MATH 125: HOPE COLLEGE



Katharine Vance
Hope College

LibreTexts™

Math 125 Hope College

This text is disseminated via the Open Education Resource (OER) LibreTexts Project (https://LibreTexts.org) and like the thousands of other texts available within this powerful platform, it is freely available for reading, printing, and "consuming."

The LibreTexts mission is to bring together students, faculty, and scholars in a collaborative effort to provide and accessible, and comprehensive platform that empowers our community to develop, curate, adapt, and adopt openly licensed resources and technologies; through these efforts we can reduce the financial burden born from traditional educational resource costs, ensuring education is more accessible for students and communities worldwide.

Most, but not all, pages in the library have licenses that may allow individuals to make changes, save, and print this book. Carefully consult the applicable license(s) before pursuing such effects. Instructors can adopt existing LibreTexts texts or Remix them to quickly build course-specific resources to meet the needs of their students. Unlike traditional textbooks, LibreTexts' web based origins allow powerful integration of advanced features and new technologies to support learning.



LibreTexts is the adaptable, user-friendly non-profit open education resource platform that educators trust for creating, customizing, and sharing accessible, interactive textbooks, adaptive homework, and ancillary materials. We collaborate with individuals and organizations to champion open education initiatives, support institutional publishing programs, drive curriculum development projects, and more.

The LibreTexts libraries are Powered by NICE CXone Expert and was supported by the Department of Education Open Textbook Pilot Project, the California Education Learning Lab, the UC Davis Office of the Provost, the UC Davis Library, the California State University Affordable Learning Solutions Program, and Merlot. This material is based upon work supported by the National Science Foundation under Grant No. 1246120, 1525057, and 1413739.

Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation nor the US Department of Education.

Have questions or comments? For information about adoptions or adaptions contact info@LibreTexts.org or visit our main website at https://LibreTexts.org.

This text was compiled on 06/09/2025

TABLE OF CONTENTS

Licensing

1: Algebra Essentials

- 1.1: Real Numbers - Algebra Essentials
- 1.2: Exponents and Scientific Notation
- 1.3: Radicals and Rational Expressions
- 1.4: Polynomials
- 1.5: Factoring Polynomials
- 1.6: Rational Expressions
- 1.7: The Rectangular Coordinate Systems and Graphs
- 1.8: Linear Equations in One Variable
- 1.9: Quadratic Equations
- 1.10: Other Types of Equations

2: Function Essentials

- 2.1: Functions and Function Notation
- 2.2: Domain and Range
- 2.3: Rates of Change and Behavior of Graphs
- 2.4: Transformation of Functions

3: Derivative Essentials

- 3.1: How do we Measure Velocity?
- 3.2: The Notion of Limit
- 3.3: The Derivative of a Function at a Point
- 3.4: The Derivative Function
- 3.5: Limits, Continuity, and Differentiability
- 3.6: The Mean Value Theorem

4: Applications of the Derivative

- 4.1: Interpretating, Estimating, and Using the Derivative
- 4.2: The Second Derivative
- 4.3: The Tangent Line Approximation
- 4.4: Elementary Derivative Rules

5: Trigonometry Essentials

- 5.1: Angles
- 5.2: Right Triangle Trigonometry
- 5.3: Unit Circle
- 5.4: The Other Trigonometric Functions
- 5.5: Graphs of the Sine and Cosine Functions

6: Differentiation Rules and Applications

- 6.1: The Sine and Cosine Function
- 6.2: The Product and Quotient Rules
- 6.3: Derivatives of Other Trigonometric Functions

- 6.4: The Chain Rule
- 6.5: Derivatives of Functions Given Implicitely

Index

Glossary

Detailed Licensing

Licensing

*A detailed breakdown of this resource's licensing can be found in **Back Matter/Detailed Licensing**.*

CHAPTER OVERVIEW

1: Algebra Essentials

1.1: Real Numbers - Algebra Essentials
1.2: Exponents and Scientific Notation
1.3: Radicals and Rational Expressions
1.4: Polynomials
1.5: Factoring Polynomials
1.6: Rational Expressions
1.7: The Rectangular Coordinate Systems and Graphs
1.8: Linear Equations in One Variable
1.9: Quadratic Equations
1.10: Other Types of Equations

1: Algebra Essentials is shared under a not declared license and was authored, remixed, and/or curated by LibreTexts.

1.1: Real Numbers - Algebra Essentials

The following text includes many examples. At the end of the text are supplemental help videos, then the homework exercises.

Learning Objectives

- Classify a real number as a natural, whole, integer, rational, or irrational number.
- Perform calculations using order of operations.
- Use the following properties of real numbers: commutative, associative, distributive, inverse, and identity.
- Evaluate algebraic expressions.
- Simplify algebraic expressions.

It is often said that mathematics is the language of science. If this is true, then the language of mathematics is numbers. The earliest use of numbers occurred 100 centuries ago in the Middle East to count, or enumerate items. Farmers, cattlemen, and tradesmen used tokens, stones, or markers to signify a single quantity—a sheaf of grain, a head of livestock, or a fixed length of cloth, for example. Doing so made commerce possible, leading to improved communications and the spread of civilization.

Three to four thousand years ago, Egyptians introduced fractions. They first used them to show reciprocals. Later, they used them to represent the amount when a quantity was divided into equal parts.

But what if there were no cattle to trade or an entire crop of grain was lost in a flood? How could someone indicate the existence of nothing? From earliest times, people had thought of a “base state” while counting and used various symbols to represent this null condition. However, it was not until about the fifth century A.D. in India that zero was added to the number system and used as a numeral in calculations.

Clearly, there was also a need for numbers to represent loss or debt. In India, in the seventh century A.D., negative numbers were used as solutions to mathematical equations and commercial debts. The opposites of the counting numbers expanded the number system even further.

Because of the evolution of the number system, we can now perform complex calculations using these and other categories of real numbers. In this section, we will explore sets of numbers, calculations with different kinds of numbers, and the use of numbers in expressions.

Classifying a Real Number

The numbers we use for counting, or enumerating items, are the natural numbers: $1, 2, 3, 4, 5$and so on. We describe them in set notation as $\{1, 2, 3, \dots\}$where the ellipsis $(\cdots)$ indicates that the numbers continue to infinity. The natural numbers are, of course, also called the counting numbers. Any time we enumerate the members of a team, count the coins in a collection, or tally the trees in a grove, we are using the set of natural numbers. The set of whole numbers is the set of natural numbers plus zero: $\{0, 1, 2, 3, \dots\}$

The set of integers adds the opposites of the natural numbers to the set of whole numbers: $\{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$ It is useful to note that the set of integers is made up of three distinct subsets: negative integers, zero, and positive integers. In this sense, the positive integers are just the natural numbers. Another way to think about it is that the natural numbers are a subset of the integers.

$$\overbrace{\cdots, -3, -2, -1}^{\text{negative integers}}, \underbrace{0}_{\text{zero}}, \overbrace{1,\ 2,\ 3,\ \cdots}^{\text{positive integers}}$$

The set of rational numbers is written as $\{\frac{m}{n} | \text{m and n are integers and } n \neq 0\}$ .Notice from the definition that rational numbers are fractions (or quotients) containing integers in both the numerator and the denominator, and the denominator is never 0. We can also see that every natural number, whole number, and integer is a rational number with a denominator of 1.

Because they are fractions, any rational number can also be expressed in decimal form. Any rational number can be represented as either:

1. a terminating decimal: $\frac{15}{8} = 1.875$, or
2. a repeating decimal: $\frac{4}{11} = 0.36363636\cdots = 0.\overline{36}$

We use a line drawn over the repeating block of numbers instead of writing the group multiple times.

Example 1.1.1: Writing Integers as Rational Numbers

Write each of the following as a rational number. Write a fraction with the integer in the numerator and 1 in the denominator.

a. 7
b. 0
c. -8

Solution

a. $7 = \frac{7}{1}$

b. $0 = \frac{0}{1}$

c. $-8 = \frac{-8}{1}$

Exercise 1.1.1

Write each of the following as a rational number.

a. 11
b. 3
c. -4

Answer

a. $\frac{11}{1}$
b. $\frac{3}{1}$
c. $-\frac{4}{1}$

Example 1.1.2: Identifying Rational Numbers

Write each of the following rational numbers as either a terminating or repeating decimal.

a. $-\frac{5}{7}$
b. $\frac{15}{5}$
c. $\frac{13}{25}$

Solution

a. a repeating decimal

b. $\frac{15}{5} = 3$ (or 3.0), a terminating decimal

c. $\frac{13}{25} = 0.52$, a terminating decimal

Exercise 1.1.2

Write each of the following rational numbers as either a terminating or repeating decimal.

a. $\frac{68}{17}$
b. $\frac{8}{13}$
c. $-\frac{13}{25}$

Answer

a. 4 (or 4.0), terminating
b. $0.\overline{615384}$, repeating
c. -0.85, terminating

Irrational Numbers

At some point in the ancient past, someone discovered that not all numbers are rational numbers. A builder, for instance, may have found that the diagonal of a square with unit sides was not 2 or even 32, but was something else. Or a garment maker might have observed that the ratio of the circumference to the diameter of a roll of cloth was a little bit more than 3, but still not a rational number. Such numbers are said to be irrational because they cannot be written as fractions. These numbers make up the set of irrational numbers. Irrational numbers cannot be expressed as a fraction of two integers. It is impossible to describe this set of numbers by a single rule except to say that a number is irrational if it is not rational. So we write this as shown.

$$\{h \mid h \text{ is not a rational number}\} \tag{1.1.1}$$

Example 1.1.3: Differentiating Rational and Irrational Numbers

Determine whether each of the following numbers is rational or irrational. If it is rational, determine whether it is a terminating or repeating decimal.

a. $\sqrt{25}$
b. $\frac{33}{9}$
c. $\sqrt{11}$
d. $\frac{17}{34}$
e. $0.3033033303333\ldots$

Solution

a. $\sqrt{25}$: This can be simplified as $\sqrt{25} = 5$. Therefore, $\sqrt{25}$ is rational.
b. $\frac{33}{9}$: Because it is a fraction, $\frac{33}{9}$ is a rational number. Next, simplify and divide.

$$\frac{33}{9} = \frac{33}{9}$$

So, $\frac{33}{9}$ is rational and a repeating decimal.

c. $\sqrt{11}$: This cannot be simplified any further. Therefore, $\sqrt{11}$ is an irrational number.
d. $\frac{17}{34}$: Because it is a fraction, $\frac{17}{34}$ is a rational number. Simplify and divide.

$$\frac{17}{34} = 0.5$$

So, $\frac{17}{34}$ is rational and a terminating decimal.
e. $0.3033033033333\ldots$ is not a terminating decimal. Also note that there is no repeating pattern because the group of $3s$ increases each time. Therefore it is neither a terminating nor a repeating decimal and, hence, not a rational number. It is an irrational number.

? Exercise 1.1.3

Determine whether each of the following numbers is rational or irrational. If it is rational, determine whether it is a terminating or repeating decimal.

a. $\frac{7}{77}$
b. $\sqrt{81}$
c. $4.27027002700027\ldots$
d. $\frac{91}{13}$
e. $\sqrt{39}$

Answer

a. rational and repeating;
b. rational and terminating;
c. irrational;
d. rational and terminating;
e. irrational

Real Numbers

Given any number n, we know that n is either rational or irrational. It cannot be both. The sets of rational and irrational numbers together make up the set of real numbers. As we saw with integers, the real numbers can be divided into three subsets: negative real numbers, zero, and positive real numbers. Each subset includes fractions, decimals, and irrational numbers according to their algebraic sign (+ or –). Zero is considered neither positive nor negative.

The real numbers can be visualized on a horizontal number line with an arbitrary point chosen as 0, with negative numbers to the left of 0 and positive numbers to the right of 0. A fixed unit distance is then used to mark off each integer (or other basic value) on either side of 0. Any real number corresponds to a unique position on the number line.The converse is also true: Each location on the number line corresponds to exactly one real number. This is known as a one-to-one correspondence. We refer to this as the real number line as shown in Figure (1.1.1.

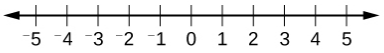


Figure 1.1.1: The real number line.

✓ Example 1.1.4: Classifying Real Numbers

Classify each number as either positive or negative and as either rational or irrational. Does the number lie to the left or the right of 0 on the number line?

a. $-\frac{10}{3}$
b. $\sqrt{5}$
c. $-\sqrt{289}$
d. -6π
e. $0.615384615384\ldots$

Solution

a. $-\frac{10}{3}$ is negative and rational. It lies to the left of 0 on the number line.
b. $\sqrt{5}$ is positive and irrational. It lies to the right of 0.
c. $-\sqrt{289} = -\sqrt{17^2} = -17$ is negative and rational. It lies to the left of 0.
d. -6π is negative and irrational. It lies to the left of 0.
e. $0.615384615384\ldots$ is a repeating decimal so it is rational and positive. It lies to the right of 0.

? Exercise 1.1.4

Classify each number as either positive or negative and as either rational or irrational. Does the number lie to the left or the right of 0 on the number line?

a. $\sqrt{73}$
b. $-11.411411411\ldots$
c. $\frac{47}{19}$
d. $-\frac{\sqrt{5}}{2}$

e. 6.210735

Answer

a. positive, irrational; right
b. negative, rational; left
c. positive, rational; right
d. negative, irrational; left
e. positive, rational; right

Sets of Numbers as Subsets

Beginning with the natural numbers, we have expanded each set to form a larger set, meaning that there is a subset relationship between the sets of numbers we have encountered so far. These relationships become more obvious when seen as a diagram, such as Figure(1.1.2).

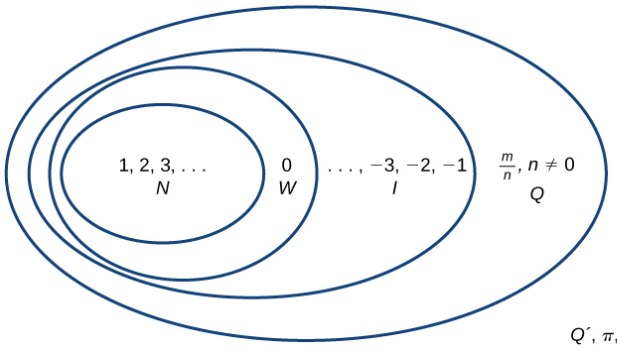


Figure 1.1.2: Sets of numbers. **N**: the set of natural numbers. **W**: the set of whole numbers. **I**: the set of integers. **Q**: the set of rational numbers. **Q′**: the set of irrational numbers

SETS OF NUMBERS

The set of natural numbers includes the numbers used for counting: $\{1, 2, 3, \dots\}$.

The set of whole numbers is the set of natural numbers plus zero: $\{0, 1, 2, 3, \dots\}$

The set of integers adds the negative natural numbers to the set of whole numbers: $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

The set of rational numbers includes fractions written as $\{\frac{m}{n} | \text{m and n are integers and } n \neq 0\}$.

The set of irrational numbers is the set of numbers that are not rational, are nonrepeating, and are nonterminating: $\{h \parallel \text{h is not a rational number}\}$.

Example 1.1.5: Differentiating the Sets of Numbers

Classify each number as being a natural number (N), whole number (W), integer (I), rational number (Q), and/or irrational number (Q′).

a. $\sqrt{36}$
b. $\frac{8}{3}$
c. $\sqrt{73}$
d. -6
e. $3.2121121112\dots$

Solution

	N	W	I	Q	Q'
a. $\sqrt{36} = 6$	X	X	X	X	
b. $\frac{8}{3} = 2.\overline{6}$				X	
c. $\sqrt{73}$					X
d. -6			X	X	
e. $3.2121121112\dots$					X

Exercise 1.1.5

Classify each number as being a natural number (N), whole number (W), integer (I), rational number (Q), and/or irrational number (Q′).

a. $-\frac{35}{7}$

b. 0
c. $\sqrt{169}$
d. $\sqrt{24}$
e. $4.763763763\ldots$

Answer

	N	W	I	Q	Q'
a. $-\frac{35}{7}$			X	X	
b. 0		X	X	X	
c. $\sqrt{169}$	X	X	X	X	
d. $\sqrt{24}$					X
e. $4.763763763\ldots$				X	

Performing Calculations Using the Order of Operations

When we multiply a number by itself, we square it or raise it to a power of 2. For example, $4^2 = 4 \times 4 = 16$. We can raise any number to any power. In general, the exponential notation an means that the number or variable a is used as a factor n times.

$$a^n = a \cdot a \cdot a \cdots a \qquad \text{n factors}$$

In this notation, a^n is read as the nth power of a, where a is called the base and n is called the exponent. A term in exponential notation may be part of a mathematical expression, which is a combination of numbers and operations. For example, $24 + 6 \times \frac{2}{3} - 4^2$ is a mathematical expression.

To evaluate a mathematical expression, we perform the various operations. However, we do not perform them in any random order. We use the order of operations. This is a sequence of rules for evaluating such expressions.

Recall that in mathematics we use parentheses ( ), brackets [ ], and braces { } to group numbers and expressions so that anything appearing within the symbols is treated as a unit. Additionally, fraction bars, radicals, and absolute value bars are treated as grouping symbols. When evaluating a mathematical expression, begin by simplifying expressions within grouping symbols.

The next step is to address any exponents or radicals. Afterward, perform multiplication and division from left to right and finally addition and subtraction from left to right.

Let's take a look at the expression provided.

$$24 + 6 \times \frac{2}{3} - 4^2$$

There are no grouping symbols, so we move on to exponents or radicals. The number 4 is raised to a power of 2, so simplify 4^2 as 16.

$$24 + 6 \times \frac{2}{3} - 4^2$$

$$24 + 6 \times \frac{2}{3} - 16$$

Next, perform multiplication or division, left to right.

$$24 + 6 \times \frac{2}{3} - 16$$

$$24 + 4 - 16$$

Lastly, perform addition or subtraction, left to right.

$$24 + 4 - 16$$

$$28 - 16$$

$$12$$

Therefore,

$$24 + 6 \times \frac{2}{3} - 4^2 = 12$$

For some complicated expressions, several passes through the order of operations will be needed. For instance, there may be a radical expression inside parentheses that must be simplified before the parentheses are evaluated. Following the order of operations ensures that anyone simplifying the same mathematical expression will get the same result.

ORDER OF OPERATIONS

Operations in mathematical expressions must be evaluated in a systematic order, which can be simplified using the acronym **PEMDAS**:

- **P**(arentheses)
- **E**(xponents)
- **M**(ultiplication) and **D**(ivision)
- **A**(ddition) and **S**(ubtraction)

HOW TO: Given a mathematical expression, simplify it using the order of operations.

1. Simplify any expressions within grouping symbols.
2. Simplify any expressions containing exponents or radicals.
3. Perform any multiplication and division in order, from left to right.
4. Perform any addition and subtraction in order, from left to right.

Example 1.1.6: Using the Order of Operations

Use the order of operations to evaluate each of the following expressions.

a. $(3\times 2)^2-4\times(6+2)$

b. $\dfrac{5^2-4}{7}-\sqrt{11-2}$

c. $6-|\,5-8\,|+3\times(4-1)$

d. $\dfrac{14-3\times 2}{2\times 5-3^2}$

e. $7\times(5\times 3)-2\times[(6-3)-4^2]+1$

Solution

a.

$$\begin{aligned}(3\times 2)^2-4\times(6+2) &= (6)^2-4\times(8) && \text{Simplify parentheses}\\ &= 36-4\times 8 && \text{Simplify exponent}\\ &= 36-32 && \text{Simplify multiplication}\\ &= 4 && \text{Simplify subtraction}\end{aligned}$$

b.

$$\begin{aligned}\frac{5^2-4}{7}-\sqrt{11-2} &= \frac{5^2-4}{7}-\sqrt{9} && \text{Simplify grouping symbols (radical)}\\ &= \frac{5^2-4}{7}-3 && \text{Simplify radical}\\ &= \frac{25-4}{7}-3 && \text{Simplify exponent}\\ &= \frac{21}{7}-3 && \text{Simplify subtraction in numerator}\\ &= 3-3 && \text{Simplify division}\\ &= 0 && \text{Simplify subtraction}\end{aligned}$$

Note that in the first step, the radical is treated as a grouping symbol, like parentheses. Also, in the third step, the fraction bar is considered a grouping symbol so the numerator is considered to be grouped.

c.

$$\begin{aligned}6-|\,5-8\,|+3\times(4-1) &= 6-|-3|+3\times 3 && \text{Simplify inside grouping symbols}\\ &= 6-3+3\times 3 && \text{Simplify absolute value}\\ &= 6-3+9 && \text{Simplify multiplication}\\ &= 3+9 && \text{Simplify subtraction}\\ &= 12 && \text{Simplify addition}\end{aligned}$$

d.

$$\begin{aligned}\frac{14-3\times 2}{2\times 5-3^2} &= \frac{14-3\times 2}{2\times 5-9} && \text{Simplify exponent}\\ &= \frac{14-6}{10-9} && \text{Simplify products}\\ &= \frac{8}{1} && \text{Simplify differences}\\ &= 8 && \text{Simplify quotient}\end{aligned}$$

In this example, the fraction bar separates the numerator and denominator, which we simplify separately until the last step.

e.

$$
\begin{aligned}
7\times(5\times3)-2\times[(6-3)-4^2]+1 &= 7\times(15)-2\times[(3)-4^2]+1 && \text{Simplify inside parentheses}\\
&= 7\times(15)-2\times(3-16)+1 && \text{Simplify exponent}\\
&= 7\times(15)-2\times(-13)+1 && \text{Subtract}\\
&= 105+26+1 && \text{Multiply}\\
&= 132 && \text{Add}
\end{aligned}
$$

? Exercise 1.1.6

Use the order of operations to evaluate each of the following expressions.

a. $\sqrt{5^2-4^2}+7\times(5-4)^2$

b. $1+\dfrac{7\times5-8\times4}{9-6}$

c. $|1.8-4.3|+0.4\times\sqrt{15+10}$

d. $\dfrac{1}{2}\times[5\times3^2-7^2]+\dfrac{1}{3}\times9^2$

e. $[(3-8^2)-4]-(3-8)$

Answer

a. 10

b. 2

c. 4.5

d. 25

e. -60

Using Properties of Real Numbers

For some activities we perform, the order of certain operations does not matter, but the order of other operations does. For example, it does not make a difference if we put on the right shoe before the left or vice-versa. However, it does matter whether we put on shoes or socks first. The same thing is true for operations in mathematics.

Commutative Properties

The commutative property of addition states that numbers may be added in any order without affecting the sum.

$$a+b=b+a \tag{1.1.2}$$

We can better see this relationship when using real numbers.

$$(-2)+7=5 \text{ and } 7+(-2)=5$$

Similarly, the **commutative property of multiplication** states that numbers may be multiplied in any order without affecting the product.

$$a\times b=b\times a \tag{1.1.3}$$

Again, consider an example with real numbers.

$$(-11)\times(-4)=44 \text{ and } (-4)\times(-11)=44$$

It is important to note that neither subtraction nor division is commutative. For example, $17-5$ is not the same as $5-17$. Similarly, $20\div5\neq5\div20$.

Associative Properties

The associative property of multiplication tells us that it does not matter how we group numbers when multiplying. We can move the grouping symbols to make the calculation easier, and the product remains the same.

$$a(bc)=(ab)c \tag{1.1.4}$$

Consider this example.

$$(3\times4)\times5=60 \text{ and } 3\times(4\times5)=60$$

The associative property of addition tells us that numbers may be grouped differently without affecting the sum.

$$a+(b+c)=(a+b)+c \tag{1.1.5}$$

This property can be especially helpful when dealing with negative integers. Consider this example.

$$[15+(-9)]+23=29 \text{ and } 15+[(-9)+23]=29$$

Are subtraction and division associative? Review these examples.

$$8-(3-15) \stackrel{?}{=} (8-3)-15$$
$$8-(-12) \stackrel{?}{=} 5-15$$
$$20 \neq 10$$
$$64 \div (8 \div 4) \stackrel{?}{=} (64 \div 8) \div 4$$
$$64 \div 2 \stackrel{?}{=} 8 \div 4$$
$$32 \neq 2$$

As we can see, neither subtraction nor division is associative.

Distributive Property

The distributive property states that the product of a factor times a sum is the sum of the factor times each term in the sum.

$$a \times (b+c) = a \times b + a \times c \tag{1.1.6}$$

This property combines both addition and multiplication (and is the only property to do so). Let us consider an example.

$$\begin{aligned} 4 \cdot [12 + (-7)] &= 4 \cdot 12 + 4 \cdot (-7) \\ &= 48 + (-28) \\ &= 20 \end{aligned}$$

Note that 4 is outside the grouping symbols, so we distribute the 4 by multiplying it by 12, multiplying it by -7, and adding the products.

Example 1.1.7

To be more precise when describing this property, we say that multiplication distributes over addition. The reverse is not true, as we can see in this example.

$$6+(3 \times 5) \stackrel{?}{=} (6+3) \times (6 \times 5)$$
$$6+(15) \stackrel{?}{=} (9) \times (11)$$
$$21 \neq 99$$

Multiplication does not distribute over subtraction, and division distributes over neither addition nor subtraction.

A special case of the distributive property occurs when a sum of terms is subtracted.

$$a-b = a+(-b) \tag{1.1.7}$$

For example, consider the difference $12-(5+3)$. We can rewrite the difference of the two terms 12 and $(5+3)$ by turning the subtraction expression into addition of the opposite. So instead of subtracting $(5+3)$, we add the opposite.

$$12+(-1) \times (5+3)$$

Now, distribute -1 and simplify the result.

$$\begin{aligned} 12-(5+3) &= 12+(-1) \times (5+3) \\ &= 12+[(-1) \times 5+(-1) \times 3] \\ &= 12+(-8) \\ &= 4 \end{aligned}$$

This seems like a lot of trouble for a simple sum, but it illustrates a powerful result that will be useful once we introduce algebraic terms. To subtract a sum of terms, change the sign of each term and add the results. With this in mind, we can rewrite the last example.

$$\begin{aligned} 12-(5+3) &= 12+(-5-3) \\ &= 12-8 \\ &= 4 \end{aligned}$$

Identity Properties

The **identity property of addition** states that there is a unique number, called the additive identity (0) that, when added to a number, results in the original number.

$$a+0 = a \tag{1.1.8}$$

The **identity property of multiplication** states that there is a unique number, called the multiplicative identity (1) that, when multiplied by a number, results in the original number.

$$a \times 1 = a \tag{1.1.9}$$

For example, we have $(-6)+0 = -6$ and $23 \times 1 = 23$. There are no exceptions for these properties; they work for every real number, including 0 and 1.

Inverse Properties

The inverse property of addition states that, for every real number a, there is a unique number, called the additive inverse (or opposite), denoted $-a$, that, when added to the original number, results in the additive identity, 0.

$$a+(-a)=0 \tag{1.1.10}$$

For example, if $a=-8$, the additive inverse is 8, since $(-8)+8=0$.

The inverse property of multiplication holds for all real numbers except 0 because the reciprocal of 0 is not defined. The property states that, for every real number a, there is a unique number, called the multiplicative inverse (or reciprocal), denoted $1a$, that, when multiplied by the original number, results in the multiplicative identity, 1.

$$a\times\frac{1}{a}=1 \tag{1.1.11}$$

For example, if $a=-\frac{2}{3}$, the reciprocal, denoted $\frac{1}{a}$, is $-\frac{3}{2}$ because

$$a\cdot\frac{1}{a}=\left(-\frac{2}{3}\right)\times\left(-\frac{3}{2}\right)=1$$

PROPERTIES OF REAL NUMBERS

The following properties hold for real numbers a, b, and c.

Table 1.1.1

	Addition	Multiplication
Commutative Property	$a+b=b+a$	$a\times b=b\times a$
Associative Property	$a+(b+c)=(a+b)+c$	$a(bc)=(ab)c$
Distributive Property	$a\times(b+c)=a\times b+a\times c$	
Identity Property	There exists a unique real number called the additive identity, 0, such that, for any real number a $a+0=a$	

There exists a unique real number called the multiplicative identity, 1, such that, for any real number a

$$a\times 1=a$$

Inverse Property

Every real number a has an additive inverse, or opposite, denoted –a, such that

$$a+(-a)=0$$

Every nonzero real number a has a multiplicative inverse, or reciprocal, denoted 1a , such that

$$a\times\left(\frac{1}{a}\right)=1$$

Example 1.1.8: Using Properties of Real Numbers

Use the properties of real numbers to rewrite and simplify each expression. State which properties apply.

a. $3\times 6+3\times 4$
b. $(5+8)+(-8)$
c. $6-(15+9)$
d. $\frac{4}{7}\times\left(\frac{2}{3}\times\frac{7}{4}\right)$
e. $100\times[0.75+(-2.38)]$

Solution

a.

$3\times 6+3\times 4$	$=3\times(6+4)$	Distributive property
	$=3\times 10$	Simplify
	$=30$	Simplify

b.

$(5+8)+(-8)$	$=5+[8+(-8)]$	Associative property of addition
	$=5+0$	Inverse property of addition
	$=5$	Identity property of addition

c.

$6-(15+9)$	$=6+[(-15)+(-9)]$	Distributive property
	$=6+(-24)$	Simplify
	$=-18$	Simplify

d.

$$\begin{aligned}
\frac{4}{7}\times\left(\frac{2}{3}\times\frac{7}{4}\right) &= \frac{4}{7}\times\left(\frac{7}{4}\times\frac{2}{3}\right) && \text{Commutative property of multiplication}\\
&= \left(\frac{4}{7}\times\frac{7}{4}\right)\times\frac{2}{3} && \text{Associative property of multiplication}\\
&= 1\times\frac{2}{3} && \text{Inverse property of multiplication}\\
&= \frac{2}{3} && \text{Identity property of multiplication}
\end{aligned}$$

e.

$$\begin{aligned}
100\times[0.75+(-2.38)] &= 100\times 0.75+100\times(-2.38) && \text{Distributive property}\\
&= 75+(-238) && \text{Simplify}\\
&= -163 && \text{Simplify}
\end{aligned}$$

? Exercise 1.1.7

Use the properties of real numbers to rewrite and simplify each expression. State which properties apply.

a. $\left(-\frac{23}{5}\right)\times\left[11\times\left(-\frac{5}{23}\right)\right]$
b. $5\times(6.2+0.4)$
c. $18-(7-15)$
d. $\frac{17}{18}+\left[\frac{4}{9}+\left(-\frac{17}{18}\right)\right]$
e. $6\times(-3)+6\times 3$

Answer

1. 11, commutative property of multiplication
2. 33, distributive property
3. 26, distributive property
4. $\frac{4}{9}$, commutative property of addition, associative property of addition, inverse property of addition, identity property of addition
5. 0, distributive property, inverse property of addition, identity property of addition

Evaluating Algebraic Expressions

So far, the mathematical expressions we have seen have involved real numbers only. In mathematics, we may see expressions such as $x+5$, $\frac{4}{3}\pi r^3$, or $\sqrt{2m^3n^2}$. In the expression $x+5$, 5 is called a **constant** because it does not vary and x is called a **variable** because it does. (In naming the variable, ignore any exponents or radicals containing the variable.) An **algebraic expression** is a collection of constants and variables joined together by the algebraic operations of addition, subtraction, multiplication, and division.

We have already seen some real number examples of exponential notation, a shorthand method of writing products of the same factor. When variables are used, the constants and variables are treated the same way.

$$\begin{aligned}
(-3)^5 &= (-3)\times(-3)\times(-3)\times(-3)\times(-3) \Rightarrow x^5 = x\times x\times x\times x\times x\\
(2\times 7)^3 &= (2\times 7)\times(2\times 7)\times(2\times 7) \qquad \Rightarrow (yz)^3 = (yz)\times(yz)\times(yz)
\end{aligned}$$

In each case, the exponent tells us how many factors of the base to use, whether the base consists of constants or variables.

Any variable in an algebraic expression may take on or be assigned different values. When that happens, the value of the algebraic expression changes. To evaluate an algebraic expression means to determine the value of the expression for a given value of each variable in the expression. Replace each variable in the expression with the given value, then simplify the resulting expression using the order of operations. If the algebraic expression contains more than one variable, replace each variable with its assigned value and simplify the expression as before.

✓ Example 1.1.9: Describing Algebraic Expressions

List the constants and variables for each algebraic expression.

a. $x+5$
b. $\frac{4}{3}\pi r^3$
c. $\sqrt{2m^3n^2}$

Solution

	Constants	Variables
a. $x+5$	5	x
b. $\frac{4}{3}\pi r^3$	$\frac{4}{3}, \pi$	r
c. $\sqrt{2m^3n^2}$	2	m,n

? Exercise 1.1.8

List the constants and variables for each algebraic expression.

a. $2(L+W)$
b. $4y^3+y$

Answer

	Constants	Variables
a. $2\pi r(r+h)$	$2,\pi$	r,h
b. $2(L+W)$	2	L, W
c. $4y^3+y$	4	y

✓ Example 1.1.10: Evaluating an Algebraic Expression at Different Values

Evaluate the expression $2x-7$ for each value for x.

a. $x=0$
b. $x=1$
c. $x=12$
d. $x=-4$

Solution

a. Substitute 0 for x.

$$\begin{aligned} 2x-7 &= 2(0)-7 \\ &= 0-7 \\ &= -7 \end{aligned}$$

b. Substitute 1 for x.

$$\begin{aligned} 2x-7 &= 2(1)-7 \\ &= 2-7 \\ &= -5 \end{aligned}$$

c. Substitute $\frac{1}{2}$ for x.

$$\begin{aligned} 2x-7 &= 2\left(\frac{1}{2}\right)-7 \\ &= 1-7 \\ &= -6 \end{aligned}$$

d. Substitute -4 for x.

$$\begin{aligned} 2x-7 &= 2(-4)-7 \\ &= -8-7 \\ &= -15 \end{aligned}$$

? Exercise 1.1.9

Evaluate the expression $11-3y$ for each value for y.

a. $y=2$
b. $y=0$
c. $y=\frac{2}{3}$
d. $y=-5$

Answer

1. 11
2. 26

Example 1.1.11: Evaluating Algebraic Expressions

Evaluate each expression for the given values.

a. $x+5$ for $x=-5$
b. $\dfrac{t}{2t-1}$ for $t=10$
c. $\dfrac{4}{3}\pi r^3$ for $r=5$
d. $a+ab+b$ for $a=11, b=-8$
e. $\sqrt{2m^3n^2}$ for $m=2, n=3$

Solution

a. Substitute -5 for x.

$$\begin{aligned} x+5 &= (-5)+5 \\ &= 0 \end{aligned}$$

b. Substitute 10 for t.

$$\begin{aligned} \frac{t}{2t-1} &= \frac{(10)}{2(10)-1} \\ &= \frac{10}{20-1} \\ &= \frac{10}{19} \end{aligned}$$

c. Substitute 5 for r.

$$\begin{aligned} \frac{4}{3}\pi r^3 &= \frac{4}{3}\pi(5)^3 \\ &= \frac{4}{3}\pi(125) \\ &= \frac{500}{3}\pi \end{aligned}$$

d. Substitute 11 for a and -8 for b.

$$\begin{aligned} a+ab+b &= (11)+(11)(-8)+(-8) \\ &= 11-88-8 \\ &= -85 \end{aligned}$$

e. Substitute 2 for m and 3 for n.

$$\begin{aligned} \sqrt{2m^3n^2} &= \sqrt{2(2)^3(3)^2} \\ &= \sqrt{2(8)(9)} \\ &= \sqrt{144} \\ &= 12 \end{aligned}$$

Exercise 1.1.10

Evaluate each expression for the given values.

a. $\dfrac{y+3}{y-3}$ for $y=5$

b. $7-2t$ for $t=-2$

c. $\dfrac{1}{3}\pi r^2$ for $r=11$

d. $(p^2q)^3$ for $p=-2, q=3$

e. $4(m-n)-5(n-m)$ for $m=\dfrac{2}{3}$ $n=\dfrac{1}{3}$

Answer

a. 4
b. 11
c. $\dfrac{121}{3}\pi$
d. 1728
e. 3

Formulas

An equation is a mathematical statement indicating that two expressions are equal. The expressions can be numerical or algebraic. The equation is not inherently true or false, but only a proposition. The values that make the equation true, the solutions, are found using the properties of real numbers and other results. For example, the equation $2x+1=7$ has the unique solution of 3 because when we substitute 3 for x in the equation, we obtain the true statement $2(3)+1=7$.

A formula is an equation expressing a relationship between constant and variable quantities. Very often, the equation is a means of finding the value of one quantity (often a single variable) in terms of another or other quantities. One of the most common examples is the formula for finding the area A of a circle in terms of the radius r of the circle: $A=\pi r^2$. For any value of r, the area A can be found by evaluating the expression πr^2.

Example 1.1.12: Using a Formula

A right circular cylinder with radius r and height h has the surface area S (in square units) given by the formula $S=2\pi r(r+h)$. See Figure 1.1.3. Find the surface area of a cylinder with radius 6 in. and height 9 in. Leave the answer in terms of π.

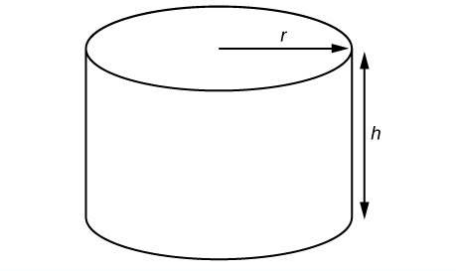


Figure 1.1.3

Evaluate the expression $2\pi r(r+h)$ for $r=6$ and $h=9$.

Solution

$$\begin{aligned} S &= 2\pi r(r+h) \\ &= 2\pi(6)[(6)+(9)] \\ &= 2\pi(6)(15) \\ &= 180\pi \end{aligned}$$

The surface area is 180π square inches.

Exercise 1.1.11

A photograph with length L and width W is placed in a matte of width 8 centimeters (cm). The area of the matte (in square centimeters, or cm^2 is found to be $A = (L + 16)(W + 16) - L \cdot W$. See Figure 1.1.4. Find the area of a matte for a photograph with length 32cm and width 24cm.

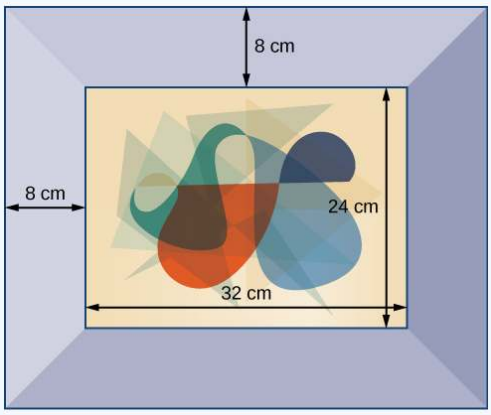


Figure 1.1.4

Answer

$1152cm^2$

Simplifying Algebraic Expressions

Sometimes we can simplify an algebraic expression to make it easier to evaluate or to use in some other way. To do so, we use the properties of real numbers. We can use the same properties in formulas because they contain algebraic expressions.

Example 1.1.13: Simplifying Algebraic Expressions

Simplify each algebraic expression.

a. $3x - 2y + x - 3y - 7$
b. $2r - 5(3 - r) + 4$
c. $\left(4t - \frac{5}{4}s\right) - \left(\frac{2}{3}t + 2s\right)$
d. $2mn - 5m + 3mn + n$

Solution

a.
$$\begin{aligned} 3x - 2y + x - 3y - 7 &= 3x + x - 2y - 3y - 7 && \text{Commutative property of addition} \\ &= 4x - 5y - 7 && \text{Simplify} \end{aligned}$$

b.
$$\begin{aligned} 2r - 5(3 - r) + 4 &= 2r - 15 + 5r + 4 && \text{Distributive property} \\ &= 2r + 5y - 15 + 4 && \text{Commutative property of addition} \\ &= 7r - 11 && \text{Simplify} \end{aligned}$$

c.
$$\begin{aligned} \left(4t - \frac{5}{4}s\right) - \left(\frac{2}{3}t + 2s\right) &= 4t - \frac{5}{4}s - \frac{2}{3}t - 2s && \text{Distributive property} \\ &= 4t - \frac{2}{3}t - \frac{5}{4}s - 2s && \text{Commutative property of addition} \\ &= \frac{10}{3}t - \frac{13}{4}s && \text{Simplify} \end{aligned}$$

d.
$$\begin{aligned} 2mn - 5m + 3mn + n &= 2mn + 3mn - 5m + n && \text{Commutative property of addition} \\ &= 5mn - 5m + n && \text{Simplify} \end{aligned}$$

Exercise 1.1.12

Simplify each algebraic expression.

a. $\frac{2}{3}y - 2\left(\frac{4}{3}y + z\right)$
b. $\frac{5}{t} - 2 - \frac{3}{t} + 1$
c. $4p(q-1) + q(1-p)$
d. $9r - (s + 2r) + (6 - s)$

Answer

a. $-2y - 2z$ or $-2(y+z)$
b. $\frac{2}{t} - 1$
c. $3pq - 4p + q$
d. $7r - 2s + 6$

Example 1.1.14: Simplifying a Formula

A rectangle with length L and width W has a perimeter P given by $P = L + W + L + W$. Simplify this expression.

Solution

$P = L + W + L + W$	
$P = L + L + W + W$	Commutative property of addition
$P = 2L + 2W$	Simplify
$P = 2(L + W)$	Distributive property

Exercise 1.1.13

If the amount P is deposited into an account paying simple interest r for time t, the total value of the deposit A is given by $A = P + Prt$. Simplify the expression. (This formula will be explored in more detail later in the course.)

Answer

$A = P(1 + rt)$

Access these online resources for additional instruction and practice with real numbers.

- Simplify an Expression
- Evaluate an Expression1
- Evaluate an Expression2

Key Concepts

- Rational numbers may be written as fractions or terminating or repeating decimals. See Example and Example.
- Determine whether a number is rational or irrational by writing it as a decimal. See Example.
- The rational numbers and irrational numbers make up the set of real numbers. See Example. A number can be classified as natural, whole, integer, rational, or irrational. See Example.
- The order of operations is used to evaluate expressions. See Example.
- The real numbers under the operations of addition and multiplication obey basic rules, known as the properties of real numbers. These are the commutative properties, the associative properties, the distributive property, the identity properties, and the inverse properties. See Example.
- Algebraic expressions are composed of constants and variables that are combined using addition, subtraction, multiplication, and division. See Example. They take on a numerical value when evaluated by replacing variables with constants. See Example,Example, and Example
- Formulas are equations in which one quantity is represented in terms of other quantities. They may be simplified or evaluated as any mathematical expression. See Example and Example.

Glossary

algebraic expression
constants and variables combined using addition, subtraction, multiplication, and division

associative property of addition
the sum of three numbers may be grouped differently without affecting the result; in symbols,a+(b+c)=(a+b)+c

associative property of multiplication
the product of three numbers may be grouped differently without affecting the result; in symbols,$a \cdot (b \cdot c) = (a \cdot b) \cdot c$

base
in exponential notation, the expression that is being multiplied

commutative property of addition
two numbers may be added in either order without affecting the result; in symbols,$a+b=b+a$

commutative property of multiplication
two numbers may be multiplied in any order without affecting the result; in symbols,$a \cdot b = b \cdot a$

constant
a quantity that does not change value

distributive property
the product of a factor times a sum is the sum of the factor times each term in the sum; in symbols,$a \cdot (b+c) = a \cdot b + a \cdot c$

equation
a mathematical statement indicating that two expressions are equal

exponent
in exponential notation, the raised number or variable that indicates how many times the base is being multiplied

exponential notation
a shorthand method of writing products of the same factor

formula
an equation expressing a relationship between constant and variable quantities

identity property of addition
there is a unique number, called the additive identity, 0, which, when added to a number, results in the original number; in symbols,$a+0=a$

identity property of multiplication
there is a unique number, called the multiplicative identity, 1, which, when multiplied by a number, results in the original number; in symbols,$a \cdot 1 = a$

integers
the set consisting of the natural numbers, their opposites, and 0:$\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$

inverse property of addition
for every real number$a$,there is a unique number, called the additive inverse (or opposite), denoted$-a$,which, when added to the original number, results in the additive identity, 0; in symbols,$a+(-a)=0$

inverse property of multiplication
for every non-zero real number$a$,there is a unique number, called the multiplicative inverse (or reciprocal), denoted$\frac{1}{a}$,which, when multiplied by the original number, results in the multiplicative identity, 1; in symbols,$a \cdot \frac{1}{a} = 1$

irrational numbers
the set of all numbers that are not rational; they cannot be written as either a terminating or repeating decimal; they cannot be expressed as a fraction of two integers

natural numbers
the set of counting numbers:$\{1,2,3,\ldots\}$

order of operations
a set of rules governing how mathematical expressions are to be evaluated, assigning priorities to operations

rational numbers
the set of all numbers of the form$\frac{m}{n}$,where$m$and$n$are integers and$n \neq 0$.Any rational number may be written as a fraction or a terminating or repeating decimal.

real number line
a horizontal line used to represent the real numbers. An arbitrary fixed point is chosen to represent 0; positive numbers lie to the right of 0 and negative numbers to the left.

real numbers
the sets of rational numbers and irrational numbers taken together

variable

a quantity that may change value

whole numbers

the set consisting of 0 plus the natural numbers: $\{0,1,2,3,\ldots\}$

Supplemental Help videos

Introduction to order of operations:



Homework Exercises 1.1

The following WeBWorK problems will be auto-graded. After submitting them, complete the problems that follow on paper to submit in class. Make sure to write down essential information from the problems and all of your work including short explanations. Leave space on your paper for instructor feedback.

Written Homework

1. What do the Associative Properties allow us to do when following the order of operations? Explain your answer with examples.

2. Simplify $-2 \times [16 \div (8-4)^2]^2$

3. Simplify $4^2 - 25 \times \frac{1}{5}$

4. Simplify $9 - 18 \div 3^2$

5. Solve $4z - 2z(1+4) - 36$ for $z = 5$

6. Simplify $8b - 4b(3) + 1$

7. Simplify $18y - 2(1+7y)$

8. Simplify $9x + 4x(2+5) - 4(2x+3x)$

9. Simplify $\frac{a}{2^3}(64) - 12a \div 6$

10. Jessica and her roommate, Adriana, have decided to share a change jar for joint expenses. Jessica put her loose change in the jar first, and then Adriana put her change in the jar. We know that it does not matter in which order the change was added to the jar. What property of addition describes this fact?

11. If a whole number is not a natural number, what must the number be?

12. Simplify the expression and state whether the number is real or not real. If it is real, state whether the number is rational or irrational: $\sqrt{-18-4(5)(-1)}$.

13. What property of real numbers would simplify the following expression: $4+7(x-1)$?

This page titled 1.1: Real Numbers - Algebra Essentials is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **1.1: Real Numbers - Algebra Essentials** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/precalculus.

1.2: Exponents and Scientific Notation

Preview Video for Sections 1.2 & 1.3

Please view, answer the "quick check" questions, and fill in your notes before class.

Learning Objectives

- Various rules of Exponents
- Scientific Notation

Mathematicians, scientists, and economists commonly encounter very large and very small numbers. But it may not be obvious how common such figures are in everyday life. For instance, a pixel is the smallest unit of light that can be perceived and recorded by a digital camera. A particular camera might record an image that is $2,048$ pixels by $1,536$ pixels, which is a very high resolution picture. It can also perceive a color depth (gradations in colors) of up to 48 bits per frame, and can shoot the equivalent of 24 frames per second. The maximum possible number of bits of information used to film a one-hour ($3,600$-second) digital film is then an extremely large number.

Using a calculator, we enter $2,048 \times 1,536 \times 48 \times 24 \times 3,600$ and press ENTER. The calculator displays $1.304596316E13$ What does this mean? The "$E13$" portion of the result represents the exponent 13 of ten, so there are a maximum of approximately 1.3×10^{13} bits of data in that one-hour film. In this section, we review rules of exponents first and then apply them to calculations involving very large or small numbers.

Using the Product Rule of Exponents

Consider the product $x^3 \times x^4$. Both terms have the same base, x, but they are raised to different exponents. Expand each expression, and then rewrite the resulting expression.

$$\begin{aligned} x^3 \times x^4 &= \overbrace{x \times x \times x}^{3 \text{ factors}} \times \overbrace{x \times x \times x \times x}^{4 \text{ factors}} \\ &= \overbrace{x \times x \times x \times x \times x \times x \times x}^{7 \text{ factors}} \\ &= x^7 \end{aligned}$$

The result is that $x^3 \times x^4 = x^{3+4} = x^7$.

Notice that the exponent of the product is the sum of the exponents of the terms. In other words, when multiplying exponential expressions with the same base, we write the result with the common base and add the exponents. This is the product rule of exponents.

$$a^m \times a^n = a^{m+n} \tag{1.2.1}$$

Now consider an example with real numbers.

$$2^3 \times 2^4 = 2^{3+4} = 2^7$$

We can always check that this is true by simplifying each exponential expression. We find that 2^3 is 8, 2^4 is 16, and 2^7 is 128. The product 8×16 equals 128, so the relationship is true. We can use the product rule of exponents to simplify expressions that are a product of two numbers or expressions with the same base but different exponents.

THE PRODUCT RULE OF EXPONENTS

For any real number a and natural numbers m and n, the product rule of exponents states that

$$a^m \times a^n = a^{m+n} \tag{1.2.2}$$

Example 1.2.1: Using the Product Rule

Write each of the following products with a single base. Do not simplify further.

a. $t^5 \times t^3$
b. $(-3)^5 \times (-3)$
c. $x^2 \times x^5 \times x^3$

Solution

Use the product rule (Equation 1.2.2) to simplify each expression.

a. $t^5 \times t^3 = t^{5+3} = t^8$
b. $(-3)^5 \times (-3) = (-3)^5 \times (-3)^1 = (-3)^{5+1} = (-3)^6$
c. $x^2 \times x^5 \times x^3$

At first, it may appear that we cannot simplify a product of three factors. However, using the associative property of multiplication, begin by simplifying the first two.

$$x^2 \times x^5 \times x^3 = (x^2 \times x^5) \times x^3 = (x^{2+5}) \times x^3 = x^7 \times x^3 = x^{7+3} = x^{10}$$

Notice we get the same result by adding the three exponents in one step.

$$x^2 \times x^5 \times x^3 = x^{2+5+3} = x^{10}$$

Exercise 1.2.1

Write each of the following products with a single base. Do not simplify further.

a. $k^6 \times k^9$
b. $\left(\frac{2}{y}\right)^4 \times \left(\frac{2}{y}\right)$
c. $t^3 \times t^6 \times t^5$

Answer a

k^{15}

Answer b

$\left(\frac{2}{y}\right)^5$

Answer c

t^{14}

Using the Quotient Rule of Exponents

The quotient rule of exponents allows us to simplify an expression that divides two numbers with the same base but different exponents. In a similar way to the product rule, we can simplify an expression such as $\frac{y^m}{y^n}$, where $m > n$. Consider the example $\frac{y^9}{y^5}$. Perform the division by canceling common factors.

$$\begin{aligned} \frac{y^9}{y^5} &= \frac{y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y}{y \cdot y \cdot y \cdot y \cdot y} \\ &= \frac{y \cdot y \cdot y \cdot y}{1} \\ &= y^4 \end{aligned}$$

Notice that the exponent of the quotient is the difference between the exponents of the divisor and dividend.

$$\frac{a^m}{a^n} = a^{m-n} \tag{1.2.3}$$

In other words, when dividing exponential expressions with the same base, we write the result with the common base and subtract the exponents.

$$\frac{y^9}{y^5} = y^{9-5} = y^4$$

For the time being, we must be aware of the condition $m > n$. Otherwise, the difference $m - n$ could be zero or negative. Those possibilities will be explored shortly. Also, instead of qualifying variables as nonzero each time, we will simplify matters and assume from here on that all variables represent nonzero real numbers.

THE QUOTIENT RULE OF EXPONENTS

For any real number a and natural numbers m and n, such that $m > n$, the quotient rule of exponents states that

$$\frac{a^m}{a^n} = a^{m-n} \tag{1.2.4}$$

Example 1.2.2: Using the Quotient Rule

Write each of the following products with a single base. Do not simplify further.

a. $\frac{(-2)^{14}}{(-2)^9}$
b. $\frac{t^{23}}{t^{15}}$
c. $\frac{(z\sqrt{2})^5}{z\sqrt{2}}$

Solution

Use the quotient rule (Equation 1.2.4) to simplify each expression.

a. $\frac{(-2)^{14}}{(-2)^9} = (-2)^{14-9} = (-2)^5$
b. $\frac{t^{23}}{t^{15}}$=t^{23−15}=t^8\)
c. $\frac{(z\sqrt{2})^5}{z\sqrt{2}} = (z\sqrt{2})^{5-1} = (z\sqrt{2})^4$

Exercise 1.2.2

Write each of the following products with a single base. Do not simplify further.

a. $\frac{s^{75}}{s^{68}}$
b. $\frac{(-3)^6}{-3}$
c. $\frac{(ef^2)^5}{(ef^2)^3}$

Answer a

s^7

Answer b

$(-3)^5$

Answer c

$(ef^2)^2$

Using the Power Rule of Exponents

Suppose an exponential expression is raised to some power. Can we simplify the result? Yes. To do this, we use the power rule of exponents. Consider the expression $(x^2)^3$. The expression inside the parentheses is multiplied twice because it has an exponent of 2. Then the result is multiplied three times because the entire expression has an exponent of 3.

$$\begin{aligned}(x^2)^3 &= (x^2)\times(x^2)\times(x^2)\\ &= x\times x\times x\times x\times x\times x\\ &= x^6\end{aligned}$$

The exponent of the answer is the product of the exponents: $(x^2)^3 = x^{2\cdot 3} = x^6$. In other words, when raising an exponential expression to a power, we write the result with the common base and the product of the exponents.

$$(a^m)^n = a^{m\cdot n} \tag{1.2.5}$$

Be careful to distinguish between uses of the product rule and the power rule. When using the product rule, different terms with the same bases are raised to exponents. In this case, you add the exponents. When using the power rule, a term in exponential notation is raised to a power. In this case, you multiply the exponents.

Table 1.2.1

Product Rule	Power Rule
$5^3\times 5^4 = 5^{3+4} = 5^7$	$(5^3)^4 = 5^{3\times 4} = 5^{12}$
$x^5\times x^2 = x^{5+2} = x^7$	$(x^5)^2 = x^{5\times 2} = x^{10}$
$(3a)^7\times(3a)^{10} = (3a)^{7+10} = (3a)^{17}$	$((3a)^7)^{10} = (3a)^{7\times 10} = (3a)^{70}$

THE POWER RULE OF EXPONENTS

For any real number a and positive integers m and n, the power rule of exponents states that

$$(a^m)^n = a^{m\cdot n} \tag{1.2.6}$$

Example 1.2.3: Using the Power Rule

Write each of the following products with a single base. Do not simplify further.

a. $(x^2)^7$
b. $((2t)^5)^3$
c. $((-3)^5)^{11}$

Solution

Use the power rule (Equation 1.2.6) to simplify each expression.

a. $(x^2)^7 = x^{2\cdot 7} = x^{14}$
b. $((2t)^5)^3 = (2t)^{5\cdot 3} = (2t)^{15}$
c. $((-3)^5)^{11} = (-3)^{5\cdot 11} = (-3)^{55}$

Exercise 1.2.3

Write each of the following products with a single base. Do not simplify further.

a. $((3y)^8)^3$
b. $(t^5)^7$
c. $((-g)^4)^4$

Answer a

$$(3y)^{24}$$

Answer b

$$t^{35}$$

Answer c

$$(-g)^{16}$$

Using the Zero Exponent Rule of Exponents

Return to the quotient rule. We made the condition that $m > n$ so that the difference $m - n$ would never be zero or negative. What would happen if $m = n$? In this case, we would use the zero exponent rule of exponents to simplify the expression to 1. To see how this is done, let us begin with an example.

$$\frac{t^8}{t^8} = 1$$

If we were to simplify the original expression using the quotient rule, we would have

$$\frac{t^8}{t^8} = t^{8-8} = t^0$$

If we equate the two answers, the result is $t^0 = 1$. This is true for any nonzero real number, or any variable representing a real number.

$$a^0 = 1$$

The sole exception is the expression 0^0. This appears later in more advanced courses, but for now, we will consider the value to be undefined.

THE ZERO EXPONENT RULE OF EXPONENTS

For any nonzero real number a, the zero exponent rule of exponents states that

$$a^0 = 1 \tag{1.2.7}$$

Example 1.2.4: Using the Zero Exponent Rule

Simplify each expression using the zero exponent rule of exponents.

a. $\frac{c^3}{c^3}$
b. $\frac{-3x^5}{x^5}$

c. $\dfrac{(j^2k)^4}{(j^2k)\times(j^2k)^3}$

d. $\dfrac{5(rs^2)^2}{(rs^2)^2}$

Solution

Use the zero exponent and other rules to simplify each expression.

a.

$$\begin{aligned}\frac{c^3}{c^3} &= c^{3-3}\\ &= c^0\\ &= 1\end{aligned}$$

b.

$$\begin{aligned}\frac{-3x^5}{x^5} &= -3\times\frac{x^5}{x^5}\\ &= -3\times x^{5-5}\\ &= -3\times x^0\\ &= -3\times 1\\ &= -3\end{aligned}$$

c.

$$\begin{aligned}\frac{(j^2k)^4}{(j^2k)\times(j^2k)^3} &= \frac{(j^2k)^4}{(j^2k)^{1+3}} && \text{Use the product rule in the denominator}\\ &= \frac{(j^2k)^4}{(j^2k)^4} && \text{Simplify}\\ &= (j^2k)^{4-4} && \text{Use the quotient rule}\\ &= (j^2k)^0 && \text{Simplify}\\ &= 1\end{aligned}$$

d.

$$\begin{aligned}\frac{5(rs^2)^2}{(rs^2)^2} &= 5(rs^2)^{2-2} && \text{Use the quotient rule}\\ &= 5(rs^2)^0 && \text{Simplify}\\ &= 5\times 1 && \text{Use the zero exponent rule}\\ &= 5 && \text{Simplify}\end{aligned}$$

? Exercise 1.2.4

Simplify each expression using the zero exponent rule of exponents.

a. $\dfrac{t^7}{t^7}$

b. $\dfrac{(de^2)^{11}}{2(de^2)^{11}}$

c. $\dfrac{w^4\times w^2}{w^6}$

d. $\dfrac{t^3\times t^4}{t^2\times t^5}$

Answer a

1

Answer b

$\frac{1}{2}$

Answer c

1

Answer d

1

Using the Negative Rule of Exponents

Another useful result occurs if we relax the condition that $m > n$ in the quotient rule even further. For example, can we simplify $\frac{t^3}{t^5}$? When $m < n$ —that is, where the difference $m - n$ is negative—we can use the negative rule of exponents to simplify the expression to its reciprocal.

Divide one exponential expression by another with a larger exponent. Use our example, $\frac{t^3}{t^5}$.

$$\begin{aligned}\frac{t^3}{t^5} &= \frac{t \times t \times t}{t \times t \times t \times t \times t} \\ &= \frac{1}{t \times t} \\ &= \frac{1}{t^2}\end{aligned}$$

If we were to simplify the original expression using the quotient rule, we would have

$$\begin{aligned}\frac{t^3}{t^5} &= t^{3-5} \\ &= t^{-2}\end{aligned}$$

Putting the answers together, we have $t^{-2} = \frac{1}{t^2}$. This is true for any nonzero real number, or any variable representing a nonzero real number.

A factor with a negative exponent becomes the same factor with a positive exponent if it is moved across the fraction bar—from numerator to denominator or vice versa.

$$a^{-n} = \frac{1}{a^n} \text{ and } a^n = \frac{1}{a^{-n}}$$

We have shown that the exponential expression an is defined when n is a natural number, 0, or the negative of a natural number. That means that an is defined for any integer n. Also, the product and quotient rules and all of the rules we will look at soon hold for any integer n.

THE NEGATIVE RULE OF EXPONENTS

For any nonzero real number a and natural number n, the negative rule of exponents states that

$$a^{-n} = \frac{1}{a^n} \tag{1.2.8}$$

Example 1.2.5: Using the Negative Exponent Rule

Write each of the following quotients with a single base. Do not simplify further. Write answers with positive exponents.

a. $\frac{\theta^3}{\theta^{10}}$

b. $\dfrac{z^2 \times z}{z^4}$

c. $\dfrac{(-5t^3)^4}{(-5t^3)^8}$

Solution

a. $\dfrac{\theta^3}{\theta^{10}} = \theta^{3-10} = \theta^{-7} = \dfrac{1}{\theta^7}$

b. $\dfrac{z^2 \times z}{z^4} = \dfrac{z^{2+1}}{z^4} = \dfrac{z^3}{z^4} = z^{3-4} = z^{-1} = \dfrac{1}{z}$

c. $\dfrac{(-5t^3)^4}{(-5t^3)^8} = (-5t^3)^{4-8} = (-5t^3)^{-4} = \dfrac{1}{(-5t^3)^4}$

Exercise 1.2.5

Write each of the following quotients with a single base. Do not simplify further. Write answers with positive exponents.

a. $\dfrac{(-3t)^2}{(-3t)^8}$

b. $\dfrac{f^{47}}{f^{49} \times f}$

c. $\dfrac{2k^4}{5k^7}$

Answer a

$$\frac{1}{(-3t)^6}$$

Answer b

$$\frac{1}{f^3}$$

Answer c

$$\frac{2}{5k^3}$$

Example 1.2.6: Using the Product and Quotient Rules

Write each of the following products with a single base. Do not simplify further. Write answers with positive exponents.

a. $b^2 \times b^{-8}$

b. $(-x)^5 \times (-x)^{-5}$

c. $\dfrac{-7z}{(-7z)^5}$

Solution

a. $b^2 \times b^{-8} = b^{2-8} = b^{-6} = \dfrac{1}{b^6}$

b. $(-x)^5 \times (-x)^{-5} = (-x)^{5-5} = (-x)^0 = 1$

c. $\dfrac{-7z}{(-7z)^5} = \dfrac{(-7z)^1}{(-7z)^5} = (-7z)^{1-5} = (-7z)^{-4} = \dfrac{1}{(-7z)^4}$

Exercise 1.2.6

Write each of the following products with a single base. Do not simplify further. Write answers with positive exponents.

a. $t^{-11} \times t^6$

b. $\dfrac{25^{12}}{25^{13}}$

Answer a

$$t^{-5} = \frac{1}{t^5}$$

Answer b

$$\frac{1}{25}$$

Finding the Power of a Product

To simplify the power of a product of two exponential expressions, we can use the power of a product rule of exponents, which breaks up the power of a product of factors into the product of the powers of the factors. For instance, consider $(pq)^3$. We begin by using the associative and commutative properties of multiplication to regroup the factors.

$$\begin{aligned}(pq)^3 &= (pq) \times (pq) \times (pq) \\ &= p \times q \times p \times q \times p \times q \\ &= p^3 \times q^3\end{aligned}$$

In other words, $(pq)^3 = p^3 \times q^3$.

THE POWER OF A PRODUCT RULE OF EXPONENTS

For any real numbers a and b and any integer n, the power of a product rule of exponents states that

$$(ab)^n = a^n b^n \tag{1.2.9}$$

Example 1.2.7: Using the Power of a Product Rule

Simplify each of the following products as much as possible using the power of a product rule. Write answers with positive exponents.

a. $(ab^2)^3$

b. $(2t)^{15}$

c. $(-2w^3)^3$

d. $\dfrac{1}{(-7z)^4}$

e. $(e^{-2}f^2)^7$

Solution

Use the product and quotient rules and the new definitions to simplify each expression.

a. $(ab^2)^3 = (a)^3 \times (b^2)^3 = a^{1\times 3} \times b^{2\times 3} = a^3 b^6$

b. $(2t)^{15} = (2)^{15} \times (t)^{15} = 2^{15} t^{15} = 32,768 t^{15}$

c. $(-2w^3)^3 = (-2)^3 \times (w^3)^3 = -8 \times w^{3\times 3} = -8w^9$

d. $\dfrac{1}{(-7z)^4} = \dfrac{1}{(-7)^4 \times (z)^4} = \dfrac{1}{2401 z^4}$

e. $(e^{-2}f^2)^7 = (e^{-2})^7 \times (f^2)^7 = e^{-2\times 7} \times f^{2\times 7} = e^{-14}f^{14} = \dfrac{f^{14}}{e^{14}}$

Exercise 1.2.7

Simplify each of the following products as much as possible using the power of a product rule. Write answers with positive exponents.

a. $(g^2h^3)^5$
b. $(5t)^3$
c. $(-3y^5)^3$
d. $\dfrac{1}{(a^6b^7)^3}$
e. $(r^3s^{-2})^4$

Answer a

$g^{10}h^{15}$

Answer b

$125t^3$

Answer c

$-27y^{15}$

Answer d

$\dfrac{1}{a^{18}b^{21}}$

Answer e

$\dfrac{r^{12}}{s^8}$

Finding the Power of a Quotient

To simplify the power of a quotient of two expressions, we can use the power of a quotient rule, which states that the power of a quotient of factors is the quotient of the powers of the factors. For example, let's look at the following example.

$$(e^{-2}f^2)^7 = \frac{f^{14}}{e^{14}} \tag{1.2.10}$$

Let's rewrite the original problem differently and look at the result.

$$\begin{aligned}(e^{-2}f^2)^7 &= \left(\frac{f^2}{e^2}\right)^7 \\ &= \frac{f^{14}}{e^{14}}\end{aligned}$$

It appears from the last two steps that we can use the power of a product rule as a power of a quotient rule.

$$\begin{aligned}(e^{-2}f^2)^7 &= \left(\frac{f^2}{e^2}\right)^7 \\ &= \frac{(f^2)^7}{(e^2)^7} \\ &= \frac{f^{2\times 7}}{e^{2\times 7}} \\ &= \frac{f^{14}}{e^{14}}\end{aligned}$$

THE POWER OF A QUOTIENT RULE OF EXPONENTS

For any real numbers a and b and any integer n, the power of a quotient rule of exponents states that

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \tag{1.2.11}$$

Example 1.2.8: Using the Power of a Quotient Rule

Simplify each of the following quotients as much as possible using the power of a quotient rule. Write answers with positive exponents.

a. $\left(\frac{4}{z^{11}}\right)^3$
b. $\left(\frac{p}{q^3}\right)^6$
c. $\left(\frac{-1}{t^2}\right)^{27}$
d. $(j^3k^{-2})^4$
e. $(m^{-2}n^{-2})^3$

Solution

a. $\left(\frac{4}{z^{11}}\right)^3 = \frac{(4)^3}{(z^{11})^3} = \frac{64}{z^{11\times 3}} = \frac{64}{z^{33}}$

b. $\left(\frac{p}{q^3}\right)^6 = \frac{(p)^6}{(q^3)^6} = \frac{p^{1\times 6}}{q^{3\times 6}} = \frac{p^6}{q^{18}}$

c. $\left(\frac{-1}{t^2}\right)^{27} = \frac{(-1)^{27}}{(t^2)^{27}} = \frac{-1}{t^{2\times 27}} = \frac{-1}{t^{54}} = -\frac{1}{t^{54}}$

d. $(j^3k^{-2})^4 = \left(\frac{j^3}{k^2}\right)^4 = \frac{(j^3)^4}{(k^2)^4} = \frac{j^{3\times 4}}{k^{2\times 4}} = \frac{j^{12}}{k^8}$

e. $(m^{-2}n^{-2})^3 = \left(\frac{1}{m^2n^2}\right)^3 = \frac{(1)^3}{(m^2n^2)^3} = \frac{1}{(m^2)^3(n^2)^3} = \frac{1}{m^{2\times 3}n^{2\times 3}} = \frac{1}{m^6n^6}$

Exercise 1.2.8

Simplify each of the following quotients as much as possible using the power of a quotient rule. Write answers with positive exponents.

a. $\left(\frac{b^5}{c}\right)^3$
b. $\left(\frac{5}{u^8}\right)^4$
c. $\left(\frac{-1}{w^3}\right)^{35}$
d. $(p^{-4}q^3)^8$
e. $(c^{-5}d^{-3})^4$

Answer a

$$\frac{b^{15}}{c^3}$$

Answer b

$$\frac{625}{u^{32}}$$

Answer c

$$\frac{-1}{w^{105}}$$

Answer d

$$\frac{q^{24}}{p^{32}}$$

Answer e

$$\frac{1}{c^{20}d^{12}}$$

Simplifying Exponential Expressions

Recall that to simplify an expression means to rewrite it by combing terms or exponents; in other words, to write the expression more simply with fewer terms. The rules for exponents may be combined to simplify expressions.

Example 1.2.9: Simplifying Exponential Expressions

Simplify each expression and write the answer with positive exponents only.

a. $(6m^2n^{-1})^3$
b. $17^5 \times 17^{-4} \times 17^{-3}$
c. $\left(\frac{u^{-1}v}{v^{-1}}\right)^2$
d. $(-2a^3b^{-1})(5a^{-2}b^2)$
e. $(x^2\sqrt{2})^4(x^2\sqrt{2})^{-4}$
f. $\frac{(3w^2)^5}{(6w^{-2})^2}$

Solution

a.

$$\begin{aligned}(6m^2n^{-1})^3 &= (6)^3(m^2)^3(n^{-1})^3 && \text{The power of a product rule}\\ &= 6^3m^{2\times3}n^{-1\times3} && \text{The power rule}\\ &= 216m^6n^{-3} && \text{The power rule}\\ &= \frac{216m^6}{n^3} && \text{The negative exponent rule}\end{aligned}$$

b.

$$\begin{aligned}17^5 \times 17^{-4} \times 17^{-3} &= 17^{5-4-3} && \text{The product rule}\\ &= 17^{-2} && \text{Simplify}\\ &= \frac{1}{17^2} \text{ or } \frac{1}{289} && \text{The negative exponent rule}\end{aligned}$$

c.

$$\begin{aligned}\left(\frac{u^{-1}v}{v^{-1}}\right)^2 &= \frac{(u^{-1}v)^2}{(v^{-1})^2} && \text{The power of a quotient rule}\\ &= \frac{u^{-2}v^2}{v^{-2}} && \text{The power of a product rule}\\ &= u^{-2}v^{2-(-2)} && \text{The quotient rule}\\ &= u^{-2}v^4 && \text{Simplify}\\ &= \frac{v^4}{u^2} && \text{The negative exponent rule}\end{aligned}$$

d.

$$\begin{aligned}\left(-2a^3b^{-1}\right)\left(5a^{-2}b^2\right) &= \left(x^2\sqrt{2}\right)^{4-4} && \text{Commutative and associative laws of multiplication}\\ &= -10\times a^{3-2}\times b^{-1+2} && \text{The product rule}\\ &= -10ab && \text{Simplify}\end{aligned}$$

e.

$$\begin{aligned}\left(x^2\sqrt{2}\right)^4\left(x^2\sqrt{2}\right)^{-4} &= \left(x^2\sqrt{2}\right)^{4-4} && \text{The product rule}\\ &= \left(x^2\sqrt{2}\right)^0 && \text{Simplify}\\ &= 1 && \text{The zero exponent rule}\end{aligned}$$

f.

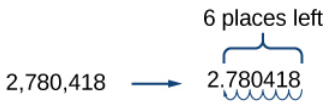
$$\begin{aligned}\frac{(3w^2)^5}{(6w^{-2})^2} &= \frac{(3)^5\times(w^2)^5}{(6)^2\times(w^{-2})^2} && \text{The power of a product rule}\\ &= \frac{3^5w^{2\times5}}{6^2w^{-2\times2}} && \text{The power rule}\\ &= \frac{243w^{10}}{36w^{-4}} && \text{Simplify}\\ &= \frac{27w^{10-(-4)}}{4} && \text{The quotient rule and reduce fraction}\\ &= \frac{27w^{14}}{4} && \text{Simplify}\end{aligned}$$

Using Scientific Notation

Recall at the beginning of the section that we found the number 1.3×10^{13} when describing bits of information in digital images. Other extreme numbers include the width of a human hair, which is about $0.00005\ m$, and the radius of an electron, which is about $0.00000000000047\ m$How can we effectively work read, compare, and calculate with numbers such as these?

A shorthand method of writing very small and very large numbers is called scientific notation, in which we express numbers in terms of exponents of 10. To write a number in scientific notation, move the decimal point to the right of the first digit in the number. Write the digits as a decimal number between 1 and 10. Count the number of places n that you moved the decimal point. Multiply the decimal number by 10 raised to a power of n. If you moved the decimal left as in a very large number, n is positive. If you moved the decimal right as in a small large number, n is negative.

For example, consider the number $2,780,418$ Move the decimal left until it is to the right of the first nonzero digit, which is 2.



We obtain $2.780418$by moving the decimal point 6 places to the left. Therefore, the exponent of 10 is 6, and it is positive because we moved the decimal point to the left. This is what we should expect for a large number.

$$2.780418\times10^6$$

Working with small numbers is similar. Take, for example, the radius of an electron, $0.00000000000047\ m$Perform the same series of steps as above, except move the decimal point to the right.



Be careful not to include the leading 0 in your count. We move the decimal point 13 places to the right, so the exponent of 10 is 13. The exponent is negative because we moved the decimal point to the right. This is what we should expect for a small number.

$$4.7 \times 10^{-13}$$

SCIENTIFIC NOTATION

A number is written in scientific notation if it is written in the form $a \times 10^n$, where $1 \leq |a| < 10$ and n is an integer.

Example 1.2.10: Converting Standard Notation to Scientific Notation

Write each number in scientific notation.

a. Distance to Andromeda Galaxy from Earth: $24,000,000,000,000,000,000,000\ m$
b. Diameter of Andromeda Galaxy: $1,300,000,000,000,000,000,000\ m$
c. Number of stars in Andromeda Galaxy: $1,000,000,000,000$
d. Diameter of electron: $0.00000000000094\ m$
e. Probability of being struck by lightning in any single year: 0.00000143

Solution

a. $24,000,000,000,000,000,000,000\ m$22 places

$2.4 \times 10^{22}\ m$

b. $1,300,000,000,000,000,000,000\ m$21 places

$1.3 \times 10^{21}\ m$

c. $1,000,000,000,000$12 places

1×10^{12}

d. $0.00000000000094\ m$13 places

$9.4 \times 10^{-13}\ m$

e. $0.00000143$6 places

1.43×10^{6}

Analysis

Observe that, if the given number is greater than 1, as in examples a–c, the exponent of 10 is positive; and if the number is less than 1, as in examples d–e, the exponent is negative.

Exercise 1.2.10

Write each number in scientific notation.

a. U.S. national debt per taxpayer (April 2014): $\$152,000$
b. World population (April 2014): $7,158,000,000$
c. World gross national income (April 2014): $\$85,500,000,000,000$
d. Time for light to travel $1\ m$: $0.00000000334\ s$
e. Probability of winning lottery (match 6 of 49 possible numbers): 0.0000000715

Answer a

$\$1.52 \times 10^5$

Answer b

7.158×10^9

Answer c

$\$8.55 \times 10^{13}$

Answer d

3.34×10^{-9}

Answer e

7.15×10^{-8}

Converting from Scientific to Standard Notation

To convert a number in scientific notation to standard notation, simply reverse the process. Move the decimal n places to the right if n is positive or n places to the left if n is negative and add zeros as needed. Remember, if n is positive, the value of the number is greater than 1, and if n is negative, the value of the number is less than one.

Example 1.2.11: Converting Scientific Notation to Standard Notation

Convert each number in scientific notation to standard notation.

a. 3.547×10^{14}
b. -2×10^6
c. 7.91×10^{-7}
d. -8.05×10^{-12}

Solution

a. 3.547×10^{14}

3.54700000000000

$\rightarrow 14$ places

$354,700,000,000,000$

b. -2×10^6

-2.000000

$\rightarrow 6$ places

$-2,000,000$

c. 7.91×10^{-7}

0000007.91

$\rightarrow 7$ places

0.000000791

d. -8.05×10^{-12}

-000000000008.05

$\rightarrow 12$ places

-0.00000000000805

? Exercise 1.2.11

Convert each number in scientific notation to standard notation.

a. 7.03×10^{5}
b. -8.16×10^{11}
c. -3.9×10^{-13}
d. 8×10^{-6}

Answer a

$$703,000$$

Answer b

$$-816,000,000,000$$

Answer c

$$-0.00000000000039$$

Answer d

$$0.000008$$

Using Scientific Notation in Applications

Scientific notation, used with the rules of exponents, makes calculating with large or small numbers much easier than doing so using standard notation. For example, suppose we are asked to calculate the number of atoms in $1\ L$ of water. Each water molecule contains 3 atoms (2 hydrogen and 1 oxygen). The average drop of water contains around 1.32×1021 molecules of water and $1\ L$ of water holds about 1.22×10^{4} average drops. Therefore, there are approximately $3 \cdot (1.32 \times 10^{21}) \cdot (1.22 \times 10^{4}) \approx 4.83 \times 10^{25}$ atoms in $1\ L$ of water. We simply multiply the decimal terms and add the exponents. Imagine having to perform the calculation without using scientific notation!

When performing calculations with scientific notation, be sure to write the answer in proper scientific notation. For example, consider the product $(7 \times 10^{4}) \cdot (5 \times 10^{6}) = 35 \times 10^{10}$. The answer is not in proper scientific notation because 35 is greater than 10. Consider 35 as 3.5×10. That adds a ten to the exponent of the answer.

$$(35) \times 10^{10} = (3.5 \times 10) \times 10^{10} = 3.5 \times (10 \times 10^{10}) = 3.5 \times 10^{11}$$

✓ Example 1.2.12: Using Scientific Notation

Perform the operations and write the answer in scientific notation.

a. $(8.14 \times 10^{-7})(6.5 \times 10^{10})$
b. $(4 \times 10^{5}) \div (-1.52 \times 10^{9})$

c. $(2.7 \times 10^5)(6.04 \times 10^{13})$
d. $(1.2 \times 10^8) \div (9.6 \times 10^5)$
e. $(3.33 \times 10^4)(-1.05 \times 10^7)(5.62 \times 10^5)$

Solutions

a.

$$\begin{aligned}(8.14 \times 10^{-7})(6.5 \times 10^{10}) &= (8.14 \times 6.5)(10^{-7} \times 10^{10}) \text{ Commutative and associative properties of multiplication} \\ &= (52.91)(10^3) \text{ Product rule of exponents} \\ &= 5.291 \times 10^4 \text{ Scientific notation}\end{aligned}$$

b.

$$\begin{aligned}(4 \times 10^5) \div (-1.52 \times 10^9) &= \left(\frac{4}{-1.52}\right)\left(\frac{10^5}{10^9}\right) \text{ Commutative and associative properties of multiplication} \\ &\approx (-2.63)(10^{-4}) \text{ Quotient rule of exponents} \\ &= -2.63 \times 10^{-4} \text{ Scientific notation}\end{aligned}$$

c.

$$\begin{aligned}(2.7 \times 10^5)(6.04 \times 10^{13}) &= (2.7 \times 6.04)(10^5 \times 10^{13}) \text{ Commutative and associative properties of multiplication} \\ &= (16.308)(10^{18}) \text{ Product rule of exponents} \\ &= 1.6308 \times 10^{19} \text{ Scientific notation}\end{aligned}$$

d.

$$\begin{aligned}(1.2 \times 10^8) \div (9.6 \times 10^5) &= \left(\frac{1.2}{9.6}\right)\left(\frac{10^8}{10^5}\right) \text{ Commutative and associative properties of multiplication} \\ &= (0.125)(10^3) \text{ Quotient rule of exponents} \\ &= 1.25 \times 10^2 \text{ Scientific notation}\end{aligned}$$

e.

$$\begin{aligned}(3.33 \times 10^4)(-1.05 \times 10^7)(5.62 \times 10^5) &= [3.33 \times (-1.05) \times 5.62](10^4 \times 10^7 \times 10^5) \\ &\approx (-19.65)(10^{16}) \\ &= -1.965 \times 10^{17}\end{aligned}$$

Exercise 1.2.12

Perform the operations and write the answer in scientific notation.

a. $(-7.5 \times 10^8)(1.13 \times 10^{-2})$
b. $(1.24 \times 10^{11}) \div (1.55 \times 10^{18})$
c. $(3.72 \times 10^9)(8 \times 10^3)$
d. $(9.933 \times 10^{23}) \div (-2.31 \times 10^{17})$
e. $(-6.04 \times 10^9)(7.3 \times 10^2)(-2.81 \times 10^2)$

Answer a

$$-8.475 \times 10^6$$

Answer b

$$8 \times 10^{-8}$$

Answer c

$$2.976 \times 10^{13}$$

Answer d

$$-4.3 \times 10^{6}$$

Answer e

$$\approx 1.24 \times 10^{15}$$

Example 1.2.13: Applying Scientific Notation to Solve Problems

In April 2014, the population of the United States was about $308,000,000$ people. The national debt was about $\$17,547,000,000,000$Write each number in scientific notation, rounding figures to two decimal places, and find the amount of the debt per U.S. citizen. Write the answer in both scientific and standard notations.

Solution

The population was $308,000,000 = 3.08 \times 10^{8}$.

The national debt was $\$17,547,000,000,000 \approx \$1.75 \times 10^{13}$.

To find the amount of debt per citizen, divide the national debt by the number of citizens.

$$\begin{aligned}(1.75 \times 10^{13}) \div (3.08 \times 10^{8}) &= \left(\frac{1.75}{3.08}\right)(10^{5}) \\ &\approx 0.57 \times 10^{5} \\ &= 5.7 \times 10^{4}\end{aligned}$$

The debt per citizen at the time was about $\$5.7 \times 10^{4}$, or $\$57,000$.

Exercise 1.2.13

An average human body contains around $30,000,000,000,000$ red blood cells. Each cell measures approximately $0.000008\ m$long. Write each number in scientific notation and find the total length if the cells were laid end-to-end. Write the answer in both scientific and standard notations.

Answer

Number of cells: 3×10^{13}; length of a cell: $8 \times 10^{-6}\ m$; total length: $2.4 \times 10^{8}\ m$ or $240,000,000\ m$.

Access these online resources for additional instruction and practice with exponents and scientific notation.

Exponential Notation

Properties of Exponents

Zero Exponent

Simplify Exponent Expressions

Quotient Rule for Exponents

Scientific Notation

Key Equations

Rules of Exponents For nonzero real numbers a and b and integers m and n	
Product rule	$a^m \cdot a^n = a^{m+n}$
Quotient rule	$\frac{a^m}{a^n} = a^{m-n}$
Power rule	$(a^m)^n = a^{m \cdot n}$
Zero exponent rule	$a^0 = 1$
Negative rule	$a^{-n} = \frac{1}{a^n}$
Power of a product rule	$(a \cdot b)^n = a^n \cdot b^n$
Power of a quotient rule	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Key Concepts

- Products of exponential expressions with the same base can be simplified by adding exponents. See Example.
- Quotients of exponential expressions with the same base can be simplified by subtracting exponents. See Example.
- Powers of exponential expressions with the same base can be simplified by multiplying exponents. See Example.
- An expression with exponent zero is defined as 1. See Example.
- An expression with a negative exponent is defined as a reciprocal. See Example and Example.
- The power of a product of factors is the same as the product of the powers of the same factors. See Example.
- The power of a quotient of factors is the same as the quotient of the powers of the same factors. See Example.
- The rules for exponential expressions can be combined to simplify more complicated expressions. See Example.
- Scientific notation uses powers of 10 to simplify very large or very small numbers. See Example and Example.
- Scientific notation may be used to simplify calculations with very large or very small numbers. See Example and Example.

Homework Exercises 1.2

WeBWorK problems:

1-5:

Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login

Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login



Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login



Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login



Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login

Written Problems:

Simplify Expressions Using the Properties for Exponents

In the following exercises, simplify each expression using the properties for exponents.

1. ⓐ $x^4 \cdot x^2$

ⓑ $8^{9x} \cdot 8^3$

ⓒ $3z^{25} \cdot 5z^8$

ⓓ $y \cdot y^3 \cdot y^5$

2. ⓐ $\dfrac{y^{20}}{y^{10}}$

ⓑ $\dfrac{7^{16}}{7^2}$

ⓒ $\dfrac{t^{10}}{t^{40}}$

ⓓ $\dfrac{8^3}{8^5}$

3. ⓐ b^{-4}

ⓑ 10^{-2}

ⓒ $\dfrac{1}{c^{-5}}$

ⓓ $\dfrac{1}{5^{-2}}$

4. ⓐ $\left(\dfrac{7}{2}\right)^{-3}$

ⓑ $\left(-\dfrac{3}{x}\right)^{-3}$

In the following exercises, simplify each expression using the Power Property.

5. ⓐ $(b^2)^7$

ⓑ $(3^8)^2$

ⓒ $(k^2)^{-5}$

In the following exercises, simplify each expression using the Quotient to a Power Property

6. ⓐ $(10k^4)^3(5k^6)^2$

ⓑ $\dfrac{(q^3)^6(q^{-2})^3}{(q^4)^8}$

7. Write each number in scientific notation.

ⓐ 57,000

ⓑ 0.026

8. Explain why $-5^3 = (-5)^3$ but $-5^4 \neq (-5)^4$.

Mixed Practice

In the following exercises, simplify each expression.

9. ⓐ $7n^{-1}$ ⓑ $(7n)^{-1}$ ⓒ $(-7n)^{-1}$

Answer

ⓐ $\frac{7}{n}$ ⓑ $\frac{1}{7n}$ ⓒ $-\frac{1}{7n}$

10. $(2m^6)^3$

Answer

$8m^{18}$

11. $(10x^2y)^3$

Answer

$1,000x^6y^3$

12. $\left(\frac{2}{3}x^2y\right)^3$

Answer

$\frac{8}{27}x^6y^3$

Use Scientific Notation

13. ⓐ 340,000

ⓑ 0.041

Answer

ⓐ 3.4×10^5 ⓑ 4.1×10^{-2}

14. ⓐ 1,290,000

ⓑ 0.00000103

Answer

ⓐ 1.29×10^6

ⓑ 1.03×10^{-6}

In the following exercises, convert each number to decimal form.

15. ⓐ -8.3×10^2

ⓑ 3.8×10^{-2}

Answer

ⓐ -830 ⓑ 0.038

This page titled 1.2: Exponents and Scientific Notation is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **1.2: Exponents and Scientific Notation** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/precalculus.
- **5.3E: Exercises** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/intermediate-algebra-2e.

1.3: Radicals and Rational Expressions

Learning Objectives

- Evaluate square roots.
- Use the product rule to simplify square roots.
- Use the quotient rule to simplify square roots.
- Add and subtract square roots.
- Rationalize denominators.
- Use rational roots.

A hardware store sells 16-ft ladders and 24-ft ladders. A window is located 12 feet above the ground. A ladder needs to be purchased that will reach the window from a point on the ground 5 feet from the building. To find out the length of ladder needed, we can draw a right triangle as shown in Figure 1.3.1, and use the Pythagorean Theorem.

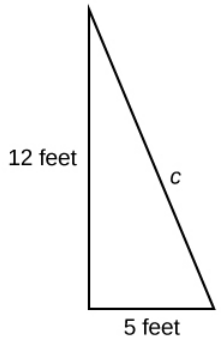


Figure 1.3.1: A right triangle

$$\begin{aligned} a^2+b^2 &= c^2 \\ 5^2+12^2 &= c^2 \\ 169 &= c^2 \end{aligned}$$

Now, we need to find out the length that, when squared, is 169, to determine which ladder to choose. In other words, we need to find a square root. In this section, we will investigate methods of finding solutions to problems such as this one.

Evaluating Square Roots

When the square root of a number is squared, the result is the original number. Since $4^2=16$, the square root of 16 is 4.The square root function is the inverse of the squaring function just as subtraction is the inverse of addition. To undo squaring, we take the square root.

In general terms, if a is a positive real number, then the square root of a is a number that, when multiplied by itself, gives a.The square root could be positive or negative because multiplying two negative numbers gives a positive number. The principal square root is the nonnegative number that when multiplied by itself equals a. The square root obtained using a calculator is the principal square root.

The principal square root of a is written as $\sqrt{a}$. The symbol is called a radical, the term under the symbol is called the radicand, and the entire expression is called a **radical expression**.



Example 1.3.1

Does $\sqrt{25} = \pm 5$?

Solution

No. Although both 5^2 and $(-5)^2$ are 25, the radical symbol implies only a *nonnegative* root, the principal square root. The principal square root of 25 is $\sqrt{25} = 5$.

Note

The principal square root of a is the nonnegative number that, when multiplied by itself, equals a. It is written as a radical expression, with a symbol called a *radical* over the term called the *radicand*: $\sqrt{a}$.

Example 1.3.2: Evaluating Square Roots

Evaluate each expression.

a. $\sqrt{\sqrt{16}}$
b. $\sqrt{49}$-$\sqrt{81}$

Solution

a. $\sqrt{\sqrt{16}} = \sqrt{4} = 2$ because $4^2 = 16$ and $2^2 = 4$
b. $\sqrt{49} - \sqrt{81} = 7 - 9 = -2$ because $7^2 = 49$ and $9^2 = 81$

Example 1.3.3:

For $\sqrt{25 + 144}$,can we find the square roots before adding?

Solution

No. $\sqrt{25} + \sqrt{144} = 5 + 12 = 17$. This is not equivalent to $\sqrt{25 + 144} = 13$. The order of operations requires us to add the terms in the radicand before finding the square root.

Exercise 1.3.1

Evaluate each expression.

a. $\sqrt{\sqrt{81}}$
b. $\sqrt{36} + \sqrt{121}$

Answer a

3

Answer b

17

Using the Product Rule to Simplify Square Roots

To simplify a square root, we rewrite it such that there are no perfect squares in the radicand. There are several properties of square roots that allow us to simplify complicated radical expressions. The first rule we will look at is the product rule for simplifying square roots, which allows us to separate the square root of a product of two numbers into the product of two separate rational expressions. For instance, we can rewrite $\sqrt{15}$ as $\sqrt{3} \times \sqrt{5}$. We can also use the product rule to express the product of multiple radical expressions as a single radical expression.

The Product Rule For Simplifying Square Roots

If a and b are nonnegative, the square root of the product ab is equal to the product of the square roots of a and b

$$\sqrt{ab} = \sqrt{a} \times \sqrt{b} \tag{1.3.1}$$

HOWTO: Given a square root radical expression, use the product rule to simplify it.

1. Factor any perfect squares from the radicand.
2. Write the radical expression as a product of radical expressions.
3. Simplify.

Example 1.3.4: Using the Product Rule to Simplify Square Roots

Simplify the radical expression.

a. $\sqrt{300}$
b. $\sqrt{162a^5b^4}$

Solution

a. $\sqrt{100 \times 3}$ Factor perfect square from radicand.

$\sqrt{100} \times \sqrt{3}$ Write radical expression as product of radical expressions.

$10\sqrt{3}$ Simplify

b. $\sqrt{81a^4b^4 \times 2a}$ Factor perfect square from radicand

$\sqrt{81a^4b^4} \times \sqrt{2a}$ Write radical expression as product of radical expressions

$9a^2b^2\sqrt{2a}$ Simplify

Exercise 1.3.2

Simplify $\sqrt{50x^2y^3z}$

Answer

$5|x||y|\sqrt{2yz}$

Notice the absolute value signs around x and y? That's because their value must be positive!

Howto: Given the product of multiple radical expressions, use the product rule to combine them into one radical expression

1. Express the product of multiple radical expressions as a single radical expression.
2. Simplify.

Example 1.3.5: Using the Product Rule to Simplify the Product of Multiple Square Roots

Simplify the radical expression.

$\sqrt{12} \times \sqrt{3}$

Solution

$\sqrt{12 \times 3}$ Express the product as a single radical expression
$\sqrt{36}$ Simplify
6

Exercise 1.3.3

Simplify $\sqrt{50x} \times \sqrt{2x}$ assuming $x > 0$.

Answer

$10|x|$

Using the Quotient Rule to Simplify Square Roots

Just as we can rewrite the square root of a product as a product of square roots, so too can we rewrite the square root of a quotient as a quotient of square roots, using the quotient rule for simplifying square roots. It can be helpful to separate the numerator and denominator of a fraction under a radical so that we can take their square roots separately. We can rewrite

$$\sqrt{\frac{5}{2}} = \frac{\sqrt{5}}{\sqrt{2}}.$$

THE QUOTIENT RULE FOR SIMPLIFYING SQUARE ROOTS

The square root of the quotient $\frac{a}{b}$ is equal to the quotient of the square roots of a and b, where $b \neq 0$.

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \tag{1.3.2}$$

Howto: Given a radical expression, use the quotient rule to simplify it

1. Write the radical expression as the quotient of two radical expressions.
2. Simplify the numerator and denominator.

Example 1.3.6: Using the Quotient Rule to Simplify Square Roots

Simplify the radical expression.

$\sqrt{\frac{5}{36}}$

Solution

$\frac{\sqrt{5}}{\sqrt{36}}$ Write as quotient of two radical expressions
$\frac{\sqrt{5}}{6}$ Simplify denominator

Exercise 1.3.4

Simplify $\sqrt{\frac{2x^2}{9y^4}}$

Answer

$\frac{x\sqrt{2}}{3y^2}$

We do not need the absolute value signs for y^2 because that term will always be nonnegative.

Example 1.3.7: Using the Quotient Rule to Simplify an Expression with Two Square Roots

Simplify the radical expression.

$\frac{\sqrt{234x^{11}y}}{\sqrt{26x^7y}}$

Solution

$\sqrt{\frac{234x^{11}y}{26x^7y}}$ Combine numerator and denominator into one radical expression

$\sqrt{9x^4}$ Simplify fraction

$3x^2$ Simplify square root

Exercise 1.3.5

Simplify $\frac{\sqrt{9a^5b^{14}}}{\sqrt{3a^4b^5}}$

Answer

$b^4\sqrt{3ab}$

Adding and Subtracting Square Roots

We can add or subtract radical expressions only when they have the same radicand and when they have the same radical type such as square roots. For example, the sum of $\sqrt{2}$ and $3\sqrt{2}$ is $4\sqrt{2}$. However, it is often possible to simplify radical expressions, and that may change the radicand. The radical expression $\sqrt{18}$ can be written with a 2 in the radicand, as $3\sqrt{2}$, so $\sqrt{2}+\sqrt{18}=\sqrt{2}+3\sqrt{2}=4\sqrt{2}$

Howto: Given a radical expression requiring addition or subtraction of square roots, solve

1. Simplify each radical expression.
2. Add or subtract expressions with equal radicands.

Example 1.3.8: Adding Square Roots

Add $5\sqrt{12}+2\sqrt{3}$.

Solution

We can rewrite $5\sqrt{12}$ as $5\sqrt{4\times3}$. According the product rule, this becomes $5\sqrt{4}\sqrt{3}$. The square root of $\sqrt{4}$ is 2, so the expression becomes $5\times2\sqrt{3}$, which is $10\sqrt{3}$. Now we can the terms have the same radicand so we can add.

$$10\sqrt{3}+2\sqrt{3}=12\sqrt{3}$$

Exercise 1.3.6

Add $\sqrt{5}+6\sqrt{20}$

Answer

$13\sqrt{5}$

Example 1.3.9: Subtracting Square Roots

Subtract $20\sqrt{72a^3b^4c}-14\sqrt{8a^3b^4c}$

Solution

Rewrite each term so they have equal radicands.

$$\begin{aligned} 20\sqrt{72a^3b^4c} &= 20\sqrt{9}\sqrt{4}\sqrt{2}\sqrt{a}\sqrt{a^2}\sqrt{(b^2)^2}\sqrt{c} \\ &= 20(3)(2)|a|b^2\sqrt{2ac} \\ &= 120|a|b^2\sqrt{2ac} \end{aligned}$$

$$\begin{aligned} 14\sqrt{8a^3b^4c} &= 14\sqrt{2}\sqrt{4}\sqrt{a}\sqrt{a^2}\sqrt{(b^2)^2}\sqrt{c} \\ &= 14(2)|a|b^2\sqrt{2ac} \\ &= 28|a|b^2\sqrt{2ac} \end{aligned}$$

Now the terms have the same radicand so we can subtract.

$$120|a|b^2\sqrt{2ac}-28|a|b^2\sqrt{2ac}=92|a|b^2\sqrt{2ac} \tag{1.3.3}$$

Exercise 1.3.7

Subtract $3\sqrt{80x}-4\sqrt{45x}$

Answer

0

Rationalizing Denominators

When an expression involving square root radicals is written in simplest form, it will not contain a radical in the denominator. We can remove radicals from the denominators of fractions using a process called rationalizing the denominator.

We know that multiplying by 1 does not change the value of an expression. We use this property of multiplication to change expressions that contain radicals in the denominator. To remove radicals from the denominators of fractions, multiply by the form of 1 that will eliminate the radical.

For a denominator containing a single term, multiply by the radical in the denominator over itself. In other words, if the denominator is $b\sqrt{c}$, multiply by $\frac{\sqrt{c}}{\sqrt{c}}$.

For a denominator containing the sum or difference of a rational and an irrational term, multiply the numerator and denominator by the conjugate of the denominator, which is found by changing the sign of the radical portion of the denominator. If the denominator is $a+b\sqrt{c}$, then the conjugate is $a-b\sqrt{c}$.

HowTo: Given an expression with a single square root radical term in the denominator, rationalize the denominator

1. Multiply the numerator and denominator by the radical in the denominator.
2. Simplify.

Example 1.3.10: Rationalizing a Denominator Containing a Single Term

Write $\frac{2\sqrt{3}}{3\sqrt{10}}$ in simplest form.

Solution

The radical in the denominator is $\sqrt{10}$. So multiply the fraction by $\frac{\sqrt{10}}{\sqrt{10}}$. Then simplify.

$$\frac{2\sqrt{3}}{3\sqrt{10}} \times \frac{\sqrt{10}}{\sqrt{10}}$$
$$\frac{2\sqrt{30}}{30}$$
$$\frac{\sqrt{30}}{15}$$

Exercise 1.3.8

Write $\frac{12\sqrt{3}}{\sqrt{2}}$ in simplest form.

Answer

$6\sqrt{6}$

How to: Given an expression with a radical term and a constant in the denominator, rationalize the denominator

1. Find the conjugate of the denominator.
2. Multiply the numerator and denominator by the conjugate.
3. Use the distributive property.
4. Simplify.

Example 1.3.11: Rationalizing a Denominator Containing Two Terms

Write $\frac{4}{1+\sqrt{5}}$ in simplest form.

Solution

Begin by finding the conjugate of the denominator by writing the denominator and changing the sign. So the conjugate of $1+\sqrt{5}$ is $1-\sqrt{5}$. Then multiply the fraction by $\frac{1-\sqrt{5}}{1-\sqrt{5}}$.

$$\frac{4}{1+\sqrt{5}} \times \frac{1-\sqrt{5}}{1-\sqrt{5}}$$
$$\frac{4-4\sqrt{5}}{-4} \quad \text{Use the distributive property}$$
$$\sqrt{5}-1 \quad \text{Simplify}$$

Exercise 1.3.9

Write $\frac{7}{2+\sqrt{3}}$ in simplest form.

Answer

$14-7\sqrt{3}$

Using Rational Roots

Although square roots are the most common rational roots, we can also find cube roots, 4th roots, 5th roots, and more. Just as the square root function is the inverse of the squaring function, these roots are the inverse of their respective power functions. These functions can be useful when we need to determine the number that, when raised to a certain power, gives a certain number.

Understanding n^{th} Roots

Suppose we know that $a^3=8$. We want to find what number raised to the 3rd power is equal to 8. Since $2^3=8$, we say that 2 is the cube root of 8.

The nth root of a is a number that, when raised to the nth power, gives a. For example, -3 is the 5th root of -243 because $(-3)^5=-243$. If a is a real number with at least one n^{th} root, then the principal n^{th} root of a is the number with the same sign as a that, when raised to the nth power, equals a.

The principal n^{th} root of a is written as $\sqrt[n]{a}$, where n is a positive integer greater than or equal to 2. In the radical expression, n is called the index of the radical.

Note: Principal n^{th} Root

If a is a real number with at least one n^{th} root, then the **principal n^{th} root** of a, written as $\sqrt[n]{a}$, is the number with the same sign as a that, when raised to the nth power, equals a. The **index** of the radical is n.

Example 1.3.12: Simplifying n^{th} Roots

Simplify each of the following:

a. $\sqrt[5]{-32}$
b. $\sqrt[4]{4}\times\sqrt[4]{10234}$
c. $-\sqrt[3]{\dfrac{8x^6}{125}}$
d. $8\sqrt[4]{3}-\sqrt[4]{48}$

Solution

a. $\sqrt[5]{-32}=-2$ because $(-2)^5=-32$

b. First, express the product as a single radical expression. $\sqrt[4]{4096}=8$ because $8^4=4096$

c.

$$\frac{-\sqrt[3]{8x^6}}{\sqrt[3]{125}} \quad \text{Write as quotient of two radical expressions}$$
$$\frac{-2x^2}{5} \quad \text{Simplify}$$

d.

$$8\sqrt[4]{3}-2\sqrt[4]{3} \quad \text{Simplify to get equal radicands}$$
$$6\sqrt[4]{3} \quad \text{Add}$$

Exercise 1.3.10

Simplify

a. $\sqrt[3]{-216}$
b. $\dfrac{3\sqrt[4]{80}}{\sqrt[4]{5}}$

c. $6\sqrt[3]{9000}+7\sqrt[3]{576}$

Answer a

-6

Answer b

6

Answer c

$88\sqrt[3]{9}$

Using Rational Exponents

Radical expressions can also be written without using the radical symbol. We can use rational (fractional) exponents. The index must be a positive integer. If the index n is even, then a cannot be negative.

$$a^{\frac{1}{n}}=\sqrt[n]{a} \tag{1.3.4}$$

We can also have rational exponents with numerators other than 1. In these cases, the exponent must be a fraction in lowest terms. We raise the base to a power and take an nth root. The numerator tells us the power and the denominator tells us the root.

$$a^{\frac{m}{n}}=(\sqrt[n]{a})^m=\sqrt[n]{a^m} \tag{1.3.5}$$

All of the properties of exponents that we learned for integer exponents also hold for rational exponents.

Note: Rational Exponents

Rational exponents are another way to express principal n^{th} roots. The general form for converting between a radical expression with a radical symbol and one with a rational exponent is

$$a^{\frac{m}{n}}=(\sqrt[n]{a})^m=\sqrt[n]{a^m} \tag{1.3.6}$$

Howto: Given an expression with a rational exponent, write the expression as a radical

1. Determine the power by looking at the numerator of the exponent.
2. Determine the root by looking at the denominator of the exponent.
3. Using the base as the radicand, raise the radicand to the power and use the root as the index.

Example 1.3.13: Writing Rational Exponents as Radicals

Write $343^{\frac{2}{3}}$ as a radical. Simplify.

Solution

The 2 tells us the power and the 3 tells us the root.

$343^{\frac{2}{3}}=(\sqrt[3]{343})^2=\sqrt[3]{343^2}$

We know that $\sqrt[3]{343}=7$ because $7^3=343$. Because the cube root is easy to find, it is easiest to find the cube root before squaring for this problem. In general, it is easier to find the root first and then raise it to a power.

$$343^{\frac{2}{3}}=(\sqrt[3]{343})^2=7^2=49 \tag{1.3.7}$$

Exercise 1.3.11

Write $9^{\frac{5}{2}}$ as a radical. Simplify.

Answer

$(\sqrt{9})^5 = 3^5 = 243$

Example 1.3.14: Writing Radicals as Rational Exponents

Write $\frac{4}{\sqrt[7]{a^2}}$ using a rational exponent.

Solution

The power is 2 and the root is 7, so the rational exponent will be $\frac{2}{7}$. We get $\frac{4}{a^{\frac{2}{7}}}$. Using properties of exponents, we get $\frac{4}{\sqrt[7]{a^2}} = 4a^{\frac{-2}{7}}$

Exercise 1.3.12

Write $x\sqrt{(5y)^9}$ using a rational exponent.

Answer

$x(5y)^{\frac{9}{2}}$

Example 1.3.15: Simplifying Rational Exponents

Simplify:

a. $5(2x^{\frac{3}{4}})(3x^{\frac{1}{5}})$

b. $\left(\frac{16}{9}\right)^{-\frac{1}{2}}$

Solution

a.

$30x^{\frac{3}{4}}\ x^{\frac{1}{5}}$ Multiply the coefficients

$30x^{\frac{3}{4}+\frac{1}{5}}$ Use properties of exponents

$30x^{\frac{19}{20}}$ Simplify

b.

https://math.libretexts.org/@go/page/106317

$\left(\frac{9}{16}\right)^{\frac{1}{2}}$ Use definition of negative exponents

$\sqrt{\frac{9}{16}}$ Rewrite as a radical

$\frac{\sqrt{9}}{\sqrt{16}}$ Use the quotient rule

$\frac{3}{4}$ Simplify

? Exercise 1.3.13

Simplify $(8x)^{\frac{1}{3}}\left(14x^{\frac{6}{5}}\right)$

Answer

$28x^{\frac{23}{15}}$

Media

Access these online resources for additional instruction and practice with radicals and rational exponents.

Radicals

Rational Exponents

Simplify Radicals

Rationalize Denominator

Key Concepts

- The principal square root of a number a is the nonnegative number that when multiplied by itself equals a. See Example.
- If a and b are nonnegative, the square root of the product ab is equal to the product of the square roots of a and b See Example and Example.
- If a and b are nonnegative, the square root of the quotient $\frac{a}{b}$ is equal to the quotient of the square roots of a and b See Example and Example.
- We can add and subtract radical expressions if they have the same radicand and the same index. See Example and Example.
- Radical expressions written in simplest form do not contain a radical in the denominator. To eliminate the square root radical from the denominator, multiply both the numerator and the denominator by the conjugate of the denominator. See Example and Example.
- The principal n^{th} root of a is the number with the same sign as a that when raised to the nth power equals a. These roots have the same properties as square roots. See Example.
- Radicals can be rewritten as rational exponents and rational exponents can be rewritten as radicals. See Example and Example.
- The properties of exponents apply to rational exponents. See Example.

Homework Exercises 1.3

WeBWorK problems:

Written problems:

Verbal

1. What does it mean when a radical does not have an index? Give an example as part of your explanation.

2. Every number will have two square roots. What is the principal square root?

3. Make up an example with a negative radicand and an odd index, and explain how to simplify it.

Numeric

4. Simplify $4^{\frac{3}{2}}$

4. Simplify $\sqrt[4]{16}$

5. Simplify $(144p^2q^6)^{\frac{1}{2}}$

6. Simplify $\sqrt{\frac{4}{121d^4}}$

7. Simplify $\sqrt[5]{1024c^{10}}$

8. Simplify $3\sqrt{44z}+\sqrt{99z}$

9. Rationalize the denominator $\frac{x-4}{\sqrt{x}-2}$

10. Rationalize the numerator $\frac{\sqrt{2+h}-\sqrt{2}}{h}$

This page titled 1.3: Radicals and Rational Expressions is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **1.3: Radicals and Rational Expressions** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/precalculus.

1.4: Polynomials

Preview Video for Sections 1.4 & 1.5

Fill in your notes (posted on moodle) as you actively go through this video.

Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login

In this section students will:

- Identify the degree and leading coefficient of polynomials.
- Add and subtract polynomials.
- Multiply polynomials.
- Use FOIL to multiply binomials.
- Perform operations with polynomials of several variables.

Earl is building a doghouse, whose front is in the shape of a square topped with a triangle. There will be a rectangular door through which the dog can enter and exit the house. Earl wants to find the area of the front of the doghouse so that he can purchase the correct amount of paint. Using the measurements of the front of the house, shown in Figure 1.4.1, we can create an expression that combines several variable terms, allowing us to solve this problem and others like it.

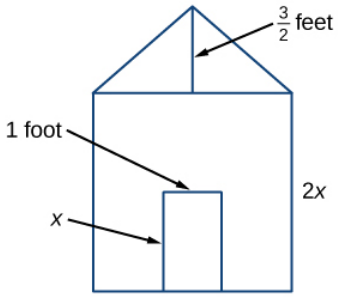


Figure 1.4.1

- First find the area of the square in square feet.

$$\begin{aligned} A &= s^2 \\ &= (2x)^2 \\ &= 4x^2 \end{aligned}$$

- Then find the area of the triangle in square feet.

$$\begin{aligned} A &= \frac{1}{2}bh \\ &= \frac{1}{2}(2x)\left(\frac{3}{2}\right) \\ &= \frac{3}{2}x \end{aligned}$$

- Next find the area of the rectangular door in square feet.

$$\begin{aligned} A &= lw \\ &= x \times 1 \\ &= x \end{aligned}$$

The area of the front of the doghouse can be found by adding the areas of the square and the triangle, and then subtracting the area of the rectangle. When we do this, we get

$$4x^2 + \frac{3}{2}x - x \ \ ft^2$$

or

$$4x^2 + \frac{1}{2}x \ \ ft^2$$

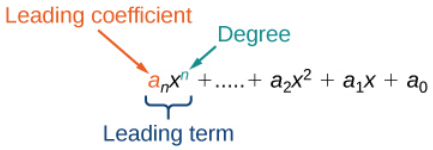
In this section, we will examine expressions such as this one, which combine several variable terms.

Identifying the Degree and Leading Coefficient of Polynomials

The formula just found is an example of a **polynomial**, which is a sum of or difference of terms, each consisting of a variable raised to a nonnegative integer power. A number multiplied by a variable raised to an exponent, such as 384π, is known as a **coefficient**. Coefficients can be positive, negative, or zero, and can be whole numbers, decimals, or fractions. Each product $a_i x^i$, such as $384\pi w$, is a **term of a polynomial**. If a term does not contain a variable, it is called a constant.

A polynomial containing only one term, such as $5x^4$, is called a **monomial**. A polynomial containing two terms, such as $2x - 9$, is called a **binomial**. A polynomial containing three terms, such as $-3x^2 + 8x - 7$, is called a **trinomial**.

We can find the **degree** of a polynomial by identifying the highest power of the variable that occurs in the polynomial. The term with the highest degree is called the **leading term** because it is usually written first. The coefficient of the leading term is called the **leading coefficient**. When a polynomial is written so that the powers are descending, we say that it is in standard form.



Polynomials

A polynomial is an expression that can be written in the form

$$a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0 \tag{1.4.1}$$

Each real number ai is called a **coefficient**. The number a_0 that is not multiplied by a variable is called aconstant. Each product $a_i x^i$ is a **term of a polynomial**. The highest power of the variable that occurs in the polynomial is called the **degree** of a polynomial. The **leading term** is the term with the highest power, and its coefficient is called the **leading coefficient.**

How to: Given a polynomial expression, identify the degree and leading coefficient.

1. Find the highest power of x to determine the degree.
2. Identify the term containing the highest power of x to find the leading term.
3. Identify the coefficient of the leading term.

Example 1.4.1: Identifying the Degree and Leading Coefficient of a Polynomial

For the following polynomials, identify the degree, the leading term, and the leading coefficient.

a. $3+2x^2-4x^3$
b. $5t^5-2t^3+7t$
c. $6p-p^3-2$

Solution

a. The highest power of x is 3, so the degree is 3. The leading term is the term containing that degree, $-4x^3$. The leading coefficient is the coefficient of that term, -4.
b. The highest power of t is 5, so the degree is 5. The leading term is the term containing that degree, $5t^5$. The leading coefficient is the coefficient of that term, 5.
c. The highest power of p is 3, so the degree is 3. The leading term is the term containing that degree, $-p^3$, The leading coefficient is the coefficient of that term, -1.

Exercise 1.4.1

Identify the degree, leading term, and leading coefficient of the polynomial $4x^2-x^6+2x-6$.

Answer

The degree is 6, the leading term is $-x^6$, and the leading coefficient is -1.

Adding and Subtracting Polynomials

We can add and subtract polynomials by combining like terms, which are terms that contain the same variables raised to the same exponents. For example, $5x^2$ and $-2x^2$ are like terms, and can be added to get $3x^2$, but $3x$ and $3x^2$ are not like terms, and therefore cannot be added.

How to: Given multiple polynomials, add or subtract them to simplify the expressions.

1. Combine like terms.
2. Simplify and write in standard form.

Example 1.4.2: Adding Polynomials

Find the sum.

$$(12x^2+9x-21)+(4x^3+8x^2-5x+20)$$

Solution

$$\begin{aligned} &4x^3+(12x^2+8x^2)+(9x-5x)+(-21+20) && \text{Combine like terms} \\ &4x^3+20x^2+4x-1 && \text{Simplify} \end{aligned}$$

Analysis

We can check our answers to these types of problems using a graphing calculator. To check, graph the problem as given along with the simplified answer. The two graphs should be equivalent. Be sure to use the same window to compare the graphs. Using different windows can make the expressions seem equivalent when they are not.

Exercise 1.4.2

Find the sum.

$$(2x^3+5x^2-x+1)+(2x^2-3x-4)$$

Answer

$2x^3+7x^2-4x-3$

Example 1.4.3: Subtracting Polynomials

Find the difference.

$$(7x^4-x^2+6x+1)-(5x^3-2x^2+3x+2)$$

Solution

$7x^4-5x^3+(-x^2+2x^2)+(6x-3x)+(1-2)$ Combine like terms

$7x^4-5x^3+x^2+3x-1$ Simplify

Analysis

Note that finding the difference between two polynomials is the same as adding the opposite of the second polynomial to the first.

Exercise 1.4.3

Find the difference.

$$(-7x^3-7x^2+6x-2)-(4x^3-6x^2-x+7)$$

Answer

$-11x^3-x^2+7x-9$

Multiplying Polynomials

Multiplying polynomials is a bit more challenging than adding and subtracting polynomials. We must use the distributive property to multiply each term in the first polynomial by each term in the second polynomial. We then combine like terms. We can also use a shortcut called the FOIL method when multiplying binomials. Certain special products follow patterns that we can memorize and use instead of multiplying the polynomials by hand each time. We will look at a variety of ways to multiply polynomials.

Multiplying Polynomials Using the Distributive Property

To multiply a number by a polynomial, we use the distributive property. The number must be distributed to each term of the polynomial. We can distribute the 2 in $2(x+7)$ to obtain the equivalent expression $2x+14$. When multiplying polynomials, the distributive property allows us to multiply each term of the first polynomial by each term of the second. We then add the products together and combine like terms to simplify.

How to: Given the multiplication of two polynomials, use the distributive property to simplify the expression.

1. Multiply each term of the first polynomial by each term of the second.
2. Combine like terms.
3. Simplify.

Example 1.4.4: Multiplying Polynomials Using the Distributive Property

Find the product.

$$(2x+1)(3x^2-x+4)$$

Solution

$2x(3x^2-x+4)+1(3x^2-x+4)$ Use the distributive property

$(6x^3-2x^2+8x)+(3x^2-x+4)$ Multiply

$6x^3+(-2x^2+3x^2)+(8x-x)+4$ Combine like terms

$6x^3+x^2+7x+4$ Simplify

Analysis

We can use a table to keep track of our work, as shown in Table 1.4.1. Write one polynomial across the top and the other down the side. For each box in the table, multiply the term for that row by the term for that column. Then add all of the terms together, combine like terms, and simplify.

Table 1.4.1

	$3x^2$	$-x$	$+4$
$2x$	$6x^3$	$-2x^2$	$8x$
$+1$	$3x^2$	$-x$	4

Exercise 1.4.4

Find the product.

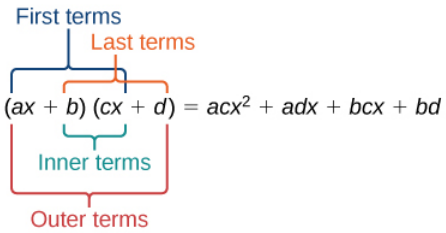
$(3x+2)(x^3-4x^2+7)$

Answer

$3x^4-10x^3-8x^2+21x+14$

Using FOIL to Multiply Binomials

A shortcut called FOIL is sometimes used to find the product of two binomials. It is called FOIL because we multiply the first terms, the outer terms, the inner terms, and then the last terms of each binomial.



The FOIL method arises out of the distributive property. We are simply multiplying each term of the first binomial by each term of the second binomial, and then combining like terms.

FOIL to simplify expression

Given two binomials, use FOIL to simplify the expression.

1. Multiply the outer terms of the binomials.
2. Multiply the last terms of each binomial.
3. Combine like terms and simplify.

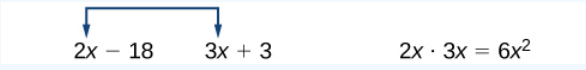
Example 1.4.5: Using FOIL to Multiply Binomials

Use FOIL to find the product.

$$(2x-10)(3x+3)$$

Solution

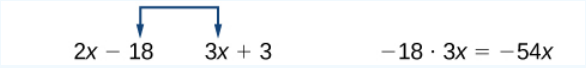
Find the product of the first terms.



Find the product of the outer terms.



Find the product of the inner terms.



Find the product of the last terms.



$$\begin{aligned} 6x^2+6x-54x-54 &\quad \text{Add the products} \\ 6x^2+(6x-54x)-54 &\quad \text{Combine like terms} \\ 6x^2-48x-54 &\quad \text{Simplify} \end{aligned}$$

? Exercise 1.4.5

Use FOIL to find the product.

$$(x+7)(3x-5)$$

Answer

$3x^2+16x-35$

Perfect Square Trinomials

Certain binomial products have special forms. When a binomial is squared, the result is called a perfect square trinomial. We can find the square by multiplying the binomial by itself. However, there is a special form that each of these perfect square trinomials takes, and memorizing the form makes squaring binomials much easier and faster. Let's look at a few perfect square trinomials to familiarize ourselves with the form.

$$(x+5)^2 = x^2+10x+25$$
$$(x-3)^2 = x^2-6x+9$$
$$(4x-1)^2 = 16x^2-8x+1$$

Notice that the first term of each trinomial is the square of the first term of the binomial and, similarly, the last term of each trinomial is the square of the last term of the binomial. The middle term is double the product of the two terms. Lastly, we see that the first sign of the trinomial is the same as the sign of the binomial.

Perfect Square Trinomials

When a binomial is squared, the result is the first term squared added to double the product of both terms and the last term squared.

$$(x+a)^2 = (x+a)(x+a) = x^2+2ax+a^2 \tag{1.4.2}$$

How to: Given a binomial, square it using the formula for perfect square trinomials.

1. Square the first term of the binomial.
2. Square the last term of the binomial.
3. For the middle term of the trinomial, double the product of the two terms.
4. Add and simplify.

Example 1.4.6: Expanding Perfect Squares

Expand $(3x-8)^2$.

Solution

Begin by squaring the first term and the last term. For the middle term of the trinomial, double the product of the two terms.

$$(3x)^2-2(3x)(8)+(-8)^2$$
$$9x^2-48x+64 \qquad \text{Simplify}$$

Exercise 1.4.6

Expand $(4x-1)^2$.

Answer

$16x^2-8x+1$

Difference of Squares

Another special product is called the difference of squares, which occurs when we multiply a binomial by another binomial with the same terms but the opposite sign. Let's see what happens when we multiply $(x+1)(x-1)$ using the FOIL method.

$$\begin{aligned}(x+1)(x-1) &= x^2-x+x-1\\ &= x^2-1\end{aligned}$$

The middle term drops out, resulting in a difference of squares. Just as we did with the perfect squares, let's look at a few examples.

$$(x+5)(x-5)=x^2-25$$
$$(x+11)(x-11)=x^2-121$$
$$(2x+3)(2x-3)=4x^2-9$$

Because the sign changes in the second binomial, the outer and inner terms cancel each other out, and we are left only with the square of the first term minus the square of the last term.

Q&A

Is there a special form for the sum of squares?

No. The difference of squares occurs because the opposite signs of the binomials cause the middle terms to disappear. There are no two binomials that multiply to equal a sum of squares.

Difference of Squares

When a binomial is multiplied by a binomial with the same terms separated by the opposite sign, the result is the square of the first term minus the square of the last term.

$$(a+b)(a-b)=a^2-b^2 \tag{1.4.3}$$

Howto: Given a binomial multiplied by a binomial with the same terms but the opposite sign, find the difference of squares.

1. Square the first term of the binomials.
2. Square the last term of the binomials.
3. Subtract the square of the last term from the square of the first term.

Example 1.4.7: Multiplying Binomials Resulting in a Difference of Squares

Multiply $(9x+4)(9x-4)$.

Solution

Square the first term to get $(9x)^2 = 81x^2$. Square the last term to get $4^2 = 16$. Subtract the square of the last term from the square of the first term to find the product of $81x^2 - 16$.

Exercise 1.4.7

Multiply $(2x+7)(2x-7)$.

Answer

$4x^2 - 49$

Performing Operations with Polynomials of Several Variables

We have looked at polynomials containing only one variable. However, a polynomial can contain several variables. All of the same rules apply when working with polynomials containing several variables. Consider an example:

$$\begin{aligned}
&(a+2b)(4a-b-c)a(4a-b-c)+2b(4a-b-c) && \text{Use the distributive property}\\
&4a^2-ab-ac+8ab-2b^2-2bc && \text{Multiply}\\
&4a^2+(-ab+8ab)-ac-2b^2-2bc && \text{Combine like terms}\\
&4a^2+7ab-ac-2bc-2b^2 && \text{Simplify}
\end{aligned}$$

Example 1.4.8: Multiplying Polynomials Containing Several Variables

Multiply $(x+4)(3x-2y+5)$.

Solution

$$\begin{aligned}
&x(3x-2y+5)+4(3x-2y+5) && \text{Use the distributive property}\\
&3x^2-2xy+5x+12x-8y+20 && \text{Multiply}\\
&3x^2-2xy+(5x+12x)-8y+20 && \text{Combine like terms}\\
&3x^2-2xy+17x-8y+20 && \text{Simplify}
\end{aligned}$$

Exercise 1.4.8

Multiply $(3x-1)(2x+7y-9)$.

Answer

$6x^2+21xy-29x-7y+9$

Media

Access these online resources for additional instruction and practice with polynomials.

1. Adding and Subtracting Polynomials
2. Multiplying Polynomials
3. Special Products of Polynomials

Key Equations

perfect square trinomial	$(x+a)^2=(x+a)(x+a)=x^2+2ax+a^2$
difference of squares	$(a+b)(a-b)=a^2-b^2$

Key Concepts

- A polynomial is a sum of terms each consisting of a variable raised to a non-negative integer power. The degree is the highest power of the variable that occurs in the polynomial. The leading term is the term containing the highest degree, and the leading coefficient is the coefficient of that term. See Example.
- We can add and subtract polynomials by combining like terms. See Example and Example.
- To multiply polynomials, use the distributive property to multiply each term in the first polynomial by each term in the second. Then add the products. See Example.
- FOIL (First, Outer, Inner, Last) is a shortcut that can be used to multiply binomials. See Example.
- Perfect square trinomials and difference of squares are special products. See Example and Example.
- Follow the same rules to work with polynomials containing several variables. See Example.

Homework Exercises 1.4

WeBWorK problems:

Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login

Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login

Written problems:

Write the given polynomials in standard form.

1. $1-x-x^2$
2. $y-5+y^2$
3. $y-3y^2+5-y^3$
4. $8-12a^2+a^3-a$
5. $2-x^2+6x-5x^3+x^4$
6. $a^3-5+a^2+2a^4-a^5+6a$

Answer

1. $-x^2-x+1$
3. $-y^3-3y^2+y+5$
5. $x^4-5x^3-x^2+6x+2$

Classify the given polynomial as a monomial, binomial, or trinomial and state the degree.

7. x^2-x+2
8. $5-10x^3$
9. $x^2y^2+5xy-6$
10. $-2x^3y^2$
11. x^4-1
12. 5

Answer

7. Trinomial; degree 2
9. Trinomial; degree 4
11. Binomial; degree 4

State whether the polynomial is linear or quadratic and give the leading coefficient.

13. $1-9x^2$
14. $10x^2$
15. $2x-3$
16. $100x$
17. $5x^2+3x-1$
18. $x-1$
19. $x-6-2x^2$
20. $1-5x$

Answer

13. Quadratic, -9
15. Linear, 2
17. Quadratic, 5
19. Quadratic, -2

Add or Subtract as indicated

21. $\left(5x^2-3x-2\right)+\left(2x^2-6x+7\right)$
22. $\left(x^2+7x-12\right)+\left(2x^2-x+3\right)$
23. $\left(x^2+5x+10\right)+\left(x^2-10\right)$
24. $\left(x^2-1\right)+(4x+2)$
25. $\left(10x^2+3x-2\right)-\left(x^2-6x+1\right)$
26. $\left(x^2-3x-8\right)-\left(2x^2-3x-8\right)$

27. $\left(\frac{2}{3}x^2+\frac{3}{4}x-1\right)-\left(\frac{1}{6}x^2+\frac{5}{2}x-\frac{1}{2}\right)$

28. $\left(\frac{4}{5}x^2-\frac{5}{8}x+\frac{10}{6}\right)-\left(\frac{3}{10}x^2-\frac{2}{3}x+\frac{3}{5}\right)$

29. $\left(x^2y^2+7xy-5\right)-\left(2x^2y^2+5xy-4\right)$

30. $\left(x^2-y^2\right)-\left(x^2+6xy+y^2\right)$

31. $\left(a^2b^2+5ab-2\right)+(7ab-2)-\left(4-a^2b^2\right)$

32. $\left(a^2+9ab-6b^2\right)-\left(a^2-b^2\right)+7ab$

33. $\left(10x^2y-8xy+5xy^2\right)-\left(x^2y-4xy\right)+\left(xy^2+4xy\right)$

34. $\left(2m^2n-6mn+9mn^2\right)-\left(m^2n+10mn\right)-m^2n$

35. $\left(8x^2y^2-5xy+2\right)-\left(x^2y^2+5\right)+(2xy-3)$

36. $\left(x^2-y^2\right)-\left(5x^2-2xy-y^2\right)-\left(x^2-7xy\right)$

37. $\left(\frac{1}{6}a^2-2ab+\frac{3}{4}b^2\right)-\left(\frac{5}{3}a^2+\frac{4}{5}b^2\right)+\frac{11}{8}ab$

38. $\left(\frac{5}{2}x^2-2y^2\right)-\left(\frac{7}{5}x^2-\frac{1}{2}xy+\frac{7}{3}y^2\right)-\frac{1}{2}xy$

39. $\left(x^{2n}+5x^n-2\right)+\left(2x^{2n}-3x^n-1\right)$

40. $\left(7x^{2n}-x^n+5\right)-\left(6x^{2n}-x^n-8\right)$

41. Subtract $4y-3$ from $y^2+7y-10$.

42. Subtract x^2+3x-2 from $2x^2+4x-1$.

43. A right circular cylinder has a height that is equal to the radius of the base, $h=r$. Find a formula for the surface area in terms of h.

44. A rectangular solid has a width that is twice the height and a length that is 3 times that of the height. Find a formula for the surface area in terms of the height.

Answers

21. $7x^2-9x+5$

23. $2x^2+5x$

25. $9x^2+9x-3$

27. $\frac{1}{2}x^2-\frac{7}{4}x-\frac{1}{2}$

29. $-x^2y^2+2xy-1$

31. $2a^2b^2+12ab-8$

33. $9x^2y+6xy^2$

35. $7x^2y^2-3xy-6$

37. $-\frac{3}{2}a^2-\frac{5}{8}ab-\frac{1}{20}b^2$

39. $-\frac{3}{2}a^2-\frac{5}{8}ab-\frac{1}{20}b^2$

41. y^2+3y-7

43. $SA=4\pi h^2$

Multiply

45. $-8x^2\cdot 2x$

46. $-10x^2y\cdot 5x^3y^2$

47. $2x(5x-1)$

48. $-4x(3x-5)$
49. $7x^2(2x-6)$
50. $-3x^2\left(x^2-x+3\right)$
51. $-5y^4\left(y^2-2y+3\right)$
52. $\frac{5}{2}a^3\left(24a^2-6a+4\right)$
53. $2xy\left(x^2-7xy+y^2\right)$
54. $-2a^2b\left(a^2-3ab+5b^2\right)$
55. $x^n\left(x^2+x+1\right)$
56. $x^n\left(x^{2n}-x^n-1\right)$
57. $(x+4)(x-5)$
58. $(x-7)(x-6)$
59. $(2x-3)(3x-1)$
60. $(9x+1)(3x+2)$
61. $\left(3x^2-y^2\right)\left(x^2-5y^2\right)$
62. $\left(5y^2-x^2\right)\left(2y^2-3x^2\right)$
63. $(3x+5)(3x-5)$
64. $(x+6)(x-6)$
65. $\left(a^2-b^2\right)\left(a^2+b^2\right)$
66. $(ab+7)(ab-7)$
67. $\left(4x-5y^2\right)\left(3x^2-y\right)$
68. $(xy+5)(x-y)$
69. $(x-5)\left(x^2-3x+8\right)$
70. $(2x-7)\left(3x^2-x+1\right)$
71. $\left(x^2+7x-1\right)\left(2x^2-3x-1\right)$
72. $\left(4x^2-x+6\right)\left(5x^2-4x-3\right)$
73. $(x+8)^2$
74. $(x-3)^2$
75. $(2x-5)^2$
76. $(3x+1)^2$
77. $(a-3b)^2$
78. $(7a-b)^2$
79. $\left(x^2+2y^2\right)^2$
80. $\left(x^2-6y\right)^2$
81. $\left(a^2-a+5\right)^2$
82. $\left(x^2-3x-1\right)^2$
83. $(x-3)^3$
84. $(x+2)^3$

85. $(3x+1)^3$

86. $(2x-3)^3$

87. $(x+2)^4$

88. $(x-3)^4$

89. $(2x-1)^4$

90. $(3x-1)^4$

91. $\left(x^{2n}+5\right)\left(x^{2n}-5\right)$

92. $\left(x^n-1\right)\left(x^{2n}+4x^n-3\right)$

93. $\left(x^{2n}-1\right)^2$

94. $\left(x^{3n}+1\right)^2$

95. Find the product of $3x-2$ and x^2-5x-2.

96. Find the product of x^2+4 and x^3-1.

97. Each side of a square measures $3x^3$ units. Determine the area in terms of x.

98. Each edge of a cube measures $2x^2$ units. Determine the volume in terms of x.

Answer

45. $-16x^3$

47. $10x^2-2x$

49. $14x^3-42x^2$

51. $-5y^6+10y^5-15y^4$

53. $2x^3y-14x^2y^2+2xy^3$

55. $x^{n+2}+x^{n+1}+x^n$

57. x^2-x-20

59. $6x^2-11x+3$

61. $3x^4-16x^2y^2+5y^4$

63. $9x^2-25$

65. a^4-b^4

67. $12x^3-15x^2y^2-4xy+5y^3$

69. $x^3-8x^2+23x-40$

71. $2x^4+11x^3-24x^2-4x+1$

73. $x^2+16x+64$

75. $4x^2-20x+25$

77. $a^2-6ab+9b^2$

79. $x^4+4x^2y^2+4y^4$

81. $a^4-2a^3+11a-10a+25$

83. $x^3-9x^2+27x-27$

85. $27x^3+27x^2+9x+1$

87. $x^4+8x^3+24x^2+32x+16$

89. $16x^4-32x^3+24x^2-8x+1$

91. $x^{4n}-25$

93. $x^{4n}-2x^{2n}+1$

95. $3x^3-17x^2+4x+4$

97. $9x^6$ square units

This page titled 1.4: Polynomials is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **1.4: Polynomials** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/precalculus.
- **1.6: Polynomials and Their Operations** by Anonymous is licensed CC BY-NC-SA 3.0. Original source: https://2012books.lardbucket.org/books/advanced-algebra/index.html.

1.5: Factoring Polynomials

Learning Objectives

In this section students will:

- Factor the greatest common factor of a polynomial.
- Factor a trinomial.
- Factor by grouping.
- Factor a perfect square trinomial.
- Factor a difference of squares.
- Factor the sum and difference of cubes.
- Factor expressions using fractional or negative exponents.

Imagine that we are trying to find the area of a lawn so that we can determine how much grass seed to purchase. The lawn is the green portion in Figure 1.5.1.

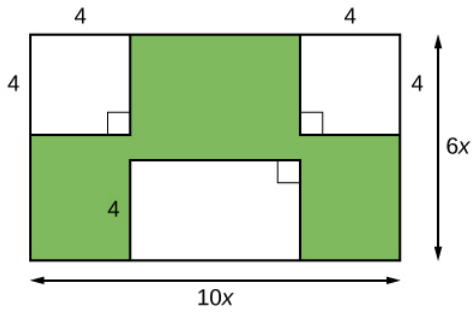


Figure 1.5.1

The area of the entire region can be found using the formula for the area of a rectangle.

$$\begin{aligned} A &= lw \\ &= 10x \times 6x \\ &= 60x^2 \ units^2 \end{aligned}$$

The areas of the portions that do not require grass seed need to be subtracted from the area of the entire region. The two square regions each have an area of $A = s^2 = 4^2 = 16 \ units^2$. The other rectangular region has one side of length $10x - 8$ and one side of length 4, giving an area of

$$A = lw = 4(10x - 8) = 40x - 32 \text{ units}^2.$$

So the region that must be subtracted has an area of

$$2(16) + 40x - 32 = 40x \text{ units}^2.$$

The area of the region that requires grass seed is found by subtracting $60x^2 - 40x \text{ units}^2$. This area can also be expressed in factored form as $20x(3x - 2) \text{ units}^2$. We can confirm that this is an equivalent expression by multiplying.

Many polynomial expressions can be written in simpler forms by factoring. In this section, we will look at a variety of methods that can be used to factor polynomial expressions.

Factoring the Greatest Common Factor of a Polynomial

When we study fractions, we learn that the greatest common factor (GCF) of two numbers is the largest number that divides evenly into both numbers. For instance, 4 is the GCF of 16 and 20 because it is the largest number that divides evenly into both 16 and 20 The GCF of polynomials works the same way: $4x$ is the GCF of $16x$ and $20x^2$ because it is the largest polynomial that divides evenly into both $16x$ and $20x^2$.

When factoring a polynomial expression, our first step should be to check for a GCF. Look for the GCF of the coefficients, and then look for the GCF of the variables.

Definition: Greatest Common Factor

The *greatest common factor* (GCF) of polynomials is the largest polynomial that divides evenly into the polynomials.

Howto: Given a polynomial expression, factor out the greatest common factor

1. Identify the GCF of the coefficients.
2. Identify the GCF of the variables.
3. Combine to find the GCF of the expression.
4. Determine what the GCF needs to be multiplied by to obtain each term in the expression.
5. Write the factored expression as the product of the GCF and the sum of the terms we need to multiply by.

