units. Notice that, with a vertical shift, the input values stay the same and only the output values change. See Table 3.4.1.

Table 3.4.1

t	0	8	10	17	19	24
$V(t)$	0	0	220	220	0	0
$S(t)$	20	20	240	240	20	20

How To...

Given a tabular function, create a new row to represent a vertical shift.

1. Identify the output row or column.
2. Determine the **magnitude** of the shift.
3. Add the shift to the value in each output cell. Add a positive value for up or a negative value for down.

Example 3.4.2: Shifting a Tabular Function Vertically

A function $f(x)$ is given in Table 3.4.2. Create a table for the function $g(x) = f(x) - 3$.

Table 3.4.2

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = f(x) - 3$ tells us that we can find the output values of g by subtracting 3 from the output values of f. For example:

$$\begin{aligned} f(x) &= 1 && \text{Given} \\ g(x) &= f(x) - 3 && \text{Given Transformation} \\ g(2) &= f(2) - 3 \\ &= 1 - 3 \\ &= -2 \end{aligned}$$

Subtracting 3 from each $f(x)$ value, we can complete a table of values for $g(x)$ as shown in Table 3.4.3.

Table 3.4.3

x	2	4	6	8
$f(x)$	1	3	7	11
$g(x)$	-2	0	4	8

Analysis

As with the earlier vertical shift, notice the input values stay the same and only the output values change.

Exercise 3.4.1

The function $h(t) = -4.9t^2 + 30t$ gives the height h of a ball (in meters) thrown upward from the ground after t seconds. Suppose the ball was instead thrown from the top of a 10-m building. Relate this new height function $b(t)$ to $h(t)$, and then find a formula for $b(t)$.

Answer

$$b(t) = h(t) + 10 = -4.9t^2 + 30t + 10$$

Identifying Horizontal Shifts

We just saw that the vertical shift is a change to the output, or outside, of the function. We will now look at how changes to input, on the inside of the function, change its graph and meaning. A shift to the input results in a movement of the graph of the function left or right in what is known as a **horizontal shift**, shown in Figure 3.4.4.

Figure 3.4.4: Horizontal shift of the function $f(x) = \sqrt[3]{x}$. Note that $h = -1$ shifts the graph to the left, that is, towards negative values of x.

For example, if $f(x) = x^2$, then $g(x) = (x-2)^2$ is a new function. Each input is reduced by 2 prior to squaring the function. The result is that the graph is shifted 2 units to the right, because we would need to increase the prior input by 2 units to yield the same output value as given in f.

Definition: Horizontal Shift

Given a function f, a new function $g(x) = f(x-h)$, where h is a constant, is a **horizontal shift** of the function f. If h is positive, the graph will shift right. If h is negative, the graph will shift left.

Example 3.4.4: Adding a Constant to an Input

Returning to our building airflow example from Figure 3.4.2, suppose that in autumn the facilities manager decides that the original venting plan starts too late, and wants to begin the entire venting program 2 hours earlier. Sketch a graph of the new function.

Solution

We can set $V(t)$ to be the original program and $F(t)$ to be the revised program.

$$V(t) = \text{ the original venting plan}$$

$$F(t) = \text{ starting 2 hrs sooner}$$

In the new graph, at each time, the airflow is the same as the original function V was 2 hours later. For example, in the original function V, the airflow starts to change at 8 a.m., whereas for the function F, the airflow starts to change at 6 a.m. The comparable function values are $V(8) = F(6)$. See Figure 3.4.5. Notice also that the vents first opened to 220ft^2 at 10 a.m. under the original plan, while under the new plan the vents reach 220ft^2 at 8 a.m., so $V(10) = F(8)$.

In both cases, we see that, because $F(t)$ starts 2 hours sooner, $h = -2$. That means that the same output values are reached when $F(t) = V(t-(-2)) = V(t+2)$.

Figure 3.4.5

Analysis

Note that $V(t+2)$ has the effect of shifting the graph to the left.

Horizontal changes or "inside changes" affect the domain of a function (the input) instead of the range and often seem counterintuitive. The new function $F(t)$ uses the same outputs as $V(t)$, but matches those outputs to inputs 2 hours earlier than those of $V(t)$. Said another way, we must add 2 hours to the input of V to find the corresponding output for F : $F(t) = V(t+2)$.

How To...

Given a tabular function, create a new row to represent a horizontal shift.

1. Identify the input row or column.
2. Determine the magnitude of the shift.
3. Add the shift to the value in each input cell.

Example 3.4.5: Shifting a Tabular Function Horizontally

A function $f(x)$ is given in Table 3.4.4. Create a table for the function $g(x) = f(x-3)$.

Table 3.4.4

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = f(x-3)$ tells us that the output values of g are the same as the output value of f when the input value is 3 less than the original value. For example, we know that $f(2) = 1$. To get the same output from the function g, we will need an input value that is 3 larger. We input a value that is 3 larger for $g(x)$ because the function takes 3 away before evaluating the function f.

$$\begin{aligned} g(5) &= f(5-3) \\ &= f(2) \\ &= 1 \end{aligned}$$

We continue with the other values to create Table 3.4.5.

Table 3.4.5

x	5	7	9	11
$x-3$	2	4	6	8
$f(x)$	1	3	7	11
$g(x)$	1	3	7	11

The result is that the function $g(x)$ has been shifted to the right by 3. Notice the output values for $g(x)$ remain the same as the output values for $f(x)$, but the corresponding input values, x, have shifted to the right by 3. Specifically, 2 shifted to 5, 4 shifted to 7, 6 shifted to 9, and 8 shifted to 11.

Analysis

Figure 3.4.6 represents both of the functions. We can see the horizontal shift in each point.

Figure 3.4.6: Graph of the points from the previous table for $f(x)$ and $g(x) = f(x-3)$

Example 3.4.6: Identifying a Horizontal Shift of a Toolkit Function

Figure 3.4.7 represents a transformation of the toolkit function $f(x) = x^2$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

Figure 3.4.7: Graph of a parabola

Solution

Notice that the graph is identical in shape to the $f(x) = x^2$ function, but the x-values are shifted to the right 2 units. The vertex used to be at $(0,0)$, but now the vertex is at $(2,0)$. The graph is the basic quadratic function shifted 2 units to the right, so

$$g(x) = f(x-2)$$

Notice how we must input the value $x = 2$ to get the output value $y = 0$; the x-values must be 2 units larger because of the shift to the right by 2 units. We can then use the definition of the $f(x)$ function to write a formula for $g(x)$ by evaluating $f(x-2)$.

$$\begin{aligned} f(x) &= x^2 \\ g(x) &= f(x-2) \\ g(x) &= f(x-2) = (x-2)^2 \end{aligned}$$

Analysis

To determine whether the shift is $+2$ or -2, consider a single reference point on the graph. For a quadratic, looking at the vertex point is convenient. In the original function, $f(0) = 0$. In our shifted function, $g(2) = 0$. To obtain the output value of 0 from the function f, we need to decide whether a plus or a minus sign will work to satisfy $g(2) = f(x - 2) = f(0) = 0$. For this to work, we will need to subtract 2 units from our input values.

Example 3.4.7: Interpreting Horizontal versus Vertical Shifts

The function $G(m)$ gives the number of gallons of gas required to drive m miles. Interpret $G(m) + 10$ and $G(m + 10)$

Solution

$G(m) + 10$ can be interpreted as adding 10 to the output, gallons. This is the gas required to drive m miles, plus another 10 gallons of gas. The graph would indicate a vertical shift.

$G(m + 10)$ can be interpreted as adding 10 to the input, miles. So this is the number of gallons of gas required to drive 10 miles more than m miles. The graph would indicate a horizontal shift.

Exercise 3.4.7

Given the function $f(x) = \sqrt{x}$, graph the original function $f(x)$ and the transformation $g(x) = f(x + 2)$ on the same axes. Is this a horizontal or a vertical shift? Which way is the graph shifted and by how many units?

Answer

The graphs of $f(x)$ and $g(x)$ are shown below. The transformation is a horizontal shift. The function is shifted to the left by 2 units.

Figure 3.4.8

Combining Vertical and Horizontal Shifts

Now that we have two transformations, we can combine them together. Vertical shifts are outside changes that affect the output $(y-)$ axis values and shift the function up or down. Horizontal shifts are inside changes that affect the input $(x-)$ axis values and shift the function left or right. Combining the two types of shifts will cause the graph of a function to shift up or down and right or left.

How To...

Given a function and both a vertical and a horizontal shift, sketch the graph.

1. Identify the vertical and horizontal shifts from the formula.
2. The vertical shift results from a constant added to the output. Move the graph up for a positive constant and down for a negative constant.
3. The horizontal shift results from a constant added to the input. Move the graph left for a positive constant and right for a negative constant.
4. Apply the shifts to the graph in either order.

Example 3.4.8: Graphing Combined Vertical and Horizontal Shifts

Given $f(x) = |x|$, sketch a graph of $h(x) = f(x + 1) - 3$.

Solution

The function f is our toolkit absolute value function. We know that this graph has a V shape, with the point at the origin. The graph of h has transformed f in two ways: $f(x + 1)$ is a change on the inside of the function, giving a horizontal shift left by 1, and the subtraction by 3 in $f(x + 1) - 3$ is a change to the outside of the function, giving a vertical shift down by 3. The transformation of the graph is illustrated in Figure 3.4.9.

Let us follow one point of the graph of $f(x) = |x|$.

- The point $(0, 0)$ is transformed first by shifting left 1 unit:$(0, 0) \rightarrow (-1, 0)$
- The point $(-1, 0)$ is transformed next by shifting down 3 units:$(-1, 0) \rightarrow (-1, -3)$

Figure 3.4.9: Graph of an absolute function, $y = |x|$, and how it was transformed to $y = |x + 1| - 3$

Figure 3.4.10 shows the graph of h.

Figure 3.4.10: The final function $y = |x + 1| - 3$.

? Exercise 3.4.8

Given $f(x) = |x|$, sketch a graph of $h(x) = f(x - 2) + 4$.

Answer

Figure 3.4.11

✓ Example 3.4.9: Identifying Combined Vertical and Horizontal Shifts

Write a formula for the graph shown in Figure 3.4.12, which is a transformation of the toolkit square root function.

Figure 3.4.12: Graph of a square root function transposed right one unit and up 2.

Solution

The graph of the toolkit function starts at the origin, so this graph has been shifted 1 to the right and up 2. In function notation, we could write that as

$$h(x) = f(x - 1) + 2$$

Using the formula for the square root function, we can write

$$h(x) = \sqrt{x - 1} + 2$$

Analysis

Note that this transformation has changed the domain and range of the function. This new graph has domain $[1, \infty)$ and range $[2, \infty)$.

? Exercise 3.4.9

Write a formula for a transformation of the toolkit reciprocal function $f(x) = \frac{1}{x}$ that shifts the function's graph one unit to the right and one unit up.

Answer

$$g(x) = \frac{1}{x - 1} + 1$$

Graphing Functions Using Reflections about the Axes

Another transformation that can be applied to a function is a reflection over the x- or y-axis. A **vertical reflection** reflects a graph vertically across the x-axis, while a **horizontal reflection** reflects a graph horizontally across the y-axis. The reflections are shown in Figure 3.4.13.

.

Figure 3.4.13: Graph of the vertical and horizontal reflection of a function.

Notice that the vertical reflection produces a new graph that is a mirror image of the base or original graph about the x-axis. The horizontal reflection produces a new graph that is a mirror image of the base or original graph about the y-axis.

Definitions: Reflections

Given a function $f(x)$, a new function $g(x) = -f(x)$ is a **vertical reflection** of the function $f(x)$, sometimes called a reflection about (or over, or through) the x-axis.

How To...

Given a function, reflect the graph both vertically and horizontally.

1. Multiply all outputs by –1 for a vertical reflection. The new graph is a reflection of the original graph about the x-axis.

Example 3.4.10: Reflecting a Graph Vertically

Reflect the graph of $s(t) = \sqrt{t}$ vertically.

Solution

a. Reflecting the graph vertically means that each output value will be reflected over the horizontal t-axis as shown in Figure **3.4.14**.

Figure **3.4.14**: Graph of the vertical reflection of the square root function.

Because each output value is the opposite of the original output value, we can write

$$V(t) = -s(t) \text{ or } V(t) = -\sqrt{t}$$

Notice that this is an outside change, or vertical shift, that affects the output $s(t)$ values, so the negative sign belongs outside of the function.

Exercise 3.4.5

Reflect the graph of $f(x) = |x - 1|$ vertically

Answer

a.

Figure **3.4.16**: Graph of a vertically reflected absolute function.

b.

Figure **3.4.17**: Graph of an absolute function translated one unit left.

Example 3.4.11: Reflecting a Tabular Function Horizontally and Vertically

A function $f(x)$ is given as Table **3.4.6**. Create a table for the functions below.

$g(x) = -f(x)$

Table **3.4.6**

x	2	4	6	8
$f(x)$	1	3	7	11

For $g(x)$, the negative sign outside the function indicates a vertical reflection, so the x-values stay the same and each output value will be the opposite of the original output value. See Table **3.4.7**.

Table 3.4.7

x	2	4	6	8
$g(x)$	-1	-3	-7	-11

Exercise 3.4.6

A function $f(x)$ is given as Table 3.4.9. Create a table for the functions below.

$g(x) = -f(x)$

Table 3.4.9

x	-2	0	2	4
$f(x)$	5	10	15	20

Answer

a. $g(x) = -f(x)$

Table 3.4.10

x	-2	0	2	4
$g(x)$	-5	-10	-15	-20

Exercise 3.4.7

Given the toolkit function $f(x) = x^2$, graph $g(x) = -f(x)$. Take note of any surprising behavior for these functions.

Answer

Figure 3.4.20: Graph of x^2 and its reflections.

Graphing Functions Using Stretches and Compressions

Adding a constant to the inputs or outputs of a function changed the position of a graph with respect to the axes, but it did not affect the shape of a graph. We now explore the effects of multiplying the inputs or outputs by some quantity.

We can transform the inside (input values) of a function or we can transform the outside (output values) of a function. Each change has a specific effect that can be seen graphically.

Vertical Stretches and Compressions

When we multiply a function by a positive constant, we get a function whose graph is stretched or compressed vertically in relation to the graph of the original function. If the constant is greater than 1, we get a **vertical stretch**; if the constant is between 0 and 1, we get a **vertical compression**. Figure 3.4.23 shows a function multiplied by constant factors 2 and 0.5 and the resulting vertical stretch and compression.

Figure 3.4.23: Vertical stretch and compression

Definitions: Vertical Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = af(x)$, where a is a constant, is a **vertical stretch** or **vertical compression** of the function $f(x)$.

- If $a > 1$, then the graph will be stretched.
- If $0 < a < 1$, then the graph will be compressed.
- If $a < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection.

How To...

Given a function, graph its vertical stretch.

a. Identify the value of a.
b. Multiply all range values by a
c. If $a > 1$, the graph is stretched by a factor of a.
d. If $0 < a < 1$, the graph is compressed by a factor of a.
e. If $a < 0$, the graph is either stretched or compressed and also reflected about the x-axis.

Example 1.5.14: Graphing a Vertical Stretch

A function $P(t)$ models the population of fruit flies. The graph is shown in Figure 3.4.24.

Figure 3.4.24: Graph to represent the growth of the population of fruit flies.

A scientist is comparing this population to another population, Q, whose growth follows the same pattern, but is twice as large. Sketch a graph of this population.

Solution

Because the population is always twice as large, the new population's output values are always twice the original function's output values. Graphically, this is shown in Figure 3.4.25.

If we choose four reference points, $(0, 1)$, $(3, 3)$, $(6, 2)$ and $(7, 0)$ we will multiply all of the outputs by 2.

The following shows where the new points for the new graph will be located.

$$(0, 1) \to (0, 2) \tag{3.4.2}$$

$$(3, 3) \to (3, 6) \tag{3.4.3}$$

$$(6, 2) \to (6, 4) \tag{3.4.4}$$

$$(7, 0) \to (7, 0) \tag{3.4.5}$$

Figure 3.4.25: Graph of the population function doubled.

Symbolically, the relationship is written as

$$Q(t) = 2P(t)$$

This means that for any input t, the value of the function Q is twice the value of the function P. Notice that the effect on the graph is a vertical stretching of the graph, where every point doubles its distance from the horizontal axis. The input values, t, stay the same while the output values are twice as large as before.

How To...

Given a tabular function and assuming that the transformation is a vertical stretch or compression, create a table for a vertical compression.

1. Determine the value of a.
2. Multiply all of the output values by a.

Example 3.4.15: Finding a Vertical Compression of a Tabular Function

A function f is given as Table 3.4.12. Create a table for the function $g(x) = \frac{1}{2}f(x)$.

Table 3.4.12

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = \frac{1}{2}f(x)$ tells us that the output values of g are half of the output values of f with the same inputs. For example, we know that $f(4) = 3$. Then

$$g(4) = \frac{1}{2}f(4) = \frac{1}{2}(3) = \frac{3}{2}$$

We do the same for the other values to produce Table 3.4.13.

Table 3.4.13

x	2	4	6	8
$g(x)$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{7}{2}$	$\frac{11}{2}$

Analysis

The result is that the function $g(x)$ has been compressed vertically by $\frac{1}{2}$. Each output value is divided in half, so the graph is half the original height.

Exercise 3.4.9

A function f is given as Table 3.4.14. Create a table for the function $g(x) = \frac{3}{4}f(x)$.

Table 3.4.14

x	2	4	6	8
$f(x)$	12	16	20	0

Answer

Table 3.4.15

x	2	4	6	8
$g(x)$	9	12	15	0

Example 3.4.16: Recognizing a Vertical Stretch

The graph in Figure 3.4.26 is a transformation of the toolkit function $f(x) = x^3$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

Figure 3.4.26: Graph of a transformation of $f(x) = x^3$

When trying to determine a vertical stretch or shift, it is helpful to look for a point on the graph that is relatively clear. In this graph, it appears that $g(2) = 2$. With the basic cubic function at the same input, $f(2) = 2^3 = 8$. Based on that, it appears that the outputs of g are $\frac{1}{4}$ the outputs of the function f because $g(2) = \frac{1}{4}f(2)$. From this we can fairly safely conclude that $g(x) = \frac{1}{4}f(x)$.

We can write a formula for g by using the definition of the function f.

$$g(x) = \frac{1}{4}f(x) = \frac{1}{4}x^3. \tag{3.4.6}$$

Exercise 3.4.1

Write the formula for the function that we get when we stretch the identity toolkit function by a factor of 3, and then shift it down by 2 units.

Answer

$g(x) = 3x - 2$

Performing a Sequence of Transformations

When combining transformations, it is very important to consider the order of the transformations. For example, vertically shifting by 3 and then vertically stretching by 2 does not create the same graph as vertically stretching by 2 and then vertically shifting by 3, because when we shift first, both the original function and the shift get stretched, while only the original function gets stretched when we stretch first.

When we see an expression such as $2f(x) + 3$, which transformation should we start with? The answer here follows nicely from the order of operations. Given the output value of $f(x)$, we first multiply by 2, causing the vertical stretch, and then add 3, causing the vertical shift. In other words, multiplication before addition.

Combining Transformations

- When combining vertical transformations written in the form $af(x) + k$, first vertically stretch by a and then vertically shift by k.
- Horizontal and vertical transformations are independent. It does not matter whether horizontal or vertical transformations are performed first.

Example 3.4.20: Finding a Triple Transformation of a Tabular Function

Given Table 3.4.18 for the function $f(x)$, create a table of values for the function $g(x) = 2f(3x) + 1$.

Table 3.4.18

x	6	12	18	24
$f(x)$	10	14	15	17

Solution

There are three steps to this transformation, and we will work from the inside out. Starting with the horizontal transformations, $f(3x)$ is a horizontal compression by $\frac{1}{3}$, which means we multiply each x-value by $\frac{1}{3}$.See Table 3.4.19.

Table 3.4.19

x	2	4	6	8
$f(3x)$	10	14	15	17

Looking now to the vertical transformations, we start with the vertical stretch, which will multiply the output values by 2. We apply this to the previous transformation. See Table 3.4.20.

Table 3.4.20

x	2	4	6	8
$2f(3x)$	20	28	30	34

Finally, we can apply the vertical shift, which will add 1 to all the output values. See Table 3.4.21.

Table 3.4.21

x	2	4	6	8
$g(x)=2f(3x)+1+1$	21	29	31	35

Key Equations

- Vertical shift $g(x)=f(x)+k$ (up for $k>0$)
- Horizontal shift $g(x)=f(x-h)$(right) for $h>0$
- Vertical reflection $g(x)=-f(x)$
- Horizontal reflection $g(x)=f(-x)$
- Vertical stretch $g(x)=af(x)$ (a>0)
- Vertical compression $g(x)=af(x)$ (0<a<1)

Key Concepts

- A function can be shifted vertically by adding a constant to the output.
- A function can be shifted horizontally by adding a constant to the input.
- Relating the shift to the context of a problem makes it possible to compare and interpret vertical and horizontal shifts.
- Vertical and horizontal shifts are often combined.
- A vertical reflection reflects a graph about the x-axis. A graph can be reflected vertically by multiplying the output by –1.
- A function presented in tabular form can also be reflected by multiplying the values in the output rows or columns accordingly.
- A function presented as an equation can be reflected by applying transformations one at a time.
- A function can be compressed or stretched vertically by multiplying the output by a constant.
- The order in which different transformations are applied does affect the final function. Both vertical and horizontal transformations must be applied in the order given. However, a vertical transformation may be combined with a horizontal transformation in any order.

Glossary

horizontal compression
a transformation that compresses a function’s graph horizontally, by multiplying the input by a constant b>1

horizontal shift
a transformation that shifts a function’s graph left or right by adding a positive or negative constant to the input

vertical compression
a function transformation that compresses the function’s graph vertically by multiplying the output by a constant 0<a<1

vertical reflection
a transformation that reflects a function’s graph across the x-axis by multiplying the output by −1

vertical shift
a transformation that shifts a function’s graph up or down by adding a positive or negative constant to the output

vertical stretch
a transformation that stretches a function’s graph vertically by multiplying the output by a constant a>1

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3.5: Absolute Value Functions

Learning Objectives

- Graph an absolute value function.
- Solve an absolute value equation.

Until the 1920s, the so-called spiral nebulae were believed to be clouds of dust and gas in our own galaxy, some tens of thousands of light years away. Then, astronomer Edwin Hubble proved that these objects are galaxies in their own right, at distances of millions of light years. Today, astronomers can detect galaxies that are billions of light years away. Distances in the universe can be measured in all directions. As such, it is useful to consider distance as an absolute value function. In this section, we will investigate **absolute value functions**.



Figure **3.5.1**: Distances in deep space can be measured in all directions. As such, it is useful to consider distance in terms of absolute values. (credit: "s58y"/Flickr)

Understanding Absolute Value

Recall that in its basic form $f(x) = |x|$, the **absolute value** function, is one of our toolkit functions. The absolute value function is commonly thought of as providing the distance the number is from zero on a number line. Algebraically, for whatever the input value is, the output is the value without regard to sign.

Absolute Value Function

The absolute value function can be defined as a piecewise function

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \tag{3.5.1}$$

Example 3.5.1: Determine a Number within a Prescribed Distance

Describe all values x within or including a distance of 4 from the number 5.

Solution

We want the distance between x and 5 to be less than or equal to 4. We can draw a number line, such as the one in , to represent the condition to be satisfied.

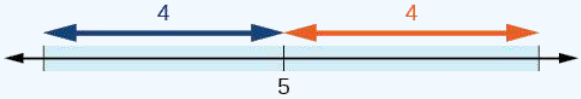


Figure **3.5.2**: Number line describing the difference of the distance of 4 away from 5.

The distance from x to 5 can be represented using the absolute value as $|x - 5|$. We want the values of x that satisfy the condition $|x - 5| \leq 4$.

Analysis

Note that

$$\begin{aligned} -4 \leq x-5 \qquad & x-5 \leq 4 \\ 1 \leq x \qquad & x \leq 9 \end{aligned}$$

So $|x-5| \leq 4$ is equivalent to $1 \leq x \leq 9$.

However, mathematicians generally prefer absolute value notation.

Exercise 3.5.1

Describe all values x within a distance of 3 from the number 2.

Answer

$|x-2| \leq 3$

Example 3.5.2: Resistance of a Resistor

Electrical parts, such as resistors and capacitors, come with specified values of their operating parameters: resistance, capacitance, etc. However, due to imprecision in manufacturing, the actual values of these parameters vary somewhat from piece to piece, even when they are supposed to be the same. The best that manufacturers can do is to try to guarantee that the variations will stay within a specified range, often ±1%, ±5%, or ±10%.

Suppose we have a resistor rated at 680 ohms, ±5%. Use the absolute value function to express the range of possible values of the actual resistance.

Solution

5% of 680 ohms is 34 ohms. The absolute value of the difference between the actual and nominal resistance should not exceed the stated variability, so, with the resistance R in ohms,

$$|R-680| \leq 34$$

Exercise 3.5.2

Students who score within 20 points of 80 will pass a test. Write this as a distance from 80 using absolute value notation.

Answer

Using the variable p for passing, $|p-80| \leq 20$

Graphing an Absolute Value Function

The most significant feature of the absolute value graph is the corner point at which the graph changes direction. This point is shown at the origin in Figure 3.5.3.

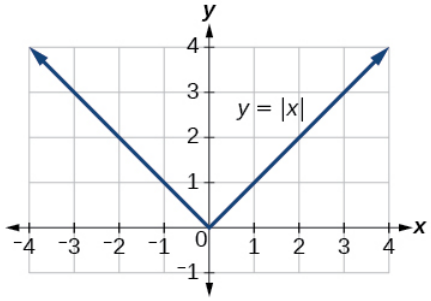


Figure 3.5.3: Graph of an absolute function.

Figure 3.5.3 shows the graph of $y = 2|x–3| + 4$. The graph of $y = |x|$ has been shifted right 3 units, vertically stretched by a factor of 2, and shifted up 4 units. This means that the corner point is located at $(3, 4)$ for this transformed function.

Figure 3.5.4: Graph of the different types of transformations for an absolute function.

Example 3.5.3: Writing an Equation for an Absolute Value Function

Write an equation for the function graphed in Figure 3.5.5.

Figure 3.5.5: Graph of an absolute function.

Solution

The basic absolute value function changes direction at the origin, so this graph has been shifted to the right 3 units and down 2 units from the basic toolkit function. See Figure 3.5.6.

Figure 3.5.6: Graph of two transformations for an absolute function at $(3, -2)$.

We also notice that the graph appears vertically stretched, because the width of the final graph on a horizontal line is not equal to 2 times the vertical distance from the corner to this line, as it would be for an unstretched absolute value function. Instead, the width is equal to 1 times the vertical distance as shown in Figure 3.5.7.

Figure 3.5.7: Graph of two transformations for an absolute function at $(3, -2)$ and the ratios between the two different transformations.

From this information we can write the equation

$$\begin{aligned} f(x) &= 2|x-3|-2, && \text{treating the stretch as a vertial stretch, or} \\ f(x) &= |2(x-3)|-2, && \text{treating the stretch as a horizontal compression.} \end{aligned}$$

Analysis

Note that these equations are algebraically equivalent—the stretch for an absolute value function can be written interchangeably as a vertical or horizontal stretch or compression.

Q & A

If we couldn't observe the stretch of the function from the graphs, could we algebraically determine it?

Answer

Yes. If we are unable to determine the stretch based on the width of the graph, we can solve for the stretch factor by putting in a known pair of values for x and $f(x)$.

$$f(x) = a|x-3|-2$$

Now substituting in the point $(1, 2)$

$$\begin{aligned} 2 &= a|1-3|-2 \\ 4 &= 2a \\ a &= 2 \end{aligned}$$

Exercise 3.5.3

Write the equation for the absolute value function that is horizontally shifted left 2 units, is vertically flipped, and vertically shifted up 3 units.

Answer

$f(x) = -|x+2|+3$

Q & A

Do the graphs of absolute value functions always intersect the vertical axis? The horizontal axis?

Answer

Yes, they always intersect the vertical axis. The graph of an absolute value function will intersect the vertical axis when the input is zero.

No, they do not always intersect the horizontal axis. The graph may or may not intersect the horizontal axis, depending on how the graph has been shifted and reflected. It is possible for the absolute value function to intersect the horizontal axis at zero, one, or two points (Figure 3.5.8).

Figure 3.5.8: (a) The absolute value function does not intersect the horizontal axis. (b) The absolute value function intersects the horizontal axis at one point. (c) The absolute value function intersects the horizontal axis at two points.

Solving an Absolute Value Equation

Now that we can graph an absolute value function, we will learn how to solve an absolute value equation. To solve an equation such as $8 = |2x - 6|$, we notice that the absolute value will be equal to 8 if the quantity inside the absolute value is 8 or -8. This leads to two different equations we can solve independently.

$$2x - 6 = 8 \quad \text{or} \quad 2x - 6 = -8$$

$$\begin{array}{ll} 2x = 14 & 2x = -2 \\ x = 7 & x = -1 \end{array}$$

Knowing how to solve problems involving absolute value functions is useful. For example, we may need to identify numbers or points on a line that are at a specified distance from a given reference point.

An **absolute value equation** is an equation in which the unknown variable appears in absolute value bars. For example,

$$\begin{aligned} |x| &= 4, \\ |2x - 1| &= 3, \\ |5x + 2| - 4 &= 9. \end{aligned}$$

Solutions to Absolute Value Equations

For real numbers A and B, an equation of the form $|A| = B$, with $B \geq 0$, will have solutions when $A = B$ or $A = -B$. If $B < 0$, the equation $|A| = B$ has no solution.

How To ...

Given the formula for an absolute value function, find the horizontal intercepts of its graph.

1. Isolate the absolute value term.
2. Use $|A| = B$ to write $A = B$ or $-A = B$, assuming $B > 0$.
3. Solve for x.

Example 3.5.4: Finding the Zeros of an Absolute Value Function

For the function $f(x) = |4x + 1| - 7$, find the values of x such that $f(x) = 0$.

Solution

$$
\begin{aligned}
0 &= |4x+1|-7 && && \text{Substitute 0 for f(x).}\\
7 &= |4x+1| && && \text{Isolate the absolute value on one side of the equation.}\\
7 &= 4x+1 && \text{or } -7 = 4x+1 && \text{Break into two separate equations and solve.}\\
6 &= 4x && -8 = 4x &&\\
x &= \frac{6}{4} = 1.5 && x = \frac{-8}{4} = -2 &&
\end{aligned}
$$

The function outputs 0 when $x = 1.5$ or $x = -2$ (Figure 3.5.9).

Figure 3.5.9: Graph of an absolute function with x-intercepts at -2 and 1.5.

Exercise 3.5.4

For the function $f(x) = |2x-1|-3$, find the values of x such that $f(x) = 0$.

Solution

$x = -1$ or $x = 2$

Q & A

Should we always expect two answers when solving $|A| = B$?

Answer

No. We may find one, two, or even no answers. For example, there is no solution to $2 + |3x-5| = 1$.

How To ...

Given an absolute value equation, solve it.

1. Isolate the absolute value term.
2. Use $|A| = B$ to write $A = B$ or $A = -B$.
3. Solve for x.

Example 3.5.5: Solving an Absolute Value Equation

Solve $1 = 4|x-2|+2$.

Solution

Isolating the absolute value on one side of the equation gives the following.

$$
\begin{aligned}
1 &= 4|x-2|+2\\
-1 &= 4|x-2|\\
-\frac{1}{4} &= |x-2|
\end{aligned}
$$

The absolute value always returns a positive value, so it is impossible for the absolute value to equal a negative value. At this point, we notice that this equation has no solutions.

Q & A

In Example 3.5.5, if $f(x) = 1$ and $g(x) = 4|x-2|+2$ were graphed on the same set of axes, would the graphs intersect?

Answer

No. The graphs of f and g would not intersect, as shown in Figure 3.5.10. This confirms, graphically, that the equation $1 = 4|x-2|+2$ has no solution.

Figure 3.5.10: Graph of $g(x) = 4|x-2|+2$ and $f(x) = 1$.

Find where the graph of the function $f(x) = -|x+2|+3$ intersects the horizontal and vertical axes.

$f(0) = 1$, so the graph intersects the vertical axis at $(0, 1)$. $f(x) = 0$ when $x = -5$ and $x = 1$ so the graph intersects the horizontal axis at $(-5, 0)$ and $(1, 0)$.

Solving an Absolute Value Inequality

Absolute value equations may not always involve equalities. Instead, we may need to solve an equation within a range of values. We would use an **absolute value inequality** to solve such an equation. An absolute value inequality is an equation of the form

$$|A| < B,\ |A| \leq B, |A| > B,$$

or

$$|A| \geq B,$$

where an expression A (and possibly but not usually B) depends on a variable x. Solving the inequality means finding the set of all x that satisfy the inequality. Usually this set will be an interval or the union of two intervals.

There are two basic approaches to solving absolute value inequalities: graphical and algebraic. The advantage of the graphical approach is we can read the solution by interpreting the graphs of two functions. The advantage of the algebraic approach is it yields solutions that may be difficult to read from the graph.

For example, we know that all numbers within 200 units of 0 may be expressed as

$$|x| < 200$$

or

$$-200 < x < 200$$

Suppose we want to know all possible returns on an investment if we could earn some amount of money within $200 of $600. We can solve algebraically for the set of values x such that the distance between x and 600 is less than 200. We represent the distance between x and 600 as $|x - 600|$.

$$|x - 600| < 200 \tag{3.5.2}$$

or

$$-200 < x - 600 < 200 \tag{3.5.3}$$

$$-200 + 600 < x - 600 + 600 < 200 + 600$$

$$400 < x < 800$$

This means our returns would be between $400 and $800.

Sometimes an absolute value inequality problem will be presented to us in terms of a shifted and/or stretched or compressed absolute value function, where we must determine for which values of the input the function's output will be negative or positive.

How To ...

Given an absolute value inequality of the form $|x - A| \leq B$ for real numbers a and b where b is positive, solve the absolute value inequality algebraically.

1. Find boundary points by solving $|x - A| = B$.
2. Test intervals created by the boundary points to determine where $|x - A| \leq B$.
3. Write the interval or union of intervals satisfying the inequality in interval, inequality, or set-builder notation.

Example 3.5.6: Solving an Absolute Value Inequality

Solve $|x - 5| \leq 4$.

Solution

With both approaches, we will need to know first where the corresponding equality is true. In this case we first will find where $|x-5|=4$. We do this because the absolute value is a function with no breaks, so the only way the function values can switch from being less than 4 to being greater than 4 is by passing through where the values equal 4. Solve $|x-5|=4$.

$$\begin{array}{lll} x-5=4 & \text{or} & x=9 \\ x-5=-4 & & x=1 \end{array}$$

After determining that the absolute value is equal to 4 at $x=1$ and $x=9$, we know the graph can change only from being less than 4 to greater than 4 at these values. This divides the number line up into three intervals:

$$x<1,\ 1<x<9,\ \text{and}\ x>9.$$

To determine when the function is less than 4, we could choose a value in each interval and see if the output is less than or greater than 4, as shown in Table 3.5.1.

Table 3.5.1

Interval test x	$f(x)$	<4 or >4	
$x<1$	0	$\|0-5\|=5$	Greater than
$1<x<9$	6	$\|6-5\|=1$	Less than
$x>9$	11	$\|11-5\|=6$	Greater than

Because $1\leq x\leq 9$ is the only interval in which the output at the test value is less than 4, we can conclude that the solution to $|x-5|\leq 4$ is $1\leq x\leq 9$, or $[1,9]$.

To use a graph, we can sketch the function $f(x)=|x-5|$. To help us see where the outputs are 4, the line $g(x)=4$ could also be sketched as in Figure 3.5.11.

Figure 3.5.11: Graph to find the points satisfying an absolute value inequality.

We can see the following:

- The output values of the absolute value are equal to 4 at $x=1$ and $x=9$.
- The graph of f is below the graph of g on $1<x<9$. This means the output values of $f(x)$ are less than the output values of $g(x)$.
- The absolute value is less than or equal to 4 between these two points, when $1\leq x\leq 9$. In interval notation, this would be the interval $[1,9]$.

Analysis

For absolute value inequalities,

$$\begin{array}{ll} |x-A|<C, & |x-A|>C, \\ -C<x-A<C, & x-A<-C \text{ or } x-A>C. \end{array}$$

The $<$ or $>$ symbol may be replaced by $\leq$ or $\geq$.

So, for this example, we could use this alternative approach.

$$\begin{array}{ll} |x-5|\leq 4 & \\ -4\leq x-5\leq 4 & \text{Rewrite by removing the absolute value bars.} \\ -4+5\leq x-5+5\leq 4+5 & \text{Isolate the x.} \\ 1\leq x\leq 9 & \end{array}$$

Exercise 3.5.5

Solve $|x+2|\leq 6$.

Answer

$-8\leq x\leq 4$

How To ...

Given an absolute value function, solve for the set of inputs where the output is positive (or negative).

1. Set the function equal to zero, and solve for the boundary points of the solution set.
2. Use test points or a graph to determine where the function's output is positive or negative.

Example 3.5.7: Using a Graphical Approach to Solve Absolute Value Inequalities

Given the function $f(x) = -\frac{1}{2}|4x - 5| + 3$, determine the x-values for which the function values are negative.

Solution

We are trying to determine where $f(x) < 0$, which is when $-\frac{1}{2}|4x - 5| + 3 < 0$. We begin by isolating the absolute value.

$$-\frac{1}{2}|4x - 5| < -3 \quad \text{Multiply both sides by –2, and reverse the inequality.}$$
$$|4x - 5| > 6$$

Next we solve for the equality $|4x - 5| = 6$.

$$4x - 5 = 6 \qquad 4x - 5 = -6$$
$$4x - 6 = 6$$

or

$$4x = -1$$
$$x = \frac{11}{4} \qquad x = -\frac{1}{4}$$

Now, we can examine the graph of f to observe where the output is negative. We will observe where the branches are below the x-axis. Notice that it is not even important exactly what the graph looks like, as long as we know that it crosses the horizontal axis at $x = -\frac{1}{4}$ and $x = \frac{11}{4}$ and that the graph has been reflected vertically. See Figure 3.5.12.

Figure 3.5.12: Graph of an absolute function with x-intercepts at -0.25 and 2.75.]

We observe that the graph of the function is below the x-axis left of $x = -\frac{1}{4}$ and right of $x = \frac{11}{4}$. This means the function values are negative to the left of the first horizontal intercept at $x = -\frac{1}{4}$, and negative to the right of the second intercept at $x = \frac{11}{4}$. This gives us the solution to the inequality.

$$x < -\frac{1}{4} \text{ or } x > 1\frac{1}{4}$$

In interval notation, this would be $(-\infty, -0.25) \cup (2.75, \infty)$.

Exercise 3.5.6

Solve $-2|k - 4| \leq -6$.

Answer

$k \leq 1$ or $k \geq 7$; in interval notation, this would be $(-\infty, 1] \cup [7, \infty)$

Key Concepts

- The absolute value function is commonly used to measure distances between points.
- Applied problems, such as ranges of possible values, can also be solved using the absolute value function.
- The graph of the absolute value function resembles a letter V. It has a corner point at which the graph changes direction.
- In an absolute value equation, an unknown variable is the input of an absolute value function.
- If the absolute value of an expression is set equal to a positive number, expect two solutions for the unknown variable.
- An absolute value equation may have one solution, two solutions, or no solutions.

- An absolute value inequality is similar to an absolute value equation but takes the form | A |<B, | A |≤B, | A |>B, or | A |≥B.It can be solved by determining the boundaries of the solution set and then testing which segments are in the set.
- Absolute value inequalities can also be solved graphically.

Glossary

absolute value equation
an equation of the form $|A| = B$, with $B \geq 0$; it will have solutions when $A = B$ or $A = -B$

absolute value inequality
a relationship in the form $|A| < B$, $|A| \leq B$, $|A| > B$, or $|A| \geq B$

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3.6: Inverse Functions

Learning Objectives

- Verify inverse functions.
- Determine the domain and range of an inverse function, and restrict the domain of a function to make it one-to-one.
- Find or evaluate the inverse of a function.
- Use the graph of a one-to-one function to graph its inverse function on the same axes.

A reversible heat pump is a climate-control system that is an air conditioner and a heater in a single device. Operated in one direction, it pumps heat out of a house to provide cooling. Operating in reverse, it pumps heat into the building from the outside, even in cool weather, to provide heating. As a heater, a heat pump is several times more efficient than conventional electrical resistance heating.

If some physical machines can run in two directions, we might ask whether some of the function "machines" we have been studying can also run backwards. Figure **3.6.1** provides a visual representation of this question. In this section, we will consider the reverse nature of functions.

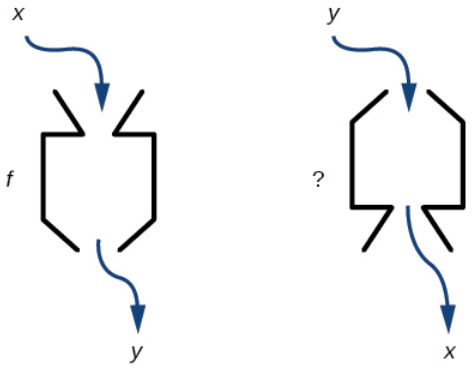


Figure* 3.6.1*: *Can a function "machine" operate in reverse?*

Verifying That Two Functions Are Inverse Functions

Suppose a fashion designer traveling to Milan for a fashion show wants to know what the temperature will be. He is not familiar with the **Celsius** scale. To get an idea of how temperature measurements are related, he asks his assistant, Betty, to convert 75 degrees **Fahrenheit** to degrees Celsius. She finds the formula

$$C = \frac{5}{9}(F - 32) \tag{3.6.1}$$

and substitutes 75 for F to calculate

$$\frac{5}{9}(75 - 32) \approx 24^\circ \tag{3.6.2}$$

Knowing that a comfortable 75 degrees Fahrenheit is about 24 degrees Celsius, he sends his assistant the week's weather forecast from Figure **3.6.2** for Milan, and asks her to convert all of the temperatures to degrees Fahrenheit.

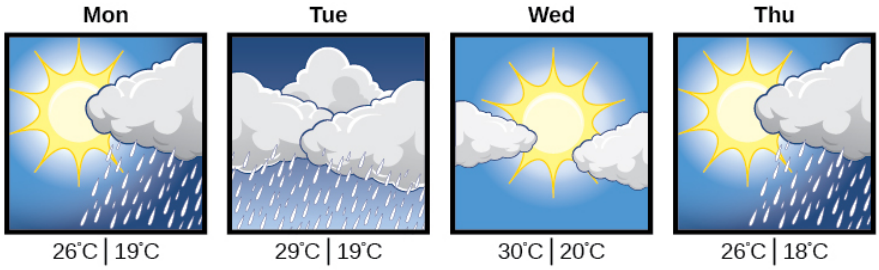


Figure **3.6.2**: A forecast of Monday's through Thursday's weather.

At first, Betty considers using the formula she has already found to complete the conversions. After all, she knows her algebra, and can easily solve the equation for F after substituting a value for C. For example, to convert 26 degrees Celsius, she could write

$$26 = \frac{5}{9}(F - 32) \tag{3.6.3}$$

$$26 \cdot \frac{9}{5} = F - 32 \tag{3.6.4}$$

$$F = 26 \cdot \frac{9}{5} + 32 \approx 79 \tag{3.6.5}$$

After considering this option for a moment, however, she realizes that solving the equation for each of the temperatures will be awfully tedious. She realizes that since evaluation is easier than solving, it would be much more convenient to have a different formula, one that takes the Celsius temperature and outputs the Fahrenheit temperature.

The formula for which Betty is searching corresponds to the idea of an **inverse function**, which is a function for which the input of the original function becomes the output of the inverse function and the output of the original function becomes the input of the inverse function.

Given a function $f(x)$, we represent its inverse as $f^{-1}(x)$, read as "f inverse of x." The raised -1 is part of the notation. It is not an exponent; it does not imply a power of -1 . In other words, $f^{-1}(x)$ does not mean $\frac{1}{f(x)}$ because $\frac{1}{f(x)}$ is the reciprocal of f and not the inverse.

The "exponent-like" notation comes from an analogy between function composition and multiplication: just as $a^{-1}a = 1$ (1 is the identity element for multiplication) for any nonzero number a, so $f^{-1} \circ f$ equals the identity function, that is,

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x \tag{3.6.6}$$

This holds for all x in the domain of f. Informally, this means that inverse functions "undo" each other. However, just as zero does not have a **reciprocal**, some functions do not have inverses.

Given a function $f(x)$, we can verify whether some other function $g(x)$ is the inverse of $f(x)$ by checking whether either $g(f(x)) = x$ or $f(g(x)) = x$ is true. We can test whichever equation is more convenient to work with because they are logically equivalent (that is, if one is true, then so is the other.)

For example, $y = 4x$ and $y = \frac{1}{4}x$ are inverse functions.

$$(f^{-1} \circ f)(x) = f^{-1}(4x) = \frac{1}{4}(4x) = x \tag{3.6.7}$$

and

$$(f \circ f^{-1})(x) = f\left(\frac{1}{4}x\right) = 4\left(\frac{1}{4}x\right) = x \tag{3.6.8}$$

A few coordinate pairs from the graph of the function $y = 4x$ are $(-2, -8)$, $(0, 0)$, and $(2, 8)$. A few coordinate pairs from the graph of the function $y = \frac{1}{4}x$ are $(-8, -2)$, $(0, 0)$, and $(8, 2)$. If we interchange the input and output of each coordinate pair of a function, the interchanged coordinate pairs would appear on the graph of the inverse function.

Definition: Inverse Function

For any **one-to-one function** $f(x) = y$, a function $f^{-1}(x)$ is an **inverse function** of f if $f^{-1}(y) = x$. This can also be written as $f^{-1}(f(x)) = x$ for all x in the domain of f. It also follows that $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1} if f^{-1} is the inverse of f.

The notation f^{-1} is read "f inverse." Like any other function, we can use any variable name as the input for f^{-1}, so we will often write $f^{-1}(x)$, which we read as "f inverse of x." Keep in mind that

$$f^{-1}(x) \neq \frac{1}{f(x)} \tag{3.6.9}$$

and not all functions have inverses.

Example 3.6.1: Identifying an Inverse Function for a Given Input-Output Pair

If for a particular one-to-one function $f(2) = 4$ and $f(5) = 12$, what are the corresponding input and output values for the inverse function?

Solution

The inverse function reverses the input and output quantities, so if

$$\begin{aligned} f(2) = 4, \text{ then } f^{-1}(4) = 2; \\ f(5) = 12, \text{ then } f^{-1}(12) = 5 \end{aligned} \tag{3.6.10}$$

.

Alternatively, if we want to name the inverse function g, then $g(4) = 2$ and $g(12) = 5$.

Analysis

Notice that if we show the coordinate pairs in a table form, the input and output are clearly reversed. See Table 3.6.1.

Table 3.6.1

$(x, f(x))$	$(x, g(x))$
$(2, 4)$	$(4, 2)$
$(5, 12)$	$(12, 5)$

Exercise 3.6.1

Given that $h^{-1}(6) = 2$, what are the corresponding input and output values of the original function h?

Answer

$h(2) = 6$

How To: Given two functions $f(x)$ and $g(x)$, test whether the functions are inverses of each other.

1. Determine whether $f(g(x)) = x$ or $g(f(x)) = x$.
2. If both statements are true, then $g = f^{-1}$ and $f = g^{-1}$. If either statement is false, then both are false, and $g \neq f^{-1}$ and $f \neq g^{-1}$.

Example 3.6.2: Testing Inverse Relationships Algebraically

If $f(x) = \frac{1}{x+2}$ and $g(x) = \frac{1}{x} - 2$, is $g = f^{-1}$?

Solution

$$g(f(x)) = \frac{1}{(\frac{1}{x+2}) - 2} \tag{3.6.11}$$

$$= x + 2 - 2 \tag{3.6.12}$$

$$= x \tag{3.6.13}$$

so

$$g = f^{-1} \text{ and } f = g^{-1} \tag{3.6.14}$$

This is enough to answer yes to the question, but we can also verify the other formula.

$$f(g(x)) = \frac{1}{\frac{1}{x} - 2 + 2} \tag{3.6.15}$$

$$= \frac{1}{\frac{1}{x}} \tag{3.6.16}$$

$$= x \tag{3.6.17}$$

Analysis

Notice the inverse operations are in reverse order of the operations from the original function.

Exercise 3.6.2

If $f(x) = x^3 - 4$ and $g(x) = \sqrt[3]{x+4}$, is $g = f^{-1}$?

Answer

Yes

Example 3.6.3: Determining Inverse Relationships for Power Functions

If $f(x) = x^3$ (the cube function) and $g(x) = \frac{1}{3}x$, is $g = f^{-1}$?

Solution

$$f(g(x)) = \frac{x^3}{27} \neq x \tag{3.6.18}$$

No, the functions are not inverses.

Analysis

The correct inverse to the cube is, of course, the cube root $\sqrt[3]{x} = x^{\frac{1}{3}}$, that is, the one-third is an exponent, not a multiplier.

Exercise 3.6.3

If $f(x) = (x-1)^3$ and $g(x) = \sqrt[3]{x} + 1$, is $g = f^{-1}$?

Answer

Yes

Finding Domain and Range of Inverse Functions

The outputs of the function f are the inputs to f^{-1}, so the range of f is also the domain of f^{-1}. Likewise, because the inputs to f are the outputs of f^{-1}, the domain of f is the range of f^{-1}. We can visualize the situation as in Figure 3.6.3.

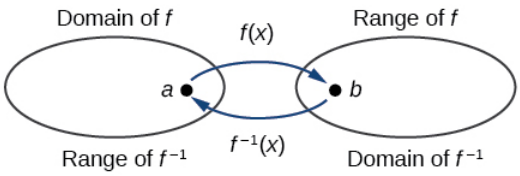


***Figure* 3.6.3:** *Domain and range of a function and its inverse.*

When a function has no inverse function, it is possible to create a new function where that new function on a limited domain does have an inverse function. For example, the inverse of $f(x) = \sqrt{x}$ is $f^{-1}(x) = x^2$, because a square "undoes" a square root; but the square is only the inverse of the square root on the domain $[0, \infty)$, since that is the range of $f(x) = \sqrt{x}$.

We can look at this problem from the other side, starting with the square (toolkit quadratic) function $f(x) = x^2$. If we want to construct an inverse to this function, we run into a problem, because for every given output of the quadratic function, there are two corresponding inputs (except when the input is 0). For example, the output 9 from the quadratic function corresponds to the inputs 3 and –3. But an output from a function is an input to its inverse; if this inverse input corresponds to more than one inverse output (input of the original function), then the "inverse" is not a function at all! To put it differently, the quadratic function is not a one-to-one function; it fails the horizontal line test, so it does not have an inverse function. In order for a function to have an inverse, it must be a one-to-one function.

In many cases, if a function is not one-to-one, we can still restrict the function to a part of its domain on which it is one-to-one. For example, we can make a restricted version of the square function $f(x) = x^2$ with its range limited to $[0, \infty)$, which is a one-to-one function (it passes the horizontal line test) and which has an inverse (the square-root function).

If $f(x) = (x-1)^2$ on $[1, \infty)$, then the inverse function is $f^{-1}(x) = \sqrt{x} + 1$.

- The domain of f = range of $f^{-1} = [1, \infty)$.
- The domain of f^{-1} = range of $f = [0, \infty)$.

Is it possible for a function to have more than one inverse?

No. If two supposedly different functions, say, g and h, both meet the definition of being inverses of another function f, then you can prove that $g = h$. We have just seen that some functions only have inverses if we restrict the domain of the original function. In these cases, there may be more than one way to restrict the domain, leading to different inverses. However, on any one domain, the original function still has only one unique inverse.

Note: Domain and Range of Inverse Functions

The range of a function $f(x)$ is the domain of the inverse function $f^{-1}(x)$.

The domain of $f(x)$ is the range of $f^{-1}(x)$.

How To: Given a function, find the domain and range of its inverse.

1. If the function is one-to-one, write the range of the original function as the domain of the inverse, and write the domain of the original function as the range of the inverse.
2. If the domain of the original function needs to be restricted to make it one-to-one, then this restricted domain becomes the range of the inverse function.

Example 3.6.4: Finding the Inverses of Toolkit Functions

Identify which of the toolkit functions besides the quadratic function are not one-to-one, and find a restricted domain on which each function is one-to-one, if any. The toolkit functions are reviewed in Table 3.6.2. We restrict the domain in such a fashion that the function assumes all y-values exactly once.

Table 3.6.2

Constant	Identity	Quadratic	Cubic	Reciprocal
$f(x) = c$	$f(x) = x$	$f(x) = x^2$	$f(x) = x^3$	$f(x) = \frac{1}{x}$
Reciprocal squared	Cube Root	Square Root	Absolute Value	
$f(x) = \frac{1}{x^2}$	$f(x) = \sqrt[3]{x}$	$f(x) = \sqrt{x}$	$f(x) = \lvert x \rvert$	

Solution

The constant function is not one-to-one, and there is no domain (except a single point) on which it could be one-to-one, so the constant function has no meaningful inverse.

The absolute value function can be restricted to the domain $[0, \infty)$,where it is equal to the identity function.

The reciprocal-squared function can be restricted to the domain $(0, \infty)$.

Analysis

We can see that these functions (if unrestricted) are not one-to-one by looking at their graphs, shown in Figure 3.6.4. They both would fail the horizontal line test. However, if a function is restricted to a certain domain so that it passes the horizontal line test, then in that restricted domain, it can have an inverse.

***Figure* 3.6.4:** *(a) Absolute value (b) Reciprocal squared*

3.6.4: The domain of function f is $(1,\infty)$ and the range of function f is $(-\infty,-2)$. Find the domain and range of the inverse function.

Solution

The domain of function f^{-1} is $(-\infty,-2)$ and the range of function f^{-1} is $(1,\infty)$.

Finding and Evaluating Inverse Functions

Once we have a one-to-one function, we can evaluate its inverse at specific inverse function inputs or construct a complete representation of the inverse function in many cases.

Inverting Tabular Functions

Suppose we want to find the inverse of a function represented in table form. Remember that the domain of a function is the range of the inverse and the range of the function is the domain of the inverse. So we need to interchange the domain and range.

Each row (or column) of inputs becomes the row (or column) of outputs for the inverse function. Similarly, each row (or column) of outputs becomes the row (or column) of inputs for the inverse function.

Example 3.6.5: Interpreting the Inverse of a Tabular Function

A function $f(t)$ is given in Table **3.6.3**, showing distance in miles that a car has traveled in t minutes. Find and interpret $f^{-1}(70)$

Table **3.6.3**

t (minutes)	30	50	70	90
$f(t)$ (miles)	20	40	60	70

The inverse function takes an output of f and returns an input for f. So in the expression $f^{-1}(70)$, 70 is an output value of the original function, representing 70 miles. The inverse will return the corresponding input of the original function f, 90 minutes, so $f^{-1}(70)=90$. The interpretation of this is that, to drive 70 miles, it took 90 minutes.

Alternatively, recall that the definition of the inverse was that if $f(a)=b$, then $f^{-1}(b)=a$. By this definition, if we are given $f^{-1}(70)=a$, then we are looking for a value a so that $f(a)=70$. In this case, we are looking for a t so that $f(t)=70$, which is when $t=90$.

Exercise 3.6.5

Using Table **3.6.4**, find and interpret (a) $f(60)$,and (b) $f^{-1}(60)$.

Table **3.6.4**

t (minutes)	30	50	60	70	90
$f(t)$ (miles)	20	40	50	60	70

Answer

$f(60)=50$. In 60 minutes, 50 miles are traveled.
$f^{-1}(60)=70$. To travel 60 miles, it will take 70 minutes.

Evaluating the Inverse of a Function, Given a Graph of the Original Function

We saw in **Functions and Function Notation** that the domain of a function can be read by observing the horizontal extent of its graph. We find the domain of the inverse function by observing the *vertical* extent of the graph of the original function, because

this corresponds to the horizontal extent of the inverse function. Similarly, we find the range of the inverse function by observing the *horizontal* extent of the graph of the original function, as this is the vertical extent of the inverse function. If we want to evaluate an inverse function, we find its input within its domain, which is all or part of the vertical axis of the original function's graph.

Given the graph of a function, evaluate its inverse at specific points.

1. Find the desired input on the y-axis of the given graph.
2. Read the inverse function's output from the x-axis of the given graph.

Example 3.6.6: Evaluating a Function and Its Inverse from a Graph at Specific Points

A function $g(x)$ is given in Figure 3.6.5. Find $g(3)$ and $g^{-1}(3)$.

.

Figure 3.6.5: Graph of $g(x)$

Solution

To evaluate $g(3)$, we find 3 on the x-axis and find the corresponding output value on the y-axis. The point $(3, 1)$ tells us that $g(3) = 1$.

To evaluate $g^{-1}(3)$, recall that by definition $g^{-1}(3)$ means the value of x for which $g(x) = 3$. By looking for the output value 3 on the vertical axis, we find the point $(5, 3)$ on the graph, which means $g(5) = 3$, so by definition, $g^{-1}(3) = 5$. See Figure 3.6.6.

Figure 3.6.6: Graph of $g(x)$.

Exercise 3.6.6

Using the graph in Figure 3.6.6, (a) find $g^{-1}(1)$,and (b) estimate $g^{-1}(4)$.

Answer a

3

Answer b

5.6

Finding Inverses of Functions Represented by Formulas

Sometimes we will need to know an inverse function for all elements of its domain, not just a few. If the original function is given as a formula— for example, y as a function of x— we can often find the inverse function by solving to obtain x as a function of y.

How To: Given a function represented by a formula, find the inverse.

1. Make sure f is a one-to-one function.
2. Solve for x
3. Interchange x and y.

Example 3.6.7: Inverting the Fahrenheit-to-Celsius Function

Find a formula for the inverse function that gives Fahrenheit temperature as a function of Celsius temperature.

$$C = \frac{5}{9}(F - 32) \tag{3.6.19}$$

Solution

$$C = \frac{5}{9}(F - 32) \tag{3.6.20}$$

$$C \cdot \frac{9}{5} = F - 32 \tag{3.6.21}$$

$$F = \frac{9}{5}C + 32 \tag{3.6.22}$$

By solving in general, we have uncovered the inverse function. If

$$C = h(F) = \frac{5}{9}(F - 32) \tag{3.6.23}$$

,

then

$$F = h^{-1}(C) = \frac{9}{5}C + 32. \tag{3.6.24}$$

In this case, we introduced a function h to represent the conversion because the input and output variables are descriptive, and writing C^{-1} could get confusing.

Exercise 3.6.7

Solve for x in terms of y given $y = \frac{1}{3}(x - 5)$

Answer

$x = 3y + 5$

Example 3.6.8: Solving to Find an Inverse Function

Find the inverse of the function $f(x) = \frac{2}{x-3} + 4$.

Solution

$$y = \frac{2}{x - 3 + 4} \qquad \text{Set up an equation.} \tag{3.6.25}$$

$$y - 4 = \frac{2}{x - 3} \qquad \text{Subtract 4 from both sides.} \tag{3.6.26}$$

$$x - 3 = \frac{2}{y - 4} \qquad \text{Multiply both sides by x−3 and divide by y−4.} \tag{3.6.27}$$

$$x = \frac{2}{y - 4} + 3 \qquad \text{Add 3 to both sides.} \tag{3.6.28}$$

So $f^{-1}(y) = \frac{2}{y-4} + 3$ or $f^{-1}(x) = \frac{2}{x-4} + 3$.

Analysis

The domain and range of f exclude the values 3 and 4, respectively. f and f^{-1} are equal at two points but are not the same function, as we can see by creating Table 3.6.5.

Table 3.6.5

x	1	2	5	$f^{-1}(y)$
$f(x)$	3	2	5	y

Example 3.6.9: Solving to Find an Inverse with Radicals

Find the inverse of the function $f(x) = 2 + \sqrt{x - 4}$.

Solution

$$y = 2 + \sqrt{x-4} \tag{3.6.29}$$
$$(y-2)^2 = x - 4 \tag{3.6.30}$$
$$x = (y-2)^2 + 4 \tag{3.6.31}$$

So $f^{-1}(x) = (x-2)^2 + 4$.

The domain of f is $[4, \infty)$. Notice that the range of f is $[2, \infty)$, so this means that the domain of the inverse function f^{-1} is also $[2, \infty)$

Analysis

The formula we found for $f^{-1}(x)$ looks like it would be valid for all real x. However, f^{-1} itself must have an inverse (namely, f) so we have to restrict the domain of f^{-1} to $[2, \infty)$ in order to make f^{-1} a one-to-one function. This domain of f^{-1} is exactly the range of f.

Exercise 3.6.8

What is the inverse of the function $f(x) = 2 - \sqrt{x}$? State the domains of both the function and the inverse function.

Answer

$f^{-1}(x) = (2-x)^2$; domain of f: $[0, \infty)$; domain of f^{-1}: $(-\infty, 2]$

Finding Inverse Functions and Their Graphs

Now that we can find the inverse of a function, we will explore the graphs of functions and their inverses. Let us return to the quadratic function $f(x) = x^2$ restricted to the domain $[0, \infty)$, on which this function is one-to-one, and graph it as in Figure 3.6.7.

***Figure* 3.6.7:** *Quadratic function with domain restricted to* $[0, \infty)$.

Restricting the domain to $[0, \infty)$ makes the function one-to-one (it will obviously pass the horizontal line test), so it has an inverse on this restricted domain.

We already know that the inverse of the toolkit quadratic function is the square root function, that is, $f^{-1}(x) = \sqrt{x}$. What happens if we graph both f and f^{-1} on the same set of axes, using the x-axis for the input to both f and f^{-1}?

We notice a distinct relationship: The graph of $f^{-1}(x)$ is the graph of $f(x)$ reflected about the diagonal line $y = x$, which we will call the identity line, shown in Figure 3.6.8.

.

***Figure* 3.6.8:** *Square and square-root functions on the non-negative domain*

This relationship will be observed for all one-to-one functions, because it is a result of the function and its inverse swapping inputs and outputs. This is equivalent to interchanging the roles of the vertical and horizontal axes.

Example 3.6.10: Finding the Inverse of a Function Using Reflection about the Identity Line

Given the graph of $f(x)$ in Figure 3.6.9, sketch a graph of $f^{-1}(x)$.

Figure 3.6.9: Graph of $f^(-1)(x)$.

This is a one-to-one function, so we will be able to sketch an inverse. Note that the graph shown has an apparent domain of $(0, \infty)$ and range of $(-\infty, \infty)$, so the inverse will have a domain of $(-\infty, \infty)$ and range of $(0, \infty)$.

If we reflect this graph over the line $y = x$, the point $(1, 0)$ reflects to $(0, 1)$ and the point $(4, 2)$ reflects to $(2, 4)$. Sketching the inverse on the same axes as the original graph gives Figure 3.6.10.

Figure 3.6.10: The function and its inverse, showing reflection about the identity line

Exercise 3.6.1

Draw graphs of the functions f and f^{-1} from Example 3.6.8.

Answer

Figure 3.6.11: Graph of $f(x)$ and $f^(-1)(x)$.

Is there any function that is equal to its own inverse?

Yes. If $f = f^{-1}$, then $f(f(x)) = x$, and we can think of several functions that have this property. The identity function does, and so does the reciprocal function, because

$$\frac{1}{\frac{1}{x}} = x \tag{3.6.32}$$

Any function $f(x) = c - x$, where c is a constant, is also equal to its own inverse.

Key Concepts

- If $g(x)$ is the inverse of $f(x)$, then $g(f(x)) = f(g(x)) = x$.
- Each of the toolkit functions has an inverse.
- For a function to have an inverse, it must be one-to-one (pass the horizontal line test).
- A function that is not one-to-one over its entire domain may be one-to-one on part of its domain.
- For a tabular function, exchange the input and output rows to obtain the inverse.
- The inverse of a function can be determined at specific points on its graph.
- To find the inverse of a formula, solve the equation $y = f(x)$ for x as a function of y. Then exchange the labels x and y.
- The graph of an inverse function is the reflection of the graph of the original function across the line $y = x$.

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CHAPTER OVERVIEW

4: Linear Functions

Recall that a function is a relation that assigns to every element in the domain exactly one element in the range. Linear functions are a specific type of function that can be used to model many real-world applications, such as plant growth over time. In this chapter, we will explore linear functions, their graphs, and how to relate them to data.

4.0: Introduction to Linear Functions

4.1: Linear Functions

4.2: Modeling with Linear Functions

Contributors

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4.0: Introduction to Linear Functions

Imagine placing a plant in the ground one day and finding that it has doubled its height just a few days later. Although it may seem incredible, this can happen with certain types of bamboo species. These members of the grass family are the fastest-growing plants in the world. One species of bamboo has been observed to grow nearly 1.5 inches every hour.1 In a twenty-four hour period, this bamboo plant grows about 36 inches, or an incredible 3 feet! A constant rate of change, such as the growth cycle of this bamboo plant, is a linear function.

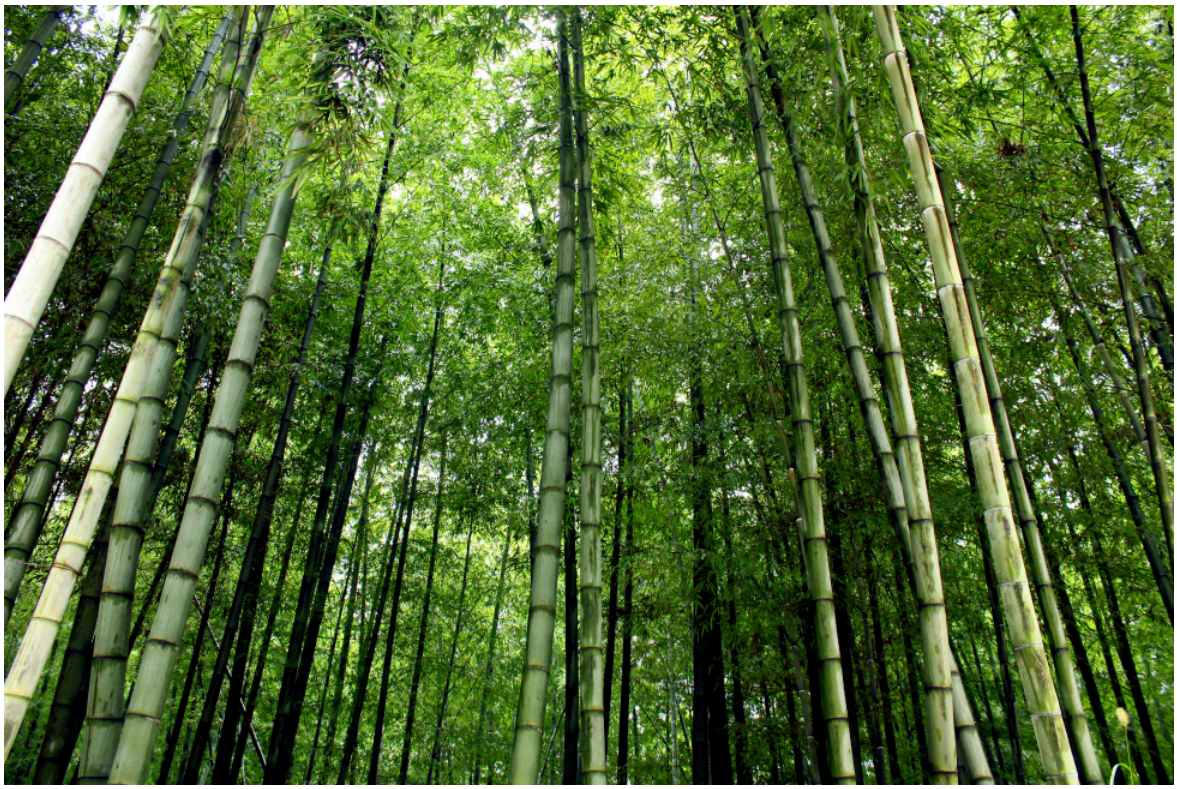


Figure* 4.0.1*: *A bamboo forest in China (credit: "JFXie"/Flickr)*

Recall from Functions and Function Notation that a function is a relation that assigns to every element in the domain exactly one element in the range. Linear functions are a specific type of function that can be used to model many real-world applications, such as plant growth over time. In this chapter, we will explore linear functions, their graphs, and how to relate them to data.

Footnotes

1 www.guinnessworldrecords.com/...growing-plant/

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4.1: Linear Functions

Learning Objectives

- Represent a linear function.
- Determine whether a linear function is increasing, decreasing, or constant.
- Interpret slope as a rate of change.
- Write and interpret an equation for a linear function.
- Graph linear functions.
- Determine whether lines are parallel or perpendicular.
- Write the equation of a line parallel or perpendicular to a given line.

Just as with the growth of a bamboo plant, there are many situations that involve constant change over time. Consider, for example, the first commercial maglev train in the world, the Shanghai MagLev Train (maglev train in the world, the Shanghai MagLev Train (Figure **4.1.1**). It carries passengers comfortably for a 30-kilometer trip from the airport to the subway station in only eight minutes.



Figure **4.1.1**: Shanghai MagLev Train (credit: “kanegen”/Flickr)

Suppose a maglev train were to travel a long distance, and that the train maintains a constant speed of 83 meters per second for a period of time once it is 250 meters from the station. How can we analyze the train’s distance from the station as a function of time? In this section, we will investigate a kind of function that is useful for this purpose, and use it to investigate real-world situations such as the train’s distance from the station at a given point in time.maglev train were to travel a long distance, and that the train maintains a constant speed of 83 meters per second for a period of time once it is 250 meters from the station. How can we analyze the train’s distance from the station as a function of time? In this section, we will investigate a kind of function that is useful for this purpose, and use it to investigate real-world situations such as the train’s distance from the station at a given point in time.

Representing Linear Functions

The function describing the train’s motion is a **linear function**, which is defined as a function with a constant rate of change, that is, a polynomial of degree 1. There are several ways to represent a linear function, including word form, function notation, tabular form, and graphical form. We will describe the train’s motion as a function using each method.

Representing a Linear Function in Word Form

Let’s begin by describing the linear function in words. For the train problem we just considered, the following word sentence may be used to describe the function relationship.

- *The train's distance from the station is a function of the time during which the train moves at a constant speed plus its original distance from the station when it began moving at constant speed.*

The speed is the rate of change. Recall that a rate of change is a measure of how quickly the dependent variable changes with respect to the independent variable. The rate of change for this example is constant, which means that it is the same for each input value. As the time (input) increases by 1 second, the corresponding distance (output) increases by 83 meters. The train began moving at this constant speed at a distance of 250 meters from the station.

Representing a Linear Function in Function Notation

Another approach to representing linear functions is by using function notation. One example of function notation is an equation written in the form known as the slope-intercept form of a line, where x is the input value, m is the rate of change, and b is the initial value of the dependent variable.

$$\begin{aligned} &\textbf{Equation form} && y = mx + b \\ &\textbf{Equation notation} && f(x) = mx + b \end{aligned}$$

In the example of the train, we might use the notation $D(t)$ in which the total distance D is a function of the time t. The rate, m, is 83 meters per second. The initial value of the dependent variable b is the original distance from the station, 250 meters. We can write a generalized equation to represent the motion of the train.

$$D(t) = 83t + 250 \tag{4.1.1}$$

Representing a Linear Function in Tabular Form

A third method of representing a linear function is through the use of a table. The relationship between the distance from the station and the time is represented in Figure 4.1.2. From the table, we can see that the distance changes by 83 meters for every 1 second increase in time.

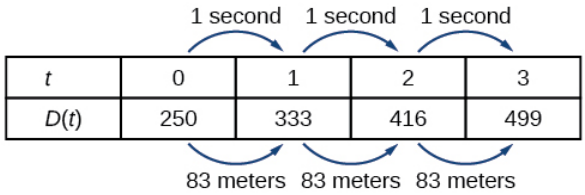


Figure 4.1.2: Tabular representation of the function D showing selected input and output values

Q/A? Can the input in the previous example be any real number?

No. The input represents time, so while nonnegative rational and irrational numbers are possible, negative real numbers are not possible for this example. The input consists of non-negative real numbers.

Representing a Linear Function in Graphical Form

Another way to represent linear functions is visually, using a graph. We can use the function relationship from above, $D(t) = 83t + 250$, to draw a graph, represented in Figure 4.1.3. Notice the graph is a line. When we plot a linear function, the graph is always a line.

The rate of change, which is constant, determines the slant, or **slope** of the line. The point at which the input value is zero is the vertical intercept, or **y-intercept**, of the line. We can see from the graph in Figure 4.1.3 that the y-intercept in the train example we just saw is $(0, 250)$ and represents the distance of the train from the station when it began moving at a constant speed.

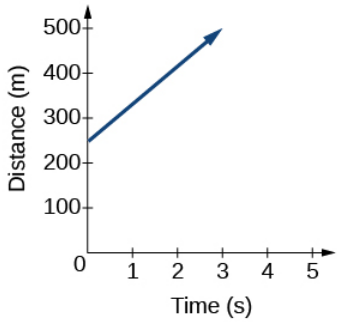


Figure 4.1.3: The graph of $D(t) = 83t + 250$. Graphs of linear functions are lines because the rate of change is constant.

Notice that the graph of the train example is restricted, but this is not always the case. Consider the graph of the line $f(x) = 2x + 1$. Ask yourself what numbers can be input to the function, that is, what is the domain of the function? The domain is comprised of all real numbers because any number may be doubled, and then have one added to the product.

Definition: Linear Function

A **linear function** is a function whose graph is a line. Linear functions can be written in the slope-intercept form of a line

$$f(x) = mx + b \tag{4.1.2}$$

where b is the initial or starting value of the function (when input, $x = 0$), and m is the constant rate of change, or **slope** of the function. The **y-intercept** is at $(0, b)$.

Example 4.1.1: Using a Linear Function to Find the Pressure on a Diver

The pressure, P, in pounds per square inch (PSI) on the diver in Figure 4.1.4 depends upon her depth below the water surface, d, in feet. This relationship may be modeled by the equation, $P(d) = 0.434d + 14.696$. Restate this function in words.



Figure 4.1.4: (credit: Ilse Reijs and Jan-Noud Hutten)Hutten)

To restate the function in words, we need to describe each part of the equation. The pressure as a function of depth equals four hundred thirty-four thousandths times depth plus fourteen and six hundred ninety-six thousandths.

Analysis

The initial value, 14.696, is the pressure in PSI on the diver at a depth of 0 feet, which is the surface of the water. The rate of change, or slope, is 0.434 PSI per foot. This tells us that the pressure on the diver increases 0.434 PSI for each foot her depth increases.

Determining whether a Linear Function Is Increasing, Decreasing, or Constant

The linear functions we used in the two previous examples increased over time, but not every linear function does. A linear function may be increasing, decreasing, or constant. For an increasing function, as with the train example, the output values increase as the input values increase. The graph of an **increasing function** has a positive slope. A line with a positive slope slants upward from left to right as in Figure 4.1.5(a). For a **decreasing function**, the slope is negative. The output values decrease as the input values increase. A line with a negative slope slants downward from left to right as in Figure 4.1.5(b). If the function is constant, the output values are the same for all input values so the slope is zero. A line with a slope of zero is horizontal as in Figure 4.1.5(c).

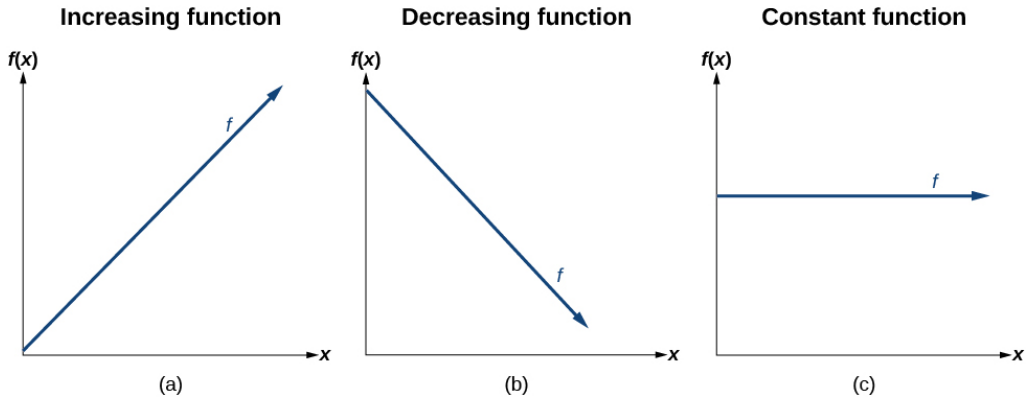


Figure 4.1.5: Three graphs depicting an increasing function, a decreasing function, and a constant function.]
Increasing and Decreasing Functions

Increasing and Decreasing Functions

The slope determines if the function is an **increasing linear function**, a **decreasing linear function**, or a constant function.

- $f(x) = mx + b$ is an increasing function if $m > 0$.
- $f(x) = mx + b$ is an decreasing function if $m < 0$.
- $f(x) = mx + b$ is a constant function if $m = 0$.

Example 4.1.2: decide whether a Function is Increasing, Decreasing, or Constant

Some recent studies suggest that a teenager sends an average of 60 texts per day. For each of the following scenarios, find the linear function that describes the relationship between the input value and the output value. Then, determine whether the graph of the function is increasing, decreasing, or constant.

a. The total number of texts a teen sends is considered a function of time in days. The input is the number of days, and output is the total number of texts sent.
b. A teen has a limit of 500 texts per month in his or her data plan. The input is the number of days, and output is the total number of texts remaining for the month.
c. A teen has an unlimited number of texts in his or her data plan for a cost of $50 per month. The input is the number of days, and output is the total cost of texting each month.

Solution

Analyze each function.

a. The function can be represented as $f(x) = 60x$ where x is the number of days. The slope, 60, is positive so the function is increasing. This makes sense because the total number of texts increases with each day.
b. The function can be represented as $f(x) = 500 - 60x$ where x is the number of days. In this case, the slope is negative so the function is decreasing. This makes sense because the number of texts remaining decreases each day and this function represents the number of texts remaining in the data plan after x days.
c. The cost function can be represented as $f(x) = 50$ because the number of days does not affect the total cost. The slope is 0 so the function is constant.

Calculating and Interpreting Slope

In the examples we have seen so far, we have had the slope provided for us. However, we often need to calculate the slope given input and output values. Given two values for the input, x_1 and x_2, and two corresponding values for the output, y_1 and y_2—which can be represented by a set of points, (x_1, y_1) and (x_2, y_2)—we can calculate the slope m, as follows

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}}$$
$$= \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

where Δy is the vertical displacement and Δx is the horizontal displacement. Note in function notation two corresponding values for the output y_1 and y_2 for the function f, $y_1 = f(x_1)$ and $y_2 = f(x_2)$, so we could equivalently write

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Figure **4.1.6** indicates how the slope of the line between the points, (x_1, y_1) and (x_2, y_2), is calculated. Recall that the slope measures steepness. The greater the absolute value of the slope, the steeper the line is.

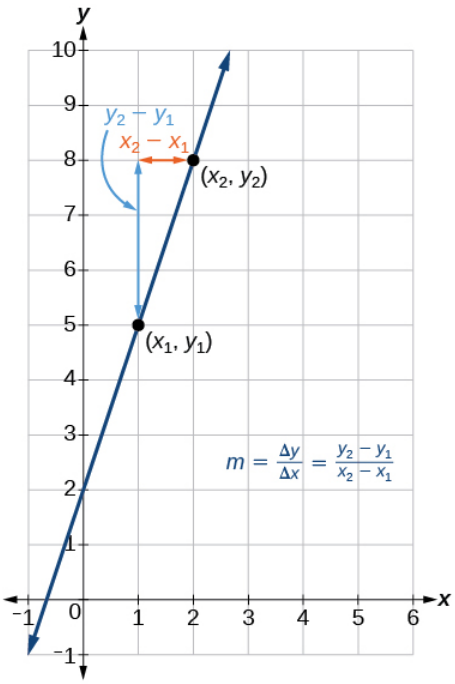


Figure **4.1.6**: The slope of a function is calculated by the change in y divided by the change in x. It does not matter which coordinate is used as the (x_2, y_2) and which is the (x_1, y_1),as long as each calculation is started with the elements from the same coordinate pair.

Q & A

Are the units for slope always $\frac{\text{units for the output}}{\text{units for the input}}$?

Yes. Think of the units as the change of output value for each unit of change in input value. An example of slope could be miles per hour or dollars per day. Notice the units appear as a ratio of units for the output per units for the input.

Note: Calculate Slope

The slope or rate of change of a function, m, can be calculated according to the following:

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \tag{4.1.3}$$

where x_1 and x_2 are input values, y_1 and y_2 are output values.

How to: Given two points from a linear function, calculate and interpret the slope.

1. Determine the units for output and input values.
2. Calculate the change of output values and change of input values.
3. Interpret the slope as the change in output values per unit of the input value.

Example 4.1.3: Finding the Slope of a Linear Function

If $f(x)$ is a linear function, and $(3,-2)$ and $(8,1)$ are points on the line, find the slope. Is this function increasing or decreasing?

Solution

The coordinate pairs are $(3,-2)$ and $(8,1)$. To find the rate of change, we divide the change in output by the change in input.

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{1-(-2)}{8-3} = \frac{3}{5} \tag{4.1.4}$$

We could also write the slope as $m = 0.6$. The function is increasing because $m > 0$.

Analysis

As noted earlier, the order in which we write the points does not matter when we compute the slope of the line as long as the first output value, or *y*-coordinate, used corresponds with the first input value, or *x*-coordinate, used.

Try it! 4.1.1

If $f(x)$ is a linear function, and $(2,3)$ and $(0,4)$ are points on the line, find the slope. Is this function increasing or decreasing?

Answer

$m = \frac{4-3}{0-2} = \frac{1}{-2} = -\frac{1}{2}$; decreasing because $m < 0$.

Example 4.1.4: Finding the Population Change from a Linear Function

The population of a city increased from 23,400 to 27,800 between 2008 and 2012. Find the change of population per year if we assume the change was constant from 2008 to 2012.

The rate of change relates the change in population to the change in time. The population increased by $27,800 - 23,400 = 4400$ people over the four-year time interval. To find the rate of change, divide the change in the number of people by the number of years.

$$\frac{4,400 \text{ people}}{4 \text{ years}} = 1,100 \frac{\text{people}}{\text{year}} \tag{4.1.5}$$

So the population increased by 1,100 people per year.

Analysis

Because we are told that the population increased, we would expect the slope to be positive. This positive slope we calculated is therefore reasonable.

? Try it! 4.1.2

The population of a small town increased from 1,442 to 1,868 between 2009 and 2012. Find the change of population per year if we assume the change was constant from 2009 to 2012.

Answer

$m = \frac{1{,}868-1{,}442}{2{,}012-2{,}009} = \frac{426}{3} = 142$ people per year

Writing the Point-Slope Form of a Linear Equation

Up until now, we have been using the slope-intercept form of a linear equation to describe linear functions. Here, we will learn another way to write a linear function, the **point-slope form**.

$$y - y_1 = m(x - x_1) \tag{4.1.6}$$

The point-slope form is derived from the slope formula.

$$\begin{aligned} m &= \frac{y-y_1}{x-x_1} && \text{assuming } x \neq x_1 \\ m(x-x_1) &= \frac{y-y_1}{x-x_1}(x-x_1) && \text{Multiply both sides by } (x-x_1). \\ m(x-x_1) &= y-y_1 && \text{Simplify} \\ y-y_1 &= m(x-x_1) && \text{Rearrange} \end{aligned}$$

Keep in mind that the slope-intercept form and the point-slope form can be used to describe the same function. We can move from one form to another using basic algebra. For example, suppose we are given an equation in point-slope form, $y-4 = -\frac{1}{2}(x-6)$. We can convert it to the slope-intercept form as shown.

$$\begin{aligned} y-4 &= -\frac{1}{2}(x-6) \\ y-4 &= -\frac{1}{2}x+3 && \text{Distribute the } -\frac{1}{2}. \\ y &= -\frac{1}{2}x+7 && \text{Add 4 to each side.} \end{aligned}$$

Therefore, the same line can be described in slope-intercept form as $y = \frac{1}{2}x + 7$.

Note: Point-Slope Form of a Linear Equation

The **point-slope** form of a linear equation takes the form

$$y - y_1 = m(x - x_1) \tag{4.1.7}$$

where m is the slope, x_1 and y_1 are the x and y coordinates of a specific point through which the line passes.

Writing the Equation of a Line Using a Point and the Slope

The point-slope form is particularly useful if we know one point and the slope of a line. Suppose, for example, we are told that a line has a slope of 2 and passes through the point $(4, 1)$. We know that $m = 2$ and that $x_1 = 4$ and $y_1 = 1$. We can substitute these values into the general point-slope equation.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= 2(x - 4) \end{aligned}$$

If we wanted to then rewrite the equation in slope-intercept form, we apply algebraic techniques.

$$\begin{aligned} y-1 &= 2(x-4) \\ y-1 &= 2x-8 && \text{Distribute the 2.} \\ y &= 2x-7 && \text{Add 1 to each side.} \end{aligned}$$

Both equations, $y - 1 = 2(x - 4)$ and $y = 2x–7$, describe the same line. See Figure 4.1.7.

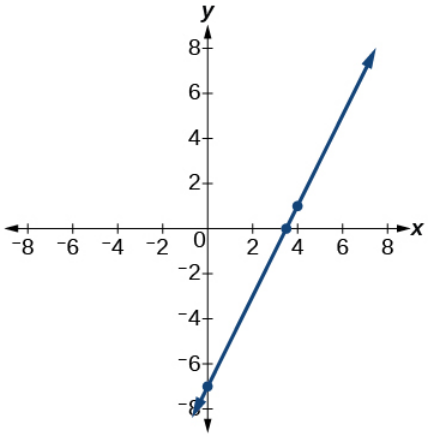


Figure 4.1.7

Example 4.1.5: Writing Linear Equations Using a Point and the Slope

Write the point-slope form of an equation of a line with a slope of 3 that passes through the point $(6, -1)$. Then rewrite it in the slope-intercept form.

Solution

Let's figure out what we know from the given information. The slope is 3, so $m = 3$. We also know one point, so we know $x_1 = 6$ and $y_1 = -1$. Now we can substitute these values into the general point-slope equation.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-1) &= 3(x - 6) && \text{Substitute known values.} \\ y + 1 &= 3(x - 6) && \text{Distribute } -1 \text{ to find point-slope form.} \end{aligned}$$

Then we use algebra to find the slope-intercept form.

$$\begin{aligned} y + 1 &= 3(x - 6) \\ y + 1 &= 3x - 18 && \text{Distribute 3.} \\ y &= 3x - 19 && \text{Simplify to slope-intercept form.} \end{aligned}$$

? Try it! 4.1.3

Write the point-slope form of an equation of a line with a slope of –2 that passes through the point $(-2, 2)$. Then rewrite it in the slope-intercept form.

Answer

$y - 2 = -2(x + 2)$; $y = -2x - 2$

Writing the Equation of a Line Using Two Points

The point-slope form of an equation is also useful if we know any two points through which a line passes. Suppose, for example, we know that a line passes through the points $(0, 1)$ and $(3, 2)$. We can use the coordinates of the two points to find the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{2 - 1}{3 - 0} \\ &= \frac{1}{3} \end{aligned}$$

Now we can use the slope we found and the coordinates of one of the points to find the equation for the line. Let use $(0, 1)$ for our point.

$$y - y_1 = m(x - x_1)$$
$$y - 1 = \frac{1}{3}(x - 0)$$

As before, we can use algebra to rewrite the equation in the slope-intercept form.

$$\begin{aligned} y - 1 &= \frac{1}{3}(x - 0) \\ y - 1 &= \frac{1}{3}x && \text{Distribute the } \frac{1}{3}. \\ y &= \frac{1}{3}x + 1 && \text{Add 1 to each side.} \end{aligned}$$

Both equations describe the line shown in Figure 4.1.8.

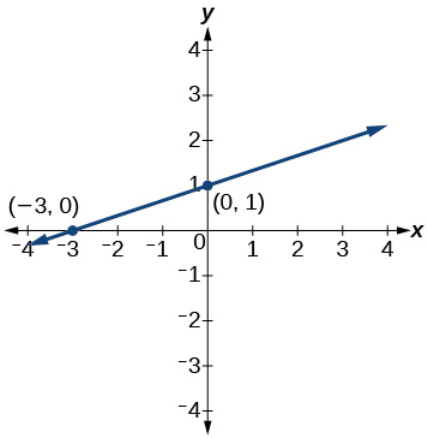


Figure 4.1.8

Example 4.1.6: Writing Linear Equations Using Two Points

Write the point-slope form of an equation of a line that passes through the points $(5, 1)$ and $(8, 7)$. Then rewrite it in the slope-intercept form.

Let's begin by finding the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{7 - 1}{8 - 5} \\ &= \frac{6}{3} \\ &= 2 \end{aligned}$$

So $m = 2$. Next, we substitute the slope and the coordinates for one of the points into the general point-slope equation. We can choose either point, but we will use $(5, 1)$.

$$y - y_1 = m(x - x_1)$$
$$y - 1 = 2(x - 5)$$

The point-slope equation of the line is $y - 1 = 2(x - 5)$. To rewrite the equation in slope-intercept form, we use algebra.

$$\begin{aligned} y - 1 &= 2(x - 5) \\ y - 1 &= 2x - 10 \\ y &= 2x - 9 \end{aligned}$$

The slope-intercept equation of the line is $y = 2x - 9$.

Try It! 4.1.4

Write the point-slope form of an equation of a line that passes through the points $(-1, 3)$ and $(0, 0)$. Then rewrite it in the slope-intercept form.

Answer

$y - 0 = -3(x - 0)$; $y = -3x$.

Writing and Interpreting an Equation for a Linear Function

Now that we have written equations for linear functions in both the slope-intercept form and the point-slope form, we can choose which method to use based on the information we are given. That information may be provided in the form of a graph, a point and a slope, two points, and so on. Look at the graph of the function f in Figure 4.1.9.

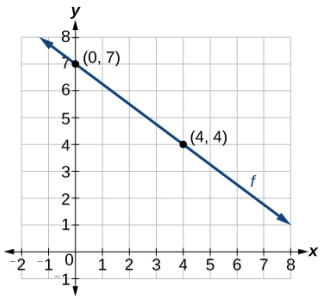


Figure 4.1.9: Graph depicting how to calculate the slope of a line.

We are not given the slope of the line, but we can choose any two points on the line to find the slope. Let's choose $(0, 7)$ and $(4, 4)$. We can use these points to calculate the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4 - 7}{4 - 0} \\ &= -\frac{3}{4} \end{aligned}$$

Now we can substitute the slope and the coordinates of one of the points into the point-slope form.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 4 &= -\frac{3}{4}(x - 4) \end{aligned}$$

If we want to rewrite the equation in the slope-intercept form, we would find

$$\begin{aligned} y - 4 &= -\frac{3}{4}(x - 4) \\ y - 4 &= -\frac{3}{4}x + 3 \\ y &= -\frac{3}{4}x + 7 \end{aligned}$$

If we wanted to find the slope-intercept form without first writing the point-slope form, we could have recognized that the line crosses the y-axis when the output value is 7. Therefore, $b = 7$. We now have the initial value b and the slope m so we can substitute m and b into the slope-intercept form of a line.

$$f(x) = mx + b$$

$m = -\frac{3}{4}$, $b = 7$

$$f(x) = -\frac{3}{4}x + 7$$

Figure 4.1.10

So the function is $f(x) = -\frac{3}{4}x + 7$, and the linear equation would be $y = -\frac{3}{4}x + 7$.

How to: Given the graph of a linear function, write an equation to represent the function.

1. Identify two points on the line.
2. Use the two points to calculate the slope.
3. Determine where the line crosses the y-axis to identify the y-intercept by visual inspection.
4. Substitute the slope and y-intercept into the slope-intercept form of a line equation.

Example 4.1.7: Writing an Equation for a Linear Function

Write an equation for a linear function given a graph of f shown in Figure 4.1.11.

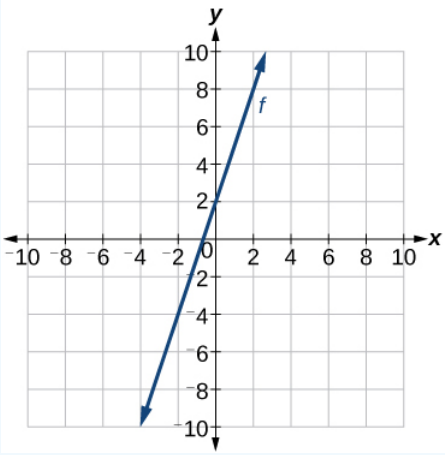


Figure 4.1.11

Solution

Identify two points on the line, such as $(0, 2)$ and $(-2, -4)$. Use the points to calculate the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4 - 2}{-2 - 0} \\ &= \frac{-6}{-2} \\ &= 3 \end{aligned}$$

Substitute the slope and the coordinates of one of the points into the point-slope form.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-4) &= 3(x - (-2)) \\ y + 4 &= 3(x + 2) \end{aligned}$$

We can use algebra to rewrite the equation in the slope-intercept form.

$$\begin{aligned} y + 4 &= 3(x + 2) \\ y + 4 &= 3x + 6 \\ y &= 3x + 2 \end{aligned}$$

Analysis

This makes sense because we can see from Figure 4.1.12 that the line crosses the y-axis at the point $(0, 2)$, which is the y-intercept, so $b = 2$.

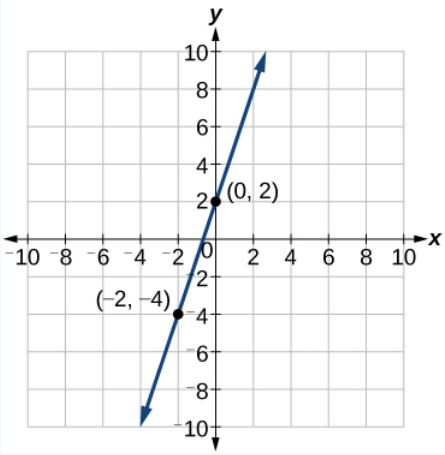


Figure 4.1.12: Graph of an increasing line with points at $(0, 2)$ and $(-2, -4)$.

Example 4.1.8: Writing an Equation for a Linear Cost Function

Suppose Ben starts a company in which he incurs a fixed cost of $1,250 per month for the overhead, which includes his office rent. His production costs are $37.50 per item. Write a linear function C where $C(x)$ is the cost for x items produced in a given month.

Solution

The fixed cost is present every month, $1,250. The costs that can vary include the cost to produce each item, which is $37.50 for Ben. The variable cost, called the marginal cost, is represented by 37.5. The cost Ben incurs is the sum of these two costs, represented by $C(x) = 1250 + 37.5x$.

Analysis

If Ben produces 100 items in a month, his monthly cost is represented by

$$\begin{aligned} C(100) &= 1250 + 37.5(100) \\ &= 5000 \end{aligned}$$

So his monthly cost would be $5,000.

Example 4.1.9: Writing an Equation for a Linear Function Given Two Points

If f is a linear function, with $f(3) = -2$, and $f(8) = 1$, find an equation for the function in slope-intercept form.

Solution

We can write the given points using coordinates.

$$\begin{aligned} f(3) = -2 &\rightarrow (3, 2) \\ f(8) = 1 &\rightarrow (8, 1) \end{aligned}$$

We can then use the points to calculate the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1 - (-2)}{8 - 3} \\ &= \frac{3}{5} \end{aligned}$$

Substitute the slope and the coordinates of one of the points into the point-slope form.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-2) &= \frac{3}{5}(x - 3) \end{aligned}$$

We can use algebra to rewrite the equation in the slope-intercept form.

$$\begin{aligned} y+2 &= \frac{3}{5}(x-3) \\ y+2 &= \frac{3}{5}x - \frac{9}{5} \\ y &= \frac{3}{5}x - \frac{19}{5} \end{aligned}$$

? Try It! 4.1.5

If $f(x)$ is a linear function, with $f(2) = -11$, and $f(4) = -25$, find an equation for the function in slope-intercept form.

Answer

$y = -7x + 3$

Modeling Real-World Problems with Linear Functions

In the real world, problems are not always explicitly stated in terms of a function or represented with a graph. Fortunately, we can analyze the problem by first representing it as a linear function and then interpreting the components of the function. As long as we know, or can figure out, the initial value and the rate of change of a linear function, we can solve many different kinds of real-world problems.

How to: Given a linear function f and the initial value and rate of change, evaluate $f(c)$.

1. Determine the initial value and the rate of change (slope).
2. Substitute the values into $f(x) = mx + b$.
3. Evaluate the function at $x = c$.

Example 4.1.10: Using a Linear Function to Determine the Number of Songs in a Music Collection

Marcus currently has 200 songs in his music collection. Every month, he adds 15 new songs. Write a formula for the number of songs, N, in his collection as a function of time, t, the number of months. How many songs will he own in a year?

Solution

The initial value for this function is 200 because he currently owns 200 songs, so $N(0) = 200$, which means that $b = 200$.

The number of songs increases by 15 songs per month, so the rate of change is 15 songs per month. Therefore we know that $m = 15$. We can substitute the initial value and the rate of change into the slope-intercept form of a line.

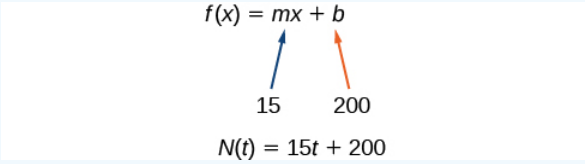


Figure 4.1.13

We can write the formula $N(t) = 15t + 200$.

With this formula, we can then predict how many songs Marcus will have in 1 year (12 months). In other words, we can evaluate the function at $t = 12$.

$$\begin{aligned} N(12) &= 15(12) + 200 \\ &= 180 + 200 \\ &= 380 \end{aligned}$$

Marcus will have 380 songs in 12 months.

Analysis

Notice that N is an increasing linear function. As the input (the number of months) increases, the output (number of songs) increases as well.

Example 4.1.11: Using a Linear Function to Calculate Salary Plus Commission

Working as an insurance salesperson, Ilya earns a base salary plus a commission on each new policy. Therefore, Ilya's weekly income, I,depends on the number of new policies, n, he sells during the week. Last week he sold 3 new policies, and earned \$760 for the week. The week before, he sold 5 new policies and earned \$920. Find an equation for $I(n)$, and interpret the meaning of the components of the equation.

Solution

The given information gives us two input-output pairs: $(3, 760)$ and $(5, 920)$. We start by finding the rate of change.

$$\begin{aligned} m &= \frac{920-760}{5-3} \\ &= \frac{\$160}{2 \text{ policies}} \\ &= \$80 \text{ per policy} \end{aligned}$$

Keeping track of units can help us interpret this quantity. Income increased by \$160 when the number of policies increased by 2, so the rate of change is \$80 per policy. Therefore, Ilya earns a commission of \$80 for each policy sold during the week.

We can then solve for the initial value.

$$\begin{aligned} I(n) &= 80n + b \\ 760 &= 80(3) + b \text{ when } n = 3, I(3) = 760 \\ 760 - 80(3) &= b \\ 520 &= b \end{aligned}$$

The value of b is the starting value for the function and represents Ilya's income when $n = 0$, or when no new policies are sold. We can interpret this as Ilya's base salary for the week, which does not depend upon the number of policies sold.

We can now write the final equation.

$$I(n) = 80n + 520$$

Our final interpretation is that Ilya's base salary is \$520 per week and he earns an additional \$80 commission for each policy sold.

Example 4.1.12: Using Tabular Form to Write an Equation for a Linear Function

Table 4.1.1 relates the number of rats in a population to time, in weeks. Use the table to write a linear equation.

Table 4.1.1

w, number of weeks	0	2	4	6
P(w), number of rats	1000	1080	1160	1240

Solution

We can see from the table that the initial value for the number of rats is 1000, so $b = 1000$.

Rather than solving for m, we can tell from looking at the table that the population increases by 80 for every 2 weeks that pass. This means that the rate of change is 80 rats per 2 weeks, which can be simplified to 40 rats per week.

$$P(w) = 40w + 1000$$

If we did not notice the rate of change from the table we could still solve for the slope using any two points from the table. For example, using $(2, 1080)$ and $(6, 1240)$

$$\begin{aligned} m &= \frac{1240 - 1080}{6 - 2} \\ &= \frac{160}{4} \\ &= 40 \end{aligned}$$

Is the initial value always provided in a table of values?

No. Sometimes the initial value is provided in a table of values, but sometimes it is not. If you see an input of 0, then the initial value would be the corresponding output. If the initial value is not provided because there is no value of input on the table equal to 0, find the slope, substitute one coordinate pair and the slope into $f(x) = mx + b$, and solve for b.

Try It 4.1.6

A new plant food was introduced to a young tree to test its effect on the height of the tree. Table 4.1.2 shows the height of the tree, in feet, x months since the measurements began. Write a linear function, $H(x)$, where x is the number of months since the start of the experiment.

Table 4.1.2

x	0	2	4	8	12
$H(x)$	12.5	13.5	14.5	16.5	18.5

Answer

$H(x) = 0.5x + 12.5$

Key Equations

- slope-intercept form of a line: $f(x) = mx + b$
- slope: $m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$
- point-slope form of a line: $y - y_1 = m(x - x_1)$

Key Concepts

- The ordered pairs given by a linear function represent points on a line.
- Linear functions can be represented in words, function notation, tabular form, and graphical form.
- The rate of change of a linear function is also known as the slope.
- An equation in the slope-intercept form of a line includes the slope and the initial value of the function.
- The initial value, or y-intercept, is the output value when the input of a linear function is zero. It is the y-value of the point at which the line crosses the y-axis.
- An increasing linear function results in a graph that slants upward from left to right and has a positive slope.
- A decreasing linear function results in a graph that slants downward from left to right and has a negative slope.
- A constant linear function results in a graph that is a horizontal line.
- Analyzing the slope within the context of a problem indicates whether a linear function is increasing, decreasing, or constant.

- The slope of a linear function can be calculated by dividing the difference between y-values by the difference in corresponding x-values of any two points on the line.
- The slope and initial value can be determined given a graph or any two points on the line.
- One type of function notation is the slope-intercept form of an equation.
- The point-slope form is useful for finding a linear equation when given the slope of a line and one point.
- The point-slope form is also convenient for finding a linear equation when given two points through which a line passes.
- The equation for a linear function can be written if the slope m and initial value b are known.
- A linear function can be used to solve real-world problems.
- A linear function can be written from tabular form.

Footnotes

1 www.chinahighlights.com/shang...glev-train.htm
2 www.cbsnews.com/8301-501465_1...ay-study-says/

Glossary

decreasing linear function

a function with a negative slope: If $f(x) = mx + b$, then $m < 0$.

increasing linear function

a function with a positive slope: If $f(x) = mx + b$, then $m > 0$.

linear function

a function with a constant rate of change that is a polynomial of degree 1, and whose graph is a straight line

point-slope form

the equation for a line that represents a linear function of the form \(y−y_1=m(x−x_1)

slope

the ratio of the change in output values to the change in input values; a measure of the steepness of a line

slope-intercept form

the equation for a line that represents a linear function in the form $f(x) = mx + b$

y-intercept

the value of a function when the input value is zero; also known as initial value

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4.2: Modeling with Linear Functions

Learning Objectives

- Build linear models from verbal descriptions.
- Model a set of data with a linear function.

Emily is a college student who plans to spend a summer in Seattle. She has saved $3,500 for her trip and anticipates spending $400 each week on rent, food, and activities. How can we write a linear model to represent her situation? What would be the x-intercept, and what can she learn from it? To answer these and related questions, we can create a model using a linear function. Models such as this one can be extremely useful for analyzing relationships and making predictions based on those relationships. In this section, we will explore examples of **linear function** models.



Figure 4.2.1: (credit: EEK Photography/Flickr)

Identifying Steps to Model and Solve Problems

When **modeling** scenarios with linear functions and solving problems involving quantities with a **constant rate of change**, we typically follow the same problem strategies that we would use for any type of function. Let's briefly review them:

Identify changing quantities, and then define descriptive variables to represent those quantities. When appropriate, sketch a picture or define a coordinate system.
Carefully read the problem to identify important information. Look for information that provides values for the variables or values for parts of the functional model, such as slope and initial value.
Carefully read the problem to determine what we are trying to find, identify, solve, or interpret.
Identify a solution pathway from the provided information to what we are trying to find. Often this will involve checking and tracking units, building a table, or even finding a formula for the function being used to model the problem.
When needed, write a formula for the function.
Solve or evaluate the function using the formula.
Reflect on whether your answer is reasonable for the given situation and whether it makes sense mathematically.
Clearly convey your result using appropriate units, and answer in full sentences when necessary.

Building Linear Models

Now let's take a look at the student in Seattle. In her situation, there are two changing quantities: time and money. The amount of money she has remaining while on vacation depends on how long she stays. We can use this information to define our variables, including units.

- Output: M, money remaining, in dollars
- Input: t, time, in weeks

So, the amount of money remaining depends on the number of weeks: $M(t)$

We can also identify the initial value and the rate of change.

- Initial Value: She saved $3,500, so $3,500 is the initial value for M.
- Rate of Change: She anticipates spending $400 each week, so –$400 per week is the rate of change, or slope.

Notice that the unit of dollars per week matches the unit of our output variable divided by our input variable. Also, because the slope is negative, the linear function is decreasing. This should make sense because she is spending money each week.

The **rate of change is constant**, so we can start with the linear model $M(t) = mt + b$. Then we can substitute the intercept and slope provided.

$$M(t) = mt + b$$

$$M(t) = -400t + 3500$$

Figure 4.2.2

To find the x-intercept, we set the output to zero, and solve for the input.

$$\begin{aligned} 0 &= -400t + 3500 \\ t &= \frac{3500}{400} \\ &= 8.75 \end{aligned}$$

The x-intercept is 8.75 weeks. Because this represents the input value when the output will be zero, we could say that Emily will have no money left after 8.75 weeks.

When modeling any real-life scenario with functions, there is typically a limited domain over which that model will be valid—almost no trend continues indefinitely. Here the domain refers to the number of weeks. In this case, it doesn't make sense to talk about input values less than zero. A negative input value could refer to a number of weeks before she saved $3,500, but the scenario discussed poses the question once she saved $3,500 because this is when her trip and subsequent spending starts. It is also likely that this model is not valid after the x-intercept, unless Emily will use a credit card and goes into debt. The domain represents the set of input values, so the reasonable domain for this function is $0 \le t \le 8.75$.

In the above example, we were given a written description of the situation. We followed the steps of modeling a problem to analyze the information. However, the information provided may not always be the same. Sometimes we might be provided with an intercept. Other times we might be provided with an output value. We must be careful to analyze the information we are given, and use it appropriately to build a linear model.

Using a Given Intercept to Build a Model

Some real-world problems provide the y-intercept, which is the constant or initial value. Once the y-intercept is known, the x-intercept can be calculated. Suppose, for example, that Hannah plans to pay off a no-interest loan from her parents. Her loan balance is $1,000. She plans to pay $250 per month until her balance is $0. The y-intercept is the initial amount of her debt, or $1,000. The rate of change, or slope, is -$250 per month. We can then use the slope-intercept form and the given information to develop a linear model.

$$\begin{aligned} f(x) &= mx + b \\ &= -250x + 1000 \end{aligned}$$

Now we can set the function equal to 0, and solve for x to find the x-intercept.

$$\begin{aligned} 0 &= -250 + 1000 \\ 1000 &= 250x \\ 4 &= x \\ x &= 4 \end{aligned}$$

The x-intercept is the number of months it takes her to reach a balance of $0. The x-intercept is 4 months, so it will take Hannah four months to pay off her loan.

Using a Given Input and Output to Build a Model

Many real-world applications are not as direct as the ones we just considered. Instead they require us to identify some aspect of a linear function. We might sometimes instead be asked to evaluate the linear model at a given input or set the equation of the linear model equal to a specified output.

Given a word problem that includes two pairs of input and output values, use the linear function to solve a problem.

1. Identify the input and output values.
2. Convert the data to two coordinate pairs.
3. Find the slope.
4. Write the linear model.
5. Use the model to make a prediction by evaluating the function at a given x-value.
6. Use the model to identify an x-value that results in a given y-value.
7. Answer the question posed.

Example 4.2.1: Using a Linear Model to Investigate a Town's Population

A town's population has been growing linearly. In 2004 the population was 6,200. By 2009 the population had grown to 8,100. Assume this trend continues.

a. Predict the population in 2013.
b. Identify the year in which the population will reach 15,000.

Solution

The two changing quantities are the population size and time. While we could use the actual year value as the input quantity, doing so tends to lead to very cumbersome equations because the y-intercept would correspond to the year 0, more than 2000 years ago!

To make computation a little nicer, we will define our input as the number of years since 2004:

- Input: t, years since 2004
- Output: $P(t)$, the town's population

To predict the population in 2013 ($t = 9$), we would first need an equation for the population. Likewise, to find when the population would reach 15,000, we would need to solve for the input that would provide an output of 15,000. To write an equation, we need the initial value and the rate of change, or slope.

To determine the rate of change, we will use the change in output per change in input.

$$m = \frac{\text{change in output}}{\text{change in input}} \tag{4.2.1}$$

The problem gives us two input-output pairs. Converting them to match our defined variables, the year 2004 would correspond to $t = 0$, giving the point $(0, 6200)$. Notice that through our clever choice of variable definition, we have "given" ourselves the y-intercept of the function. The year 2009 would correspond to $t = 5$, giving the point $(5, 8100)$.

The two coordinate pairs are $(0, 6200)$ and $(5, 8100)$. Recall that we encountered examples in which we were provided two points earlier in the chapter. We can use these values to calculate the slope.

$$\begin{aligned} m &= \frac{8100 - 6200}{5 - 0} \\ &= \frac{1900}{5} \\ &= 380 \text{ people per year} \end{aligned}$$

We already know the y-intercept of the line, so we can immediately write the equation:

$$P(t) = 380t + 6200 \tag{4.2.2}$$

To predict the population in 2013, we evaluate our function at $t = 9$.

$$\begin{aligned} P(9) &= 380(9) + 6,200 \\ &= 9,620 \end{aligned}$$

If the trend continues, our model predicts a population of 9,620 in 2013.

To find when the population will reach 15,000, we can set $P(t) = 15000$ and solve for t.

$$\begin{aligned} 15000 &= 380t + 6200 \\ 8800 &= 380t \\ t &\approx 23.158 \end{aligned}$$

Our model predicts the population will reach 15,000 in a little more than 23 years after 2004, or somewhere around the year 2027.

Exercise 4.2.1A

A company sells doughnuts. They incur a fixed cost of $25,000 for rent, insurance, and other expenses. It costs $0.25 to produce each doughnut.

a. Write a linear model to represent the cost C of the company as a function of x, the number of doughnuts produced.
b. Find and interpret the y-intercept.

Solution

a. $C(x) = 0.25x + 25,000$ b. The y-intercept is $(0, 25,000)$. If the company does not produce a single doughnut, they still incur a cost of $25,000.

Exercise 4.2.1B

A city's population has been growing linearly. In 2008, the population was 28,200. By 2012, the population was 36,800. Assume this trend continues.

a. Predict the population in 2014.
b. Identify the year in which the population will reach 54,000.

Solution

a. 41,100 b. 2020

Using a Diagram to Model a Problem

It is useful for many real-world applications to draw a picture to gain a sense of how the variables representing the input and output may be used to answer a question. To draw the picture, first consider what the problem is asking for. Then, determine the input and the output. The diagram should relate the variables. Often, geometrical shapes or figures are drawn. Distances are often traced out. If a right triangle is sketched, the Pythagorean Theorem relates the sides. If a rectangle is sketched, labeling width and height is helpful.

Example 4.2.2: Using a Diagram to Model Distance Walked

Anna and Emanuel start at the same intersection. Anna walks east at 4 miles per hour while Emanuel walks south at 3 miles per hour. They are communicating with a two-way radio that has a range of 2 miles. How long after they start walking will they fall out of radio contact?