Example 1.5.1: Factoring the Greatest Common Factor

Factor $6x^3y^3+45x^2y^2+21xy$.

Solution

First, find the GCF of the expression. The GCF of 6, 45, and 21 is 3. The GCF of x^3,x^2, and x is x. (Note that the GCF of a set of expressions in the form x^n will always be the exponent of lowest degree.) And the GCF of y^3,y^2, and y is y. Combine these to find the GCF of the polynomial, $3xy$.

Next, determine what the GCF needs to be multiplied by to obtain each term of the polynomial. We find that

- $3xy(2x^2y^2)=6x^3y^3$,
- $3xy(15xy)=45x^2y^2$, and
- $3xy(7)=21xy$.

Finally, write the factored expression as the product of the GCF and the sum of the terms we needed to multiply by.

$$(3xy)(2x^2y^2+15xy+7)$$

Analysis

After factoring, we can check our work by multiplying. Use the distributive property to confirm that

$$(3xy)(2x^2y^2+15xy+7)=6x^3y^3+45x^2y^2+21xy$$

Exercise 1.5.1

Factor $x(b^2-a)+6(b^2-a)$ by pulling out the GCF.

Answer

$(b^2-a)(x+6)$

Factoring a Trinomial with Leading Coefficient 1

Although we should always begin by looking for a GCF, pulling out the GCF is not the only way that polynomial expressions can be factored. The polynomial x^2+5x+6 has a GCF of 1, but it can be written as the product of the factors $(x+2)$ and $(x+3)$.

Trinomials of the form x^2+bx+c can be factored by finding two numbers with a product of c and a sum of b. The trinomial $x^2+10x+16$, for example, can be factored using the numbers 2 and 8 because the product of those numbers is 16 and their sum is 10. The trinomial can be rewritten as the product of $(x+2)$ and $(x+8)$.

FACTORING A TRINOMIAL WITH LEADING COEFFICIENT 1

A trinomial of the form x^2+bx+c can be written in factored form as $(x+p)(x+q)$ where $pq=c$ and $p+q=b$.

Q&A: Can every trinomial be factored as a product of binomials?

No. Some polynomials cannot be factored. These polynomials are said to be *prime*.

Howto: Given a trinomial in the form $x^2 + bx + c$, factor it

1. List factors of c.
2. Find p and q, a pair of factors of c with a sum of b.
3. Write the factored expression $(x+p)(x+q)$.

Example 1.5.2: Factoring a Trinomial with Leading Coefficient 1

Factor $x^2+2x-15$.

Solution

We have a trinomial with leading coefficient 1, $b=2$, and $c=-15$. We need to find two numbers with a product of -15 and a sum of 2. In Table 1.5.1, we list factors until we find a pair with the desired sum.

Table 1.5.1

Factors of −15	Sum of Factors
1,−15	−14
−1,15	14
3,−5	−2
−3,5	

Now that we have identified p and q as -3 and 5, write the factored form as $(x-3)(x+5)$.

Analysis

We can check our work by multiplying. Use FOIL to confirm that $(x-3)(x+5)=x^2+2x-15$.

Q&A: Does the order of the factors matter?

No. Multiplication is commutative, so the order of the factors does not matter.

Exercise 1.5.2

Factor x^2-7x+6 .

Answer

$(x-6)(x-1)$

Factoring by Grouping

Trinomials with leading coefficients other than 1 are slightly more complicated to factor. For these trinomials, we can **factor by grouping** by dividing the x term into the sum of two terms, factoring each portion of the expression separately, and then factoring out the GCF of the entire expression. The trinomial $2x^2+5x+3$ can be rewritten as $(2x+3)(x+1)$ using this process. We begin by rewriting the original expression as $2x^2+2x+3x+3$ and then factor each portion of the expression to obtain $2x(x+1)+3(x+1)$. We then pull out the GCF of $(x+1)$ to find the factored expression.

Factor by Grouping

To factor a trinomial in the form ax^2+bx+c by grouping, we find two numbers with a product of ac and a sum of b. We use these numbers to divide the x term into the sum of two terms and factor each portion of the expression separately, then factor out the GCF of the entire expression.

Howto: Given a trinomial in the form ax^2+bx+c, factor by grouping.

1. List factors of ac.
2. Find p and q, a pair of factors of ac with a sum of b.
3. Rewrite the original expression as $ax^2+px+qx+c$.
4. Pull out the GCF of ax^2+px.
5. Pull out the GCF of $qx+c$.
6. Factor out the GCF of the expression.

Example 1.5.3: Factoring a Trinomial by Grouping

Factor $5x^2+7x-6$ by grouping.

Solution

We have a trinomial with $a=5, b=7$, and $c=-6$. First, determine $ac=-30$. We need to find two numbers with a product of -30 and a sum of 7. In the table below, we list factors until we find a pair with the desired sum.

Table 1.5.2

Factors of −30	Sum of Factors
1,−30	−29
−1,30	29
2,−15	−13
−2,15	13
3,−10	−7
−3,10	7

So $p=-3$ and $q=10$.

$5x^2-3x+10x-6$ Rewrite the original expression as $ax^2+px+qx+c$.

$x(5x-3)+2(5x-3)$ Factor out the GCF of each part

$(5x-3)(x+2)$ Factor out the GCF of the expression.

Analysis

We can check our work by multiplying. Use FOIL to confirm that $(5x-3)(x+2)=5x^2+7x-6$.

Exercise 1.5.3

Factor:

a. $2x^2+9x+9$
b. $6x^2+x-1$

Answer a

$(2x+3)(x+3)$

Answer b

$(3x-1)(2x+1)$

Factoring a Perfect Square Trinomial

A perfect square trinomial is a trinomial that can be written as the square of a binomial. Recall that when a binomial is squared, the result is the square of the first term added to twice the product of the two terms and the square of the last term.

$$a^2+2ab+b^2=(a+b)^2 \tag{1.5.1}$$

and

$$a^2-2ab+b^2=(a-b)^2 \tag{1.5.2}$$

We can use this equation to factor any perfect square trinomial.

Perfect Square Trinomials

A perfect square trinomial can be written as the square of a binomial:

$$a^2+2ab+b^2=(a+b)^2 \tag{1.5.3}$$

Howto: Given a perfect square trinomial, factor it into the square of a binomial

1. Confirm that the first and last term are perfect squares.
2. Confirm that the middle term is twice the product of ab.
3. Write the factored form as $(a+b)^2$.

Example 1.5.4: Factoring a Perfect Square Trinomial

Factor $25x^2+20x+4$.

Solution

Notice that $25x^2$ and 4 are perfect squares because $25x^2=(5x)^2$ and $4=2^2$. Then check to see if the middle term is twice the product of $5x$ and 2. The middle term is, indeed, twice the product: $2(5x)(2)=20x$. Therefore, the trinomial is a perfect square trinomial and can be written as $(5x+2)^2$.

Exercise 1.5.4

Factor $49x^2-14x+1$.

Answer

$(7x-1)^2$

Factoring a Difference of Squares

A difference of squares is a perfect square subtracted from a perfect square. Recall that a difference of squares can be rewritten as factors containing the same terms but opposite signs because the middle terms cancel each other out when the two factors are multiplied.

$$a^2-b^2=(a+b)(a-b) \tag{1.5.4}$$

We can use this equation to factor any differences of squares.

Differences of Squares

A difference of squares can be rewritten as two factors containing the same terms but opposite signs.

$$a^2 - b^2 = (a+b)(a-b) \tag{1.5.5}$$

Howto: Given a difference of squares, factor it into binomials

1. Confirm that the first and last term are perfect squares.
2. Write the factored form as $(a+b)(a-b)$.

Example 1.5.5: Factoring a Difference of Squares

Factor $9x^2 - 25$.

Solution

Notice that $9x^2$ and 25 are perfect squares because $9x^2 = (3x)^2$ and $25 = 5^2$. The polynomial represents a difference of squares and can be rewritten as $(3x+5)(3x-5)$.

Exercise 1.5.5

Factor $81y^2 - 100$.

Answer

$(9y+10)(9y-10)$

Q&A: Is there a formula to factor the sum of squares?

No. A sum of squares cannot be factored.

Factoring the Sum and Difference of Cubes

Now, we will look at two new special products: the sum and difference of cubes. Although the sum of squares cannot be factored, the sum of cubes can be factored into a binomial and a trinomial.

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2) \tag{1.5.6}$$

Similarly, the sum of cubes can be factored into a binomial and a trinomial, but with different signs.

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2) \tag{1.5.7}$$

We can use the acronym SOAP to remember the signs when factoring the sum or difference of cubes. The first letter of each word relates to the signs: Same Opposite Always Positive. For example, consider the following example.

$$x^3 - 2^3 = (x-2)(x^2 + 2x + 4) \tag{1.5.8}$$

The sign of the first 2 is the same as the sign between $x^3 - 2^3$. The sign of the $2x$ term is opposite the sign between $x^3 - 2^3$. And the sign of the last term, 4, is always positive.

Sum and Difference of Cubes

We can factor the sum of two cubes as

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2) \tag{1.5.9}$$

We can factor the difference of two cubes as

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2) \tag{1.5.10}$$

Howto: Given a sum of cubes or difference of cubes, factor it

1. Confirm that the first and last term are cubes, a^3+b^3 or a^3-b^3.
2. For a sum of cubes, write the factored form as $(a+b)(a^2-ab+b^2)$. For a difference of cubes, write the factored form as $(a-b)(a^2+ab+b^2)$.

Example 1.5.6: Factoring a Sum of Cubes

Factor x^3+512.

Solution

Notice that x^3 and 512 are cubes because $8^3=512$. Rewrite the sum of cubes as $(x+8)(x^2-8x+64)$.

Analysis

After writing the sum of cubes this way, we might think we should check to see if the trinomial portion can be factored further. However, the trinomial portion cannot be factored, so we do not need to check.

Exercise 1.5.6

Factor the sum of cubes: $216a^3+b^3$.

Answer

$(6a+b)(36a^2-6ab+b^2)$

Example 1.5.7: Factoring a Difference of Cubes

Factor $8x^3-125$.

Solution

Notice that $8x^3$ and 125 are cubes because $8x^3=(2x)^3$ and $125=5^3$. Write the difference of cubes as $(2x-5)(4x^2+10x+25)$.

Analysis

Just as with the sum of cubes, we will not be able to further factor the trinomial portion.

Exercise 1.5.7

Factor the difference of cubes: $1000x^3-1$

Answer

$(10x-1)(100x^2+10x+1)$

Factoring Expressions with Fractional or Negative Exponents

Expressions with fractional or negative exponents can be factored by pulling out a GCF. Look for the variable or exponent that is common to each term of the expression and pull out that variable or exponent raised to the lowest power. These expressions follow the same factoring rules as those with integer exponents. For instance, $2x^{\frac{1}{4}}+5x^{\frac{3}{4}}$ can be factored by pulling out $x^{\frac{1}{4}}$ and being rewritten as $x^{\frac{1}{4}}(2+5x^{\frac{1}{2}})$.

Example 1.5.8: Factoring an Expression with Fractional or Negative Exponents

Factor $3x(x+2)^{-\frac{1}{3}}+4(x+2)^{\frac{2}{3}}$.

Solution

Factor out the term with the lowest value of the exponent. In this case, that would be $(x+2)^{-\frac{1}{3}}$.

$$(x+2)^{-\frac{1}{3}}(3x+4(x+2)) \qquad \text{Factor out the GCF}$$
$$(x+2)^{-\frac{1}{3}}(3x+4x+8) \qquad \text{Simplify}$$
$$(x+2)^{-\frac{1}{3}}(7x+8)$$

Exercise 1.5.8

Factor $2(5a-1)^{\frac{3}{4}}+7a(5a-1)^{-\frac{1}{4}}$.

Answer

$$(5a-1)^{-\frac{1}{4}}(17a-2)$$

Media

Access these online resources for additional instruction and practice with factoring polynomials.

1. Identify GCF
2. Factor Trinomials when a Equals 1
3. Factor Trinomials when a is not equal to 1
4. Factor Sum or Difference of Cubes

Key Equations

difference of squares	$a^2-b^2=(a+b)(a-b)$
perfect square trinomial	$a^2+2ab+b^2=(a+b)^2$
sum of cubes	$a^3+b^3=(a+b)(a^2-ab+b^2)$
difference of cubes	$a^3-b^3=(a-b)(a^2+ab+b^2)$

- The greatest common factor, or GCF, can be factored out of a polynomial. Checking for a GCF should be the first step in any factoring problem. See Example.
- Trinomials with leading coefficient 1 can be factored by finding numbers that have a product of the third term and a sum of the second term. See Example.
- Trinomials can be factored using a process called factoring by grouping. See Example.
- Perfect square trinomials and the difference of squares are special products and can be factored using equations. See Example and Example.
- The sum of cubes and the difference of cubes can be factored using equations. See Example and Example.
- Polynomials containing fractional and negative exponents can be factored by pulling out a GCF. See Example.

Homework Exercises 1.5

WeBWorK Problems:

https://math.libretexts.org/@go/page/106321

Written Problems:

Find the greatest common factor of each term to factor the following expressions.

1. $14x + 4xy - 18xy^2$

2. $6xh - h^2$

3. $-2x^2 - 10x$

4. $x^2(7 - x) + 3(7 - x)$

Factor the following trinomials, if possible. If it is not possible, make the expression a function of *x* and graph it to show if the expression is irreducible (prime) or if there is another reason it is not factorable by the methods of this section.

5. $6n^2 - 19n - 11$

6. $10h^2 - 9h - 9$

7. $3t^2 - t + 1$

8. $2y^2 - y - 15$

Factor the following difference of squares.

9. $4m^2 - 9$

10. $144b^2 - 25c^2$

11. $81y^4 - 256$

Mixed Practice: Factor the following expressions, if possible.

12. $m^2 - 20m + 100$

13. $25x^4 - 1$

14. $a^2 - 8a + 16$

15. $x(5b + 1) - 2(5b + 1)$

This page titled 1.5: Factoring Polynomials is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **1.5: Factoring Polynomials** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/precalculus.

1.6: Rational Expressions

Preview Video for Section 1.6.



Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login

Learning Objectives

In this section students will:

- Simplify rational expressions.
- Multiply rational expressions.
- Divide rational expressions.
- Add and subtract rational expressions.
- Simplify complex rational expressions.

A pastry shop has fixed costs of $\$280$ per week and variable costs of $\$9$ per box of pastries. The shop's costs per week in terms of x, the number of boxes made, is $280+9x$. We can divide the costs per week by the number of boxes made to determine the cost per box of pastries.

$$\frac{280+9x}{x}$$

Notice that the result is a polynomial expression divided by a second polynomial expression. In this section, we will explore quotients of polynomial expressions.

Simplifying Rational Expressions

The quotient of two polynomial expressions is called a rational expression. We can apply the properties of fractions to rational expressions, such as simplifying the expressions by canceling common factors from the numerator and the denominator. To do this, we first need to factor both the numerator and denominator. Let's start with the rational expression shown.

$$\frac{x^2+8x+16}{x^2+11x+28}$$

We can factor the numerator and denominator to rewrite the expression.

$$\frac{(x+4)^2}{(x+4)(x+7)}$$

Then we can simplify that expression by canceling the common factor $(x+4)$.

$$\frac{x+4}{x+7}$$

Howto: Given a rational expression, simplify it

1. Factor the numerator and denominator.
2. Cancel any common factors.

Example 1.6.1: Simplifying Rational Expressions

Simplify $\dfrac{x^2-9}{x^2+4x+3}$

Solution

$$\frac{(x+3)(x-3)}{(x+3)(x+1)} \qquad \text{Factor the numerator and the denominator}$$

$$\frac{x-3}{x+1} \qquad \text{Cancel common factor } (x+3)$$

Analysis

We can cancel the common factor because any expression divided by itself is equal to 1.

Q&A

Can the x^2 term be cancelled in the last example?

No. A factor is an expression that is multiplied by another expression. The x^2 term is not a factor of the numerator or the denominator.

Exercise 1.6.1

Simplify $\dfrac{x-6}{x^2-36}$

Answer

$$\frac{1}{x+6}$$

Multiplying Rational Expressions

Multiplication of rational expressions works the same way as multiplication of any other fractions. We multiply the numerators to find the numerator of the product, and then multiply the denominators to find the denominator of the product. Before multiplying, it is helpful to factor the numerators and denominators just as we did when simplifying rational expressions. We are often able to simplify the product of rational expressions.

Howto: Given two rational expressions, multiply them

1. Factor the numerator and denominator.
2. Multiply the numerators.
3. Multiply the denominators.
4. Simplify.

Example 1.6.2: Multiplying Rational Expressions

Multiply the rational expressions and show the product in simplest form:

$$\frac{(x+5)(x-1)}{3(x+6)} \times \frac{(2x-1)}{(x+5)}$$

Solution

$\frac{(x+5)(x-1)}{3(x+6)} \times \frac{(2x-1)}{(x+5)}$	Factor the numerator and denominator.
$\frac{(x+5)(x-1)(2x-1)}{3(x+6)(x+5)}$	Multiply numerators and denominators
$\frac{(x-1)(2x-1)}{3(x+6)}$	Cancel common factors to simplify

? Exercise 1.6.2

Multiply the rational expressions and show the product in simplest form:

$$\frac{x^2+11x+30}{x^2+5x+6} \times \frac{x^2+7x+12}{x^2+8x+16}$$

Answer

$$\frac{(x+5)(x+6)}{(x+2)(x+4)}$$

Dividing Rational Expressions

Division of rational expressions works the same way as division of other fractions. To divide a rational expression by another rational expression, multiply the first expression by the reciprocal of the second. Using this approach, we would rewrite $\frac{1}{x} \div \frac{x^2}{3}$ as the product $\frac{1}{x} \cdot \frac{3}{x^2}$. Once the division expression has been rewritten as a multiplication expression, we can multiply as we did before.

$$\frac{1}{x} \cdot \frac{3}{x^2} = \frac{3}{x^3}$$

Howto: Given two rational expressions, divide them

1. Rewrite as the first rational expression multiplied by the reciprocal of the second.
2. Factor the numerators and denominators.
3. Multiply the numerators.
4. Multiply the denominators.
5. Simplify.

✓ Example 1.6.3: Dividing Rational Expressions

Divide the rational expressions and express the quotient in simplest form:

$$\frac{2x^2+x-6}{x^2-1} \div \frac{x^2-4}{x^2+2x+1}$$

Solution

$$\frac{2x^2+x-6}{x^2-1} \div \frac{x^2-4}{x^2+2x+1}$$

$$\frac{2x^2+x-6}{x^2-1} \times \frac{x^2+2x+1}{x^2-4}$$ Rewrite as a multiplication problem

$$\frac{(2x-3)(x+2)}{(x-1)(x+1)} \times \frac{(x+1)(x+1)}{(x-2)(x+2)}$$ Factor the numerator and denominator.

$$\frac{(2x-3)(x+2)(x+1)(x+1)}{(x-1)(x+1)(x-2)(x+2)}$$ Multiply numerators and denominators

$$\frac{(2x-3)(x+1)}{(x-1)(x-2)}$$ Cancel common factors to simplify

? Exercise 1.6.3

Divide the rational expressions and express the quotient in simplest form:

$$\frac{9x^2-16}{3x^2+17x-28} \div \frac{3x^2-2x-8}{x^2+5x-14}$$

Answer

1

Adding and Subtracting Rational Expressions

Adding and subtracting rational expressions works just like adding and subtracting numerical fractions. To add fractions, we need to find a common denominator. Let's look at an example of fraction addition.

$$\begin{aligned}\frac{5}{24}+\frac{1}{40} &= \frac{25}{120}+\frac{3}{120}\\ &= \frac{28}{120}\\ &= \frac{7}{30}\end{aligned}$$

We have to rewrite the fractions so they share a common denominator before we are able to add. We must do the same thing when adding or subtracting rational expressions.

The easiest common denominator to use will be the **least common denominator**, or LCD. The LCD is the smallest multiple that the denominators have in common. To find the LCD of two rational expressions, we factor the expressions and multiply all of the distinct factors. For instance, if the factored denominators were $(x+3)(x+4)$ and $(x+4)(x+5)$, then the LCD would be $(x+3)(x+4)(x+5)$.

Once we find the LCD, we need to multiply each expression by the form of 1 that will change the denominator to the LCD. We would need to multiply the expression with a denominator of $(x+3)(x+4)$ by $\frac{x+5}{x+5}$ and the expression with a denominator of $(x+4)(x+5)$ by $\frac{x+3}{x+3}$.

Howto: Given two rational expressions, add or subtract them

1. Factor the numerator and denominator.
2. Find the LCD of the expressions.
3. Multiply the expressions by a form of 1 that changes the denominators to the LCD.
4. Add or subtract the numerators.
5. Simplify.

Example 1.6.4: Adding Rational Expressions

Add the rational expressions:

$$\frac{5}{x}+\frac{6}{y}$$

Solution

First, we have to find the LCD. In this case, the LCD will be xy. We then multiply each expression by the appropriate form of 1 to obtain xy as the denominator for each fraction.

$$\frac{5}{x}\times\frac{y}{y}+\frac{6}{y}\times\frac{x}{x}$$
$$\frac{5y}{xy}+\frac{6x}{xy}$$

Now that the expressions have the same denominator, we simply add the numerators to find the sum.

$$\frac{6x+5y}{xy}$$

Analysis

Multiplying by $\frac{y}{y}$ or $\frac{x}{x}$ does not change the value of the original expression because any number divided by itself is 1, and multiplying an expression by 1 gives the original expression.

Example 1.6.5: Subtracting Rational Expressions

Subtract the rational expressions:

$$\frac{6}{x^2+4x+4}-\frac{2}{x^2-4} \tag{1.6.1}$$

Solution

$\frac{6}{(x+2)^2}-\frac{2}{(x+2)(x-2)}$	Factor
$\frac{6}{(x+2)^2}\times\frac{x-2}{x-2}-\frac{2}{(x+2)(x-2)}\times\frac{x+2}{x+2}$	Multiply each fraction to get LCD as denominator
$\frac{6(x-2)}{(x+2)^2(x-2)}-\frac{2(x+2)}{(x+2)^2(x-2)}$	Multiply
$\frac{6x-12-(2x+4)}{(x+2)^2(x-2)}$	Apply distributive property
$\frac{4x-16}{(x+2)^2(x-2)}$	Subtract
$\frac{4(x-4)}{(x+2)^2(x-2)}$	Simplify

Q&A

Do we have to use the LCD to add or subtract rational expressions?

No. Any common denominator will work, but it is easiest to use the LCD.

Exercise 1.6.4

Subtract the rational expressions: $\dfrac{3}{x+5}-\dfrac{1}{x-3}$

Answer

$$\frac{2(x-7)}{(x+5)(x-3)}$$

Simplifying Complex Rational Expressions

A complex rational expression is a rational expression that contains additional rational expressions in the numerator, the denominator, or both. We can simplify complex rational expressions by rewriting the numerator and denominator as single rational expressions and dividing. The complex rational expression $\dfrac{a}{\dfrac{1}{b}+c}$ can be simplified by rewriting the numerator as the fraction $\dfrac{a}{1}$ and combining the expressions in the denominator as $\dfrac{1+bc}{b}$. We can then rewrite the expression as a multiplication problem using the reciprocal of the denominator. We get $\dfrac{a}{1}\cdot\dfrac{b}{1+bc}$, which is equal to $\dfrac{ab}{1+bc}$.

Howto: Given a complex rational expression, simplify it

1. Combine the expressions in the numerator into a single rational expression by adding or subtracting.
2. Combine the expressions in the denominator into a single rational expression by adding or subtracting.
3. Rewrite as the numerator divided by the denominator.
4. Rewrite as multiplication.
5. Multiply.
6. Simplify.

Example 1.6.6: Simplifying Complex Rational Expressions

Simplify: $\dfrac{y+\dfrac{1}{x}}{\dfrac{x}{y}}$

Solution

Begin by combining the expressions in the numerator into one expression.

$$y\times\frac{x}{x}+\frac{1}{x} \qquad \text{Multiply by } \frac{x}{x} \text{ to get LCD as denominator}$$

$$\frac{xy}{x}+\frac{1}{x}$$

$$\frac{xy+1}{x} \qquad \text{Add numerators}$$

Now the numerator is a single rational expression and the denominator is a single rational expression.

We can rewrite this as division, and then multiplication.

$$\frac{\frac{xy+1}{x}}{\frac{x}{y}}$$

$$\frac{xy+1}{x} \div \frac{x}{y}$$

$$\frac{xy+1}{x} \times \frac{y}{x} \quad \text{Rewrite as multiplication}$$

$$\frac{y(xy+1)}{x^2} \quad \text{Multiply}$$

? Exercise 1.6.5

Simplify: $\dfrac{\frac{x}{y} - \frac{y}{x}}{y}$

Answer

$$\frac{x^2 - y^2}{xy^2}$$

Q&A

Can a complex rational expression always be simplified?

Yes. We can always rewrite a complex rational expression as a simplified rational expression.

Media

Access these online resources for additional instruction and practice with rational expressions.

1. Simplify Rational Expressions
2. Multiply and Divide Rational Expressions
3. Add and Subtract Rational Expressions
4. Simplify a Complex Fraction

Key Concepts

- Rational expressions can be simplified by cancelling common factors in the numerator and denominator. See Example.
- We can multiply rational expressions by multiplying the numerators and multiplying the denominators. See Example.
- To divide rational expressions, multiply by the reciprocal of the second expression. See Example.
- Adding or subtracting rational expressions requires finding a common denominator. See Example and Example.
- Complex rational expressions have fractions in the numerator or the denominator. These expressions can be simplified. See Example.

Homework Exercises 1.6

WeBWorK Problems:

Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login



Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login



Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login



Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login

Written Problems:

A simplified rational expression is a single fraction with no common factors. Simplify the following rational expressions.

1. $\frac{x^2-16}{x^2-5x+4}$
2. $\frac{3y^2-7y-6}{2y^2-3y-9} \div \frac{y^2+y-2}{2y^2+y-3}$
3. $\frac{t^2-1}{3t} \times \frac{6}{t^2}$
4. $\frac{4}{x} + \frac{10}{y}$
5. $\frac{12}{2q} - \frac{6}{3p}$
6. $\frac{c+2}{3} - \frac{c-4}{4}$
7. $\frac{\frac{2}{a}+\frac{7}{b}}{b}$
8. $\frac{\frac{x}{4}-\frac{p}{8}}{p}$
9. $\frac{x}{\frac{3}{x}-\frac{x}{7}}$
10. $\frac{3x}{x+1} - \frac{2x+1}{x-2}$

This page titled 1.6: Rational Expressions is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **1.6: Rational Expressions** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/precalculus.

1.7: The Rectangular Coordinate Systems and Graphs

Learning Objectives

- Plot ordered pairs in a Cartesian coordinate system.
- Graph equations by plotting points.
- Graph equations with a graphing utility.
- Find x-intercepts and y-intercepts.
- Use the distance formula.
- Use the midpoint formula.

Tracie set out from Elmhurst, IL, to go to Franklin Park. On the way, she made a few stops to do errands. Each stop is indicated by a red dot in Figure 1.7.1. Laying a rectangular coordinate grid over the map, we can see that each stop aligns with an intersection of grid lines. In this section, we will learn how to use grid lines to describe locations and changes in locations.

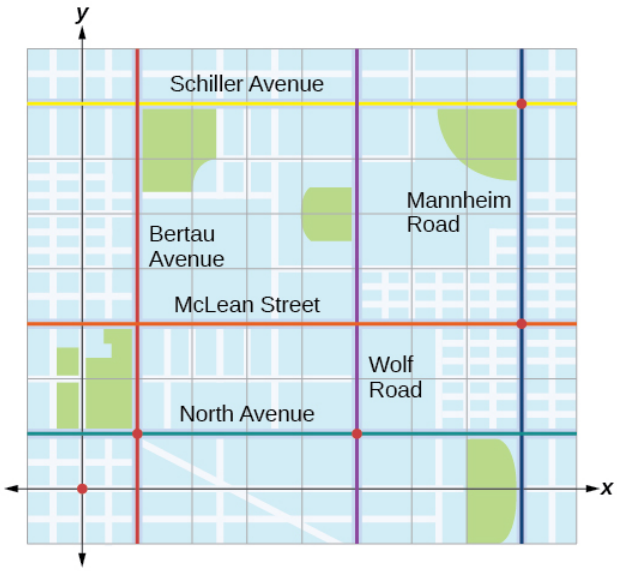


Figure 1.7.1

Plotting Ordered Pairs in the Cartesian Coordinate System

An old story describes how seventeenth-century philosopher/mathematician René Descartes invented the system that has become the foundation of algebra while sick in bed. According to the story, Descartes was staring at a fly crawling on the ceiling when he realized that he could describe the fly's location in relation to the perpendicular lines formed by the adjacent walls of his room. He viewed the perpendicular lines as horizontal and vertical axes. Further, by dividing each axis into equal unit lengths, Descartes saw that it was possible to locate any object in a two-dimensional plane using just two numbers—the displacement from the horizontal axis and the displacement from the vertical axis.

While there is evidence that ideas similar to Descartes' grid system existed centuries earlier, it was Descartes who introduced the components that comprise the Cartesian coordinate system, a grid system having perpendicular axes. Descartes named the horizontal axis the x-axis and the vertical axis the y-axis.

The Cartesian coordinate system, also called the *rectangular coordinate system*, is based on a two-dimensional plane consisting of the x-axis and the y-axis. Perpendicular to each other, the axes divide the plane into four sections. Each section is called a quadrant; the quadrants are numbered counterclockwise as shown in Figure 1.7.2.

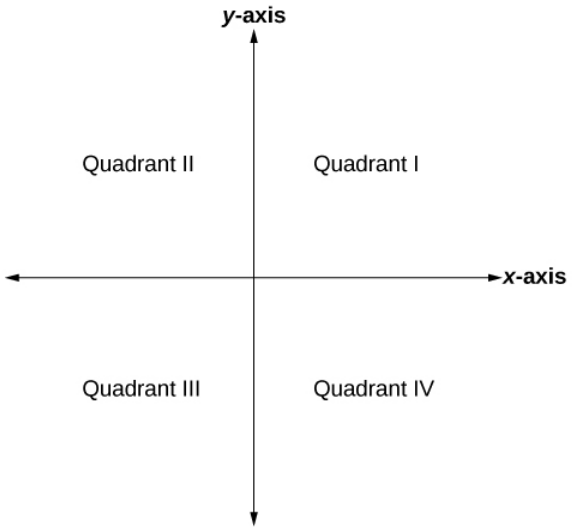


Figure 1.7.2

The center of the plane is the point at which the two axes cross. It is known as the origin, or point $(0, 0)$. From the origin, each axis is further divided into equal units: increasing, positive numbers to the right on the x-axis and up the y-axis; decreasing, negative numbers to the left on the x-axis and down the y-axis. The axes extend to positive and negative infinity as shown by the arrowheads in Figure 1.7.3.

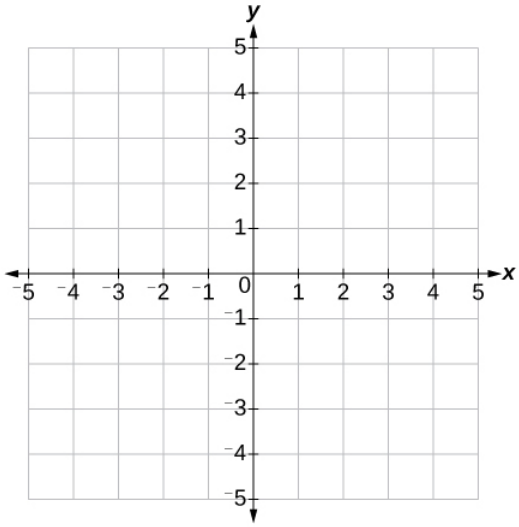


Figure 1.7.3

Each point in the plane is identified by its x-coordinate, or horizontal displacement from the origin, and its y-coordinate, or vertical displacement from the origin. Together, we write them as an ordered pair indicating the combined distance from the origin in the form (x, y). An ordered pair is also known as a coordinate pair because it consists of x- and y-coordinates. For example, we can represent the point $(3, -1)$ in the plane by moving three units to the right of the origin in the horizontal direction, and one unit down in the vertical direction. See Figure 1.7.4.

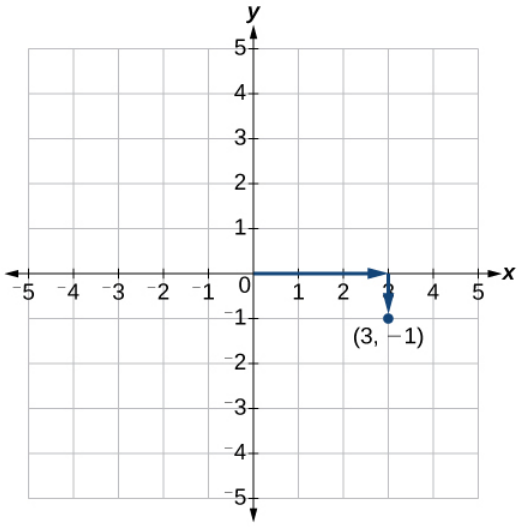


Figure 1.7.4

When dividing the axes into equally spaced increments, note that the x-axis may be considered separately from the y-axis. In other words, while the x-axis may be divided and labeled according to consecutive integers, the y-axis may be divided and labeled by increments of 2, or 10, or 100. In fact, the axes may represent other units, such as years against the balance in a savings account, or quantity against cost, and so on. Consider the rectangular coordinate system primarily as a method for showing the relationship between two quantities.

Cartesian Coordinate System

A two-dimensional plane where the

- x-axis is the horizontal axis
- y-axis is the vertical axis

A point in the plane is defined as an ordered pair, (x, y), such that x is determined by its horizontal distance from the origin and y is determined by its vertical distance from the origin.

Example 1.7.1: Plotting Points in a Rectangular Coordinate System

Plot the points $(-2, 4)$, $(3, 3)$, and $(0, -3)$ in the plane.

Solution

To plot the point $(-2, 4)$, begin at the origin. The x-coordinate is -2, so move two units to the left. The y-coordinate is 4, so then move four units up in the positive y direction.

To plot the point $(3, 3)$, begin again at the origin. The x-coordinate is 3, so move three units to the right. The y-coordinate is also 3, so move three units up in the positive y direction.

To plot the point $(0, -3)$, begin again at the origin. The x-coordinate is 0. This tells us not to move in either direction along the x-axis. The y-coordinate is -3, so move three units down in the negative y direction. See the graph in Figure $1.7.5$.

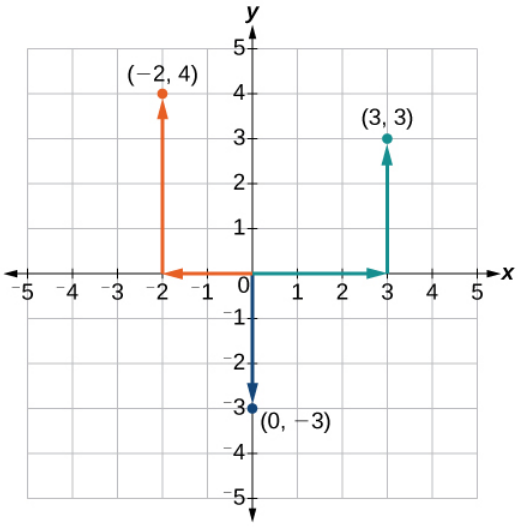


Figure 1.7.5

Analysis

Note that when either coordinate is zero, the point must be on an axis. If the x-coordinate is zero, the point is on the y-axis. If the y-coordinate is zero, the point is on the x-axis.

Graphing Equations by Plotting Points

We can plot a set of points to represent an equation. When such an equation contains both an x variable and a y variable, it is called an **equation in two variables**. Its graph is called a **graph in two variables**. Any graph on a two-dimensional plane is a graph in two variables.

Suppose we want to graph the equation $y = 2x - 1$. We can begin by substituting a value for x into the equation and determining the resulting value of y. Each pair of x- and y-values is an ordered pair that can be plotted. Table $1.7.1$ lists values of x from -3 to 3 and the resulting values for y.

Table 1.7.1

x	$y = 2x - 1$	(x, y)
-3	$y = 2(-3) - 1 = -7$	$(-3, -7)$
-2	$y = 2(-2) - 1 = -5$	$(-2, -5)$
-1	$y = 2(-1) - 1 = -3$	$(-1, -3)$
0	$y = 2(0) - 1 = -1$	$(0, -1)$
1	$y = 2(1) - 1 = 1$	$(1, 1)$
2	$y = 2(2) - 1 = 3$	$(2, 3)$
3	$y = 2(3) - 1 = 5$	$(3, 5)$

We can plot the points in the table. The points for this particular equation form a line, so we can connect them (Figure $1.7.6$). This is not true for all equations.

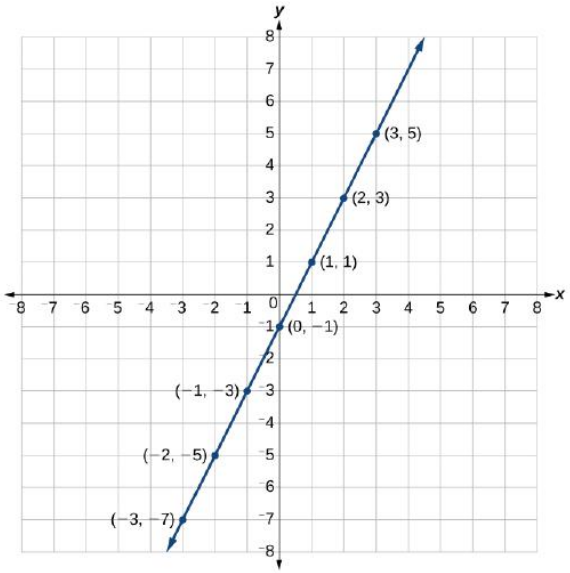


Figure 1.7.6

Note that the x-values chosen are arbitrary, regardless of the type of equation we are graphing. Of course, some situations may require particular values of x to be plotted in order to see a particular result. Otherwise, it is logical to choose values that can be calculated easily, and it is always a good idea to choose values that are both negative and positive. There is no rule dictating how many points to plot, although we need at least two to graph a line. Keep in mind, however, that the more points we plot, the more accurately we can sketch the graph.

Howto: Given an equation, graph by plotting points

1. Make a table with one column labeled x, a second column labeled with the equation, and a third column listing the resulting ordered pairs.
2. Enter x-values down the first column using positive and negative values. Selecting the x-values in numerical order will make the graphing simpler.
3. Select x-values that will yield y-values with little effort, preferably ones that can be calculated mentally.
4. Plot the ordered pairs.
5. Connect the points if they form a line.

Example 1.7.2: Graphing an Equation in Two Variables by Plotting Points

Graph the equation $y = -x + 2$ by plotting points.

Solution

First, we construct a table similar to Table 1.7.2. Choose x values and calculate y.

Table 1.7.2

x	$y = -x + 2$	(x, y)
-5	$y = -(-5) + 2 = 7$	$(-5, 7)$
-3	$y = -(-3) + 2 = 5$	$(-3, 5)$
-1	$y = -(-1) + 2 = 3$	$(-1, 3)$
0	$y = -(0) + 2 = 2$	$(0, 2)$
1	$y = -(1) + 2 = 1$	$(1, 1)$
3	$y = -(3) + 2 = -1$	$(3, -1)$
5	$y = -(5) + 2 = -3$	$(5, -3)$

Now, plot the points. Connect them if they form a line. See Figure 1.7.7.

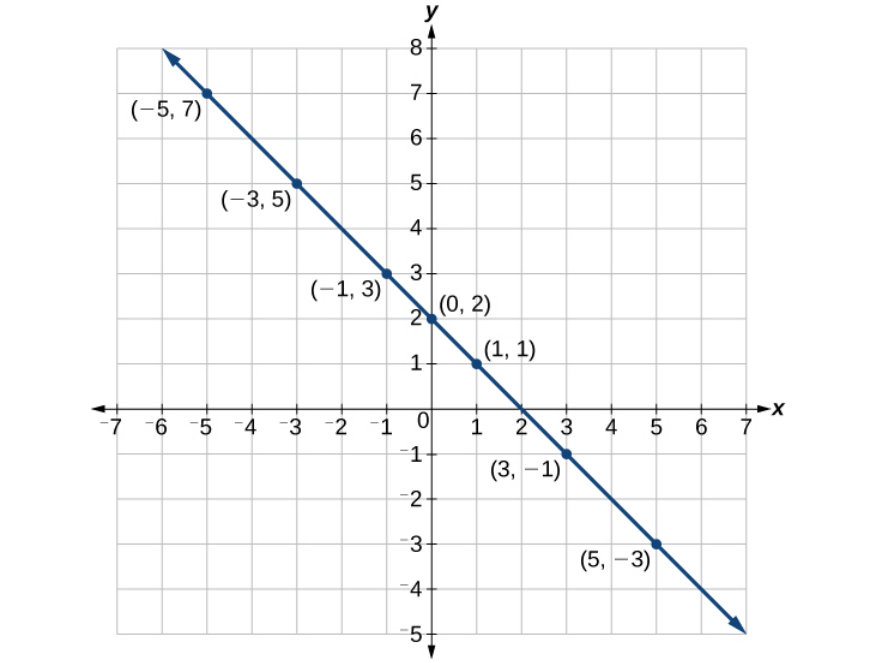


Figure 1.7.7

? Exercise 1.7.1

Construct a table and graph the equation by plotting points: $y = \frac{1}{2}x + 2$.

Answer

Please see Table 1.7.3and graph below.

Table 1.7.3

x	$y = 12x + 2$	(x, y)
-2	$y = 12(-2) + 2 = 1$	$(-2, 1)$
-1	$y = 12(-1) + 2 = 32$	$(-1, 32)$
0	$y = 12(0) + 2 = 2$	$(0, 2)$
1	$y = 12(1) + 2 = 52$	$(1, 52)$
2	$y = 12(2) + 2 = 3$	$(2, 3)$

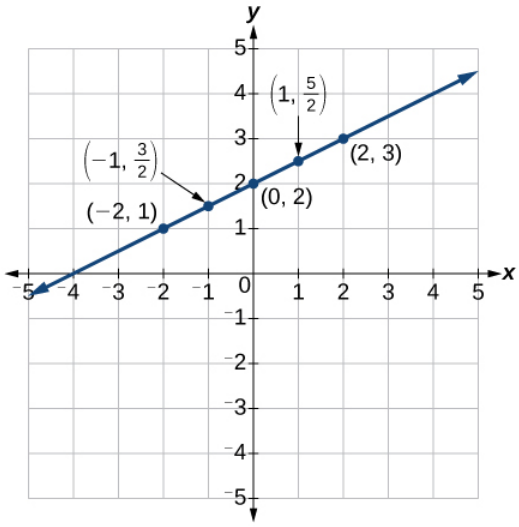
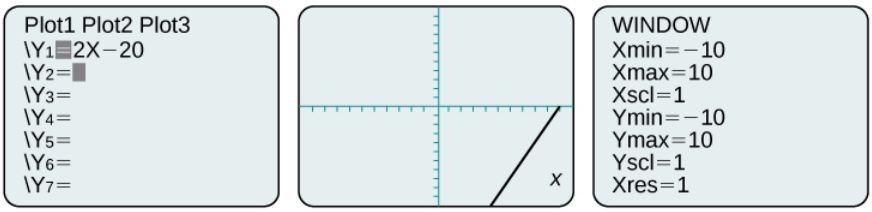


Figure 1.7.8

Graphing Equations with a Graphing Utility

Most graphing calculators require similar techniques to graph an equation. The equations sometimes have to be manipulated so they are written in the style $y =$_____ . The TI-84 Plus, and many other calculator makes and models, have a mode function, which allows the window (the screen for viewing the graph) to be altered so the pertinent parts of a graph can be seen.

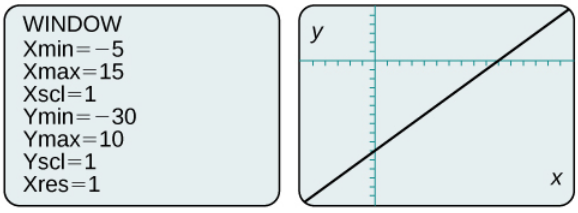
For example, the equation $y = 2x - 20$ has been entered in the TI-84 Plus shown in Figure $1.7.9a$. In Figure $1.7.9b$, the resulting graph is shown. Notice that we cannot see on the screen where the graph crosses the axes. The standard window screen on the TI-84 Plus shows $-10 \leq x \leq 10$, and $-10 \leq y \leq 10$. See Figure \(\PageIndex{9c}\).



(a) (b) (c)

Figure 1.7.9: a. Enter the equation. b. This is the graph in the original window. c. These are the original settings.

By changing the window to show more of the positive x-axis and more of the negative y-axis, we have a much better view of the graph and the x- and y-intercepts. See Figure $1.7.10a$ and Figure $1.7.10b$



(a) (b)

Figure 1.7.10: a. This screen shows the new window settings. b. We can clearly view the intercepts in the new window.

Example 1.7.3: Using a Graphing Utility to Graph an Equation

Use a graphing utility to graph the equation: $y = -\frac{2}{3}x - \frac{4}{3}$.

Solution

Enter the equation in the $y=$ function of the calculator. Set the window settings so that both the x- and y- intercepts are showing in the window. See Figure 1.7.11.

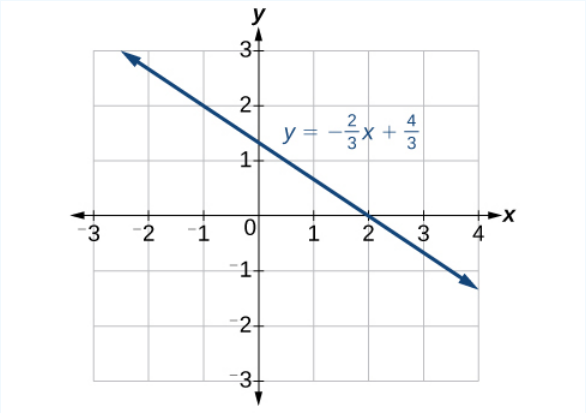


Figure 1.7.11

Finding x-intercepts and y-intercepts

The **intercepts** of a graph are points at which the graph crosses the axes. The x-intercept is the point at which the graph crosses the \(x\)-axis. At this point, the y-coordinate is zero. The y-intercept is the point at which the graph crosses the y-axis. At this point, the x-coordinate is zero.

To determine the x-intercept, we set y equal to zero and solve for x. Similarly, to determine the y-intercept, we set x equal to zero and solve for y. For example, lets find the intercepts of the equation $y=3x-1$.

To find the x-intercept, set $y=0$.

$$\begin{aligned} y &= 3x-1 \\ 0 &= 3x-1 \\ 1 &= 3x \\ \frac{1}{3} &= x \end{aligned}$$

x–intercept: $\left(\frac{1}{3},0\right)$

To find the y-intercept, set $x=0$.

$$\begin{aligned} y &= 3x-1 \\ y &= 3(0)-1 \\ y &= -1 \end{aligned}$$

y–intercept: $(0,-1)$

We can confirm that our results make sense by observing a graph of the equation as in Figure 1.7.12 Notice that the graph crosses the axes where we predicted it would.

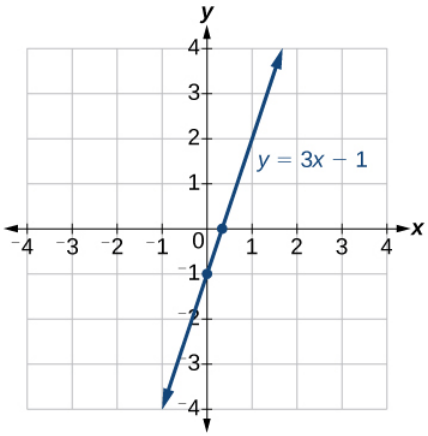


Figure 1.7.12

Howto: GIVEN AN EQUATION, FIND THE INTERCEPTS

1. Find the x-intercept by setting $y = 0$ and solving for x.
2. Find the y-intercept by setting $x = 0$ and solving for y.

Example 1.7.4: Finding the Intercepts of the Given Equation

Find the intercepts of the equation $y = -3x - 4$. Then sketch the graph using only the intercepts.

Solution

Set $y = 0$ to find the x-intercept.

$$\begin{aligned} y &= -3x - 4 \\ 0 &= -3x - 4 \\ 4 &= -3x \\ \frac{4}{3} &= x \end{aligned}$$

x−intercept: $\left(-\frac{4}{3}, 0\right)$

Set $x = 0$ to find the y-intercept.

$$\begin{aligned} y &= -3x - 4 \\ y &= -3(0) - 4 \\ y &= -4 \end{aligned}$$

y−intercept: $(0, -4)$

Plot both points, and draw a line passing through them as in Figure 1.7.13.

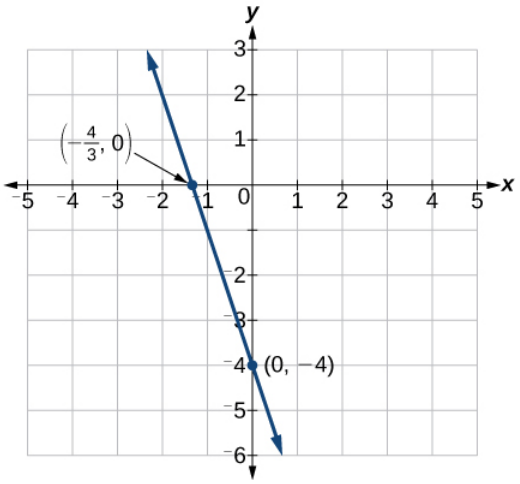


Figure 1.7.13

Exercise 1.7.2

Find the intercepts of the equation and sketch the graph: $y = -\frac{3}{4}x + 3$.

Answer

x-intercept is $(4, 0)$; y-intercept is $(0, 3)$

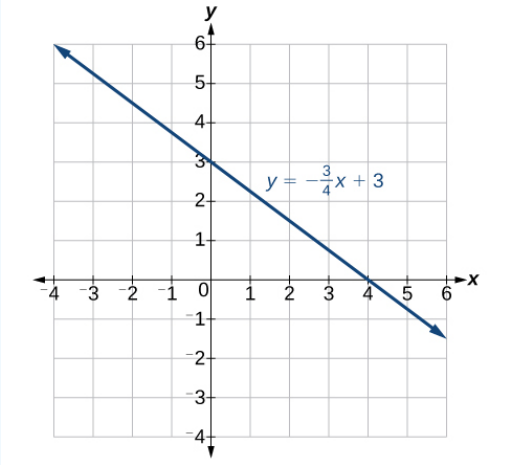


Figure 1.7.14

Using the Distance Formula

Derived from the **Pythagorean Theorem**, the **distance formula** is used to find the distance between two points in the plane. The Pythagorean Theorem, $a^2 + b^2 = c^2$, is based on a right triangle where a and b are the lengths of the legs adjacent to the right angle, and c is the length of the hypotenuse. See Figure 1.7.15.

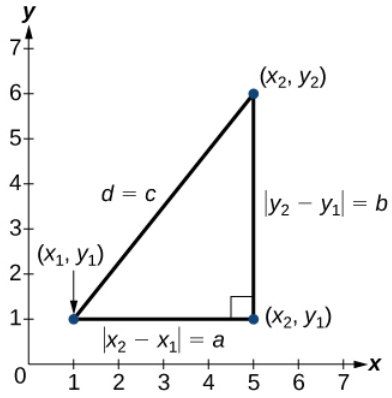


Figure 1.7.15

The relationship of sides $|x_2 - x_1|$ and $|y_2 - y_1|$ to side d is the same as that of sides a and b to side c. We use the absolute value symbol to indicate that the length is a positive number because the absolute value of any number is positive. (For example, $|-3| = 3$.) The symbols $|x_2 - x_1|$ and $|y_2 - y_1|$ indicate that the lengths of the sides of the triangle are positive. To find the length c, take the square root of both sides of the Pythagorean Theorem.

$$c^2 = a^2 + b^2 \rightarrow c = \sqrt{a^2 + b^2} \tag{1.7.1}$$

It follows that the distance formula is given as

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \rightarrow d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1.7.2}$$

We do not have to use the absolute value symbols in this definition because any number squared is positive.

distance between two points

Given endpoints (x_1, y_1) and (x_2, y_2), the distance between two points is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1.7.3}$$

Example 1.7.5: Finding the Distance between Two Points

Find the distance between the points $(-3, -1)$ and $(2, 3)$.

Solution

Let us first look at the graph of the two points. Connect the points to form a right triangle as in Figure 1.7.16

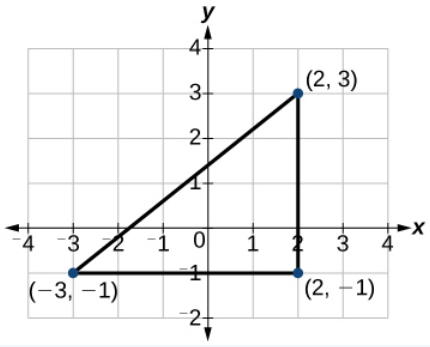


Figure 1.7.16

Then, calculate the length of d using the distance formula.

$$
\begin{aligned}
d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= \sqrt{(2 - (-3))^2 + (3 - (-1))^2} \\
&= \sqrt{(5)^2 + (4)^2} \\
&= \sqrt{25 + 16} \\
&= \sqrt{41}
\end{aligned}
$$

Exercise 1.7.3

Find the distance between two points: $(1, 4)$ and $(11, 9)$.

Answer

$\sqrt{125} = 5\sqrt{5}$

Example 1.7.6: Finding the Distance between Two Locations

Let's return to the situation introduced at the beginning of this section.

Tracie set out from Elmhurst, IL, to go to Franklin Park. On the way, she made a few stops to do errands. Each stop is indicated by a red dot in Figure 1.7.1. Find the total distance that Tracie traveled. Compare this with the distance between her starting and final positions.

Solution

The first thing we should do is identify ordered pairs to describe each position. If we set the starting position at the origin, we can identify each of the other points by counting units east (right) and north (up) on the grid. For example, the first stop is 1 block east and 1 block north, so it is at $(1, 1)$. The next stop is 5 blocks to the east, so it is at $(5, 1)$. After that, she traveled 3 blocks east and 2 blocks north to $(8, 3)$. Lastly, she traveled 4 blocks north to $(8, 7)$. We can label these points on the grid as in Figure 1.7.17.

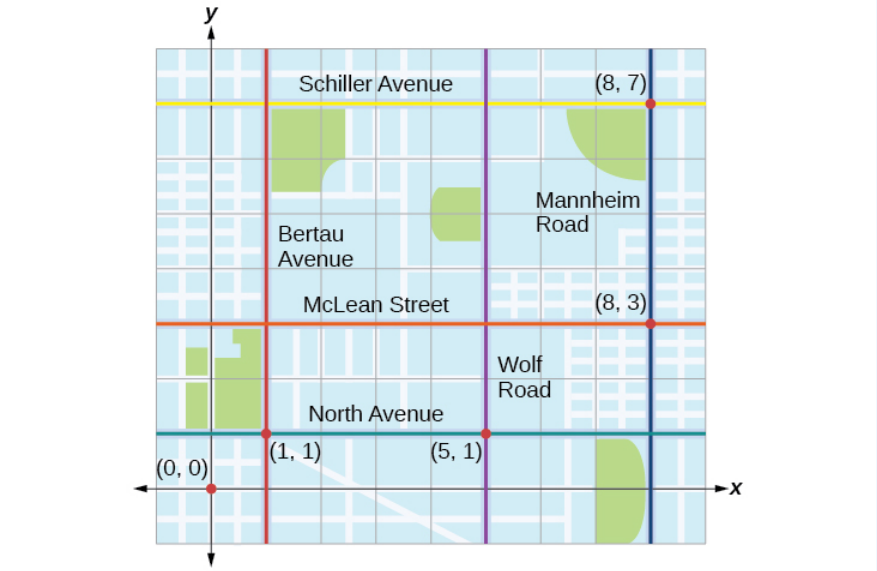


Figure 1.7.17

Next, we can calculate the distance. Note that each grid unit represents $1,000$ feet.

- From her starting location to her first stop at $(1, 1)$, Tracie might have driven north $1,000$ feet and then east $1,000$ feet, or vice versa. Either way, she drove $2,000$ feet to her first stop.
- Her second stop is at $(5, 1)$. So from $(1, 1)$ to $(5, 1)$, Tracie drove east $4,000$ feet.

- Her third stop is at $(8, 3)$. There are a number of routes from $(5, 1)$ to $(8, 3)$. Whatever route Tracie decided to use, the distance is the same, as there are no angular streets between the two points. Let's say she drove east $3,000$ feet and then north $2,000$ feet for a total of $5,000$ feet.
- Tracie's final stop is at $(8, 7)$. This is a straight drive north from $(8, 3)$ for a total of $4,000$ feet.

Next, we will add the distances listed in Table 1.7.4.

Table 1.7.4

From/To	Number of Feet Driven
$(0, 0)$ to $(1, 1)$	$2,000$
$(1, 1)$ to $(5, 1)$	$4,000$
$(5, 1)$ to $(8, 3)$	$5,000$
$(8, 3)$ to $(8, 7)$	$4,000$
Total	$15,000$

The total distance Tracie drove is $15,000$ feet, or 2.84 miles. This is not, however, the actual distance between her starting and ending positions. To find this distance, we can use the distance formula between the points $(0, 0)$ and $(8, 7)$.

$$\begin{aligned} d &= \sqrt{(0-8)^2 + (7-0)^2} \\ &= \sqrt{64+49} \\ &= \sqrt{113} \\ &= 10.63 \text{ units} \end{aligned}$$

At $1,000$ feet per grid unit, the distance between Elmhurst, IL, to Franklin Park is $10,630.14$ feet, or 2.01 miles. The distance formula results in a shorter calculation because it is based on the hypotenuse of a right triangle, a straight diagonal from the origin to the point $(8, 7)$. Perhaps you have heard the saying "as the crow flies," which means the shortest distance between two points because a crow can fly in a straight line even though a person on the ground has to travel a longer distance on existing roadways.

Using the Midpoint Formula

When the endpoints of a line segment are known, we can find the point midway between them. This point is known as the midpoint and the formula is known as the **midpoint formula**. Given the endpoints of a line segment, (x_1, y_1) and (x_2, y_2), the midpoint formula states how to find the coordinates of the midpoint M.

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \tag{1.7.4}$$

A graphical view of a midpoint is shown in Figure 1.7.18. Notice that the line segments on either side of the midpoint are congruent.

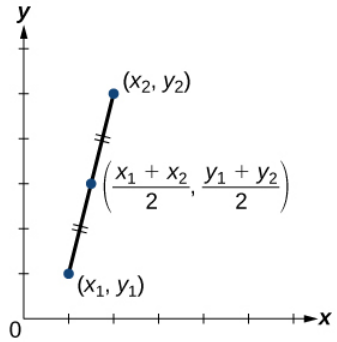


Figure 1.7.18

Example 1.7.7: Finding the Midpoint of the Line Segment

Find the midpoint of the line segment with the endpoints $(7, -2)$ and $(9, 5)$.

Solution

Use the formula to find the midpoint of the line segment.

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{7+9}{2}, \frac{-2+5}{2}\right)$$
$$= \left(8, \frac{3}{2}\right)$$

Exercise 1.7.4

Find the midpoint of the line segment with endpoints $(-2, -1)$ and $(-8, 6)$.

Answer

$$\left(-5, \frac{5}{2}\right)$$

Example 1.7.8: Finding the Center of a Circle

The diameter of a circle has endpoints $(-1, -4)$ and $(5, -4)$. Find the center of the circle.

Solution

The center of a circle is the center, or midpoint, of its diameter. Thus, the midpoint formula will yield the center point.

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{-1+5}{2}, \frac{-4-4}{2})\right)$$
$$= \left(\frac{4}{2}, -\frac{8}{2}\right)$$
$$= (2, 4)$$

Media

Access these online resources for additional instruction and practice with the Cartesian coordinate system.

1. Plotting points on the coordinate plane
2. Find x and y intercepts based on the graph of a line

Key Concepts

- We can locate, or plot, points in the Cartesian coordinate system using ordered pairs, which are defined as displacement from the x-axis and displacement from the y-axis. See Example.
- An equation can be graphed in the plane by creating a table of values and plotting points. See Example.
- Using a graphing calculator or a computer program makes graphing equations faster and more accurate. Equations usually have to be entered in the form $y =$_____. See Example.
- Finding the x- and y-intercepts can define the graph of a line. These are the points where the graph crosses the axes. See Example.
- The distance formula is derived from the Pythagorean Theorem and is used to find the length of a line segment. See Example and Example.
- The midpoint formula provides a method of finding the coordinates of the midpoint dividing the sum of the x-coordinates and the sum of the y-coordinates of the endpoints by 2. See Example and Example.

Homework Exercises 1.7

WeBWorK Problems: 1-4

https://math.libretexts.org/@go/page/106325

https://math.libretexts.org/@go/page/106325

https://math.libretexts.org/@go/page/106325

https://math.libretexts.org/@go/page/106325

Written Problems:

1. Describe the process for finding the *x*-intercept and the *y*-intercept of a graph algebraically.

2. Find the *x*-intercept and the *y*-intercept without graphing and write the coordinates of each intercept for the function: $3x + 8y = 9$.

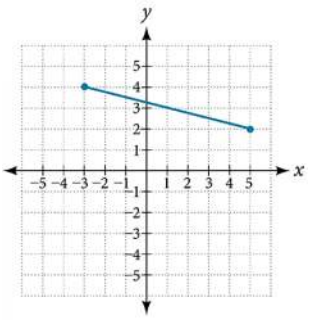
3. Solve the equation for *y* in terms of *x*: $x - 2y = 8$.

4. Find the distance between the two points: (-4, 1) and (3, -4).

5. Find the coordinates of the midpoint of the line segment that joins the two points: (-4, 1) and (3, -4).

6. Find and plot the *x*- and *y*-intercepts, and graph the straight line based on those two points for the following function: $y = \frac{x-3}{2}$

7. For each of the following exercises, use the graph in the figure below.



a. Find the distance between the two endpoints using the distance formula. Round to three decimal places.

b. Find the coordinates of the midpoint of the line segment connecting the two points.

c. Find the distance that (-3, 4) is from the origin.

d. Find the exact distance that (5, 2) is from the origin. Then round this answer to three decimal places.

e. Which point is closer to the origin?

This page titled 1.7: The Rectangular Coordinate Systems and Graphs is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **2.1: The Rectangular Coordinate Systems and Graphs** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/precalculus.

1.8: Linear Equations in One Variable

Learning Objectives

- Solve equations in one variable algebraically.
- Solve a rational equation.
- Find a linear equation.
- Given the equations of two lines, determine whether their graphs are parallel or perpendicular.
- Write the equation of a line parallel or perpendicular to a given line.

Caroline is a full-time college student planning a spring break vacation. To earn enough money for the trip, she has taken a part-time job at the local bank that pays $\$15.00/hr$, and she opened a savings account with an initial deposit of $\$400$ on January 15. She arranged for direct deposit of her payroll checks. If spring break begins March 20 and the trip will cost approximately $\$2,500$, how many hours will she have to work to earn enough to pay for her vacation? If she can only work 4 hours per day, how many days per week will she have to work? How many weeks will it take? In this section, we will investigate problems like this and others, which generate graphs like the line in Figure 1.8.1.

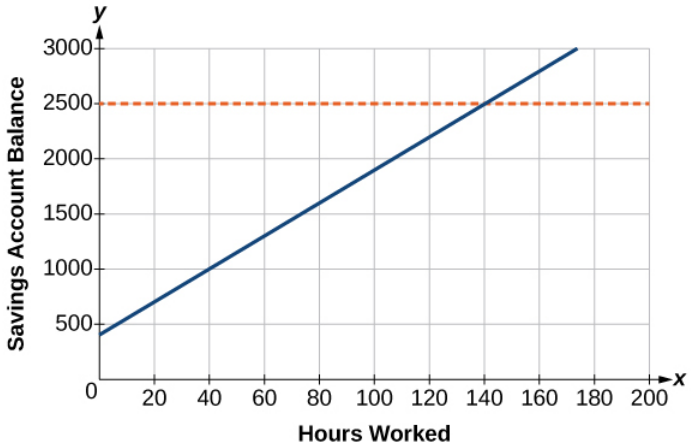


Figure 1.8.1

Solving Linear Equations in One Variable

A *linear equation* is an equation of a straight line, written in one variable. The only power of the variable is 1. Linear equations in one variable may take the form $ax+b=0$ and are solved using basic algebraic operations. We begin by classifying linear equations in one variable as one of three types: identity, conditional, or inconsistent.

- An **identity equation** is true for all values of the variable. Here is an example of an identity equation:

$$3x=2x+x$$

  The *solution set* consists of all values that make the equation true. For this equation, the solution set is all real numbers because any real number substituted for x will make the equation true.
- A **conditional equation** is true for only some values of the variable. For example, if we are to solve the equation $5x+2=3x-6$, we have the following:

$$\begin{aligned} 5x+2 &= 3x-6 \\ 2x &= -8 \\ x &= -4 \end{aligned}$$

  The solution set consists of one number: -4. It is the only solution and, therefore, we have solved a conditional equation.
- An **inconsistent equation** results in a false statement. For example, if we are to solve $5x-15=5(x-4)$, we have the following:

$$\begin{aligned} 5x-15 &= 5x-20 \\ 5x-15-5x &= 5x-20-5x \\ -15 &\neq -20 \end{aligned}$$

Indeed, $-15 \neq -20$. There is no solution because this is an inconsistent equation.

Solving linear equations in one variable involves the fundamental properties of equality and basic algebraic operations. A brief review of those operations follows.

LINEAR EQUATION IN ONE VARIABLE

A linear equation in one variable can be written in the form

$$ax+b=0 \tag{1.8.1}$$

where a and b are real numbers, $a \neq 0$.

Howto: Given a linear equation in one variable, use algebra to solve it

The following steps are used to manipulate an equation and isolate the unknown variable, so that the last line reads $x =$_____, if x is the unknown. There is no set order, as the steps used depend on what is given:

1. We may add, subtract, multiply, or divide an equation by a number or an expression as long as we do the same thing to both sides of the equal sign. Note that we cannot divide by zero.
2. Apply the distributive property as needed: $a(b+c)=ab+ac$.
3. Isolate the variable on one side of the equation.
4. When the variable is multiplied by a coefficient in the final stage, multiply both sides of the equation by the reciprocal of the coefficient.

Example 1.8.1: Solving an Equation in One Variable

Solve the following equation: $2x+7=19$.

Solution

This equation can be written in the form $ax+b=0$ by subtracting 19 from both sides. However, we may proceed to solve the equation in its original form by performing algebraic operations.

$$\begin{aligned} 2x+7 &= 19 \\ 2x &= 12 \qquad \text{Subtract 7 from both sides} \\ x &= 6 \qquad \text{Multiply both sides by } \frac{1}{2} \text{ or divide by 2} \end{aligned}$$

The solution is 6.

Exercise 1.8.1

Solve the linear equation in one variable: $2x+1=-9$.

Answer

$x=-5$

Example 1.8.2: Solving When the Variable Appears on Both Sides

Solve the following equation: $4(x-3)+12=15-5(x+6)$.

Solution

Apply standard algebraic properties.

$$\begin{aligned} 4(x-3)+12 &= 15-5(x+6) \\ 4x-12+12 &= 15-5x-30 \quad \text{Apply the distributive property} \\ 4x &= -15-5x \quad \text{Combine like terms} \\ 9x &= -15 \quad \text{Place x terms on one side and simplify} \\ x &= -\frac{15}{9} \quad \text{Multiply both sides by } \frac{1}{9}\text{, the reciprocal of 9} \\ x &= -\frac{5}{3} \end{aligned}$$

Analysis

This problem requires the distributive property to be applied twice, and then the properties of algebra are used to reach the final line, $x=-\frac{5}{3}$.

? Exercise 1.8.2

Solve the equation in one variable: $-2(3x-1)+x=14-x$.

Answer

$x=-3$

Solving a Rational Equation

In this section, we look at rational equations that, after some manipulation, result in a linear equation. If an equation contains at least one rational expression, it is a considered a **rational equation**. Recall that a **rational number** is the ratio of two numbers, such as $\frac{2}{3}$ or $\frac{7}{2}$. A rational expression is the ratio, or quotient, of two polynomials. Here are three examples.

$$\frac{x+1}{x^2-4}$$

$$\frac{1}{x-3}$$

or

$$\frac{4}{x^2+x-2}$$

Rational equations have a variable in the denominator in at least one of the terms. Our goal is to perform algebraic operations so that the variables appear in the numerator. In fact, we will eliminate all denominators by multiplying both sides of the equation by the **least common denominator** (LCD). Finding the LCD is identifying an expression that contains the highest power of all of the factors in all of the denominators. We do this because when the equation is multiplied by the LCD, the common factors in the LCD and in each denominator will equal one and will cancel out.

✔ Example 1.8.3: Solving a Rational Equation

Solve the rational equation:

$$\frac{7}{2x}-\frac{5}{3x}=\frac{22}{3}$$

Solution

We have three denominators; $2x$,$3x$, and 3. The LCD must contain $2x$,$3x$, and 3. An LCD of $6x$ contains all three denominators. In other words, each denominator can be divided evenly into the LCD. Next, multiply both sides of the equation by the LCD $6x$.

$$
\begin{aligned}
(6x)\left[\frac{7}{2x}-\frac{5}{3x}\right] &= \left[\frac{22}{3}\right](6x) \\
(6x)\left(\frac{7}{2x}\right)-(6x)\left(\frac{5}{3x}\right) &= \left(\frac{22}{3}\right)(6x) && \text{Use the distributive property. Cancel out the common factors} \\
3(7)-2(5) &= 22(2x) && \text{Multiply remaining factors by each numerator.} \\
21-10 &= 44x \\
11 &= 44x \\
\frac{11}{44} &= x \\
\frac{1}{4} &= x
\end{aligned}
$$

A common mistake made when solving rational equations involves finding the LCD when one of the denominators is a binomial—two terms added or subtracted—such as $(x+1)$. Always consider a binomial as an individual factor—the terms cannot be separated. For example, suppose a problem has three terms and the denominators are x, $x-1$, and $3x-3$. First, factor all denominators. We then have x, $(x-1)$, and $3(x-1)$ as the denominators. (Note the parentheses placed around the second denominator.) Only the last two denominators have a common factor of $(x-1)$. The x in the first denominator is separate from the x in the $(x-1)$ denominators. An effective way to remember this is to write factored and binomial denominators in parentheses, and consider each parentheses as a separate unit or a separate factor. The LCD in this instance is found by multiplying together the x, one factor of $(x-1)$, and the 3. Thus, the LCD is the following:

$$x(x-1)3 = 3x(x-1)$$

So, both sides of the equation would be multiplied by $3x(x-1)$. Leave the LCD in factored form, as this makes it easier to see how each denominator in the problem cancels out.

Another example is a problem with two denominators, such as x and x^2+2x. Once the second denominator is factored as $x^2+2x=x(x+2)$, there is a common factor of x in both denominators and the LCD is $x(x+2)$.

Sometimes we have a rational equation in the form of a proportion; that is, when one fraction equals another fraction and there are no other terms in the equation.

$$\frac{a}{b}=\frac{c}{d} \tag{1.8.2}$$

We can use another method of solving the equation without finding the LCD: cross-multiplication. We multiply terms by crossing over the equal sign.

$$\text{If } \frac{a}{b}=\frac{c}{d}, \text{ then } \frac{a}{b}\times\frac{c}{d}.$$

Multiply a(d) and b(c), which results in $ad=bc$.

Any solution that makes a denominator in the original expression equal zero must be excluded from the possibilities.

RATIONAL EQUATIONS

A r*ational equation* contains at least one rational expression where the variable appears in at least one of the denominators.

Howto: Given a rational equation, solve it.

1. Factor all denominators in the equation.
2. Find and exclude values that set each denominator equal to zero.
3. Find the LCD.
4. Multiply the whole equation by the LCD. If the LCD is correct, there will be no denominators left.
5. Solve the remaining equation.
6. Make sure to check solutions back in the original equations to avoid a solution producing zero in a denominator

Example 1.8.4: Solving a Rational Equation without Factoring

Solve the following rational equation:

$$\frac{2}{x}-\frac{3}{2}=\frac{7}{2x}$$

Solution

We have three denominators: x, 2, and $2x$. No factoring is required. The product of the first two denominators is equal to the third denominator, so, the LCD is $2x$. Only one value is excluded from a solution set, 0. Next, multiply the whole equation (both sides of the equal sign) by $2x$.

$$\begin{aligned} 2x\left[\frac{2}{x}-\frac{3}{2}\right] &= \left[\frac{7}{2x}\right](2x) \\ 2x\left(\frac{2}{x}\right)-2x\left(\frac{3}{2}\right) &= \left(\frac{7}{2x}\right)(2x) \qquad \text{Distribute } 2x \\ 2(2)-3x &= 7 \qquad \text{Denominators cancel out.} \\ 4-3x &= 7 \\ -3x &= 3 \\ x &= -1 \text{ or } \{-1\} \end{aligned}$$

The proposed solution is -1, which is not an excluded value, so the solution set contains one number, $x=-1$, or $\{-1\}$ written in set notation.

Exercise 1.8.4

Solve the rational equation:

$$\frac{2}{3x}=\frac{1}{4}-\frac{1}{6x}$$

Answer

$$x=\frac{10}{3}$$

Example 1.8.5: Solving a Rational Equation by Factoring the Denominator

Solve the following rational equation:

$$\frac{1}{x}=\frac{1}{10}-\frac{3}{4x}$$

Solution

First find the common denominator. The three denominators in factored form are $x, 10=2\cdot 5$, and $4x=2\cdot 2\cdot x$. The smallest expression that is divisible by each one of the denominators is $20x$. Only $x=0$ is an excluded value. Multiply the whole equation by $20x$.

$$\begin{aligned} 20x\left(\frac{1}{x}\right) &= \left(\frac{1}{10}-\frac{3}{4x}\right)20x \\ 20 &= 2x-15 \\ 35 &= 2x \\ \frac{35}{2} &= x \end{aligned}$$

The solution is $\frac{35}{2}$.

Exercise 1.8.5

Solve the rational equation:

$$-\frac{5}{2x}+\frac{3}{4x}=-\frac{7}{4}$$

Answer

$x=1$

Example 1.8.6: Solving Rational Equations with a Binomial in the Denominator

Solve the following rational equations and state the excluded values:

a. $\dfrac{3}{x-6}=\dfrac{5}{x}$

b. $\dfrac{x}{x-3}=\dfrac{5}{x-3}-\dfrac{1}{2}$

c. $\dfrac{x}{x-2}=\dfrac{5}{x-2}-\dfrac{1}{2}$

Solution

a.

The denominators x and $x-6$ have nothing in common. Therefore, the LCD is the product $x(x-6)$. However, for this problem, we can cross-multiply.

$$\begin{aligned}\frac{3}{x-6}&=\frac{5}{x}\\ 3x&=5(x-6) \qquad \text{Distribute.}\\ 3x&=5x-30\\ -2x&=-30\\ x&=15\end{aligned}$$

The solution is 15. The excluded values are 6 and 0.

b.

The LCD is $2(x-3)$. Multiply both sides of the equation by $2(x-3)$.

$$\begin{aligned}2(x-3)\left[\frac{x}{x-3}\right]&=\left[\frac{5}{x-3}-\frac{1}{2}\right]2(x-3)\\ \frac{2(x-3)x}{x-3}&=\frac{2(x-3)5}{x-3}-\frac{2(x-3)}{2}\\ 2x&=10-(x-3)\\ 2x&=13-x\\ 3x&=13\\ x&=\frac{13}{3}\end{aligned}$$

The solution is $\dfrac{13}{3}$. The excluded value is 3.

c.

The least common denominator is $2(x-2)$. Multiply both sides of the equation by $x(x-2)$.

$$2(x-2)\left[\frac{x}{x-2}\right] = \left[\frac{5}{x-2}-\frac{1}{2}\right]2(x-2)$$
$$2x = 10-(x-2)$$
$$2x = 12-x$$
$$3x = 12$$
$$x = 4$$

The solution is 4. The excluded value is 2.

Exercise 1.8.6

Solve $\frac{-3}{2x+1} = \frac{4}{3x+1}$. State the excluded values.

Answer

$x = -\frac{7}{17}$. Excluded values are $x = -12$ and $x = -13$.

Example 1.8.7: Solving a Rational Equation with Factored Denominators and Stating Excluded Values

Solve the rational equation after factoring the denominators: $\frac{2}{x+1} - \frac{1}{x-1} = \frac{2x}{x^2-1}$. State the excluded values.

Solution

We must factor the denominator x^2-1. We recognize this as the difference of squares, and factor it as $(x-1)(x+1)$. Thus, the LCD that contains each denominator is $(x-1)(x+1)$. Multiply the whole equation by the LCD, cancel out the denominators, and solve the remaining equation.

$$(x+1)(x-1)\left[\frac{2}{x+1}-\frac{1}{x-1}\right] = \left[\frac{2x}{x^2-1}\right](x+1)(x-1)$$
$$2(x-1)-(x+1) = 2x$$
$$2x-2-x-1 = 2x \text{ Distribute the negative sign}$$
$$-3-x = 0$$
$$x = -3$$

The solution is -3. The excluded values are 1 and -1.

Exercise 1.8.7

Solve the rational equation:

$$\frac{2}{x-2}+\frac{1}{x+1} = \frac{1}{x^2-x-2}$$

Answer

$x = \frac{1}{3}$

Finding a Linear Equation

Perhaps the most familiar form of a linear equation is the *slope-intercept form*, written as

$$y = mx+b \tag{1.8.3}$$

where $m =$ slope and $b = $ y−intercept. Let us begin with the slope.

The **slope** of a line refers to the ratio of the vertical change in y over the horizontal change in x between any two points on a line. It indicates the direction in which a line slants as well as its steepness. Slope is sometimes described as rise over run.

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{1.8.4}$$

If the slope is positive, the line slants to the right. If the slope is negative, the line slants to the left. As the slope increases, the line becomes steeper. Some examples are shown in Figure 1.8.2. The lines indicate the following slopes: $m = -3$, $m = 2$, and $m = \frac{1}{3}$.

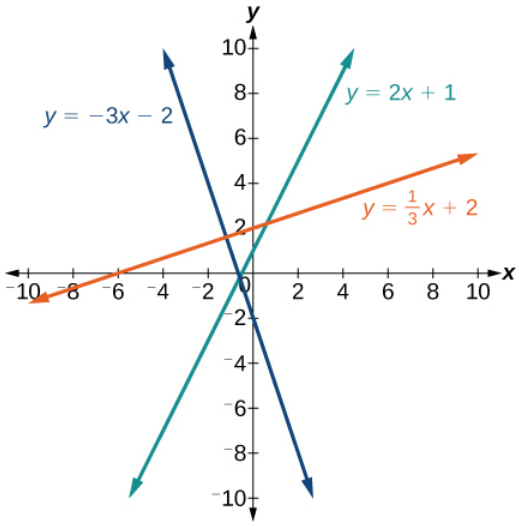


Figure 1.8.2

THE SLOPE OF A LINE

The slope of a line, m, represents the change in y over the change in x. Given two points, (x_1, y_1) and (x_2, y_2), the following formula determines the slope of a line containing these points:

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{1.8.5}$$

Example 1.8.8: Finding the Slope of a Line Given Two Points

Find the slope of a line that passes through the points $(2, -1)$ and $(-5, 3)$.

Solution

We substitute the y-values and the x-values into the formula.

$$\begin{aligned} m &= \frac{3 - (-1)}{-5 - 2} \\ &= \frac{4}{-7} \\ &= -\frac{4}{7} \end{aligned}$$

The slope is $-\frac{4}{7}$

Analysis

It does not matter which point is called (x_1, y_1) or (x_2, y_2). As long as we are consistent with the order of the y terms and the order of the x terms in the numerator and denominator, the calculation will yield the same result.

Exercise 1.8.8

Find the slope of the line that passes through the points $(-2, 6)$ and $(1, 4)$.

Answer

$$m = -\frac{2}{3}$$

Example 1.8.9: Identifying the Slope and y-intercept of a Line Given an Equation

Identify the slope and y-intercept, given the equation $y = -\frac{3}{4}x - 4$.

Solution

As the line is in $y = mx + b$ form, the given line has a slope of $m = -\frac{3}{4}$. The y-intercept is $b = -4$.

Analysis

The y-intercept is the point at which the line crosses the y-axis. On the y-axis, $x = 0$. We can always identify the y-intercept when the line is in slope-intercept form, as it will always equal b. Or, just substitute $x = 0$ and solve for y.

The Point-Slope Formula

Given the slope and one point on a line, we can find the equation of the line using the point-slope formula.

$$y - y_1 = m(x - x_1) \tag{1.8.6}$$

This is an important formula, as it will be used in other areas of college algebra and often in calculus to find the equation of a tangent line. We need only one point and the slope of the line to use the formula. After substituting the slope and the coordinates of one point into the formula, we simplify it and write it in slope-intercept form.

THE POINT-SLOPE FORMULA

Given one point and the slope, the point-slope formula will lead to the equation of a line:

$$y - y_1 = m(x - x_1) \tag{1.8.7}$$

Example 1.8.10: Finding the Equation of a Line Given the Slope and One Point

Write the equation of the line with slope $m = -3$ and passing through the point $(4, 8)$. Write the final equation in slope-intercept form.

Solution

Using the point-slope formula, substitute -3 for m and the point $(4, 8)$ for (x_1, y_1).

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 8 &= -3(x - 4) \\ y - 8 &= -3x + 12 \\ y &= -3x + 20 \end{aligned}$$

Analysis

Note that any point on the line can be used to find the equation. If done correctly, the same final equation will be obtained.

Exercise 1.8.10

Given $m = 4$, find the equation of the line in slope-intercept form passing through the point $(2, 5)$.

Answer

$y = 4x - 3$

Example 1.8.11: Finding the Equation of a Line Passing Through Two Given Points

Find the equation of the line passing through the points $(3, 4)$ and $(0, -3)$. Write the final equation in slope-intercept form.

Solution

First, we calculate the slope using the slope formula and two points.

$$\begin{aligned} m &= \frac{-3-4}{0-3} \\ m &= \frac{-7}{-3} \\ m &= \frac{7}{3} \end{aligned}$$

Next, we use the point-slope formula with the slope of $\frac{7}{3}$, and either point. Let's pick the point $(3, 4)$ for (x_1, y_1).

$$\begin{aligned} y - 4 &= \frac{7}{3}(x-3) \\ y - 4 &= \frac{7}{3}x - 7 \\ y &= \frac{7}{3}x - 3 \end{aligned}$$

In slope-intercept form, the equation is written as $y = \frac{7}{3}x - 3$

Analysis

To prove that either point can be used, let us use the second point $(0, -3)$ and see if we get the same equation.

$$\begin{aligned} y - (-3) &= \frac{7}{3}(x-0) \\ y + 3 &= \frac{7}{3}x \\ y &= \frac{7}{3}x - 3 \end{aligned}$$

We see that the same line will be obtained using either point. This makes sense because we used both points to calculate the slope.

Standard Form of a Line

Another way that we can represent the equation of a line is in **standard form**. Standard form is given as

$$Ax + By = C \tag{1.8.8}$$

where A, B, and C are integers. The x- and y-terms are on one side of the equal sign and the constant term is on the other side.

Example 1.8.12: Finding the Equation of a Line and Writing It in Standard Form

Find the equation of the line with $m = -6$ and passing through the point $\left(\frac{1}{4}, -2\right)$. Write the equation in standard form.

Solution

We begin using the point-slope formula.

$$\begin{aligned} y-(-2) &= -6\left(x-\frac{1}{4}\right) \\ y+2 &= -6x+\frac{3}{2} \end{aligned}$$

From here, we multiply through by 2, as no fractions are permitted in standard form, and then move both variables to the left aside of the equal sign and move the constants to the right.

$$\begin{aligned} 2(y+2) &= \left(-6x+\frac{3}{2}\right)2 \\ 2y+4 &= -12x+3 \\ 12x+2y &= -1 \end{aligned}$$

This equation is now written in standard form.

? Exercise 1.8.12

Find the equation of the line in standard form with slope $m=-\frac{1}{3}$ and passing through the point $(1,13)$.

Answer

$x+3y=2$

Vertical and Horizontal Lines

The equations of vertical and horizontal lines do not require any of the preceding formulas, although we can use the formulas to prove that the equations are correct. The equation of a **vertical line** is given as

$$x=c \tag{1.8.9}$$

where c is a constant. The slope of a vertical line is undefined, and regardless of the y-value of any point on the line, the x-coordinate of the point will be c.

Suppose that we want to find the equation of a line containing the following points: $(-3,-5)$,$(-3,1)$,$(-3,3)$, and $(-3,5)$. First, we will find the slope.

$$m=\frac{5-3}{-3-(-3)}=\frac{2}{0}$$

Zero in the denominator means that the slope is undefined and, therefore, we cannot use the point-slope formula. However, we can plot the points. Notice that all of the x-coordinates are the same and we find a vertical line through $x=-3$. See Figure 1.8.3.

The equation of a **horizontal line** is given as

$$y=c \tag{1.8.10}$$

where c is a constant. The slope of a horizontal line is zero, and for any x-value of a point on the line, the y-coordinate will be c.

Suppose we want to find the equation of a line that contains the following set of points: $(-2,-2)$,$(0,-2)$,$(3,-2)$, and $(5,-2)$. We can use the point-slope formula. First, we find the slope using any two points on the line.

$$\begin{aligned} m &= \frac{-2-(-2)}{0-(-2)} \\ &= \frac{0}{2} \\ &= 0 \end{aligned}$$

Use any point for (x_1,y_1) in the formula, or use the y-intercept.

$$\begin{aligned} y-(-2) &= 0(x-3) \\ y+2 &= 0 \\ y &= -2 \end{aligned}$$

The graph is a horizontal line through $y = -2$. Notice that all of the y-coordinates are the same. See Figure 1.8.3.

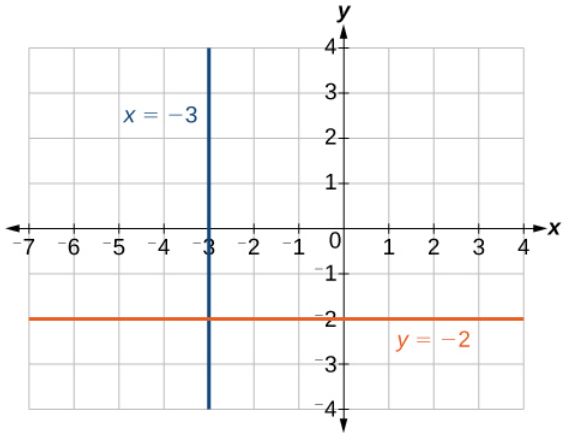


Figure 1.8.3: The line x = −3 is a vertical line. The line y = −2 is a horizontal line

Example 1.8.13: Finding the Equation of a Line Passing Through the Given Points

Find the equation of the line passing through the given points: $(1, -3)$ and $(1, 4)$.

Solution

The x-coordinate of both points is 1. Therefore, we have a vertical line, $x = 1$.

Exercise 1.8.13

Find the equation of the line passing through $(-5, 2)$ and $(2, 2)$.

Answer

Horizontal line: $y = 2$

Determining Whether Graphs of Lines are Parallel or Perpendicular

Parallel lines have the same slope and different y-intercepts. Lines that are parallel to each other will never intersect. For example, Figure 1.8.4 shows the graphs of various lines with the same slope, $m = 2$.

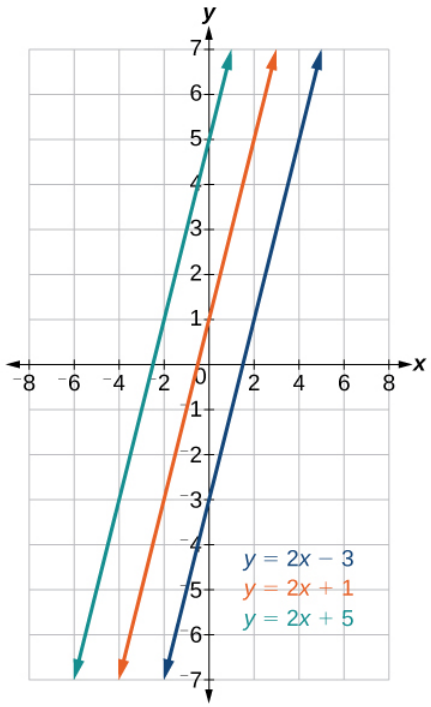


Figure 1.8.4: Parallel lines

All of the lines shown in the graph are parallel because they have the same slope and different y-intercepts.

Lines that are **perpendicular** intersect to form a 90° -angle. The slope of one line is the negative **reciprocal** of the other. We can show that two lines are perpendicular if the product of the two slopes is -1 : $m_1 \cdot m_2 = -1$. For example, Figure 1.8.5 shows the graph of two perpendicular lines. One line has a slope of 3; the other line has a slope of $-\frac{1}{3}$.

$$m_1 \cdot m_2 = -1$$
$$3 \cdot \left(-\frac{1}{3}\right) = -1$$

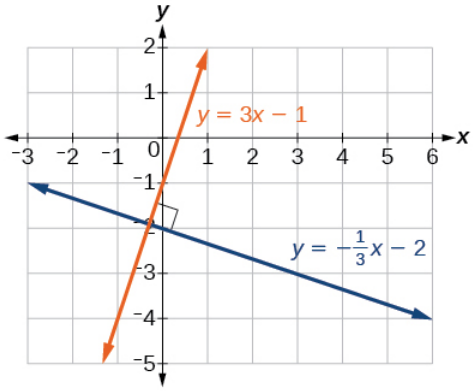


Figure 1.8.5: Perpendicular lines

✔ Example 1.8.14: Graphing Two Equations, and Determining Whether the Lines are Parallel, Perpendicular, or Neither

Graph the equations of the given lines, and state whether they are parallel, perpendicular, or neither: $3y = -4x + 3$ and $3x - 4y = 8$.

Solution

The first thing we want to do is rewrite the equations so that both equations are in slope-intercept form.

First equation:

$$\begin{aligned} 3y &= -4x+3 \\ y &= -\frac{4}{3}x+1 \end{aligned}$$

Second equation:

$$\begin{aligned} 3x-4y &= 8 \\ -4y &= -3x+8 \\ y &= \frac{3}{4}x-2 \end{aligned}$$

See the graph of both lines in Figure 1.8.6.

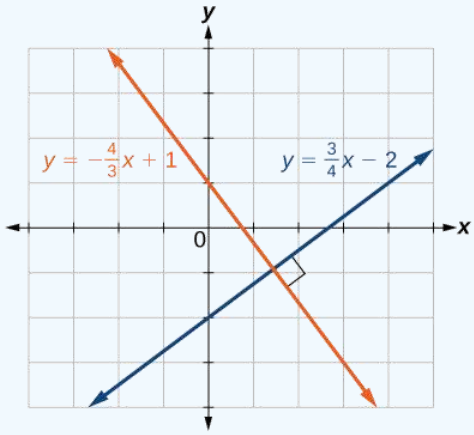


Figure 1.8.6

From the graph, we can see that the lines appear perpendicular, but we must compare the slopes.

$$\begin{aligned} m_1 &= -\frac{4}{3} \\ m_2 &= \frac{3}{4} \\ m_1 \cdot m_2 &= \left(-\frac{4}{3}\right)\left(\frac{3}{4}\right) \\ &= -1 \end{aligned}$$

The slopes are negative reciprocals of each other, confirming that the lines are perpendicular.

Exercise 1.8.14

Graph the two lines and determine whether they are parallel, perpendicular, or neither: $2y-x=10$ and $2y=x+4$.

Answer

Parallel lines: equations are written in slope-intercept form.

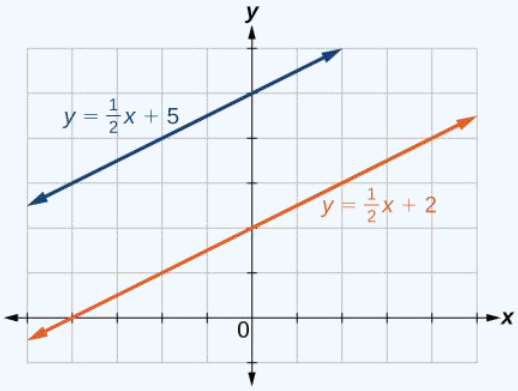


Figure 1.8.7

Writing the Equations of Lines Parallel or Perpendicular to a Given Line

As we have learned, determining whether two lines are parallel or perpendicular is a matter of finding the slopes. To write the equation of a line parallel or perpendicular to another line, we follow the same principles as we do for finding the equation of any line. After finding the slope, use the **point-slope formula** to write the equation of the new line.

Given an equation for a line, write the equation of a line parallel or perpendicular to it.

1. Find the slope of the given line. The easiest way to do this is to write the equation in slope-intercept form.
2. Use the slope and the given point with the point-slope formula.
3. Simplify the line to slope-intercept form and compare the equation to the given line.

Example 1.8.15: Writing the Equation of a Line Parallel to a Given Line Passing Through a Given Point

Write the equation of line parallel to a $5x + 3y = 1$ and passing through the point $(3, 5)$.

Solution

First, we will write the equation in slope-intercept form to find the slope.

$$\begin{aligned} 5x + 3y &= 1 \\ 3y &= -5x + 1 \\ y &= -\frac{5}{3}x + \frac{1}{3} \end{aligned}$$

The slope is $m = -\frac{5}{3}$. The y-intercept is 13, but that really does not enter into our problem, as the only thing we need for two lines to be parallel is the same slope. The one exception is that if the y-intercepts are the same, then the two lines are the same line. The next step is to use this slope and the given point with the point-slope formula.

$$\begin{aligned} y - 5 &= -\frac{5}{3}(x - 3) \\ y - 5 &= -\frac{5}{3}x + 5 \\ y &= -\frac{5}{3}x + 10 \end{aligned}$$

The equation of the line is $y = -\frac{5}{3}x + 10$. See Figure $1.8.8$.

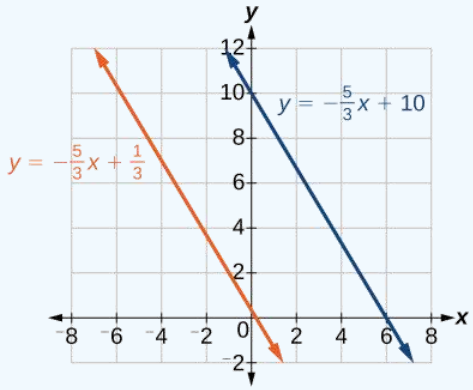


Figure 1.8.8

? Exercise 1.8.15

Find the equation of the line parallel to $5x = 7 + y$ and passing through the point $(-1, -2)$.

Answer

$y = 5x + 3$

✓ Example 1.8.16: Finding the Equation of a Line Perpendicular to a Given Line Passing Through a Given Point

Find the equation of the line perpendicular to $5x - 3y + 4 = 0$ $(-4, 1)$.

Solution

The first step is to write the equation in slope-intercept form.

$$\begin{aligned} 5x - 3y + 4 &= 0 \\ -3y &= -5x - 4 \\ y &= \frac{5}{3}x + \frac{4}{3} \end{aligned}$$

We see that the slope is $m = \frac{5}{3}$. This means that the slope of the line perpendicular to the given line is the negative reciprocal, or $-\frac{3}{5}$. Next, we use the point-slope formula with this new slope and the given point.

$$\begin{aligned} y - 1 &= -\frac{3}{5}(x - (-4)) \\ y - 1 &= -\frac{3}{5}x - \frac{12}{5} \\ y &= -\frac{3}{5}x - \frac{12}{5} + \frac{5}{5} \\ y &= -\frac{3}{5}x - \frac{7}{5} \end{aligned}$$

Media

Access these online resources for additional instruction and practice with linear equations.

1. Solving rational equations
2. Equation of a line given two points
3. Finding the equation of a line perpendicular to another line through a given point
4. Finding the equation of a line parallel to another line through a given point

Key Concepts

- We can solve linear equations in one variable in the form $ax+b=0$ using standard algebraic properties. See Example and Example.
- A rational expression is a quotient of two polynomials. We use the LCD to clear the fractions from an equation. See Example and Example.
- All solutions to a rational equation should be verified within the original equation to avoid an undefined term, or zero in the denominator. See Example and Example.
- Given two points, we can find the slope of a line using the slope formula. See Example.
- We can identify the slope and y-intercept of an equation in slope-intercept form. See Example.
- We can find the equation of a line given the slope and a point. See Example.
- We can also find the equation of a line given two points. Find the slope and use the point-slope formula. See Example.
- The standard form of a line has no fractions. See Example.
- Horizontal lines have a slope of zero and are defined as $y=c$, where c is a constant.
- Vertical lines have an undefined slope (zero in the denominator), and are defined as $x=c$, where c is a constant. See Example.
- Parallel lines have the same slope and different y-intercepts. See Example.
- Perpendicular lines have slopes that are negative reciprocals of each other unless one is horizontal and the other is vertical. See Example.

Contributors and Attributions

The OpenStax College name, OpenStax College logo, OpenStax College book covers, OpenStax CNX name, and OpenStax CNX logo are not subject to the creative commons license and may not be reproduced without the prior and express written consent of Rice University. For questions regarding this license, please contact partners@openstaxcollege.org. If you use this textbook as a bibliographic reference, then you should cite it as follows: OpenStax College, Models and Applications. OpenStax CNX. Apr 1, 2015 http://cnx.org/contents/c49b7453-abd...baac08df9ebc@4. If you redistribute this textbook in a print format, then you must include on every physical page the following attribution: "Download for free at http://cnx.org/contents/c49b7453-abd...baac08df9ebc@4." If you redistribute part of this textbook, then you must retain in every digital format page view (including but not limited to EPUB, PDF, and HTML) and on every physical printed page the following attribution: "Download for free at http://cnx.org/contents/c49b7453-abd...baac08df9ebc@4."

Homework Exercises 1.8

WeBWorK Problems: 1-8

https://math.libretexts.org/@go/page/106327

Written Problems:

Verbal

1. What does it mean when we say that two lines are parallel?

2. How do we recognize when an equation, for example, $y = 4x + 3$, will be a straight line (linear) when graphed?

3. What is the "Point-Slope" formula for the equation of a line?

Algebraic

In exercises 4-6, solve the equations for *x*.

4. $\frac{1}{2} - \frac{1}{3}x = \frac{4}{3}$

5. $\frac{x+2}{4} - \frac{x-1}{3} = 2$

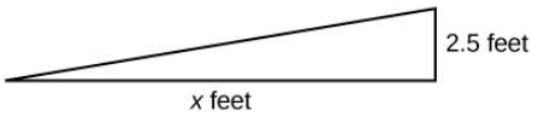
6. $\frac{1}{x} = \frac{1}{5} + \frac{3}{2x}$

7. Find the equation of the line given the slope is $\frac{3}{4}$ and it passes through the point (1, 4).

8. Find the equation of the line that passes through the points (5, 4) and (7, 9).

Application

9. The slope for a wheelchair ramp for a home has to be $\frac{1}{12}$. If the vertical distance from the ground to the door bottom is 2.5 ft, find the distance the ramp has to extend from the home in order to comply with the needed slope.



This page titled 1.8: Linear Equations in One Variable is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **2.2: Linear Equations in One Variable** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/precalculus.

1.9: Quadratic Equations

Preview Video for Sections 1.9 & 1.10

Note the section numbers and date are incorrect on the video (but correct on the notes handout)

Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login

Learning Objectives

- Solve quadratic equations by factoring.
- Solve quadratic equations by the square root property.
- Solve quadratic equations by completing the square.
- Solve quadratic equations by using the quadratic formula.

The computer monitor on the left in Figure 1.9.1 is a 23.6-inch model and the one on the right is a 27-inch model. Proportionally, the monitors appear very similar. If there is a limited amount of space and we desire the largest monitor possible, how do we decide which one to choose? In this section, we will learn how to solve problems such as this using four different methods.



Figure 1.9.1

Solving Quadratic Equations by Factoring

An equation containing a second-degree polynomial is called a **quadratic equation**. For example, equations such as $2x^2+3x-1=0$ and $x^2-4=0$ are quadratic equations. They are used in countless ways in the fields of engineering, architecture, finance, biological science, and, of course, mathematics.

Often the easiest method of solving a quadratic equation is **factoring**. Factoring means finding expressions that can be multiplied together to give the expression on one side of the equation.

If a quadratic equation can be factored, it is written as a product of linear terms. Solving by factoring depends on the zero-product property, which states that if $a\cdot b=0$, then $a=0$ or $b=0$, where a and b are real numbers or algebraic expressions. In other words, if the product of two numbers or two expressions equals zero, then one of the numbers or one of the expressions must equal zero because zero multiplied by anything equals zero.

Multiplying the factors expands the equation to a string of terms separated by plus or minus signs. So, in that sense, the operation of multiplication undoes the operation of factoring. For example, expand the factored expression $(x-2)(x+3)$ by multiplying the two factors together.

$$\begin{aligned}(x-2)(x+3) &= x^2+3x-2x-6\\ &= x^2+x-6\end{aligned}$$

The product is a quadratic expression. Set equal to zero, $x^2+x-6=0$ is a quadratic equation. If we were to factor the equation, we would get back the factors we multiplied.

The process of factoring a quadratic equation depends on the leading coefficient, whether it is 1 or another integer. We will look at both situations; but first, we want to confirm that the equation is written in standard form, $ax^2+bx+c=0$, where a, b, and c are real numbers, and $a\neq 0$. The equation $x^2+x-6=0$ is in standard form.

We can use the zero-product property to solve quadratic equations in which we first have to factor out the greatest common factor(GCF), and for equations that have special factoring formulas as well, such as the difference of squares, both of which we will see later in this section.

ZERO-PRODUCT PROPERTY AND QUADRATIC EQUATIONS

The **zero-product property** states

$$\text{If } a\cdot b=0\text{, then } a=0 \text{ or } b=0,$$

where a and b are real numbers or algebraic expressions.

A quadratic equation is an equation containing a second-degree polynomial; for example

$$ax^2+bx+c=0 \tag{1.9.1}$$

where a, b, and c are real numbers, and if $a\neq 0$, it is in standard form.

Solving Quadratics with a Leading Coefficient of 1

In the quadratic equation $x^2+x-6=0$, the leading coefficient, or the coefficient of x^2, is 1. We have one method of factoring quadratic equations in this form.

How to: Factor a quadratic equation with the leading coefficient of 1

1. Find two numbers whose product equals c and whose sum equals b.
2. Use those numbers to write two factors of the form $(x+k)$ or $(x-k)$, where k is one of the numbers found in step 1. Use the numbers exactly as they are. In other words, if the two numbers are 1 and -2, the factors are $(x+1)(x-2)$.
3. Solve using the zero-product property by setting each factor equal to zero and solving for the variable.

Example 1.9.1: Solving a Quadratic with Leading Coefficient of 1

Factor and solve the equation: $x^2+x-6=0$.

Solution

To factor $x^2+x-6=0$, we look for two numbers whose product equals -6 and whose sum equals 1. Begin by looking at the possible factors of -6.

$$1\cdot(-6)$$

$$(-6)\cdot 1$$

$$2\cdot(-3)$$

$$3\cdot(-2)$$

The last pair, $3\cdot(-2)$ sums to 1, so these are the numbers. Note that only one pair of numbers will work. Then, write the factors.

$$(x-2)(x+3)=0$$

To solve this equation, we use the zero-product property. Set each factor equal to zero and solve.

$$\begin{aligned}(x-2)(x+3)&=0\\(x-2)&=0\\x&=2\\(x+3)&=0\\x&=-3\end{aligned}$$

The two solutions are 2 and -3. We can see how the solutions relate to the graph in Figure 1.9.2. The solutions are the x-intercepts of $x^2+x-6=0$.

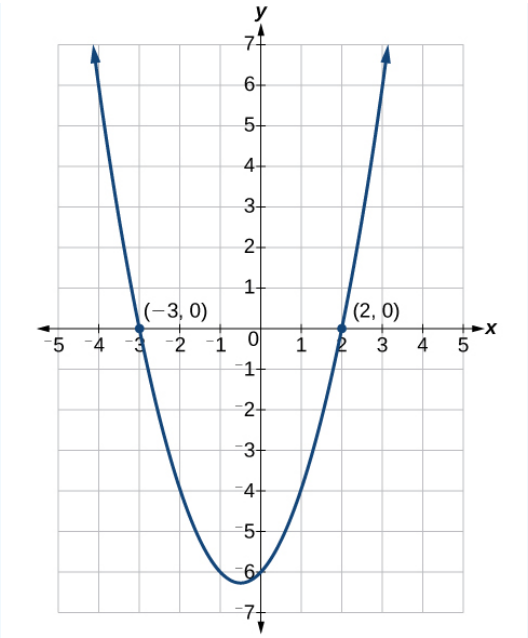


Figure 1.9.2

Exercise 1.9.1

Factor and solve the quadratic equation: $x^2 - 5x - 6 = 0$.

Answer

$(x-6)(x+1) = 0$, $x = 6, x = -1$

Example 1.9.2: Solve the Quadratic Equation by Factoring

Solve the quadratic equation by factoring: $x^2 + 8x + 15 = 0$.

Solution

Find two numbers whose product equals 15 and whose sum equals 8. List the factors of 15.

$$1 \cdot 15$$

$$3 \cdot 5$$

$$(-1) \cdot (-15)$$

$$(-3) \cdot (-5)$$

The numbers that add to 8 are 3 and 5. Then, write the factors, set each factor equal to zero, and solve.

$$\begin{aligned} (x+3)(x+5) &= 0 \\ (x+3) &= 0 \\ x &= -3 \\ (x+5) &= 0 \\ x &= -5 \end{aligned}$$

The solutions are -3 and -5.

Exercise 1.9.2

Solve the quadratic equation by factoring: $x^2 - 4x - 21 = 0$.

Answer

$(x-7)(x+3) = 0$, $x = 7, x = -3$

Example 1.9.3: Using Zero-Product Property to Solve a Quadratic Equation

Solve the difference of squares equation using the zero-product property: $x^2 - 9 = 0$.

Solution

Recognizing that the equation represents the difference of squares, we can write the two factors by taking the square root of each term, using a minus sign as the operator in one factor and a plus sign as the operator in the other. Solve using the zero-factor property.

$$\begin{aligned} x^2 - 9 &= 0 \\ (x-3)(x+3) &= 0 \\ x - 3 &= 0 \\ x &= 3 \\ (x+3) &= 0 \\ x &= -3 \end{aligned}$$

The solutions are 3 and -3.

Exercise 1.9.3

Solve by factoring: $x^2 - 25 = 0$.

Answer

$(x+5)(x-5) = 0, x = -5, x = 5$

Factoring and Solving a Quadratic Equation of Higher Order

When the leading coefficient is not 1, we factor a quadratic equation using the method called *grouping*, which requires four terms.

Grouping: Steps for factoring quadratic equations

With the equation in standard form, let's review the grouping procedures

1. With the quadratic in standard form, $ax^2 + bx + c = 0$, multiply $a \cdot c$.
2. Find two numbers whose product equals ac and whose sum equals b.
3. Rewrite the equation replacing the bx term with two terms using the numbers found in step 1 as coefficients of x.
4. Factor the first two terms and then factor the last two terms. The expressions in parentheses must be exactly the same to use grouping.
5. Factor out the expression in parentheses.
6. Set the expressions equal to zero and solve for the variable.

Example 1.9.4: Solving a Quadratic Equation Using Grouping

Use grouping to factor and solve the quadratic equation: $4x^2 + 15x + 9 = 0$.

Solution

First, multiply ac : $4(9) = 36$. Then list the factors of 36.

$$1 \cdot 36$$

$$2 \cdot 18$$

$$3 \cdot 12$$

$$4 \cdot 9$$

$$6 \cdot 6$$

The only pair of factors that sums to 15 is $3 + 12$. Rewrite the equation replacing the b term, $15x$, with two terms using 3 and 12 as coefficients of x. Factor the first two terms, and then factor the last two terms.

$$
\begin{aligned}
4x^2+3x+12x+9 &= 0 \\
x(4x+3)+3(4x+3) &= 0 \\
(4x+3)(x+3) &= 0 \qquad \text{Solve using the zero-product property} \\
(4x+3) &= 3 \\
x &= -\frac{3}{4} \\
(x+3) &= 0 \\
x &= -3
\end{aligned}
$$

The solutions are $-\frac{3}{4}$, and -3. See Figure 1.9.3.

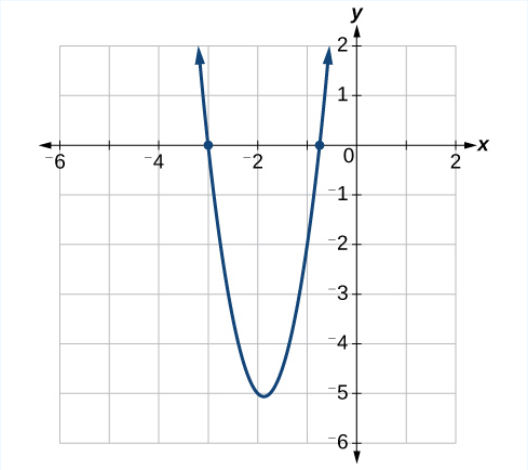


Figure 1.9.3

? Exercise 1.9.4

Solve using factoring by grouping: $12x^2+11x+2=0$.

Answer

$(3x+2)(4x+1)=0$, $x=-\frac{2}{3}$, $x=-\frac{1}{4}$

✓ Example 1.9.5: Solving a Higher Degree Quadratic Equation by Factoring

Solve the equation by factoring: $-3x^3-5x^2-2x=0$.

Solution

This equation does not look like a quadratic, as the highest power is 3, not 2. Recall that the first thing we want to do when solving any equation is to factor out the GCF, if one exists. And it does here. We can factor out $-x$ from all of the terms and then proceed with grouping.

$$
\begin{aligned}
-3x^3-5x^2-2x &= 0 \\
-x(3x^2+5x+2) &= 0 \\
-x(3x^2+3x+2x+2) &= 0 \qquad \text{Use grouping on the expression in parentheses} \\
-x[3x(x+1)+2(x+1)] &= 0 \\
-x(3x+2)(x+1) &= 0
\end{aligned}
$$

Now, we use the zero-product property. Notice that we have three factors.

$$
\begin{aligned}
-x &= 0 \\
x &= 0 \\
3x+2 &= 0 \\
x &= -\frac{2}{3} \\
x+1 &= 0 \\
x &= -1
\end{aligned}
$$

The solutions are 0, $-\frac{2}{3}$, and -1.

Exercise 1.9.5

Solve by factoring: $x^3+11x^2+10x=0$.

Answer

$x=0, x=-10, x=-1$

Using the Square Root Property

When there is no linear term in the equation, another method of solving a quadratic equation is by using the **square root property**, in which we isolate the x^2 term and take the square root of the number on the other side of the equals sign. Keep in mind that sometimes we may have to manipulate the equation to isolate the x^2 term so that the square root property can be used.

THE SQUARE ROOT PROPERTY

With the x^2 term isolated, the square root property states that:

$$\text{if } x^2=k\text{, then } x=\pm\sqrt{k}$$

where k is a nonzero real number.

Howto: Given a quadratic equation with an x^2 term but no x term, use the square root property to solve it

1. Isolate the x^2 term on one side of the equal sign.
2. Take the square root of both sides of the equation, putting a $\pm$ sign before the expression on the side opposite the squared term.
3. Simplify the numbers on the side with the $\pm$ sign.

Example 1.9.6: Solving a Simple Quadratic Equation Using the Square Root Property

Solve the quadratic using the square root property: $x^2=8$.

Solution

Take the square root of both sides, and then simplify the radical. Remember to use a $\pm$ sign before the radical symbol.

$$\begin{aligned} x^2 &= 8 \\ x &= \pm\sqrt{8} \\ &= \pm 2\sqrt{2} \end{aligned}$$

The solutions are $2\sqrt{2}, -2\sqrt{2}$

Example 1.9.7: Solving a Quadratic Equation Using the Square Root Property

Solve the quadratic equation: $4x^2+1=7$.

Solution

First, isolate the x^2 term. Then take the square root of both sides.

$$\begin{aligned} 4x^2+1 &= 7 \\ 4x^2 &= 6 \\ x^2 &= \frac{6}{4} \\ x &= \pm\frac{\sqrt{6}}{2} \end{aligned}$$

The solutions are $\frac{\sqrt{6}}{2}$, and $-\frac{\sqrt{6}}{2}$.

? Exercise 1.9.6

Solve the quadratic equation using the square root property: $3(x-4)^2=15$.

Answer

$x=4\pm\sqrt{5}$

Completing the Square

Not all quadratic equations can be factored or can be solved in their original form using the square root property. In these cases, we may use a method for solving a **quadratic equation** known as **completing the square**. Using this method, we add or subtract terms to both sides of the equation until we have a perfect square trinomial on one side of the equal sign. We then apply the square root property. To complete the square, the leading coefficient, a, must equal 1. If it does not, then divide the entire equation by a. Then, we can use the following procedures to solve a quadratic equation by completing the square.

We will use the example $x^2+4x+1=0$ to illustrate each step.

Given a quadratic equation that cannot be factored, and with $a=1$, first add or subtract the constant term to the right sign of the equal sign.

$$\begin{aligned} x^2+4x+1 &= 0 \\ x^2+4x &= -1 \qquad \text{Multiply the b term by } \frac{1}{2} \text{ and square it.} \\ \frac{1}{2}(4) &= 2 \\ 2^2 &= 4 \qquad \text{Add } \left(\frac{1}{2}\right)^2 \text{ to both sides of the equal sign and simplify the right side. We have} \\ x^2+4x+4 &= -1+4 \\ x^2+4x+4 &= 3 \qquad \text{The left side of the equation can now be factored as a perfect square.} \\ (x+2)^2 &= 3 \\ \sqrt{(x+2)^2} &= \pm\sqrt{3} \qquad \text{Use the square root property and solve.} \\ \sqrt{(x+2)^2} &= \pm\sqrt{3} \\ x+2 &= \pm\sqrt{3} \\ x &= -2\pm\sqrt{3} \end{aligned}$$

The solutions are $-2+\sqrt{3}$, and $-2-\sqrt{3}$.

✓ Example 1.9.8: Solving a Quadratic by Completing the Square

Solve the quadratic equation by completing the square: $x^2-3x-5=0$.

Solution

First, move the constant term to the right side of the equal sign.

$$\begin{aligned} x^2-3x &= 5 \qquad \text{Then, take } \frac{1}{2} \text{ of the b term and square it.} \\ \frac{1}{2}(-3) &= -\frac{3}{2} \\ \left(-\frac{3}{2}\right)^2 &= \frac{9}{4} \\ x^2-3x+\left(-\frac{3}{2}\right)^2 &= 5+\left(-\frac{3}{2}\right)^2 \qquad \text{Add the result to both sides of the equal sign.} \\ x^2-3x+\frac{9}{4} &= 5+\frac{9}{4} \end{aligned}$$

Factor the left side as a perfect square and simplify the right side.

$$\begin{aligned} \left(x-\frac{3}{2}\right)^2 &= \frac{29}{4} \\ (x-\frac{3}{2}) &= \pm\frac{\sqrt{29}}{2} \qquad \text{Use the square root property and solve.} \\ x &= \frac{3}{2}\pm\frac{\sqrt{29}}{2} \end{aligned}$$

The solutions are $\frac{3}{2}+\frac{\sqrt{29}}{2}$, and $\frac{3}{2}-\frac{\sqrt{29}}{2}$

? Exercise 1.9.7

Solve by completing the square: $x^2-6x=13$.

Answer

$x=3\pm\sqrt{22}$

Using the Quadratic Formula

The fourth method of solving a **quadratic equation** is by using the **quadratic formula**, a formula that will solve all quadratic equations. Although the quadratic formula works on any quadratic equation in standard form, it is easy to make errors in substituting the values into the formula. Pay close attention when substituting, and use parentheses when inserting a negative number.

We can derive the quadratic formula by **completing the square**. We will assume that the leading coefficient is positive; if it is negative, we can multiply the equation by -1 and obtain a positive a. Given $ax^2+bx+c=0, a\neq 0$, we will complete the square as follows:

First, move the constant term to the right side of the equal sign:

$$ax^2+bx=-c$$

As we want the leading coefficient to equal 1, divide through by a:

$$x^2+\frac{b}{a}x=-\frac{c}{a}$$

Then, find $\frac{1}{2}$ of the middle term, and add $(\frac{1}{2}\frac{b}{a})^2=\frac{b^2}{4a^2}$ to both sides of the equal sign:

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}$$

Next, write the left side as a perfect square. Find the common denominator of the right side and write it as a single fraction:

$$(x+\frac{b}{2a})^2=\frac{b^2-4ac}{4a^2}$$

Now, use the square root property, which gives

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a}$$

Finally, add $-\frac{b}{2a}$ to both sides of the equation and combine the terms on the right side. Thus,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

THE QUADRATIC FORMULA

Written in standard form, $ax^2+bx+c=0$, any quadratic equation can be solved using the **quadratic formula**:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \tag{1.9.2}$$

where a, b, and c are real numbers and $a\neq 0$.

How to

Given a quadratic equation, solve it using the quadratic formula

1. Make sure the equation is in standard form: $ax^2+bx+c=0$.
2. Make note of the values of the coefficients and constant term, a, b, and c.

3. Carefully substitute the values noted in step 2 into the equation. To avoid needless errors, use parentheses around each number input into the formula.
4. Calculate and solve.

Example 1.9.9: Solve the Quadratic Equation Using the Quadratic Formula

Solve the quadratic equation: $x^2+5x+1=0$.

Solution

Identify the coefficients: $a=1, b=5, c=1$. Then use the quadratic formula.

$$\begin{aligned} x &= \frac{-(5)\pm\sqrt{(5)^2-4(1)(1)}}{2(1)} \\ &= \frac{-5\pm\sqrt{25-4}}{2} \\ &= \frac{-5\pm\sqrt{21}}{2} \end{aligned}$$

Example 1.9.10: Solving a Quadratic Equation with the Quadratic Formula

Use the quadratic formula to solve $x^2+x+2=0$.

Solution

First, we identify the coefficients: $a=1$,$b=1$, and $c=2$.

Substitute these values into the quadratic formula.

$$\begin{aligned} x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ &= \frac{-(1)\pm\sqrt{(1)^2-4(1)(2)}}{2(1)} \\ &= \frac{-1\pm\sqrt{1-8}}{2} \\ &= \frac{-1\pm\sqrt{-7}}{2} \\ &= \frac{-1\pm i\sqrt{7}}{2} \end{aligned}$$

Exercise 1.9.8

Solve the quadratic equation using the quadratic formula: $9x^2+3x-2=0$.

Answer

$x=-\frac{2}{3}, x=\frac{1}{3}$

The Discriminant

The **quadratic formula** not only generates the solutions to a quadratic equation, it tells us about the nature of the solutions when we consider the **discriminant**, or the expression under the radical, b^2-4ac . The discriminant tells us whether the solutions are real numbers or complex numbers, and how many solutions of each type to expect. Table 1.9.1 relates the value of the discriminant to the solutions of a quadratic equation.

Table 1.9.1

Value of Discriminant	Results
$b^2-4ac=0$	One rational solution (double solution)

Value of Discriminant	Results
$b^2-4ac>0$, perfect square	Two rational solutions
$b^2-4ac>0$, not a perfect square	Two irrational solutions
$b^2-4ac<0$	Two complex solutions

THE DISCRIMINANT

For $ax^2+bx+c=0$, where a, b, and c are real numbers, the **discriminant** is the expression under the radical in the quadratic formula: b^2-4ac. It tells us whether the solutions are real numbers or complex numbers and how many solutions of each type to expect.

Example 1.9.11: Using the Discriminant to Find the Nature of the Solutions to a Quadratic Equation

Use the discriminant to find the nature of the solutions to the following quadratic equations:

a. $x^2+4x+4=0$
b. $8x^2+14x+3=0$
c. $3x^2-5x-2=0$
d. $3x^2-10x+15=0$

Solution

Calculate the discriminant b^2-4ac for each equation and state the expected type of solutions.

a.

$x^2+4x+4=0$

$b^2-4ac=(4)^2-4(1)(4)=0$ There will be one rational double solution.

b.

$8x^2+14x+3=0$

$b^2-4ac=(14)^2-4(8)(3)=100$ As 100 is a perfect square, there will be two rational solutions.

c.

$3x^2-5x-2=0$

$b^2-4ac=(-5)^2-4(3)(-2)=49$ As 49 is a perfect square, there will be two rational solutions.

d.

$3x^2-10x+15=0$

$b^2-4ac=(-10)^2-4(3)(15)=-80$ There will be two complex solutions.

Using the Pythagorean Theorem

One of the most famous formulas in mathematics is the **Pythagorean Theorem**. It is based on a right triangle, and states the relationship among the lengths of the sides as $a^2+b^2=c^2$, where a and b refer to the legs of a right triangle adjacent to the 90° angle, and c refers to the hypotenuse. It has immeasurable uses in architecture, engineering, the sciences, geometry, trigonometry, and algebra, and in everyday applications.

We use the Pythagorean Theorem to solve for the length of one side of a triangle when we have the lengths of the other two. Because each of the terms is squared in the theorem, when we are solving for a side of a triangle, we have a quadratic equation. We can use the methods for solving quadratic equations that we learned in this section to solve for the missing side.

The Pythagorean Theorem is given as

$$a^2+b^2=c^2 \tag{1.9.3}$$

where a and b refer to the legs of a right triangle adjacent to the 90° angle, and c refers to the hypotenuse, as shown in .

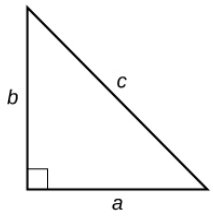


Figure 1.9.4

Example 1.9.12: Finding the Length of the Missing Side of a Right Triangle

Find the length of the missing side of the right triangle in Figure 1.9.5.

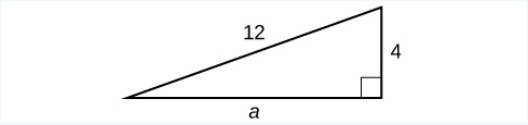


Figure 1.9.5

Solution

As we have measurements for side b and the hypotenuse, the missing side is a.

$$\begin{aligned} a^2+b^2 &= c^2 \\ a^2+(4)^2 &= (12)^2 \\ a^2+16 &= 144 \\ a^2 &= 128 \\ a &= \sqrt{128} \\ &= 8\sqrt{2} \end{aligned}$$

Exercise 1.9.9

Use the Pythagorean Theorem to solve the right triangle problem: Leg a measures 4 units, leg b measures 3 units. Find the length of the hypotenuse.

Answer

5 units

Media

Access these online resources for additional instruction and practice with quadratic equations.

1. Solving Quadratic Equations by Factoring
2. The Zero-Product Property
3. Completing the Square
4. Quadratic Formula with Two Rational Solutions
5. Length of a leg of a right triangle

Key Equations

quadratic formula	$x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

Key Concepts

- Many quadratic equations can be solved by factoring when the equation has a leading coefficient of 1 or if the equation is a difference of squares. The zero-factor property is then used to find solutions. See Example, Example, and Example.
- Many quadratic equations with a leading coefficient other than 1 can be solved by factoring using the grouping method. See Example and Example.
- Another method for solving quadratics is the square root property. The variable is squared. We isolate the squared term and take the square root of both sides of the equation. The solution will yield a positive and negative solution. See Example and Example.
- Completing the square is a method of solving quadratic equations when the equation cannot be factored. See Example.

- A highly dependable method for solving quadratic equations is the quadratic formula, based on the coefficients and the constant term in the equation. See Example.
- The discriminant is used to indicate the nature of the roots that the quadratic equation will yield: real or complex, rational or irrational, and how many of each. See Example.
- The Pythagorean Theorem, among the most famous theorems in history, is used to solve right-triangle problems and has applications in numerous fields. Solving for the length of one side of a right triangle requires solving a quadratic equation. See Example.

Homework Exercises 1.9

WeBWorK Problems:



Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login



Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login

Written Problems:

Verbal:

1. How do we recognize when an equation is quadratic?

2. When we solve a quadratic equation by factoring, why do we move all terms to one side, having zero on the other side?

Solve by factoring:

3. $4x^2 = 9$

4. $4x^2 = 5x$

5. $\frac{x}{3} - \frac{9}{x} = 2$

Solve by using the square root property:

6. $(x-1)^2 = 25$

Solve by using the quadratic formula:

7. $2x^2 + 5x + 3 = 0$

8. $4 + \frac{1}{x} - \frac{1}{x^2} = 0$

Application

9. A formula for the normal systolic blood pressure for a man age A, measured in mmHg, is given as $P = 0.006A^2 - 0.02A + 120$. Find the age to the nearest year of a man whose normal blood pressure measures 125 mmHg. You may solve either by using the quadratic formula, then round to the nearest year, or by graphing.

This page titled 1.9: Quadratic Equations is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **2.5: Quadratic Equations** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/precalculus.

1.10: Other Types of Equations

Learning Objectives

- Solve equations involving rational exponents.
- Solve equations using factoring.
- Solve radical equations.
- Solve absolute value equations.
- Solve other types of equations.

We have solved linear equations, rational equations, and quadratic equations using several methods. However, there are many other types of equations, and we will investigate a few more types in this section. We will look at equations involving rational exponents, polynomial equations, radical equations, absolute value equations, equations in quadratic form, and some rational equations that can be transformed into quadratics. Solving any equation, however, employs the same basic algebraic rules. We will learn some new techniques as they apply to certain equations, but the algebra never changes.

Solving Equations Involving Rational Exponents

Rational exponents are exponents that are fractions, where the numerator is a power and the denominator is a root. For example, $16^{\frac{1}{2}}$ is another way of writing $\sqrt{16}$; $8^{\frac{1}{3}}$ is another way of writing $\sqrt[3]{8}$. The ability to work with rational exponents is a useful skill, as it is highly applicable in calculus.

We can solve equations in which a variable is raised to a rational exponent by raising both sides of the equation to the reciprocal of the exponent. The reason we raise the equation to the reciprocal of the exponent is because we want to eliminate the exponent on the variable term, and a number multiplied by its reciprocal equals 1. For example,

$$\frac{2}{3}\left(\frac{3}{2}\right)=1$$

$$3\left(\frac{1}{3}\right)=1,$$

and so on.

RATIONAL EXPONENTS

A *rational exponent* indicates a power in the numerator and a root in the denominator. There are multiple ways of writing an expression, a variable, or a number with a rational exponent:

$$a^{\frac{m}{n}}=\left(a^{\frac{1}{n}}\right)^{m}=a^{m\frac{1}{n}}=\sqrt[n]{a^{m}}=(\sqrt[n]{a})^{m} \tag{1.10.1}$$

Example 1.10.1: Evaluating a Number Raised to a Rational Exponent

Evaluate $8^{\frac{2}{3}}$

Solution

Whether we take the root first or the power first depends on the number. It is easy to find the cube root of 8, so rewrite $8^{\frac{2}{3}}$ as $\left(8^{\frac{1}{3}}\right)^{2}$.

$$\begin{aligned}\left(8^{\frac{1}{3}}\right)^{2}&=(2)^{2}\\&=4\end{aligned}$$

Exercise 1.10.1

Evaluate $64^{-\frac{1}{3}}$

Answer

$\frac{1}{4}$

Example 1.10.2: Solve the Equation Including a Variable Raised to a Rational Exponent

Solve the equation in which a variable is raised to a rational exponent: $x^{\frac{5}{4}} = 32$.

Solution

The way to remove the exponent on x is by raising both sides of the equation to a power that is the reciprocal of $\frac{5}{4}$, which is $\frac{4}{5}$.

$$\begin{aligned} x^{\frac{5}{4}} &= 32 \\ \left(x^{\frac{5}{4}}\right)^{\frac{4}{5}} &= (32)^{\frac{4}{5}} \\ x &= (2)^4 \\ &= 16 \end{aligned}$$

Exercise 1.10.2

Solve the equation $x^{\frac{3}{2}} = 125$.

Answer

25

Example 1.10.3: Solving an Equation Involving Rational Exponents and Factoring

Solve $3x^{\frac{3}{4}} = x^{\frac{1}{2}}$.

Solution

This equation involves rational exponents as well as factoring rational exponents. Let us take this one step at a time. First, put the variable terms on one side of the equal sign and set the equation equal to zero.

$$\begin{aligned} 3x^{\frac{3}{4}} - \left(x^{\frac{1}{2}}\right) &= x^{\frac{1}{2}} - \left(x^{\frac{1}{2}}\right) \\ 3x^{\frac{3}{4}} - x^{\frac{1}{2}} &= 0 \end{aligned}$$

Now, it looks like we should factor the left side, but what do we factor out? We can always factor the term with the lowest exponent. Rewrite $x^{\frac{1}{2}}$ as $x^{\frac{2}{4}}$. Then, factor out $x^{\frac{2}{4}}$ from both terms on the left.

$$\begin{aligned} 3x^{\frac{3}{4}} - x^{\frac{1}{2}} &= 0 \\ x^{\frac{2}{4}}\left(3x^{\frac{1}{4}} - 1\right) &= 0 \end{aligned}$$

Where did $x^{\frac{1}{4}}$ come from? Remember, when we multiply two numbers with the same base, we add the exponents. Therefore, if we multiply $x^{\frac{2}{4}}$ back in using the distributive property, we get the expression we had before the factoring, which is what should happen. We need an exponent such that when added to $\frac{2}{4}$ equals $\frac{3}{4}$. Thus, the exponent on x in the parentheses is $\frac{1}{4}$.

Let us continue. Now we have two factors and can use the zero factor theorem.

$$\begin{aligned} x^{\frac{2}{4}}\left(3x^{\frac{1}{4}}-1\right) &= 0 \\ x^{\frac{2}{4}} &= 0 \\ x &= 0 \\ 3x^{\frac{1}{4}}-1 &= 0 \\ 3x^{\frac{1}{4}} &= 1 \\ x^{\frac{1}{4}} &= \frac{1}{3}, \qquad \text{Divide both sides by } 3. \\ \left(x^{\frac{1}{4}}\right)^4 &= \left(\frac{1}{3}\right)^4, \qquad \text{Raise both sides to the reciprocal of } \frac{1}{4} \\ x &= \frac{1}{81} \end{aligned}$$

The two solutions are 0 and $\frac{1}{81}$.

Exercise 1.10.3

Solve: $(x+5)^{\frac{3}{2}}=8$.

Answer

-1

Solving Equations Using Factoring

We have used factoring to solve quadratic equations, but it is a technique that we can use with many types of polynomial equations, which are equations that contain a string of terms including numerical coefficients and variables. When we are faced with an equation containing polynomials of degree higher than 2, we can often solve them by factoring.

POLYNOMIAL EQUATIONS

A polynomial of degree n is an expression of the type

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \tag{1.10.2}$$

where n is a positive integer and $a_n, \ldots, a_0$ are real numbers and $a_n \neq 0$.

Setting the polynomial equal to zero gives a **polynomial equation**. The total number of solutions (real and complex) to a polynomial equation is equal to the highest exponent n.

Example 1.10.4: Solving a Polynomial by Factoring

Solve the polynomial by factoring: $5x^4 = 80x^2$.

Solution

First, set the equation equal to zero. Then factor out what is common to both terms, the GCF.

$$\begin{aligned} 5x^4 - 80x^2 &= 0 \\ 5x^2(x^2 - 16) &= 0 \end{aligned}$$

Notice that we have the difference of squares in the factor $x^2 - 16$, which we will continue to factor and obtain two solutions. The first term, $5x^2$, generates, technically, two solutions as the exponent is 2, but they are the same solution.

$$\begin{aligned} 5x^2 &= 0 \\ x &= 0 \\ x^2 - 16 &= 0 \\ (x+4)(x-4) &= 0 \\ x &= 4 \\ x &= -4 \end{aligned}$$

The solutions are 0 (double solution), 4, and -4.

Analysis

We can see the solutions on the graph in Figure 1.10.1. The x-coordinates of the points where the graph crosses the x-axis are the solutions—the x-intercepts. Notice on the graph that at the solution 0, the graph touches the x-axis and bounces back. It does not cross the x-axis. This is typical of double solutions.

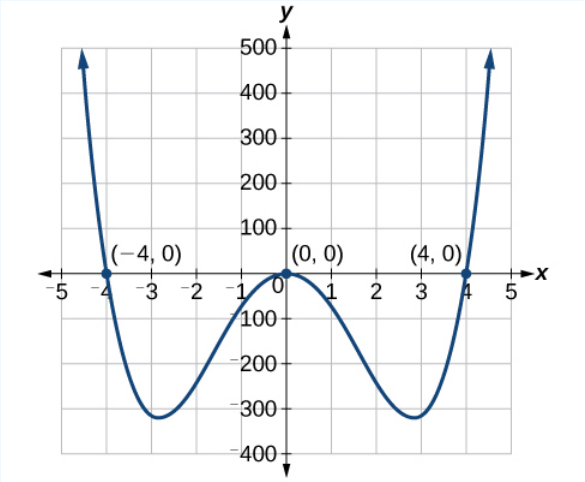


Figure 1.10.1

? Exercise 1.10.4

Solve by factoring: $12x^4 = 3x^2$.

Answer

$x = 0, x = 12, x = -12$

✔ Example 1.10.5: Solve a Polynomial by Grouping

Solve a polynomial by grouping: $x^3 + x^2 - 9x - 9 = 0$.