Solution

In essence, we can partially answer this question by saying they will fall out of radio contact when they are 2 miles apart, which leads us to ask a new question:

"How long will it take them to be 2 miles apart?"

In this problem, our changing quantities are time and position, but ultimately we need to know how long will it take for them to be 2 miles apart. We can see that time will be our input variable, so we'll define our input and output variables.

- Input: t, time in hours.
- Output: $A(t)$, distance in miles, and $E(t)$, distance in miles

Because it is not obvious how to define our output variable, we'll start by drawing a picture such as Figure **4.2.3**.

Figure **4.2.3**

- Initial Value: They both start at the same intersection so when $t = 0$, the distance traveled by each person should also be 0. Thus the initial value for each is 0.
- Rate of Change: Anna is walking 4 miles per hour and Emanuel is walking 3 miles per hour, which are both rates of change. The slope for A is 4 and the slope for E is 3.

Using those values, we can write formulas for the distance each person has walked.

$$A(t) = 4t \tag{4.2.3}$$

$$E(t) = 3t \tag{4.2.4}$$

For this problem, the distances from the starting point are important. To notate these, we can define a coordinate system, identifying the "starting point" at the intersection where they both started. Then we can use the variable, A, which we introduced above, to represent Anna's position, and define it to be a measurement from the starting point in the eastward direction. Likewise, can use the variable, E, to represent Emanuel's position, measured from the starting point in the southward direction. Note that in defining the coordinate system, we specified both the starting point of the measurement and the direction of measure.

We can then define a third variable, D, to be the measurement of the distance between Anna and Emanuel. Showing the variables on the diagram is often helpful, as we can see from Figure **4.2.4**.

Recall that we need to know how long it takes for D, the distance between them, to equal 2 miles. Notice that for any given input t, the outputs $A(t)$, $E(t)$, and $D(t)$ represent distances.

Figure **4.2.4**: We can use the Pythagorean Theorem because we have drawn a right angle.

Using the Pythagorean Theorem, we get:

$$\begin{aligned} d(t)^2 &= A(t)^2 + E(t)^2 \\ &= (4t)^2 + (3t)^2 \\ &= 16t^2 + 9t^2 \\ &= 25t^2 \\ D(t) &= \pm\sqrt{25t^2} \qquad \text{Solve for } D(t) \text{ using the square root} \\ &= \pm 5|t| \end{aligned}$$

In this scenario we are considering only positive values of t, so our distance $D(t)$ will always be positive. We can simplify this answer to $D(t) = 5t$. This means that the distance between Anna and Emanuel is also a linear function. Because D is a linear function, we can now answer the question of when the distance between them will reach 2 miles. We will set the output $D(t) = 2$ and solve for t.

$$\begin{aligned} D(t) &= 2 \\ 5t &= 2 \\ t &= \frac{2}{5} = 0.4 \end{aligned}$$

They will fall out of radio contact in 0.4 hours, or 24 minutes.

Should I draw diagrams when given information based on a geometric shape?

Yes. Sketch the figure and label the quantities and unknowns on the sketch.

Example 4.2.3: Using a Diagram to Model Distance between Cities

There is a straight road leading from the town of Westborough to Agritown 30 miles east and 10 miles north. Partway down this road, it junctions with a second road, perpendicular to the first, leading to the town of Eastborough. If the town of Eastborough is located 20 miles directly east of the town of Westborough, how far is the road junction from Westborough?

Solution

It might help here to draw a picture of the situation. See Figure 4.2.5. It would then be helpful to introduce a coordinate system. While we could place the origin anywhere, placing it at Westborough seems convenient. This puts Agritown at coordinates $(30, 10)$, and Eastborough at $(20, 0)$.

Figure 4.2.5

Using this point along with the origin, we can find the slope of the line from Westborough to Agritown:

$$m = \frac{10-0}{30-0} = \frac{1}{3} \tag{4.2.5}$$

The equation of the road from Westborough to Agritown would be

$$W(x) = \frac{1}{3}x \tag{4.2.6}$$

From this, we can determine the perpendicular road to Eastborough will have slope $m = -3$. Because the town of Eastborough is at the point $(20, 0)$, we can find the equation:

$$\begin{aligned} E(x) &= -3x + b \\ 0 &= -3(20) + b \qquad \text{Substitute in } (20, 0) \\ b &= 60 \\ E(x) &= -3x + 60 \end{aligned}$$

We can now find the coordinates of the junction of the roads by finding the intersection of these lines. Setting them equal,

$$\begin{aligned} \frac{1}{3}x &= -3x + 60 \\ \frac{10}{3}x &= 60 \\ 10x &= 180 \\ x &= 18 \qquad \text{Substituting this back into } W(x) \\ y &= W(18) \\ &= \frac{1}{3}(18) \\ &= 6 \end{aligned}$$

The roads intersect at the point $(18, 6)$. Using the distance formula, we can now find the distance from Westborough to the junction.

$$\begin{aligned} \text{distance} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(18-0)^2 + (6-0)^2} \\ &\approx 18.743 \text{miles} \end{aligned}$$

Analysis

One nice use of linear models is to take advantage of the fact that the graphs of these functions are lines. This means real-world applications discussing maps need linear functions to model the distances between reference points.

Exercise 4.2.2

There is a straight road leading from the town of Timpson to Ashburn 60 miles east and 12 miles north. Partway down the road, it junctions with a second road, perpendicular to the first, leading to the town of Garrison. If the town of Garrison is located 22 miles directly east of the town of Timpson, how far is the road junction from Timpson?

Solution

21.15 miles

Building Systems of Linear Models

Real-world situations including two or more linear functions may be modeled with a **system of linear equations**. Remember, when solving a system of linear equations, we are looking for points the two lines have in common. Typically, there are three types of answers possible, as shown in Figure **4.2.6**.

Figure **4.2.6**

Given a situation that represents a system of linear equations, write the system of equations and identify the solution.

a. Identify the input and output of each linear model.
b. Identify the slope and y-intercept of each linear model.
c. Find the solution by setting the two linear functions equal to another and solving for x,or find the point of intersection on a graph.

Example **4.2.4**: Building a System of Linear Models to Choose a Truck Rental Company

Jamal is choosing between two truck-rental companies. The first, Keep on Trucking, Inc., charges an up-front fee of $20, then 59 cents a mile[1]. The second, Move It Your Way, charges an up-front fee of $16, then 63 cents a mile. When will Keep on Trucking, Inc. be the better choice for Jamal?

Solution

The two important quantities in this problem are the cost and the number of miles driven. Because we have two companies to consider, we will define two functions.

- Input: d, distance driven in miles
- Outputs: $K(d)$: cost, in dollars, for renting from Keep on Trucking

  $M(d)$: cost, in dollars, for renting from Move It Your Way
- Initial Value: Up-front fee: $K(0) = 20$ and $M(0) = 16$
- Rate of Change: $K(d) = \frac{\$0.59}{\text{mile}}$ and $P(d) = \frac{\$0.63}{\text{mile}}$

A linear function is of the form $f(x) = mx + b$. Using the rates of change and initial charges, we can write the equations

$$K(d) = 0.59d + 20$$

$$M(d) = 0.63d + 16$$

Using these equations, we can determine when Keep on Trucking, Inc. will be the better choice. Because all we have to make that decision from is the costs, we are looking for when Keep on Trucking, Inc. will cost less, or when $K(d) < M(d)$. The solution pathway will lead us to find the equations for the two functions, find the intersection, and then see where the $K(d)$ function is smaller.

These graphs are sketched in Figure **4.2.7**, with $K(d)$ in blue.

Figure **4.2.7**

To find the intersection, we set the equations equal and solve:

$$\begin{aligned} K(d) &= M(d) \\ 0.59d + 20 &= 0.63d + 16 \\ 4 &= 0.04d \\ 100 &= d \\ d &= 100 \end{aligned}$$

This tells us that the cost from the two companies will be the same if 100 miles are driven. Either by looking at the graph, or noting that $K(d)$ is growing at a slower rate, we can conclude that Keep on Trucking, Inc. will be the cheaper price when more than 100 miles are driven, that is $d > 100$.

Key Concepts

- We can use the same problem strategies that we would use for any type of function.
- When modeling and solving a problem, identify the variables and look for key values, including the slope and y-intercept.
- Draw a diagram, where appropriate.
- Check for reasonableness of the answer.
- Linear models may be built by identifying or calculating the slope and using the y-intercept.
- The x-intercept may be found by setting $y = 0$, which is setting the expression $mx + b$ equal to 0.
- The point of intersection of a system of linear equations is the point where the x- and y-values are the same.
- A graph of the system may be used to identify the points where one line falls below (or above) the other line.

Footnotes

1 Rates retrieved Aug 2, 2010 from www.budgettruck.com and http://www.uhaul.com/

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CHAPTER OVERVIEW

5: Polynomial and Rational Functions

5.0: Prelude to Polynomial and Rational Functions
5.1: Quadratic Functions
5.2: Quadratic inequalities
5.3: Applications with quadratic functions
5.4: Power Functions and Polynomial Functions
5.5: Dividing Polynomials
5.6: Inverses and Radical Functions

Contributors

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5.0: Prelude to Polynomial and Rational Functions

Digital photography has dramatically changed the nature of photography. No longer is an image etched in the emulsion on a roll of film. Instead, nearly every aspect of recording and manipulating images is now governed by mathematics. An image becomes a series of numbers, representing the characteristics of light striking an image sensor. When we open an image file, software on a camera or computer interprets the numbers and converts them to a visual image. Photo editing software uses complex polynomials to transform images, allowing us to manipulate the image in order to crop details, change the color palette, and add special effects. Inverse functions make it possible to convert from one file format to another. In this chapter, we will learn about these concepts and discover how mathematics can be used in such applications.

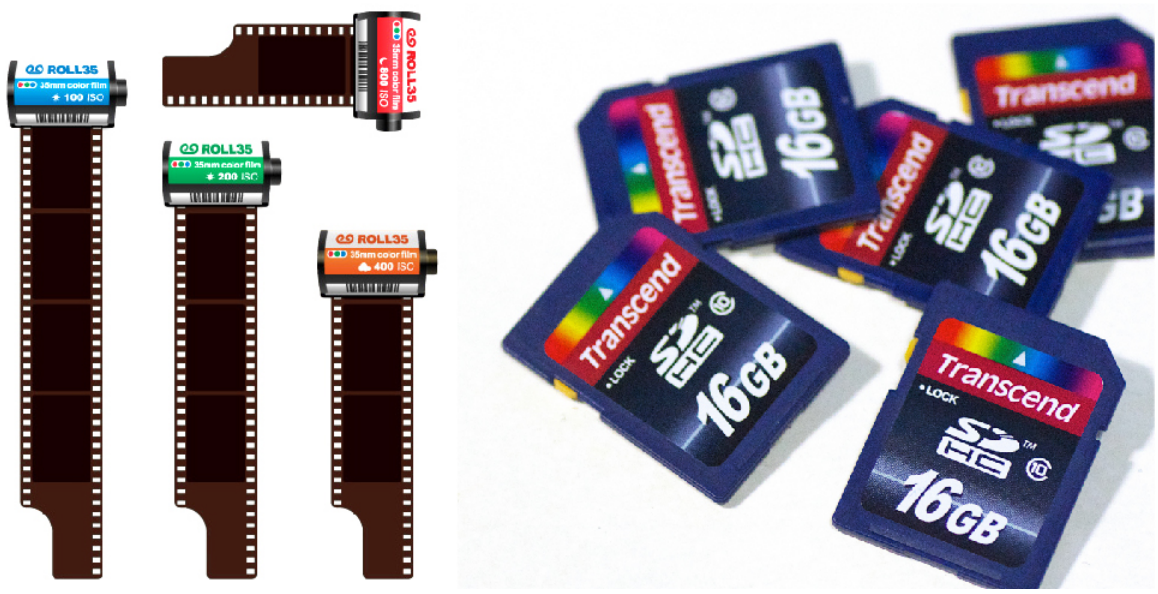


Figure* 5.0.1*: *35-mm film, once the standard for capturing photographic images, has been made largely obsolete by digital photography. (credit "film": modification of work by Horia Varlan; credit "memory cards": modification of work by Paul Hudson)*

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5.1: Quadratic Functions

Learning Objectives

- Recognize characteristics of parabolas.
- Understand how the graph of a parabola is related to its quadratic function.
- Determine a quadratic function's minimum or maximum value.
- Solve problems involving a quadratic function's minimum or maximum value.

Curved antennas, such as the ones shown in Figure 5.1.1, are commonly used to focus microwaves and radio waves to transmit television and telephone signals, as well as satellite and spacecraft communication. The cross-section of the antenna is in the shape of a parabola, which can be described by a quadratic function.

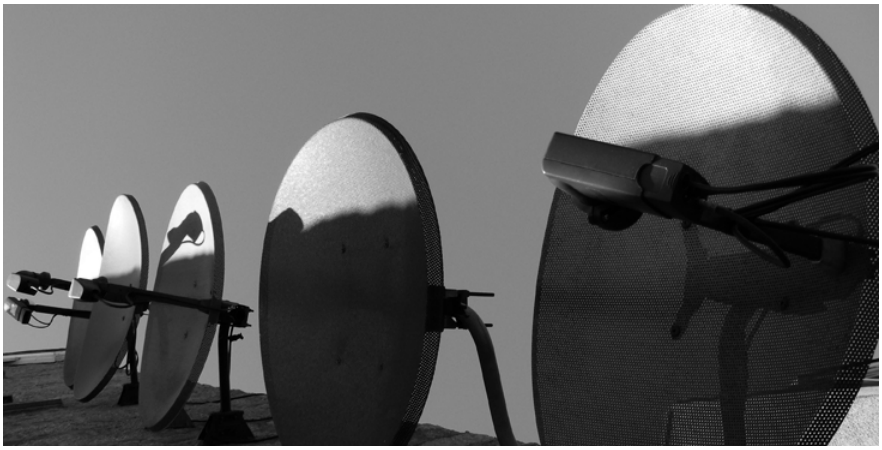


Figure 5.1.1: *An array of satellite dishes. (credit: Matthew Colvin de Valle, Flickr)*

In this section, we will investigate quadratic functions, which frequently model problems involving area and projectile motion. Working with quadratic functions can be less complex than working with higher degree functions, so they provide a good opportunity for a detailed study of function behavior.

Recognizing Characteristics of Parabolas

The graph of a quadratic function is a U-shaped curve called a parabola. One important feature of the graph is that it has an extreme point, called the **vertex**. If the parabola opens up, the vertex represents the lowest point on the graph, or the **minimum value** of the quadratic function. If the parabola opens down, the vertex represents the highest point on the graph, or the **maximum value**. In either case, the vertex is a turning point on the graph. The graph is also symmetric with a vertical line drawn through the vertex, called the **axis of symmetry**. These features are illustrated in Figure 5.1.2.

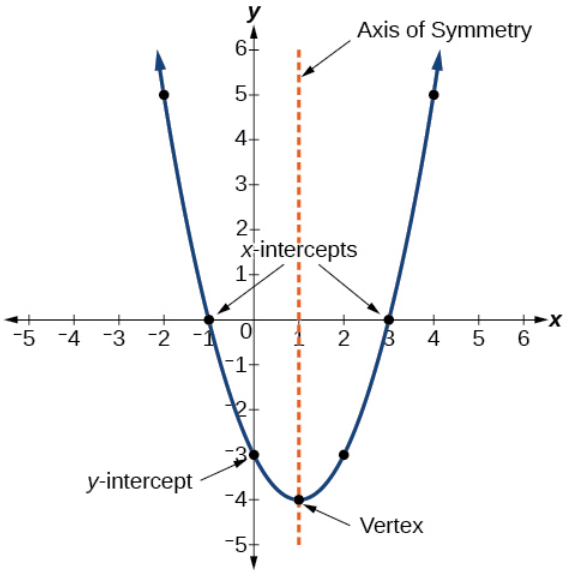


Figure 5.1.2: Graph of a parabola showing where the x and y intercepts, vertex, and axis of symmetry are.

The y-intercept is the point at which the parabola crosses the y-axis. The x-intercepts are the points at which the parabola crosses the x-axis. If they exist, the x-intercepts represent the **zeros**, or **roots**, of the quadratic function, the values of x at which $y = 0$.

Example 5.1.1: Identifying the Characteristics of a Parabola

Determine the vertex, axis of symmetry, zeros, and y-intercept of the parabola shown in Figure 5.1.3.

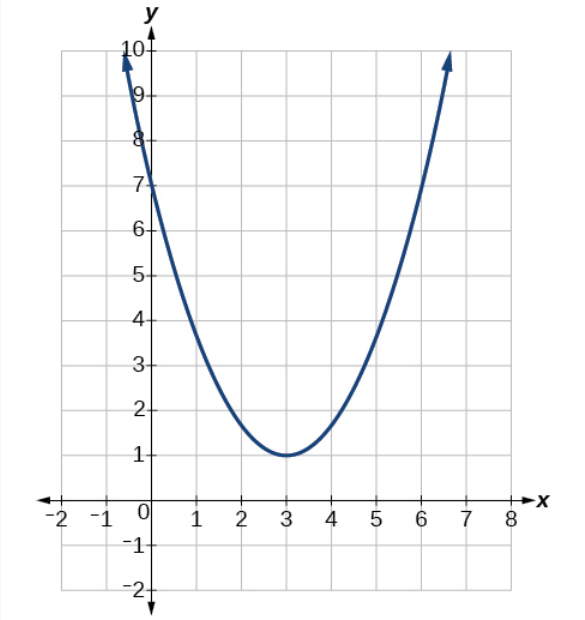


Figure 5.1.3.

Solution

The vertex is the turning point of the graph. We can see that the vertex is at $(3, 1)$. Because this parabola opens upward, the axis of symmetry is the vertical line that intersects the parabola at the vertex. So the axis of symmetry is $x = 3$. This parabola does not cross the x-axis, so it has no zeros. It crosses the y-axis at $(0, 7)$ so this is the y-intercept.

Understanding How the Graphs of Parabolas are Related to Their Quadratic Functions

The **general form of a quadratic function** presents the function in the form

$$f(x) = ax^2 + bx + c \tag{5.1.1}$$

where a, b, and c are real numbers and $a \neq 0$. If $a > 0$, the parabola opens upward. If $a < 0$, the parabola opens downward. We can use the general form of a parabola to find the equation for the axis of symmetry.

The axis of symmetry is defined by $x = -\frac{b}{2a}$. If we use the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, to solve $ax^2 + bx + c = 0$ for the x-intercepts, or zeros, we find the value of x halfway between them is always $x = -\frac{b}{2a}$, the equation for the axis of symmetry.

Figure 5.1.4 represents the graph of the quadratic function written in general form as $y = x^2 + 4x + 3$. In this form, $a = 1$, $b = 4$, and $c = 3$. Because $a > 0$, the parabola opens upward. The axis of symmetry is $x = -\frac{4}{2(1)} = -2$. This also makes sense because we can see from the graph that the vertical line $x = -2$ divides the graph in half. The vertex always occurs along the axis of symmetry. For a parabola that opens upward, the vertex occurs at the lowest point on the graph, in this instance, $(-2, -1)$. The x-intercepts, those points where the parabola crosses the x-axis, occur at $(-3, 0)$ and $(-1, 0)$.

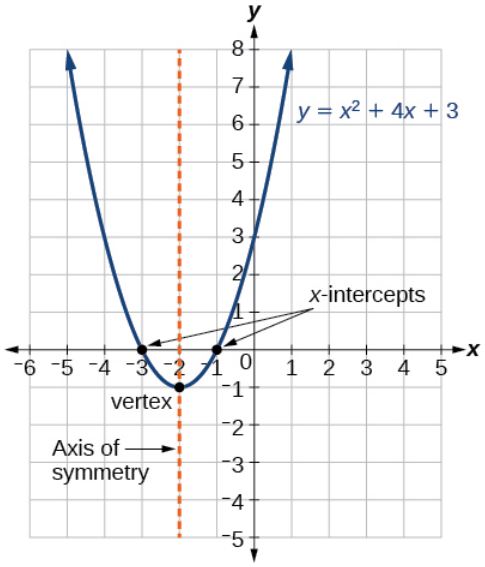


Figure 5.1.4: Graph of a parabola showing where the x and y intercepts, vertex, and axis of symmetry are for the function $y = x^2 + 4x + 3$.

The **standard form of a quadratic function** presents the function in the form

$$f(x) = a(x - h)^2 + k \tag{5.1.2}$$

where (h, k) is the vertex. Because the vertex appears in the standard form of the quadratic function, this form is also known as the **vertex form of a quadratic function**.

As with the general form, if $a > 0$, the parabola opens upward and the vertex is a minimum. If $a < 0$, the parabola opens downward, and the vertex is a maximum. Figure 5.1.5 represents the graph of the quadratic function written in standard form as $y = -3(x + 2)^2 + 4$. Since $x - h = x + 2$ in this example, $h = -2$. In this form, $a = -3$, $h = -2$, and $k = 4$. Because $a < 0$, the parabola opens downward. The vertex is at $(-2, 4)$.

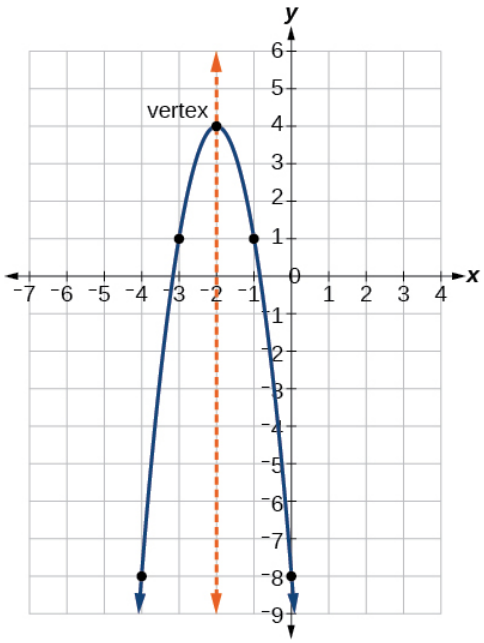


Figure 5.1.5: Graph of a parabola showing where the x and y intercepts, vertex, and axis of symmetry are for the function $y = -3(x+2)^2+4$.

The standard form is useful for determining how the graph is transformed from the graph of $y = x^2$. Figure 5.1.6 is the graph of this basic function.

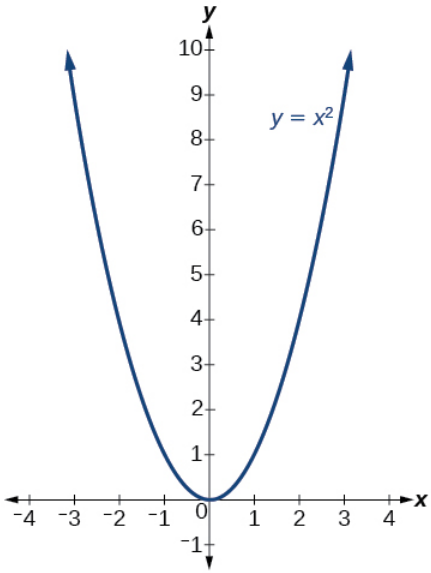


Figure 5.1.6: Graph of $y = x^2$.

If $k > 0$, the graph shifts upward, whereas if $k < 0$, the graph shifts downward. In Figure 5.1.5, $k > 0$, so the graph is shifted 4 units upward. If $h > 0$, the graph shifts toward the right and if $h < 0$, the graph shifts to the left. In Figure 5.1.5, $h < 0$, so the graph is shifted 2 units to the left. The magnitude of a indicates the stretch of the graph. If $|a| > 1$, the point associated with a particular x-value shifts farther from the x-axis, so the graph appears to become narrower, and there is a vertical stretch. But if $|a| < 1$, the point associated with a particular x-value shifts closer to the x-axis, so the graph appears to become wider, but in fact there is a vertical compression. In Figure 5.1.5, $|a| > 1$, so the graph becomes narrower.

The standard form and the general form are equivalent methods of describing the same function. We can see this by expanding out the general form and setting it equal to the standard form.

$$a(x-h)^2+k=ax^2+bx+c$$
$$ax^2-2ahx+(ah^2+k)=ax^2+bx+c$$

For the linear terms to be equal, the coefficients must be equal.

$$-2ah=b, \text{ so } h=-\frac{b}{2a}.$$

This is the **axis of symmetry** we defined earlier. Setting the constant terms equal:

$$\begin{aligned} ah^2+k &= c \\ k &= c-ah^2 \\ &= c-a\cdot\left(-\frac{b}{2a}\right)^2 \\ &= c-\frac{b^2}{4a} \end{aligned}$$

In practice, though, it is usually easier to remember that k is the output value of the function when the input is h, so $f(h)=k$.

Definitions: Forms of Quadratic Functions

A quadratic function is a function of degree two. The graph of a *quadratic function* is a parabola.

- The *general form* of a quadratic function is $f(x)=ax^2+bx+c$ where a, b, and c are real numbers and $a\neq 0$.
- The standard form of a quadratic function is $f(x)=a(x-h)^2+k$.
- The vertex (h,k) is located at

$$h=-\frac{b}{2a},\ k=f(h)=f\left(\frac{-b}{2a}\right). \tag{5.1.3}$$

HOWTO: Write a quadratic function in a general form

Given a graph of a quadratic function, write the equation of the function in general form.

1. Identify the horizontal shift of the parabola; this value is h. Identify the vertical shift of the parabola; this value is k.
2. Substitute the values of the horizontal and vertical shift for h and k. in the function $f(x)=a(x–h)^2+k$.
3. Substitute the values of any point, other than the vertex, on the graph of the parabola for x and $f(x)$.
4. Solve for the stretch factor, $|a|$.
5. If the parabola opens up, $a>0$. If the parabola opens down, $a<0$ since this means the graph was reflected about the x-axis.
6. Expand and simplify to write in general form.

Example 5.1.2: Writing the Equation of a Quadratic Function from the Graph

Write an equation for the quadratic function g in Figure 5.1.7 as a transformation of $f(x)=x^2$, and then expand the formula, and simplify terms to write the equation in general form.

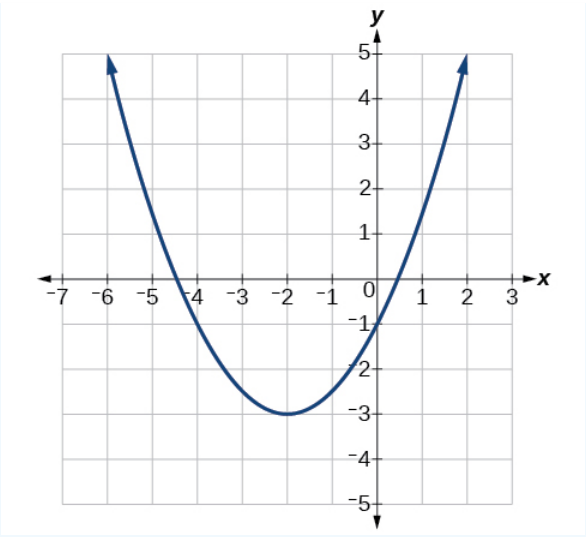


Figure 5.1.7: Graph of a parabola with its vertex at $(-2,-3)$.

Solution

We can see the graph of g is the graph of $f(x)=x^2$ shifted to the left 2 and down 3, giving a formula in the form $g(x)=a(x+2)^2-3$.

Substituting the coordinates of a point on the curve, such as $(0,-1)$, we can solve for the stretch factor.

$$-1=a(0+2)^2-3 \tag{5.1.4}$$
$$2=4a \tag{5.1.5}$$
$$a=\frac{1}{2} \tag{5.1.6}$$

In standard form, the algebraic model for this graph is $g(x)=\frac{1}{2}(x+2)^2-3$.

To write this in general polynomial form, we can expand the formula and simplify terms.

$$g(x)=\frac{1}{2}(x+2)^2-3 \tag{5.1.7}$$
$$=\frac{1}{2}(x+2)(x+2)-3 \tag{5.1.8}$$
$$=\frac{1}{2}(x^2+4x+4)-3 \tag{5.1.9}$$
$$=\frac{1}{2}x^2+2x+2-3 \tag{5.1.10}$$
$$=\frac{1}{2}x^2+2x-1 \tag{5.1.11}$$

Notice that the horizontal and vertical shifts of the basic graph of the quadratic function determine the location of the vertex of the parabola; the vertex is unaffected by stretches and compressions.

Analysis

We can check our work using the table feature on a graphing utility. First enter $\mathbf{Y1}=\frac{1}{2}(x+2)^2-3$. Next, select **TBLSET**, then use $\mathbf{TblStart}=-6$ and $\mathbf{\Delta Tbl}=2$, and select **TABLE**. See Table 5.1.1

Table 5.1.1

x	-6	-4	-2	0	2
y	-5	-1	-3	-1	5

The ordered pairs in the table correspond to points on the graph.

Exercise 5.1.2

A coordinate grid has been superimposed over the quadratic path of a basketball in Figure 5.1.8. Find an equation for the path of the ball. Does the shooter make the basket?

Figure 5.1.8: *Stop motioned picture of a boy throwing a basketball into a hoop to show the parabolic curve it makes. (credit: modification of work by Dan Meyer)*

Answer

The path passes through the origin and has vertex at $(-4, 7)$, so $h(x) = -\frac{7}{16}(x+4)^2 + 7$. To make the shot, $h(-7.5)$ would need to be about 4 but $h(-7.5) \approx 1.64$; he doesn't make it.

How To: Given a quadratic function in general form, find the vertex of the parabola.

1. Identify a, b, and c.
2. Find h, the x-coordinate of the vertex, by substituting a and b into $h = -\frac{b}{2a}$.
3. Find k, the y-coordinate of the vertex, by evaluating $k = f(h) = f\left(-\frac{b}{2a}\right)$.

Example 5.1.3: Finding the Vertex of a Quadratic Function

Find the vertex of the quadratic function $f(x) = 2x^2 - 6x + 7$. Rewrite the quadratic in standard form (vertex form).

Solution

The horizontal coordinate of the vertex will be at

$$h = -\frac{b}{2a} \tag{5.1.12}$$
$$= -\frac{-6}{2(2)} \tag{5.1.13}$$
$$= \frac{6}{4} \tag{5.1.14}$$
$$= \frac{3}{2} \tag{5.1.15}$$

The vertical coordinate of the vertex will be at

$$k = f(h) \tag{5.1.16}$$
$$= f\left(\frac{3}{2}\right) \tag{5.1.17}$$
$$= 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) + 7 \tag{5.1.18}$$
$$= \frac{5}{2} \tag{5.1.19}$$

Rewriting into standard form, the stretch factor will be the same as the a in the original quadratic.

$$f(x) = ax^2 + bx + c \tag{5.1.20}$$
$$f(x) = 2x^2 - 6x + 7$$

Using the vertex to determine the shifts,

$$f(x) = 2\left(x - \frac{3}{2}\right)^2 + \frac{5}{2} \tag{5.1.21}$$

Analysis

One reason we may want to identify the vertex of the parabola is that this point will inform us what the maximum or minimum value of the function is, (k),and where it occurs, (h).

Exercise 5.1.3

Given the equation $g(x) = 13 + x^2 - 6x$, write the equation in general form and then in standard form.

Answer

$g(x) = x^2 - 6x + 13$ in general form; $g(x) = (x-3)^2 + 4$ in standard form.

Finding the Domain and Range of a Quadratic Function

Any number can be the input value of a quadratic function. Therefore, the domain of any quadratic function is all real numbers. Because parabolas have a maximum or a minimum point, the range is restricted. Since the vertex of a parabola will be either a maximum or a minimum, the range will consist of all y-values greater than or equal to the y-coordinate at the turning point or less than or equal to the y-coordinate at the turning point, depending on whether the parabola opens up or down.

Definition: Domain and Range of a Quadratic Function

The domain of any **quadratic function** is all real numbers.

The range of a quadratic function written in general form $f(x) = ax^2 + bx + c$ with a positive a value is $f(x) \geq f(-\frac{b}{2a})$, or $[f(-\frac{b}{2a}), \infty)$; the range of a quadratic function written in general form with a negative a value is $f(x) \leq f(-\frac{b}{2a})$, or $(-\infty, f(-\frac{b}{2a})]$.

The range of a quadratic function written in standard form $f(x) = a(x-h)^2 + k$ with a positive a value is $f(x) \geq k$; the range of a quadratic function written in standard form with a negative a value is $f(x) \leq k$.

How To: Given a quadratic function, find the domain and range.

1. Identify the domain of any quadratic function as all real numbers.
2. Determine whether a is positive or negative. If a is positive, the parabola has a minimum. If a is negative, the parabola has a maximum.
3. Determine the maximum or minimum value of the parabola, k.
4. If the parabola has a minimum, the range is given by $f(x) \geq k$, or $[k, \infty)$. If the parabola has a maximum, the range is given by $f(x) \leq k$, or $(-\infty, k]$.

Example 5.1.4: Finding the Domain and Range of a Quadratic Function

Find the domain and range of $f(x) = -5x^2 + 9x - 1$.

Solution

As with any quadratic function, the domain is all real numbers.

Because a is negative, the parabola opens downward and has a maximum value. We need to determine the maximum value. We can begin by finding the x-value of the vertex.

$$h = -\frac{b}{2a} \tag{5.1.22}$$
$$= -\frac{9}{2(-5)} \tag{5.1.23}$$
$$= \frac{9}{10} \tag{5.1.24}$$

The maximum value is given by $f(h)$.

$$f(\frac{9}{10}) = 5(\frac{9}{10})^2 + 9(\frac{9}{10}) - 1 \tag{5.1.25}$$
$$= \frac{61}{20} \tag{5.1.26}$$

The range is $f(x) \leq \frac{61}{20}$, or $(-\infty, \frac{61}{20}]$.

Exercise 5.1.4

Find the domain and range of $f(x) = 2\left(x - \frac{4}{7}\right)^2 + \frac{8}{11}$.

Answer

The domain is all real numbers. The range is $f(x) \geq \frac{8}{11}$, or $\left[\frac{8}{11}, \infty\right)$.

Determining the Maximum and Minimum Values of Quadratic Functions

The output of the quadratic function at the vertex is the maximum or minimum value of the function, depending on the orientation of the **parabola**. We can see the maximum and minimum values in Figure 5.1.9.

Figure 5.1.9: Minimum and maximum of two quadratic functions.

There are many real-world scenarios that involve finding the maximum or minimum value of a quadratic function, such as applications involving area and revenue.

Example 5.1.5: Finding the Maximum Value of a Quadratic Function

A backyard farmer wants to enclose a rectangular space for a new garden within her fenced backyard. She has purchased 80 feet of wire fencing to enclose three sides, and she will use a section of the backyard fence as the fourth side.

a. Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length L.
b. What dimensions should she make her garden to maximize the enclosed area?

Solution

Let's use a diagram such as Figure 5.1.10 to record the given information. It is also helpful to introduce a temporary variable, W, to represent the width of the garden and the length of the fence section parallel to the backyard fence.

Figure 5.1.10: Diagram of the garden and the backyard.

a. We know we have only 80 feet of fence available, and $L + W + L = 80$, or more simply, $2L + W = 80$. This allows us to represent the width, W, in terms of L.

$$W = 80 - 2L \tag{5.1.27}$$

Now we are ready to write an equation for the area the fence encloses. We know the area of a rectangle is length multiplied by width, so

$$A = LW = L(80 - 2L) \tag{5.1.28}$$
$$A(L) = 80L - 2L^2 \tag{5.1.29}$$

This formula represents the area of the fence in terms of the variable length L. The function, written in general form, is

$$A(L) = -2L^2 + 80L \tag{5.1.30}$$

.

The quadratic has a negative leading coefficient, so the graph will open downward, and the vertex will be the maximum value for the area. In finding the vertex, we must be careful because the equation is not written in standard polynomial form with decreasing powers. This is why we rewrote the function in general form above. Since a is the coefficient of the squared term, $a = -2$, $b = 80$, and $c = 0$.

To find the vertex:

$$h = -\frac{80}{2(-2)} \qquad k = A(20) \tag{5.1.31}$$
$$= 20 \qquad \text{and} \qquad = 80(20) - 2(20)^2 \tag{5.1.32}$$
$$= 800 \tag{5.1.33}$$

The maximum value of the function is an area of 800 square feet, which occurs when $L = 20$ feet. When the shorter sides are 20 feet, there is 40 feet of fencing left for the longer side. To maximize the area, she should enclose the garden so the two shorter sides have length 20 feet and the longer side parallel to the existing fence has length 40 feet.

Analysis

This problem also could be solved by graphing the quadratic function. We can see where the maximum area occurs on a graph of the quadratic function in Figure 5.1.11.

Figure 5.1.11: Graph of the parabolic function $A(L) = -2L^2 + 80L$

How To: Given an application involving revenue, use a quadratic equation to find the maximum.

1. Write a quadratic equation for revenue.
2. Find the vertex of the quadratic equation.
3. Determine the y-value of the vertex.

Example 5.1.6: Finding Maximum Revenue

The unit price of an item affects its supply and demand. That is, if the unit price goes up, the demand for the item will usually decrease. For example, a local newspaper currently has 84,000 subscribers at a quarterly charge of $30. Market research has suggested that if the owners raise the price to $32, they would lose 5,000 subscribers. Assuming that subscriptions are linearly related to the price, what price should the newspaper charge for a quarterly subscription to maximize their revenue?

Solution

Revenue is the amount of money a company brings in. In this case, the revenue can be found by multiplying the price per subscription times the number of subscribers, or quantity. We can introduce variables, p for price per subscription and Q for quantity, giving us the equation $\text{Revenue} = pQ$.

Because the number of subscribers changes with the price, we need to find a relationship between the variables. We know that currently $p = 30$ and $Q = 84,000$. We also know that if the price rises to $32, the newspaper would lose 5,000 subscribers, giving a second pair of values, $p = 32$ and $Q = 79,000$. From this we can find a linear equation relating the two quantities. The slope will be

$$m = \frac{79,000 - 84,000}{32 - 30} \tag{5.1.34}$$
$$= -\frac{5,000}{2} \tag{5.1.35}$$
$$= -2,500 \tag{5.1.36}$$

This tells us the paper will lose 2,500 subscribers for each dollar they raise the price. We can then solve for the y-intercept.

$$Q = -2500p + b \quad \text{Substitute in the point } Q = 84,000 \text{ and } p = 30 \tag{5.1.37}$$
$$84,000 = -2500(30) + b \quad \text{Solve for } b \tag{5.1.38}$$
$$b = 159,000 \tag{5.1.39}$$

This gives us the linear equation $Q = -2,500p + 159,000$ relating cost and subscribers. We now return to our revenue equation.

$$\text{Revenue} = pQ \tag{5.1.40}$$
$$\text{Revenue} = p(-2,500p + 159,000) \tag{5.1.41}$$
$$\text{Revenue} = -2,500p^2 + 159,000p \tag{5.1.42}$$

We now have a quadratic function for revenue as a function of the subscription charge. To find the price that will maximize revenue for the newspaper, we can find the vertex.

$$h = -\frac{159,000}{2(-2,500)} \tag{5.1.43}$$
$$= 31.8 \tag{5.1.44}$$

The model tells us that the maximum revenue will occur if the newspaper charges $31.80 for a subscription. To find what the maximum revenue is, we evaluate the revenue function.

$$\text{maximum revenue} = -2,500(31.8)^2 + 159,000(31.8) \tag{5.1.45}$$
$$= 2,528,100 \tag{5.1.46}$$

Analysis

This could also be solved by graphing the quadratic as in Figure 5.1.12. We can see the maximum revenue on a graph of the quadratic function.

Figure 5.1.12: Graph of the parabolic function

Finding the x- and y-Intercepts of a Quadratic Function

Much as we did in the application problems above, we also need to find intercepts of quadratic equations for graphing parabolas. Recall that we find the y-intercept of a quadratic by evaluating the function at an input of zero, and we find the x-intercepts at locations where the output is zero. Notice in Figure 5.1.13 that the number of x-intercepts can vary depending upon the location of the graph.

Figure 5.1.13: Number of x-intercepts of a parabola.

How To: Given a quadratic function $f(x)$, find the y- and x-intercepts.

1. Evaluate $f(0)$ to find the y-intercept.
2. Solve the quadratic equation $f(x) = 0$ to find the x-intercepts.

Example 5.1.7: Finding the y- and x-Intercepts of a Parabola

Find the y- and x-intercepts of the quadratic $f(x) = 3x^2 + 5x - 2$.

Solution

We find the y-intercept by evaluating $f(0)$.

$$f(0) = 3(0)^2 + 5(0) - 2 \tag{5.1.47}$$
$$= -2 \tag{5.1.48}$$

So the y-intercept is at $(0, -2)$.

For the x-intercepts, we find all solutions of $f(x) = 0$.

$$0 = 3x^2 + 5x - 2 \tag{5.1.49}$$

In this case, the quadratic can be factored easily, providing the simplest method for solution.

$$0 = (3x - 1)(x + 2) \tag{5.1.50}$$

$$0 = 3x - 1 \qquad 0 = x + 2 \tag{5.1.51}$$
$$x = \frac{1}{3} \quad \text{or} \quad x = -2 \tag{5.1.52}$$

So the x-intercepts are at $(\frac{1}{3}, 0)$ and $(-2, 0)$.

Analysis

By graphing the function, we can confirm that the graph crosses the y-axis at $(0, -2)$. We can also confirm that the graph crosses the x-axis at $\left(\frac{1}{3}, 0\right)$ and $(-2, 0)$. See Figure 5.1.14.

Figure 5.1.14: Graph of a parabola.

Rewriting Quadratics in Standard Form

In Example 5.1.7, the quadratic was easily solved by factoring. However, there are many quadratics that cannot be factored. We can solve these quadratics by first rewriting them in standard form.

How To: Given a quadratic function, find the x-intercepts by rewriting in standard form.

1. Substitute a and b into $h = -\frac{b}{2a}$.
2. Substitute $x = h$ into the general form of the quadratic function to find k.
3. Rewrite the quadratic in standard form using h and k.
4. Solve for when the output of the function will be zero to find the x-intercepts.

Example 5.1.8: Finding the x-Intercepts of a Parabola

Find the x-intercepts of the quadratic function $f(x) = 2x^2 + 4x - 4$.

Solution

We begin by solving for when the output will be zero.

$$0 = 2x^2 + 4x - 4$$

Because the quadratic is not easily factorable in this case, we solve for the intercepts by first rewriting the quadratic in standard form.

$$f(x) = a(x - h)^2 + k$$

We know that $a = 2$. Then we solve for h and k.

$$\begin{aligned} h &= -\frac{b}{2a} & k &= f(-1) \\ &= -\frac{4}{2(2)} & &= 2(-1)^2 + 4(-1) - 4 \\ &= -1 & &= -6 \end{aligned}$$

So now we can rewrite in standard form.

$$f(x) = 2(x + 1)^2 - 6$$

We can now solve for when the output will be zero.

$$\begin{aligned} 0 &= 2(x + 1)^2 - 6 \\ 6 &= 2(x + 1)^2 \\ 3 &= (x + 1)^2 \\ x + 1 &= \pm\sqrt{3} \\ x &= -1 \pm \sqrt{3} \end{aligned}$$

The graph has x-intercepts at $(-1 - \sqrt{3}, 0)$ and $(-1 + \sqrt{3}, 0)$.

Analysis

We can check our work by graphing the given function on a graphing utility and observing the x-intercepts. See Figure 5.1.15.

Figure 5.1.15: Graph of a parabola which has the following x-intercepts: $(-2.732, 0)$ and $(0.732, 0)$.

Exercise 5.1.1

In Try It 5.1.1, we found the standard and general form for the function $g(x) = 13 + x^2 - 6x$. Now find the y- and x-intercepts (if any).

Answer

y-intercept at $(0, 13)$, No x-intercepts

Example 5.1.9: Solving a Quadratic Equation with the Quadratic Formula

Solve $x^2 + x + 2 = 0$.

Solution

Let's begin by writing the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

When applying the quadratic formula, we identify the coefficients a, b and c. For the equation $x^2+x+2=0$, we have $a=1$, $b=1$, and $c=2$. Substituting these values into the formula we have:

$$\begin{aligned} x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ &= \frac{-1\pm\sqrt{1^2-4\cdot 1\cdot (2)}}{2\cdot 1} \\ &= \frac{-1\pm\sqrt{1-8}}{2} \\ &= \frac{-1\pm\sqrt{-7}}{2} \\ &= \frac{-1\pm i\sqrt{7}}{2} \end{aligned}$$

The solutions to the equation are $x=\frac{-1+i\sqrt{7}}{2}$ and $x=\frac{-1-i\sqrt{7}}{2}$ or $x=-\frac{1}{2}+\frac{i\sqrt{7}}{2}$ and $x=\frac{-1}{2}-\frac{i\sqrt{7}}{2}$.

Example 5.1.10: Applying the Vertex and x-Intercepts of a Parabola

A ball is thrown upward from the top of a 40 foot high building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation $H(t)=-16t^2+80t+40$.

When does the ball reach the maximum height?
What is the maximum height of the ball?
When does the ball hit the ground?

The ball reaches the maximum height at the vertex of the parabola.

$$h=-\frac{80}{2(-16)} \tag{5.1.53}$$
$$=\frac{80}{32} \tag{5.1.54}$$
$$=\frac{5}{2} \tag{5.1.55}$$
$$=2.5 \tag{5.1.56}$$

The ball reaches a maximum height after 2.5 seconds.

To find the maximum height, find the y-coordinate of the vertex of the parabola.

$$k=H(-\frac{b}{2a}) \tag{5.1.57}$$
$$=H(2.5) \tag{5.1.58}$$
$$=-16(2.5)^2+80(2.5)+40 \tag{5.1.59}$$
$$=140 \tag{5.1.60}$$

The ball reaches a maximum height of 140 feet.

To find when the ball hits the ground, we need to determine when the height is zero, $H(t)=0$.

We use the quadratic formula.

$$t=\frac{-80\pm\sqrt{80^2-4(-16)(40)}}{2(-16)} \tag{5.1.61}$$
$$=\frac{-80\pm\sqrt{8960}}{-32} \tag{5.1.62}$$

Because the square root does not simplify nicely, we can use a calculator to approximate the values of the solutions.

$$t=\frac{-80-\sqrt{8960}}{-32}\approx 5.458 \text{ or } t=\frac{-80+\sqrt{8960}}{-32}\approx -0.458 \tag{5.1.63}$$

The second answer is outside the reasonable domain of our model, so we conclude the ball will hit the ground after about 5.458 seconds. See Figure 5.1.16.

Figure 5.1.16

5.1.5: A rock is thrown upward from the top of a 112-foot high cliff overlooking the ocean at a speed of 96 feet per second. The rock's height above ocean can be modeled by the equation $H(t) = -16t^2 + 96t + 112$.

a. When does the rock reach the maximum height?
b. What is the maximum height of the rock?
c. When does the rock hit the ocean?

Solution

a. 3 seconds b. 256 feet c. 7 seconds

Key Equations

- general form of a quadratic function: $f(x) = ax^2 + bx + c$
- the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- standard form of a quadratic function: $f(x) = a(x - h)^2 + k$

Key Concepts

- A polynomial function of degree two is called a quadratic function.
- The graph of a quadratic function is a parabola. A parabola is a U-shaped curve that can open either up or down.
- The axis of symmetry is the vertical line passing through the vertex. The zeros, or x-intercepts, are the points at which the parabola crosses the x-axis. The y-intercept is the point at which the parabola crosses the y-axis.
- Quadratic functions are often written in general form. Standard or vertex form is useful to easily identify the vertex of a parabola. Either form can be written from a graph.
- The vertex can be found from an equation representing a quadratic function. .
- The domain of a quadratic function is all real numbers. The range varies with the function.
- A quadratic function's minimum or maximum value is given by the y-value of the vertex.
- The minimum or maximum value of a quadratic function can be used to determine the range of the function and to solve many kinds of real-world problems, including problems involving area and revenue.
- Some quadratic equations must be solved by using the quadratic formula.
- The vertex and the intercepts can be identified and interpreted to solve real-world problems.

Glossary

axis of symmetry
a vertical line drawn through the vertex of a parabola around which the parabola is symmetric; it is defined by $x = -\frac{b}{2a}$.

general form of a quadratic function
the function that describes a parabola, written in the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and a≠0.

standard form of a quadratic function
the function that describes a parabola, written in the form $f(x) = a(x - h)^2 + k$, where (h, k) is the vertex.

vertex
the point at which a parabola changes direction, corresponding to the minimum or maximum value of the quadratic function

vertex form of a quadratic function
another name for the standard form of a quadratic function

zeros
in a given function, the values of x at which $y = 0$, also called roots

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5.2: Quadratic inequalities

We can think about an earlier chapter where we obtained solutions for linear inequalities where the solutions are intervals of numbers. With quadratic inequalities, it is similar to linear inequalities, but, now, instead of lines, we have parabolas. Hence, we have two methods for solving quadratic inequalities: graphing or algebraically.

Solving Quadratic Inequalities by Graphing

Note

We can use the zeros (or roots) of a graph of a quadratic equation to solve for quadratic inequalities. Recall, the zeros (or roots) of a graph are the x-intercepts, i.e., $y = 0$.

Example 11.5.1

Solve $x^2 - 3x - 4 \geq 0$ by using the graphing method.

Solution

We first begin by graphing $x^2 - 3x - 4$ as we would in the previous section. Recall, the x-coordinate of the vertex is $-\frac{b}{2a}$. Hence,

$$-\frac{b}{2a} = -\frac{-3}{2(1)} = \frac{3}{2} = 1.5$$

The y-coordinate of the vertex is $x^2 - 3x - 4$ evaluated for $x = 1.5$:

$$(1.5)^2 - 3(1.5) - 4 = -6.25$$

The vertex is at $(1.5, -6.25)$. The y-intercept is when $x = 0$:

$$0^2 - 3(0) - 4 = -4$$

and the x-intercepts are when $x^2 - 3x - 4 = 0$. Solving for the roots we get

$$\begin{aligned} x^2 - 3x - 4 &= 0 \\ (x-4)(x+1) &= 0 \\ x - 4 = 0 \quad &\text{or} \quad x + 1 = 0 \\ x = 4 \quad &\text{or} \quad x = -1 \end{aligned} \tag{5.2.1}$$

Thus, the intercepts are $(0, -4)$, $(4, 0)$, and $(-1, 0)$ with a vertex at $(1.5, -6.25)$. Now we draw the graph:

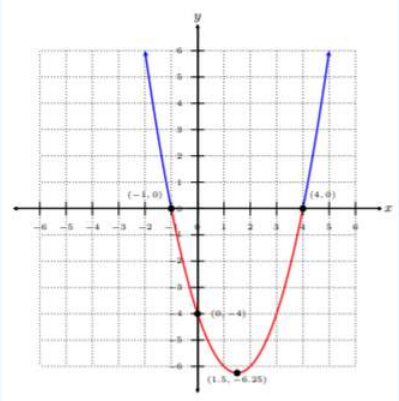


Figure 11.5.1

Given $x^2 - 3x - 4 \geq 0$ says we are specifically looking for the values of $x^2 - 3x - 4$ in which are greater than or equal to zero. In other words, we are looking for all y-values that are above the x-axis because that is where all the y-values are positive.

Looking at the graph above, these are all the values on the blue parts of the graph. Thus, using the graphing method, the solution is $(-\infty, -1] \cup [4, \infty)$. Note, we use brackets since it is $\geq$.

Notice, we ignore the red part of the parabola since this is where $x^2 - 3x - 4$ is negative, i.e., all the y-values are negative, because we were only looking for when $x^2 - 3x - 4 \geq 0$, the parts of the graph above the x-axis.

Solving Quadratic Inequalities Algebraically

The next method to solving quadratic inequalities is algebraically. Hence, there is a little more work involved since the method is algebraic. We must follow an order of steps to obtain the correct solution. The good news is that these two methods are similar where we always use the x-intercepts to determine the intervals.

Steps to solving quadratic inequalities

Step 1. Rewrite the inequality so that $ax^2 + bx + c$ is on one side and zero is on the other.

Step 2. Determine where the *inequality is zero* using any method appropriate.

Step 3. Use the x-values obtained in the previous step to label on a number line.

Step 4. Take test values to observe where the inequality is true.

- If the inequality is < 0 or ≤ 0, then the inequality is true where the test values are negative.
- If the inequality is > 0 or ≥ 0, then the inequality is true where the test values are positive.

Step 5. Write the solution in interval notation.

Example 11.5.2

Solve algebraically: $-x^2 + 8 > 2x$

Solution

Step 1. We rewrite $-x^2 + 8 > 2x$ so that zero is on one side:

$$-x^2 - 2x + 8 > 0$$

Step 2. We set the left side equal to zero to obtain the roots (or zeros):

$$\begin{aligned} -x^2 - 2x + 8 &= 0 \\ x^2 + 2x - 8 &= 0 \\ (x+4)(x-2) &= 0 \\ x + 4 = 0 \quad &\text{or} \quad x - 2 = 0 \\ x = -4 \quad &\text{or} \quad x = 2 \end{aligned} \tag{5.2.2}$$

Step 3. Label -4 and 2 on a blank number line:

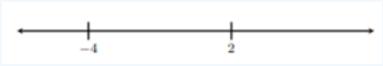


Figure 11.5.2

Step 4. We take test values on each side of -4 and 2. Let's choose fairly easy numbers such as -5, 0, and 3. We plug these numbers into $-x^2 - 2x + 8$ and determine whether the value is positive or negative:



Figure 11.5.3

$$\begin{aligned} -(-5)^2 - 2(-5) + 8 = -7 < 0 \quad &\Longrightarrow \quad \text{letting } x = -5 \\ -(0)^2 - 2(0) + 8 = 8 > 0 \quad &\Longrightarrow \quad \text{letting } x = 0 \\ -(3)^2 - 2(3) + 8 = -7 < 0 \quad &\Longrightarrow \quad \text{letting } x = 3 \end{aligned} \tag{5.2.3}$$

Step 5. Since $-x^2 - 2x + 8 > 0$ (from **Step 1.**), then we are looking for where the test values are positive. Looking at the number line above, we see these are the values in between -4 and 2. Thus, the solution is $(-4, 2)$. Note, we use parenthesis since the inequality symbol is $>$.

Quadratic Inequalities Homework

Solve the inequality. Write the solution in interval notation.

Exercise 11.5.1

$x^2 - 9x + 18 > 0$

Exercise 11.5.2

$x^2 - 2x - 24 \leq 0$

Exercise 11.5.3

$x^2 - 2x - 3 < 0$

Exercise 11.5.4

$x^2 + 10x + 24 \geq 0$

Exercise 11.5.5

$x^2 - 4x + 4 > 0$

Exercise 11.5.6

$x^2 + 2x \geq 8$

Exercise 11.5.7

$x^2 - 4x \leq -3$

Exercise 11.5.8

$3x^2 + 7x - 20 \leq 0$

Exercise 11.5.9

$4x^2 + 11x - 20 \geq 0$

Exercise 11.5.10

$3x^2 + 2x - 1 < 0$

Exercise 11.5.11

$-4x^2 + 7x \geq 0$

Exercise 11.5.12

$x^2 + 6x \geq 0$

Exercise 11.5.13

$x^2 - 12x + 36 < 0$

Exercise 11.5.14

$x^2 - 2x + 1 \geq 0$

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5.3: Applications with quadratic functions

There are many applications involving quadratic functions that it is almost challenging to pick the few functions to discuss for this section. Yet, we only choose a few applications that are common in most algebra classes. First, we must start with defining the ***extreme value*** of a quadratic function.

Find the Extreme Value

Definition: Extreme Value

The ***extreme value*** of a quadratic function $f(x)$ is either the *minimum* or *maximum* value of the quadratic function $f(x)$. The extreme value is given as

$$f\left(-\frac{b}{2a}\right)$$

The *minimum* value is located at the lowest point of an upwards parabola and the *maximum* value is located at the highest point of a downwards parabola.

Note

Notice in the definition that the extreme value is the y-coordinate of the vertex. Recall, the vertex is

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$

and we can see the y-coordinate is just the extreme value. Hence, when we use the words vertex, minimum or maximum value, or extreme value, they are all associated with the vertex of a parabola.