Solution

This polynomial consists of 4 terms, which we can solve by grouping. Grouping procedures require factoring the first two terms and then factoring the last two terms. If the factors in the parentheses are identical, we can continue the process and solve, unless more factoring is suggested.

$$\begin{aligned} x^3+x^2-9x-9 &= 0 \\ x^2(x+1)-9(x+1) &= 0 \\ (x^2-9)(x+1) &= 0 \end{aligned}$$

The grouping process ends here, as we can factor x^2-9 using the difference of squares formula.

$$\begin{aligned} (x^2-9)(x+1) &= 0 \\ (x-3)(x+3)(x+1) &= 0 \\ x &= 3 \\ x &= -3 \\ x &= -1 \end{aligned}$$

The solutions are 3, -3, and -1. Note that the highest exponent is 3 and we obtained 3 solutions. We can see the solutions, the x-intercepts, on the graph in Figure $1.10.2$

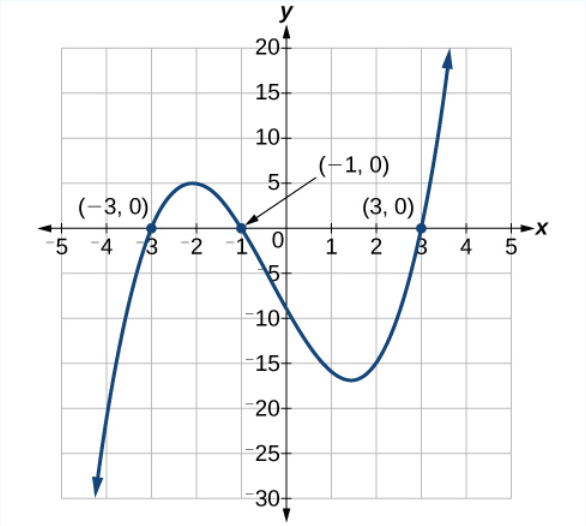


Figure 1.10.2

Analysis

We looked at solving quadratic equations by factoring when the leading coefficient is 1. When the leading coefficient is not 1, we solved by grouping. Grouping requires four terms, which we obtained by splitting the linear term of quadratic equations. We can also use grouping for some polynomials of degree higher than 2, as we saw here, since there were already four terms.

Solving Radical Equations

Radical equations are equations that contain variables in the **radicand** (the expression under a radical symbol), such as

$$\sqrt{3x+18} = x$$

$$\sqrt{x+3} = x-3$$

$$\sqrt{x+5} - \sqrt{x-3} = 2$$

Radical equations may have one or more radical terms, and are solved by eliminating each radical, one at a time. We have to be careful when solving radical equations, as it is not unusual to find **extraneous solutions**, roots that are not, in fact, solutions to the equation. These solutions are not due to a mistake in the solving method, but result from the process of raising both sides of an equation to a power. However, checking each answer in the original equation will confirm the true solutions.

RADICAL EQUATIONS

An equation containing terms with a variable in the radicand is called a **radical equation.**

Howto: Given a radical equation, solve it

1. Isolate the radical expression on one side of the equal sign. Put all remaining terms on the other side.
2. If the radical is a square root, then square both sides of the equation. If it is a cube root, then raise both sides of the equation to the third power. In other words, for an nth root radical, raise both sides to the nth power. Doing so eliminates the radical symbol.
3. Solve the remaining equation.
4. If a radical term still remains, repeat steps 1–2.
5. Confirm solutions by substituting them into the original equation.

Example 1.10.6: Solving an Equation with One Radical

Solve $\sqrt{15-2x}=x$.

Solution

The radical is already isolated on the left side of the equal side, so proceed to square both sides.

$$\begin{aligned}\sqrt{15-2x} &= x\\ (\sqrt{15-2x})^2 &= (x)^2\\ 15-2x &= x^2\end{aligned}$$

We see that the remaining equation is a quadratic. Set it equal to zero and solve.

$$\begin{aligned}0 &= x^2+2x-15\\ 0 &= (x+5)(x-3)\\ x &= -5\\ x &= 3\end{aligned}$$

The proposed solutions are -5 and 3. Let us check each solution back in the original equation. First, check $x=-5$.

$$\begin{aligned}\sqrt{15-2x} &= x\\ \sqrt{15-2(-5)} &= -5\\ \sqrt{25} &= -5\\ 5 &\neq -5\end{aligned}$$

This is an extraneous solution. While no mistake was made solving the equation, we found a solution that does not satisfy the original equation.

Check $x=3$.

$$\begin{aligned}\sqrt{15-2x} &= x\\ \sqrt{15-2(3)} &= 3\\ \sqrt{9} &= 3\\ 3 &= 3\end{aligned}$$

The solution is 3.

Exercise 1.10.5

Solve the radical equation: $\sqrt{x+3}=3x-1$

Answer

$x=1$, extraneous solution $x=-\frac{2}{9}$

Example 1.10.7: Solving a Radical Equation Containing Two Radicals

Solve $\sqrt{2x+3}+\sqrt{x-2}=4$

Solution

As this equation contains two radicals, we isolate one radical, eliminate it, and then isolate the second radical.

$$\sqrt{2x+3}+\sqrt{x-2}=4$$

$$\begin{aligned}\sqrt{2x+3} &= 4-\sqrt{x-2} && \text{Subtract } \sqrt{x-2} \text{ from both sides}\\ (\sqrt{2x+3})^2 &= (4-\sqrt{x-2})^2 && \text{Square both sides}\end{aligned}$$

Use the perfect square formula to expand the right side: $(a-b)^2=a^2-2ab+b^2$.

$$\begin{aligned}2x+3 &= (4)^2-2(4)\sqrt{x-2}+(\sqrt{x-2})^2\\ 2x+3 &= 16-8\sqrt{x-2}+(x-2)\\ 2x+3 &= 14+x-8\sqrt{x-2} && \text{Combine like terms}\\ x-11 &= -8\sqrt{x-2} && \text{Isolate the second radical}\\ (x-11)^2 &= (-8\sqrt{x-2})^2 && \text{Square both sides}\\ x^2-22x+121 &= 64(x-2)\end{aligned}$$

Now that both radicals have been eliminated, set the quadratic equal to zero and solve.

$$\begin{aligned}x^2-22x+121 &= 64x-128\\ x^2-86x+249 &= 0\\ (x-3)(x-83) &= 0\\ x &= 3\\ x &= 83\end{aligned}$$

The proposed solutions are 3 and 83. Check each solution in the original equation.

$$\begin{aligned}\sqrt{2x+3}+\sqrt{x-2} &= 4\\ \sqrt{2x+3} &= 4-\sqrt{x-2}\\ \sqrt{2(3)+3} &= 4-\sqrt{(3)-2}\\ \sqrt{9} &= 4-\sqrt{1}\\ 3 &= 3\end{aligned}$$

One solution is 3.

Check $x=83$.

$$\begin{aligned}\sqrt{2x+3}+\sqrt{x-2} &= 4\\ \sqrt{2x+3} &= 4-\sqrt{x-2}\\ \sqrt{2(83)+3} &= 4-\sqrt{(83)-2}\\ \sqrt{169} &= 4-\sqrt{81}\\ 13 &\neq -5\end{aligned}$$

The only solution is 3. We see that $x=83$ is an extraneous solution.

Exercise 1.10.6

Solve the equation with two radicals: $\sqrt{3x+7}+\sqrt{x+2}=1$

Answer

$x=-2$, extraneous solution $x=-1$

Solving an Absolute Value Equation

Next, we will learn how to solve an **absolute value equation**. To solve an equation such as $|2x-6|=8$, we notice that the absolute value will be equal to 8 if the quantity inside the absolute value bars is 8 or -8. This leads to two different equations we can solve independently.

$$\begin{aligned} 2x-6 &= 8 \\ 2x &= 14 \\ x &= 7 \end{aligned}$$

OR

$$\begin{aligned} 2x-6 &= -8 \\ 2x &= -2 \\ x &= -1 \end{aligned}$$

Knowing how to solve problems involving absolute value functions is useful. For example, we may need to identify numbers or points on a line that are at a specified distance from a given reference point.

ABSOLUTE VALUE EQUATIONS

The absolute value of x is written as $|x|$. It has the following properties:

If $x \geq 0$, then $|x| = x$.If $x < 0$, then $x = -x$.

For real numbers A and B, an equation of the form $|A| = B$, with $B \geq 0$, will have solutions when $A = B$ or $A = -B$. If $B < 0$, the equation $|A| = B$ has no solution.

An **absolute value equation** in the form $|ax+b|=c$ has the following properties:

- If $c<0, |ax+b|=c$ has no solution.
- If $c=0, |ax+b|=c$ has one solution.
- If $c>0, |ax+b|=c$ has two solutions.

How to

Given an absolute value equation, solve it.

1. Isolate the absolute value expression on one side of the equal sign.
2. If $c>0$, write and solve two equations: $ax+b=c$ and $ax+b=-c$.

Example 1.10.8: Solving Absolute Value Equations

Solve the following absolute value equations:

a. $|6x+4|=8$
b. $|3x+4|=-9$
c. $|3x-5|-4=6$
d. $|-5x+10|=0$

Solution

a. $|6x+4|=8$

Write two equations and solve each:

$$\begin{aligned} 6x+4 &= 8 \\ 6x &= 4 \\ x &= \frac{2}{3} \end{aligned}$$

OR

$$\begin{aligned} 6x+4 &= -8 \\ 6x &= -12 \\ x &= -2 \end{aligned}$$

The two solutions are $\frac{2}{3}$ and -2.

b. $|3x+4| = -9$

There is no solution as an absolute value cannot be negative.

c. $|3x-5|-4 = 6$

Isolate the absolute value expression and then write two equations.

$$\begin{aligned} |3x-5|-4 &= 6 \\ |3x-5| &= 10 \\ 3x-5 &= 10 \\ 3x &= 15 \\ x &= 5 \end{aligned}$$

OR

$$\begin{aligned} 3x-5 &= -10 \\ 3x &= -5 \\ x &= \frac{5}{3} \end{aligned}$$

There are two solutions: 5, and $-\frac{5}{3}$.

d. $|-5x+10| = 0$

The equation is set equal to zero, so we have to write only one equation.

$$\begin{aligned} -5x+10 &= 0 \\ -5x &= -10 \\ x &= 2 \end{aligned}$$

There is one solution: \(2\).

? Exercise 1.10.7

Solve the absolute value equation: $|1-4x|+8 = 13$.

Answer

$x = -1, x = \frac{3}{2}$

Solving Other Types of Equations

There are many other types of equations in addition to the ones we have discussed so far. We will see more of them throughout the text. Here, we will discuss equations that are in quadratic form, and rational equations that result in a quadratic.

Solving Equations in Quadratic Form

Equations in quadratic form are equations with three terms. The first term has a power other than 2. The middle term has an exponent that is one-half the exponent of the leading term. The third term is a constant. We can solve equations in this form as if they were quadratic. A few examples of these equations include $x^4-5x^2+4=0$,$x^6+7x^3-8=0$, and $x^{\frac{2}{3}}+4x^{\frac{1}{3}}+2=0$. In each one, doubling the exponent of the middle term equals the exponent on the leading term. We can solve these equations by substituting a variable for the middle term.

QUADRATIC FORM

If the exponent on the middle term is one-half of the exponent on the leading term, we have **an equation in quadratic form**, which we can solve as if it were a quadratic. We substitute a variable for the middle term to solve equations in quadratic form.

Howto: Given an equation quadratic in form, solve it

1. Identify the exponent on the leading term and determine whether it is double the exponent on the middle term.
2. If it is, substitute a variable, such as u, for the variable portion of the middle term.
3. Rewrite the equation so that it takes on the standard form of a quadratic.
4. Solve using one of the usual methods for solving a quadratic.
5. Replace the substitution variable with the original term.
6. Solve the remaining equation.

Example 1.10.9: Solving a Fourth-degree Equation in Quadratic Form

Solve this fourth-degree equation: $3x^4 - 2x^2 - 1 = 0$.

Solution

This equation fits the main criteria, that the power on the leading term is double the power on the middle term. Next, we will make a substitution for the variable term in the middle. Let $u = x^2$. Rewrite the equation in u.

$$3u^2 - 2u - 1 = 0$$

Now solve the quadratic.

$$\begin{aligned} 3u^2 - 2u - 1 &= 0 \\ (3u+1)(u-1) &= 0 \end{aligned}$$

Solve each factor and replace the original term for u.

$$\begin{aligned} 3u + 1 &= 0 \\ 3u &= -1 \\ u &= -\frac{1}{3} \\ x^2 &= -\frac{1}{3} \\ x &= \pm i\sqrt{\frac{1}{3}} \\ u - 1 &= 0 \\ u &= 1 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

The solutions are $x = \pm i\sqrt{\frac{1}{3}}$ and $x = \pm 1$

Exercise 1.10.8

Solve using substitution: $x^4 - 8x^2 - 9 = 0$.

Answer

$x = -3, 3, -i, i$

Example 1.10.10: Solving an Equation in Quadratic Form Containing a Binomial

Solve the equation in quadratic form: $(x+2)^2+11(x+2)-12=0$.

Solution

This equation contains a binomial in place of the single variable. The tendency is to expand what is presented. However, recognizing that it fits the criteria for being in quadratic form makes all the difference in the solving process. First, make a substitution, letting $u=x+2$. Then rewrite the equation in u.

$$\begin{aligned} u^2+11u-12 &= 0 \\ (u+12)(u-1) &= 0 \end{aligned}$$

Solve using the zero-factor property and then replace u with the original expression.

$$\begin{aligned} u+12 &= 0 \\ u &= -12 \\ x+2 &= -12 \\ x &= -14 \end{aligned}$$

The second factor results in

$$\begin{aligned} u-1 &= 0 \\ u &= 1 \\ x+2 &= 1 \\ x &= -1 \end{aligned}$$

We have two solutions: -14, and -1.

Exercise 1.10.9

Solve: $(x-5)^2-4(x-5)-21=0$.

Answer

$x=2, x=12$

Solving Rational Equations Resulting in a Quadratic

Earlier, we solved rational equations. Sometimes, solving a rational equation results in a quadratic. When this happens, we continue the solution by simplifying the quadratic equation by one of the methods we have seen. It may turn out that there is no solution.

Example 1.10.11: Solving a Rational Equation Leading to a Quadratic

Solve the following rational equation: $\dfrac{-4x}{x-1}+\dfrac{4}{x+1}=\dfrac{-8}{x^2-1}$

Solution

We want all denominators in factored form to find the LCD. Two of the denominators cannot be factored further. However, $x^2-1=(x+1)(x-1)$. Then, the LCD is $(x+1)(x-1)$. Next, we multiply the whole equation by the LCD.

$$
\begin{aligned}
(x+1)(x-1)\left(\frac{-4x}{x-1}+\frac{4}{x+1}\right) &= \left(\frac{-8}{x^2-1}\right)(x+1)(x-1) \\
-4x(x+1)+4(x-1) &= -8 \\
-4x^2-4x+4x-4 &= -8 \\
-4x^2+4 &= 0 \\
-4(x^2-1) &= 0 \\
-4(x+1)(x-1) &= 0 \\
x &= -1 \\
x &= 1
\end{aligned}
$$

In this case, either solution produces a zero in the denominator in the original equation. Thus, there is no solution.

? Exercise 1.10.10

Solve $\frac{3x+2}{x-2}+\frac{1}{x}=\frac{-2}{x^2-2x}$

Answer

$x=-1, x=0$ is not a solution.

Media

Access these online resources for additional instruction and practice with different types of equations.

1. Rational Equation with no Solution
2. Solving equations with rational exponents using reciprocal powers
3. Solving radical equations part 1 of 2
4. Solving radical equations part 2 of 2

Key Concepts

- Rational exponents can be rewritten several ways depending on what is most convenient for the problem. To solve, both sides of the equation are raised to a power that will render the exponent on the variable equal to 1. See Example, Example, and Example.
- Factoring extends to higher-order polynomials when it involves factoring out the GCF or factoring by grouping. See Example and Example.
- We can solve radical equations by isolating the radical and raising both sides of the equation to a power that matches the index. See Example and Example.
- To solve absolute value equations, we need to write two equations, one for the positive value and one for the negative value. See Example.
- Equations in quadratic form are easy to spot, as the exponent on the first term is double the exponent on the second term and the third term is a constant. We may also see a binomial in place of the single variable. We use substitution to solve. See Example and Example.
- Solving a rational equation may also lead to a quadratic equation or an equation in quadratic form. See Example.

Homework Exercises 1.10

WeBWorK Problems:

Written Problems:

Verbal:

1. Explain why possible solutions *must* be checked in radical equations.

2. Explain how to change a rational exponent into the correct radical expression.

3. Why are there two solutions to $x^2 = 4$? If you take the square root of both sides of an equation to solve for a squared expression, what do you need to remember to do?

Solve algebraically:

4. $x^{\frac{2}{3}} = 16$

5. $(x-1)^{\frac{3}{4}} = 8$

6. $\sqrt{3x-1} - 2 = 0$

7. $\sqrt{12-x} = x$

8. $x^4 - 10x^2 + 9 = 0$

Application

9. A model for body surface area, *BSA*, is $BSA = \sqrt{\frac{wh}{3600}}$, where *w* = weight in kg and *h* = height in cm. Find the height of a 72 kg person to the nearest cm whose *BSA* = 1.8.

This page titled 1.10: Other Types of Equations is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **2.6: Other Types of Equations** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/precalculus.

CHAPTER OVERVIEW

2: Function Essentials

2.1: Functions and Function Notation
2.2: Domain and Range
2.3: Rates of Change and Behavior of Graphs
2.4: Transformation of Functions

2: Function Essentials is shared under a not declared license and was authored, remixed, and/or curated by LibreTexts.

2.1: Functions and Function Notation

Preview Video for section 2.1



Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login

Learning Objectives

- Determine whether a relation represents a function.
- Find the value of a function.
- Determine whether a function is one-to-one.
- Use the vertical line test to identify functions.
- Graph the functions listed in the library of functions.

A jetliner changes altitude as its distance from the starting point of a flight increases. The weight of a growing child increases with time. In each case, one quantity depends on another. There is a relationship between the two quantities that we can describe, analyze, and use to make predictions. In this section, we will analyze such relationships.

Determining Whether a Relation Represents a Function

A relation is a set of ordered pairs. The set of the first components of each ordered pair is called the domain and the set of the second components of each ordered pair is called the range. Consider the following set of ordered pairs. The first numbers in each pair are the first five natural numbers. The second number in each pair is twice that of the first.

$$\{(1,2),(2,4),(3,6),(4,8),(5,10)\} \tag{1.1.1}$$

The domain is $\{1,2,3,4,5\}$ The range is $\{2,4,6,8,10\}$

Note that each value in the domain is also known as an **input** value, or **independent variable**, and is often labeled with the lowercase letter x. Each value in the range is also known as an output value, or **dependent variable**, and is often labeled lowercase letter y.

A function f is a relation that assigns a single value in the range to each value in the domain. In other words, no x-values are repeated. For our example that relates the first five **natural numbers** to numbers double their values, this relation is a function because each element in the domain, {1, 2, 3, 4, 5}, is paired with exactly one element in the range, $\{2,4,6,8,10\}$

Now let's consider the set of ordered pairs that relates the terms "even" and "odd" to the first five natural numbers. It would appear as

$$\{(\text{odd},1),(\text{even},2),(\text{odd},3),(\text{even},4),(\text{odd},5)\} \tag{1.1.2}$$

Notice that each element in the domain, {even, odd} is not paired with exactly one element in the range, $\{1,2,3,4,5\}$ For example, the term "odd" corresponds to three values from the range, $\{1,3,5\}$,and the term "even" corresponds to two values from the range, $\{2,4\}$. This violates the definition of a function, so this relation is not a function.

Figure 2.1.1 compares relations that are functions and not functions.

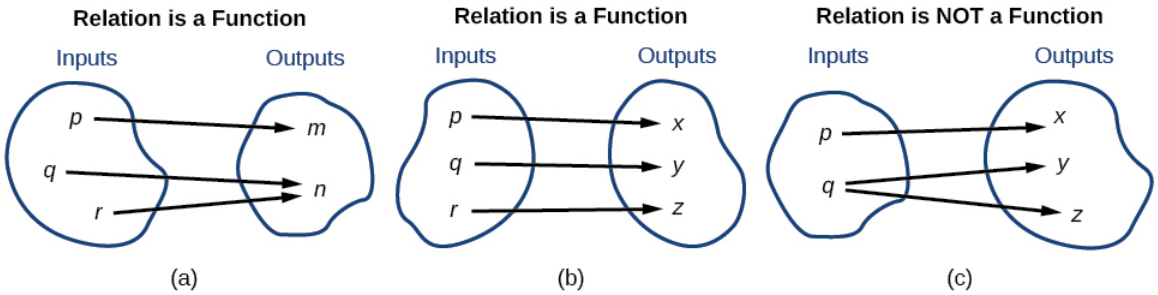


Figure 2.1.1: (a) This relationship is a function because each input is associated with a single output. Note that input q and r both give output n. (b) This relationship is also a function. In this case, each input is associated with a single output. (c) This relationship is not a function because input q is associated with two different outputs.

Function

A **function** is a relation in which each possible input value leads to exactly one output value. We say "the output is a function of the input."

The **input** values make up the **domain**, and the **output** values make up the **range**.

How To: Given a relationship between two quantities, determine whether the relationship is a function

1. Identify the input values.
2. Identify the output values.
3. If each input value leads to only one output value, classify the relationship as a function. If any input value leads to two or more outputs, do not classify the relationship as a function.

Example 2.1.1: Determining If Menu Price Lists Are Functions

The coffee shop menu, shown in Figure 2.1.2 consists of items and their prices.

a. Is price a function of the item?
b. Is the item a function of the price?

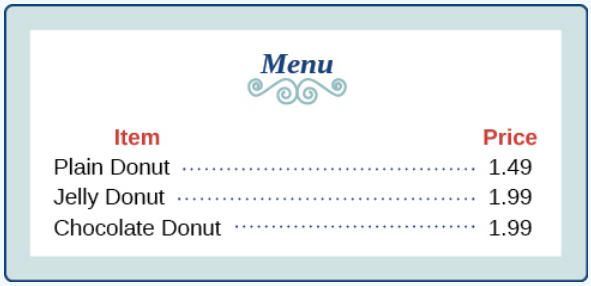


Figure 2.1.2: A menu of donut prices from a coffee shop where a plain donut is $1.49 and a jelly donut and chocolate donut are $1.99.

Solution

a. Let's begin by considering the input as the items on the menu. The output values are then the prices. See Figure 2.1.3.



Figure 2.1.3: A menu of donut prices from a coffee shop where a plain donut is $1.49 and a jelly donut and chocolate donut are $1.99.

Each item on the menu has only one price, so the price is a function of the item.

a. Two items on the menu have the same price. If we consider the prices to be the input values and the items to be the output, then the same input value could have more than one output associated with it. See Figure 2.1.4.

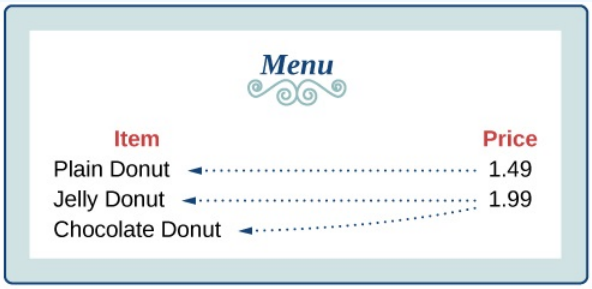


Figure 2.1.4: Association of the prices to the donuts.

Therefore, the item is a not a function of price.

Example 2.1.2: Determining If Class Grade Rules Are Functions

In a particular math class, the overall percent grade corresponds to a grade point average. Is grade point average a function of the percent grade? Is the percent grade a function of the grade point average? Table 2.1.1 shows a possible rule for assigning grade points.

Table 2.1.1: Class grade points.

Percent grade	0–56	57–61	62–66	67–71	72–77	78–86	87–91	92–100
Grade point average	0.0	1.0	1.5	2.0	2.5	3.0	3.5	4.0

Solution

For any percent grade earned, there is an associated grade point average, so the grade point average is a function of the percent grade. In other words, if we input the percent grade, the output is a specific grade point average.

In the grading system given, there is a range of percent grades that correspond to the same grade point average. For example, students who receive a grade point average of 3.0 could have a variety of percent grades ranging from 78 all the way to 86. Thus, percent grade is not a function of grade point average.

Exercise 2.1.2

Table 2.1.2 lists the five greatest baseball players of all time in order of rank.

Table 2.1.2: Five greatest baseball players.

Player	Rank

Player	Rank
Babe Ruth	1
Willie Mays	2
Ty Cobb	3
Walter Johnson	4
Hank Aaron	5

a. Is the rank a function of the player name?
b. Is the player name a function of the rank?

Answer a

Yes

Answer b

yes. (Note: If two players had been tied for, say, 4th place, then the name would not have been a function of rank.)

Using Function Notation

Once we determine that a relationship is a function, we need to display and define the functional relationships so that we can understand and use them, and sometimes also so that we can program them into computers. There are various ways of representing functions. A standard function notation is one representation that facilitates working with functions.

To represent "height is a function of age," we start by identifying the descriptive variables h for height and a for age. The letters f, g,and h are often used to represent functions just as we use x, y,and z to represent numbers and A, B, and C to represent sets.

$$\begin{aligned} &h \text{ is } f \text{ of } a && \text{We name the function } f\text{; height is a function of age.} \\ &h = f(a) && \text{We use parentheses to indicate the function input.} \\ &f(a) && \text{We name the function } f \text{ ; the expression is read as "} f \text{ of } a\text{."} \end{aligned} \tag{2.1.1}$$

Remember, we can use any letter to name the function; the notation $h(a)$ shows us that h depends on a. The value a must be put into the function h to get a result. The parentheses indicate that age is input into the function; they do not indicate multiplication.

We can also give an algebraic expression as the input to a function. For example $f(a+b)$ means "first add a and b, and the result is the input for the function f." The operations must be performed in this order to obtain the correct result.

Function Notation

The notation $y = f(x)$ defines a function named f. This is read as "y is a function of x." The letter x represents the input value, or independent variable. The letter y, or $f(x)$, represents the output value, or dependent variable.

Example 2.1.3: Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month.

Solution

Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month.

The number of days in a month is a function of the name of the month, so if we name the function f, we write $\text{days} = f(\text{month})$ or $d = f(m)$. The name of the month is the input to a "rule" that associates a specific number (the output) with each input.

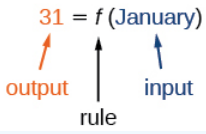


Figure 2.1.5: The function $31 = f(January)$ where 31 is the output, f is the rule, and January is the input.

For example, $f(\text{March}) = 31$, because March has 31 days. The notation $d = f(m)$ reminds us that the number of days, d (the output), is dependent on the name of the month, m (the input).

Analysis

Note that the inputs to a function do not have to be numbers; function inputs can be names of people, labels of geometric objects, or any other element that determines some kind of output. However, most of the functions we will work with in this book will have numbers as inputs and outputs.

Example 2.1.3B: Interpreting Function Notation

A function $N = f(y)$ gives the number of police officers, N, in a town in year y. What does $f(2005) = 300$ represent?

Solution

When we read $f(2005) = 300$, we see that the input year is 2005. The value for the output, the number of police officers (N), is 300. Remember, $N = f(y)$. The statement $f(2005) = 300$ tells us that in the year 2005 there were 300 police officers in the town.

Exercise 2.1.3

Use function notation to express the weight of a pig in pounds as a function of its age in days d.

Answer

$w = f(d)$

Q&A

Instead of a notation such as $y = f(x)$, could we use the same symbol for the output as for the function, such as $y = y(x)$, meaning "y is a function of x?"

Yes, this is often done, especially in applied subjects that use higher math, such as physics and engineering. However, in exploring math itself we like to maintain a distinction between a function such as f, which is a rule or procedure, and the output y we get by applying f to a particular input x. This is why we usually use notation such as $y = f(x)$, $P = W(d)$, and so on.

Representing Functions Using Tables

A common method of representing functions is in the form of a table. The table rows or columns display the corresponding input and output values. In some cases, these values represent all we know about the relationship; other times, the table provides a few select examples from a more complete relationship.

Table 2.1.3 lists the input number of each month ($\text{January} = 1$, $\text{February} = 2$, and so on) and the output value of the number of days in that month. This information represents all we know about the months and days for a given year (that is not a leap year). Note that, in this table, we define a days-in-a-month function f where $D = f(m)$ identifies months by an integer rather than by name.

Table 2.1.3: Months and number of days per month.

Month number, m (input)	1	2	3	4	5	6	7	8	9	10	11	12
Days in month, D (output)	31	28	31	30	31	30	31	31	30	31	30	31

Table 2.1.4 defines a function $Q = g(n)$ Remember, this notation tells us that g is the name of the function that takes the input n and gives the output Q.

Table 2.1.4: Function $Q = g(n)$

n	1	2	3	4	5
Q	8	6	7	6	8

Table 2.1.5 displays the age of children in years and their corresponding heights. This table displays just some of the data available for the heights and ages of children. We can see right away that this table does not represent a function because the same input value, 5 years, has two different output values, 40 in. and 42 in.

Table 2.1.5: Age of children and their corresponding heights.

Age in years, a (input)	5	5	6	7	8	9	10
Height in inches, h (output)	40	42	44	47	50	52	54

How To: Given a table of input and output values, determine whether the table represents a function

1. Identify the input and output values.
2. Check to see if each input value is paired with only one output value. If so, the table represents a function.

Example 2.1.5: Identifying Tables that Represent Functions

Which table, Table 2.1.6, Table 2.1.7, or Table 2.1.8, represents a function (if any)?

Table 2.1.6

Input	Output
2	1
5	3
8	6

Table 2.1.7

Input	Output
-3	5
0	1
4	5

Table 2.1.8

Input	Output
1	0
5	2
5	4

Solution

Table 2.1.6 and Table 2.1.7 define functions. In both, each input value corresponds to exactly one output value. Table 2.1.8 does not define a function because the input value of 5 corresponds to two different output values.

When a table represents a function, corresponding input and output values can also be specified using function notation.

The function represented by Table 2.1.6 can be represented by writing

$$f(2) = 1,\ f(5) = 3, \text{ and } f(8) = 6$$

Similarly, the statements

$$g(-3) = 5,\ g(0) = 1, \text{ and } g(4) = 5$$

represent the function in Table 2.1.7.

Table 2.1.8 cannot be expressed in a similar way because it does not represent a function.

Exercise 2.1.5

Does Table 2.1.9 represent a function?

Table 2.1.9

Input	Output
1	10
2	100
3	1000

Answer

yes

Finding Input and Output Values of a Function

When we know an input value and want to determine the corresponding output value for a function, we evaluate the function. Evaluating will always produce one result because each input value of a function corresponds to exactly one output value.

When we know an output value and want to determine the input values that would produce that output value, we set the output equal to the function's formula and solve for the input. Solving can produce more than one solution because different input values can produce the same output value.

Evaluation of Functions in Algebraic Forms

When we have a function in formula form, it is usually a simple matter to evaluate the function. For example, the function $f(x) = 5 - 3x^2$ can be evaluated by squaring the input value, multiplying by 3, and then subtracting the product from 5.

How To: Given the formula for a function, evaluate.

Given the formula for a function, evaluate.

1. Replace the input variable in the formula with the value provided.
2. Calculate the result.

Example 2.1.6A: Evaluating Functions at Specific Values

1. Evaluate $f(x) = x^2 + 3x - 4$ at

   a. 2
   b. a
   c. $a + h$
   d. Evaluate $\frac{f(a+h)-f(a)}{h}$

Solution

Replace the x in the function with each specified value.

a. Because the input value is a number, 2, we can use simple algebra to simplify.

$$\begin{aligned} f(2) &= 2^2 + 3(2) - 4 \\ &= 4 + 6 - 4 \\ &= 6 \end{aligned}$$

b. In this case, the input value is a letter so we cannot simplify the answer any further.

$$f(a) = a^2 + 3a - 4$$

c. With an input value of $a + h$, we must use the distributive property.

$$\begin{aligned} f(a+h) &= (a+h)^2 + 3(a+h) - 4 \\ &= a^2 + 2ah + h^2 + 3a + 3h - 4 \end{aligned}$$

d. In this case, we apply the input values to the function more than once, and then perform algebraic operations on the result. We already found that

$$f(a+h) = a^2 + 2ah + h^2 + 3a + 3h - 4$$

and we know that

$$f(a) = a^2 + 3a - 4$$

Now we combine the results and simplify.

$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= \frac{(a^2 + 2ah + h^2 + 3a + 3h - 4) - (a^2 + 3a - 4)}{h} \\ &= \frac{(2ah + h^2 + 3h)}{h} \\ &= \frac{h(2a + h + 3)}{h} && \text{Factor out h.} \\ &= 2a + h + 3 && \text{Simplify.} \end{aligned}$$

Example 2.1.6B: Evaluating Functions

Given the function $h(p) = p^2 + 2p$, evaluate $h(4)$.

Solution

To evaluate $h(4)$, we substitute the value 4 for the input variable p in the given function.

$$\begin{aligned} h(p) &= p^2 + 2p \\ h(4) &= (4)^2 + 2(4) \\ &= 16 + 8 \\ &= 24 \end{aligned}$$

Therefore, for an input of 4, we have an output of 24.

Exercise 2.1.6

Given the function $g(m) = \sqrt{m-4}$, evaluate $g(5)$.

Answer

$g(5) = 1$

Example 2.1.7: Solving Functions

Given the function $h(p) = p^2 + 2p$, solve for $h(p) = 3$.

Solution

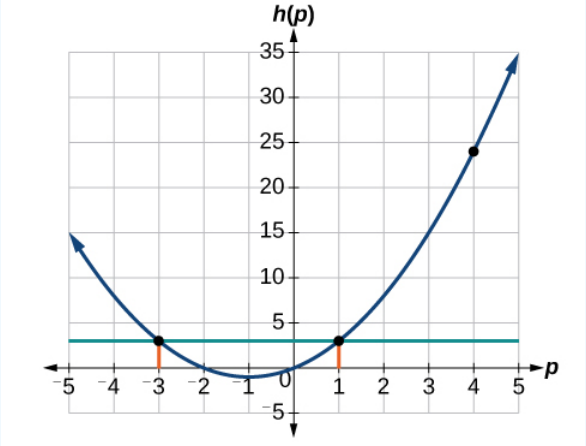
$$\begin{aligned} h(p) &= 3 & \\ p^2 + 2p &= 3 & \text{Substitute the original function} \\ p^2 + 2p - 3 &= 0 & \text{Subtract 3 from each side.} \\ (p+3)(p-1) &= 0 & \text{Factor.} \end{aligned}$$

If $(p+3)(p-1) = 0$, either $(p+3) = 0$ or $(p-1) = 0$ (or both of them equal 0). We will set each factor equal to 0 and solve for p in each case.

$$(p+3) = 0,\ p = -3$$

$$(p-1) = 0,\ p = 1$$

This gives us two solutions. The output $h(p) = 3$ when the input is either $p = 1$ or $p = -3$. We can also verify by graphing as in Figure 2.1.6. The graph verifies that $h(1) = h(-3) = 3$ and $h(4) = 24$.



p	−3	−2	0	1	4
h(p)	3	0	0	3	24

Figure 2.1.6: Graph of $h(p) = p^2 + 2p$

Exercise 2.1.7

Given the function $g(m) = \sqrt{m-4}$, solve $g(m) = 2$.

Answer

$m = 8$

Evaluating Functions Expressed in Formulas

Some functions are defined by mathematical rules or procedures expressed in **equation** form. If it is possible to express the function output with a formula involving the input quantity, then we can define a function in algebraic form. For example, the equation $2n + 6p = 12$ expresses a functional relationship between n and p. We can rewrite it to decide if p is a function of n.

How to: Given a function in equation form, write its algebraic formula.

1. Solve the equation to isolate the output variable on one side of the equal sign, with the other side as an expression that involves only the input variable.
2. Use all the usual algebraic methods for solving equations, such as adding or subtracting the same quantity to or from both sides, or multiplying or dividing both sides of the equation by the same quantity.

Example 2.1.8A: Finding an Equation of a Function

Express the relationship $2n + 6p = 12$ as a function $p = f(n)$, if possible.

Solution

To express the relationship in this form, we need to be able to write the relationship where p is a function of n, which means writing it as $p = [\text{expression involving } n]$.

$$\begin{aligned} 2n + 6p &= 12 \\ 6p &= 12 - 2n && \text{Subtract 2n from both sides.} \\ p &= \frac{12-2n}{6} && \text{Divide both sides by 6 and simplify.} \\ p &= \frac{12}{6} - \frac{2n}{6} \\ p &= 2 - \frac{1}{3}n \end{aligned}$$

Therefore, p as a function of n is written as

$$p = f(n) = 2 - \frac{1}{3}n$$

Analysis

It is important to note that not every relationship expressed by an equation can also be expressed as a function with a formula.

Example 2.1.8B: Expressing the Equation of a Circle as a Function

Does the equation $x^2 + y^2 = 1$ represent a function with x as input and y as output? If so, express the relationship as a function $y = f(x)$.

Solution

First we subtract x^2 from both sides.

$$y^2 = 1 - x^2$$

We now try to solve for y in this equation.

$$y = \pm\sqrt{1 - x^2}$$

$$\text{so, } y = \sqrt{1-x^2} \text{ and } y = -\sqrt{1-x^2}$$

We get two outputs corresponding to the same input, so this relationship cannot be represented as a single function $y = f(x)$.

Exercise 2.1.8

If $x - 8y^3 = 0$, express y as a function of x.

Answer

$$y = f(x) = \frac{\sqrt[3]{x}}{2}$$

Q & A

Are there relationships expressed by an equation that do represent a function but which still cannot be represented by an algebraic formula?

Yes, this can happen. For example, given the equation $x = y + 2^y$, if we want to express y as a function of x, there is no simple algebraic formula involving only x that equals y. However, each x does determine a unique value for y, and there are mathematical procedures by which y can be found to any desired accuracy. In this case, we say that the equation gives an implicit (implied) rule for y as a function of x, even though the formula cannot be written explicitly.

Evaluating a Function Given in Tabular Form

As we saw above, we can represent functions in tables. Conversely, we can use information in tables to write functions, and we can evaluate functions using the tables. For example, how well do our pets recall the fond memories we share with them? There is an urban legend that a goldfish has a memory of 3 seconds, but this is just a myth. Goldfish can remember up to 3 months, while the beta fish has a memory of up to 5 months. And while a puppy's memory span is no longer than 30 seconds, the adult dog can remember for 5 minutes. This is meager compared to a cat, whose memory span lasts for 16 hours.

The function that relates the type of pet to the duration of its memory span is more easily visualized with the use of a table (Table 2.1.10).

Table 2.1.10

Pet Memory	span in hours
Puppy	0.008
Adult Dog	0.083
Cat	3
Goldfish	2160
Beta Fish	3600

At times, evaluating a function in table form may be more useful than using equations. Here let us call the function P. The domain of the function is the type of pet and the range is a real number representing the number of hours the pet's memory span lasts. We can evaluate the function P at the input value of "goldfish." We would write $P(goldfish) = 2160$. Notice that, to evaluate the function in table form, we identify the input value and the corresponding output value from the pertinent row of the table. The tabular form for function P seems ideally suited to this function, more so than writing it in paragraph or function form.

How To: Given a function represented by a table, identify specific output and input values

1. Find the given input in the row (or column) of input values.
2. Identify the corresponding output value paired with that input value.
3. Find the given output values in the row (or column) of output values, noting every time that output value appears.
4. Identify the input value(s) corresponding to the given output value.

Example 2.1.9: Evaluating and Solving a Tabular Function

Using Table 2.1.11,

a. Evaluate $g(3)$.
b. Solve $g(n) = 6$.

Table 2.1.11

n	1	2	3	4	5
$g(n)$	8	6	7	6	8

Solution

a. Evaluating $g(3)$ means determining the output value of the function g for the input value of $n = 3$. The table output value corresponding to $n = 3$ is 7, so $g(3) = 7$.
b. Solving $g(n) = 6$ means identifying the input values, n,that produce an output value of 6. Table 2.1.12 shows two solutions: 2 and 4.

Table 2.1.12

n	1	2	3	4	5
$g(n)$	8	6	7	6	8

When we input 2 into the function g, our output is 6. When we input 4 into the function g, our output is also 6.

Exercise 2.1.1

Using Table 2.1.12, evaluate $g(1)$.

Answer

$g(1) = 8$

Finding Function Values from a Graph

Evaluating a function using a graph also requires finding the corresponding output value for a given input value, only in this case, we find the output value by looking at the graph. Solving a function equation using a graph requires finding all instances of the given output value on the graph and observing the corresponding input value(s).

Example 2.1.10: Reading Function Values from a Graph

Given the graph in Figure 2.1.7,

a. Evaluate $f(2)$.
b. Solve $f(x) = 4$.

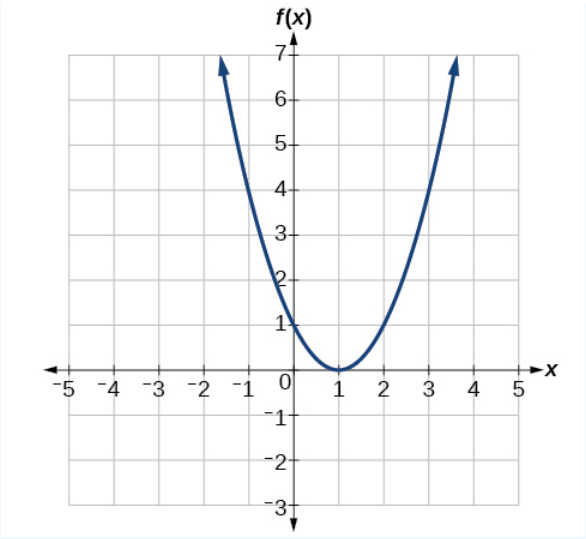
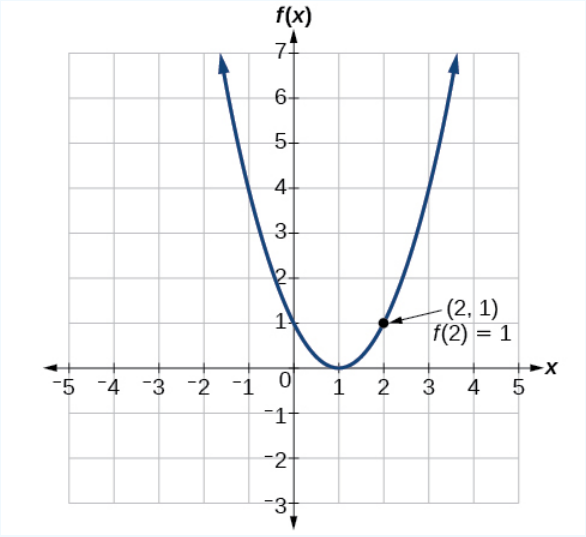


Figure 2.1.7: Graph of a positive parabola centered at $(1, 0)$.

Solution

To evaluate $f(2)$, locate the point on the curve where $x = 2$, then read the y-coordinate of that point. The point has coordinates $(2, 1)$, so $f(2) = 1$. See Figure 2.1.8.



2.1.8: Graph of a positive parabola centered at $(1, 0)$ with the labeled point $(2, 1)$ where $f(2) = 1$.

To solve $f(x) = 4$, we find the output value 4 on the vertical axis. Moving horizontally along the line $y = 4$, we locate two points of the curve with output value 4: $(-1, 4)$ and $(3, 4)$. These points represent the two solutions to $f(x) = 4$: -1 or 3. This means $f(-1) = 4$ and $f(3) = 4$, or when the input is -1 or 3, the output is 4. See Figure 2.1.9.

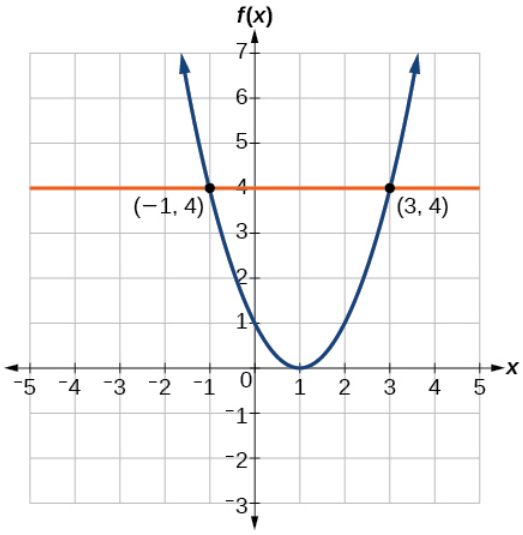


Figure 2.1.9: Graph of an upward-facing parabola with a vertex at $(0,1)$ and labeled points at $(-1,4)$ and $(3,4)$. A line at $y=4$ intersects the parabola at the labeled points.

Exercise 2.1.10

Given the graph in Figure 2.1.7, solve $f(x)=1$.

Answer

$x=0$ or $x=2$

Determining Whether a Function is One-to-One

Some functions have a given output value that corresponds to two or more input values. For example, in the stock chart shown in the Figure at the beginning of this chapter, the stock price was $1000 on five different dates, meaning that there were five different input values that all resulted in the same output value of $1000.

However, some functions have only one input value for each output value, as well as having only one output for each input. We call these functions one-to-one functions. As an example, consider a school that uses only letter grades and decimal equivalents, as listed in Table 2.1.13.

Table 2.1.13: Letter grades and decimal equivalents.

Letter Grade	Grade Point Average
A	4.0
B	3.0
C	2.0
D	1.0

This grading system represents a one-to-one function, because each letter input yields one particular grade point average output and each grade point average corresponds to one input letter.

To visualize this concept, let's look again at the two simple functions sketched in Figures 2.1.1*a* and 2.1.1*b*. The function in part (a) shows a relationship that is not a one-to-one function because inputs q and r both give output n. The function in part (b) shows a relationship that is a one-to-one function because each input is associated with a single output.

One-to-One Functions

A **one-to-one function** is a function in which each output value corresponds to exactly one input value.

Example 2.1.11: Determining Whether a Relationship Is a One-to-One Function

Is the area of a circle a function of its radius? If yes, is the function one-to-one?

Solution

A circle of radius r has a unique area measure given by $A = \pi r^2$, so for any input, r, there is only one output, A. The area is a function of radius r.

If the function is one-to-one, the output value, the area, must correspond to a unique input value, the radius. Any area measure A is given by the formula $A = \pi r^2$. Because areas and radii are positive numbers, there is exactly one solution: $\sqrt{\frac{A}{\pi}}$. So the area of a circle is a one-to-one function of the circle's radius.

Exercise 2.1.11A

a. Is a balance a function of the bank account number?
b. Is a bank account number a function of the balance?
c. Is a balance a one-to-one function of the bank account number?

Answer

a. yes, because each bank account has a single balance at any given time;

b. no, because several bank account numbers may have the same balance;

c. no, because the same output may correspond to more than one input.

Exercise 2.1.11B

Evaluate the following:

a. If each percent grade earned in a course translates to one letter grade, is the letter grade a function of the percent grade?
b. If so, is the function one-to-one?

Answer

a. Yes, letter grade is a function of percent grade;
b. No, it is not one-to-one. There are 100 different percent numbers we could get but only about five possible letter grades, so there cannot be only one percent number that corresponds to each letter grade.

Using the Vertical Line Test

As we have seen in some examples above, we can represent a function using a graph. Graphs display a great many input-output pairs in a small space. The visual information they provide often makes relationships easier to understand. By convention, graphs are typically constructed with the input values along the horizontal axis and the output values along the vertical axis.

The most common graphs name the input value x and the output y, and we say y is a function of x, or $y = f(x)$ when the function is named f. The graph of the function is the set of all points (x, y) in the plane that satisfies the equation $y = f(x)$. If the function is defined for only a few input values, then the graph of the function is only a few points, where the x-coordinate of each point is an input value and the y-coordinate of each point is the corresponding output value. For example, the black dots on the graph in Figure 2.1.10 tell us that $f(0) = 2$ and $f(6) = 1$. However, the set of all points (x, y) satisfying $y = f(x)$ is a curve. The curve shown includes $(0, 2)$ and $(6, 1)$ because the curve passes through those points

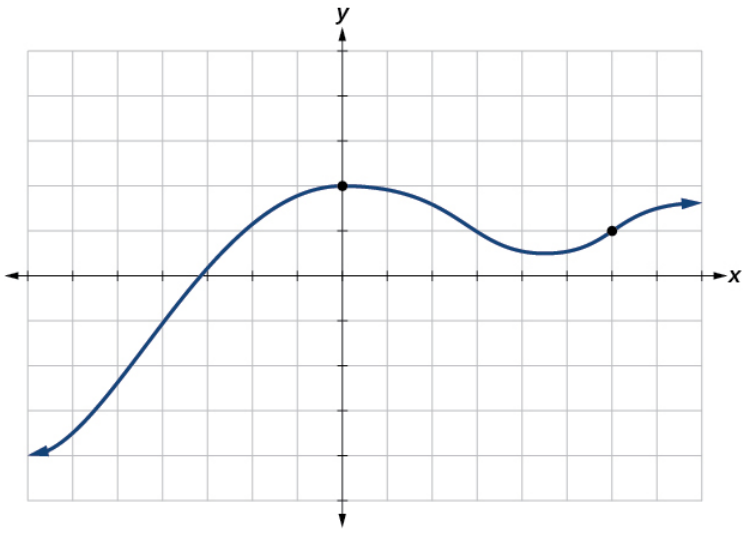


Figure 2.1.10: Graph of a polynomial.

The vertical line test can be used to determine whether a graph represents a function. If we can draw any vertical line that intersects a graph more than once, then the graph does not define a function because a function has only one output value for each input value. See Figure 2.1.11.

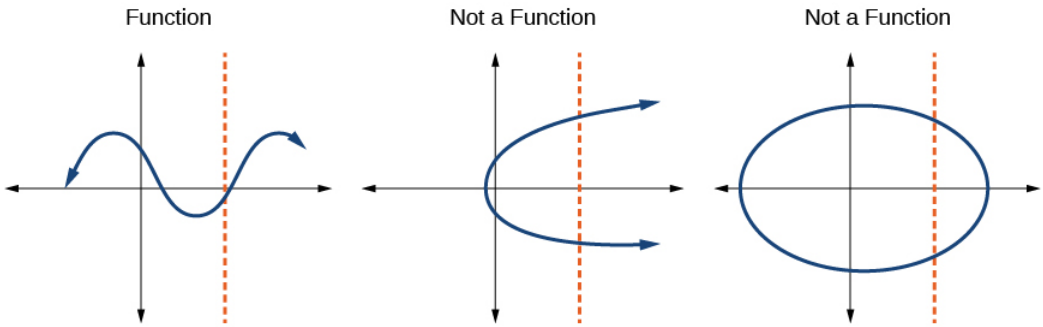


Figure 2.1.11: Three graphs visually showing what is and is not a function.

Howto: Given a graph, use the vertical line test to determine if the graph represents a function

1. Inspect the graph to see if any vertical line drawn would intersect the curve more than once.
2. If there is any such line, determine that the graph does not represent a function.

Example 2.1.12: Applying the Vertical Line Test

Which of the graphs in Figure 2.1.12represent(s) a function $y = f(x)$?

https://math.libretexts.org/@go/page/106334

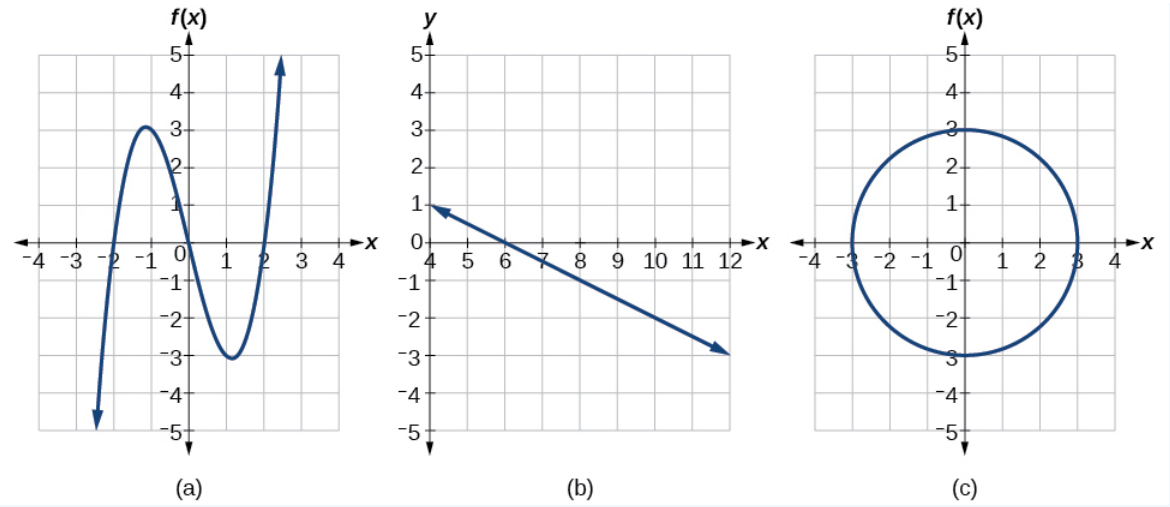


Figure 2.1.12: Graph of a polynomial (a), a downward-sloping line (b), and a circle (c).

Solution

If any vertical line intersects a graph more than once, the relation represented by the graph is not a function. Notice that any vertical line would pass through only one point of the two graphs shown in parts (a) and (b) of Figure 2.1.12 From this we can conclude that these two graphs represent functions. The third graph does not represent a function because, at most x-values, a vertical line would intersect the graph at more than one point, as shown in Figure 2.1.13

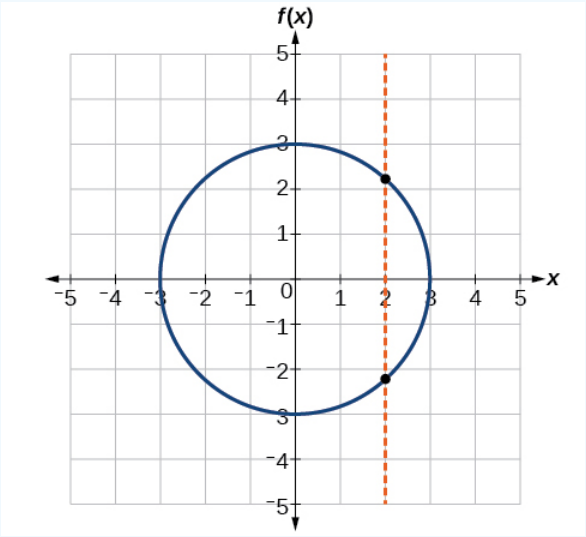


Figure 2.1.13: Graph of a circle.

Exercise 2.1.12

Does the graph in Figure 2.1.14represent a function?

https://math.libretexts.org/@go/page/106334

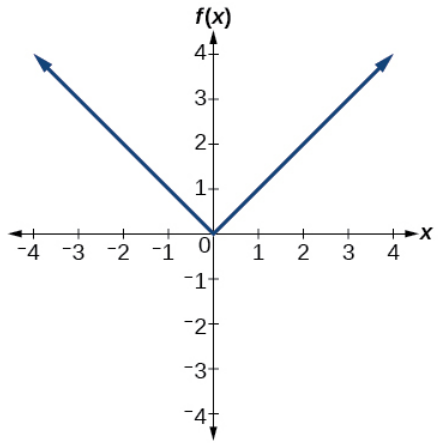


Figure 2.1.14: Graph of absolute value function.

Answer

yes

Using the Horizontal Line Test

Once we have determined that a graph defines a function, an easy way to determine if it is a one-to-one function is to use the **horizontal line test**. Draw horizontal lines through the graph. If any horizontal line intersects the graph more than once, then the graph does not represent a one-to-one function.

Howto: Given a graph of a function, use the horizontal line test to determine if the graph represents a one-to-one function

1. Inspect the graph to see if any horizontal line drawn would intersect the curve more than once.
2. If there is any such line, determine that the function is not one-to-one.

Example 2.1.13: Applying the Horizontal Line Test

Consider the functions shown in Figure $2.1.12a$ and Figure $2.1.12b$ Are either of the functions one-to-one?

Solution

The function in Figure $2.1.12a$ is not one-to-one. The horizontal line shown in Figure $2.1.15$ intersects the graph of the function at two points (and we can even find horizontal lines that intersect it at three points.)

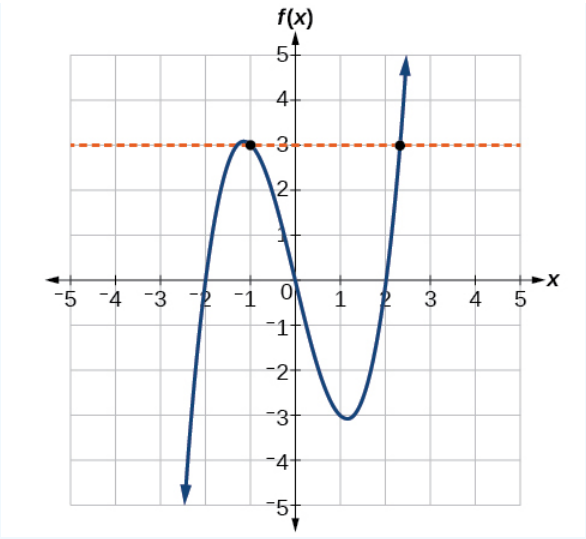


Figure 2.1.15: Graph of a polynomial with a horizontal line crossing through 2 points

The function in Figure $2.1.12b$is one-to-one. Any horizontal line will intersect a diagonal line at most once.

Exercise 2.1.13

Is the graph shown in Figure $2.1.13$one-to-one?

Answer

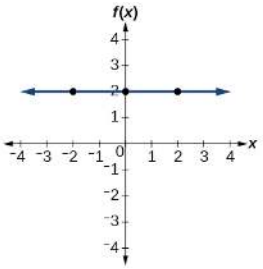
No, because it does not pass the horizontal line test.

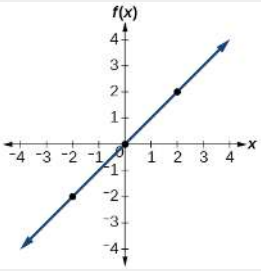
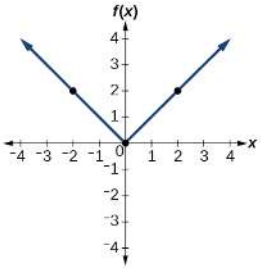
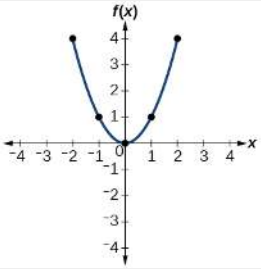
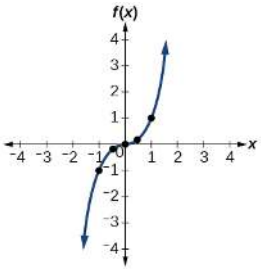
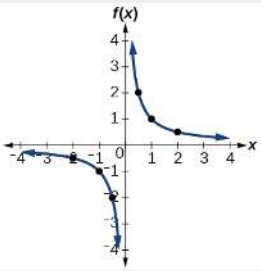
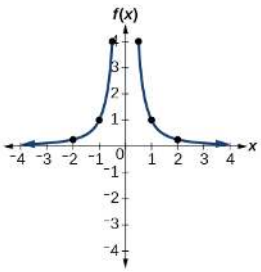
Identifying Basic Toolkit Functions

In this text, we will be exploring functions—the shapes of their graphs, their unique characteristics, their algebraic formulas, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of building-block elements. We call these our "toolkit functions," which form a set of basic named functions for which we know the graph, formula, and special properties. Some of these functions are programmed to individual buttons on many calculators. For these definitions we will use x as the input variable and $y = f(x)$ as the output variable.

We will see these toolkit functions, combinations of toolkit functions, their graphs, and their transformations frequently throughout this book. It will be very helpful if we can recognize these toolkit functions and their features quickly by name, formula, graph, and basic table properties. The graphs and sample table values are included with each function shown in Table $2.1.14$

Table 2.1.14: Toolkit Functions

Name	Function	Graph
Constant	$f(x) = c$ where c is a constant	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>2</td></tr><tr><td>0</td><td>2</td></tr><tr><td>2</td><td>2</td></tr></table>

Name	Function	Graph
Identity	$f(x) = x$	
Absolute Value	$f(x) = \|x\|$	
Quadratic	$f(x) = x^2$	
Cubic	$f(x) = x^3$	
reciprocal	$f(x) = \frac{1}{x}$	
Reciprocal squared	$f(x) = \frac{1}{x^2}$	

x	f(x)
−2	−2
0	0
2	2

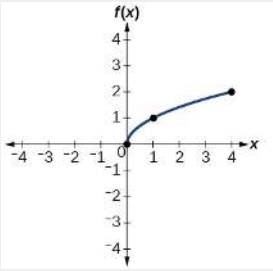
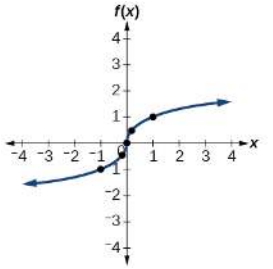
x	f(x)
−2	2
0	0
2	2

x	f(x)
−2	4
−1	1
0	0
1	1
2	4

x	f(x)
−1	−1
−0.5	−0.125
0	0
0.5	0.125
1	1

x	f(x)
−2	−0.5
−1	−1
−0.5	−2
0.5	2
1	1
2	0.5

x	f(x)
−2	0.25
−1	1
−0.5	4
0.5	4
1	1
2	0.25

Name	Function	Graph
Square root	$f(x) = \sqrt{x}$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td></tr><tr><td>4</td><td>2</td></tr></table>
Cube root	$f(x) = \sqrt[3]{x}$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−1</td><td>−1</td></tr><tr><td>−0.125</td><td>−0.5</td></tr><tr><td>0</td><td>0</td></tr><tr><td>0.125</td><td>0.5</td></tr><tr><td>1</td><td>1</td></tr></table>

Key Equations

- Constant function $f(x) = c$, where c is a constant
- Identity function $f(x) = x$
- Absolute value function $f(x) = |x|$
- Quadratic function $f(x) = x^2$
- Cubic function $f(x) = x^3$
- Reciprocal function $f(x) = \dfrac{1}{x}$
- Reciprocal squared function $f(x) = \frac{1}{x^2}$
- Square root function $f(x) = \sqrt{x}$
- Cube root function $f(x) = 3\sqrt{x}$

Key Concepts

- A relation is a set of ordered pairs. A function is a specific type of relation in which each domain value, or input, leads to exactly one range value, or output.
- Function notation is a shorthand method for relating the input to the output in the form $y = f(x)$.
- In tabular form, a function can be represented by rows or columns that relate to input and output values.
- To evaluate a function, we determine an output value for a corresponding input value. Algebraic forms of a function can be evaluated by replacing the input variable with a given value.
- To solve for a specific function value, we determine the input values that yield the specific output value.
- An algebraic form of a function can be written from an equation.
- Input and output values of a function can be identified from a table.
- Relating input values to output values on a graph is another way to evaluate a function.
- A function is one-to-one if each output value corresponds to only one input value.
- A graph represents a function if any vertical line drawn on the graph intersects the graph at no more than one point.
- The graph of a one-to-one function passes the horizontal line test.

Footnotes

1 http://www.baseball-almanac.com/lege.../lisn100.shtml. Accessed 3/24/2014.
2 www.kgbanswers.com/how-long-i...y-span/4221590. Accessed 3/24/2014.

Glossary

dependent variable
an output variable

domain
the set of all possible input values for a relation

function
a relation in which each input value yields a unique output value

horizontal line test
a method of testing whether a function is one-to-one by determining whether any horizontal line intersects the graph more than once

independent variable
an input variable

input
each object or value in a domain that relates to another object or value by a relationship known as a function

one-to-one function
a function for which each value of the output is associated with a unique input value

output
each object or value in the range that is produced when an input value is entered into a function

range
the set of output values that result from the input values in a relation

relation
a set of ordered pairs

vertical line test
a method of testing whether a graph represents a function by determining whether a vertical line intersects the graph no more than once

Homework Exercises 2.1

WeBWorK Problems:

Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login

Login with LibreOne to view this question
NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.
Login

Login with LibreOne to view this question
NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.
Login

Login with LibreOne to view this question
NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.
Login

Written Problems:

Verbal

1. What is the difference between the input and the output of a function?

2. How can you determine if a relation is a one-to-one function?

Algebraic

3. Given the function $p(c) = c^2 + c$:

a. Evaluate *p*(-3)

b. Evaluate *p*(1+*h*)

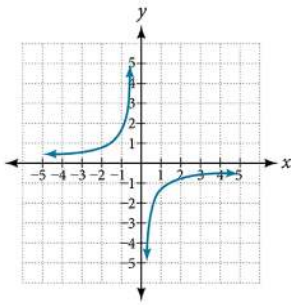
c. Solve *p(c)* = 2

4. Given the function $f(x) = 3^x$, evaluate: $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, and $f(2)$.

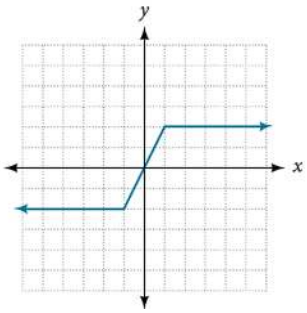
5. Given $f(x) = 3x - 2$ and $g(x) = 5 - x^2$, evaluate: $3f(1) - 4g(-2)$.

Determine which graphs show relations that are functions:

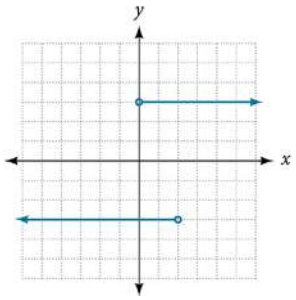
6.



7.



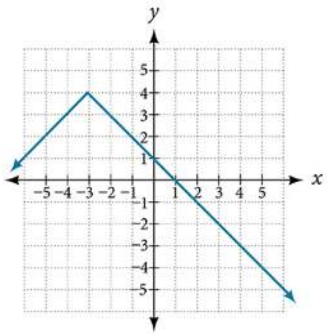
8.



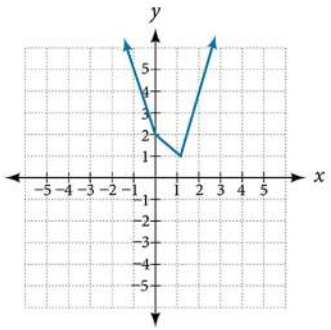
9. Given the following graph,

a. Evaluate $f(-4)$

b. Solve $f(x) = 2$



10. Determine if the function shown in the graph below is a one-to-one function.



11. Is the function $f(x) = 3(x-5)^2 + 7$ a one-to-one function? Why or why not?

Numeric

12. Determine if the following set of points represents a function: $(-1, -1), (0, -1), (1, 2)$ Why or why not?

Application

13. The number of cubic yards of dirt, *D*, needed to cover a garden with area *a* square feet is given by $D = g(a)$.

a. A garden with area $5000\ ft^2$ requires $50\ yd^3$ of dirt. Express this information in function notation.

b. Explain the meaning of the statement $g(100) = 1$.

This page titled 2.1: Functions and Function Notation is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **1.1: Functions and Function Notation** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/precalculus.

2.2: Domain and Range

Preview Video for Section 2.2

Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login

Learning Objectives

- Find the domain of a function defined by an equation.
- Graph piecewise-defined functions.

If you're in the mood for a scary movie, you may want to check out one of the five most popular horror movies of all time—I am Legend, Hannibal, The Ring, The Grudge, and The Conjuring. Figure 2.2.1 shows the amount, in dollars, each of those movies grossed when they were released as well as the ticket sales for horror movies in general by year. Notice that we can use the data to create a function of the amount each movie earned or the total ticket sales for all horror movies by year. In creating various functions using the data, we can identify different independent and dependent variables, and we can analyze the data and the functions to determine the domain and range. In this section, we will investigate methods for determining the domain and range of functions such as these.

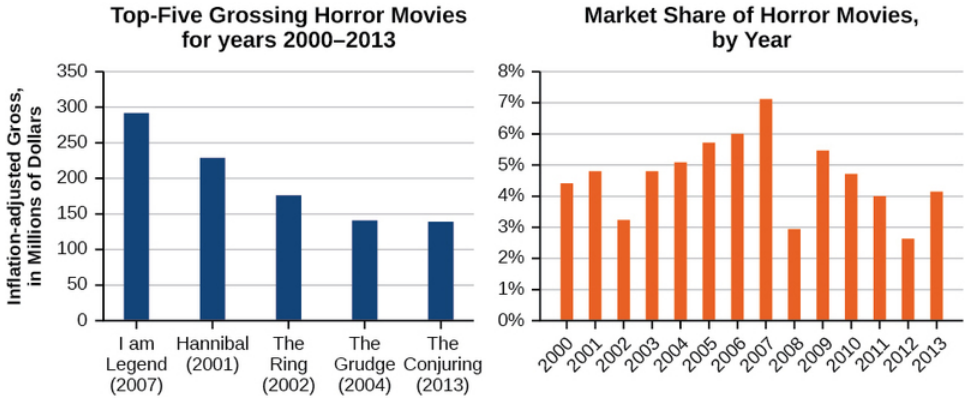


Figure 2.2.1: Graph of the Top-Five Grossing Horror Movies for years 2000-2003, and a Graph of the Market Share of Horror Movies by Year. Based on data compiled by www.the-numbers.com.

Finding the Domain of a Function Defined by an Equation

In Functions and Function Notation, we were introduced to the concepts of **domain and range**. In this section, we will practice determining domains and ranges for specific functions. Keep in mind that, in determining domains and ranges, we need to consider what is physically possible or meaningful in real-world examples, such as tickets sales and year in the horror movie example above. We also need to consider what is mathematically permitted. For example, we cannot include any input value that leads us to take an even root of a negative number if the domain and range consist of real numbers. Or in a function expressed as a formula, we cannot include any input value in the domain that would lead us to divide by 0.

We can visualize the domain as a “holding area” that contains “raw materials” for a “function machine” and the range as another “holding area” for the machine’s products (Figure 2.2.2).

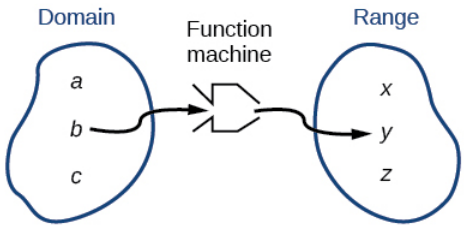


Figure 2.2.2: Diagram of how a function relates two relations

We can write the **domain and range** in **interval notation**, which uses values within brackets to describe a set of numbers. In interval notation, we use a square bracket [when the set includes the endpoint and a parenthesis (to indicate that the endpoint is either not included or the interval is unbounded. For example, if a person has $100 to spend, he or she would need to express the interval that is more than 0 and less than or equal to 100 and write $(0, 100]$. We will discuss interval notation in greater detail later.

Let’s turn our attention to finding the domain of a function whose equation is provided. Oftentimes, finding the domain of such functions involves remembering three different forms. First, if the function has no denominator or an even root, consider whether the domain could be all real numbers. Second, if there is a denominator in the function’s equation, exclude values in the domain that force the denominator to be zero. Third, if there is an even root, consider excluding values that would make the radicand negative.

Before we begin, let us review the conventions of interval notation:

- The smallest term from the interval is written first.
- The largest term in the interval is written second, following a comma.
- Parentheses, (or), are used to signify that an endpoint is not included, called exclusive.
- Brackets, [or], are used to indicate that an endpoint is included, called inclusive.

See Figure 2.2.3 for a summary of interval notation.

Inequality	Interval Notation	Graph on Number Line	Description
$x > a$	(a, ∞)		x is greater than a
$x < a$	$(-\infty, a)$		x is less than a
$x \geq a$	$[a, \infty)$		x is greater than or equal to a
$x \leq a$	$(-\infty, a]$		x is less than or equal to a
$a < x < b$	(a, b)		x is strictly between a and b
$a \leq x < b$	$[a, b)$		x is between a and b, to include a
$a < x \leq b$	$(a, b]$		x is between a and b, to include b
$a \leq x \leq b$	$[a, b]$		x is between a and b, to include a and b

Figure 2.2.3: Summary of interval notation.

Example 2.2.1: Finding the Domain of a Function as a Set of Ordered Pairs

Find the domain of the following function: $\{(2, 10), (3, 10), (4, 20), (5, 30), (6, 40)\}$

Solution

First identify the input values. The input value is the first coordinate in an ordered pair. There are no restrictions, as the ordered pairs are simply listed. The domain is the set of the first coordinates of the ordered pairs.

$$\{2, 3, 4, 5, 6\}$$

Exercse 2.2.1

Find the domain of the function:

$$\{(-5, 4), (0, 0), (5, -4), (10, -8), (15, -12)\}$$

Answer

$\{-5, 0, 5, 10, 15\}$

How To: Given a function written in equation form, find the domain.

1. Identify the input values.
2. Identify any restrictions on the input and exclude those values from the domain.
3. Write the domain in interval form, if possible.

Example 2.2.2: Finding the Domain of a Function

Find the domain of the function $f(x) = x^2 - 1$.

Solution

The input value, shown by the variable x in the equation, is squared and then the result is lowered by one. Any real number may be squared and then be lowered by one, so there are no restrictions on the domain of this function. The domain is the set of real numbers.

In interval form, the domain of f is $(-\infty, \infty)$.

Exercse 2.2.2

Find the domain of the function:

$$f(x) = 5 - x + x^3$$

Answer

$(-\infty, \infty)$

Howto: Given a function written in an equation form that includes a fraction, find the domain

1. Identify the input values.
2. Identify any restrictions on the input. If there is a denominator in the function's formula, set the denominator equal to zero and solve for x . If the function's formula contains an even root, set the radicand greater than or equal to 0, and then solve.
3. Write the domain in interval form, making sure to exclude any restricted values from the domain.

Example 2.2.3: Finding the Domain of a Function Involving a Denominator

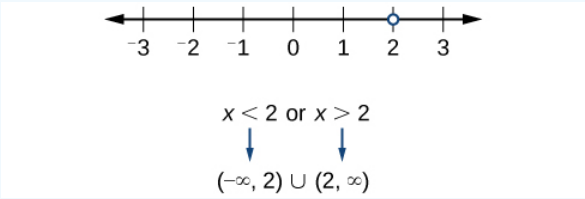
Find the domain of the function $f(x) = \dfrac{x+1}{2-x}$.

Solution

When there is a denominator, we want to include only values of the input that do not force the denominator to be zero. So, we will set the denominator equal to 0 and solve for x.

$$\begin{aligned} 2 - x &= 0 \\ -x &= -2 \\ x &= 2 \end{aligned}$$

Now, we will exclude 2 from the domain. The answers are all real numbers where $x < 2$ or $x > 2$. We can use a symbol known as the union, $\cup$,to combine the two sets. In interval notation, we write the solution:$(-\infty, 2) \cup (2, \infty)$.



In interval form, the domain of f is $(-\infty, 2) \cup (2, \infty)$.

Exercse 2.2.3

Find the domain of the function:

$$f(x)=\frac{1+4x}{2x-1}$$

Answer

$$(-\infty,\frac{1}{2})\cup(\frac{1}{2},\infty)$$

How To: Given a function written in equation form including an even root, find the domain.

1. Identify the input values.
2. Since there is an even root, exclude any real numbers that result in a negative number in the radicand. Set the radicand greater than or equal to zero and solve for x.
3. The solution(s) are the domain of the function. If possible, write the answer in interval form.

Example 2.2.4: Finding the Domain of a Function with an Even Root

Find the domain of the function:

$$f(x)=\sqrt{7-x}.$$

Solution

When there is an even root in the formula, we exclude any real numbers that result in a negative number in the radicand.

Set the radicand greater than or equal to zero and solve for x.

$$\begin{aligned}7-x &\geq 0\\ -x &\geq -7\\ x &\leq 7\end{aligned}$$

Now, we will exclude any number greater than 7 from the domain. The answers are all real numbers less than or equal to 7, or $(-\infty,7]$.

Exercse 2.2.4

Find the domain of the function

$$f(x)=\sqrt{5+2x}.$$

Answer

$$[-2.5,\infty)$$

Q&A: Can there be functions in which the domain and range do not intersect at all?

Yes. For example, the function $f(x)=-\frac{1}{\sqrt{x}}$ *has the set of all positive real numbers as its domain but the set of all negative real numbers as its range. As a more extreme example, a function's inputs and outputs can be completely different categories (for example, names of weekdays as inputs and numbers as outputs, as on an attendance chart), in such cases the domain and range have no elements in common.*

Using Notations to Specify Domain and Range

In the previous examples, we used inequalities and lists to describe the domain of functions. We can also use inequalities, or other statements that might define sets of values or data, to describe the behavior of the variable in set-builder notation. For example,

$\{x|10 \leq x < 30\}$ describes the behavior of x in set-builder notation. The braces $\{\}$ are read as "the set of," and the vertical bar $|$ is read as "such that," so we would read $\{x|10 \leq x < 30\}$ as "the set of x-values such that 10 is less than or equal to x, and x is less than 30."

Figure 2.2.4 compares inequality notation, set-builder notation, and interval notation.

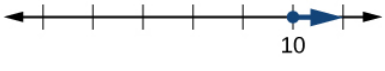
	Inequality Notation	Set-builder Notation	Interval Notation
	$5 < h \leq 10$	$\{h \mid 5 < h \leq 10\}$	$(5, 10]$
	$5 \leq h < 10$	$\{h \mid 5 \leq h < 10\}$	$[5, 10)$
	$5 < h < 10$	$\{h \mid 5 < h < 10\}$	$(5, 10)$
	$h < 10$	$\{h \mid h < 10\}$	$(-\infty, 10)$
	$h \geq 10$	$\{h \mid h \geq 10\}$	$[10, \infty)$
	All real numbers	$\mathbb{R}$	$(-\infty, \infty)$

Figure 2.2.4: Summary of notations for inequalities, set-builder, and interval.

To combine two intervals using inequality notation or set-builder notation, we use the word "or." As we saw in earlier examples, we use the union symbol, $\cup$,to combine two unconnected intervals. For example, the union of the sets $\{2, 3, 5\}$ and $\{4, 6\}$ is the set $\{2, 3, 4, 5, 6\}$ It is the set of all elements that belong to one or the other (or both) of the original two sets. For sets with a finite number of elements like these, the elements do not have to be listed in ascending order of numerical value. If the original two sets have some elements in common, those elements should be listed only once in the union set. For sets of real numbers on intervals, another example of a union is

$$\{x||x| \geq 3\} = (-\infty, -3] \cup [3, \infty) \tag{2.2.1}$$

Set-Builder Notation and Interval Notation

Set-builder notation is a method of specifying a set of elements that satisfy a certain condition. It takes the form $\{x|$ statement about x$\}$ which is read as, "the set of all x such that the statement about x is true." For example,

$$\{x|4 < x \leq 12\}$$

Interval notation is a way of describing sets that include all real numbers between a lower limit that may or may not be included and an upper limit that may or may not be included. The endpoint values are listed between brackets or parentheses. A square bracket indicates inclusion in the set, and a parenthesis indicates exclusion from the set. For example,

$$(4, 12]$$

Given a line graph, describe the set of values using interval notation.

1. Identify the intervals to be included in the set by determining where the heavy line overlays the real line.
2. At the left end of each interval, use [with each end value to be included in the set (solid dot) or (for each excluded end value (open dot).
3. At the right end of each interval, use] with each end value to be included in the set (filled dot) or) for each excluded end value (open dot).
4. Use the union symbol $\cup$ to combine all intervals into one set.

Example 2.2.5: Describing Sets on the Real-Number Line

Describe the intervals of values shown in Figure 2.2.5 using inequality notation, set-builder notation, and interval notation.



Figure 2.2.5: Line graph of $1 \leq x \leq 3$ or $x > 5$.

Solution

To describe the values, x, included in the intervals shown, we would say, "x is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5."

Inequality

$$1 \leq x \leq 3 \text{ or } x > 5$$

Set-builder Notation

$$\{x | 1 \leq x \leq 3 \text{ or } x > 5\}$$

Interval notation

$$[1, 3] \cup (5, \infty)$$

Remember that, when writing or reading interval notation, using a square bracket means the boundary is included in the set. Using a parenthesis means the boundary is not included in the set.

Exercse 2.2.5

Given Figure 2.2.6, specify the graphed set in

a. words
b. set-builder notation
c. interval notation



Figure 2.2.6: Line graph of $-2 \leq x$, $-1 \leq x < 3$.leq}x\), $-1 \leq x < 3$.

Answer a

Values that are less than or equal to –2, or values that are greater than or equal to –1 and less than 3;

Answer b

$\{x | x \leq -2 or -1 \leq x < 3\}$

Answer c

$(-\infty, -2] \cup [-1, 3)$

Finding Domain and Range from Graphs

Another way to identify the domain and range of functions is by using graphs. Because the domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the x-axis. The range is the set of possible output

values, which are shown on the y-axis. Keep in mind that if the graph continues beyond the portion of the graph we can see, the domain and range may be greater than the visible values. See Figure 2.2.7.

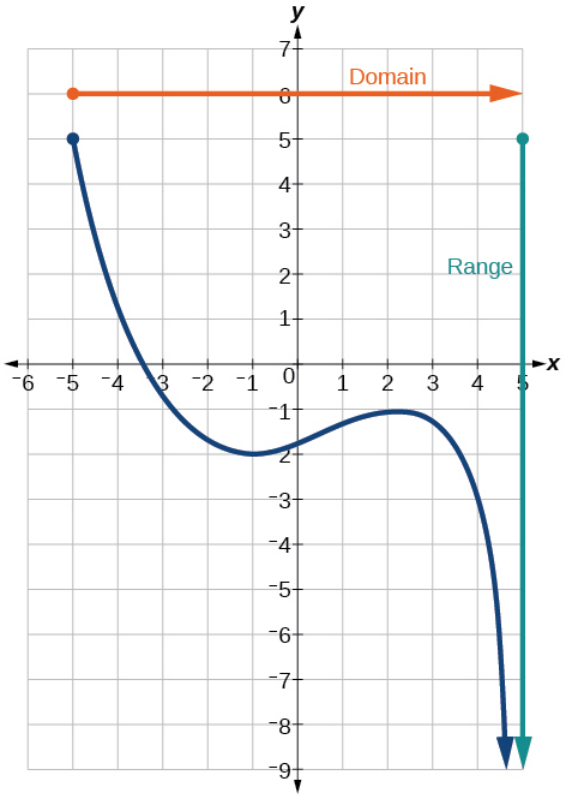


Figure 2.2.7: Graph of a polynomial that shows the x-axis is the domain and the y-axis is the range

We can observe that the graph extends horizontally from −5 to the right without bound, so the domain is $[-5, \infty)$. The vertical extent of the graph is all range values 5 and below, so the range is $(-\infty, 5]$. Note that the domain and range are always written from smaller to larger values, or from left to right for domain, and from the bottom of the graph to the top of the graph for range.

Example 2.2.6A: Finding Domain and Range from a Graph

Find the domain and range of the function f whose graph is shown in Figure 1.2.8.

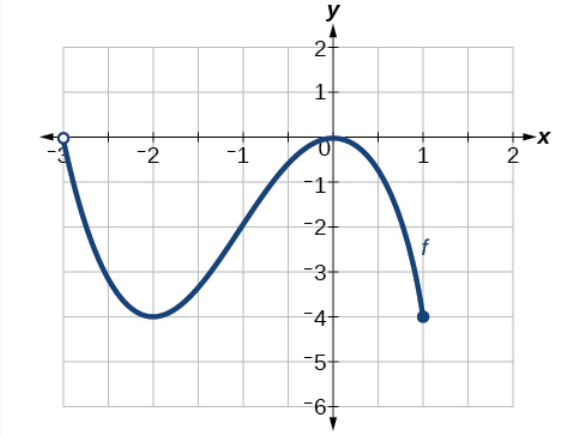


Figure 2.2.8: Graph of a function from (-3, 1].

Solution

We can observe that the horizontal extent of the graph is –3 to 1, so the domain of f is $(-3, 1]$.

The vertical extent of the graph is 0 to –4, so the range is $[-4, 0)$. See Figure 2.2.9.

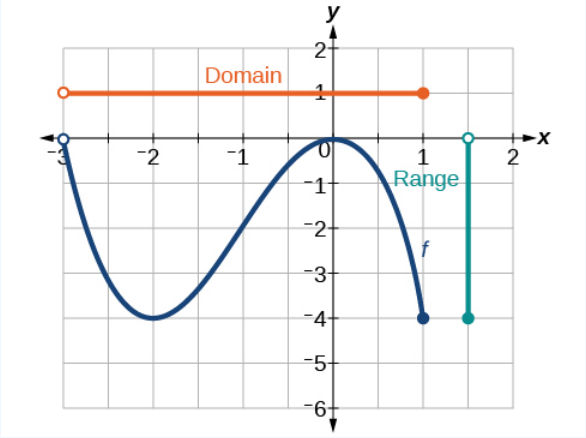


Figure 2.2.9: Graph of the previous function shows the domain and range

Example 2.2.6B: Finding Domain and Range from a Graph of Oil Production

Find the domain and range of the function f whose graph is shown in Figure 2.2.10.

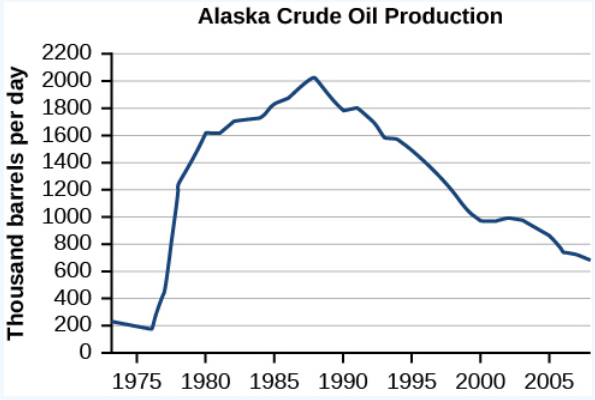


Figure 2.2.10: Graph of the Alaska Crude Oil Production where the y-axis is thousand barrels per day and the -axis is the years(credit: modification of work by the U.S. Energy Information Administration)

Solution

The input quantity along the horizontal axis is “years,” which we represent with the variable t for time. The output quantity is “thousands of barrels of oil per day,” which we represent with the variable b for barrels. The graph may continue to the left and right beyond what is viewed, but based on the portion of the graph that is visible, we can determine the domain as $1973 \leq t \leq 2008$ and the range as approximately $180 \leq b \leq 2010$.

In interval notation, the domain is $[1973, 2008]$ and the range is about $[180, 2010]$. For the domain and the range, we approximate the smallest and largest values since they do not fall exactly on the grid lines.

Exercse 2.2.6

Given Figure 2.2.11, identify the domain and range using interval notation.

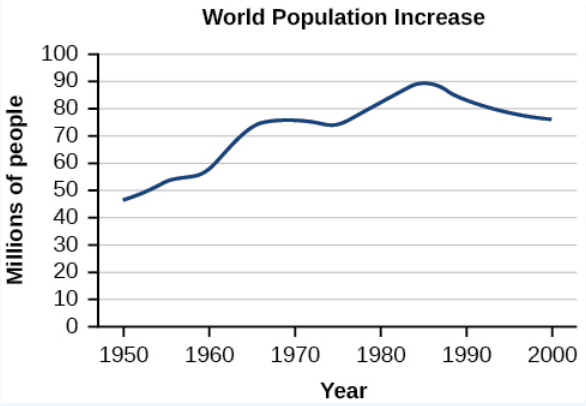


Figure 2.2.11: Graph of World Population Increase where the y-axis represents millions of people and the x-axis represents the year.

Answer

domain =$[1950, 2002]$

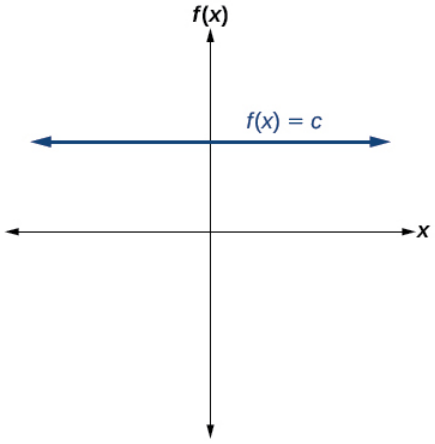
range = $[47,000,000, 89,000,000]$

Can a function's domain and range be the same?

Yes. For example, the domain and range of the cube root function are both the set of all real numbers.

Finding Domains and Ranges of the Toolkit Functions

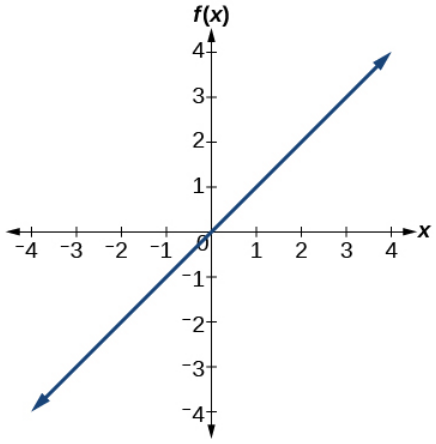
We will now return to our set of toolkit functions to determine the domain and range of each.



Domain: $(-\infty, \infty)$
Range: $[c,c]$

Figure 2.2.12: Constant function $f(x) = c$.

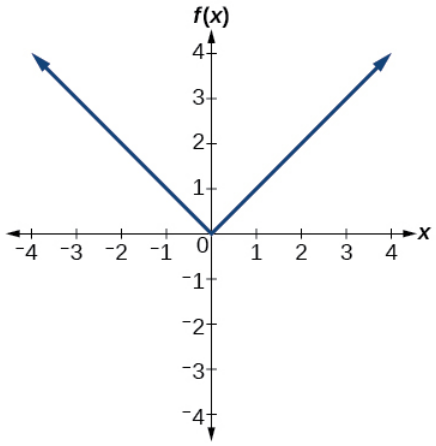
For the **constant function** $f(x) = c$, the domain consists of all real numbers; there are no restrictions on the input. The only output value is the constant c, so the range is the set $\{c\}$ that contains this single element. In interval notation, this is written as $[c, c]$, the interval that both begins and ends with c.



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Figure 2.2.13: *Identity function f(x)=x.*

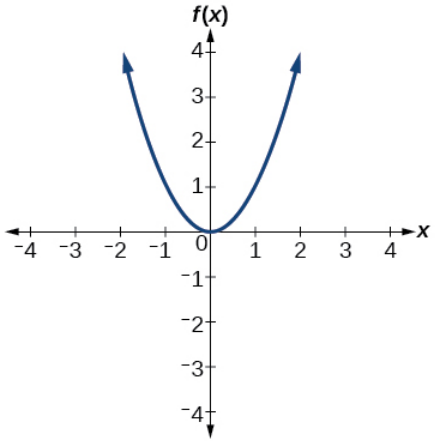
For the **identity function** $f(x) = x$, there is no restriction on x. Both the domain and range are the set of all real numbers.



Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

Figure 2.2.14: Absolute function $f(x) = |x|$.

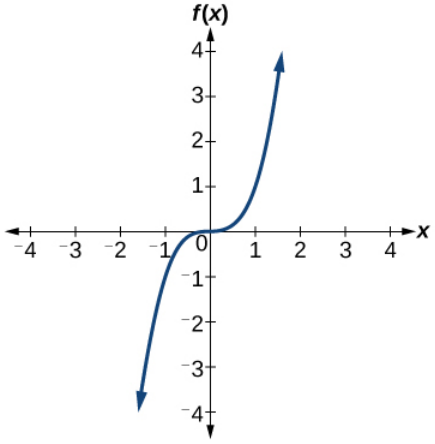
For the **absolute value function** $f(x) = |x|$, there is no restriction on x. However, because absolute value is defined as a distance from 0, the output can only be greater than or equal to 0.



Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

Figure 2.2.15: Quadratic function $f(x) = x^2$.

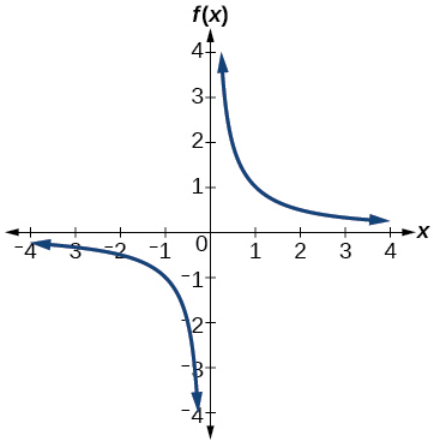
For the **quadratic function** $f(x) = x^2$, the domain is all real numbers since the horizontal extent of the graph is the whole real number line. Because the graph does not include any negative values for the range, the range is only nonnegative real numbers.



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Figure 2.2.16: Cubic function $f(x) - x^3$.

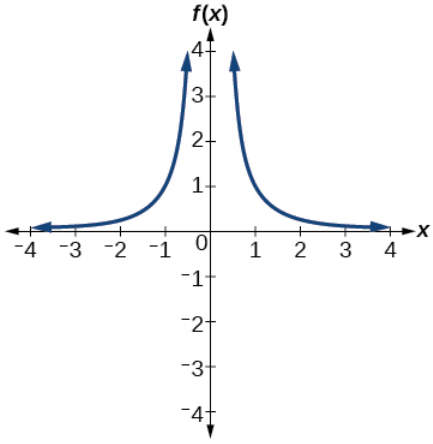
For the **cubic function** $f(x) = x^3$, the domain is all real numbers because the horizontal extent of the graph is the whole real number line. The same applies to the vertical extent of the graph, so the domain and range include all real numbers.



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$

Figure 2.2.17: Reciprocal function $f(x) = \frac{1}{x}$.

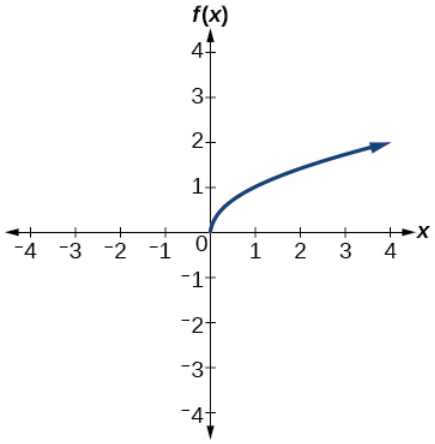
For the **reciprocal function** $f(x) = \frac{1}{x}$, we cannot divide by 0, so we must exclude 0 from the domain. Further, 1 divided by any value can never be 0, so the range also will not include 0. In set-builder notation, we could also write $\{x | x \neq 0\}$,the set of all real numbers that are not zero.



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(0, \infty)$

Figure 2.2.18: Reciprocal squared function $f(x) = \frac{1}{x^2}$

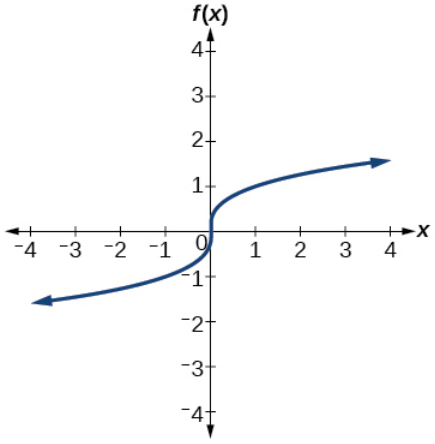
For the **reciprocal squared function** $f(x) = \frac{1}{x^2}$,we cannot divide by 0, so we must exclude 0 from the domain. There is also no x that can give an output of 0, so 0 is excluded from the range as well. Note that the output of this function is always positive due to the square in the denominator, so the range includes only positive numbers.



Domain: [0, ∞)
Range: [0, ∞)

Figure* 2.2.19*: *Square root function* $f(x) = \sqrt{(x)}$.

For the **square root function** $f(x) = \sqrt{x}$, we cannot take the square root of a negative real number, so the domain must be 0 or greater. The range also excludes negative numbers because the square root of a positive number x is defined to be positive, even though the square of the negative number $-\sqrt{x}$ also gives us x.



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Figure 2.2.20: Cube root function $f(x) = \sqrt[3]{x}$.

For the **cube root function** $f(x) = \sqrt[3]{x}$, the domain and range include all real numbers. Note that there is no problem taking a cube root, or any odd-integer root, of a negative number, and the resulting output is negative (it is an odd function).

How To: Given the formula for a function, determine the domain and range.

1. Exclude from the domain any input values that result in division by zero.
2. Exclude from the domain any input values that have nonreal (or undefined) number outputs.
3. Use the valid input values to determine the range of the output values.
4. Look at the function graph and table values to confirm the actual function behavior.

Finding the Domain and Range Using Toolkit Functions

Find the domain and range of $f(x) = 2x^3 - x$.

Solution

There are no restrictions on the domain, as any real number may be cubed and then subtracted from the result.

The domain is $(-\infty, \infty)$ and the range is also $(-\infty, \infty)$.

Example 2.2.7B: Finding the Domain and Range

Find the domain and range of $f(x) = \frac{2}{x+1}$.

Solution

We cannot evaluate the function at -1 because division by zero is undefined. The domain is $(-\infty, -1) \cup (-1, \infty)$. Because the function is never zero, we exclude 0 from the range. The range is $(-\infty, 0) \cup (0, \infty)$.

Example 2.2.7C: Finding the Domain and Range

Find the domain and range of $f(x) = 2\sqrt{x+4}$.

Solution

We cannot take the square root of a negative number, so the value inside the radical must be nonnegative.

$x + 4 \geq 0$ when $x \geq -4$

The domain of $f(x)$ is $[-4, \infty)$.

We then find the range. We know that $f(-4) = 0$, and the function value increases as x increases without any upper limit. We conclude that the range of f is $[0, \infty)$.

Analysis

Figure $2.2.19$represents the function f.

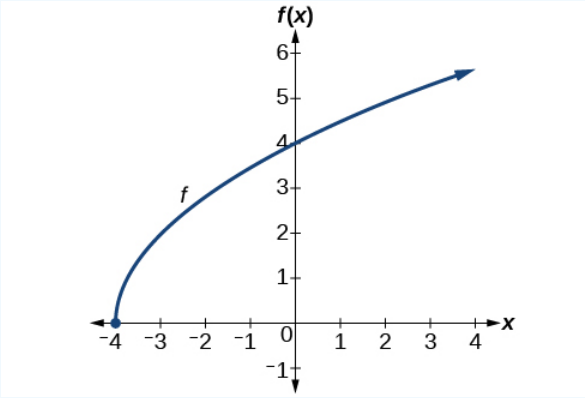


Figure 2.2.19: Graph of a square root function at $(-4, 0)$.

Exercise 2.2.7

Find the domain and range of

$f(x) = \sqrt{-2-x}$.

Answer

domain: $(-\infty, -2]$

range: $[0, \infty)$

Graphing Piecewise-Defined Functions

Sometimes, we come across a function that requires more than one formula in order to obtain the given output. For example, in the toolkit functions, we introduced the absolute value function $f(x) = |x|$. With a domain of all real numbers and a range of values greater than or equal to 0, **absolute value** can be defined as the **magnitude**, or **modulus**, of a real number value regardless of sign. It is the distance from 0 on the number line. All of these definitions require the output to be greater than or equal to 0.

If we input 0, or a positive value, the output is the same as the input.

$$f(x) = x \text{ if } x \geq 0$$

If we input a negative value, the output is the opposite of the input.

$$f(x) = -x \text{ if } x < 0$$

Because this requires two different processes or pieces, the absolute value function is an example of a **piecewise function**. A piecewise function is a function in which more than one formula is used to define the output over different pieces of the domain.

We use piecewise functions to describe situations in which a rule or relationship changes as the input value crosses certain "boundaries." For example, we often encounter situations in business for which the cost per piece of a certain item is discounted once the number ordered exceeds a certain value. Tax brackets are another real-world example of piecewise functions. For example, consider a simple tax system in which incomes up to $10,000 are taxed at 10%, and any additional income is taxed at 20%. The tax on a total income S would be $0.1S$ if $S \leq \$10,000$ and $\$1000 + 0.2(S - \$10,000)$ if $S > \$10,000$.

Piecewise Function

A piecewise function is a function in which more than one formula is used to define the output. Each formula has its own domain, and the domain of the function is the union of all these smaller domains. We notate this idea like this:

$$f(x) = \begin{cases} \text{formula 1} & \text{if x is in domain 1} \\ \text{formula 2} & \text{if x is in domain 2} \\ \text{formula 3} & \text{if x is in domain 3} \end{cases}$$

In piecewise notation, the absolute value function is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

How To: Given a piecewise function, write the formula and identify the domain for each interval.

1. Identify the intervals for which different rules apply.
2. Determine formulas that describe how to calculate an output from an input in each interval.
3. Use braces and if-statements to write the function.

Example 2.2.8A: Writing a Piecewise Function

A museum charges $5 per person for a guided tour with a group of 1 to 9 people or a fixed $50 fee for a group of 10 or more people. Write a **function** relating the number of people, n, to the cost, C.

Solution

Two different formulas will be needed. For n-values under 10, $C = 5n$. For values of n that are 10 or greater, $C = 50$.

$$C(n) = \begin{cases} 5n & \text{if } n < 10 \\ 50 & \text{if } n \geq 10 \end{cases}$$

Analysis

The function is represented in Figure 2.2.20. The graph is a diagonal line from $n = 0$ to $n = 10$ and a constant after that. In this example, the two formulas agree at the meeting point where $n = 10$, but not all piecewise functions have this property.

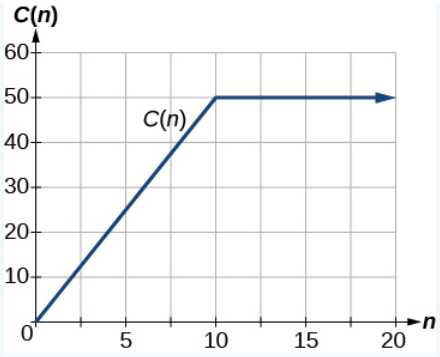


Figure 2.2.20

Example 2.2.8B: Working with a Piecewise Function

A cell phone company uses the function below to determine the cost, C, in dollars for g gigabytes of data transfer.

$$C(g) = \begin{cases} 25 & \text{if } 0 < g < 2 \\ 25 + 10(g-2) & \text{if } g \geq 2 \end{cases}$$

Find the cost of using 1.5 gigabytes of data and the cost of using 4 gigabytes of data.

Soltuion

To find the cost of using 1.5 gigabytes of data, $C(1.5)$, we first look to see which part of the domain our input falls in. Because 1.5 is less than 2, we use the first formula.

$$C(1.5) = \$25$$

To find the cost of using 4 gigabytes of data, C(4), we see that our input of 4 is greater than 2, so we use the second formula.

$$C(4) = 25 + 10(4-2) = \$45$$

Analysis

The function is represented in Figure 2.2.21. We can see where the function changes from a constant to a shifted and stretched identity at $g = 2$. We plot the graphs for the different formulas on a common set of axes, making sure each formula is applied on its proper domain.

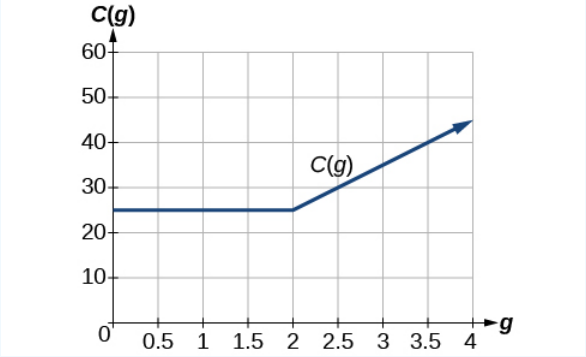


Figure 2.2.21

How To: Given a piecewise function, sketch a graph.

1. Indicate on the x-axis the boundaries defined by the intervals on each piece of the domain.
2. For each piece of the domain, graph on that interval using the corresponding equation pertaining to that piece. Do not graph two functions over one interval because it would violate the criteria of a function.

Example 2.2.8C: Graphing a Piecewise Function

Sketch a graph of the function.

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ 3 & \text{if } 1 < x \leq 2 \\ x & \text{if } x > 2 \end{cases}$$

Solution

Each of the component functions is from our library of toolkit functions, so we know their shapes. We can imagine graphing each function and then limiting the graph to the indicated domain. At the endpoints of the domain, we draw open circles to indicate where the endpoint is not included because of a less-than or greater-than inequality; we draw a closed circle where the endpoint is included because of a less-than-or-equal-to or greater-than-or-equal-to inequality.

Figure 2.2.20 shows the three components of the piecewise function graphed on separate coordinate systems.

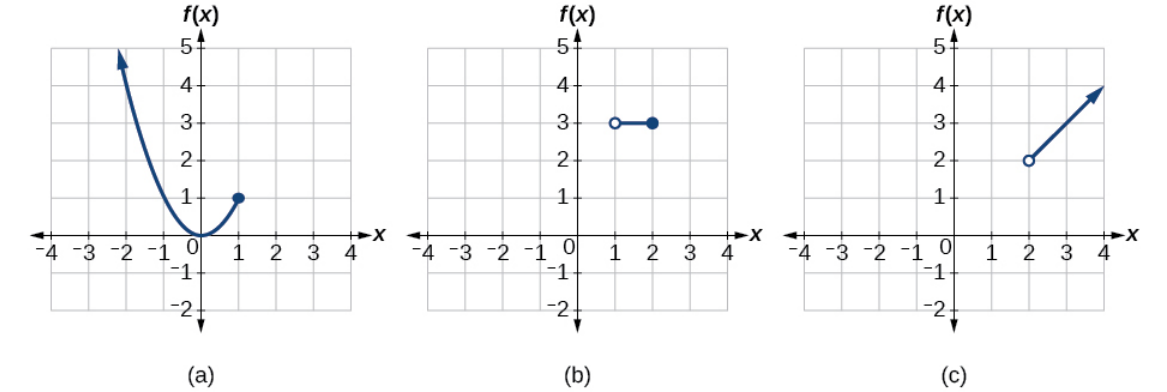


Figure 2.2.20: *Graph of each part of the piece-wise function f(x)*

(a) $f(x) = x^2$ if $x \leq 1$; (b) $f(x) = 3$ if $1 < x \leq 2$; (c) $f(x) = x$ if $x > 2$

Now that we have sketched each piece individually, we combine them in the same coordinate plane. See Figure 2.2.21.

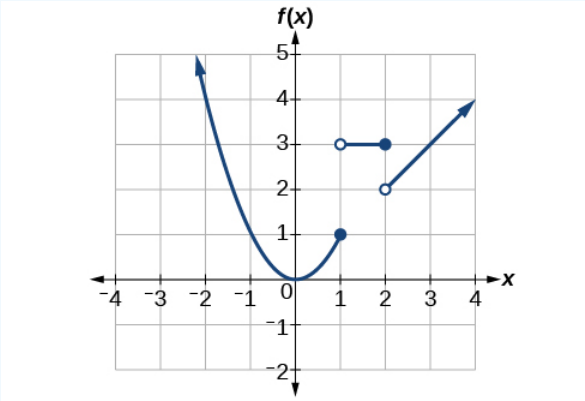


Figure 2.2.21: Graph of the entire function.

Analysis

Note that the graph does pass the vertical line test even at $x = 1$ and $x = 2$ because the points $(1, 3)$ and $(2, 2)$ are not part of the graph of the function, though $(1, 1)$ and $(2, 3)$ are.

Exercise 2.2.8

Graph the following piecewise function.

https://math.libretexts.org/@go/page/106336

$$f(x) = \begin{cases} x^3 & \text{if } x < -1 \\ -2 & \text{if } -1 < x < 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$$

Answer

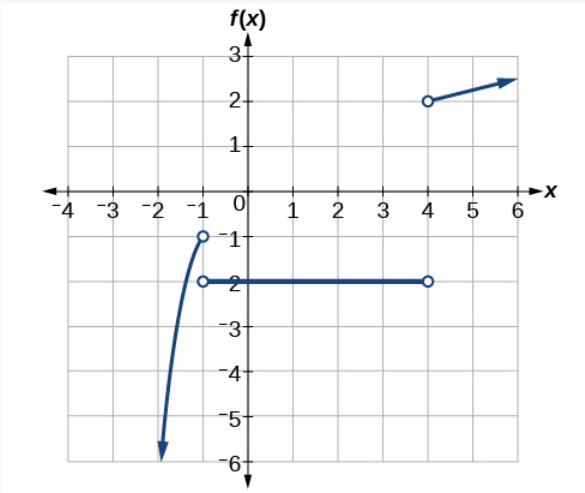


Figure 2.2.22

Can more than one formula from a piecewise function be applied to a value in the domain?

No. Each value corresponds to one equation in a piecewise formula.

Key Concepts

- The domain of a function includes all real input values that would not cause us to attempt an undefined mathematical operation, such as dividing by zero or taking the square root of a negative number.
- The domain of a function can be determined by listing the input values of a set of ordered pairs.
- The domain of a function can also be determined by identifying the input values of a function written as an equation.
- Interval values represented on a number line can be described using inequality notation, set-builder notation, and interval notation.
- For many functions, the domain and range can be determined from a graph.
- An understanding of toolkit functions can be used to find the domain and range of related functions.
- A piecewise function is described by more than one formula.
- A piecewise function can be graphed using each algebraic formula on its assigned subdomain.

Footnotes

1 The Numbers: Where Data and the Movie Business Meet. “Box Office History for Horror Movies.” http://www.the-numbers.com/market/genre/Horror. Accessed 3/24/2014

2 www.eia.gov/dnav/pet/hist/Lea...s=MCRFPAK2&f=A.

Glossary

interval notation

a method of describing a set that includes all numbers between a lower limit and an upper limit; the lower and upper values are listed between brackets or parentheses, a square bracket indicating inclusion in the set, and a parenthesis indicating exclusion

piecewise function

a function in which more than one formula is used to define the output

set-builder notation

a method of describing a set by a rule that all of its members obey; it takes the form {x| statement about x}

Homework Exercises 2.2

WeBWorK Problems:



Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login



Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login



Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login



Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login

Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login

Written Problems:

In creating various functions using the data, we can identify different independent and dependent variables, and we can analyze the data and the functions to determine the domain and range. In this section, we will investigate methods for determining the domain and range of functions.

Verbal

1) Why does the domain differ for different functions?

Answer

The domain of a function depends upon what values of the independent variable make the function undefined or imaginary.

2) How do we determine the domain of a function defined by an equation?

3) Explain why the domain of $f(x)=\sqrt[3]{x}$ is different from the domain of $f(x)=\sqrt{x}$.

Answer

There is no restriction on x for $f(x)=\sqrt[3]{x}$ because you can take the cube root of any real number. So the domain is all real numbers, $(-\infty,\infty)$. When dealing with the set of real numbers, you cannot take the square root of negative numbers. So x-values are restricted for $f(x)=\sqrt{x}$ to nonnegative numbers and the domain is $[0,\infty)$.

4) When describing sets of numbers using interval notation, when do you use a parenthesis and when do you use a bracket?

5) How do you graph a piecewise function?

Answer

Graph each formula of the piecewise function over its corresponding domain. Use the same scale for the x-axis and y-axis for each graph. Indicate inclusive endpoints with a solid circle and exclusive endpoints with an open circle. Use an arrow to indicate $-\infty$ or ∞. Combine the graphs to find the graph of the piecewise function.

For the exercises 6-25, find the domain of each function using interval notation.

6) $f(x) = -2x(x-1)(x-2)$

7) $f(x) = 5 - 2x^2$

Answer

$(-\infty, \infty)$

8) $f(x) = 3\sqrt{x-2}$

9) $f(x) = 3 - \sqrt{6-2x}$

Answer

$(-\infty, 3]$

10) $f(x) = \sqrt{4-3x}$

11) $f(x) = \sqrt{x^2+4}$

Answer

$(-\infty, \infty)$

12) $f(x) = \sqrt[3]{1-2x}$

13) $f(x) = \sqrt[3]{x-1}$

Answer

$(-\infty, \infty)$

14) $f(x) = \dfrac{9}{x-6}$

15) $f(x) = \dfrac{3x+1}{4x+2}$

Answer

$\left(-\infty, -\dfrac{1}{2}\right) \cup \left(-\dfrac{1}{2}, \infty\right)$

16) $f(x) = \dfrac{\sqrt{x+4}}{x-4}$

17) $f(x) = \dfrac{x-3}{x^2+9x-22}$

Answer

$(-\infty, -11) \cup (-11, 2) \cup (2, \infty)$

18) $f(x) = \dfrac{1}{x^2-x-6}$

19) $f(x) = \dfrac{2x^3-250}{x^2-2x-15}$

Answer

$(-\infty, -3) \cup (-3, 5) \cup (5, \infty)$

20) $\frac{5}{\sqrt{x-3}}$

21) $\frac{2x+1}{\sqrt{5-x}}$

Answer

$(-\infty, 5)$

22) $\frac{\sqrt{x-4}}{\sqrt{x-6}}$

23) $\frac{\sqrt{x-6}}{\sqrt{x-4}}$

Answer

$[6, \infty)$

24) $f(x) = \frac{x}{x}$

25) $f(x) = \frac{x^2 - 9x}{x^2 - 81}$

Answer

$(-\infty, -9) \cup (-9, 9) \cup (9, \infty)$

26) Find the domain of the function $f(x) = \sqrt{2x^3 - 50x}$ by:

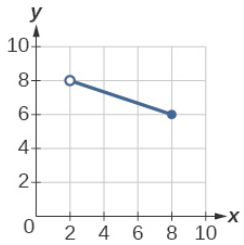
a. using algebra

b. graphing the function in the radicand and determining intervals on the x-axis for which the radicand is nonnegative.

Graphical

For the exercises 27-37, write the domain and range of each function using interval notation.

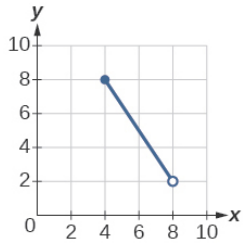
27)



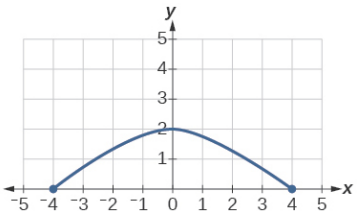
Answer

domain: $(2, 8]$, range $[6, 8)$

28)



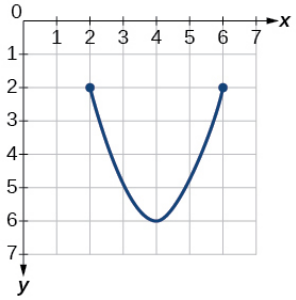
29)



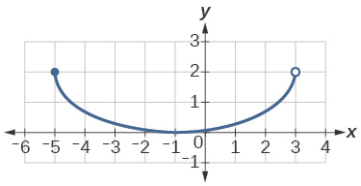
Answer

domain: $[-4, 4]$, range: $[0, 2]$

30)



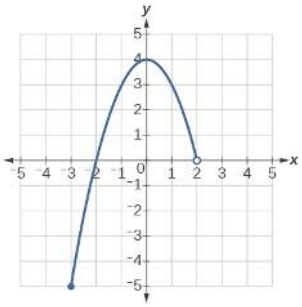
31)



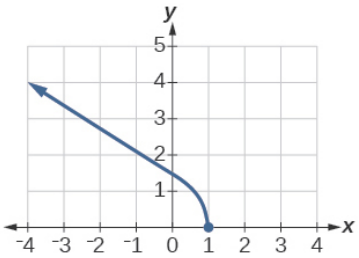
Answer

domain: $[-5, 3)$, range: $[0, 2]$

32)



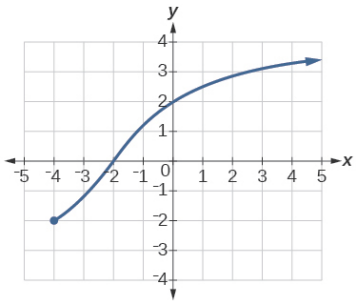
33)



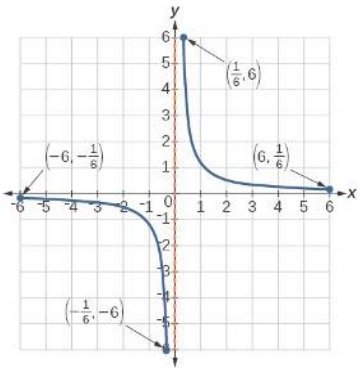
Answer

domain: $(-\infty, 1]$, range: $[0, \infty)$

34)



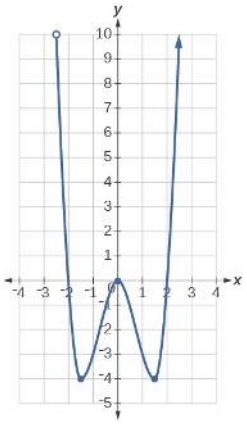
35)



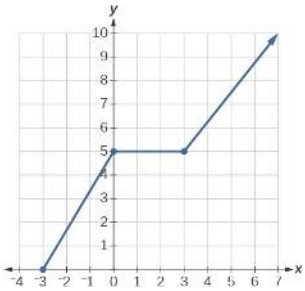
Answer

domain: $[-6, -16] \cup [16, 6]$, range: $[-6, -16] \cup [16, 6]$

36)



37)



Answer

domain: $[-3, \infty)$; range: $[0, \infty)$

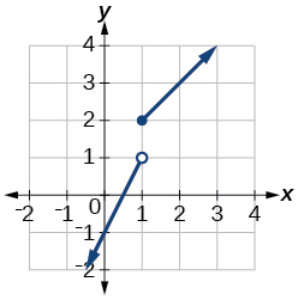
For the exercises 38-45, sketch a graph of the piecewise function. Write the domain in interval notation.

38) $f(x) = \begin{cases} x+1 & \text{if } x < -2 \\ -2x-3 & \text{if } x \geq -2 \end{cases}$

39) $f(x) = \begin{cases} 2x-1 & \text{if } x < 1 \\ 1+x & \text{if } x \geq 1 \end{cases}$

Answer

domain: $(-\infty, \infty)$

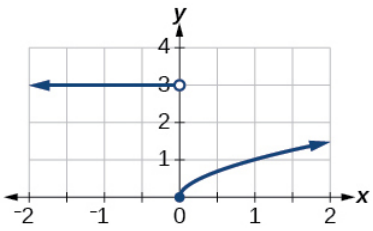


40) $f(x) = \begin{cases} x+1 & \text{if } x < 0 \\ x-1 & \text{if } x > 0 \end{cases}$

41) $f(x) = \begin{cases} 3 & \text{if } x < 0 \\ \sqrt{x} & \text{if } x \geq 0 \end{cases}$

Answer

domain: $(-\infty, \infty)$

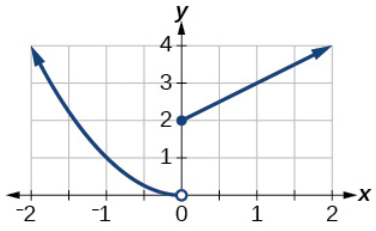


42) $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ 1-x & \text{if } x > 0 \end{cases}$

43) $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x+2 & \text{if } x \geq 0 \end{cases}$

Answer

domain: $(-\infty, \infty)$

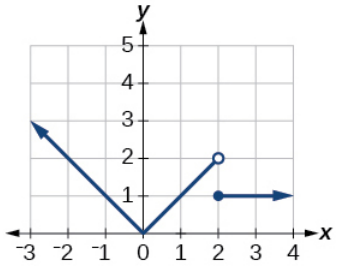


44) $f(x) = \begin{cases} x+1 & \text{if } x < 1 \\ x^3 & \text{if } x \geq 1 \end{cases}$

45) $f(x) = \begin{cases} |x| & \text{if } x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$

Answer

domain: $(-\infty, \infty)$



Numeric

For the exercises 46-48, given each function f,evaluate $f(-3)$, $f(-2)$, $f(-1)$, and $f(0)$.

46) $f(x) = \begin{cases} x+1 & \text{if } x < -2 \\ -2x-3 & \text{if } x \geq -2 \end{cases}$

47) $f(x) = \begin{cases} 1 & \text{if } x \leq -3 \\ 0 & \text{if } x > -3 \end{cases}$

Answer

$f(-3) = 1$; $f(-2) = 0$; $f(-1) = 0$; $f(0) = 0$

48) $f(x) = \begin{cases} -2x^2+3 & \text{if } x \leq -1 \\ 5x-7 & \text{if } x > -1 \end{cases}$

For the exercises 49-51, given each function f, evaluate $f(-1)$, $f(0)$, $f(2)$, and $f(4)$.

49) $f(x) = \begin{cases} 7x+3 & \text{if } x < 0 \\ 7x+6 & \text{if } x \geq 0 \end{cases}$

Answer

$f(-1) = -4$; $f(0) = 6$; $f(2) = 20$; $f(4) = 34$

50) $f(x) = \begin{cases} x^2-2 & \text{if } x < 2 \\ 4+|x-5| & \text{if } x \geq 2 \end{cases}$

51) $f(x) = \begin{cases} 5x & \text{if } x < 0 \\ 3 & \text{if } 0 \geq x \leq 2 \\ x^2 & \text{if } x > 3 \end{cases}$

Answer

$f(-1) = -5$; $f(0) = 3$; $f(2) = 3$; $f(4) = 16$

For the exercises 52-54, write the domain for the piecewise function in interval notation.

52) $f(x) = \begin{cases} x+1 & \text{if } x < -2 \\ -2x-3 & \text{if } x \geq -2 \end{cases}$

53) $f(x) = \begin{cases} x^2-2 & \text{if } x < 1 \\ -x^2+2 & \text{if } x > 1 \end{cases}$

Answer

domain: $(-\infty, 1) \cup (1, \infty)$

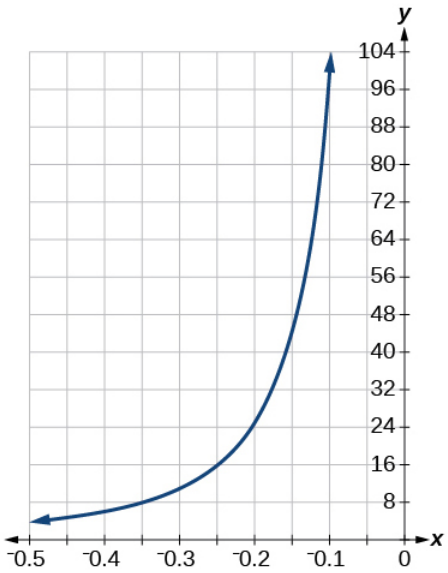
54) $f(x) = \begin{cases} x^2-3 & \text{if } x < 0 \\ -3x^2 & \text{if } x \geq 2 \end{cases}$

Technology

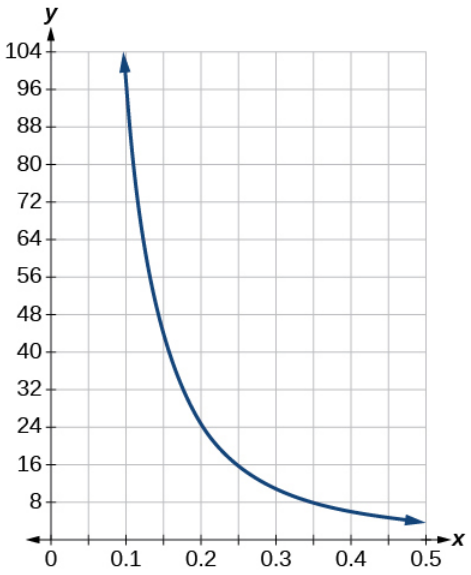
55) Graph $y = \dfrac{1}{x^2}$ on the viewing window $[-0.5, -0.1]$ and $[0.1, 0.5]$. Determine the corresponding range for the viewing window. Show the graphs.

Answer

window: $[-0.5, -0.1]$; range: $[4, 100]$



window: $[0.1, 0.5]$; range: $[4, 100]$



56) Graph $y = \frac{1}{x}$ on the viewing window $[-0.5, -0.1]$and $[0.1, 0.5]$. Determine the corresponding range for the viewing window. Show the graphs.

Extension

57) Suppose the range of a function f is $[-5, 8]$. What is the range of $|f(x)|$?

Answer

$[0, 8]$

58) Create a function in which the range is all nonnegative real numbers.

59) Create a function in which the domain is $x > 2$.

Answer

Many answers. One function is $f(x) = \frac{1}{\sqrt{x-2}}$.

Real-World Applications

60) The height h of a projectile is a function of the time t it is in the air. The height in feet for t seconds is given by the function $h(t) = -16t^2 + 96t$. What is the domain of the function? What does the domain mean in the context of the problem?

Answer

The domain is $[0, 6]$; it takes 6 seconds for the projectile to leave the ground and return to the ground

61) The cost in dollars of making x items is given by the function $C(x) = 10x + 500$.

a. The fixed cost is determined when zero items are produced. Find the fixed cost for this item.

b. What is the cost of making 25 items?

c. Suppose the maximum cost allowed is $\$1500$. What are the domain and range of the cost function, $C(x)$?

This page titled 2.2: Domain and Range is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **1.2: Domain and Range** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/precalculus.
- **1.E: Functions (Exercises)** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/precalculus.

2.3: Rates of Change and Behavior of Graphs

No preview video for this section, but there are supplemental videos after the text.

Learning Objectives

- Find the average rate of change of a function.
- Use a graph to determine where a function is increasing, decreasing, or constant.
- Use a graph to locate local maxima and local minima.
- Use a graph to locate the absolute maximum and absolute minimum.

Gasoline costs have experienced some wild fluctuations over the last several decades. Table 2.3.1 lists the average cost, in dollars, of a gallon of gasoline for the years 2005–2012. The cost of gasoline can be considered as a function of year.

Table 2.3.1

y	2005	2006	2007	2008	2009	2010	2011	2012
$C(y)$	2.31	2.62	2.84	3.30	2.41	2.84	3.58	3.68

If we were interested only in how the gasoline prices changed between 2005 and 2012, we could compute that the cost per gallon had increased from $2.31 to $3.68, an increase of $1.37. While this is interesting, it might be more useful to look at how much the price changed per year. In this section, we will investigate changes such as these.

Finding the Average Rate of Change of a Function

The price change per year is a **rate of change** because it describes how an output quantity changes relative to the change in the input quantity. We can see that the price of gasoline in Table 2.3.1 did not change by the same amount each year, so the rate of change was not constant. If we use only the beginning and ending data, we would be finding the **average rate of change** over the specified period of time. To find the average rate of change, we divide the change in the output value by the change in the input value.

$$\begin{aligned}\text{Average rate of change} &= \frac{\text{Change in output}}{\text{Change in input}} \\ &= \frac{\Delta y}{\Delta x} \\ &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{f(x_2) - f(x_1)}{x_2 - x_1}\end{aligned}$$

The Greek letter Δ (delta) signifies the change in a quantity; we read the ratio as "delta-y over delta-x" or "the change in y divided by the change in x." Occasionally we write Δf instead of Δy, which still represents the change in the function's output value resulting from a change to its input value. It does not mean we are changing the function into some other function.

In our example, the gasoline price increased by $1.37 from 2005 to 2012. Over 7 years, the average rate of change was

$$\frac{\Delta y}{\Delta x} = \frac{\$1.37}{7\text{years}} \approx 0.196 \text{ dollars per year.} \tag{2.3.1}$$

On average, the price of gas increased by about 19.6¢ each year. Other examples of rates of change include:

- A population of rats increasing by 40 rats per week
- A car traveling 68 miles per hour (distance traveled changes by 68 miles each hour as time passes)
- A car driving 27 miles per gallon (distance traveled changes by 27 miles for each gallon)
- The current through an electrical circuit increasing by 0.125 amperes for every volt of increased voltage
- The amount of money in a college account decreasing by $4,000 per quarter

Definition: Rate of Change

A **rate of change** describes how an output quantity changes relative to the change in the input quantity. The units on a rate of change are "output units per input units."

The average rate of change between two input values is the total change of the function values (output values) divided by the change in the input values.

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \tag{2.3.2}$$

How To...

Given the value of a function at different points, calculate the average rate of change of a function for the interval between two values x_1 and x_2.

1. Calculate the difference $y_2 - y_1 = \Delta y$.
2. Calculate the difference $x_2 - x_1 = \Delta x$.
3. Find the ratio $\frac{\Delta y}{\Delta x}$.

Example 2.3.1: Computing an Average Rate of Change

Using the data in Table 2.3.1, find the average rate of change of the price of gasoline between 2007 and 2009.

Solution

In 2007, the price of gasoline was \$2.84. In 2009, the cost was \$2.41. The average rate of change is

$$\begin{aligned} \frac{\Delta y}{\Delta x} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{\$2.41 - \$2.84}{2009 - 2007} \\ &= \frac{-\$0.43}{2 \text{ years}} \\ &= -\$0.22 \text{ per year} \end{aligned}$$

Analysis

Note that a decrease is expressed by a negative change or "negative increase." A rate of change is negative when the output decreases as the input increases or when the output increases as the input decreases.

Exercise 2.3.1

Using the data in Table 2.3.1, find the average rate of change between 2005 and 2010.

Solution

$$\frac{\$2.84 - \$2.315}{5 \text{ years}} = \frac{\$0.535}{5 \text{ years}} = \$0.106 \text{per year.}$$

Example 2.3.2: Computing Average Rate of Change from a Graph

Given the function $g(t)$ shown in Figure 2.3.1, find the average rate of change on the interval $[-1, 2]$.

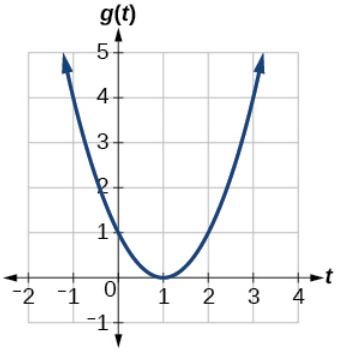


Figure 2.3.1: Graph of a parabola.

Solution

At $t=-1$, Figure 2.3.2 shows $g(-1)=4$. At $t=2$,the graph shows $g(2)=1$.

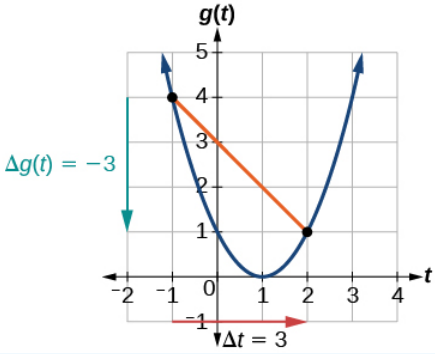


Figure 2.3.2: Graph of a parabola with a line from points (-1, 4) and (2, 1) to show the changes for g(t) and t.

The horizontal change $\Delta t=3$ is shown by the red arrow, and the vertical change $\Delta g(t)=-3$ is shown by the turquoise arrow. The output changes by –3 while the input changes by 3, giving an average rate of change of

$$\frac{1-4}{2-(-1)}=\frac{-3}{3}=-1 \tag{2.3.3}$$

Analysis

Note that the order we choose is very important. If, for example, we use $\frac{y_2-y_1}{x_1-x_2}$, we will not get the correct answer. Decide which point will be 1 and which point will be 2, and keep the coordinates fixed as (x_1,y_1) and (x_2,y_2).

Example 2.3.3: Computing Average Rate of Change from a Table

After picking up a friend who lives 10 miles away, Anna records her distance from home over time. The values are shown in Table 2.3.2. Find her average speed over the first 6 hours.

Table 2.3.2

t (hours)	0	1	2	3	4	5	6	7
$D(t)$ (miles)	10	55	90	153	214	240	292	300

Solution

Here, the average speed is the average rate of change. She traveled 292 miles in 6 hours, for an average speed of

$$\frac{292-10}{6-0}=\frac{282}{6}$$
$$=47$$

The average speed is about 47 miles per hour.

Analysis

Because the speed is not constant, the average speed depends on the interval chosen. For the interval $[2,3]$, the average speed is 63 miles per hour.

Example 2.3.4: Computing Average Rate of Change for a Function Expressed as a Formula

Compute the average rate of change of $f(x)=x^2-\frac{1}{x}$ on the interval $[2,4]$.

Solution

We can start by computing the function values at each endpoint of the interval.

$$\begin{aligned} f(2) &= 2^2-\frac{1}{2} & f(4) &= 4^2-\frac{1}{4} \\ &= 4-\frac{1}{2} & &= 16-\frac{1}{4} \\ &= 72 & &= \frac{63}{4} \end{aligned}$$

Now we compute the average rate of change.

$$\begin{aligned} \text{Average rate of change} &= \frac{f(4)-f(2)}{4-2} \\ &= \frac{\frac{63}{4}-\frac{7}{2}}{4-2} \\ &= \frac{\frac{49}{4}}{2} \\ &= \frac{49}{8} \end{aligned}$$

Exercise 2.3.2

Find the average rate of change of $f(x)=x-2\sqrt{x}$ on the interval $[1,9]$.

Solution

$\frac{1}{2}$

Example 2.3.5: Finding the Average Rate of Change of a Force

The electrostatic force F, measured in newtons, between two charged particles can be related to the distance between the particles d,in centimeters, by the formula $F(d)=\frac{2}{d^2}$. Find the average rate of change of force if the distance between the particles is increased from 2 cm to 6 cm.

Solution

We are computing the average rate of change of $F(d)=\dfrac{2}{d^2}$ on the interval $[2,6]$.

$$\begin{aligned}
\text{Average rate of change} &= \frac{F(6)-F(2)}{6-2} \\
&= \frac{\frac{2}{6^2}-\frac{2}{2^2}}{6-2} && \text{Simplify} \\
&= \frac{\frac{2}{36}-\frac{2}{4}}{4} \\
&= \frac{-\frac{16}{36}}{4} && \text{Combine numerator terms.} \\
&= -\frac{1}{9} && \text{Simplify}
\end{aligned}$$

The average rate of change is $-\frac{1}{9}$ newton per centimeter.

Example 2.3.6: Finding an Average Rate of Change as an Expression

Find the average rate of change of $g(t)=t^2+3t+1$ on the interval $[0,a]$. The answer will be an expression involving a.

Solution

We use the average rate of change formula.

$$\begin{aligned}
\text{Average rate of change} &= \frac{g(a)-g(0)}{a-0} && \text{Evaluate.} \\
&= \frac{(a^2+3a+1)-(0^2+3(0)+1)}{a-0} && \text{Simplify.} \\
&= \frac{a^2+3a+1-1}{a} && \text{Simplify and factor.} \\
&= \frac{a(a+3)}{a} && \text{Divide by the common factor a.} \\
&= a+3
\end{aligned}$$

This result tells us the average rate of change in terms of a between $t=0$ and any other point $t=a$. For example, on the interval $[0,5]$, the average rate of change would be $5+3=8$.

Exercise 2.3.3

Find the average rate of change of $f(x)=x^2+2x-8$ on the interval $[5,a]$.

Solution

$a+7$

Using a Graph to Determine Where a Function is Increasing, Decreasing, or Constant

As part of exploring how functions change, we can identify intervals over which the function is changing in specific ways. We say that a function is increasing on an interval if the function values increase as the input values increase within that interval. Similarly, a function is decreasing on an interval if the function values decrease as the input values increase over that interval. The average rate of change of an increasing function is positive, and the average rate of change of a decreasing function is negative. Figure 2.3.3 shows examples of increasing and decreasing intervals on a function.

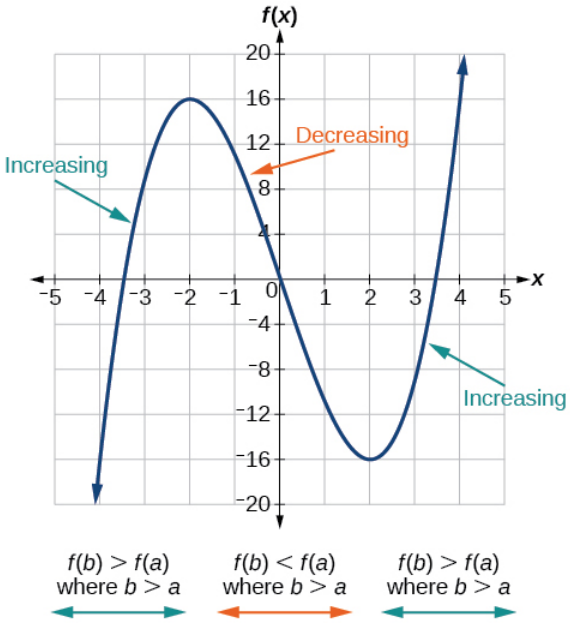


Figure 2.3.3: The function $f(x) = x^3 - 12x$ is increasing on $(-\infty, -2) \cup (2, \infty)$ and is decreasing on $(-2, 2)$.

While some functions are increasing (or decreasing) over their entire domain, many others are not. A value of the input where a function changes from increasing to decreasing (as we go from left to right, that is, as the input variable increases) is called a **local maximum**. If a function has more than one, we say it has local maxima. Similarly, a value of the input where a function changes from decreasing to increasing as the input variable increases is called a **local minimum**. The plural form is "local minima." Together, local maxima and minima are called **local extrema**, or local extreme values, of the function. (The singular form is "extremum.") Often, the term local is replaced by the term relative. In this text, we will use the term local.

Clearly, a function is neither increasing nor decreasing on an interval where it is constant. A function is also neither increasing nor decreasing at extrema. Note that we have to speak of local extrema, because any given local extremum as defined here is not necessarily the highest maximum or lowest minimum in the function's entire domain.

For the function whose graph is shown in Figure 2.3.4, the local maximum is 16, and it occurs at $x = -2$. The local minimum is -16 and it occurs at $x = 2$.

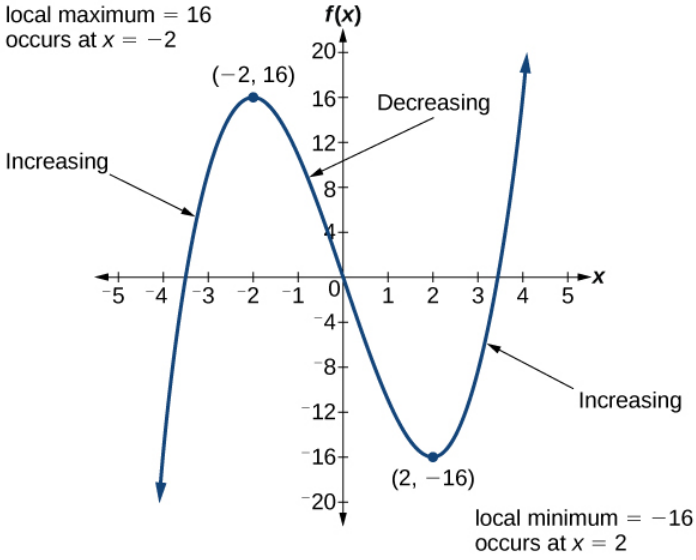


Figure 2.3.4: Graph of a polynomial that shows the increasing and decreasing intervals and local maximum.maximum

To locate the local maxima and minima from a graph, we need to observe the graph to determine where the graph attains its highest and lowest points, respectively, within an open interval. Like the summit of a roller coaster, the graph of a function is higher at a

local maximum than at nearby points on both sides. The graph will also be lower at a local minimum than at neighboring points. Figure 2.3.5 illustrates these ideas for a local maximum.

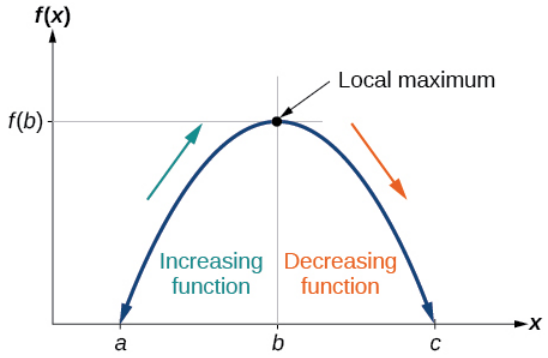


Figure 2.3.5: Definition of a local maximum

These observations lead us to a formal definition of local extrema.

Local Minima and Local Maxima

- A function f is an **increasing function** on an open interval if $f(b) > f(a)$ for every a, b interval where $b > a$.
- A function f is a **decreasing function** on an open interval if $f(b) < f(a)$ for every a, b interval where $b > a$.

A function f has a local maximum at a point b in an open interval (a, c) if $f(b)$ is greater than or equal to $f(x)$ for every point x (x does not equal b) in the interval. Likewise, f has a local minimum at a point b in (a, c) if $f(b)$ is less than or equal to $f(x)$ for every x (x does not equal b) in the interval.

Example 2.3.7 Finding Increasing and Decreasing Intervals on a Graph

Given the function $p(t)$ in Figure 2.3.6, identify the intervals on which the function appears to be increasing.

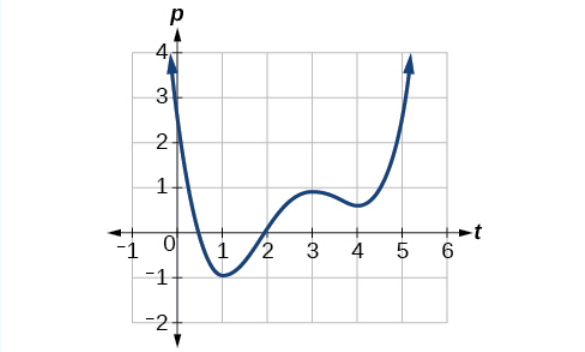


Figure 2.3.6: Graph of a polynomial.

Solution

We see that the function is not constant on any interval. The function is increasing where it slants upward as we move to the right and decreasing where it slants downward as we move to the right. The function appears to be increasing from $t = 1$ to $t = 3$ and from $t = 4$ on.

In interval notation, we would say the function appears to be increasing on the interval $(1, 3)$ and the interval $(4, \infty)$.

Analysis

Notice in this example that we used open intervals (intervals that do not include the endpoints), because the function is neither increasing nor decreasing at $t = 1$, $t = 3$, and $t = 4$. These points are the local extrema (two minima and a maximum).

Example 2.3.8: Finding Local Extrema from a Graph

Graph the function $f(x) = \frac{2}{x} + \frac{x}{3}$. Then use the graph to estimate the local extrema of the function and to determine the intervals on which the function is increasing.

Solution

Using technology, we find that the graph of the function looks like that in Figure 2.3.7. It appears there is a low point, or local minimum, between $x = 2$ and $x = 3$, and a mirror-image high point, or local maximum, somewhere between $x = -3$ and $x = -2$

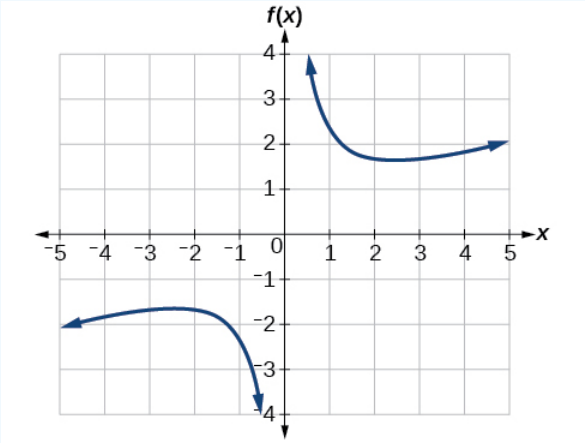


Figure 2.3.7: Graph of a reciprocal function.

Analysis

Most graphing calculators and graphing utilities can estimate the location of maxima and minima. Figure 2.3.8 provides screen images from two different technologies, showing the estimate for the local maximum and minimum.

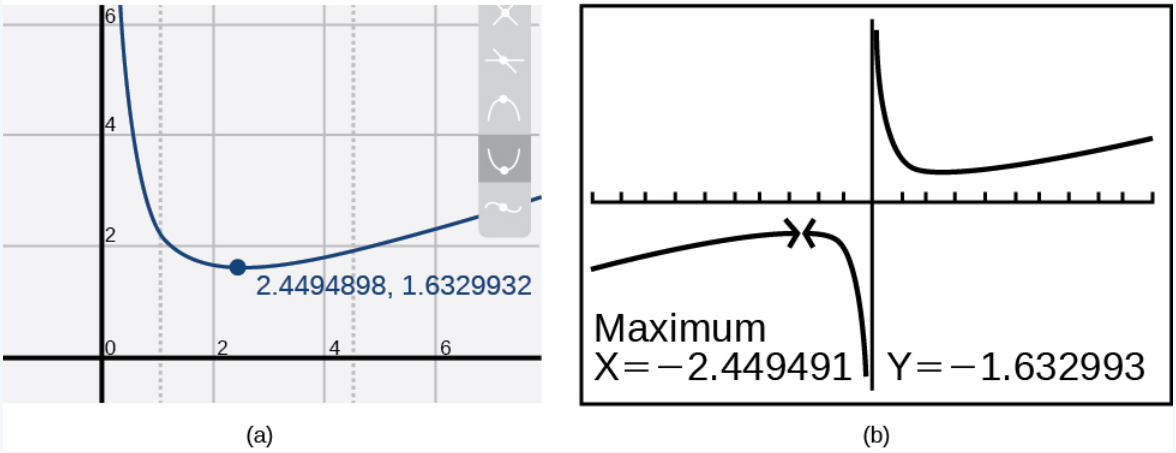


Figure 2.3.8: Graph of the reciprocal function on a graphing calculator.

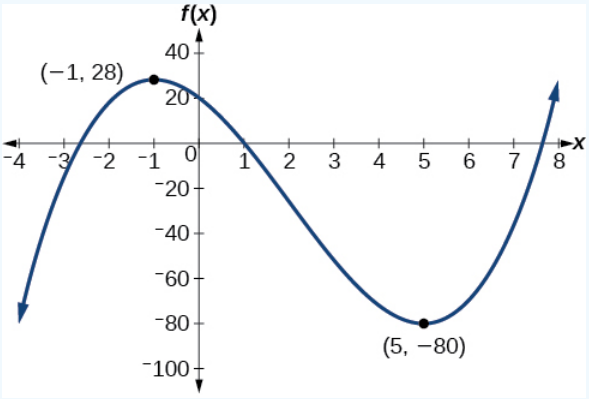
Based on these estimates, the function is increasing on the interval $(-\infty, -2.449)$ and $(2.449, \infty)$. Notice that, while we expect the extrema to be symmetric, the two different technologies agree only up to four decimals due to the differing approximation algorithms used by each. (The exact location of the extrema is at $\pm\sqrt{6}$, but determining this requires calculus.)

Exercise 2.3.8

Graph the function $f(x) = x^3 - 6x^2 - 15x + 20$ to estimate the local extrema of the function. Use these to determine the intervals on which the function is increasing and decreasing.

Solution

The local maximum appears to occur at $(-1, 28)$, and the local minimum occurs at $(5, -80)$. The function is increasing on $(-\infty, -1) \cup (5, \infty)$ and decreasing on $(-1, 5)$.



Graph of a polynomial with a local maximum at (-1, 28) and local minimum at (5, -80).

Example 2.3.9: Finding Local Maxima and Minima from a Graph

For the function f whose graph is shown in Figure 2.3.9, find all local maxima and minima.

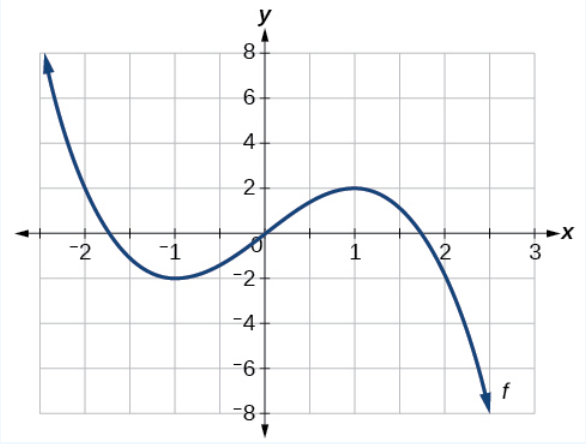


Figure 2.3.9: Graph of a polynomial.

Solution

Observe the graph of f. The graph attains a local maximum at $x = 1$ because it is the highest point in an open interval around $x = 1$.The local maximum is the y-coordinate at $x = 1$, which is 2.

The graph attains a local minimum at $x = -1$ because it is the lowest point in an open interval around $x = -1$. The local minimum is the y-coordinate at $x = -1$, which is −2.

Analyzing the Toolkit Functions for Increasing or Decreasing Intervals

We will now return to our toolkit functions and discuss their graphical behavior in Figure 2.3.10, Figure 2.3.11, and Figure 2.3.12

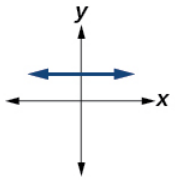
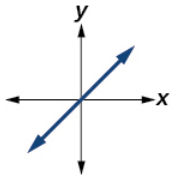
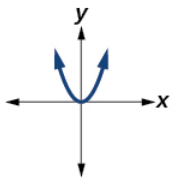
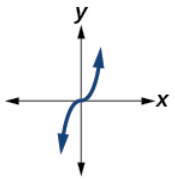
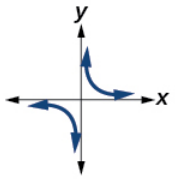
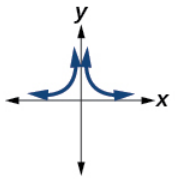
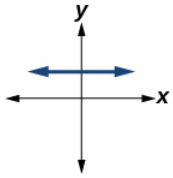
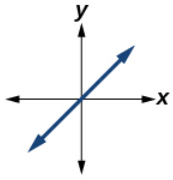
Function	Increasing/Decreasing	Example
Constant Function $f(x) = c$	Neither increasing nor decreasing	
Identity Function $f(x) = x$	Increasing	
Quadratic Function $f(x) = x^2$	Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$ Minimum at $x = 0$	

Figure 2.3.10

Function	Increasing/Decreasing	Example
Cubic Function $f(x) = x^3$	Increasing	
Reciprocal $f(x) = \frac{1}{x}$	Decreasing $(-\infty, 0) \cup (0, \infty)$	
Reciprocal Squared $f(x) = \frac{1}{x^2}$	Increasing on $(-\infty, 0)$ Decreasing on $(0, \infty)$	

.

Figure 2.3.11

Function	Increasing/Decreasing	Example
Constant Function $f(x) = c$	Neither increasing nor decreasing	
Identity Function $f(x) = x$	Increasing	

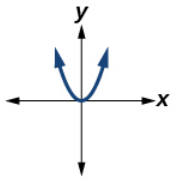
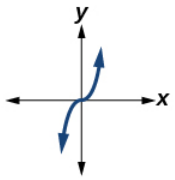
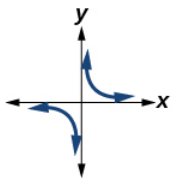
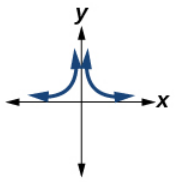
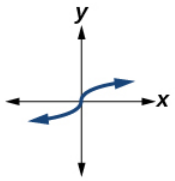
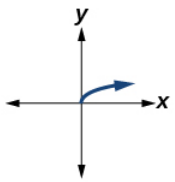
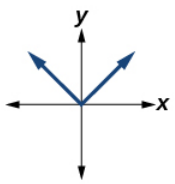
Quadratic Function $f(x) = x^2$	Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$ Minimum at $x = 0$	
Cubic Function $f(x) = x^3$	Increasing	
Reciprocal $f(x) = \frac{1}{x}$	Decreasing $(-\infty, 0) \cup (0, \infty)$	
Reciprocal Squared $f(x) = \frac{1}{x^2}$	Increasing on $(-\infty, 0)$ Decreasing on $(0, \infty)$	
Cube Root $f(x) = \sqrt[3]{x}$	Increasing	
Square Root $f(x) = \sqrt{x}$	Increasing on $(0, \infty)$	
Absolute Value $f(x) = \lvert x \rvert$	Increasing on $(0, \infty)$ Decreasing on $(\infty, 0)$	

***Figure* 2.3.12**

Use A Graph to Locate the Absolute Maximum and Absolute Minimum

There is a difference between locating the highest and lowest points on a graph in a region around an open interval (locally) and locating the highest and lowest points on the graph for the entire domain. The y-coordinates (output) at the highest and lowest points are called the **absolute maximum** and **absolute minimum**, respectively. To locate absolute maxima and minima from a graph, we need to observe the graph to determine where the graph attains it highest and lowest points on the domain of the function (Figure 2.3.13).

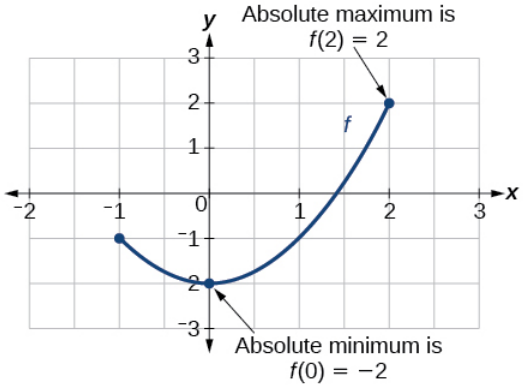


Figure 2.3.13: Graph of a segment of a parabola with an absolute minimum at (0, -2) and absolute maximum at (2, 2).

Not every function has an absolute maximum or minimum value. The toolkit function $f(x) = x^3$ is one such function.

Absolute Maxima and Minima

- The **absolute maximum** of f at $x = c$ is $f(c)$ where $f(c) \geq f(x)$ for all x in the domain of f.
- The **absolute minimum** of f at $x = d$ is $f(d)$ where $f(d) \leq f(x)$ for all x in the domain of f.

Example 2.3.10: Finding Absolute Maxima and Minima from a Graph

For the function f shown in Figure 2.3.14, find all absolute maxima and minima.

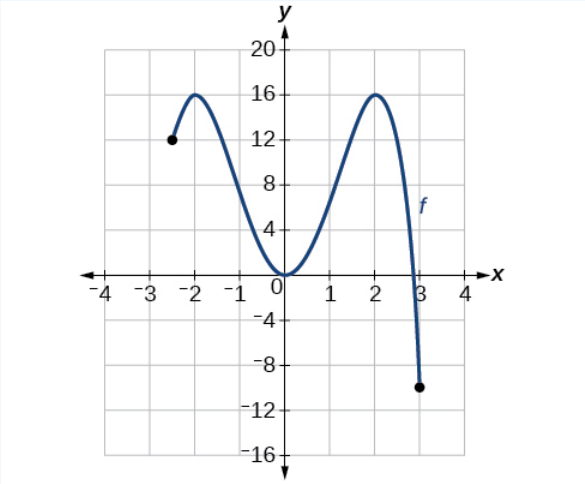


Figure 2.3.14: Graph of a polynomial.

Solution

Observe the graph of f. The graph attains an absolute maximum in two locations, $x = -2$ and $x = 2$, because at these locations, the graph attains its highest point on the domain of the function. The absolute maximum is the y-coordinate at $x = -2$ and $x = 2$, which is 16.

The graph attains an absolute minimum at x=3, because it is the lowest point on the domain of the function's graph. The absolute minimum is the y-coordinate at x=3,which is−10.

Key Equations

- Average rate of change: $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

Key Concepts

- A rate of change relates a change in an output quantity to a change in an input quantity. The average rate of change is determined using only the beginning and ending data. See Example.

- Identifying points that mark the interval on a graph can be used to find the average rate of change. See Example.
- Comparing pairs of input and output values in a table can also be used to find the average rate of change. See Example.
- An average rate of change can also be computed by determining the function values at the endpoints of an interval described by a formula. See Example and Example.
- The average rate of change can sometimes be determined as an expression. See Example.
- A function is increasing where its rate of change is positive and decreasing where its rate of change is negative. See Example.
- A local maximum is where a function changes from increasing to decreasing and has an output value larger (more positive or less negative) than output values at neighboring input values.
- A local minimum is where the function changes from decreasing to increasing (as the input increases) and has an output value smaller (more negative or less positive) than output values at neighboring input values.
- Minima and maxima are also called extrema.
- We can find local extrema from a graph. See Example and Example.
- The highest and lowest points on a graph indicate the maxima and minima. See Example.

Supplemental Videos

Average Rate of Change:



Increasing, Decreasing, and Extrema:



Homework Exercises 2.3

WeBWorK Problems:

Written Problems:

Make sure to include sketches of the graphs with your work.

In this section, we will investigate changes in functions. For example, a rate of change relates a change in an output quantity to a change in an input quantity. The average rate of change is determined using only the beginning and ending data. Identifying points that mark the interval on a graph can be used to find the average rate of change. Comparing pairs of input and output values in a table can also be used to find the average rate of change.

Verbal

1) Can the average rate of change of a function be constant?

Answer

Yes, the average rate of change of all linear functions is constant.

2) If a function f is increasing on (a, b) and decreasing on (b, c), then what can be said about the local extremum of f on (a, c)?

3) How are the absolute maximum and minimum similar to and different from the local extrema?

Answer

The absolute maximum and minimum relate to the entire graph, whereas the local extrema relate only to a specific region around an open interval.

4) How does the graph of the absolute value function compare to the graph of the quadratic function, $y = x^2$, in terms of increasing and decreasing intervals?

Algebraic

For the exercises 5-15, find the average rate of change of each function on the interval specified for real numbers b or h.

5) $f(x) = 4x^2 - 7$ on $[1, b]$

Answer

$4(b+1)$

6) $g(x) = 2x^2 - 9$ on $[4, b]$

7) $p(x) = 3x + 4$ on $[2, 2 + h]$

Answer

3

8) $k(x) = 4x - 2$ on $[3, 3 + h]$

9) $f(x) = 2x^2 + 1$ on $[x, x + h]$

Answer

$4x + 2h$

10) $g(x) = 3x^2 - 2$ on $[x, x + h]$

11) $a(t) = \dfrac{1}{t+4}$ on $[9, 9 + h]$

Answer

$$\frac{-1}{13(13+h)}$$

12) $b(x) = 1x + 3$ on $[1, 1 + h]$

13) $j(x) = 3x^3$ on $[1, 1 + h]$

Answer

$3h^2 + 9h + 9$

14) $r(t) = 4t^3$ on $[2, 2 + h]$

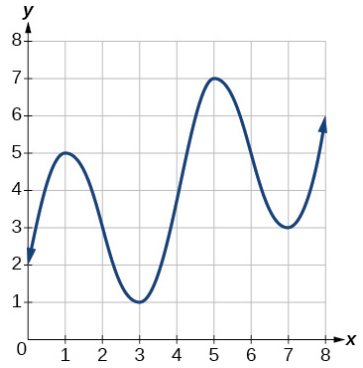
15) $\dfrac{f(x+h) - f(x)}{h}$ given $f(x) = 2x^2 - 3x$ on $[x, x + h]$

Answer

$4x + 2h - 3$

Graphical

For the exercises 16-17, consider the graph of f shown in the Figure below



Graph of a polynomial.

16) Estimate the average rate of change from $x = 1$ to $x = 4$.

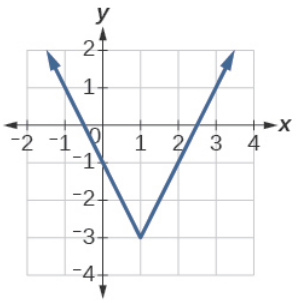
17) Estimate the average rate of change from $x = 2$ to $x = 5$.

Answer

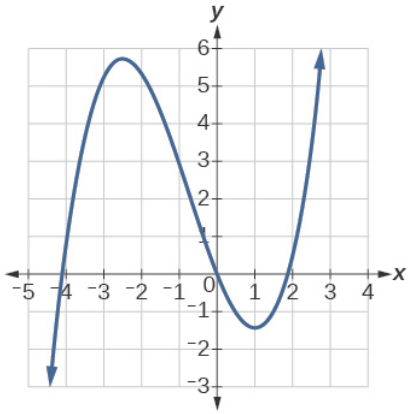
$\frac{4}{3}$

For the following exercises, use the graph of each function to estimate the intervals on which the function is increasing or decreasing.

18)



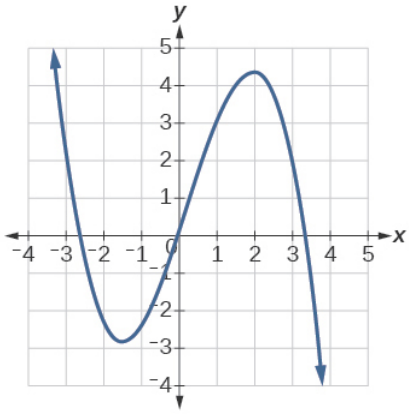
19)



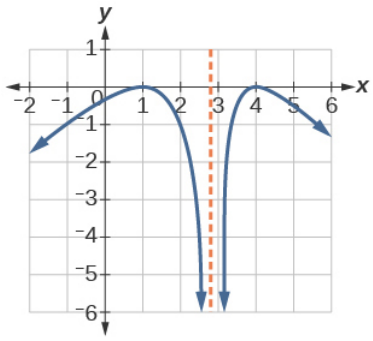
Answer

increasing on $(-\infty, -2.5) \cup (1, \infty)$, decreasing on $(-2.5, 1)$

20)



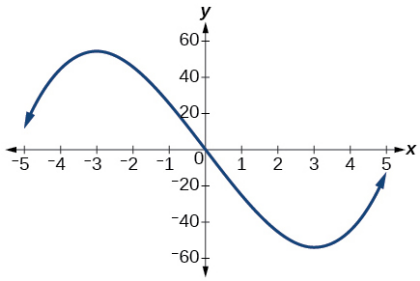
21)



Answer

increasing on $(-\infty, 1) \cup (3, 4)$, decreasing on $(1, 3) \cup (4, \infty)$

For the exercises 22-23, consider the graph shown in the Figure below.



Graph of a cubic function.

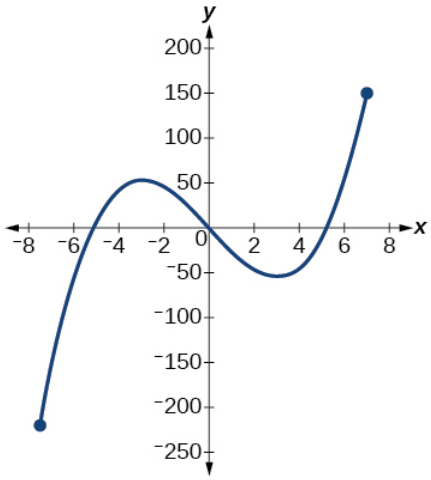
22) Estimate the intervals where the function is increasing or decreasing.

23) Estimate the point(s) at which the graph of f has a local maximum or a local minimum.

Answer

local maximum: $(-3, 60)$, local minimum: $(3, -60)$

For the exercises 24-25, consider the graph in the Figure below.



Graph of a cubic function.

24) If the complete graph of the function is shown, estimate the intervals where the function is increasing or decreasing.

25) If the complete graph of the function is shown, estimate the absolute maximum and absolute minimum.

Answer

absolute maximum at approximately $(7, 150)$, absolute minimum at approximately $(-7.5, -220)$

Numeric

26) The Table below gives the annual sales (in millions of dollars) of a product from 1998 to 2006. What was the average rate of change of annual sales (a) between 2001 and 2002, and (b) between 2001 and 2004?

Year	Sales (millions of dollars)
1998	201
1999	219
2000	233
2001	243
2002	249
2003	251
2004	249
2005	243
2006	233

27) The Table below gives the population of a town (in thousands) from 2000 to 2008. What was the average rate of change of population (a) between 2002 and 2004, and (b) between 2002 and 2006?

Year	Population (thousands)
2000	87
2001	84
2002	83
2003	80
2004	77
2005	76
2006	78
2007	81
2008	85

Answer

a. -3000; b. -1250

For the exercises 28-34, find the average rate of change of each function on the interval specified.

28) $f(x) = x^2$ on $[1, 5]$

29) $h(x) = 5 - 2x^2$ on $[-2, 4]$

Answer

-4

30) $q(x) = x^3$ on $[-4, 2]$

31) $g(x) = 3x^3 - 1$ on $[-3, 3]$

Answer

27

32) $y = \dfrac{1}{x}$ on $[1, 3]$

33) $p(t) = \dfrac{(t^2 - 4)(t + 1)}{t^2 + 3}$ on $[-3, 1]$

Answer

-0.167

34) $k(t) = 6t^2 + \dfrac{4}{t^3}$ on $[-1, 3]$

Technology

For the exercises 35-40, use a graphing utility to estimate the local extrema of each function and to estimate the intervals on which the function is increasing and decreasing.

35) $f(x) = x^4 - 4x^3 + 5$

Answer

Local minimum at $(3, -22)$, decreasing on $(-\infty, 3)$, increasing on $(3, \infty)$

36) $h(x) = x^5 + 5x^4 + 10x^3 + 10x^2 - 1$

37) $g(t) = t\sqrt{t + 3}$

Answer

Local minimum at $(-2, -2)$, decreasing on $(-3, -2)$, increasing on $(-2, \infty)$

38) $k(t) = 3t^{\frac{2}{3}} - t$

39) $m(x) = x^4 + 2x^3 - 12x^2 - 10x + 4$

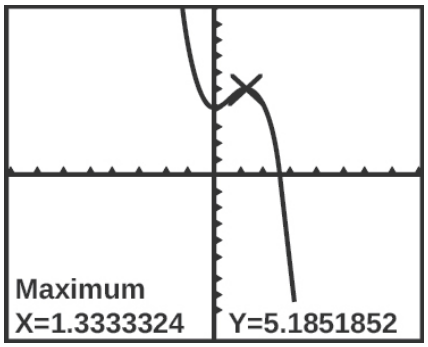
Answer

Local maximum at $(-0.5, 6)$, local minima at $(-3.25, -47)$ and $(2.1, -32)$, decreasing on $(-\infty, -3.25)$ and $(-0.5, 2.1)$, increasing on $(-3.25, -0.5)$and $(2.1, \infty)$

40) $n(x) = x^4 - 8x^3 + 18x^2 - 6x + 2$

Extension

41) The graph of the function f is shown in the Figure below:



Graph of f(x) on a graphing calculator.

Based on the calculator screen shot, the point $(1.333, 5.185)$is which of the following?

a. a relative (local) maximum of the function
b. the vertex of the function
c. the absolute maximum of the function
d. a zero of the function

Answer

a

42) Let $f(x) = \frac{1}{x}$. Find a number c such that the average rate of change of the function f on the interval $(1, c)$ is $-\frac{1}{4}$.

43) Let $f(x) = \frac{1}{x}$. Find the number b such that the average rate of change of f on the interval $(2, b)$ is $-\frac{1}{10}$.

Answer

$b = 5$

Real-World Applications

44) At the start of a trip, the odometer on a car read $21,395$. At the end of the trip, 13.5 hours later, the odometer read $22,125$. Assume the scale on the odometer is in miles. What is the average speed the car traveled during this trip?

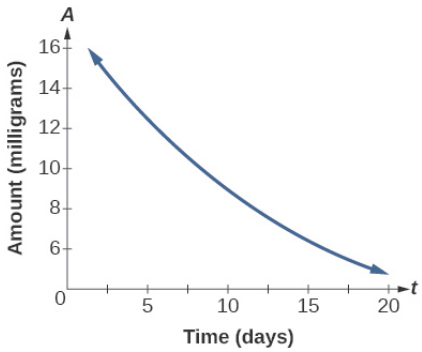
45) A driver of a car stopped at a gas station to fill up his gas tank. He looked at his watch, and the time read exactly 3:40 p.m. At this time, he started pumping gas into the tank. At exactly 3:44, the tank was full and he noticed that he had pumped 10.7 gallons. What is the average rate of flow of the gasoline into the gas tank?

Answer

2.7 gallons per minute

46) Near the surface of the moon, the distance that an object falls is a function of time. It is given by $d(t) = 2.6667t^2$, where t is in seconds and $d(t)$ is in feet. If an object is dropped from a certain height, find the average velocity of the object from $t = 1$ to $t = 2$.

47) The graph in the Figure below illustrates the decay of a radioactive substance over t days.



Graph of an exponential function.

Use the graph to estimate the average decay rate from $t = 5$ to $t = 15$.

Answer

approximately -0.6 milligrams per day

This page titled 2.3: Rates of Change and Behavior of Graphs is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **1.3: Rates of Change and Behavior of Graphs** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/precalculus.
- **1.E: Functions (Exercises)** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/precalculus.

2.4: Transformation of Functions

Preview Video for Section 2.4.

Note the section number and day are incorrect on the video, but correct on the notes handout.

Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login

Learning Objectives

- Graph functions using vertical and horizontal shifts.
- Graph functions using reflections about the x-axis and the y-axis.
- Determine whether a function is even, odd, or neither from its graph.
- Graph functions using compressions and stretches.
- Combine transformations.

We all know that a flat mirror enables us to see an accurate image of ourselves and whatever is behind us. When we tilt the mirror, the images we see may shift horizontally or vertically. But what happens when we bend a flexible mirror? Like a carnival funhouse mirror, it presents us with a distorted image of ourselves, stretched or compressed horizontally or vertically. In a similar way, we can distort or transform mathematical functions to better adapt them to describing objects or processes in the real world. In this section, we will take a look at several kinds of transformations.



Figure 2.4.1: (credit: "Misko"/Flickr)

Often when given a problem, we try to model the scenario using mathematics in the form of words, tables, graphs, and equations. One method we can employ is to adapt the basic graphs of the toolkit functions to build new models for a given scenario. There are systematic ways to alter functions to construct appropriate models for the problems we are trying to solve.

Identifying Vertical Shifts

One simple kind of transformation involves shifting the entire graph of a function up, down, right, or left. The simplest shift is a vertical shift, moving the graph up or down, because this transformation involves adding a positive or negative constant to the function. In other words, we add the same constant to the output value of the function regardless of the input. For a function $g(x) = f(x) + k$, the function $f(x)$ is shifted vertically k units. See Figure 2.4.2 for an example.

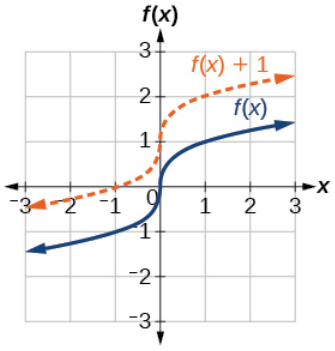


Figure 2.4.2: Vertical shift by $k = 1$ of the cube root function $f(x) = \sqrt[3]{x}$.

To help you visualize the concept of a vertical shift, consider that $y = f(x)$. Therefore, $f(x) + k$ is equivalent to $y + k$. Every unit of y is replaced by $y + k$, so the y-value increases or decreases depending on the value of k. The result is a shift upward or downward.

Definition: Vertical Shift

Given a function $f(x)$, a new function $g(x) = f(x) + k$, where k is a constant, is a **vertical shift** of the function $f(x)$. All the output values change by k units. If k is positive, the graph will shift up. If k is negative, the graph will shift down.

Example 2.4.1: Adding a Constant to a Function

To regulate temperature in a green building, airflow vents near the roof open and close throughout the day. Figure 2.4.3 shows the area of open vents V (in square feet) throughout the day in hours after midnight, t. During the summer, the facilities manager decides to try to better regulate temperature by increasing the amount of open vents by 20 square feet throughout the day and night. Sketch a graph of this new function.

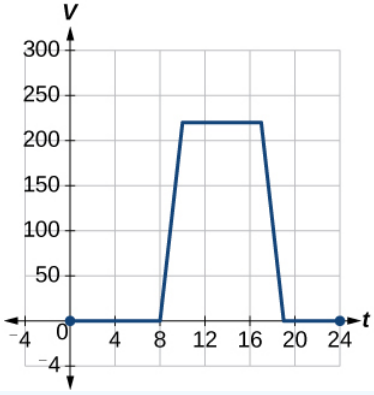


Figure 2.4.3

Solution

We can sketch a graph of this new function by adding 20 to each of the output values of the original function. This will have the effect of shifting the graph vertically up, as shown in Figure 2.4.4.

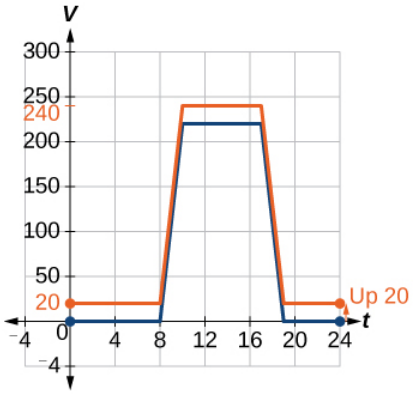


Figure 2.4.4

Notice that in Figure 2.4.4, for each input value, the output value has increased by 20, so if we call the new function $S(t)$,we could write

$$S(t) = V(t) + 20 \tag{2.4.1}$$

This notation tells us that, for any value of t,$S(t)$ can be found by evaluating the function V at the same input and then adding 20 to the result. This defines S as a transformation of the function V, in this case a vertical shift up 20 units. Notice that, with a vertical shift, the input values stay the same and only the output values change. See Table 2.4.1.

Table 2.4.1

t	0	8	10	17	19	24
$V(t)$	0	0	220	220	0	0
$S(t)$	20	20	240	240	20	20

How To...

Given a tabular function, create a new row to represent a vertical shift.

1. Identify the output row or column.
2. Determine the **magnitude** of the shift.
3. Add the shift to the value in each output cell. Add a positive value for up or a negative value for down.

Example 2.4.2: Shifting a Tabular Function Vertically

A function $f(x)$ is given in Table 2.4.2. Create a table for the function $g(x) = f(x) - 3$.

Table 2.4.2

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = f(x) - 3$ tells us that we can find the output values of g by subtracting 3 from the output values of f. For example:

$$\begin{aligned} f(x) &= 1 && \text{Given} \\ g(x) &= f(x) - 3 && \text{Given Transformation} \\ g(2) &= f(2) - 3 \\ &= 1 - 3 \\ &= -2 \end{aligned}$$

Subtracting 3 from each $f(x)$ value, we can complete a table of values for $g(x)$ as shown in Table 2.4.3.

Table 2.4.3

x	2	4	6	8
$f(x)$	1	3	7	11
$g(x)$	-2	0	4	8

Analysis

As with the earlier vertical shift, notice the input values stay the same and only the output values change.

Exercise 2.4.1

The function $h(t) = -4.9t^2 + 30t$ gives the height h of a ball (in meters) thrown upward from the ground after t seconds. Suppose the ball was instead thrown from the top of a 10-m building. Relate this new height function $b(t)$ to $h(t)$, and then find a formula for $b(t)$.

Answer

$b(t) = h(t) + 10 = -4.9t^2 + 30t + 10$

Identifying Horizontal Shifts

We just saw that the vertical shift is a change to the output, or outside, of the function. We will now look at how changes to input, on the inside of the function, change its graph and meaning. A shift to the input results in a movement of the graph of the function left or right in what is known as a **horizontal shift**, shown in Figure 2.4.4.

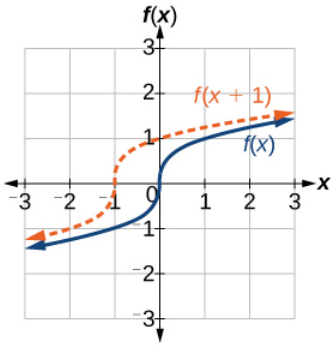


Figure 2.4.4: Horizontal shift of the function $f(x) = \sqrt[3]{x}$. Note that $h = -1$ shifts the graph to the left, that is, towards negative values of x.

For example, if $f(x) = x^2$, then $g(x) = (x-2)^2$ is a new function. Each input is reduced by 2 prior to squaring the function. The result is that the graph is shifted 2 units to the right, because we would need to increase the prior input by 2 units to yield the same output value as given in f.

Definition: Horizontal Shift

Given a function f, a new function $g(x) = f(x-h)$, where h is a constant, is a **horizontal shift** of the function f. If h is positive, the graph will shift right. If h is negative, the graph will shift left.

Example 2.4.4: Adding a Constant to an Input

Returning to our building airflow example from Figure 2.4.2, suppose that in autumn the facilities manager decides that the original venting plan starts too late, and wants to begin the entire venting program 2 hours earlier. Sketch a graph of the new function.

Solution

We can set $V(t)$ to be the original program and $F(t)$ to be the revised program.

$$V(t) = \text{ the original venting plan}$$

$$F(t) = \text{ starting 2 hrs sooner}$$

In the new graph, at each time, the airflow is the same as the original function V was 2 hours later. For example, in the original function V, the airflow starts to change at 8 a.m., whereas for the function F, the airflow starts to change at 6 a.m. The comparable function values are $V(8) = F(6)$. See Figure 2.4.5. Notice also that the vents first opened to 220ft^2 at 10 a.m. under the original plan, while under the new plan the vents reach 220ft^2 at 8 a.m., so $V(10) = F(8)$.

In both cases, we see that, because $F(t)$ starts 2 hours sooner, $h = -2$. That means that the same output values are reached when $F(t) = V(t-(-2)) = V(t+2)$.

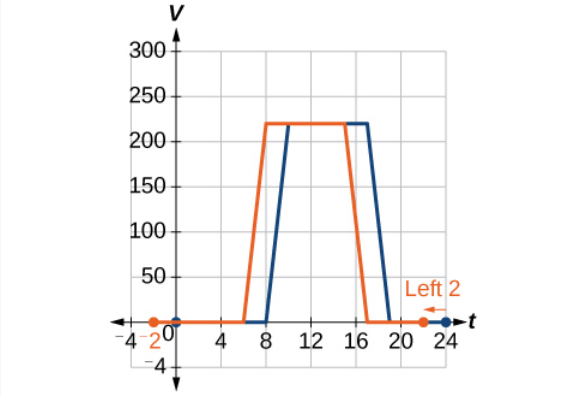


Figure 2.4.5

Analysis

Note that $V(t+2)$ has the effect of shifting the graph to the left.

Horizontal changes or "inside changes" affect the domain of a function (the input) instead of the range and often seem counterintuitive. The new function $F(t)$ uses the same outputs as $V(t)$, but matches those outputs to inputs 2 hours earlier than those of $V(t)$. Said another way, we must add 2 hours to the input of V to find the corresponding output for F : $F(t) = V(t+2)$.

How To...

Given a tabular function, create a new row to represent a horizontal shift.

1. Identify the input row or column.
2. Determine the magnitude of the shift.
3. Add the shift to the value in each input cell.

Example 2.4.5: Shifting a Tabular Function Horizontally

A function $f(x)$ is given in Table 2.4.4. Create a table for the function $g(x) = f(x-3)$.

Table 2.4.4

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = f(x-3)$ tells us that the output values of g are the same as the output value of f when the input value is 3 less than the original value. For example, we know that $f(2) = 1$. To get the same output from the function g, we will need an input value that is 3 larger. We input a value that is 3 larger for $g(x)$ because the function takes 3 away before evaluating the function f.

$$\begin{aligned} g(5) &= f(5-3) \\ &= f(2) \\ &= 1 \end{aligned}$$

We continue with the other values to create Table 2.4.5.

Table 2.4.5

x	5	7	9	11
$x-3$	2	4	6	8
$f(x)$	1	3	7	11
$g(x)$	1	3	7	11

The result is that the function $g(x)$ has been shifted to the right by 3. Notice the output values for $g(x)$ remain the same as the output values for $f(x)$, but the corresponding input values, x, have shifted to the right by 3. Specifically, 2 shifted to 5, 4 shifted to 7, 6 shifted to 9, and 8 shifted to 11.

Analysis

Figure 2.4.6 represents both of the functions. We can see the horizontal shift in each point.

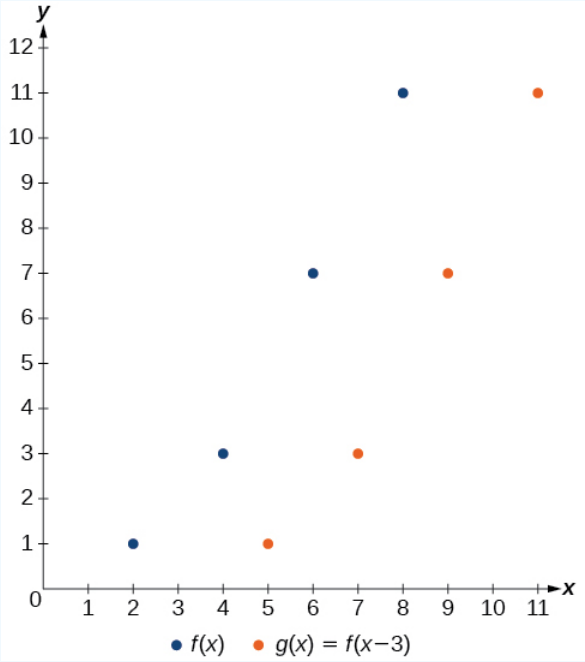


Figure 2.4.6: Graph of the points from the previous table for $f(x)$ and $g(x) = f(x-3)$

Example 2.4.6: Identifying a Horizontal Shift of a Toolkit Function

Figure 2.4.7 represents a transformation of the toolkit function $f(x) = x^2$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

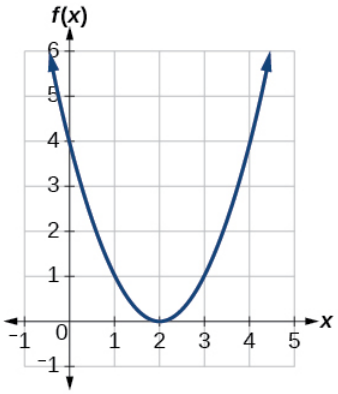


Figure 2.4.7: Graph of a parabola

Solution

Notice that the graph is identical in shape to the $f(x)=x^2$ function, but the x-values are shifted to the right 2 units. The vertex used to be at $(0,0)$, but now the vertex is at $(2,0)$. The graph is the basic quadratic function shifted 2 units to the right, so

$$g(x)=f(x-2)$$

Notice how we must input the value $x=2$ to get the output value $y=0$; the x-values must be 2 units larger because of the shift to the right by 2 units. We can then use the definition of the $f(x)$ function to write a formula for $g(x)$ by evaluating $f(x-2)$.

$$\begin{aligned} f(x) &= x^2 \\ g(x) &= f(x-2) \\ g(x) &= f(x-2)=(x-2)^2 \end{aligned}$$

Analysis

To determine whether the shift is $+2$ or -2, consider a single reference point on the graph. For a quadratic, looking at the vertex point is convenient. In the original function, $f(0)=0$. In our shifted function, $g(2)=0$. To obtain the output value of 0 from the function f, we need to decide whether a plus or a minus sign will work to satisfy $g(2)=f(x-2)=f(0)=0$. For this to work, we will need to subtract 2 units from our input values.

Example 2.4.7: Interpreting Horizontal versus Vertical Shifts

The function $G(m)$ gives the number of gallons of gas required to drive m miles. Interpret $G(m)+10$ and $G(m+10)$

Solution

$G(m)+10$ can be interpreted as adding 10 to the output, gallons. This is the gas required to drive m miles, plus another 10 gallons of gas. The graph would indicate a vertical shift.

$G(m+10)$ can be interpreted as adding 10 to the input, miles. So this is the number of gallons of gas required to drive 10 miles more than m miles. The graph would indicate a horizontal shift.

Exercise 2.4.7

Given the function $f(x)=\sqrt{x}$, graph the original function $f(x)$ and the transformation $g(x)=f(x+2)$ on the same axes. Is this a horizontal or a vertical shift? Which way is the graph shifted and by how many units?

Answer

The graphs of $f(x)$ and $g(x)$ are shown below. The transformation is a horizontal shift. The function is shifted to the left by 2 units.

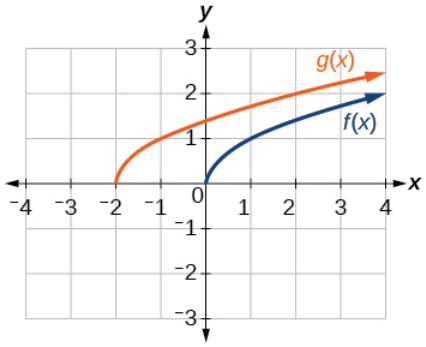


Figure 2.4.8

Combining Vertical and Horizontal Shifts

Now that we have two transformations, we can combine them together. Vertical shifts are outside changes that affect the output $(y-)$ axis values and shift the function up or down. Horizontal shifts are inside changes that affect the input $(x-)$ axis values and shift the function left or right. Combining the two types of shifts will cause the graph of a function to shift up or down and right or left.

How To...

Given a function and both a vertical and a horizontal shift, sketch the graph.

1. Identify the vertical and horizontal shifts from the formula.
2. The vertical shift results from a constant added to the output. Move the graph up for a positive constant and down for a negative constant.
3. The horizontal shift results from a constant added to the input. Move the graph left for a positive constant and right for a negative constant.
4. Apply the shifts to the graph in either order.

Example 2.4.8: Graphing Combined Vertical and Horizontal Shifts

Given $f(x)=|x|$, sketch a graph of $h(x)=f(x+1)-3$.

Solution

The function f is our toolkit absolute value function. We know that this graph has a V shape, with the point at the origin. The graph of h has transformed f in two ways: $f(x+1)$ is a change on the inside of the function, giving a horizontal shift left by 1, and the subtraction by 3 in $f(x+1)-3$ is a change to the outside of the function, giving a vertical shift down by 3. The transformation of the graph is illustrated in Figure 2.4.9.

Let us follow one point of the graph of $f(x)=|x|$.

- The point $(0,0)$ is transformed first by shifting left 1 unit:$(0,0)\rightarrow(-1,0)$
- The point $(-1,0)$ is transformed next by shifting down 3 units:$(-1,0)\rightarrow(-1,-3)$

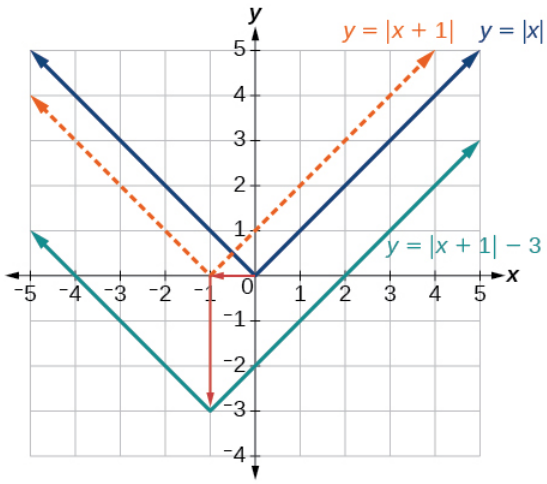


Figure 2.4.9: Graph of an absolute function, $y = |x|$, and how it was transformed to $y = |x + 1| - 3$

Figure 2.4.10 shows the graph of h.

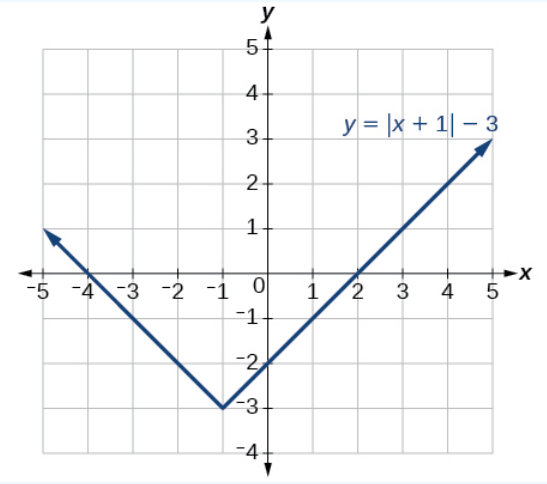


Figure 2.4.10: The final function $y = |x + 1| - 3$.

Exercise 2.4.8

Given $f(x) = |x|$, sketch a graph of $h(x) = f(x - 2) + 4$.

Answer

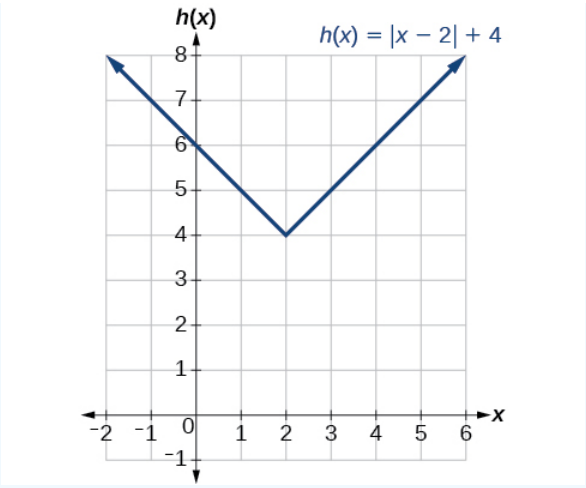


Figure 2.4.11

Example 2.4.9: Identifying Combined Vertical and Horizontal Shifts

Write a formula for the graph shown in Figure 2.4.12, which is a transformation of the toolkit square root function.

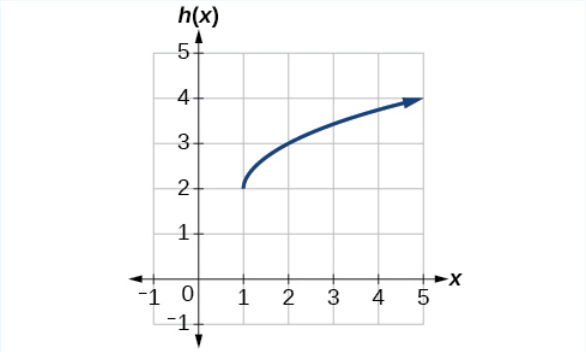


Figure 2.4.12: Graph of a square root function transposed right one unit and up 2.

Solution

The graph of the toolkit function starts at the origin, so this graph has been shifted 1 to the right and up 2. In function notation, we could write that as

$$h(x) = f(x-1)+2$$

Using the formula for the square root function, we can write

$$h(x) = \sqrt{x-1}+2$$

Analysis

Note that this transformation has changed the domain and range of the function. This new graph has domain $[1, \infty)$ and range $[2, \infty)$.

Exercise 2.4.9

Write a formula for a transformation of the toolkit reciprocal function $f(x) = \frac{1}{x}$ that shifts the function's graph one unit to the right and one unit up.

Answer

$$g(x) = \frac{1}{x-1}+1$$

Graphing Functions Using Reflections about the Axes

Another transformation that can be applied to a function is a reflection over the x- or y-axis. A **vertical reflection** reflects a graph vertically across the x-axis, while a **horizontal reflection** reflects a graph horizontally across the y-axis. The reflections are shown in Figure 2.4.13.

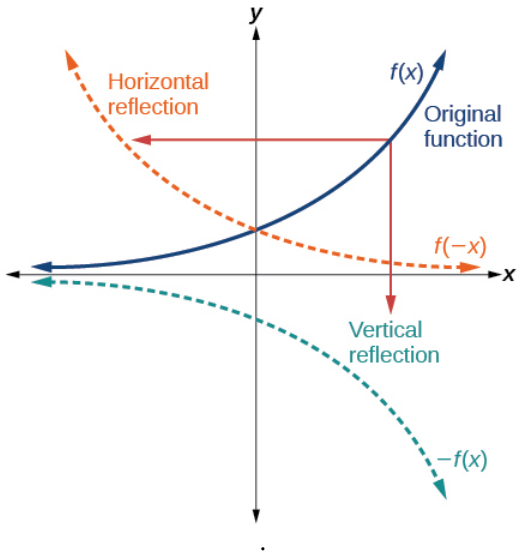


Figure 2.4.13: Graph of the vertical and horizontal reflection of a function.

Notice that the vertical reflection produces a new graph that is a mirror image of the base or original graph about the x-axis. The horizontal reflection produces a new graph that is a mirror image of the base or original graph about the y-axis.

Definitions: Reflections

Given a function $f(x)$, a new function $g(x) = -f(x)$ is a **vertical reflection** of the function $f(x)$, sometimes called a reflection about (or over, or through) the x-axis.

Given a function $f(x)$, a new function $g(x) = f(-x)$ is a **horizontal reflection** of the function $f(x)$, sometimes called a reflection about the y-axis.

How To...

Given a function, reflect the graph both vertically and horizontally.

1. Multiply all outputs by –1 for a vertical reflection. The new graph is a reflection of the original graph about the x-axis.
2. Multiply all inputs by –1 for a horizontal reflection. The new graph is a reflection of the original graph about the y-axis.

Example 2.4.10: Reflecting a Graph Horizontally and Vertically

Reflect the graph of $s(t) = \sqrt{t}$ (a) vertically and (b) horizontally.

Solution

a. Reflecting the graph vertically means that each output value will be reflected over the horizontal t-axis as shown in Figure 2.4.14

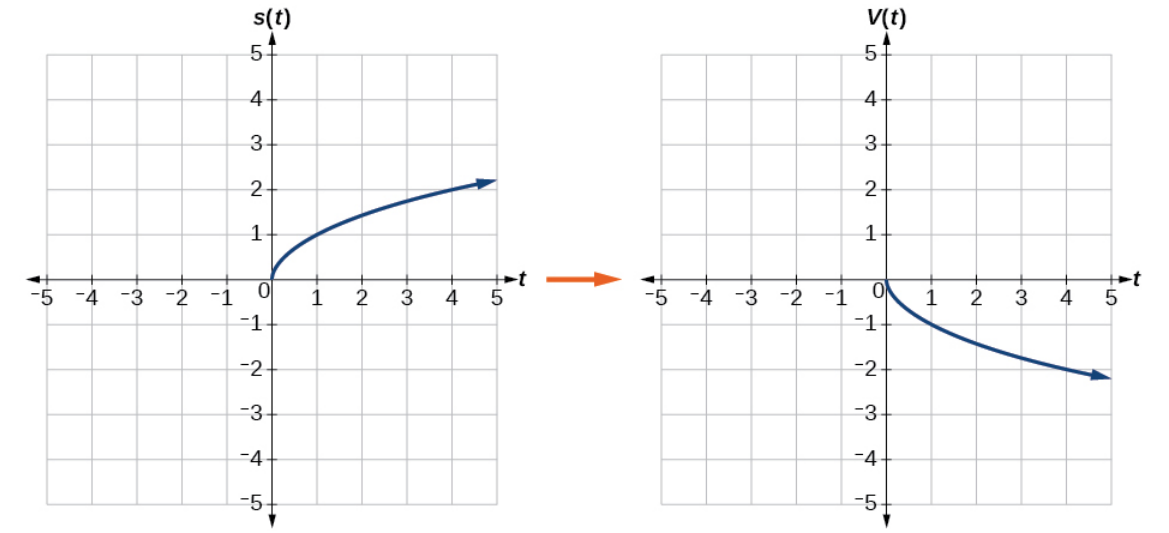


Figure 2.4.14: Graph of the vertical reflection of the square root function.

Because each output value is the opposite of the original output value, we can write

$$V(t) = -s(t) \text{ or } V(t) = -\sqrt{t}$$

Notice that this is an outside change, or vertical shift, that affects the output $s(t)$ values, so the negative sign belongs outside of the function.

b. Reflecting horizontally means that each input value will be reflected over the vertical axis as shown in Figure 2.4.15.

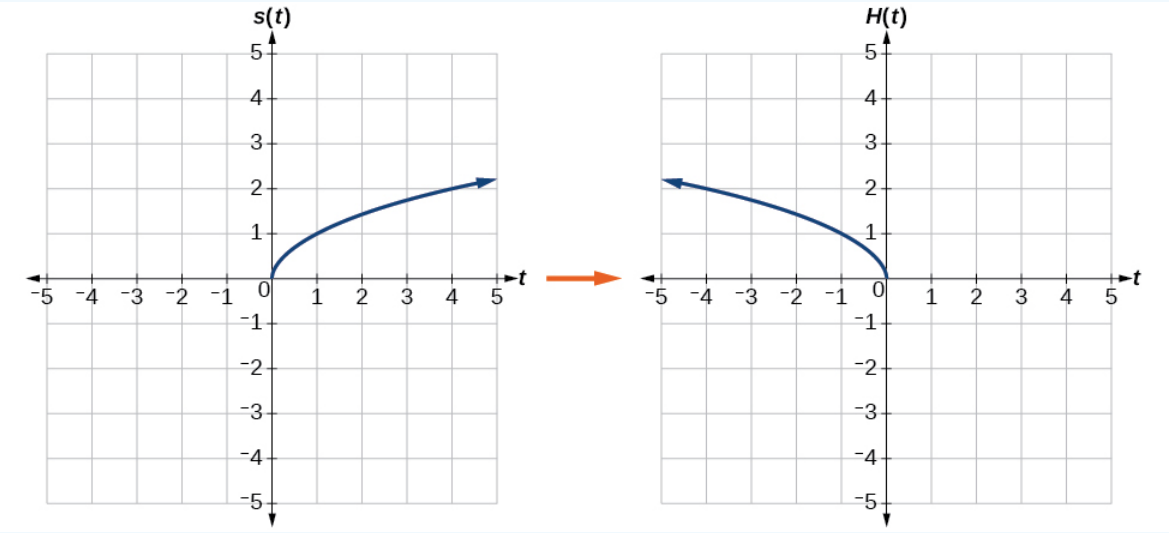


Figure 2.4.15: Horizontal reflection of the square root function

Because each input value is the opposite of the original input value, we can write

$$H(t) = s(-t) \text{ or } H(t) = \sqrt{-t}$$

Notice that this is an inside change or horizontal change that affects the input values, so the negative sign is on the inside of the function.

Note that these transformations can affect the domain and range of the functions. While the original square root function has domain $[0, \infty)$ and range $[0, \infty)$, the vertical reflection gives the $V(t)$ function the range $(-\infty, 0]$ and the horizontal reflection gives the $H(t)$ function the domain $(-\infty, 0]$.

Exercise 2.4.5

Reflect the graph of $f(x) = |x - 1|$ (a) vertically and (b) horizontally.

Answer

a.

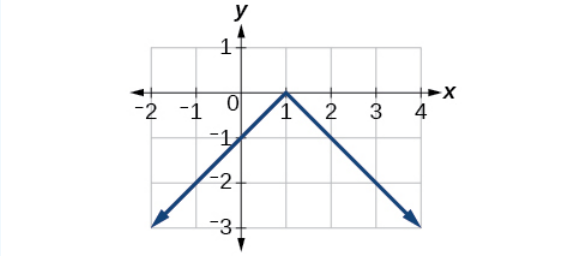


Figure 2.4.16: Graph of a vertically reflected absolute function.

b.

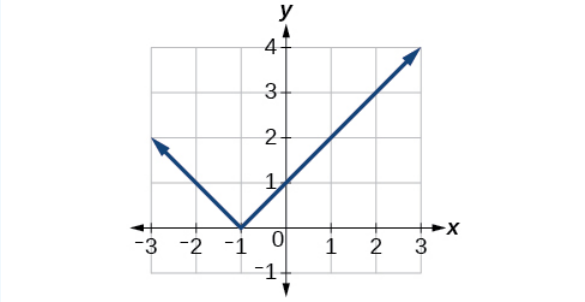


Figure 2.4.17: Graph of an absolute function translated one unit left.

Example 2.4.11: Reflecting a Tabular Function Horizontally and Vertically

A function $f(x)$ is given as Table 2.4.6. Create a table for the functions below.

a. $g(x) = -f(x)$
b. $h(x) = f(-x)$

Table 2.4.6

x	2	4	6	8
$f(x)$	1	3	7	11

a. For $g(x)$, the negative sign outside the function indicates a vertical reflection, so the x-values stay the same and each output value will be the opposite of the original output value. See Table 2.4.7.

Table 2.4.7

x	2	4	6	8
$g(x)$	-1	-3	-7	-11

b. For $h(x)$, the negative sign inside the function indicates a horizontal reflection, so each input value will be the opposite of the original input value and the $h(x)$ values stay the same as the $f(x)$ values. See Table 2.4.8.

Table 2.4.8

x	-2	-4	-6	-8
$h(x)$	1	3	7	11

Exercise 2.4.6

A function $f(x)$ is given as Table 2.4.9. Create a table for the functions below.

a. $g(x) = -f(x)$
b. $h(x) = f(-x)$

Table 2.4.9

x	-2	0	2	4
$f(x)$	5	10	15	20

Answer

a. $g(x) = -f(x)$

Table 2.4.10

x	-2	0	2	4
$g(x)$	-5	-10	-15	-20

b. $h(x) = f(-x)$

Table 2.4.11

x	-2	0	2	-4
$h(x)$	15	10	5	20

Example 2.4.12: Applying a Learning Model Equation

A common model for learning has an equation similar to $k(t) = -2^{-t} + 1$, where k is the percentage of mastery that can be achieved after t practice sessions. This is a transformation of the function $f(t) = 2^t$ shown in Figure 2.4.18. Sketch a graph of $k(t)$.

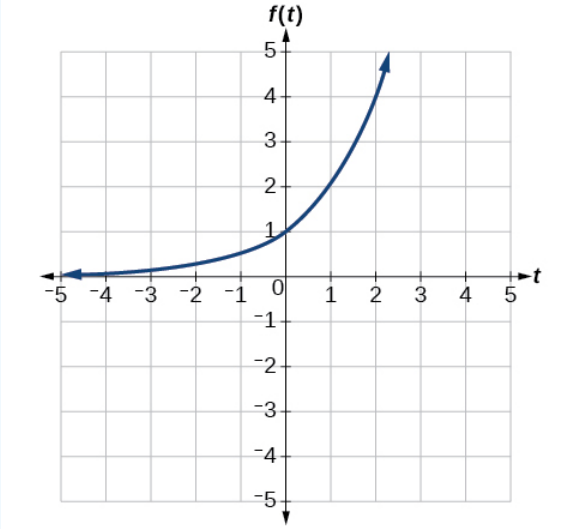


Figure 2.4.18: Graph of $k(t)$

Solution

This equation combines three transformations into one equation.

- A horizontal reflection: $f(-t) = 2^{-t}$

- A vertical reflection: $-f(-t) = -2^{-t}$
- A vertical shift: $-f(-t)+1 = -2^{-t}+1$

We can sketch a graph by applying these transformations one at a time to the original function. Let us follow two points through each of the three transformations. We will choose the points $(0, 1)$ and $(1, 2)$.

- First, we apply a horizontal reflection: $(0, 1)$ $(-1, 2)$.
- Then, we apply a vertical reflection: $(0, -1)$ $(-1, -2)$.
- Finally, we apply a vertical shift: $(0, 0)$ $(-1, -1)$.

This means that the original points, $(0, 1)$ and $(1, 2)$ become $(0, 0)$ and $(-1, -1)$ after we apply the transformations.

In Figure 2.4.19, the first graph results from a horizontal reflection. The second results from a vertical reflection. The third results from a vertical shift up 1 unit.

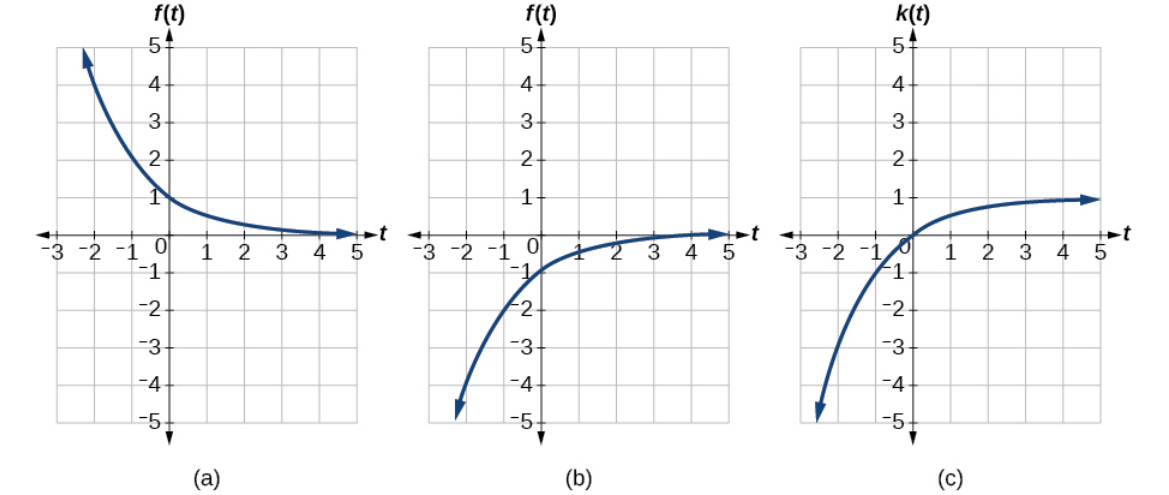


Figure 2.4.19: Graphs of all the transformations.

Analysis

As a model for learning, this function would be limited to a domain of $t \geq 0$, with corresponding range $[0, 1)$.

Exercise 2.4.7

Given the toolkit function $f(x) = x^2$, graph $g(x) = -f(x)$ and $h(x) = f(-x)$. Take note of any surprising behavior for these functions.

Answer

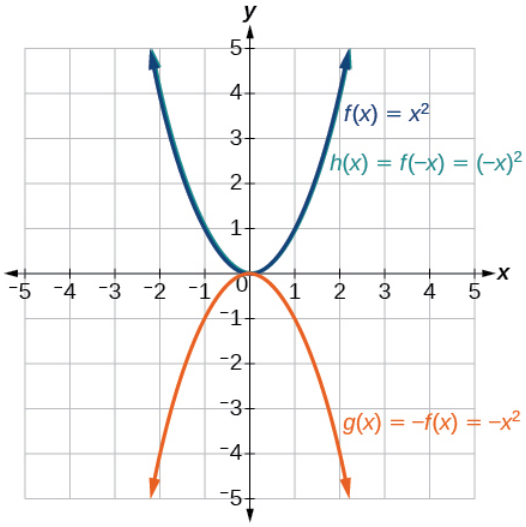


Figure 2.4.20: Graph of x^2 and its reflections.

Notice: $g(x) = f(-x)$ looks the same as $f(x)$.

Determining Even and Odd Functions

Some functions exhibit symmetry so that reflections result in the original graph. For example, horizontally reflecting the toolkit functions $f(x) = x^2$ or $f(x) = |x|$ will result in the original graph. We say that these types of graphs are symmetric about the y-axis. Functions whose graphs are symmetric about the y-axis are called **even functions**.

If the graphs of $f(x) = x^3$ or $f(x) = \frac{1}{x}$ were reflected over both axes, the result would be the original graph, as shown in Figure 2.4.21.

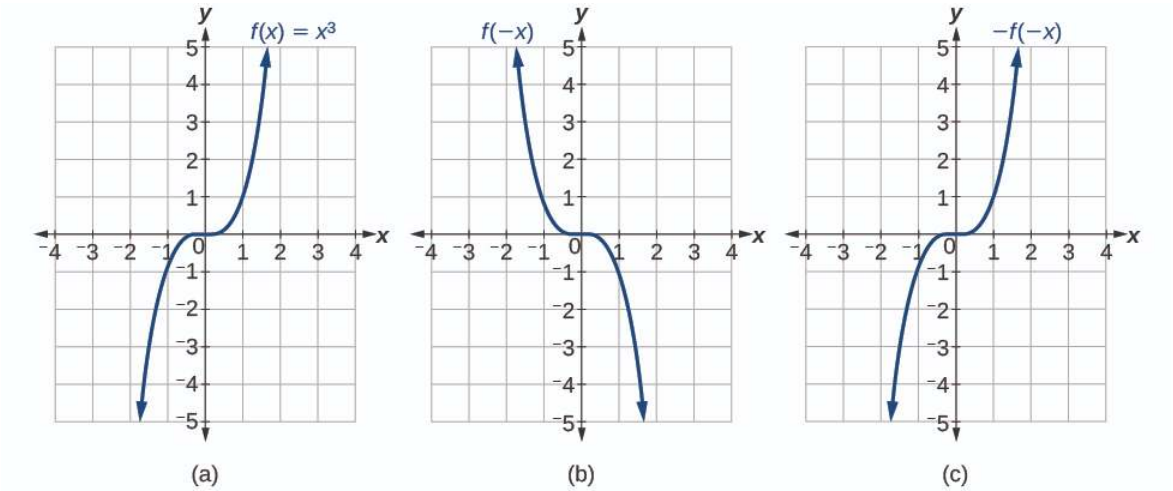


Figure 2.4.21: (a) The cubic toolkit function (b) Horizontal reflection of the cubic toolkit function (c) Horizontal and vertical reflections reproduce the original cubic function.

We say that these graphs are symmetric about the origin. A function with a graph that is symmetric about the origin is called an **odd function**.

Note: A function can be neither even nor odd if it does not exhibit either symmetry. For example, $f(x) = 2^x$ is neither even nor odd. Also, the only function that is both even and odd is the constant function $f(x) = 0$.

Definitions: Even and Odd Functions

A function is called an **even function** if for every input x

$f(x) = f(-x)$

The graph of an even function is symmetric about the y-axis.

A function is called an **odd function** if for every input x

$f(x) = -f(-x)$

The graph of an odd function is symmetric about the origin.

How To...

Given the formula for a function, determine if the function is even, odd, or neither.

1. Determine whether the function satisfies $f(x) = f(-x)$. If it does, it is even.
2. Determine whether the function satisfies $f(x) = -f(-x)$. If it does, it is odd.
3. If the function does not satisfy either rule, it is neither even nor odd.

Example 2.4.13: Determining whether a Function Is Even, Odd, or Neither

Is the function $f(x) = x^3 + 2x$ even, odd, or neither?

Solution

Without looking at a graph, we can determine whether the function is even or odd by finding formulas for the reflections and determining if they return us to the original function. Let's begin with the rule for even functions.

$$f(-x) = (-x)^3 + 2(-x) = -x^3 - 2x$$

This does not return us to the original function, so this function is not even. We can now test the rule for odd functions.

$$-f(-x) = -(-x^3 - 2x) = x^3 + 2x$$

Because $-f(-x) = f(x)$, this is an odd function.

Analysis

Consider the graph of f in Figure 2.4.22 Notice that the graph is symmetric about the origin. For every point (x, y) on the graph, the corresponding point $(-x, -y)$ is also on the graph. For example, $(1, 3)$ is on the graph of f, and the corresponding point $(-1, -3)$ is also on the graph.

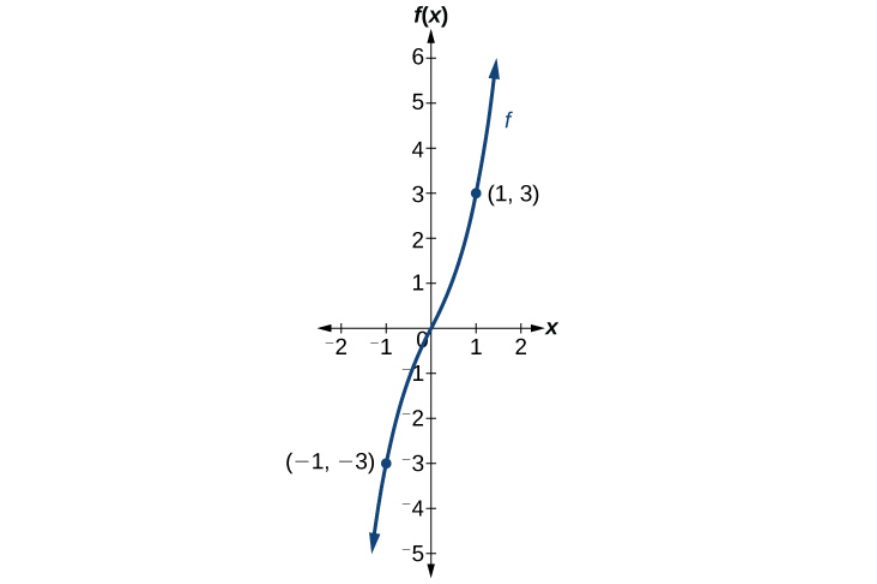


Figure 2.4.22: Graph of $f(x)$ with labeled points at $(1, 3)$ and $(-1, -3)$.

Exercise 2.4.8

Is the function $f(s) = s^4 + 3s^2 + 7$ even, odd, or neither?

Answer

even

Graphing Functions Using Stretches and Compressions

Adding a constant to the inputs or outputs of a function changed the position of a graph with respect to the axes, but it did not affect the shape of a graph. We now explore the effects of multiplying the inputs or outputs by some quantity.

We can transform the inside (input values) of a function or we can transform the outside (output values) of a function. Each change has a specific effect that can be seen graphically.

Vertical Stretches and Compressions

When we multiply a function by a positive constant, we get a function whose graph is stretched or compressed vertically in relation to the graph of the original function. If the constant is greater than 1, we get a **vertical stretch**; if the constant is between 0 and 1, we get a **vertical compression**. Figure 2.4.23 shows a function multiplied by constant factors 2 and 0.5 and the resulting vertical stretch and compression.

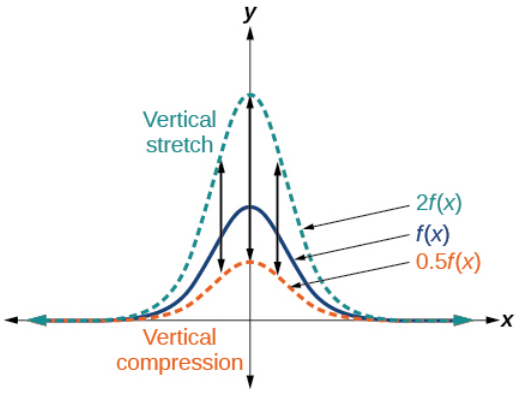


Figure 2.4.23: Vertical stretch and compression

Definitions: Vertical Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = af(x)$, where a is a constant, is a **vertical stretch** or **vertical compression** of the function $f(x)$.

- If $a > 1$, then the graph will be stretched.
- If $0 < a < 1$, then the graph will be compressed.
- If $a < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection.

How To...

Given a function, graph its vertical stretch.

a. Identify the value of a.
b. Multiply all range values by a
c. If $a > 1$, the graph is stretched by a factor of a.
d. If $0 < a < 1$, the graph is compressed by a factor of a.
e. If $a < 0$, the graph is either stretched or compressed and also reflected about the x-axis.

Example 1.5.14: Graphing a Vertical Stretch

A function $P(t)$ models the population of fruit flies. The graph is shown in Figure $2.4.24$

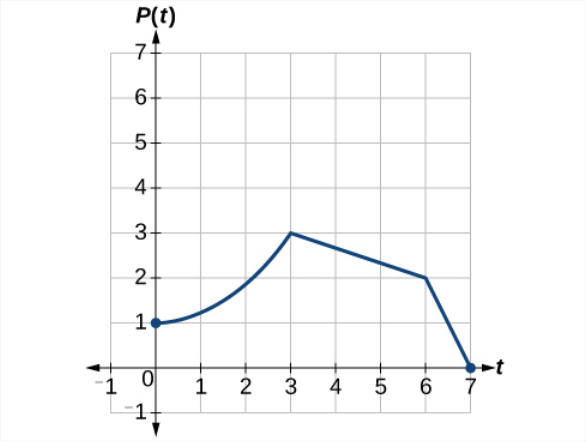


Figure 2.4.24: Graph to represent the growth of the population of fruit flies.

A scientist is comparing this population to another population, Q, whose growth follows the same pattern, but is twice as large. Sketch a graph of this population.

Solution

Because the population is always twice as large, the new population's output values are always twice the original function's output values. Graphically, this is shown in Figure $2.4.25$

If we choose four reference points, $(0, 1)$, $(3, 3)$, $(6, 2)$ and $(7, 0)$ we will multiply all of the outputs by 2.

The following shows where the new points for the new graph will be located.

$$(0, 1) \to (0, 2) \tag{2.4.2}$$
$$(3, 3) \to (3, 6) \tag{2.4.3}$$
$$(6, 2) \to (6, 4) \tag{2.4.4}$$
$$(7, 0) \to (7, 0) \tag{2.4.5}$$

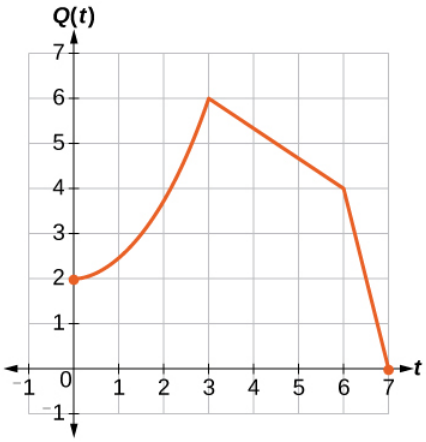


Figure 2.4.25: Graph of the population function doubled.

Symbolically, the relationship is written as

$$Q(t) = 2P(t)$$

This means that for any input t, the value of the function Q is twice the value of the function P. Notice that the effect on the graph is a vertical stretching of the graph, where every point doubles its distance from the horizontal axis. The input values, t, stay the same while the output values are twice as large as before.

How To...

Given a tabular function and assuming that the transformation is a vertical stretch or compression, create a table for a vertical compression.

1. Determine the value of a.
2. Multiply all of the output values by a.

Example 2.4.15: Finding a Vertical Compression of a Tabular Function

A function f is given as Table 2.4.12 Create a table for the function $g(x) = \frac{1}{2} f(x)$.

Table 2.4.12

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = \frac{1}{2} f(x)$ tells us that the output values of g are half of the output values of f with the same inputs. For example, we know that $f(4) = 3$. Then

$$g(4) = \frac{1}{2} f(4) = \frac{1}{2}(3) = \frac{3}{2}$$

We do the same for the other values to produce Table 2.4.13

Table 2.4.13

x	2	4	6	8
$g(x)$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{7}{2}$	$\frac{11}{2}$

Analysis

The result is that the function $g(x)$ has been compressed vertically by $\frac{1}{2}$. Each output value is divided in half, so the graph is half the original height.

Exercise 2.4.9

A function f is given as Table 2.4.14 Create a table for the function $g(x) = \frac{3}{4}f(x)$.

Table 2.4.14

x	2	4	6	8
$f(x)$	12	16	20	0

Answer

Table 2.4.15

x	2	4	6	8
$g(x)$	9	12	15	0

Example 2.4.16: Recognizing a Vertical Stretch

The graph in Figure 2.4.26 is a transformation of the toolkit function $f(x) = x^3$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

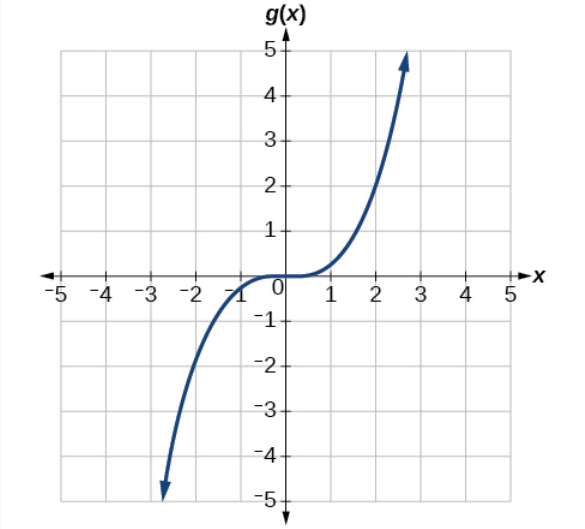


Figure 2.4.26: Graph of a transformation of $f(x) = x^3$

When trying to determine a vertical stretch or shift, it is helpful to look for a point on the graph that is relatively clear. In this graph, it appears that $g(2) = 2$. With the basic cubic function at the same input, $f(2) = 2^3 = 8$. Based on that, it appears that the outputs of g are $\frac{1}{4}$ the outputs of the function f because $g(2) = \frac{1}{4}f(2)$. From this we can fairly safely conclude that $g(x) = \frac{1}{4}f(x)$.

We can write a formula for g by using the definition of the function f.

$$g(x) = \frac{1}{4}f(x) = \frac{1}{4}x^3. \tag{2.4.6}$$

Exercise 2.4.1

Write the formula for the function that we get when we stretch the identity toolkit function by a factor of 3, and then shift it down by 2 units.

Answer

$g(x) = 3x - 2$

Horizontal Stretches and Compressions

Now we consider changes to the inside of a function. When we multiply a function's input by a positive constant, we get a function whose graph is stretched or compressed horizontally in relation to the graph of the original function. If the constant is between 0 and 1, we get a **horizontal stretch**; if the constant is greater than 1, we get a **horizontal compression** of the function.

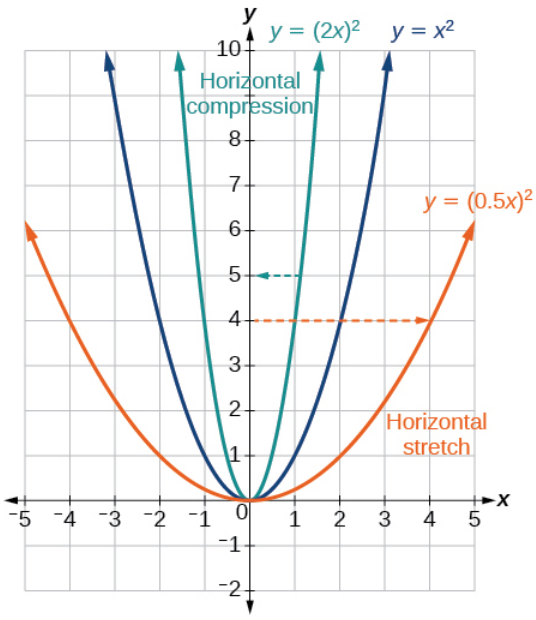


Figure 2.4.27: Graph of the vertical stretch and compression of x^2.

Given a function $y = f(x)$, the form $y = f(bx)$ results in a horizontal stretch or compression. Consider the function $y = x^2$. Observe Figure 2.4.27. The graph of $y = (0.5x)^2$ is a horizontal stretch of the graph of the function $y = x^2$ by a factor of 2. The graph of $y = (2x)^2$ is a horizontal compression of the graph of the function $y = x^2$ by a factor of 2.

Definitions: Horizontal Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = f(bx)$, where b is a constant, is a **horizontal stretch** or **horizontal compression** of the function $f(x)$.

- If $b > 1$, then the graph will be compressed by $\frac{1}{b}$.
- If $0 < b < 1$, then the graph will be stretched by $\frac{1}{b}$.
- If $b < 0$, then there will be combination of a horizontal stretch or compression with a horizontal reflection.

How To...

Given a description of a function, sketch a horizontal compression or stretch.

1. Write a formula to represent the function.
2. Set $g(x) = f(bx)$ where $b > 1$ for a compression or $0 < b < 1$ for a stretch.

Example 2.4.17: Graphing a Horizontal Compression

Suppose a scientist is comparing a population of fruit flies to a population that progresses through its lifespan twice as fast as the original population. In other words, this new population, R, will progress in 1 hour the same amount as the original population does in 2 hours, and in 2 hours, it will progress as much as the original population does in 4 hours. Sketch a graph of this population.

Solution

Symbolically, we could write

$$R(1) = P(2), \tag{2.4.7}$$
$$R(2) = P(4), \quad \text{and in general,} \tag{2.4.8}$$
$$R(t) = P(2t). \tag{2.4.9}$$

See Figure 2.4.28 for a graphical comparison of the original population and the compressed population.

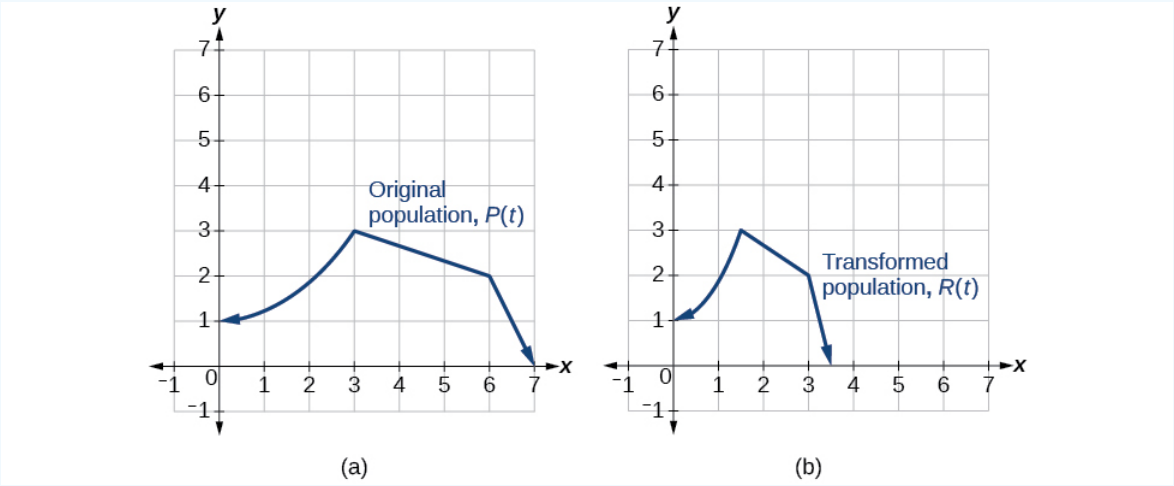


Figure 2.4.28: (a) Original population graph (b) Compressed population graph

Analysis

Note that the effect on the graph is a horizontal compression where all input values are half of their original distance from the vertical axis.

Example 2.4.18: Finding a Horizontal Stretch for a Tabular Function

A function $f(x)$ is given as Table 2.4.16. Create a table for the function $g(x) = f(\frac{1}{2}x)$.

Table 2.4.16

x	2	4	6	8
$f(x)$	1	3	7	11

The formula $g(x) = f(\frac{1}{2}x)$ tells us that the output values for g are the same as the output values for the function f at an input half the size. Notice that we do not have enough information to determine $g(2)$ because $g(2) = f(\frac{1}{2} \cdot 2) = f(1)$, and we do not have a value for $f(1)$ in our table. Our input values to g will need to be twice as large to get inputs for f that we can evaluate. For example, we can determine $g(4)$.

$$g(4) = f(\frac{1}{2} \cdot 4) = f(2) = 1 \tag{2.4.10}$$

We do the same for the other values to produce Table 2.4.17.

Table 2.4.17

x	4	8	12	16

$g(x)$	1	3	7	11

Figure 2.4.29 shows the graphs of both of these sets of points.

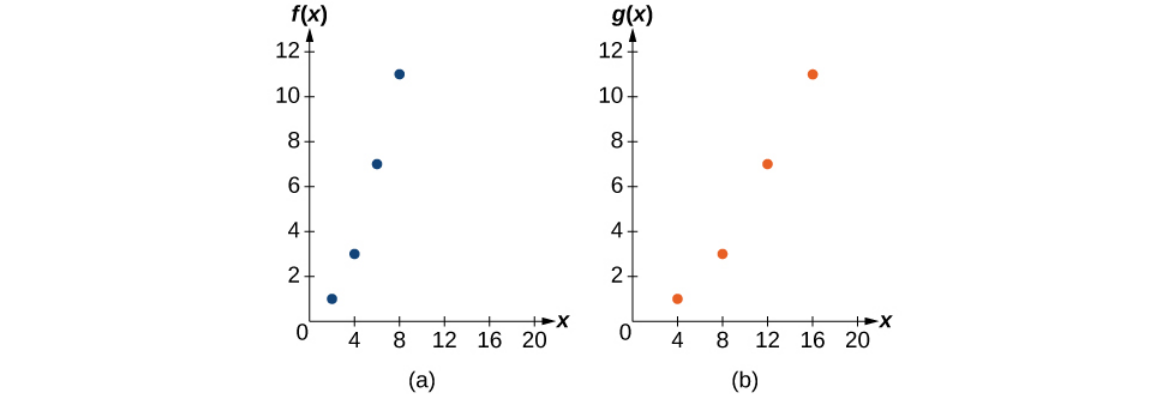


Figure 2.4.29: Graph of the previous table.

Analysis

Because each input value has been doubled, the result is that the function $g(x)$ has been stretched horizontally by a factor of 2.

Example 2.4.19: Recognizing a Horizontal Compression on a Graph

Relate the function $g(x)$ to $f(x)$ in Figure 2.4.30.

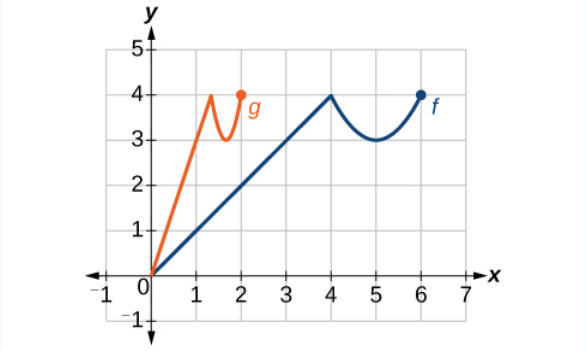


Figure 2.4.30: Graph of $f(x)$ being vertically compressed to $g(x)$.

Solution

The graph of $g(x)$ looks like the graph of $f(x)$ horizontally compressed. Because $f(x)$ ends at (6,4) and $g(x)$ ends at (2,4), we can see that the x-values have been compressed by $\frac{1}{3}$, because $6(\frac{1}{3}) = 2$. We might also notice that $g(2) = f(6)$ and $g(1) = f(3)$. Either way, we can describe this relationship as $g(x) = f(3x)$. This is a horizontal compression by $\frac{1}{3}$.

Analysis

Notice that the coefficient needed for a horizontal stretch or compression is the reciprocal of the stretch or compression. So to stretch the graph horizontally by a scale factor of 4, we need a coefficient of $\frac{1}{4}$ in our function: $f(\frac{1}{4}x)$. This means that the input values must be four times larger to produce the same result, requiring the input to be larger, causing the horizontal stretching.

Exercise 2.4.11

Write a formula for the toolkit square root function horizontally stretched by a factor of 3.

Answer

$g(x) = f(\frac{1}{3}x)$, so using the square root function we get $g(x) = \sqrt{\frac{1}{3}x}$

Performing a Sequence of Transformations

When combining transformations, it is very important to consider the order of the transformations. For example, vertically shifting by 3 and then vertically stretching by 2 does not create the same graph as vertically stretching by 2 and then vertically shifting by 3, because when we shift first, both the original function and the shift get stretched, while only the original function gets stretched when we stretch first.

When we see an expression such as $2f(x)+3$, which transformation should we start with? The answer here follows nicely from the order of operations. Given the output value of $f(x)$, we first multiply by 2, causing the vertical stretch, and then add 3, causing the vertical shift. In other words, multiplication before addition.

Horizontal transformations are a little trickier to think about. When we write $g(x) = f(2x+3)$, for example, we have to think about how the inputs to the function g relate to the inputs to the function f. Suppose we know $f(7) = 12$. What input to g would produce that output? In other words, what value of x will allow $g(x) = f(2x+3) = 12$? We would need $2x+3 = 7$. To solve for x, we would first subtract 3, resulting in a horizontal shift, and then divide by 2, causing a horizontal compression.

This format ends up being very difficult to work with, because it is usually much easier to horizontally stretch a graph before shifting. We can work around this by factoring inside the function.

$$f(bx+p) = f(b(x+\frac{p}{b}))$$

Let's work through an example.

$$f(x) = (2x+4)^2$$

We can factor out a 2.

$$f(x) = (2(x+2))^2$$

Now we can more clearly observe a horizontal shift to the left 2 units and a horizontal compression. Factoring in this way allows us to horizontally stretch first and then shift horizontally.

Combining Transformations

- When combining vertical transformations written in the form $af(x)+k$, first vertically stretch by a and then vertically shift by k.
- When combining horizontal transformations written in the form $f(bx+h)$, first horizontally shift by h and then horizontally stretch by $\frac{1}{b}$.
- When combining horizontal transformations written in the form $f(b(x+h))$, first horizontally stretch by $\frac{1}{b}$ and then horizontally shift by h.
- Horizontal and vertical transformations are independent. It does not matter whether horizontal or vertical transformations are performed first.

Example 2.4.20: Finding a Triple Transformation of a Tabular Function

Given Table 2.4.18 for the function $f(x)$, create a table of values for the function $g(x) = 2f(3x)+1$.

Table 2.4.18

x	6	12	18	24
$f(x)$	10	14	15	17

Solution

There are three steps to this transformation, and we will work from the inside out. Starting with the horizontal transformations, $f(3x)$ is a horizontal compression by $\frac{1}{3}$, which means we multiply each x-value by $\frac{1}{3}$. See Table 2.4.19

Table 2.4.19

x	2	4	6	8
$f(3x)$	10	14	15	17

Looking now to the vertical transformations, we start with the vertical stretch, which will multiply the output values by 2. We apply this to the previous transformation. See Table 2.4.20

Table 2.4.20

x	2	4	6	8
$2f(3x)$	20	28	30	34

Finally, we can apply the vertical shift, which will add 1 to all the output values. See Table 2.4.21

Table 2.4.21

x	2	4	6	8
$g(x)=2f(3x)+1+1$	21	29	31	35

Example 2.4.21: Finding a Triple Transformation of a Graph

Use the graph of $f(x)$ in Figure 2.4.31 to sketch a graph of $k(x)=f\left(\frac{1}{2}x+1\right)-3$.

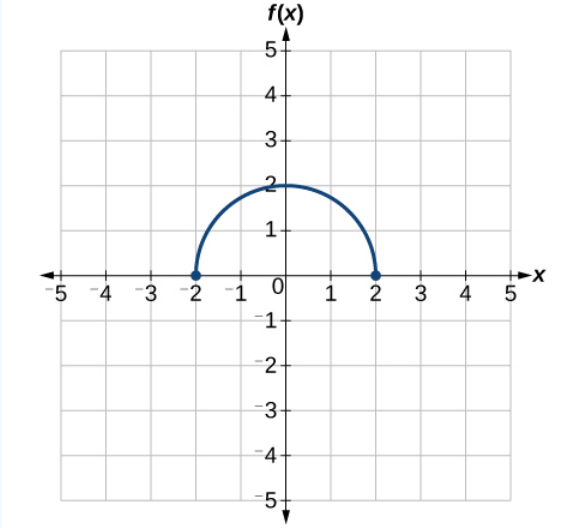


Figure 2.4.31: Graph of a half-circle.

To simplify, let's start by factoring out the inside of the function.

$$f\left(\frac{1}{2}x+1\right)-3=f\left(\frac{1}{2}(x+2)\right)-3 \tag{2.4.11}$$

By factoring the inside, we can first horizontally stretch by 2, as indicated by the $\frac{1}{2}$ on the inside of the function. Remember that twice the size of 0 is still 0, so the point $(0,2)$ remains at $(0,2)$ while the point $(2,0)$ will stretch to $(4,0)$. See Figure 2.4.32

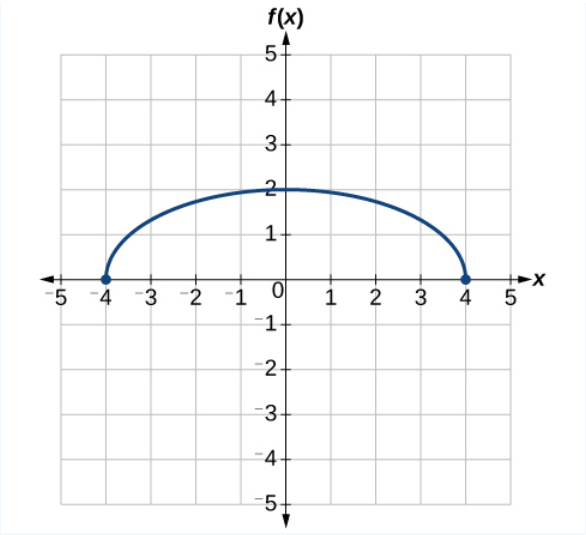


Figure 2.4.32: Graph of a vertically stretch half-circle.

Next, we horizontally shift left by 2 units, as indicated by $x+2$. See Figure 2.4.33

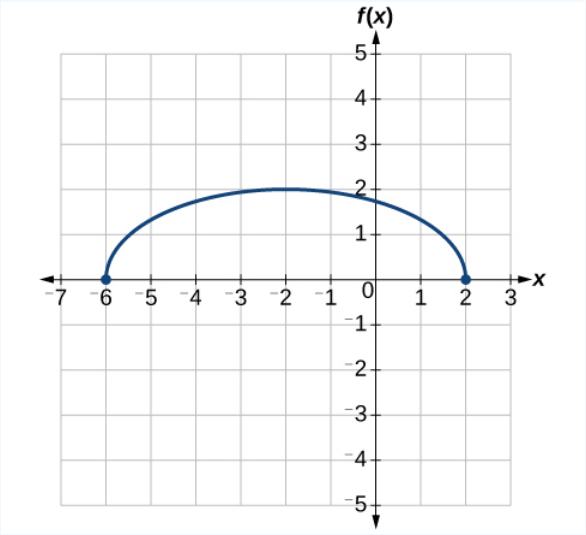


Figure 2.4.33: Graph of a vertically stretch and translated half-circle.

Last, we vertically shift down by 3 to complete our sketch, as indicated by the -3 on the outside of the function. See Figure 2.4.34

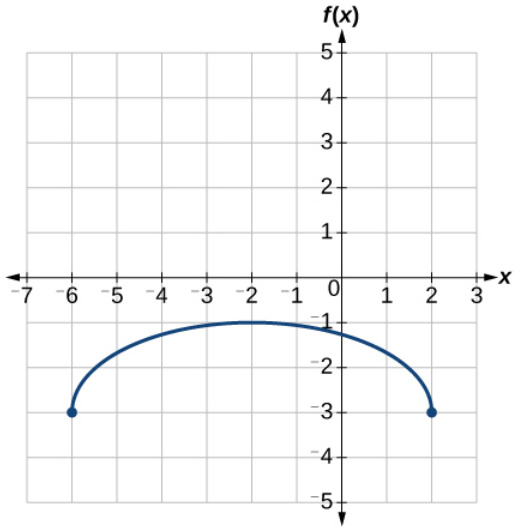


Figure 2.4.34: Graph of a vertically stretch and translated half-circle.

Key Equations

- Vertical shift $g(x) = f(x) + k$ (up for $k > 0$)
- Horizontal shift $g(x) = f(x - h)$ (right) for $h > 0$
- Vertical reflection $g(x) = -f(x)$
- Horizontal reflection $g(x) = f(-x)$
- Vertical stretch $g(x) = af(x)$ (a>0)
- Vertical compression $g(x) = af(x)$ (0<a<1)
- Horizontal stretch $g(x) = f(bx)(0 < b < 1)$
- Horizontal compression $g(x) = f(bx)$ (b>1)

Key Concepts

- A function can be shifted vertically by adding a constant to the output.
- A function can be shifted horizontally by adding a constant to the input.
- Relating the shift to the context of a problem makes it possible to compare and interpret vertical and horizontal shifts.
- Vertical and horizontal shifts are often combined.
- A vertical reflection reflects a graph about the x-axis. A graph can be reflected vertically by multiplying the output by –1.
- A horizontal reflection reflects a graph about the y-axis. A graph can be reflected horizontally by multiplying the input by –1.
- A graph can be reflected both vertically and horizontally. The order in which the reflections are applied does not affect the final graph.
- A function presented in tabular form can also be reflected by multiplying the values in the input and output rows or columns accordingly.
- A function presented as an equation can be reflected by applying transformations one at a time.
- Even functions are symmetric about the y-axis, whereas odd functions are symmetric about the origin.
- Even functions satisfy the condition $f(x) = f(-x)$.
- Odd functions satisfy the condition $f(x) = -f(-x)$.
- A function can be odd, even, or neither.
- A function can be compressed or stretched vertically by multiplying the output by a constant.
- A function can be compressed or stretched horizontally by multiplying the input by a constant.
- The order in which different transformations are applied does affect the final function. Both vertical and horizontal transformations must be applied in the order given. However, a vertical transformation may be combined with a horizontal transformation in any order.

Glossary

even function

a function whose graph is unchanged by horizontal reflection, $f(x) = f(-x)$, and is symmetric about the y-axis

horizontal compression
a transformation that compresses a function's graph horizontally, by multiplying the input by a constant b>1

horizontal reflection
a transformation that reflects a function's graph across the y-axis by multiplying the input by −1

horizontal shift
a transformation that shifts a function's graph left or right by adding a positive or negative constant to the input

horizontal stretch
a transformation that stretches a function's graph horizontally by multiplying the input by a constant 0<b<1

odd function
a function whose graph is unchanged by combined horizontal and vertical reflection, $f(x) = -f(-x)$, and is symmetric about the origin

vertical compression
a function transformation that compresses the function's graph vertically by multiplying the output by a constant 0<a<1

vertical reflection
a transformation that reflects a function's graph across the x-axis by multiplying the output by −1

vertical shift
a transformation that shifts a function's graph up or down by adding a positive or negative constant to the output

vertical stretch
a transformation that stretches a function's graph vertically by multiplying the output by a constant a>1

Supplemental Resources

This is a link to another open-source text. It includes nice animations to see shifts.

1.8: Transformations of Functions

Homework Exercises 2.4

WeBWorK Problems:



Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login



Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login

Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login

Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login

Written Problems:

Often when given a problem, we try to model the scenario using mathematics in the form of words, tables, graphs, and equations. One method we can employ is to adapt the basic graphs of the toolkit functions to build new models for a given scenario. There are systematic ways to alter functions to construct appropriate models for the problems we are trying to solve.

Verbal

1) When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal shift from a vertical shift?

Answer

A horizontal shift results when a constant is added to or subtracted from the input. A vertical shifts results when a constant is added to or subtracted from the output.

2) When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal stretch from a vertical stretch?

3) When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal compression from a vertical compression?

Answer

A horizontal compression results when a constant greater than 1 is multiplied by the input. A vertical compression results when a constant between 0 and 1 is multiplied by the output.

4) When examining the formula of a function that is the result of multiple transformations, how can you tell a reflection with respect to the x-axis from a reflection with respect to the y-axis?

5) How can you determine whether a function is odd or even from the formula of the function?

Answer

For a function f, substitute (−x) for (x) in $f(x)$. Simplify. If the resulting function is the same as the original function, $f(-x)=f(x)$, then the function is even. If the resulting function is the opposite of the original function, $f(-x)=-f(x)$, then the original function is odd. If the function is not the same or the opposite, then the function is neither odd nor even.

Algebraic

6) Write a formula for the function obtained when the graph of $f(x)=\sqrt{x}$ is shifted up 1 unit and to the left 2 units.

7) Write a formula for the function obtained when the graph of $f(x)=|x|$ is shifted down 3 units and to the right 1 unit.

Answer

$g(x)=|x-1|-3$

8) Write a formula for the function obtained when the graph of $f(x)=\dfrac{1}{x}$ is shifted down 4 units and to the right 3 units.

9) Write a formula for the function obtained when the graph of $f(x)=\dfrac{1}{x^2}$ is shifted up 2 units and to the left 4 units.

Answer

$g(x)=\dfrac{1}{(x+4)^2}+2$

For the exercises 10-19, describe how the graph of the function is a transformation of the graph of the original function f.

10) $y=f(x-49)$

11) $y=f(x+43)$

Answer

The graph of $f(x+43)$ is a horizontal shift to the left 43 units of the graph of f.

12) $y=f(x+3)$

13) $y=f(x-4)$

Answer

The graph of $f(x-4)$ is a horizontal shift to the right 4 units of the graph of f.

14) $y=f(x)+5$

15) $y = f(x) + 8$

Answer

The graph of $f(x) + 8$ is a vertical shift up 8 units of the graph of f.

16) $y = f(x) - 2$

17) $y = f(x) - 7$

Answer

The graph of $f(x) - 7$ is a vertical shift down 7 units of the graph of f.

18) $y = f(x - 2) + 3$

19) $y = f(x + 4) - 1$

Answer

The graph of $f(x + 4) - 1$ is a horizontal shift to the left 4 units and a vertical shift down 1 unit of the graph of f.

For the exercises 20-23, determine the interval(s) on which the function is increasing and decreasing.

20) $f(x) = 4(x + 1)^2 - 5$

21) $g(x) = 5(x + 3)^2 - 2$

Answer

decreasing on $(-\infty, -3)$ and increasing on $(-3, \infty)$

22) $a(x) = \sqrt{-x + 4}$

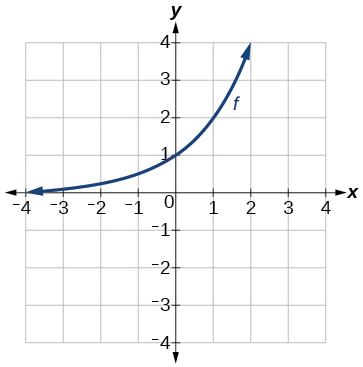
23) $k(x) = -3\sqrt{x} - 1$

Answer

decreasing on $(0, \infty)$

Graphical

For the exercises 24-26, use the graph of $f(x) = 2^x$ shown in the Figure below to sketch a graph of each transformation of $f(x)$.

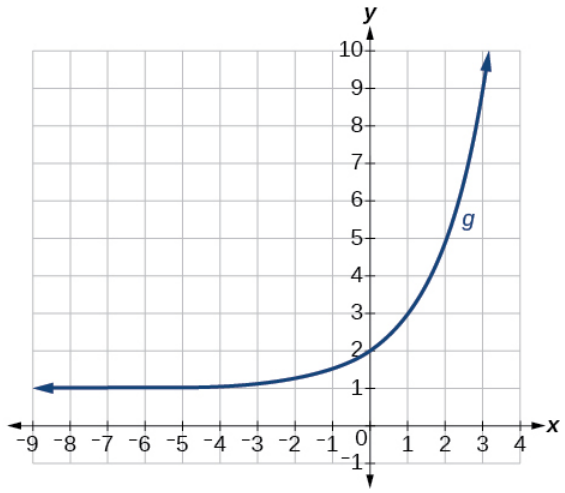


Graph of $f(x)$.

24) $h(x) = 2^x - 3$

25) $g(x) = 2^x + 1$

Answer



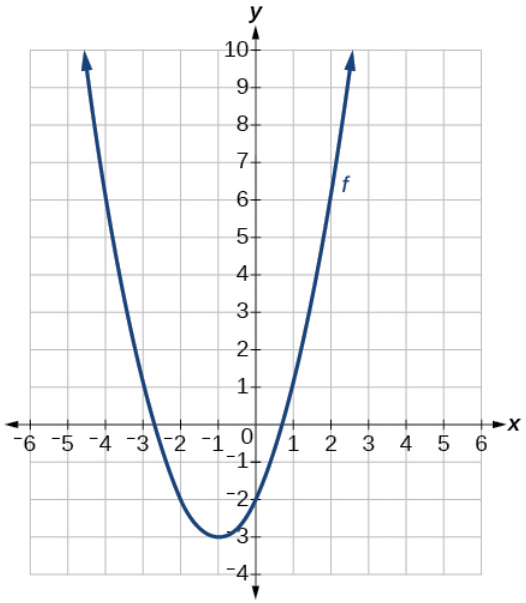
Graph of $g(x)$

26) $w(x) = 2^{x-1}$

For the exercises 27-30, sketch a graph of the function as a transformation of the graph of one of the toolkit functions.

27) $f(t) = (t+1)^2 - 3$

Answer

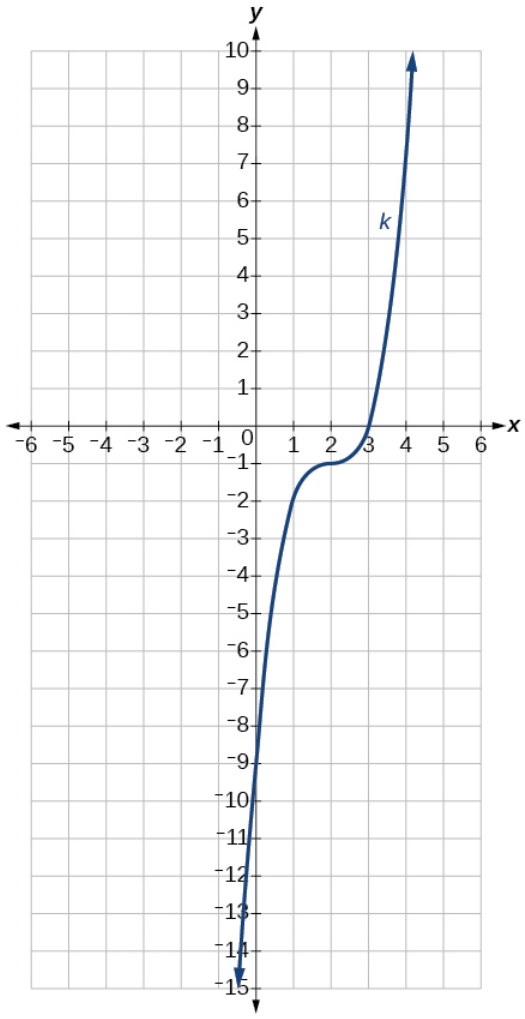


Graph of $f(t)$.

28) $h(x) = |x-1| + 4$

29) $k(x) = (x-2)^3 - 1$

Answer



Graph of $k(x)$.

30) $m(t) = 3 + \sqrt{t+2}$

Numeric

31) Tabular representations for the functions f, g, and h are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

x	-2	-1	0	1	2
$f(x)$	-2	-1	-3	1	2

x	-1	0	1	2	3
$g(x)$	-2	-1	-3	1	2

x	-2	-1	0	1	2
$h(x)$	-1	0	-2	2	3

Answer

$g(x) = f(x-1)$, $h(x) = f(x)+1$

32) Tabular representations for the functions f, g, and h are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

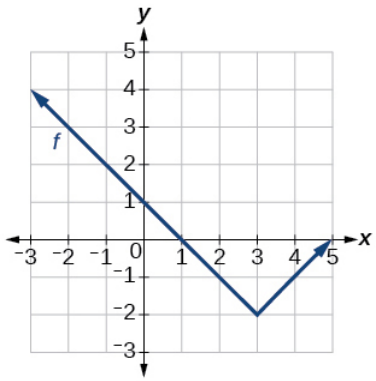
x	-2	-1	0	1	2
$f(x)$	-1	-3	4	2	1

x	-3	-2	-1	0	1
$g(x)$	-1	-3	4	2	1

x	-2	-1	0	1	2
$h(x)$	-2	-4	3	1	0

For the exercises 33-40, write an equation for each graphed function by using transformations of the graphs of one of the toolkit functions.

33)

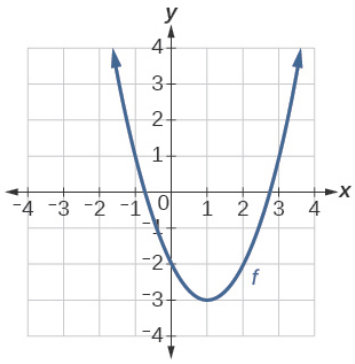


Graph of an absolute function.

Answer

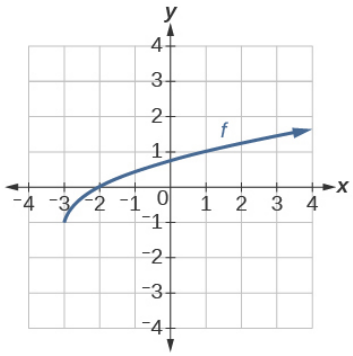
$f(x) = |x - 3| - 2$

34)



Graph of a parabola.

35)

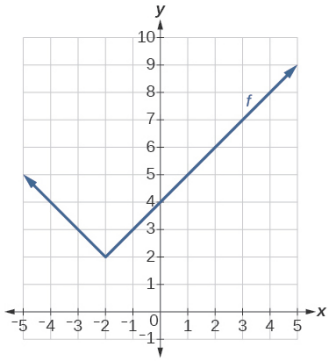


Graph of a square root function.

Answer

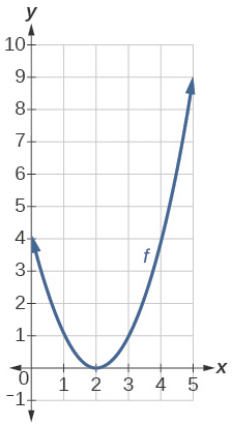
$f(x) = \sqrt{x+3} - 1$

36)



Graph of an absolute function.

37)

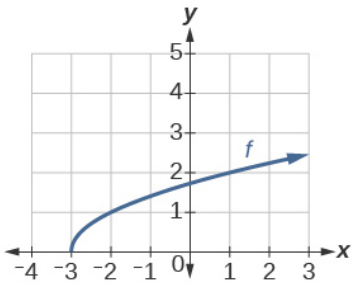


Graph of a parabola

Answer

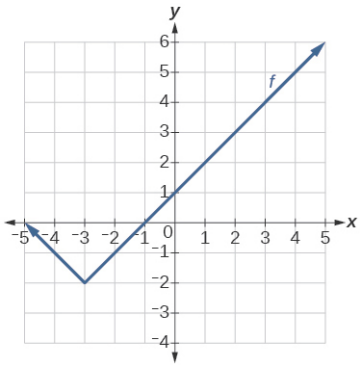
$f(x) = (x-2)^2$

38)



Graph of a square root function.

39)

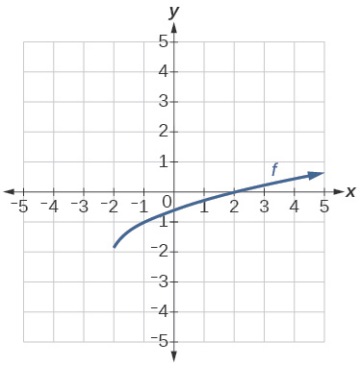


Graph of an absolute function.

Answer

$f(x) = |x + 3| - 2$

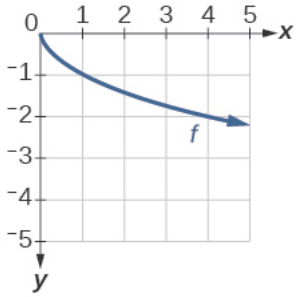
40)



Graph of a square root function.

For the exercises 41-42, use the graphs of transformations of the square root function to find a formula for each of the functions.

41)

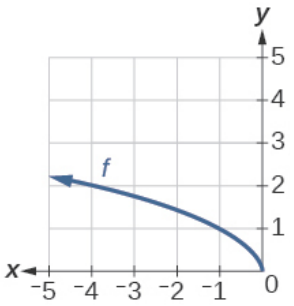


Graph of a square root function.

Answer

$f(x) = -\sqrt{x}$

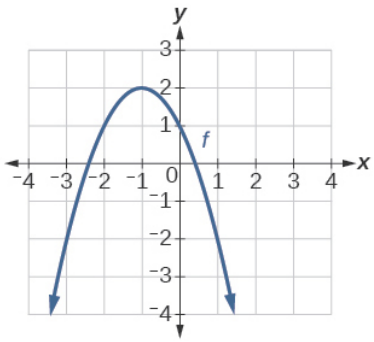
42)



Graph of a square root function.

For the exercises 43-46, use the graphs of the transformed toolkit functions to write a formula for each of the resulting functions.

43)



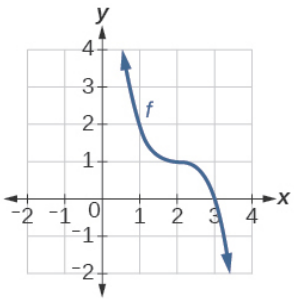
Graph of a parabola.

Answer

$f(x) = -(x+1)^2 + 2$

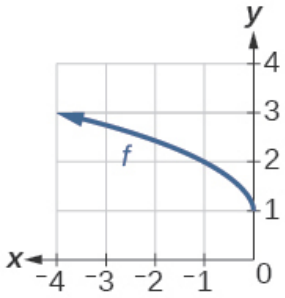
44)

https://math.libretexts.org/@go/page/106340



Graph of a cubic function.

45)

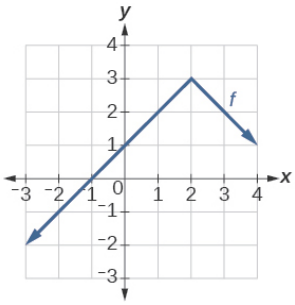


Graph of a square root function.

Answer

$f(x) = \sqrt{-x} + 1$

46)



Graph of an absolute function.

For the exercises 47-52, determine whether the function is odd, even, or neither.

47) $f(x) = 3x^4$

Answer

even

48) $g(x) = \sqrt{x}$

49) $h(x) = \dfrac{1}{x} + 3x$

Answer

odd

50) $f(x) = (x-2)^2$

51) $g(x) = 2x^4$

Answer

even

52) $h(x) = 2x - x^3$

For the exercises 53-62, describe how the graph of each function is a transformation of the graph of the original function f.

53) $g(x) = -f(x)$

Answer

The graph of g is a vertical reflection (across the x-axis) of the graph of f.

54) $g(x) = f(-x)$

55) $g(x) = 4f(x)$

Answer

The graph of g is a vertical stretch by a factor of 4 of the graph of f.

56) $g(x) = 6f(x)$

57) $g(x) = f(5x)$

Answer

The graph of g is a horizontal compression by a factor of 15 of the graph of f.

58) $g(x) = f(2x)$

59) $g(x) = f\left(\frac{1}{3}x\right)$

Answer

The graph of g is a horizontal stretch by a factor of 3 of the graph of f.

60) $g(x) = f\left(\frac{1}{5}x\right)$

61) $g(x) = 3f(-x)$

Answer

The graph of g is a horizontal reflection across the y-axis and a vertical stretch by a factor of 3 of the graph of f.

62) $g(x) = -f(3x)$

For the exercises 63-68, write a formula for the function g that results when the graph of a given toolkit function is transformed as described.

63) The graph of $f(x) = |x|$ is reflected over the y-axis and horizontally compressed by a factor of $\frac{1}{4}$.

Answer

$g(x) = |-4x|$

64) The graph of $f(x) = \sqrt{x}$ is reflected over the x-axis and horizontally stretched by a factor of 2.

65) The graph of $f(x) = \frac{1}{x^2}$ is vertically compressed by a factor of $\frac{1}{3}$, then shifted to the left 2 units and down 3 units.

Answer

$$g(x) = \frac{1}{3(x+2)^2} - 3$$

66) The graph of $f(x) = \frac{1}{x}$ is vertically stretched by a factor of 8, then shifted to the right 4 units and up 2 units.

67) The graph of $f(x) = x^2$ is vertically compressed by a factor of $\frac{1}{2}$, then shifted to the right 5 units and up 1 unit.

Answer

$$g(x) = \frac{1}{2}(x-5)^2 + 1$$

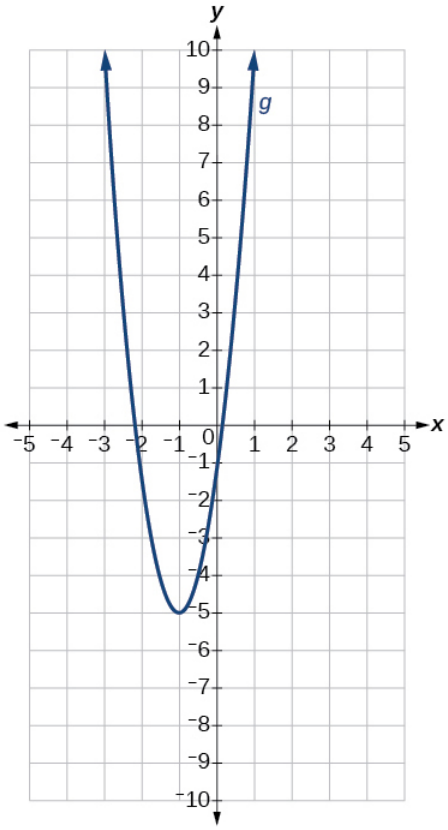
68) The graph of $f(x) = x^2$ is horizontally stretched by a factor of 3, then shifted to the left 4 units and down 3 units.

For the exercises 69-77, describe how the formula is a transformation of a toolkit function. Then sketch a graph of the transformation.

69) $g(x) = 4(x+1)^2 - 5$

Answer

The graph of the function $f(x) = x^2$ is shifted to the left 1 unit, stretched vertically by a factor of 4, and shifted down 5 units.



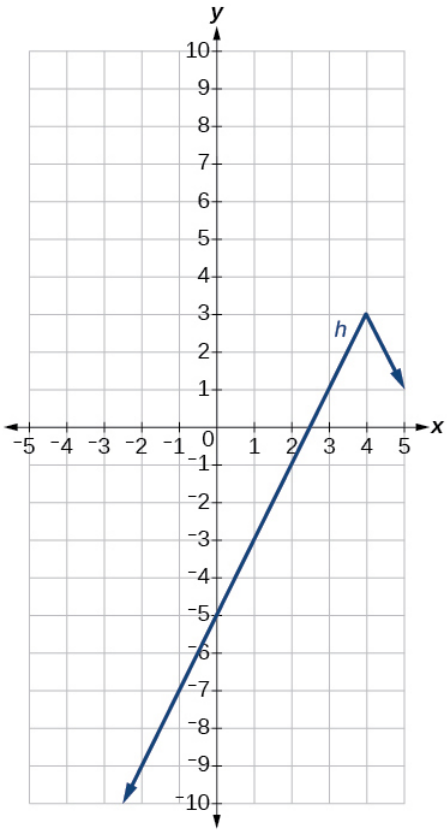
Graph of a parabola.

70) $g(x) = 5(x+3)^2 - 2$

71) $h(x) = -2|x-4| + 3$

Answer

The graph of $f(x) = |x|$ is stretched vertically by a factor of 2, shifted horizontally 4 units to the right, reflected across the horizontal axis, and then shifted vertically 3 units up.



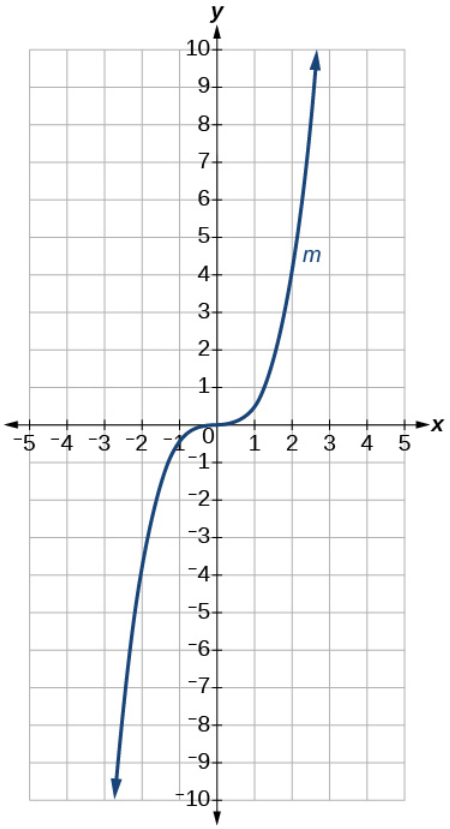
Graph of an absolute value function.

72) $k(x) = -3\sqrt{x} - 1$

73) $m(x) = \frac{1}{2}x^3$

Answer

The graph of the function $f(x) = x^3$ is compressed vertically by a factor of $\frac{1}{2}$.



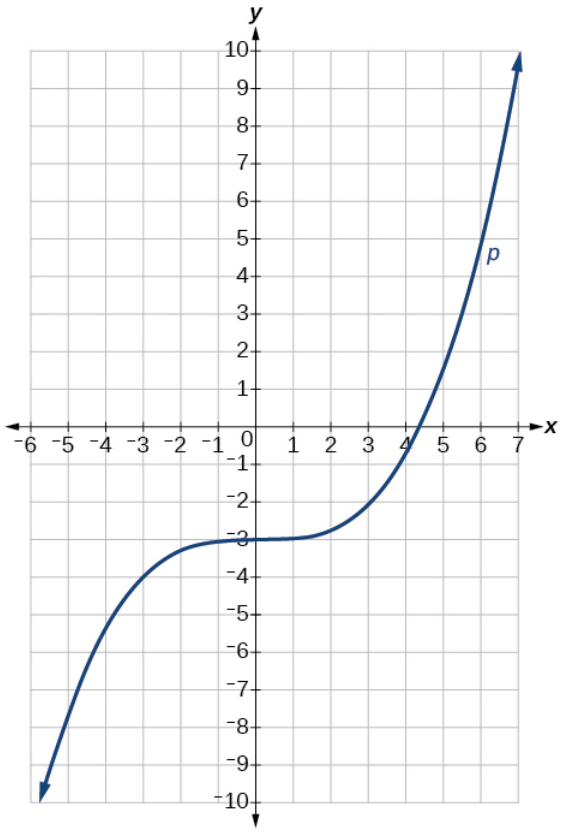
Graph of a cubic function.

74) $n(x) = \frac{1}{3}|x-2|$

75) $p(x) = (\frac{1}{3}x)^3 - 3$

Answer

The graph of the function is stretched horizontally by a factor of 3 and then shifted vertically downward by 3 units.



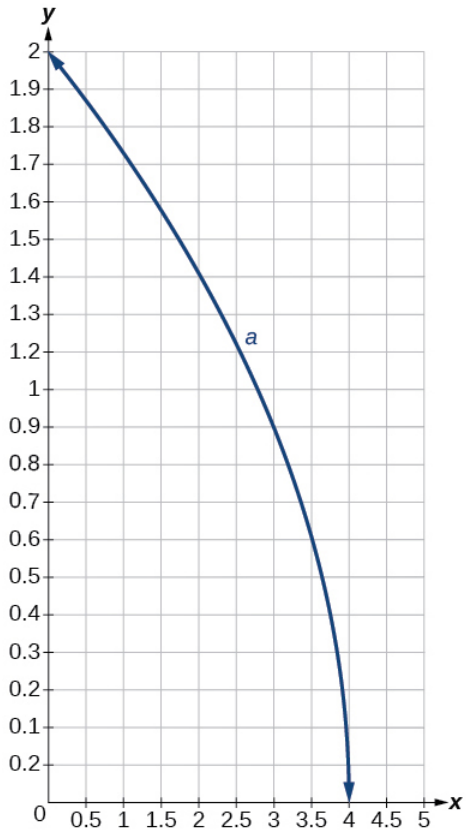
Graph of a cubic function.

76) $q(x) = (\frac{1}{4}x)^3 + 1$

77) $a(x) = \sqrt{-x+4}$

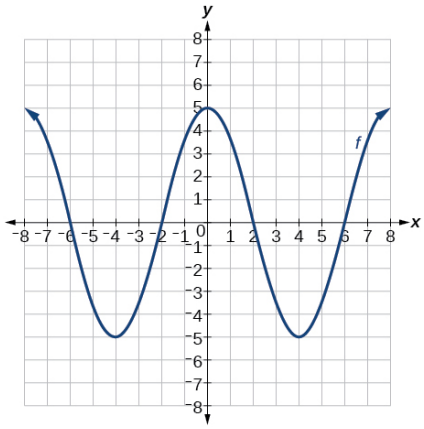
Answer

The graph of $f(x) = \sqrt{x}$ is shifted right 4 units and then reflected across the vertical line $x = 4$.



Graph of a square root function.

For the exercises 78-81, use the graph in the Figure below to sketch the given transformations.

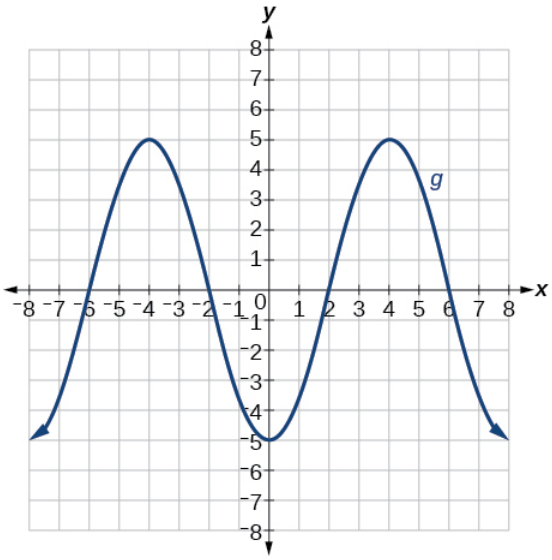


Graph of a polynomial.

78) $g(x) = f(x) - 2$

79) $g(x) = -f(x)$

Answer

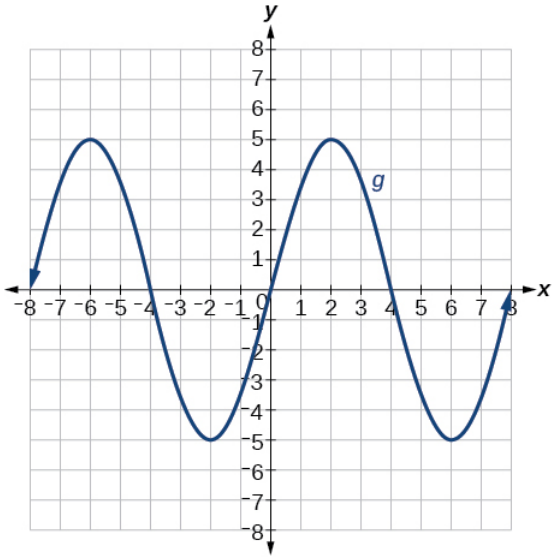


Graph of a polynomial.

80) $g(x) = f(x+1)$

81) $g(x) = f(x-2)$

Answer



Graph of a polynomial.

This page titled 2.4: Transformation of Functions is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **1.5: Transformation of Functions** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/precalculus.
- **1.E: Functions (Exercises)** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/precalculus.

CHAPTER OVERVIEW

3: Derivative Essentials

3.1: How do we Measure Velocity?
3.2: The Notion of Limit
3.3: The Derivative of a Function at a Point
3.4: The Derivative Function
3.5: Limits, Continuity, and Differentiability
3.6: The Mean Value Theorem

3: Derivative Essentials is shared under a not declared license and was authored, remixed, and/or curated by LibreTexts.

3.1: How do we Measure Velocity?

Preview Video

This optional summary video from the textbook author might be helpful to use as a preview. Other, more detailed, supplemental videos for this section are posted at the end of the text.



Learning Objectives

In this section, we strive to understand the ideas generated by the following important questions:

- How is the average velocity of a moving object connected to the values of its position function?
- How do we interpret the average velocity of an object geometrically with regard to the graph of its position function?
- How is the notion of instantaneous velocity connected to average velocity?

Calculus can be viewed broadly as the study of change. A natural and important question to ask about any changing quantity is "how fast is the quantity changing?" It turns out that in order to make the answer to this question precise, substantial mathematics is required.