Example 11.6.1

Find the vertex and the extreme value of the function $f(x) = x^2 - 4x + 5$.

Solution

Notice $a = 1$, which means $a > 0$. Hence, $f(x)$ is an upwards parabola and, from the definition, we expect $f(x)$ to have a minimum value. Let's use the formula to find the vertex, where $a = 1$ and $b = -4$.

$$\begin{aligned} x &= -\frac{b}{2a} && \text{Plug-n-chug} \\ x &= -\frac{-4}{2(1)} && \text{Simplify} \\ x &= 2 && \text{The } x\text{-coordinate of the vertex} \end{aligned}$$

Next, we find the y-coordinate of the vertex by obtaining $f(2)$.

$$\begin{aligned} f(x) &= x^2 - 4x + 5 && \text{Plug-n-chug} \\ f(2) &= (2)^2 - 4(2) + 5 && \text{Evaluate} \\ f(2) &= 1 && \text{The } y\text{-coordinate of the vertex} \end{aligned}$$

Hence, the vertex is $(2, 1)$. Next, we find the extreme value of $f(x)$. From the vertex calculation, we see $f\left(-\frac{b}{2a}\right) = f(2) = 1$. Thus, the extreme value is 1, the y-coordinate of the vertex.

Example 11.6.2

Find the vertex and the extreme value of the function $q(n) = -3n^2 - 5n + 3$.

Solution

Notice $a = -3$, which means $a < 0$. Hence, $q(n)$ is an downwards parabola and, from the definition, we expect $q(n)$ to have a maximum value. Let's use the formula to find the vertex, where $a = -3$ and $b = -5$.

$$\begin{aligned} n &= -\frac{b}{2a} && \text{Plug-n-chug} \\ n &= -\frac{-5}{2(-3)} && \text{Simplify} \\ n &= -\frac{5}{6} && \text{The } x\text{-coordinate of the vertex} \end{aligned}$$

Next, we find the y-coordinate of the vertex by obtaining $q\left(-\frac{5}{6}\right)$.

$$\begin{aligned} q(n) &= -3n^2 - 5n + 3 && \text{Plug-n-chug} \\ q\left(-\frac{5}{6}\right) &= -3\left(-\frac{5}{6}\right)^2 - 5\left(-\frac{5}{6}\right) + 3 && \text{Evaluate} \\ q\left(-\frac{5}{6}\right) &= \frac{61}{12} && \text{The } y\text{-coordinate of the vertex} \end{aligned}$$

Hence, the vertex is $\left(-\frac{5}{6}, \frac{61}{12}\right)$. Next, we find the extreme value of $q(n)$. From the vertex calculation, we see $q\left(-\frac{b}{2a}\right) = q\left(-\frac{5}{6}\right) = \frac{61}{12}$. Thus, the extreme value is $\frac{61}{12}$, the y-coordinate of the vertex.

It may seem a little redundant to find the vertex and the extreme value, but, remember, the goal is to apply this concept to real-world applications with quadratic functions. Let's take a look at a few applications where we find the extreme value in context of a real-world model.

Projectile Motion

Example 11.6.3

A rocket is launched at $t = 0$ seconds. Its height, in meters above sea-level, as a function of time is given by $h(t) = -4.9t^2 + 46t + 157$. At what time does the rocket reach its maximum height? At what height does the rocket reach its maximum height above the water? *Round the answers to 2 decimal places.*

Solution

When we read the word *maximum*, we should think about the vertex of $h(t)$. Since we need to find the time in which the maximum height occurs, then we can find the x-coordinate of the vertex.

$$\begin{aligned} t &= -\frac{b}{2a} \\ t &= -\frac{46}{2(-4.9)} \\ t &\approx 4.69 \end{aligned} \tag{5.3.1}$$

Thus, the maximum height occurs after 4.69 seconds. Next, we find the height of the rocket when it reaches its maximum height above the water. Since we need to find the maximum height, then we need to find the y-coordinate of the vertex, or $h(4.69)$.

$$h(4.69) = -4.9(4.69)^2 + 46(4.69) + 157 \approx 264.96$$

Thus, the maximum height of the rocket is 264.96 meters after 4.69 seconds the rocket is launched.

Revenue and Cost Functions

Example 11.6.4

The revenue, $R(x)$, of producing and selling x *Awesome Hearing Aids* is modeled by the function $R(x) = -6x^2 + 108x$. What is the maximum revenue?

Solution

To find the maximum revenue, we need to find $R\left(-\frac{b}{2a}\right)$, where $a=-6$ and $b=108$. Let's plug-n-chug this into $R(x)$ to find the maximum revenue.

$$\begin{aligned} R(x) &= -6x^2+108x \\ R\left(-\frac{b}{2a}\right) &= -6\left(-\frac{b}{2a}\right)^2+108\left(-\frac{b}{2a}\right) \\ R\left(-\frac{108}{2(-6)}\right) &= -6\left(-\frac{108}{2(-6)}\right)^2+108\left(-\frac{108}{2(-6)}\right) \\ R\left(-\frac{108}{2(-6)}\right) &= 486 \end{aligned} \tag{5.3.2}$$

Thus, the maximum revenus is \$486.

Example 11.6.5

The cost, $C(x)$, of producing x *Totally Cool Coolers* is modeled by the function $C(x)=0.005x^2-0.3x+17$. What is the minimum cost?

Solution

To find the minimum cost, we need to find $C\left(-\frac{b}{2a}\right)$, where $a=0.005$ and $b=-0.3$. Let's plug-n-chug this into $C(x)$ to find the minimum cost.

$$\begin{aligned} C(x) &= 0.005x^2-0.3x+17 \\ C\left(-\frac{b}{2a}\right) &= 0.005\left(-\frac{b}{2a}\right)^2-0.3\left(-\frac{b}{2a}\right)+17 \\ C\left(-\frac{-0.3}{2(0.005)}\right) &= 0.005\left(-\frac{-0.3}{2(0.005)}\right)^2-0.3\left(-\frac{-0.3}{2(0.005)}\right)+17 \\ C\left(-\frac{-0.3}{2(0.005)}\right) &= 12.5 \end{aligned} \tag{5.3.3}$$

Note

From all the examples, we see the variety of methods in obtaining the extreme value of a quadratic function. We can either graph the function, find each coordinate of the vertex, or directly calculate the extreme value. It is at the discretion of the student to use any method. However, directly calculating the extreme value is recommended when only the extreme value is needed.

Applications with Quadratic Functions Homework

Exercise 11.6.1

Find the vertex and the extreme value of the function

$$f(x)=2x^2-5x-4$$

What is the vertex? What is the extreme value?

Exercise 11.6.2

Find the vertex and the extreme value of the function

$$f(x)=2x^2-2x-5$$

What is the vertex? What is the extreme value?

Exercise 11.6.3

Find the vertex and the extreme value of the function

$$f(x) = x^2 + 2x + 24$$

What is the vertex? What is the extreme value?

Exercise 11.6.4

A rocket is launched at $t = 0$ seconds. Its height, in meters above sea-level, as a function of time is given by

$$h(t) = -4.9t^2 + 271t + 150$$

At what time does the rocket reach its maximum height? At what height does the rocket reach its maximum height above the water? *Round your answer to 2 decimal places.*

Exercise 11.6.5

A rocket is launched at $t = 0$ seconds. Its height, in meters above sea-level, as a function of time is given by

$$h(t) = -4.9t^2 + 190t + 395$$

At what time does the rocket reach its maximum height? At what height does the rocket reach its maximum height above the water? *Round your answer to 2 decimal places.*

Exercise 11.6.6

A rocket is launched at $t = 0$ seconds. Its height, in meters above sea-level, as a function of time is given by

$$h(t) = -4.9t^2 + 223t + 129$$

At what time does the rocket reach its maximum height? At what height does the rocket reach its maximum height above the water? *Round your answer to 2 decimal places.*

Exercise 11.6.7

The cost, $C(x)$, of producing x *Totally Cool Coolers* is modeled by the function

$$C(x) = 0.005x^2 - 0.25x + 12$$

How many coolers need to be produced and sold in order to minimize the cost? What is the cost?

Exercise 11.6.8

The cost, $C(x)$, of producing x *Totally Cool Coolers* is modeled by the function

$$C(x) = 0.005x^2 - 0.45x + 25$$

How many coolers need to be produced and sold in order to minimize the cost? What is the cost?

Exercise 11.6.9

The revenue, $R(x)$, of producing and selling x *Awesome Hearing Aids* is modeled by the function

$$R(x) = -5x^2 + 105x$$

How many hearing aids need to be produced and sold in order to maximize the revenue? What is the revenue?

Exercise 11.6.10

The revenue, $R(x)$, of producing and selling x *Awesome Hearing Aids* is modeled by the function

$$R(x) = -2x^2 + 62x$$

How many hearing aids need to be produced and sold in order to maximize the revenue? What is the revenue?

Exercise 11.6.11

The revenue, $R(x)$, of producing and selling x *Awesome Hearing Aids* is modeled by the function

$$R(x) = -4x^2 + 76x$$

How many hearing aids need to be produced and sold in order to maximize the revenue? What is the revenue

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5.4: Power Functions and Polynomial Functions

Learning Objectives

- Identify power functions.
- Identify polynomial functions.
- Identify the degree and leading coefficient of polynomial functions.

Suppose a certain species of bird thrives on a small island. Its population over the last few years is shown in Table 5.4.1.

Table 5.4.1

Year	2009	2010	2011	2012	2013
Bird Population	800	897	992	1,083	1,169

The population can be estimated using the function $P(t) = -0.3t^3 + 97t + 800$, where $P(t)$ represents the bird population on the island t years after 2009. We can use this model to estimate the maximum bird population and when it will occur. We can also use this model to predict when the bird population will disappear from the island. In this section, we will examine functions that we can use to estimate and predict these types of changes.



Figure 5.4.1: (credit: Jason Bay, Flickr)

Identifying Power Functions

In order to better understand the bird problem, we need to understand a specific type of function. A power function is a function with a single term that is the product of a real number, a coefficient, and a variable raised to a fixed real number. (A number that multiplies a variable raised to an exponent is known as a coefficient.)

As an example, consider functions for area or volume. The function for the area of a circle with radius r is

$$A(r) = \pi r^2$$

and the function for the volume of a sphere with radius r is

$$V(r) = \frac{4}{3}\pi r^3$$

Both of these are examples of power functions because they consist of a coefficient, π or $\frac{4}{3}\pi$, multiplied by a variable r raised to a power.

Definition: Power Function

A *power function* is a function that can be represented in the form

$$f(x) = kx^p \tag{5.4.1}$$

where k and p are real numbers, and k is known as the *coefficient*.

Q&A: Is $f(x) = 2^x$ a power function?

No. A power function contains a variable base raised to a fixed power (Equation 5.4.1). This function has a constant base raised to a variable power. This is called an exponential function, not a power function. This function will be discussed later.

Example 5.4.1: Identifying Power Functions

Which of the following functions are power functions?

$f(x) = 1$	Constant function
$f(x) = x$	Identify function
$f(x) = x^2$	Quadratic function
$f(x) = x^3$	Cubic function
$f(x) = \frac{1}{x}$	Reciprocal function
$f(x) = \frac{1}{x^2}$	Reciprocal squared function
$f(x) = \sqrt{x}$	Square root function
$f(x) = \sqrt[3]{x}$	Cube root function

Solution

All of the listed functions are power functions.

The constant and identity functions are power functions because they can be written as $f(x) = x^0$ and $f(x) = x^1$ respectively.

The quadratic and cubic functions are power functions with whole number powers $f(x) = x^2$ and $f(x) = x^3$.

The **reciprocal** and reciprocal squared functions are power functions with negative whole number powers because they can be written as $f(x) = x^{-1}$ and $f(x) = x^{-2}$.

The square and **cube root** functions are power functions with fractional powers because they can be written as $f(x) = x^{1/2}$ or $f(x) = x^{1/3}$.

Exercise 5.4.1

Which functions are power functions?

- $f(x) = 2x^2 \cdot 4x^3$
- $g(x) = -x^5 + 5x^3 - 4x$
- $h(x) = \frac{2x^5 - 1}{3x^2 + 4}$

Answer

$f(x)$ is a power function because it can be written as $f(x) = 8x^5$. The other functions are not power functions.

Identifying Polynomial Functions

An oil pipeline bursts in the Gulf of Mexico, causing an oil slick in a roughly circular shape. The slick is currently 24 miles in radius, but that radius is increasing by 8 miles each week. We want to write a formula for the area covered by the oil slick by combining two functions. The radius r of the spill depends on the number of weeks w that have passed. This relationship is linear.

$$r(w) = 24 + 8w$$

We can combine this with the formula for the area A of a circle.

$$A(r) = \pi r^2$$

Composing these functions gives a formula for the area in terms of weeks.

$$\begin{aligned} A(w) &= A(r(w)) \\ &= A(24 + 8w) \\ &= \pi(24 + 8w)^2 \end{aligned}$$

Multiplying gives the formula.

$$A(w) = 576\pi + 384\pi w + 64\pi w^2$$

This formula is an example of a polynomial function. A polynomial function consists of either zero or the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power.

Definition: Polynomial Functions

Let n be a non-negative integer. A *polynomial function* is a function that can be written in the form

$$f(x) = a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0 \tag{5.4.2}$$

This is called the general form of a polynomial function. Each a_i is a coefficient and can be any real number. Each product $a_i x^i$ is a *term of a polynomial function*.

Example 5.4.4: Identifying Polynomial Functions

Which of the following are polynomial functions?

- $f(x) = 2x^3 \cdot 3x + 4$
- $g(x) = -x(x^2 - 4)$
- $h(x) = 5\sqrt{x} + 2$

Solution

The first two functions are examples of polynomial functions because they can be written in the form of Equation 5.4.2, where the powers are non-negative integers and the coefficients are real numbers.

- $f(x)$ can be written as $f(x) = 6x^4 + 4$.
- $g(x)$ can be written as $g(x) = -x^3 + 4x$.
- $h(x)$ cannot be written in this form and is therefore not a polynomial function.

Identifying the Degree and Leading Coefficient of a Polynomial Function

Because of the form of a polynomial function, we can see an infinite variety in the number of terms and the power of the variable. Although the order of the terms in the polynomial function is not important for performing operations, we typically arrange the terms in descending order of power, or in general form. The degree of the polynomial is the highest power of the variable that occurs in the polynomial; it is the power of the first variable if the function is in general form. The leading term is the term containing the highest power of the variable, or the term with the highest degree. The leading coefficient is the coefficient of the leading term.

Terminology of Polynomial Functions

We often rearrange polynomials so that the powers are descending.

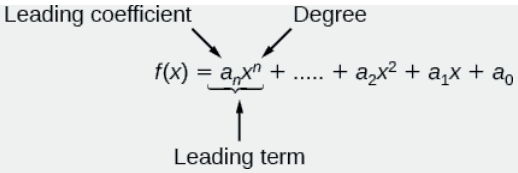


Figure 5.4.7

When a polynomial is written in this way, we say that it is in *general form*.

How To: Given a polynomial function, identify the degree and leading coefficient

1. Find the highest power of x to determine the degree function.
2. Identify the term containing the highest power of x to find the leading term.
3. Identify the coefficient of the leading term.

Example 5.4.5: Identifying the Degree and Leading Coefficient of a Polynomial Function

Identify the degree, leading term, and leading coefficient of the following polynomial functions.

$$f(x) = 3 + 2x^2 - 4x^3$$

$$g(t) = 5t^5 - 2t^3 + 7t$$

$$h(p) = 6p - p^3 - 2$$

Solution

For the function $f(x)$, the highest power of x is 3, so the degree is 3. The leading term is the term containing that degree, $-4x^3$. The leading coefficient is the coefficient of that term, −4.

For the function $g(t)$, the highest power of t is 5, so the degree is 5. The leading term is the term containing that degree, $5t^5$. The leading coefficient is the coefficient of that term, 5.

For the function $h(p)$, the highest power of p is 3, so the degree is 3. The leading term is the term containing that degree, $-p^3$; the leading coefficient is the coefficient of that term, −1.

Exercise 5.4.3

Identify the degree, leading term, and leading coefficient of the polynomial $f(x) = 4x^2 - x^6 + 2x - 6$.

Answer

The degree is **6**. The leading term is $-x^6$. The leading coefficient is **−1**.

Key Equations

- general form of a polynomial function: $f(x) = a_n x^n + a_{n-1} x^{n-1} ... + a_2 x^2 + a_1 x + a_0$

Key Concepts

- A power function is a variable base raised to a number power.
- A polynomial function is the sum of terms, each of which consists of a transformed power function with positive whole number power.
- The degree of a polynomial function is the highest power of the variable that occurs in a polynomial. The term containing the highest power of the variable is called the leading term. The coefficient of the leading term is called the leading coefficient.

Glossary

coefficient

a nonzero real number that is multiplied by a variable raised to an exponent (only the number factor is the coefficient)

degree

the highest power of the variable that occurs in a polynomial

leading coefficient

the coefficient of the leading term

leading term

the term containing the highest power of the variable

polynomial function

a function that consists of either zero or the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power.

power function

a function that can be represented in the form $f(x) = kx^p$ where k is a constant, the base is a variable, and the exponent, p, is a constant

term of a polynomial function

any $a_i x^i$ of a polynomial function in the form $f(x) = a_n x^n + a_{n-1} x^{n-1} ... + a_2 x^2 + a_1 x + a_0$

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5.5: Dividing Polynomials

Learning Objectives

- Use long division to divide polynomials.

The exterior of the Lincoln Memorial in Washington, D.C., is a large rectangular solid with length 61.5 meters (m), width 40 m, and height 30 m.[1]

We can easily find the volume using elementary geometry.

$$\begin{aligned} V &= l \cdot w \cdot h \\ &= 61.5 \cdot 40 \cdot 30 \\ &= 73,800 \end{aligned}$$

So the volume is $73,800$ cubic meters (m^3).



Figure 5.5.1: Lincoln Memorial, Washington, D.C. (credit: Ron Cogswell, Flickr)

Suppose we knew the volume, length, and width. We could divide to find the height.

$$\begin{aligned} h &= \frac{V}{l \cdot w} \\ &= \frac{73,800}{61.5 \cdot 40} \\ &= 30 \end{aligned}$$

As we can confirm from the dimensions above, the height is 30 m. We can use similar methods to find any of the missing dimensions. We can also use the same method if any or all of the measurements contain variable expressions. For example, suppose the volume of a rectangular solid is given by the polynomial $3x^4 - 3x^3 - 33x^2 + 54x$. The length of the solid is given by $3x$; the width is given by $x - 2$.

To find the height of the solid, we can use polynomial division, which is the focus of this section.

Using Long Division to Divide Polynomials

We are familiar with the **long division** algorithm for ordinary arithmetic. We begin by dividing into the digits of the dividend that have the greatest place value. We divide, multiply, subtract, include the digit in the next place value position, and repeat. For example, let's divide 178 by 3 using long division.

Long Division

$$\begin{array}{r} 59 \\ 3\overline{)178} \\ \underline{-15}\downarrow \\ 28 \\ \underline{-27} \\ 1 \end{array}$$

Step 1: $5 \times 3 = 15$ and $17 - 15 = 2$
Step 2: Bring down the 8
Step 3: $9 \times 3 = 27$ and $28 - 27 = 1$

Answer: $59\,R\,1$ or $59\frac{1}{3}$

Another way to look at the solution is as a sum of parts. This should look familiar, since it is the same method used to check division in elementary arithmetic.

$$\begin{aligned} \text{dividend} &= (\text{divisor}\cdot\text{quotient}) + \text{remainder} \\ 178 &= (3\cdot 59) + 1 \\ &= 177 + 1 \\ &= 178 \end{aligned}$$

We call this the **Division Algorithm** and will discuss it more formally after looking at an example.

Division of polynomials that contain more than one term has similarities to long division of whole numbers. We can write a polynomial dividend as the product of the divisor and the quotient added to the remainder. The terms of the polynomial division correspond to the digits (and place values) of the whole number division. This method allows us to divide two polynomials. For example, if we were to divide $2x^3-3x^2+4x+5$ by $x+2$ using the long division algorithm, it would look like this:

$$x+2\overline{)2x^3-3x^2+4x+5}$$ Set up the division problem.

$$\begin{array}{r} 2x^2 \\ x+2\overline{)2x^3-3x^2+4x+5} \end{array}$$ $2x^3$ divided by x is $2x^2$.

$$\begin{array}{r} 2x^2 \\ x+2\overline{)2x^3-3x^2+4x+5} \\ \underline{-(2x^3+4x^2)} \\ -7x^2+4x \end{array}$$ Multiply $x+2$ by $2x^2$ and subtract. Bring down the next term.

$$\begin{array}{r} 2x^2-7x \\ x+2\overline{)2x^3-3x^2+4x+5} \\ \underline{-(2x^3+4x^2)} \\ -7x^2+4x \\ \underline{-(-7x^2-14x)} \\ 18x+5 \end{array}$$ $-7x^2$ divided by x is $-7x$. Multiply $x+2$ by $-7x$. Subtract and bring down the next term.

$$\begin{array}{r} 2x^2-7x+18 \\ x+2\overline{)2x^3-3x^2+4x+5} \\ \underline{-(2x^3+4x^2)} \\ -7x^2+4x \\ \underline{-(-7x^2-14x)} \\ 18x+5 \\ \underline{-(18x+36)} \\ -31 \end{array}$$ $18x$ divided by x is 18. Multiply $x+2$ by 18. Subtract.

We have found

$$\frac{2x^3-3x^2+4x+5}{x+2} = 2x^2-7x+18-\frac{31}{x+2}$$

or

$$2x^3-3x^2+4x+5 = (x+2)(2x^2-7x+18)-31$$

We can identify the dividend, the **divisor**, the **quotient**, and the **remainder**.

Writing the result in this manner illustrates the Division Algorithm.

The Division Algorithm

The Division Algorithm states that, given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)q(x) + r(x) \tag{5.5.1}$$

$q(x)$ is the quotient and $r(x)$ is the remainder. The remainder is either equal to zero or has degree strictly less than $d(x)$.

If $r(x) = 0$, then $d(x)$ divides evenly into $f(x)$. This means that, in this case, both $d(x)$ and $q(x)$ are factors of $f(x)$.

Given a polynomial and a binomial, use long division to divide the polynomial by the binomial

1. Set up the division problem.
2. Determine the first term of the quotient by dividing the leading term of the dividend by the leading term of the divisor.
3. Multiply the answer by the divisor and write it below the like terms of the dividend.
4. Subtract the bottom **binomial** from the top binomial.
5. Bring down the next term of the dividend.
6. Repeat steps 2–5 until reaching the last term of the dividend.
7. If the remainder is non-zero, express as a fraction using the divisor as the denominator.

Example 5.5.1: Using Long Division to Divide a Second-Degree Polynomial

Divide $5x^2 + 3x - 2$ by $x + 1$.

Solution

$x+1\overline{)5x^2+3x-2}$	Set up division problem.
$\phantom{x+1\overline{)}}5x$ $x+1\overline{)5x^2+3x-2}$	$5x^2$ divided by x is $5x$.
$\phantom{x+1\overline{)}}5x$ $x+1\overline{)5x^2+3x-2}$	Multiply $x+1$ by $5x$.
$\phantom{x+1\overline{)}}5x$ $x+1\overline{)5x^2+3x-2}$ $\underline{-(5x^2+5x)}$	Subtract.
$\phantom{x+1\overline{)}}5x$ $x+1\overline{)5x^2+3x-2}$ $\underline{-(5x^2+5x)}$ $-2x-2$	Bring down the next term.
$\phantom{x+1\overline{)}}5x-2$ $x+1\overline{)5x^2+3x-2}$ $\underline{-(5x^2+5x)}$ $-2x-2$	$-2x$ divided by x is -2.
$\phantom{x+1\overline{)}}5x-2$ $x+1\overline{)5x^2+3x-2}$ $\underline{-(5x^2+5x)}$ $-2x-2$ $\underline{-(-2x-2)}$	Multiply $x+1$ by -2.
$\phantom{x+1\overline{)}}5x-2$ $x+1\overline{)5x^2+3x-2}$ $\underline{-(5x^2+5x)}$ $-2x-2$ $\underline{-(-2x-2)}$ 0	Subtract.

The quotient is $5x - 2$. The remainder is 0. We write the result as

$$\frac{5x^2+3x-2}{x+1} = 5x - 2$$

or

$$5x^2 + 3x - 2 = (x+1)(5x-2)$$

Analysis

This division problem had a remainder of 0. This tells us that the dividend is divided evenly by the divisor, and that the divisor is a factor of the dividend.

Example 5.5.2: Using Long Division to Divide a Third-Degree Polynomial

Divide $6x^3+11x^2-31x+15$ by $3x-2$.

Solution

$$
\begin{array}{rl}
 & 2x^2+\ 5x-\ 7 \\
 & \overline{\qquad\qquad\quad 3x} \\
-2\big) & 6x^3+11x^2-31x+15 \\
-(6x^3-4x^2) & \\
\hline
 & 15x^2-31x \\
-(15x^2-10x) & \\
\hline
 & -21x+15 \\
-(-21x+14) & \\
\hline
 & 1
\end{array}
$$

$6x^3$ divided by $3x$ is $2x^2$.

Multiply $3x-2$ by $2x^2$.

Subtract. Bring down next term. $15x^2$ divided by $3x$ is $5x$.

Multiply $3x-2$ by $5x$.

Subtract. Bring down next term. $-21x$ divided by $3x$ is -7.

Multiply $3x-2$ by -7.

Subtract. The remainder is 1.

There is a remainder of 1. We can express the result as:

$$\frac{6x^3+11x^2-31x+15}{3x-2}=2x^2+5x-7+\frac{1}{3x-2}$$

Analysis

We can check our work by using the Division Algorithm to rewrite the solution. Then multiply.

$$(3x-2)(2x^2+5x-7)+1=6x^3+11x^2-31x+15$$

Notice, as we write our result,

- the dividend is $6x^3+11x^2-31x+15$
- the divisor is $3x-2$
- the quotient is $2x^2+5x-7$
- the remainder is 1

Try It! 5.5.2

Divide $16x^3-12x^2+20x-3$ by $4x+5$.

Solution

$$4x^2-8x+15-\frac{78}{4x+5}$$

Using Polynomial Division to Solve Application Problems

Polynomial division can be used to solve a variety of application problems involving expressions for area and volume. We looked at an application at the beginning of this section. Now we will solve that problem in the following example.

Example 5.5.6: Using Polynomial Division in an Application Problem

The volume of a rectangular solid is given by the polynomial $3x^4-3x^3-33x^2+54x$. The length of the solid is given by $3x$ and the width is given by $x-2$. Find the height h of the solid.

Solution

There are a few ways to approach this problem. We need to divide the expression for the volume of the solid by the expressions for the length and width. Let us create a sketch as in Figure 5.5.3. Let h equal the height of the box.

Figure 5.5.3

We can now write an equation by substituting the known values into the formula for the volume of a rectangular solid.

$$V = l \cdot w \cdot h$$
$$3x^4 - 3x^3 - 33x^2 + 54x = 3x \cdot (x-2) \cdot h$$

To solve for h, first divide both sides by $3x$.

$$\frac{3x \cdot (x-2) \cdot h}{3x} = \frac{3x^4 - 3x^3 - 33x^2 + 54x}{3x}$$

$$(x-2)h = \frac{x^3 - x^2 - 11x + 18}{x-2}$$

Now solve for h using synthetic division.

$$h = \frac{x^3 - x^2 - 11x + 18}{x-2}$$

Erroneous nesting of equation structures

The quotient is $x^2 + x - 9$ and the remainder is 0. The height of the solid is $x^2 + x - 9$.

? Try It! 5.5.4

The area of a rectangle is given by $3x^3 + 14x^2 - 23x + 6$. The width of the rectangle is given by $x + 6$. Find an expression for the length of the rectangle.

Solution

$3x^2 - 4x + 1$

Key Equations

Division Algorithm $f(x) = d(x)q(x) + r(x)$ where $q(x) \neq 0$

Key Concepts

- Polynomial long division can be used to divide a polynomial by any polynomial with equal or lower degree.
- The Division Algorithm tells us that a polynomial dividend can be written as the product of the divisor and the quotient added to the remainder.
- Polynomial division can be used to solve application problems, including area and volume.

Footnotes

[1] National Park Service. "Lincoln Memorial Building Statistics." www.nps.gov/linc/historycultu...statistics.htm. Accessed 4/3/2014

Glossary

Division Algorithm

given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that $f(x) = d(x)q(x) + r(x)$ where $q(x)$ is the quotient and $r(x)$ is the remainder. The remainder is either equal to zero or has degree strictly less than $d(x)$.

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5.6: Inverses and Radical Functions

Learning Objectives

In this section, you will:

- Find the inverse of an invertible polynomial function.

A mound of gravel is in the shape of a cone with the height equal to twice the radius.



Figure 5.6.1

The volume is found using a formula from elementary geometry.

$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi r^2 (2r) \\ &= \frac{2}{3}\pi r^3 \end{aligned}$$

We have written the volume V in terms of the radius r. However, in some cases, we may start out with the volume and want to find the radius. For example: A customer purchases 100 cubic feet of gravel to construct a cone shape mound with a height twice the radius. What are the radius and height of the new cone? To answer this question, we use the formula

$$r = \sqrt[3]{\frac{3V}{2\pi}}$$

This function is the inverse of the formula for V in terms of r.

In this section, we will explore the inverses of polynomial and rational functions and in particular the radical functions we encounter in the process.

Finding the Inverse of a Polynomial Function

Two functions f and g are inverse functions if for every coordinate pair in f, (a, b), there exists a corresponding coordinate pair in the inverse function, g, (b, a). In other words, the coordinate pairs of the inverse functions have the input and output interchanged. Only one-to-one functions have inverses. Recall that a one-to-one function has a unique output value for each input value and passes the horizontal line test.

For example, suppose a water runoff collector is built in the shape of a parabolic trough as shown in Figure 5.6.2. We can use the information in the figure to find the surface area of the water in the trough as a function of the depth of the water.

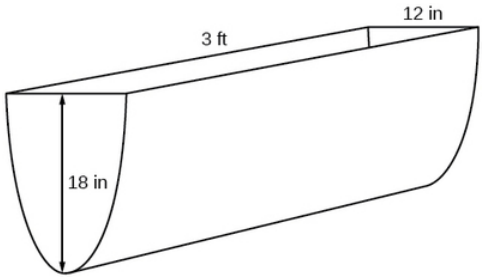


Figure 5.6.2

Because it will be helpful to have an equation for the parabolic cross-sectional shape, we will impose a coordinate system at the cross section, with x measured horizontally and y measured vertically, with the origin at the vertex of the parabola (Figure 5.6.3).

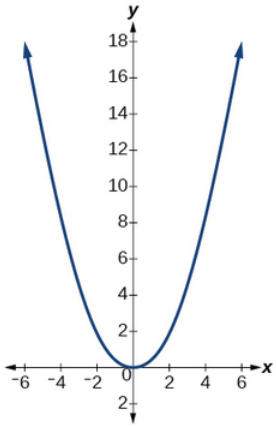


Figure 5.6.3

From this we find an equation for the parabolic shape. We placed the origin at the vertex of the parabola, so we know the equation will have form $y(x) = ax^2$. Our equation will need to pass through the point $(6, 18)$, from which we can solve for the stretch factor a.

$$\begin{aligned} 18 &= a6^2 \\ a &= \frac{18}{36} \\ &= \frac{1}{2} \end{aligned}$$

Our parabolic cross section has the equation

$$y(x) = \frac{1}{2}x^2$$

We are interested in the *surface area* of the water, so we must determine the width at the top of the water as a function of the water depth. For any depth y, the width will be given by $2x$, so we need to solve the equation above for x and find the inverse function. However, notice that the original function is not one-to-one, and indeed, given any output there are two inputs that produce the same output, one positive and one negative.

To find an inverse, we can restrict our original function to a limited domain on which it *is* one-to-one. In this case, it makes sense to restrict ourselves to positive x values. On this domain, we can find an inverse by solving for the input variable:

$$\begin{aligned} y &= \frac{1}{2}x^2 \\ 2y &= x^2 \\ x &= \pm\sqrt{2y} \end{aligned}$$

This is not a function as written. We are limiting ourselves to positive x values, so we eliminate the negative solution, giving us the inverse function we're looking for.

$$y = \frac{x^2}{2},\ x > 0$$

Because x is the distance from the center of the parabola to either side, the entire width of the water at the top will be $2x$. The trough is 3 feet (36 inches) long, so the surface area will then be:

$$\begin{aligned}\text{Area} &= l \cdot w \\ &= 36 \cdot 2x \\ &= 72x \\ &= 72\sqrt{2y}\end{aligned}$$

This example illustrates two important points:

1. When finding the inverse of a quadratic, we have to limit ourselves to a domain on which the function is one-to-one.
2. The inverse of a quadratic function is a square root function. Both are toolkit functions and different types of power functions.

Functions involving roots are often called **radical functions**. While it is not possible to find an inverse of most polynomial functions, some basic polynomials do have inverses. Such functions are called **invertible functions**, and we use the notation $f^{-1}(x)$.

Warning: $f^{-1}(x)$ is not the same as the reciprocal of the function $f(x)$. This use of "–1" is reserved to denote inverse functions. To denote the reciprocal of a function $f(x)$, we would need to write:

$$(f(x))^{-1} = \frac{1}{f(x)}. \tag{5.6.1}$$

An important relationship between inverse functions is that they "undo" each other. If f^{-1} is the inverse of a function f, then f is the inverse of the function f^{-1}. In other words, whatever the function f does to x, f^{-1} undoes it—and vice-versa.

$$f^{-1}(f(x)) = x, \text{ for all } x \text{ in the domain of } f$$

and

$$f(f^{-1}(x)) = x, \text{ for all } x \text{ in the domain of } f^{-1}$$

Note that the inverse switches the domain and range of the original function.

VERIFYING TWO FUNCTIONS ARE INVERSES OF ONE ANOTHER

Two functions, f and g, are inverses of one another if for all x in the domain of f and g,

$$g(f(x)) = f(g(x)) = x$$

Howto: Given a polynomial function, find the inverse of the function by restricting the domain in such a way that the new function is one-to-one

1. Replace $f(x)$ with y.
2. Interchange x and y.
3. Solve for y, and rename the function $f^{-1}(x)$.

Example 5.6.1: Verifying Inverse Functions

Show that $f(x) = \frac{1}{x+1}$ and $f^{-1}(x) = \frac{1}{x} - 1$ are inverses, for $x \neq 0, -1$.

Solution

We must show that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

$$f^{-1}(f(x)) = f^{-1}\left(\frac{1}{x+1}\right)$$
$$= \frac{1}{\frac{1}{x+1}} - 1$$
$$= (x+1) - 1$$
$$= x$$

and

$$f(f^{-1}(x)) = f(\frac{1}{x-1})$$
$$= \frac{1}{\left(\frac{1}{x-1}\right)+1}$$
$$= \frac{1}{\frac{1}{x}}$$
$$= x$$

Therefore, $f(x) = \frac{1}{x+1}$ and $f^{-1}(x) = \frac{1}{x} - 1$ are inverses.

? Exercise 5.6.1

Show that $f(x) = \frac{x+5}{3}$ and $f^{-1}(x) = 3x - 5$ are inverses.

Answer a

$f^{-1}(f(x)) = f^{-1}(\frac{x+5}{3}) = 3(\frac{x+5}{3}) - 5 = (x-5) + 5 = x$

Answer b

$f(f^{-1}(x)) = f(3x-5) = \frac{(3x-5)+5}{3} = \frac{3x}{3} = x$

✓ Example 5.6.2: Finding the Inverse of a Cubic Function

Find the inverse of the function $f(x) = 5x^3 + 1$.

Solution

This is a transformation of the basic cubic toolkit function, and based on our knowledge of that function, we know it is one-to-one. Solving for the inverse by solving for x.

$$y = 5x^3 + 1$$
$$x = 5y^3 + 1$$
$$x - 1 = 5y^3$$
$$\frac{x-1}{5} = y^3$$
$$f^{-1}(x) = \sqrt[3]{\frac{x-1}{5}}$$

Analysis

Look at the graph of f and f^{-1}. Notice that one graph is the reflection of the other about the line $y = x$. This is always the case when graphing a function and its inverse function.

Also, since the method involved interchanging x and y, notice corresponding points. If (a, b) is on the graph of f,then (b, a) is on the graph of f^{-1}. Since $(0, 1)$ is on the graph of f, then $(1, 0)$ is on the graph of f^{-1}. Similarly, since $(1, 6)$ is on the graph of f,then $(6, 1)$ is on the graph of f^{-1} (Figure 5.6.9).

Figure 5.6.4

Exercise 5.6.2

Find the inverse function of $f(x) = \sqrt[3]{x+4}$.

Answer

$f^{-1}(x) = x^3 - 4$

Finding Inverses

Notice that the functions from previous examples were all polynomials, and their inverses were radical functions. If we want to find the inverse of a radical function, we will need to restrict the domain of the answer because the range of the original function is limited.

How to: Given a radical function, find the inverse

1. Determine the range of the original function.
2. Replace $f(x)$ with y, then solve for x.
3. If necessary, restrict the domain of the inverse function to the range of the original function.

Example 5.6.5: Finding the Inverse of a Radical Function

Restrict the domain of the function $f(x) = \sqrt{x-4}$ and then find the inverse.

Solution

Note that the original function has range $f(x) \geq 0$. Replace $f(x)$ with y, then solve for x.

$$y = \sqrt{x-4} \text{ Replace } f(x) \text{ with } y.$$

$$x = \sqrt{y-4} \text{ Interchange } x \text{ and } y.$$

$$x = \sqrt{y-4} \text{ Square each side.}$$

$$x^2 = y - 4 \text{ Add 4.}$$

$$x^2 + 4 = y \text{ Rename the function } f^{-1}(x).$$

$$f^{-1}(x) = x^2 + 4$$

Recall that the domain of this function must be limited to the range of the original function.

$$f^{-1}(x) = x^2 + 4, \; x \geq 0$$

Analysis

Notice in Figure 5.6.8 that the inverse is a reflection of the original function over the line $y = x$. Because the original function has only positive outputs, the inverse function has only nonnegative inputs.

Figure 5.6.8

Exercise 5.6.5

Restrict the domain and then find the inverse of the function $f(x) = \sqrt{2x+3}$.

Answer

$f^{-1}(x) = \frac{x^2-3}{2}, \; x \geq 0$

Solving Applications of Radical Functions

Radical functions are common in physical models, as we saw in the section opener. We now have enough tools to be able to solve the problem posed at the start of the section.

Example 5.6.6: Solving an Application with a Cubic Function

A mound of gravel is in the shape of a cone with the height equal to twice the radius. The volume of the cone in terms of the radius is given by

$$V = \frac{2}{3}\pi r^3$$

Find the inverse of the function $V = \frac{2}{3}\pi r^3$ that determines the volume V of a cone and is a function of the radius r. Then use the inverse function to calculate the radius of such a mound of gravel measuring 100 cubic feet. Use $\pi = 3.14$.

Solution

Start with the given function for V. Notice that the meaningful domain for the function is $r > 0$ since negative radii would not make sense in this context nor would a radius of 0. Also note the range of the function (hence, the domain of the inverse function) is $V > 0$. Solve for r in terms of V, using the method outlined previously. Note that in real-world applications, we do not swap the variables when finding inverses. Instead, we change which variable is considered to be the independent variable.

$$V = \frac{2}{3}\pi r^3$$

Solve for r^3.

$$r^3 = \frac{3V}{2\pi}$$

Solve for r.

$$r = \sqrt[3]{\frac{3V}{2\pi}}$$

This is the result stated in the section opener. Now evaluate this for $V = 100$ and $\pi = 3.14$.

$$\begin{aligned} r &= \sqrt[3]{\frac{3V}{2\pi}} \\ &= \sqrt[3]{\frac{3 \cdot 100}{2 \cdot 3.14}} \\ &\approx \sqrt[3]{47.7707} \\ &\approx 3.63 \end{aligned}$$

Therefore, the radius is about 3.63 ft.

Determining the Domain of a Radical Function Composed with Other Functions

When radical functions are composed with other functions, determining domain can become more complicated.

Example 5.6.7: Finding the Domain of a Radical Function Composed with a Rational Function

Find the domain of the function:

$$f(x) = \sqrt{\frac{(x+2)(x-3)}{(x-1)}}.$$

Solution

Because a square root is only defined when the quantity under the radical is non-negative, we need to determine where

$$\frac{(x+2)(x-3)}{(x-1)} \geq 0.$$

The output of a rational function can change signs (change from positive to negative or vice versa) at *x*-intercepts and at vertical asymptotes. For this equation, the graph could change signs at $x = -2$, 1, and 3.

To determine the intervals on which the rational expression is positive, we could test some values in the expression or sketch a graph. While both approaches work equally well, for this example we will use a graph as shown in Figure 5.6.9.

Figure 5.6.9

This function has two *x*-intercepts, both of which exhibit linear behavior near the *x*-intercepts. There is one vertical asymptote, corresponding to a linear factor; this behavior is similar to the basic reciprocal toolkit function, and there is no horizontal asymptote because the degree of the numerator is larger than the degree of the denominator. There is a *y*-intercept at $(0, \sqrt{6})$.

From the *y*-intercept and *x*-intercept at $x = -2$, we can sketch the left side of the graph. From the behavior at the asymptote, we can sketch the right side of the graph.

From the graph, we can now tell on which intervals the outputs will be non-negative, so that we can be sure that the original function $f(x)$ will be defined. $f(x)$ has domain $-2 \leq x < 1$ or $x \geq 3$, or in interval notation, $[-2, 1) \cup [3, \infty)$.

Finding Inverses of Rational Functions

As with finding inverses of quadratic functions, it is sometimes desirable to find the *inverse of a rational function*, particularly of rational functions that are the ratio of linear functions, such as in concentration applications.

Example 5.6.8: Finding the Inverse of a Rational Function

The function

$$C = \frac{20 + 0.4n}{100 + n} \tag{5.6.2}$$

represents the concentration C of an acid solution after n mL of 40% solution has been added to 100 mL of a 20% solution. First, find the inverse of the function; that is, find an expression for n in terms of C. Then use your result to determine how much of the 40% solution should be added so that the final mixture is a 35% solution.

Solution

We first want the inverse of the function in order to determine how many mL we need for a given concentration. We will solve for n in terms of C.

$$\begin{aligned} C &= \frac{20 + 0.4n}{100 + n} \\ C(100 + n) &= 20 + 0.4n \\ 100C + Cn &= 20 + 0.4n \\ 100C - 20 &= 0.4n - Cn \\ 100C - 20 &= (0.4 - C)n \\ n &= \frac{100C - 20}{0.4 - C} \end{aligned}$$

Now evaluate this function at 35%, which is $C = 0.35$.

$$\begin{aligned} n &= \frac{100(0.35) - 20}{0.4 - 0.35} \\ &= \frac{15}{0.05} \\ &= 300 \end{aligned}$$

We can conclude that 300 mL of the 40% solution should be added.

Exercise 5.6.8

Find the inverse of the function $f(x) = \frac{x+3}{x-2}$.

Answer

$f^{-1}(x) = \frac{2x+3}{x-1}$

Media

Access these online resources for additional instruction and practice with inverses and radical functions.

- Graphing the Basic Square Root Function
- Find the Inverse of a Square Root Function
- Find the Inverse of a Rational Function
- Find the Inverse of a Rational Function and an Inverse Function Value
- Inverse Functions

Key Concepts

- The inverse of a quadratic function is a square root function.
- If f^{-1} is the inverse of a function f, then f is the inverse of the function f^{-1}. See Example 5.6.1.
- While it is not possible to find an inverse of most polynomial functions, some basic polynomials are invertible. See Example 5.6.2.
- When finding the inverse of a radical function, we need a restriction on the domain of the answer. See Example 5.6.5 and 5.6.7.
- Inverse and radical and functions can be used to solve application problems. See Examples 5.6.6 and 5.6.8.

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CHAPTER OVERVIEW

6: Exponential and Logarithmic Functions

In this chapter, we will explore exponential functions, which can be used for, among other things, modeling growth patterns such as those found in bacteria. We will also investigate logarithmic functions, which are closely related to exponential functions. Both types of functions have numerous real-world applications when it comes to modeling and interpreting data.

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6.4: Graphs of Logarithmic Functions
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6.0: Prelude to Exponential and Logarithmic Functions

Focus in on a square centimeter of your skin. Look closer. Closer still. If you could look closely enough, you would see hundreds of thousands of microscopic organisms. They are bacteria, and they are not only on your skin, but in your mouth, nose, and even your intestines. In fact, the bacterial cells in your body at any given moment outnumber your own cells. But that is no reason to feel bad about yourself. While some bacteria can cause illness, many are healthy and even essential to the body.

Bacteria commonly reproduce through a process called binary fission, during which one bacterial cell splits into two. When conditions are right, bacteria can reproduce very quickly. Unlike humans and other complex organisms, the time required to form a new generation of bacteria is often a matter of minutes or hours, as opposed to days or years. For simplicity's sake, suppose we begin with a culture of one bacterial cell that can divide every hour. Table **6.0.1** shows the number of bacterial cells at the end of each subsequent hour. We see that the single bacterial cell leads to over one thousand bacterial cells in just ten hours! And if we were to extrapolate the table to twenty-four hours, we would have over 16 million!

Table **6.0.1**

Hour	0	1	2	3	4	5	6	7	8	9	10
Bacteria	1	2	4	8	16	32	64	128	256	512	1024

In this chapter, we will explore exponential functions, which can be used for, among other things, modeling growth patterns such as those found in bacteria. We will also investigate logarithmic functions, which are closely related to exponential functions. Both types of functions have numerous real-world applications when it comes to modeling and interpreting data.

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6.1: Exponential Functions

Learning Objectives

- Evaluate exponential functions.
- Find the equation of an exponential function.
- Use compound interest formulas.
- Evaluate exponential functions with base e.

India is the second most populous country in the world with a population of about 1.25 billion people in 2013. The population is growing at a rate of about $.2\%$ each year. If this rate continues, the population of India will exceed China's population by the year 2031. When populations grow rapidly, we often say that the growth is "exponential," meaning that something is growing very rapidly. To a mathematician, however, the term *exponential growth* has a very specific meaning. In this section, we will take a look at *exponential functions*, which model this kind of rapid growth.

Identifying Exponential Functions

When exploring linear growth, we observed a constant rate of change—a constant number by which the output increased for each unit increase in input. For example, in the equation $f(x) = 3x + 4$, the slope tells us the output increases by 3 each time the input increases by 1. The scenario in the India population example is different because we have a *percent* change per unit time (rather than a constant change) in the number of people.

Defining an Exponential Function

A study found that the percent of the population who are vegans in the United States doubled from 2009 to 2011. In 2011, 2.5% of the population was vegan, adhering to a diet that does not include any animal products—no meat, poultry, fish, dairy, or eggs. If this rate continues, vegans will make up 10% of the U.S. population in 2015, 40% in 2019, and 80% in 2050.

What exactly does it mean to *grow exponentially*? What does the word *double* have in common with *percent increase*? People toss these words around errantly. Are these words used correctly? The words certainly appear frequently in the media.

- **Percent change** refers to a *change* based on a *percent* of the original amount.
- **Exponential growth** refers to an *increase* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* increase of the original amount over time.
- **Exponential decay** refers to a *decrease* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* decrease of the original amount over time.

For us to gain a clear understanding of **exponential growth**, let us contrast exponential growth with **linear growth**. We will construct two functions. The first function is exponential. We will start with an input of 0, and increase each input by 1. We will double the corresponding consecutive outputs. The second function is linear. We will start with an input of 0, and increase each input by 1. We will add 2 to the corresponding consecutive outputs (Table $6.1.1$).

From Table $6.1.1$ we can infer that for these two functions, exponential growth dwarfs linear growth.

- **Exponential growth** refers to the original value from the range increases by the *same percentage* over equal increments found in the domain.
- **Linear growth** refers to the original value from the range increases by the *same amount* over equal increments found in the domain.

Table $6.1.1$

x	$f(x)=2^x$	$g(x)=2x$
0	1	0
1	2	2
2	4	4
3	8	6
4	16	8
5	32	10
6	64	12

Apparently, the difference between “the same percentage” and “the same amount” is quite significant. For exponential growth, over equal increments, the constant multiplicative rate of change resulted in doubling the output whenever the input increased by one. For linear growth, the constant additive rate of change over equal increments resulted in adding 2 to the output whenever the input was increased by one.

The general form of the **exponential function** is $f(x)=ab^x$, where a is any nonzero number, b is a positive real number not equal to 1.

- If $b>1$,the function grows at a rate proportional to its size.
- If $0<b<1$, the function decays at a rate proportional to its size.

Let’s look at the function $f(x)=2^x$ from our example. We will create a table (Table $6.1.2$) to determine the corresponding outputs over an interval in the domain from -3 to 3.

Table $6.1.2$

x	-3	-2	-1	0	1	2	3
$f(x)=2^x$	$2^{-3}=\frac{1}{8}$	$2^{-2}=\frac{1}{4}$	$2^{-1}=\frac{1}{2}$	$2^0=1$	$2^1=2$	$2^2=4$	$2^3=8$

Let us examine the graph of f by plotting the ordered pairs from Table $6.1.2$ and then make a few observations $6.1.1$.

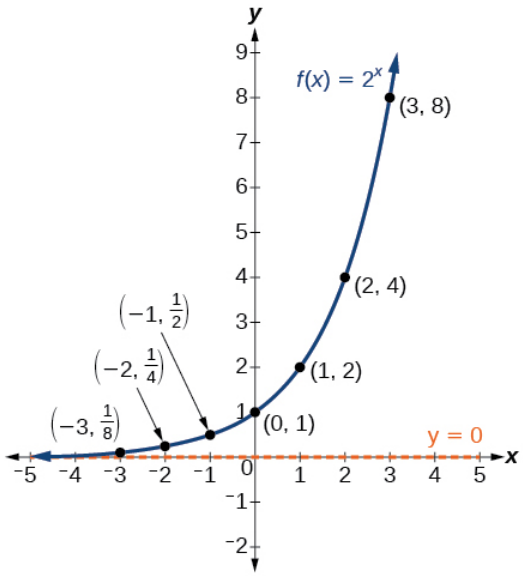


Figure $6.1.1$

Let’s define the behavior of the graph of the exponential function $f(x)=2^x$ and highlight some its key characteristics.

- the domain is $(-\infty, \infty)$,
- the range is $(0, \infty)$,
- as $x \to \infty$, $f(x) \to \infty$,
- as $x \to -\infty$, $f(x) \to 0$,
- $f(x)$ is always increasing,
- the graph of $f(x)$ will never touch the x-axis because base two raised to any exponent never has the result of zero.
- $y = 0$ is the horizontal asymptote.
- the y-intercept is 1.

Definition: Exponential Functions

For any real number x,an exponential function is a function with the form

$$f(x) = ab^x \tag{6.1.1}$$

where

- a is a non-zero real number called the initial value and
- b is any positive real number such that $b \neq 1$.
- The domain of f is all real numbers.
- The range of f is all positive real numbers if $a > 0$.
- The range of f is all negative real numbers if $a < 0$.
- The y-intercept is $(0, a)$,and the horizontal asymptote is $y = 0$.

Example 6.1.1: Identifying Exponential Functions

Which of the following equations are *not* exponential functions?

- $f(x) = 4^{3(x-2)}$
- $g(x) = x^3$
- $h(x) = \left(\frac{1}{3}\right)^x$
- $j(x) = (-2)^x$

Solution

By definition, an exponential function has a constant as a base and an independent variable as an exponent. Thus, $g(x) = x^3$ does not represent an exponential function because the base is an independent variable. In fact, $g(x) = x^3$ is a power function.

Recall that the base b of an exponential function is always a positive constant, and $b \neq 1$. Thus, $j(x) = (-2)^x$ does not represent an exponential function because the base, -2, is less than 0.

Exercise 6.1.1

Which of the following equations represent exponential functions?

- $f(x) = 2x^2 - 3x + 1$
- $g(x) = 0.875^x$
- $h(x) = 1.75x + 2$
- $j(x) = 1095.6^{-2x}$

Answer

$g(x) = 0.875^x$ and $j(x) = 1095.6^{-2x}$ represent exponential functions.

Evaluating Exponential Functions

Recall that the base of an exponential function must be a positive real number other than 1.Why do we limit the base bb to positive values? To ensure that the outputs will be real numbers. Observe what happens if the base is not positive:

- Let $b=-9$ and $x=\frac{1}{2}$. Then $f(x)=f\left(\frac{1}{2}\right)=(-9)^{\frac{1}{2}}=\sqrt{-9}$,which is not a real number.

Why do we limit the base to positive values other than 1? Because base 1 results in the constant function. Observe what happens if the base is 1:

- Let $b=1$. Then $f(x)=1^x=1$ for any value of x.

To evaluate an exponential function with the form $f(x)=b^x$,we simply substitute x with the given value, and calculate the resulting power. For example:

Let $f(x)=2^x$. What is $f(3)$?

$$\begin{aligned} f(x)&=2^x \\ f(3)&=2^3 && \text{Substitute } x=3 \\ &=8 && \text{Evaluate the power} \end{aligned}$$

To evaluate an exponential function with a form other than the basic form, it is important to follow the order of operations. For example:

Let $f(x)=30(2)^x$. What is $f(3)$?

$$\begin{aligned} f(x)&=30(2)^x \\ f(3)&=30(2)^3 && \text{Substitute } x=3 \\ &=30(8) && \text{Simplify the power first} \\ &=240 && \text{Multiply} \end{aligned}$$

Note that if the order of operations were not followed, the result would be incorrect:

$$f(3)=30(2)^3\neq 60^3=216,000$$

Example 6.1.2: Evaluating Exponential Functions

Let $f(x)=5(3)^{x+1}$. Evaluate $f(2)$ without using a calculator.

Solution

Follow the order of operations. Be sure to pay attention to the parentheses.

$$\begin{aligned} f(x)&=5(3)^{x+1} \\ f(2)&=5(3)^{2+1} && \text{Substitute } x=2 \\ &=5(3)^3 && \text{Add the exponents} \\ &=5(27) && \text{Simplify the power} \\ &=135 && \text{Multiply} \end{aligned}$$

Exercise 6.1.2

Let $f(x)=8(1.2)^{x-5}$. Evaluate $f(3)$ using a calculator. Round to four decimal places.

Answer

5.5556

Defining Exponential Growth

Because the output of exponential functions increases very rapidly, the term “exponential growth” is often used in everyday language to describe anything that grows or increases rapidly. However, exponential growth can be defined more precisely in a

mathematical sense. If the growth rate is proportional to the amount present, the function models exponential growth.

Definition: Exponential Growth

A function that models exponential growth grows by a rate proportional to the amount present. For any real number x and any positive real numbers a and b such that $b \neq 1$,an exponential growth function has the form

$$f(x) = ab^x \tag{6.1.2}$$

where

- a is the initial or starting value of the function.
- b is the growth factor or growth multiplier per unit x.

In more general terms, we have an *exponential function*, in which a constant base is raised to a variable exponent. To differentiate between linear and exponential functions, let's consider two companies, A and B. Company A has 100 stores and expands by opening 50 new stores a year, so its growth can be represented by the function $A(x) = 100 + 50x$. Company B has 100 stores and expands by increasing the number of stores by 50% each year, so its growth can be represented by the function $B(x) = 100(1 + 0.5)^x$.

A few years of growth for these companies are illustrated in Table $6.1.3$.

Table $6.1.3$

Year, x	Stores, Company A	Stores, Company B
0	$100 + 50(0) = 100$	$100(1 + 0.5)^0 = 100$
1	$100 + 50(1) = 150$	$100(1 + 0.5)^1 = 150$
2	$100 + 50(2) = 200$	$100(1 + 0.5)^2 = 225$
3	$100 + 50(3) = 250$	$100(1 + 0.5)^3 = 337.5$
x	$A(x) = 100 + 50x$	$B(x) = 100(1 + 0.5)^x$

The graphs comparing the number of stores for each company over a five-year period are shown in Figure $6.1.2$. We can see that, with exponential growth, the number of stores increases much more rapidly than with linear growth.