We begin with a familiar problem: a ball being tossed straight up in the air from an initial height. From this elementary scenario, we will ask questions about how the ball is moving. These questions will lead us to begin investigating ideas that will be central throughout our study of differential calculus and that have wide-ranging consequences. In a great deal of our thinking about calculus, we will be well-served by remembering this first example and asking ourselves how the various (sometimes abstract) ideas we are considering are related to the simple act of tossing a ball straight up in the air.

Preview Activity 3.1.1:

Suppose that the height s of a ball (in feet) at time t (in seconds) is given by the formula $s(t) = 64 - 16(t-1)^2$.

a. Construct an accurate graph of $y = s(t)$ on the time interval $0 \le t \le 3$. Label at least six distinct points on the graph, including the three points that correspond to when the ball was released, when the ball reaches its highest point, and when the ball lands.

b. In everyday language, describe the behavior of the ball on the time interval $0 < t < 1$ and on time interval $1 < t < 3$. What occurs at the instant $t = 1$?

c. Consider the expression $AV_{[0.5,1]} = \dfrac{s(1) - s(0.5)}{1 - 0.5}$.

Compute the value of $AV_{[0.5,1]}$. What does this value measure geometrically? What does this value measure physically? In particular, what are the units on $AV_{[0.5,1]}$?

Position and Average Velocity

Any moving object has a *position* that can be considered a function of *time*. When this motion is along a straight line, the position is given by a single variable, and we usually let this position be denoted by $s(t)$, which reflects the fact that position is a function of time. For example, we might view $s(t)$ as telling the mile marker of a car traveling on a straight highway at time t in hours; similarly, the function s described in Preview Activity 1.1.1 is a position function, where position is measured vertically relative to the ground.

Not only does such a moving object have a position associated with its motion, but on any time interval, the object has an *average velocity*. Think, for example, about driving from one location to another: the vehicle travels some number of miles over a certain time interval (measured in hours), from which we can compute the vehicle's average velocity. In this situation, average velocity is the number of miles traveled divided by the time elapsed, which of course is given in *miles per hour*. Similarly, the calculation of $AV_{[0.5,1]}$ in Preview Activity 1.1.1 found the average velocity of the ball on the time interval $[0.5, 1]$, measured in feet per second.

In general, we make the following definition:

Average Velocity

For an object moving in a straight line whose position at time t is given by the function $s(t)$, the average velocity of the object on the interval from $t = a$ to $t = b$, denoted $AV_{[a,b]}$, is given by the formula

$$AV_{[a,b]} = \frac{s(b) - s(a)}{b - a}$$

Note well: the units on $AV_{[a,b]}$ are "units of s per unit of t," such as "miles per hour" or "feet per second."

Activity 3.1.2:

The following questions concern the position function given by $s(t) = 64 - 16(t - 1)^2$, which is the same function considered in Preview Activity 3.1.1

a. Compute the average velocity of the ball on each of the following time intervals: $[0.4, 0.8], [0.7, 0.8], [0.79, 0.8], [0.799, 0.8], [0.8, 1.2], [0.8, 0.9], [0.8, 0.81], [0.8, 0.801]$. Include units for each value.

b. On the provided graph in Figure 3.1.1, sketch the line that passes through the points $A = (0.4, s(0.4))$ and $B = (0.8, s(0.8))$. What is the meaning of the slope of this line? In light of this meaning, what is a geometric way to interpret each of the values computed in the preceding question?

c. Use a graphing utility to plot the graph of $s(t) = 64 - 16(t - 1)^2$ on an interval containing the value $t = 0.8$. Then, zoom in repeatedly on the point $(0.8, s(0.8))$. What do you observe about how the graph appears as you view it more and more closely?

d. What do you conjecture is the velocity of the ball at the instant $t = 0.8$? Why?

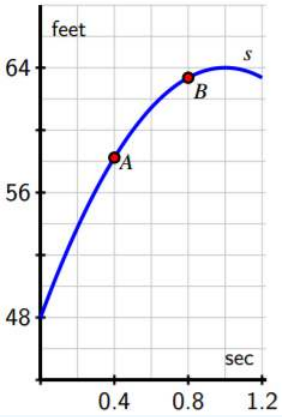


***Figure* 3.1.1**: *A partial plot of* $s(t) = 64 - 16(t-1)^2$.

Instantaneous Velocity

Whether driving a car, riding a bike, or throwing a ball, we have an intuitive sense that any moving object has a velocity at any given moment – a number that measures how fast the 4 object is moving *right now*. For instance, a car's speedometer tells the driver what appears to be the car's velocity at any given instant. In fact, the posted velocity on a speedometer is really an average velocity that is computed over a very small time interval (by computing how many revolutions the tires have undergone to compute distance traveled), since velocity fundamentally comes from considering a change in position divided by a change in time. But if we let the time interval over which average velocity is computed become shorter and shorter, then we can progress from average velocity to *instantaneous* velocity.

Informally, we define the i*nstantaneous velocity* of a moving object at time $t = a$ to be the value that the average velocity approaches as we take smaller and smaller intervals of time containing $t = a$ to compute the average velocity. We will develop a more formal definition of this momentarily, one that will end up being the foundation of much of our work in first semester calculus. For now, it is fine to think of instantaneous velocity this way: take average velocities on smaller and smaller time intervals, and if those average velocities approach a single number, then that number will be the instantaneous velocity at that point.

Activity 3.1.3:

Each of the following questions concern $s(t) = 64 - 16(t-1)^2$, the position function from Preview Activity 3.1.1.

a. Compute the average velocity of the ball on the time interval $[1.5, 2]$. What is different between this value and the average velocity on the interval $[0, 0.5]$?
b. Use appropriate computing technology to estimate the instantaneous velocity of the ball at $t = 1.5$. Likewise, estimate the instantaneous velocity of the ball at $t = 2$. Which value is greater?
c. How is the sign of the instantaneous velocity of the ball related to its behavior at a given point in time? That is, what does positive instantaneous velocity tell you the ball is doing? Negative instantaneous velocity?
d. Without doing any computations, what do you expect to be the instantaneous velocity of the ball at $t = 1$? Why?

At this point we have started to see a close connection between average velocity and instantaneous velocity, as well as how each is connected not only to the physical behavior of the moving object but also to the geometric behavior of the graph of the position function. In order to make the link between average and instantaneous velocity more formal, we will introduce the notion of *limit* in Section 1.2. As a preview of that concept, we look at a way to consider the limiting value of average velocity through the introduction of a parameter. Note that if we desire to know the instantaneous velocity at $t = a$ of a moving object with position function s, we are interested in computing average velocities on the interval $[a, b]$ for smaller and smaller intervals. One way to visualize this is to think of the value b as being $b = a + h$, where h is a small number that is allowed to vary. Thus, we observe that the average velocity of the object on the interval $[a, a + h]$ is

$$AV_{[a,a+h]} = \frac{s(a+h) - s(a)}{h},$$

with the denominator being simply h because $(a+h)-a=h$. Initially, it is fine to think of h being a small positive real number; but it is important to note that we allow h to be a small negative number, too, as this enables us to investigate the average velocity of the moving object on intervals prior to $t=a$, as well as following $t=a$. When $h<0$, $AV_{[a,a+h]}$ measures the average velocity on the interval $[a+h,a]$.

To find the instantaneous velocity at $t=a$, we investigate what happens as the value of h approaches zero.

Example 3.1.2: Computing instantaneous velocity for a falling ball.

The position function for a falling ball is given by $s(t)=16-16t^2$ (where s is measured in feet and t in seconds).

a. Find an expression for the average velocity of the ball on a time interval of the form $[0.5,0.5+h]$ where $-0.5<h<0.5$ and $h\neq 0$.

b. Use this expression to compute the average velocity on $[0.5,0.75]$and $[0.4,0.5]$.

c. Make a conjecture about the instantaneous velocuty at $t=0.5$.

Answer

a. We make the assumptions that $-0.5<h<0.5$ and $h\neq 0$ because h cannot be zero (otherwise there is no interval on which to compute average velocity) and because the function only makes sense on the time interval $0\leq t\leq 1$, as this is the duration of time during which the ball is falling. We want to compute and simplify

$$AV_{[0.5,0.5+h]}=\frac{s(0.5+h)-s(0.5)}{(0.5+h)-0.5}.$$

We start by finding $s(0.5+h)$. To do so, we follow the rule that defines the function s.

$$\begin{aligned} s(0.5+h) &= 16-16(0.5+h)^2 \\ &= 16-16\left(0.25+h+h^2\right) \\ &= 16-4-16h-16h^2 \\ &= 12-16h-16h^2 \end{aligned}$$

Now, returning to our computation of the average velocity, we find that

$$\begin{aligned} AV_{[0.5,0.5+h]} &= \frac{s(0.5+h)-s(0.5)}{(0.5+h)-0.5} \\ &= \frac{\left(12-16h-16h^2\right)-\left(16-16(0.5)^2\right)}{0.5+h-0.5} \\ &= \frac{12-16h-16h^2-12}{h} \\ &= \frac{-16h-16h^2}{h}. \end{aligned}$$

At this point, we note two things: first, the expression for average velocity clearly depends on h, which it must, since as h changes the average velocity will change. Further, we note that since h can never equal zero, we may remove the common factor of h from the numerator and denominator. It follows that

$$AV_{[0.5,0.5+h]}=-16-16h.$$

b. From this expression we can compute the average for any small positive or negative value of h. For instance, to obtain the average velocity on $[0.5,0.75]$, we let $h=0.25$, and the average velocity is $-16-16(0.25)=-20\text{ft}/\text{sec}$. To get the average velocity on $[0.4,0.5]$, we let $h=-0.1$, and compute the average velocity as

$$-16-16(-0.1)=-14.4\text{ft}/\text{sec}$$

c. We can even explore what happens to $AV_{[0.5,0.5+h]}$ as h gets closer and closer to zero. As h approaches zero, $-16h$ will also approach zero, so it appears that the instantaneous velocity of the ball at $t=0.5$ should be $-16\text{ft}/\text{sec}$.

Activity 3.1.4:

For the function given by $s(t) = 64 - 16(t-1)^2$ from Preview Activity 1.1.1, find the most simplified expression you can for the average velocity of the ball on the interval $[2, 2+h]$. Use your result to compute the average velocity on $[1.5, 2]$ and to estimate the instantaneous velocity at $t = 2$. Finally, compare your earlier work in Activity 3.1.1

Summary

In this section, we encountered the following important ideas:

- For an object moving in a straight line with position function $s(t)$, the *average velocity of the object on the interval from* $t = a$ *to* $t = b$, denoted $AV_{[a,b]}$, is given by the formula
$$AV_{[a,b]} = \frac{s(b) - s(a)}{b - a}.$$
- The average velocity on $[a, b]$ can be viewed geometrically as the slope of the line between the points $(a, s(a))$ and $(b, s(b))$ on the graph of $y = s(t)$, as shown in Figure 3.1.2.

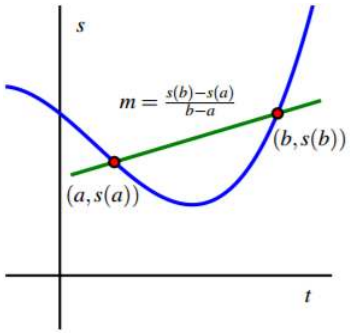


***Figure* 3.1.3:** *The graph of position function s together with the line through* $(a, s(a))$ *and* $(b, s(b))$ *whose slope is* $m = \frac{s(b) - s(a)}{b - a}$. *The line's slope is the average rate of change of* s *on the interval* $[a, b]$.

- Given a moving object whose position at time t is given by a function s, the average velocity of the object on the time interval $[a, b]$ is given by $AV_{[a,b]} = \frac{s(b) - s(a)}{b - a}$. Viewing the interval $[a, b]$ as having the form $[a, a+h]$, we equivalently compute average velocity by the formula $AV_{[a,a+h]} = \frac{s(a+h) - s(a)}{h}$.
- The instantaneous velocity of a moving object at a fixed time is estimated by considering average velocities on shorter and shorter time intervals that contain the instant of interest.

Supplemental videos





Homework Exercises 3.1

WeBWorK Problems:

1.

2.

3.

4.

5.

6.

Written Problems:

1. If the figure below does not show up, please put the graph in a utility like Desmos.com. Make sure you are in radian mode. You may also use the app: https://www.desmos.com/calculator/tspc2wuxdu

A bungee jumper dives from a tower at time $t = 0$. Her height h (measured in feet) at time t (in seconds) is given by the graph in Figure 1.1.4. In this problem, you may base your answers on estimates from the graph or use the fact that the jumper's height function is given by $s(t) = 100\cos(0.75t) \cdot e^{-0.2t} + 100$.

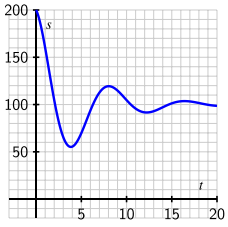


Figure 1.1.4. A bungee jumper's height function.

a. What is the change in vertical position of the bungee jumper between $t = 0$ and $t = 15$?
b. Estimate the jumper's average velocity on each of the following time intervals: $[0, 15], [0, 2], [1, 6]$, and $[8, 10]$. Include units on your answers.
c. On what time interval(s) do you think the bungee jumper achieves her greatest average velocity? Why?

d. Estimate the jumper's instantaneous velocity at $t = 5$. Show your work and explain your reasoning, and include units on your answer.
e. Among the average and instantaneous velocities you computed in earlier questions, which are positive and which are negative? What does negative velocity indicate?

2.

According to the U.S. census, the population of the city of Grand Rapids, MI, was 181,843 in 1980; 189,126 in 1990; and 197,800 in 2000.

a. Between 1980 and 2000, by how many people did the population of Grand Rapids grow?
b. In an average year between 1980 and 2000, by how many people did the population of Grand Rapids grow?
c. Just like we can find the average velocity of a moving body by computing change in position over change in time, we can compute the average rate of change of any function f. In particular, the *average rate of change* of a function f over an interval $[a, b]$ is the quotient

$$\frac{f(b) - f(a)}{b - a}.$$

What does the quantity $\frac{f(b)-f(a)}{b-a}$ measure on the graph of $y = f(x)$ over the interval $[a, b]$?
d. Let $P(t)$ represent the population of Grand Rapids at time t, where t is measured in years from January 1, 1980. What is the average rate of change of P on the interval $t = 0$ to $t = 20$? What are the units on this quantity?
e. If we assume the population of Grand Rapids is growing at a rate of approximately 4% per decade, we can model the population function with the formula

$$P(t) = 181843(1.04)^{t/10}.$$

Use this formula to compute the average rate of change of the population on the intervals $[5, 10], [5, 9], [5, 8], [5, 7]$, and $[5, 6]$.
f. How fast do you think the population of Grand Rapids was changing on January 1, 1985? Said differently, at what rate do you think people were being added to the population of Grand Rapids as of January 1, 1985? How many additional people should the city have expected in the following year? Why?

This page titled 3.1: How do we Measure Velocity? is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Matthew Boelkins, David Austin & Steven Schlicker (ScholarWorks @Grand Valley State University) via source content that was edited to the style and standards of the LibreTexts platform.

- **1.1: How do we Measure Velocity?** is licensed CC BY-SA 4.0. Original source: https://activecalculus.org/single.
- **1.1: How do we Measure Velocity?** by Matthew Boelkins, David Austin & Steven Schlicker is licensed CC BY-SA 4.0. Original source: https://activecalculus.org/single.

3.2: The Notion of Limit

Preview Video

This optional summary video from the textbook author might be helpful to use as a preview. Other, more detailed, supplemental videos for this section are posted at the end of the text.



Learning Objectives

In this section, we strive to understand the ideas generated by the following important questions:

- What is the mathematical notion of limit and what role do limits play in the study of functions?
- What is the meaning of the notation $\lim_{x \to a} f(x) = L$?
- How do we go about determining the value of the limit of a function at a point?
- How do we manipulate average velocity to compute instantaneous velocity?

In Section 1.1 we used a function, $s(t)$, to model the location of a moving object at a given time. Functions are at the heart of mathematics: a function is a process or rule that associates each individual input to exactly one corresponding output. Students learn in courses prior to calculus that there are many different ways to represent functions, including through formulas, graphs, tables, and even words. For example, the squaring function can be thought of in any of these ways. In words, the squaring function takes any real number x and computes its square. The formulaic and graphical representations go hand in hand, as $y = f(x) = x^2$ is one of the simplest curves to graph. Finally, we can also partially represent this function through a table of values, essentially by listing some of the ordered pairs that lie on the curve, such as $(-2, 4)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$ and $(2, 4)$.

Functions are especially important in calculus because they often model important phenomena – the location of a moving object at a given time, the rate at which an automobile is consuming gasoline at a certain velocity, the reaction of a patient to the size of a dose of a drug – and calculus can be used to study how these output quantities change in response to changes in the input variable. Moreover, thinking about concepts like average and instantaneous velocity leads us naturally from an initial function to a related, sometimes more complicated function. As one example of this, think about the falling ball whose position function is given by $s(t) = 64 - 16t^2$ and the average velocity of the ball on the interval $[1, x]$. Observe that

$$AV_{[1,x]} = \frac{s(x) - s(1)}{x - 1} = \frac{(64 - 16x^2) - (64 - 16)}{x - 1} = \frac{16 - 16x^2}{x - 1}.$$

Now, two things are essential to note: this average velocity depends on x (indeed, $AV_{[1,x]}$ is a function of x), and our most focused interest in this function occurs near $x = 1$, which is where the function is not defined. Said differently, the function $g(x) = \frac{16 - 16x^2}{x - 1}$ tells us the average velocity of the ball on the interval from $t = 1$ to $t = x$, and if we are interested 12 in the instantaneous velocity of the ball when $t = 1$, we'd like to know what happens to $g(x)$ as x gets closer and closer to 1. At the same time, $g(1)$ is not defined, because it leads to the quotient 0/0.

This is where the idea of *limits* comes in. By using a limit, we'll be able to allow x to get arbitrarily close, but not equal, to 1 and fully understand the behavior of $g(x)$ near this value. We'll develop key language, notation, and conceptual understanding in what

follows, but for now we consider a preliminary activity that uses the graphical interpretation of a function to explore points on a graph where interesting behavior occurs.

Preview Activity 3.2.1:

Suppose that g is the function given by the graph below. Use the graph to answer each of the following questions.

a. Determine the values $g(-2), g(-1), g(0), g(1)$, and $g(2)$, if defined. If the function value is not defined, explain what feature of the graph tells you this.
b. For each of the values $a = -1$, $a = 0$, and $a = 2$, complete the following sentence: "As x gets closer and closer (but not equal) to a, $g(x)$ gets as close as we want to _____."
c. What happens as x gets closer and closer (but not equal) to $a = 1$? Does the function $g(x)$ get as close as we would like to a single value?

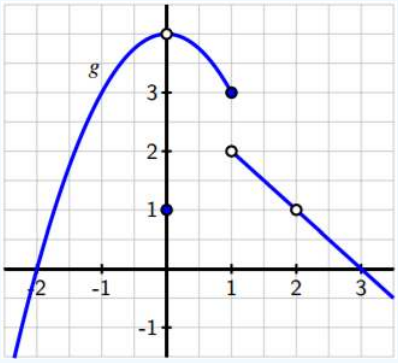


Figure 1.2.1. Graph of $y = g(x)$ for Preview Activity 1.2.1.

The Notion of Limit

Limits can be thought of as a way to study the tendency or trend of a function as the input variable approaches a fixed value, or even as the input variable increases or decreases without bound. We put off the study of the latter idea until further along in the course when we will have some helpful calculus tools for understanding the end behavior of functions. Here, we focus on what it means to say that "a function f has limit L as x approaches a." To begin, we think about a recent example.

In Preview Activity 1.2, you saw that for the given function g, as x gets closer and closer (but not equal) to 0, $g(x)$ gets as close as we want to the value 4. At first, this may feel counterintuitive, because the value of $g(0)$ is 1, not 4. By their very definition, limits regard the behavior of a function *arbitrarily close to* a fixed input, but the value of the function at the fixed input does not matter. More formally[1], we say the following.

Definition 1.2.2. (Limits)

Given a function f, a fixed input $x = a$, and a real number L, we say that f ***has limit*** L ***as*** x ***approaches*** a, and write

$$\lim_{x \to a} f(x) = L$$

provided that we can make $f(x)$ as close to L as we like by taking x sufficiently close (but not equal) to a. If we cannot make $f(x)$ as close to a single value as we would like as x approaches a, then we say that f ***does not have a limit as*** x ***approaches*** a.

Example 3.2.3

For the function g pictured in Figure 1.2.1, we make the following observations:

$$\lim_{x \to -1} g(x) = 3, \ \lim_{x \to 0} g(x) = 4, \ \textit{and} \ \lim_{x \to 2} g(x) = 1,$$

When working from a graph, it suffices to ask if the function *approaches* a single value from each side of the fixed input. The function value *at* the fixed input is irrelevant. This reasoning explains the values of the three limits stated above.

However, g does not have a limit as $x \to 1$. There is a jump in the graph at $x = 1$. If we approach $x = 1$ from the left, the function values tend to get close to 3, but if we approach $x = 1$ from the right, the function values get close to 2. There is no single number that all of these function values approach. This is why the limit of g does not exist at $x = 1$.

For any function f, there are typically three ways to answer the question "does f have a limit at $x = a$, and if so, what is the limit?" The first is to reason graphically as we have just done with the example from Preview Activity 1.2. If we have a formula for $f(x)$, there are two additional possibilities:

1. Evaluate the function at a sequence of inputs that approach a on either side, typically using some sort of computing technology, and ask if the sequence of outputs seems to approach a single value;
2. Use the algebraic form of the function to understand the trend in its output as the input values approach a.

The first approach only produces an approximation of the value of the limit, while the latter can often be used to determine the limit exactly. The following example demonstrates both of these approaches, while also using the graphs of the respective functions to help confirm our conclusions.

[1]What follows here is not what mathematicians consider the formal definition of a limit. To be completely precise, it is necessary to quantify both what it means to say "as close to L as we like" and "sufficiently close to a." That can be accomplished through what is traditionally called the epsilon-delta definition of limits. The definition presented here is sufficient for the purposes of this text.

Example 3.2.4:

For each of the following functions, we'd like to know whether or not the function has a limit at the stated a-values. Use both numerical and algebraic approaches to investigate and, if possible, estimate or determine the value of the limit. Compare the results with a careful graph of the function on an interval containing the points of interest.

a. $f(x) = \frac{4-x^2}{x+2}$; $a = -1,\ a = -2$
b. $g(x) = \sin\left(\frac{\pi}{x}\right)$; $a = 3,\ a = 0$

Answer

We first construct a graph of f along with tables of values near $a = -1$ and $a = -2$.

x	$f(x)$
-0.9	2.9
-0.99	2.99
-0.999	2.999
-0.9999	2.9999
-1.1	3.1
-1.01	3.01
-1.001	3.001
-1.0001	3.0001

x	$f(x)$
-1.9	3.9
-1.99	3.99
-1.999	3.999
-1.9999	3.9999
-2.1	4.1
-2.01	4.01
-2.001	4.001
-2.0001	4.0001

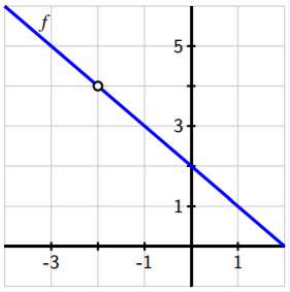


Figure 1.6: Tables and graph for $f(x) = \frac{4-x^2}{x+2}$.

From the left table, it appears that we can make f as close as we want to 3 by taking x sufficiently close to -1, which suggests that $\lim_{x\to -1} f(x) = 3$. This is also consistent with the graph of f. To see this a bit more rigorously and from an algebraic point of view, consider the formula for f : $f(x) = \frac{4-x^2}{x+2}$. The numerator and denominator are each polynomial functions, which are among the most well-behaved functions that exist. Formally, such functions are *continuous*[2] , which means that the limit of the function at any point is equal ([2]see Section 1.7 for more on the notion of continuity) to its function value. Here, it follows that as $x \to -1$, $(4-x^2) \to (4-(-1)^2) = 3$, and $(x+2) \to (-1+2) = 1$, so as $x \to -1$, the numerator of f tends to 3 and the denominator tends to 1, hence $\lim_{x\to -1} f(x) = \frac{3}{1} = 3$.

The situation is more complicated when $x \to -2$, due in part to the fact that $f(-2)$ is not defined. If we attempt to use a similar algebraic argument regarding the numerator and denominator, we observe that as $x \to -2, (4-x^2) \to (4-(-2)^2) = 0$, and $(x+2) \to (-2+2) = 0$, so as $x \to -2$, the numerator of f tends to 0 and

the denominator tends to 0. We call 0/0 an *indeterminate form* and will revisit several important issues surrounding such quantities later in the course. For now, we simply observe that this tells us there is somehow more work to do. From the table and the graph, it appears that f should have a limit of 4 at $x = -2$. To see algebraically why this is the case, let's work directly with the form of $f(x)$. Observe that

$$\begin{aligned}\lim_{x\to -2} f(x) &= \lim_{x\to -2} \frac{4-x^2}{x+2} \\ &= \lim_{x\to -2} \frac{(2-x)(2+x)}{x+2}.\end{aligned} \tag{3.2.1}$$

At this point, it is important to observe that since we are taking the limit as $x \to -2$, we are considering x values that are close, but not equal, to -2. Since we never actually allow x to equal -2, the quotient $\frac{2+x}{x+2}$ has value 1 for every possible value of x. Thus, we can simplify the most recent expression above, and now find that

$$\lim_{x\to -2} f(x) = \lim_{x\to -2} 2-x. \tag{3.2.2}$$

Because $2-x$ is simply a linear function, this limit is now easy to determine, and its value clearly is 4. Thus, from several points of view we've seen that $\lim_{x\to -2} f(x) = 4$.

Next we turn to the function g, and construct two tables and a graph.

x	$g(x)$
2.9	0.84864
2.99	0.86428
2.999	0.86585
2.9999	0.86601
3.1	0.88351
3.01	0.86777
3.001	0.86620
3.0001	0.86604

x	$g(x)$
-0.1	0
-0.01	0
-0.001	0
-0.0001	0
0.1	0
0.01	0
0.001	0
0.0001	0

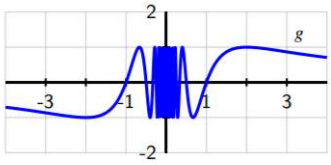


Figure 1.7: Tables and graph for $g(x) = \sin(\frac{\pi}{x})$.

First, as $x \to 3$, it appears from the data (and the graph) that the function is approaching approximately 0.866025. To be precise, we have to use the fact that $\frac{\pi}{x} \to \frac{\pi}{3}$, and thus we find that $g(x) = \sin(\frac{\pi}{x}) \to \sin(\frac{\pi}{3})$ as $x \to 3$. The exact value of $\sin(\frac{\pi}{3})$ is $\frac{\sqrt{3}}{2}$, which is approximately 0.8660254038. Thus, we see that

$$\lim_{x\to 3} g(x) = \frac{\sqrt{3}}{2} \tag{3.2.3}$$

.

As $x \to 0$, we observe that $\frac{\pi}{x}$ does not behave in an elementary way. When x is positive and approaching zero, we are dividing by smaller and smaller positive values, and $\frac{\pi}{x}$ increases without bound. When x is negative and approaching zero, $\frac{\pi}{x}$ decreases without bound. In this sense, as we get close to $x = 0$, the inputs to the sine function are growing rapidly, and this leads to wild oscillations in the graph of g. It is an instructive exercise to plot the function $g(x) = \sin(\frac{\pi}{x})$ with a graphing utility and then zoom in on $x = 0$. Doing so shows that the function never settles down to a single value near the origin and suggests that g does not have a limit at $x = 0$.

How do we reconcile this with the righthand table above, which seems to suggest that the limit of g as x approaches 0 may in fact be 0? Here we need to recognize that the data misleads us because of the special nature of the sequence {0.1, 0.01, 0.001, . . .}: when we evaluate $g(10^{-k})$, we get $g(10^{-k}) = \sin\frac{\pi}{10^{-k}} = sin(\frac{10^k}{\pi}) = 0$ for each positive integer value of k. But if we take a different sequence of values approaching zero, say $\{0.3, 0.03, 0.003, \ldots\}$, then we find that

$$g(3\cdot 10^{-k}) = \sin(\frac{\pi}{3\cdot 10^{-k}}) = \sin(\frac{10^k\pi}{3}) = -\frac{\sqrt{3}}{2} \approx -0.866025. \tag{3.2.4}$$

That sequence of data would suggest that the value of the limit is $\frac{\sqrt{3}}{2}$. Clearly the function cannot have two different values for the limit, and this shows that g has no limit as $x \to 0$.

An important lesson to take from Example 1.2.4. is that tables can be misleading when determining the value of a limit. While a table of values is useful for investigating the possible value of a limit, we should also use other tools to confirm the value, if we think the table suggests the limit exists.

Activity 3.2.2:

Estimate the value of each of the following limits by constructing appropriate tables of values. Then determine the exact value of the limit by using algebra to simplify the function. Finally, plot each function on an appropriate interval to check your result visually.

a. $\lim_{x\to 1} \frac{x^2-1}{x-1}$
b. $\lim_{x\to 0} \frac{(2+x)^3-8}{x}$
c. $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$

This concludes a rather lengthy introduction to the notion of limits. It is important to remember that our primary motivation for considering limits of functions comes from our interest in studying the rate of change of a function. To that end, we close this section by revisiting our previous work with average and instantaneous velocity and highlighting the role that limits play.

Instantaneous Velocity

Suppose that we have a moving object whose position at time t is given by a function s. We know that the average velocity of the object on the time interval $[a, b]$ is $AV_{[a,b]} = \frac{s(b)-s(a)}{b-a}$. We define the *instantaneous velocity* at a to be the limit of average velocity as b approaches a. Note particularly that as $b \to a$, the length of the time interval gets shorter and shorter (while always including a). In Section 1.3, we will introduce a helpful shorthand notation to represent the instantaneous rate of change. For now, we will write $IV_{t=a}$ for the instantaneous velocity at $t = a$, and thus

$$IV_{t=a} = \lim_{b\to a} AV_{[a,b]} = \lim_{b\to a} \frac{s(b)-s(a)}{b-a}.$$

Equivalently, if we think of the changing value b as being of the form $b = a + h$, where h is some small number, then we may instead write

$$IV_{t=a} = \lim_{h\to 0} AV_{[a,a+h]} = \lim_{h\to 0} \frac{s(a+h)-s(a)}{h}.$$

Again, the most important idea here is that to compute instantaneous velocity, we take a limit of average velocities as the time interval shrinks.

Activity 3.2.3

Consider a moving object whose position function is given by $s(t) = t^2$, where s is measured in meters and t is measured in minutes.

a. Determine the most simplified expression for the average velocity of the object on the interval $[3, 3 + h]$, where $h > 0$.
b. Determine the average velocity of the object on the interval $[3, 3.2]$. Include units on your answer.
c. Determine the instantaneous velocity of the object when $t = 3$. Include units on your answer.

The closing activity of this section asks you to make some connections among average velocity, instantaneous velocity, and slopes of certain lines.

Activity 3.2.4

For the moving object whose position s at time t is given by the graph below, answer each of the following questions. Assume that s is measured in feet and t is measured in seconds.

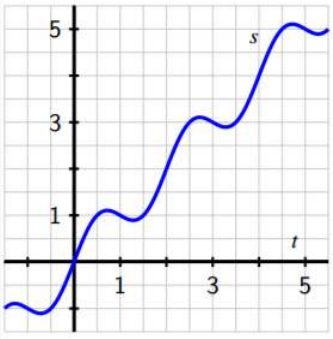


Figure 1.2.11. Plot of the position function $y = s(t)$ in Activity 1.2.4.

a. Use the graph to estimate the average velocity of the object on each of the following intervals: $[0.5, 1]$, $[1.5, 2.5]$, $[0, 5]$ Draw each line whose slope represents the average velocity you seek.
b. How could you use average velocities or slopes of lines to estimate the instantaneous velocity of the object at a fixed time?
c. Use the graph to estimate the instantaneous velocity of the object when $t = 2$. Should this instantaneous velocity at $t = 2$ be greater or less than the average velocity on $[1.5, 2.5]$that you computed in (a)? Why?

Summary

In this section, we encountered the following important ideas:

- Limits enable us to examine trends in function behavior near a specific point. In particular, taking a limit at a given point asks if the function values nearby tend to approach a particular fixed value.
- When we write $\lim_{x \to a} f(x) = L$, we read this as saying "the limit of f as x approaches a is L," and this means that we can make the value of $f(x)$ as close to L as we want by taking x sufficiently close (but not equal) to a.
- If we desire to know $\lim_{x \to a} f(x)$ for a given value of a and a known function f, we can estimate this value from the graph of f or by generating a table of function values that result from a sequence of x-values that are closer and closer to a. If we want the exact value of the limit, we need to work with the function algebraically and see if we can use familiar properties of known, basic functions to understand how different parts of the formula for f change as $x \to a$.
- The instantaneous velocity of a moving object at a fixed time is found by taking the limit of average velocities of the object over shorter and shorter time intervals that all contain the time of interest.

Supplemental videos





Homework Exercises 3.2

WeBWorK Problems:

1.

2.

3.

4.

https://math.libretexts.org/@go/page/106344

5.

https://math.libretexts.org/@go/page/106344

6.

https://math.libretexts.org/@go/page/106344

7.

https://math.libretexts.org/@go/page/106344

8.

Written Problems:

1. You will need to sketch one graph for the function f(x) and one graph for the function g(x) described below:

For each of the following prompts, sketch a graph on the provided axes of a function that has the stated properties.

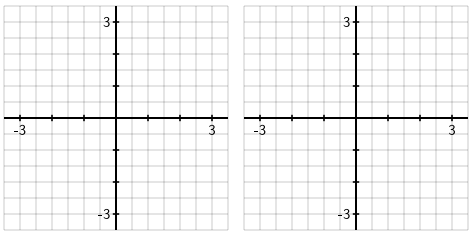


Figure 1.2.12. Axes for plotting $y = f(x)$ in (a) and $y = g(x)$ in (b).

1. $y = f(x)$ such that
   - $f(-2) = 2$ and $\lim_{x \to -2} f(x) = 1$
   - $f(-1) = 3$ and $\lim_{x \to -1} f(x) = 3$
   - $f(1)$ is not defined and $\lim_{x \to 1} f(x) = 0$
   - $f(2) = 1$ and $\lim_{x \to 2} f(x)$ does not exist.
2. $y = g(x)$ such that
   - $g(-2) = 3,\ g(-1) = -1,\ g(1) = -2,$ and $g(2) = 3$
   - At $x = -2, -1, 1$ and 2, g has a limit, and its limit equals the value of the function at that point.
   - $g(0)$ is not defined and $\lim_{x \to 0} g(x)$ does not exist.

https://math.libretexts.org/@go/page/106344

2.

A bungee jumper dives from a tower at time $t = 0$. Her height s in feet at time t in seconds is given by $s(t) = 100\cos(0.75t) \cdot e^{-0.2t} + 100$.

1. Write an expression for the average velocity of the bungee jumper on the interval $[1, 1 + h]$.
2. Use computing technology to estimate the value of the limit as $h \to 0$ of the quantity you found in (a).
3. What is the meaning of the value of the limit in (b)? What are its units?

This page titled 3.2: The Notion of Limit is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Matthew Boelkins, David Austin & Steven Schlicker (ScholarWorks @Grand Valley State University) via source content that was edited to the style and standards of the LibreTexts platform.

- **1.2: The Notion of Limit** is licensed CC BY-SA 4.0. Original source: https://activecalculus.org/single.
- **1.2: The Notion of Limit** by Matthew Boelkins, David Austin & Steven Schlicker is licensed CC BY-SA 4.0. Original source: https://activecalculus.org/single.

3.3: The Derivative of a Function at a Point

Preview Video

This optional summary video from the textbook author might be helpful to use as a preview. Other, more detailed, supplemental videos for this section are posted at the end of the text.



Learning Objectives

In this section, we strive to understand the ideas generated by the following important questions:

- How is the average rate of change of a function on a given interval defined, and what does this quantity measure?
- How is the instantaneous rate of change of a function at a particular point defined? How is the instantaneous rate of change linked to average rate of change?
- What is the derivative of a function at a given point? What does this derivative value measure? How do we interpret the derivative value graphically?
- How are limits used formally in the computation of derivatives?

Introduction

An idea that sits at the foundations of calculus is the *instantaneous rate of change* of a function. This rate of change is always considered with respect to change in the input variable, often at a particular fixed input value. This is a generalization of the notion of instantaneous velocity and essentially allows us to consider the question "how do we measure how fast a particular function is changing at a given point?" When the original function represents the position of a moving object, this instantaneous rate of change is precisely velocity, and might be measured in units such as feet per second. But in other contexts, instantaneous rate of change could measure the number of cells added to a bacteria culture per day, the number of additional gallons of gasoline consumed by going one mile per additional mile per hour in a car's velocity, or the number of dollars added to a mortgage payment for each percentage increase in interest rate. Regardless of the presence of a physical or practical interpretation of a function, the instantaneous rate of change may also be interpreted geometrically in connection to the function's graph, and this connection is also foundational to many of the main ideas in calculus.

In what follows, we will introduce terminology and notation that makes it easier to talk about the instantaneous rate of change of a function at a point. In addition, just as instantaneous velocity is defined in terms of average velocity, the more general instantaneous rate of change will be connected to the more general average rate of change. Recall that for a moving object with position function s, its average velocity on the time interval $t = a$ to $t = a + h$ is given by the quotient

$$AV_{[a,a+h]} = \frac{s(a+h)-s(a)}{h}. \tag{3.3.1}$$

In a similar way, we make the following definition for an arbitrary function $y = f(x)$.

Definition 1.2

For a function f, the *average rate of change* of f on the interval $[a, a+h]$ is given by the value

$$AV_{[a,a+h]} = \frac{f(a+h)-f(a)}{h}. \tag{3.3.2}$$

Equivalently, if we want to consider the average rate of change of f on $[a, b]$, we compute

$$AV_{[a,b]} = \frac{f(b)-f(a)}{b-a}. \tag{3.3.3}$$

It is essential to understand how the average rate of change of f on an interval is connected to its graph.

Preview Activity 3.3.1:

Suppose that f is the function given by the graph below and that a and $a+h$ are the input values as labeled on the x-axis. Use the graph in Figure 3.3.1 to answer the following questions.

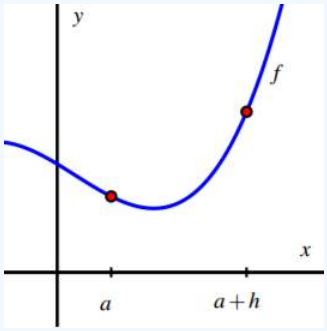


Figure 3.3.1: Plot of $y = f(x)$ for Preview Activity 1.3.

a. Locate and label the points $(a, f(a))$ and $(a+h, f(a+h))$ on the graph.
b. Construct a right triangle whose hypotenuse is the line segment from $(a, f(a))$ to $(a+h, f(a+h))$. What are the lengths of the respective legs of this triangle?
c. What is the slope of the line that connects the points $(a, f(a))$ and $(a+h, f(a+h))$?
d. Write a meaningful sentence that explains how the average rate of change of the function on a given interval and the slope of a related line are connected.

The Derivative of a Function at a Point

Just as we defined instantaneous velocity in terms of average velocity, we now define the instantaneous rate of change of a function at a point in terms of the average rate of change of the function f over related intervals. In addition, we give a special name to "the instantaneous rate of change of f at a," calling this quantity "the *derivative* of f at a," with this value being represented by the shorthand notation $f'(a)$. Specifically, we make the following definition.

Definition 1.3

Let f be a function and $x = a$ a value in the function's domain. We *define the derivative of f with respect to x evaluated at $x = a$*, denoted $f'(a)$, by the formula

$$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}, \tag{3.3.4}$$

provided this limit exists. Aloud, we read the symbol $f'(a)$ as either "f -prime at a" or "the derivative of f evaluated at $x = a$." Much of the next several chapters will be devoted to understanding, computing, applying, and interpreting derivatives. For now, we make the following important notes.

- The derivative of f at the value $x = a$ is defined as the limit of the average rate of change of f on the interval $[a, a+h]$ as $h \to 0$. It is possible for this limit not to exist, so not every function has a derivative at every point.
- We say that a function that has a derivative at $x = a$ is *differentiable* at $x = a$.
- The derivative is a generalization of the instantaneous velocity of a position function: when $y = s(t)$ is a position function of a moving body, $s'(a)$ tells us the instantaneous velocity of the body at time $t = a$.
- Because the units on $\frac{f(a+h)-f(a)}{h}$ are "units of f per unit of x," the derivative has these very same units. For instance, if s measures position in feet and t measures time in seconds, the units on $s'(a)$ are feet per second.
- Because the quantity $\frac{f(a+h)-f(a)}{h}$ represents the slope of the line through $(a, f(a))$ and $(a+h, f(a+h))$, when we compute the derivative we are taking the limit of a collection of slopes of lines, and thus the derivative itself represents the slope of a particularly important line.

While all of the above ideas are important and we will add depth and perspective to them through additional time and study, for now it is most essential to recognize how the derivative of a function at a given value represents the slope of a certain line. Thus, we expand upon the last bullet item above.

As we move from an average rate of change to an instantaneous one, we can think of one point as "sliding towards" another. In particular, provided the function has a derivative at $(a, f(a))$, the point $(a+h, f(a+h))$ will approach $(a, f(a))$ as $h \to 0$. Because this process of taking a limit is a dynamic one, it can be helpful to use computing technology to visualize what the limit is accomplishing. While there are many different options[3] , one of the best is a java applet in which the user is able to control the point that is moving. See the examples referenced in the footnote here, or consider building your own, perhaps using the fantastic free program Geogebra[4] .

In Figure 3.3.2, we provide a sequence of figures with several different lines through the points $(a, f(a))$ and $(a+h, f(a+h))$ that are generated by different values of h. These lines (shown in the first three figures in magenta), are often called secant lines to the curve $y = f(x)$. A secant line to a curve is simply a line that passes through two points that lie on the curve. For each such line, the slope of the secant line is $m = \frac{f(a+h)-f(a)}{h}$, where the value of h depends on the location of the point we choose. We can see in the diagram how, as $h \to 0$, the secant lines start to approach a single line that passes through the point $(a, f(a))$. In the situation where the limit of the slopes of the secant lines exists, we say that the resulting value is the slope of the tangent line to the curve. This tangent line (shown in the right-most figure in green) to the graph of $y = f(x)$ at the point $(a, f(a))$ is the line through $(a, f(a))$ whose slope is $m = f'(a)$.

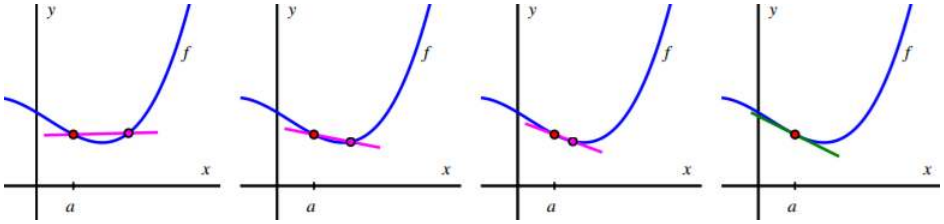


Figure 3.3.2: A sequence of secant lines approaching the tangent line to f at $(a, f(a))$.

As we will see in subsequent study, the existence of the tangent line at $x = a$ is connected to whether or not the function f looks like a straight line when viewed up close at $(a, f(a))$, which can also be seen in Figure 3.3.3, where we combine the four graphs in Figure 3.3.2 into the single one on the left, and then we zoom in on the box centered at $(a, f(a))$, with that view expanded on the right (with two of the secant lines omitted). Note how the tangent line sits relative to the curve $y = f(x)$ at $(a, f(a))$ and how closely it resembles the curve near $x = a$.

[3]For a helpful collection of java applets, consider the work of David Austin of Grand Valley State University at http://gvsu.edu/s/5r, and the particularly relevant example at http://gvsu.edu/s/5s. For applets that have been built in Geogebra, a nice example is the work of Marc Renault of Shippensburg University at http://gvsu.edu/s/5p, with the example at http://gvsu.edu/s/5q being especially fitting for our work in this section. There are scores of other examples posted by other authors on the internet.

[4]Available for free download from http://geogebra.org.

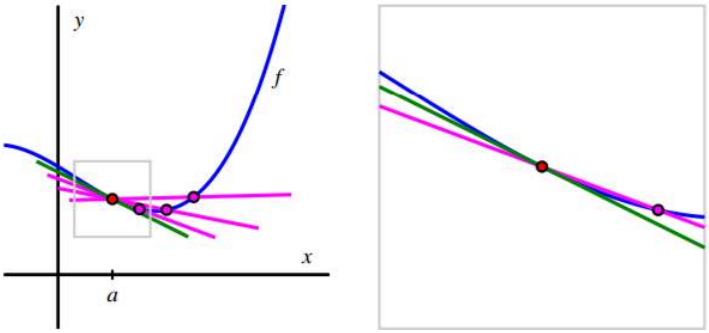


Figure 3.3.3: A sequence of secant lines approaching the tangent line to f at $(a, f(a))$. At right, we zoom in on the point $(a, f(a))$. The slope of the tangent line (in green) to f at $(a, f(a))$ is given by $f'(a)$.

At this time, it is most important to note that $f'(a)$, the instantaneous rate of change of f with respect to x at $x = a$, also measures the slope of the tangent line to the curve $y = f(x)$ at $(a, f(a))$. The following example demonstrates several key ideas involving the derivative of a function.

Example 3.3.1:

Example 1.3. For the function given by $f(x) = x - x^2$, use the limit definition of the derivative to compute $f'(2)$. In addition, discuss the meaning of this value and draw a labeled graph that supports your explanation.

Solution. From the limit definition, we know that

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}. \tag{3.3.5}$$

Now we use the rule for f, and observe that $f(2) = 2 - 2^2 = -2$ and $f(2+h) = (2+h) - (2+h)^2$. Substituting these values into the limit definition, we have that

$$f'(2) = \lim_{h \to 0} \frac{(2+h) - (2+h)^2 - (-2)}{h}. \tag{3.3.6}$$

Observe that with h in the denominator and our desire to let $h \to 0$, we have to wait to take the limit (that is, we wait to actually let h approach 0). Thus, we do additional

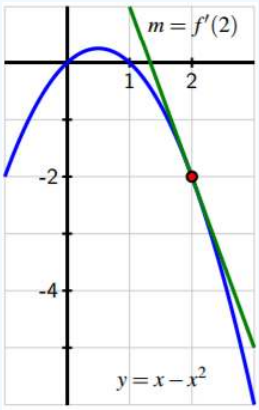


Figure 3.3.4: The tangent line to $y = x - x^2$ at the point $(2, -2)$.

algebra. Expanding and distributing in the numerator,

$$f'(2) = \lim_{h \to 0} \frac{2 + h - 4 - 4h - h^2 + 2}{h}. \tag{3.3.7}$$

Combining like terms, we have

$$f'(2) = \lim_{h \to 0} \frac{-3h - h^2}{h}. \tag{3.3.8}$$

Next, we observe that there is a common factor of h in both the numerator and denominator, which allows us to simplify and find that

$$f'(2) = \lim_{h \to 0}(-3 - h). \tag{3.3.9}$$

Finally, we are able to take the limit as $h \to 0$, and thus conclude that $f'(2) = -3$.

Now, we know that $f'(2)$ represents the slope of the tangent line to the curve $y = x - x^2$ at the point $(2, -2)$; $f'(2)$ is also the instantaneous rate of change of f at the point $(2, -2)$. Graphing both the function and the line through $(2, -2)$ with slope $m = f'(2) = -3$, we indeed see that by calculating the derivative, we have found the slope of the tangent line at this point, as shown in Figure 1.3.

The following activities will help you explore a variety of key ideas related to derivatives.

Activity 3.3.2:

Consider the function f whose formula is $f(x) = 3 - 2x$.

a. What familiar type of function is f? What can you say about the slope of f at every value of x?
b. Compute the average rate of change of f on the intervals $[1, 4]$, $[3, 7]$ and $[5, 5 + h]$; simplify each result as much as possible. What do you notice about these quantities?
c. Use the limit definition of the derivative to compute the exact instantaneous rate of change of f with respect to x at the value $a = 1$. That is, compute $f'(1)$ using the limit definition. Show your work. Is your result surprising?
d. Without doing any additional computations, what are the values of $f'(2)$, $f'(\pi)$, and $f'(-\sqrt{2})$? Why?

Activity 3.3.3:

A water balloon is tossed vertically in the air from a window. The balloon's height in feet at time t in seconds after being launched is given by $s(t) = -16t^2 + 16t + 32$. Use this function to respond to each of the following questions.

a. Sketch an accurate, labeled graph of s on the axes provided in Figure 3.3.5. You should be able to do this without using computing technology.

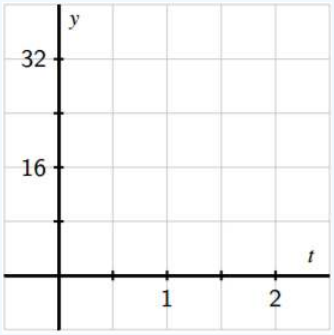


Figure 3.3.5: Axes for plotting $y = s(t)$ in Activity 1.8

1. Compute the average rate of change of s on the time interval $[1, 2]$. Include units on your answer and write one sentence to explain the meaning of the value you found.
2. Use the limit definition to compute the instantaneous rate of change of s with respect to time, t, at the instant $a = 1$. Show your work using proper notation, include units on your answer, and write one sentence to explain the meaning of the value you found.
3. On your graph in (a), sketch two lines: one whose slope represents the average rate of change of s on $[1, 2]$, the other whose slope represents the instantaneous rate of change of s at the instant $a = 1$. Label each line clearly.
4. For what values of a do you expect $s'(a)$ to be positive? Why? Answer the same questions when "positive" is replaced by "negative" and "zero."

Activity 3.3.4:

A rapidly growing city in Arizona has its population P at time t, where t is the number of decades after the year 2010, modeled by the formula $P(t) = 25000e^{t/5}$. Use this function to respond to the following questions.

a. Sketch an accurate graph of P for $t = 0$ to $t = 5$ on the axes provided in Figure 3.3.6. Label the scale on the axes carefully.

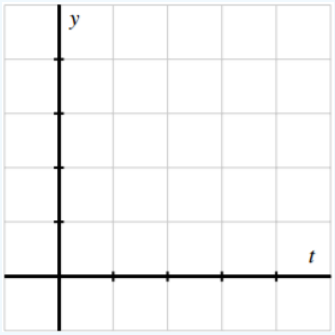


Figure 3.3.6: Axes for plotting $y = P(t)$ in Activity 1.9.

1. Compute the average rate of change of P between 2030 and 2050. Include units on your answer and write one sentence to explain the meaning (in everyday language) of the value you found.
2. Use the limit definition to write an expression for the instantaneous rate of change of P with respect to time, t, at the instant $a = 2$. Explain why this limit is difficult to evaluate exactly.
3. Estimate the limit in (c) for the instantaneous rate of change of P at the instant $a = 2$ by using several small h values. Once you have determined an accurate 30 estimate of $P'(2)$, include units on your answer, and write one sentence (using everyday language) to explain the meaning of the value you found.
4. On your graph above, sketch two lines: one whose slope represents the average rate of change of P on $[2, 4]$, the other whose slope represents the instantaneous rate of change of P at the instant $a = 2$.
5. In a carefully-worded sentence, describe the behavior of $P'(a)$ as a increases in value. What does this reflect about the behavior of the given function P?

Summary

In this section, we encountered the following important ideas:

- The average rate of change of a function f on the interval $[a, b]$ is $\frac{f(b)-f(a)}{b-a}$. The units on the average rate of change are units of f per unit of x, and the numerical value of the average rate of change represents the slope of the secant line between the points $(a, f(a))$ and $(b, f(b))$ on the graph of $y = f(x)$. If we view the interval as being $[a, a+h]$ instead of [a, b], the meaning is still the same, but the average rate of change is now computed by $\frac{f(a+h)-f(a)}{h}$.
- The instantaneous rate of change with respect to x of a function f at a value $x = a$ is denoted $f'(a)$ (read "the derivative of f evaluated at a" or "f -prime at a") and is defined by the formula

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \tag{3.3.10}$$

provided the limit exists. Note particularly that the instantaneous rate of change at $x = a$ is the limit of the average rate of change on $[a, a+h]$ as $h \to 0$.

- Provided the derivative $f'(a)$ exists, its value tells us the instantaneous rate of change of f with respect to x at $x = a$, which geometrically is the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$. We even say that $f'(a)$ is the *slope of the curve* $y = f(x)$ at the point $(a, f(a))$.
- Limits are the link between average rate of change and instantaneous rate of change: they allow us to move from the rate of change over an interval to the rate of change at a single point.

Supplemental videos





Homework Exercises 3.3

WeBWorK Problems:

1.

2.

3.

4.

Written Problems:

1a. Write the general form of the limit definition of the derivative of f at $x = a$.

b. Consider $f(x) = 6^x$ at $x = 3$. Revise your general form for $f'(3)$.

c. Notice that we can't use algebra to simplify this expression in order to find the exact value. Estimate this derivative with $h \to 0$.

2.

For each of the following prompts, sketch a graph on the provided axes in Figure 1.3.13 of a function that has the stated properties.

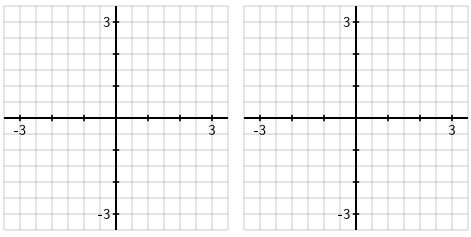


Figure 1.3.13. Axes for plotting $y = f(x)$ in (a) and $y = g(x)$ in (b).

1. $y = f(x)$ such that
   - the average rate of change of f on $[-3, 0]$ is -2 and the average rate of change of f on $[1, 3]$ is 0.5, and
   - the instantaneous rate of change of f at $x = -1$ is -1 and the instantaneous rate of change of f at $x = 2$ is 1.

2. $y = g(x)$ such that
   - $\frac{g(3)-g(-2)}{5} = 0$ and $\frac{g(1)-g(-1)}{2} = -1$, and
   - $g'(2) = 1$ and $g'(-1) = 0$

3.

Suppose that the population, P, of China (in billions) can be approximated by the function $P(t) = 1.15(1.014)^t$ where t is the number of years since the start of 1993.

1. According to the model, what was the total change in the population of China between January 1, 1993 and January 1, 2000? What will be the average rate of change of the population over this time period? Is this average rate of change greater or less than the instantaneous rate of change of the population on January 1, 2000? Explain and justify, being sure to include proper units on all your answers.
2. According to the model, what is the average rate of change of the population of China in the ten-year period starting on January 1, 2012?
3. Write an expression involving limits that, if evaluated, would give the exact instantaneous rate of change of the population on today's date. Then estimate the value of this limit (discuss how you chose to do so) and explain the meaning (including units) of the value you have found.
4. Find an equation for the tangent line to the function $y = P(t)$ at the point where the t-value is given by today's date.

This page titled 3.3: The Derivative of a Function at a Point is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Matthew Boelkins, David Austin & Steven Schlicker (ScholarWorks @Grand Valley State University) via source content that was edited to the style and standards of the LibreTexts platform.

- **1.3: The Derivative of a Function at a Point** is licensed CC BY-SA 4.0. Original source: https://activecalculus.org/single.
- **1.3: The Derivative of a Function at a Point** by Matthew Boelkins, David Austin & Steven Schlicker is licensed CC BY-SA 4.0. Original source: https://activecalculus.org/single.

3.4: The Derivative Function

Preview Video

This optional summary video from the textbook author might be helpful to use as a preview. Other, more detailed, supplemental videos for this section are posted at the end of the text.



Section 3.4

Learning Objectives

In this section, we strive to understand the ideas generated by the following important questions:

- How does the limit definition of the derivative of a function f lead to an entirely new (but related) function f'?
- What is the difference between writing $f'(a)$ and $f'(x)$?
- How is the graph of the derivative function $f'(x)$ connected to the graph of $f(x)$?
- What are some examples of functions f for which f' is not defined at one or more points?

Given a function $y = f(x)$, we now know that if we are interested in the instantaneous rate of change of the function at $x = a$, or equivalently the slope of the tangent line to $y = f(x)$ at $x = a$, we can compute the value $f'(a)$. In all of our examples to date, we have arbitrarily identified a particular value of a as our point of interest: $a = 1$, $a = 3$, etc. But it is not hard to imagine that we will often be interested in the derivative value for more than just one a-value, and possibly for many of them. In this section, we explore how we can move from computing simply $f'(1)$ or $f'(3)$ to working more generally with $f'(a)$, and indeed $f'(x)$. Said differently, we will work toward understanding how the so-called process of "taking the derivative" generates a new function that is derived from the original function $y = f(x)$. The following preview activity starts us down this path.

Preview Activity 3.4.1

Consider the function $f(x) = 4x - x^2$.

a. Use the limit definition to compute the following derivative values: $f'(0)$, $f'(1)$, $f'(2)$, and $f'(3)$.
b. Observe that the work to find $f'(a)$ is the same, regardless of the value of a. Based on your work in (a), what do you conjecture is the value of $f'(4)$? How about $f'(5)$? (Note: you should not use the limit definition of the derivative to find either value.)
c. Conjecture a formula for $f'(a)$ that depends only on the value a. That is, in the same way that we have a formula for $f(x)$ (recall $f(x) = 4x - x^2$), see if you can use your work above to guess a formula for $f'(a)$ in terms of a.

The Derivative is Itself a Function

In your work in Preview Activity 1.4 with $f(x) = 4x - x^2$, you may have found several patterns. One comes from observing that $f'(0) = 4$, $f'(1) = 2$, $f'(2) = 0$, and $f'(3) = -2$. That sequence of values leads us naturally to conjecture that $f'(4) = -4$ and $f'(5) = -6$. Even more than these individual numbers, if we consider the role of 0, 1, 2, and 3 in the process of computing the value of the derivative through the limit definition, we observe that the particular number has very little effect on our work. To see this more clearly, we compute $f'(a)$, where a represents a number to be named later. Following the now standard process of using the limit definition of the derivative,

$$(f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \tag{3.4.1}$$

$$= \lim_{h\to 0} \frac{4(a+h) - (a+h)^2 - (4a - a^2)}{h} \tag{3.4.2}$$

$$= \lim_{h\to 0} \frac{4a + 4h - a^2 - 2ha - h^2 - 4a + a^2}{h} \tag{3.4.3}$$

$$= \lim_{h\to 0} \frac{4h - 2ha - h^2}{h} \tag{3.4.4}$$

$$= \lim_{h\to 0} \frac{h(4 - 2a - h)}{h} \tag{3.4.5}$$

$$= \lim_{h\to 0} (4 - 2a - h). \tag{3.4.6}$$

Here we observe that neither 4 nor $2a$ depend on the value of h, so as

$$h \to 0,\ (4 - 2a - h) \to (4 - 2a). \tag{3.4.7}$$

Thus, $f'(a) = 4 - 2a$.

This observation is consistent with the specific values we found above: e.g., $f'(3) = 4 - 2(3) = -2$. And indeed, our work with a confirms that while the particular value of a at which we evaluate the derivative affects the value of the derivative, that value has almost no bearing on the process of computing the derivative. We note further that the letter being used is immaterial: whether we call it a, x, or anything else, the derivative at a given value is simply given by "4 minus 2 times the value." We choose to use x for consistency with the original function given by $y = f(x)$, as well as for the purpose of graphing the derivative function, and thus we have found that for the function $f(x) = 4x - x^2$, it follows that $f'(x) = 4 - 2x$.

Because the value of the derivative function is so closely linked to the graphical behavior of the original function, it makes sense to look at both of these functions plotted on the same domain. In Figure 1.18, on the left we show a plot of $f(x) = 4x - x^2$ together with a selection of tangent lines at the points we've considered above. On the right, we show a plot of $f'(x) = 4 - 2x$ with emphasis on the heights of the derivative graph at the same selection of points. Notice the connection between colors in the left and right graph: the green tangent line on the original graph is tied to the green point on the right graph in the following way: *the slope of the tangent line* at a point on the lefthand graph is the

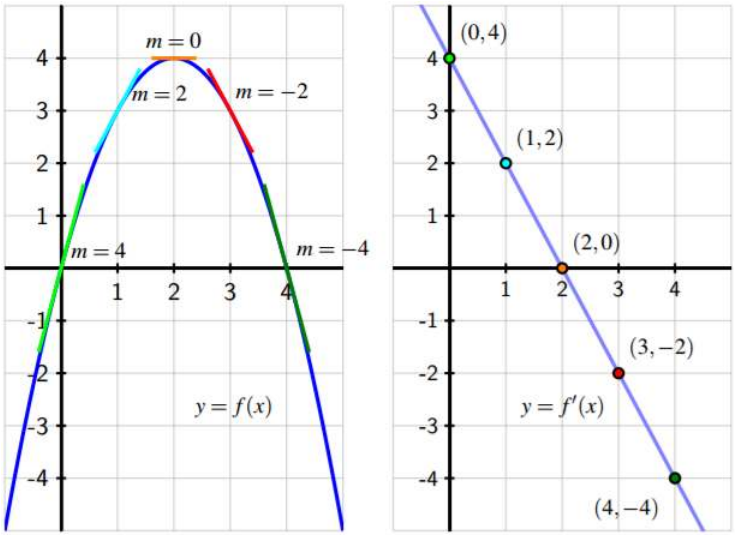


Figure 1.18: The graphs of $f(x) = 4x - x^2$ (at left) and $f'(x) = 4 - 2x$ (at right). Slopes on the graph of f correspond to heights on the graph of f'.

same as the *height* at the corresponding point on the righthand graph. That is, at each respective value of x, the slope of the tangent line to the original function at that x-value is the same as the height of the derivative function at that x-value. Do note, however, that the units on the vertical axes are different: in the left graph, the vertical units are simply the output units of f. On the righthand graph of $y = f'(x)$, the units on the vertical axis are units of f per unit of x.

Of course, this relationship between the graph of a function $y = f(x)$ and its derivative is a dynamic one. An excellent way to explore how the graph of $f(x)$ generates the graph of $f'(x)$ is through a java applet. See, for instance, the applets at http://gvsu.edu/s/5C or http://gvsu.edu/s/5D, via the sites of Austin and Renault[5] .

In Section 1.3 when we first defined the derivative, we wrote the definition in terms of a value a to find $f'(a)$. As we have seen above, the letter a is merely a placeholder, and it often makes more sense to use x instead. For the record, here we restate the definition of the derivative.

Definition 1.4

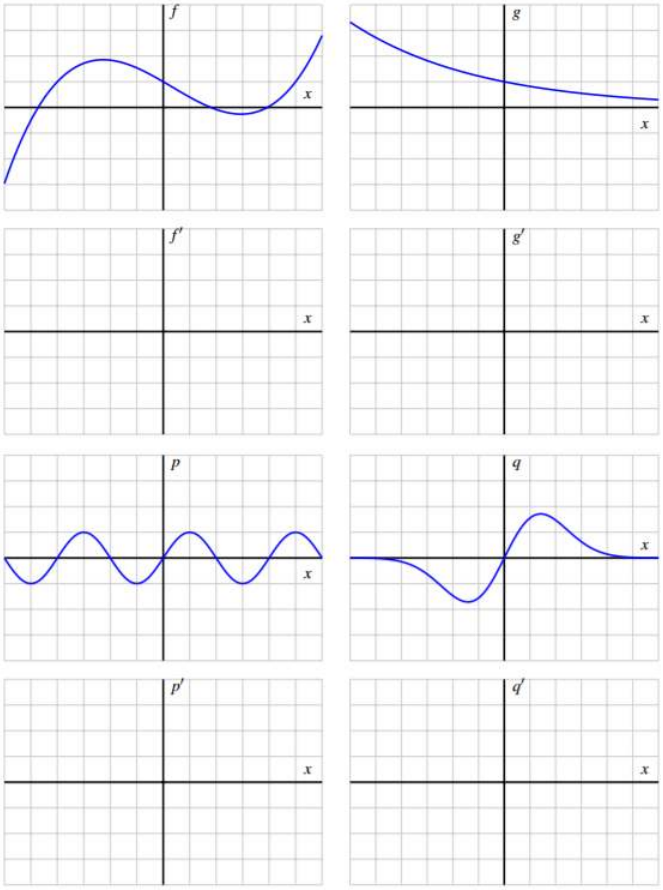
Let f be a function and x a value in the function's domain. We define the derivative of f with respect to x at the value x, denoted $f'(x)$, by the formula $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, provided this limit exists.

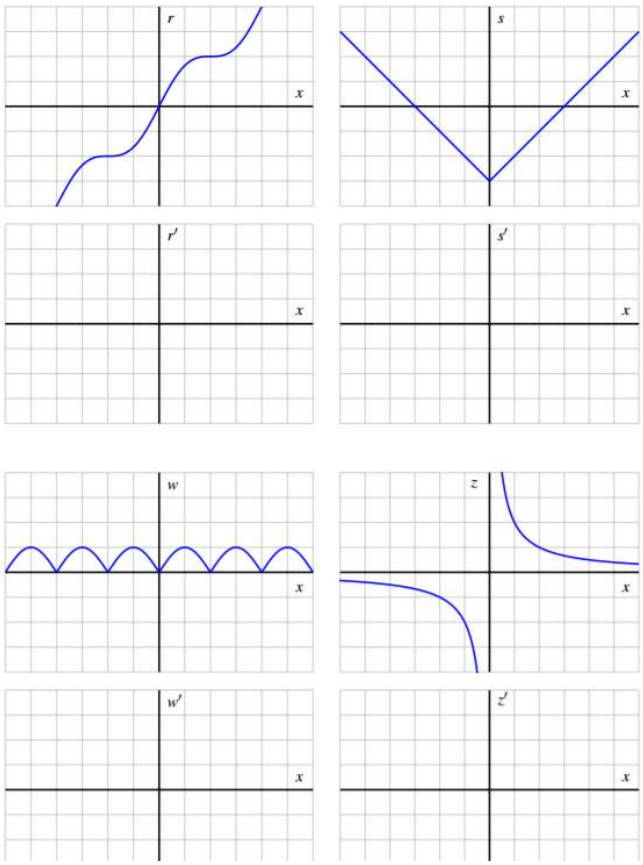
[5]David Austin, http://gvsu.edu/s/5r; Marc Renault, http://gvsu.edu/s/5p.

We now may take two different perspectives on thinking about the derivative function: given a graph of $y = f(x)$, how does this graph lead to the graph of the derivative function $y = f'(x)$? and given a formula for $y = f(x)$, how does the limit definition of the derivative generate a formula for $y = f'(x)$? Both of these issues are explored in the following activities.

Exercise 3.4.1

For each given graph of $y = f(x)$, sketch an approximate graph of its derivative function, $y = f'(x)$, on the axes immediately below. The scale of the grid for the graph of f is 1×1; assume the horizontal scale of the grid for the graph of f' is identical to that for f. If necessary, adjust and label the vertical scale on the axes for f'.


f
x
g
x
f′
x
g′
x
p
x
q
x
p′
x
q′
x


r
x
s
x
r′
x
s′
x
w
x
z
x
w′
x
z′
x

When you are finished with all 8 graphs, write several sentences that describe your overall process for sketching the graph of the derivative function, given the graph the original function. What are the values of the derivative function that you tend to identify first? What do you do thereafter? How do key traits of the graph of the derivative function exemplify properties of the graph of the original function? C For a dynamic investigation that allows you to experiment with graphing f' when given the graph of f, see http://gvsu.edu/s/8y. 6 6Marc Renault, Calculus Applets Using Geogebra. 39

Now, recall the opening example of this section: we began with the function $y = f(x) = 4x - x^2$ and used the limit definition of the derivative to show that $f'(a) = 4 - 2a$, or equivalently that $f'(x) = 4 - 2x$. We subsequently graphed the functions f and f' as shown in Figure 1.18. Following Activity 1.10, we now understand that we could have constructed a fairly accurate graph of $f'(x)$ without knowing a formula for either f or f'. At the same time, it is ideal to know a formula for the derivative function whenever it is possible to find one. In the next activity, we further explore the more algebraic approach to finding $f'(x)$: given a formula for $y = f(x)$, the limit definition of the derivative will be used to develop a formula for $f'(x)$.

Activity 3.4.2

For each of the listed functions, determine a formula for the derivative function. For the first two, determine the formula for the derivative by thinking about the nature of the given function and its slope at various points; do not use the limit definition. For the latter four, use the limit definition. Pay careful attention to the function names and independent variables. It is important to be comfortable with using letters other than f and x. For example, given a function $p(z)$, we call its derivative $p'(z)$.

a. $f(x) = 1$
b. $g(t) = t$
c. $p(z) = z^2$
d. $q(s) = s^3$
e. $F(t) = \frac{1}{t}$
f. $G(y) = \sqrt{y}$

Summary

In this section, we encountered the following important ideas:

- The limit definition of the derivative, $f'(x) = lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$, produces a value for each x at which the derivative is defined, and this leads to a new function whose formula is $y = f'(x)$. Hence we talk both about a given function f and its derivative f'. It is especially important to note that taking the derivative is a process that starts with a given function (f) and produces a new, related function (f').
- There is essentially no difference between writing $f'(a)$ (as we did regularly in Section 1.3) and writing $f'(x)$. In either case, the variable is just a placeholder that is used to define the rule for the derivative function.
- Given the graph of a function $y = f(x)$, we can sketch an approximate graph of its derivative $y = f'(x)$ by observing that heights on the derivative's graph correspond to slopes on the original function's graph.
- In Activity 1.10, we encountered some functions that had sharp corners on their graphs, such as the shifted absolute value function. At such points, the derivative fails to exist, and we say that f is not differentiable there. For now, it suffices to understand this as a consequence of the jump that must occur in the derivative function at a sharp corner on the graph of the original function.

Supplemental videos





Homework Exercises 3.4

WeBWorK Problems:

1.

2.

3.

https://math.libretexts.org/@go/page/106346

4.

5.

6.

7.

https://math.libretexts.org/@go/page/106346

Written Problems:

1.

For each graph that provides an original function $y = f(x)$ in Figure 1.4.5, your task is to sketch an approximate graph of its derivative function, $y = f'(x)$, on the axes immediately below. View the scale of the grid for the graph of f as being 1×1, and assume the horizontal scale of the grid for the graph of f' is identical to that for f. If you need to adjust the vertical scale on the axes for the graph of f', you should label that accordingly.

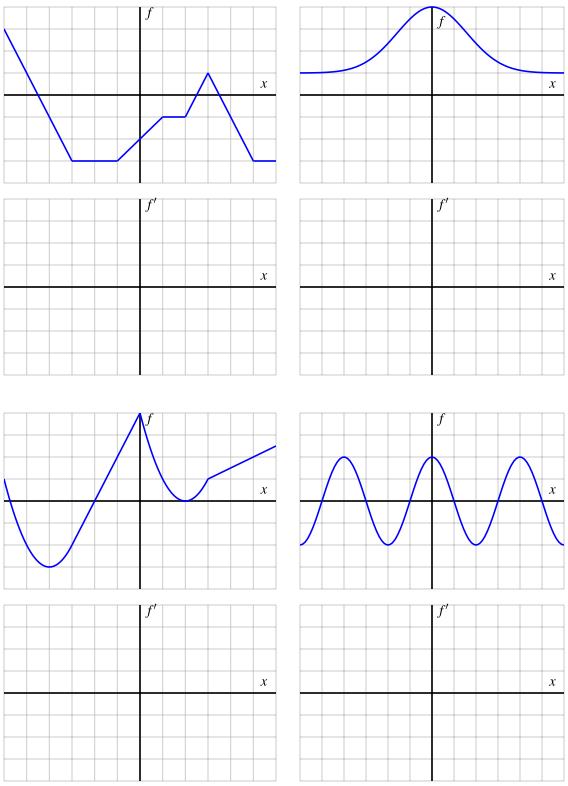
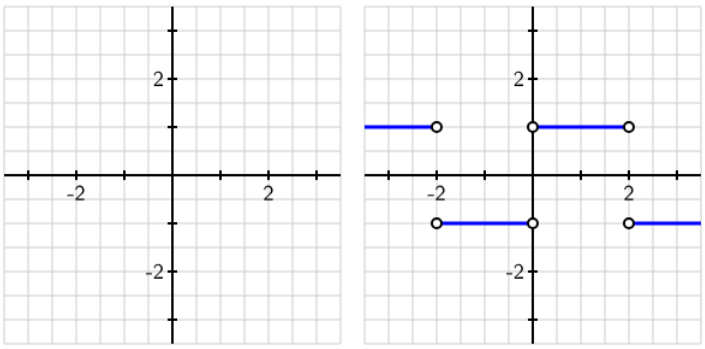


Figure 1.4.5. Graphs of $y = f(x)$ and grids for plotting the corresponding graph of $y = f'(x)$.

2. Let $g(x)$ be a continuous function (that is, one with no jumps or holes in the graph) and suppose that a graph of y=g′(x) is given by the graph on the right in the figure below. Please sketch the following on your paper and label them as g and g'



a. Observe that for every value of x such that 0<x<2, the value of g′(x) is constant. What does this tell you about the behavior of the graph of y=g(x) on this interval?

b. On what intervals other than 0<x<2 do you expect y=g(x) to be a linear function? Why?

c. At which values of x is g′(x) not defined? What behavior does this lead you to expect to see in the graph of y=g(x)?

d. Suppose that g(0)=1. Sketch an accurate graph of y=g(x).

3. Consider the function $g(x) = x^2 - x + 3$

a. Use the limit definition of the derivative to determine a formula for $g'(x)$.

b. Use a graphing utility to plot both $y=g(x)$ and your result for $y=g'(x)$. Sketch these on your paper. Annotate your graphs to show that the graphs makes sense in terms of slopes.

4. Let f be a function with the following properties:

- f is differentiable at every value of x (that is, f has a derivative at every point)
- $f(-2)=1$
- $f'(-2)=-2$
- $f'(-1)=-1$
- $f'(0)=0$
- $f'(1)=1$
- $f'(2)=2$.

Sketch a possible graph of $y=f(x)$. Explain why your graph meets the stated criteria.

This page titled 3.4: The Derivative Function is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Matthew Boelkins, David Austin & Steven Schlicker (ScholarWorks @Grand Valley State University) via source content that was edited to the style and standards of the LibreTexts platform.

- **1.4: The Derivative Function** is licensed CC BY-SA 4.0. Original source: https://activecalculus.org/single.
- **1.4: The Derivative Function** by Matthew Boelkins, David Austin & Steven Schlicker is licensed CC BY-SA 4.0. Original source: https://activecalculus.org/single.

3.5: Limits, Continuity, and Differentiability

Preview Video

This optional summary video from the textbook author might be helpful to use as a preview. Other, more detailed, supplemental videos for this section are posted at the end of the text.



Learning Objectives

In this section, we strive to understand the ideas generated by the following important questions:

- What does it mean graphically to say that f has limit L as $x \to a$? How is this connected to having a left-hand limit at $x = a$ and having a right-hand limit at $x = a$?
- What does it mean to say that a function f is continuous at $x = a$? What role do limits play in determining whether or not a function is continuous at a point?
- What does it mean graphically to say that a function f is differentiable at $x = a$? How is this connected to the function being locally linear?
- How are the characteristics of a function having a limit, being continuous, and being differentiable at a given point related to one another?

Introduction

In Section 1.2, we learned about how the concept of limits can be used to study the trend of a function near a fixed input value. As we study such trends, we are fundamentally interested in knowing how well-behaved the function is at the given point, say $x = a$. In this present section, we aim to expand our perspective and develop language and understanding to quantify how the function acts and how its value changes near a particular point. Beyond thinking about whether or not the function has a limit L at $x = a$, we will also consider the value of the function $f(a)$ and how this value is related to $lim_{x\to a} f(x)$, as well as whether or not the function has a derivative $f'(a)$ at the point of interest. Throughout, we will build on and formalize ideas that we have encountered in several settings.

We begin to consider these issues through the following preview activity that asks you to consider the graph of a function with a variety of interesting behaviors.

Preview Activity 3.5.1

A function f defined on $-4 < x < 4$ is given by the graph in Figure 1.7.1. Use the graph to answer each of the following questions. Note: to the right of $x = 2$, the graph of f is exhibiting infinite oscillatory behavior similar to the function $\sin(\frac{\pi}{x})$ that we encountered in the key example early in Section 1.2.

(a) For each of the values a = −3, −2, −1, 0, 1, 2, 3, determine whether or not $\lim_{x\to a} f(x)$ exists. If the function has a limit L at a given point, state the value of the limit using the notation $\lim_{x\to a} f(x) = L$. If the function does not have a limit at a given point, write a sentence to explain why.

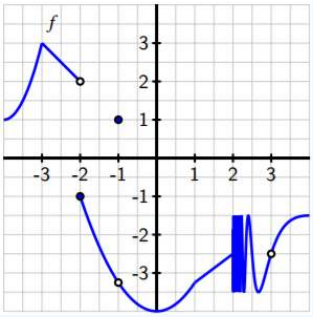


***Figure* 3.5.1**: *The graph of* $y = f(x)$.

(b) For each of the values of a from part (a) where f has a limit, determine the value of $f(a)$ at each such point. In addition, for each such a value, does $f(a)$ have the same value as $\lim_{x\to a} f(x)$?

(c) For each of the values a = −3, −2, −1, 0, 1, 2, 3, determine whether or not $f'(a)$ exists. In particular, based on the given graph, ask yourself if it is reasonable to say that f has a tangent line at $(a, f(a))$ for each of the given a-values. If so, visually estimate the slope of the tangent line to find the value of $f'(a)$.

Having a limit at a point

In Section 1.2, we first encountered limits and learned that we say that f has limit L as x approaches a and write $lim_{x\to a} f(x) = L$ provided that we can make the value of f (x) as close to L as we like by taking x sufficiently close (but not equal to) a. Here, we expand further on this definition and focus in more depth on what it means for a function not to have a limit at a given value.

Essentially there are two behaviors that a function can exhibit at a point where it fails to have a limit. In Figure 1.7.2, at left we see a function f whose graph shows a jump at a = 1. In particular, if we let x approach 1 from the left side, the value of f approaches 2, while if we let x go to 1 from the right, the value of f tends to 3. Because the value of f does not approach a single number as x gets arbitrarily close to 1 from both sides, we know that f does not have a limit at a = 1.

Since f does approach a single value on each side of a = 1, we can introduce the notion of *left* and *right* (or *one-sided*) *limits*. We say that f *has limit* L_1 *as* x *approaches* a *from the left* and write

$$\lim_{x\to a^-} f(x) = L_1 \tag{3.5.1}$$

provided that we can make the value of $f(x)$ as close to L_1 as we like by taking x sufficiently close to a while always having $x < a$. In this case, we call L_1 the left-hand limit of f as x approaches a. Similarly, we say L_2 is *the right-hand limit of* f *as* x *approaches* a and write

$$\lim_{x\to a^+} f(x) = L_2 \tag{3.5.2}$$

provided that we can make the value of $f(x)$ as close to L_2 as we like by taking x sufficiently close to a while always having $x > a$. In the graph of the function f in Figure 1.7.2, we see that

$$\lim_{x\to 1^-} f(x) = 2 \text{ and } \lim_{x\to 1^+} f(x) = 3 \tag{3.5.3}$$

and precisely because the left and right limits are not equal, the overall limit of f as $x \to 1$ fails to exist.

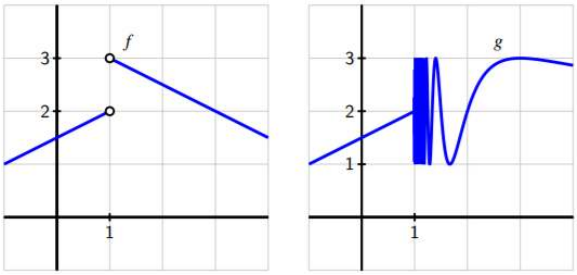


***Figure* 3.5.2**: *Functions f and g that each fail to have a limit at a = 1.*

For the function g pictured at right in Figure 1.7.2, the function fails to have a limit at $a = 1$ for a different reason. While the function does not have a jump in its graph at $a = 1$, it is still not the case that g approaches a single value as x approaches 1. In particular, due to the infinitely oscillating behavior of g to the right of $a = 1$, we say that the right-hand limit of g as $x \to 1^+$ does not exist, and thus $lim_{x\to 1} g(x)$ does not exist. To summarize, anytime either a left- or right-hand limit fails to exist or the left- and right-hand limits are not equal to each other, the overall limit will not exist. Said differently,

A function f has limit L as $x \to a$ if and only if

$$lim_{x\to a^-} f(x) = L = lim_{x\to a^+} f(x). \tag{3.5.4}$$

That is, a function has a limit at $x = a$ if and only if both the left- and right-hand limits at $x = a$ exist and share the same value.

In Preview Activity 1.7, the function f given in Figure 1.7.1 only fails to have a limit at two values: at $a = -2$ (where the left- and right-hand limits are 2 and −1, respectively) and at $x = 2$, where $lim_{x\to 2^+} f(x)$ does not exist). Note well that even at values like a = −1 and a = 0 where there are holes in the graph, the limit still exists.

Activity 3.5.2

Consider a function that is piecewise-defined according to the formula

$$f(x) = \begin{cases} 3(x+2)+2 & \text{for } -3 < x < -2 \\ \frac{2}{3}(x+2)+1 & \text{for } -2 \le x < -1 \\ \frac{2}{3}(x+2)+1 & \text{for } -1 < x < 1 \\ 2 & \text{for } x = 1 \\ 4-x & \text{for } x > 1 \end{cases} \tag{3.5.5}$$

Use the given formula to answer the following questions.

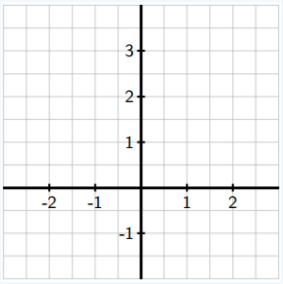


***Figure* 3.5.3**: *Axes for plotting the function $y = f(x)$ in Activity 1.18.*

(a) For each of the values a = −2, −1, 0, 1, 2, compute $f(a)$.

(b) For each of the values a = −2, −1, 0, 1, 2, determine $lim_{x\to a^-} f(x)$ and $lim_{x\to a^+} f(x)$.

(c) For each of the values a = −2, −1, 0, 1, 2, determine $\lim_{x\to a} f(x)$. If the limit fails to exist, explain why by discussing the left- and right-hand limits at the relevant a-value.

(d) For which values of a is the following statement true?

$$\lim_{x\to a} f(x) \neq f(a) \tag{3.5.6}$$

(e) On the axes provided in Figure 1.7.3, sketch an accurate, labeled graph of $y = f(x)$. Be sure to carefully use open circles (◦) and filled circles (•) to represent key points on the graph, as dictated by the piecewise formula.

Being continuous at a point

Intuitively, a function is continuous if we can draw it without ever lifting our pencil from the page. Alternatively, we might say that the graph of a continuous function has no jumps or holes in it. We first consider three specific situations in Figure 1.7.4 where all three functions have a limit at $a = 1$, and then work to make the idea of continuity more precise.

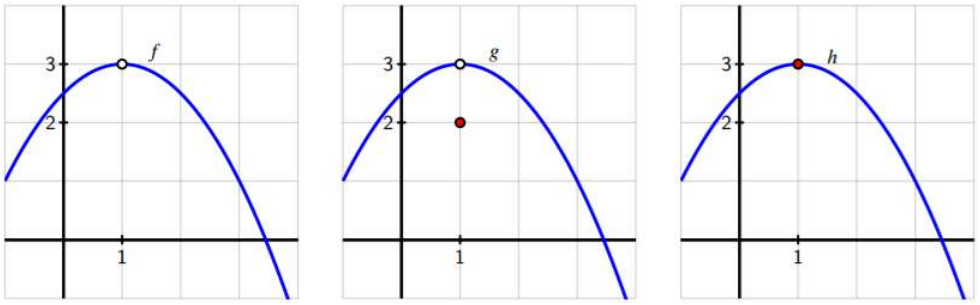


Figure 3.5.4: *Functions f ,g, and h that demonstrate subtly different behaviors at $a = 1$.*

Note that f (1) is not defined, which leads to the resulting hole in the graph of f at $a = 1$. We will naturally say that f is *not continuous* at $a = 1$. For the next function g in in Figure 1.7.4, we observe that while $lim_{x\to 1} g(x) = 3$, the value of g (1) = 2, and thus the limit does not equal the function value. Here, too, we will say that g is *not continuous*, even though the function is defined at $a = 1$. Finally, the function h appears to be the most well-behaved of all three, since at $a = 1$ its limit and its function value agree. That is,

$$\lim_{x\to 1} h(x) = 3 = h(1). \tag{3.5.7}$$

With no hole or jump in the graph of h at $a = 1$, we desire to say that h is *continuous* there. More formally, we make the following definition.

Definition 1.7

A function f is *continuous* at $x = a$ provided that

(a) f has a limit as $x \to a$,

(b) f is defined at $x = a$, and

(c) $\lim_{x\to a} f(x) = f(a)$.

Conditions (a) and (b) are technically contained implicitly in (c), but we state them explicitly to emphasize their individual importance. In words, (c) essentially says that a function is continuous at $x = a$ provided that its limit as $x \to a$ exists and equals its function value at $x = a$. If a function is continuous at every point in an interval $[a, b]$, we say the function is "continuous on $[a, b]$." If a function is continuous at every point in its domain, we simply say the function is "continuous." Thus, continuous functions are particularly nice: to evaluate the limit of a continuous function at a point, all we need to do is evaluate the function.

Activity 3.5.3

This activity builds on your work in Preview Activity 1.7, using the same function f as given by the graph that is repeated in Figure 1.7.5

(a) At which values of a does $\lim_{x\to a} f(x)$ not exist?

(b) At which values of a is $f(a)$ not defined?

(c) At which values of a does f have a limit, but $\lim_{x\to a} f(x) \neq f(a)$?

(d) State all values of a for which f is not continuous at $x = a$.

(e) Which condition is stronger, and hence implies the other: f has a limit at $x = a$ or f is continuous at $x = a$? Explain, and hence complete the following sentence: "If f at $x = a$, then f at $x = a$," where you complete the blanks with *has a limit* and *is continuous*, using each phrase once.

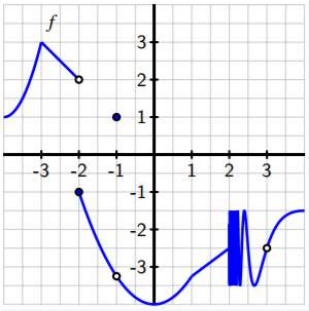


***Figure* 3.5.5:** *The graph of $y = f(x)$ for Activity 1.19.*

Being differentiable at a point

We recall that a function f is said to be differentiable at $x = a$ whenever $f'(a)$ exists. Moreover, for $f'(a)$ to exist, we know that the function $y = f(x)$ must have a tangent line at the point $(a, f(a))$, since $f'(a)$ is precisely the slope of this line. In order to even ask if f has a tangent line at $(a, f(a))$, it is necessary that f be continuous at $x = a$: if f fails to have a limit at $x = a$, if $f(a)$ is not defined, or if $f(a)$ does not equal the value of $\lim_{x\to a} f(x)$, then it doesn't even make sense to talk about a tangent line to the curve at this point.

Indeed, it can be proved formally that if a function f is differentiable at x = a, then it must be continuous at $x = a$. So, if f is not continuous at $x = a$, then it is automatically the case that f is not differentiable there. For example, in Figure 1.7.4 from our early discussion of continuity, both f and g fail to be differentiable at $x = 1$ because neither function is continuous at $x = 1$. But can a function fail to be differentiable at a point where the function is continuous?

In Figure 1.7.6, we revisit the situation where a function has a sharp corner at a point, something we encountered several times in Section 1.4. For the pictured function f, we observe that f is clearly continuous at $a = 1$, since $\lim_{x\to 1} f(x) = 1 = f(1)$.

But the function f in Figure 1.7.6 is not differentiable at $a = 1$ because $f'(1)$ fails to exist. One way to see this is to observe that $f'(x) = -1$ for every value of x that is less than 1, while $f'(x) = -1$ for every value of x that is greater than 1. That makes it seem that either +1 or −1 would be equally good candidates for the value of the derivative at $x = 1$. Alternately, we could use the limit definition of the derivative to attempt to compute $f'(x) = -1$, and discover that the derivative does not exist. A similar problem will be investigated in Activity 1.20. Finally, we can also see visually that the function f in Figure 1.7.6 does not have a tangent line. When we zoom in on (1, 1) on the graph of f, no matter how closely we examine the function, it will always look like a "V", and never like a single line, which

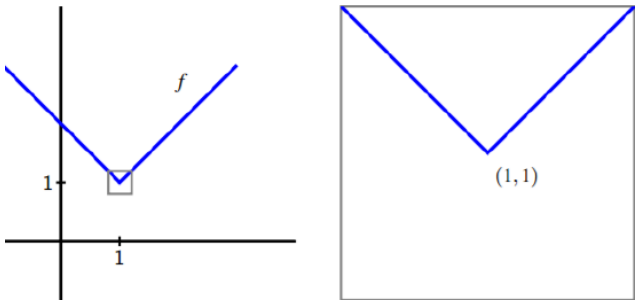


***Figure* 3.5.6**: *A function f that is continuous at $a = 1$ but not differentiable at $a = 1$; at right, we zoom in on the point $(1, 1)$ in a magnified version of the box in the left-hand plot.*

tells us there is no possibility for a tangent line there.

To make a more general observation, if a function does have a tangent line at a given point, when we zoom in on the point of tangency, the function and the tangent line should appear essentially indistinguishable[7] . Conversely, if we have a function such that when we zoom in on a point the function looks like a single straight line, then the function should have a tangent line there, and thus be differentiable. Hence, a function that is differentiable at $x = a$ will, up close, look more and more like its tangent line at $(a, f(a))$, and thus we say that a function is differentiable at $x = a$ is *locally linear*.

To summarize the preceding discussion of differentiability and continuity, we make several important observations.

- If f is differentiable at $x = a$, then f is continuous at $x = a$. Equivalently, if f fails to be continuous at $x = a$, then f will not be differentiable at $x = a$.
- A function can be continuous at a point, but not be differentiable there. In particular, a function f is not differentiable at $x = a$ if the graph has a sharp corner (or cusp) at the point (a, f (a)).
- If f is differentiable at $x = a$, then f is locally linear at $x = a$. That is, when a function is differentiable, it looks linear when viewed up close because it resembles its tangent line there.

Activity 3.5.4

In this activity, we explore two different functions and classify the points at which each is not differentiable. Let g be the function given by the rule $g(x) = |x|$, and let $f$be the function that we have previously explored in Preview Activity 1.7, whose graph is given again in Figure 1.41.

(a) Reasoning visually, explain why g is differentiable at every point x such that $x \neq 0$.

(b) Use the limit definition of the derivative to show that $g'(0) = \lim_{h \to 0} \frac{|h|}{h}$.

(c) Explain why $g'(0)$ fails to exist by using small positive and negative values of h.

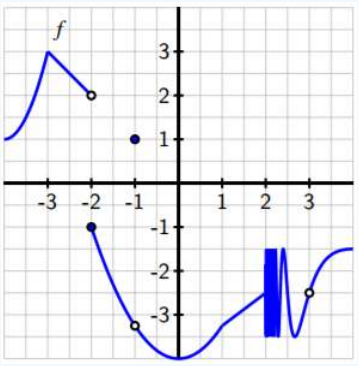


***Figure* 3.5.7**: *The graph of $y = f(x)$ for Activity 1.20.*

(d) State all values of a for which f is not differentiable at $x = a$. For each, provide a reason for your conclusion.

(e) True or false: if a function p is differentiable at $x = b$, then $\lim_{x \to b} p(x)$must exist. Why?

Summary

In this section, we encountered the following important ideas:

- A function f has limit L as $x \to a$ if and only if f has a left-hand limit at $x = a$, has a right-hand limit at $x = a$, and the left- and right-hand limits are equal. Visually, this means that there can be a hole in the graph at $x = a$, but the function must approach the same single value from either side of $x = a$.
- A function f is continuous at $x = a$ whenever $f(a)$ is defined, f has a limit as $x \to a$, and the value of the limit and the value of the function agree. This guarantees that there is not a hole or jump in the graph of f at $x = a$.
- A function f is differentiable at $x = a$ whenever $f'(a)$ exists, which means that f has a tangent line at $(a, f(a))$ and thus f is locally linear at the value $x = a$. Informally, this means that the function looks like a line when viewed up close at $(a, f(a))$ and that there is not a corner point or cusp at $(a, f(a))$.
- Of the three conditions discussed in this section (having a limit at $x = a$, being continuous at $x = a$, and being differentiable at $x = a$), the strongest condition is being differentiable, and the next strongest is being continuous. In particular, if f is differentiable at $x = a$, then f is also continuous at $x = a$, and if f is continuous at $x = a$, then f has a limit at $x = a$.

[7]See, for instance, http://gvsu.edu/s/6J for an applet (due to David Austin, GVSU) where zooming in shows the increasing similarity between the tangent line and the curve.

Contributors and Attributions

- Matt Boelkins (Grand Valley State University), David Austin (Grand Valley State University), Steve Schlicker (Grand Valley State University)

Supplemental videos







Homework Exercises 3.5

WeBWorK Problems:

1.

2.

3.

4.

5.

6.

7.

8.

Written Problems:

1.

Consider the graph of the function $y = p(x)$ that is provided in Figure 1.7.10. Assume that each portion of the graph of p is a straight line, as pictured.

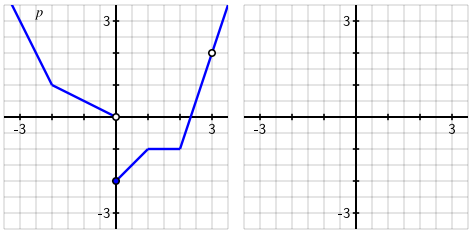


Figure 1.7.10. At left, the piecewise linear function $y = p(x)$. At right, axes for plotting $y = p'(x)$.

1. State all values of a for which $\lim_{x \to a} p(x)$ does not exist.
2. State all values of a for which p is not continuous at a.
3. State all values of a for which p is not differentiable at $x = a$.
4. On the axes provided in Figure 1.7.10, sketch an accurate graph of $y = p'(x)$.

2. For each description below, sketch an example of a graph of a function (all are different functions). If such a function is not possible, explain why.

a. f is continuous at a = 2, but is not differentiable at a = 2.

b. g is differentiable at a = 3, but does not have a limit at a = 3.

c. h has a limit at a = -2, is defined at a = -2, but is not continuous at a = -2.

d. p(-1) = 3, $\lim_{x \to -1} p(x) = 2$

e. $r(0) = 1$, $r'(0) = 0$

f. $\lim_{x \to 1} q(x) = q(1)$ and $q'(1)$ DNE.

This page titled 3.5: Limits, Continuity, and Differentiability is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Matthew Boelkins, David Austin & Steven Schlicker (ScholarWorks @Grand Valley State University) via source content that was edited to the style and standards of the LibreTexts platform.

- **1.7: Limits, Continuity, and Differentiability** is licensed CC BY-SA 4.0. Original source: https://activecalculus.org/single.
- **1.7: Limits, Continuity, and Differentiability** by Matthew Boelkins, David Austin & Steven Schlicker is licensed CC BY-SA 4.0. Original source: https://activecalculus.org/single.

3.6: The Mean Value Theorem

Preview Video 3.6

Fill out your first page of notes for this section as you view the following video and answer the questions.

Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login

- Explain the meaning of Rolle's theorem.
- Describe the significance of the Mean Value Theorem.
- State three important consequences of the Mean Value Theorem.

The *Mean Value Theorem* is one of the most important theorems in calculus. We look at some of its implications at the end of this section. First, let's start with a special case of the Mean Value Theorem, called Rolle's theorem.

Rolle's Theorem

Informally, **Rolle's theorem** states that if the outputs of a differentiable function f are equal at the endpoints of an interval, then there must be an interior point c where $f'(c) = 0$. Figure $3.6.1$ illustrates this theorem.

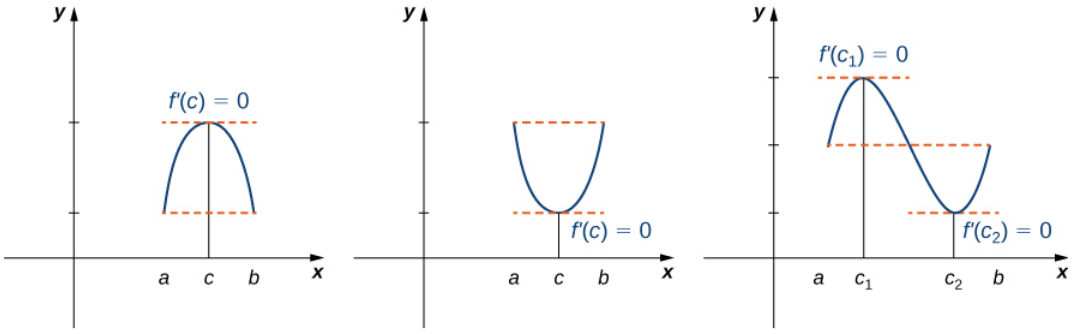


Figure $3.6.1$: If a differentiable function f satisfies $f(a) = f(b)$, then its derivative must be zero at some point(s) between a and b.

Rolle's Theorem

Let f be a continuous function over the closed interval $[a, b]$ and differentiable over the open interval (a, b) such that $f(a) = f(b)$. There then exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

Proof

Let $k = f(a) = f(b)$. We consider three cases:

1. $f(x) = k$ for all $x \in (a, b)$.
2. There exists $x \in (a, b)$ such that $f(x) > k$.
3. There exists $x \in (a, b)$ such that $f(x) < k$.

Case 1: If $f(x) = k$ for all $x \in (a, b)$, then $f'(x) = 0$ for all $x \in (a, b)$.

Case 2: Since f is a continuous function over the closed, bounded interval $[a, b]$, by the extreme value theorem, it has an absolute maximum. Also, since there is a point $x \in (a, b)$ such that $f(x) > k$, the absolute maximum is greater than k. Therefore, the absolute maximum does not occur at either endpoint. As a result, the absolute maximum must occur at an interior point $c \in (a, b)$. Because f has a maximum at an interior point c, and f is differentiable at c, by Fermat's theorem, $f'(c) = 0$.

Case 3: The case when there exists a point $x \in (a, b)$ such that $f(x) < k$ is analogous to case 2, with maximum replaced by minimum.

□

An important point about Rolle's theorem is that the differentiability of the function f is critical. If f is not differentiable, even at a single point, the result may not hold. For example, the function $f(x) = |x| - 1$ is continuous over $[-1, 1]$ and $f(-1) = 0 = f(1)$, but $f'(c) \neq 0$ for any $c \in (-1, 1)$ as shown in the following figure.

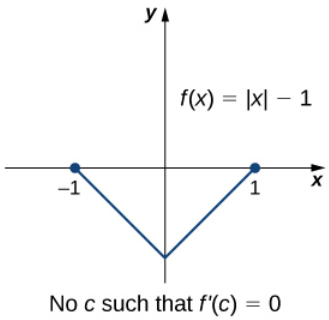


Figure 3.6.2: Since $f(x) = |x| - 1$ is not differentiable at $x = 0$, the conditions of Rolle's theorem are not satisfied. In fact, the conclusion does not hold here; there is no $c \in (-1, 1)$ such that $f'(c) = 0$.

Let's now consider functions that satisfy the conditions of Rolle's theorem and calculate explicitly the points c where $f'(c) = 0$.

Example 3.6.1: Using Rolle's Theorem

For each of the following functions, verify that the function satisfies the criteria stated in Rolle's theorem and find all values c in the given interval where $f'(c) = 0$.

a. $f(x) = x^2 + 2x$ over $[-2, 0]$
b. $f(x) = x^3 - 4x$ over $[-2, 2]$

Solution

a. Since f is a polynomial, it is continuous and differentiable everywhere. In addition, $f(-2) = 0 = f(0)$. Therefore, f satisfies the criteria of Rolle's theorem. We conclude that there exists at least one value $c \in (-2, 0)$ such that $f'(c) = 0$. Since $f'(x) = 2x + 2 = 2(x + 1)$, we see that $f'(c) = 2(c + 1) = 0$ implies $c = -1$ as shown in the following graph.

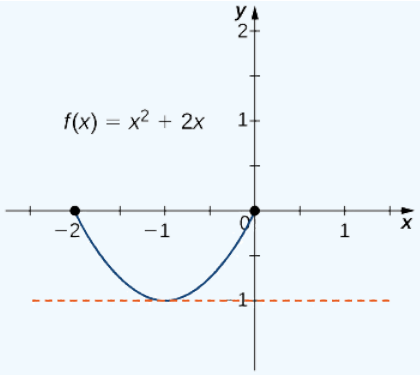


Figure 3.6.3: This function is continuous and differentiable over $[-2,0]$, $f'(c) = 0$ when $c = -1$.

b. As in part a. f is a polynomial and therefore is continuous and differentiable everywhere. Also, $f(-2) = 0 = f(2)$. That said, f satisfies the criteria of Rolle's theorem. Differentiating, we find that $f'(x) = 3x^2 - 4$. Therefore, $f'(c) = 0$ when $x = \pm\frac{2}{\sqrt{3}}$. Both points are in the interval $[-2, 2]$, and, therefore, both points satisfy the conclusion of Rolle's theorem as shown in the following graph.

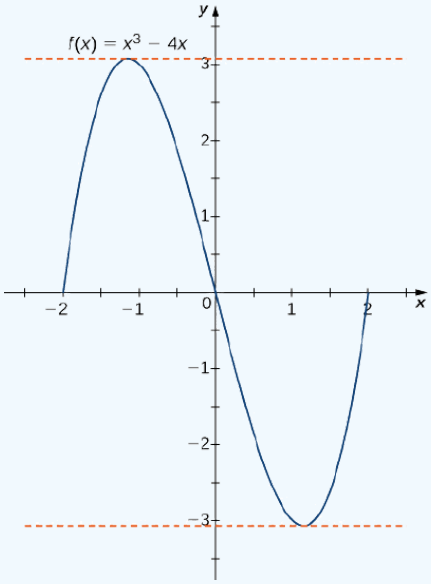


Figure 3.6.4: For this polynomial over $[-2, 2]$, $f'(c) = 0$ at $x = \pm 2/\sqrt{3}$.

? Exercise 3.6.1

Verify that the function $f(x) = 2x^2 - 8x + 6$ defined over the interval $[1, 3]$ satisfies the conditions of Rolle's theorem. Find all points c guaranteed by Rolle's theorem.

Hint

Find all values c, where $f'(c) = 0$.

Answer

$c = 2$

The Mean Value Theorem and Its Meaning

Rolle's theorem is a special case of the Mean Value Theorem. In Rolle's theorem, we consider differentiable functions f that are zero at the endpoints. The Mean Value Theorem generalizes Rolle's theorem by considering functions that are not necessarily zero at the endpoints. Consequently, we can view the Mean Value Theorem as a slanted version of Rolle's theorem (Figure 3.6.5). The

Mean Value Theorem states that if f is continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b), then there exists a point $c \in (a, b)$ such that the tangent line to the graph of f at c is parallel to the secant line connecting $(a, f(a))$ and $(b, f(b))$.

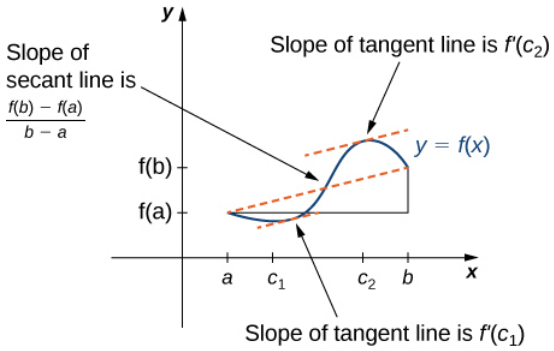


Figure 3.6.5: The Mean Value Theorem says that for a function that meets its conditions, at some point the tangent line has the same slope as the secant line between the ends. For this function, there are two values c_1 and c_2 such that the tangent line to f at c_1 and c_2 has the same slope as the secant line.

Mean Value Theorem

Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b). Then, there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{3.6.1}$$

Proof

The proof follows from Rolle's theorem by introducing an appropriate function that satisfies the criteria of Rolle's theorem. Consider the line connecting $(a, f(a))$ and $(b, f(b))$. Since the slope of that line is

$$\frac{f(b) - f(a)}{b - a} \tag{3.6.2}$$

and the line passes through the point $(a, f(a))$, the equation of that line can be written as

$$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a). \tag{3.6.3}$$

Let $g(x)$ denote the vertical difference between the point $(x, f(x))$ and the point (x, y) on that line. Therefore,

$$g(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right]. \tag{3.6.4}$$

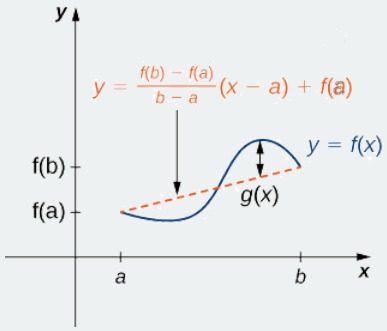


Figure 3.6.6: The value $g(x)$ is the vertical difference between the point $(x, f(x))$ and the point (x, y) on the secant line connecting $(a, f(a))$ and $(b, f(b))$.

Since the graph of f intersects the secant line when $x = a$ and $x = b$, we see that $g(a) = 0 = g(b)$. Since f is a differentiable function over (a, b), g is also a differentiable function over (a, b). Furthermore, since f is continuous over $[a, b]$, g is also

continuous over $[a, b]$. Therefore, g satisfies the criteria of Rolle's theorem. Consequently, there exists a point $c \in (a, b)$ such that $g'(c) = 0$. Since

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}, \tag{3.6.5}$$

we see that

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}. \tag{3.6.6}$$

Since $g'(c) = 0$, we conclude that

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{3.6.7}$$

□

In the next example, we show how the Mean Value Theorem can be applied to the function $f(x) = \sqrt{x}$ over the interval $[0, 9]$. The method is the same for other functions, although sometimes with more interesting consequences.

Example 3.6.2: Verifying that the Mean Value Theorem Applies

For $f(x) = \sqrt{x}$ over the interval $[0, 9]$, show that f satisfies the hypothesis of the Mean Value Theorem, and therefore there exists at least one value $c \in (0, 9)$ such that $f'(c)$ is equal to the slope of the line connecting $(0, f(0))$ and $(9, f(9))$. Find these values c guaranteed by the Mean Value Theorem.

Solution

We know that $f(x) = \sqrt{x}$ is continuous over $[0, 9]$ and differentiable over $(0, 9)$. Therefore, f satisfies the hypotheses of the Mean Value Theorem, and there must exist at least one value $c \in (0, 9)$ such that $f'(c)$ is equal to the slope of the line connecting $(0, f(0))$ and $(9, f(9))$ (Figure 3.6.7). To determine which value(s) of c are guaranteed, first calculate the derivative of f. The derivative $f'(x) = \frac{1}{(2\sqrt{x})}$. The slope of the line connecting $(0, f(0))$ and $(9, f(9))$ is given by

$$\frac{f(9) - f(0)}{9 - 0} = \frac{\sqrt{9} - \sqrt{0}}{9 - 0} = \frac{3}{9} = \frac{1}{3}.$$

We want to find c such that $f'(c) = \frac{1}{3}$. That is, we want to find c such that

$$\frac{1}{2\sqrt{c}} = \frac{1}{3}.$$

Solving this equation for c, we obtain $c = \frac{9}{4}$. At this point, the slope of the tangent line equals the slope of the line joining the endpoints.

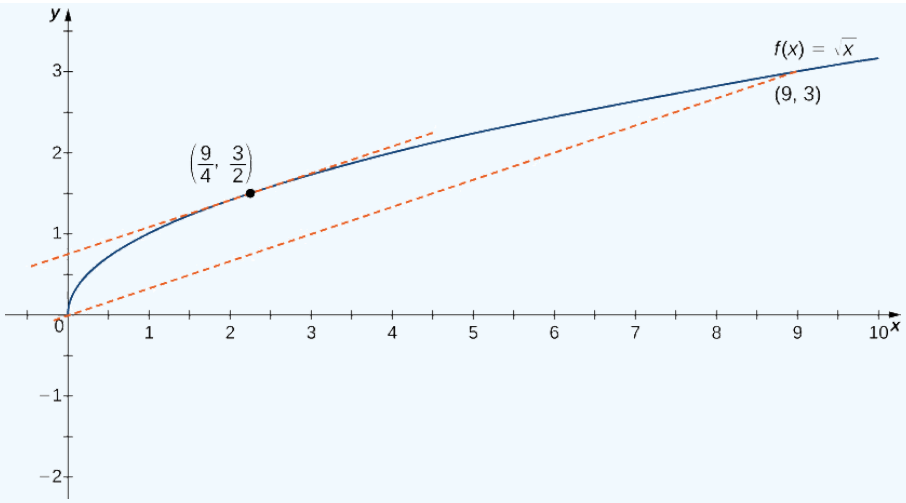


Figure 3.6.7: The slope of the tangent line at $c = 9/4$ is the same as the slope of the line segment connecting (0,0) and (9,3).

One application that helps illustrate the Mean Value Theorem involves velocity. For example, suppose we drive a car for 1 h down a straight road with an average velocity of 45 mph. Let $s(t)$ and $v(t)$ denote the position and velocity of the car, respectively, for $0 \le t \le 1$ h. Assuming that the position function $s(t)$ is differentiable, we can apply the Mean Value Theorem to conclude that, at some time $c \in (0, 1)$, the speed of the car was exactly

$$v(c) = s'(c) = \frac{s(1) - s(0)}{1 - 0} = 45 \text{ mph}.$$

Example 3.6.3: Mean Value Theorem and Velocity

If a rock is dropped from a height of 100 ft, its position t seconds after it is dropped until it hits the ground is given by the function $s(t) = -16t^2 + 100$.

a. Determine how long it takes before the rock hits the ground.
b. Find the average velocity v_{avg} of the rock for when the rock is released and the rock hits the ground.
c. Find the time t guaranteed by the Mean Value Theorem when the instantaneous velocity of the rock is v_{avg}.

Solution

a. When the rock hits the ground, its position is $s(t) = 0$. Solving the equation $-16t^2 + 100 = 0$ for t, we find that $t = \pm\frac{5}{2} sec$. Since we are only considering $t \ge 0$, the ball will hit the ground $\frac{5}{2}$ sec after it is dropped.

b. The average velocity is given by

$$v_{avg} = \frac{s(5/2) - s(0)}{5/2 - 0} = \frac{0 - 100}{5/2} = -40 \text{ ft/sec}.$$

c. The instantaneous velocity is given by the derivative of the position function. Therefore, we need to find a time t such that $v(t) = s'(t) = v_{avg} = -40$ ft/sec. Since $s(t)$ is continuous over the interval $[0, 5/2]$ and differentiable over the interval $(0, 5/2)$, by the Mean Value Theorem, there is guaranteed to be a point $c \in (0, 5/2)$ such that

$$s'(c) = \frac{s(5/2) - s(0)}{5/2 - 0} = -40.$$

Taking the derivative of the position function $s(t)$, we find that $s'(t) = -32t$. Therefore, the equation reduces to $s'(c) = -32c = -40$. Solving this equation for c, we have $c = \frac{5}{4}$. Therefore, $\frac{5}{4}$ sec after the rock is dropped, the instantaneous velocity equals the average velocity of the rock during its free fall: -40 ft/sec.

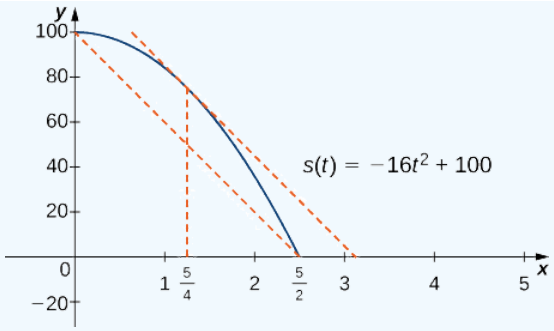


Figure 3.6.8: At time $t = 5/4$ sec, the velocity of the rock is equal to its average velocity from the time it is dropped until it hits the ground.

Exercise 3.6.2

Suppose a ball is dropped from a height of 200 ft. Its position at time t is $s(t) = -16t^2 + 200$. Find the time t when the instantaneous velocity of the ball equals its average velocity.

Hint

First, determine how long it takes for the ball to hit the ground. Then, find the average velocity of the ball from the time it is dropped until it hits the ground.

Answer

$\frac{5}{2\sqrt{2}}$ sec

Corollaries of the Mean Value Theorem

Let's now look at three corollaries of the Mean Value Theorem. These results have important consequences, which we use in upcoming sections.

At this point, we know the derivative of any constant function is zero. The Mean Value Theorem allows us to conclude that the converse is also true. In particular, if $f'(x) = 0$ for all x in some interval I, then $f(x)$ is constant over that interval. This result may seem intuitively obvious, but it has important implications that are not obvious, and we discuss them shortly.

Corollary 1: Functions with a Derivative of Zero

Let f be differentiable over an interval I. If $f'(x) = 0$ for all $x \in I$, then $f(x) =$ constant for all $x \in I$.

Proof

Since f is differentiable over I, f must be continuous over I. Suppose $f(x)$ is not constant for all x in I. Then there exist $a, b \in I$, where $a \neq b$ and $f(a) \neq f(b)$. Choose the notation so that $a < b$. Therefore,

$$\frac{f(b) - f(a)}{b - a} \neq 0.$$

Since f is a differentiable function, by the Mean Value Theorem, there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Therefore, there exists $c \in I$ such that $f'(c) \neq 0$, which contradicts the assumption that $f'(x) = 0$ for all $x \in I$.

□

From "Corollary 1: Functions with a Derivative of Zero," it follows that if two functions have the same derivative, they differ by, at most, a constant.

Corollary 2: Constant Difference Theorem

If f and g are differentiable over an interval I and $f'(x) = g'(x)$ for all $x \in I$, then $f(x) = g(x) + C$ for some constant C.

Proof

Let $h(x) = f(x) - g(x)$. Then, $h'(x) = f'(x) - g'(x) = 0$ for all $x \in I$. By Corollary 1, there is a constant C such that $h(x) = C$ for all $x \in I$. Therefore, $f(x) = g(x) + C$ for all $x \in I$.

□

The third corollary of the Mean Value Theorem discusses when a function is increasing and when it is decreasing. Recall that a function f is increasing over I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, whereas f is decreasing over I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$. Using the Mean Value Theorem, we can show that if the derivative of a function is positive, then the function is increasing; if the derivative is negative, then the function is decreasing (Figure 3.6.9). We make use of this fact in the next section, where we show how to use the derivative of a function to locate local maximum and minimum values of the function, and how to determine the shape of the graph.

This fact is important because it means that for a given function f, if there exists a function F such that $F'(x) = f(x)$; then, the only other functions that have a derivative equal to f are $F(x) + C$ for some constant C. We discuss this result in more detail later in the chapter.

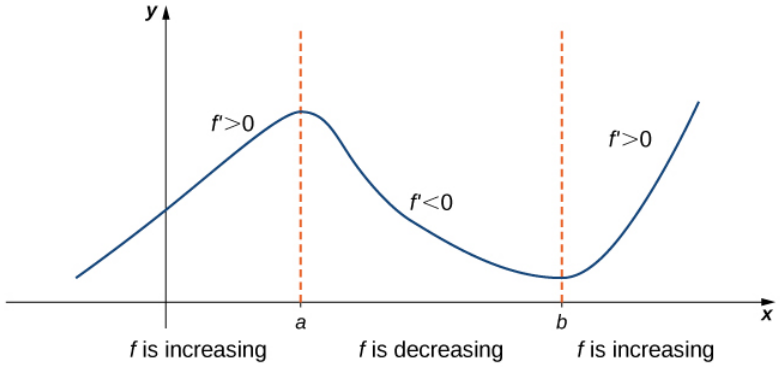


Figure 3.6.9: If a function has a positive derivative over some interval I, then the function increases over that interval I; if the derivative is negative over some interval I, then the function decreases over that interval I.

Corollary 3: Increasing and Decreasing Functions

Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b).

i. If $f'(x) > 0$ for all $x \in (a, b)$, then f is an increasing function over $[a, b]$.
ii. If $f'(x) < 0$ for all $x \in (a, b)$, then f is a decreasing function over $[a, b]$.

Proof

We will prove i.; the proof of ii. is similar. Suppose f is not an increasing function on I. Then there exist a and b in I such that $a < b$, but $f(a) \geq f(b)$. Since f is a differentiable function over I, by the Mean Value Theorem there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Since $f(a) \geq f(b)$, we know that $f(b) - f(a) \leq 0$. Also, $a < b$ tells us that $b - a > 0$. We conclude that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \leq 0.$$

However, $f'(x) > 0$ for all $x \in I$. This is a contradiction, and therefore f must be an increasing function over I.

□

Key Concepts

- If f is continuous over $[a,b]$ and differentiable over (a,b) and $f(a)=f(b)$, then there exists a point $c\in(a,b)$ such that $f'(c)=0$. This is Rolle's theorem.
- If f is continuous over $[a,b]$ and differentiable over (a,b), then there exists a point $c\in(a,b)$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}.$$

  This is the Mean Value Theorem.
- If $f'(x)=0$ over an interval I, then f is constant over I.
- If two differentiable functions f and g satisfy $f'(x)=g'(x)$ over I, then $f(x)=g(x)+C$ for some constant C.
- If $f'(x)>0$ over an interval I, then f is increasing over I. If $f'(x)<0$ over I, then f is decreasing over I.

Glossary

mean value theorem

if f is continuous over $[a,b]$ and differentiable over (a,b), then there exists $c\in(a,b)$ such that $f'(c)=\frac{f(b)-f(a)}{b-a}$

rolle's theorem

if f is continuous over $[a,b]$ and differentiable over (a,b), and if $f(a)=f(b)$, then there exists $c\in(a,b)$ such that $f'(c)=0$

Homework Exercises 3.6

1) Why do you need continuity to apply the Mean Value Theorem? Construct a counterexample. In other words, sketch a graph of a function that is not continuous on an interval and does not have a tangent line parallel to the secant line.

2) Why do you need differentiability to apply the Mean Value Theorem? Find a counterexample. Please sketch a different example than the one given in the Answer.

Answer

One example is $f(x)=|x|+3, -2\le x\le 2$

3) When are Rolle's theorem and the Mean Value Theorem equivalent?

4) If you have a function with a discontinuity, is it still possible to have $f'(c)(b-a)=f(b)-f(a)$? Draw such an example or prove why not.

Answer

Yes, but the Mean Value Theorem still does not apply

In exercises 5 - 9, determine over what intervals (if any) the Mean Value Theorem applies. Justify your answer.

5) $y=\sin(\pi x)$

6) $y=\dfrac{1}{x^3}$

Answer

$(-\infty,0),(0,\infty)$

7) $y=\sqrt{4-x^2}$

8) $y=\sqrt{x^2-4}$

Answer

$(-\infty,-2),(2,\infty)$

9) $y = \ln(3x - 5)$

In exercises 10 - 13, graph the functions on a calculator and draw the secant line that connects the endpoints. Estimate the number of points c such that $f'(c)(b-a) = f(b) - f(a)$. . Note that you can modify this formula to make it look like the result of the MVT. Show on your sketch of the graph the approximate values of c.

10) $y = 3x^3 + 2x + 1$ over $[-1, 1]$

Answer

2 points

11) $y = \tan\left(\frac{\pi}{4}x\right)$ over $\left[-\frac{3}{2}, \frac{3}{2}\right]$

12) $y = x^2 \cos(\pi x)$ over $[-2, 2]$

Answer

5 points

13) $y = x^6 - \frac{3}{4}x^5 - \frac{9}{8}x^4 + \frac{15}{16}x^3 + \frac{3}{32}x^2 + \frac{3}{16}x + \frac{1}{32}$ over $[-1, 1]$

In exercises 14 - 19, use the Mean Value Theorem and find all points $0 < c < 2$ such that $f(2) - f(0) = f'(c)(2-0)$.

14) $f(x) = x^3$

Answer

$c = \frac{2\sqrt{3}}{3}$

15) $f(x) = \sin(\pi x)$

16) $f(x) = \cos(2\pi x)$

Answer

$c = \frac{1}{2}, 1, \frac{3}{2}$

17) $f(x) = 1 + x + x^2$

18) $f(x) = (x-1)^{10}$

Answer

$c = 1$

19) $f(x) = (x-1)^9$

In exercises 20 - 23, show there is no c such that $f(1) - f(-1) = f'(c)(2)$. Explain why the Mean Value Theorem does not apply over the interval $[-1, 1]$.

20) $f(x) = \left|x - \frac{1}{2}\right|$

Answer

Not differentiable

21) $f(x) = \dfrac{1}{x^2}$

22) $f(x) = \sqrt{|x|}$

Answer

Not differentiable

23) $f(x) = \lfloor x \rfloor$ (Hint: This is called the floor function and it is defined so that $f(x)$ is the largest integer less than or equal to x.)

In exercises 24 - 34, determine whether the Mean Value Theorem applies for the functions over the given interval $[a, b]$. Justify your answer.

24) $y = e^x$ over $[0, 1]$

Answer

Yes

25) $y = \ln(2x + 3)$ over $[-\frac{3}{2}, 0]$

26) $f(x) = \tan(2\pi x)$ over $[0, 2]$

Answer

The Mean Value Theorem does not apply since the function is discontinuous at $x = \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}$.

27) $y = \sqrt{9 - x^2}$ over $[-3, 3]$

28) $y = \dfrac{1}{|x + 1|}$ over $[0, 3]$

Answer

Yes

29) $y = x^3 + 2x + 1$ over $[0, 6]$

30) $y = \dfrac{x^2 + 3x + 2}{x}$ over $[-1, 1]$

Answer

The Mean Value Theorem does not apply; discontinuous at $x = 0$.

31) $y = \dfrac{x}{\sin(\pi x) + 1}$ over $[0, 1]$

32) $y = \ln(x + 1)$ over $[0, e - 1]$

Answer

Yes

33) $y = x \sin(\pi x)$ over $[0, 2]$

34) $y = 5 + |x|$ over $[-1, 1]$

Answer

The Mean Value Theorem does not apply; not differentiable at $x = 0$.

For exercises 35 - 37, consider the roots of each equation.

35) Show that the equation $y = x^3 + 3x^2 + 16$ has exactly one real root. What is it?

36) Find the conditions for exactly one root (double root) for the equation $y = x^2 + bx + c$

Answer

$b = \pm 2\sqrt{c}$

37) Find the conditions for $y = e^x - b$ to have one root. Is it possible to have more than one root?

In exercises 38 - 42, use a calculator to graph the function over the interval $[a, b]$ and graph the secant line from a to b. Use the calculator to estimate all values of c as guaranteed by the Mean Value Theorem. Then, find the exact value of c, if possible, or write the final equation and use a calculator to estimate to four digits.

38) [T] $y = \tan(\pi x)$ over $\left[-\frac{1}{4}, \frac{1}{4}\right]$

Answer

$c \approx \pm 0.1533$

$c = \pm \frac{1}{\pi} \cos^{-1}\left(\frac{\sqrt{\pi}}{2}\right)$

39) [T] $y = \dfrac{1}{\sqrt{x+1}}$ over $[0, 3]$

40) [T] $y = |x^2 + 2x - 4|$ over $[-4, 0]$

Answer

The Mean Value Theorem does not apply.

41) [T] $y = x + \dfrac{1}{x}$ over $\left[\frac{1}{2}, 4\right]$

42) [T] $y = \sqrt{x+1} + \dfrac{1}{x^2}$ over $[3, 8]$

Answer

$\dfrac{1}{2\sqrt{c+1}} - \dfrac{2}{c^3} = \dfrac{521}{2880}$

$c \approx 3.133, 5.867$

43) At 10:17 a.m., you pass a police car at 55 mph that is stopped on the freeway. You pass a second police car at 55 mph at 10:53 a.m., which is located 39 mi from the first police car. If the speed limit is 60 mph, can the police cite you for speeding?

44) Two cars drive from one stoplight to the next, leaving at the same time and arriving at the same time. Is there ever a time when they are going the same speed? Prove or disprove.

Answer

Yes

45) Show that $y = \sec^2 x$ and $y = \tan^2 x$ have the same derivative. What can you say about $y = \sec^2 x - \tan^2 x$?

46) Show that $y = \csc^2 x$ and $y = \cot^2 x$ have the same derivative. What can you say about $y = \csc^2 x - \cot^2 x$?

Answer

It is constant.

This page titled 3.6: The Mean Value Theorem is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **4.4: The Mean Value Theorem** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.
- **4.4E: Exercises for Section 4.4** is licensed CC BY-NC-SA 4.0.

CHAPTER OVERVIEW

4: Applications of the Derivative

4.1: Interpretating, Estimating, and Using the Derivative
4.2: The Second Derivative
4.3: The Tangent Line Approximation
4.4: Elementary Derivative Rules

4: Applications of the Derivative is shared under a not declared license and was authored, remixed, and/or curated by LibreTexts.

4.1: Interpretating, Estimating, and Using the Derivative

Preview Video

This optional summary video from the textbook author might be helpful to use as a preview. Other, more detailed, supplemental videos for this section are posted at the end of the text.



Learning Objectives

In this section, we strive to understand the ideas generated by the following important questions:

- In contexts other than the position of a moving object, what does the derivative of a function measure?
- What are the units on the derivative function f', and how are they related to the units of the original function f?
- What is a central difference, and how can one be used to estimate the value of the derivative at a point from given function data?
- Given the value of the derivative of a function at a point, what can we infer about how the value of the function changes nearby?

Introduction

An interesting and powerful feature of mathematics is that it can often be thought of both in abstract terms and in applied ones. For instance, calculus can be developed almost entirely as an abstract collection of ideas that focus on properties of arbitrary functions. At the same time, calculus can also be very directly connected to our experience of physical reality by considering functions that represent meaningful processes. We have already seen that for a position function $y = s(t)$, say for a ball being tossed straight up in the air, the ball's velocity at time t is given by $v(t) = s'(t)$, the derivative of the position function.

Further, recall that if $s(t)$ is measured in feet at time t, the units on $v(t) = s'(t)$ are feet per second. In what follows in this section, we investigate several different functions, each with specific physical meaning, and think about how the units on the independent variable, dependent variable, and the derivative function add to our understanding. To start, we consider the familiar problem of a position function of a moving object.

Preview Activity 4.1.1

One of the longest stretches of straight (and flat) road in North America can be found on the Great Plains in the state of North Dakota on state highway 46, which lies just south of the interstate highway I-94 and runs through the town of Gackle. A car leaves town (at time $t = 0$) and heads east on highway 46; its position in miles from Gackle at time t in minutes is given by the graph of the function in Figure 1.22. Three important points are labeled on the graph; where the curve looks linear, assume that it is indeed a straight line.

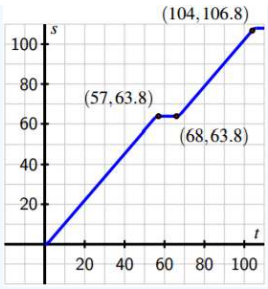


Figure 1.22: The graph of $y = s(t)$, the position of the car along highway 46, which tells its distance in miles from Gackle, ND, at time t in minutes.

a. In everyday language, describe the behavior of the car over the provided time interval. In particular, discuss what is happening on the time intervals [57, 68] and [68, 104].
b. Find the slope of the line between the points (57, 63.8) and (104, 106.8). What are the units on this slope? What does the slope represent?
c. Find the average rate of change of the car's position on the interval [68, 104]. Include units on your answer.
d. Estimate the instantaneous rate of change of the car's position at the moment $t = 80$. Write a sentence to explain your reasoning and the meaning of this value.

Units of the derivative function

As we now know, the derivative of the function f at a fixed value x is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{4.1.1}$$

,

and this value has several different interpretations. If we set $x = a$, one meaning of $f'(a)$ is the slope of the tangent line at the point $(a, (f(a))$.

In alternate notation, we also sometimes equivalently write $\frac{df}{dx}$ or $\frac{dy}{dx}$ instead of $f'(x)$, and these notations helps us to further see the units (and thus the meaning) of the derivative as it is viewed as *the instantaneous rate of change of f with respect to x*. Note that the units on the slope of the secant line, $\frac{f(x+h)-f(x)}{h}$, are "units of f per unit of x." Thus, when we 45 take the limit to get $f'(x)$, we get these same units on the derivative $f'(x)$: units of f per unit of x. Regardless of the function f under consideration (and regardless of the variables being used), it is helpful to remember that the units on the derivative function are "units of output per unit of input," in terms of the input and output of the original function.

For example, say that we have a function $y = P(t)$, where P measures the population of a city (in thousands) at the start of year t (where $t = 0$ corresponds to 2010 AD), and we are told that $P'(2) = 21.37$. What is the meaning of this value? Well, since P is measured in thousands and t is measured in years, we can say that the instantaneous rate of change of the city's population with respect to time at the start of 2012 is 21.37 thousand people per year. We therefore expect that in the coming year, about 21,370 people will be added to the city's population.

Toward more accurate derivative estimates

Toward more accurate derivative estimates It is also helpful to recall, as we first experienced in Section 1.3, that when we want to estimate the value of $f'(x)$ at a given x, we can use the *difference quotient* $\frac{f(x+h)-f(x)}{h}$ with a relatively small value of h. In doing so, we should use both positive and negative values of h in order to make sure we account for the behavior of the function on both sides of the point of interest. To that end, we consider the following brief example to demonstrate the notion of a *central difference* and its role in estimating derivatives.

Example 4.1.1

Suppose that $y = f(x)$ is a function for which three values are known: $f(1) = 2.5$, $f(2) = 3.25$, and $f(3) = 3.625$. Estimate $f'(2)$.

Solution.

We know that $f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$. But since we don't have a graph for $y = f(x)$ nor a formula for the function, we can neither sketch a tangent line nor evaluate the limit exactly. We can't even use smaller and smaller values of h to estimate the limit. Instead, we have just two choices: using $h = -1$ or $h = 1$, depending on which point we pair with (2, 3.25). So, one estimate is

$$f'(2) \approx \frac{f(1)-f(2)}{1-2} = \frac{2.5-3.25}{-1} = 0.75. \tag{4.1.2}$$

The other is

$$f'(2) \approx \frac{f(3)-f(2)}{3-2} = \frac{3.625-3.25}{1} = 0.375. \tag{4.1.3}$$

Since the first approximation looks only backward from the point (2, 3.25) and the second approximation looks only forward from (2, 3.25), it makes sense to average these two values in order to account for behavior on both sides of the point of interest. Doing so, we find that

$$f'(2) \approx \frac{0.75+0.375}{2} = 0.5625. \tag{4.1.4}$$

The intuitive approach to average the two estimates found in Example 1.4 is in fact the best possible estimate to $f'(2)$ when we have just two function values for f on opposite sides of the point of interest. To see why, we think about the diagram in Figure 1.23, which

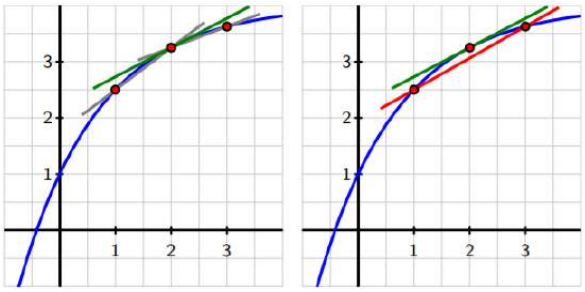


Figure 1.23: At left, the graph of $y = f(x)$ along with the secant line through (1, 2.5) and (2, 3.25), the secant line through (2, 3.25) and (3, 3.625), as well as the tangent line. At right, the same graph along with the secant line through (1, 2.5) and (3, 3.625), plus the tangent line.

shows a possible function $y = f(x)$ that satisfies the data given in Example 1.4. On the left, we see the two secant lines with slopes that come from computing the *backward difference* $\frac{f(1)-f(2)}{1-2} = 0.75$ and from the *forward difference* $\frac{f(3)-f(2)}{3-2} = 0.375$. Note how the first such line's slope over-estimates the slope of the tangent line at $(2, f(2))$, while the second line's slope underestimates $f'(2)$. On the right, however, we see the secant line whose slope is given by the *central difference*

$$\frac{f(3)-f(1)}{3-1} = \frac{3.625-2.5}{2} = \frac{1.125}{2} = 0.5625. \tag{4.1.5}$$

Note that this central difference has the exact same value as the average of the forward difference and backward difference (and it is straightforward to explain why this always holds), and moreover that the central difference yields a very good approximation to the derivative's value, in part because the secant line that uses both a point before and after the point of tangency yields a line that is closer to being parallel to the tangent line.

In general, the central difference approximation to the value of the first derivative is given by

$$f'(a) \approx \frac{f(a+h)-f(a-h)}{2h} \tag{4.1.6}$$

,

and this quantity measures the slope of the secant line to $y=f(x)$ through the points $(a-h, f(a-h))$ and $(a+h, f(a+h))$. Anytime we have symmetric data surrounding a point at which we desire to estimate the derivative, the central difference is an ideal choice for so doing.

The following activities will further explore the meaning of the derivative in several different contexts while also viewing the derivative from graphical, numerical, and algebraic perspectives.

Activity 4.1.2

A potato is placed in an oven, and the potato's temperature F (in degrees Fahrenheit) at various points in time is taken and recorded in the following table. Time t is measured in minutes.

t	$F(t)$
0	70
15	180.5
30	251
45	296
60	324.5
75	342.8
90	354.5

a. Use a central difference to estimate the instantaneous rate of change of the temperature of the potato at $t=30$. Include units on your answer.
b. Use a central difference to estimate the instantaneous rate of change of the temperature of the potato at $t=60$. Include units on your answer.
c. Without doing any calculation, which do you expect to be greater: $f'(75)$ or $f'(90)$? Why?
d. Suppose it is given that $F(64)=330.28$ and $f'(64)=1.341$. What are the units on these two quantities? What do you expect the temperature of the potato to be when $t=65$? when $t=66$? Why?
e. Write a couple of careful sentences that describe the behavior of the temperature of the potato on the time interval $[0,90]$, as well as the behavior of the instantaneous rate of change of the temperature of the potato on the same time interval.

Activity 4.1.3

A company manufactures rope, and the total cost of producing r feet of rope is $C(r)$ dollars.

a. What does it mean to say that $C(2000)=800$?
b. What are the units of $C'(r)$?
c. Suppose that $C(2000)=800$ and $C'(2000)=0.35$. Estimate $C(2100)$, and justify your estimate by writing at least one sentence that explains your thinking.
d. Which of the following statements do you think is true, and why?
    1. $C'(2000) < C'(3000)$
    2. $C'(2000) = C'(3000)$
    3. $C'(2000) > C'(3000)$
e. Suppose someone claims that $C'(5000)=-0.1$. What would the practical meaning of this derivative value tell you about the approximate cost of the next foot of rope? Is this possible? Why or why not?

Activity 4.1.4

Researchers at a major car company have found a function that relates gasoline consumption to speed for a particular model of car. In particular, they have determined that the consumption C, in **liters per kilometer**, at a given speed s, is given by a function $C = f(s)$, where s is the car's speed in **kilometers per hour**.

a. Data provided by the car company tells us that $f(80) = 0.015$, $f(90) = 0.02$, and $f(100) = 0.027$. Use this information to estimate the instantaneous rate of change of fuel consumption with respect to speed at $s = 90$. Be as accurate as possible, use proper notation, and include units on your answer.
b. By writing a complete sentence, interpret the meaning (in the context of fuel consumption) of "$f(80) = 0.015$."
c. Write at least one complete sentence that interprets the meaning of the value of $f'(90)$ that you estimated in (a).

In Section 1.4, we learned how use to the graph of a given function f to plot the graph of its derivative, f'. It is important to remember that when we do so, not only does the scale on the vertical axis often have to change to accurately represent f', but the units on that axis also differ. For example, suppose that $P(t) = 400 - 330e^{-0.03t}$ tells us the temperature in degrees Fahrenheit of a potato in an oven at time t in minutes. In Figure 1.24, we sketch the graph of P on the left and the graph of P' on the right.

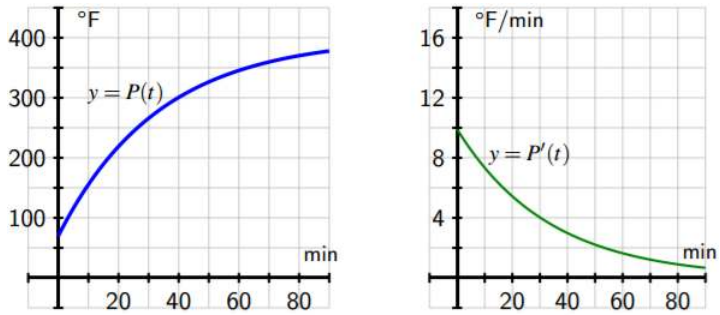


Figure 1.24: Plot of $P(t) = 400 - 330e^{-0.03t}$ at left, and its derivative $P'(t)$ at right.

Note how not only are the vertical scales different in size, but different in units, as the units of P are $^\circ F$, while those of P' are $^\circ F/min$. In all cases where we work with functions that have an applied context, it is helpful and instructive to think carefully about units involved and how they further inform the meaning of our computations.

Summary

In this section, we encountered the following important ideas:

- Regardless of the context of a given function $y = f(x)$, the derivative always measures the instantaneous rate of change of the output variable with respect to the input variable.
- The units on the derivative function $y = f'(x)$ are units of f per unit of x. Again, this measures how fast the output of the function f changes when the input of the function changes.
- The central difference approximation to the value of the first derivative is given by

$$f'(a) \approx \frac{f(a+h) - f(a-h)}{2h}, \tag{4.1.7}$$

  and this quantity measures the slope of the secant line to $y = f(x)$ through the points $(a-h, f(a-h))$ and $(a+h, f(a+h))$. The central difference generates a good approximation of the derivative's value any time we have symmetric data surrounding a point of interest.
- Knowing the derivative and function values at a single point enables us to estimate other function values nearby. If, for example, we know that $f'(7) = 2$, then we know that at x=7, the function f is increasing at an instantaneous rate of 2 units of output for every one unit of input. Thus, we expect $f(8)$ to be approximately 2 units greater than $f(7)$. The value is approximate because we don't know that the rate of change stays the same as x changes.

Supplemental videos





Homework Exercises 4.1

WeBWorK Problems:

1.

2.

3.

Written Problems:

1. The temperature change T (in Fahrenheit degrees), in a patient, that is generated by a dose d (in milliliters), of a drug, is given by the function $T = f(d)$.

a. What does it mean to say $f(50)=0.75$? Write a complete sentence to explain, using correct units.

b. A person's sensitivity, s, to the drug is defined by the function $s(d) = f'(d)$. What are the units of sensitivity?

c. Suppose that $f'(50) = -0.02$. Write a complete sentence to explain the meaning of this value, including units. Then explain how you can estimate $f(51)$.

2. The value, V, of a particular automobile (in dollars) depends on the number of miles, m, the car has been driven, according to the function $V=f(m)$.

a. Suppose that $f(40000)=15500$ and $f(55000)=13200$. What is the average rate of change of f on the interval [40000,55000], and what are the units on this value?

b. In addition to the information given in (a), say that $f(70000)=11100$. Determine the best possible estimate of $f'(55000)$ and write one sentence to explain the meaning of your result, including units on your answer.

c. Which value do you expect to be greater: $f'(30000)$ or $f'(80000)$? Why?

d. Write a sentence to describe the long-term behavior of the function $V=f(m)$, plus another sentence to describe the long-term behavior of f'. Provide your discussion in practical terms regarding the value of the car and the rate at which that value is changing.

This page titled 4.1: Interpretating, Estimating, and Using the Derivative is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Matthew Boelkins, David Austin & Steven Schlicker (ScholarWorks @Grand Valley State University) via source content that was edited to the style and standards of the LibreTexts platform.

- **1.5: Interpretating, Estimating, and Using the Derivative** is licensed CC BY-SA 4.0. Original source: https://activecalculus.org/single.

4.2: The Second Derivative

Preview Video

This optional summary video from the textbook author might be helpful to use as a preview. Other, more detailed, supplemental videos for this section are posted at the end of the text.



Learning Objectives

In this section, we strive to understand the ideas generated by the following important questions:

- How does the derivative of a function tell us whether the function is increasing or decreasing at a point or on an interval?
- What can we learn by taking the derivative of the derivative (to achieve the second derivative) of a function f?
- What does it mean to say that a function is concave up or concave down? How are these characteristics connected to certain properties of the derivative of the function?
- What are the units of the second derivative? How do they help us understand the rate of change of the rate of change?

Introduction

Given a differentiable function $y = f(x)$, we know that its derivative, $y = f'(x)$, is a related function whose output at a value $x = a$ tells us the slope of the tangent line to $y = f(x)$ at the point $(a, f(a))$. That is, heights on the derivative graph tell us the values of slopes on the original function's graph. Therefore, the derivative tells us important information about the function f.

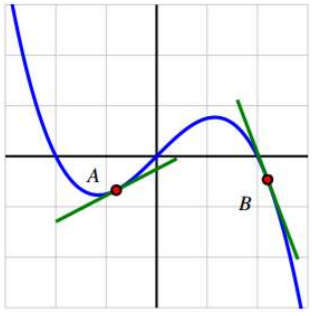


Figure 1.25: Two tangent lines on a graph demonstrate how the slope of the tangent line tells us whether the function is rising or falling, as well as whether it is doing so rapidly or slowly.

At any point where $f'(x)$ is positive, it means that the slope of the tangent line to f is positive, and therefore the function f is increasing (or rising) at that point. Similarly, if $f'(a)$ is negative, we know that the graph of f is decreasing (or falling) at that

point.

In the next part of our study, we work to understand not only whether the function f is increasing or decreasing at a point or on an interval, but also how the function f is increasing or decreasing. Comparing the two tangent lines shown in Figure 1.25, we see that at point A, the value of $f'(x)$ is positive and relatively close to zero, which coincides with the graph rising slowly. By contrast, at point B, the derivative is negative and relatively large in absolute value, which is tied to the fact that f is decreasing rapidly at B. It also makes sense to not only ask whether the value of the derivative function is positive or negative and whether the derivative is large or small, but also to ask "how is the derivative changing?"

We also now know that the derivative, $y = f'(x)$, is itself a function. This means that we can consider taking its derivative – the derivative of the derivative – and therefore ask questions like "what does the derivative of the derivative tell us about how the original function behaves?" As we have done regularly in our work to date, we start with an investigation of a familiar problem in the context of a moving object.

Preview Activity 4.2.1

The position of a car driving along a straight road at time t in minutes is given by the function $y = s(t)$ that is pictured in Figure 1.26. The car's position function has units measured in thousands of feet. For instance, the point (2, 4) on the graph indicates that after 2 minutes, the car has traveled 4000 feet.

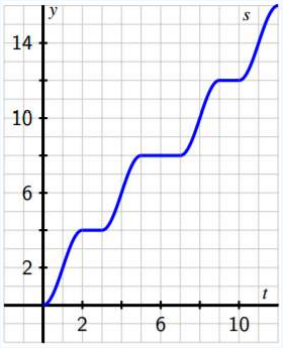


Figure 1.26: The graph of $y = s(t)$, the position of the car (measured in thousands of feet from its starting location) at time t in minutes.

a. In everyday language, describe the behavior of the car over the provided time interval. In particular, you should carefully discuss what is happening on each of the time intervals [0, 1], [1, 2], [2, 3], [3, 4], and [4, 5], plus provide commentary overall on what the car is doing on the interval [0, 12].
b. On the lefthand axes provided in Figure 1.27, sketch a careful, accurate graph of $y = s'(t)$.
c. What is the meaning of the function $y = s'(t)$ in the context of the given problem? What can we say about the car's behavior when $s'(t)$ is positive? when $s'(t)$ is zero? When $s'(t)$ is negative?
d. Rename the function you graphed in (b) to be called $y = v(t)$. Describe the behavior of v in words, using phrases like "v is increasing on the interval . . ." and "v is constant on the interval . . .."
e. Sketch a graph of the function $y = v'(t)$ on the righthand axes provide in Figure 1.27. Write at least one sentence to explain how the behavior of $v'(t)$ is connected to the graph of $y = v(t)$.

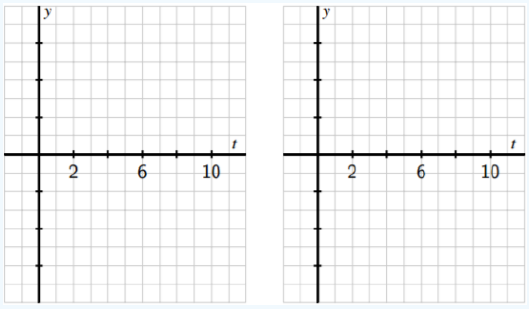


Figure 1.27: Axes for plotting $y = v(t) = s'(t)$ and $y = v'(t)$.

Increasing, decreasing, or neither

When we look at the graph of a function, there are features that strike us naturally, and common language can be used to name these features. In many different settings so far, we have intuitively used the words increasing and decreasing to describe a function's graph. Here we connect these terms more formally to a function's behavior on an interval of input values.

Definition 1.5

Given a function $f(x)$ defined on the interval (a,b), we say that f is increasing on (a,b)provided that for all x, y in the interval (a,b), if $x<y$, then $f(x)<f(y)$. Similarly, we say that f *is decreasing on* (a,b) provided that for all x, y in the interval (a,b), if $x<y$, then $f(x)>f(y)$.

Simply put, an increasing function is one that is rising as we move from left to right along the graph, and a decreasing function is one that falls as the value of the input increases. For a function that has a derivative, we can use the sign of the derivative to determine whether or not the function is increasing or decreasing.

Let f be a function that is differentiable on an interval (a,b). We say that f is increasing on (a,b) if and only if $f'(x)>0$ for every x such that $a<x<b$; similarly, f is decreasing on(a,b) if and only if $f'(x)<0$. If $f'(a)=0$, then we say f is neither increasing nor decreasing at $x=a$.

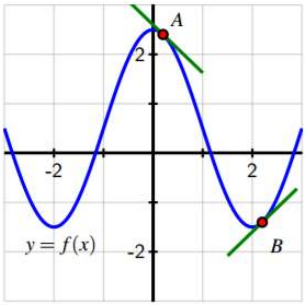


Figure 1.28: A function that is decreasing on the intervals $-3<x<-2$ and $0<x<2$ and increasing on $-2<x<0$ and $2<x<3$.

For example, the function pictured in Figure 1.28 is increasing on the entire interval $-2<x<0$. Note that at both $x=\pm 2$ and $x=0$, we say that f is neither increasing nor decreasing, because $f'(x)=0$ at these values.

The Second Derivative

For any function, we are now accustomed to investigating its behavior by thinking about its derivative. Given a function f, its derivative is a new function, one that is given by the rule

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}. \tag{4.2.1}$$

Because f' is itself a function, it is perfectly feasible for us to consider the derivative of the derivative, which is the new function $y=[f'(x)]'$. We call this resulting function *the second derivative* of $y=f(x)$, and denote the second derivative by $y=f''(x)$. Due to the presence of multiple possible derivatives, we will sometimes call f' "the first derivative" of f, rather than simply "the derivative" of f. Formally, the second derivative is defined by the limit definition of the derivative of the first derivative:

$$f''(x)=\lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h}. \tag{4.2.2}$$

We note that all of the established meaning of the derivative function still holds, so when we compute $y=f''(x)$, this new function measures slopes of tangent lines to the curve $y=f'(x)$, as well as the instantaneous rate of change of $y=f'(x)$. In other words, just as the first derivative measures the rate at which the original function changes, the second derivative measures the rate at which the first derivative changes. This means that the second derivative tracks the instantaneous rate of change of the instantaneous rate of change of f. That is, the second derivative will help us to understand how the rate of change of the original function is itself changing.

Concavity

In addition to asking *whether* a function is increasing or decreasing, it is also natural to inquire *how* a function is increasing or decreasing. To begin, there are three basic behaviors that an increasing function can demonstrate on an interval, as pictured in Figure 1.29: the function can increase more and more rapidly, increase at the same rate, or increase in a way that is slowing down. Fundamentally, we are beginning to think about how a particular curve bends, with the natural comparison being made to lines, which don't bend at all. More than this, we want to understand how the bend in a function's graph is tied to behavior characterized by the first derivative of the function.

For the leftmost curve in Figure 1.29, picture a sequence of tangent lines to the curve. As we move from left to right, the slopes of those tangent lines will increase. Therefore, the rate of change of the pictured function is increasing, and this explains why we say this function is *increasing at an increasing rate*. For the rightmost graph in Figure 1.29, observe that as x increases, the function increases but the slope of the tangent line decreases, hence this function is *increasing at a decreasing rate*.

Of course, similar options hold for how a function can decrease. Here we must be extra careful with our language, since decreasing functions involve negative slopes, and negative numbers present an interesting situation in the tension between common language and mathematical language. For example, it can be tempting to say that "-100 is bigger than -2." But we must remember that when we say one number is greater than another, this describes how the numbers lie on a number line: $x < y$ provided that x lies to the left of y. So of course, -100 is less than -2. Informally, it might be helpful to say that

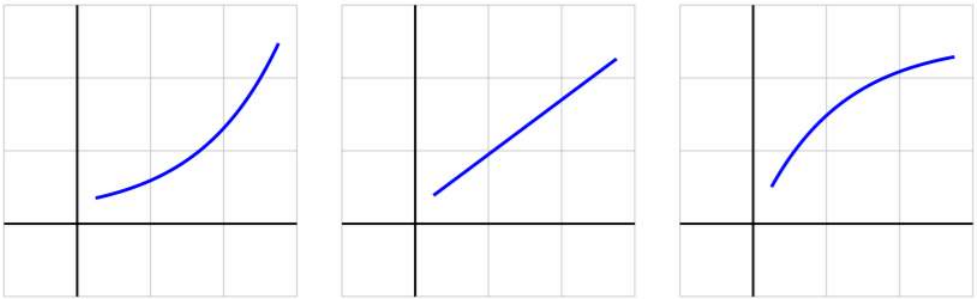


Figure 1.29: Three functions that are all increasing, but doing so at an increasing rate, at a constant rate, and at a decreasing rate, respectively.

"-100 is more negative than -2." This leads us to note particularly that when a function's values are negative, and those values subsequently get more negative, the function must be decreasing.

Now consider the three graphs shown in Figure 1.30. Clearly the middle graph demonstrates the behavior of a function decreasing at a constant rate. If we think about a sequence of tangent lines to the first curve that progress from left to right, we see that the slopes of these lines get less and less negative as we move from left to right. That means that the values of the first derivative, while all negative, are increasing, and thus we say that the leftmost curve is *decreasing at an increasing rate*.

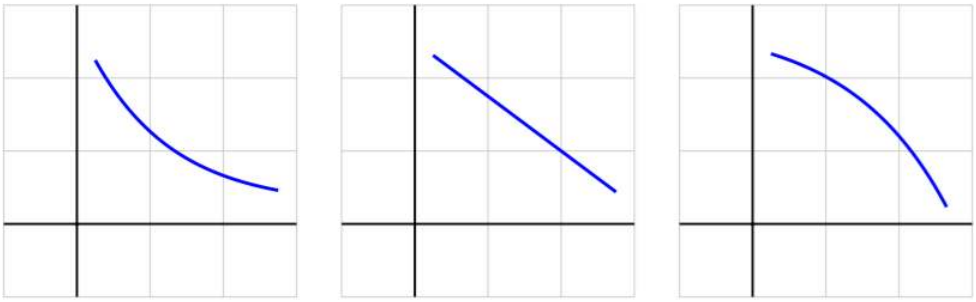


Figure 1.30: From left to right, three functions that are all decreasing, but doing so in different ways.

This leaves only the rightmost curve in Figure 1.30 to consider. For that function, the slope of the tangent line is negative throughout the pictured interval, but as we move from left to right, the slopes get more and more negative. Hence the slope of the curve is decreasing, and we say that the function is *decreasing at a decreasing rate*.

This leads us to introduce the notion of concavity which provides simpler language to describe some of these behaviors. Informally, when a curve opens up on a given interval, like the upright parabola $y = x^2$ or the exponential growth function $y = e^x$, we say that

the curve is concave up on that interval. Likewise, when a curve opens down, such as the parabola $y = -x^2$ or the opposite of the exponential function $y = -e^x$, we say that the function is concave down. This behavior is linked to both the first and second derivatives of the function.

In Figure 1.31, we see two functions along with a sequence of tangent lines to each. On the lefthand plot where the function is concave up, observe that the tangent lines to the curve always lie below the curve itself and that, as we move from left to right, the slope of the tangent line is increasing. Said differently, the function f is concave up on the interval shown because its derivative, f', is increasing on that interval. Similarly, on the righthand plot in Figure 1.31, where the function shown is concave down, there we see that the tangent lines always lie above the curve and that the value of the slope of the tangent line is decreasing as we move from left to right. Hence, what makes f concave down on the interval is the fact that its derivative, f', is decreasing.

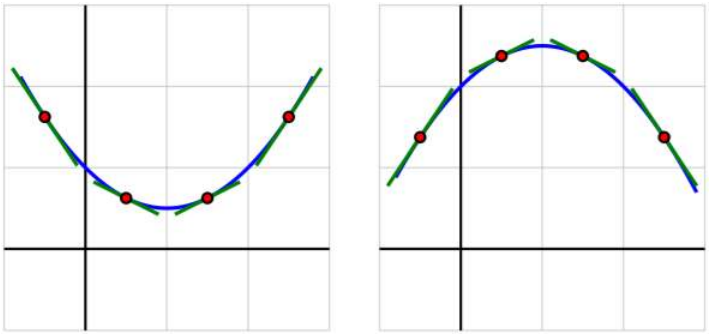


Figure 1.31: At left, a function that is concave up; at right, one that is concave down.

We state these most recent observations formally as the definitions of the terms *concave up* and *concave down*.

Definition 1.6

Let f be a differentiable function on an interval (a, b). Then f is concave up on (a, b) if and only if f' is increasing on (a, b); f is concave down on (a, b) if and only if f' is decreasing on (a, b).

The following activities lead us to further explore how the first and second derivatives of a function determine the behavior and shape of its graph. We begin by revisiting Preview Activity 1.6.

Activity 4.2.2

The position of a car driving along a straight road at time t in minutes is given by the function $y = s(t)$ that is pictured in Figure 1.32. The car's position function has units measured in thousands of feet. Remember that you worked with this function and sketched graphs of $y = v(t) = s'(t)$ and $y = v'(t)$ in Preview Activity 1.6.

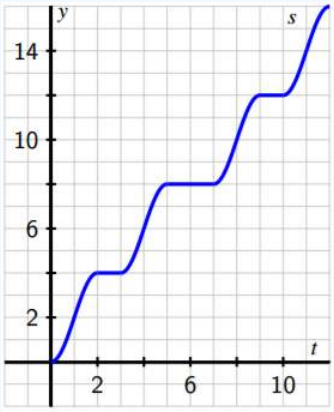


Figure 1.32: The graph of $y = s(t)$, the position of the car (measured in thousands of feet from its starting location) at time t in minutes.

a. On what intervals is the position function $y = s(t)$ increasing? decreasing? Why?
b. On which intervals is the velocity function $y = v(t) = s'(t)$ increasing? decreasing? neither? Why?

c. Acceleration is defined to be the instantaneous rate of change of velocity, as the acceleration of an object measures the rate at which the velocity of the object is changing. Say that the car's acceleration function is named $a(t)$. How is $a(t)$ computed from $v(t)$? How is $a(t)$ computed from $s(t)$? Explain.
d. What can you say about s'' whenever s' is increasing? Why?
e. Using only the words increasing, decreasing, constant, concave up, concave down, and linear, complete the following sentences. For the position function s with velocity v and acceleration a,
   - on an interval where v is positive, s is .
   - on an interval where v is negative, s is .
   - on an interval where v is zero, s is .
   - on an interval where a is positive, v is .
   - on an interval where a is negative, v is .
   - on an interval where a is zero, v is .
   - on an interval where a is positive, s is .
   - on an interval where a is negative, s is .
   - on an interval where a is zero, s is .

The context of position, velocity, and acceleration is an excellent one in which to understand how a function, its first derivative, and its second derivative are related to one another. In Activity 1.15, we can replace s, v, and a with an arbitrary function f and its derivatives $f'$and f'', and essentially all the same observations hold. In particular, note that f' is increasing if and only if f is concave up, and similarly f' is increasing if and only if f'' is positive. Likewise, f' is decreasing if and only if f is concave down, and f' is decreasing if and only if f'' is negative.

Activity 4.2.3:

A potato is placed in an oven, and the potato's temperature F (in degrees Fahrenheit) at various points in time is taken and recorded in the following table. Time t is measured in minutes. In Activity 1.12, we computed approximations to $F'(30)$ and $F'(60)$ using central differences. Those values and more are provided in the second table below, along with several others computed in the same way.

t	$F(t)$
0	70
15	180.5
30	251
45	296
60	24.5
75	342.8
90	354.5

t	$F'(t)$
0	NA
15	6.03
30	3.85
45	2.45
60	1.56
75	1.00
90	NA

a. What are the units on the values of $F'(t)$?
b. Use a central difference to estimate the value of $F''(30)$.
c. What is the meaning of the value of $F''(30)$ that you have computed in (b) in terms of the potato's temperature? Write several careful sentences that discuss, with appropriate units, the values of $F(30)$, $F'(30)$, and $F''(30)$, and explain the overall behavior of the potato's temperature at this point in time.
d. Overall, is the potato's temperature increasing at an increasing rate, increasing at a constant rate, or increasing at a decreasing rate? Why?

Activity 4.2.4:

This activity builds on our experience and understanding of how to sketch the graph of f' given the graph of

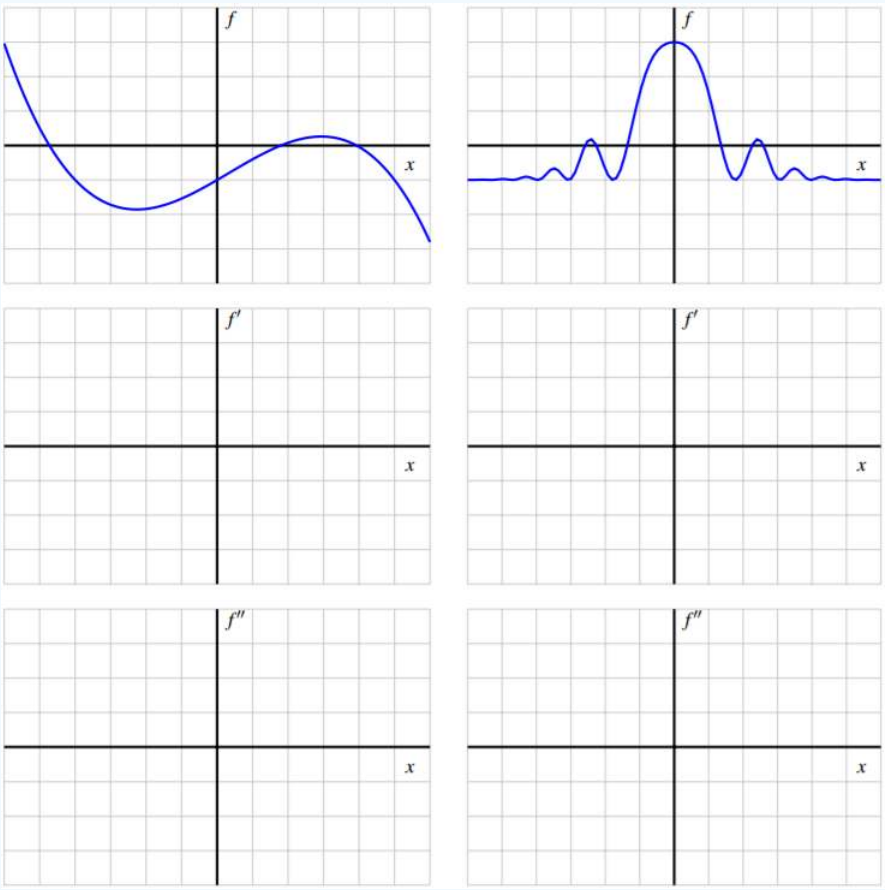


Figure 1.33: Two given functions f, with axes provided for plotting f' and f'' below.

In Figure 1.33, given the respective graphs of two different functions f, sketch the corresponding graph of f' on the first axes below, and then sketch f'' on the second set of axes. In addition, for each, write several careful sentences in the spirit of those in Activity 1.15 that connect the behaviors of f, f', and f''. For instance, write something such as

f' is on the interval , which is connected to the fact that f is on the same interval , and f'' is on the interval as well

but of course with the blanks filled in. Throughout, view the scale of the grid for the graph of f as being 1×1, and assume the horizontal scale of the grid for the graph of f' is identical to that for f. If you need to adjust the vertical scale on the axes for the graph of f'or f'', you should label that accordingly.

Summary

In this section, we encountered the following important ideas:

- A differentiable function f is increasing at a point or on an interval whenever its first derivative is positive, and decreasing whenever its first derivative is negative.
- By taking the derivative of the derivative of a function f, we arrive at the second derivative, f''. The second derivative measures the instantaneous rate of change of the first derivative, and thus the sign of the second derivative tells us whether or not the slope of the tangent line to f is increasing or decreasing.
- A differentiable function is concave up whenever its first derivative is increasing (or equivalently whenever its second derivative is positive), and concave down whenever its first derivative is decreasing (or equivalently whenever its second derivative is negative). Examples of functions that are everywhere concave up are $y = x^2$ and $y = e^x$; examples of functions that are everywhere concave down are $y = -x^2$ and $y = -e^x$.
- The units on the second derivative are "units of output per unit of input per unit of input." They tell us how the value of the derivative function is changing in response to changes in the input. In other words, the second derivative tells us the rate of change of the rate of change of the original function.

Supplemental videos





Homework Exercises 4.2

WeBWorK Problems:

1.

2.

3.

4.

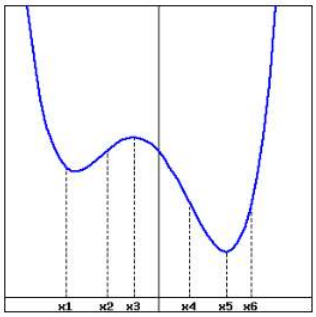
5.

6. The following is the graph of *f*.

7.

Written Problems:

1. The graph of *f* ′ (***not f***) is given below.



a. Note that f' is positive everywhere on this graph. What does that mean about the function *f* ?

b. At which of the marked values of *x* is $f(x)$ the greatest? The least?

c. Sketch a graph of $f''(x)$ and determine at which of the marked values of *x* is $f''(x)$ the greatest and the least.

2. Suppose that $y = f(x)$ is a twice-differentiable function such that f'' is continuous for which the following information is known: *f*(2)=−3, $f'(2) = 1.5$, $f''(2) = -0.25$.

1. Is *f* increasing or decreasing near *x* =2? Is *f* concave up or concave down near *x* =2?
2. Do you expect *f*(2.1) to be greater than −3, equal to −3, or less than −3? Why?
3. Do you expect *f* ′(2.1) to be greater than 1.5, equal to 1.5, or less than 1.5? Why?
4. Sketch a graph of $y = f(x)$ near (2, *f*(2)) and include a graph of the tangent line.

3. A bungee jumper's height *h* (in feet) at time *t* (in seconds) is given in part by the table:

t	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0

$h(t)$	200	184.2	159.9	131.9	104.7	81.8	65.5	56.8	55.5	60.4	69.8

t	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0
$h(t)$	81.6	93.7	104.4	112.6	117.7	119.4	118.2	114.8	110.0	104.7

1. Use the given data to estimate $h'(4.5)$, $h'(5)$, and $h'(5.5)$. At which of these times is the bungee jumper rising most rapidly?
2. Use the given data and your work in (a) to estimate $h''(5)$.
3. What physical property of the bungee jumper does the value of $h''(5)$ measure? What are its units?
4. Based on the data, on what approximate time intervals is the function $y = h(t)$ concave down? What is happening to the velocity of the bungee jumper on these time intervals?

This page titled 4.2: The Second Derivative is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Matthew Boelkins, David Austin & Steven Schlicker (ScholarWorks @Grand Valley State University) via source content that was edited to the style and standards of the LibreTexts platform.

- **1.6: The Second Derivative** is licensed CC BY-SA 4.0. Original source: https://activecalculus.org/single.

4.3: The Tangent Line Approximation

Preview Video

This optional summary video from the textbook author might be helpful to use as a preview. Other, more detailed, supplemental videos for this section are posted at the end of the text.



Learning Objectives

In this section, we strive to understand the ideas generated by the following important questions:

- What is the formula for the general tangent line approximation to a differentiable function $y = f(x)$ at the point $(a, f(a))$?
- What is the principle of local linearity and what is the local linearization of a differentiable function f at a point $(a, f(a))$?
- How does knowing just the tangent line approximation tell us information about the behavior of the original function itself near the point of approximation? How does knowing the second derivative's value at this point provide us additional knowledge of the original function's behavior?

Among all functions, linear functions are simplest. One of the powerful consequences of a function $y = f(x)$ being differentiable at a point $(a, f(a))$ is that, up close, the function$y = f(x)$ is locally linear and looks like its tangent line at that point. In certain circumstances, this allows us to approximate the original function f with a simpler function L that is linear: this can be advantageous when we have limited information about f or when f is computationally or algebraically complicated. We will explore all of these situations in what follows.

It is essential to recall that when$f$ is differentiable at $x = a$, the value of $f'(a)$ provides the slope of the tangent line to $y = f(x)$ at the point $(a, f(a))$. By knowing both a point on the line and the slope of the line we are thus able to find the equation of the tangent line. Preview Activity $4.3.1$ will refresh these concepts through a key example and set the stage for further study.

Preview Activity 4.3.1

Consider the function $y = g(x) = -x^2 + 3x + 2$

a. Use the limit definition of the derivative to compute a formula for $y = g'(x)$.
b. Determine the slope of the tangent line to $y = g(x)$ at the value x = 2.
c. Compute g (2).
d. Find an equation for the tangent line to $y = g(x)$ at the point (2,g(2)). Write your result in point-slope form[8].

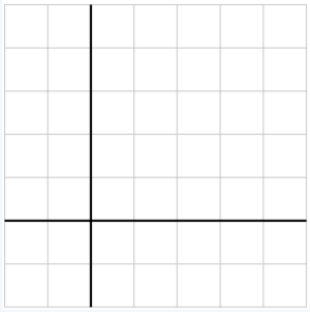


***Figure* 4.3.1**: *Axes for plotting $y = g(x)$ and its tangent line to the point (2,g(2))).*

The Tangent Line

Given a function f that is differentiable at $x = a$, we know that we can determine the slope of the tangent line to $y = f(x)$ at $(a, f(a))$ by computing $f'(a)$. The resulting tangent line through $(a, f(a))$ with slope $m = f'(a)$ has its equation in point-slope form given by

$$y - f(a) = f'(a)(x - a), \tag{4.3.1}$$

which we can also express as $y = f'(a)(x - a) + f(a)$. Note well: there is a major difference between $f(a)$ and $f(x)$ in this context. The former is a constant that results from using the given fixed value of a, while the latter is the general expression for the rule that defines the function. The same is true for $f'(a)$ and $f'(x)$: we must carefully distinguish between these expressions. Each time we find the tangent line, we need to evaluate the function and its derivative at a fixed a-value.

In Figure 1.8.2, we see a labeled plot of the graph of a function f and its tangent line at the point $(a, f(a))$. Notice how when we zoom in we see the local linearity of f more clearly highlighted as the function and its tangent line are nearly indistinguishable up close. This can also be seen dynamically in the java applet at http://gvsu.edu/s/6J.

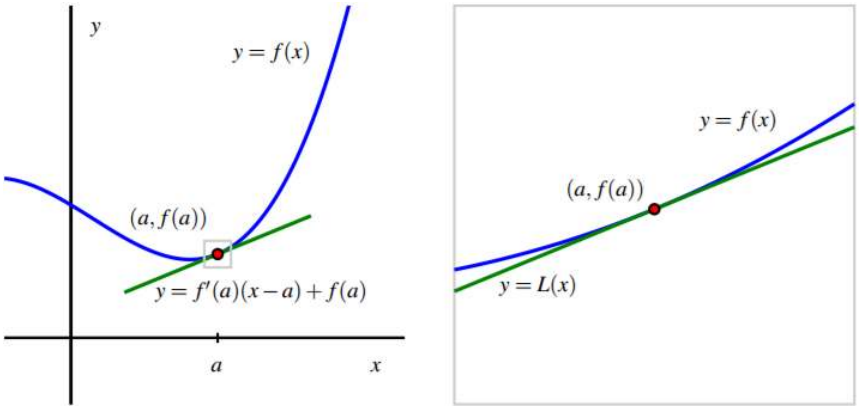


***Figure* 4.3.2**: *A function $y = f(x)$ and its tangent line at the point $(a, f(a))$: at left, from a distance, and at right, up close. At right, we label the tangent line function by $y = L(x)$ and observe that for x near $(a, f(a)) \approx L(x)$.*

The Local Linearization

A slight change in perspective and notation will enable us to be more precise in discussing how the tangent line to $y = f(x)$ at $(a, f(a))$ approximates f near $x = a$. Taking the equation for the tangent line and solving for y, we observe that the tangent line is given by

$$y = f'(a)(x - a) + f(a) \tag{4.3.2}$$

and moreover that this line is itself a function of x. Replacing the variable y with the expression $L(x)$, we call

$$L(x) = f'(a)(x - a) + f(a) \tag{4.3.3}$$

the *local linearization* of f at the point$(a, f(a))$. In this notation, it is particularly important to observe that $L(x)$ is nothing more than a new name for the tangent line, and that for x close to a, we have that $f(x) \approx L(x)$.

Say, for example, that we know that a function $y = f(x)$ has its tangent line approximation given by $L(x) = 3 - 2(x - 1)$ at the point (1, 3), but we do not know anything else about the function f. If we are interested in estimating a value of f (x) for x near 1, such as $f(1.2)$, we can use the fact that $f(1.2) \approx L(1.2)$ and hence

$$f(1.2) \approx L(1.2) = 3 - 2(1.2 - 1) = 3 - 2(0.2) = 2.6 \tag{4.3.4}$$

Again, much of the new perspective here is only in notation since $y = L(x)$ is simply a new name for the tangent line function. In light of this new notation and our observations above, we note that since $L(x) = f(a) + f'(a)(x - a)$ and $L(x) \approx f(x)$ for x near a, it also follows that we can write

$$f(x) \approx f(a) + f'(a)(x - a) \text{ for } x \text{ near } a \tag{4.3.5}$$

The next activity explores some additional important properties of the local linearization $y = L(x)$ to a function f at given a-value.

Activity 4.3.2

Suppose it is known that for a given differentiable function $y = g(x)$, its local linearization at the point where $a = -1$ is given by $L(x) = -2 + 3(x + 1)$.

a. Compute the values of $L(-1)$ and $L'(-1)$.
b. What must be the values of $g(-1)$ and $g'(-1)$? Why?
c. Do you expect the value of $g(-1.03)$ to be greater than or less than the value of $g(-1)$? Why?
d. Use the local linearization to estimate the value of $g(-1.03)$.
e. Suppose that you also know that $g''(-1) = 2$. What does this tell you about the graph of $y = g(x)$ at $a = -1$?
f. For x near -1, sketch the graph of the local linearization $y = L(x)$ as well as a possible graph of $y = g(x)$ on the axes provided in Figure 1.8.3.

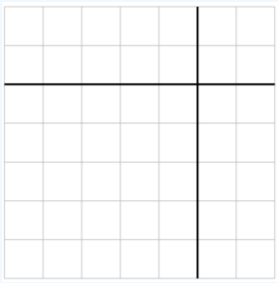


***Figure* 4.3.3**: Axes for plotting $y = L(x)$ *and* $y = g(x)$.

As we saw in the example provided by Activity 1.8.2, the local linearization $y = L(x)$ is a linear function that shares two important values with the function $y = L(x)$ that it is derived from. In particular, observe that since $L(x) = f(a) + f'(a)(x - a)$, it follows that $L(a) = f(a)$. In addition, since L is a linear function, its derivative is its slope. Hence, $L'(x) = f'(a)$ for every value of x, and specifically $L'(x) = f'(a)$. Therefore, we see that L is a linear function that has both the same value and the same slope as the function f at the point $(a, f(a))$.

In situations where we know the linear approximation $y = L(x)$, we therefore know the original function's value and slope at the point of tangency. What remains unknown, however, is the shape of the function f at the point of tangency. There are essentially four possibilities, as enumerated in Figure 1.8.4.

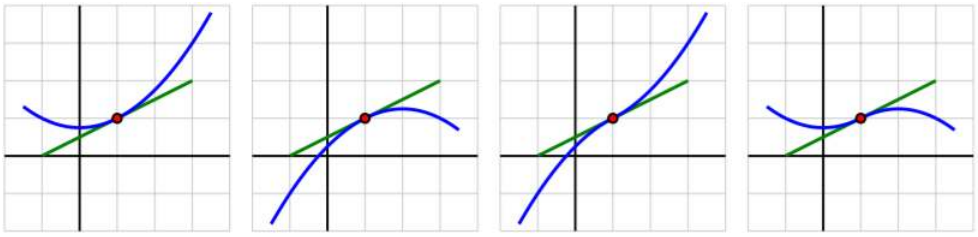


***Figure* 4.3.4:** *Four possible graphs for a nonlinear differentiable function and how it can be situated relative to its tangent line at a point.*

These stem from the fact that there are three options for the value of the second derivative: either $f''(a) < 0, f''(a) = 0,$ or $f''(a) > 0$. If $f''(a) > 0$., then we know the graph of f is concave up, and we see the first possibility on the left, where the tangent line lies entirely below the curve. If $f''(a) < 0$, then we find ourselves in the second situation (from left) where f is concave down and the tangent line lies above the curve. In the situation where $f''(a) = 0$ and f'' changes sign at $x = a$, the concavity of the graph will change, and we will see either the third or fourth option[9] . A fifth option (that is not very interesting) can occur, which is where the function f is linear, and so $f(x) = L(x)$ for all values of x.

The plots in Figure 1.8.4 highlight yet another important thing that we can learn from the concavity of the graph near the point of tangency: whether the tangent line lies above or below the curve itself. This is key because it tells us whether or not the tangent line approximation's values will be too large or too small in comparison to the true value of f. For instance, in the first situation in the leftmost plot in Figure 1.8.4 where $f''(a) > 0$, since the tangent line falls below the curve, we know that $L(x) \leq f(x)$ for all values of x near a. We explore these ideas further in the following activity.

Activity 4.3.3

This activity concerns a function $f(x)$ about which the following information is known:

- f is a differentiable function defined at every real number x
- $f(2) = -1$
- $y = f'(x)$ has its graph given in Figure 1.8.5

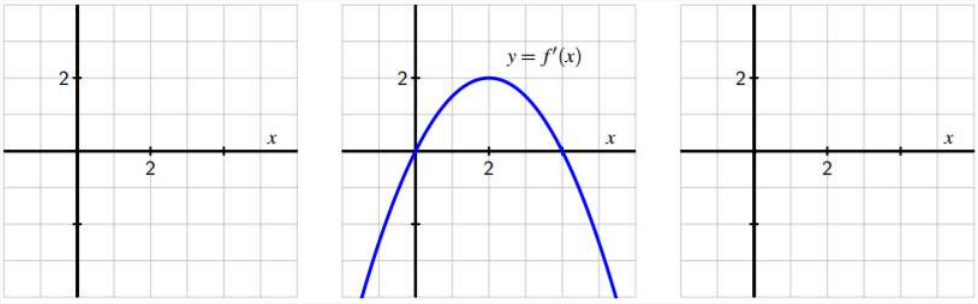


***Figure* 4.3.5:** *At center, a graph of $y = f'(x)$; at left, axes for plotting $y = f(x)$; at right, axes for plotting $y = f''(x)$.*

Your task is to determine as much information as possible about f(especially near the value $a = 2$) by responding to the questions below.

(a) Find a formula for the tangent line approximation, $L(x)$, to f at the point $(2, -1)$.

(b) Use the tangent line approximation to estimate the value of $f(2.07)$. Show your work carefully and clearly.

(c) Sketch a graph of $y = f''(x)$ on the righthand grid in Figure 1.8.5; label it appropriately.

(d) Is the slope of the tangent line to $y = f(x)$ increasing, decreasing, or neither when $x = 2$? Explain.

(e) Sketch a possible graph of $y = f(x)$ near $x = 2$ on the lefthand grid in Figure 1.8.5. Include a sketch of $y = L(x)$ (found in part (a)). Explain how you know the graph of $y = f(x)$ looks like you have drawn it.

(f) Does your estimate in (b) over- or under-estimate the true value of $f(2.07)$? Why?

The idea that a differentiable function looks linear and can be well-approximated by a linear function is an important one that finds wide application in calculus. For example, by approximating a function with its local linearization, it is possible to develop an effective algorithm to estimate the zeroes of a function. Local linearity also helps us to make further sense of certain challenging limits. For instance, we have seen that a limit such as

$$\lim_{x \to 0} \frac{\sin(x)}{x} \tag{4.3.6}$$

x is indeterminate because both its numerator and denominator tend to 0. While there is no algebra that we can do to simplify $\frac{\sin(x)}{x}$, it is straightforward to show that the linearization of $f(x) = sin(x)$ at the point $(0, 0)$ is given by $L(x) = x$. Hence, for values of x near 0, $sin(x) \approx x$. As such, for values of x near 0,

$$\frac{\sin(x)}{x} \approx \frac{x}{x} = 1, \tag{4.3.7}$$

which makes plausible the fact that

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1 \tag{4.3.8}$$

These ideas and other applications of local linearity will be explored later on in our work.

Summary

In this section, we encountered the following important ideas:

- The tangent line to a differentiable function $y = f(x)$ at the point $(a, f(a))$ is given in point-slope form by the equation
$$y - f(a) = f'(a)(x - a). \tag{4.3.9}$$
- The principle of local linearity tells us that if we zoom in on a point where a function $y = f(x)$ is differentiable, the function should become indistinguishable from its tangent line. That is, a differentiable function looks linear when viewed up close. We rename the tangent line to be the function $y = L(x)$ where $L(x) = f(a) + f'(a)(x - a)$ and note that $f(x) \approx L(x)$ for all x near $x = a$.
- If we know the tangent line approximation $L(x) = f(a) + f'(a)(x - a)$, then because $L(a) = f(a)$ and $L'(a) = f'(a)$, we also know both the value and the derivative of the function $y = f(x)$ at the point where $x = a$. In other words, the linear approximation tells us the height and slope of the original function. If, in addition, we know the value of $f''(a)$, we then know whether the tangent line lies above or below the graph of $y = f(x)$ depending on the concavity of f.

[8]Recall that a line with slope m that passes through (x_0, y_0) has equation $y - y_0 = m\,(x - x_0)$, and this is the *point-slope form* of the equation.

[9]It is possible to have $f''(a) = 0$ and have f'' not change sign at $x = a$, in which case the graph will look like one of the first two options.

Supplemental videos







Homework Exercises 4.3

WeBWorK Problems:

1. Note that you can't see the scale for the y-axis, but you can figure it out!

2.

3.

4.

Written Problems:

1. A certain function $y = p(x)$ has its local linearization at a =3 given by $L(x) = -2x + 5$.

a. What are the values of $p(3)$ and $p'(3)$? Why?

b. Estimate the value of $p(2.79)$.

c. Suppose that $p''(3)=0$ and you know that $p''(x)<0$ for $x < 3$. Is your estimate in (b) too large or too small?

d. Suppose that $p''(x)>0$ for $x > 3$. Use this fact and the additional information above to sketch an accurate graph of $y = p(x)$ near x =3. Include a sketch of $y = L(x)$ in your work.

2. An object moving along a straight line path has a differentiable position function $y = s(t)$; $s(t)$ measures the object's position relative to the origin at time t. It is known that at time t =9 seconds, the object's position is $s(9)$=4 feet (i.e., 4 feet to the right of the origin). Furthermore, the object's instantaneous velocity at t =9 is −1.2 feet per second, and its acceleration at the same instant is 0.08 feet per second per second.

a. Use local linearity to estimate the position of the object at t =9.34.

b. Is your estimate likely too large or too small? Why?

c. In everyday language, describe the behavior of the moving object at t=9. Is it moving toward the origin or away from it? Is its velocity increasing or decreasing? Include a sketch of the graph of the position function in your explanation.

This page titled 4.3: The Tangent Line Approximation is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Matthew Boelkins, David Austin & Steven Schlicker (ScholarWorks @Grand Valley State University) via source content that was edited to the style and standards of the LibreTexts platform.

- **1.8: The Tangent Line Approximation** is licensed CC BY-SA 4.0. Original source: https://activecalculus.org/single.

4.4: Elementary Derivative Rules

Preview Video

This optional summary video from the textbook author might be helpful to use as a preview. Other, more detailed, supplemental videos for this section are posted at the end of the text.



Learning Objectives

In this section, we strive to understand the ideas generated by the following important questions:

- What are alternate notations for the derivative?
- How can we sometimes use the algebraic structure of a function $f(x)$ to easily compute a formula for $f'(x)$?
- What is the derivative of a power function of the form $f(x) = x^n$? What is the derivative of an exponential function of form $f(x) = a^x$?
- If we know the derivative of $y = f(x)$, how is the derivative of $y = kf(x)$ computed, where k is a constant?
- If we know the derivatives of $y = f(x)$ and $y = g(x)$, how is the derivative of $y = f(x) + g(x)$ computed?

In Chapter 1, we developed the concept of the derivative of a function. We now know that the derivative f' of a function f measures the instantaneous rate of change of f with respect to x as well as the slope of the tangent line to $y = f(x)$ at any given value of x. To date, we have focused primarily on interpreting the derivative graphically or, in the context of functions in a physical setting, as a meaningful rate of change. To actually calculate the value of the derivative at a specific point, we have typically relied on the limit definition of the derivative.

In this present chapter, we will investigate how the limit definition of the derivative,

$$f'(x) = \lim_{h \to 0} f(x+h) - f(x)h, \tag{4.4.1}$$

leads to interesting patterns and rules that enable us to quickly find a formula for $f'(x)$ based on the formula for $f(x)$ *without using* the limit definition directly. For example, we already know that if $f(x) = x$, then it follows that $f'(x) = 1$. While we could use the limit definition of the derivative to confirm this, we know it to be true because $f(x)$ is a linear function with slope 1 at every value of x. One of our goals is to be able to take standard functions, say ones such as

$$g(x) = 4x^7 - \sin(x) + 3e^x, \tag{4.4.2}$$

and, based on the algebraic form of the function, be able to apply shortcuts to almost immediately determine the formula for $g'(x)$.

Preview Activity 4.4.1

Functions of the form $f(x) = x^n$, where $n = 1, 2, 3, \ldots$, are often called *power functions*. The first two questions below revisit work we did earlier in Chapter 1, and the following questions extend those ideas to higher powers of x.

a. Use the limit definition of the derivative to find $f'(x)$ for $f(x) = x^2$.
b. Use the limit definition of the derivative to find $f'(x)$ for $f(x) = x^3$.
c. Use the limit definition of the derivative to find $f'(x)$ for $f(x) = x^4$. (Hint: $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$. Apply this rule to $(x+h)^4$ within the limit definition.)
d. Based on your work in (a), (b), and (c), what do you conjecture is the derivative of $f(x) = x^5$? Of $f(x) = x^{13}$?
e. Conjecture a formula for the derivative of $f(x) = x^n$ that holds for any positive integer n. That is, given $f(x) = x^n$ where n is a positive integer, what do you think is the formula for $f'(x)$?

Some Key Notation

In addition to our usual f' notation for the derivative, there are other ways to symbolically denote the derivative of a function, as well as the instruction to take the derivative. We know that if we have a function, say $f(x) = x^2$, that we can denote its derivative by $f'(x)$, and we write $f'(x) = 2x$. Equivalently, if we are thinking more about the relationship between y and x, we sometimes denote the derivative of y with respect to x with the symbol

$$\frac{dy}{dx} \tag{4.4.3}$$

which we read "dee-y dee-x." This notation comes from the fact that the derivative is related to the slope of a line, and slope is measured by $\frac{\Delta y}{\Delta x}$. Note that while we read $\frac{\Delta y}{\Delta x}$ as "change in y over change in x," for the derivative symbol $\frac{dy}{dx}$, we view this is a single symbol, not a quotient of two quantities[1]. For example, if $y = x^2$, we'll write that the derivative is $\frac{dy}{dx} = 2x$.

Furthermore, we use a variant of $\frac{dy}{dx}$ notation to convey the instruction to take the derivative of a certain quantity with respect to a given variable. In particular, if we write

$$\frac{d}{dx}[] \tag{4.4.4}$$

this means "take the derivative of the quantity in $[]$ with respect to x." To continue our example above with the squaring function, here we may write $\frac{d}{dx}\left[x^2\right] = 2x$.

It is important to note that the independent variable can be different from x. If we have $f(z) = z^2$, we then write $f'(z) = 2z$. Similarly, if $y = t^2$, we can say $\frac{dy}{dt} = 2$. And changing the variable and derivative notation once more, it is also true that $\frac{d}{dq}\left[q^2\right] = 2q$. This notation may also be applied to second derivatives:

$$f''(z) = \frac{d}{dz}\left[\frac{df}{dz}\right] = \frac{d^2f}{dz^2}. \tag{4.4.5}$$

In what follows, we'll be working to widely expand our repertoire of functions for which we can quickly compute the corresponding derivative formula

Constant, Power, and Exponential Functions

So far, we know the derivative formula for two important classes of functions: constant functions and power functions. For the first kind, observe that if $f(x) = c$ is a constant function, then its graph is a horizontal line with slope zero at every point. Thus, $\frac{d}{dx}[c] = 0$. We summarize this with the following rule.

Constant Functions:

For any real number c, if $f(x) = c$, then $f'(x) = 0$.

Thus, if $f(x) = 7$, then $f'(x) = 0$.Similarly, $\frac{d}{dx}[\sqrt{3}] = 0$.

For power functions, from your work in Preview Activity 2.1, you have conjectured that for any positive integer n, if $f(x) = x^n$, then $f'(x) = nx^{n-1}$. Not only can this rule be formally proved to hold for any positive integer n, but also for any nonzero real number (positive or negative).

Power Functions

For any nonzero real number, if $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

As we next turn to thinking about derivatives of combinations of basic functions, it will be instructive to have one more type of basic function whose derivative formula we know. For now, we simply state this rule without explanation or justification; we will explore why this rule is true in one of the exercises at the end of this section, plus we will encounter graphical reasoning for why the rule is plausible in Preview Activity 2.2.

Exponential Functions

For any positive real number a, if $f(x) = a^x$, then $f'(x) = a^x ln(a)$.

For instance, this rule tells us that if

$$f(x) = 2x \tag{4.4.6}$$

then

$$f'(x) = 2^x \ln(2). \tag{4.4.7}$$

Similarly, for

$$p(t) = 10^t \tag{4.4.8}$$

$$p'(t) = 10^t \ln(10). \tag{4.4.9}$$

It is especially important to note that when $a = e$, where e is the base of the natural logarithm function, we have that

$$\frac{d}{dx}[e^x] = e^x \ln(e) = e^x \tag{4.4.10}$$

since $\ln(e) = 1$. This is an extremely important property of the function e^x : its derivative function is itself!

Finally, note carefully the distinction between power functions and exponential functions: in power functions, the variable is in the base, as in x^2, while in exponential functions, the variable is in the power, as in 2^x. As we can see from the rules, this makes a big difference in the form of the derivative.

The following activity will check your understanding of the derivatives of the three basic types of functions noted above.

Activity 4.4.1

Use the three rules above to determine the derivative of each of the following functions. For each, state your answer using full and proper notation, labeling the derivative with its name. For example, if you are given a function $h(z)$, you should write " $h'(z) =$ " or " $\frac{dh}{dz} =$ " as part of your response.

a. $f(t) = \pi$
b. $g(z) = 7z$
c. $h(w) = w^{3/4}$
d. $p(x) = 3^{1/2}$
e. $r(t) = (\sqrt{2})^t$
f. $\frac{d}{dq}\left[q^{-1}\right]$
g. $m(t) = \frac{1}{t^3}$

Constant Multiples and Sums of Functions

Of course, most of the functions we encounter in mathematics are more complicated than being simply constant, a power of a variable, or a base raised to a variable power. In this section and several following, we will learn how to quickly compute the derivative of a function constructed as an algebraic combination of basic functions. For instance, we'd like to be able to understand how to take the derivative of a polynomial function such as

$$p(t) = 3t^5 - 7t^4 + t^2 - 9, \tag{4.4.11}$$

which is a function made up of constant multiples and sums of powers of t. To that end, we develop two new rules: the *Constant Multiple Rule* and the *Sum Rule*.

Say we have a function $y = f(x)$ whose derivative formula is known. How is the derivative of $y = kf(x)$ related to the derivative of the original function? Recall that when we multiply a function by a constant k, we vertically stretch the graph by a factor of $|k|$ (and reflect the graph across $y = 0$ if $k < 0$). This vertical stretch affects the slope of the graph, making the slope of the function $y = kf(x)$ be k times as steep as the slope of $y = f(x)$. In terms of the derivative, this is essentially saying that when we multiply a function by a factor of k, we change the value of its derivative by a factor of k as well. Thus[2] , the *Constant Multiple Rule* holds:

The Constant Multiple Rule

For any real number k, if $f(x)$ is a differentiable function with derivative $f'(x)$, then $\frac{d}{dx}[kf(x)] = kf'(x)$.

In words, this rule says that "the derivative of a constant times a function is the constant times the derivative of the function." For example, if

$$g(t) = 3 \cdot 5^t, \tag{4.4.12}$$

we have

$$g'(t) = 3 \cdot 5t \ln(5). \tag{4.4.13}$$

Similarly,

$$\frac{d}{dz}[5z^{-2}] = 5(-2z^{-3}). \tag{4.4.14}$$

Next we examine what happens when we take a sum of two functions. If we have $y = f(x)$ and $y = g(x)$, we can compute a new function $y = (f + g)(x)$ by adding the outputs of the two functions: $(f + g)(x) = f(x) + g(x)$. Not only does this result in the value of the new function being the sum of the values of the two known functions, but also the slope of the new function is the sum of the slopes of the known functions. Therefore[3], we arrive at the following Sum Rule for derivatives:

The Sum Rule

If $f(x)$ and $g(x)$ are differentiable functions with derivatives $f'(x)$ and $g'(x)$ respectively, then $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$.

In words, the Sum Rule tells us that "the derivative of a sum is the sum of the derivatives." It also tells us that any time we take a sum of two differentiable functions, the result must also be differentiable. Furthermore, because we can view the difference function

$$y = (f - g)(x) = f(x) - g(x) \text{ as } y = f(x) + (-1 \cdot g(x)), \tag{4.4.15}$$

the Sum Rule and Constant Multiple Rules together tell us that

$$\frac{d}{dx}[f(x) + (-1 \cdot g(x))] = f'(x) - g'(x), \tag{4.4.16}$$

or that "the derivative of a difference is the difference of the derivatives." Hence we can now compute derivatives of sums and differences of elementary functions. For instance,

$$\frac{d}{dw}\left(2^w + w^2\right) = 2^w \ln(2) + 2w, \tag{4.4.17}$$

and if

$$h(q) = 3q^6 - 4q^{-3}, \tag{4.4.18}$$

then

$$h'(q) = 3\left(6q^5\right) - 4\left(-3q^{-4}\right) = 18q^5 + 12q^{-4}. \tag{4.4.19}$$

Activity 4.4.2

Use only the rules for constant, power, and exponential functions, together with the Constant Multiple and Sum Rules, to compute the derivative of each function below with respect to the given independent variable. Note well that we do not yet know any rules for how to differentiate the product or quotient of functions. This means that you may have to do some algebra first on the functions below before you can actually use existing rules to compute the desired derivative formula. In each case, label the derivative you calculate with its name using proper notation such as $f'(x)$, $h'(z)$, dr/dt, etc.

a. $f(x) = x^{5/3} - x^4 + 2^x$
b. $g(x) = 14e^x + 3x^5 - x$
c. $h(z) = \sqrt{z} + \frac{1}{z^4} + 5^2$
d. $r(t) = \sqrt{53}t^7 - \pi e^t + e^4$
e. $s(y) = \left(y^2 + 1\right)\left(y^2 - 1\right)$
f. $q(x) = \frac{x^3 - x + 2}{x}$
g. $p(a) = 3a^4 - 2a^3 + 7a^2 - a + 12$

In the same way that we have shortcut rules to help us find derivatives, we introduce some language that is simpler and shorter. Often, rather than say "take the derivative of f," we'll instead say simply "differentiate f." This phrasing is tied to the notion of having a derivative to begin with: if the derivative exists at a point, we say "f is differentiable," which is tied to the fact that f can be differentiated.

As we work more and more with the algebraic structure of functions, it is important to strive to develop a big picture view of what we are doing. Here, we can note several general observations based on the rules we have so far. One is that the derivative of any polynomial function will be another polynomial function, and that the degree of the derivative is one less than the degree of the original function. For instance, if

$$p(t) = 7t^5 - 4t^3 + 8t \tag{4.4.20}$$

p is a degree 5 polynomial, and its derivative,

$$p'(t) = 35t^4 - 12t^2 + 8, \tag{4.4.21}$$

is a degree 4 polynomial. Additionally, the derivative of any exponential function is another exponential function: for example, if $g(z) = 7 \cdot 2^z$, then $g'(z) = 7 \cdot 2^z \ln(2)$, which is also exponential.

Furthermore, while our current emphasis is on learning shortcut rules for finding derivatives without directly using the limit definition, we should be certain not to lose sight of the fact that all of the meaning of the derivative still holds that we developed in Chapter 1. That is, anytime we compute a derivative, that derivative measures the instantaneous rate of change of the original function, as well as the slope of the tangent line at any selected point on the curve. The following activity asks you to combine the just-developed derivative rules with some key perspectives that we studied in Chapter 1.

Activity 4.4.3

Each of the following questions asks you to use derivatives to answer key questions about functions. Be sure to think carefully about each question and to use proper notation in your responses.

a. Find the slope of the tangent line to $h(z) = \sqrt{z} + \frac{1}{z}$ at the point where $z = 4$.

b. A population of cells is growing in such a way that its total number in millions is given by the function $P(t) = 2(1.37)^t + 32$, where t is measured in days.

c. i. Determine the instantaneous rate at which the population is growing on day 4, and include units on your answer.

ii. Is the population growing at an increasing rate or growing at a decreasing rate on day 4? Explain.

d. Find an equation for the tangent line to the curve $p(a) = 3a^4 - 2a^3 + 7a^2 - a + 12$ at the point where $a = -1$.

e. What is the difference between being asked to find the slope of the tangent line (asked in (a)) and the *equation* of the tangent line (asked in (c))?

Summary

In this section, we encountered the following important ideas:

- Given a differentiable function $y = f(x)$, we can express the derivative of f in several different notations: $f'(x)$, $\frac{df}{dx}$, $\frac{dy}{dx}$, and $\frac{d}{dx}[f(x)]$.
- The limit definition of the derivative leads to patterns among certain families of functions that enable us to compute derivative formulas without resorting directly to the limit definition. For example, if f is a power function of the form $f(x) = x^n$, then $f'(x) = nx^{n-1}$ for any real number n other than 0. This is called the Rule for Power Functions.
- We have stated a rule for derivatives of exponential functions in the same spirit as the rule for power functions: for any positive real number a, if $f(x) = a^x$, then $f'(x) = a^x \ln(a)$.
- If we are given a constant multiple of a function whose derivative we know, or a sum of functions whose derivatives we know, the Constant Multiple and Sum Rules make it straightforward to compute the derivative of the overall function. More formally, if $f(x)$ and $g(x)$ are differentiable with derivatives $f'(x)$ and $g'(x)$ and a and b are constants, then

$$\frac{d}{dx}[af(x) + bg(x)] = af'(x) + bg'(x) \tag{4.4.22}$$

.

---

[1]That is, we do not say “dee-y over dee-x.”

[2]The Constant Multiple Rule can be formally proved as a consequence of properties of limits, using the limit definition of the derivative.

[3]Like the Constant Multiple Rule, the Sum Rule can be formally proved as a consequence of properties of limits, using the limit definition of the derivative.

Supplemental videos







Homework Exercises 4.4

WeBWorK Problems:

1.

2.

3.

4.

5.

https://math.libretexts.org/@go/page/106354

6.

7.

8.

9.

10.

https://math.libretexts.org/@go/page/106354

Written Problems:

1.

Let f and g be differentiable functions for which the following information is known: $f(2) = 5,\ g(2) = -3,\ f'(2) = -1/2,\ g'(2) = 2$.

1. Let h be the new function defined by the rule $h(x) = 3f(x) - 4g(x)$. Determine $h(2)$ and $h'(2)$.
2. Find an equation for the tangent line to $y = h(x)$ at the point $(2, h(2))$.
3. Let p be the function defined by the rule $p(x) = -2f(x) + \frac{1}{2}g(x)$. Is p increasing, decreasing, or neither at $a = 2$? Why?
4. Estimate the value of $p(2.03)$ by using the local linearization of p at the point $(2, p(2))$.

2. The height of a sand dune (in centimeters) is represented by $h(t) = 600 - 7t^2$ cm, where t is measured in years since 1995. Find $h(4)$ and $h'(4)$, and carefully explain what each means in terms of the sand dune.

This page titled 4.4: Elementary Derivative Rules is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Matthew Boelkins, David Austin & Steven Schlicker (ScholarWorks @Grand Valley State University) via source content that was edited to the style and standards of the LibreTexts platform.

- **2.1: Elementary Derivative Rules** is licensed CC BY-SA 4.0. Original source: https://activecalculus.org/single.
- **2.1: Elementary Derivative Rules** by Matthew Boelkins, David Austin & Steven Schlicker is licensed CC BY-SA 4.0. Original source: https://activecalculus.org/single.

CHAPTER OVERVIEW

5: Trigonometry Essentials

5.1: Angles
5.2: Right Triangle Trigonometry
5.3: Unit Circle
5.4: The Other Trigonometric Functions
5.5: Graphs of the Sine and Cosine Functions

5: Trigonometry Essentials is shared under a not declared license and was authored, remixed, and/or curated by LibreTexts.

5.1: Angles

Preview Video for sections 5.1 & 5.2

Please fill out your notes as you view the video and answer the questions.

Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login

Learning Objectives

- Draw angles in standard position.
- Convert between degrees and radians.
- Find coterminal angles.
- Find the length of a circular arc.
- Use linear and angular speed to describe motion on a circular path.

A golfer swings to hit a ball over a sand trap and onto the green. An airline pilot maneuvers a plane toward a narrow runway. A dress designer creates the latest fashion. What do they all have in common? They all work with angles, and so do all of us at one time or another. Sometimes we need to measure angles exactly with instruments. Other times we estimate them or judge them by eye. Either way, the proper angle can make the difference between success and failure in many undertakings. In this section, we will examine properties of angles.

Drawing Angles in Standard Position

Properly defining an angle first requires that we define a *ray*. A ray consists of one point on a line and all points extending in one direction from that point. The first point is called the endpoint of the ray. We can refer to a specific ray by stating its *endpoint* and any other point on it. The ray in Figure 5.1.1 can be named as ray EF, or in symbol form $\overrightarrow{EF}$.

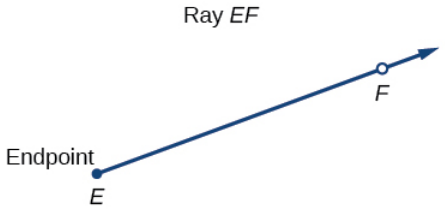


Figure 5.1.1

An *angle* is the union of two rays having a common endpoint. The endpoint is called the **vertex** of the angle, and the two rays are the sides of the angle. The angle in Figure 5.1.2 is formed from $\overrightarrow{ED}$ and $\overrightarrow{EF}$. Angles can be named using a point on each ray and the vertex, such as angle *DEF*, or in symbol form $\angle DEF$.

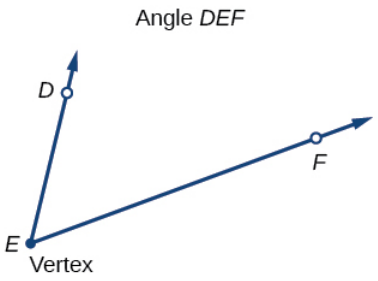


Figure 5.1.2

Greek letters are often used as variables for the measure of an angle. Table 5.1.1 is a list of Greek letters commonly used to represent angles, and a sample angle is shown in Figure 5.1.3.

Table 5.1.1

θ	φ or ϕ	α	β	γ
theta	phi	alpha	beta	gamma

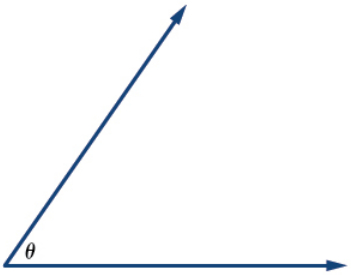


***Figure* 5.1.3**: *Angle theta, shown as* $\angle\theta$

Angle creation is a dynamic process. We start with two rays lying on top of one another. We leave one fixed in place, and rotate the other. The fixed ray is the **initial side,** and the rotated ray is the **terminal side**. In order to identify the different sides, we indicate the rotation with a small arc and arrow close to the vertex as in Figure 5.1.4.

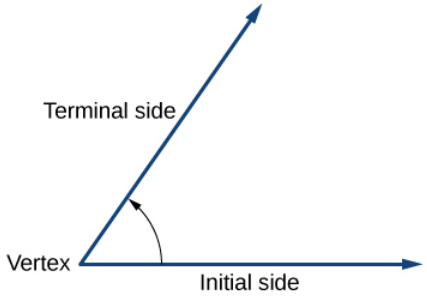


Figure 5.1.4

As we discussed at the beginning of the section, there are many applications for angles, but in order to use them correctly, we must be able to measure them. The **measure of an angle** is the amount of rotation from the initial side to the terminal side. Probably the most familiar unit of angle measurement is the degree. One **degree** is $\frac{1}{360}$ of a circular rotation, so a complete circular rotation contains 360 degrees. An angle measured in degrees should always include the unit “degrees” after the number, or include the degree symbol °. For example, 90 degrees = 90°.

To formalize our work, we will begin by drawing angles on an *x*-*y* coordinate plane. Angles can occur in any position on the coordinate plane, but for the purpose of comparison, the convention is to illustrate them in the same position whenever possible. An angle is in **standard position** if its vertex is located at the origin, and its initial side extends along the positive *x*-axis. See Figure 5.1.5.

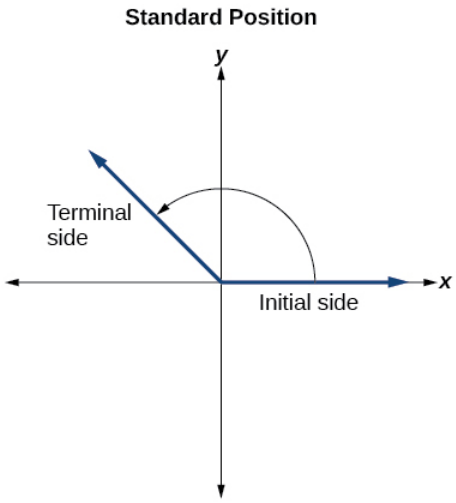


Figure 5.1.5

If the angle is measured in a counterclockwise direction from the initial side to the terminal side, the angle is said to be a **positive angle**. If the angle is measured in a clockwise direction, the angle is said to be a **negative angle**.

Drawing an angle in standard position always starts the same way—draw the initial side along the positive *x*-axis. To place the terminal side of the angle, we must calculate the fraction of a full rotation the angle represents. We do that by dividing the angle measure in degrees by 360°. For example, to draw a 90° angle, we calculate that $\frac{90^\circ}{360^\circ} = \frac{1}{4}$. So, the terminal side will be one-fourth of the way around the circle, moving counterclockwise from the positive *x*-axis. To draw a 360° angle, we calculate that $\frac{360^\circ}{360^\circ} = 1$. So the terminal side will be 1 complete rotation around the circle, moving counterclockwise from the positive *x*-axis. In this case, the initial side and the terminal side overlap. See Figure 5.1.6.

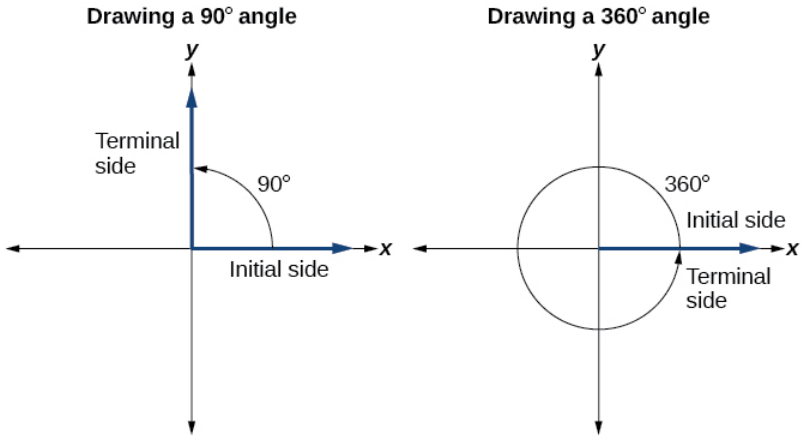


Figure 5.1.6

Since we define an angle in **standard position** by its initial side, we have a special type of angle whose terminal side lies on an axis, a **quadrantal angle**. This type of angle can have a measure of 0°, 90°, 180°, 270° or 360°. See Figure 5.1.7.

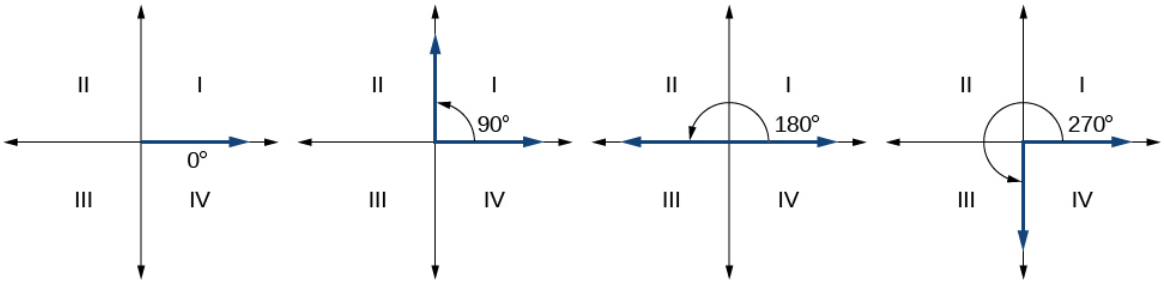


Figure 5.1.7: Quadrantal angles are angles in standard position whose terminal side lies along an axis. Examples are shown.

QUADRANTAL ANGLES

Quadrantal angles are angles in standard position whose terminal side lies on an axis, including 0°, 90°, 180°, 270°, or 360°.

Given an angle measure in degrees, draw the angle in standard position

1. Express the angle measure as a fraction of 360°.
2. Reduce the fraction to simplest form.
3. Draw an angle that contains that same fraction of the circle, beginning on the positive *x*-axis and moving counterclockwise for positive angles and clockwise for negative angles.

Example 5.1.1: Drawing an Angle in Standard Position Measured in Degrees

1. Sketch an angle of 30° in standard position.
2. Sketch an angle of −135° in standard position.

Solution

1. Divide the angle measure by 360°.

$$\frac{30^\circ}{360^\circ} = \frac{1}{12} \tag{5.1.1}$$

To rewrite the fraction in a more familiar fraction, we can recognize that

$$\frac{1}{12} = \frac{1}{3}\left(\frac{1}{4}\right) \tag{5.1.2}$$

One-twelfth equals one-third of a quarter, so by dividing a quarter rotation into thirds, we can sketch a line at 30° as in Figure 5.1.8.

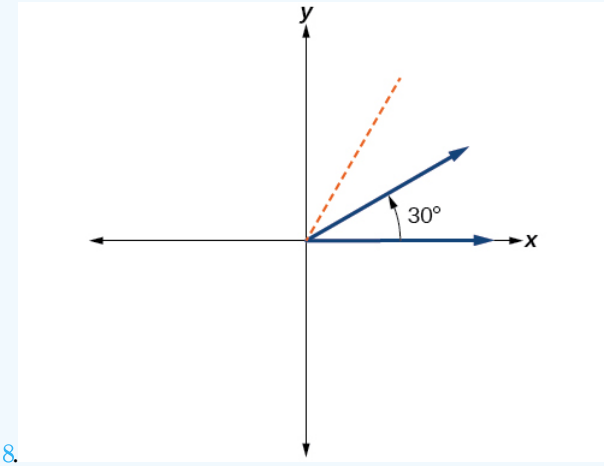


***Figure* 5.1.8**

2. Divide the angle measure by 360°.

$$\frac{-135^\circ}{360^\circ} = -\frac{3}{8} \tag{5.1.3}$$

In this case, we can recognize that

$$-\frac{3}{8} = -\frac{3}{2}\left(\frac{1}{4}\right) \tag{5.1.4}$$

Negative three-eighths is one and one-half times a quarter, so we place a line by moving clockwise one full quarter and one-half of another quarter, as in Figure 5.1.9.

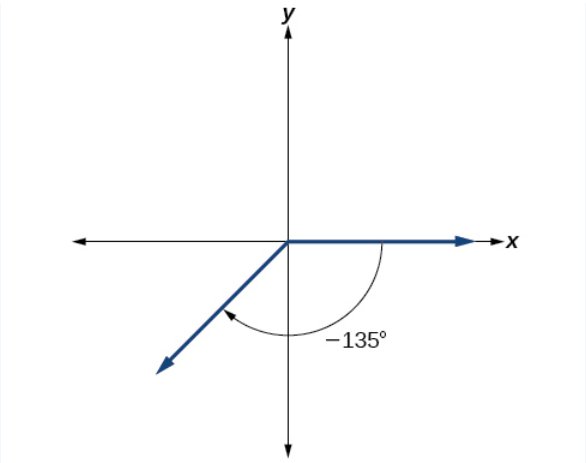


***Figure* 5.1.9**

5.1.1:

Show an angle of 240° on a circle in standard position.

Solution

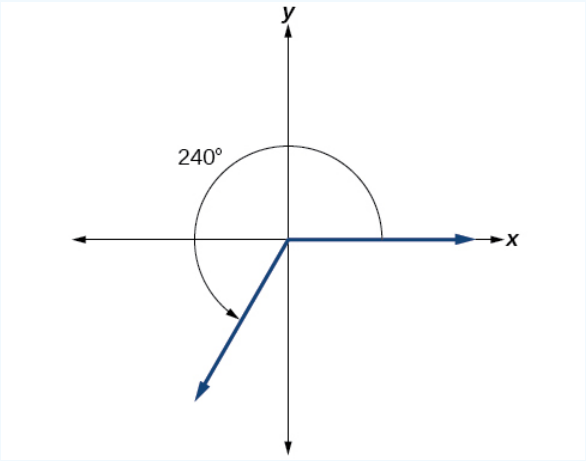


Figure 5.1.10

Converting Between Degrees and Radians

Dividing a circle into 360 parts is an arbitrary choice, although it creates the familiar degree measurement. We may choose other ways to divide a circle. To find another unit, think of the process of drawing a circle. Imagine that you stop before the circle is completed. The portion that you drew is referred to as an arc. An **arc** may be a portion of a full circle, a full circle, or more than a full circle, represented by more than one full rotation. The length of the arc around an entire circle is called the **circumference** of that circle.

The circumference of a circle is $C = 2\pi r$. If we divide both sides of this equation by r, we create the ratio of the circumference to the radius, which is always 2π regardless of the length of the radius. So the circumference of any circle is $2\pi \approx 6.28$ times the length of the radius. That means that if we took a string as long as the radius and used it to measure consecutive lengths around the circumference, there would be room for six full string-lengths and a little more than a quarter of a seventh, as shown in Figure 5.1.11.

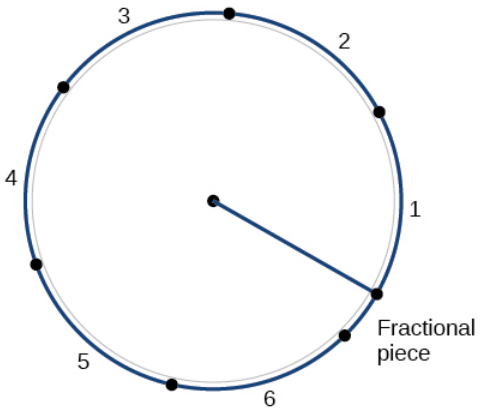


Figure 5.1.11

This brings us to our new angle measure. One **radian** is the measure of a central angle of a circle that intercepts an arc equal in length to the radius of that circle. A central angle is an angle formed at the center of a circle by two radii. Because the total circumference equals 2π times the radius, a full circular rotation is 2π radians. So

$$2\pi \text{ radians } = 360^\circ \tag{5.1.5}$$

$$\pi \text{ radians } = \frac{360^\circ}{2} = 180^\circ \tag{5.1.6}$$

$$1 \text{ radian } = \frac{180^\circ}{\pi} \approx 57.3^\circ \tag{5.1.7}$$

See Figure 5.1.12. Note that when an angle is described without a specific unit, it refers to radian measure. For example, an angle measure of 3 indicates 3 radians. In fact, radian measure is dimensionless, since it is the quotient of a length (circumference) divided by a length (radius) and the length units cancel out.

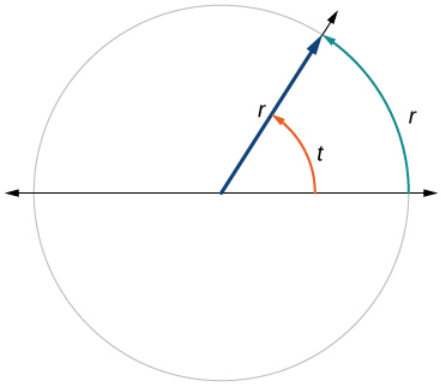


Figure 5.1.12: The angle t sweeps out a measure of one radian. Note that the length of the intercepted arc is the same as the length of the radius of the circle.

Relating Arc Lengths to Radius

An **arc length** s is the length of the curve along the arc. Just as the full circumference of a circle always has a constant ratio to the radius, the arc length produced by any given angle also has a constant relation to the radius, regardless of the length of the radius.

This ratio, called the **radian measure**, is the same regardless of the radius of the circle—it depends only on the angle. This property allows us to define a measure of any angle as the ratio of the arc length s s to the radius r. See Figure 5.1.13.

$$s = r\theta \tag{5.1.8}$$

$$\theta = \frac{s}{r} \tag{5.1.9}$$

If $s = r$, then $\theta = \frac{r}{r} = 1$ radian.

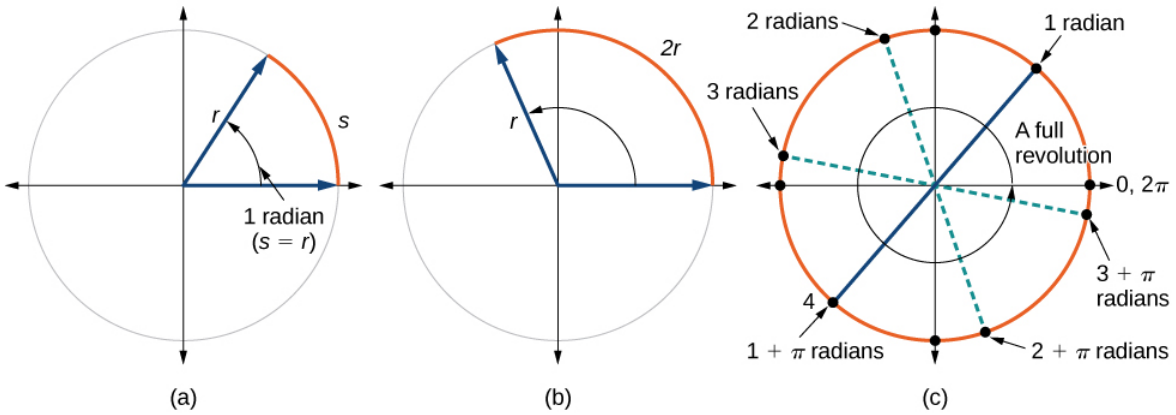


Figure 5.1.13: (a) In an angle of 1 radian, the arc length s equals the radius r. (b) An angle of 2 radians has an arc length $s = 2r$. (c) A full revolution is 2π or about 6.28 radians.

To elaborate on this idea, consider two circles, one with radius 2 and the other with radius 3. Recall the circumference of a circle is $C = 2\pi r$,where r is the radius. The smaller circle then has circumference $2\pi(2) = 4\pi$ and the larger has circumference $2\pi(3) = 6\pi$. Now we draw a 45° angle on the two circles, as in Figure 5.1.14.

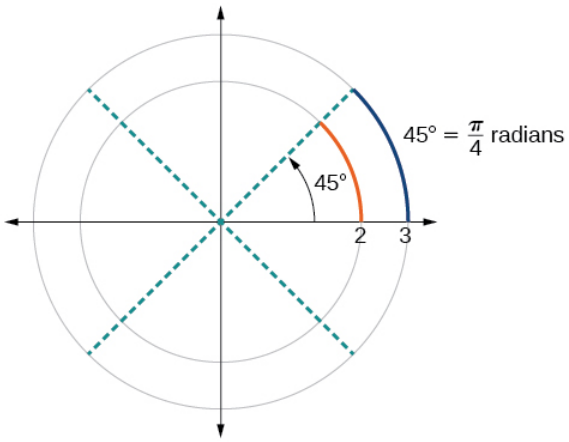


Figure 5.1.14: A 45° angle contains one-eighth of the circumference of a circle, regardless of the radius.

Notice what happens if we find the ratio of the arc length divided by the radius of the circle.

$$\text{Smaller circle: } \frac{\frac{1}{2}\pi}{2} = \frac{1}{4}\pi \tag{5.1.10}$$

$$\text{Larger circle: } \frac{\frac{3}{4}\pi}{3} = \frac{1}{4}\pi \tag{5.1.11}$$

Since both ratios are $\frac{1}{4}\pi$, the angle measures of both circles are the same, even though the arc length and radius differ.

RADIANS

One **radian** is the measure of the central angle of a circle such that the length of the arc between the initial side and the terminal side is equal to the radius of the circle. A full revolution (360°) equals 2π radians. A half revolution (180°) is equivalent to π radians.

The **radian measure** of an angle is the ratio of the length of the arc subtended by the angle to the radius of the circle. In other words, if s is the length of an arc of a circle, and r is the radius of the circle, then the central angle containing that arc measures $\frac{s}{r}$ radians. In a circle of radius 1, the radian measure corresponds to the length of the arc.

A measure of 1 radian looks to be about 60°. Is that correct?

Yes. It is approximately 57.3°. Because 2π *radians equals 360°,* 1 *radian equals* $\frac{360^\circ}{2\pi} \approx 57.3^\circ$.

Using Radians

Because **radian** measure is the ratio of two lengths, it is a unitless measure. For example, in Figure 5.1.14, suppose the radius were 2 inches and the distance along the arc were also 2 inches. When we calculate the radian measure of the angle, the "inches" cancel, and we have a result without units. Therefore, it is not necessary to write the label "radians" after a radian measure, and if we see an angle that is not labeled with "degrees" or the degree symbol, we can assume that it is a radian measure.

Considering the most basic case, the **unit circle** (a circle with radius 1), we know that 1 rotation equals 360 degrees, 360°. We can also track one rotation around a circle by finding the circumference, $C = 2\pi r$, and for the unit circle $C = 2\pi$. These two different ways to rotate around a circle give us a way to convert from degrees to radians.

$$\begin{aligned} 1\text{rotation} &= 360^\circ &&= 2\pi \quad \text{radians} \\ \frac{1}{2}\text{rotation} &= 180^\circ &&= \pi \quad \text{radians} \\ \frac{1}{4}\text{rotation} &= 90^\circ &&= \frac{\pi}{2} \quad \text{radians} \end{aligned} \tag{5.1.12}$$

Identifying Special Angles Measured in Radians

In addition to knowing the measurements in degrees and radians of a quarter revolution, a half revolution, and a full revolution, there are other frequently encountered angles in one revolution of a circle with which we should be familiar. It is common to encounter multiples of 30, 45, 60, and 90 degrees. These values are shown in Figure 5.1.15. Memorizing these angles will be very useful as we study the properties associated with angles.

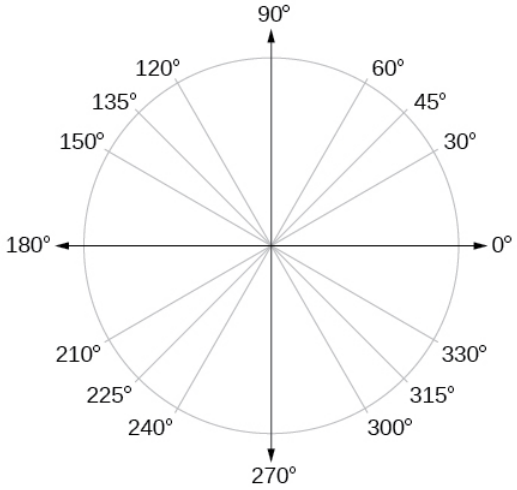


Figure 5.1.15: Commonly encountered angles measured in degrees

Now, we can list the corresponding radian values for the common measures of a circle corresponding to those listed in Figure 5.1.15, which are shown in Figure 5.1.16. Be sure you can verify each of these measures.

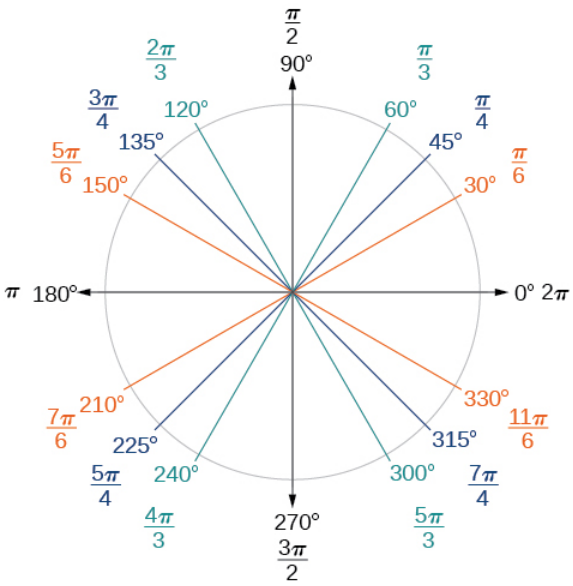


Figure 5.1.16: Commonly encountered angles measured in radians

Example 5.1.2: Finding a Radian Measure

Find the radian measure of one-third of a full rotation.

Solution

For any circle, the arc length along such a rotation would be one-third of the circumference. We know that

$$1 \text{ rotation} = 2\pi r \tag{5.1.13}$$

So,

$$s = \frac{1}{3}(2\pi r) \tag{5.1.14}$$

$$= \frac{2\pi r}{3} \tag{5.1.15}$$

The radian measure would be the arc length divided by the radius.

$$\text{radian measure} = \frac{\frac{2\pi r}{3}}{r} \tag{5.1.16}$$

$$= \frac{2\pi r}{3r} \tag{5.1.17}$$

$$= \frac{2\pi}{3} \tag{5.1.18}$$

5.1.2

Find the radian measure of three-fourths of a full rotation.

Solution

$$\frac{3\pi}{2} \tag{5.1.19}$$

Converting between Radians and Degrees

Because degrees and radians both measure angles, we need to be able to convert between them. We can easily do so using a proportion.

$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \tag{5.1.20}$$

This proportion shows that the measure of angle θ in degrees divided by 180 equals the measure of angle θ in radians divided by π. Or, phrased another way, degrees is to 180 as radians is to π.

$$\frac{\text{Degrees}}{180} = \frac{Radians}{\pi} \tag{5.1.21}$$

CONVERTING BETWEEN RADIANS AND DEGREES

To convert between degrees and radians, use the proportion

$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \tag{5.1.22}$$

Example 5.1.3: Converting Radians to Degrees

Convert each radian measure to degrees.

1. $\frac{\pi}{6}$
2. 3

Solution

Because we are given radians and we want degrees, we should set up a proportion and solve it.

1. We use the proportion, substituting the given information.

$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \tag{5.1.23}$$

$$\frac{\theta}{180} = \frac{\frac{\pi}{6}}{\pi} \tag{5.1.24}$$

$$\theta = \frac{180}{6} \tag{5.1.25}$$

$$\theta = 30^\circ \tag{5.1.26}$$

2. We use the proportion, substituting the given information.

$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \tag{5.1.27}$$

$$\frac{\theta}{180} = \frac{3}{\pi} \tag{5.1.28}$$

$$\theta = \frac{3(180)}{\pi} \tag{5.1.29}$$

$$\theta \approx 172^\circ \tag{5.1.30}$$

5.1.3

Convert $-\frac{3\pi}{4}$ radians to degrees.

Solution

-135°

Example 5.1.4: Converting Degrees to Radians

Convert 15 degrees to radians.

Solution

In this example, we start with degrees and want radians, so we again set up a proportion and solve it, but we substitute the given information into a different part of the proportion.

$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \tag{5.1.31}$$

$$\frac{15}{180} = \frac{\theta^R}{\pi} \tag{5.1.32}$$

$$\frac{15\pi}{180} = \theta^R \tag{5.1.33}$$

$$\frac{\pi}{12} = \theta^R \tag{5.1.34}$$

Analysis

Another way to think about this problem is by remembering that $30^\circ = \frac{\pi}{6}$. Because $15^\circ = \frac{1}{2}(30^\circ)$, ,we can find that $\frac{1}{2}(\frac{\pi}{6})$ is $\frac{\pi}{12}$.

5.1.4

Convert 126° to radians.

Solution

$\frac{7\pi}{10}$

Finding Coterminal Angles

Converting between degrees and radians can make working with angles easier in some applications. For other applications, we may need another type of conversion. Negative angles and angles greater than a full revolution are more awkward to work with than those in the range of 0° to 360°, or 0 to 2π. It would be convenient to replace those out-of-range angles with a corresponding angle within the range of a single revolution.

It is possible for more than one angle to have the same terminal side. Look at Figure 5.1.17. The angle of 140° is a **positive angle**, measured counterclockwise. The angle of –220° is a **negative angle**, measured clockwise. But both angles have the same terminal side. If two angles in standard position have the same terminal side, they are **coterminal** angles. Every angle greater than 360° or less than 0° is coterminal with an angle between 0° and 360°, and it is often more convenient to find the coterminal angle within the range of 0° to 360° than to work with an angle that is outside that range.

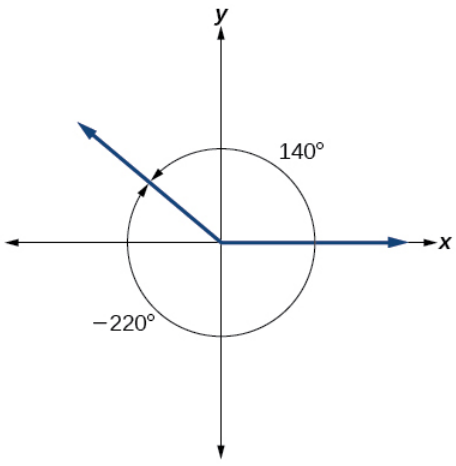


Figure 5.1.17: An angle of 140° and an angle of –220° are coterminal angles.

Any angle has infinitely many **coterminal angles** because each time we add 360° to that angle—or subtract 360° from it—the resulting value has a terminal side in the same location. For example, 100° and 460° are coterminal for this reason, as is −260°. Recognizing that any angle has infinitely many coterminal angles explains the repetitive shape in the graphs of trigonometric functions.

An angle's reference angle is the measure of the smallest, positive, acute angle t formed by the terminal side of the angle t and the horizontal axis. Thus positive reference angles have terminal sides that lie in the first quadrant and can be used as models for angles in other quadrants. See Figure 5.1.18 for examples of reference angles for angles in different quadrants.

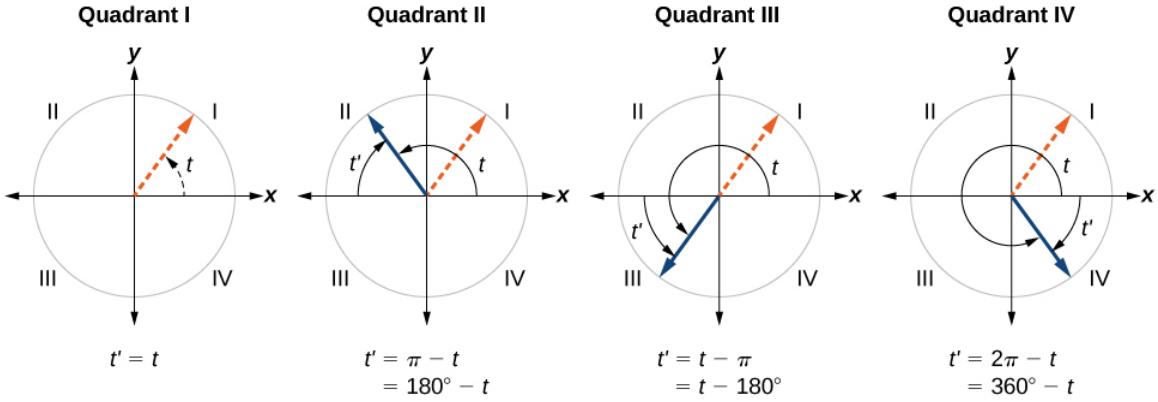


Figure 5.1.18

COTERMINAL AND REFERENCE ANGLES

- *Coterminal angles* are two angles in standard position that have the same terminal side.
- An angle's *reference angle* is the size of the smallest acute angle, t', formed by the terminal side of the angle t and the horizontal axis.

Given an angle greater than 360°, find a coterminal angle between 0° and 360°

1. Subtract 360° from the given angle.
2. If the result is still greater than 360°, subtract 360° again till the result is between 0° and 360°.
3. The resulting angle is coterminal with the original angle.

Example 5.1.5: Finding an Angle Coterminal with an Angle of Measure Greater Than 360°

Find the least positive angle θ that is coterminal with an angle measuring 800°, where $0^\circ \le \theta < 360^\circ$.

Solution

An angle with measure 800° is coterminal with an angle with measure 800 − 360 = 440°, but 440° is still greater than 360°, so we subtract 360° again to find another coterminal angle: 440 − 360 = 80°.

The angle $\theta = 80^\circ$ is coterminal with 800°. To put it another way, 800° equals 80° plus two full rotations, as shown in Figure 5.1.19.

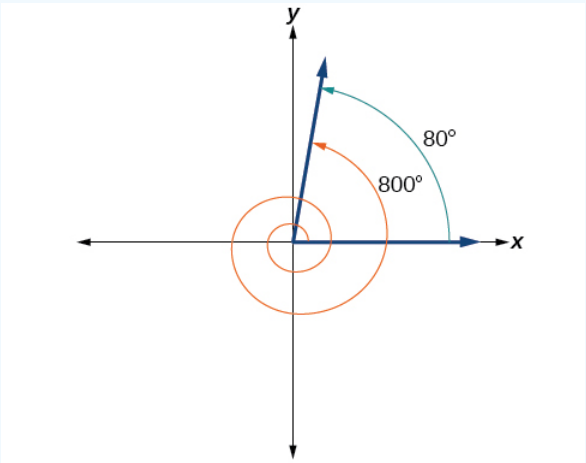


Figure 5.1.19

5.1.5

Find an angle α that is coterminal with an angle measuring 870°, where $0^\circ \le \alpha < 360^\circ$.

Solution

$\alpha = 150^\circ$

Given an angle with measure less than 0°, find a coterminal angle having a measure between 0° and 360°.

1. Add 360° to the given angle.
2. If the result is still less than 0°, add 360° again until the result is between 0° and 360°.
3. The resulting angle is coterminal with the original angle.

Example 5.1.6: Finding an Angle Coterminal with an Angle Measuring Less Than 0°

Show the angle with measure −45° on a circle and find a positive coterminal angle α such that 0° ≤ α < 360°.

Solution

Since 45° is half of 90°, we can start at the positive horizontal axis and measure clockwise half of a 90° angle.

Because we can find coterminal angles by adding or subtracting a full rotation of 360°, we can find a positive coterminal angle here by adding 360°:

$$-45^\circ + 360^\circ = 315^\circ \tag{5.1.35}$$

We can then show the angle on a circle, as in Figure 5.1.20.

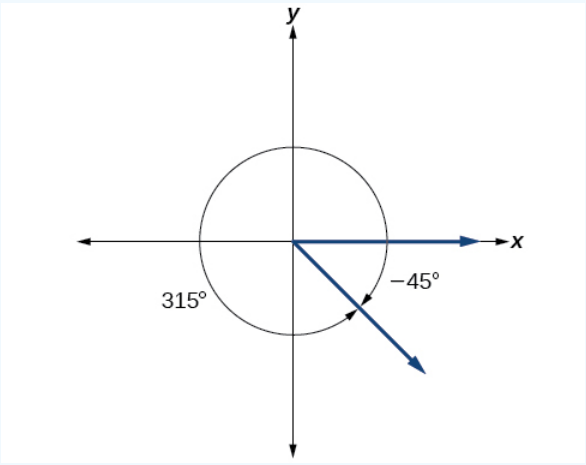


Figure 5.1.20

5.1.6

Find an angle β that is coterminal with an angle measuring −300° such that $0^\circ \le \beta < 360^\circ$.

Solution

$\beta = 60^\circ$

Finding Coterminal Angles Measured in Radians

We can find **coterminal** angles measured in radians in much the same way as we have found them using degrees. In both cases, we find coterminal angles by adding or subtracting one or more full rotations.

Given an angle greater than 2π, find a coterminal angle between 0 and 2π.

1. Subtract 2π from the given angle.
2. If the result is still greater than 2π, subtract 2π again until the result is between 0 and 2π.
3. The resulting angle is coterminal with the original angle.

Example 5.1.7: Finding Coterminal Angles Using Radians

Find an angle β that is coterminal with $\frac{19\pi}{4}$, where $0 \leq \beta < 2\pi$.

Solution

When working in degrees, we found coterminal angles by adding or subtracting 360 degrees, a full rotation. Likewise, in radians, we can find coterminal angles by adding or subtracting full rotations of 2π radians:

$$\frac{19\pi}{4} - 2\pi = \frac{19\pi}{4} - \frac{8\pi}{4} \tag{5.1.36}$$

$$= \frac{11\pi}{4} \tag{5.1.37}$$

The angle $\frac{11\pi}{4}$ is coterminal, but not less than 2π, so we subtract another rotation:

$$\frac{11\pi}{4} - 2\pi = \frac{11\pi}{4} - \frac{8\pi}{4} \tag{5.1.38}$$

$$= \frac{3\pi}{4} \tag{5.1.39}$$

The angle $\frac{3\pi}{4}$ is coterminal with $\frac{19\pi}{4}$, as shown in Figure 5.1.21.

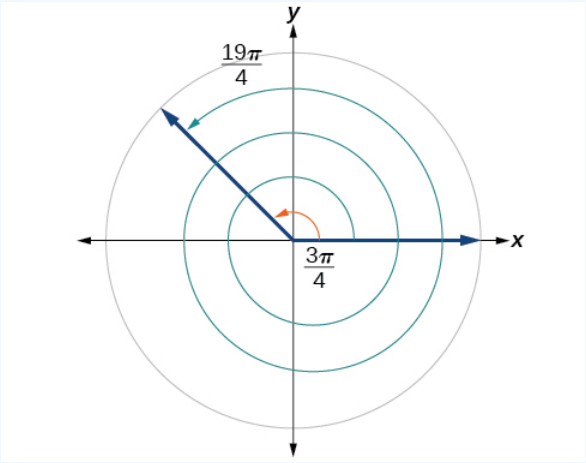


Figure 5.1.21

5.1.7

Find an angle of measure θ that is coterminal with an angle of measure $-\frac{17\pi}{6}$ where $0 \leq \theta < 2\pi$.

Solution

$\frac{7\pi}{6}$

Determining the Length of an Arc

Recall that the **radian measure** θ of an angle was defined as the ratio of the **arc length** s of a circular arc to the radius r of the circle, $\theta = \frac{s}{r}$. From this relationship, we can find arc length along a circle, given an angle.

ARC LENGTH ON A CIRCLE

In a circle of radius r, the length of an arc s subtended by an angle with measure θ in radians, shown in Figure 5.1.22, is

$$s = r\theta \tag{5.1.40}$$

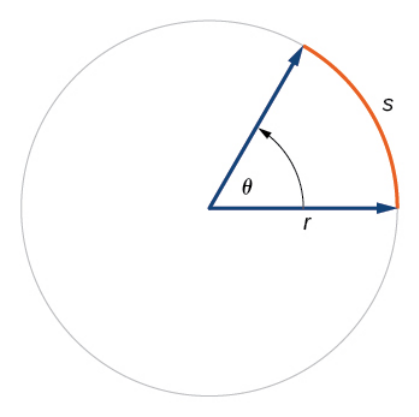


Figure 5.1.22

Given a circle of radius r, calculate the length s of the arc subtended by a given angle of measure θ.

1. If necessary, convert θ to radians.
2. Multiply the radius r by the radian measure of θ : $s = r\theta$.

Example 5.1.8: Finding the Length of an Arc

Assume the orbit of Mercury around the sun is a perfect circle. Mercury is approximately 36 million miles from the sun.

1. In one Earth day, Mercury completes 0.0114 of its total revolution. How many miles does it travel in one day?
2. Use your answer from part (a) to determine the radian measure for Mercury's movement in one Earth day.

Solution

1. Let's begin by finding the circumference of Mercury's orbit.

$$C = 2\pi r \tag{5.1.41}$$
$$= 2\pi(36 \text{ million miles}) \tag{5.1.42}$$
$$\approx 226 \text{ million miles} \tag{5.1.43}$$

Since Mercury completes 0.0114 of its total revolution in one Earth day, we can now find the distance traveled:

$$(0.0114)226 \text{ million miles} = 2.58 \text{ million miles} \tag{5.1.44}$$

2. Now, we convert to radians:

$$\text{radian} = \frac{\text{arc length}}{\text{radius}} \tag{5.1.45}$$
$$= \frac{2.58 \text{ million miles}}{36 \text{ million miles}} \tag{5.1.46}$$
$$= 0.0717 \tag{5.1.47}$$

5.1.8

Find the arc length along a circle of radius 10 units subtended by an angle of 215°.

Solution

$$\frac{215\pi}{18} = 37.525 \text{ units} \tag{5.1.48}$$

Finding the Area of a Sector of a Circle

In addition to arc length, we can also use angles to find the area of a **sector of a circle**. A sector is a region of a circle bounded by two radii and the intercepted arc, like a slice of pizza or pie. Recall that the area of a circle with radius r can be found using the formula $A = \pi r^2$. If the two radii form an angle of θ, measured in radians, then $\frac{\theta}{2\pi}$ is the ratio of the angle measure to the measure of a full rotation and is also, therefore, the ratio of the area of the sector to the area of the circle. Thus, the **area of a sector** is the fraction $\frac{\theta}{2\pi}$ multiplied by the entire area. (Always remember that this formula only applies if θ is in radians.)

$$\text{Area of sector} = \left(\frac{\theta}{2\pi}\right)\pi r^2 \tag{5.1.49}$$

$$= \frac{\theta \pi r^2}{2\pi} \tag{5.1.50}$$

$$= \frac{1}{2}\theta r^2 \tag{5.1.51}$$

AREA OF A SECTOR

The area of a **sector of a circle** with radius r subtended by an angle θ, measured in radians, is

$$A = \frac{1}{2}\theta r^2 \tag{5.1.52}$$

See Figure 5.1.23.

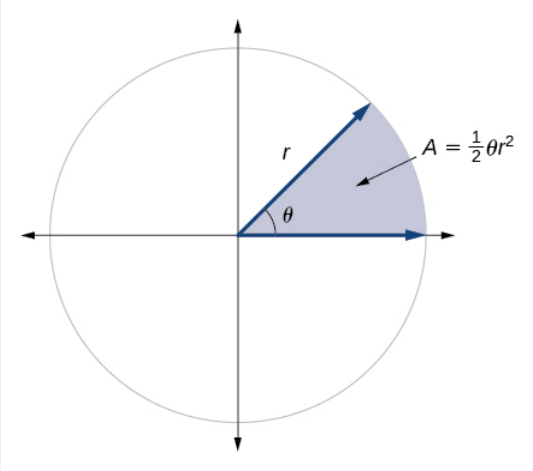


Figure 5.1.23: The area of the sector equals half the square of the radius times the central angle measured in radians.

Given a circle of radius r, find the area of a sector defined by a given angle θ.

1. If necessary, convert θ to radians.
2. Multiply half the radian measure of θ by the square of the radius r : $A = \frac{1}{2}\theta r^2$.

Example 5.1.9: Finding the Area of a Sector

An automatic lawn sprinkler sprays a distance of 20 feet while rotating 30 degrees, as shown in Figure 5.1.24. What is the area of the sector of grass the sprinkler waters?

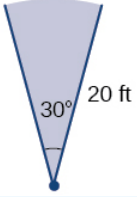


Figure 5.1.24: The sprinkler sprays 20 ft within an arc of 30°.

Solution

First, we need to convert the angle measure into radians. Because 30 degrees is one of our special angles, we already know the equivalent radian measure, but we can also convert:

$$30 \text{ degrees} = 30 \cdot \frac{\pi}{180} \tag{5.1.53}$$

$$= \frac{\pi}{6} \text{ radians} \tag{5.1.54}$$

The area of the sector is then

$$\text{Area} = \frac{1}{2}(\frac{\pi}{6})(20)^2 \tag{5.1.55}$$

$$\approx 104.72 \tag{5.1.56}$$

So the area is about 104.72 ft^2.

5.1.9

In central pivot irrigation, a large irrigation pipe on wheels rotates around a center point. A farmer has a central pivot system with a radius of 400 meters. If water restrictions only allow her to water 150 thousand square meters a day, what angle should she set the system to cover? Write the answer in radian measure to two decimal places.

Solution

1.88

Use Linear and Angular Speed to Describe Motion on a Circular Path

In addition to finding the area of a sector, we can use angles to describe the speed of a moving object. An object traveling in a circular path has two types of speed. **Linear speed** is speed along a straight path and can be determined by the distance it moves along (its **displacement**) in a given time interval. For instance, if a wheel with radius 5 inches rotates once a second, a point on the edge of the wheel moves a distance equal to the circumference, or 10π inches, every second. So the linear speed of the point is 10π in./s. The equation for linear speed is as follows where v is linear speed, s is displacement, and t is time.

$$v = \frac{s}{t} \tag{5.1.57}$$

Angular speed results from circular motion and can be determined by the angle through which a point rotates in a given time interval. In other words, angular speed is angular rotation per unit time. So, for instance, if a gear makes a full rotation every 4 seconds, we can calculate its angular speed as $\frac{360 \text{ degrees}}{4 \text{ seconds}} = 90$ degrees per second. Angular speed can be given in radians per second, rotations per minute, or degrees per hour for example. The equation for angular speed is as follows, where ω (read as omega) is angular speed, θ is the angle traversed, and t is time.

$$\omega = \frac{\theta}{t} \tag{5.1.58}$$

Combining the definition of angular speed with the arc length equation, $s = r\theta$, we can find a relationship between angular and linear speeds. The angular speed equation can be solved for θ, giving $\theta = \omega t$. Substituting this into the arc length equation gives:

$$s = r\theta \tag{5.1.59}$$

$$= r\omega t \tag{5.1.60}$$

Substituting this into the linear speed equation gives:

$$v = \frac{s}{t} \qquad = \frac{r\omega t}{t} = r\omega \tag{5.1.61}$$

ANGULAR AND LINEAR SPEED

As a point moves along a circle of radius r, its **angular speed**, ω, is the angular rotation θ per unit time, t.

$$\omega = \frac{\theta}{t} \tag{5.1.62}$$

The **linear speed.** v, of the point can be found as the distance traveled, arc length s, per unit time, t.

$$v = \frac{s}{t} \tag{5.1.63}$$

When the angular speed is measured in radians per unit time, linear speed and angular speed are related by the equation

$$v = r\omega \tag{5.1.64}$$

This equation states that the angular speed in radians, ω, representing the amount of rotation occurring in a unit of time, can be multiplied by the radius r to calculate the total arc length traveled in a unit of time, which is the definition of linear speed.

Given the amount of angle rotation and the time elapsed, calculate the angular speed

1. If necessary, convert the angle measure to radians.
2. Divide the angle in radians by the number of time units elapsed: $\omega = \frac{\theta}{t}$.
3. The resulting speed will be in radians per time unit.

Example 5.1.10: Finding Angular Speed

A water wheel, shown in Figure 5.1.25, completes 1 rotation every 5 seconds. Find the angular speed in radians per second.

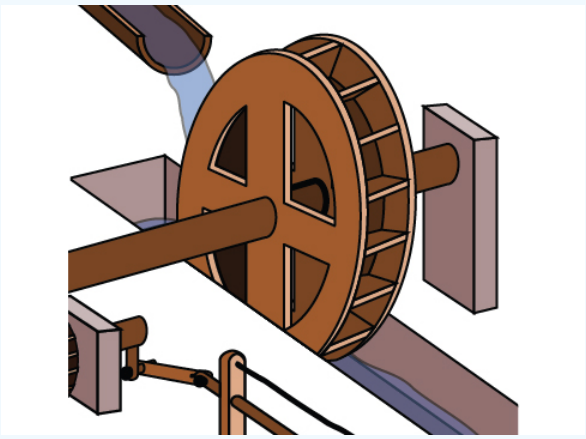


Figure 5.1.25

Solution

The wheel completes 1 rotation, or passes through an angle of 2π radians in 5 seconds, so the angular speed would be $\omega = \frac{2\pi}{5} \approx 1.257$ radians per second.

5.1.10

An old vinyl record is played on a turntable rotating clockwise at a rate of 45 rotations per minute. Find the angular speed in radians per second.

Solution

$-\frac{3\pi}{2}$ rad/s

Given the radius of a circle, an angle of rotation, and a length of elapsed time, determine the linear speed

1. Convert the total rotation to radians if necessary.
2. Divide the total rotation in radians by the elapsed time to find the angular speed: apply $\omega = \frac{\theta}{t}$.
3. Multiply the angular speed by the length of the radius to find the linear speed, expressed in terms of the length unit used for the radius and the time unit used for the elapsed time: apply $v = r\omega$.

Example 5.1.11: Finding a Linear Speed

A bicycle has wheels 28 inches in diameter. A tachometer determines the wheels are rotating at 180 RPM (revolutions per minute). Find the speed the bicycle is traveling down the road.

Solution

Here, we have an angular speed and need to find the corresponding linear speed, since the linear speed of the outside of the tires is the speed at which the bicycle travels down the road.

We begin by converting from rotations per minute to radians per minute. It can be helpful to utilize the units to make this conversion:

$$180 \frac{\text{rotations}}{\text{minute}} \cdot \frac{2\pi \text{ radians}}{\text{rotation}} = 360\pi \frac{\text{radians}}{\text{minute}} \tag{5.1.65}$$

Using the formula from above along with the radius of the wheels, we can find the linear speed:

$$v = (14 \text{ inches})(360\pi \frac{\text{radians}}{\text{minute}}) \tag{5.1.66}$$

$$= 5040\pi \frac{\text{inches}}{\text{minute}} \tag{5.1.67}$$

Remember that radians are a unitless measure, so it is not necessary to include them.unitless measure, so it is not necessary to include them.

Finally, we may wish to convert this linear speed into a more familiar measurement, like miles per hour.

$$5040\pi \frac{\text{inches}}{\text{minute}} \cdot \frac{1 \text{ feet}}{12 \text{ inches}} \cdot \frac{1 \text{ mile}}{5280 \text{ feet}} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}} \approx 14.99 \text{ miles per hour (mph)} \tag{5.1.68}$$

5.1.11

A satellite is rotating around Earth at 0.25 radians per hour at an altitude of 242 km above Earth. If the radius of Earth is 6378 kilometers, find the linear speed of the satellite in kilometers per hour.

Solution

1655 kilometers per hour

Media

Access these online resources for additional instruction and practice with angles, arc length, and areas of sectors.

- Angles in Standard Position
- Angle of Rotation
- Coterminal Angles
- Determining Coterminal Angles
- Positive and Negative Coterminal Angles
- Radian Measure
- Coterminal Angles in Radians
- Arc Length and Area of a Sector

Key Equations

arc length	$s = r\theta$
area of a sector	$A = \frac{1}{2}\theta r^2$
angular speed	$\omega = \frac{\theta}{t}$
linear speed	$v = \frac{s}{t}$
linear speed related to angular speed	$v = r\omega$

Key Concepts

- An angle is formed from the union of two rays, by keeping the initial side fixed and rotating the terminal side. The amount of rotation determines the measure of the angle.
- An angle is in standard position if its vertex is at the origin and its initial side lies along the positive *x*-axis. A positive angle is measured counterclockwise from the initial side and a negative angle is measured clockwise.
- To draw an angle in standard position, draw the initial side along the positive *x*-axis and then place the terminal side according to the fraction of a full rotation the angle represents. See **Example**.
- In addition to degrees, the measure of an angle can be described in radians. See **Example**.
- To convert between degrees and radians, use the proportion $\frac{\theta}{180} = \frac{\theta^R}{\pi}$. See Example and Example.
- Two angles that have the same terminal side are called coterminal angles.
- We can find coterminal angles by adding or subtracting 360° or 2π. See Example and Example.
- Coterminal angles can be found using radians just as they are for degrees. See **Example**.
- The length of a circular arc is a fraction of the circumference of the entire circle. See **Example**.
- The area of sector is a fraction of the area of the entire circle. See **Example**.
- An object moving in a circular path has both linear and angular speed.
- The angular speed of an object traveling in a circular path is the measure of the angle through which it turns in a unit of time. See **Example**.
- The linear speed of an object traveling along a circular path is the distance it travels in a unit of time. See **Example**.

Glossary

angle
the union of two rays having a common endpoint

angular speed
the angle through which a rotating object travels in a unit of time

arc length
the length of the curve formed by an arc

area of a sector
area of a portion of a circle bordered by two radii and the intercepted arc; the fraction $\frac{\theta}{2\pi}$ multiplied by the area of the entire circle

coterminal angles
description of positive and negative angles in standard position sharing the same terminal side

degree
a unit of measure describing the size of an angle as one-360th of a full revolution of a circle

initial side
the side of an angle from which rotation begins

linear speed

the distance along a straight path a rotating object travels in a unit of time; determined by the arc length

measure of an angle
the amount of rotation from the initial side to the terminal side

negative angle
description of an angle measured clockwise from the positive *x*-axis

positive angle
description of an angle measured counterclockwise from the positive *x*-axis

quadrantal angle
an angle whose terminal side lies on an axis

radian measure
the ratio of the arc length formed by an angle divided by the radius of the circle

radian
the measure of a central angle of a circle that intercepts an arc equal in length to the radius of that circle

ray
one point on a line and all points extending in one direction from that point; one side of an angle

reference angle
the measure of the acute angle formed by the terminal side of the angle and the horizontal axis

standard position
the position of an angle having the vertex at the origin and the initial side along the positive *x*-axis

terminal side
the side of an angle at which rotation ends

vertex
the common endpoint of two rays that form an angle

Homework Exercises 5.1

WeBWorK Problems:

1.



Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login

2.



Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login

3.



Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login

4.



Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login

5.



Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login

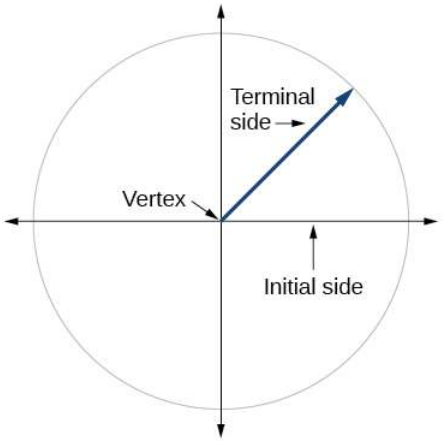
Written Problems:

In this section, we will examine properties of angles.

Verbal

1) Draw an angle in standard position. Label the vertex, initial side, and terminal side.

Answer



2) Explain why there are an infinite number of angles that are coterminal to a certain angle.

3) State what a positive or negative angle signifies, and explain how to draw each.

Answer

Whether the angle is positive or negative determines the direction. A positive angle is drawn in the counterclockwise direction, and a negative angle is drawn in the clockwise direction.

4) How does radian measure of an angle compare to the degree measure? Include an explanation of 1 radian in your paragraph.

5) Explain the differences between linear speed and angular speed when describing motion along a circular path.

Answer

Linear speed is a measurement found by calculating distance of an arc compared to time. Angular speed is a measurement found by calculating the angle of an arc compared to time.

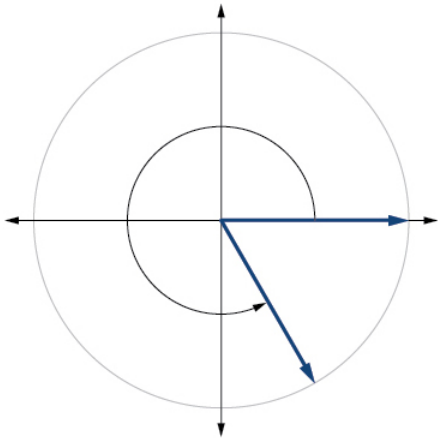
Graphical

For the exercises 6-21, draw an angle in standard position with the given measure.

6) 30°

7) 300°

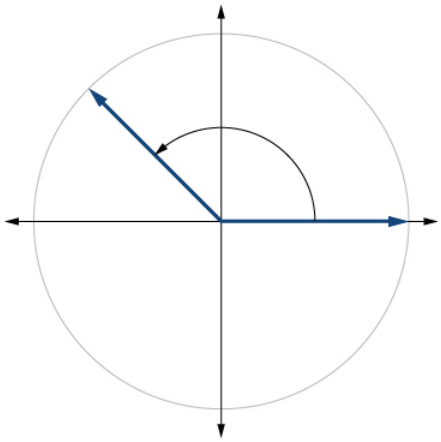
Answer



8) -80°

9) 135°

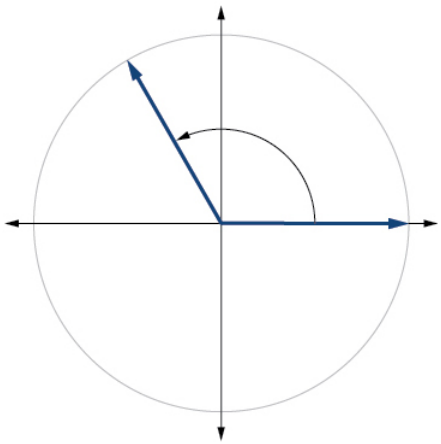
Answer



10) -150°

11) $\frac{2\pi}{3}$

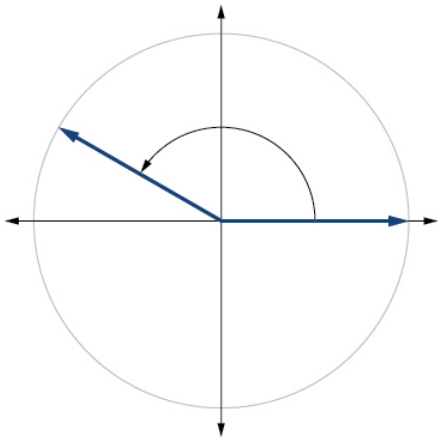
Answer



12) $\frac{7\pi}{4}$

13) $\frac{5\pi}{6}$

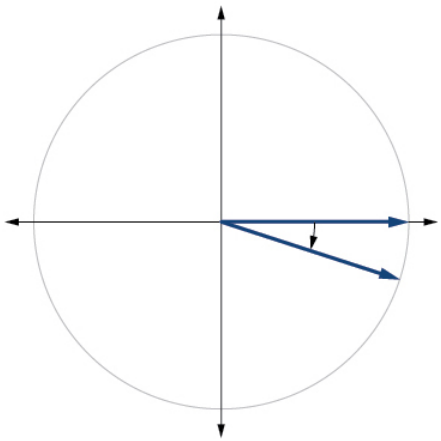
Answer



14) $\frac{\pi}{2}$

15) $-\frac{\pi}{10}$

Answer

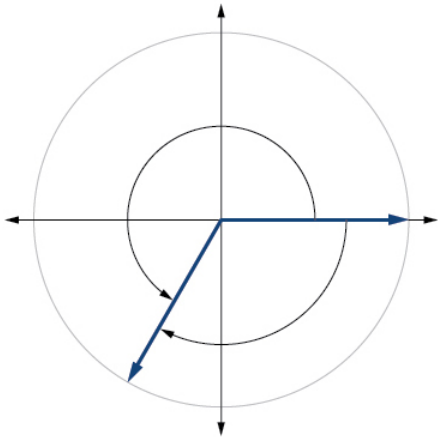


16) 415°

17) -120°

Answer

240°

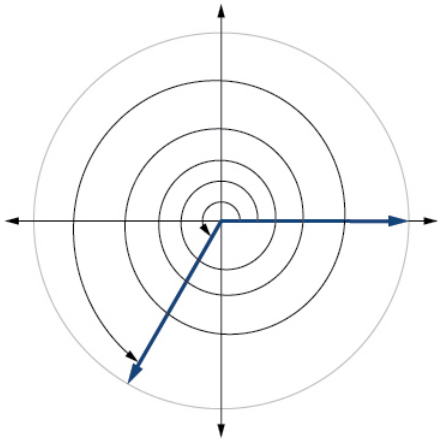


18) -315°

19) $\frac{22\pi}{3}$

Answer

$\frac{4\pi}{3}$

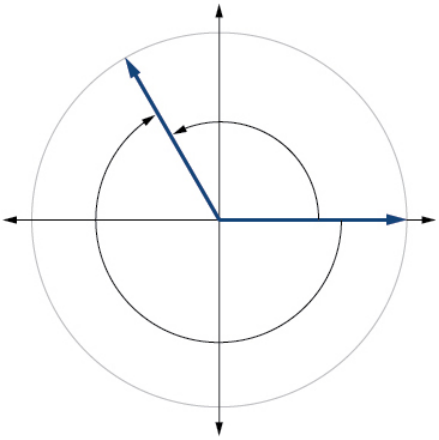


20) $-\frac{\pi}{6}$

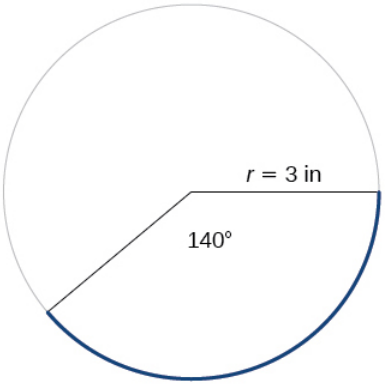
21) $-\frac{4\pi}{3}$

Answer

$\frac{2\pi}{3}$



For the exercises 22-23, refer to Figure below. Round to two decimal places.



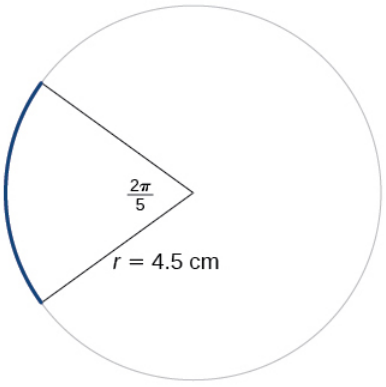
22) Find the arc length.

23) Find the area of the sector.

Answer

$\frac{27\pi}{2} \approx 11.00 \text{ in}^2$

For the exercises 24-25, refer to Figure below. Round to two decimal places.



24) Find the arc length.

25) Find the area of the sector.

Answer

$\frac{81\pi}{20} \approx 12.72 \text{ cm}^2$

Algebraic

For the exercises 26-32, convert angles in radians to degrees.

26) $\frac{3\pi}{4}$ radians

27) $\frac{\pi}{9}$ radians

Answer

20°

28) $-\frac{5\pi}{4}$ radians

29) $\frac{\pi}{3}$ radians

Answer

60°

30) $-\frac{7\pi}{3}$ radians

31) $-\frac{5\pi}{12}$ radians

Answer

-75°

32) $\frac{11\pi}{6}$ radians

For the exercises 33-39, convert angles in degrees to radians.

33) 90°

Answer

$\frac{\pi}{2}$ radians

34) 100°

35) -540°

Answer

-3π radians

36) -120°

37) 180°

Answer

π radians

38) -315°

39) 150°

Answer

$\frac{5\pi}{6}$ radians

For the exercises 40-45, use to given information to find the length of a circular arc. Round to two decimal places.

40) Find the length of the arc of a circle of radius 12 inches subtended by a central angle of $\frac{\pi}{4}$ radians.

41) Find the length of the arc of a circle of radius 5.02 miles subtended by the central angle of $\frac{\pi}{3}$.

Answer

$\frac{5.02\pi}{3} \approx 5.26$ miles

42) Find the length of the arc of a circle of diameter 14 meters subtended by the central angle of $\frac{5\pi}{6}$.

43) Find the length of the arc of a circle of radius 10 centimeters subtended by the central angle of 50°.

Answer

$\frac{25\pi}{9} \approx 8.73$ centimeters

44) Find the length of the arc of a circle of radius 5 inches subtended by the central angle of 220^{circ}.

45) Find the length of the arc of a circle of diameter 12 meters subtended by the central angle is 63^{circ}.

Answer

$\frac{21\pi}{10} \approx 6.60$ meters

For the exercises 46-49, use the given information to find the area of the sector. Round to four decimal places.

46) A sector of a circle has a central angle of 45° and a radius 6 cm.

47) A sector of a circle has a central angle of 30° and a radius of 20 cm.

Answer

$104.7198\ cm^2$

48) A sector of a circle with diameter 10 feet and an angle of $\frac{\pi}{2}$ radians.

49) A sector of a circle with radius of 0.7 inches and an angle of π radians.

Answer

$0.7697\ in^2$

For the exercises 50-53, find the angle between 0° and 360° that is coterminal to the given angle.

50) -40°

51) -110°

Answer

250°

52) 700°

53) 1400°

Answer

320°

For the exercises 54-57, find the angle between 0 and 2π in radians that is coterminal to the given angle.

54) $-\frac{\pi}{9}$

55) $\frac{10\pi}{3}$

Answer

$\frac{4\pi}{3}$

56) $\frac{13\pi}{6}$

57) $\frac{44\pi}{9}$

Answer

$\frac{8\pi}{9}$

Real-World Applications

58) A truck with 32-inch diameter wheels is traveling at 60 mi/h. Find the angular speed of the wheels in rad/min. How many revolutions per minute do the wheels make?

59) A bicycle with 24-inch diameter wheels is traveling at 15 mi/h. Find the angular speed of the wheels in rad/min. How many revolutions per minute do the wheels make?

Answer

$1320\text{rad}\ 210.085\text{RPM}$

60) A wheel of radius 8 inches is rotating $15^\circ/s$. What is the linear speed v, the angular speed in RPM, and the angular speed in rad/s?

61) A wheel of radius 14 inches is rotating 0.5rad/s. What is the linear speed v, the angular speed in RPM, and the angular speed in deg/s?

Answer

7 in./s, 4.77RPM, 28.65deg/s

62) A CD has diameter of 120 millimeters. When playing audio, the angular speed varies to keep the linear speed constant where the disc is being read. When reading along the outer edge of the disc, the angular speed is about 200 RPM (revolutions per minute). Find the linear speed.

63) When being burned in a writable CD-R drive, the angular speed of a CD is often much faster than when playing audio, but the angular speed still varies to keep the linear speed constant where the disc is being written. When writing along the outer edge of the disc, the angular speed of one drive is about 4800 RPM (revolutions per minute). Find the linear speed if the CD has diameter of 120 millimeters.

Answer

$1,809,557.37\ \text{mm/min} = 30.16\ \text{m/s}$

64) A person is standing on the equator of Earth (radius 3960 miles). What are his linear and angular speeds?

65) Find the distance along an arc on the surface of Earth that subtends a central angle of 5 minutes $(1 \text{ minute} = \frac{1}{60} \text{ degree})$. The radius of Earth is 3960 miles.

Answer

5.76 miles

66) Find the distance along an arc on the surface of Earth that subtends a central angle of 7 minutes $(1 \text{ minute} = \frac{1}{60} \text{ degree})$. The radius of Earth is 3960 miles.

67) Consider a clock with an hour hand and minute hand. What is the measure of the angle the minute hand traces in 20 minutes?

Answer

120°

Extensions

68) Two cities have the same longitude. The latitude of city A is 9.00 degrees north and the latitude of city B is 30.00 degree north. Assume the radius of the earth is 3960 miles. Find the distance between the two cities.

69) A city is located at 40 degrees north latitude. Assume the radius of the earth is 3960 miles and the earth rotates once every 24 hours. Find the linear speed of a person who resides in this city.

Answer

794 miles per hour

70) A city is located at 75 degrees north latitude. Assume the radius of the earth is 3960 miles and the earth rotates once every 24 hours. Find the linear speed of a person who resides in this city.

71) Find the linear speed of the moon if the average distance between the earth and moon is 239,000 miles, assuming the orbit of the moon is circular and requires about 28 days. Express answer in miles per hour.

Answer

2,234 miles per hour

72) A bicycle has wheels 28 inches in diameter. A tachometer determines that the wheels are rotating at 180 RPM (revolutions per minute). Find the speed the bicycle is traveling down the road.

73) A car travels 3 miles. Its tires make 2640 revolutions. What is the radius of a tire in inches?

Answer

11.5 inches

74) A wheel on a tractor has a 24-inch diameter. How many revolutions does the wheel make if the tractor travels 4 miles?

This page titled 5.1: Angles is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **5.1: Angles** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/precalculus.
- **5.E: Trigonometric Functions (Exercises)** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/precalculus.

5.2: Right Triangle Trigonometry

Learning Objectives

- Use right triangles to evaluate trigonometric functions.
- Find function values for $30°(\frac{\pi}{6}), 45°(\frac{\pi}{4})$, and $60°(\frac{\pi}{3})$.
- Use equal cofunctions of complementary angles.
- Use the definitions of trigonometric functions of any angle.
- Use right-triangle trigonometry to solve applied problems.

Mt. Everest, which straddles the border between China and Nepal, is the tallest mountain in the world. Measuring its height is no easy task and, in fact, the actual measurement has been a source of controversy for hundreds of years. The measurement process involves the use of triangles and a branch of mathematics known as trigonometry. In this section, we will define a new group of functions known as trigonometric functions, and find out how they can be used to measure heights, such as those of the tallest mountains.

We have previously defined the sine and cosine of an angle in terms of the coordinates of a point on the unit circle intersected by the terminal side of the angle:

$$\cos t = x$$
$$\sin t = y$$

In this section, we will see another way to define trigonometric functions using properties of *right triangles*.

Using Right Triangles to Evaluate Trigonometric Functions

In earlier sections, we used a unit circle to define the *trigonometric functions*. In this section, we will extend those definitions so that we can apply them to right triangles. The value of the sine or cosine function of t is its value at t radians. First, we need to create our right triangle. Figure $5.2.1$ shows a point on a unit circle of radius 1. If we drop a vertical line segment from the point (x, y) to the *x*-axis, we have a right triangle whose vertical side has length y and whose horizontal side has length x. We can use this right triangle to redefine sine, cosine, and the other trigonometric functions as ratios of the sides of a right triangle.

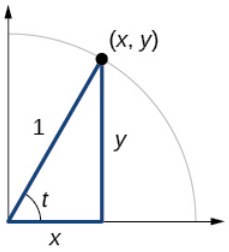


Figure 5.2.1

We know

$$\cos t = \frac{x}{1} = x \tag{5.2.1}$$

Likewise, we know

$$\sin t = \frac{y}{1} = y \tag{5.2.2}$$

These ratios still apply to the sides of a right triangle when no unit circle is involved and when the triangle is not in standard position and is not being graphed using (x, y) coordinates. To be able to use these ratios freely, we will give the sides more general names: Instead of x,we will call the side between the given angle and the right angle the **adjacent side** to angle t. (Adjacent means "next to.") Instead of y,we will call the side most distant from the given angle the **opposite side** from angle t. And instead of 1,we will call the side of a right triangle opposite the right angle the **hypotenuse**. These sides are labeled in Figure $5.2.2$.