Figure $6.1.2$: The graph shows the numbers of stores Companies A and B opened over a five-year period.

Notice that the domain for both functions is $[0, \infty)$,and the range for both functions is $[100, \infty)$. After year 1, Company B always has more stores than Company A.

Now we will turn our attention to the function representing the number of stores for Company B, $B(x) = 100(1 + 0.5)^x$. In this exponential function, 100 represents the initial number of stores, 0.50 represents the growth rate, and $1 + 0.5 = 1.5$ represents the growth factor. Generalizing further, we can write this function as $B(x) = 100(1.5)^x$,where 100 is the initial value, 1.5 is called the *base*, and x is called the *exponent*.

Example $6.1.3$: Evaluating a Real-World Exponential Model

At the beginning of this section, we learned that the population of India was about 1.25 billion in the year 2013, with an annual growth rate of about 1.2%. This situation is represented by the growth function $P(t) = 1.25(1.012)^t$, where t is the number of years since 2013. To the nearest thousandth, what will the population of India be in 2031?

Solution

To estimate the population in 2031, we evaluate the models for $t = 18$, because 2031 is 18 years after 2013. Rounding to the nearest thousandth,

$$P(18) = 1.25(1.012)^{18} \approx 1.549$$

There will be about 1.549 billion people in India in the year 2031.

? Exercise 6.1.3

The population of China was about 1.39 billion in the year 2013, with an annual growth rate of about 0.6%. This situation is represented by the growth function $P(t) = 1.39(1.006)^t$, where t is the number of years since 2013.To the nearest thousandth, what will the population of China be for the year 2031? How does this compare to the population prediction we made for India in Example $6.1.3$?

Answer

About 1.548 billion people; by the year 2031, India's population will exceed China's by about 0.001 billion, or 1 million people.

Finding Equations of Exponential Functions

In the previous examples, we were given an exponential function, which we then evaluated for a given input. Sometimes we are given information about an exponential function without knowing the function explicitly. We must use the information to first write the form of the function, then determine the constants a, a and b, b,and evaluate the function.

How to: Given two data points, write an exponential model

1. If one of the data points has the form $(0, a)$, then a is the initial value. Using a, substitute the second point into the equation $f(x) = a(b)^x$, and solve for b.
2. If neither of the data points have the form $(0, a)$, substitute both points into two equations with the form $f(x) = a(b)^x$. Solve the resulting system of two equations in two unknowns to find a and b.
3. Using the a and b found in the steps above, write the exponential function in the form $f(x) = a(b)^x$.

Example 6.1.4: Writing an Exponential Model When the Initial Value Is Known

In 2006, 80 deer were introduced into a wildlife refuge. By 2012, the population had grown to 180 deer. The population was growing exponentially. Write an algebraic function $N(t)$ representing the population (N) of deer over time t.

Solution

We let our independent variable t be the number of years after 2006. Thus, the information given in the problem can be written as input-output pairs: (0, 80) and (6, 180). Notice that by choosing our input variable to be measured as years after 2006, we have given ourselves the initial value for the function, $a = 80$. We can now substitute the second point into the equation $N(t) = 80b^t$ to find b:

$$\begin{aligned} N(t) &= 80b^t \\ 180 &= 80b^6 && \text{Substitute using point } (6, 180) \\ \frac{9}{4} &= b^6 && \text{Divide and write in lowest terms} \\ b &= \left(\frac{9}{4}\right)^{\frac{1}{6}} && \text{Isolate b using properties of exponents} \\ b &\approx 1.1447 && \text{Round to 4 decimal places} \end{aligned}$$

> *Unless otherwise stated, do not round any intermediate calculations. Then round the final answer to four places for the remainder of this section.*

The exponential model for the population of deer is $N(t) = 80(1.1447)^t$. (Note that this exponential function models short-term growth. As the inputs gets large, the output will get increasingly larger, so much so that the model may not be useful in the long term.)

We can graph our model to observe the population growth of deer in the refuge over time. Notice that the graph in Figure $6.1.3$ passes through the initial points given in the problem, $(0, 80)$ and $(6, 180)$. We can also see that the domain for the function is $[0, \infty)$,and the range for the function is $[80, \infty)$.

Figure 6.1.3: Graph showing the population of deer over time, $N(t) = 80(1.1447)^t$, t years after 2006

Exercise 6.1.4

A wolf population is growing exponentially. In 2011, 129 wolves were counted. By 2013, the population had reached 236 wolves. What two points can be used to derive an exponential equation modeling this situation? Write the equation representing the population N of wolves over time t.

Answer

$(0, 129)$ and $(2, 236)$; $N(t) = 129(1.3526)^t$

Example 6.1.5: Writing an Exponential Model When the Initial Value is Not Known

Find an exponential function that passes through the points $(-2, 6)$ and $(2, 1)$.

Solution

Because we don't have the initial value, we substitute both points into an equation of the form $f(x) = ab^x$, and then solve the system for a and b.

- Substituting $(-2, 6)$ gives $6 = ab^{-2}$
- Substituting $(2, 1)$ gives $1 = ab^2$

Use the first equation to solve for a in terms of b:

$$\begin{aligned} 6 &= ab^{-2} \\ \frac{6}{b^{-2}} &= a \quad \text{Divide} \\ a &= 6b^2 \quad \text{Use properties of exponents to rewrite the denominator} \end{aligned}$$

Substitute a in the second equation, and solve for b:

$$\begin{aligned} 1 &= ab^2 \\ 1 &= 6b^2b^2 \\ &= 6b^4 \quad \text{Substitute a} \\ b &= \left(\frac{1}{6}\right)^{\frac{1}{4}} \quad \text{Round 4 decimal places rewrite the denominator} \\ b &\approx 0.6389 \end{aligned}$$

Use the value of b in the first equation to solve for the value of a:

$$\begin{aligned} a &= 6b^2 \\ &\approx 6(0.6389)^2 \\ &\approx 2.4492 \end{aligned}$$

Thus, the equation is $f(x) = 2.4492(0.6389)^x$.

We can graph our model to check our work. Notice that the graph in Figure 6.1.4 passes through the initial points given in the problem, $(-2, 6)$ and $(2, 1)$. The graph is an example of an exponential decay function.

Figure 6.1.4: The graph of $f(x) = 2.4492(0.6389)^x$ models exponential decay.

Exercise 6.1.5

Given the two points $(1, 3)$ and $(2, 4.5)$,find the equation of the exponential function that passes through these two points.

Answer

$f(x) = 2(1.5)^x$

Q&A: Do two points always determine a unique exponential function?

Yes, provided the two points are either both above the x-axis or both below the x-axis and have different x-coordinates. But keep in mind that we also need to know that the graph is, in fact, an exponential function. Not every graph that looks exponential really is exponential. We need to know the graph is based on a model that shows the same percent growth with each unit increase in x, which in many real world cases involves time.

How to: Given the graph of an exponential function, write its equation

1. First, identify two points on the graph. Choose the y-intercept as one of the two points whenever possible. Try to choose points that are as far apart as possible to reduce round-off error.
2. If one of the data points is the y-intercept $(0, a)$, then a is the initial value. Using a, substitute the second point into the equation $f(x) = a(b)^x$, and solve for b
3. If neither of the data points have the form $(0, a)$, substitute both points into two equations with the form $f(x) = a(b)^x$. Solve the resulting system of two equations in two unknowns to find a and b.
4. Write the exponential function, $f(x) = a(b)^x$.

Example 6.1.6: Writing an Exponential Function Given Its Graph

Find an equation for the exponential function graphed in Figure **6.1.5**.

Figure **6.1.5**

Solution

We can choose the y-intercept of the graph, $(0, 3)$, as our first point. This gives us the initial value, $a = 3$. Next, choose a point on the curve some distance away from $(0, 3)$ that has integer coordinates. One such point is $(2, 12)$.

$y = ab^x$	Write the general form of an exponential equation
$y = 3b^x$	Substitute the initial value 3 for a
$12 = 3b^2$	Substitute in 12 for y and 2 for x
$4 = b^2$	Divide by 3
$b = \pm 2$	Take the square root

Because we restrict ourselves to positive values of b, we will use $b = 2$. Substitute a and b into the standard form to yield the equation $f(x) = 3(2)^x$.

Exercise 6.1.6

Find an equation for the exponential function graphed in Figure **6.1.6**.

Figure **6.1.6**

Answer

$f(x) = \sqrt{2}(\sqrt{2})^x$. Answers may vary due to round-off error. The answer should be very close to $1.4142(1.4142)^x$.

How to: Given two points on the curve of an exponential function, use a graphing calculator to find the equation

1. Press **[STAT].**
2. Clear any existing entries in columns **L1** or **L2.**
3. In **L1**, enter the *x*-coordinates given.
4. In **L2**, enter the corresponding *y*-coordinates.
5. Press **[STAT]** again. Cursor right to **CALC**, scroll down to **ExpReg (Exponential Regression)**, and press **[ENTER].**
6. The screen displays the values of *a* and *b* in the exponential equation $y = a \cdot b^x$.

Example 6.1.7: Using a Graphing Calculator to Find an Exponential Function

Use a graphing calculator to find the exponential equation that includes the points $(2, 24.8)$ and $(5, 198.4)$.

Solution

Follow the guidelines above. First press **[STAT]**, **[EDIT]**, **[1: Edit…],** and clear the lists **L1** and **L2**. Next, in the **L1** column, enter the x-coordinates, 2 and 5. Do the same in the **L2** column for the y-coordinates, 24.8 and 198.4.

Now press **[STAT]**, **[CALC]**, **[0: ExpReg]** and press **[ENTER]**. The values $a = 6.2$ and $b = 2$ will be displayed. The exponential equation is $y = 6.2 \cdot 2^x$.

Exercise 6.1.7

Use a graphing calculator to find the exponential equation that includes the points $(3, 75.98)$ and $(6, 481.07)$.

Answer

$y \approx 12 \cdot 1.85^x$

Applying the Compound-Interest Formula

Savings instruments in which earnings are continually reinvested, such as mutual funds and retirement accounts, use **compound interest**. The term *compounding* refers to interest earned not only on the original value, but on the accumulated value of the account.

The **annual percentage rate (APR)** of an account, also called the **nominal rate**, is the yearly interest rate earned by an investment account. The term *nominal* is used when the compounding occurs a number of times other than once per year. In fact, when interest is compounded more than once a year, the effective interest rate ends up being *greater* than the nominal rate! This is a powerful tool for investing.

We can calculate the compound interest using the compound interest formula, which is an exponential function of the variables time t, principal P, $APR\, r$, and number of compounding periods in a year n:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

For example, observe Table 6.1.4, which shows the result of investing $\$1,000$ at 10% for one year. Notice how the value of the account increases as the compounding frequency increases.

Table 6.1.4

Frequency	Value after 1 year
Annually	$1100
Semiannually	$1102.50
Quarterly	$1103.81
Monthly	$1104.71
Daily	$1105.16

Definition: Compound Interest

Compound interest can be calculated using the formula

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} \tag{6.1.3}$$

where

- $A(t)$ is the account value,
- t is measured in years,
- P is the starting amount of the account, often called the principal, or more generally present value,
- r is the annual percentage rate (APR) expressed as a decimal, and
- n is the number of compounding periods in one year.

Example 6.1.8: Calculating Compound Interest

If we invest $\$3,000$ in an investment account paying 3% interest compounded quarterly, how much will the account be worth in 10 years?

Solution

Because we are starting with $\$3,000$, $P = 3000$. Our interest rate is 3%, so $r = 0.03$. Because we are compounding quarterly, we are compounding 4 times per year, so $n = 4$. We want to know the value of the account in 10 years, so we are looking for $A(10)$,the value when $t = 10$.

$$\begin{aligned} A(t) &= P\left(1+\frac{r}{n}\right)^{nt} && \text{Use the compound interest formula} \\ A(10) &= 3000\left(1+\frac{0.03}{4}\right)^{(4)\cdot(10)} && \text{Substitute using given values} \\ &\approx \$4045.05 && \text{Round to two decimal places} \end{aligned}$$

The account will be worth about $\$4,045.05$ in 10 years.

Exercise 6.1.8

An initial investment of $\$100,000$ at 12% interest is compounded weekly (use 52 weeks in a year). What will the investment be worth in 30 years?

Answer

about $\$3,644,675.88$

Example 6.1.9: Using the Compound Interest Formula to Solve for the Principal

A 529 Plan is a college-savings plan that allows relatives to invest money to pay for a child's future college tuition; the account grows tax-free. Lily wants to set up a 529 account for her new granddaughter and wants the account to grow to $\$40,000$ over 18 years. She believes the account will earn 6% compounded semi-annually (twice a year). To the nearest dollar, how much will Lily need to invest in the account now?

Solution

The nominal interest rate is 6%, so $r = 0.06$. Interest is compounded twice a year, so $k = 2$.

We want to find the initial investment, P, needed so that the value of the account will be worth $\$40,000$ in 18 years. Substitute the given values into the compound interest formula, and solve for P.

$$\begin{aligned} A(t) &= P\left(1+\frac{r}{n}\right)^{nt} && \text{Use the compound interest formula} \\ 40,000 &= P\left(1+\frac{0.06}{2}\right)^{2(18)} && \text{Substitute using given values } A, r, n, t \\ 40,000 &= P(1.03)^{36} && \text{Simplify} \\ \frac{40,000}{(1.03)^{36}} &= P && \text{Isolate } P \\ P &\approx \$13,801 && \text{Divide and round to the nearest dollar} \end{aligned}$$

Lily will need to invest $\$13,801$ to have $\$40,000$ in 18 years.

Exercise 6.1.9

Refer to Example **6.1.9**. To the nearest dollar, how much would Lily need to invest if the account is compounded quarterly?

Answer

$13,693

Evaluating Functions with Base e

As we saw earlier, the amount earned on an account increases as the compounding frequency increases. Table **6.1.5** shows that the increase from annual to semi-annual compounding is larger than the increase from monthly to daily compounding. This might lead us to ask whether this pattern will continue.

Examine the value of **$1** invested at **100%** interest for **1** year, compounded at various frequencies, listed in Table **6.1.5**.

Table **6.1.5**

Frequency	$A(t)=\left(1+\frac{1}{n}\right)^n$	Value
Annually	$\left(1+\frac{1}{1}\right)^1$	**$2**
Semiannually	$\left(1+\frac{1}{2}\right)^2$	**$2.25**
Quarterly	$\left(1+\frac{1}{4}\right)^4$	**$2.441406**
Monthly	$\left(1+\frac{1}{12}\right)^{12}$	**$2.613035**
Daily	$\left(1+\frac{1}{365}\right)^{365}$	**$2.714567**
Hourly	$\left(1+\frac{1}{8760}\right)^{8760}$	**$2.718127**
Once per minute	$\left(1+\frac{1}{525600}\right)^{525600}$	**$2.718279**
Once per second	$\left(1+\frac{1}{31536000}\right)^{31536000}$	**$2.718282**

These values appear to be approaching a limit as n increases without bound. In fact, as n gets larger and larger, the expression $\left(1+\frac{1}{n}\right)^n$ approaches a number used so frequently in mathematics that it has its own name: the letter e. This value is an irrational number, which means that its decimal expansion goes on forever without repeating. Its approximation to six decimal places is shown below.

Definition: The Number e

The letter e represents the irrational number

$$\left(1+\frac{1}{n}\right)^n \tag{6.1.4}$$

as n increases without bound

The letter e is used as a base for many real-world exponential models. To work with base e, we use the approximation, $e \approx 2.718282$. The constant was named by the Swiss mathematician Leonhard Euler (1707–1783) who first investigated and discovered many of its properties.

Example 6.1.10: Using a Calculator to Find Powers of e

Calculate $e^{3.14}$. Round to five decimal places.

Solution

On a calculator, press the button labeled [e^x]. The window shows [$e^($]. Type 3.14 and then close parenthesis, [)]. Press [ENTER]. Rounding to 5 decimal places, $e^{3.14} \approx 23.10387$. Caution: Many scientific calculators have an "Exp" button, which is used to enter numbers in scientific notation. It is not used to find powers of e.

Exercise 6.1.10

Use a calculator to find $e^{-0.5}$. Round to five decimal places.

Answer

$e^{-0.5} \approx 0.60653$

Investigating Continuous Growth

So far we have worked with rational bases for exponential functions. For most real-world phenomena, however, e is used as the base for exponential functions. Exponential models that use e as the base are called continuous growth or decay models. We see these models in finance, computer science, and most of the sciences, such as physics, toxicology, and fluid dynamics.

Definition: The Continuous Growth/decay Formula

For all real numbers t,and all positive numbers a and r, continuous growth or decay is represented by the formula

$$A(t) = ae^{rt} \tag{6.1.5}$$

where

- a is the initial value,
- r is the continuous growth rate per unit time,
- t is the elapsed time.

If $r > 0$, then the formula represents continuous growth. If $r < 0$, then the formula represents continuous decay.

For business applications, the continuous growth formula is called the continuous compounding formula and takes the form

$$A(t) = Pe^{rt} \tag{6.1.6}$$

where

- P is the principal or the initial invested,
- r is the growth or interest rate per unit time,
- t is the period or term of the investment.

How to: Given the initial value, rate of growth or decay, and time t, solve a continuous growth or decay function

1. Use the information in the problem to determine a, the initial value of the function.
2. Use the information in the problem to determine the growth rate r.
   - If the problem refers to continuous growth, then $r > 0$.
   - If the problem refers to continuous decay, then $r < 0$.
3. Use the information in the problem to determine the time t.
4. Substitute the given information into the continuous growth formula and solve for $A(t)$.

Example 6.1.11: Calculating Continuous Growth

A person invested $\$1,000$ in an account earning a nominal 10% per year compounded continuously. How much was in the account at the end of one year?

Solution

Since the account is growing in value, this is a continuous compounding problem with growth rate $r = 0.10$. The initial investment was $\$1,000$, so $P = 1000$. We use the continuous compounding formula to find the value after $t = 1$ year:

$$\begin{aligned} A(t) &= Pe^{rt} && \text{Use the continuous compounding formula} \\ &= 1000(e)^{0.1} && \text{Substitute known values for } P, r, t \\ &\approx 1105.17 && \text{Use a calculator to approximate} \end{aligned}$$

The account is worth $\$1,105.17$ after one year.

Exercise 6.1.11

A person invests $\$100,000$ at a nominal 12% interest per year compounded continuously. What will be the value of the investment in 30 years?

Answer

$\$3,659,823.44$

Example 6.1.12: Calculating Continuous Decay

$Radon-222$ decays at a continuous rate of 17.3% per day. How much will $100mg$ of $Radon-222$ decay to in 3 days?

Solution

Since the substance is decaying, the rate, 17.3%, is negative. So, $r = -0.173$. The initial amount of $Radon-222$ was 100 mg, so $a = 100$. We use the continuous decay formula to find the value after $t = 3$ days:

$$\begin{aligned} A(t) &= ae^{rt} && \text{Use the continuous growth formula} \\ &= 100e6-0.173(3) && \text{Substitute known values for } a, r, t \\ &\approx 59.5115 && \text{Use a calculator to approximate} \end{aligned}$$

So 59.5115 mg of $Radon-222$ will remain.

Exercise 6.1.12

Using the data in Example $6.1.12$, how much $Radon-222$ will remain after one year?

Answer

$3.77E-26$ (This is calculator notation for the number written as 3.77×10^{-26} in scientific notation. While the output of an exponential function is never zero, this number is so close to zero that for all practical purposes we can accept zero as the answer.)

Media

Access these online resources for additional instruction and practice with exponential functions.

- Exponential Growth Function
- Compound Interest

Key Equations

definition of the exponential function	$f(x) = b^x$, where $b > 0$, $b \neq 1$
definition of exponential growth	$f(x) = ab^x$, where $a > 0$, $b > 0$, $b \neq 1$
compound interest formula	$A(t) = P(1 + \frac{r}{n})^{nt}$, where $A(t)$ is the account value at time t t is the number of years P is the initial investment, often called the principal r is the annual percentage rate (APR), or nominal rate n is the number of compounding periods in one year
continuous growth formula	$A(t) = ae^{rt}$, where t is the number of unit time periods of growth a is the starting amount (in the continuous compounding formula a is replaced with P, the principal) e is the mathematical constant, $e \approx 2.718282$

Key Concepts

- An exponential function is defined as a function with a positive constant other than 1 raised to a variable exponent. See Example.
- A function is evaluated by solving at a specific value. See Example and Example.
- An exponential model can be found when the growth rate and initial value are known. See Example.
- An exponential model can be found when the two data points from the model are known. See Example.
- An exponential model can be found using two data points from the graph of the model. See Example.
- An exponential model can be found using two data points from the graph and a calculator. See Example.
- The value of an account at any time t can be calculated using the compound interest formula when the principal, annual interest rate, and compounding periods are known. See Example.
- The initial investment of an account can be found using the compound interest formula when the value of the account, annual interest rate, compounding periods, and life span of the account are known. See Example.
- The number e is a mathematical constant often used as the base of real world exponential growth and decay models. Its decimal approximation is $e \approx 2.718282$.
- Scientific and graphing calculators have the key [ex] or [$exp(x)$] for calculating powers of e. See Example.
- Continuous growth or decay models are exponential models that use e as the base. Continuous growth and decay models can be found when the initial value and growth or decay rate are known. See Example and Example.

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6.2: Graphs of Exponential Functions

Learning Objectives

- Graph exponential functions.
- Graph exponential functions using transformations.

As we discussed in the previous section, exponential functions are used for many real-world applications such as finance, forensics, computer science, and most of the life sciences. Working with an equation that describes a real-world situation gives us a method for making predictions. Most of the time, however, the equation itself is not enough. We learn a lot about things by seeing their pictorial representations, and that is exactly why graphing exponential equations is a powerful tool. It gives us another layer of insight for predicting future events.

Graphing Exponential Functions

Before we begin graphing, it is helpful to review the behavior of exponential growth. Recall the table of values for a function of the form $f(x) = b^x$ whose base is greater than one. We'll use the function $f(x) = 2^x$. Observe how the output values in Table 6.2.1 change as the input increases by 1.

Table 6.2.1

x	-3	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Each output value is the product of the previous output and the base, 2. We call the base 2 the *constant ratio*. In fact, for any exponential function with the form $f(x) = ab^x$, b is the constant ratio of the function. This means that as the input increases by 1, the output value will be the product of the base and the previous output, regardless of the value of a.

Notice from the table that

- the output values are positive for all values of x;
- as x increases, the output values increase without bound; and
- as x decreases, the output values grow smaller, approaching zero.

Figure 6.2.1 shows the exponential growth function $f(x) = 2^x$.

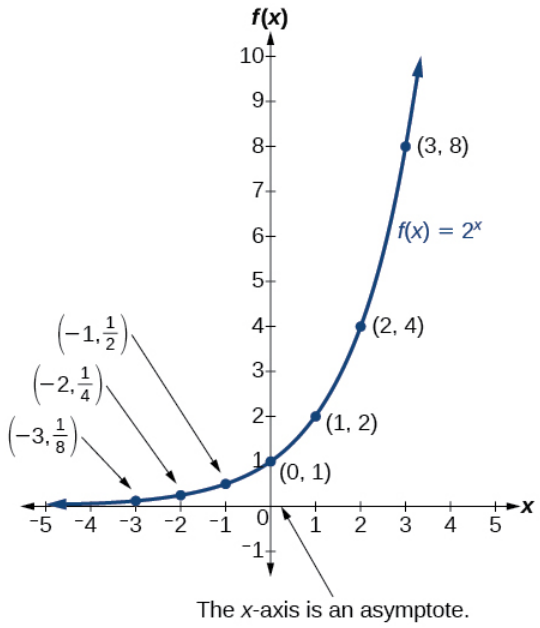


Figure 6.2.1: Notice that the graph gets close to the x-axis, but never touches it.

The domain of $f(x) = 2^x$ is all real numbers, the range is $(0, \infty)$, and the horizontal asymptote is $y = 0$.

To get a sense of the behavior of exponential decay, we can create a table of values for a function of the form $f(x) = b^x$ whose base is between zero and one. We'll use the function $g(x) = \left(\frac{1}{2}\right)^x$. Observe how the output values in Table 6.2.2 change as the input increases by 1.

Table 6.2.2

x	-3	-2	-1	0	1	2	3
$g(x) = \left(\frac{1}{2}\right)^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

Again, because the input is increasing by 1, each output value is the product of the previous output and the base, or constant ratio $\frac{1}{2}$.

Notice from the table that

- the output values are positive for all values of x;
- as x increases, the output values grow smaller, approaching zero; and
- as x decreases, the output values grow without bound.

Figure 6.2.2 shows the exponential decay function, $g(x) = \left(\frac{1}{2}\right)^x$.

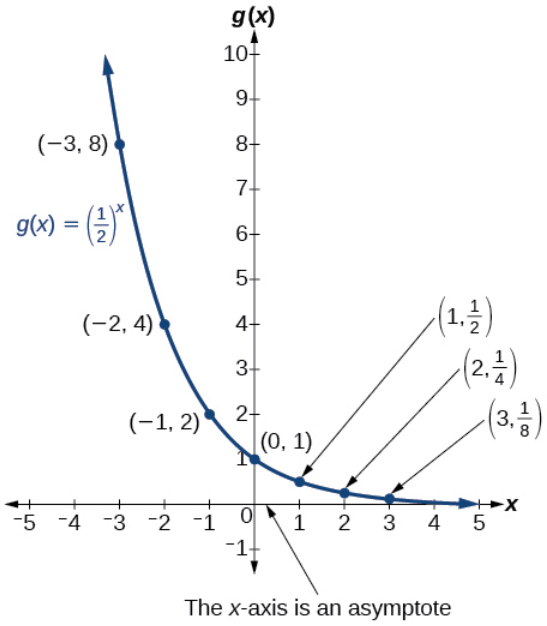


Figure **6.2.2**

The domain of $g(x) = (\frac{1}{2})^x$ is all real numbers, the range is $(0, \infty)$,and the horizontal asymptote is $y = 0$.

CHARACTERISTICS OF THE GRAPH OF THE PARENT FUNCTION $F(X) = b^x$

An exponential function with the form $f(x) = b^x$, $b > 0$, $b \neq 1$,has these characteristics:

- one-to-one function
- horizontal asymptote: $y = 0$
- domain: $(-\infty, \infty)$
- range: $(0, \infty)$
- *x*-intercept: none
- *y*-intercept: $(0, 1)$
- increasing if $b > 1$
- decreasing if $b < 1$

Figure **6.2.3** compares the graphs of exponential growth and decay functions.



1 and the second graph is of the same function when b is 0

Figure 6.2.3

How to: Given an exponential function of the form $f(x) = b^x$,graph the function

1. Create a table of points.
2. Plot at least **3** point from the table, including the *y*-intercept $(0, 1)$.
3. Draw a smooth curve through the points.
4. State the domain, $(-\infty, \infty)$,the range, $(0, \infty)$,and the horizontal asymptote, $y = 0$.

Example 6.2.1: Sketching the Graph of an Exponential Function of the Form $f(x) = b^x$

Sketch a graph of $f(x) = 0.25^x$. State the domain, range, and asymptote.

Solution

Before graphing, identify the behavior and create a table of points for the graph.

Table 6.2.3

x	-3	-2	-1	0	1	2	3
$f(x) = 0.25^x$	64	16	4	1	0.25	\(0.0625\0	0.015625

- Since $b = 0.25$ is between zero and one, we know the function is decreasing. The left tail of the graph will increase without bound, and the right tail will approach the asymptote $y = 0$.
- Create a table of points as in Table 6.2.3.
- Plot the *y*-intercept, $(0, 1)$,along with two other points. We can use $(-1, 4)$ and $(1, 0.25)$.

Draw a smooth curve connecting the points as in Figure 6.2.4.

Figure 6.2.4

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

? Exercise 6.2.1

Sketch the graph of $f(x) = 4^x$. State the domain, range, and asymptote.

Answer

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

Graphing Transformations of Exponential Functions

Transformations of exponential graphs behave similarly to those of other functions. Just as with other parent functions, we can apply the four types of transformations—shifts, reflections, stretches, and compressions—to the parent function $f(x) = b^x$ without loss of shape. For instance, just as the quadratic function maintains its parabolic shape when shifted, reflected, stretched, or compressed, the exponential function also maintains its general shape regardless of the transformations applied.

Graphing a Vertical Shift

The first transformation occurs when we add a constant d to the parent function $f(x) = b^x$, giving us a vertical shiftd d units in the same direction as the sign. For example, if we begin by graphing a parent function, $f(x) = 2^x$, we can then graph two vertical shifts alongside it, using $d = 3$: the upward shift, $g(x) = 2^x + 3$ and the downward shift, $h(x) = 2^x - 3$. Both vertical shifts are shown in Figure 6.2.5.

Figure 6.2.5

Observe the results of shifting $f(x) = 2^x$ vertically:

- The domain, $(-\infty, \infty)$ remains unchanged.
- When the function is shifted up 3 units to $g(x) = 2^x + 3$:
  - The *y*-intercept shifts up 3 units to $(0, 4)$.
  - The asymptote shifts up 3 units to $y = 3$.
  - The range becomes $(3, \infty)$.
- When the function is shifted down 3 units to $h(x) = 2^x - 3$:
  - The *y*-intercept shifts down 3 units to $(0, -2)$.
  - The asymptote also shifts down 3 units to $y = -3$.
  - The range becomes $(-3, \infty)$.

Graphing a Horizontal Shift

The next transformation occurs when we add a constant c to the input of the parent function $f(x) = b^x$, giving us a horizontal shift c units in the *opposite* direction of the sign. For example, if we begin by graphing the parent function $f(x) = 2^x$, we can then graph

two horizontal shifts alongside it, using $c = 3$: the shift left, $g(x) = 2^{x+3}$, and the shift right, $h(x) = 2^{x-3}$. $h(x) = 2^{x-3}$. Both horizontal shifts are shown in Figure 6.2.6.

Figure 6.2.6

Observe the results of shifting $f(x) = 2^x$ horizontally:

- The domain, $(-\infty, \infty)$,remains unchanged.
- The asymptote, $y = 0$,remains unchanged.
- The *y*-intercept shifts such that:
  - When the function is shifted left 3 units to $g(x) = 2^{x+3}$,the *y*-intercept becomes $(0, 8)$. This is because $2^{x+3} = (8)2^x$,so the initial value of the function is 8.
  - When the function is shifted right 3 units to $h(x) = 2^{x-3}$,the *y*-intercept becomes $(0, \frac{1}{8})$. Again, see that $2^{x-3} = (\frac{1}{8})2^x$,so the initial value of the function is $\frac{1}{8}$.

SHIFTS OF THE PARENT FUNCTION $F(X) = b^x$

For any constants c and d,the function $f(x) = b^{x+c} + d$ shifts the parent function $f(x) = b^x$

- vertically d units, in the *same* direction of the sign of d.
- horizontally c units, in the *opposite* direction of the sign of c.
- The *y*-intercept becomes $(0, b^c + d)$.
- The horizontal asymptote becomes $y = d$.
- The range becomes (d, ∞).
- The domain, $(-\infty, \infty)$,remains unchanged.

How to: Given an exponential function with the form $f(x) = b^{x+c} + d$, graph the translation

1. Draw the horizontal asymptote $y = d$.
2. Identify the shift as $(-c, d)$. Shift the graph of $f(x) = b^x$ left c units if c is positive, and right c units if c is negative.
3. Shift the graph of $f(x) = b^x$ up d units if d is positive, and down d units if d is negative.
4. State the domain, $(-\infty, \infty)$,the range, (d, ∞),and the horizontal asymptote $y = d$.

Example 6.2.2: Graphing a Shift of an Exponential Function

Graph $f(x) = 2^{x+1} - 3$. State the domain, range, and asymptote.

Solution

We have an exponential equation of the form $f(x) = b^{x+c} + d$, with $b = 2$, $c = 1$, and $d = -3$.

Draw the horizontal asymptote $y = d$, so draw $y = -3$.

Identify the shift as $(-c, d)$, so the shift is $(-1, -3)$.

Shift the graph of $f(x) = b^x$ left 1 units and down 3 units.

Figure 6.2.7

The domain is $(-\infty, \infty)$; the range is $(-3, \infty)$; the horizontal asymptote is $y = -3$.

Exercise 6.2.2

Graph $f(x) = 2^{x-1} + 3$. State domain, range, and asymptote.

Answer

The domain is $(-\infty, \infty)$; the range is $(3, \infty)$; the horizontal asymptote is $y = 3$.

How to: Given an equation of the form $f(x) = b^{x+c} + d$ for x, use a graphing calculator to approximate the solution

- Press **[Y=]**. Enter the given exponential equation in the line headed "$\mathbf{Y_1}$=".
- Enter the given value forf(x) f(x) in the line headed "$\mathbf{Y_2}$=".
- Press **[WINDOW]**. Adjust the *y*-axis so that it includes the value entered for "$\mathbf{Y_2}$=".
- Press **[GRAPH]** to observe the graph of the exponential function along with the line for the specified value off(x). f(x).
- To find the value ofx, x,we compute the point of intersection. Press **[2ND]** then **[CALC]**. Select "intersect" and press **[ENTER]** three times. The point of intersection gives the value of *x* for the indicated value of the function.

Example 6.2.3: Approximating the Solution of an Exponential Equation

Solve $42 = 1.2(5)^x + 2.8$ graphically. Round to the nearest thousandth.

Solution

Press **[Y=]** and enter $1.2(5)^x + 2.8$ next to $\mathbf{Y_1}$=. Then enter 42 next to **Y2**=. For a window, use the values -3 to 3 for x and -5 to 55 for y. Press **[GRAPH]**. The graphs should intersect somewhere near $x = 2$.

For a better approximation, press **[2ND]** then **[CALC]**. Select **[5: intersect]** and press **[ENTER]** three times. The *x*-coordinate of the point of intersection is displayed as 2.1661943. (Your answer may be different if you use a different window or use a different value for **Guess?**) To the nearest thousandth, $x \approx 2.166$.

Exercise 6.2.3

Solve $4 = 7.85(1.15)^x - 2.27$ graphically. Round to the nearest thousandth.

Answer

$x \approx -1.608$

Graphing a Stretch or Compression

While horizontal and vertical shifts involve adding constants to the input or to the function itself, a **stretch** or **compression** occurs when we multiply the parent function $f(x) = b^x$ by a constant $|a| > 0$. For example, if we begin by graphing the parent function $f(x) = 2^x$,we can then graph the stretch, using $a = 3$,to get $g(x) = 3(2)^x$ as shown on the left in Figure 6.2.8, and the compression, using $a = \frac{1}{3}$,to get $h(x) = \frac{1}{3}(2)^x$ as shown on the right in Figure 6.2.8.

Figure 6.2.8: (a) $g(x) = 3(2)^x$ stretches the graph of $f(x) = 2^x$ vertically by a factor of 3. (b) $h(x) = \frac{1}{3}(2)^x$ compresses the graph of $f(x) = 2^x$ vertically by a factor of $\frac{1}{3}$.

STRETCHES AND COMPRESSIONS OF THE PARENT FUNCTION $F(X) = B^X$

For any factor $a > 0$,the function $f(x) = a(b)^x$

- is stretched vertically by a factor of a if $|a| > 1$.
- is compressed vertically by a factor of a if $|a| < 1$.
- has a *y*-intercept of $(0, a)$.
- has a horizontal asymptote at $y = 0$, a range of $(0, \infty)$, and a domain of $(-\infty, \infty)$,which are unchanged from the parent function.

Example 6.2.4: Graphing the Stretch of an Exponential Function

Sketch a graph of $f(x) = 4\left(\frac{1}{2}\right)^x$. State the domain, range, and asymptote.

Solution

Before graphing, identify the behavior and key points on the graph.

- Since $b = \frac{1}{2}$ is between zero and one, the left tail of the graph will increase without bound as x decreases, and the right tail will approach the x-axis as x increases.
- Since $a = 4$,the graph of $f(x) = \left(\frac{1}{2}\right)^x$ will be stretched by a factor of 4.
- Create a table of points as shown in Table 6.2.4.

Table 6.2.4

x	-3	-2	-1	0	1	2	3
$f(x) = 4\left(\frac{1}{2}\right)^x$	32	16	8	4	2	1	0.5

- Plot the y-intercept, $(0, 4)$,along with two other points. We can use $(-1, 8)$ and $(1, 2)$.

Draw a smooth curve connecting the points, as shown in Figure 6.2.9.

Figure 6.2.9

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

Exercise 6.2.4

Sketch the graph of $f(x) = \frac{1}{2}(4)^x$. State the domain, range, and asymptote.

Answer

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

Graphing Reflections

In addition to shifting, compressing, and stretching a graph, we can also reflect it about the x-axis or the y-axis. When we multiply the parent function $f(x) = b^x$ by -1,we get a reflection about the x-axis. When we multiply the input by -1,we get a reflection about the y-axis. For example, if we begin by graphing the parent function $f(x) = 2^x$, we can then graph the two reflections alongside it. The reflection about the x-axis, $g(x) = -2^x$,is shown on the left side of Figure 6.2.10, and the reflection about the y-axis $h(x) = 2^{-x}$, is shown on the right side of Figure 6.2.10.

Figure 6.2.10: (a) $g(x) = -2^x$ reflects the graph of $f(x) = 2^x$ about the x-axis. (b) $g(x) = 2^{-x}$ reflects the graph of $f(x) = 2^x$ about the y-axis.

REFLECTIONS OF THE PARENT FUNCTION $F(X) = B^X$

The function $f(x) = -b^x$

- reflects the parent function $f(x) = b^x$ about the x-axis.
- has a y-intercept of $(0, -1)$.
- has a range of $(-\infty, 0)$
- has a horizontal asymptote at $y = 0$ and domain of $(-\infty, \infty)$,which are unchanged from the parent function.

The function $f(x) = b^{-x}$

- reflects the parent function $f(x) = b^x$ about the y-axis.

- has a *y*-intercept of $(0, 1)$, a horizontal asymptote at $y = 0$, a range of $(0, \infty)$, and a domain of $(-\infty, \infty)$, which are unchanged from the parent function.

Example 6.2.5: Writing and Graphing the Reflection of an Exponential Function

Find and graph the equation for a function, $g(x)$,that reflects $f(x) = (\frac{1}{4})^x$ about the *x*-axis. State its domain, range, and asymptote.

Solution

Since we want to reflect the parent function $f(x) = (\frac{1}{4})^x$ about the *x*-axis, we multiply $f(x)$ by -1 to get, $g(x) = -(\frac{1}{4})^x$. Next we create a table of points as in Table 6.2.5.

Table 6.2.5

x	-3	-2	-1	0	1	2	3
$g(x) = -(\frac{1}{4})^x$	-64	-16	-4	-1	-0.25	-0.0625	-0.0156

Plot the *y*-intercept, $(0, -1)$,along with two other points. We can use $(-1, -4)$ and $(1, -0.25)$.

Draw a smooth curve connecting the points:

Figure 6.2.11

The domain is $(-\infty, \infty)$; the range is $(-\infty, 0)$; the horizontal asymptote is $y = 0$.

Exercise 6.2.5

Find and graph the equation for a function, $g(x)$, that reflects $f(x) = 1.25^x$ about the *y*-axis. State its domain, range, and asymptote.

Answer

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

Summarizing Translations of the Exponential Function

Now that we have worked with each type of translation for the exponential function, we can summarize them in **Table 6.2.6** to arrive at the general equation for translating exponential functions.

1, and notes the following changes: the reflected function is decreasing as x moves from 0 to infinity, the asymptote remains x=0, the x-intercept remains (1, 0), the key point changes to (b^(-1), 1), the domain remains (0, infinity), and the range remains (-infinity, infinity). The second column shows the left shift of the equation g(x)=log_b(x) when b>1, and notes the following changes: the reflected function is decreasing as x moves from 0 to infinity, the asymptote remains x=0, the x-intercept changes to (-1, 0), the key point changes to (-b, 1), the domain changes to (-infinity, 0), and the range remains (-infinity, infinity).">

Table 6.2.6: Translations of the Parent Function $f(x) = b^x$

Translation	Form
Shift • Horizontally c units to the left • Vertically d units up	$f(x) = b^{x+c} + d$
Stretch and Compress • Stretch if $\|a\| > 1$ • Compression if $0 < \|a\| < 1$	$f(x) = ab^x$
Reflect about the x-axis	$f(x) = -b^x$
Reflect about the y-axis	$f(x) = b^{-x} = (\frac{1}{b})^x$
General equation for all translations	$f(x) = ab^{x+c} + d$

TRANSLATIONS OF EXPONENTIAL FUNCTIONS

A translation of an exponential function has the form

$$f(x) = ab^{x+c} + d$$

Where the parent function, $y = b^x, b > 1$,is

- shifted horizontally c units to the left.
- stretched vertically by a factor of $|a|$ if $|a| > 0$.
- compressed vertically by a factor of $|a|$ if $0 < |a| < 1$.
- shifted vertically d units.
- reflected about the *x*-axis when $a < 0$.

Note the order of the shifts, transformations, and reflections follow the order of operations.

Example 6.2.6: Writing a Function from a Description

Write the equation for the function described below. Give the horizontal asymptote, the domain, and the range.

$f(x) = e^x$ is vertically stretched by a factor of 2, reflected across the *y*-axis, and then shifted up 4 units.

Solution

We want to find an equation of the general form f(x)=abx+c+d. f(x)=abx+c+d. We use the description provided to finda, a, b,b, c,c, and d. d.

- We are given the parent function $f(x) = e^x$, so $b = e$.
- The function is stretched by a factor of 2, so $a = 2$.
- The function is reflected about the *y*-axis. We replace x with $-x$ to get: e^{-x}.
- The graph is shifted vertically 4 units, so $d = 4$.

Substituting in the general form we get,

$$f(x) = ab^{x+c} + d$$
$$= 2e^{-x+0} + 4$$
$$= 2e^{-x} + 4$$

The domain is $(-\infty, \infty)$; the range is $(4, \infty)$; the horizontal asymptote is $y = 4$.

Exercise 6.2.6

Write the equation for function described below. Give the horizontal asymptote, the domain, and the range.

$f(x) = e^x$ is compressed vertically by a factor of $\frac{1}{3}$, reflected across the x-axis and then shifted down 2 units.

Answer

$f(x) = -\frac{1}{3}e^x - 2$; the domain is $(-\infty, \infty)$; the range is $(-\infty, 2)$; the horizontal asymptote is $y = 2$.

Media

Access this online resource for additional instruction and practice with graphing exponential functions.

- Graph Exponential Functions

Key Equations

General Form for the Translation of the Parent Function $f(x) = b^x$	$f(x) = ab^{x+c} + d$

Key Concepts

- The graph of the function $f(x) = b^x$ has a y-intercept at $(0, 1)$,domain $(-\infty, \infty)$,range $(0, \infty)$, and horizontal asymptote $y = 0$. See Example.
- If $b > 1$,the function is increasing. The left tail of the graph will approach the asymptote $y = 0$, and the right tail will increase without bound.
- If $0 < b < 1$, the function is decreasing. The left tail of the graph will increase without bound, and the right tail will approach the asymptote $y = 0$.
- The equation $f(x) = b^x + d$ represents a vertical shift of the parent function $f(x) = b^x$.
- The equation $f(x) = b^{x+c}$ represents a horizontal shift of the parent function $f(x) = b^x$. See Example.
- Approximate solutions of the equation $f(x) = b^{x+c} + d$ can be found using a graphing calculator. See Example.
- The equation $f(x) = ab^x$, where $a > 0$, represents a vertical stretch if $|a| > 1$ or compression if $0 < |a| < 1$ of the parent function $f(x) = b^x$. See Example.
- When the parent function $f(x) = b^x$ is multiplied by -1,the result, $f(x) = -b^x$, is a reflection about the x-axis. When the input is multiplied by -1,the result, $f(x) = b^{-x}$, is a reflection about the y-axis. See Example.
- All translations of the exponential function can be summarized by the general equation $f(x) = ab^{x+c} + d$. See Table.
- Using the general equation $f(x) = ab^{x+c} + d$, we can write the equation of a function given its description. See Example.

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6.3: Logarithmic Functions

Learning Objectives

- Convert from logarithmic to exponential form.
- Convert from exponential to logarithmic form.
- Evaluate logarithms.
- Use common logarithms.
- Use natural logarithms.

In 2010, a major earthquake struck Haiti, destroying or damaging over 285,000 homes. One year later, another, stronger earthquake devastated Honshu, Japan, destroying or damaging over 332,000 buildings, like those shown in Figure **6.3.1**. Even though both caused substantial damage, the earthquake in 2011 was 100 times stronger than the earthquake in Haiti. How do we know? The magnitudes of earthquakes are measured on a scale known as the Richter Scale. The Haitian earthquake registered a 7.0 on the Richter Scale whereas the Japanese earthquake registered a 9.0.



Figure **6.3.1**: Devastation of March 11, 2011 earthquake in Honshu, Japan. (credit: Daniel Pierce).

The Richter Scale is a base-ten logarithmic scale. In other words, an earthquake of magnitude 8 is not twice as great as an earthquake of magnitude 4. It is

$$10^{8-4} = 10^4 = 10,000$$

times as great! In this lesson, we will investigate the nature of the Richter Scale and the base-ten function upon which it depends.

Converting from Logarithmic to Exponential Form

In order to analyze the magnitude of earthquakes or compare the magnitudes of two different earthquakes, we need to be able to convert between logarithmic and exponential form. For example, suppose the amount of energy released from one earthquake were 500 times greater than the amount of energy released from another. We want to calculate the difference in magnitude. The equation that represents this problem is $10^x = 500$, where x represents the difference in magnitudes on the Richter Scale. How would we solve for x?

We have not yet learned a method for solving exponential equations. None of the algebraic tools discussed so far is sufficient to solve $10^x = 500$. We know that $10^2 = 100$ and $10^3 = 1000$, so it is clear that x must be some value between 2 and 3, since $y = 10^x$ is increasing. We can examine a graph, as in Figure **6.3.1**, to better estimate the solution.

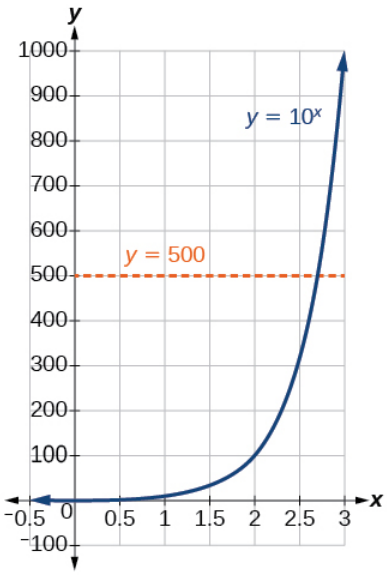


Figure 6.3.2

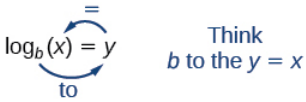
Estimating from a graph, however, is imprecise. To find an algebraic solution, we must introduce a new function. Observe that the graph in Figure 6.3.2 passes the horizontal line test. The exponential function $y = b^x$ is one-to-one, so its inverse, $x = b^y$ is also a function. As is the case with all inverse functions, we simply interchange x and y and solve for y to find the inverse function. To represent y as a function of x, we use a logarithmic function of the form $y = \log_b(x)$. The base b **logarithm** of a number is the exponent by which we must raise b to get that number.

We read a logarithmic expression as, "The logarithm with base b of x is equal to y," or, simplified, "log base b of x is y." We can also say, "b raised to the power of y is x," because logs are exponents. For example, the base 2 logarithm of 32 is 5, because 5 is the exponent we must apply to 2 to get 32. Since $2^5 = 32$, we can write $\log_2 32 = 5$. We read this as "log base 2 of 32 is 5."

We can express the relationship between logarithmic form and its corresponding exponential form as follows:

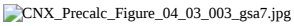
$$\log_b(x) = y \Leftrightarrow b^y = x, b > 0, b \neq 1 \tag{6.3.1}$$

Note that the base b is always positive.



Because logarithm is a function, it is most correctly written as $\log_b(x)$, using parentheses to denote function evaluation, just as we would with $f(x)$. However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written without parentheses, as $\log_b x$. Note that many calculators require parentheses around the x.

We can illustrate the notation of logarithms as follows:



Notice that, comparing the logarithm function and the exponential function, the input and the output are switched. This means $y = \log^b(x)$ and $y = b^x$ are inverse functions.

DEFINITION OF THE LOGARITHMIC FUNCTION

A logarithm base b of a positive number x satisfies the following definition.

For $x > 0$, $b > 0$, $b \neq 1$,

$$y = \log_b(x) \text{ is equivalent to } b^y = x \tag{6.3.2}$$

where,

- we read $\log_b(x)$ as, “the logarithm with base b of x” or the “log base b of x."
- the logarithm y is the exponent to which b must be raised to get x.

Also, since the logarithmic and exponential functions switch the x and y values, the domain and range of the exponential function are interchanged for the logarithmic function. Therefore,

- the domain of the logarithm function with base b is $(0, \infty)$.
- the range of the logarithm function with base b is $(-\infty, \infty)$.

Q&A: Can we take the logarithm of a negative number?

No. Because the base of an exponential function is always positive, no power of that base can ever be negative. We can never take the logarithm of a negative number. Also, we cannot take the logarithm of zero. Calculators may output a log of a negative number when in complex mode, but the log of a negative number is not a real number.

How to: Given an equation in logarithmic form $\log_b(x) = y$, convert it to exponential form

1. Examine the equation $y = \log_b x$ and identify b, y,and x.
2. Rewrite $\log_b x = y$ as $b^y = x$.

Example 6.3.1: Converting from Logarithmic Form to Exponential Form

Write the following logarithmic equations in exponential form.

a. $\log_6(\sqrt{6}) = \frac{1}{2}$
b. $\log_3(9) = 2$

Solution

First, identify the values of b, y,and x. Then, write the equation in the form $b^y = x$.

a. $\log_6(\sqrt{6}) = \frac{1}{2}$

Here, $b = 6$, $y = \frac{1}{2}$,and $x = \sqrt{6}$. Therefore, the equation $\log_6(\sqrt{6}) = \frac{1}{2}$ is equivalent to

$$6^{\frac{1}{2}} = \sqrt{6}$$

b. $\log_3(9) = 2$

Here, $b = 3$, $y = 2$,and $x = 9$. Therefore, the equation $\log_3(9) = 2$ is equivalent to

$$3^2 = 9$$

Exercise 6.3.1

Write the following logarithmic equations in exponential form.

a. $\log_{10}(1,000,000) = 6$
b. $\log_5(25) = 2$

Answer a

$\log_{10}(1,000,000) = 6$ is equivalent to $10^6 = 1,000,000$

Answer b

$\log_5(25) = 2$ is equivalent to $5^2 = 25$

Converting from Exponential to Logarithmic Form

To convert from exponents to logarithms, we follow the same steps in reverse. We identify the base b,exponent x,and output y. Then we write $x = \log_b(y)$.

Example 6.3.2: Converting from Exponential Form to Logarithmic Form

Write the following exponential equations in logarithmic form.

a. $2^3 = 8$
b. $5^2 = 25$
c. $10^{-4} = \dfrac{1}{10,000}$

Solution

First, identify the values of b, y,and x. Then, write the equation in the form $x = \log_b(y)$.

a. $2^3 = 8$

Here, $b = 2$, $x = 3$,and $y = 8$. Therefore, the equation $2^3 = 8$ is equivalent to $\log_2(8) = 3$.

b. $5^2 = 25$

Here, $b = 5$, $x = 2$,and $y = 25$. Therefore, the equation $5^2 = 25$ is equivalent to $\log_5(25) = 2$.

c. $10^{-4} = \dfrac{1}{10,000}$

Here, $b = 10$, $x = -4$,and $y = \dfrac{1}{10,000}$. Therefore, the equation $10^{-4} = \dfrac{1}{10,000}$ is equivalent to $\log_{10}\left(\dfrac{1}{10,000}\right) = -4$.

Exercise 6.3.2

Write the following exponential equations in logarithmic form.

a. $3^2 = 9$
b. $5^3 = 125$
c. $2^{-1} = \dfrac{1}{2}$

Answer a

$3^2 = 9$ is equivalent to $\log_3(9) = 2$

Answer b

$5^3 = 125$ is equivalent to $\log_5(125) = 3$

Answer c

$2^{-1} = \dfrac{1}{2}$ is equivalent to $\log_2\left(\dfrac{1}{2}\right) = -1$

Evaluating Logarithms

Knowing the squares, cubes, and roots of numbers allows us to evaluate many logarithms mentally. For example, consider $\log_2 8$. We ask, "To what exponent must 2 be raised in order to get 8?" Because we already know $2^3 = 8$, it follows that $\log_2 8 = 3$.

Now consider solving $\log_7 49$ and $\log_3 27$ mentally.

- We ask, "To what exponent must 7 be raised in order to get 49?" We know $7^2 = 49$. Therefore, $\log_7 49 = 2$
- We ask, "To what exponent must 3 be raised in order to get 27?" We know $3^3 = 27$. Therefore, $\log_3 27 = 3$

Even some seemingly more complicated logarithms can be evaluated without a calculator. For example, let's evaluate $\log_{\frac{2}{3}} \frac{4}{9}$ mentally.

- We ask, "To what exponent must $\frac{2}{3}$ be raised in order to get $\frac{4}{9}$?" We know $2^2 = 4$ and $3^2 = 9$, so

$$\left(\frac{2}{3}\right)^2 = \frac{4}{9}.$$

Therefore,

$$\log_{\frac{2}{3}}\left(\frac{4}{9}\right) = 2.$$

How to: Given a logarithm of the form $y = \log_b(x)$,evaluate it mentally

1. Rewrite the argument x as a power of b: $b^y = x$.
2. Use previous knowledge of powers of b identify y by asking, "To what exponent should b be raised in order to get x?"

Example 6.3.3: Solving Logarithms Mentally

Solve $y = \log_4(64)$ without using a calculator.

Solution

First we rewrite the logarithm in exponential form: $4^y = 64$. Next, we ask, "To what exponent must 4 be raised in order to get 64?"

We know

$$4^3 = 64$$

Therefore,

$$\log_4(64) = 3$$

Exercise 6.3.3

Solve $y = \log_{121}(11)$ without using a calculator.

Answer

$\log_{121}(11) = \frac{1}{2}$ (recalling that $\sqrt{121} = (121)^{\frac{1}{2}} = 11$)

Example 6.3.4: Evaluating the Logarithm of a Reciprocal

Evaluate $y = \log_3\left(\frac{1}{27}\right)$ without using a calculator.

Solution

First we rewrite the logarithm in exponential form: $3^y = \frac{1}{27}$. Next, we ask, "To what exponent must 3 be raised in order to get $\frac{1}{27}$?"

We know $3^3 = 27$,but what must we do to get the reciprocal, $\frac{1}{27}$? Recall from working with exponents that $b^{-a} = \frac{1}{b^a}$. We use this information to write

$$\begin{aligned} 3^{-3} &= \frac{1}{3^3} \\ &= \frac{1}{27} \end{aligned}$$

Therefore, $\log_3\left(\frac{1}{27}\right) = -3$.

Exercise 6.3.4

Evaluate $y = \log_2\left(\frac{1}{32}\right)$ without using a calculator.

Answer

$$\log_2\left(\frac{1}{32}\right) = -5$$

Using Common Logarithms

Sometimes we may see a logarithm written without a base. In this case, we assume that the base is 10. In other words, the expression $\log(x)$ means $\log_{10}(x)$. We call a base -10 logarithm a *common logarithm*. Common logarithms are used to measure the Richter Scale mentioned at the beginning of the section. Scales for measuring the brightness of stars and the pH of acids and bases also use common logarithms.

DEFINITION OF THE COMMON LOGARITHM

A common logarithm is a logarithm with base 10. We write $\log_{10}(x)$ simply as $\log(x)$. The common logarithm of a positive number x satisfies the following definition.

For $x > 0$,

$$y = \log(x) \text{ is equivalent to } 10^y = x \tag{6.3.3}$$

We read $\log(x)$ as, "the logarithm with base 10 of x " or "log base 10 of x."

The logarithm y is the exponent to which 10 must be raised to get x.