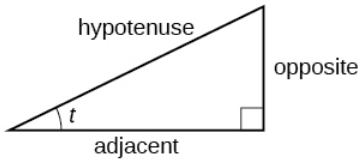


Figure 5.2.2: The sides of a right triangle in relation to angle t.

Understanding Right Triangle Relationships

Given a right triangle with an acute angle of t,

$$\sin(t) = \frac{\text{opposite}}{\text{hypotenuse}} \tag{5.2.3}$$

$$\cos(t) = \frac{\text{adjacent}}{\text{hypotenuse}} \tag{5.2.4}$$

$$\tan(t) = \frac{\text{opposite}}{\text{adjacent}} \tag{5.2.5}$$

A common mnemonic for remembering these relationships is SohCahToa, formed from the first letters of "Sine is opposite over hypotenuse, Cosine is adjacent over hypotenuse, Tangent is opposite over adjacent."

how to: Given the side lengths of a right triangle and one of the acute angles, find the sine, cosine, and tangent of that angle

1. Find the sine as the ratio of the opposite side to the hypotenuse.
2. Find the cosine as the ratio of the adjacent side to the hypotenuse.
3. Find the tangent is the ratio of the opposite side to the adjacent side.

Example 5.2.1: Evaluating a Trigonometric Function of a Right Triangle

Given the triangle shown in Figure 5.2.3, find the value of $\cos\alpha$.

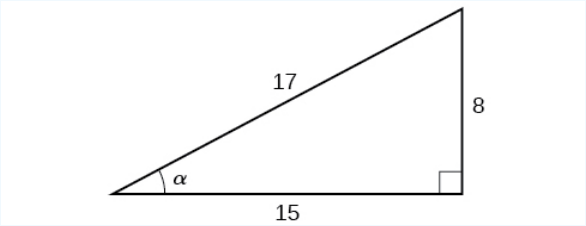


Figure 5.2.3

Solution

The side adjacent to the angle is 15, and the hypotenuse of the triangle is 17, so via Equation 5.2.4:

$$\cos(\alpha) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$= \frac{15}{17}$$

Exercise 5.2.1

Given the triangle shown in Figure 5.2.4, find the value of $\sin t$.

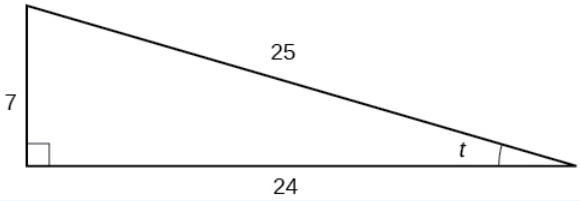


Figure 5.2.4

Answer

$\frac{7}{25}$

Relating Angles and Their Functions

When working with right triangles, the same rules apply regardless of the orientation of the triangle. In fact, we can evaluate the six trigonometric functions of either of the two acute angles in the triangle in Figure 5.2.5. The side opposite one acute angle is the side adjacent to the other acute angle, and vice versa.

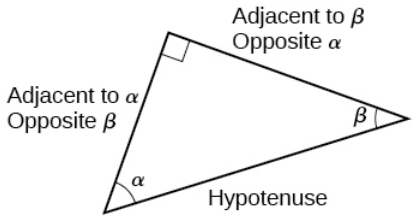


Figure 5.2.5: The side adjacent to one angle is opposite the other.

We will be asked to find all six trigonometric functions for a given angle in a triangle. Our strategy is to find the sine, cosine, and tangent of the angles first. Then, we can find the other trigonometric functions easily because we know that the reciprocal of sine is cosecant, the reciprocal of cosine is secant, and the reciprocal of tangent is cotangent.

how to: Given the side lengths of a right triangle, evaluate the six trigonometric functions of one of the acute angles

1. If needed, draw the right triangle and label the angle provided.
2. Identify the angle, the adjacent side, the side opposite the angle, and the hypotenuse of the right triangle.
3. Find the required function:
   - sine as the ratio of the opposite side to the hypotenuse
   - cosine as the ratio of the adjacent side to the hypotenuse
   - tangent as the ratio of the opposite side to the adjacent side
   - secant as the ratio of the hypotenuse to the adjacent side
   - cosecant as the ratio of the hypotenuse to the opposite side
   - cotangent as the ratio of the adjacent side to the opposite side

Example 5.2.2: Evaluating Trigonometric Functions of Angles Not in Standard Position

Using the triangle shown in Figure 5.2.6, evaluate $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, $\sec\alpha$, $\csc\alpha$, and $\cot\alpha$.

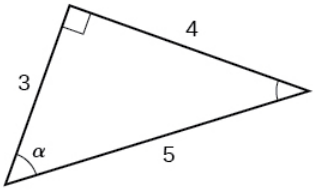


Figure 5.2.6

Solution

$$\sin\alpha = \frac{\text{opposite } \alpha}{\text{hypotenuse}} = \frac{4}{5}$$
$$\cos\alpha = \frac{\text{adjacent to } \alpha}{\text{hypotenuse}} = \frac{3}{5}$$
$$\tan\alpha = \frac{\text{opposite } \alpha}{\text{adjacent to } \alpha} = \frac{4}{3}$$
$$\sec\alpha = \frac{\text{hypotenuse}}{\text{adjacent to } \alpha} = \frac{5}{3}$$
$$\csc\alpha = \frac{\text{hypotenuse}}{\text{opposite } \alpha} = \frac{5}{4}$$
$$\cot\alpha = \frac{\text{adjacent to } \alpha}{\text{opposite } \alpha} = \frac{3}{4}$$

Exercise 5.2.2

Using the triangle shown in Figure 5.2.7, evaluate $\sin t$, $\cos t$, $\tan t$, $\sec t$, $\csc t$, and $\cot t$.

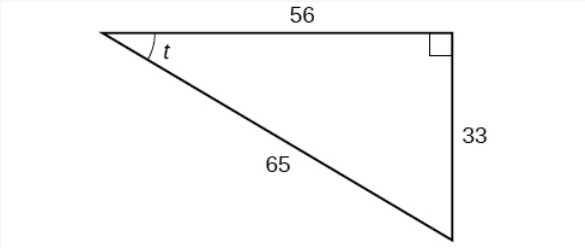


Figure 5.2.7

Answer

$$\sin t = \frac{33}{65}, \cos t = \frac{56}{65}, \tan t = \frac{33}{56},$$

$$\sec t = \frac{65}{56}, \csc t = \frac{65}{33}, \cot t = \frac{56}{33}$$

Finding Trigonometric Functions of Special Angles Using Side Lengths

We have already discussed the trigonometric functions as they relate to the **special angles** on the unit circle. Now, we can use those relationships to evaluate triangles that contain those special angles. We do this because when we evaluate the special angles in trigonometric functions, they have relatively friendly values, values that contain either no or just one square root in the ratio. Therefore, these are the angles often used in math and science problems. We will use multiples of $30^\circ, 60^\circ,$ and 45°, however, remember that when dealing with right triangles, we are limited to angles between 0° and 90°.

Suppose we have a $30^\circ, 60^\circ, 90^\circ$ triangle, which can also be described as a $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$ triangle. The sides have lengths in the relation $s, \sqrt{3}s, 2s$. The sides of a $45^\circ, 45^\circ, 90^\circ$ triangle, which can also be described as a $\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}$ triangle, have lengths in the relation $s, s, \sqrt{2}s$. These relations are shown in Figure 5.2.8.

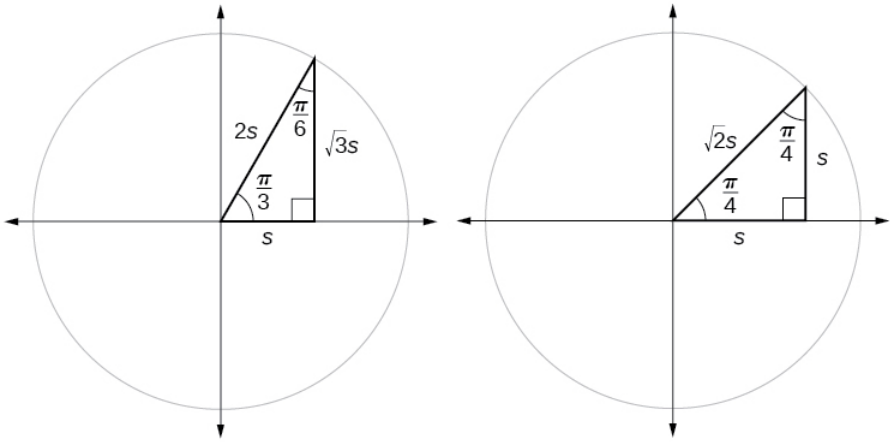


Figure 5.2.8: Side lengths of special triangles

We can then use the ratios of the side lengths to evaluate trigonometric functions of special angles.

Given trigonometric functions of a special angle, evaluate using side lengths.

1. Use the side lengths shown in Figure 5.2.8 for the special angle you wish to evaluate.
2. Use the ratio of side lengths appropriate to the function you wish to evaluate.

Example 5.2.3: Evaluating Trigonometric Functions of Special Angles Using Side Lengths

Find the exact value of the trigonometric functions of $\frac{\pi}{3}$, using side lengths.

Solution

$$\sin(\frac{\pi}{3}) = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}s}{2s} = \frac{\sqrt{3}}{2}$$
$$\cos(\frac{\pi}{3}) = \frac{\text{adj}}{\text{hyp}} = \frac{s}{2s} = \frac{1}{2}$$
$$\tan(\frac{\pi}{3}) = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{3}s}{s} = \sqrt{3}$$
$$\sec(\frac{\pi}{3}) = \frac{\text{hyp}}{\text{adj}} = \frac{2s}{s} = 2$$
$$\csc(\frac{\pi}{3}) = \frac{\text{hyp}}{\text{opp}} = \frac{2s}{\sqrt{3}s} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$
$$\cot(\frac{\pi}{3}) = \frac{\text{adj}}{\text{opp}} = \frac{s}{\sqrt{3}s} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Exercise 5.2.3

Find the exact value of the trigonometric functions of $\frac{\pi}{4}$ using side lengths.

Answer

$\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}, \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}, \tan(\frac{\pi}{4}) = 1,$
$\sec(\frac{\pi}{4}) = \sqrt{2}, \csc(\frac{\pi}{4}) = \sqrt{2}, \cot(\frac{\pi}{4}) = 1$

Using Equal Cofunction of Complements

If we look more closely at the relationship between the sine and cosine of the special angles relative to the unit circle, we will notice a pattern. In a right triangle with angles of $\frac{\pi}{6}$ and $\frac{\pi}{3}$, we see that the sine of $\frac{\pi}{3}$, namely $\frac{\sqrt{3}}{2}$, is also the cosine of $\frac{\pi}{6}$, while the

sine of $\frac{\pi}{6}$, namely $\frac{1}{2}$, is also the cosine of $\frac{\pi}{3}$ (Figure 5.2.9).

$$\sin\frac{\pi}{3} = \cos\frac{\pi}{6} = \frac{\sqrt{3}s}{2s} = \frac{\sqrt{3}}{2}$$
$$\sin\frac{\pi}{6} = \cos\frac{\pi}{3} = \frac{s}{2s} = \frac{1}{2}$$

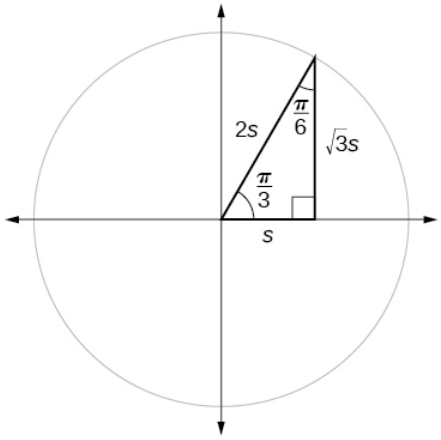
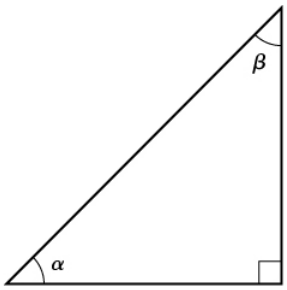


Figure 5.2.9: The sine of $\frac{\pi}{3}$ equals the cosine of $\frac{\pi}{6}$ and vice versa.

This result should not be surprising because, as we see from Figure 5.2.9, the side opposite the angle of $\frac{\pi}{3}$ is also the side adjacent to $\frac{\pi}{6}$, so $\sin(\frac{\pi}{3})$ and $\cos(\frac{\pi}{6})$ are exactly the same ratio of the same two sides, $\sqrt{3}s$ and $2s$. Similarly, $\cos(\frac{\pi}{3})$ and $\sin(\frac{\pi}{6})$ are also the same ratio using the same two sides, s and $2s$.

The interrelationship between the sines and cosines of $\frac{\pi}{6}$ and $\frac{\pi}{3}$ also holds for the two acute angles in any right triangle, since in every case, the ratio of the same two sides would constitute the sine of one angle and the cosine of the other. Since the three angles of a triangle add to π, and the right angle is $\frac{\pi}{2}$, the remaining two angles must also add up to $\frac{\pi}{2}$. That means that a right triangle can be formed with any two angles that add to $\frac{\pi}{2}$—in other words, any two complementary angles. So we may state a *cofunction identity*: If any two angles are complementary, the sine of one is the cosine of the other, and vice versa. This identity is illustrated in Figure 5.2.10.



$\sin\alpha = \cos\beta$
$\sin\beta = \cos\alpha$

Figure 5.2.10: Cofunction identity of sine and cosine of complementary angles

Using this identity, we can state without calculating, for instance, that the sine of $\frac{\pi}{12}$ equals the cosine of $\frac{5\pi}{12}$, and that the sine of $\frac{5\pi}{12}$ equals the cosine of $\frac{\pi}{12}$. We can also state that if, for a certain angle t, $\cos t = \frac{5}{13}$, then $\sin(\frac{\pi}{2} - t) = \frac{5}{13}$ as well.

COFUNCTION IDENTITIES

The **cofunction identities** in radians are listed in Table 5.2.1.

Table 5.2.1

$\cos t = \sin(\frac{\pi}{2} - t)$	$\sin t = \cos(\frac{\pi}{2} - t)$
$\tan t = \cot(\frac{\pi}{2} - t)$	$\cot t = \tan(\frac{\pi}{2} - t)$

$\sec t = \csc(\frac{\pi}{2} - t)$	$\csc t = \sec(\frac{\pi}{2} - t)$

how to: Given the sine and cosine of an angle, find the sine or cosine of its complement.

1. To find the sine of the complementary angle, find the cosine of the original angle.
2. To find the cosine of the complementary angle, find the sine of the original angle.

Example 5.2.4: Using Cofunction Identities

If $\sin t = \frac{5}{12}$, find $(\cos\frac{\pi}{2} - t)$.

Solution

According to the cofunction identities for sine and cosine,

$$\sin t = \cos(\frac{\pi}{2} - t).$$

So

$$\cos(\frac{\pi}{2} - t) = \frac{5}{12}.$$

Exercise 5.2.4

If $\csc(\frac{\pi}{6}) = 2$, find $\sec(\frac{\pi}{3})$.

Solution

2

Using Trigonometric Functions

In previous examples, we evaluated the sine and cosine in triangles where we knew all three sides. But the real power of right-triangle trigonometry emerges when we look at triangles in which we know an angle but do not know all the sides.

how to: Given a right triangle, the length of one side, and the measure of one acute angle, find the remaining sides

1. For each side, select the trigonometric function that has the unknown side as either the numerator or the denominator. The known side will in turn be the denominator or the numerator.
2. Write an equation setting the function value of the known angle equal to the ratio of the corresponding sides.
3. Using the value of the trigonometric function and the known side length, solve for the missing side length.

Example 5.2.5: Finding Missing Side Lengths Using Trigonometric Ratios

Find the unknown sides of the triangle in Figure 5.2.11.

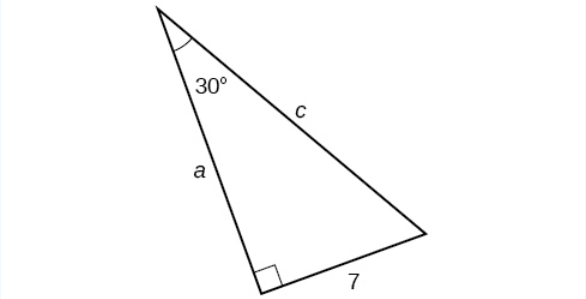


Figure 5.2.11

Solution

We know the angle and the opposite side, so we can use the tangent to find the adjacent side.

$$\tan(30^\circ) = \frac{7}{a}$$

We rearrange to solve for a.

$$a = \frac{7}{\tan(30^\circ)} \tag{5.2.6}$$

$$= 12.1 \tag{5.2.7}$$

We can use the sine to find the hypotenuse.

$$\sin(30^\circ) = \frac{7}{c}$$

Again, we rearrange to solve for c.

$$c = \frac{7}{\sin(30^\circ)} = 14$$

Exercise 5.2.5:

A right triangle has one angle of $\frac{\pi}{3}$ and a hypotenuse of 20. Find the unknown sides and angle of the triangle.

Answer

$\text{adjacent} = 10; \text{opposite} = 10\sqrt{3}$; missing angle is $\frac{\pi}{6}$

Using Right Triangle Trigonometry to Solve Applied Problems

Right-triangle trigonometry has many practical applications. For example, the ability to compute the lengths of sides of a triangle makes it possible to find the height of a tall object without climbing to the top or having to extend a tape measure along its height. We do so by measuring a distance from the base of the object to a point on the ground some distance away, where we can look up to the top of the tall object at an angle. The **angle of elevation** of an object above an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. The right triangle this position creates has sides that represent the unknown height, the measured distance from the base, and the angled line of sight from the ground to the top of the object. Knowing the measured distance to the base of the object and the angle of the line of sight, we can use trigonometric functions to calculate the unknown height. Similarly, we can form a triangle from the top of a tall object by looking downward. The **angle of depression** of an object below an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. See Figure 5.2.12

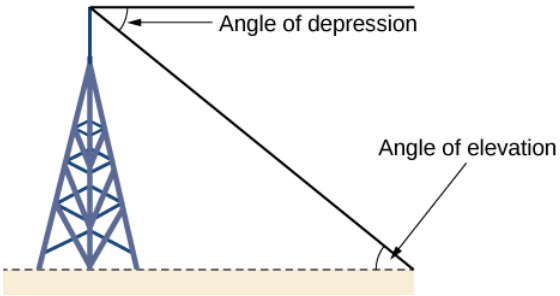


Figure 5.2.12

how to: Given a tall object, measure its height indirectly

1. Make a sketch of the problem situation to keep track of known and unknown information.
2. Lay out a measured distance from the base of the object to a point where the top of the object is clearly visible.
3. At the other end of the measured distance, look up to the top of the object. Measure the angle the line of sight makes with the horizontal.

4. Write an equation relating the unknown height, the measured distance, and the tangent of the angle of the line of sight.
5. Solve the equation for the unknown height.

Example 5.2.6: Measuring a Distance Indirectly

To find the height of a tree, a person walks to a point 30 feet from the base of the tree. She measures an angle of 57° 57° between a line of sight to the top of the tree and the ground, as shown in Figure 5.2.13. Find the height of the tree.

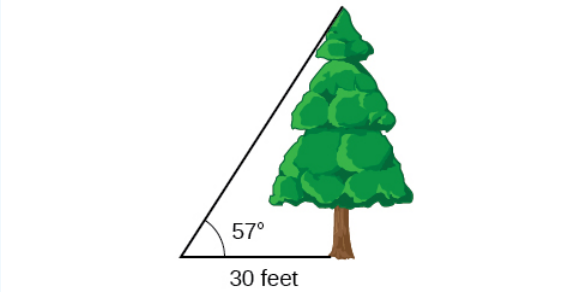


Figure 5.2.13

Solution

We know that the angle of elevation is 57° and the adjacent side is 30 ft long. The opposite side is the unknown height.

The trigonometric function relating the side opposite to an angle and the side adjacent to the angle is the tangent. So we will state our information in terms of the tangent of 57°, letting h be the unknown height.

$$\begin{aligned} \tan\theta &= \frac{\text{opposite}}{\text{adjacent}} \\ \tan(57^\circ) &= \frac{h}{30} && \text{Solve for } h. \\ h &= 30\tan(57^\circ) && \text{Multiply.} \\ h &\approx 46.2 && \text{Use a calculator.} \end{aligned} \tag{5.2.8}$$

The tree is approximately 46 feet tall.

Exercise 5.2.6:

How long a ladder is needed to reach a windowsill 50 feet above the ground if the ladder rests against the building making an angle of $\frac{5\pi}{12}$ with the ground? Round to the nearest foot.

Answer

About 52 ft

media:

Access these online resources for additional instruction and practice with right triangle trigonometry.

- Finding Trig Functions on Calculator
- Finding Trig Functions Using a Right Triangle
- Relate Trig Functions to Sides of a Right Triangle
- Determine Six Trig Functions from a Triangle
- Determine Length of Right Triangle Side

Visit this website for additional practice questions from Learningpod.

Key Equations

Cofunction Identities

$$\begin{aligned}\cos t &= \sin\left(\frac{\pi}{2} - t\right)\\ \sin t &= \cos\left(\frac{\pi}{2} - t\right)\\ \tan t &= \cot\left(\frac{\pi}{2} - t\right)\\ \cot t &= \tan\left(\frac{\pi}{2} - t\right)\\ \sec t &= \csc\left(\frac{\pi}{2} - t\right)\\ \csc t &= \sec\left(\frac{\pi}{2} - t\right)\end{aligned}$$

Key Concepts

- We can define trigonometric functions as ratios of the side lengths of a right triangle. See Example.
- The same side lengths can be used to evaluate the trigonometric functions of either acute angle in a right triangle. See Example.
- We can evaluate the trigonometric functions of special angles, knowing the side lengths of the triangles in which they occur. See Example.
- Any two complementary angles could be the two acute angles of a right triangle.
- If two angles are complementary, the cofunction identities state that the sine of one equals the cosine of the other and vice versa. See Example.
- We can use trigonometric functions of an angle to find unknown side lengths.
- Select the trigonometric function representing the ratio of the unknown side to the known side. See Example.
- Right-triangle trigonometry permits the measurement of inaccessible heights and distances.
- The unknown height or distance can be found by creating a right triangle in which the unknown height or distance is one of the sides, and another side and angle are known. See Example.

Glossary

adjacent side
in a right triangle, the side between a given angle and the right angle

angle of depression
the angle between the horizontal and the line from the object to the observer's eye, assuming the object is positioned lower than the observer

angle of elevation
the angle between the horizontal and the line from the object to the observer's eye, assuming the object is positioned higher than the observer

opposite side
in a right triangle, the side most distant from a given angle

hypotenuse
the side of a right triangle opposite the right angle

Homework Exercises 5.2

WeBWorK Problems: (only 3)

1.

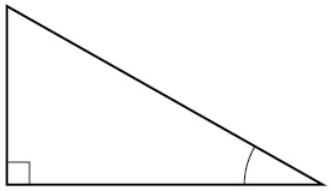
2.

3.

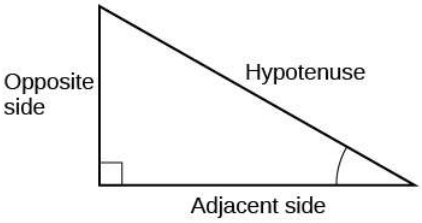
Written Problems:

Verbal

1) For the given right triangle, label the adjacent side, opposite side, and hypotenuse for the indicated angle.



Answer



2) When a right triangle with a hypotenuse of 1 is placed in the unit circle, which sides of the triangle correspond to the x- and y-coordinates?

3) The tangent of an angle compares which sides of the right triangle?

Answer

The tangent of an angle is the ratio of the opposite side to the adjacent side.

4) What is the relationship between the two acute angles in a right triangle?

5) Explain the cofunction identity.

Answer

For example, the sine of an angle is equal to the cosine of its complement; the cosine of an angle is equal to the sine of its complement.

Algebraic

For the exercises 6-9, use cofunctions of complementary angles.

6) $\cos(34^\circ) = \sin(\_\_^\circ)$

7) $\cos(\frac{\pi}{3}) = \sin(\_\_\_)$

Answer

$\frac{\pi}{6}$

8) $\csc(21^\circ) = \sec(\_\_\_^\circ)$

9) $\tan(\frac{\pi}{4}) = \cot(\_\_)$

Answer

$\frac{\pi}{4}$

For the exercises 10-16, find the lengths of the missing sides if side a is opposite angle A, side b is opposite angle B, and side c is the hypotenuse.

10) $\cos B = \frac{4}{5}, a = 10$

11) $\sin B = \frac{1}{2}, a = 20$

Answer

$b = \frac{20\sqrt{3}}{3}, c = \frac{40\sqrt{3}}{3}$

12) $\tan A = \frac{5}{12}, b = 6$

13) $\tan A = 100, b = 100$

Answer

$a = 10,000, c = 10,000.5$

14) $\sin B = \frac{1}{\sqrt{3}}, a = 2$

15) $a = 5, \measuredangle A = 60^\circ$

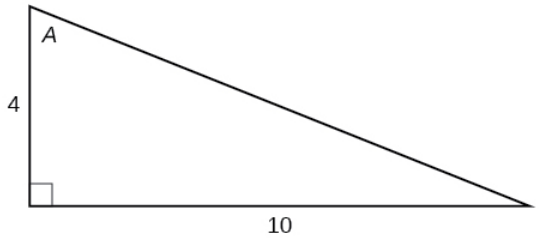
Answer

$b = \frac{5\sqrt{3}}{3}, c = \frac{10\sqrt{3}}{3}$

16) $c = 12, \angle A = 45^\circ$

Graphical

For the exercises 17-22, use Figure below to evaluate each trigonometric function of angle A.



17) $\sin A$

Answer

$\frac{5\sqrt{29}}{29}$

18) $\cos A$

19) $\tan A$

Answer

$\frac{5}{2}$

20) $\csc A$

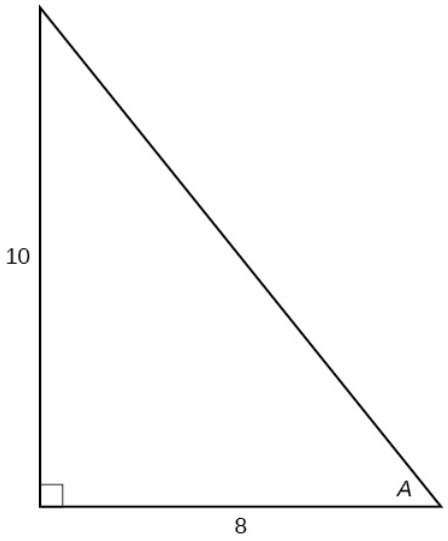
21) $\sec A$

Answer

$\frac{\sqrt{29}}{2}$

22) $\cot A$

For the exercises 23-,28 use Figure below to evaluate each trigonometric function of angle A.



23) $\sin A$

Answer

$\frac{5\sqrt{41}}{41}$

24) $\cos A$

25) $\tan A$

Answer

$\frac{5}{4}$

26) $\csc A$

27) $\sec A$

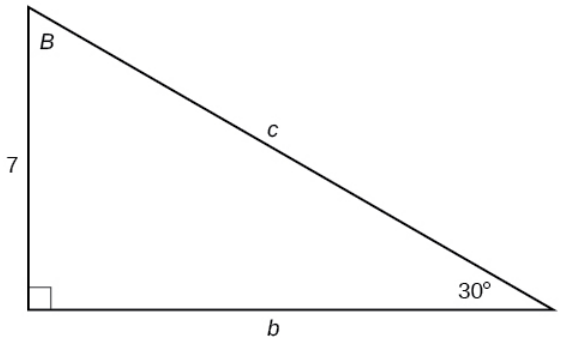
Answer

$\frac{\sqrt{41}}{4}$

28) $\cot A$

For the exercises 29-31, solve for the unknown sides of the given triangle.

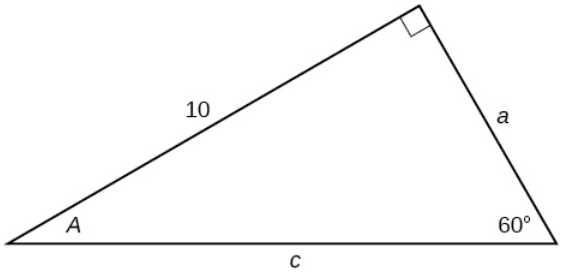
29)



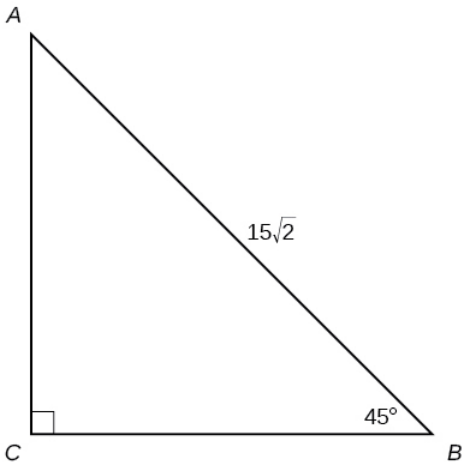
Answer

$c = 14, b = 7\sqrt{3}$

30)



31)



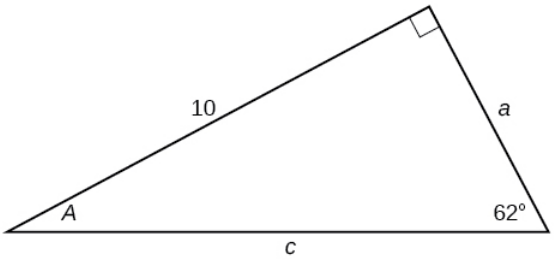
Answer

$a = 15, b = 15$

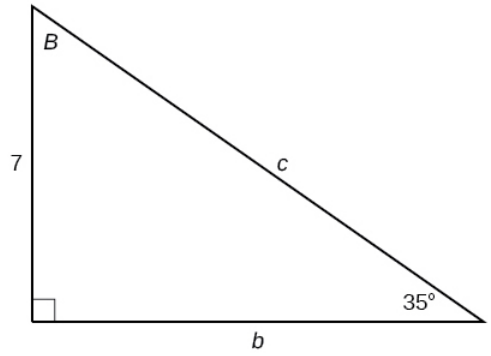
Technology

For the exercises 32-41, use a calculator to find the length of each side to four decimal places.

32)



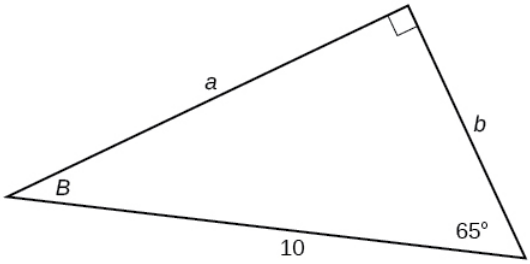
33)



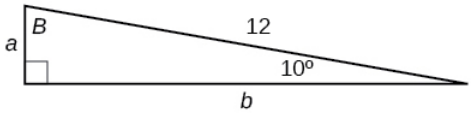
Answer

$b = 9.9970, c = 12.2041$

34)



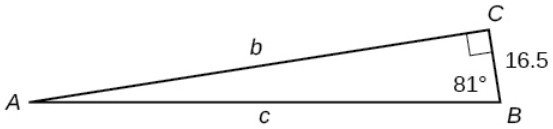
35)



Answer

$a = 2.0838, b = 11.8177$

36)



37) $b = 15, \angle B = 15^\circ$

Answer

$a = 55.9808, c = 57.9555$

38) $c = 200, \angle B = 5^\circ$

39) $c = 50, \angle B = 21^\circ$

Answer

$a = 46.6790, b = 17.9184$

40) $a = 30, \angle A = 27^\circ$

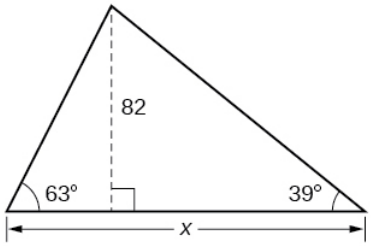
41) $b = 3.5, \angle A = 78^\circ$

Answer

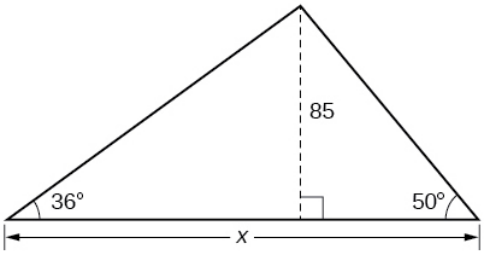
$a = 16.4662, c = 16.8341$

Extensions

42) Find x.



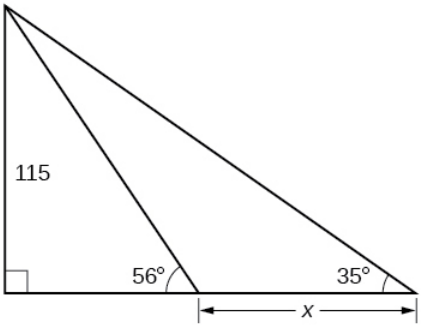
43) Find x.



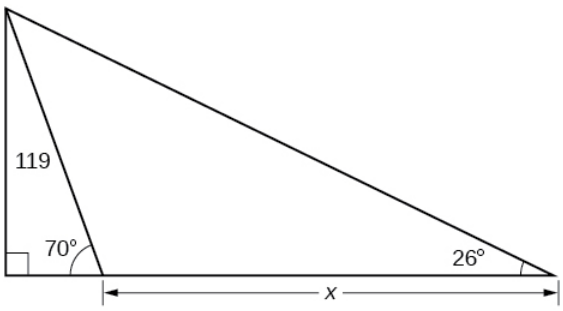
Answer

188.3159

44) Find x.



45) Find x.



Answer

200.6737

46) A radio tower is located 400 feet from a building. From a window in the building, a person determines that the angle of elevation to the top of the tower is 36°, and that the angle of depression to the bottom of the tower is 23°. How tall is the tower?

47) A radio tower is located 325 feet from a building. From a window in the building, a person determines that the angle of elevation to the top of the tower is 43°, and that the angle of depression to the bottom of the tower is 31°. How tall is the tower?

Answer

498.3471ft

48) A 200-foot tall monument is located in the distance. From a window in a building, a person determines that the angle of elevation to the top of the monument is 15°, and that the angle of depression to the bottom of the tower is 2°. How far is the person from the monument?

49) A 400-foot tall monument is located in the distance. From a window in a building, a person determines that the angle of elevation to the top of the monument is 18°, and that the angle of depression to the bottom of the monument is 3°. How far is the person from the monument?

Answer

1060.09ft

50) There is an antenna on the top of a building. From a location 300 feet from the base of the building, the angle of elevation to the top of the building is measured to be 40°. From the same location, the angle of elevation to the top of the antenna is measured to be 43°. Find the height of the antenna.

51) There is lightning rod on the top of a building. From a location 500 feet from the base of the building, the angle of elevation to the top of the building is measured to be 36°. From the same location, the angle of elevation to the top of the lightning rod is measured to be 38°. Find the height of the lightning rod.

Answer

27.372ft

Real-World Applications

52) A 33-ft ladder leans against a building so that the angle between the ground and the ladder is 80°. How high does the ladder reach up the side of the building?

53) A 23-ft ladder leans against a building so that the angle between the ground and the ladder is 80°. How high does the ladder reach up the side of the building?

Answer

22.6506ft

54) The angle of elevation to the top of a building in New York is found to be 9 degrees from the ground at a distance of 1 mile from the base of the building. Using this information, find the height of the building.

55) The angle of elevation to the top of a building in Seattle is found to be 2 degrees from the ground at a distance of 2 miles from the base of the building. Using this information, find the height of the building.

Answer

368.7633ft

56) Assuming that a 370-foot tall giant redwood grows vertically, if I walk a certain distance from the tree and measure the angle of elevation to the top of the tree to be 60°, how far from the base of the tree am I?

This page titled 5.2: Right Triangle Trigonometry is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **5.4: Right Triangle Trigonometry** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/precalculus.

- **5.E: Trigonometric Functions (Exercises)** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/precalculus.

5.3: Unit Circle

Preview Video Section 5.3

Please fill in your notes as you view the video and answer the questions.

Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login

Learning Objectives

- Find function values for the sine and cosine of $30°$ or $(\frac{\pi}{6})$,$45°$ or $(\frac{\pi}{4})$,and $60°$ or $(\frac{\pi}{3})$.
- Identify the domain and range of sine and cosine functions.
- Find reference angles.
- Use reference angles to evaluate trigonometric functions.

Looking for a thrill? Then consider a ride on the Singapore Flyer, the world's tallest Ferris wheel. Located in Singapore, the Ferris wheel soars to a height of 541 feet—a little more than a tenth of a mile! Described as an observation wheel, riders enjoy spectacular views as they travel from the ground to the peak and down again in a repeating pattern. In this section, we will examine this type of revolving motion around a circle. To do so, we need to define the type of circle first, and then place that circle on a coordinate system. Then we can discuss circular motion in terms of the coordinate pairs.



Figure 5.3.1: The Singapore Flyer is the world's tallest Ferris wheel. (credit: "Vibin JK"/Flickr)

Finding Function Values for the Sine and Cosine

To define our trigonometric functions, we begin by drawing a unit circle, a circle centered at the origin with radius 1, as shown in Figure 5.3.2. The angle (in radians) that t intercepts forms an arc of length s. Using the formula $s = rt$, and knowing that $r = 1$, we see that for a **unit circle**, $s = t$.

Recall that the *x*- and *y*-axes divide the coordinate plane into four quarters called quadrants. We label these quadrants to mimic the direction a positive angle would sweep. The four quadrants are labeled I, II, III, and IV.

For any angle t, we can label the intersection of the terminal side and the unit circle as by its coordinates, (x, y). The coordinates x and y will be the outputs of the trigonometric functions $f(t) = \cos t$ and $f(t) = \sin t$, respectively. This means $x = \cos t$ and $y = \sin t$.

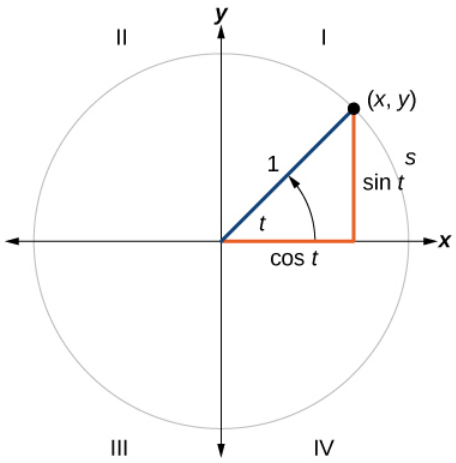


Figure 5.3.2: Unit circle where the central angle is t radians

UNIT CIRCLE

A unit circle has a center at $(0, 0)$ and radius 1. The length of the intercepted arc is equal to the radian measure of the central angle t.

Let (x, y) be the endpoint on the unit circle of an arc of arc length s. The (x, y) coordinates of this point can be described as functions of the angle.

Defining Sine and Cosine Functions

Now that we have our unit circle labeled, we can learn how the (x, y) coordinates relate to the **arc length** and **angle**. The **sine function** relates a real number t to the y-coordinate of the point where the corresponding angle intercepts the unit circle. More precisely, the sine of an angle t equals the y-value of the endpoint on the unit circle of an arc of length t. In Figure 5.3.3, the sine is equal to y. Like all functions, the sine function has an input and an output. Its input is the measure of the angle; its output is the y-coordinate of the corresponding point on the unit circle.

The **cosine function** of an angle t equals the x-value of the endpoint on the unit circle of an arc of length t. In Figure 5.3.1, the cosine is equal to x.

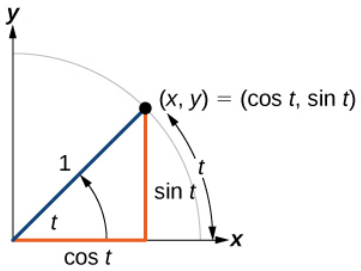


Figure 5.3.3

Because it is understood that sine and cosine are functions, we do not always need to write them with parentheses: $\sin t$ is the same as $\sin(t)$ and $\cos t$ is the same as $\cos(t)$. Likewise, $\cos^2 t$ is a commonly used shorthand notation for $(\cos(t))^2$. Be aware that many calculators and computers do not recognize the shorthand notation. When in doubt, use the extra parentheses when entering calculations into a calculator or computer.

SINE AND COSINE FUNCTIONS

If t is a real number and a point (x, y) on the unit circle corresponds to an angle of t, then

$$\cos t = x \tag{5.3.1}$$
$$\sin t = y \tag{5.3.2}$$

How To: Given a point $P(x, y)$ on the unit circle corresponding to an angle of t, find the sine and cosine

1. The sine of t is equal to the y-coordinate of point P: $\sin t = y$.
2. The cosine of t is equal to the x-coordinate of point P: $\cos t = x$.

Example 5.3.1: Finding Function Values for Sine and Cosine

Point P is a point on the unit circle corresponding to an angle of t, as shown in Figure 5.3.4. Find $\cos(t)$ and $\sin(t)$.

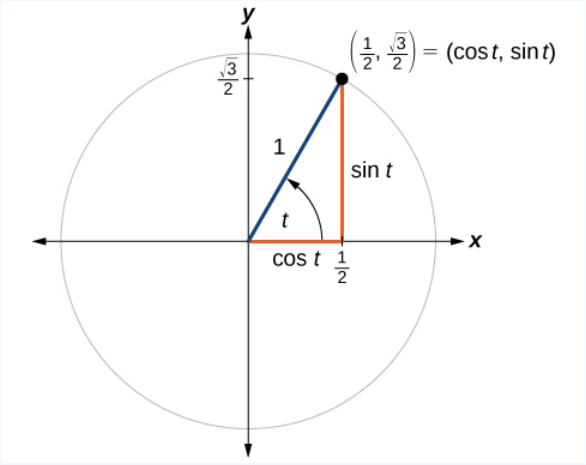


Figure 5.3.4

Solution

We know that $\cos t$ is the x-coordinate of the corresponding point on the unit circle and $\sin t$ is the y-coordinate of the corresponding point on the unit circle. So:

$$x = \cos t = \frac{1}{2}$$
$$y = \sin t = \frac{\sqrt{3}}{2}$$

Try It 5.3.1

A certain angle t corresponds to a point on the unit circle at $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ as shown in Figure 5.3.5. Find $\cos t$ and $\sin t$.

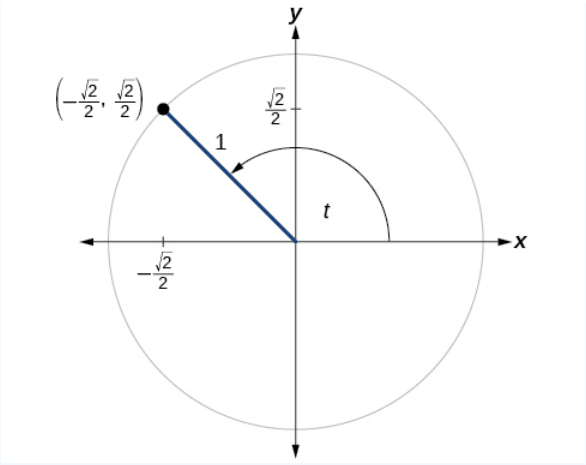


Figure 5.3.5

Answer

$$\cos(t) = -\frac{\sqrt{2}}{2}, \sin(t) = \frac{\sqrt{2}}{2}$$

Finding Sines and Cosines of Angles on an Axis

For quadrantral angles, the corresponding point on the unit circle falls on the x- or y-axis. In that case, we can easily calculate cosine and sine from the values of x and y.

Example 5.3.2: Calculating Sines and Cosines along an Axis

Find $\cos(90^\circ)$ and $\sin(90^\circ)$.

Solution

Moving 90° counterclockwise around the unit circle from the positive x-axis brings us to the top of the circle, where the (x, y) coordinates are (0, 1), as shown in Figure 5.3.6.

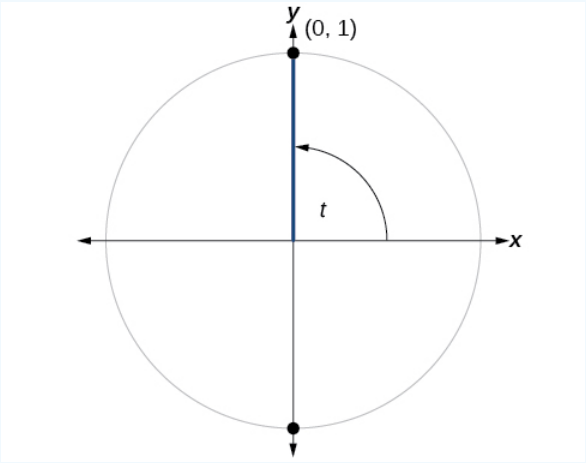


Figure 5.3.6

Using our definitions of cosine and sine,

$$x = \cos t = \cos(90^\circ) = 0$$
$$y = \sin t = \sin(90^\circ) = 1$$

The cosine of 90° is 0; the sine of 90° is 1.

Try It 5.3.2

Find cosine and sine of the angle π.

Answer

$\cos(\pi) = -1,\ \sin(\pi) = 0$

The Pythagorean Identity

Now that we can define sine and cosine, we will learn how they relate to each other and the unit circle. Recall that the equation for the unit circle is $x^2 + y^2 = 1$. Because $x = \cos t$ and $y = \sin t$, we can substitute for x and y to get $\cos^2 t + \sin^2 t = 1$. This equation, $\cos^2 t + \sin^2 t = 1$, is known as the **Pythagorean Identity**. See Figure 5.3.7.

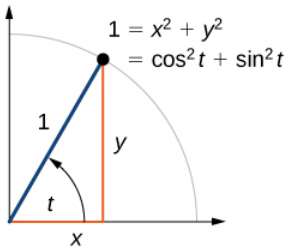


Figure 5.3.7

We can use the Pythagorean Identity to find the cosine of an angle if we know the sine, or vice versa. However, because the equation yields two solutions, we need additional knowledge of the angle to choose the solution with the correct sign. If we know the quadrant where the angle is, we can easily choose the correct solution.

PYTHAGOREAN IDENTITY

The **Pythagorean Identity** states that, for any real number t,

$$\cos^2 t + \sin^2 t = 1 \tag{5.3.3}$$

How To: Given the sine of some angle t and its quadrant location, find the cosine of t

1. Substitute the known value of $\sin(t)$ into the Pythagorean Identity.
2. Solve for $\cos(t)$.
3. Choose the solution with the appropriate sign for the x-values in the quadrant where t is located.

Example 5.3.3: Finding a Cosine from a Sine or a Sine from a Cosine

If $\sin(t) = \frac{3}{7}$ and t is in the second quadrant, find $\cos(t)$.

Solution

If we drop a vertical line from the point on the unit circle corresponding to t, we create a right triangle, from which we can see that the Pythagorean Identity is simply one case of the Pythagorean Theorem. See Figure 5.3.8.

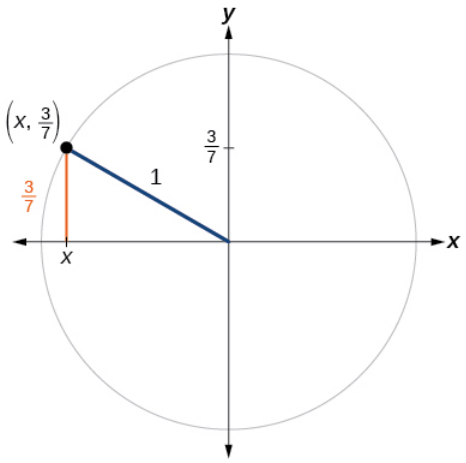


Figure 5.3.8

Substituting the known value for sine into the Pythagorean Identity,

$$\begin{aligned}\cos^2(t)+\sin^2(t) &= 1\\ \cos^2(t)+\frac{9}{49} &= 1\\ \cos^2(t) &= \frac{40}{49}\\ \cos(t) &= \pm\sqrt{\frac{40}{49}} = \pm\frac{\sqrt{40}}{7} = \pm\frac{2\sqrt{10}}{7}\end{aligned}$$

Because the angle is in the second quadrant, we know the x-value is a negative real number, so the cosine is also negative. So

$$\cos(t) = -\frac{2\sqrt{10}}{7}$$

Try It 5.3.3

If $\cos(t) = \frac{24}{25}$ and t is in the fourth quadrant, find $\sin(t)$.

Answer

$$\sin(t) = -\frac{7}{25}$$

Finding Sines and Cosines of Special Angles

We have already learned some properties of the special angles, such as the conversion from radians to degrees. We can also calculate sines and cosines of the special angles using the **Pythagorean Identity** and our knowledge of triangles.

Finding Sines and Cosines of 45° Angles

First, we will look at angles of 45° or $\frac{\pi}{4}$, as shown in Figure 5.3.9. A $45^\circ-45^\circ-90^\circ$ triangle is an isosceles triangle, so the x- and y-coordinates of the corresponding point on the circle are the same. Because the x- and y-values are the same, the sine and cosine values will also be equal.

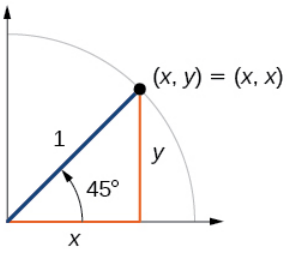


Figure 5.3.9

At $t = \frac{\pi}{4}$, which is 45 degrees, the radius of the unit circle bisects the first **quadrantal angle**. This means the radius lies along the line $y = x$. A unit circle has a radius equal to 1. So, the right triangle formed below the line $y = x$ has sides x and y (with $y = x$), and a radius = 1. See Figure 5.3.10

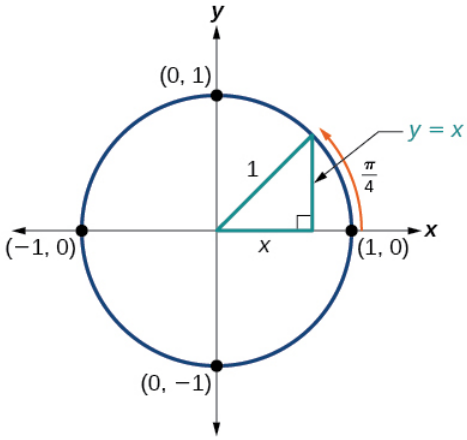


Figure 5.3.10

From the Pythagorean Theorem we get

$$x^2 + y^2 = 1$$

Substituting $y = x$, we get

$$x^2 + x^2 = 1$$

Combining like terms we get

$$2x^2 = 1$$

And solving for x, we get

$$x^2 = \frac{1}{2}$$
$$x = \pm\frac{1}{\sqrt{2}}$$

In quadrant I, $x = \frac{1}{\sqrt{2}}$.

At $t = \frac{\pi}{4}$ or 45 degrees,

$$(x, y) = (x, x) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$
$$x = \frac{1}{\sqrt{2}},\ y = \frac{1}{\sqrt{2}}$$
$$\cos t = \frac{1}{\sqrt{2}},\ \sin t = \frac{1}{\sqrt{2}}$$

If we then rationalize the denominators, we get

$$\begin{aligned}\cos t &= \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{2} \\ \sin t &= \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

Therefore, the (x, y) coordinates of a point on a circle of radius 1 at an angle of 45° are $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Finding Sines and Cosines of 30° and 60° Angles

Next, we will find the cosine and sine at an angle of 30°, or $\frac{\pi}{6}$. First, we will draw a triangle inside a circle with one side at an angle of 30°, and another at an angle of -30°, as shown in Figure 5.3.11. If the resulting two right triangles are combined into one large triangle, notice that all three angles of this larger triangle will be 60°, as shown in Figure 5.3.12

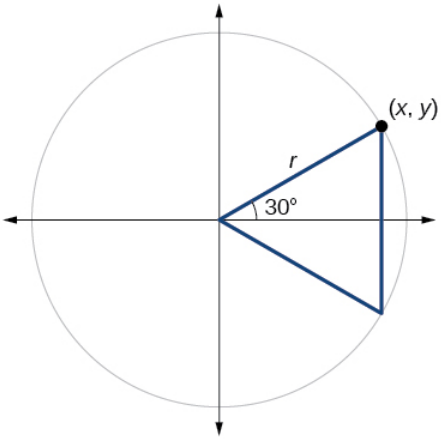


Figure 5.3.11

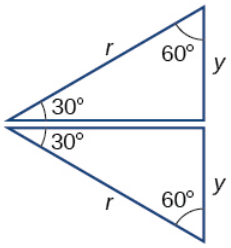


Figure 5.3.12

Because all the angles are equal, the sides are also equal. The vertical line has length $2y$, and since the sides are all equal, we can also conclude that $r = 2y$ or $y = \frac{1}{2}r$. Since $\sin t = y$,

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

And since $r = 1$ in our **unit circle**,

$$\begin{aligned}\sin\left(\frac{\pi}{6}\right) &= \frac{1}{2}(1) \\ &= \frac{1}{2}\end{aligned}$$

Using the Pythagorean Identity, we can find the cosine value.

$$\cos^2\left(\frac{\pi}{6}\right)+\sin^2\left(\frac{\pi}{6}\right)=1$$

$$\cos^2\left(\frac{\pi}{6}\right)+\left(\frac{1}{2}\right)^2=1$$

$$\cos^2\left(\frac{\pi}{6}\right)=\frac{3}{4}$$ Use the square root property.

$$\cos\left(\frac{\pi}{6}\right)=\pm\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{2}$$ Since y is positive, choose the positive root.

The (x,y) coordinates for the point on a circle of radius 1 at an angle of 30° are $\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$. At $t=\frac{\pi}{3}$ (60°), the radius of the unit circle, 1, serves as the hypotenuse of a 30-60-90 degree right triangle, BAD, as shown in Figure 5.3.13. Angle A has measure 60°. 60°. At point B, we draw an angle ABC with measure of 60°. We know the angles in a triangle sum to 180°, so the measure of angle C is also 60°. Now we have an equilateral triangle. Because each side of the equilateral triangle ABC is the same length, and we know one side is the radius of the unit circle, all sides must be of length 1.

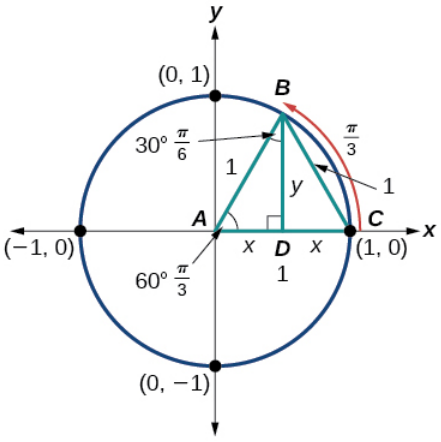


Figure 5.3.13

The measure of angle ABD is 30°. So, if double, angle ABC is 60°. BD is the perpendicular bisector of AC, so it cuts AC in half. This means that AD is $\frac{1}{2}$ the radius, or $\frac{1}{2}$. Notice that AD is the x-coordinate of point B, which is at the intersection of the 60° angle and the unit circle. This gives us a triangle BAD with hypotenuse of 1 and side x of length $\frac{1}{2}$.

From the Pythagorean Theorem, we get

$$x^2+y^2=1$$

Substituting $x=\frac{1}{2}$, we get

$$\left(\frac{1}{2}\right)^2+y^2=1$$

Solving for y, we get

$$\frac{1}{4}+y^2=1$$

$$y^2=1-\frac{1}{4}$$

$$y^2=\frac{3}{4}$$

$$y=\pm\frac{\sqrt{3}}{2}$$

Since $t=\frac{\pi}{3}$ has the terminal side in quadrant I where the y-coordinate is positive, we choose $y=\frac{\sqrt{3}}{2}$, the positive value.

At $t = \frac{\pi}{3}$ (60°), the (x, y) coordinates for the point on a circle of radius 1 at an angle of 60° are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, so we can find the sine and cosine.

$$(x, y) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

$$x = \frac{1}{2},\ y = \frac{\sqrt{3}}{2}$$

$$\cos t = \frac{1}{2},\ \sin t = \frac{\sqrt{3}}{2}$$

We have now found the cosine and sine values for all of the most commonly encountered angles in the first quadrant of the unit circle. Table 5.3.1 summarizes these values.

Table 5.3.1

Angle	0	$\frac{\pi}{6}$, or 30	$\frac{\pi}{4}$, or 45°	$\frac{\pi}{3}$, or 60°	$\frac{\pi}{2}$, or 90°
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Figure 5.3.14 shows the common angles in the first quadrant of the unit circle.

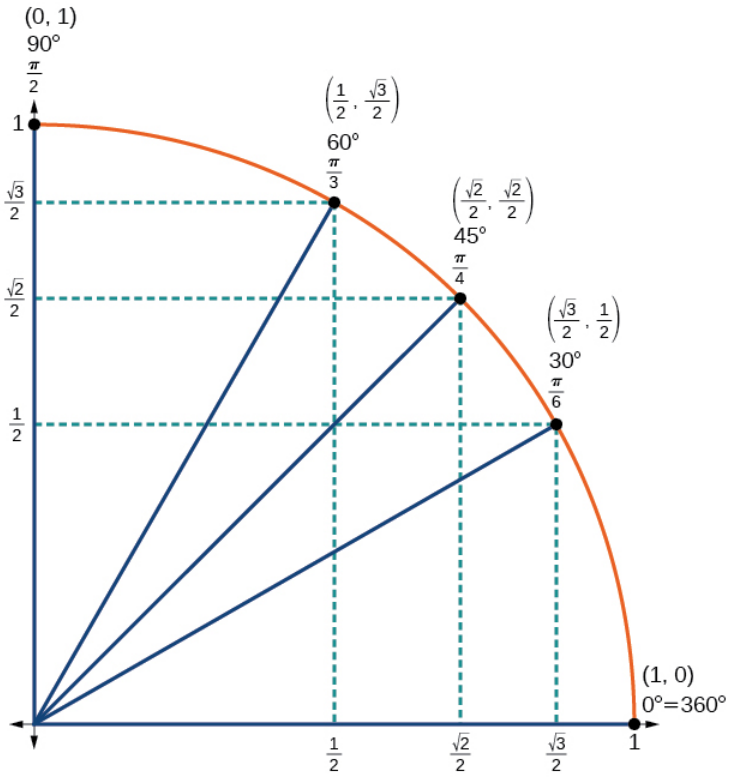


Figure 5.3.14

Using a Calculator to Find Sine and Cosine

To find the cosine and sine of angles other than the **special angles**, we turn to a computer or calculator. **Be aware**: Most calculators can be set into "degree" or "radian" mode, which tells the calculator the units for the input value. When we evaluate $\cos(30)$ on our calculator, it will evaluate it as the cosine of 30 degrees if the calculator is in degree mode, or the cosine of 30 radians if the calculator is in radian mode.

How To: Given an angle in radians, use a graphing calculator to find the cosine

1. If the calculator has degree mode and radian mode, set it to radian mode.
2. Press the COS key.
3. Enter the radian value of the angle and press the close-parentheses key ")".
4. Press ENTER.

Example 5.3.4: Using a Graphing Calculator to Find Sine and Cosine

Evaluate $\cos\left(\frac{5\pi}{3}\right)$ using a graphing calculator or computer.

Solution

Enter the following keystrokes:

$$\text{COS}(5 \times \pi \div 3) \text{ ENTER}$$

$$\cos\left(\frac{5\pi}{3}\right) = 0.5$$

Analysis

We can find the cosine or sine of an angle in degrees directly on a calculator with degree mode. For calculators or software that use only radian mode, we can find the sign of 20°, for example, by including the conversion factor to radians as part of the input:

$$\text{SIN}(20 \times \pi \div 180) \text{ ENTER}$$

Try It 5.3.4

Evaluate $\sin\left(\frac{\pi}{3}\right)$.

Answer

approximately 0.866025403

Identifying the Domain and Range of Sine and Cosine Functions

Now that we can find the sine and cosine of an angle, we need to discuss their domains and ranges. What are the domains of the sine and cosine functions? That is, what are the smallest and largest numbers that can be inputs of the functions? Because angles smaller than 0 and angles larger than 2π can still be graphed on the unit circle and have real values of x, y, and r, there is no lower or upper limit to the angles that can be inputs to the sine and cosine functions. The input to the sine and cosine functions is the rotation from the positive x-axis, and that may be any real number.

What are the ranges of the sine and cosine functions? What are the least and greatest possible values for their output? We can see the answers by examining the **unit circle**, as shown in Figure 5.3.15. The bounds of the x-coordinate are $[-1, 1]$. The bounds of the y-coordinate are also $[-1, 1]$. Therefore, the range of both the sine and cosine functions is $[-1, 1]$.

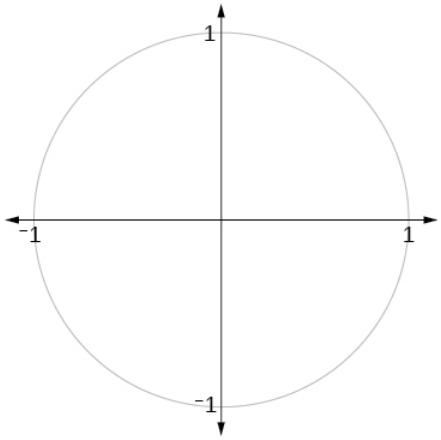


Figure 5.3.15

Finding Reference Angles

We have discussed finding the sine and cosine for angles in the first quadrant, but what if our angle is in another quadrant? For any given angle in the first quadrant, there is an angle in the second quadrant with the same sine value. Because the sine value is the y-coordinate on the unit circle, the other angle with the same sine will share the same y-value, but have the opposite x-value. Therefore, its cosine value will be the opposite of the first angle's cosine value.

Likewise, there will be an angle in the fourth quadrant with the same cosine as the original angle. The angle with the same cosine will share the same x-value but will have the opposite y-value. Therefore, its sine value will be the opposite of the original angle's sine value.

As shown in Figure 5.3.16, angle α has the same sine value as angle t; the cosine values are opposites. Angle β has the same cosine value as angle t; the sine values are opposites.

$$\sin(t) = \sin(\alpha) \quad \text{and} \quad \cos(t) = -\cos(\alpha)$$
$$\sin(t) = -\sin(\beta) \quad \text{and} \quad \cos(t) = \cos(\beta)$$

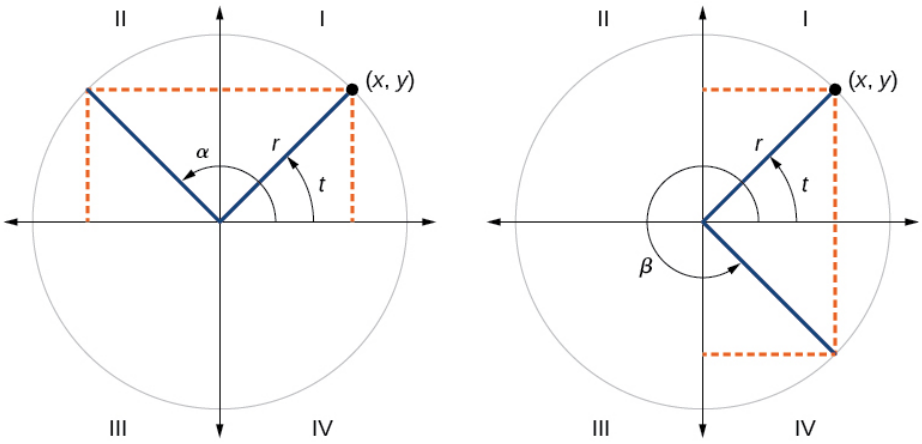


Figure 5.3.16

Recall that an angle's reference angle is the acute angle, t, formed by the terminal side of the angle t and the horizontal axis. A reference angle is always an angle between 0 and 90°, or 0 and $\frac{\pi}{2}$ radians. As we can see from Figure 5.3.17, for any angle in quadrants II, III, or IV, there is a reference angle in quadrant I.

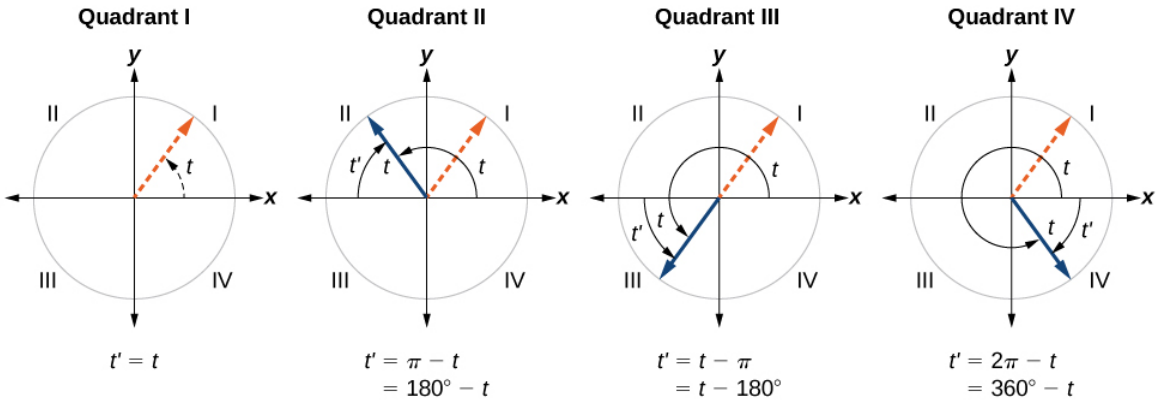


Figure 5.3.17

How To: Given an angle between 0 and 2π, find its reference angle

1. An angle in the first quadrant is its own reference angle.
2. For an angle in the second or third quadrant, the reference angle is $|\pi - t|$ or $|180^\circ - t|$.
3. For an angle in the fourth quadrant, the reference angle is $2\pi - t$ or $360^\circ - t$.
4. If an angle is less than 0 or greater than 2π, add or subtract 2π as many times as needed to find an equivalent angle between 0 and 2π.

Example 5.3.5: Finding a Reference Angle

Find the reference angle of 225° as shown in Figure 5.3.18

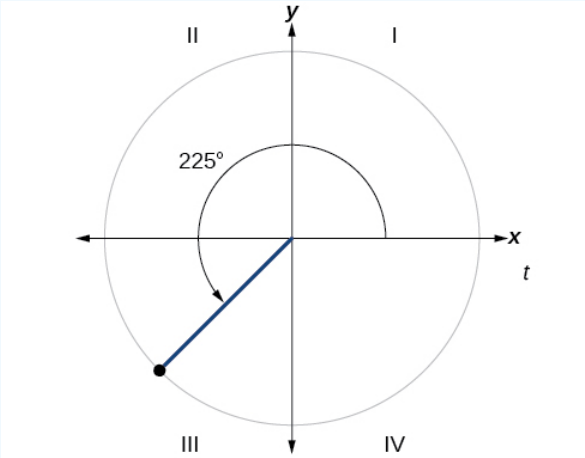


Figure 5.3.18

Solution

Because 225° is in the third quadrant, the reference angle is

$$|(180^\circ - 225^\circ)| = |-45^\circ| = 45^\circ$$

Try It 5.3.5

Find the reference angle of $\frac{5\pi}{3}$.

Answer

(\dfrac{π}{3}\)

Using Reference Angles

Now let's take a moment to reconsider the Ferris wheel introduced at the beginning of this section. Suppose a rider snaps a photograph while stopped twenty feet above ground level. The rider then rotates three-quarters of the way around the circle. What is the rider's new elevation? To answer questions such as this one, we need to evaluate the sine or cosine functions at angles that are greater than 90 degrees or at a **negative angle**. Reference angles make it possible to evaluate trigonometric functions for angles outside the first quadrant. They can also be used to find (x, y) coordinates for those angles. We will use the **reference angle** of the angle of rotation combined with the quadrant in which the terminal side of the angle lies.

Using Reference Angles to Evaluate Trigonometric Functions

We can find the cosine and sine of any angle in any quadrant if we know the cosine or sine of its reference angle. The absolute values of the cosine and sine of an angle are the same as those of the reference angle. The sign depends on the quadrant of the original angle. The cosine will be positive or negative depending on the sign of the x-values in that quadrant. The sine will be positive or negative depending on the sign of the y-values in that quadrant.

USING REFERENCE ANGLES TO FIND COSINE AND SINE

Angles have cosines and sines with the same absolute value as cosines and sines of their reference angles. The sign (positive or negative) can be determined from the quadrant of the angle.

How To: Given an angle in standard position, find the reference angle, and the cosine and sine of the original angle

1. Measure the angle between the terminal side of the given angle and the horizontal axis. That is the reference angle.
2. Determine the values of the cosine and sine of the reference angle.
3. Give the cosine the same sign as the x-values in the quadrant of the original angle.
4. Give the sine the same sign as the y-values in the quadrant of the original angle.

Example 5.3.6: Using Reference Angles to Find Sine and Cosine

1. Using a reference angle, find the exact value of $\cos(150^\circ)$ and $\sin(150^\circ)$.
2. Using the reference angle, find $\cos\dfrac{5\pi}{4}$ and $\sin\frac{5\pi}{4}$.

Solution

This tells us that 150° has the same sine and cosine values as 30°, except for the sign. We know that

Since 150° is in the second quadrant, the x-coordinate of the point on the circle is negative, so the cosine value is negative. The y-coordinate is positive, so the sine value is positive.

$\dfrac{5\pi}{4}$ is in the third quadrant. Its reference angle is $\left|\dfrac{5\pi}{4}-\pi\right|=\dfrac{\pi}{4}$. The cosine and sine of $\dfrac{\pi}{4}$ are both $\dfrac{\sqrt{2}}{2}$. In the third quadrant, both x and y are negative, so:

$$\cos\frac{5\pi}{4}=-\frac{\sqrt{2}}{2} \text{ and } \sin\frac{5\pi}{4}=-\frac{\sqrt{2}}{2}$$

Try It 5.3.6

1. Use the reference angle of 315° to find $\cos(315^\circ)$ and $\sin(315^\circ)$.
2. Use the reference angle of $-\dfrac{\pi}{6}$ to find $\cos\left(-\dfrac{\pi}{6}\right)$ and $\sin\left(-\dfrac{\pi}{6}\right)$

Answer

1. $\cos(315^\circ)=\dfrac{\sqrt{2}}{2}, \sin(315^\circ)=\dfrac{-\sqrt{2}}{2}$
2. $\cos\left(-\dfrac{\pi}{6}\right)=\dfrac{\sqrt{3}}{2}, \sin\left(-\dfrac{\pi}{6}\right)=-\frac{1}{2}$

Using Reference Angles to Find Coordinates

Now that we have learned how to find the cosine and sine values for special angles in the first quadrant, we can use symmetry and reference angles to fill in cosine and sine values for the rest of the special angles on the unit circle. They are shown in Figure 5.3.19. Take time to learn the (x, y) coordinates of all of the major angles in the first quadrant.

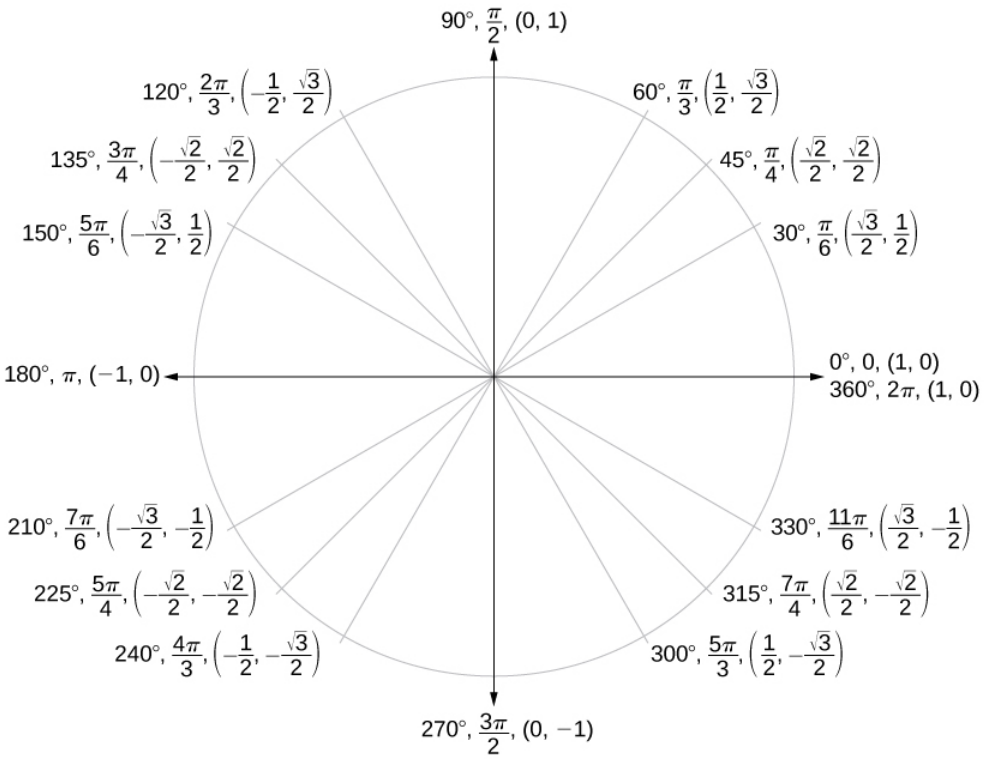


Figure 5.3.19: Special angles and coordinates of corresponding points on the unit circle

In addition to learning the values for special angles, we can use reference angles to find (x, y) coordinates of any point on the unit circle, using what we know of reference angles along with the **identities**

$$x = \cos t$$
$$y = \sin t$$

First we find the reference angle corresponding to the given angle. Then we take the sine and cosine values of the **reference angle**, and give them the signs corresponding to the y- and x-values of the quadrant.

How To: Given the angle of a point on a circle and the radius of the circle, find the (x, y) coordinates of the point

1. Find the reference angle by measuring the smallest angle to the x-axis.
2. Find the cosine and sine of the reference angle.
3. Determine the appropriate signs for x and y in the given quadrant.

Example 5.3.7: Using the Unit Circle to Find Coordinates

Find the coordinates of the point on the unit circle at an angle of $\frac{7\pi}{6}$.

Solution

We know that the angle $\frac{7\pi}{6}$ is in the third quadrant.

First, let's find the reference angle by measuring the angle to the x-axis. To find the reference angle of an angle whose terminal side is in quadrant III, we find the difference of the angle and π.

$$\frac{7\pi}{6} - \pi = \frac{\pi}{6}$$

Next, we will find the cosine and sine of the reference angle:

$$\cos\left(\frac{\pi}{6}\right) = \frac{3}{2} \qquad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

We must determine the appropriate signs for x and y in the given quadrant. Because our original angle is in the third quadrant, where both x and y are negative, both cosine and sine are negative.

$$\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$
$$\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$$

Now we can calculate the (x, y) coordinates using the identities $x = \cos\,\theta$ and $y = \sin\,\theta$.

The coordinates of the point are $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ on the unit circle.

Try It 5.3.7

Find the coordinates of the point on the unit circle at an angle of $\frac{5\pi}{3}$.

Answer

$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

media

Access these online resources for additional instruction and practice with sine and cosine functions.

- Trigonometric Functions Using the Unit Circle
- Sine and Cosine from the Unit Circle
- Sine and Cosine from the Unit Circle and Multiples of Pi Divided by Six
- Sine and Cosine from the Unit Circle and Multiples of Pi Divided by Four
- Trigonometric Functions Using Reference Angles

Key Equations

Cosine	$\cos\,t = x$
Sine	$\sin\,t = y$
Pythagorean Identity	$\cos^2 t + \sin^2 t = 1$

Key Concepts

- Finding the function values for the sine and cosine begins with drawing a unit circle, which is centered at the origin and has a radius of 1 unit.
- Using the unit circle, the sine of an angle t equals the y-value of the endpoint on the unit circle of an arc of length t whereas the cosine of an angle t equals the x-value of the endpoint. See Example.
- The sine and cosine values are most directly determined when the corresponding point on the unit circle falls on an axis. See Example.
- When the sine or cosine is known, we can use the Pythagorean Identity to find the other. The Pythagorean Identity is also useful for determining the sines and cosines of special angles. See Example.

- Calculators and graphing software are helpful for finding sines and cosines if the proper procedure for entering information is known. See Example.
- The domain of the sine and cosine functions is all real numbers.
- The range of both the sine and cosine functions is $[-1, 1]$.
- The sine and cosine of an angle have the same absolute value as the sine and cosine of its reference angle.
- The signs of the sine and cosine are determined from the x- and y-values in the quadrant of the original angle.
- An angle's reference angle is the size angle, t, formed by the terminal side of the angle t and the horizontal axis. See Example.
- Reference angles can be used to find the sine and cosine of the original angle. See Example.
- Reference angles can also be used to find the coordinates of a point on a circle. See Example.

Glossary

cosine function

the x-value of the point on a unit circle corresponding to a given angle

Pythagorean Identity

a corollary of the Pythagorean Theorem stating that the square of the cosine of a given angle plus the square of the sine of that angle equals 1

sine function

the y-value of the point on a unit circle corresponding to a given angle

unit circle

a circle with a center at $(0, 0)$ and radius 1.

Homework Exercises 5.3

Written Problems:

Please draw a unit circle and mark the angle for problems 10-49.

Verbal

1) Describe the unit circle.

Answer

The unit circle is a circle of radius 1 centered at the origin.

2) What do the x- and y-coordinates of the points on the unit circle represent?

3) Discuss the difference between a coterminal angle and a reference angle.

Answer

Coterminal angles are angles that share the same terminal side. A reference angle is the size of the smallest acute angle, t, formed by the terminal side of the angle t and the horizontal axis.

4) Explain how the cosine of an angle in the second quadrant differs from the cosine of its reference angle in the unit circle.

5) Explain how the sine of an angle in the second quadrant differs from the sine of its reference angle in the unit circle.

Answer

The sine values are equal.

Algebraic

For the exercises 6-9, use the given sign of the sine and cosine functions to find the quadrant in which the terminal point determined by t lies.

6) $\sin(t) < 0$ and $\cos(t) < 0$

7) $\sin(t) > 0$ and $\cos(t) > 0$

Answer

I

8) $\sin(t) > 0$ and $\cos(t) < 0$

9) $\sin(t) < 0$ and $\cos(t) > 0$

Answer

IV

For the exercises 10-22, find the exact value of each trigonometric function.

10) $\sin\frac{\pi}{2}$

11) $\sin\frac{\pi}{3}$

Answer

$\frac{\sqrt{3}}{2}$

12) $\cos\frac{\pi}{2}$

13) $\cos\frac{\pi}{3}$

Answer

$\frac{1}{2}$

14) $\sin\frac{\pi}{4}$

15) $\cos\frac{\pi}{4}$

Answer

$\frac{\sqrt{2}}{2}$

16) $\sin\frac{\pi}{6}$

17) $\sin\pi$

Answer

0

18) $\sin\frac{3\pi}{2}$

19) $\cos\pi$

Answer

-1

20) $\cos 0$

21) $cos\frac{\pi}{6}$

Answer

$\frac{\sqrt{3}}{2}$

22) $\sin 0$

Numeric

For the exercises 23-33, state the reference angle for the given angle.

23) 240°

Answer

60°

24) -170°

25) 100°

Answer

80°

26) -315°

27) 135°

Answer

45°

28) $\frac{5\pi}{4}$

29) $\frac{2\pi}{3}$

Answer

$\frac{\pi}{3}$

30) $\frac{5\pi}{6}$

31) $-\frac{11\pi}{3}$

Answer

$\frac{\pi}{3}$

32) $\frac{-7\pi}{4}$

33) $\frac{-\pi}{8}$

Answer

$\frac{\pi}{8}$

For the exercises 34-49, find the reference angle, the quadrant of the terminal side, and the sine and cosine of each angle. If the angle is not one of the angles on the unit circle, use a calculator and round to three decimal places.

34) 225°

35) 300°

Answer

60°, Quadrant IV, $\sin(300^\circ) = -\frac{\sqrt{3}}{2}, \cos(300^\circ) = \frac{1}{2}$

36) 320°

37) 135°

Answer

45°, Quadrant II, $\sin(135^\circ) = \frac{\sqrt{2}}{2}, \cos(135^\circ) = -\frac{\sqrt{2}}{2}$

38) 210°

39) 120°

Answer

60°, Quadrant II, $\sin(120^\circ) = \frac{\sqrt{3}}{2}, \cos(120^\circ) = -\frac{1}{2}$

40) 250°

41) 150°

Answer

30°, Quadrant II, $\sin(150^\circ) = \frac{1}{2}, \cos(150^\circ) = -\frac{\sqrt{3}}{2}$

42) $\frac{5\pi}{4}$

43) $\frac{7\pi}{6}$

Answer

$\frac{\pi}{6}$, Quadrant III, $\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}, \cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

44) $\frac{5\pi}{3}$

45) $\frac{3\pi}{4}$

Answer

$\frac{\pi}{4}$, Quadrant II, $\sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}, \cos\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{2}}{2}$

46) $\frac{4\pi}{3}$

47) $\frac{2\pi}{3}$

Answer

$\frac{\pi}{3}$, Quadrant II, $\sin\left(\frac{2\pi}{3}\right)=\frac{\sqrt{3}}{2}$, $\cos\left(\frac{2\pi}{3}\right)=-\frac{1}{2}$

48) $\frac{5\pi}{6}$

49) $\frac{7\pi}{4}$

Answer

$\frac{\pi}{4}$, Quadrant IV, $\sin\left(\frac{7\pi}{4}\right)=-\frac{\sqrt{2}}{2}$, $\cos\left(\frac{7\pi}{4}\right)=\frac{\sqrt{2}}{2}$

For the exercises 50-59, find the requested value.

50) If $\cos(t)=\frac{1}{7}$ and t is in the 4th quadrant, find $\sin(t)$.

51) If $\cos(t)=\frac{2}{9}$ and t is in the 1st quadrant, find $\sin(t)$.

Answer

$\frac{\sqrt{77}}{9}$

52) If $\sin(t)=\frac{3}{8}$ and t is in the 2nd quadrant, find $\cos(t)$.

53) If $\sin(t)=-\frac{1}{4}$ and t is in the 3rd quadrant, find $\cos(t)$.

Answer

$-\frac{\sqrt{15}}{4}$

54) Find the coordinates of the point on a circle with radius 15 corresponding to an angle of 220°.

55) Find the coordinates of the point on a circle with radius 20 corresponding to an angle of 120°.

Answer

$(-10, 10\sqrt{3})$

56) Find the coordinates of the point on a circle with radius 8 corresponding to an angle of $\frac{7\pi}{4}$.

57) Find the coordinates of the point on a circle with radius 16 corresponding to an angle of $\frac{5\pi}{9}$.

Answer

$(-2.778, 15.757)$

58) State the domain of the sine and cosine functions.

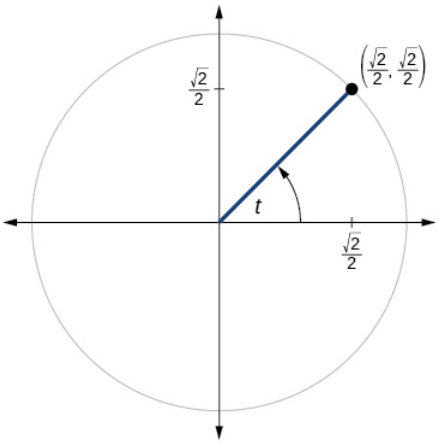
59) State the range of the sine and cosine functions.

Answer

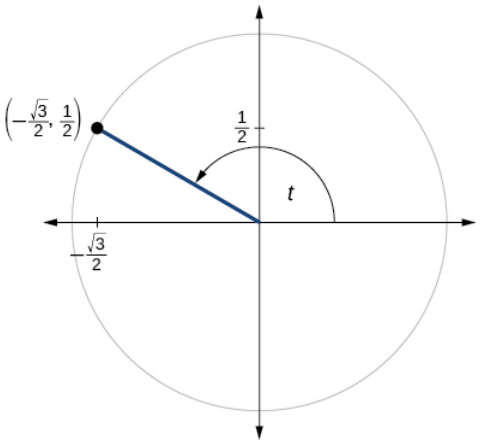
$[-1, 1]$

For the exercises 60-79, use the given point on the unit circle to find the value of the sine and cosine of t.

60)



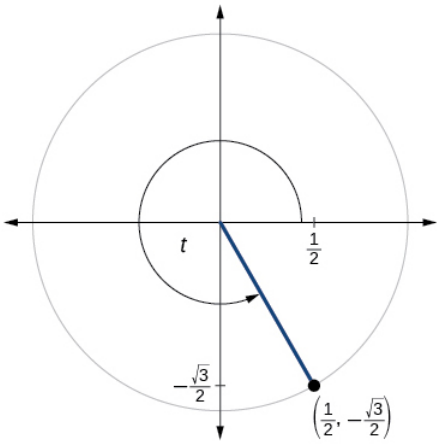
61)



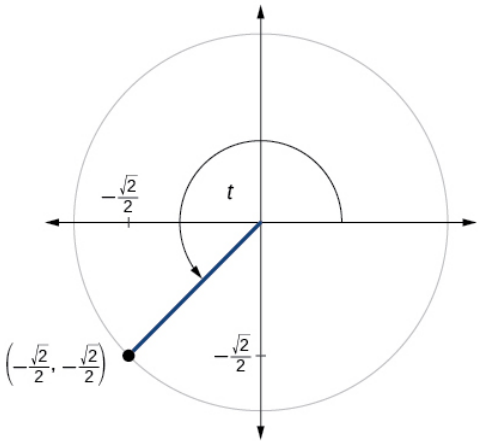
Answer

$$\sin t = \frac{1}{2}, \cos t = -\frac{\sqrt{3}}{2}$$

62)



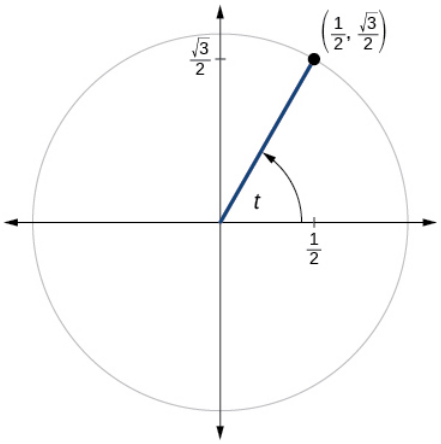
63)



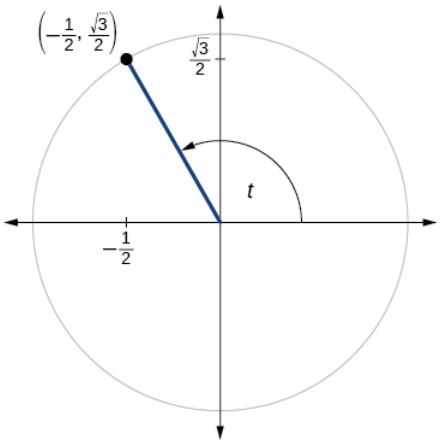
Answer

$$\sin t = -\frac{\sqrt{2}}{2}, \cos t = -\frac{\sqrt{2}}{2}$$

64)



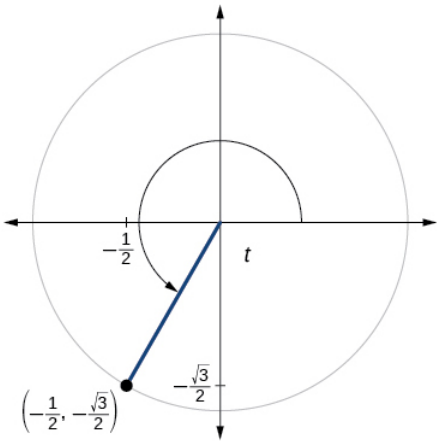
65)



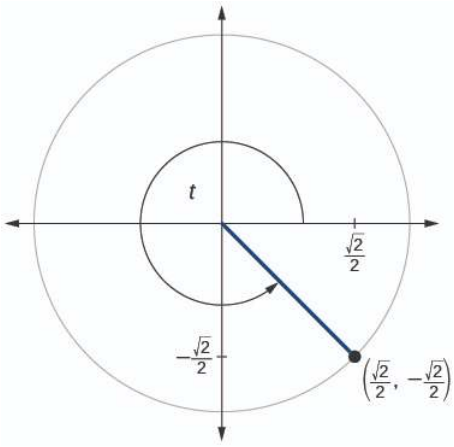
Answer

$$\sin t = \frac{\sqrt{3}}{2}, \cos t = -\frac{1}{2}$$

66)



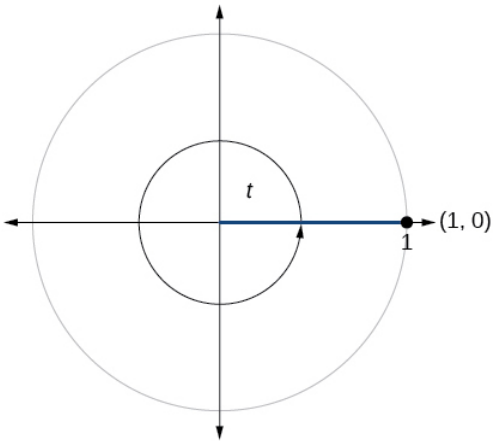
67)



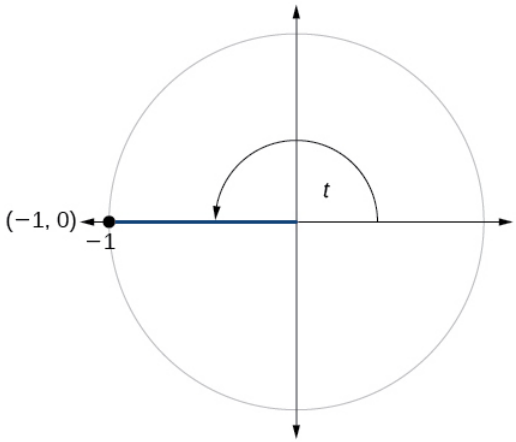
Answer

$$\sin t = -\frac{\sqrt{2}}{2}, \cos t = \frac{\sqrt{2}}{2}$$

68)



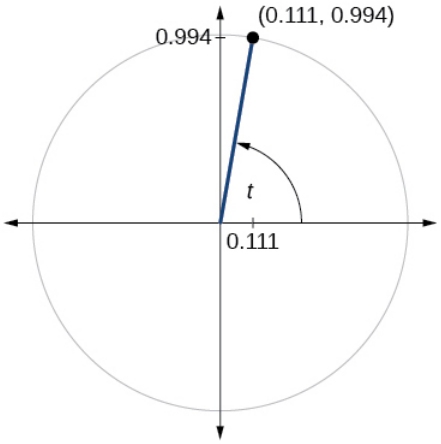
69)



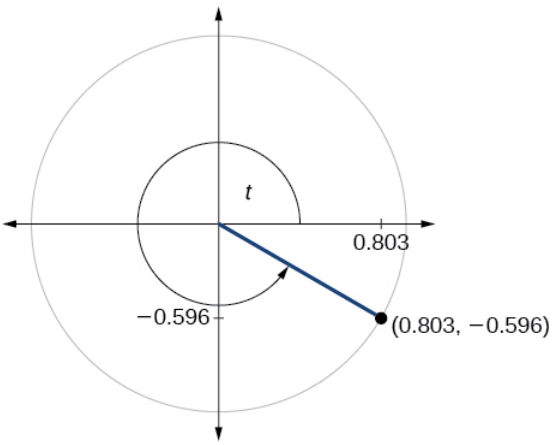
Answer

$\sin t = 0, \cos t = -1$

70)



71)

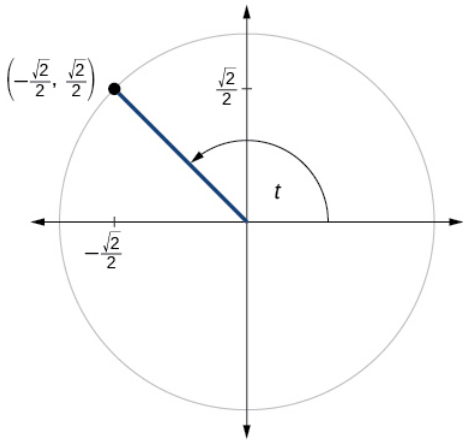


Answer

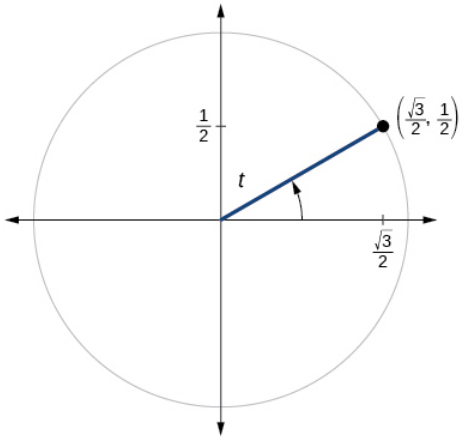
$\sin t = -0.596, \cos t = 0.803$

https://math.libretexts.org/@go/page/106361

72)



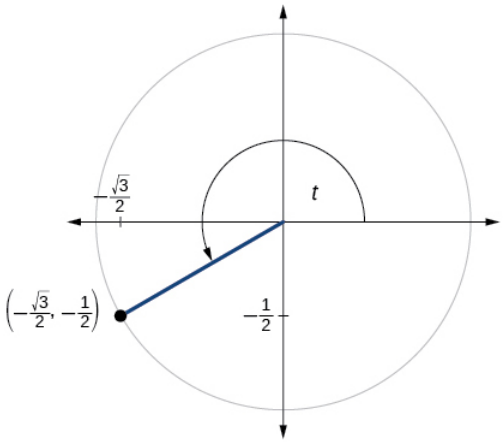
73)



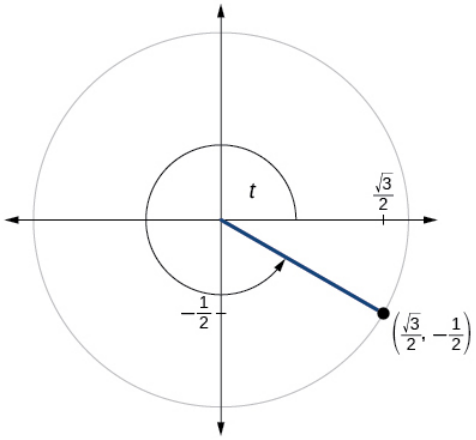
Answer

$$\sin t = \frac{1}{2}, \cos t = \frac{\sqrt{3}}{2}$$

74)



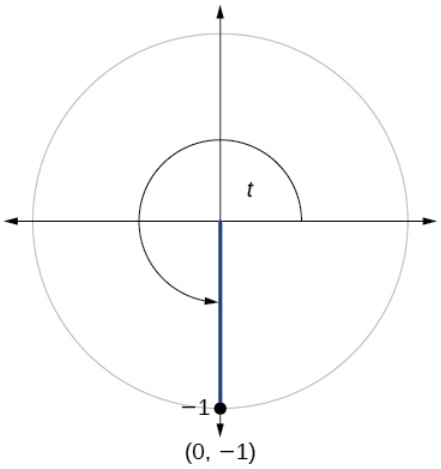
75)



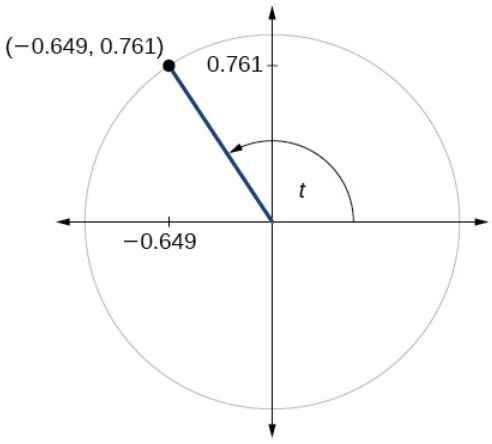
Answer

$\sin t = -\frac{1}{2}, \cos t = \frac{\sqrt{3}}{2}$

76)



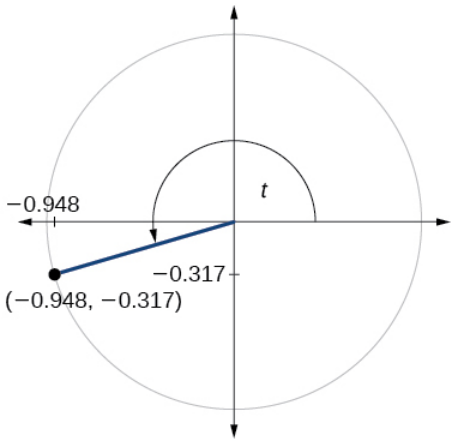
77)



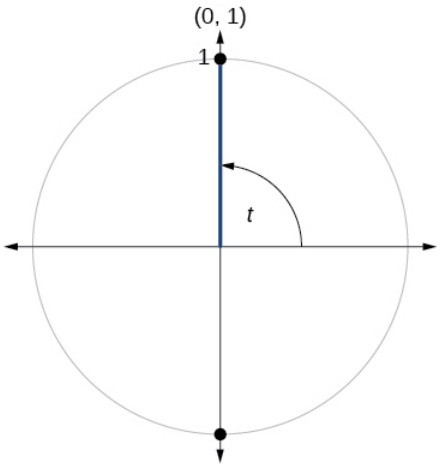
Answer

$\sin t = 0.761, \cos t = -0.649$

78)



79)



Answer

$\sin t = 1, \cos t = 0$

Technology

For the exercises 80-89, use a graphing calculator to evaluate.

80) $\sin \frac{5\pi}{9}$

81) $cos \frac{5\pi}{9}$

Answer

-0.1736

82) $\sin \frac{\pi}{10}$

83) $\cos \frac{\pi}{10}$

Answer

0.9511

84) $\sin\frac{3\pi}{4}$

85) $\cos\frac{3\pi}{4}$

Answer

-0.7071

86) $\sin 98^\circ$

87) $\cos 98^\circ$

Answer

-0.1392

88) $\cos 310^\circ$

89) $\sin 310^\circ$

Answer

-0.7660

Extensions

For the exercises 90-99, evaluate.

90) $\sin\left(\frac{11\pi}{3}\right)\cos\left(\frac{-5\pi}{6}\right)$

91) $\sin\left(\frac{3\pi}{4}\right)\cos\left(\frac{5\pi}{3}\right)$

Answer

$\frac{\sqrt{2}}{4}$

92) $\sin\left(-\frac{4\pi}{3}\right)\cos\left(\frac{\pi}{2}\right)$

93) $\sin\left(\frac{-9\pi}{4}\right)\cos\left(\frac{-\pi}{6}\right)$

Answer

$-\frac{\sqrt{6}}{4}$

94) $\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{-\pi}{3}\right)$

95) $\sin\left(\frac{7\pi}{4}\right)\cos\left(\frac{-2\pi}{3}\right)$

Answer

$\frac{\sqrt{2}}{4}$

96) $\cos\left(\frac{5\pi}{6}\right)\cos\left(\frac{2\pi}{3}\right)$

97) $\cos\left(\frac{-\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)$

Answer

$\frac{\sqrt{2}}{4}$

98) $\sin\left(\frac{-5\pi}{4}\right)\sin\left(\frac{11\pi}{6}\right)$

99) $\sin(\pi)\sin\left(\frac{\pi}{6}\right)$

Answer

0

Real-World Applications

For the exercises 100-104, use this scenario: A child enters a carousel that takes one minute to revolve once around. The child enters at the point $(0,1)$, that is, on the due north position. Assume the carousel revolves counter clockwise.

100) What are the coordinates of the child after 45 seconds?

101) What are the coordinates of the child after 90 seconds?

Answer

$(0,-1)$

102) What is the coordinates of the child after 125 seconds?

103) When will the child have coordinates $(0.707,-0.707)$if the ride lasts 6 minutes? (There are multiple answers.)

Answer

37.5seconds, 97.5seconds, 157.5seconds, 217.5seconds, 277.5seconds, 337.5seconds

104) When will the child have coordinates $(-0.866,-0.5)$if the ride last 6 minutes?

This page titled 5.3: Unit Circle is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **5.2: Unit Circle - Sine and Cosine Functions** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/precalculus.
- **5.E: Trigonometric Functions (Exercises)** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/precalculus.

5.4: The Other Trigonometric Functions

Preview Video section 5.4

Please fill in your notes as you view the video and answer the questions.

Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login

Learning Objectives

- Find exact values of the trigonometric functions secant, cosecant, tangent, and cotangent of $\frac{\pi}{3}$, $\frac{\pi}{4}$, and $\frac{\pi}{6}$.
- Use reference angles to evaluate the trigonometric functions secant, tangent, and cotangent.
- Use properties of even and odd trigonometric functions.
- Recognize and use fundamental identities.
- Evaluate trigonometric functions with a calculator.

A wheelchair ramp that meets the standards of the Americans with Disabilities Act must make an angle with the ground whose tangent is $\frac{1}{12}$ or less, regardless of its length. A tangent represents a ratio, so this means that for every 1 inch of rise, the ramp must have 12 inches of run. Trigonometric functions allow us to specify the shapes and proportions of objects independent of exact dimensions. We have already defined the sine and cosine functions of an angle. Though sine and cosine are the trigonometric functions most often used, there are four others. Together they make up the set of six trigonometric functions. In this section, we will investigate the remaining functions.

Finding Exact Values of the Trigonometric Functions Secant, Cosecant, Tangent, and Cotangent

To define the remaining functions, we will once again draw a unit circle with a point (x, y) corresponding to an angle of t,as shown in Figure 5.4.1. As with the sine and cosine, we can use the (x, y) coordinates to find the other functions.

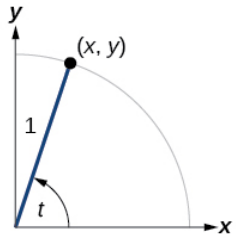


Figure 5.4.1

The first function we will define is the tangent. The **tangent** of an angle is the ratio of the *y*-value to the *x*-value of the corresponding point on the unit circle. In Figure 5.4.1, the tangent of angle t is equal to $\frac{y}{x}, x \neq 0$. Because the *y*-value is equal to the sine of t, and the *x*-value is equal to the cosine of t, the tangent of angle t can also be defined as $\frac{\sin t}{\cos t}, \cos t \neq 0$. The tangent function is abbreviated as tan. The remaining three functions can all be expressed as reciprocals of functions we have already defined.

- The **secant** function is the reciprocal of the cosine function. In Figure 5.4.1, the secant of angle t is equal to $\frac{1}{\cos t} = \frac{1}{x}, x \neq 0$. The secant function is abbreviated as sec.
- The **cotangent** function is the reciprocal of the tangent function. In Figure 5.4.1, the cotangent of angle t is equal to $\frac{\cos t}{\sin t} = \frac{x}{y}, y \neq 0$. The cotangent function is abbreviated as cot.
- The **cosecant** function is the reciprocal of the sine function. In Figure 5.4.1, the cosecant of angle t is equal to $\frac{1}{\sin t} = \frac{1}{y}, y \neq 0$. The cosecant function is abbreviated as csc.

TANGENT, SECANT, COSECANT, AND COTANGENT FUNCTIONS

If t is a real number and (x, y) is a point where the terminal side of an angle of t radians intercepts the unit circle, then

$$\tan t = \frac{y}{x}, x \neq 0 \tag{5.4.1}$$

$$\sec t = \frac{1}{x}, x \neq 0 \tag{5.4.2}$$

$$\csc t = \frac{1}{y}, y \neq 0 \tag{5.4.3}$$

$$\cot t = \frac{x}{y}, y \neq 0 \tag{5.4.4}$$

Example 5.4.1: Finding Trigonometric Functions from a Point on the Unit Circle

The point $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$ is on the unit circle, as shown in Figure 5.4.2. Find $\sin t$, $\cos t$, $\tan t$, $\sec t$, $\csc t$, and $\cot t$.

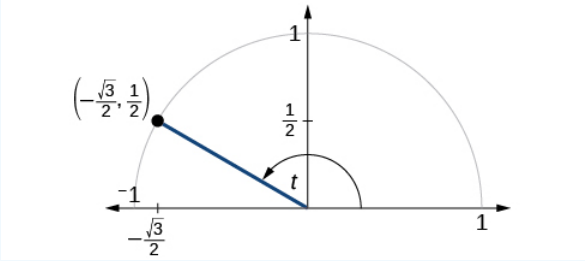


Figure 5.4.2

Solution

Because we know the (x, y) coordinates of the point on the unit circle indicated by angle t, we can use those coordinates to find the six functions:

$$\sin t = y = \frac{1}{2}$$

$$\cos t = x = -\frac{\sqrt{3}}{2}$$

$$\tan t = \frac{y}{x} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{2}\left(-\frac{2}{\sqrt{3}}\right) = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

$$\sec t = \frac{1}{x} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

$$\csc t = \frac{1}{y} = \frac{1}{\frac{1}{2}} = 2$$

$$\cot t = \frac{x}{y} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\frac{\sqrt{3}}{2}\left(\frac{2}{1}\right) = -\sqrt{3}$$

Exercise 5.4.1:

The point $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ is on the unit circle, as shown in Figure 5.4.3. Find $\sin t, \cos t, \tan t, \sec t, \csc t$, and $\cot t$.

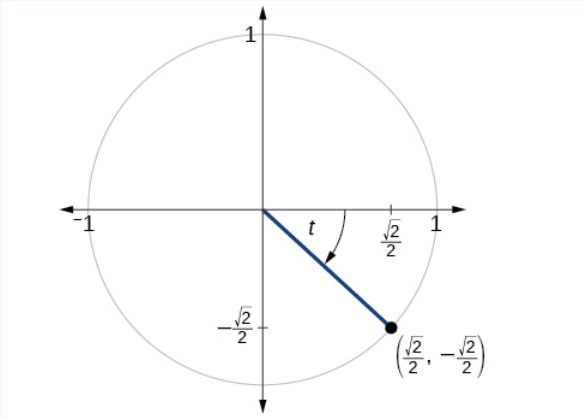


Figure 5.4.3

Solution

$\sin t = -\frac{\sqrt{2}}{2}, \cos t = \frac{\sqrt{2}}{2}, \tan t = -1, \sec t = \sqrt{2}, \csc t = -\sqrt{2}, \cot t = -1$

Example 5.4.2: Finding the Trigonometric Functions of an Angle

Find $\sin t, \cos t, \tan t, \sec t, \csc t$, and $\cot t$ when $t = \frac{\pi}{6}$.

Solution

We have previously used the properties of equilateral triangles to demonstrate that $\sin\frac{\pi}{6} = \frac{1}{2}$ and $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$. We can use these values and the definitions of tangent, secant, cosecant, and cotangent as functions of sine and cosine to find the remaining function values.

$$\begin{aligned}
\tan\frac{\pi}{6} &= \frac{\sin\frac{\pi}{6}}{\cos\frac{\pi}{6}} \\
&= \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \\
\sec\frac{\pi}{6} &= \frac{1}{\cos\frac{\pi}{6}} \\
&= \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \\
\csc\frac{\pi}{6} &= \frac{1}{\sin\frac{\pi}{6}} = \frac{1}{\frac{1}{2}} = 2 \\
\cot\frac{\pi}{6} &= \frac{\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}} \\
&= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}
\end{aligned}$$

Exercise 5.4.2:

Find $\sin t, \cos t, \tan t, \sec t, \csc t$, and $\cot t$ when $t = \frac{\pi}{3}$.

Solution

$$\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2} \tag{5.4.5}$$
$$\cos\frac{\pi}{3}=\frac{1}{2} \tag{5.4.6}$$
$$\tan\frac{\pi}{3}=\sqrt{3} \tag{5.4.7}$$
$$\sec\frac{\pi}{3}=2 \tag{5.4.8}$$
$$\csc\frac{\pi}{3}=\frac{2\sqrt{3}}{3} \tag{5.4.9}$$
$$\cot\frac{\pi}{3}=\frac{\sqrt{3}}{3} \tag{5.4.10}$$

Because we know the sine and cosine values for the common first-quadrant angles, we can find the other function values for those angles as well by setting x x equal to the cosine and y y equal to the sine and then using the definitions of tangent, secant, cosecant, and cotangent. The results are shown in Table 5.4.1.

Table 5.4.1

Angle	**0**	$\frac{\pi}{6}$, or 30°	$\frac{\pi}{4}$, or 45°	$\frac{\pi}{3}$, or 60°	$\frac{\pi}{2}$, or 90°
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
Tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined
Secant	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	Undefined
Cosecant	Undefined	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
Cotangent	Undefined	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

Using Reference Angles to Evaluate Tangent, Secant, Cosecant, and Cotangent

We can evaluate trigonometric functions of angles outside the first quadrant using reference angles as we have already done with the sine and cosine functions. The procedure is the same: Find the **reference angle** formed by the terminal side of the given angle with the horizontal axis. The trigonometric function values for the original angle will be the same as those for the reference angle, except for the positive or negative sign, which is determined by *x*- and *y*-values in the original quadrant. Figure 5.4.4 shows which functions are positive in which quadrant.

To help us remember which of the six trigonometric functions are positive in each quadrant, we can use the mnemonic phrase “A Smart Trig Class.” Each of the four words in the phrase corresponds to one of the four quadrants, starting with quadrant I and rotating counterclockwise. In quadrant I, which is “**A**,” **a**ll of the six trigonometric functions are positive. In quadrant II, “**S**mart,” only **s**ine and its reciprocal function, cosecant, are positive. In quadrant III, “**T**rig,” only **t**angent and its reciprocal function, cotangent, are positive. Finally, in quadrant IV, “**C**lass,” only **c**osine and its reciprocal function, secant, are positive.

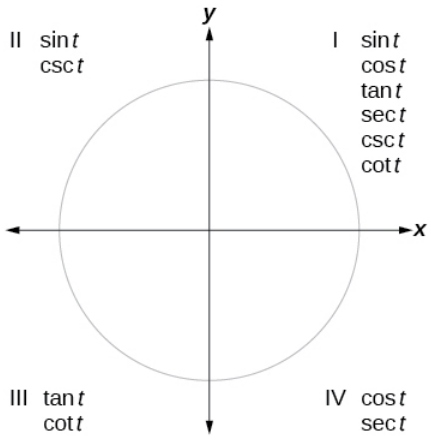


Figure 5.4.4

HOWTO: Given an angle not in the first quadrant, use reference angles to find all six trigonometric functions

1. Measure the angle formed by the terminal side of the given angle and the horizontal axis. This is the reference angle.
2. Evaluate the function at the reference angle.
3. Observe the quadrant where the terminal side of the original angle is located. Based on the quadrant, determine whether the output is positive or negative.

Example 5.4.3: Using Reference Angles to Find Trigonometric Functions

Use reference angles to find all six trigonometric functions of $-\frac{5\pi}{6}$.

Solution

The angle between this angle's terminal side and the *x*-axis is $\frac{\pi}{6}$, so that is the reference angle. Since $-\frac{5\pi}{6}$ is in the third quadrant, where both x and y are negative, cosine, sine, secant, and cosecant will be negative, while tangent and cotangent will be positive.

$$\cos(-\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2}, \sin(-\frac{5\pi}{6}) = -\frac{1}{2}, \tan(-\frac{5\pi}{6}) = \frac{\sqrt{3}}{3} \tag{5.4.11}$$

$$\sec(-\frac{5\pi}{6}) = -\frac{2\sqrt{3}}{3}, \csc(-\frac{5\pi}{6}) = -2, \cot(-\frac{5\pi}{6}) = \sqrt{3} \tag{5.4.12}$$

Exercise 5.4.3

Use reference angles to find all six trigonometric functions of $-\frac{7\pi}{4}$.

Solution

$\sin(-\frac{7\pi}{4}) = \frac{\sqrt{2}}{2}, \cos(\frac{-7\pi}{4}) = \frac{\sqrt{2}}{2}, \tan(\frac{-7\pi}{4}) = 1,$

$\sec(\frac{-7\pi}{4}) = \sqrt{2}, \csc(\frac{-7\pi}{4}) = \sqrt{2}, \cot(\frac{-7\pi}{4}) = 1$

Using Even and Odd Trigonometric Functions

To be able to use our six trigonometric functions freely with both positive and negative angle inputs, we should examine how each function treats a negative input. As it turns out, there is an important difference among the functions in this regard.

Consider the function $f(x) = x^2$, shown in Figure 5.4.5. The graph of the function is symmetrical about the *y*-axis. All along the curve, any two points with opposite *x*-values have the same function value. This matches the result of calculation: $(4)^2 = (-4)^2, (-5)^2 = (5)^2$, and so on. So $f(x) = x^2$ is an **even function**, a function such that two inputs that are opposites have the same output. That means $f(-x) = f(x)$.

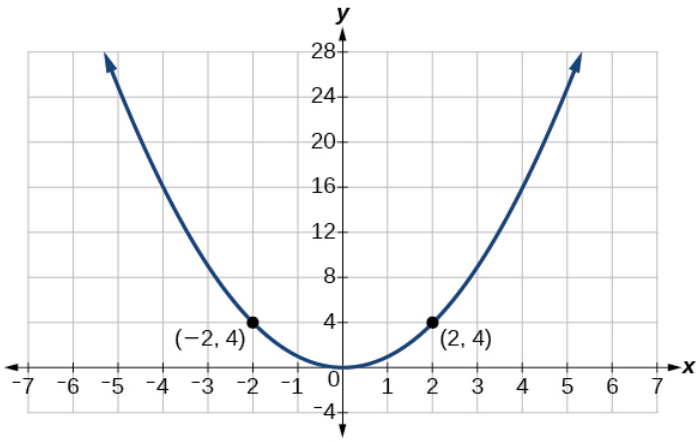


Figure 5.4.5: The function $f(x) = x^2$ is an even function.

Now consider the function $f(x) = x^3$, shown in Figure 5.4.6. The graph is not symmetrical about the *y*-axis. All along the graph, any two points with opposite *x*-values also have opposite *y*-values. So $f(x) = x^3$ is an **odd function**, one such that two inputs that are opposites have outputs that are also opposites. That means $f(-x) = -f(x)$.

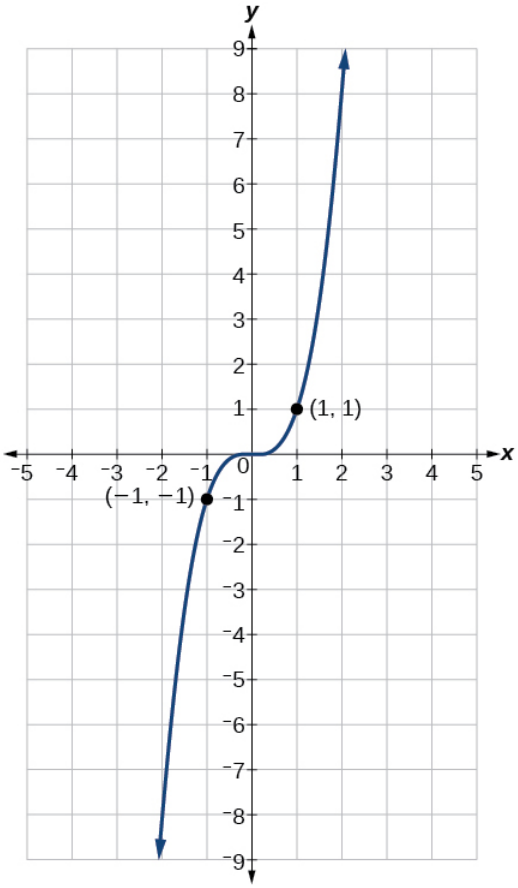


Figure 5.4.6: The function $f(x) = x^3$ is an odd function.

We can test whether a trigonometric function is even or odd by drawing a **unit circle** with a positive and a negative angle, as in Figure 5.4.7. The sine of the positive angle is y. The sine of the negative angle is $-y$. The **sine function**, then, is an odd function. We can test each of the six trigonometric functions in this fashion. The results are shown in Table 5.4.2.

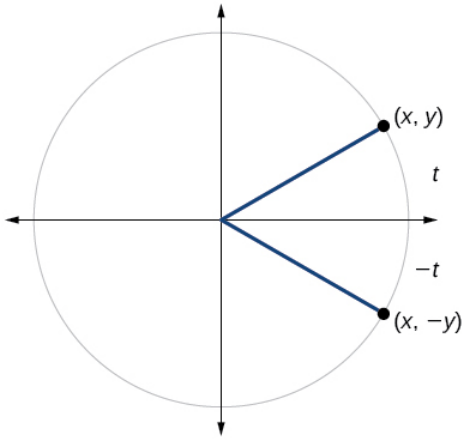


Figure 5.4.7

Table 5.4.2

$\sin t = y$ (5.4.13)	$\cos t = x$ (5.4.16)	$\tan(t) = \frac{y}{x}$ (5.4.19)
$\sin(-t) = -y$ (5.4.14)	$\cos(-t) = x$ (5.4.17)	$\tan(-t) = -\frac{y}{x}$ (5.4.20)
$\sin t \neq sin(-t)$ (5.4.15)	$\cos t = \cos(-t)$ (5.4.18)	$\tan t \neq \tan(-t)$ (5.4.21)
$\sec t = \frac{1}{x}$ (5.4.22)	$\csc t = \frac{1}{y}$ (5.4.25)	$\cot t = \frac{x}{y}$ (5.4.28)
$\sec(-t) = \frac{1}{x}$ (5.4.23)	$\csc(-t) = \frac{1}{-y}$ (5.4.26)	$\cot(-t) = \frac{x}{-y}$ (5.4.29)
$\sec t = \sec(-t)$ (5.4.24)	$\csc t \neq \csc(-t)$ (5.4.27)	$\cot t \neq \cot(-t)$ (5.4.30)

EVEN AND ODD TRIGONOMETRIC FUNCTIONS

- An **even function** is one in which $f(-x) = f(x)$.
- An **odd function** is one in which $f(-x) = -f(x)$.

Cosine and secant are even:

$$\cos(-t) = \cos t \quad (5.4.31)$$
$$\sec(-t) = \sec t \quad (5.4.32)$$

Sine, tangent, cosecant, and cotangent are odd:

$$\sin(-t) = -\sin t \quad (5.4.33)$$
$$\tan(-t) = -\tan t \quad (5.4.34)$$
$$\csc(-t) = -\csc t \quad (5.4.35)$$
$$\cot(-t) = -\cot t \quad (5.4.36)$$

Example 5.4.4: Using Even and Odd Properties of Trigonometric Functions

If the secant of angle t is 2, what is the secant of $-t$?

Solution

Secant is an even function. The secant of an angle is the same as the secant of its opposite. So if the secant of angle t is 2, the secant of $-t$ is also 2.

Exercise 5.4.4:

If the cotangent of angle t is $\sqrt{3}$, what is the cotangent of $-t$?

Solution

$-\sqrt{3}$

Recognizing and Using Fundamental Identities

We have explored a number of properties of trigonometric functions. Now, we can take the relationships a step further, and derive some fundamental identities. Identities are statements that are true for all values of the input on which they are defined. Usually, identities can be derived from definitions and relationships we already know. For example, the Pythagorean Identity we learned earlier was derived from the Pythagorean Theorem and the definitions of sine and cosine.

FUNDAMENTAL IDENTITIES

We can derive some useful **identities** from the six trigonometric functions. The other four trigonometric functions can be related back to the sine and cosine functions using these basic relationships:

$$\tan t = \frac{\sin t}{\cos t} \tag{5.4.37}$$

$$\sec t = \frac{1}{\cos t} \tag{5.4.38}$$

$$\csc t = \frac{1}{\sin t} \tag{5.4.39}$$

$$\cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t} \tag{5.4.40}$$

Example 5.4.5: Using Identities to Evaluate Trigonometric Functions

a. Given $\sin(45^\circ) = \frac{\sqrt{2}}{2}, \cos(45^\circ) = \frac{\sqrt{2}}{2}$, evaluate $\tan(45^\circ)$.

b. Given $\sin(\frac{5\pi}{6}) = \frac{1}{2}, \cos(\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2}$, evaluate $\sec(\frac{5\pi}{6})$.

Solution

Because we know the sine and cosine values for these angles, we can use identities to evaluate the other functions.

a.
$$\begin{aligned}\tan(45^\circ) &= \frac{\sin(45^\circ)}{\cos(45^\circ)} \\ &= \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} \\ &= 1\end{aligned}$$

b.
$$\begin{aligned}\sec(\frac{5\pi}{6}) &= \frac{1}{\cos(\frac{5\pi}{6})} \\ &= \frac{1}{-\frac{\sqrt{3}}{2}} \\ &= \frac{-2\sqrt{3}}{1} \\ &= \frac{-2}{\sqrt{3}} \\ &= -\frac{2\sqrt{3}}{3}\end{aligned}$$

Exercise 5.4.5

Evaluate $\csc(\frac{7\pi}{6})$.

Solution

-2

Example 5.4.6: Using Identities to Simplify Trigonometric Expressions

Simplify $\frac{\sec t}{\tan t}$.

Solution

We can simplify this by rewriting both functions in terms of sine and cosine.

$$\begin{aligned} \frac{\sec t}{\tan t} &= \frac{1/\cos t}{\sin t/\cos t} && \text{To divide the functions, we multiply by the reciprocal.} \\ &= \frac{1}{\cos t}\frac{\cos t}{\sin t} && \text{Divide out the cosines.} \\ &= \frac{1}{\sin t} && \text{Simplify and use the identity.} \\ &= \csc t \end{aligned} \tag{5.4.41}$$

By showing that $\frac{\sec t}{\tan t}$ can be simplified to $\csc t$,we have, in fact, established a new identity.

$$\frac{\sec t}{\tan t} = \csc t$$

Exercise 5.4.6

Simplify $(\tan t)(\cos t)$.

Solution

$\sin t$

Alternate Forms of the Pythagorean Identity

We can use these fundamental identities to derive alternative forms of the **Pythagorean Identity**, $\cos^2 t + \sin^2 t = 1$. One form is obtained by dividing both sides by $\cos^2 t$:

$$\frac{\cos^2 t}{\cos^2 t} + \frac{\sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t} \tag{5.4.42}$$

$$1 + \tan^2 t = \sec^2 t \tag{5.4.43}$$

The other form is obtained by dividing both sides by $\sin^2 t$:

$$\frac{\cos^2 t}{\sin^2 t} + \frac{\sin^2 t}{\sin^2 t} = \frac{1}{\sin^2 t} \tag{5.4.44}$$

$$\cot^2 t + 1 = \csc^2 t \tag{5.4.45}$$

ALTERNATE FORMS OF THE PYTHAGOREAN IDENTITY

$$1 + \tan^2 t = \sec^2 t \tag{5.4.46}$$

$$\cot^2 t + 1 = \csc^2 t \tag{5.4.47}$$

Example 5.4.7: Using Identities to Relate Trigonometric Functions

If cos(t)=1213 cos(t)=1213 and t t is in quadrant IV, as shown in Figure 5.4.8, find the values of the other five trigonometric functions.

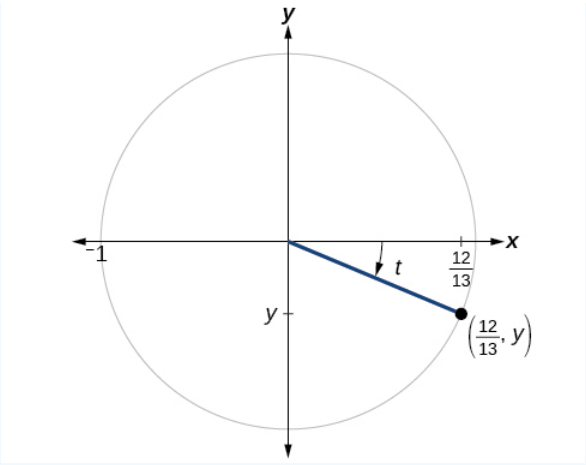


Figure 5.4.8

Solution

We can find the sine using the Pythagorean Identity, $\cos^2 t + \sin^2 t = 1$, and the remaining functions by relating them to sine and cosine.

$$(\frac{12}{13})^2 + \sin^2 t = 1 \tag{5.4.48}$$

$$\sin^2 t = 1 - (\frac{12}{13})^2 \tag{5.4.49}$$

$$\sin^2 t = 1 - \frac{144}{169} \tag{5.4.50}$$

$$\sin^2 t = \frac{25}{169} \tag{5.4.51}$$

$$\sin t = \pm\sqrt{\frac{25}{169}} \tag{5.4.52}$$

$$\sin t = \pm\frac{\sqrt{25}}{\sqrt{169}} \tag{5.4.53}$$

$$\sin t = \pm\frac{5}{13} \tag{5.4.54}$$

The sign of the sine depends on the *y*-values in the quadrant where the angle is located. Since the angle is in quadrant IV, where the *y*-values are negative, its sine is negative, $-\frac{5}{13}$.

The remaining functions can be calculated using identities relating them to sine and cosine.

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{5}{13}}{\frac{12}{13}} = -\frac{5}{12} \tag{5.4.55}$$

$$\sec t = \frac{1}{\cos t} = \frac{1}{\frac{12}{13}} = \frac{13}{12} \tag{5.4.56}$$

$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{5}{13}} = -\frac{13}{5} \tag{5.4.57}$$

$$\cot t = \frac{1}{\tan t} = \frac{1}{-\frac{5}{12}} = -\frac{12}{5} \tag{5.4.58}$$

Exercise 5.4.7:

If $\sec(t) = -\frac{17}{8}$ and $0 < t < \pi$, find the values of the other five functions.

Solution

$\cos t = -\frac{8}{17}, \sin t = \frac{15}{17}, \tan t = -\frac{15}{8}$

$\csc t = \frac{17}{15}, \cot t = -\frac{8}{15}$

As we discussed in the chapter opening, a function that repeats its values in regular intervals is known as a **periodic function**. The trigonometric functions are periodic. For the four trigonometric functions, sine, cosine, cosecant and secant, a revolution of one circle, or 2π,will result in the same outputs for these functions. And for tangent and cotangent, only a half a revolution will result in the same outputs.

Other functions can also be periodic. For example, the lengths of months repeat every four years. If x x represents the length time, measured in years, and $f(x)$ represents the number of days in February, then $f(x+4) = f(x)$. This pattern repeats over and over through time. In other words, every four years, February is guaranteed to have the same number of days as it did 4 years earlier. The positive number 4 is the smallest positive number that satisfies this condition and is called the period. A **period** is the shortest interval over which a function completes one full cycle—in this example, the period is 4 and represents the time it takes for us to be certain February has the same number of days.

PERIOD OF A FUNCTION

The **period** P of a repeating function f f is the number representing the interval such that $f(x+P) = f(x)$ for any value of x.

The period of the cosine, sine, secant, and cosecant functions is 2π.

The period of the tangent and cotangent functions is π.

Example 5.4.8: Finding the Values of Trigonometric Functions

Find the values of the six trigonometric functions of angle t based on Figure 5.4.9.

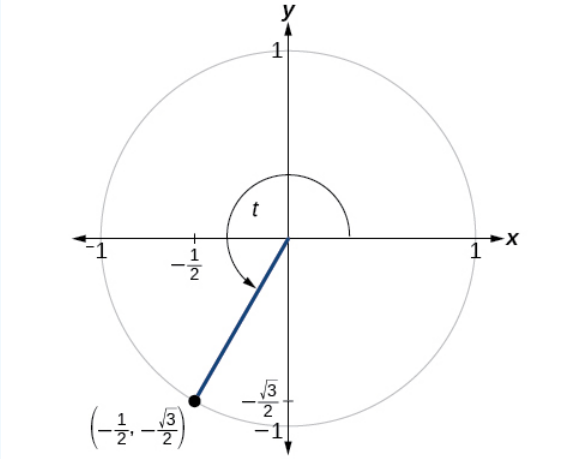


Figure 5.4.9

Solution

$$\sin t = y = -\frac{\sqrt{3}}{2}$$
$$\cos t = x = -\frac{1}{2}$$
$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3}$$
$$\sec t = \frac{1}{\cos t} = \frac{1}{-\frac{1}{2}} = -2$$
$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2\sqrt{3}}{3}$$
$$\cot t = \frac{1}{\tan t} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Exercise 5.4.8

Find the values of the six trigonometric functions of angle t based on Figure 5.4.10.

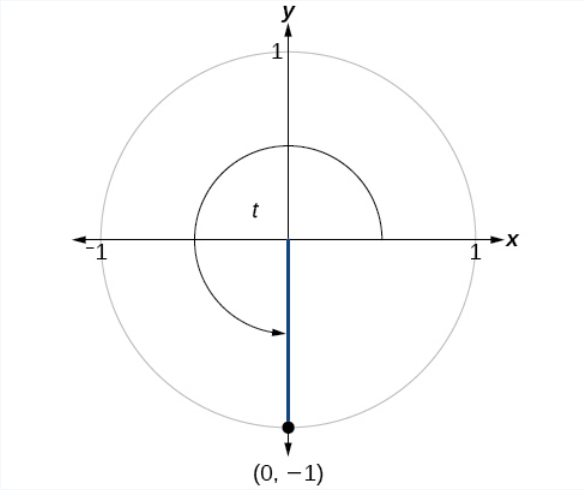


Figure 5.4.10

Solution

$$\sin t = -1, \cos t = 0, \tan t = \text{Undefined} \quad (5.4.59)$$
$$(5.4.60)$$
$$sect = \text{Undefined}, \csc t = -1, \cot t = 0 \quad (5.4.61)$$

Example 5.4.9: Finding the Value of Trigonometric Functions

If $\sin(t) = -\frac{\sqrt{3}}{2}$ and $\cos(t) = \frac{1}{2}$,find $\sec(t), \csc(t), \tan(t), \cot(t)$.

Solution

$$\sec t = \frac{1}{\cos t} = \frac{1}{\frac{1}{2}} = 2 \tag{5.4.62}$$

$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{\sqrt{3}}{2}} - \frac{2\sqrt{3}}{3} \tag{5.4.63}$$

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3} \tag{5.4.64}$$

$$\cot t = \frac{1}{\tan t} = \frac{1}{-\sqrt{3}} = -\frac{\sqrt{3}}{3} \tag{5.4.65}$$

Exercise 5.4.9:

If $\sin(t) = \frac{\sqrt{2}}{2}$ and $\cos(t) = \frac{\sqrt{2}}{2}$, find $\sec(t), \csc(t), \tan(t)$, and $\cot(t)$.

Solution

$\sec t = \sqrt{2}, \csc t = \sqrt{2}, \tan t = 1, \cot t = 1$

Evaluating Trigonometric Functions with a Calculator

We have learned how to evaluate the six trigonometric functions for the common first-quadrant angles and to use them as reference angles for angles in other quadrants. To evaluate trigonometric functions of other angles, we use a scientific or graphing calculator or computer software. If the calculator has a degree mode and a radian mode, confirm the correct mode is chosen before making a calculation.

Evaluating a tangent function with a scientific calculator as opposed to a graphing calculator or computer algebra system is like evaluating a sine or cosine: Enter the value and press the TAN key. For the reciprocal functions, there may not be any dedicated keys that say CSC, SEC, or COT. In that case, the function must be evaluated as the reciprocal of a sine, cosine, or tangent.

If we need to work with degrees and our calculator or software does not have a degree mode, we can enter the degrees multiplied by the conversion factor $\frac{\pi}{180}$ to convert the degrees to radians. To find the secant of 30°, we could press

$$(\text{for a scientific calculator}) : \frac{1}{30 \times \frac{\pi}{180}} \text{COS} \tag{5.4.66}$$

or

$$(\text{for a graphing calculator}) : \frac{1}{\cos(\frac{30\pi}{180})} \tag{5.4.67}$$

how to: Given an angle measure in radians, use a scientific calculator to find the cosecant

1. If the calculator has degree mode and radian mode, set it to radian mode.
2. Enter: 1 /
3. Enter the value of the angle inside parentheses.
4. Press the SIN key.
5. Press the = key.

how to: Given an angle measure in radians, use a graphing utility/calculator to find the cosecant

1. If the graphing utility has degree mode and radian mode, set it to radian mode.
2. Enter: 1 /
3. Press the SIN key.
4. Enter the value of the angle inside parentheses.
5. Press the ENTER key.

Example 5.4.10: Evaluating the Cosecant Using Technology

Evaluate the cosecant of $\frac{5\pi}{7}$.

Solution

For a scientific calculator, enter information as follows:

$$1/(5\times\pi/7)\text{SIN}= \tag{5.4.68}$$

$$\csc(\frac{5\pi}{7})\approx 1.279 \tag{5.4.69}$$

Exercise 5.4.10:

Evaluate the cotangent of $-\frac{\pi}{8}$.

≈ -2.414

media

Access these online resources for additional instruction and practice with other trigonometric functions.

- Determining Trig Function Values
- More Examples of Determining Trig Functions
- Pythagorean Identities
- Trig Functions on a Calculator

Key Equations

Tangent function	$\tan t = \frac{\sin t}{\cos t}$
Secant function	$\sec t = \frac{1}{\cos t}$
Cosecant function	$\csc t = \frac{1}{\sin t}$
Cotangent function	$\cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t}$

Key Concepts

- The tangent of an angle is the ratio of the *y*-value to the *x*-value of the corresponding point on the unit circle.
- The secant, cotangent, and cosecant are all reciprocals of other functions. The secant is the reciprocal of the cosine function, the cotangent is the reciprocal of the tangent function, and the cosecant is the reciprocal of the sine function.
- The six trigonometric functions can be found from a point on the unit circle. See Example.
- Trigonometric functions can also be found from an angle. See Example.
- Trigonometric functions of angles outside the first quadrant can be determined using reference angles. See Example.
- A function is said to be even if $f(-x)=f(x)$ and odd if $f(-x)=-f(x)$.
- Cosine and secant are even; sine, tangent, cosecant, and cotangent are odd.
- Even and odd properties can be used to evaluate trigonometric functions. See Example.
- The Pythagorean Identity makes it possible to find a cosine from a sine or a sine from a cosine.
- Identities can be used to evaluate trigonometric functions. See Example and Example.
- Fundamental identities such as the Pythagorean Identity can be manipulated algebraically to produce new identities. See Example.
- The trigonometric functions repeat at regular intervals.
- The period P of a repeating function f f is the smallest interval such that $f(x+P)=f(x)$ for any value of x.
- The values of trigonometric functions of special angles can be found by mathematical analysis.
- To evaluate trigonometric functions of other angles, we can use a calculator or computer software. See Example.

Glossary

cosecant
the reciprocal of the sine function: on the unit circle, $\csc t = \frac{1}{y}, y \neq 0$

cotangent
the reciprocal of the tangent function: on the unit circle, $\cot t = \frac{x}{y}, y \neq 0$

identities
statements that are true for all values of the input on which they are defined

period
the smallest interval P of a repeating function f such that $f(x+P) = f(x)$

secant
the reciprocal of the cosine function: on the unit circle, $\sec t = \frac{1}{x}, x \neq 0$

tangent
the quotient of the sine and cosine: on the unit circle, $\tan t = \frac{y}{x}, x \neq 0$

Homework Exercises 5.4

Verbal

1) On an interval of $[0, 2\pi)$, can the sine and cosine values of a radian measure ever be equal? If so, where?