How to: Given a common logarithm of the form $y = \log(x)$, evaluate it mentally

1. Rewrite the argument x as a power of 10: $10^y = x$.
2. Use previous knowledge of powers of 10 to identify y by asking, "To what exponent must 10 be raised in order to get x?"

Example 6.3.5: Finding the Value of a Common Logarithm Mentally

Evaluate $y = \log(1000)$ without using a calculator.

Solution

First we rewrite the logarithm in exponential form: $10^y = 1000$. Next, we ask, "To what exponent must 10 be raised in order to get 1000?" We know

$$10^3 = 1000$$

Therefore, $\log(1000) = 3$.

Exercise 6.3.5

Evaluate $y = \log(1,000,000)$.

Answer

$$\log(1,000,000) = 6$$

How to: Given a common logarithm with the form $y = \log(x)$,evaluate it using a calculator

1. Press **[LOG]**.
2. Enter the value given for x,followed by **[ )]**.
3. Press **[ENTER]**.

Example 6.3.6: Finding the Value of a Common Logarithm Using a Calculator

Evaluate $y = \log(321)$ to four decimal places using a calculator.

Solution

- Press **[LOG]**.
- Enter 321, followed by **[ )]**.
- Press **[ENTER]**.

Rounding to four decimal places, $\log(321) \approx 2.5065$.

Analysis

Note that $10^2 = 100$ and that $10^3 = 1000$. Since 321 is between 100 and 1000, we know that $\log(321)$ must be between $\log(100)$ and $\log(1000)$. This gives us the following:

$$100 < 321 < 1000$$

$$2 < 2.5065 < 3$$

Exercise 6.3.6

Evaluate $y = \log(123)$ to four decimal places using a calculator.

Answer

$\log(123) \approx 2.0899$

Example 6.3.7: Rewriting and Solving a Real-World Exponential Model

The amount of energy released from one earthquake was 500 times greater than the amount of energy released from another. The equation $10^x = 500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

Solution

We begin by rewriting the exponential equation in logarithmic form.

$$10^x = 500$$

$$\log(500) = x \text{ Use the definition of the common log.}$$

Next we evaluate the logarithm using a calculator:

- Press **[LOG]**.
- Enter 500,followed by **[ )]**.
- Press **[ENTER]**.
- To the nearest thousandth, $\log(500) \approx 2.699$.

The difference in magnitudes was about 2.699.

Exercise 6.3.7

The amount of energy released from one earthquake was $8,500$ times greater than the amount of energy released from another. The equation $10^x = 8500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

Answer

The difference in magnitudes was about 3.929.

Using Natural Logarithms

The most frequently used base for logarithms is e. Base e logarithms are important in calculus and some scientific applications; they are called **natural logarithms**. The base e logarithm, $\log_e(x)$, has its own notation, $\ln(x)$. Most values of $\ln(x)$ can be found only using a calculator. The major exception is that, because the logarithm of 1 is always 0 in any base, $\ln 1 = 0$. For other natural logarithms, we can use the $\ln$ key that can be found on most scientific calculators. We can also find the natural logarithm of any power of e using the inverse property of logarithms.

DEFINITION OF THE NATURAL LOGARITHM

A natural logarithm is a logarithm with base e. We write $\log_e(x)$ simply as $\ln(x)$. The natural logarithm of a positive number x satisfies the following definition.

For $x > 0$,

$$y = \ln(x) \text{ is equivalent to } e^y = x$$

We read $\ln(x)$ as, "the logarithm with base e of x" or "the natural logarithm of x."

The logarithm y is the exponent to which e must be raised to get x.

Since the functions $y = e^x$ and $y = \ln(x)$ are inverse functions, $\ln(e^x) = x$ for all x and $e^{\ln(x)} = x$ for $x > 0$.

How to: Given a natural logarithm with the form $y = \ln(x)$, evaluate it using a calculator

1. Press **[LN]**.
2. Enter the value given for x, followed by **[ )]**.
3. Press **[ENTER]**.

Example 6.3.8: Evaluating a Natural Logarithm Using a Calculator

Evaluate $y = \ln(500)$ to four decimal places using a calculator.

Solution

- Press **[LN]**.
- Enter 500, followed by **[ )]**.
- Press **[ENTER]**.

Rounding to four decimal places, $\ln(500) \approx 6.2146$

Exercise 6.3.8

Evaluate $\ln(-500)$.

Answer

It is not possible to take the logarithm of a negative number in the set of real numbers.

Media

Access this online resource for additional instruction and practice with logarithms.

- Introduction to Logarithms

Key Equations

Definition of the logarithmic function	For $x > 0$, $b > 0$, $b \neq 1$, $y = \log_b(x)$ if and only if $b^y = x$.
Definition of the common logarithm	For $x > 0$, $y = \log(x)$ if and only if $10^y = x$.
Definition of the natural logarithm	For $x > 0$, $y = \ln(x)$ if and only if $e^y = x$.

Key Concepts

- The inverse of an exponential function is a logarithmic function, and the inverse of a logarithmic function is an exponential function.
- Logarithmic equations can be written in an equivalent exponential form, using the definition of a logarithm. See Example **6.3.1**.
- Exponential equations can be written in their equivalent logarithmic form using the definition of a logarithm See Example **6.3.2**.
- Logarithmic functions with base b can be evaluated mentally using previous knowledge of powers of b. See Example **6.3.3** and Example **6.3.4**.
- Common logarithms can be evaluated mentally using previous knowledge of powers of **10**. See Example **6.3.5**.
- When common logarithms cannot be evaluated mentally, a calculator can be used. See Example **6.3.6**.
- Real-world exponential problems with base **10** can be rewritten as a common logarithm and then evaluated using a calculator. See Example **6.3.7**.
- Natural logarithms can be evaluated using a calculator Example **6.3.8.**

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6.4: Graphs of Logarithmic Functions

Learning Objectives

- Identify the domain of a logarithmic function.
- Graph logarithmic functions.

In the Section on Graphs of Exponential Functions, we saw how creating a graphical representation of an exponential model gives us another layer of insight for predicting future events. How do logarithmic graphs give us insight into situations? Because every logarithmic function is the inverse function of an exponential function, we can think of every output on a logarithmic graph as the input for the corresponding inverse exponential equation. In other words, logarithms give the *cause* for an *effect*.

To illustrate, suppose we invest $\$2500$ in an account that offers an annual interest rate of 5%, compounded continuously. We already know that the balance in our account for any year t can be found with the equation $A = 2500e^{0.05t}$.

But what if we wanted to know the year for any balance? We would need to create a corresponding new function by interchanging the input and the output; thus we would need to create a logarithmic model for this situation. By graphing the model, we can see the output (year) for any input (account balance). For instance, what if we wanted to know how many years it would take for our initial investment to double? Figure 6.4.1 shows this point on the logarithmic graph.

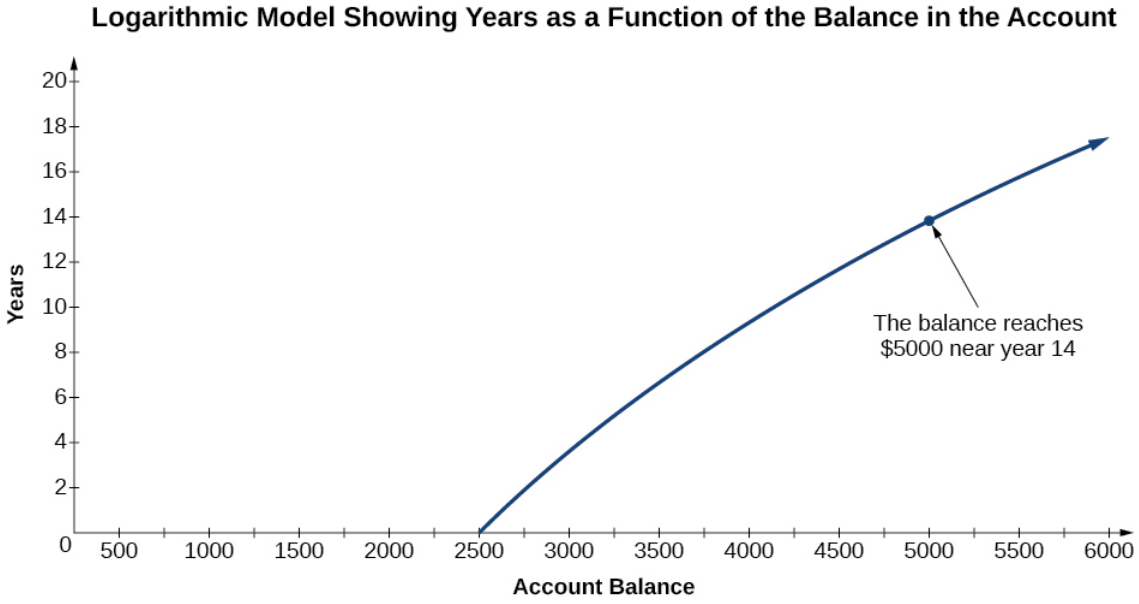


Figure 6.4.1

In this section we will discuss the values for which a logarithmic function is defined, and then turn our attention to graphing the family of logarithmic functions.

Finding the Domain of a Logarithmic Function

Before working with graphs, we will take a look at the domain (the set of input values) for which the logarithmic function is defined.

Recall that the exponential function is defined as $y = b^x$ for any real number x and constant $b > 0$, $b \neq 1$, where

- The domain of y is $(-\infty, \infty)$.
- The range of y is $(0, \infty)$.

In the last section we learned that the logarithmic function $y = \log_b(x)$ is the inverse of the exponential function $y = b^x$. So, as inverse functions:

- The domain of $y = \log_b(x)$ is the range of $y = b^x$: $(0, \infty)$.
- The range of $y = \log_b(x)$ is the domain of $y = b^x$: $(-\infty, \infty)$.

Transformations of the parent function $y = \log_b(x)$ behave similarly to those of other functions. Just as with other parent functions, we can apply the four types of transformations—shifts, stretches, compressions, and reflections—to the parent function without loss of shape.

In Graphs of Exponential Functions we saw that certain transformations can change the *range* of $y = b^x$. Similarly, applying transformations to the parent function $y = \log_b(x)$ can change the *domain*. When finding the domain of a logarithmic function, therefore, it is important to remember that the domain consists *only of positive real numbers*. That is, the argument of the logarithmic function must be greater than zero.

For example, consider $f(x) = \log_4(2x - 3)$. This function is defined for any values of x such that the argument, in this case $2x - 3$,is greater than zero. To find the domain, we set up an inequality and solve for x:

$$\begin{aligned} 2x - 3 &> 0 && \text{Show the argument greater than zero} \\ 2x &> 3 && \text{Add 3} \\ x &> 1.5 && \text{Divide by 2} \end{aligned}$$

In interval notation, the domain of $f(x) = \log_4(2x - 3)$ is $(1.5, \infty)$.

Given a logarithmic function, identify the domain

1. Set up an inequality showing the argument greater than zero.
2. Solve for x.
3. Write the domain in interval notation.

Example 6.4.1: Identifying the Domain of a Logarithmic Shift

What is the domain of $f(x) = \log_2(x + 3)$?

Solution

The logarithmic function is defined only when the input is positive, so this function is defined when $x + 3 > 0$. Solving this inequality,

$$\begin{aligned} x + 3 &> 0 && \text{The input must be positive} \\ x &> -3 && \text{Subtract 3} \end{aligned}$$

The domain of $f(x) = \log_2(x + 3)$ is $(-3, \infty)$.

Exercise 6.4.1

What is the domain of $f(x) = \log_5(x - 2) + 1$?

Answer

$(2, \infty)$

Example 6.4.2: Identifying the Domain of a Logarithmic Shift and Reflection

What is the domain of $f(x) = \log(5 - 2x)$?

Solution

The logarithmic function is defined only when the input is positive, so this function is defined when $5–2x > 0$. Solving this inequality,

$$\begin{aligned} 5 - 2x &> 0 && \text{The input must be positive} \\ -2x &> -5 && \text{Subtract 5} \\ x &< \frac{5}{2} && \text{Divide by -2 and switch the inequality} \end{aligned}$$

The domain of $f(x) = \log(5 - 2x)$ is $\left(-\infty, \frac{5}{2}\right)$.

? Exercise 6.4.2

What is the domain of $f(x) = \log(x - 5) + 2$?

Answer

$(5, \infty)$

Graphing Logarithmic Functions

Now that we have a feel for the set of values for which a logarithmic function is defined, we move on to graphing logarithmic functions. The family of logarithmic functions includes the parent function $y = \log_b(x)$ along with all its transformations: shifts, stretches, compressions, and reflections.

We begin with the parent function $y = \log_b(x)$. Because every logarithmic function of this form is the inverse of an exponential function with the form $y = b^x$, their graphs will be reflections of each other across the line $y = x$. To illustrate this, we can observe the relationship between the input and output values of $y = 2^x$ and its equivalent $x = \log_2(y)$ in Table **6.4.1**.

Table **6.4.1**

x	-3	-2	-1	0	1	2	3
$2^x = y$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$\log_2(y) = x$	-3	-2	-1	0	1	2	3

Using the inputs and outputs from Table **6.4.1**, we can build another table to observe the relationship between points on the graphs of the inverse functions $f(x) = 2^x$ and $g(x) = \log_2(x)$. See Table **6.4.2**.

Table **6.4.2**

$f(x) = 2^x$	$\left(-3, \frac{1}{8}\right)$	$\left(-2, \frac{1}{4}\right)$	$\left(-1, \frac{1}{2}\right)$	$(0, 1)$	$(1, 2)$	$(2, 4)$	$(3, 8)$
$g(x) = \log_2(x)$	$\left(\frac{1}{8}, -3\right)$	$\left(\frac{1}{4}, -2\right)$	$\left(\frac{1}{2}, -1\right)$	$(1, 0)$	$(2, 1)$	$(4, 2)$	$(8, 3)$

As we'd expect, the x- and y-coordinates are reversed for the inverse functions. Figure **6.4.2** shows the graph of f and g.

Figure **6.4.2**: Notice that the graphs of $f(x) = 2^x$ and $g(x) = \log_2(x)$ are reflections about the line $y = x$.

Observe the following from the graph:

- $f(x) = 2^x$ has a y-intercept at $(0, 1)$ and $g(x) = \log_2(x)$ has an x- intercept at $(1, 0)$.
- The domain of $f(x) = 2^x$, $(-\infty, \infty)$, is the same as the range of $g(x) = \log_2(x)$.
- The range of $f(x) = 2^x$, $(0, \infty)$, is the same as the domain of $g(x) = \log_2(x)$.

CHARACTERISTICS OF THE GRAPH OF THE PARENT FUNCTION, $F(X) = LOG_B(X)$

For any real number x and constant $b > 0$, $b \neq 1$, we can see the following characteristics in the graph of $f(x) = \log_b(x)$:

- one-to-one function
- vertical asymptote: $x = 0$
- domain: $(0, \infty)$
- range: $(-\infty, \infty)$
- x-intercept: $(1, 0)$ and key point $(b, 1)$
- y-intercept: none
- increasing if $b > 1$

- decreasing if $0 < b < 1$

See Figure 6.4.3.

Figure 6.4.4 shows how changing the base b in $f(x) = \log_b(x)$ can affect the graphs. Observe that the graphs compress vertically as the value of the base increases. (Note: recall that the function $\ln(x)$ has base $e \approx 2.718$.)

Figure 6.4.4: The graphs of three logarithmic functions with different bases, all greater than 1.

Given a logarithmic function with the form $f(x) = \log_b(x)$, graph the function.

1. Draw and label the vertical asymptote, $x = 0$.
2. Plot the x-intercept, $(1, 0)$.
3. Plot the key point $(b, 1)$.
4. Draw a smooth curve through the points.
5. State the domain, $(0, \infty)$,the range, $(-\infty, \infty)$, and the vertical asymptote, $x = 0$.

Example 6.4.3: Graphing a Logarithmic Function with the Form $f(x) = log_b(x)$

Graph $f(x) = \log_5(x)$. State the domain, range, and asymptote.

Solution

Before graphing, identify the behavior and key points for the graph.

- Since $b = 5$ is greater than one, we know the function is increasing. The left tail of the graph will approach the vertical asymptote $x = 0$, and the right tail will increase slowly without bound.
- The x-intercept is $(1, 0)$.
- The key point $(5, 1)$ is on the graph.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points (see Figure 6.4.5).

Figure 6.4.5

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Exercise 6.4.3

Graph $f(x) = \log_{\frac{1}{5}}(x)$. State the domain, range, and asymptote.

Answer

Figure 6.4.6

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Graphing Transformations of Logarithmic Functions

As we mentioned in the beginning of the section, transformations of logarithmic graphs behave similarly to those of other parent functions. We can shift, stretch, compress, and reflect the parent function $y = \log_b(x)$ without loss of shape.

Graphing a Horizontal Shift of $f(x) = log_b(x)$

When a constant c is added to the input of the parent function $f(x) = \log_b(x)$, the result is a horizontal shift c units in the *opposite* direction of the sign on c. To visualize horizontal shifts, we can observe the general graph of the parent function $f(x) = \log_b(x)$ and for $c > 0$ alongside the shift left, $g(x) = \log_b(x + c)$, and the shift right, $h(x) = \log_b(x - c)$. See Figure 6.4.7.

Figure 6.4.7

HORIZONTAL SHIFTS OF THE PARENT FUNCTION $Y = LOG_B(X)$

For any constant c,the function $f(x) = \log_b(x + c)$

- shifts the parent function $y = \log_b(x)$ left c units if $c > 0$.
- shifts the parent function $y = \log_b(x)$ right c units if $c < 0$.
- has the vertical asymptote $x = -c$.
- has domain $(-c, \infty)$.
- has range $(-\infty, \infty)$.

Given a logarithmic function with the form $f(x) = \log_b(x + c)$, graph the translation.

1. Identify the horizontal shift:
   - If $c > 0$,shift the graph of $f(x) = \log_b(x)$ left c units.
   - If $c < 0$,shift the graph of $f(x) = \log_b(x)$ right c units.
2. Draw the vertical asymptote $x = -c$.
3. Identify three key points from the parent function. Find new coordinates for the shifted functions by subtracting c from the x coordinate.
4. Label the three points.
5. The Domain is $(-c, \infty)$,the range is $(-\infty, \infty)$, and the vertical asymptote is $x = -c$.

Example 6.4.4: Graphing a Horizontal Shift of the Parent Function $y = log_b(x)$

Sketch the horizontal shift $f(x) = \log_3(x - 2)$ alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

Solution

Since the function is $f(x) = \log_3(x - 2)$, we notice $x + (-2) = x–2$.

Thus $c = -2$, so $c < 0$. This means we will shift the function $f(x) = \log_3(x)$ right 2 units.

The vertical asymptote is $x = -(-2)$ or $x = 2$.

Consider the three key points from the parent function, $\left(\frac{1}{3}, -1\right)$, $(1, 0)$,and $(3, 1)$.

The new coordinates are found by adding 2 to the x coordinates.

Label the points $\left(\frac{7}{3}, -1\right)$, $(3, 0)$,and $(5, 1)$.

The domain is $(2, \infty)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 2$.

Figure 6.4.8

Exercise 6.4.4

Sketch a graph of $f(x) = \log_3(x + 4)$ alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

Answer

Figure 6.4.9

The domain is $(-4, \infty)$,the range $(-\infty, \infty)$,and the asymptote $x = –4$.

Graphing a Vertical Shift of $y = log_b(x)$

When a constant d is added to the parent function $f(x) = \log_b(x)$,the result is a vertical shift d units in the direction of the sign on d. To visualize vertical shifts, we can observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the shift up, $g(x) = \log_b(x) + d$ and the shift down, $h(x) = \log_b(x) - d$.See Figure 6.4.10.

Figure 6.4.10

VERTICAL SHIFTS OF THE PARENT FUNCTION $Y = LOG_B(X)$

For any constant d,the function $f(x) = \log_b(x) + d$

- shifts the parent function $y = \log_b(x)$ up d units if $d > 0$.
- shifts the parent function $y = \log_b(x)$ down d units if $d < 0$.
- has the vertical asymptote $x = 0$.
- has domain $(0, \infty)$.
- has range $(-\infty, \infty)$.

Given a logarithmic function with the form $f(x) = \log_b(x) + d$, graph the translation.

1. Identify the vertical shift:
   - If $d > 0$, shift the graph of $f(x) = \log_b(x)$ up d units.
   - If $d < 0$, shift the graph of $f(x) = \log_b(x)$ down d units.
2. Draw the vertical asymptote $x = 0$.
3. Identify three key points from the parent function. Find new coordinates for the shifted functions by adding d to the y coordinate.
4. Label the three points.
5. The domain is $(0, \infty)$, the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

Example 6.4.5: Graphing a Vertical Shift of the Parent Function $y = log_b(x)$

Sketch a graph of $f(x) = \log_3(x) - 2$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution

Since the function is $f(x) = \log_3(x) - 2$,we will notice $d = -2$. Thus $d < 0$.

This means we will shift the function $f(x) = \log_3(x)$ down 2 units.

The vertical asymptote is $x = 0$.

Consider the three key points from the parent function, $\left(\frac{1}{3}, -1\right)$, $(1, 0)$,and $(3, 1)$.

The new coordinates are found by subtracting 2 from the y coordinates.

Label the points $\left(\frac{1}{3}, -3\right)$, $(1, -2)$, and $(3, -1)$.

The domain is $(0, \infty)$,the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Figure 6.4.11

The domain is $(0, \infty)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

Exercise 6.4.5

Sketch a graph of $f(x) = \log_2(x) + 2$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Answer

Figure 6.4.12

The domain is $(0,\infty)$,the range is $(-\infty,\infty)$, and the vertical asymptote is $x=0$.

Graphing Stretches and Compressions of $y=log_b(x)$

When the parent function $f(x)=\log_b(x)$ is multiplied by a constant $a>0$, the result is a vertical stretch or compression of the original graph. To visualize stretches and compressions, we set $a>1$ and observe the general graph of the parent function $f(x)=\log_b(x)$ alongside the vertical stretch, $g(x)=a\log_b(x)$ and the vertical compression, $h(x)=\frac{1}{a}\log_b(x)$.See Figure 6.4.13.

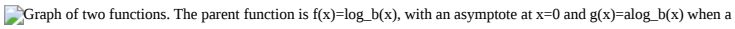

Graph of two functions. The parent function is f(x)=log_b(x), with an asymptote at x=0 and g(x)=alog_b(x) when a 1 is the translation function with an asymptote at x=0. The graph note the intersection of the two lines at (1, 0). This shows the translation of a vertical stretch." src="/@api/deki/files/12175/fig_6.5.13.jpg">

Figure 6.4.13

VERTICAL STRETCHES AND COMPRESSIONS OF THE PARENT FUNCTION $Y=LOG_B(X)$

For any constant $a>1$,the function $f(x)=a\log_b(x)$

- stretches the parent function $y=\log_b(x)$ vertically by a factor of a if $a>1$.
- compresses the parent function $y=\log_b(x)$ vertically by a factor of a if $0<a<1$.
- has the vertical asymptote $x=0$.
- has the x-intercept $(1,0)$.
- has domain $(0,\infty)$.
- has range $(-\infty,\infty)$.

Given a logarithmic function with the form $f(x)=a\log_b(x)$, $a>0$,graph the translation.

1. Identify the vertical stretch or compressions:
   - If $|a|>1$, the graph of $f(x)=\log_b(x)$ is stretched by a factor of a units.
   - If $|a|<1$, the graph of $f(x)=\log_b(x)$ is compressed by a factor of a units.
2. Draw the vertical asymptote $x=0$.
3. Identify three key points from the parent function. Find new coordinates for the shifted functions by multiplying they y coordinates by a.
4. Label the three points.
5. The domain is $(0,\infty)$, the range is $(-\infty,\infty)$, and the vertical asymptote is $x=0$.

Example 6.4.6: Graphing a Stretch or Compression of the Parent Function $y=log_b(x)$

Sketch a graph of $f(x)=2\log_4(x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution

Since the function is $f(x)=2\log_4(x)$,we will notice $a=2$.

This means we will stretch the function $f(x)=\log_4(x)$ by a factor of 2.

The vertical asymptote is $x=0$.

Consider the three key points from the parent function, $\left(\frac{1}{4},-1\right)$, $(1,0)$, and $(4,1)$.

The new coordinates are found by multiplying the y coordinates by 2.

Label the points $\left(\frac{1}{4},-2\right)$, $(1,0)$, and $(4,2)$.

The domain is $(0,\infty)$, the range is $(-\infty,\infty)$, and the vertical asymptote is $x=0$. See Figure 6.4.14.

Figure 6.4.14

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Exercise 6.4.6

Sketch a graph of $f(x) = \frac{1}{2}\log_4(x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Answer

Figure 6.4.15

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

Example 6.4.7: Combining a Shift and a Stretch

Sketch a graph of $f(x) = 5\log(x + 2)$. State the domain, range, and asymptote.

Solution

Remember: what happens inside parentheses happens first. First, we move the graph left 2 units, then stretch the function vertically by a factor of 5, as in Figure 6.4.16. The vertical asymptote will be shifted to $x = -2$. The x-intercept will be $(-1, 0)$. The domain will be $(-2, \infty)$. Two points will help give the shape of the graph: $(-1, 0)$ and $(8, 5)$. We chose $x = 8$ as the x-coordinate of one point to graph because when $x = 8$, $x + 2 = 10$, the base of the common logarithm.

Figure 6.4.16

The domain is $(-2, \infty)$, the range is $(-\infty, \infty)$,and the vertical asymptote is $x = -2$.

Exercise 6.4.7

Sketch a graph of the function $f(x) = 3\log(x - 2) + 1$. State the domain, range, and asymptote.

Answer

Figure 6.4.17

The domain is $(2, \infty)$,the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 2$.

Graphing Reflections of $f(x) = log_b(x)$

When the parent function $f(x) = \log_b(x)$ is multiplied by -1,the result is a reflection about the x-axis. When the *input* is multiplied by -1,the result is a reflection about the y-axis. To visualize reflections, we restrict $b > 1$, and observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the reflection about the x-axis, $g(x) = -\log_b(x)$ and the reflection about the y-axis, $h(x) = \log_b(-x)$.

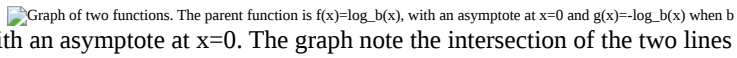

Graph of two functions. The parent function is f(x)=log_b(x), with an asymptote at x=0 and g(x)=-log_b(x) when b
1 is the translation function with an asymptote at x=0. The graph note the intersection of the two lines at (1, 0). This shows the translation of a reflection about the x-axis." src="/@api/deki/files/12180/fig_6.5.18.jpg">

Figure 6.4.18

REFLECTIONS OF THE PARENT FUNCTION $Y = LOG_B(X)$

The function $f(x) = -\log_b(x)$

- reflects the parent function $y = \log_b(x)$ about the x-axis.
- has domain, $(0, \infty)$, range, $(-\infty, \infty)$, and vertical asymptote, $x = 0$, which are unchanged from the parent function.

The function $f(x) = \log_b(-x)$

- reflects the parent function $y = \log_b(x)$ about the y-axis.
- has domain $(-\infty, 0)$.
- has range, $(-\infty, \infty)$, and vertical asymptote, $x = 0$, which are unchanged from the parent function.

Given a logarithmic function with the parent function $f(x) = \log_b(x)$, graph a translation.

Table 6.4.3

If $f(x) = -\log_b(x)$	If $f(x) = \log_b(-x)$
1. Draw the vertical asymptote, $x = 0$.	1. Draw the vertical asymptote, $x = 0$.
2. Plot the x-intercept, $(1, 0)$.	2. Plot the x-intercept, $(1, 0)$.
3. Reflect the graph of the parent function $f(x) = \log_b(x)$ about the x-axis.	3. Reflect the graph of the parent function $f(x) = \log_b(x)$ about the y-axis.
4. Draw a smooth curve through the points.	4. Draw a smooth curve through the points.
5. State the domain, $(0, \infty)$, the range, $(-\infty, \infty)$, and the vertical asymptote $x = 0$.	5. State the domain, $(-\infty, 0)$, the range, $(-\infty, \infty)$, and the vertical asymptote $x = 0$.

Example 6.4.8: Graphing a Reflection of a Logarithmic Function

Sketch a graph of $f(x) = \log(-x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution

Before graphing $f(x) = log(-x)$, $f(x) = log(-x)$,identify the behavior and key points for the graph.

- Since $b = 10$ is greater than one, we know that the parent function is increasing. Since the *input* value is multiplied by -1, $f(x)$ is a reflection of the parent graph about the y-axis. Thus, $f(x) = \log(-x)$ will be decreasing as x moves from negative infinity to zero, and the right tail of the graph will approach the vertical asymptote $x = 0$.
- The x-intercept is $(-1, 0)$.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points.

Figure 6.4.19

The domain is $(-\infty, 0)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Exercise 6.4.8

Graph $f(x) = -\log(-x)$. State the domain, range, and asymptote.

Answer

Figure 6.4.20

The domain is $(-\infty, 0)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

Given a logarithmic equation, use a graphing calculator to approximate solutions.

1. Press **[Y=]**. Enter the given logarithm equation or equations as $\mathbf{Y_1}$= and, if needed, $\mathbf{Y_2}$=.
2. Press **[GRAPH]** to observe the graphs of the curves and use **[WINDOW]** to find an appropriate view of the graphs, including their point(s) of intersection.
3. To find the value of x, we compute the point of intersection. Press **[2ND]** then **[CALC]**. Select "intersect" and press **[ENTER]** three times. The point of intersection gives the value of x,for the point(s) of intersection.

Example 6.4.9: Approximating the Solution of a Logarithmic Equation

Solve $4\ln(x) + 1 = -2\ln(x - 1)$ graphically. Round to the nearest thousandth.

Solution

Press **[Y=]** and enter $4\ln(x)+1$ next to Y_1=. Then enter $-2\ln(x-1)$ next to Y_2=. For a window, use the values 0 to 5 for $x\backslash 0 \text{ and } \backslash(-10$ to 10 for y. Press **[GRAPH]**. The graphs should intersect somewhere a little to right of $x=1$.

For a better approximation, press **[2ND]** then **[CALC]**. Select **[5: intersect]** and press **[ENTER]** three times. The x-coordinate of the point of intersection is displayed as 1.3385297. (Your answer may be different if you use a different window or use a different value for **Guess?**) So, to the nearest thousandth, $x \approx 1.339$.

? Exercise 6.4.9

Solve $5\log(x+2)=4-\log(x)$ graphically. Round to the nearest thousandth.

Answer

$x \approx 3.049$

Summarizing Translations of the Logarithmic Function

Now that we have worked with each type of translation for the logarithmic function, we can summarize each in Table **6.4.4** to arrive at the general equation for translating exponential functions.

1 and a compression is 0<|a|<1. Note its form is f(x)=alog_b(x). The third translation is a reflection about the x-axis with the form f(x) = -log_b(x). The fourth translation is a reflection about the y-axis with the form f(x)=log_b(-x). The general equation for all translations is f(x)=alog_b(x+c)+d.">

***Table* 6.4.4**

Translations of the Parent Function $y=\log_b(x)$	
Translation	**Form**
Shift Horizontally c units to the left Vertically d units up	$y=\log_b(x+c)+d$
Stretch and Compress Stretch if $\lvert a\rvert>1$ Compression if $\lvert a\rvert<1$	$y=a\log_b(x)$
Reflect about the x-axis	$y=-\log_b(x)$
Reflect about the y-axis	$y=\log_b(-x)$
General equation for all translations	$y=a\log_b(x+c)+d$

TRANSLATIONS OF LOGARITHMIC FUNCTIONS

All translations of the parent logarithmic function, $y=\log_b(x)$, have the form

$f(x)=a\log_b(x+c)+d$

where the parent function, $y=\log_b(x)$, $b>1$,is

- shifted vertically up d units.
- shifted horizontally to the left c units.
- stretched vertically by a factor of $|a|$ if $|a|>0$.
- compressed vertically by a factor of $|a|$ if $0<|a|<1$.
- reflected about the x-axis when $a<0$.

For $f(x)=\log(-x)$, the graph of the parent function is reflected about the y-axis.

Example 6.4.10: Finding the Vertical Asymptote of a Logarithm Graph

What is the vertical asymptote of $f(x) = -2\log_3(x+4) + 5$?

Solution

The vertical asymptote is at $x = -4$.

Analysis

The coefficient, the base, and the upward translation do not affect the asymptote. The shift of the curve 4 units to the left shifts the vertical asymptote to $x = -4$.

Exercise 6.4.10

What is the vertical asymptote of $f(x) = 3 + \ln(x-1)$?

Answer

$x = 1$

Example 6.4.11: Finding the Equation from a Graph

Find a possible equation for the common logarithmic function graphed in Figure 6.4.21.

Figure 6.4.21

Solution

This graph has a vertical asymptote at $x = -2$ and has been vertically reflected. We do not know yet the vertical shift or the vertical stretch. We know so far that the equation will have form:

$f(x) = -a\log(x+2) + k$

It appears the graph passes through the points $(-1, 1)$ and $(2, -1)$. Substituting $(-1, 1)$,

$$\begin{aligned} 1 &= -a\log(-1+2) + k && \text{Substitute}(-1,1) \\ 1 &= -a\log(1) + k && \text{Arithmetic} \\ 1 &= k\log(1) \\ &= 0 \end{aligned}$$

Next, substituting in $(2, -1)$,

$$\begin{aligned} -1 &= -a\log(2+2) + 1 && \text{Substitute}(2,-1) \\ -2 &= -a\log(4) && \text{Arithmetic} \\ a &= \frac{2}{\log(4)} && \text{Solve for a} \end{aligned}$$

This gives us the equation $f(x) = -\frac{2}{\log(4)}\log(x+2) + 1$.

Analysis

We can verify this answer by comparing the function values in Table 6.4.5 with the points on the graph in Figure 6.4.21.

Table 6.4.5

x	−1	0	1	2	3
$f(x)$	1	0	−0.58496	−1	−1.3219
x	4	5	6	7	8
$f(x)$	−1.5850	−1.8074	−2	−2.1699	−2.3219

? Exercise 6.4.11

Give the equation of the natural logarithm graphed in Figure **6.4.22**.

Figure **6.4.22**

Answer

$$f(x) = 2\ln(x+3) - 1$$

Media: Is it possible to tell the domain and range and describe the end behavior of a function just by looking at the graph?

Yes, if we know the function is a general logarithmic function. For example, look at the graph in Figure **6.4.22**. The graph approaches $x = -3$ (or thereabouts) more and more closely, so $x = -3$ is, or is very close to, the vertical asymptote. It approaches from the right, so the domain is all points to the right, $x|x > -3$. The range, as with all general logarithmic functions, is all real numbers. And we can see the end behavior because the graph goes down as it goes left and up as it goes right. The end behavior is that as $x \to -3^+$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$.

Media

Access these online resources for additional instruction and practice with graphing logarithms.

- Graph an Exponential Function and Logarithmic Function
- Match Graphs with Exponential and Logarithmic Functions
- Find the Domain of Logarithmic Functions

Key Equations

General Form for the Translation of the Parent Logarithmic Function $f(x) = \log_b(x)$	$f(x) = a\log_b(x+c) + d$

Key Concepts

- To find the domain of a logarithmic function, set up an inequality showing the argument greater than zero, and solve for x. See Example **6.4.1** and Example **6.4.2**
- The graph of the parent function $f(x) = \log_b(x)$ has an x-intercept at $(1, 0)$,domain $(0, \infty)$,range $(-\infty, \infty)$,vertical asymptote $x = 0$,and
- - if $b > 1$,the function is increasing.
  - if $0 < b < 1$, the function is decreasing.

  See Example **6.4.3**.
- The equation $f(x) = \log_b(x+c)$ shifts the parent function $y = \log_b(x)$ horizontally
  - left c units if $c > 0$.
  - right c units if $c < 0$.

  See Example **6.4.4**.
- The equation $f(x) = \log_b(x) + d$ shifts the parent function $y = \log_b(x)$ vertically
  - up d units if $d > 0$.
  - down d units if $d < 0$.

  See Example **6.4.5**.
- For any constant $a > 0$, the equation $f(x) = a\log_b(x)$
  - stretches the parent function $y = \log_b(x)$ vertically by a factor of a if $|a| > 1$.
  - compresses the parent function $y = \log_b(x)$ vertically by a factor of a if $|a| <1$.

  See Example **6.4.6** and Example **6.4.7**.

- When the parent function $y = \log_b(x)$ is multiplied by -1, the result is a reflection about the *x*-axis. When the input is multiplied by -1, the result is a reflection about the *y*-axis.
    - The equation $f(x) = -\log_b(x)$ represents a reflection of the parent function about the *x*-axis.
    - The equation $f(x) = \log_b(-x)$ represents a reflection of the parent function about the *y*-axis.

    See Example **6.4.8**.
    - A graphing calculator may be used to approximate solutions to some logarithmic equations See Example **6.4.9**.
- All translations of the logarithmic function can be summarized by the general equation $f(x) = a\log_b(x + c) + d$. See Table **6.4.4**.
- Given an equation with the general form $f(x) = a\log_b(x + c) + d$,we can identify the vertical asymptote $x = -c$ for the transformation. See Example **6.4.10**.
- Using the general equation $f(x) = a\log_b(x + c) + d$, we can write the equation of a logarithmic function given its graph. See Example **6.4.11**.

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6.5: Logarithmic Properties

Learning Objectives

- Use the product rule for logarithms.
- Use the quotient rule for logarithms.
- Use the power rule for logarithms.
- Expand logarithmic expressions.
- Condense logarithmic expressions.
- Use the change-of-base formula for logarithms.

In chemistry, the pH scale is used as a measure of the acidity or alkalinity of a substance. Substances with a pH less than 7 are considered acidic, and substances with a pH greater than 7 are said to be alkaline. Our bodies, for instance, must maintain a pH close to 7.35 in order for enzymes to work properly. To get a feel for what is acidic and what is alkaline, consider the following pH levels of some common substances:

- Battery acid: 0.8
- Stomach acid: 2.7
- Orange juice: 3.3
- Pure water: 7 at $25°C$
- Human blood: 7.35
- Fresh coconut: 7.8
- Sodium hydroxide (lye): 14

To determine whether a solution is acidic or alkaline, we find its pH, which is a measure of the number of active positive hydrogen ions in the solution. The pH is defined by the following formula, where $[H^+]$ is the concentration of hydrogen ions in the solution

$$pH = -\log([H^+]) \tag{6.5.1}$$

$$= \log\left(\frac{1}{[H^+]}\right) \tag{6.5.2}$$

The equivalence of Equations 6.5.1 and 6.5.2 is one of the logarithm properties we will examine in this section.

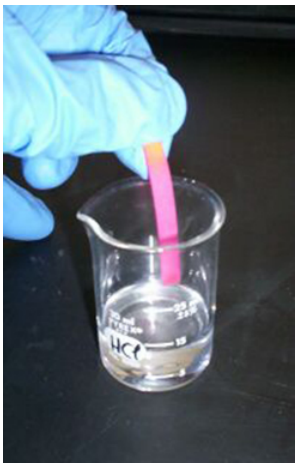


Figure 6.5.1: The pH of hydrochloric acid is tested with litmus paper. (credit: David Berardan).

Using the Product Rule for Logarithms

Recall that the logarithmic and exponential functions "undo" each other. This means that logarithms have similar properties to exponents. Some important properties of logarithms are given here. First, the following properties are easy to prove.

$$\log_b 1 = 0$$

$$\log_b b = 1$$

For example, $\log_5 1 = 0$ since $5^0 = 1$. And $\log_5 5 = 1$ since $5^1 = 5$.

Next, we have the inverse property.

$$\log_b(b^x) = x$$
$$b^{\log_b x} = x,\ x > 0$$

For example, to evaluate $\log(100)$, we can rewrite the logarithm as $\log_{10}(10^2)$, and then apply the inverse property $\log_b(b^x) = x$ to get $\log_{10}(10^2) = 2$.

To evaluate $e^{\ln(7)}$, we can rewrite the logarithm as $e^{\log_e 7}$, and then apply the inverse property $b^{\log_b x} = x$ to get $e^{\log_e 7} = 7$.

Finally, we have the one-to-one property.

$$\log_b M = \log_b N \text{ if and only if } M = N \tag{6.5.3}$$

We can use the one-to-one property to solve the equation $\log_3(3x) = \log_3(2x + 5)$ for x. Since the bases are the same, we can apply the one-to-one property by setting the arguments equal and solving for x:

$3x = 2x + 5$ Set the arguments equal.

$x = 5$ Subtract $2x$.

But what about the equation $\log_3(3x) + \log_3(2x + 5) = 2$? The one-to-one property does not help us in this instance. Before we can solve an equation like this, we need a method for combining terms on the left side of the equation.

Recall that we use the *product rule of exponents* to combine the product of exponents by adding: $x^a x^b = x^{a+b}$. We have a similar property for logarithms, called the **product rule for logarithms**, which says that the logarithm of a product is equal to a sum of logarithms. Because logs are exponents, and we multiply like bases, we can add the exponents. We will use the inverse property to derive the product rule below.

Given any real number x and positive real numbers M, N, and b, where $b \neq 1$, we will show

$\log_b(MN) = \log_b(M) + \log_b(N)$.

Let $m = \log_b M$ and $n = \log_b N$. In exponential form, these equations are $b^m = M$ and $b^n = N$. It follows that

$$\begin{aligned} \log_b(MN) &= \log_b(b^m b^n) && \text{Substitute for M and N} \\ &= \log_b(b^{m+n}) && \text{Apply the product rule for exponents} \\ &= m + n && \text{Apply the inverse property of logs} \\ &= \log_b(M) + \log_b(N) && \text{Substitute for m and n} \end{aligned}$$

Note that repeated applications of the product rule for logarithms allow us to simplify the logarithm of the product of any number of factors. For example, consider $\log_b(wxyz)$. Using the product rule for logarithms, we can rewrite this logarithm of a product as the sum of logarithms of its factors:

$\log_b(wxyz) = \log_b w + \log_b x + \log_b y + \log_b z$

The Product Rule for Logarithms

The product rule for logarithms can be used to simplify a logarithm of a product by rewriting it as a sum of individual logarithms.

$$\log_b(MN) = \log_b(M) + \log_b(N) \text{ for } b > 0 \tag{6.5.4}$$

How to: Given the logarithm of a product, use the product rule of logarithms to write an equivalent sum of logarithms

1. Factor the argument completely, expressing each whole number factor as a product of primes.
2. Write the equivalent expression by summing the logarithms of each factor.

Example 6.5.1: Using the Product Rule for Logarithms

Expand $\log_3(30x(3x+4))$.

Solution

We begin by factoring the argument completely, expressing **30** as a product of primes.

$\log_3(30x(3x+4)) = \log_3(2\cdot 3\cdot 5\cdot x\cdot(3x+4))$

Next we write the equivalent equation by summing the logarithms of each factor.

$\log_3(30x(3x+4)) = \log_3(2)+\log_3(3)+\log_3(5)+\log_3(x)+\log_3(3x+4)$

Exercise 6.5.1

Expand $\log_b(8k)$.

Answer

$$\log_b 2+\log_b 2+\log_b 2+\log_b k = 3\log_b 2+\log_b k$$

Using the Quotient Rule for Logarithms

For quotients, we have a similar rule for logarithms. Recall that we use the *quotient rule of exponents* to combine the quotient of exponents by subtracting: $x^{\frac{a}{b}} = x^{a-b}$. The **quotient rule for logarithms** says that the logarithm of a quotient is equal to a difference of logarithms.

The Quotient Rule for Logarithms

The quotient rule for logarithms can be used to simplify a logarithm or a quotient by rewriting it as the difference of individual logarithms.

$$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N \tag{6.5.5}$$

Just as with the product rule, we can use the inverse property to derive the quotient rule.

Proof

Given any real number x and positive real numbers M, N, and b, b, where $b\neq 1$, we will show

$\log_b\left(\frac{M}{N}\right) = \log_b(M)-\log_b(N)$.

Let $m=\log_b M$ and $n=\log_b N$. In exponential form, these equations are $b^m = M$ and $b^n = N$. It follows that

$$\begin{aligned}\log_b\left(\frac{M}{N}\right) &= \log_b\left(\frac{b^m}{b^n}\right) && \text{Substitute for M and N}\\ &= \log_b(b^{m-n}) && \text{Apply the quotient rule for exponents}\\ &= m-n && \text{Apply the inverse property of logs}\\ &= \log_b(M)-\log_b(N) && \text{Substitute for m and n}\end{aligned}$$

For example, to expand $\log\left(\frac{2x^2+6x}{3x+9}\right)$, we must first express the quotient in lowest terms. Factoring and canceling we get,

$$\begin{aligned}\log\left(\frac{2x^2+6x}{3x+9}\right) &= \log\left(\frac{2x(x+3)}{3(x+3)}\right) && \text{Factor the numerator and denominator}\\ &= \log\left(\frac{2x}{3}\right) && \text{Cancel the common factors}\end{aligned}$$

Next we apply the quotient rule by subtracting the logarithm of the denominator from the logarithm of the numerator. Then we apply the product rule.

$$\begin{aligned}\log\left(\frac{2x}{3}\right) &= \log(2x) - \log(3) \\ &= \log(2) + \log(x) - \log(3)\end{aligned}$$

How to: Given the logarithm of a quotient, use the quotient rule of logarithms to write an equivalent difference of logarithms

1. Express the argument in lowest terms by factoring the numerator and denominator and canceling common terms.
2. Write the equivalent expression by subtracting the logarithm of the denominator from the logarithm of the numerator.
3. Check to see that each term is fully expanded. If not, apply the product rule for logarithms to expand completely.

Example 6.5.2: Using the Quotient Rule for Logarithms

Expand $log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right)$.

Solution

First we note that the quotient is factored and in lowest terms, so we apply the quotient rule.

$$\log_2(\frac{15x(x-1)}{(3x+4)(2-x)}) = \log_2(15x(x-1)) - \log_2((3x+4)(2-x))$$

Notice that the resulting terms are logarithms of products. To expand completely, we apply the product rule, noting that the prime factors of the factor 15 are 3 and 5.

$$\begin{aligned}\log_2(15x(x-1)) - \log_2((3x+4)(2-x)) &= [\log_2(3) + \log_2(5) + \log_2(x) + \log_2(x-1)] \\ &\quad - [\log_2(3x+4) + \log_2(2-x)] \\ &= \log_2(3) + \log_2(5) + \log_2(x) + \log_2(x-1) \\ &\quad - \log_2(3x+4) - \log_2(2-x)\end{aligned}$$

Analysis

There are exceptions to consider in this and later examples. First, because denominators must never be zero, this expression is not defined for $x = -\frac{4}{3}$ and $x = 2$. Also, since the argument of a logarithm must be positive, we note as we observe the expanded logarithm, that $x > 0$, $x > 1$, $x > -\frac{4}{3}$, and $x < 2$. Combining these conditions is beyond the scope of this section, and we will not consider them here or in subsequent exercises.

Exercise 6.5.2

Expand $\log_3\left(\frac{7x^2+21x}{7x(x-1)(x-2)}\right)$.

Answer

$$\log_3(x+3) - \log_3(x-1) - \log_3(x-2)$$

Using the Power Rule for Logarithms

We've explored the product rule and the quotient rule, but how can we take the logarithm of a power, such as x^2? One method is as follows:

$$\begin{aligned}\log_b(x^2) &= \log_b(x \cdot x) \\ &= \log_b x + \log_b x \\ &= 2\log_b x\end{aligned}$$

Notice that we used the **product rule for logarithms** to find a solution for the example above. By doing so, we have derived the **power rule for logarithms**, which says that the log of a power is equal to the exponent times the log of the base. Keep in mind that, although the input to a logarithm may not be written as a power, we may be able to change it to a power. For example,

$$100 = 10^2 \quad \sqrt{3} = 3^{\frac{1}{2}} \quad \frac{1}{e} = e^{-1}$$

The Power Rule for Logarithms

The power rule for logarithms can be used to simplify the logarithm of a power by rewriting it as the product of the exponent times the logarithm of the base.

$$\log_b(M^n) = n\log_b M \tag{6.5.6}$$

How to: Given the logarithm of a power, use the power rule of logarithms to write an equivalent product of a factor and a logarithm

1. Express the argument as a power, if needed.
2. Write the equivalent expression by multiplying the exponent times the logarithm of the base.

Example 6.5.3: Expanding a Logarithm with Powers

Expand $\log_2 x^5$.

Solution

The argument is already written as a power, so we identify the exponent, 5, and the base, x, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.

$\log_2(x^5) = 5\log_2 x$

Exercise 6.5.3

Expand $\ln x^2$.

Answer

$2\ln x$

Example 6.5.4: Rewriting an Expression as a Power before Using the Power Rule

Expand $\log_3(25)$ using the power rule for logs.

Solution

Expressing the argument as a power, we get $\log_3(25) = \log_3(5^2)$.

Next we identify the exponent, 2, and the base, 5, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.

$\log_3(52) = 2\log_3(5)$

Exercise 6.5.4

Expand $\ln\left(\frac{1}{x^2}\right)$.

Answer

$-2\ln(x)$

Example 6.5.5: Using the Power Rule in Reverse

Rewrite $4\ln(x)$ using the power rule for logs to a single logarithm with a leading coefficient of 1.

Solution

Because the logarithm of a power is the product of the exponent times the logarithm of the base, it follows that the product of a number and a logarithm can be written as a power. For the expression $4\ln(x)$, we identify the factor, 4, as the exponent and the argument, x, as the base, and rewrite the product as a logarithm of a power: $4\ln(x) = \ln(x^4)$.

Exercise 6.5.5

Rewrite $2\log_3 4$ using the power rule for logs to a single logarithm with a leading coefficient of 1.

Answer

$\log_3 16$

Expanding Logarithmic Expressions

Taken together, the product rule, quotient rule, and power rule are often called "laws of logs." Sometimes we apply more than one rule in order to simplify an expression. For example:

$$\begin{aligned}\log_b\left(\frac{6x}{y}\right) &= \log_b(6x) - \log_b y \\ &= \log_b 6 + \log_b x - \log_b y\end{aligned}$$

We can use the power rule to expand logarithmic expressions involving negative and fractional exponents. Here is an alternate proof of the quotient rule for logarithms using the fact that a reciprocal is a negative power:

$$\begin{aligned}\log_b\left(\frac{A}{C}\right) &= \log_b(AC^{-1}) \\ &= \log_b(A) + \log_b(C^{-1}) \\ &= \log_b A + (-1)\log_b C \\ &= \log_b A - \log_b C\end{aligned}$$

We can also apply the product rule to express a sum or difference of logarithms as the logarithm of a product.

With practice, we can look at a logarithmic expression and expand it mentally, writing the final answer. Remember, however, that we can only do this with products, quotients, powers, and roots—never with addition or subtraction inside the argument of the logarithm.

Example 6.5.6: Expanding Logarithms Using Product, Quotient, and Power Rules

Rewrite $ln\left(\frac{x^4y}{7}\right)$ as a sum or difference of logs.

Solution

First, because we have a quotient of two expressions, we can use the quotient rule:

$$\ln\left(\frac{x^4y}{7}\right) = \ln(x^4y) - \ln(7)$$

Then seeing the product in the first term, we use the product rule:

$$\ln(x^4y) - \ln(7) = \ln(x^4) + \ln(y) - \ln(7)$$

Finally, we use the power rule on the first term:

$$\ln(x^4) + \ln(y) - \ln(7) = 4\ln(x) + \ln(y) - \ln(7)$$

? Exercise 6.5.6

Expand $\log\left(\frac{x^2y^3}{z^4}\right)$.

Answer

$$2\log x + 3\log y - 4\log z$$

✓ Example 6.5.7: Using the Power Rule for Logarithms to Simplify the Logarithm of a Radical Expression

Expand $\log(\sqrt{x})$.

Solution

$$\begin{aligned}\log(\sqrt{x}) &= \log x^{\left(\frac{1}{2}\right)} \\ &= \frac{1}{2}\log x\end{aligned}$$

? Exercise 6.5.7

Expand $\ln(\sqrt[3]{x^2})$.

Answer

$$\frac{2}{3}\ln x$$

Q&A: Can we expand $\ln(x^2+y^2)$?

No. There is no way to expand the logarithm of a sum or difference inside the argument of the logarithm.

✓ Example 6.5.8: Expanding Complex Logarithmic Expressions

Expand $\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right)$.

Solution

We can expand by applying the Product and Quotient Rules.

$$\begin{aligned}\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right) &= \log_6 64 + \log_6 x^3 + \log_6(4x+1) - \log_6(2x-1) \\ &\qquad \text{Apply the Quotient Rule} \\ &= \log_6 26 + \log_6 x^3 + \log_6(4x+1) - \log_6(2x-1) \\ &\qquad \text{Simplify by writing 64 as } 2^6 \\ &= 6\log_6 2 + 3\log_6 x + \log_6(4x+1) - \log_6(2x-1) \\ &\qquad \text{Apply the Power Rule}\end{aligned}$$

? Exercise 6.5.8

Expand $\ln\left(\frac{\sqrt{(x-1)(2x+1)^2}}{(x^2-9)}\right)$.

Answer

$$\frac{1}{2}\ln(x-1) + \ln(2x+1) - \ln(x+3) - \ln(x-3)$$

Condensing Logarithmic Expressions

We can use the rules of logarithms we just learned to condense sums, differences, and products with the same base as a single logarithm. It is important to remember that the logarithms must have the same base to be combined. We will learn later how to change the base of any logarithm before condensing.

How to: Given a sum, difference, or product of logarithms with the same base, write an equivalent expression as a single logarithm

1. Apply the power property first. Identify terms that are products of factors and a logarithm, and rewrite each as the logarithm of a power.
2. Next apply the product property. Rewrite sums of logarithms as the logarithm of a product.
3. Apply the quotient property last. Rewrite differences of logarithms as the logarithm of a quotient.

Example 6.5.9: Using the Product and Quotient Rules to Combine Logarithms

Write $\log_3(5)+\log_3(8)-\log_3(2)$ as a single logarithm.

Solution

Using the product and quotient rules

$$\log_3(5)+\log_3(8)=\log_3(5\cdot 8)=\log_3(40)$$

This reduces our original expression to

$$\log_3(40)-\log_3(2)$$

Then, using the quotient rule

$$\log_3(40)-\log_3(2)=\log_3\left(\frac{40}{2}\right)=\log_3(20)$$

Exercise 6.5.9

Condense $\log_3-\log_4+\log_5-\log_6$.

Answer

$\log\left(\frac{3\cdot 5}{4\cdot 6}\right)$; can also be written $\log\left(\frac{5}{8}\right)$ by reducing the fraction to lowest terms.

Example 6.5.10: Condensing Complex Logarithmic Expressions

Condense $\log_2(x^2)+\frac{1}{2}\log_2(x-1)-3\log_2((x+3)^2)$.

Solution

We apply the power rule first:

$$\log_2(x^2)+\frac{1}{2}\log_2(x-1)-3\log_2((x+3)^2)=\log_2(x^2)+\log_2(\sqrt{x-1})-\log_2((x+3)^6)$$

Next we apply the product rule to the sum:

$$\log_2(x^2)+\log_2(\sqrt{x-1})-\log_2((x+3)^6)=\log_2(x^2\sqrt{x-1})-\log_2((x+3)^6)$$

Finally, we apply the quotient rule to the difference:

$$\log_2(x^2\sqrt{x-1})-\log_2((x+3)^6)=\log_2\frac{x^2\sqrt{x-1}}{(x+3)^6}$$

Example 6.5.11: Rewriting as a Single Logarithm

Rewrite $2\log x - 4\log(x+5) + \frac{1}{x}\log(3x+5)$ as a single logarithm.

Solution

We apply the power rule first:

$$2\log x - 4\log(x+5) + \frac{1}{x}\log(3x+5) = \log(x^2) - \log(x+5)^4 + \log((3x+5)^{x^{-1}})$$

Next we rearrange and apply the product rule to the sum:

$$\begin{aligned} \log(x^2) - \log(x+5)^4 + \log((3x+5)^{x^{-1}}) &= \log(x^2) + \log((3x+5)^{x^{-1}} - \log(x+5)^4 \\ &= \log(x^2(3x+5)^{x^{-1}}) - \log(x+5)^4 \\ &= \log\frac{x^2(3x+5)^{x^{-1}}}{(x+5)^4} \end{aligned}$$

Apply the quotient rule to the difference

Exercise 6.5.10

Rewrite $\log(5) + 0.5\log(x) - \log(7x-1) + 3\log(x-1)$ as a single logarithm.

Answer

$$\log\frac{5(x-1)^3\sqrt{x}}{(7x-1)}$$

Exercise 6.5.11

Condense $4(3\log(x) + \log(x+5) - \log(2x+3))$.

Answer

$\log\frac{x^{12}(x+5)^4}{(2x+3)^4}$; this answer could also be written $\log\left(\frac{x^3(x+5)}{(2x+3)}\right)^4$

Example 6.5.12: Applying of the Laws of Logs

Recall that, in chemistry, $pH = -\log[H+]$. If the concentration of hydrogen ions in a liquid is doubled, what is the effect on pH?

Solution

Suppose C is the original concentration of hydrogen ions, and P is the original pH of the liquid. Then $P = -\log(C)$. If the concentration is doubled, the new concentration is $2C$. Then the pH of the new liquid is

$pH = -\log(2C)$

Using the product rule of logs

$pH = -\log(2C) = -(\log(2) + \log(C)) = -\log(2) - \log(C)$

Since $P = -\log(C)$,the new pH is

$pH = P - \log(2) \approx P - 0.301$

Exercise 6.5.12

When the concentration of hydrogen ions is doubled, the pH decreases by about 0.301.

How does the pH change when the concentration of positive hydrogen ions is decreased by half?

Answer

The pH increases by about 0.301.

Using the Change-of-Base Formula for Logarithms

Most calculators can evaluate only common and natural logs. In order to evaluate logarithms with a base other than 10 ore, e,we use the **change-of-base formula** to rewrite the logarithm as the quotient of logarithms of any other base; when using a calculator, we would change them to common or natural logs.

To derive the change-of-base formula, we use the **one-to-one** property and **power rule for logarithms**.

Given any positive real numbers M, b, and n, where $n \neq 1$ and $b \neq 1$,we show

$$\log_b M = \frac{\log_n M}{\log_n b}$$

Let $y = \log_b M$. By taking the log base n of both sides of the equation, we arrive at an exponential form, namely $b^y = M$. It follows that

$$\begin{aligned} \log_n(b^y) &= \log_n M && \text{Apply the one-to-one property} \\ y\log_n b &= \log_n M && \text{Apply the power rule for logarithms} \\ y &= \frac{\log_n M}{\log_n b} && \text{Isolate y} \\ \log_b M &= \frac{\log_n M}{\log_n b} && \text{Substitute for y} \end{aligned}$$

For example, to evaluate $\log_5 36$ using a calculator, we must first rewrite the expression as a quotient of common or natural logs. We will use the common log.

$$\begin{aligned} \log_5 36 &= \frac{\log(36)}{\log(5)} && \text{Apply the change of base formula using base 10} \\ &\approx 2.2266 && \text{Use a calculator to evaluate to 4 decimal places} \end{aligned}$$

THE CHANGE-OF-BASE FORMULA

The change-of-base formula can be used to evaluate a logarithm with any base.

For any positive real numbers M, b, and n, where $n \neq 1$ and $b \neq 1$,

$$\log_b M = \frac{\log_n M}{\log_n b} \tag{6.5.7}$$

It follows that the change-of-base formula can be used to rewrite a logarithm with any base as the quotient of common or natural logs.

$$\log_b M = \frac{\ln M}{\ln b} \tag{6.5.8}$$

and

$$\log_b M = \frac{\log M}{\log b} \tag{6.5.9}$$

How to: Given a logarithm with the form $\log_b M$, use the change-of-base formula to rewrite it as a quotient of logs with any positive base n, where $n \neq 1$

1. Determine the new base n, remembering that the common log, $\log(x)$, has base 10, and the natural log, $\ln(x)$,has base e.
2. Rewrite the log as a quotient using the change-of-base formula
   - The numerator of the quotient will be a logarithm with base n and argument M.
   - The denominator of the quotient will be a logarithm with base n and argument b.

Example 6.5.13: Changing Logarithmic Expressions to Expressions Involving Only Natural Logs

Change $\log_5 3$ to a quotient of natural logarithms.

Solution

Because we will be expressing $\log_5 3$ as a quotient of natural logarithms, the new base, $n = e$.

We rewrite the log as a quotient using the change-of-base formula. The numerator of the quotient will be the natural log with argument 3. The denominator of the quotient will be the natural log with argument 5.

$$\log_b M = \frac{\ln M}{\ln b}$$

$$\log_5 3 = \frac{\ln 3}{\ln 5}$$

Exercise 6.5.13

Change $\log 0.58$ to a quotient of natural logarithms.

Answer

$$\frac{\ln 8}{\ln 0.5}$$

Q&A: Can we change common logarithms to natural logarithms?

Yes. Remember that $\log 9$ *means* $\log_{10} 9$. *So,* $\log 9 = \frac{\ln 9}{\ln 10}$.

Example 6.5.14: Using the Change-of-Base Formula with a Calculator

Evaluate $\log_2(10)$ using the change-of-base formula with a calculator.

Solution

According to the change-of-base formula, we can rewrite the log base 2 as a logarithm of any other base. Since our calculators can evaluate the natural log, we might choose to use the natural logarithm, which is the log base e.

$$\log_2 10 = \frac{\ln 10}{\ln 2} \quad \text{Apply the change of base formula using base } e$$

$$\approx 3.3219 \quad \text{Use a calculator to evaluate to 4 decimal places}$$

Exercise 6.5.14

Evaluate $\log_5(100)$ using the change-of-base formula.

Answer

$$\frac{\ln 100}{\ln 5} \approx \frac{4.6051}{1.6094} = 2.861$$

Media

Access these online resources for additional instruction and practice with laws of logarithms.

- The Properties of Logarithms
- Expand Logarithmic Expressions
- Evaluate a Natural Logarithmic Expression

Key Equations

The Product Rule for Logarithms	$\log_b(MN) = \log_b(M) + \log_b(N)$
The Quotient Rule for Logarithms	$\log_b(\frac{M}{N}) = \log_b M - \log_b N$
The Power Rule for Logarithms	$\log_b(M^n) = n\log_b M$
The Change-of-Base Formula	$\log_b M = \frac{\log_n M}{\log_n b}$ $n > 0$, $n \neq 1$, $b \neq 1$

Key Concepts

- We can use the product rule of logarithms to rewrite the log of a product as a sum of logarithms. See Example **6.5.1**.
- We can use the quotient rule of logarithms to rewrite the log of a quotient as a difference of logarithms. See Example **6.5.2**.
- We can use the power rule for logarithms to rewrite the log of a power as the product of the exponent and the log of its base. See Example **6.5.3**, Example **6.5.4**, and Example **6.5.5**.
- We can use the product rule, the quotient rule, and the power rule together to combine or expand a logarithm with a complex input. See Example **6.5.6**, Example **6.5.7**, and Example **6.5.8**.
- The rules of logarithms can also be used to condense sums, differences, and products with the same base as a single logarithm. See Example **6.5.9**, Example **6.5.10**, Example **6.5.11**, and Example **6.5.12**.
- We can convert a logarithm with any base to a quotient of logarithms with any other base using the change-of-base formula. See Example **6.5.13**.
- The change-of-base formula is often used to rewrite a logarithm with a base other than 10 and e as the quotient of natural or common logs. That way a calculator can be used to evaluate. See Example **6.5.14**.

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6.6: Exponential and Logarithmic Equations

Learning Objectives

- Use like bases to solve exponential equations.
- Use logarithms to solve exponential equations.
- Use the definition of a logarithm to solve logarithmic equations.
- Use the one-to-one property of logarithms to solve logarithmic equations.
- Solve applied problems involving exponential and logarithmic equations.

In 1859, an Australian landowner named Thomas Austin released 24 rabbits into the wild for hunting. Because Australia had few predators and ample food, the rabbit population exploded. In fewer than ten years, the rabbit population numbered in the millions.



Figure **6.6.1**: Wild rabbits in Australia. The rabbit population grew so quickly in Australia that the event became known as the "rabbit plague." (credit: Richard Taylor, Flickr)

Uncontrolled population growth, as in the wild rabbits in Australia, can be modeled with exponential functions. Equations resulting from those exponential functions can be solved to analyze and make predictions about exponential growth. In this section, we will learn techniques for solving exponential functions.

Using Like Bases to Solve Exponential Equations

The first technique involves two functions with like bases. Recall that the one-to-one property of exponential functions tells us that, for any real numbers b, S, and T, where $b > 0$, $b \neq 1$, $b^S = b^T$ if and only if $S = T$.

In other words, when an **exponential equation** has the same base on each side, the exponents must be equal. This also applies when the exponents are algebraic expressions. Therefore, we can solve many exponential equations by using the rules of exponents to rewrite each side as a power with the same base. Then, we use the fact that exponential functions are one-to-one to set the exponents equal to one another, and solve for the unknown.

For example, consider the equation $3^{4x-7} = \dfrac{3^{2x}}{3}$. To solve for x, we use the division property of exponents to rewrite the right side so that both sides have the common base, 3. Then we apply the one-to-one property of exponents by setting the exponents equal to one another and solving for x:

$$
\begin{aligned}
3^{4x-7} &= \frac{3^{2x}}{3} && \\
3^{4x-7} &= \frac{3^{2x}}{3^1} && \text{Rewrite 3 as } 3^1 \\
3^{4x-7} &= 3^{2x-1} && \text{Use the division property of exponents} \\
4x-7 &= 2x-1 && \text{Apply the one-to-one property of exponents} \\
2x &= 6 && \text{Subtract 2x and add 7 to both sides} \\
x &= 3 && \text{Divide by 3}
\end{aligned}
$$

USING THE ONE-TO-ONE PROPERTY OF EXPONENTIAL FUNCTIONS TO SOLVE EXPONENTIAL EQUATIONS

For any algebraic expressions S and T, and any positive real number $b \neq 1$,

$$b^S = b^T \text{ if and only if } S = T \tag{6.6.1}$$

How to: Given an exponential equation with the form $b^S = b^T$, where S and T are algebraic expressions with an unknown, solve for the unknown.

1. Use the rules of exponents to simplify, if necessary, so that the resulting equation has the form $b^S = b^T$.
2. Use the one-to-one property to set the exponents equal.
3. Solve the resulting equation, $S = T$, for the unknown.

Example 6.6.1: Solving an Exponential Equation with a Common Base

Solve $2^{x-1} = 2^{2x-4}$.

Solution

$2^{x-1} = 2^{2x-4}$	The common base is 2
$x - 1 = 2x - 4$	By the one-to-one property the exponents must be equal
$x = 3$	Solve for x

Exercise 6.6.1

Solve $5^{2x} = 5^{3x+2}$.

Answer

$x = -2$

Rewrite Equations So All Powers Have the Same Base

Sometimes the *common base* for an exponential equation is not explicitly shown. In these cases, we simply rewrite the terms in the equation as powers with a common base, and solve using the one-to-one property.

For example, consider the equation $256 = 4^{x-5}$. We can rewrite both sides of this equation as a power of 2. Then we apply the rules of exponents, along with the one-to-one property, to solve for x:

$256 = 4^{x-5}$	
$2^8 = (2^2)^{x-5}$	Rewrite each side as a power with base 2
$2^8 = 2^{2x-10}$	Use the one-to-one property of exponents
$8 = 2x - 10$	Apply the one-to-one property of exponents
$18 = 2x$	Add 10 to both sides
$x = 9$	Divide by 2

How to: Given an exponential equation with unlike bases, use the one-to-one property to solve it.

1. Rewrite each side in the equation as a power with a common base.
2. Use the rules of exponents to simplify, if necessary, so that the resulting equation has the form $b^S = b^T$.
3. Use the one-to-one property to set the exponents equal.
4. Solve the resulting equation, $S = T$, for the unknown.

Example 6.6.2: Solving Equations by Rewriting Them to Have a Common Base

Solve $8^{x+2} = 16^{x+1}$.

Solution

$8^{x+2} = 16^{x+1}$	
$(2^3)^{x+2} = (2^4)^{x+1}$	Write 8 and 16 as powers of 2
$2^{3x+6} = 2^{4x+4}$	To take a power of a power, multiply exponents
$3x + 6 = 4x + 4$	Use the one-to-one property to set the exponents equal
$x = 2$	Solve for x

Exercise 6.6.2

Solve $5^{2x} = 25^{3x+2}$.

Answer

$x = -1$

Example 6.6.3: Solving Equations by Rewriting Roots with Fractional Exponents to Have a Common Base

Solve $2^{5x} = \sqrt{2}$.

Solution

$2^{5x} = 2^{\frac{1}{2}}$	Write the square root of 2 as a power of 2
$5x = \frac{1}{2}$	Use the one-to-one property
$x = \frac{1}{10}$	Solve for x

Exercise 6.6.3

Solve $5^x = \sqrt{5}$.

Answer

$x = \frac{1}{2}$

Q&A: Do all exponential equations have a solution? If not, how can we tell if there is a solution during the problem-solving process?

No. Recall that the range of an exponential function is always positive. While solving the equation, we may obtain an expression that is undefined.

Example 6.6.4: Solving an Equation with Positive and Negative Powers

Solve $3^{x+1} = -2$.

Solution

This equation has no solution. There is no real value of x that will make the equation a true statement because any power of a positive number is positive.

Analysis

Figure 6.6.2 shows that the two graphs do not cross so the left side is never equal to the right side. Thus the equation has no solution.

Figure 6.6.2

Exercise 6.6.4

Solve $2^x = -100$.

Answer

The equation has no solution.

Solving Exponential Equations Using Logarithms

Sometimes the terms of an exponential equation cannot be rewritten with a common base. In these cases, we solve by taking the logarithm of each side. Recall, since $\log(a) = \log(b)$ is equivalent to $a = b$, we may apply logarithms with the same base on both sides of an exponential equation.

How to: Given an exponential equation in which a common base cannot be found, solve for the unknown

1. Apply the logarithm of both sides of the equation.
   - If one of the terms in the equation has base 10, use the common logarithm.
   - If none of the terms in the equation has base 10, use the natural logarithm.
2. Use the rules of logarithms to solve for the unknown.

Example 6.6.5: Solving an Equation Containing Powers of Different Bases

Solve $5^{x+2} = 4^x$.

Solution

$5^{x+2} = 4^x$	There is no easy way to get the powers to have the same base
$\ln 5^{x+2} = \ln 4^x$	Take ln of both sides
$(x+2)\ln 5 = x \ln 4$	Use laws of logs
$x \ln 5 + 2 \ln 5 = x \ln 4$	Use the distributive law
$x \ln 5 - x \ln 4 = -2 \ln 5$	Get terms containing x on one side, terms without x on the other
$x(\ln 5 - \ln 4) = -2 \ln 5$	On the left hand side, factor out an x
$x \ln\left(\frac{5}{4}\right) = \ln\left(\frac{1}{25}\right)$	Use the laws of logs
$x = \frac{\ln\left(\frac{1}{25}\right)}{\ln\left(\frac{5}{4}\right)}$	Divide by the coefficient of x

Exercise 6.6.5

Solve $2^x = 3^{x+1}$.

Answer

$$x = \frac{\ln 3}{\ln\left(\frac{2}{3}\right)}$$

Q&A: Is there any way to solve $2^x = 3^x$?

Yes. The solution is 0.

Equations Containing e

One common type of exponential equations are those with base e. This constant occurs again and again in nature, in mathematics, in science, in engineering, and in finance. When we have an equation with a base e on either side, we can use the natural logarithm to solve it.

How to: Given an equation of the form $y = Ae^{kt}$, solve for t.

1. Divide both sides of the equation by A.
2. Apply the natural logarithm of both sides of the equation.
3. Divide both sides of the equation by k.

Example 6.6.6: Solve an Equation of the Form $y = Ae^{kt}$

Solve $100 = 20e^{2t}$.

Solution

$$\begin{aligned} 100 &= 20e^{2t} && \\ 5 &= e^{2t} && \text{Divide by the coefficient of the power} \\ \ln 5 &= 2t && \text{Take ln of both sides. Use the fact that } ln(x) \text{ and } e^x \text{ are inverse functions} \\ t &= \frac{\ln 5}{2} && \text{Divide by the coefficient of t} \end{aligned}$$

Analysis

Using laws of logs, we can also write this answer in the form $t = \ln \sqrt{5}$. If we want a decimal approximation of the answer, we use a calculator.

Exercise 6.6.6

Solve $3e^{0.5t} = 11$.

Answer

$$t = 2\ln\left(\frac{11}{3}\right) \text{ or } \ln\left(\frac{11}{3}\right)^2$$

Q&A: Does every equation of the form $y = Ae^{kt}$ have a solution?

No. There is a solution when $k \neq 0$,and when y and A are either both 0 or neither 0, and they have the same sign. An example of an equation with this form that has no solution is $2 = -3e^t$.

Example 6.6.7: Solving an Equation That Can Be Simplified to the Form $y = Ae^{kt}$

Solve $4e^{2x} + 5 = 12$.

Solution

$4e^{2x}+5=12$	
$4e^{2x}=7$	Combine like terms
$e^{2x}=\frac{7}{4}$	Divide by the coefficient of the power
$2x=\ln\left(\frac{7}{4}\right)$	Take ln of both sides
$x=\frac{1}{2}\ln\left(\frac{7}{4}\right)$	Solve for x

? Exercise 6.6.7

Solve $3+e^{2t}=7e^{2t}$.

Answer

$$t=\ln\left(\frac{1}{\sqrt{2}}\right)=-\frac{1}{2}\ln(2)$$

Extraneous Solutions

Sometimes the methods used to solve an equation introduce an **extraneous solution**, which is a solution that is correct algebraically but does not satisfy the conditions of the original equation. One such situation arises in solving when the logarithm is taken on both sides of the equation. In such cases, remember that the argument of the logarithm must be positive. If the number we are evaluating in a logarithm function is negative, there is no output.

✓ Example 6.6.8: Solving Exponential Functions in Quadratic Form

Solve $e^{2x}-e^{x}=56$.

Solution

$e^{2x}-e^{x}=56$	
$e^{2x}-e^{x}-56=0$	Get one side of the equation equal to zero
$(e^{x}+7)(e^{x}-8)=0$	Factor by the FOIL method
$e^{x}+7=0$	or
$e^{x}-8=0$	If a product is zero, then one factor must be zero
$e^{x}=-7$	or
$e^{x}=8$	Isolate the exponentials
$e^{x}=8$	Reject the equation in which the power equals a negative number
$x=\ln 8$	Solve the equation in which the power equals a positive number

Analysis

When we plan to use factoring to solve a problem, we always get zero on one side of the equation, because zero has the unique property that when a product is zero, one or both of the factors must be zero. We reject the equation $e^{x}=-7$ because a positive number never equals a negative number. The solution $\ln(-7)$ is not a real number, and in the real number system this solution is rejected as an extraneous solution.

? Exercise 6.6.8

Solve $e^{2x}=e^{x}+2$.

Answer

$$x=\ln 2$$

Q&A: Does every logarithmic equation have a solution?

No. Keep in mind that we can only apply the logarithm to a positive number. Always check for extraneous solutions.

Using the Definition of a Logarithm to Solve Logarithmic Equations

We have already seen that every logarithmic equation $\log_b(x) = y$ is equivalent to the exponential equation $b^y = x$. We can use this fact, along with the rules of logarithms, to solve logarithmic equations where the argument is an algebraic expression.

For example, consider the equation $\log_2(2) + \log_2(3x - 5) = 3$. To solve this equation, we can use rules of logarithms to rewrite the left side in compact form and then apply the definition of logs to solve for x:

$$\begin{aligned} \log_2(2) + \log_2(3x-5) &= 3 \\ \log_2(2(3x-5)) &= 3 && \text{Apply the product rule of logarithms} \\ \log_2(6x-10) &= 3 && \text{Distribute} \\ 2^3 &= 6x-10 && \text{Apply the definition of a logarithm} \\ 8 &= 6x-10 && \text{Calculate } 2^3 \\ 18 &= 6x && \text{Add 10 to both sides} \\ x &= 3 && \text{Divide by 6} \end{aligned}$$

USING THE DEFINITION OF A LOGARITHM TO SOLVE LOGARITHMIC EQUATIONS

For any algebraic expression S and real numbers b and c,where $b > 0$, $b \neq 1$,

$$\log_b(S) = c \text{ if and only if } b^c = S \tag{6.6.2}$$

Example 6.6.9: Using Algebra to Solve a Logarithmic Equation

Solve $2\ln x + 3 = 7$.

Solution

$$\begin{aligned} 2\ln x + 3 &= 7 \\ 2\ln x &= 4 && \text{Subtract 3} \\ \ln x &= 2 && \text{Divide by 2} \\ x &= e^2 && \text{Rewrite in exponential form} \end{aligned}$$

Exercise 6.6.9

Solve $6 + \ln x = 10$.

Answer

$x = e^4$

Example 6.6.10: Using Algebra Before and After Using the Definition of the Natural Logarithm

Solve $2\ln(6x) = 7$.

Solution

$$\begin{aligned} 2\ln(6x) &= 7 \\ \ln(6x) &= \frac{7}{2} && \text{Divide by 2} \\ 6x &= e^{\left(\frac{7}{2}\right)} && \text{Use the definition of ln} \\ x &= \frac{1}{6}e^{\left(\frac{7}{2}\right)} && \text{Divide by 6} \end{aligned}$$

? Exercise 6.6.10

Solve $2\ln(x+1)=10$.

Answer

$x=e^5-1$

✓ Example 6.6.11: Using a Graph to Understand the Solution to a Logarithmic Equation

Solve $\ln x=3$.

Solution

$$\begin{aligned}\ln x &= 3\\ x &= e^3 \quad \text{Use the definition of the natural logarithm}\end{aligned}$$

Figure 6.6.3 represents the graph of the equation. On the graph, the x-coordinate of the point at which the two graphs intersect is close to 20. In other words $e^3\approx 20$. A calculator gives a better approximation: $e^3\approx 20.0855$.

Figure 6.6.3: The graphs of $y=\ln x$ and $y=3$ cross at the point $(e^3,3)$, which is approximately $(20.0855,3)$.

? Exercise 6.6.11

Use a graphing calculator to estimate the approximate solution to the logarithmic equation $2^x=1000$ to 2 decimal places.

Answer

$x\approx 9.97$

Using the One-to-One Property of Logarithms to Solve Logarithmic Equations

As with exponential equations, we can use the one-to-one property to solve logarithmic equations. The one-to-one property of logarithmic functions tells us that, for any real numbers $x>0$, $S>0$, $T>0$ and any positive real number b, where $b\neq 1$,

$\log_b S=\log_b T$ if and only if $S=T$.

For example,

If $\log_2(x-1)=\log_2(8)$, then $x-1=8$.

So, if $x-1=8$, then we can solve for x,and we get $x=9$. To check, we can substitute $x=9$ into the original equation: $\log_2(9-1)=\log_2(8)=3$. In other words, when a logarithmic equation has the same base on each side, the arguments must be equal. This also applies when the arguments are algebraic expressions. Therefore, when given an equation with logs of the same base on each side, we can use rules of logarithms to rewrite each side as a single logarithm. Then we use the fact that logarithmic functions are one-to-one to set the arguments equal to one another and solve for the unknown.

For example, consider the equation $\log(3x-2)-\log(2)=\log(x+4)$. To solve this equation, we can use the rules of logarithms to rewrite the left side as a single logarithm, and then apply the one-to-one property to solve for x:

$$\begin{aligned}\log(3x-2)-\log(2)&=\log(x+4)\\ \log\left(\frac{3x-2}{2}\right)&=\log(x+4) \quad \text{Apply the quotient rule of logarithms}\\ \frac{3x-2}{2}&=x+4 \quad \text{Apply the one to one property of a logarithm}\\ 3x-2&=2x+8 \quad \text{Multiply both sides of the equation by 2}\\ x&=10 \quad \text{Subtract 2x and add 2}\end{aligned}$$

To check the result, substitute $x=10$ into $\log(3x-2)-\log(2)=\log(x+4)$.

$$\log(3(10)-2)-\log(2)=\log((10)+4)$$
$$\log(28)-\log(2)=\log(14)$$
$$\log\left(\frac{28}{2}\right)=\log(14) \qquad \text{The solution checks}$$

USING THE ONE-TO-ONE PROPERTY OF LOGARITHMS TO SOLVE LOGARITHMIC EQUATIONS

For any algebraic expressions S and T and any positive real number b, where $b \neq 1$,

$$b^S = b^T \text{ if and only if } S = T \tag{6.6.3}$$

Note, when solving an equation involving logarithms, always check to see if the answer is correct or if it is an extraneous solution.

How to: Given an equation containing logarithms, solve it using the one-to-one property

1. Use the rules of logarithms to combine like terms, if necessary, so that the resulting equation has the form $\log_b S = \log_b T$.
2. Use the one-to-one property to set the arguments equal.
3. Solve the resulting equation, $S = T$, for the unknown.

Example 6.6.12: Solving an Equation Using the One-to-One Property of Logarithms

Solve $\ln(x^2) = \ln(2x+3)$.

Solution

$$\begin{aligned} \ln(x^2) &= \ln(2x+3) \\ x^2 &= 2x+3 \quad \text{Use the one-to-one property of the logarithm} \\ x^2-2x-3 &= 0 \quad \text{Get zero on one side before factoring} \\ (x-3)(x+1) &= 0 \quad \text{Factor using FOIL} \\ x-3 &= 0 \quad \text{or } x+1=0 \text{ If a product is zero, one of the factors must be zero} \\ x &= 3 \quad \text{or} \\ x &= -11 \quad \text{Solve for x} \end{aligned}$$

Analysis

There are two solutions: 3 or -1. The solution -1 is negative, but it checks when substituted into the original equation because the argument of the logarithm functions is still positive.

Exercise 6.6.12

Solve $\ln(x^2) = \ln 1$.

Answer

$x = 1$ or $x = -1$

Solving Applied Problems Using Exponential and Logarithmic Equations

In previous sections, we learned the properties and rules for both exponential and logarithmic functions. We have seen that any exponential function can be written as a logarithmic function and vice versa. We have used exponents to solve logarithmic equations and logarithms to solve exponential equations. We are now ready to combine our skills to solve equations that model real-world situations, whether the unknown is in an exponent or in the argument of a logarithm.

One such application is in science, in calculating the time it takes for half of the unstable material in a sample of a radioactive substance to decay, called its **half-life**. Table 6.6.1 lists the half-life for several of the more common radioactive substances.

Table 6.6.1

Substance	Use	Half-life
gallium-67	nuclear medicine	80 hours
cobalt-60	manufacturing	5.3 years
technetium-99m	nuclear medicine	6 hours
americium-241	construction	432 years
carbon-14	archeological dating	5,715 years
uranium-235	atomic power	703,800,000 years

We can see how widely the half-lives for these substances vary. Knowing the half-life of a substance allows us to calculate the amount remaining after a specified time. We can use the formula for radioactive decay:

$$A(t) = A_0 e^{\frac{\ln(0.5)}{T}t} \tag{6.6.4}$$

$$A(t) = A_0 e^{\frac{\ln(0.5)t}{T}} \tag{6.6.5}$$

$$A(t) = A_0 (e^{\ln(0.5)})^{\frac{t}{T}} \tag{6.6.6}$$

$$A(t) = A_0 \left(\frac{1}{2}\right)^{\frac{t}{T}} \tag{6.6.7}$$

where

- A_0 is the amount initially present
- T is the half-life of the substance
- t is the time period over which the substance is studied
- y is the amount of the substance present after time t

Example 6.6.13: Using the Formula for Radioactive Decay to Find the Quantity of a Substance

How long will it take for ten percent of a 1000-gram sample of uranium-235 to decay?

Solution

$$\begin{aligned}
y &= 1000e^{\frac{\ln(0.5)}{703,800,000}t} \\
900 &= 1000e^{\frac{\ln(0.5)}{703,800,000}t} && \text{After 10\% decays, 900 grams are left} \\
0.9 &= e^{\frac{\ln(0.5)}{703,800,000}t} && \text{Divide by 1000} \\
\ln(0.9) &= \ln\left(e^{\frac{\ln(0.5)}{703,800,000}t}\right) && \text{Take ln of both sides} \\
\ln(0.9) &= \frac{\ln(0.5)}{703,800,000}t && \ln(e^M) = M \\
t &= 703,800,000 \times \frac{\ln(0.9)}{\ln(0.5)} && \text{years Solve for t} \\
t &\approx 106,979,777 && \text{years}
\end{aligned}$$

Analysis

Ten percent of 1000 grams is 100 grams. If 100 grams decay, the amount of uranium-235 remaining is 900 grams.

Exercise 6.6.13

How long will it take before twenty percent of our 1000-gram sample of uranium-235 has decayed?

Answer

$t = 703,800,000 \times \frac{\ln(0.8)}{\ln(0.5)}$ years $\approx$ 226,572,993 years.

Media

Access these online resources for additional instruction and practice with exponential and logarithmic equations.

- Solving Logarithmic Equations
- Solving Exponential Equations with Logarithms

Key Equations

One-to-one property for exponential functions	For any algebraic expressions S and T and any positive real number b, where $b^S = b^T$ if and only if $S = T$.
Definition of a logarithm	For any algebraic expression S and positive real numbers b and c, where $b \neq 1$, $\log_b(S) = c$ if and only if $b^c = S$.
One-to-one property for logarithmic functions	For any algebraic expressions S and T and any positive real number b, where $b \neq 1$, $\log_b S = \log_b T$ if and only if $S = T$.

Key Concepts

- We can solve many exponential equations by using the rules of exponents to rewrite each side as a power with the same base. Then we use the fact that exponential functions are one-to-one to set the exponents equal to one another and solve for the unknown.
- When we are given an exponential equation where the bases are explicitly shown as being equal, set the exponents equal to one another and solve for the unknown. See Example **6.6.1**.
- When we are given an exponential equation where the bases are *not* explicitly shown as being equal, rewrite each side of the equation as powers of the same base, then set the exponents equal to one another and solve for the unknown. See Example **6.6.2**, Example **6.6.3**, and Example **6.6.4**.
- When an exponential equation cannot be rewritten with a common base, solve by taking the logarithm of each side. See Example **6.6.5**.
- We can solve exponential equations with base e,by applying the natural logarithm of both sides because exponential and logarithmic functions are inverses of each other. See Example **6.6.6** and Example **6.6.7**.
- After solving an exponential equation, check each solution in the original equation to find and eliminate any extraneous solutions. See Example **6.6.8**.
- When given an equation of the form $\log_b(S) = c$, where S is an algebraic expression, we can use the definition of a logarithm to rewrite the equation as the equivalent exponential equation $b^c = S$, and solve for the unknown. See Example **6.6.9** and Example **6.6.10**.
- We can also use graphing to solve equations with the form $\log_b(S) = c$. We graph both equations $y = \log_b(S)$ and $y = c$ on the same coordinate plane and identify the solution as the x-value of the intersecting point. See Example **6.6.11**.
- When given an equation of the form $\log_b S = \log_b T$, where S and T are algebraic expressions, we can use the one-to-one property of logarithms to solve the equation $S = T$ for the unknown. See Example **6.6.12**.
- Combining the skills learned in this and previous sections, we can solve equations that model real world situations, whether the unknown is in an exponent or in the argument of a logarithm. See Example **6.6.13**.

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6.7: Exponential and Logarithmic Models

Learning Objectives

- Model exponential growth and decay.
- Use Newton's Law of Cooling.
- Use logistic-growth models.

We have already explored some basic applications of exponential and logarithmic functions. In this section, we explore some important applications in more depth, including radioactive isotopes and Newton's Law of Cooling.

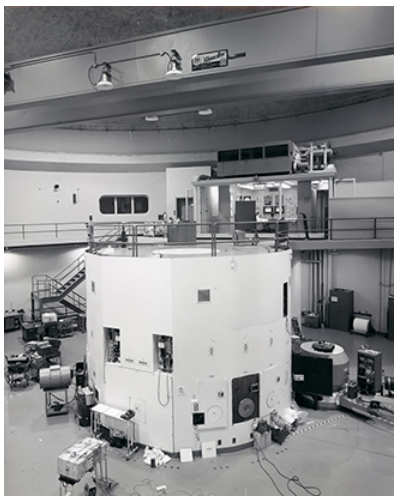


Figure **6.7.1**: A nuclear research reactor inside the Neely Nuclear Research Center on the Georgia Institute of Technology campus (credit: Georgia Tech Research Institute)

Modeling Exponential Growth and Decay

In real-world applications, we need to model the behavior of a function. In mathematical modeling, we choose a familiar general function with properties that suggest that it will model the real-world phenomenon we wish to analyze. In the case of rapid growth, we may choose the exponential growth function:

$$y = A_0 e^{kt} \tag{6.7.1}$$

where A_0 is equal to the value at time zero, e is Euler's constant, and k is a positive constant that determines the rate (percentage) of growth. We may use the exponential growth function in applications involving **doubling time**, the time it takes for a quantity to double. Such phenomena as wildlife populations, financial investments, biological samples, and natural resources may exhibit growth based on a doubling time. In some applications, however, as we will see when we discuss the logistic equation, the logistic model sometimes fits the data better than the exponential model.

On the other hand, if a quantity is falling rapidly toward zero, without ever reaching zero, then we should probably choose the **exponential decay** model. Again, we have the form $y = A_0 e^{kt}$ where A_0 is the starting value, and e is Euler's constant. Now k is a negative constant that determines the rate of decay. We may use the exponential decay model when we are calculating half-life, or the time it takes for a substance to exponentially decay to half of its original quantity. We use half-life in applications involving radioactive isotopes.

In our choice of a function to serve as a mathematical model, we often use data points gathered by careful observation and measurement to construct points on a graph and hope we can recognize the shape of the graph. Exponential growth and decay graphs have a distinctive shape, as we can see in Figure **6.7.2** and Figure **6.7.3**. It is important to remember that, although parts of each of the two graphs seem to lie on the x-axis, they are really a tiny distance above the x-axis.

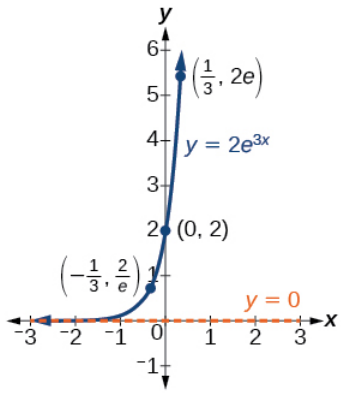


Figure 6.7.2: A graph showing exponential growth. The equation is $y = 2e^{3x}$.

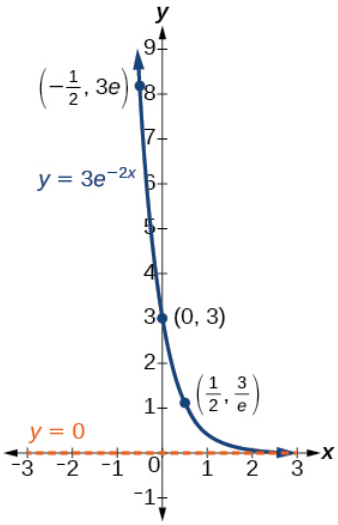


Figure 6.7.3: A graph showing exponential decay. The equation is $y = 3e^{-2x}$.

Exponential growth and decay often involve very large or very small numbers. To describe these numbers, we often use orders of magnitude. The **order of magnitude** is the power of ten, when the number is expressed in scientific notation, with one digit to the left of the decimal. For example, the distance to the nearest star, **Proxima Centauri**, measured in kilometers, is $40,113,497,200,000$ kilometers. Expressed in scientific notation, this is 4.01134972×1013. So, we could describe this number as having order of magnitude 1013.

$y = a_0e^{kt}$

An exponential function with the form $y = A_0e^{kt}$ has the following characteristics:

- one-to-one function
- horizontal asymptote: $y = 0$
- domain: $(-\infty, \infty)$
- range: $(0, \infty)$
- x intercept: none
- y-intercept: $(0, A_0)$
- increasing if $k > 0$ (see Figure 6.7.4)
- decreasing if $k < 0$ (see Figure 6.7.4)

Figure 6.7.4: An exponential function models exponential growth when $k > 0$ and exponential decay when $k < 0$.

Example 6.7.1: Graphing Exponential Growth

A population of bacteria doubles every hour. If the culture started with 10 bacteria, graph the population as a function of time.

Solution

When an amount grows at a fixed percent per unit time, the growth is exponential. To find A_0 we use the fact that A_0 is the amount at time zero, so $A_0 = 10$. To find k, use the fact that after one hour $(t = 1)$ the population doubles from 10 to 20.The formula is derived as follows

$$\begin{aligned} 20 &= 10e^{k\cdot 1} \\ 2 &= e^k \quad \text{Divide by 10} \\ \ln 2 &= k \quad \text{Take the natural logarithm} \end{aligned}$$

so $k = \ln(2)$. Thus the equation we want to graph is $y = 10e^{(\ln 2)t} = 10(e^{\ln 2})^t = 10 \cdot 2^t$. The graph is shown in Figure 6.7.5.

Figure 6.7.5: The graph of $y = 10e^{(\ln 2)t}$

Analysis

The population of bacteria after ten hours is $10,240$. We could describe this amount is being of the order of magnitude 10^4. The population of bacteria after twenty hours is $10,485,760$ which is of the order of magnitude 10^7, so we could say that the population has increased by three orders of magnitude in ten hours.

Half-Life

We now turn to **exponential decay**. One of the common terms associated with exponential decay, as stated above, is **half-life**, the length of time it takes an exponentially decaying quantity to decrease to half its original amount. Every radioactive isotope has a half-life, and the process describing the exponential decay of an isotope is called radioactive decay.

To find the half-life of a function describing exponential decay, solve the following equation:

$$\frac{1}{2}A_0 = A_0e^{kt}$$

We find that the half-life depends only on the constant k and not on the starting quantity A_0.

The formula is derived as follows

$$\begin{aligned} \frac{1}{2}A_0 &= A_0e^{kt} \\ \frac{1}{2} &= e^{kt} \quad \text{Divide by } A_0 \\ \ln\left(\frac{1}{2}\right) &= ktv \quad \text{Take the natural log} \\ -\ln(2) &= kt \quad \text{Apply laws of logarithms} \\ -\ln(2)k &= t \quad \text{Divide by k} \end{aligned}$$

Since t, the time, is positive, k must, as expected, be negative. This gives us the half-life formula

$$t = -\frac{\ln(2)}{k} \tag{6.7.2}$$

Given the half-life, find the decay rate.

1. Write $A = A_0e^{kt}$.
2. Replace A by $\frac{1}{2}A_0$ and replace t by the given half-life.
3. Solve to find k. Express k as an exact value (do not round).

Note: It is also possible to find the decay rate using $k = -\ln(2)t$.

Example 6.7.2: Finding the Function that Describes Radioactive Decay

The half-life of carbon-14 is $5,730$ years. Express the amount of carbon-14 remaining as a function of time, t.

Solution

This formula is derived as follows.

$$\begin{aligned} A &= A_0 e^{kt} && \text{The continuous growth formula} \\ 0.5A_0 &= A_0 e^{k \cdot 5730} && \text{Substitute the half-life for t and } 0.5A_0 \text{ for } f(t) \\ 0.5 &= e^{5730k} && \text{Divide by } A_0 \\ \ln(0.5) &= 5730k && \text{Take the natural log of both sides} \\ k &= \frac{\ln(0.5)}{5730} && \text{Divide by the coefficient of k} \\ A &= A_0 e^{\left(\frac{\ln(0.5)}{5730}\right)t} && \text{Substitute for r in the continuous growth formula} \end{aligned}$$

The function that describes this continuous decay is $f(t) = A_0 e^{\left(\frac{\ln(0.5)}{5730}\right)t}$. We observe that the coefficient of t, $\frac{\ln(0.5)}{5730} \approx -1.2097 \times 10^{-4}$ is negative, as expected in the case of exponential decay.

Exercise 6.7.1

The half-life of plutonium-244 is $80,000,000$ years. Find function gives the amount of carbon-14 remaining as a function of time, measured in years.

Answer

$f(t) = A_0 e^{-0.0000000087t}$

Radiocarbon Dating

The formula for radioactive decay is important in **radiocarbon dating**, which is used to calculate the approximate date a plant or animal died. Radiocarbon dating was discovered in 1949 by Willard Libby, who won a Nobel Prize for his discovery. It compares the difference between the ratio of two isotopes of carbon in an organic artifact or fossil to the ratio of those two isotopes in the air. It is believed to be accurate to within about 1% error for plants or animals that died within the last $60,000$ years.

Carbon-14 is a radioactive isotope of carbon that has a half-life of $5,730$ years. It occurs in small quantities in the carbon dioxide in the air we breathe. Most of the carbon on Earth is carbon-12, which has an atomic weight of 12 and is not radioactive. Scientists have determined the ratio of carbon-14 to carbon-12 in the air for the last $60,000$ years, using tree rings and other organic samples of known dates—although the ratio has changed slightly over the centuries.

As long as a plant or animal is alive, the ratio of the two isotopes of carbon in its body is close to the ratio in the atmosphere. When it dies, the carbon-14 in its body decays and is not replaced. By comparing the ratio of carbon-14 to carbon-12 in a decaying sample to the known ratio in the atmosphere, the date the plant or animal died can be approximated.

Since the half-life of carbon-14 is $5,730$ years, the formula for the amount of carbon-14 remaining after t years is

$$A \approx A_0 e^{\left(\frac{\ln(0.5)}{5730}\right)t}$$

where

- A is the amount of carbon-14 remaining
- A_0 is the amount of carbon-14 when the plant or animal began decaying.

This formula is derived as follows:

$$
\begin{aligned}
A &= A_0 e^{kt} && \text{The continuous growth formula} \\
0.5A_0 &= A_0 e^{k \cdot 5730} && \text{Substitute the half-life for t and } 0.5A_0 \text{ for f(t)} \\
0.5 &= e^{5730k} && \text{Divide by } A_0 \\
\ln(0.5) &= 5730k && \text{Take the natural log of both sides} \\
k &= \frac{\ln(0.5)}{5730} && \text{Divide by the coefficient of k} \\
A &= A_0 e^{\left(\frac{\ln(0.5)}{5730}\right)t} && \text{Substitute for r in the continuous growth formula}
\end{aligned}
$$

To find the age of an object, we solve this equation for t:

$$t = \frac{\ln\left(\frac{A}{A_0}\right)}{-0.000121}$$

Out of necessity, we neglect here the many details that a scientist takes into consideration when doing carbon-14 dating, and we only look at the basic formula. The ratio of carbon-14 to carbon-12 in the atmosphere is approximately 0.0000000001. Let r be the ratio of carbon-14 to carbon-12 in the organic artifact or fossil to be dated, determined by a method called liquid scintillation. From the equation $A \approx A_0 e^{-0.000121t}$ we know the ratio of the percentage of carbon-14 in the object we are dating to the percentage of carbon-14 in the atmosphere is $r = \frac{A}{A_0} \approx e^{-0.000121t}$. We solve this equation for t, to get

$$t = \frac{\ln(r)}{-0.000121}$$

Given the percentage of carbon-14 in an object, determine its age.

1. Express the given percentage of carbon-14 as an equivalent decimal, k.
2. Substitute for k in the equation $t = \frac{\ln(r)}{-0.000121}$ and solve for the age, t.

Example 6.7.3: Finding the Age of a Bone

A bone fragment is found that contains 20% of its original carbon-14. To the nearest year, how old is the bone?

Solution

We substitute $20\% = 0.20$ for k in the equation and solve for t:

$$
\begin{aligned}
t &= \frac{\ln(r)}{-0.000121} && \text{Use the general form of the equation} \\
&= \frac{\ln(0.20)}{-0.000121} && \text{Substitute for r} \\
&\approx 13301 && \text{Round to the nearest year}
\end{aligned}
$$

The bone fragment is about $13,301$ years old.

Analysis

The instruments that measure the percentage of carbon-14 are extremely sensitive and, as we mention above, a scientist will need to do much more work than we did in order to be satisfied. Even so, carbon dating is only accurate to about 1%, so this age should be given as $13,301$ years $\pm 1\%$ or $13,301$ years ± 133 years.

Exercise 6.7.2

Cesium-137 has a half-life of about 30 years. If we begin with 200 mg of cesium-137, will it take more or less than 230 years until only 1 milligram remains?

Answer

less than 230 years, 229.3157 to be exact

Calculating Doubling Time

For decaying quantities, we determined how long it took for half of a substance to decay. For growing quantities, we might want to find out how long it takes for a quantity to double. As we mentioned above, the time it takes for a quantity to double is called the **doubling time**.

Given the basic **exponential growth** equation $A = A_0e^{kt}$, doubling time can be found by solving for when the original quantity has doubled, that is, by solving $2A_0 = A_0e^{kt}$.

The formula is derived as follows:

$$\begin{aligned} 2A_0 &= A_0e^{kt} \\ 2 &= e^{kt} && \text{Divide by } A_0 \\ \ln 2 &= kt && \text{Take the natural logarithm} \\ t &= \frac{\ln 2}{k} && \text{Divide by the coefficient of t} \end{aligned}$$

Thus the doubling time is

$$t = \frac{\ln 2}{k} \tag{6.7.3}$$

Example 6.7.4: Finding a Function That Describes Exponential Growth

According to Moore's Law, the doubling time for the number of transistors that can be put on a computer chip is approximately two years. Give a function that describes this behavior.

Solution

The formula is derived as follows:

$$\begin{aligned} t &= \frac{\ln 2}{k} && \text{The doubling time formula} \\ 2 &= \frac{\ln 2}{k} && \text{Use a doubling time of two years} \\ k &= \frac{\ln 2}{2} && \text{Multiply by k and divide by 2} \\ A &= A_0e^{\frac{\ln 2}{2}t} && \text{Substitute k into the continuous growth formula} \end{aligned}$$

The function is $A = A_0e^{\frac{\ln 2}{2}t}$.

Exercise 6.7.3

Recent data suggests that, as of 2013, the rate of growth predicted by Moore's Law no longer holds. Growth has slowed to a doubling time of approximately three years. Find the new function that takes that longer doubling time into account.

Answer

$f(t) = A_0e^{\frac{\ln 2}{3}t}$

Using Newton's Law of Cooling

Exponential decay can also be applied to temperature. When a hot object is left in surrounding air that is at a lower temperature, the object's temperature will decrease exponentially, leveling off as it approaches the surrounding air temperature. On a graph of the temperature function, the leveling off will correspond to a horizontal asymptote at the temperature of the surrounding air. Unless the room temperature is zero, this will correspond to a **vertical shift** of the generic **exponential decay** function. This translation leads to **Newton's Law of Cooling**, the scientific formula for temperature as a function of time as an object's temperature is equalized with the ambient temperature

$$T(t) = ae^{kt} + T_s$$

This formula is derived as follows:

$$\begin{aligned} T(t) &= Ab^{ct} + T_s \\ T(t) &= Ae^{\ln(b^{ct})} + T_s && \text{Laws of logarithms} \\ T(t) &= Ae^{ct\ln b} + T_s && \text{Laws of logarithms} \\ T(t) &= Ae^{kt} + T_s && \text{Rename the constant c } lnb\text{, calling it } k \end{aligned}$$

NEWTON'S LAW OF COOLING

The temperature of an object, T, in surrounding air with temperature T_s will behave according to the formula

$$T(t) = Ae^{kt} + T_s \tag{6.7.4}$$

where

- t is time
- A is the difference between the initial temperature of the object and the surroundings
- k is a constant, the continuous rate of cooling of the object

Given a set of conditions, apply Newton's Law of Cooling.

1. Set T_s equal to the y-coordinate of the horizontal asymptote (usually the ambient temperature).
2. Substitute the given values into the continuous growth formula $T(t) = Ae^{kt} + T_s$ to find the parameters A and k.
3. Substitute in the desired time to find the temperature or the desired temperature to find the time.

Example 6.7.5: Using Newton's Law of Cooling

A cheesecake is taken out of the oven with an ideal internal temperature of $165°F$, and is placed into a $35°F$ refrigerator. After 10 minutes, the cheesecake has cooled to $150°F$. If we must wait until the cheesecake has cooled to $70°F$ before we eat it, how long will we have to wait?

Solution

Because the surrounding air temperature in the refrigerator is 35 degrees, the cheesecake's temperature will decay exponentially toward 35, following the equation

$$T(t) = Ae^{kt} + 35$$

We know the initial temperature was 165, so $T(0) = 165$.

$$\begin{aligned} 165 &= Ae^{k0} + 35 && \text{Substitute } (0, 165) \\ A &= 130 && \text{Solve for A} \end{aligned}$$

We were given another data point, $T(10) = 150$, which we can use to solve for k.

$$\begin{aligned} 150 &= 130e^{k10} + 35 && \text{Substitute } (10, 150) \\ 115 &= 130e^{k10} && \text{Subtract } 35 \\ \frac{115}{130} &= e^{10k} && \text{Divide by } 130 \\ \ln\left(\frac{115}{130}\right) &= 10k && \text{Take the natural log of both sides} \\ k &= \frac{\ln\left(\frac{115}{130}\right)}{10} \\ &= -0.0123 && \text{Divide by the coefficient of k} \end{aligned}$$

This gives us the equation for the cooling of the cheesecake: $T(t) = 130e^{-0.0123t} + 35$.

Now we can solve for the time it will take for the temperature to cool to 70 degrees.

$$
\begin{aligned}
70 &= 130e^{-0.0123t} + 35 && \text{Substitute in 70 for } T(t)\\
35 &= 130e^{-0.0123t} && \text{Subtract 35}\\
\frac{35}{130} &= e^{-0.0123t} && \text{Divide by 130}\\
\ln\left(\frac{35}{130}\right) &= -0.0123t && \text{Take the natural log of both sides}\\
t &= \frac{\ln\left(\frac{35}{130}\right)}{-0.0123}\\
&\approx 106.68 && \text{Divide by the coefficient of t}
\end{aligned}
$$

It will take about 107 minutes, or one hour and 47 minutes, for the cheesecake to cool to $70°F$.

? Exercise 6.7.4

A pitcher of water at 40 degrees Fahrenheit is placed into a 70 degree room. One hour later, the temperature has risen to 45 degrees. How long will it take for the temperature to rise to 60 degrees?

Answer

6.026 hours

Using Logistic Growth Models

Exponential growth cannot continue forever. Exponential models, while they may be useful in the short term, tend to fall apart the longer they continue. Consider an aspiring writer who writes a single line on day one and plans to double the number of lines she writes each day for a month. By the end of the month, she must write over 17 billion lines, or one-half-billion pages. It is impractical, if not impossible, for anyone to write that much in such a short period of time. Eventually, an exponential model must begin to approach some limiting value, and then the growth is forced to slow. For this reason, it is often better to use a model with an upper bound instead of an exponential growth model, though the **exponential growth** model is still useful over a short term, before approaching the limiting value.

The **logistic growth model** is approximately exponential at first, but it has a reduced rate of growth as the output approaches the model's upper bound, called the **carrying capacity**. For constants a, b, and c, the logistic growth of a population over time x is represented by the model

$$f(x) = \frac{c}{1 + ae^{-bx}}$$

The graph in Figure 6.7.6 shows how the growth rate changes over time. The graph increases from left to right, but the growth rate only increases until it reaches its point of maximum growth rate, at which point the rate of increase decreases.

Figure 6.7.6

LOGISTIC GROWTH

The logistic growth model is

$$f(x) = \frac{c}{1 + ae^{-bx}} \tag{6.7.5}$$

where

- $\frac{c}{1+a}$ is the initial value
- c is the *carrying capacity*, or *limiting value*
- b is a constant determined by the rate of growth.

Example 6.7.6: Using the Logistic-Growth Model

An influenza epidemic spreads through a population rapidly, at a rate that depends on two factors: The more people who have the flu, the more rapidly it spreads, and also the more uninfected people there are, the more rapidly it spreads. These two factors make the logistic model a good one to study the spread of communicable diseases. And, clearly, there is a maximum value for the number of people infected: the entire population.

For example, at time $t = 0$ there is one person in a community of $1,000$ people who has the flu. So, in that community, at most $1,000$ people can have the flu. Researchers find that for this particular strain of the flu, the logistic growth constant is $b = 0.6030$. Estimate the number of people in this community who will have had this flu after ten days. Predict how many people in this community will have had this flu after a long period of time has passed.

Solution

We substitute the given data into the logistic growth model

$$f(x) = \frac{c}{1 + ae^{-bx}}$$

Because at most $1,000$ people, the entire population of the community, can get the flu, we know the limiting value is $c = 1000$. To find a,we use the formula that the number of cases at time $t = 0$ is $\frac{c}{1+a} = 1$, from which it follows that $a = 999$.This model predicts that, after ten days, the number of people who have had the flu is $f(x) = \frac{1000}{1 + 999e^{-0.6030x}} \approx 293.8$. Because the actual number must be a whole number (a person has either had the flu or not) we round to 294. In the long term, the number of people who will contract the flu is the limiting value, $c = 1000$.

Analysis

Remember that, because we are dealing with a virus, we cannot predict with certainty the number of people infected. The model only approximates the number of people infected and will not give us exact or actual values.

The graph in Figure 6.7.7 gives a good picture of how this model fits the data.

Figure 6.7.7: The graph of $f(x) = \frac{1000}{1 + 999e^{-0.6030x}}$

Exercise 6.7.5

Using the model in the Example above, estimate the number of cases of flu on day 15.

Answer

895 cases on day 15? Exercise 6.7.7

Media

Access these online resources for additional instruction and practice with exponential and logarithmic models.

- Logarithm Application – pH
- Exponential Model – Age Using Half-Life
- Newton's Law of Cooling
- Exponential Growth Given Doubling Time
- Exponential Growth – Find Initial Amount Given Doubling Time

Key Equations

Half-life formula	If $A = A_0e^{kt}$, $k < 0$, the half-life is $t = -\frac{\ln(2)}{k}$.
Carbon-14 dating	$t = \frac{\ln(\frac{A}{A_0})}{-0.000121}$. A_0 A is the amount of carbon-14 when the plant or animal died t is the amount of carbon-14 remaining today is the age of the fossil in years
Doubling time formula	If $A = A_0e^{kt}$, $k > 0$, the doubling time is $t = \frac{\ln 2}{k}$
Newton's Law of Cooling	$T(t) = Ae^{kt} + T_s$, where T_s is the ambient temperature, $A = T(0) - T_s$, and k is the continuous rate of cooling.

Key Concepts

- The basic exponential function is $f(x) = ab^x$. If $b > 1$,we have exponential growth; if $0 < b < 1$, we have exponential decay.
- We can also write this formula in terms of continuous growth as $A = A_0e^{kx}$, where A_0 is the starting value. If A_0 is positive, then we have exponential growth when $k > 0$ and exponential decay when $k < 0$. See Example 6.7.1.
- In general, we solve problems involving exponential growth or decay in two steps. First, we set up a model and use the model to find the parameters. Then we use the formula with these parameters to predict growth and decay. See Example 6.7.2.
- We can find the age, t, of an organic artifact by measuring the amount, k, of carbon-14 remaining in the artifact and using the formula $t = \frac{\ln(k)}{-0.000121}$ to solve for t. See Example 6.7.3.
- Given a substance's doubling time or half-time, we can find a function that represents its exponential growth or decay. See Example 6.7.4.
- We can use Newton's Law of Cooling to find how long it will take for a cooling object to reach a desired temperature, or to find what temperature an object will be after a given time. See Example 6.7.5.
- We can use logistic growth functions to model real-world situations where the rate of growth changes over time, such as population growth, spread of disease, and spread of rumors. See Example 6.7.6.

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CHAPTER OVERVIEW

7: Systems of Equations and Inequalities

In this chapter, we will investigate matrices and their inverses, and various ways to use matrices to solve systems of equations. First, however, we will study systems of equations on their own: linear and nonlinear, and then partial fractions.

7.0: Prelude to Systems of Equations and Inequalities

7.1: Systems of Linear Equations - Two Variables

7.2: Systems of Linear Equations with Three Variables

7.3: Systems of Nonlinear Equations and Inequalities - Two Variables

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7.0: Prelude to Systems of Equations and Inequalities



Figure **7.0.1**: Enigma machines like this one, once owned by Italian dictator Benito Mussolini, were used by government and military officials for enciphering and deciphering top-secret communications during World War II. (credit: Dave Addey, Flickr)

By 1943, it was obvious to the Nazi regime that defeat was imminent unless it could build a weapon with unlimited destructive power, one that had never been seen before in the history of the world. In September, Adolf Hitler ordered German scientists to begin building an atomic bomb. Rumors and whispers began to spread from across the ocean. Refugees and diplomats told of the experiments happening in Norway. However, Franklin D. Roosevelt wasn't sold, and even doubted British Prime Minister Winston Churchill's warning. Roosevelt wanted undeniable proof. Fortunately, he soon received the proof he wanted when a group of mathematicians cracked the "Enigma" code, proving beyond a doubt that Hitler was building an atomic bomb. The next day, Roosevelt gave the order that the United States begin work on the same.

The Enigma is perhaps the most famous cryptographic device ever known. It stands as an example of the pivotal role cryptography has played in society. Now, technology has moved cryptanalysis to the digital world.

Many ciphers are designed using invertible matrices as the method of message transference, as finding the inverse of a matrix is generally part of the process of decoding. In addition to knowing the matrix and its inverse, the receiver must also know the key that, when used with the matrix inverse, will allow the message to be read.

In this chapter, we will investigate matrices and their inverses, and various ways to use matrices to solve systems of equations. First, however, we will study systems of equations on their own: linear and nonlinear, and then partial fractions. We will not be breaking any secret codes here, but we will lay the foundation for future courses.

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7.1: Systems of Linear Equations - Two Variables

Learning Objectives

- Solve systems of equations by graphing.
- Solve systems of equations by substitution.
- Solve systems of equations by addition.
- Identify inconsistent systems of equations containing two variables.
- Express the solution of a system of dependent equations containing two variables.

A skateboard manufacturer introduces a new line of boards. The manufacturer tracks its costs, which is the amount it spends to produce the boards, and its revenue, which is the amount it earns through sales of its boards. How can the company determine if it is making a profit with its new line? How many skateboards must be produced and sold before a profit is possible? In this section, we will consider linear equations with two variables to answer these and similar questions.



Figure 7.1.1

Introduction to Systems of Equations

In order to investigate situations such as that of the skateboard manufacturer, we need to recognize that we are dealing with more than one variable and likely more than one equation. **A system of linear equations** consists of two or more linear equations made up of two or more variables such that all equations in the system are considered simultaneously. To find the unique solution to a system of linear equations, we must find a numerical value for each variable in the system that will satisfy all equations in the system at the same time. Some linear systems may not have a solution and others may have an infinite number of solutions. In order for a linear system to have a unique solution, there must be at least as many equations as there are variables. Even so, this does not guarantee a unique solution.

In this section, we will look at systems of linear equations in two variables, which consist of two equations that contain two different variables. For example, consider the following system of linear equations in two variables.

$$\begin{aligned} 2x+y&=15 \\ 3x-y&=5 \end{aligned}$$

The *solution* to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. In this example, the ordered pair $(4,7)$ is the solution to the system of linear equations. We can verify the solution by substituting the values into each equation to see if the ordered pair satisfies both equations. Shortly we will investigate methods of finding such a solution if it exists.

$$\begin{aligned} 2(4)+(7)&=15 \text{ True} \\ 3(4)-(7)&=5 \text{ True} \end{aligned}$$

In addition to considering the number of equations and variables, we can categorize systems of linear equations by the number of solutions. A consistent system of equations has at least one solution. A consistent system is considered to be an **independent system** if it has a single solution, such as the example we just explored. The two lines have different slopes and intersect at one point in the plane. A consistent system is considered to be a **dependent system** if the equations have the same slope and the same

y-intercepts. In other words, the lines coincide so the equations represent the same line. Every point on the line represents a coordinate pair that satisfies the system. Thus, there are an infinite number of solutions.

Another type of system of linear equations is an **inconsistent system**, which is one in which the equations represent two parallel lines. The lines have the same slope and different y-intercepts. There are no points common to both lines; hence, there is no solution to the system.

TYPES OF LINEAR SYSTEMS

There are three types of systems of linear equations in two variables, and three types of solutions.

- An independent system has exactly one solution pair (x, y). The point where the two lines intersect is the only solution.
- An inconsistent system has no solution. Notice that the two lines are parallel and will never intersect.
- A dependent system has infinitely many solutions. The lines are coincident. They are the same line, so every coordinate pair on the line is a solution to both equations.

Figure **7.1.2** compares graphical representations of each type of system.

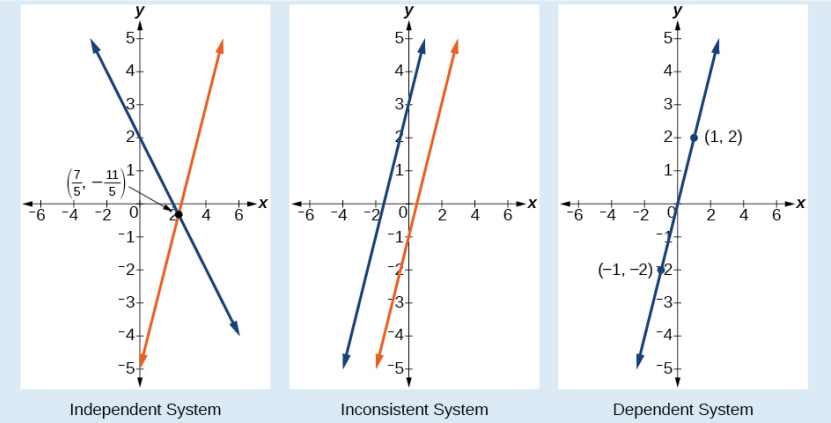


Figure **7.1.2**

Given a system of linear equations and an ordered pair, determine whether the ordered pair is a solution

1. Substitute the ordered pair into each equation in the system.
2. Determine whether true statements result from the substitution in both equations; if so, the ordered pair is a solution.

Example **7.1.1**: Determining Whether an Ordered Pair Is a Solution to a System of Equations

Determine whether the ordered pair $(5, 1)$ is a solution to the given system of equations.

$$x + 3y = 8$$
$$2x - 9 = y$$

Solution

Substitute the ordered pair $(5, 1)$ into both equations.

$$(5) + 3(1) = 8$$
$$8 = 8 \text{ True}$$
$$2(5) - 9 = (1)$$
$$1 = 1 \text{ True}$$

The ordered pair $(5, 1)$ satisfies both equations, so it is the solution to the system.

Analysis

We can see the solution clearly by plotting the graph of each equation. Since the solution is an ordered pair that satisfies both equations, it is a point on both of the lines and thus the point of intersection of the two lines. See Figure **7.1.3**.

Figure 7.1.3

? Exercise 7.1.1

Determine whether the ordered pair $(8, 5)$ is a solution to the following system.

$$\begin{aligned} 5x - 4y &= 20 \\ 2x + 1 &= 3y \end{aligned}$$

Answer

Not a solution.

Solving Systems of Equations by Graphing

There are multiple methods of solving systems of linear equations. For a system of linear equations in two variables, we can determine both the type of system and the solution by graphing the system of equations on the same set of axes.

✓ Example 7.1.2: Solving a System of Equations in Two Variables by Graphing

Solve the following system of equations by graphing. Identify the type of system.

$$\begin{aligned} 2x + y &= -8 \\ x - y &= -1 \end{aligned}$$

Solution

Solve the first equation for y.

$$\begin{aligned} 2x + y &= -8 \\ y &= -2x - 8 \end{aligned}$$

Solve the second equation for y.

$$\begin{aligned} x - y &= -1 \\ y &= x + 1 \end{aligned}$$

Graph both equations on the same set of axes as in Figure 7.1.4.

Figure 7.1.4

The lines appear to intersect at the point $(-3, -2)$. We can check to make sure that this is the solution to the system by substituting the ordered pair into both equations.

$$\begin{aligned} 2(-3) + (-2) &= -8 \\ -8 &= -8 \text{ True} \\ (-3) - (-2) &= -1 \\ -1 &= -1 \text{ True} \end{aligned}$$

The solution to the system is the ordered pair $(-3, -2)$,so the system is independent.

Exercise 7.1.2

Solve the following system of equations by graphing.

$$\begin{aligned} 2x-5y&=-25 \\ -4x+5y&=35 \end{aligned}$$

Answer

The solution to the system is the ordered pair $(-5,3)$.

Figure 7.1.5

Q&A

Can graphing be used if the system is inconsistent or dependent?

Yes, in both cases we can still graph the system to determine the type of system and solution. If the two lines are parallel, the system has no solution and is inconsistent. If the two lines are identical, the system has infinite solutions and is a dependent system.

Solving Systems of Equations by Substitution

Solving a linear system in two variables by graphing works well when the solution consists of integer values, but if our solution contains decimals or fractions, it is not the most precise method. We will consider two more methods of solving a **system of linear equations** that are more precise than graphing. One such method is solving a system of equations by the **substitution method**, in which we solve one of the equations for one variable and then substitute the result into the second equation to solve for the second variable. Recall that we can solve for only one variable at a time, which is the reason the substitution method is both valuable and practical.

How to: Given a system of two equations in two variables, solve using the substitution method.

1. Solve one of the two equations for one of the variables in terms of the other.
2. Substitute the expression for this variable into the second equation, then solve for the remaining variable.
3. Substitute that solution into either of the original equations to find the value of the first variable. If possible, write the solution as an ordered pair.
4. Check the solution in both equations.

Example 7.1.3: Solving a System of Equations in Two Variables by Substitution

Solve the following system of equations by substitution.

$$\begin{aligned} -x+y&=-5 \\ 2x-5y&=1 \end{aligned}$$

Solution

First, we will solve the first equation for y.

$$\begin{aligned} -x+y&=-5 \\ y&=x-5 \end{aligned}$$

Now we can substitute the expression $x-5$ for y in the second equation.

$$\begin{aligned} 2x-5y&=1 \\ 2x-5(x-5)&=1 \\ 2x-5x+25&=1 \\ -3x&=-24 \\ x&=8 \end{aligned}$$

Now, we substitute $x = 8$ into the first equation and solve for y.

$$\begin{aligned} -(8) + y &= -5 \\ y &= 3 \end{aligned}$$

Our solution is $(8, 3)$.

Check the solution by substituting $(8, 3)$ into both equations.

$$\begin{aligned} -x + y &= -5 \\ -(8) + (3) &= -5 \text{ True} \\ 2x - 5y &= 1 \\ 2(8) - 5(3) &= 1 \text{ True} \end{aligned}$$

Exercise 7.1.3

Solve the following system of equations by substitution.

$$\begin{aligned} x &= y + 3 \\ 4 &= 3x - 2y \end{aligned}$$

Answer

$(-2, -5)$

Q&A

Can the substitution method be used to solve any linear system in two variables?

Yes, but the method works best if one of the equations contains a coefficient of 1 *or* –1 *so that we do not have to deal with fractions.*

Solving Systems of Equations in Two Variables by the Addition Method

A third method of solving systems of linear equations is the addition method. In this method, we add two terms with the same variable, but opposite coefficients, so that the sum is zero. Of course, not all systems are set up with the two terms of one variable having opposite coefficients. Often we must adjust one or both of the equations by multiplication so that one variable will be eliminated by addition.

How to: Given a system of equations, solve using the addition method.

1. Write both equations with *x*- and *y*-variables on the left side of the equal sign and constants on the right.
2. Write one equation above the other, lining up corresponding variables. If one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, add the equations together, eliminating one variable. If not, use multiplication by a nonzero number so that one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, then add the equations to eliminate the variable.
3. Solve the resulting equation for the remaining variable.
4. Substitute that value into one of the original equations and solve for the second variable.
5. Check the solution by substituting the values into the other equation.

Example 7.1.4: Solving a System by the Addition Method

Solve the given system of equations by addition.

$$\begin{aligned} x + 2y &= -1 \\ -x + y &= 3 \end{aligned}$$

Solution

Both equations are already set equal to a constant. Notice that the coefficient of x in the second equation, -1, is the opposite of the coefficient of x in the first equation, 1. We can add the two equations to eliminate x without needing to multiply by a constant.

$$\begin{aligned} x+2y &= -1 \\ \underline{-x+y} &= \underline{3} \\ 3y &= 2 \end{aligned}$$

Now that we have eliminated x, we can solve the resulting equation for y.

$$\begin{aligned} 3y &= 2 \\ y &= \frac{2}{3} \end{aligned}$$

Then, we substitute this value for y into one of the original equations and solve for x.

$$\begin{aligned} -x+y &= 3 \\ -x+\frac{2}{3} &= 3 \\ -x &= 3-\frac{2}{3} \\ -x &= \frac{7}{3} \\ x &= -\frac{7}{3} \end{aligned}$$

The solution to this system is $\left(-\frac{7}{3}, \frac{2}{3}\right)$.

Check the solution in the first equation.

$$\begin{aligned} x+2y &= -1 \\ \left(-\frac{7}{3}\right)+2\left(\frac{2}{3}\right) &= \\ -\frac{7}{3}+\frac{4}{3} &= -\frac{3}{3} \\ -1 &= -1 \qquad \text{True} \end{aligned}$$

Analysis

We gain an important perspective on systems of equations by looking at the graphical representation. See Figure 7.1.6 to find that the equations intersect at the solution. We do not need to ask whether there may be a second solution because observing the graph confirms that the system has exactly one solution.

Figure 7.1.6

Example 7.1.5: Using the Addition Method When Multiplication of One Equation Is Required

Solve the given system of equations by the addition method.

$$\begin{aligned} 3x+5y &= -11 \\ x-2y &= 11 \end{aligned}$$

Solution

Adding these equations as presented will not eliminate a variable. However, we see that the first equation has $3x$ in it and the second equation has x. So if we multiply the second equation by -3,the x-terms will add to zero.

$$\begin{aligned} x-2y &= 11 \\ -3(x-2y) &= -3(11) \qquad \text{Multiply both sides by } -3. \\ -3x+6y &= -33 \qquad \text{Use the distributive property.} \end{aligned}$$

Now, let's add them.

$$\begin{aligned} 3x+5y &= -11 \\ \underline{-3x+6y} &= \underline{-33} \\ 11y &= -44 \\ y &= -4 \end{aligned}$$

For the last step, we substitute $y=-4$ into one of the original equations and solve for x.

$$\begin{aligned} 3x+5y &= -11 \\ 3x+5(-4) &= -11 \\ 3x-20 &= -11 \\ 3x &= 9 \\ x &= 3 \end{aligned}$$

Our solution is the ordered pair $(3,-4)$. See Figure 7.1.7. Check the solution in the original second equation.

$$\begin{aligned} x-2y &= 11 \\ (3)-2(-4) &= 3+8 \\ &= 11 \qquad \text{True} \end{aligned}$$

Figure 7.1.7

? Exercise 7.1.4

Solve the system of equations by addition.

$$\begin{aligned} 2x-7y &= 2 \\ 3x+y &= -20 \end{aligned}$$

Answer

$(-6,-2)$

✓ Example 7.1.6: Using the Addition Method When Multiplication of Both Equations Is Required

Solve the given system of equations in two variables by addition.

$$\begin{aligned} 2x+3y &= -16 \\ 5x-10y &= 30 \end{aligned}$$

Solution

One equation has $2x$ and the other has $5x$. The least common multiple is $10x$ so we will have to multiply both equations by a constant in order to eliminate one variable. Let's eliminate x by multiplying the first equation by -5 and the second equation by 2.

$$\begin{aligned} -5(2x+3y) &= -5(-16) \\ -10x-15y &= 80 \\ 2(5x-10y) &= 2(30) \\ 10x-20y &= 60 \end{aligned}$$

Then, we add the two equations together.

$$\begin{aligned} -10x-15y &= 80 \\ \underline{10x-20y} &= \underline{60} \\ -35y &= 140 \\ y &= -4 \end{aligned}$$

Substitute $y=-4$ into the original first equation.

$$\begin{aligned} 2x+3(-4) &= -16 \\ 2x-12 &= -16 \\ 2x &= -4 \\ x &= -2 \end{aligned}$$

The solution is $(-2,-4)$. Check it in the other equation.

$$\begin{aligned} 5x-10y &= 30 \\ 5(-2)-10(-4) &= 30 \\ -10+40 &= 30 \\ 30 &= 30 \end{aligned}$$

See Figure 7.1.8.

Figure 7.1.8

Example 7.1.7: Using the Addition Method in Systems of Equations Containing Fractions

Solve the given system of equations in two variables by addition.

$$\begin{aligned} \frac{x}{3}+\frac{y}{6} &= 3 \\ \frac{x}{2}-\frac{y}{4} &= 1 \end{aligned}$$

Solution

First clear each equation of fractions by multiplying both sides of the equation by the least common denominator.

$$\begin{aligned} 6\left(\frac{x}{3}+\frac{y}{6}\right) &= 6(3) \\ 2x+y &= 18 \\ 4\left(\frac{x}{2}-\frac{y}{4}\right) &= 4(1) \\ 2x-y &= 4 \end{aligned}$$

Now multiply the second equation by -1 so that we can eliminate the x-variable.

$$\begin{aligned} -1(2x-y) &= -1(4) \\ -2x+y &= -4 \end{aligned}$$

Add the two equations to eliminate the x-variable and solve the resulting equation.

$$\begin{aligned} 2x+y &= 18 \\ -2x+y &= -4 \\ 2y &= 14 \\ y &= 7 \end{aligned}$$

Substitute $y=7$ into the first equation.

$$\begin{aligned} 2x+(7) &= 18 \\ 2x &= 11 \\ x &= \frac{11}{2} \\ &= 7.5 \end{aligned}$$

The solution is $\left(\frac{11}{2},7\right)$. Check it in the other equation.

$$\begin{aligned} \frac{x}{2}-\frac{y}{4} &= 1 \\ \frac{\frac{11}{2}}{2}-\frac{7}{4} &= 1 \\ \frac{11}{4}-\frac{7}{4} &= 1 \\ \frac{4}{4} &= 1 \end{aligned}$$

? Exercise 7.1.5

Solve the system of equations by addition.

$$2x + 3y = 8$$
$$3x + 5y = 10$$

Answer

$(10, -4)$

Identifying Inconsistent Systems of Equations Containing Two Variables

Now that we have several methods for solving systems of equations, we can use the methods to identify inconsistent systems. Recall that an inconsistent system consists of parallel lines that have the same slope but different y-intercepts. They will never intersect. When searching for a solution to an inconsistent system, we will come up with a false statement, such as $12 = 0$.

✓ Example 7.1.8: Solving an Inconsistent System of Equations

Solve the following system of equations.

$$x = 9 - 2y$$
$$x + 2y = 13$$

Solution

We can approach this problem in two ways. Because one equation is already solved for x, the most obvious step is to use substitution.

$$\begin{aligned} x + 2y &= 13 \\ (9 - 2y) + 2y &= 13 \\ 9 + 0y &= 13 \\ 9 &= 13 \end{aligned}$$

Clearly, this statement is a contradiction because $9 \neq 13$. Therefore, the system has no solution.

The second approach would be to first manipulate the equations so that they are both in slope-intercept form. We manipulate the first equation as follows.

$$\begin{aligned} x &= 9 - 2y \\ 2y &= -x + 9 \\ y &= -\frac{1}{2}x + \frac{9}{2} \end{aligned}$$

We then convert the second equation expressed to slope-intercept form.

$$\begin{aligned} x + 2y &= 13 \\ 2y &= -x + 13 \\ y &= -\frac{1}{2}x + \frac{13}{2} \end{aligned}$$

Comparing the equations, we see that they have the same slope but different y-intercepts. Therefore, the lines are parallel and do not intersect.

$$y = -\frac{1}{2}x + \frac{9}{2}$$
$$y = -\frac{1}{2}x + \frac{13}{2}$$

Analysis

Writing the equations in slope-intercept form confirms that the system is inconsistent because all lines will intersect eventually unless they are parallel. Parallel lines will never intersect; thus, the two lines have no points in common. The graphs of the equations in this example are shown in Figure 7.1.9.

Figure 7.1.9

Exercise 7.1.6

Solve the following system of equations in two variables.

$$2y - 2x = 2$$
$$2y - 2x = 6$$

Answer

No solution. It is an inconsistent system.

Expressing the Solution of a System of Dependent Equations Containing Two Variables

Recall that a dependent system of equations in two variables is a system in which the two equations represent the same line. Dependent systems have an infinite number of solutions because all of the points on one line are also on the other line. After using substitution or addition, the resulting equation will be an identity, such as $0 = 0$.

Example 7.1.9: Finding a Solution to a Dependent System of Linear Equations

Find a solution to the system of equations using the **addition method**.

$$x + 3y = 2$$
$$3x + 9y = 6$$

Solution

With the addition method, we want to eliminate one of the variables by adding the equations. In this case, let's focus on eliminating x. If we multiply both sides of the first equation by -3, then we will be able to eliminate the x-variable.

$$\begin{aligned} x + 3y &= 2 \\ (-3)(x + 3y) &= (-3)(2) \\ -3x - 9y &= -6 \end{aligned}$$

Now add the equations.

$$\begin{aligned} -3x - 9y &= -6 \\ \underline{+\ 3x + 9y} &= \underline{6} \\ 0 &= 0 \end{aligned}$$

We can see that there will be an infinite number of solutions that satisfy both equations.

Analysis

If we rewrote both equations in the slope-intercept form, we might know what the solution would look like before adding. Let's look at what happens when we convert the system to slope-intercept form.

$$\begin{aligned} x + 3y &= 2 \\ 3y &= -x + 2 \\ y &= -\frac{1}{3}x + \frac{2}{3} \\ 3x + 9y &= 6 \\ 9y &= -3x + 6 \\ y &= -\frac{3}{9}x + \frac{6}{9} \\ y &= -\frac{1}{3}x + \frac{2}{3} \end{aligned}$$

See Figure 7.1.10. Notice the results are the same. The general solution to the system is $\left(x, -\frac{1}{3}x + \frac{2}{3}\right)$.

Figure 7.1.10

? Exercise 7.1.7

Solve the following system of equations in two variables.

$$y - 2x = 5$$
$$-3y + 6x = -15$$

Answer

The system is dependent so there are infinite solutions of the form $(x, 2x + 5)$.

Using Systems of Equations to Investigate Profits

Using what we have learned about systems of equations, we can return to the skateboard manufacturing problem at the beginning of the section. The skateboard manufacturer's revenue function is the function used to calculate the amount of money that comes into the business. It can be represented by the equation $R = xp$, where x=quantity and p=price. The revenue function is shown in orange in Figure 7.1.11.

The cost function is the function used to calculate the costs of doing business. It includes fixed costs, such as rent and salaries, and variable costs, such as utilities. The cost function is shown in blue in Figure 7.1.11. The x-axis represents quantity in hundreds of units. The y-axis represents either cost or revenue in hundreds of dollars.

Figure 7.1.11

The point at which the two lines intersect is called the break-even point. We can see from the graph that if 700 units are produced, the cost is $\$3,300$ and the revenue is also $\$3,300$. In other words, the company breaks even if they produce and sell 700 units. They neither make money nor lose money.

The shaded region to the right of the break-even point represents quantities for which the company makes a profit. The shaded region to the left represents quantities for which the company suffers a loss. The profit function is the revenue function minus the cost function, written as $P(x) = R(x) - C(x)$. Clearly, knowing the quantity for which the cost equals the revenue is of great importance to businesses.

✓ Example 7.1.10: Finding the Break-Even Point and the Profit Function Using Substitution

Given the cost function $C(x) = 0.85x + 35,000$ and the revenue function $R(x) = 1.55x$, find the break-even point and the profit function.

Solution

Write the system of equations using y to replace function notation.

$$y = 0.85x + 35,000$$
$$y = 1.55x$$

Substitute the expression $0.85x + 35,000$ from the first equation into the second equation and solve for x.

$$\begin{aligned} 0.85x + 35,000 &= 1.55x \\ 35,000 &= 0.7x \\ 50,000 &= x \end{aligned}$$

Then, we substitute $x = 50,000$ into either the cost function or the revenue function.

$$1.55(50,000) = 77,500$$

The break-even point is $(50,000, 77,500)$.

The profit function is found using the formula $P(x) = R(x) - C(x)$.

$$\begin{aligned} P(x) &= 1.55x - (0.85x + 35,000) \\ &= 0.7x - 35,000 \end{aligned}$$

The profit function is $P(x) = 0.7x - 35,000$.

Analysis

The cost to produce $50,000$ units is $\$77,500$, and the revenue from the sales of $50,000$ units is also $\$77,500$. To make a profit, the business must produce and sell more than $50,000$ units. See Figure 7.1.12.

Figure 7.1.12

We see from the graph in Figure 7.1.13 that the profit function has a negative value until $x = 50,000$, when the graph crosses the x-axis. Then, the graph emerges into positive y-values and continues on this path as the profit function is a straight line. This illustrates that the break-even point for businesses occurs when the profit function is 0. The area to the left of the break-even point represents operating at a loss.

Figure 7.1.13

Example 7.1.11: Writing and Solving a System of Equations in Two Variables

The cost of a ticket to the circus is $25.00 for children and $50.00 for adults. On a certain day, attendance at the circus is 2,000 and the total gate revenue is $70,000. How many children and how many adults bought tickets?

Solution

Let c = the number of children and a = the number of adults in attendance.

The total number of people is 2,000. We can use this to write an equation for the number of people at the circus that day.

$$c + a = 2,000$$

The revenue from all children can be found by multiplying $25.00 by the number of children, $25c$. The revenue from all adults can be found by multiplying $50.00 by the number of adults, $50a$. The total revenue is $70,000. We can use this to write an equation for the revenue.

$$25c + 50a = 70,000$$

We now have a system of linear equations in two variables.

$$c + a = 2,000$$

$$25c + 50a = 70,000$$

In the first equation, the coefficient of both variables is 1. We can quickly solve the first equation for either c or a. We will solve for a.

$$\begin{aligned} c + a &= 2,000 \\ a &= 2,000 - c \end{aligned}$$

Substitute the expression $2,000 - c$ in the second equation fora a and solve for c.

$$\begin{aligned} 25c + 50(2,000 - c) &= 70,000 \\ 25c + 100,000 - 50c &= 70,000 \\ -25c &= -30,000 \\ c &= 1,200 \end{aligned}$$

Substitute $c = 1,200$ into the first equation to solve for a.

$$\begin{aligned} 1,200 + a &= 2,000 \\ a &= 800 \end{aligned}$$

We find that 1,200 children and 800 adults bought tickets to the circus that day.

Exercise 7.1.8

Meal tickets at the circus cost $4.00 for children and $12.00 for adults. If 1,650 meal tickets were bought for a total of, $14,200, how many children and how many adults bought meal tickets?

Answer

700 children, 950 adults

Media

Access these online resources for additional instruction and practice with systems of linear equations.

- Solving Systems of Equations Using Substitution
- Solving Systems of Equations Using Elimination
- Applications of Systems of Equations

Key Concepts

- A system of linear equations consists of two or more equations made up of two or more variables such that all equations in the system are considered simultaneously.
- The solution to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. See Example **7.1.1**.
- Systems of equations are classified as independent with one solution, dependent with an infinite number of solutions, or inconsistent with no solution.
- One method of solving a system of linear equations in two variables is by graphing. In this method, we graph the equations on the same set of axes. See Example **7.1.2**.
- Another method of solving a system of linear equations is by substitution. In this method, we solve for one variable in one equation and substitute the result into the second equation. See Example **7.1.3**.
- A third method of solving a system of linear equations is by addition, in which we can eliminate a variable by adding opposite coefficients of corresponding variables. See Example **7.1.4**.
- It is often necessary to multiply one or both equations by a constant to facilitate elimination of a variable when adding the two equations together. See Example **7.1.5**, Example **7.1.6**, and Example **7.1.7**.
- Either method of solving a system of equations results in a false statement for inconsistent systems because they are made up of parallel lines that never intersect. See Example **7.1.8**.
- The solution to a system of dependent equations will always be true because both equations describe the same line. See Example **7.1.9**.
- Systems of equations can be used to solve real-world problems that involve more than one variable, such as those relating to revenue, cost, and profit. See Example **7.1.10** and Example **7.1.11**.

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7.2: Systems of Linear Equations with Three Variables

Learning Objectives

- Solve systems of three equations in three variables.
- Identify inconsistent systems of equations containing three variables.
- Express the solution of a system of dependent equations containing three variables.

John received an inheritance of $\$12,000$ that he divided into three parts and invested in three ways: in a money-market fund paying 3% annual interest; in municipal bonds paying 4% annual interest; and in mutual funds paying 7% annual interest. John invested $\$4,000$ more in municipal funds than in municipal bonds. He earned $\$670$ in interest the first year. How much did John invest in each type of fund?



Figure 7.2.1: (credit: "Elembis," Wikimedia Commons)

Understanding the correct approach to setting up problems such as this one makes finding a solution a matter of following a pattern. We will solve this and similar problems involving three equations and three variables in this section. Doing so uses similar techniques as those used to solve systems of two equations in two variables. However, finding solutions to systems of three equations requires a bit more organization and a touch of visual gymnastics.

Solving Systems of Three Equations in Three Variables

In order to solve systems of equations in three variables, known as three-by-three systems, the primary tool we will be using is called Gaussian elimination, named after the prolific German mathematician Karl Friedrich Gauss. While there is no definitive order in which operations are to be performed, there are specific guidelines as to what type of moves can be made. We may number the equations to keep track of the steps we apply. The goal is to eliminate one variable at a time to achieve upper triangular form, the ideal form for a three-by-three system because it allows for straightforward back-substitution to find a solution (x, y, z), which we call an ordered triple. A system in upper triangular form looks like the following:

$$\begin{aligned} Ax + By + Cz &= D \\ Ey + Fz &= G \\ Hz &= K \end{aligned}$$

The third equation can be solved for z,and then we back-substitute to find y and x. To write the system in upper triangular form, we can perform the following operations:

1. Interchange the order of any two equations.
2. Multiply both sides of an equation by a nonzero constant.
3. Add a nonzero multiple of one equation to another equation.

The **solution set** to a three-by-three system is an ordered triple (x, y, z). Graphically, the ordered triple defines the point that is the intersection of three planes in space. You can visualize such an intersection by imagining any corner in a rectangular room. A corner is defined by three planes: two adjoining walls and the floor (or ceiling). Any point where two walls and the floor meet represents the intersection of three planes.

NUMBER OF POSSIBLE SolutionS

Figure 7.2.2 and Figure 7.2.3 illustrate possible solution scenarios for three-by-three systems.

- Systems that have a single solution are those which, after elimination, result in a solution set consisting of an ordered triple (x, y, z). Graphically, the ordered triple defines a point that is the intersection of three planes in space.
- Systems that have an infinite number of solutions are those which, after elimination, result in an expression that is always true, such as $0 = 0$. Graphically, an infinite number of solutions represents a line or coincident plane that serves as the intersection of three planes in space.
- Systems that have no solution are those that, after elimination, result in a statement that is a contradiction, such as $3 = 0$. Graphically, a system with no solution is represented by three planes with no point in common.

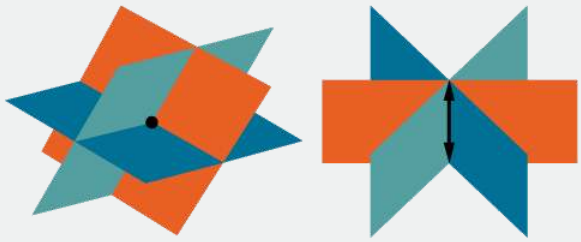


Figure 7.2.2: (a)Three planes intersect at a single point, representing a three-by-three system with a single solution. (b) Three planes intersect in a line, representing a three-by-three system with infinite solutions.



Figure 7.2.3: All three figures represent three-by-three systems with no solution. (a) The three planes intersect with each other, but not at a common point. (b) Two of the planes are parallel and intersect with the third plane, but not with each other. (c) All three planes are parallel, so there is no point of intersection.

Example 7.2.1: Determining Whether an Ordered Triple Is a Solution to a System

Determine whether the ordered triple $(3, -2, 1)$ is a solution to the system.

$$\begin{aligned} x + y + z &= 2 \\ 6x - 4y + 5z &= 31 \\ 5x + 2y + 2z &= 13 \end{aligned}$$

Solution

We will check each equation by substituting in the values of the ordered triple for x, y, and z.

$$\begin{array}{lll} x + y + z = 2 & 6x - 4y + 5z = 31 & 5x + 2y + 2z = 13 \\ (3) + (-2) + (1) = 2 & 6(3) - 4(-2) + 5(1) = 31 & 5(3) + 2(-2) + 2(1) = 13 \\ \text{True} & 18 + 8 + 5 = 31 & 15 - 4 + 2 = 13 \\ & \text{True} & \text{True} \end{array}$$

The ordered triple $(3, -2, 1)$ is indeed a solution to the system.

How to: Given a linear system of three equations, solve for three unknowns

1. Pick any pair of equations and solve for one variable.
2. Pick another pair of equations and solve for the same variable.
3. You have created a system of two equations in two unknowns. Solve the resulting two-by-two system.

4. Back-substitute known variables into any one of the original equations and solve for the missing variable.

Example 7.2.2: Solving a System of Three Equations in Three Variables by Elimination

Find a solution to the following system:

$$\begin{aligned} x-2y+3z &= 9 \ (1)\\ -x+3y-z &= -6 \ (2)\\ 2x-5y+5z &= 17 \ (3) \end{aligned}$$

Solution

There will always be several choices as to where to begin, but the most obvious first step here is to eliminate x by adding equations (1) and (2).

$$\begin{aligned} x-2y+3z &= 9 \ \ (1)\\ \underline{-x+3y-z} &\underline{= -6 \ \ (2)}\\ y+2z &= 3 \ \ (4) \end{aligned}$$

The second step is multiplying equation (1) by -2 and adding the result to equation (3). These two steps will eliminate the variable x.

$$\begin{aligned} -2x+4y-6z &= -18 \ (1) \quad \text{multiplied by } -2\\ \underline{2x-5y+5z} &\underline{= 17 \ (3)}\\ -y-z &= -1 \ (5) \end{aligned}$$

In equations (4) and (5), we have created a new two-by-two system. We can solve for z by adding the two equations.

$$\begin{aligned} y+2z &= 3 \ (4)\\ \underline{-y-z} &\underline{= -1 \ (5)}\\ z &= 2 \ (6) \end{aligned}$$

Choosing one equation from each new system, we obtain the upper triangular form:

$$\begin{aligned} x-2y+3z &= 9 \ (1)\\ y+2z &= 3 \ (4)\\ z &= 2 \ (6) \end{aligned}$$

Next, we back-substitute $z=2$ into equation (4) and solve for y.

$$\begin{aligned} y+2(2) &= 3\\ y+4 &= 3\\ y &= -1 \end{aligned}$$

Finally, we can back-substitute $z=2$ and $y=-1$ into equation (1). This will yield the solution for x.

$$\begin{aligned} x-2(-1)+3(2) &= 9\\ x+2+6 &= 9\\ x &= 1 \end{aligned}$$

The solution is the ordered triple $(1,-1,2)$. See Figure 7.2.4.

Figure 7.2.4

Example 7.2.3: Solving a Real-World Problem Using a System of Three Equations in Three Variables

In the problem posed at the beginning of the section, John invested his inheritance of $\$12,000$ in three different funds: part in a money-market fund paying 3% interest annually; part in municipal bonds paying 4% annually; and the rest in mutual funds

paying 7% annually. John invested $\$4,000$ more in mutual funds than he invested in municipal bonds. The total interest earned in one year was $\$670$. How much did he invest in each type of fund?

Solution

To solve this problem, we use all of the information given and set up three equations. First, we assign a variable to each of the three investment amounts:

$$x = \text{amount invested in money-market fund}$$
$$y = \text{amount invested in municipal bonds}$$
$$z = \text{amount invested in mutual funds}$$

The first equation indicates that the sum of the three principal amounts is $\$12,000$.

$$x + y + z = 12,000$$

We form the second equation according to the information that John invested $\$4,000$ more in mutual funds than he invested in municipal bonds.

$$z = y + 4,000$$

The third equation shows that the total amount of interest earned from each fund equals $\$670$.

$$0.03x + 0.04y + 0.07z = 670$$

Then, we write the three equations as a system.

$$\begin{aligned} x + y + z &= 12,000 \\ -y + z &= 4,000 \\ 0.03x + 0.04y + 0.07z &= 670 \end{aligned}$$

To make the calculations simpler, we can multiply the third equation by 100. Thus,

$$\begin{aligned} x + y + z &= 12,000 \quad (1) \\ -y + z &= 4,000 \quad (2) \\ 3x + 4y + 7z &= 67,000 \quad (3) \end{aligned}$$

Step 1. Interchange equation (2) and equation (3) so that the two equations with three variables will line up.

$$\begin{aligned} x + y + z &= 12,000 \\ 3x + 4y + 7z &= 67,000 \\ -y + z &= 4,000 \end{aligned}$$

Step 2. Multiply equation (1) by -3 and add to equation (2). Write the result as row 2.

$$\begin{aligned} x + y + z &= 12,000 \\ y + 4z &= 31,000 \\ -y + z &= 4,000 \end{aligned}$$

Step 3. Add equation (2) to equation (3) and write the result as equation (3).

$$\begin{aligned} x + y + z &= 12,000 \\ y + 4z &= 31,000 \\ 5z &= 35,000 \end{aligned}$$

Step 4. Solve for z in equation (3). Back-substitute that value in equation (2) and solve for y. Then, back-substitute the values for z and y into equation (1) and solve for x.

$$5z = 35,000$$
$$z = 7,000$$

$$y + 4(7,000) = 31,000$$
$$y = 3,000$$

$$x + 3,000 + 7,000 = 12,000$$
$$x = 2,000$$

John invested $\$2,000$ in a money-market fund, $\$3,000$ in municipal bonds, and $\$7,000$ in mutual funds.

? Exercise 7.2.1

Solve the system of equations in three variables.

$$2x + y - 2z = -1$$
$$3x - 3y - z = 5$$
$$x - 2y + 3z = 6$$

Answer

$(1, -1, 1)$

Identifying Inconsistent Systems of Equations Containing Three Variables

Just as with systems of equations in two variables, we may come across an inconsistent system of equations in three variables, which means that it does not have a solution that satisfies all three equations. The equations could represent three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location. The process of elimination will result in a false statement, such as $3 = 7$ or some other contradiction.

✓ Example 7.2.4: Solving an Inconsistent System of Three Equations in Three Variables

Solve the following system.

$$x - 3y + z = 4 \quad (7.2.1)$$
$$-x + 2y - 5z = 3 \quad (7.2.2)$$
$$5x - 13y + 13z = 8 \quad (7.2.3)$$

Solution

Looking at the coefficients of x, we can see that we can eliminate x by adding Equation 7.2.1 to Equation 7.2.2.

$$x - 3y + z = 4 \quad (1)$$
$$\underline{-x + 2y - 5z = 3 \quad (2)}$$
$$-y - 4z = 7 \quad (4)$$

Next, we multiply equation (1) by -5 and add it to equation (3).

$$-5x + 15y - 5z = -20 \quad (1) \quad \text{multiplied by } -5$$
$$\underline{5x - 13y + 13z = 8 \quad (3)}$$
$$2y + 8z = -12 \quad (5)$$

Then, we multiply equation (4) by 2 and add it to equation (5).

$$-2y - 8z = 14 \quad (4) \quad \text{multiplied by 2}$$
$$\underline{2y + 8z = -12 \quad (5)}$$
$$0 = 2$$

The final equation $0 = 2$ is a contradiction, so we conclude that the system of equations in inconsistent and, therefore, has no solution.

Analysis

In this system, each plane intersects the other two, but not at the same location. Therefore, the system is inconsistent.

? Exercise 7.2.2

Solve the system of three equations in three variables.

$$\begin{aligned} x + y + z &= 2 \\ y - 3z &= 1 \\ 2x + y + 5z &= 0 \end{aligned}$$

Answer

No solution.

Expressing the Solution of a System of Dependent Equations Containing Three Variables

We know from working with systems of equations in two variables that a dependent system of equations has an infinite number of solutions. The same is true for dependent systems of equations in three variables. An infinite number of solutions can result from several situations. The three planes could be the same, so that a solution to one equation will be the solution to the other two equations. All three equations could be different but they intersect on a line, which has infinite solutions. Or two of the equations could be the same and intersect the third on a line.

✓ Example 7.2.5: Finding the Solution to a Dependent System of Equations

Find the solution to the given system of three equations in three variables.

$$\begin{aligned} 2x + y - 3z &= 0 \quad (1) \\ 4x + 2y - 6z &= 0 \quad (2) \\ x - y + z &= 0 \quad (3) \end{aligned}$$

Solution

First, we can multiply equation (1) by -2 and add it to equation (2).

$$\begin{aligned} -4x - 2y + 6z &= 0 (1) \quad \text{multiplied by } -2 \\ \underline{4x + 2y - 6z} &= \underline{0 (2)} \\ 0 &= 0 \end{aligned}$$

We do not need to proceed any further. The result we get is an identity, $0 = 0$, which tells us that this system has an infinite number of solutions. There are other ways to begin to solve this system, such as multiplying equation (3) by -2, and adding it to equation (1). We then perform the same steps as above and find the same result, $0 = 0$.

When a system is dependent, we can find general expressions for the solutions. Adding equations (1) and (3), we have

$$\begin{aligned} 2x + y - 3z &= 0 \\ x - y + z &= 0 \\ 3x - 2z &= 0 \end{aligned}$$

We then solve the resulting equation for z.

$$\begin{aligned} 3x - 2z &= 0 \\ z &= \frac{3}{2}x \end{aligned}$$

We back-substitute the expression for z into one of the equations and solve for y.

$$2x + y - 3(\frac{3}{2}x) = 0$$
$$2x + y - \frac{9}{2}x = 0$$
$$y = \frac{9}{2}x - 2x$$
$$y = \frac{5}{2}x$$

So the general solution is $\left(x, \frac{5}{2}x, \frac{3}{2}x\right)$. In this solution, x can be any real number. The values of y and z are dependent on the value selected for x.

Analysis

As shown in Figure **7.2.5**, two of the planes are the same and they intersect the third plane on a line. The solution set is infinite, as all points along the intersection line will satisfy all three equations.

Figure **7.2.5**

Q&A: Does the generic solution to a dependent system always have to be written in terms of x?

No, you can write the generic solution in terms of any of the variables, but it is common to write it in terms of x and if needed x and y.

? Exercise 7.2.3:

Solve the following system.

$$x + y + z = 7$$
$$3x - 2y - z = 4$$
$$x + 6y + 5z = 24$$

Answer

Infinite number of solutions of the form $(x, 4x - 11, -5x + 18)$.

Media

Access these online resources for additional instruction and practice with systems of equations in three variables.

- Ex 1: System of Three Equations with Three Unknowns Using Elimination
- Ex. 2: System of Three Equations with Three Unknowns Using Elimination

Key Concepts

- A solution set is an ordered triple {(x,y,z)} that represents the intersection of three planes in space. See Example **7.2.1**.
- A system of three equations in three variables can be solved by using a series of steps that forces a variable to be eliminated. The steps include interchanging the order of equations, multiplying both sides of an equation by a nonzero constant, and adding a nonzero multiple of one equation to another equation. See Example **7.2.2**.

- Systems of three equations in three variables are useful for solving many different types of real-world problems. See Example **7.2.3**.
- A system of equations in three variables is inconsistent if no solution exists. After performing elimination operations, the result is a contradiction. See Example **7.2.4**.
- Systems of equations in three variables that are inconsistent could result from three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location.
- A system of equations in three variables is dependent if it has an infinite number of solutions. After performing elimination operations, the result is an identity. See Example **7.2.5**.
- Systems of equations in three variables that are dependent could result from three identical planes, three planes intersecting at a line, or two identical planes that intersect the third on a line.

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7.3: Systems of Nonlinear Equations and Inequalities - Two Variables

Learning Objectives

- Solve a system of nonlinear equations using substitution.
- Solve a system of nonlinear equations using elimination.

Halley's Comet (Figure **7.3.1**) orbits the sun about once every **75** years. Its path can be considered to be a very elongated ellipse. Other comets follow similar paths in space. These orbital paths can be studied using systems of equations. These systems, however, are different from the ones we considered in the previous section because the equations are not linear.



Figure* 7.3.1*: *Halley's Comet (credit: "NASA Blueshift"/Flickr)*

In this section, we will consider the intersection of a parabola and a line, a circle and a line, and a circle and an ellipse. The methods for solving systems of nonlinear equations are similar to those for linear equations.

Solving a System of Nonlinear Equations Using Substitution

A system of nonlinear equations is a system of two or more equations in two or more variables containing at least one equation that is not linear. Recall that a linear equation can take the form $Ax + By + C = 0$. Any equation that cannot be written in this form in nonlinear. The substitution method we used for linear systems is the same method we will use for nonlinear systems. We solve one equation for one variable and then substitute the result into the second equation to solve for another variable, and so on. There is, however, a variation in the possible outcomes.

Intersection of a Parabola and a Line

There are three possible types of solutions for a system of nonlinear equations involving a parabola and a line.

Possible Types of Solutions For Points of Intersection of a Parabola and a Line

Figure **7.3.2** illustrates possible solution sets for a system of equations involving a parabola and a line.

- No solution - The line will never intersect the parabola.
- One solution - The line is tangent to the parabola and intersects the parabola at exactly one point.
- Two solutions - The line crosses on the inside of the parabola and intersects the parabola at two points.

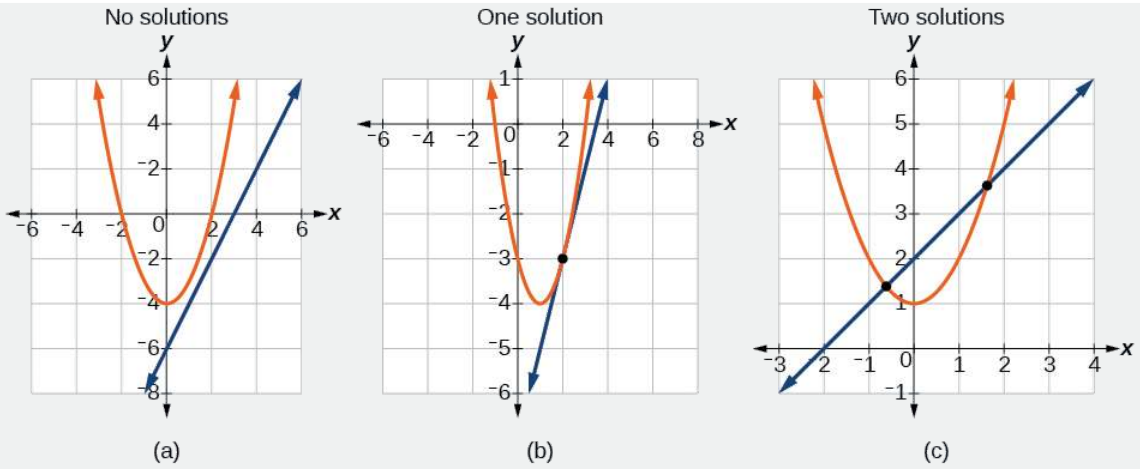


***Figure* 7.3.2**

How to: Given a system of equations containing a line and a parabola, find the solution

1. Solve the linear equation for one of the variables.
2. Substitute the expression obtained in step one into the parabola equation.
3. Solve for the remaining variable.
4. Check your solutions in both equations.

Example 7.3.1: Solving a System of Nonlinear Equations Representing a Parabola and a Line

Solve the system of equations.

$$\begin{aligned} x - y &= -1 \\ y &= x^2 + 1 \end{aligned}$$

Solution

Solve the first equation for x and then substitute the resulting expression into the second equation.

$$\begin{aligned} x - y &= -1 \\ x &= y - 1 \qquad \text{Solve for } x. \end{aligned}$$

$$\begin{aligned} y &= x^2 + 1 \\ y &= (y-1)^2 + 1 \qquad \text{Substitute expression for } x. \end{aligned}$$

Expand the equation and set it equal to zero.

$$\begin{aligned} y &= (y-1)^2 + 1 \\ &= (y^2 - 2y + 1) + 1 \\ &= y^2 - 2y + 2 \\ 0 &= y^2 - 3y + 2 \\ &= (y-2)(y-1) \end{aligned}$$

Solving for y gives $y = 2$ and $y = 1$. Next, substitute each value for y into the first equation to solve for x. Always substitute the value into the linear equation to check for extraneous solutions.

$$\begin{aligned} x - y &= -1 \\ x - (2) &= -1 \\ x &= 1 \\ x - (1) &= -1 \\ x &= 0 \end{aligned}$$

The solutions are $(1,2)$ and $(0,1)$,which can be verified by substituting these (x,y) values into both of the original equations (Figure 7.3.3).

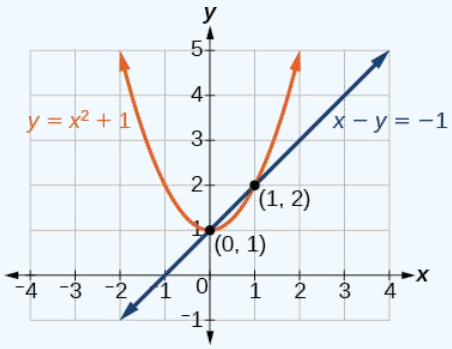


***Figure* 7.3.3**

Q&A: Could we have substituted values for y into the second equation to solve for x in last example?

Yes, but because x is squared in the second equation this could give us extraneous solutions for x.

For $y = 1$

$$\begin{aligned} y &= x^2 + 1 \\ y &= x^2 + 1 \\ x^2 &= 0 \\ x &= \pm\sqrt{0} = 0 \end{aligned}$$

This gives us the same value as in the solution.

For $y = 2$

$$\begin{aligned} y &= x^2 + 1 \\ 2 &= x^2 + 1 \\ x^2 &= 1 \\ x &= \pm\sqrt{1} = \pm 1 \end{aligned}$$

Notice that -1 is an extraneous solution.

Exercise 7.3.1

Solve the given system of equations by substitution.

$$\begin{aligned} 3x - y &= -2 \\ 2x^2 - y &= 0 \end{aligned}$$

Answer

$\left(-\frac{1}{2}, \frac{1}{2}\right)$ and $(2, 8)$

Intersection of a Circle and a Line

Just as with a parabola and a line, there are three possible outcomes when solving a system of equations representing a circle and a line.

POSSIBLE TYPES OF SolutionS FOR THE POINTS OF INTERSECTION OF A CIRCLE AND A LINE

Figure 7.3.4 illustrates possible solution sets for a system of equations involving a circle and a line.

- No solution - The line does not intersect the circle.

- One solution - The line is tangent to the circle and intersects the circle at exactly one point.
- Two solutions - The line crosses the circle and intersects it at two points.

***Figure* 7.3.4**

How to: Given a system of equations containing a line and a circle, find the solution

1. Solve the linear equation for one of the variables.
2. Substitute the expression obtained in step one into the equation for the circle.
3. Solve for the remaining variable.
4. Check your solutions in both equations.

Example 7.3.2: Finding the Intersection of a Circle and a Line by Substitution

Find the intersection of the given circle and the given line by substitution.

$$\begin{aligned} x^2+y^2&=5\\ y&=3x-5 \end{aligned}$$

Solution

One of the equations has already been solved for y. We will substitute $y=3x-5$ into the equation for the circle.

$$\begin{aligned} x^2+(3x-5)^2&=5\\ x^2+9x^2-30x+25&=5\\ 10x^2-30x+20&=0 \end{aligned}$$

Now, we factor and solve for x.

$$\begin{aligned} 10(x2-3x+2)&=0\\ 10(x-2)(x-1)&=0\\ x&=2\\ x&=1 \end{aligned}$$

Substitute the two x-values into the original linear equation to solve for y.

$$\begin{aligned} y&=3(2)-5\\ &=1\\ y&=3(1)-5\\ &=-2 \end{aligned}$$

The line intersects the circle at $(2,1)$ and $(1,-2)$,which can be verified by substituting these (x,y) values into both of the original equations (Figure 7.3.5).

***Figure* 7.3.5**

Exercise 7.3.2

Solve the system of nonlinear equations.

$$\begin{aligned} x^2+y^2&=10\\ x-3y&=-10 \end{aligned}$$

Answer

$(-1,3)$

Solving a System of Nonlinear Equations Using Elimination

We have seen that substitution is often the preferred method when a system of equations includes a linear equation and a nonlinear equation. However, when both equations in the system have like variables of the second degree, solving them using elimination by addition is often easier than substitution. Generally, elimination is a far simpler method when the system involves only two equations in two variables (a two-by-two system), rather than a three-by-three system, as there are fewer steps. As an example, we will investigate the possible types of solutions when solving a system of equations representing a circle and an ellipse.

POSSIBLE TYPES OF SolutionS FOR THE POINTS OF INTERSECTION OF A CIRCLE AND AN ELLIPSE

Figure **7.3.6** illustrates possible solution sets for a system of equations involving a circle and an ellipse.

- No solution - The circle and ellipse do not intersect. One shape is inside the other or the circle and the ellipse are a distance away from the other.
- One solution - The circle and ellipse are tangent to each other, and intersect at exactly one point.
- Two solutions - The circle and the ellipse intersect at two points.
- Three solutions - The circle and the ellipse intersect at three points.
- Four solutions - The circle and the ellipse intersect at four points.

***Figure* 7.3.6**

Example **7.3.3**: Solving a System of Nonlinear Equations Representing a Circle and an Ellipse

Solve the system of nonlinear equations.

$$\begin{aligned} x^2+y^2 &= 26 \quad (1)\\ 3x^2+25y^2 &= 100 \quad (2)\end{aligned}$$

Solution

Let's begin by multiplying equation (1) by -3,and adding it to equation (2).

$$\begin{aligned} (-3)(x^2+y^2) &= (-3)(26)\\ -3x^2-3y^2 &= -78\\ 3x^2+25y^2 &= 100\\ \hline 22y^2 &= 22\end{aligned}$$

After we add the two equations together, we solve for y.

$$\begin{aligned} y^2 &= 1\\ y &= \pm\sqrt{1} = \pm 1\end{aligned}$$

Substitute $y=\pm 1$ into one of the equations and solve for x.

$$\begin{aligned} x^2+(1)^2 &= 26\\ x^2+1 &= 26\\ x^2 &= 25\\ x &= \pm\sqrt{25} = \pm 5\\ x^2+(-1)^2 &= 26\\ x^2+1 &= 26\\ x^2 &= \pm\sqrt{25} = \pm 5\end{aligned}$$

There are four solutions: $(5,1)$, $(-5,1)$, $(5,-1)$,and $(-5,-1)$. See Figure **7.3.7**.

***Figure* 7.3.7**

Exercise 7.3.3

Find the solution set for the given system of nonlinear equations.

$$4x^2 + y^2 = 13$$
$$x^2 + y^2 = 10$$

Answer

$(1, 3), (1, -3), (-1, 3), (-1, -3)$

Media

Access these online resources for additional instruction and practice with nonlinear equations.

- Solve a System of Nonlinear Equations Using Substitution
- Solve a System of Nonlinear Equations Using Elimination

Key Concepts

- There are three possible types of solutions to a system of equations representing a line and a parabola: (1) no solution, the line does not intersect the parabola; (2) one solution, the line is tangent to the parabola; and (3) two solutions, the line intersects the parabola in two points. See Example **7.3.1**.
- There are three possible types of solutions to a system of equations representing a circle and a line: (1) no solution, the line does not intersect the circle; (2) one solution, the line is tangent to the parabola; (3) two solutions, the line intersects the circle in two points. See Example **7.3.2**.
- There are five possible types of solutions to the system of nonlinear equations representing an ellipse and a circle: (1) no solution, the circle and the ellipse do not intersect; (2) one solution, the circle and the ellipse are tangent to each other; (3) two solutions, the circle and the ellipse intersect in two points; (4) three solutions, the circle and ellipse intersect in three places; (5) four solutions, the circle and the ellipse intersect in four points. See Example **7.3.3**.

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