Answer

Yes, when the reference angle is $\frac{\pi}{4}$ and the terminal side of the angle is in quadrants I and III. Thus, at $x = \frac{\pi}{4}, \frac{5\pi}{4}$, the sine and cosine values are equal.

2) What would you estimate the cosine of π degrees to be? Explain your reasoning.

3) For any angle in quadrant II, if you knew the sine of the angle, how could you determine the cosine of the angle?

Answer

Substitute the sine of the angle in for y in the Pythagorean Theorem $x^2 + y^2 = 1$. Solve for x and take the negative solution.

4) Describe the secant function.

5) Tangent and cotangent have a period of π. What does this tell us about the output of these functions?

Answer

The outputs of tangent and cotangent will repeat every π units.

Algebraic

For the exercises 6-17, find the exact value of each expression.

6) $\tan\frac{\pi}{6}$

7) $\sec\frac{\pi}{6}$

Answer

$\frac{2\sqrt{3}}{3}$

8) $\csc\frac{\pi}{6}$

9) $\cot\frac{\pi}{6}$

Answer

$\sqrt{3}$

10) $\tan\frac{\pi}{4}$

11) $\sec\frac{\pi}{4}$

Answer

$\sqrt{2}$

12) $\csc\frac{\pi}{4}$

13) $\cot\frac{\pi}{4}$

Answer

1

14) $\tan\frac{\pi}{3}$

15) $\sec\frac{\pi}{3}$

Answer

2

16) $\csc\frac{\pi}{3}$

17) $\cot\frac{\pi}{3}$

Answer

$\frac{\sqrt{3}}{3}$

For the exercises 18-48, use reference angles to evaluate the expression.

18) $\tan\frac{5\pi}{6}$

19) $\sec\frac{7\pi}{6}$

Answer

$-\frac{2\sqrt{3}}{3}$

20) $\csc\frac{11\pi}{6}$

21) $\cot\frac{13\pi}{6}$

Answer

$\sqrt{3}$

22) $\tan \frac{7\pi}{4}$

23) $\sec \frac{3\pi}{4}$

Answer

$-\sqrt{2}$

24) $\csc \frac{5\pi}{4}$

25) $\cot \frac{11\pi}{4}$

Answer

-1

26) $\tan \frac{8\pi}{3}$

27) $\sec \frac{4\pi}{3}$

Answer

-2

28) $\csc \frac{2\pi}{3}$

29) $\cot \frac{5\pi}{3}$

Answer

$-\frac{\sqrt{3}}{3}$

30) $\tan 225^\circ$

31) $\sec 300^\circ$

Answer

2

32) $\csc 150^\circ$

33) $\cot 240^\circ$

Answer

$\frac{\sqrt{3}}{3}$

34) $\tan 330^\circ$

35) $\sec 120^\circ$

Answer

-2

36) $\csc 210^\circ$

37) $\cot 315^\circ$

Answer

-1

38) If $\sin t = \frac{3}{4}$, and t is in quadrant II, find $\cos t, \sec t, \csc t, \tan t, \cot t$.

39) If $\cos t = -\frac{1}{3}$, and t is in quadrant III, find $\sin t, \sec t, \csc t, \tan t, \cot t$.

Answer

If $\sin t = -\frac{2\sqrt{2}}{3}, \sec t = -3, \csc t = -\csc t = -\frac{3\sqrt{2}}{4}, \tan t = 2\sqrt{2}, \cot t = \frac{\sqrt{2}}{4}$

40) If $\tan t = \frac{12}{5}$, and $0 \le t < \frac{\pi}{2}$, find $\sin t, \cos t, \sec t, \csc t$, and $\cot t$.

41) If $\sin t = \frac{\sqrt{3}}{2}$ and $\cos t = \frac{1}{2}$, find $\sec t, \csc t, \tan t$, and $\cot t$.

Answer

$\sec t = 2, \csc t = \csc t = \frac{2\sqrt{3}}{3}, \tan t = \sqrt{3}, \cot t = \frac{\sqrt{3}}{3}$

42) If $\sin 40^\circ \approx 0.643 \ \cos 40^\circ \approx 0.766 \ \sec 40^\circ, \csc 40^\circ, \tan 40^\circ,$ and $\cot 40.^\circ$

43) If $\sin t = \frac{\sqrt{2}}{2}$, what is the $\sin(-t)$?

Answer

$-\frac{\sqrt{2}}{2}$

44) If $\cos t = \frac{1}{2}$, what is the $\cos(-t)$?

45) If $\sec t = 3.1$, what is the $\sec(-t)$?

Answer

3.1

46) If $\csc t = 0.34$, what is the $\csc(-t)$?

47) If $\tan t = -1.4$, what is the $\tan(-t)$?

Answer

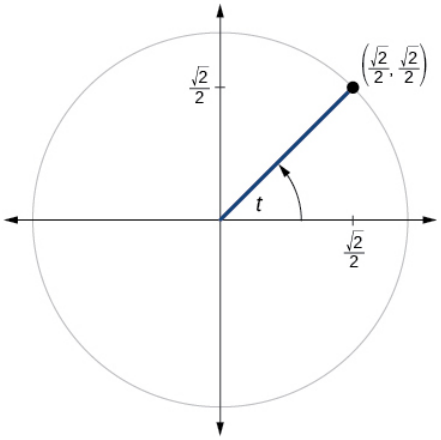
1.4

48) If $\cot t = 9.23$, what is the $\cot(-t)$?

Graphical

For the exercises 49-51, use the angle in the unit circle to find the value of the each of the six trigonometric functions.

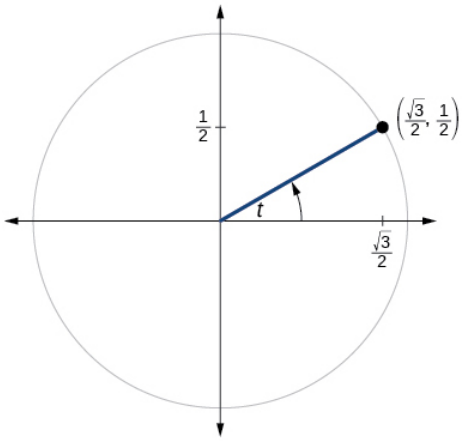
49)



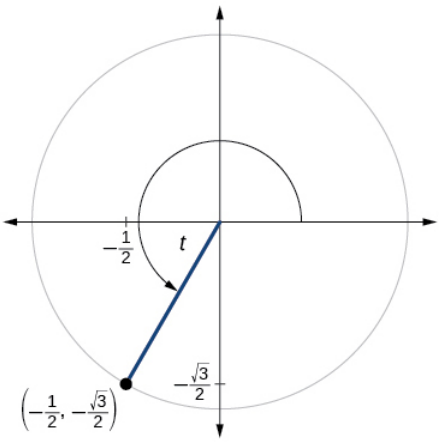
Answer

$$\sin t = \frac{\sqrt{2}}{2}, \cos t = \frac{\sqrt{2}}{2}, \tan t = 1, \cot t = 1, \sec t = \sqrt{2}, \csc t = \csc t = \sqrt{2}$$

50)



51)



Answer

$$\sin t = -\frac{\sqrt{3}}{2}, \cos t = -\frac{1}{2}, \tan t = \sqrt{3}, \cot t = \frac{\sqrt{3}}{3}, \sec t = -2, \csc t = -\csc t = -\frac{2\sqrt{3}}{3}$$

Technology

For the exercises 52-61, use a graphing calculator to evaluate.

52) $\csc \frac{5\pi}{9}$

53) $\cot \frac{4\pi}{7}$

Answer

-0.228

54) $\sec \frac{\pi}{10}$

55) $\tan \frac{5\pi}{8}$

Answer

-2.414

56) $\sec \frac{3\pi}{4}$

57) $\csc \frac{\pi}{4}$

Answer

1.414

58) $\tan 98^\circ$

59) $\cot 33^\circ$

Answer

1.540

60) $\cot 140^\circ$

61) $\sec 310^\circ$

Answer

1.556

Extensions

For the exercises 62-69, use identities to evaluate the expression.

62) If $\tan(t) \approx 2.7$, and $\sin(t) \approx 0.94$, find $\cos(t)$.

63) If $\tan(t) \approx 1.3$, and $\cos(t) \approx 0.61$, find $\sin(t)$.

Answer

$\sin(t) \approx 0.79$

64) If $\csc(t) \approx 3.2$, and $\csc(t) \approx 3.2$, and $\cos(t) \approx 0.95$, find $\tan(t)$.

65) If $\cot(t) \approx 0.58$, and $\cos(t) \approx 0.5$, find $\csc(t)$.

Answer

$\csc(t) \approx 1.16$

66) Determine whether the function $f(x) = 2\sin x \cos x$ is even, odd, or neither.

67) Determine whether the function $f(x) = 3\sin^2 x \cos x + \sec x$ is even, odd, or neither.

Answer

even

68) Determine whether the function $f(x) = \sin x - 2\cos^2 x$ is even, odd, or neither.

69) Determine whether the function $f(x) = \csc^2 x + \sec x$ is even, odd, or neither.

Answer

even

For the exercises 70-71, use identities to simplify the expression.

70) $\csc t \tan t$

71) $\dfrac{\sec t}{\csc t}$

Answer

$\dfrac{\sin t}{\cos t} = \tan t$

Real-World Applications

72) The amount of sunlight in a certain city can be modeled by the function $h = 15\cos\left(\dfrac{1}{600}d\right)$, where h represents the hours of sunlight, and d is the day of the year. Use the equation to find how many hours of sunlight there are on February 10, the 42nd day of the year. State the period of the function.

73) The amount of sunlight in a certain city can be modeled by the function $h = 16\cos\left(\dfrac{1}{500}d\right)$, where h represents the hours of sunlight, and d is the day of the year. Use the equation to find how many hours of sunlight there are on September 24, the 267th day of the year. State the period of the function.

Answer

13.77hours, period: 1000π

74) The equation $P = 20\sin(2\pi t) + 100$ models the blood pressure, P, where t represents time in seconds.

a. Find the blood pressure after 15 seconds.
b. What are the maximum and minimum blood pressures?

75) The height of a piston, h, in inches, can be modeled by the equation $y = 2\cos x + 6$, where x represents the crank angle. Find the height of the piston when the crank angle is 55°.

Answer

7.73inches

76) The height of a piston, h,in inches, can be modeled by the equation $y = 2\cos x + 5$, where x represents the crank angle. Find the height of the piston when the crank angle is 55°.

This page titled 5.4: The Other Trigonometric Functions is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **5.3: The Other Trigonometric Functions** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/precalculus.

- **5.E: Trigonometric Functions (Exercises)** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/precalculus.

5.5: Graphs of the Sine and Cosine Functions

Preview Video section 5.5

Please fill in your notes as you view the video and answer the questions.

Login with LibreOne to view this question

NOTE: If you typically access ADAPT assignments through an LMS like Canvas, you should open this page there.

Login

Learning Objectives

- Graph variations of $y = \sin(x)$ and $y = \cos(x)$.
- Use phase shifts of sine and cosine curves.

White light, such as the light from the sun, is not actually white at all. Instead, it is a composition of all the colors of the rainbow in the form of waves. The individual colors can be seen only when white light passes through an optical prism that separates the waves according to their wavelengths to form a rainbow.



Figure 5.5.1: Light can be separated into colors because of its wavelike properties. (credit: "wonderferret"/ Flickr)

Light waves can be represented graphically by the sine function. In the chapter on Trigonometric Functions, we examined trigonometric functions such as the sine function. In this section, we will interpret and create graphs of sine and cosine functions.

Graphing Sine and Cosine Functions

Recall that the sine and cosine functions relate real number values to the x- and y-coordinates of a point on the unit circle. So what do they look like on a graph on a coordinate plane? Let's start with the sine function. We can create a table of values and use them to sketch a graph. Table 5.5.1 lists some of the values for the sine function on a unit circle.

Table 5.5.1

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Plotting the points from the table and continuing along the *x*-axis gives the shape of the sine function. See Figure 5.5.2.

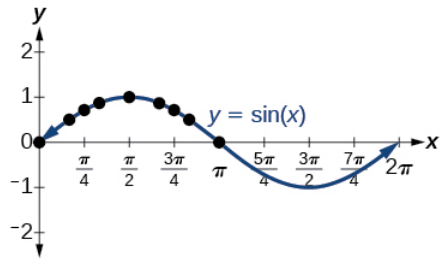


Figure 5.5.2: The sine function

Notice how the sine values are positive between 0 and π, which correspond to the values of the sine function in quadrants I and II on the unit circle, and the sine values are negative between π and 2π, which correspond to the values of the sine function in quadrants III and IV on the unit circle. See Figure 5.5.3.

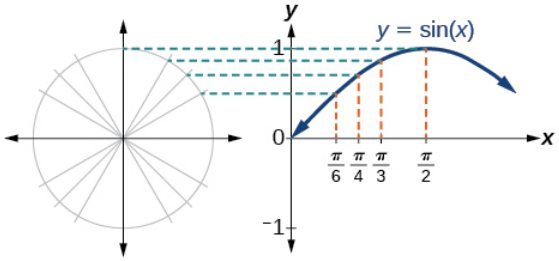


Figure 5.5.3: Plotting values of the sine function

Now let's take a similar look at the **cosine function**. Again, we can create a table of values and use them to sketch a graph. Table 5.5.2 lists some of the values for the cosine function on a unit circle.

Table 5.5.2

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

As with the sine function, we can plots points to create a graph of the cosine function as in Figure 5.5.4.

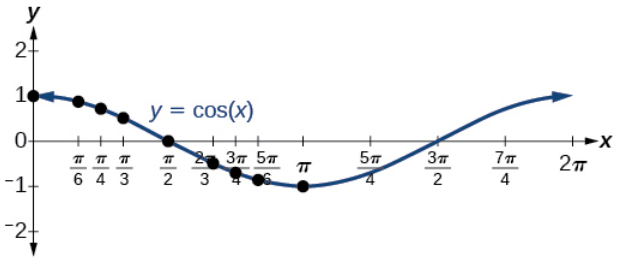


Figure 5.5.4: The cosine function

Because we can evaluate the sine and cosine of any real number, both of these functions are defined for all real numbers. By thinking of the sine and cosine values as coordinates of points on a unit circle, it becomes clear that the range of both functions must be the interval $[-1, 1]$.

In both graphs, the shape of the graph repeats after 2π, which means the functions are periodic with a period of 2π. A **periodic function** is a function for which a specific **horizontal shift**, P, results in a function equal to the original function:

$f(x+P)=f(x)$ for all values of x in the domain of f. When this occurs, we call the smallest such horizontal shift with $P>0$ the **period** of the function. Figure $5.5.5$ shows several periods of the sine and cosine functions.

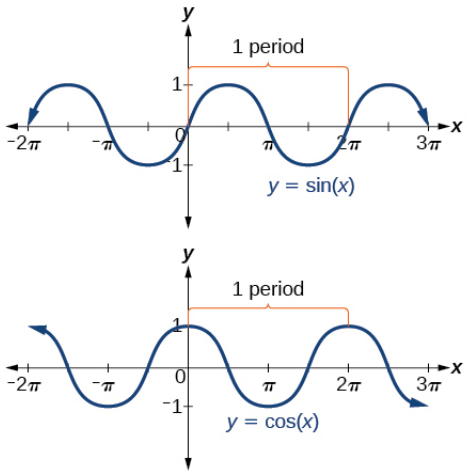


Figure 5.5.5

Looking again at the sine and cosine functions on a domain centered at the y-axis helps reveal symmetries. As we can see in Figure $5.5.6$, the sine function is symmetric about the origin. Recall from The Other Trigonometric Functions that we determined from the unit circle that the sine function is an odd function because $\sin(-x)=-\sin\ x$. Now we can clearly see this property from the graph.

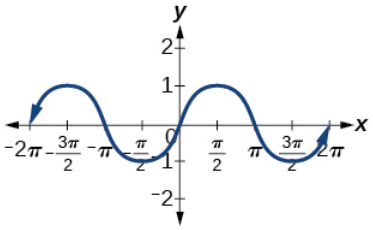


Figure 5.5.6: Odd symmetry of the sine function

Figure $5.5.7$ shows that the cosine function is symmetric about the y-axis. Again, we determined that the cosine function is an even function. Now we can see from the graph that $\cos(-x)=\cos\ x$.

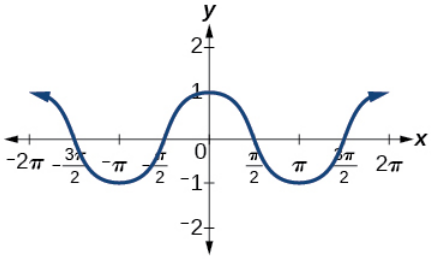


Figure 5.5.7: Even symmetry of the cosine function

CHARACTERISTICS OF SINE AND COSINE FUNCTIONS

The sine and cosine functions have several distinct characteristics:

- They are periodic functions with a period of 2π.
- The domain of each function is $(-\infty,\infty)$ and the range is $[-1,1]$.
- The graph of $y=\sin\ x$ is symmetric about the origin, because it is an odd function.
- The graph of $y=\cos\ x$ is symmetric about they- y-axis, because it is an even function.

Investigating Sinusoidal Functions

As we can see, sine and cosine functions have a regular period and range. If we watch ocean waves or ripples on a pond, we will see that they resemble the sine or cosine functions. However, they are not necessarily identical. Some are taller or longer than others. A function that has the same general shape as a sine or **cosine function** is known as a **sinusoidal function**. The general forms of sinusoidal functions are

$$y = A\sin(Bx - C) + D \tag{5.5.1}$$

and

$$y = A\cos(Bx - C) + D \tag{5.5.2}$$

Determining the Period of Sinusoidal Functions

Looking at the forms of sinusoidal functions, we can see that they are transformations of the sine and cosine functions. We can use what we know about transformations to determine the period.

In the general formula, B is related to the period by $P = \dfrac{2\pi}{|B|}$. If $|B| > 1$, then the period is less than 2π and the function undergoes a horizontal compression, whereas if $|B| < 1$, then the period is greater than 2π and the function undergoes a horizontal stretch. For example, $f(x) = \sin(x)$, $B = 1$, so the period is 2π,which we knew. If $f(x) = \sin(2x)$, then $B = 2$, so the period is π and the graph is compressed. If $f(x) = \sin\left(\dfrac{x}{2}\right)$, then $B = \dfrac{1}{2}$, so the period is 4π and the graph is stretched. Notice in Figure 5.5.8 how the period is indirectly related to $|B|$.

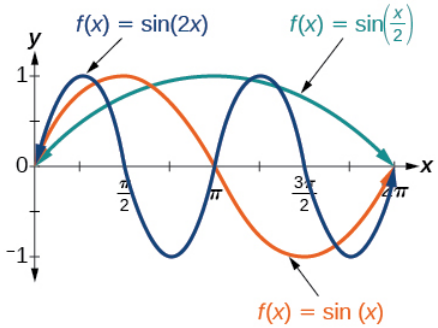


Figure 5.5.8

PERIOD OF SINUSOIDAL FUNCTIONS

If we let $C = 0$ and $D = 0$ in the general form equations of the sine and cosine functions, we obtain the forms

- $y = A\sin(Bx)$
- $y = A\cos(Bx)$

The period is $\dfrac{2\pi}{|B|}$.

Example 5.5.1: Identifying the Period of a Sine or Cosine Function

Determine the period of the function $f(x) = \sin\left(\dfrac{\pi}{6}x\right)$.

Solution

Let's begin by comparing the equation to the general form $y = A\sin(Bx)$.

In the given equation, $B = \dfrac{\pi}{6}$, so the period will be

$$P = \frac{2\pi}{|B|}$$
$$= \frac{2\pi}{\frac{\pi}{6}}$$
$$= 2\pi \cdot \frac{6}{\pi}$$
$$= 12$$

? Exercise 5.5.1

Determine the period of the function $g(x) = \cos(\frac{x}{3})$.

Answer

6π

Determining Amplitude

Returning to the general formula for a sinusoidal function, we have analyzed how the variable B relates to the period. Now let's turn to the variable A so we can analyze how it is related to the **amplitude**, or greatest distance from rest. A represents the vertical stretch factor, and its absolute value $|A|$ is the amplitude. The local maxima will be a distance $|A|$ above the vertical **midline** of the graph, which is the line $x = D$; because $D = 0$ in this case, the midline is the *x*-axis. The local minima will be the same distance below the midline. If $|A| > 1$, the function is stretched. For example, the amplitude of $f(x) = 4sinx$ is twice the amplitude of

$$f(x) = 2\sin x$$

If $|A| < 1$, the function is compressed. Figure 5.5.9 compares several sine functions with different amplitudes.

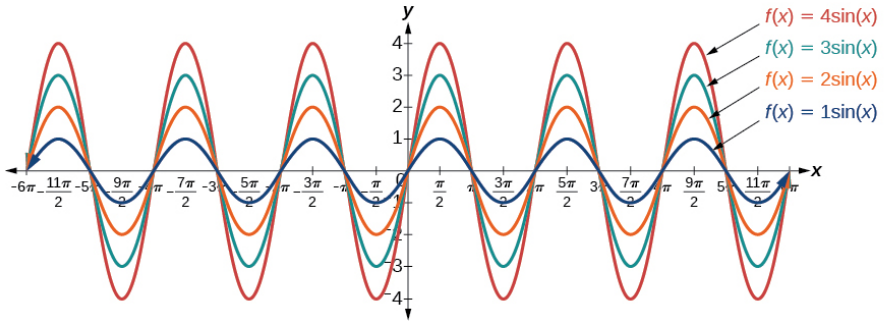


Figure 5.5.9

AMPLITUDE OF SINUSOIDAL FUNCTIONS

If we let $C = 0$ and $D = 0$ in the general form equations of the sine and cosine functions, we obtain the forms

$$y = A\sin(Bx) \text{ and } y = A\cos(Bx) \tag{5.5.3}$$

The amplitude is A, and the vertical height from the midline is $|A|$. In addition, notice in the example that

$$|A| = amplitude = \frac{1}{2}\,|\,maximum - minimum| \tag{5.5.4}$$

✓ Example 5.5.2: Identifying the Amplitude of a Sine or Cosine Function

What is the amplitude of the sinusoidal function $f(x) = -4\sin(x)$? Is the function stretched or compressed vertically?

Solution

Let's begin by comparing the function to the simplified form $y = A\sin(Bx)$.

In the given function, $A = -4$, so the amplitude is $|A| = |-4| = 4$. The function is stretched.

Analysis

The negative value of A results in a reflection across the x-axis of the sine function, as shown in Figure 5.5.10.

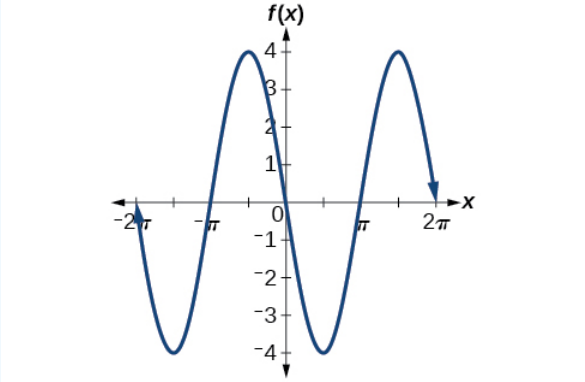


Figure 5.5.10

Exercise 5.5.2

What is the amplitude of the sinusoidal function $f(x) = \frac{1}{2}\sin(x)$? Is the function stretched or compressed vertically?

Answer

$\frac{1}{2}$ compressed

Analyzing Graphs of Variations of $y = \sin\ x$ and $y = \cos\ x$

Now that we understand how A and B relate to the general form equation for the sine and cosine functions, we will explore the variables C and D. Recall the general form:

$$y = A\sin(Bx - C) + D \quad \text{and} \quad y = A\cos(Bx - C) + D \tag{5.5.5}$$

or

$$y = A\sin\left(B\left(x - \frac{C}{B}\right)\right) + D \quad \text{and} \quad y = A\cos\left(B\left(x - \frac{C}{B}\right)\right) + D \tag{5.5.6}$$

The value $\frac{C}{B}$ for a sinusoidal function is called the **phase shift**, or the horizontal displacement of the basic sine or cosine function. If $C > 0$, the graph shifts to the right. If $C < 0$, the graph shifts to the left. The greater the value of $|C|$, the more the graph is shifted. Figure 5.5.11 shows that the graph of $f(x) = \sin(x - \pi)$ shifts to the right by π units, which is more than we see in the graph of $f(x) = \sin\left(x - \frac{\pi}{4}\right)$, which shifts to the right by $\frac{\pi}{4}$ units.

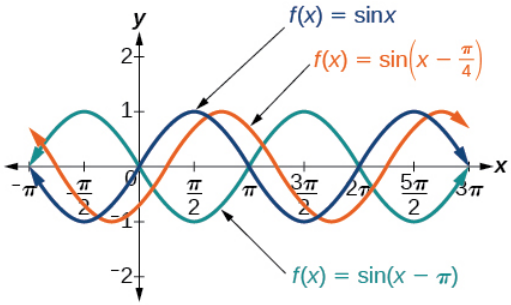


Figure 5.5.11

While C relates to the horizontal shift, D indicates the vertical shift from the midline in the general formula for a sinusoidal function. See Figure 5.5.12. The function $y = \cos(x) + D$ has its midline at $y = D$.

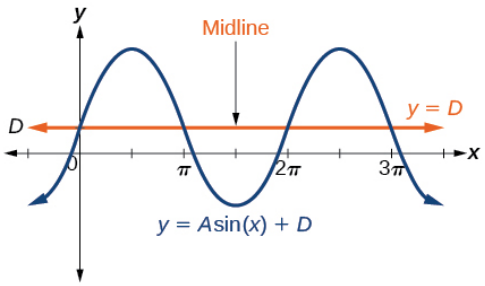


Figure 5.5.12

Any value of D other than zero shifts the graph up or down. Figure 5.5.13 compares $f(x) = \sin x$ with $f(x) = \sin x + 2$, which is shifted 2 units up on a graph.

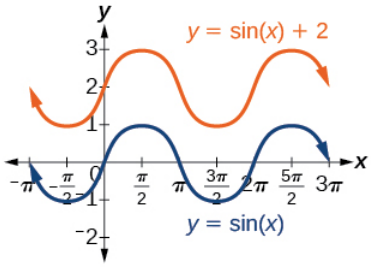


Figure 5.5.13

VARIATIONS OF SINE AND COSINE FUNCTIONS

Given an equation in the form $f(x) = A\sin(Bx - C) + D$ or $f(x) = A\cos(Bx - C) + D$, $\frac{C}{B}$ is the **phase shift** and D is the **vertical shift.**

Example 5.5.3: Identifying the Phase Shift of a Function

Determine the direction and magnitude of the phase shift for $f(x) = \sin\left(x + \frac{\pi}{6}\right) - 2$.

Solution

Let's begin by comparing the equation to the general form $y = A\sin(Bx - C) + D$.

In the given equation, notice that $B = 1$ and $C = -\frac{\pi}{6}$. So the phase shift is

$$\begin{aligned} \frac{C}{B} &= -\frac{\frac{\pi}{6}}{1} \\ &= -\frac{\pi}{6} \end{aligned}$$

or $\frac{\pi}{6}$ units to the left.

Analysis

We must pay attention to the sign in the equation for the general form of a sinusoidal function. The equation shows a minus sign before C. Therefore $f(x) = \sin(x + \frac{\pi}{6}) - 2$ can be rewritten as $f(x) = \sin\left(x - \left(-\frac{\pi}{6}\right)\right) - 2$. If the value of C is negative, the shift is to the left.

Exercise 5.5.3

Determine the direction and magnitude of the phase shift for $f(x) = 3\cos\left(x - \frac{\pi}{2}\right)$.

Answer

$\frac{\pi}{2}$; right

Example 5.5.4: Identifying the Vertical Shift of a Function

Determine the direction and magnitude of the vertical shift for $f(x) = \cos(x) - 3$.

Solution

Let's begin by comparing the equation to the general form $y = A\cos(Bx - C) + D$.

In the given equation, $D = -3$ so the shift is 3 units downward.

Exercise 5.5.4

Determine the direction and magnitude of the vertical shift for $f(x) = 3\sin(x) + 2$.

Answer

2 units up

How to: Given a sinusoidal function in the form $f(x) = A\sin(Bx - C) + D$,identify the midline, amplitude, period, and phase shift

1. Determine the amplitude as $|A|$.
2. Determine the period as $P = \frac{2\pi}{|B|}$.
3. Determine the phase shift as $\frac{C}{B}$.
4. Determine the midline as $y = D$.

Example 5.5.5: Identifying the Variations of a Sinusoidal Function from an Equation

Determine the midline, amplitude, period, and phase shift of the function $y = 3\sin(2x) + 1$.

Solution

Let's begin by comparing the equation to the general form $y = A\sin(Bx - C) + D$.

$A = 3$, so the amplitude is $|A| = 3$.

Next, $B = 2$, so the period is $P = \dfrac{2\pi}{|B|} = \dfrac{2\pi}{2} = \pi$.

There is no added constant inside the parentheses, so $C = 0$ and the phase shift is $\dfrac{C}{B} = \dfrac{0}{2} = 0$.

Finally, $D = 1$, so the midline is $y = 1$.

Analysis

Inspecting the graph, we can determine that the period is π, the midline is $y = 1$, and the amplitude is 3. See Figure 5.5.14

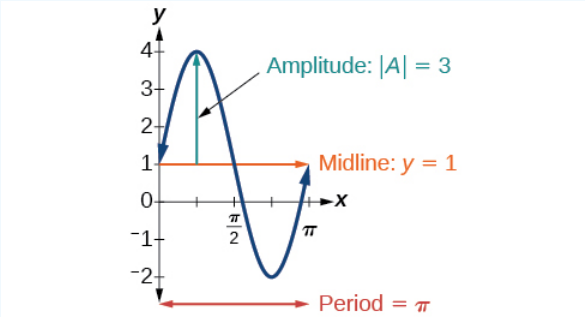


Figure 5.5.14

https://math.libretexts.org/@go/page/106365

Exercise 5.5.5

Determine the midline, amplitude, period, and phase shift of the function $y = \frac{1}{2}\cos\left(\frac{x}{3} - \frac{\pi}{3}\right)$.

Answer

midline: $y = 0$; amplitude: $|A| = \frac{1}{2}$; period: $P = \frac{2\pi}{|B|} = 6\pi$; phase shift: $\frac{C}{B} = \pi$

Example 5.5.6: Identifying the Equation for a Sinusoidal Function from a Graph

Determine the formula for the cosine function in Figure 5.5.15.

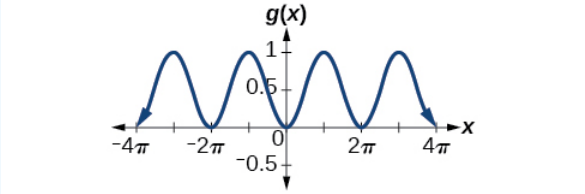


Figure 5.5.15

Solution

To determine the equation, we need to identify each value in the general form of a sinusoidal function.

$$y = A\sin(Bx - C) + D$$

$$y = A\cos(Bx - C) + D$$

The graph could represent either a sine or a **cosine function** that is shifted and/or reflected. When $x = 0$, the graph has an extreme point, $(0, 0)$. Since the cosine function has an extreme point for $x = 0$, let us write our equation in terms of a cosine function.

Let's start with the midline. We can see that the graph rises and falls an equal distance above and below $y = 0.5$. This value, which is the midline, is D in the equation, so $D = 0.5$.

The greatest distance above and below the midline is the amplitude. The maxima are 0.5 units above the midline and the minima are 0.5 units below the midline. So $|A| = 0.5$. Another way we could have determined the amplitude is by recognizing that the difference between the height of local maxima and minima is 1, so $|A| = \frac{1}{2} = 0.5$. Also, the graph is reflected about the x-axis so that $A = -0.5$.

The graph is not horizontally stretched or compressed, so $B = 1$; and the graph is not shifted horizontally, so $C = 0$.

Putting this all together,

$$g(x) = -0.5\cos(x) + 0.5$$

Exercise 5.5.6

Determine the formula for the sine function in Figure 5.5.16.

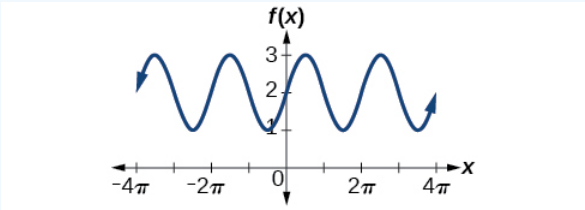


Figure 5.5.16

Answer

$f(x) = \sin(x) + 2$

Example 5.5.7: Identifying the Equation for a Sinusoidal Function from a Graph

Determine the equation for the sinusoidal function in Figure 5.5.17.

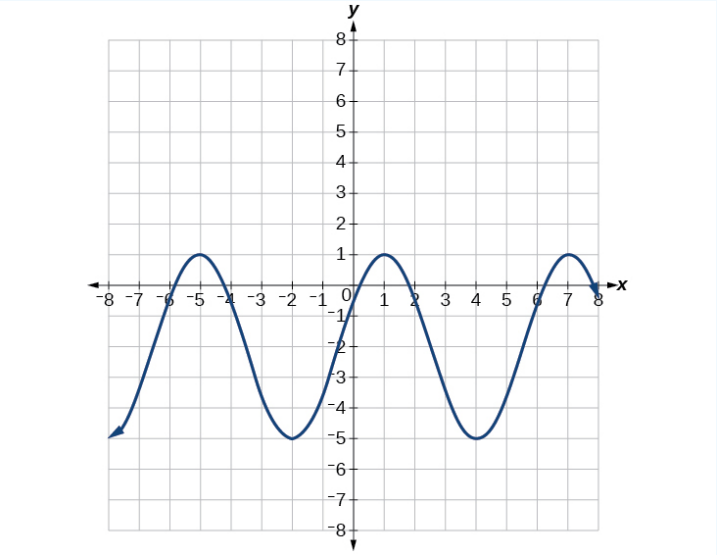


Figure 5.5.17

Solution

With the highest value at 1 and the lowest value at -5, the midline will be halfway between at -2. So $D=-2$.

The distance from the midline to the highest or lowest value gives an amplitude of $|A|=3$.

The period of the graph is 6, which can be measured from the peak at $x=1$ to the next peak at $x=7$,or from the distance between the lowest points. Therefore, $P=\dfrac{2\pi}{|B|}=6$. Using the positive value for B,we find that

$$\begin{aligned} B &= \frac{2\pi}{P} \\ &= \frac{2\pi}{6} \\ &= \frac{\pi}{3} \end{aligned}$$

So far, our equation is either $y=3\sin\left(\frac{\pi}{3}x-C\right)-2$ or $y=3\cos\left(\frac{\pi}{3}x-C\right)-2$.For the shape and shift, we have more than one option. We could write this as any one of the following:

- a cosine shifted to the right
- a negative cosine shifted to the left
- a sine shifted to the left
- a negative sine shifted to the right

While any of these would be correct, the cosine shifts are easier to work with than the sine shifts in this case because they involve integer values. So our function becomes

$$\begin{aligned} y &= 3\cos\left(\frac{\pi}{3}x-\frac{\pi}{3}\right)-2 \qquad \text{or} \\ y &= -3\cos\left(\frac{\pi}{3}x+\frac{2\pi}{3}\right)-2 \end{aligned}$$

Again, these functions are equivalent, so both yield the same graph.

Exercise 5.5.7

Write a formula for the function graphed in Figure 5.5.18

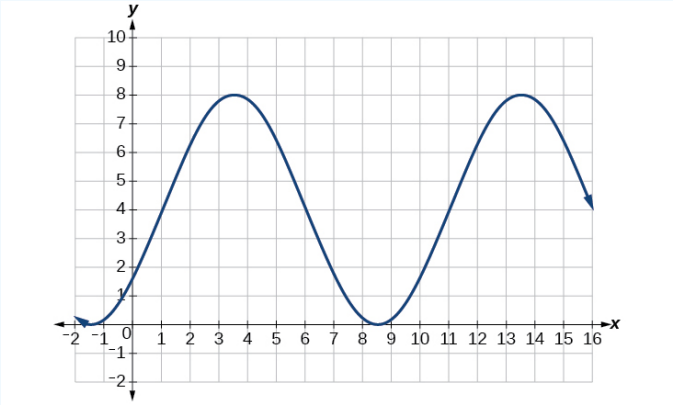


Figure 5.5.18

Answer

two possibilities: $y = 4\sin\left(\frac{\pi}{5}x - \frac{\pi}{5}\right) + 4$ or $y = -4\sin\left(\frac{\pi}{5}x + \frac{4\pi}{5}\right) + 4$

Graphing Variations of $y = \sin\ x$ and $y = \cos\ x$

Throughout this section, we have learned about types of variations of sine and cosine functions and used that information to write equations from graphs. Now we can use the same information to create graphs from equations.

Instead of focusing on the general form equations

$$y = A\sin(Bx - C) + D \text{ and } y = A\cos(Bx - C) + D$$

we will let $C = 0$ and $D = 0$ and work with a simplified form of the equations in the following examples.

Given the function $y = A\sin(Bx)$, sketch its graph.

1. Identify the amplitude, $|A|$.
2. Identify the period, $P = \frac{2\pi}{|B|}$.
3. Start at the origin, with the function increasing to the right if A is positive or decreasing if A is negative.
4. At $x = \frac{\pi}{2|B|}$ there is a local maximum for $A > 0$ or a minimum for $A < 0$, with $y = A$.
5. The curve returns to the *x*-axis at $x = \frac{\pi}{|B|}$.
6. There is a local minimum for $A > 0$ (maximum for $A < 0$) at $x = \frac{3\pi}{2|B|}$ with $y = -A$.
7. The curve returns again to the *x*-axis at $x = \frac{\pi}{2|B|}$.

Example 5.5.8: Graphing a Function and Identifying the Amplitude and Period

Sketch a graph of $f(x) = -2\sin\left(\frac{\pi x}{2}\right)$.

Solution

Let's begin by comparing the equation to the form $y = A\sin(Bx)$.

- *Step 1.* We can see from the equation that $A = -2$, so the amplitude is 2.

  $|A| = 2$

- *Step 2.* The equation shows that $B = \frac{\pi}{2}$, so the period is

$$\begin{aligned} P &= \frac{2\pi}{\frac{\pi}{2}} \\ &= 2\pi \cdot \frac{2}{\pi} \\ &= 4 \end{aligned}$$

- *Step 3.* Because A is negative, the graph descends as we move to the right of the origin.
- *Step 4.* The x-intercepts are at the beginning of one period, $x = 0$, the horizontal midpoints are at $x = 2$ and at the end of one period at $x = 4$.

The quarter points include the minimum at $x = 1$ and the maximum at $x = 3$. A local minimum will occur 2 units below the midline, at $x = 1$, and a local maximum will occur at 2 units above the midline, at $x = 3$. Figure 5.5.19 shows the graph of the function.

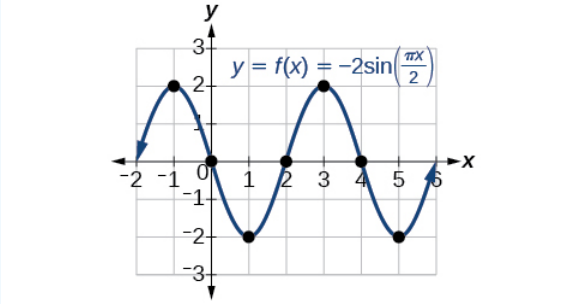


Figure 5.5.19

? Exercise 5.5.8

Sketch a graph of $g(x) = -0.8\cos(2x)$. Determine the midline, amplitude, period, and phase shift.

Answer

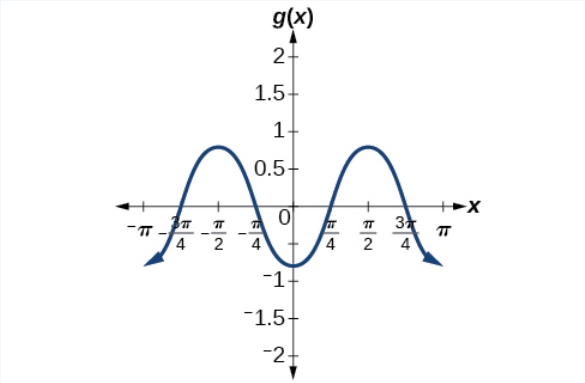


Figure 5.5.20

midline: $y = 0$; amplitude: $|A| = 0.8$; period: $P = \frac{2\pi}{|B|} = \pi$; phase shift: $\frac{C}{B} = 0$ or none

How to: Given a sinusoidal function with a phase shift and a vertical shift, sketch its graph

1. Express the function in the general form $y = A\sin(Bx - C) + D$ or $y = A\cos(Bx - C) + D$.
2. Identify the amplitude, $|A|$.
3. Identify the period, $P = \frac{2\pi}{|B|}$.

4. Identify the phase shift, $\frac{C}{B}$.

5. Draw the graph of $f(x) = A\sin(Bx)$ shifted to the right or left by $\frac{C}{B}$ and up or down by D.

Example 5.5.9: Graphing a Transformed Sinusoid

Sketch a graph of $f(x) = 3\sin\left(\frac{\pi}{4}x - \frac{\pi}{4}\right)$.

Solution

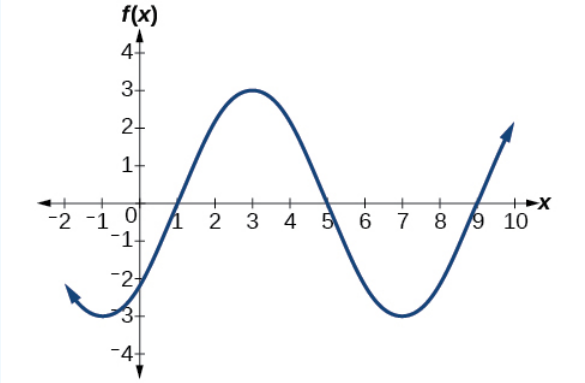


Figure 5.5.21: A horizontally compressed, vertically stretched, and horizontally shifted sinusoid

- *Step 1.* The function is already written in general form: $f(x) = 3\sin\left(\frac{\pi}{4}x - \frac{\pi}{4}\right)$.This graph will have the shape of a sine function, starting at the midline and increasing to the right.
- *Step 2.* $|A| = |3| = 3$. The amplitude is 3.
- *Step 3.* Since $|B| = |\frac{\pi}{4}| = \frac{\pi}{4}$, we determine the period as follows.

$$\begin{aligned} P &= \frac{2\pi}{|B|} \\ &= \frac{2\pi}{\frac{\pi}{4}} \\ &= 2\pi \cdot \frac{4}{\pi} \\ &= 8 \end{aligned}$$

The period is 8.

- *Step 4.* Since $C = \frac{\pi}{4}$, the phase shift is

$$\frac{C}{B} = \frac{\frac{\pi}{4}}{\frac{\pi}{4}} = 1 \tag{5.5.7}$$

.

The phase shift is 1 unit.

- *Step 5.* Figure 5.5.21 shows the graph of the function.

Exercise 5.5.9

Draw a graph of $g(x) = -2\cos\left(\frac{\pi}{3}x + \frac{\pi}{6}\right)$. Determine the midline, amplitude, period, and phase shift.

Answer

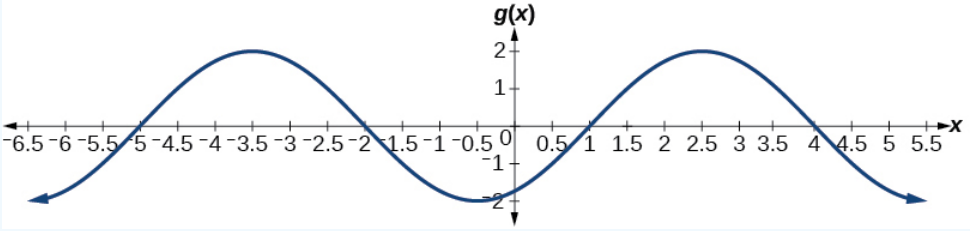


Figure 5.5.22

midline: $y = 0$; amplitude: $|A| = 2$; period: $P = \dfrac{2\pi}{|B|} = 6$; phase shift: $\dfrac{C}{B} = -\dfrac{1}{2}$

Example 5.5.10: Identifying the Properties of a Sinusoidal Function

Given $y = -2cos\left(\dfrac{\pi}{2}x + \pi\right) + 3$, determine the amplitude, period, phase shift, and horizontal shift. Then graph the function.

Solution

Begin by comparing the equation to the general form.

$$y = A\cos(Bx - C) + D$$

- *Step 1.* The function is already written in general form.
- *Step 2.* Since $A = -2$, the amplitude is $|A| = 2$.
- *Step 3.* $|B| = \dfrac{\pi}{2}$, so the period is $P = \dfrac{2\pi}{|B|} = \dfrac{2\pi}{\frac{pi}{2}} = 2\pi \cdot \dfrac{2}{\pi} = 4$. The period is 4.
- *Step 4.* $C = -\pi$,so we calculate the phase shift as $\dfrac{C}{B} = -\dfrac{\pi}{\frac{\pi}{2}} = -\pi \cdot \dfrac{2}{\pi} = -2$. The phase shift is -2.
- *Step 5.* $D = 3$,so the midline is $y = 3$, and the vertical shift is up 3.

Since A is negative, the graph of the cosine function has been reflected about the x-axis.

Figure 5.5.23shows one cycle of the graph of the function.

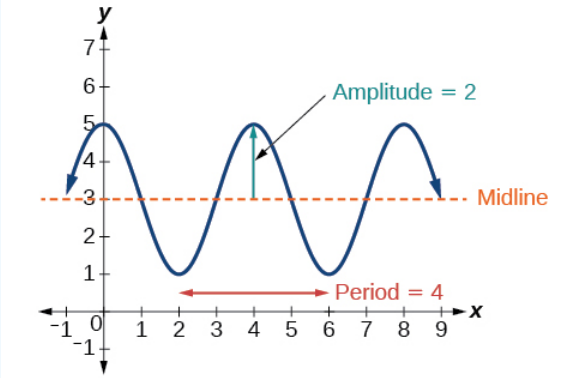


Figure 5.5.23

Using Transformations of Sine and Cosine Functions

We can use the transformations of sine and cosine functions in numerous applications. As mentioned at the beginning of the chapter,**circular motion** can be modeled using either the sine or **cosine function**.

Example 5.5.11: Finding the Vertical Component of Circular Motion

A point rotates around a circle of radius 3 centered at the origin. Sketch a graph of the y-coordinate of the point as a function of the angle of rotation.

Solution

Recall that, for a point on a circle of radius r, the y-coordinate of the point is $y = r\sin(x)$, so in this case, we get the equation $y(x) = 3\sin(x)$. The constant 3 causes a vertical stretch of the y-values of the function by a factor of 3, which we can see in the graph in Figure 5.5.24

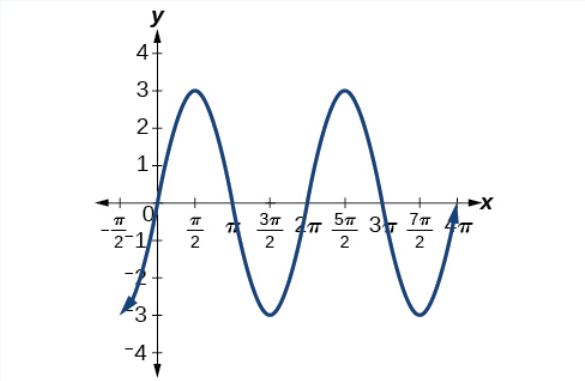


Figure 5.5.24

Analysis

Notice that the period of the function is still 2π; as we travel around the circle, we return to the point $(3, 0)$ for $x = 2\pi, 4\pi, 6\pi$,...Because the outputs of the graph will now oscillate between -3 and 3, the amplitude of the sine wave is 3.

Exercise 5.5.10

What is the amplitude of the function $f(x) = 7\cos(x)$? Sketch a graph of this function.

Answer

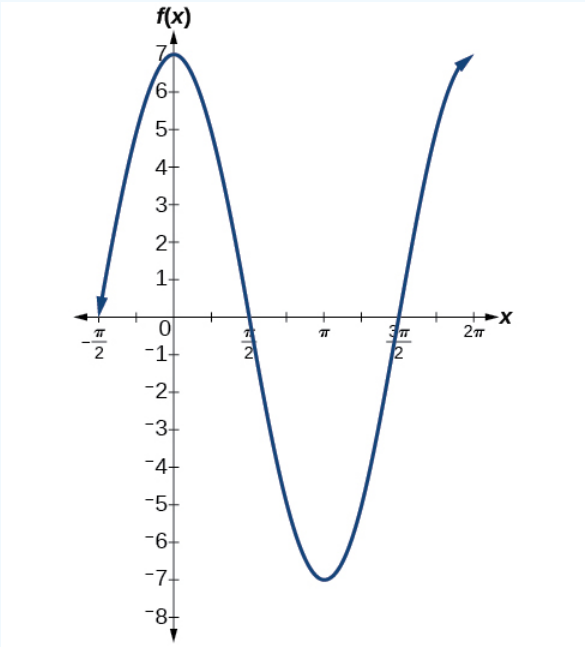


Figure 5.5.25

https://math.libretexts.org/@go/page/106365

Example 5.5.12: Finding the Vertical Component of Circular Motion

A circle with radius 3 ft is mounted with its center 4 ft off the ground. The point closest to the ground is labeled P, as shown in Figure 5.5.26. Sketch a graph of the height above the ground of the point P as the circle is rotated; then find a function that gives the height in terms of the angle of rotation.

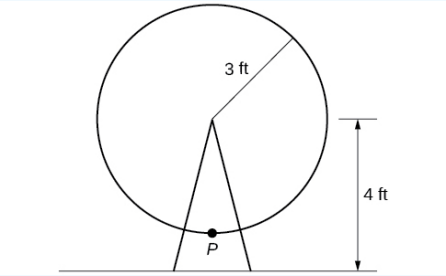


Figure 5.5.26

Solution

Sketching the height, we note that it will start 1 ft above the ground, then increase up to 7 ft above the ground, and continue to oscillate 3 ft above and below the center value of 4 ft, as shown in Figure 5.5.27.

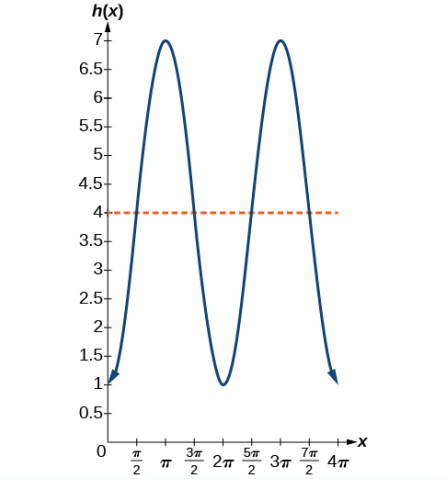


Figure 5.5.27

Although we could use a transformation of either the sine or cosine function, we start by looking for characteristics that would make one function easier to use than the other. Let's use a cosine function because it starts at the highest or lowest value, while a **sine function** starts at the middle value. A standard cosine starts at the highest value, and this graph starts at the lowest value, so we need to incorporate a vertical reflection.

Second, we see that the graph oscillates 3 above and below the center, while a basic cosine has an amplitude of 1, so this graph has been vertically stretched by 3, as in the last example.

Finally, to move the center of the circle up to a height of 4, the graph has been vertically shifted up by 4. Putting these transformations together, we find that

$$y = -3\cos(x) + 4$$

Exercise 5.5.11

A weight is attached to a spring that is then hung from a board, as shown in Figure 5.5.28. As the spring oscillates up and down, the position y of the weight relative to the board ranges from -1 in. (at time $x = 0$) to -7 in. (at time $x = \pi$) below the board. Assume the position of y is given as a sinusoidal function of x. Sketch a graph of the function, and then find a cosine function that gives the position y in terms of x.

https://math.libretexts.org/@go/page/106365

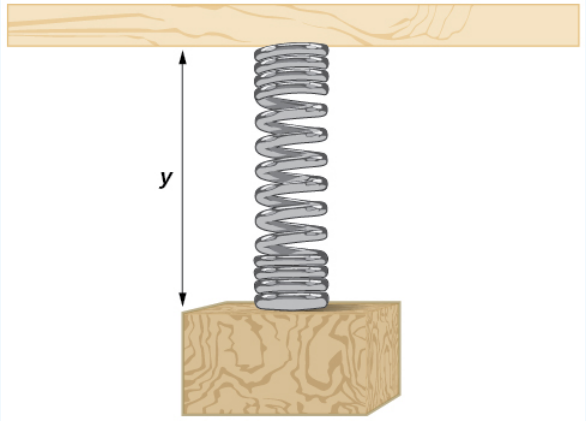


Figure 5.5.28

Answer

$y = 3\cos(x) - 4$

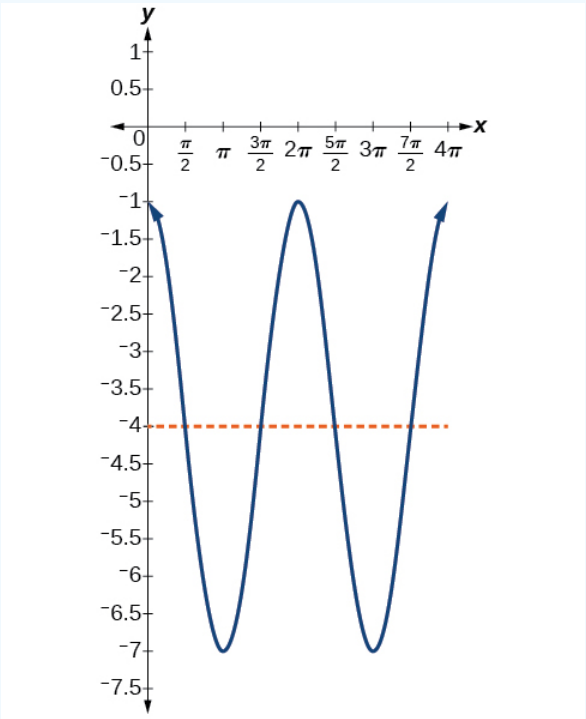


Figure 5.5.29

Example 5.5.13: Determining a Rider's Height on a Ferris Wheel

The London Eye is a huge Ferris wheel with a diameter of 135 meters (443 feet). It completes one rotation every 30 minutes. Riders board from a platform 2 meters above the ground. Express a rider's height above ground as a function of time in minutes.

Solution

With a diameter of 135 m, the wheel has a radius of 67.5 m. The height will oscillate with amplitude 67.5 m above and below the center.

Passengers board 2 m above ground level, so the center of the wheel must be located $67.5 + 2 = 69.5$ m above ground level. The midline of the oscillation will be at 69.5 m.

The wheel takes 30 minutes to complete 1 revolution, so the height will oscillate with a period of 30 minutes.

Lastly, because the rider boards at the lowest point, the height will start at the smallest value and increase, following the shape of a vertically reflected cosine curve.

- Amplitude: 67.5, so $A = 67.5$
- Midline: 69.5, so $D = 69.5$
- Period:30, so $B = \dfrac{2\pi}{30} = \dfrac{\pi}{15}$
- Shape: $-\cos(t)$

An equation for the rider's height would be

$$y = -67.5\cos\left(\frac{\pi}{15}t\right) + 69.5$$

where t is in minutes and y is measured in meters.

Media

Access these online resources for additional instruction and practice with graphs of sine and cosine functions.

- Amplitude and Period of Sine and Cosine
- Translations of Sine and Cosine
- Graphing Sine and Cosine Transformations
- Graphing the Sine Function

Key Equations

Sinusoidal functions	$f(x) = A\sin(Bx - C) + D$ $f(x) = A\cos(Bx - C) + D$

Key Concepts

- Periodic functions repeat after a given value. The smallest such value is the period. The basic sine and cosine functions have a period of 2π.
- The function $\sin x$ is odd, so its graph is symmetric about the origin. The function $\cos x$ is even, so its graph is symmetric about the *y*-axis.
- The graph of a sinusoidal function has the same general shape as a sine or cosine function.
- In the general formula for a sinusoidal function, the period is $P = \dfrac{2\pi}{|B|}$. See Example $5.5.1$.
- In the general formula for a sinusoidal function, $|A|$ represents amplitude. If $|A| > 1$, the function is stretched, whereas if $|A| < 1$, the function is compressed. See Example $5.5.2$.
- The value $\dfrac{C}{B}$ in the general formula for a sinusoidal function indicates the phase shift. See Example $5.5.3$.
- The value D in the general formula for a sinusoidal function indicates the vertical shift from the midline. See Example $5.5.4$.
- Combinations of variations of sinusoidal functions can be detected from an equation. See Example $5.5.5$.
- The equation for a sinusoidal function can be determined from a graph. See Example $5.5.6$ and Example $5.5.7$.
- A function can be graphed by identifying its amplitude and period. See Example $5.5.8$ and Example $5.5.9$.
- A function can also be graphed by identifying its amplitude, period, phase shift, and horizontal shift. See Example $5.5.10$.
- Sinusoidal functions can be used to solve real-world problems. See Example $5.5.11$, Example $5.5.12$, and Example $5.5.13$.

Homework Exercises 5.5

WeBWorK Problems:

none

Written Problems:

Please do #28 instead of #30 from the assignment.

In the chapter on Trigonometric Functions, we examined trigonometric functions such as the sine function. In this section, we will interpret and create graphs of sine and cosine functions

Verbal

1) Why are the sine and cosine functions called periodic functions?

Answer

The sine and cosine functions have the property that $f(x+P)=f(x)$ for a certain P. This means that the function values repeat for every P units on the x-axis.

2) How does the graph of $y=\sin x$ compare with the graph of $y=\cos x$? Explain how you could horizontally translate the graph of $y=\sin x$ to obtain $y=\cos x$.

3) For the equation $A\cos(Bx+C)+D$, what constants affect the range of the function and how do they affect the range?

Answer

The absolute value of the constant A (amplitude) increases the total range and the constant D (vertical shift) shifts the graph vertically.

4) How does the range of a translated sine function relate to the equation $y=A\sin(Bx+C)+D$?

5) How can the unit circle be used to construct the graph of $f(t)=\sin t$?

Answer

At the point where the terminal side of t intersects the unit circle, you can determine that the $\sin t$ equals the y-coordinate of the point.

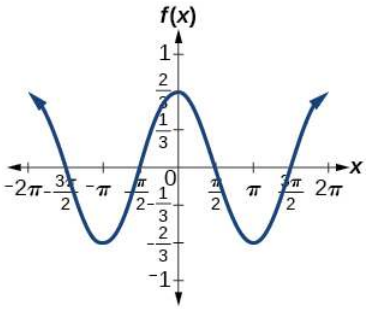
Graphical

For the following exercises, graph two full periods of each function and state the amplitude, period, and midline. State the maximum and minimum y-values and their corresponding x-values on one period for $x>0$. Round answers to two decimal places if necessary.

6) $f(x)=2\sin x$

7) $f(x)=\frac{2}{3}\cos x$

Answer

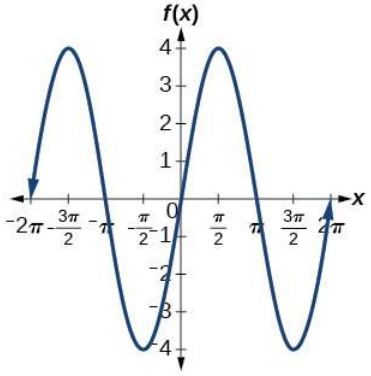


amplitude: $\frac{2}{3}$; period: 2π; midline: $y=0$; maximum: $y=\frac{2}{3}$ occurs at $x=0$; minimum: $y=-\frac{2}{3}$ occurs at $x=\pi$; for one period, the graph starts at 0 and ends at 2π.

8) $f(x)=-3\sin x$

9) $f(x) = 4\sin x$

Answer

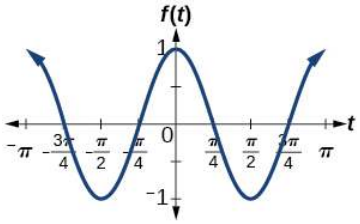


amplitude: 4; period: 2π; midline: $y = 0$; maximum $y = 4$ occurs at $x = \frac{\pi}{2}$; minimum: $y = -4$ occurs at $x = \frac{3\pi}{2}$; one full period occurs from $x = 0$ to $x = 2\pi$

10) $f(x) = 2\cos x$

11) $f(x) = \cos(2x)$

Answer

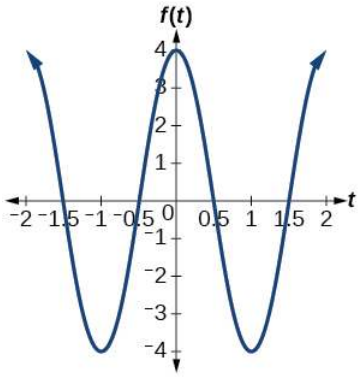


amplitude: 1; period: π; midline: $y = 0$; maximum: $y = 1$ occurs at $x = \pi$; minimum: $y = -1$ occurs at $x = \frac{\pi}{2}$; one full period is graphed from $x = 0$ to $x = \pi$

12) $f(x) = 2\sin\left(\frac{1}{2}x\right)$

13) $f(x) = 4\cos(\pi x)$

Answer

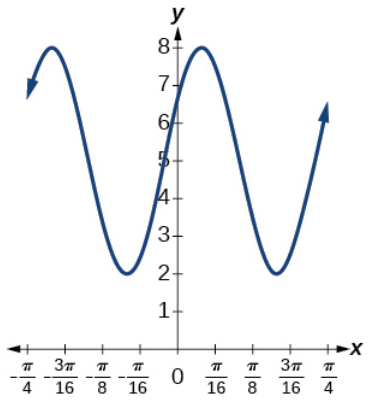


amplitude: 4; period: 2; midline: $y = 0$; maximum: $y = 4$ occurs at $x = 0$; minimum: $y = -4$ occurs at $x = 1$

14) $f(x) = 3\cos\left(\frac{6}{5}x\right)$

15) $y = 3\sin(8(x+4)) + 5$

Answer

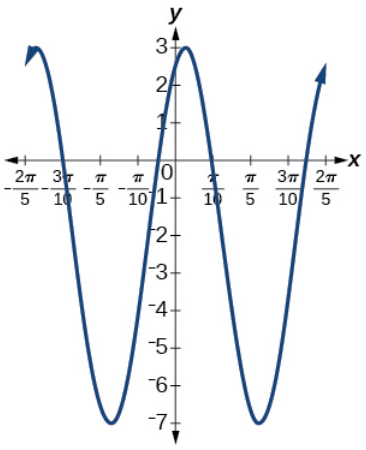


amplitude: 3; period: $\frac{\pi}{4}$; midline: $y = 5$;
maximum: $y = 8$ occurs at $x = -4 + \frac{21\pi}{16} \approx 0.123$;
minimum: $y = 2$ occurs at $x = -4 + \frac{23\pi}{16} \approx 0.516$;
horizontal shift: -4; vertical translation 5;
one period occurs from $x = -4 + \frac{22\pi}{16} \approx 0.320$ to $x = -4 + \frac{26\pi}{16} \approx 1.105$

16) $y = 2\sin(3x - 21) + 4$

17) $y = 5\sin(5x + 20) - 2$

Answer



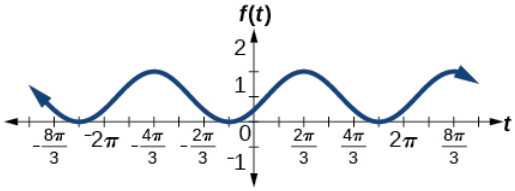
amplitude: 5; period: $\frac{2\pi}{5}$; midline: $y = -2$;
maximum: $y = 3$ occurs at $x = -4 + \frac{13\pi}{10} \approx 0.084$;
minimum: $y = -7$ occurs at $x = -4 + \frac{15\pi}{10} \approx 0.712$;
phase shift: -4; vertical translation: -2;
one full period can be graphed on $x = -4 + \frac{7\pi}{5} \approx 0.398$ to$x = -4 + \frac{9\pi}{5} \approx 1.655$

For the following exercises, graph one full period of each function, starting at $x = 0$.
For each function, state the amplitude, period, and midline.
State the maximum and minimum y-values and their corresponding x-values on one period for $x > 0$.
State the phase shift and vertical translation, if applicable.
Round answers to two decimal places if necessary.

18) $f(t) = 2\sin\left(t - \frac{5\pi}{6}\right)$

19) $f(t) = -\cos\left(t + \frac{\pi}{3}\right) + 1$

Answer

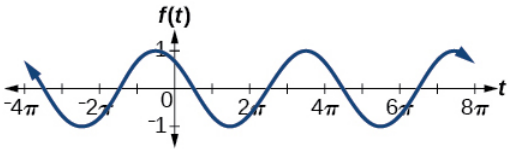


amplitude: 1; period: 2π; midline: $y = 1$;
maximum: $y = 2$ occurs at $t = \frac{2\pi}{3} \approx 2.094$;
minimum: $y = 0$ occurs at $t = \frac{2\pi}{3} \approx 5.24$;
phase shift: $-\frac{\pi}{3}$; vertical translation: 1;
one full period is from $t = \frac{2\pi}{3} \approx 2.094$ to $t = \frac{8\pi}{3} \approx 8.378$

20) $f(t) = 4\cos\left(2\left(t + \frac{\pi}{4}\right)\right) - 3$

21) $f(t) = -\sin\left(\frac{1}{2}t + \frac{5\pi}{3}\right)$

Answer



amplitude: 1; period: 4π; midline: $y = 0$;
maximum: $y = 1$ occurs at $t = \frac{11\pi}{3} \approx 11.52$;
minimum: $y = -1$ occurs at $t = \frac{5\pi}{3} \approx 5.24$;
phase shift: $-\dfrac{10\pi}{3}$; vertical shift: 0;
one full period is from $t = \frac{2\pi}{3} \approx 2.094$ to $t = \frac{14\pi}{3} \approx 14.661$

22) $f(x) = 4\sin\left(\dfrac{\pi}{2}(x-3)\right) + 7$

23) Determine the amplitude, midline, period, and an equation involving the sine function for the graph shown in Figure below.

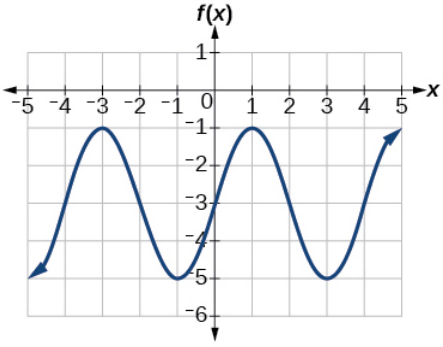


Figure 5.5.23

Answer

23. amplitude: 2; midline: $y = -3$ period: 4; equation: $f(x) = 2\sin\left(\dfrac{\pi}{2}x\right) - 3$

24) Determine the amplitude, midline, period, and an equation involving the cosine function for the graph shown in Figure below.

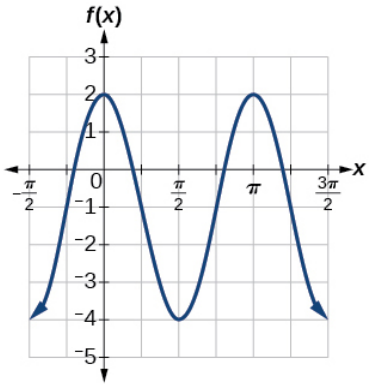


Figure 5.5.24

25) Determine the amplitude, midline, period, and an equation involving the cosine function for the graph shown in Figure below.

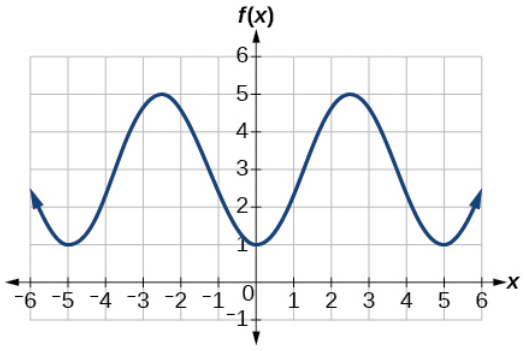


Figure 5.5.25

Answer

25. amplitude: 2; period: 5; midline: $y = 3$ equation: $f(x) = -2\cos\left(\frac{2\pi}{5}x\right) + 3$

26) Determine the amplitude, midline, period, and an equation involving the sine function for the graph shown in Figure below.

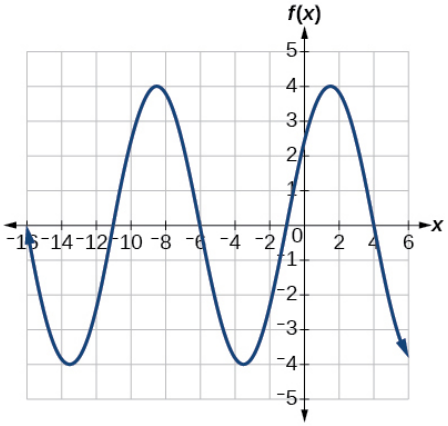


Figure 5.5.26

27) Determine the amplitude, midline, period, and an equation involving the cosine function for the graph shown in Figure below.



Figure 5.5.27

Answer

27. amplitude: 4; period: 2; midline: $y = 0$; equation: $f(x) = -4\cos\left(\pi\left(x - \frac{\pi}{2}\right)\right)$

28) Determine the amplitude, midline, period, and an equation involving the sine function for the graph shown in Figure below.

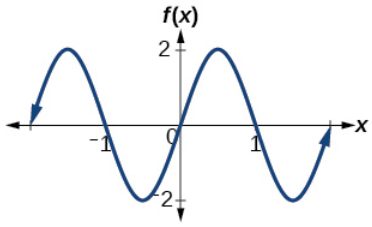


Figure 5.5.28

29) Determine the amplitude, midline, period, and an equation involving the cosine function for the graph shown in Figure below.

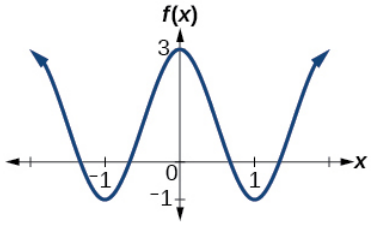


Figure 5.5.29

Answer

29. amplitude: 2; period: 2; midline $y = 1$ equation: $f(x) = 2\cos(\pi x) + 1$

30) Determine the amplitude, midline, period, and an equation involving the sine function for the graph shown in Figure below.



Figure 5.5.30

Algebraic

For the following exercises, let $f(x) = \sin x$.

31) On $[0, 2\pi)$, solve $f(x) = 0$.

32) On $[0, 2\pi)$, solve $f(x) = \dfrac{1}{2}$.

Answer

$\dfrac{\pi}{6}, \dfrac{5\pi}{6}$

33) Evaluate $f\left(\dfrac{\pi}{2}\right)$.

34) On $[0, 2\pi)$, $f(x) = \dfrac{\sqrt{2}}{2}$. Find all values of x.

Answer

$\dfrac{\pi}{4}, \dfrac{3\pi}{4}$

35) On $[0, 2\pi)$, the maximum value(s) of the function occur(s) at what x-value(s)?

36) On $[0, 2\pi)$, the minimum value(s) of the function occur(s) at what x-value(s)?

Answer

$\dfrac{3\pi}{2}$

37) Show that $f(-x) = -f(x)$. This means that $f(x) = \sin x$ is an odd function and possesses symmetry with respect to _____.

For the following exercises, let $f(x) = \cos x$

38) On $[0, 2\pi)$, solve the equation $f(x) = \cos x = 0$

Answer

$\dfrac{\pi}{2}, \dfrac{3\pi}{2}$

39) On $[0, 2\pi)$, solve $f(x) = \dfrac{1}{2}$.

40) On $[0, 2\pi)$, find the x-intercepts of $f(x) = \cos x$.

Answer

$\dfrac{\pi}{2}, \dfrac{3\pi}{2}$

41) On $[0, 2\pi)$, find the x-values at which the function has a maximum or minimum value.

42) On $[0, 2\pi)$, solve the equation $f(x) = \dfrac{\sqrt{3}}{2}$.

Answer

$\dfrac{\pi}{6}, \dfrac{11\pi}{6}$

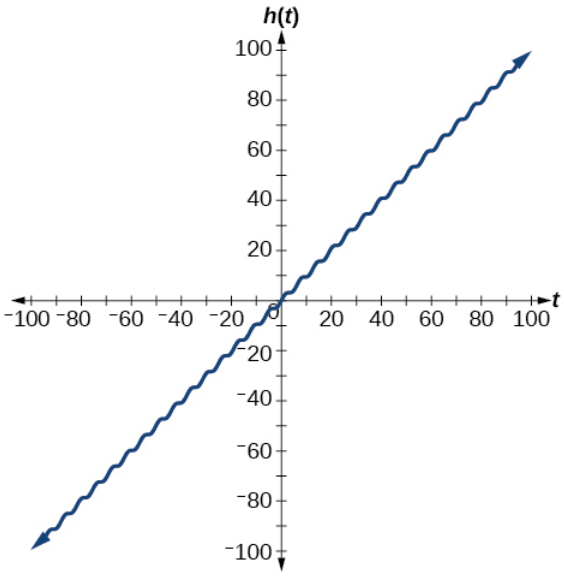
Technology

43) Graph $h(x) = x + \sin x$ on $[0, 2\pi]$. Explain why the graph appears as it does.

44) Graph $h(x) = x + \sin x$ on $[-100, 100]$. Did the graph appear as predicted in the previous exercise?

Answer

The graph appears linear. The linear functions dominate the shape of the graph for large values of x.

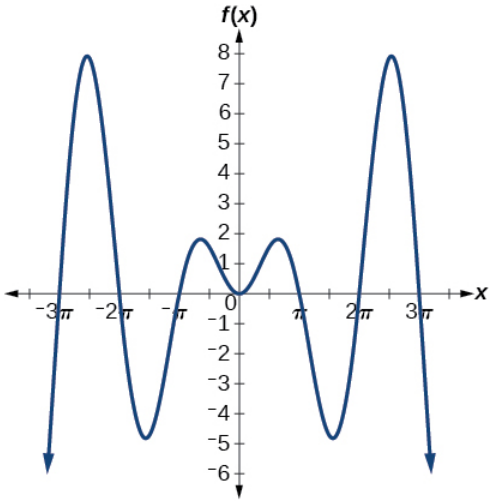


45) Graph $f(x) = x \sin x$ on $[0, 2\pi]$ and verbalize how the graph varies from the graph of $f(x) = x \sin x$.

46) Graph $f(x) = x \sin x$ on the window $[-10, 10]$ and explain what the graph shows.

Answer

The graph is symmetric with respect to the y-axis and there is no amplitude because the function is not periodic.



47) Graph $f(x) = \dfrac{\sin x}{x}$ on the window $[-5\pi, 5\pi]$and explain what the graph shows.

Real-World Applications

48) A Ferris wheel is 25 meters in diameter and boarded from a platform that is 1 meter above the ground. The six o'clock position on the Ferris wheel is level with the loading platform. The wheel completes 1 full revolution in 10 minutes. The function $h(t)$ gives a person's height in meters above the ground t minutes after the wheel begins to turn

a. Find the amplitude, midline, and period of $h(t)$.
b. Find a formula for the height function $h(t)$.
c. How high off the ground is a person after 5 minutes?

Answer

a. Amplitude: 12.5; period: 10; midline: $y = 13.5$
b. $h(t) = 12.5 \sin\left(\dfrac{\pi}{5}(t - 2.5)\right) + 13.5$
c. 26 ft

This page titled 5.5: Graphs of the Sine and Cosine Functions is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **8.1: Graphs of the Sine and Cosine Functions** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/precalculus.
- **6.E: Periodic Functions (Exercises)** by OpenStax is licensed CC BY 4.0.

CHAPTER OVERVIEW

6: Differentiation Rules and Applications

6.1: The Sine and Cosine Function
6.2: The Product and Quotient Rules
6.3: Derivatives of Other Trigonometric Functions
6.4: The Chain Rule
6.5: Derivatives of Functions Given Implicitely

6: Differentiation Rules and Applications is shared under a not declared license and was authored, remixed, and/or curated by LibreTexts.

6.1: The Sine and Cosine Function

(There are no videos for this section.)

Learning Objectives

In this section, we strive to understand the ideas generated by the following important questions:

- What is a graphical justification for why $\frac{d}{dx}\left[a^x\right] = a^x \ln(a)$?
- What do the graphs of $y = \sin(x)$ and $y = \cos(x)$ suggest as formulas for their respective derivatives?
- Once we know the derivatives of $\sin(x)$ and $\cos(x)$, how do previous derivative rules work when these functions are involved?

Throughout Chapter 2, we will be working to develop shortcut derivative rules that will help us to bypass the limit definition of the derivative in order to quickly determine the formula for $f'(x)$ when we are given a formula for $f(x)$. In Section 2.1, we learned the rule for power functions, that if $f(x) = x^n$, then $f'(x) = nx^{n-1}$, and justified this in part due to results from different n-values when applying the limit definition of the derivative. We also stated the rule for exponential functions, that if a is a positive real number and $f(x) = a^x$, then f'(x) = a x ln(a). Later in this present section, we are going to work to conjecture formulas for the sine and cosine functions, primarily through a graphical argument. To help set the stage for doing so, the following preview activity asks you to think about exponential functions and why it is reasonable to think that the derivative of an exponential function is a constant times the exponential function itself.

Preview Activity 6.1.1

Consider the function $g(x) = 2^x$, which is graphed in Figure 6.1.1.

a. At each of $x = -2, -1, 0, 1, 2$, use a straightedge to sketch an accurate tangent line to $y = g(x)$.
b. Use the provided grid to estimate the slope of the tangent line you drew at each point in (a).
c. Use the limit definition of the derivative to estimate $g'(0)$ by using small values of h, and compare the result to your visual estimate for the slope of the tangent line to $y = g(x)$ at $x = 0$ in (b).
d. Based on your work in (a), (b), and (c), sketch an accurate graph of $y = g'(x)$ on the axes adjacent to the graph of $y = g(x)$.
e. Write at least one sentence that explains why it is reasonable to think that $g'(x) = cg(x)$, where c is a constant. In addition, calculate $ln(2)$, and then discuss how this value, combined with your work above, reasonably suggests that $g'(x) = 2^x ln(2)$.

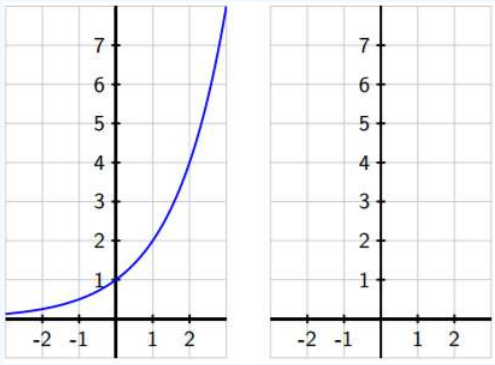


Figure 6.1.1: At left, the graph of y = g(x) = 2 x . At right, axes for plotting y = g 0 (x).

The sine and cosine functions

The sine and cosine functions are among the most important functions in all of mathematics. Sometimes called the circular functions due to their genesis in the unit circle, these periodic functions play a key role in modeling repeating phenomena such as the location of a point on a bicycle tire, the behavior of an oscillating mass attached to a spring, tidal elevations, and more. Like polynomial and exponential functions, the sine and cosine functions are considered basic functions, ones that are often used in the

building of more complicated functions. As such, we would like to know formulas for $\frac{d}{dx}[\sin(x)]$ and $\frac{a}{dx}[\cos(x)]$, and the next two activities lead us to that end.

Activity 6.1.1

Consider the function $f(x) = \sin(x)$, which is graphed in Figure 6.1.2 below. Note carefully that the grid in the diagram does not have boxes that are 1 × 1, but rather approximately 1.57 × 1, as the horizontal scale of the grid is $\pi/2$ units per box.

a. At each of $x = -2\pi, -\frac{3\pi}{2}, -\pi, -\frac{\pi}{2}$,0, (\frac { \pi } { 2 }) , (\pi), (\frac { 3 \pi } { 2 })\, 2π) $useastraightedgetosketchanaccuratetangentlineto\(y = f(x)$.
b. Use the provided grid to estimate the slope of the tangent line you drew at each point. Pay careful attention to the scale of the grid.
c. Use the limit definition of the derivative to estimate $f'(0)$ by using small values of h, and compare the result to your visual estimate for the slope of the tangent line to $y = f(x)$ at $x = 0$ in (b). Using periodicity, what does this result suggest about $f'(2\pi)$? about $f'(-2\pi)$?
d. Based on your work in (a), (b), and (c), sketch an accurate graph of $y = f'(x)$ on the axes adjacent to the graph of $y = f(x)$.
e. What familiar function do you think is the derivative of $f(x) = \sin(x)$?

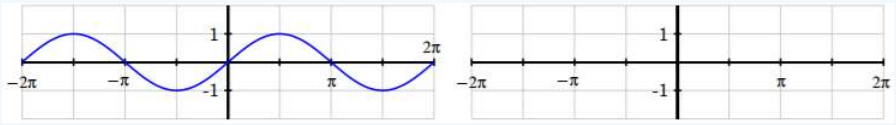


Figure 6.1.2: At left, the graph of y = f (x) = \sin(x).

Activity 6.1.2

Consider the function $g(x) = \cos(x)$, which is graphed in Figure 6.1.3 below. Note carefully that the grid in the diagram does not have boxes that are 1 × 1, but rather approximately 1.57 × 1, as the horizontal scale of the grid is $\pi/2$ units per box.

a. At each of $x = -2\pi, -\frac{3\pi}{2}, -\pi, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ use a straightedge to sketch an accurate tangent line to y = g(x).
b. Use the provided grid to estimate the slope of the tangent line you drew at each point. Again, note the scale of the axes and grid.
c. Use the limit definition of the derivative to estimate $g'(\pi^2)$ by using small values of h, and compare the result to your visual estimate for the slope of the tangent line to $y = g(x)$ at \(x = \frac{\pi}{2}) in (b). Using periodicity, what does this result suggest about \(g ' (\frac{− 3π}{ 2})? can symmetry on the graph help you estimate other slopes easily?
d. Based on your work in (a), (b), and (c), sketch an accurate graph of $y = g'(x)$ on the axes adjacent to the graph of $y = g(x)$.
e. What familiar function do you think is the derivative of $g(x) = \cos(x)$?

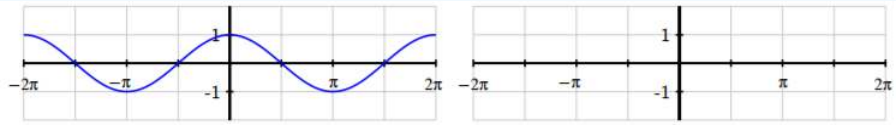


Figure 6.1.3: At left, the graph of y = g(x) = \cos(x).

The results of the two preceding activities suggest that the sine and cosine functions not only have the beautiful interrelationships that are learned in a course in trigonometry – connections such as the identities

$$\sin^2(x) + \cos^2(x) = 1 \tag{6.1.1}$$

and

$$\cos\left(x - \frac{\pi}{2}\right) = \sin(x) \tag{6.1.2}$$

– but that they are even further linked through calculus, as the derivative of each involves the other. The following rules summarize the results of the activities[4] .

Sine and Cosine Functions

For all real numbers x,

$$\frac{d}{dx}[\sin(x)] = \cos(x) \tag{6.1.3}$$

and

$$\frac{d}{dx}[\cos(x)] = -\sin(x) \tag{6.1.4}$$

We have now added two additional functions to our library of basic functions whose derivatives we know: power functions, exponential functions, and the sine and cosine functions. The constant multiple and sum rules still hold, of course, and all of the inherent meaning of the derivative persists, regardless of the functions that are used to constitute a given choice of $f(x)$. The following activity puts our new knowledge of the derivatives of $\sin(x)$ and $\cos(x)$ to work.

Activity 6.1.3

Answer each of the following questions. Where a derivative is requested, be sure to label the derivative function with its name using proper notation.

a. Determine the derivative of $h(t) = 3cos(t) - 4sin(t)$.
b. Find the exact slope of the tangent line to $y = f(x) = 2x + \frac{\sin(x)}{2}$ at the point where $x = \frac{\pi}{6}$.
c. Find the equation of the tangent line to $y = g(x) = x^2 + 2\cos(x)$ at the point where $x = \pi^2$.
d. Determine the derivative of $p(z) = z^4 + 4^z + 4cos(z) - sin(\frac{\pi}{2})$.
e. The function $P(t) = 24 + 8sin(t)$ represents a population of a particular kind of animal that lives on a small island, where P is measured in hundreds and t is measured in decades since January 1, 2010. What is the instantaneous rate of change of P on January 1, 2030? What are the units of this quantity? Write a sentence in everyday language that explains how the population is behaving at this point in time.

Summary

In this section, we encountered the following important ideas:

- If we consider the graph of an exponential function $f(x) = a^x$ (where a > 1), the graph of f'(x) behaves similarly, appearing exponential and as a possibly scaled version of the original function a x . For f (x) = 2 x , careful analysis of the graph and its slopes suggests that d dx [2 x] = 2 x ln(2), which is a special case of the rule we stated in Section 2.1.
- By carefully analyzing the graphs of $y = \sin(x)$ and $y = \cos(x)$, plus using the limit definition of the derivative at select points, we found that

$$\frac{d}{dx}[\sin(x)] = \cos(x) \tag{6.1.5}$$

and

$$\frac{d}{dx}[\cos(x)] = -\sin(x). \tag{6.1.6}$$

- We note that all previously encountered derivative rules still hold, but now may also be applied to functions involving the sine and cosine, plus all of the established meaning of the derivative applies to these trigonometric functions as well.

Contributors and Attributions

- Matt Boelkins (Grand Valley State University), David Austin (Grand Valley State University), Steve Schlicker (Grand Valley State University)

[4]These two rules may be formally proved using the limit definition of the derivative and the expansion identities for $\sin(x+h)$ and $\cos(x+h)$.

Homework Exercises 6.1

? 1

Suppose that $V(t) = 24 \cdot 1.07^t + 6\sin(t)$ represents the value of a person's investment portfolio in thousands of dollars in year t, where $t = 0$ corresponds to January 1, 2010.

1. At what instantaneous rate is the portfolio's value changing on January 1, 2012? Include units on your answer.
2. Determine the value of $V''(2)$. What are the units on this quantity and what does it tell you about how the portfolio's value is changing?
3. On the interval $0 \le t \le 20$, graph the function $V(t) = 24 \cdot 1.07^t + 6\sin(t)$ and describe its behavior in the context of the problem. Then, compare the graphs of the functions $A(t) = 24 \cdot 1.07^t$ and $V(t) = 24 \cdot 1.07^t + 6\sin(t)$, as well as the graphs of their derivatives $A'(t)$ and $V'(t)$. What is the impact of the term $6\sin(t)$ on the behavior of the function $V(t)$?

? 2.

Let $f(x) = 3\cos(x) - 2\sin(x) + 6$.

1. Determine the exact slope of the tangent line to $y = f(x)$ at the point where $a = \frac{\pi}{4}$.
2. Determine the tangent line approximation to $y = f(x)$ at the point where $a = \pi$.
3. At the point where $a = \frac{\pi}{2}$, is f increasing, decreasing, or neither?
4. At the point where $a = \frac{3\pi}{2}$, does the tangent line to $y = f(x)$ lie above the curve, below the curve, or neither? How can you answer this question without even graphing the function or the tangent line?

? 3

In this exercise, we explore how the limit definition of the derivative more formally shows that $\frac{d}{dx}[\sin(x)] = \cos(x)$. Letting $f(x) = \sin(x)$, note that the limit definition of the derivative tells us that

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}.$$

1. Recall the trigonometric identity for the sine of a sum of angles α and β: $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$. Use this identity and some algebra to show that
$$f'(x) = \lim_{h \to 0} \frac{\sin(x)(\cos(h) - 1) + \cos(x)\sin(h)}{h}.$$
2. Next, note that as h changes, x remains constant. Explain why it therefore makes sense to say that
$$f'(x) = \sin(x) \cdot \lim_{h \to 0} \frac{\cos(h) - 1}{h} + \cos(x) \cdot \lim_{h \to 0} \frac{\sin(h)}{h}.$$
3. Finally, use small values of h to estimate the values of the two limits in (c):
$$\lim_{h \to 0} \frac{\cos(h) - 1}{h} \text{ and } \lim_{h \to 0} \frac{\sin(h)}{h}.$$
4. What do your results in (b) and (c) thus tell you about $f'(x)$?
5. By emulating the steps taken above, use the limit definition of the derivative to argue convincingly that $\frac{d}{dx}[\cos(x)] = -\sin(x)$.

This page titled 6.1: The Sine and Cosine Function is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Matthew Boelkins, David Austin & Steven Schlicker (ScholarWorks @Grand Valley State University) via source content that was edited to the style and standards of the LibreTexts platform.

- **2.2: The Sine and Cosine Function** is licensed CC BY-SA 4.0. Original source: https://activecalculus.org/single.
- **2.2: The Sine and Cosine Function** by Matthew Boelkins, David Austin & Steven Schlicker is licensed CC BY-SA 4.0. Original source: https://activecalculus.org/single.

6.2: The Product and Quotient Rules

Motivating Questions

- How does the algebraic structure of a function guide us in computing its derivative using shortcut rules?
- How do we compute the derivative of a product of two basic functions in terms of the derivatives of the basic functions?
- How do we compute the derivative of a quotient of two basic functions in terms of the derivatives of the basic functions?
- How do the product and quotient rules combine with the sum and constant multiple rules to expand the library of functions we can differentiate quickly?

So far, we can differentiate power functions (x^n), exponential functions (a^x), and the two fundamental trigonometric functions ($\sin(x)$ and $\cos(x)$). With the sum rule and constant multiple rules, we can also compute the derivative of combined functions.

Example 6.2.1

Differentiate

$$f(x) = 7x^{11} - 4 \cdot 9^x + \pi \sin(x) - \sqrt{3}\cos(x)$$

Because f is a sum of basic functions, we can now quickly say that $f'(x) = 77x^{10} - 4 \cdot 9^x \ln(9) + \pi \cos(x) + \sqrt{3}\sin(x)$.

What about a product or quotient of two basic functions, such as

$$p(z) = z^3 \cos(z),$$

or

$$q(t) = \frac{\sin(t)}{2^t}?$$

While the derivative of a sum is the sum of the derivatives, it turns out that the rules for computing derivatives of products and quotients are more complicated.

Preview Activity 6.2.1

Let f and g be the functions defined by $f(t) = 2t^2$ and $g(t) = t^3 + 4t$.

1. Determine $f'(t)$ and $g'(t)$.
2. Let $p(t) = 2t^2(t^3 + 4t)$ and observe that $p(t) = f(t) \cdot g(t)$. Rewrite the formula for p by distributing the $2t^2$ term. Then, compute $p'(t)$ using the sum and constant multiple rules.
3. True or false: $p'(t) = f'(t) \cdot g'(t)$.
4. Let $q(t) = \frac{t^3+4t}{2t^2}$ and observe that $q(t) = \frac{g(t)}{f(t)}$. Rewrite the formula for q by dividing each term in the numerator by the denominator and simplify to write q as a sum of constant multiples of powers of t. Then, compute $q'(t)$ using the sum and constant multiple rules.
5. True or false: $q'(t) = \frac{g'(t)}{f'(t)}$.

6.2.1: The product rule

As part (b) of Preview Activity 6.2.1 shows, it is not true in general that the derivative of a product of two functions is the product of the derivatives of those functions. To see why this is the case, we consider an example involving meaningful functions.

Say that an investor is regularly purchasing stock in a particular company. Let $N(t)$ represent the number of shares owned on day t, where $t = 0$ represents the first day on which shares were purchased. Let $S(t)$ give the value of one share of the stock on day t; note that the units on $S(t)$ are dollars per share. To compute the total value of the stock on day t, we take the product

$$V(t) = N(t) \text{ shares} \cdot S(t) \text{ dollars per share}.$$

Observe that over time, both the number of shares and the value of a given share will vary. The derivative $N'(t)$ measures the rate at which the number of shares is changing, while $S'(t)$ measures the rate at which the value per share is changing. How do these respective rates of change affect the rate of change of the total value function?

To help us understand the relationship among changes in N, S, and V, let's consider some specific data.

- Suppose that on day 100, the investor owns 520 shares of stock and the stock's current value is \$27.50 per share. This tells us that $N(100) = 520$ and $S(100) = 27.50$.
- On day 100, the investor purchases an additional 12 shares (so the number of shares held is rising at a rate of 12 shares per day).
- On that same day the price of the stock is rising at a rate of 0.75 dollars per share per day.

In calculus notation, the latter two facts tell us that $N'(100) = 12$ (shares per day) and $S'(100) = 0.75$ (dollars per share per day). At what rate is the value of the investor's total holdings changing on day 100?

Observe that the increase in total value comes from two sources: the growing number of shares, and the rising value of each share. If only the number of shares is increasing (and the value of each share is constant), the rate at which which total value would rise is the product of the current value of the shares and the rate at which the number of shares is changing. That is, the rate at which total value would change is given by

$$S(100) \cdot N'(100) = 27.50 \, \frac{\text{dollars}}{\text{share}} \cdot 12 \, \frac{\text{shares}}{\text{day}} = 330 \, \frac{\text{dollars}}{\text{day}}.$$

Note particularly how the units make sense and show the rate at which the total value V is changing, measured in dollars per day.

If instead the number of shares is constant, but the value of each share is rising, the rate at which the total value would rise is the product of the number of shares and the rate of change of share value. The total value is rising at a rate of

$$N(100) \cdot S'(100) = 520 \text{ shares} \cdot 0.75 \, \frac{\text{dollars per share}}{\text{day}} = 390 \, \frac{\text{dollars}}{\text{day}}.$$

Of course, when both the number of shares and the value of each share are changing, we have to include both of these sources. In that case the rate at which the total value is rising is

$$V'(100) = S(100) \cdot N'(100) + N(100) \cdot S'(100) = 330 + 390 = 720 \, \frac{\text{dollars}}{\text{day}}.$$

We expect the total value of the investor's holdings to rise by about \$720 on the 100th day.[1]

While this example highlights why the product rule is true, there are some subtle issues to recognize. For one, if the stock's value really does rise exactly \$0.75 on day 100, and the number of shares really rises by 12 on day 100, then we'd expect that $V(101) = N(101) \cdot S(101) = 532 \cdot 28.25 = 15029$. If, as noted above, we expect the total value to rise by \$720, then with $V(100) = N(100) \cdot S(100) = 520 \cdot 27.50 = 14300$, then it seems we should find that $V(101) = V(100) + 720 = 15020$. Why do the two results differ by 9? One way to understand why this difference occurs is to recognize that $N'(100) = 12$ represents an *instantaneous* rate of change, while our (informal) discussion has also thought of this number as the total change in the number of shares over the course of a single day. The formal proof of the product rule reconciles this issue by taking the limit as the change in the input tends to zero.

Next, we expand our perspective from the specific example above to the more general and abstract setting of a product p of two differentiable functions, f and g. If $P(x) = f(x) \cdot g(x)$, our work above suggests that $P'(x) = f(x)g'(x) + g(x)f'(x)$. Indeed, a formal proof using the limit definition of the derivative can be given to show that the following rule, called the *product rule*, holds in general.

Product Rule

If f and g are differentiable functions, then their product $P(x) = f(x) \cdot g(x)$ is also a differentiable function, and

$$P'(x) = f(x)g'(x) + g(x)f'(x).$$

In light of the earlier example involving shares of stock, the product rule also makes sense intuitively: the rate of change of P should take into account both how fast f and g are changing, as well as how large f and g are at the point of interest. In words the product rule says: if P is the product of two functions f (the first function) and g (the second), then "the derivative of P is the first

times the derivative of the second, plus the second times the derivative of the first." It is often a helpful mental exercise to say this phrasing aloud when executing the product rule.

Example 6.2.2

If $P(z) = z^3 \cdot \cos(z)$, we can use the product rule to differentiate P. The first function is z^3 and the second function is $\cos(z)$. By the product rule, P' will be given by the first, z^3, times the derivative of the second, $-\sin(z)$, plus the second, $\cos(z)$, times the derivative of the first, $3z^2$. That is,

$$P'(z) = z^3(-\sin(z)) + \cos(z)3z^2 = -z^3\sin(z) + 3z^2\cos(z).$$

Activity 6.2.2

Use the product rule to answer each of the questions below. Throughout, be sure to carefully label any derivative you find by name. It is not necessary to algebraically simplify any of the derivatives you compute.

1. Let $m(w) = 3w^{17}4^w$. Find $m'(w)$.
2. Let $h(t) = (\sin(t) + \cos(t))t^4$. Find $h'(t)$.
3. Determine the slope of the tangent line to the curve $y = f(x)$ at the point where $a = 1$ if f is given by the rule $f(x) = e^x \sin(x)$.
4. Find the tangent line approximation $L(x)$ to the function $y = g(x)$ at the point where $a = -1$ if g is given by the rule $g(x) = (x^2 + x)2^x$.

6.2.2: The quotient rule

Because quotients and products are closely linked, we can use the product rule to understand how to take the derivative of a quotient. Let $Q(x)$ be defined by $Q(x) = f(x)/g(x)$, where f and g are both differentiable functions. It turns out that Q is differentiable everywhere that $g(x) \neq 0$. We would like a formula for Q' in terms of f, g, f', and g'. Multiplying both sides of the formula $Q = f/g$ by g, we observe that

$$f(x) = Q(x) \cdot g(x).$$

Now we can use the product rule to differentiate f.

$$f'(x) = Q(x)g'(x) + g(x)Q'(x).$$

We want to know a formula for Q', so we solve this equation for $Q'(x)$.

$$Q'(x)g(x) = f'(x) - Q(x)g'(x)$$

and dividing both sides by $g(x)$, we have

$$Q'(x) = \frac{f'(x) - Q(x)g'(x)}{g(x)}.$$

Finally, we recall that $Q(x) = \frac{f(x)}{g(x)}$. Substituting this expression in the preceding equation, we have

$$\begin{aligned} Q'(x) &= \frac{f'(x) - \frac{f(x)}{g(x)}g'(x)}{g(x)} \\ &= \frac{f'(x) - \frac{f(x)}{g(x)}g'(x)}{g(x)} \cdot \frac{g(x)}{g(x)} \\ &= \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}. \end{aligned}$$

This calculation gives us the *quotient rule*.

Quotient Rule

If f and g are differentiable functions, then their quotient $Q(x) = \frac{f(x)}{g(x)}$ is also a differentiable function for all x where $g(x) \neq 0$ and

$$Q'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}.$$

As with the product rule, it can be helpful to think of the quotient rule verbally. If a function Q is the quotient of a top function f and a bottom function g, then Q' is given by "the bottom times the derivative of the top, minus the top times the derivative of the bottom, all over the bottom squared."

Example 6.2.3

If $Q(t) = \sin(t)/2^t$, we call $\sin(t)$ the top function and 2^t the bottom function. By the quotient rule, Q' is given by the bottom, 2^t, times the derivative of the top, $\cos(t)$, minus the top, $\sin(t)$, times the derivative of the bottom, $2^t \ln(2)$, all over the bottom squared, $(2^t)^2$. That is,

$$Q'(t) = \frac{2^t \cos(t) - \sin(t)2^t \ln(2)}{(2^t)^2}.$$

In this particular example, it is possible to simplify $Q'(t)$ by removing a factor of 2^t from both the numerator and denominator, so that

$$Q'(t) = \frac{\cos(t) - \sin(t)\ln(2)}{2^t}.$$

In general, we must be careful in doing any such simplification, as we don't want to execute the quotient rule correctly but then make an algebra error.

Activity 6.2.3

Use the quotient rule to answer each of the questions below. Throughout, be sure to carefully label any derivative you find by name. That is, if you're given a formula for $f(x)$, clearly label the formula you find for $f'(x)$. It is not necessary to algebraically simplify any of the derivatives you compute.

1. Let $r(z) = \frac{3^z}{z^4+1}$. Find $r'(z)$.
2. Let $v(t) = \frac{\sin(t)}{\cos(t)+t^2}$. Find $v'(t)$.
3. Determine the slope of the tangent line to the curve $R(x) = \dfrac{x^2 - 2x - 8}{x^2 - 9}$ at the point where $x = 0$.
4. When a camera flashes, the intensity I of light seen by the eye is given by the function

$$I(t) = \frac{100t}{e^t},$$

where I is measured in candles and t is measured in milliseconds. Compute $I'(0.5)$, $I'(2)$, and $I'(5)$; include appropriate units on each value; and discuss the meaning of each.

6.2.3: Combining rules

In order to apply the derivative shortcut rules correctly we must recognize the fundamental structure of a function.

Example 6.2.4

Determine the derivative of the function

$$f(x) = x\sin(x) + \frac{x^2}{\cos(x)+2}.$$

How do we decide which rules to apply? Our first task is to recognize the structure of the function. This function f is a sum of two slightly less complicated functions, so we can apply the sum rule[2] to get

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[x\sin(x) + \frac{x^2}{\cos(x)+2}\right] \\ &= \frac{d}{dx}[x\sin(x)] + \frac{d}{dx}\left[\frac{x^2}{\cos(x)+2}\right] \end{aligned}$$

When taking a derivative that involves the use of multiple derivative rules, it is often helpful to use the notation $\frac{d}{dx}[\ ]$ to wait to apply subsequent rules. This is demonstrated in each of the two examples presented here.

Now, the left-hand term above is a product, so the product rule is needed there, while the right-hand term is a quotient, so the quotient rule is required. Applying these rules respectively, we find that

$$\begin{aligned} f'(x) &= (x\cos(x)+\sin(x)) + \frac{(\cos(x)+2)2x - x^2(-\sin(x))}{(\cos(x)+2)^2} \\ &= x\cos(x)+\sin(x) + \frac{2x\cos(x)+4x+x^2\sin(x)}{(\cos(x)+2)^2}. \end{aligned}$$

Example 6.2.5

Differentiate

$$s(y) = \frac{y\cdot 7^y}{y^2+1}.$$

The function s is a quotient of two simpler functions, so the quotient rule will be needed. To begin, we set up the quotient rule and use the notation $\frac{d}{dy}$ to indicate the derivatives of the numerator and denominator. Thus,

$$s'(y) = \frac{(y^2+1)\cdot\frac{d}{dy}[y\cdot 7^y] - y\cdot 7^y\cdot\frac{d}{dy}[y^2+1]}{(y^2+1)^2}.$$

Now, there remain two derivatives to calculate. The first one, $\frac{d}{dy}[y\cdot 7^y]$ calls for use of the product rule, while the second, $\frac{d}{dy}[y^2+1]$ needs only the sum rule. Applying these rules, we now have

$$s'(y) = \frac{(y^2+1)[y\cdot 7^y\ln(7)+7^y\cdot 1] - y\cdot 7^y[2y]}{(y^2+1)^2}.$$

While some simplification is possible, we are content to leave $s'(y)$ in its current form.

Success in applying derivative rules begins with recognizing the structure of the function, followed by the careful and diligent application of the relevant derivative rules. The best way to become proficient at this process is to do a large number of examples.

Activity 6.2.4

Use relevant derivative rules to answer each of the questions below. Throughout, be sure to use proper notation and carefully label any derivative you find by name.

1. Let $f(r) = (5r^3+\sin(r))(4^r - 2\cos(r))$. Find $f'(r)$.
2. Let $p(t) = \dfrac{\cos(t)}{t^6\cdot 6^t}$. Find $p'(t)$.
3. Let $g(z) = 3z^7e^z - 2z^2\sin(z) + \frac{z}{z^2+1}$. Find $g'(z)$.

4. A moving particle has its position in feet at time t in seconds given by the function $s(t) = \frac{3\cos(t) - \sin(t)}{e^t}$. Find the particle's instantaneous velocity at the moment $t = 1$.
5. Suppose that $f(x)$ and $g(x)$ are differentiable functions and it is known that $f(3) = -2$, $f'(3) = 7$, $g(3) = 4$, and $g'(3) = -1$. If $p(x) = f(x) \cdot g(x)$ and $q(x) = \frac{f(x)}{g(x)}$, calculate $p'(3)$ and $q'(3)$.

As the algebraic complexity of the functions we are able to differentiate continues to increase, it is important to remember that all of the derivative's meaning continues to hold. Regardless of the structure of the function f, the value of $f'(a)$ tells us the instantaneous rate of change of f with respect to x at the moment $x = a$, as well as the slope of the tangent line to $y = f(x)$ at the point $(a, f(a))$.

6.2.4: Summary

- If a function is a sum, product, or quotient of simpler functions, then we can use the sum, product, or quotient rules to differentiate it in terms of the simpler functions and their derivatives.
- The product rule tells us that if P is a product of differentiable functions f and g according to the rule $P(x) = f(x)g(x)$, then
$$P'(x) = f(x)g'(x) + g(x)f'(x).$$
- The quotient rule tells us that if Q is a quotient of differentiable functions f and g according to the rule $Q(x) = \frac{f(x)}{g(x)}$, then
$$Q'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}.$$
- Along with the constant multiple and sum rules, the product and quotient rules enable us to compute the derivative of any function that consists of sums, constant multiples, products, and quotients of basic functions. For instance, if F has the form
$$F(x) = \frac{2a(x) - 5b(x)}{c(x) \cdot d(x)},$$
then F is a quotient, in which the numerator is a sum of constant multiples and the denominator is a product. This, the derivative of F can be found by applying the quotient rule and then using the sum and constant multiple rules to differentiate the numerator and the product rule to differentiate the denominator.

6.2.5: Video: Example using both rules



6.2.6: WeBWorK Homework

1.
2.
3.
4.

5.

6.

7.

8.

6.2.7: Written Homework:

1. Let f and g be differentiable functions for which the following information is known: $f(2)=5$, $g(2)=-3$, $f'(2)=-1/2$, $g'(2)=2$.

a. Let h be the new function defined by the rule $h(x)=g(x)\cdot f(x)$. Determine $h(2)$ and $h'(2)$.

b. Find an equation for the tangent line to $y=h(x)$ at the point $(2,h(2))$ (where h is the function defined in (a)).

c. Let r be the function defined by the rule $r(x)=g(x)f(x)$. Is r increasing, decreasing, or neither at $a=2$? Why?

d. Estimate the value of $r(2.06)$ (where r is the function defined in (c)) by using the local linearization of r at the point $(2,r(2))$.

This page titled 6.2: The Product and Quotient Rules is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Matthew Boelkins, David Austin & Steven Schlicker (ScholarWorks @Grand Valley State University) via source content that was edited to the style and standards of the LibreTexts platform.

- **2.3: The Product and Quotient Rules** by Matthew Boelkins, David Austin & Steven Schlicker is licensed CC BY-SA 4.0. Original source: https://activecalculus.org/single.

6.3: Derivatives of Other Trigonometric Functions

Preview Video

This optional summary video from the textbook author might be helpful to use as a preview. Other, more detailed, supplemental videos for this section are posted at the end of the text.



Learning Objectives

In this section, we strive to understand the ideas generated by the following important questions:

- What are the derivatives of the tangent, cotangent, secant, and cosecant functions?
- How do the derivatives of $\tan(x)$, $\cot(x)$, $\sec(x)$, and $\csc(x)$ combine with other derivative rules we have developed to expand the library of functions we can quickly differentiate?

One of the powerful themes in trigonometry is that the entire subject emanates from a very simple idea: locating a point on the unit circle.

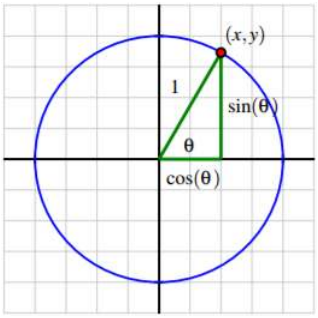


Figure* 6.3.1*: *The unit circle and the definition of the sine and cosine functions.*

Because each angle θ corresponds to one and only one point (x, y) on the unit circle, the x- and y-coordinates of this point are each functions of θ. Indeed, this is the very definition of $\cos(\theta)$ and $\sin(\theta)$: $\cos(\theta)$ is the x-coordinate of the point on the unit circle corresponding to the angle θ, and $\sin(\theta)$ is the y-coordinate. From this simple definition, all of trigonometry is founded. For instance, the fundamental trigonometric identity,

$$\sin^2(\theta) + \cos^2(\theta) = 1 \tag{6.3.1}$$

is a restatement of the Pythagorean Theorem, applied to the right triangle shown in Figure 6.3.1

We recall as well that there are four other trigonometric functions, each defined in terms of the sine and/or cosine functions. These six trigonometric functions together offer us a wide range of flexibility in problems involving right triangles. The tangent function is defined by

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}, \tag{6.3.2}$$

while the cotangent function is its reciprocal:

$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}. \tag{6.3.3}$$

The secant function is the reciprocal of the cosine function,

$$\sec(\theta) = \frac{1}{\cos(\theta)}, \tag{6.3.4}$$

and the cosecant function is the reciprocal of the sine function,

$$\csc(\theta) = \frac{1}{\sin(\theta)}. \tag{6.3.5}$$

Because we know the derivatives of the sine and cosine function, and the other four trigonometric functions are defined in terms of these familiar functions, we can now develop shortcut differentiation rules for the tangent, cotangent, secant, and cosecant functions. In this section's preview activity, we work through the steps to find the derivative of $y = tan(x)$.

Preview Activity 6.3.1

Consider the function $f(x) = tan(x)$, and remember that $tan(x) = \sin(x)\cos(x)$.

a. What is the domain of f?
b. Use the quotient rule to show that one expression for $f'(x)$ is

$$f'(x) = \frac{\cos(x)\cos(x) + \sin(x)\sin(x)}{\cos^2(x)} \tag{6.3.6}$$

.
c. What is the Fundamental Trigonometric Identity? How can this identity be used to find a simpler form for f' (x)?
d. Recall that sec(x) = 1 cos(x) . How can we express $f'(x)$ in terms of the secant function?
e. For what values of x is f' (x) defined? How does this set compare to the domain of f?

Derivatives of the cotangent, secant, and cosecant functions

In Preview Activity 2.4, we found that the derivative of the tangent function can be expressed in several ways, but most simply in terms of the secant function. Next, we develop the derivative of the cotangent function.

Let $g(x) = cot(x)$. To find $g'(x)$, we observe that $g(x) = \cos(x)\sin(x)$ and apply the quotient rule. Hence

$$g'(x) = \frac{\sin(x)(-\sin(x)) - \cos(x)\cos(x)}{\sin^2(x)} \tag{6.3.7}$$

$$= -\frac{sin^2(x) + cos^2(x)}{\sin^2(x)} \tag{6.3.8}$$

By the Fundamental Trigonometric Identity, we see that

$$g'(x) = -\frac{1}{\sin^2(x)} \tag{6.3.9}$$

recalling Equation 6.3.5, it follows that we can most simply express g' by the rule

$$g'(x) = -\csc^2(x). \tag{6.3.10}$$

Note that neither g nor g' is defined when $\sin(x) = 0$, which occurs at every integer multiple of π. Hence we have the following rule.

Cotangent Function

For all real numbers x such that $x \neq k\pi$, where $k = 0, \pm 1, \pm 2, \ldots$,

$$\frac{d}{dx}[\cot(x)] = -\csc^2(x). \tag{6.3.11}$$

Observe that the shortcut rule for the cotangent function is very similar to the rule we discovered in Preview Activity 2.4 for the tangent function.

Tangent Function

For all real numbers x such that $x \neq \frac{(2k+1)\pi}{2}$, where $k = \pm 1, \pm 2, \ldots$,

$$\frac{d}{dx}[\tan(x)] = \sec^2(x). \tag{6.3.12}$$

In the next two activities, we develop the rules for differentiating the secant and cosecant functions.

Activity 6.3.1

Let h(x) = sec(x) and recall that sec(x) = 1 cos(x) .

a. What is the domain of h?
b. Use the quotient rule to develop a formula for h 0 (x) that is expressed completely in terms of sin(x) and cos(x).
c. How can you use other relationships among trigonometric functions to write h 0 (x) only in terms of tan(x) and sec(x)? (d) What is the domain of h 0 ? How does this compare to the domain of h? C 116 Activity 2.11. Let p(x) = csc(x) and recall that csc(x) = 1 sin(x) . (a) What is the domain of p? (b) Use the quotient rule to develop a formula for p 0 (x) that is expressed completely in terms of sin(x) and cos(x). (c) How can you use other relationships among trigonometric functions to write p 0 (x) only in terms of cot(x) and csc(x)?
d. What is the domain of p 0 ? How does this compare to the domain of p?

The quotient rule has thus enabled us to determine the derivatives of the tangent, cotangent, secant, and cosecant functions, expanding our overall library of basic functions we can differentiate. Moreover, we observe that just as the derivative of any polynomial function is a polynomial, and the derivative of any exponential function is another exponential function, so it is that the derivative of any basic trigonometric function is another function that consists of basic trigonometric functions. This makes sense because all trigonometric functions are periodic, and hence their derivatives will be periodic, too.

As has been and will continue to be the case throughout our work in Chapter 2, the derivative retains all of its fundamental meaning as an instantaneous rate of change and as the slope of the tangent line to the function under consideration. Our present work primarily expands the list of functions for which we can quickly determine a formula for the derivative. Moreover, with the addition of tan(x), cot(x), sec(x), and csc(x) to our library of basic functions, there are many more functions we can differentiate through the sum, constant multiple, product, and quotient rules.

Activity 6.3.2

Answer each of the following questions. Where a derivative is requested, be sure to label the derivative function with its name using proper notation.

a. Let f (x) = 5 sec(x) - 2 csc(x). Find the slope of the tangent line to f at the point where x = π 3 .
b. Let p(z) = z 2 sec(z) - z cot(z). Find the instantaneous rate of change of p at the point where z = π 4 .
c. Let h(t) = tan(t) t 2 + 1 - 2e t cos(t). Find h 0 (t).
d. Let g(r) = r sec(r) 5 r . Find g' (r).
e. When a mass hangs from a spring and is set in motion, the object's position oscillates in a way that the size of the oscillations decrease. This is usually called 117 a damped oscillation. Suppose that for a particular object, its displacement from equilibrium (where the object sits at rest) is modeled by the function s(t) = 15 sin(t) e t . Assume that s is measured in inches and t in seconds. Sketch a graph of this function for t ≥ 0 to see how it represents the situation described. Then

compute ds/dt, state the units on this function, and explain what it tells you about the object's motion. Finally, compute and interpret s 0 (2).

Summary

In this section, we encountered the following important ideas:

- The derivatives of the other four trigonometric functions are d dx [tan(x)] = sec2 (x), d dx [cot(x)] = - csc2 (x), d dx [sec(x)] = sec(x)tan(x), and d dx [csc(x)] = - csc(x) cot(x). Each derivative exists and is defined on the same domain as the original function. For example, both the tangent function and its derivative are defined for all real numbers x such that x , kπ 2 , where k = ±1, ±2, ...
- The above four rules for the derivatives of the tangent, cotangent, secant, and cosecant can be used along with the rules for power functions, exponential functions, and the sine and cosine, as well as the sum, constant multiple, product, and quotient rules, to quickly differentiate a wide range of different functions.

Supplemental video



Homework Exercises 6.3

WeBWorK Problems:

1.

2.

https://math.libretexts.org/@go/page/106370

3.

https://math.libretexts.org/@go/page/106370

Written Problems:

1. Find the equation of the tangent line to the curve $y = 3tan(x)$ at the point $(\frac{pi}{4}, 3)$. You may leave your answer in point slope form without simplifying to $y = mx + b$.

2. An object moving vertically has its height at time t (measured in feet, with time in seconds) given by the function $h(t) = 3 + \frac{2cos(t)}{1.2^t}$.

a. What is a function that describes the object's instantaneous velocity at any time?

b. Find the approximate (to two decimal places) instantaneous velocity of the object at $t = 2$ and at $t = 2.1$

c. Describe in everyday language the behaviour of the object at the instant $t = 2$. Make sure to include the speed, direction, and whether it is slowing down or speeding up. It might be helpful to look at the graph of $h(t)$ at $t = 2$. Also note what the sign of the second derivative of $h(t)$ should be.

This page titled 6.3: Derivatives of Other Trigonometric Functions is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Matthew Boelkins, David Austin & Steven Schlicker (ScholarWorks @Grand Valley State University) via source content that was edited to the style and standards of the LibreTexts platform.

- **2.4: Derivatives of Other Trigonometric Functions** is licensed CC BY-SA 4.0. Original source: https://activecalculus.org/single.

6.4: The Chain Rule

Preview

Here is a good review of Compositions of Functions, if needed:

3.4: Composition of Functions

Preview Video:

This optional summary video from the textbook author might be helpful to use as a preview. Other, more detailed, supplemental videos for this section are posted at the end of the text.



Learning Objectives

In this section, we strive to understand the ideas generated by the following important questions:

- What is a composite function and how do we recognize its structure algebraically?
- Given a composite function $C(x) = f(g(x))$ that is built from differentiable functions f and g, how do we compute $C'(x)$ in terms of f, g, f', and g'?
- What is the statement of the Chain Rule?

In addition to learning how to differentiate a variety of basic functions, we have also been developing our ability to understand how to use rules to differentiate certain algebraic combinations of them. For example, we not only know how to take the derivative of $f(x) = sin(x)$ and $g(x) = x^2$, but now we can quickly find the derivative of each of the following combinations of f and g:

$$s(x) = 3x^2 - 5\sin(x) \tag{6.4.1}$$

$$p(x) = x^2 \sin(x), \tag{6.4.2}$$

and

$$q(x) = \sin(x)x^2. \tag{6.4.3}$$

Finding s' uses the sum and constant multiple rules, determining p' requires the product rule, and q' can be attained with the quotient rule. Again, we note the importance of recognizing the algebraic structure of a given function in order to find its derivative:

$$s(x) = 3g(x) - 5f(x) \tag{6.4.4}$$

$$p(x) = g(x) \cdot f(x), \tag{6.4.5}$$

and

$$q(x) = \frac{f(x)}{g(x)}. \tag{6.4.6}$$

There is one more natural way to algebraically combine basic functions, and that is by composing them. For instance, let's consider the function

$$C(x) = \sin(x^2) \tag{6.4.7}$$

and observe that any input x passes through a chain of functions. In particular, in the process that defines the function $C(x)$, x is first squared, and then the sine of the result is taken. Using an arrow diagram,

$$x \to x^2 \to \sin(x^2). \tag{6.4.8}$$

In terms of the elementary functions f and g, we observe that x is first input in the function g, and then the result is used as the input in f. Said differently, we can write

$$C(x) = f(g(x)) = \sin(x^2) \tag{6.4.9}$$

and say that C is the *composition* of f and g. We will refer to g, the function that is first applied to x, as the *inner function*, while f, the function that is applied to the result, is the *outer function*.

The main question that we answer in the present section is: given a composite function $C(x) = f(g(x))$ that is built from differentiable functions f and g, how do we compute C'(x) in terms of f, g, f', and g'? In the same way that the rate of change of a product of two functions,

$$p(x) = f(x) \cdot g(x), \tag{6.4.10}$$

depends on the behavior of both f and g, it makes sense intuitively that the rate of change of a composite function $C(x) = f(g(x))$ will also depend on some combination of f and g and their derivatives. The rule that describes how to compute C' in terms of f and g and their derivatives will be called the **chain rule**. But before we can learn what the chain rule says and why it works, we first need to be comfortable decomposing composite functions so that we can correctly identify the inner and outer functions, as we did in the example above with

$$C(x) = \sin(x^2). \tag{6.4.11}$$

Preview Activity 6.4.1:

For each function given below, identify its fundamental algebraic structure. In particular, is the given function a sum, product, quotient, or composition of basic functions? If the function is a composition of basic functions, state a formula for the inner function g and the outer function f so that the overall composite function can be written in the form $f(g(x))$. If the function is a sum, product, or quotient of basic functions, use the appropriate rule to determine its derivative.

a. $h(x) = tan(2^x)$
b. $p(x) = 2^x \tan(x)$
c. $r(x) = (\tan(x))^2$
d. $m(x) = e^{\tan(x)}$
e. $w(x) = \sqrt{x} + \tan(x)$
f. $z(x) = \sqrt{\tan(x)}$

The Chain Rule

One of the challenges of differentiating a composite function is that it often cannot be written in an alternate algebraic form. For instance, the function

$$C(x) = \sin(x^2) \tag{6.4.12}$$

cannot be expanded or otherwise rewritten, so it presents no alternate approaches to taking the derivative. But other composite functions can be expanded or simplified, and these present a way to begin to explore how the chain rule might have to work. To that end, we consider two examples of composite functions that present alternate means of finding the derivative.

Example 6.4.1: Derivative of a Composite Function of Polynomials

Let $f(x) = -4x + 7$ and $g(x) = 3x - 5$. Determine a formula for $C(x) = f(g(x))$ and compute $C'(x)$. How is C' related to f and g and their derivatives?

Solution

By the rules given for f and g,

$$\begin{aligned} C(x) &= f(g(x)) \\ &= f(3x-5) \\ &= -4(3x-5)+7 \\ &= -12x+20+7 \\ &= -12x+27. \end{aligned} \tag{6.4.13}$$

Thus, $C'(x) = -12$. Noting that $f'(x) = -4$ and $g'(x) = 3$, we observe that C' appears to be the product of f' and g'.

From one perspective, Example 6.4.1 may be too elementary. Linear functions are the simplest of all functions, and perhaps composing linear functions (which yields another linear function) does not exemplify the true complexity that is involved with differentiating a composition of more complicated functions. At the same time, we should remember the perspective that any differentiable function is locally linear, so any function with a derivative behaves like a line when viewed up close. From this point of view, the fact that the derivatives of f and g are multiplied to find the derivative of their composition turns out to be a key insight. We now consider a second example involving a nonlinear function to gain further understanding of how differentiating a composite function involves the basic functions that combine to form it.

Example 6.4.2: Derivative of a Composite Function involving trigonometric functiosn

Let $C(x) = \sin(2x)$. Use the double angle identity to rewrite C as a product of basic functions, and use the product rule to find C'. Rewrite C' in the simplest form possible.

Solution

By the double angle identity for the sine function

$$C(x) = \sin(2x) = 2\sin(x)\cos(x). \tag{6.4.14}$$

Applying the product rule and simplifying,

$$C'(x) = 2\sin(x)(-\sin(x)) + \cos(x)(2\cos(x)) = 2(\cos^2(x) - \sin^2(x)). \tag{6.4.15}$$

Next, we recall that one of the double angle identities for the cosine function tells us that

$$\cos(2x) = \cos^2(x) - \sin^2(x). \tag{6.4.16}$$

Substituting this result in our expression for $C'(x)$, we now have that $C'(x) = 2\cos(2x)$.

So from Example 6.4.2, we see that if $C(x) = \sin(2x)$, then $C'(x) = 2\cos(2x)$. Letting $g(x) = 2x$ and $f(x) = \sin(x)$, we observe that $C(x) = f(g(x))$. Moreover, with $g'(x) = 2$ and $f'(x) = \cos(x)$, it follows that we can view the structure of $C'(x)$ as

$$C'(x) = 2\cos(2x) = g'(x)f'(g(x)). \tag{6.4.17}$$

In this example, we see that for the composite function $C(x) = f(g(x))$, the derivative C' is (as in the example involving linear functions) constituted by multiplying the derivatives of f and g, but with the special condition that f' is evaluated at $g(x)$, rather than at x.

It makes sense intuitively that these two quantities are involved in understanding the rate of change of a composite function: if we are considering $C(x) = f(g(x))$ and asking how fast C is changing at a given x value as x changes, it clearly matters how fast g is

changing at x, as well as how fast f is changing at the value of $g(x)$. It turns out that this structure holds not only for the functions in Examples 6.4.1 and 6.4.2, but indeed for all differentiable functions as is stated in the Chain Rule (Like other differentiation rules, the Chain Rule can be proved formally using the limit definition of the derivative).

Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function C defined by $C(x) = f(g(x))$ is differentiable at x and

$$C'(x) = f'(g(x))g'(x). \tag{6.4.18}$$

As with the product and quotient rules, it is often helpful to think verbally about what the chain rule says:

> *If C is a composite function defined by an outer function f and an inner function g, then C' is given by the derivative of the outer function, evaluated at the inner function, times the derivative of the inner function.*

At least initially in working particular examples requiring the chain rule, it can also be helpful to clearly identify the inner function g and outer function f, compute their derivatives individually, and then put all of the pieces together to generate the derivative of the overall composite function. To see what we mean by this, consider the function

$$r(x) = (\tan(x))^2. \tag{6.4.19}$$

The function r is composite, with inner function $g(x) = \tan(x)$ and outer function $f(x) = x^2$. Organizing the key information involving f, g, and their derivatives, we have

- $f(x) = x^2$
- $g(x) = \tan(x)$
- $f'(x) = 2x$
- $g'(x) = \sec^2(x)$
- $f'(g(x)) = 2\tan(x)$

Applying the chain rule (Equation 6.4.18), which tells us that

$$r'(x) = f'(g(x))g'(x), \tag{6.4.20}$$

we find that for $r(x) = (\tan(x))^2$, its derivative is

$$r'(x) = 2\tan(x)\sec^2(x). \tag{6.4.21}$$

As a side note, we remark that another way to write $r(x)$ is $r(x) = \tan^2(x)$. Observe that in this format, the composite nature of the function is more implicit, but this is common notation for powers of trigonometric functions: $cos^4(x)$, $sin^5(x)$, and $\sec^2(x)$ are all composite functions, with the outer function a power function and the inner function a trigonometric one.

The chain rule now substantially expands the library of functions we can differentiate, as the following activity demonstrates.

Activity 6.4.1: Inner vs. Outer Functions

For each function given below, identify an inner function g and outer function f to write the function in the form $f(g(x))$. Then, determine $f'(x)$, $g'(x)$, and $f'(g(x))$, and finally apply the chain rule (Equation 6.4.18) to determine the derivative of the given function.

a. $h(x) = \cos(x^4)$
b. $p(x) = \sqrt{\tan(x)}$
c. $s(x) = 2^{\sin(x)}$
d. $z(x) = \cot^5(x)$
e. $m(x) = (\sec(x) + e^x)^9$

Using multiple rules simultaneously The chain rule now joins the sum, constant multiple, product, and quotient rules in our collection of the different techniques for finding the derivative of a function through understanding its algebraic structure and the basic functions that constitute it. It takes substantial practice to get comfortable with navigating multiple rules in a single problem; using proper notation and taking a few extra steps can be particularly helpful as well. We demonstrate with an example and then provide further opportunity for practice in the following activity.

Example 6.4.3: A more Complex Composite Function

Find a formula for the derivative of

$$h(t) = 3^{t^2+2t}\sec^4(t). \tag{6.4.22}$$

Solution

We first observe that the most basic structure of h is that it is the product of two functions:

$$h(t) = a(t) \cdot b(t) \tag{6.4.23}$$

where $a(t) = 3^{t^2+2t}$ and $b(t) = \sec^4(t)$. Therefore, we see that we will need to use the product rule to differentiate h. When it comes time to differentiate a and b in their roles in the product rule, we observe that since each is a composite function, the chain rule will be needed. We therefore begin by working separately to compute $a'(t)$ and $b'(t)$.

Writing

$$a(t) = f(g(t)) = 3^{t^2+2t} \tag{6.4.24}$$

and finding the derivatives of f and g, we have

- \(f (t) = 3^t
- $g(t) = t^2 + 2t$
- $f'(t) = 3^t \ln(3)$
- $g'(t) = 2t + 2$
- $f'(g(t)) = 3^{t^2+2t} \ln(3)$

Thus, by the chain rule, it follows that

$$a'(t) = f'(g(t))g'(t) = 3^{t^2+2t} \ln(3)(2t + 2). \tag{6.4.25}$$

Turning next to b, we write

$$b(t) = r(s(t)) = \sec^4(t) \tag{6.4.26}$$

and find the derivatives of r and g. Doing so,

- $r(t) = t^4$
- $s(t) = \sec(t)$
- $r'(t) = 4t^3$
- $s'(t) = \sec(t)\tan(t)$
- $r'(s(t)) = 4\sec^3(t)$

By the chain rule (Equation 6.4.18), we now know that

$$b'(t) = r'(s(t))s'(t) = 4\sec^3(t)\sec(t)\tan(t) = 4\sec^4(t)\tan(t). \tag{6.4.27}$$

Now we are finally ready to compute the derivative of the overall function h. Recalling that

$$h(t) = 3^{t^2+2t}\sec^4(t), \tag{6.4.28}$$

by the product rule we have

$$h'(t) = 3t2 + 2t\frac{d}{dt}[\sec^4(t)] + \sec^4(t)\frac{d}{dt}\left[3^{t^2+2t}\right]. \tag{6.4.29}$$

From our work above with a and b, we know the derivatives of 3^{t^2+2t} and $\sec^4(t)$, and therefore

$$h'(t) = 3^{t^2+2t} 4\sec^4(t)\tan(t) + \sec^4(t) 3^{t^2+2t} \ln(3)(2t+2). \tag{6.4.30}$$

Activity 6.4.2

For each of the following functions, find the function's derivative. State the rule(s) you use, label relevant derivatives appropriately, and be sure to clearly identify your overall answer.

a. $p(r) = 4\sqrt{r^6 + 2e^r}$
b. $m(v) = \sin(v^2)\cos(v^3)$
c. $h(y) = \dfrac{\cos(10y)}{e^{4y}+1}$
d. $s(z) = 2^{z^2\sec(z)}$
e. $c(x) = \sin(e^{x^2})$

The chain rule now adds substantially to our ability to do different familiar problems that involve derivatives. Whether finding the equation of the tangent line to a curve, the instantaneous velocity of a moving particle, or the instantaneous rate of change of a certain quantity, if the function under consideration involves a composition of other functions, the chain rule is indispensable.

Activity 6.4.3

Use known derivative rules, including the chain rule, as needed to answer each of the following questions.

a. Find an equation for the tangent line to the curve $y = \sqrt{e^x + 3}$ at the point where $x = 0$.
b. If

$$s(t) = \frac{1}{(t^2+1)^3} \tag{6.4.31}$$

represents the position function of a particle moving horizontally along an axis at time t (where s is measured in inches and t in seconds), find the particle's instantaneous velocity at $t = 1$. Is the particle moving to the left or right at that instant?

c. At sea level, air pressure is 30 inches of mercury. At an altitude of h feet above sea level, the air pressure, P, in inches of mercury, is given by the function

$$P = 30e^{-0.0000323h}. \tag{6.4.32}$$

Compute dP/dh and explain what this derivative function tells you about air pressure, including a discussion of the units on dP/dh. In addition, determine how fast the air pressure is changing for a pilot of a small plane passing through an altitude of 1000 feet.

d. Suppose that $f(x)$ and $g(x)$ are differentiable functions and that the following information about them is known:

x	f (x)	f'(x)	g(x)	g'(x)
-1	2	-5	-3	4
2	-3	4	-1	2

If $C(x)$ is a function given by the formula $f(g(x))$, determine $C'(2)$. In addition, if $D(x)$ is the function $f(f(x))$, find $D'(-1)$.

The Composite Version of Basic Function Rules

As we gain more experience with differentiating complicated functions, we will become more comfortable in the process of simply writing down the derivative without taking multiple steps. We demonstrate part of this perspective here by showing how we can find a composite rule that corresponds to two of our basic functions. For instance, we know that

$$\frac{d}{dx}[\sin(x)] = \cos(x). \tag{6.4.33}$$

If we instead want to know

$$\frac{d}{dx}[\sin(u(x))], \tag{6.4.34}$$

where u is a differentiable function of x, then this requires the chain rule with the sine function as the outer function. Applying the chain rule (Equation 6.4.18),

$$\frac{d}{dx}[\sin(u(x))] = \cos(u(x)) \cdot u'(x). \tag{6.4.35}$$

Similarly, since

$$\frac{d}{dx}[a^x] = a^x \ln(a), \tag{6.4.36}$$

it follows by the chain rule that

$$\frac{d}{dx}\left[a^{u(x)}\right] = a^{u(x)} \ln(a) \cdot u'(x). \tag{6.4.37}$$

In the process of getting comfortable with derivative rules, an excellent exercise is to write down a list of all basic functions whose derivatives are known, list those derivatives, and then write the corresponding chain rule for the composite version with the inner function being an unknown function $u(x)$ and the outer function being the known basic function. These versions of the chain rule are particularly simple when the inner function is linear, since the derivative of a linear function is a constant. For instance,

$$\frac{d}{dx}(5x+7)^{10} = 10(5x+7)^9 \cdot 5 \tag{6.4.38}$$

$$\frac{d}{dx}[\tan(17x)] = 17\sec^2(17x), \tag{6.4.39}$$

and

$$\frac{d}{dx}\left[e^{-3x}\right] = -3e^{-3x}. \tag{6.4.40}$$

Summary

In this section, we encountered the following important ideas:

- A composite function is one where the input variable x first passes through one function, and then the resulting output passes through another. For example, the function

  $$h(x) = 2^{\sin(x)} \tag{6.4.41}$$

  is composite since

  $$x \to \sin(x) \to 2^{\sin(x)}. \tag{6.4.42}$$

- Given a composite function $C(x) = f(g(x))$ that is built from differentiable functions f and g, the chain rule tells us that we compute $C'(x)$ in terms of f, g, f', and g' according to the formula

  $$C'(x) = f'(g(x))g'(x). \tag{6.4.43}$$

Supplemental Videos

Example of the Chain rule - polynomials



Example of Chain Rule with Radicals



Example of Chain Rule with Trig Functions



Example of Chain Rule with Exponential Functions



Examples mixing rules



Example of Chain rule when given the graphs



Homework Exercises 6.4

WeBWorK Problems:

1.

https://math.libretexts.org/@go/page/106371

2.

3.

4.

5.

Written Problems:

1. If a tank holds 4000 gallons of water, which drains from the bottom of the tank in 40 minutes, then Torricelli's Law gives the volume V of water remaining in the tank after t minutes as

$$V = 4000(1 - \tfrac{t}{40})^2, \, 0 \leq t \leq 40 \, .$$

a. Find the rate at which the water is draining out of the tank after t minutes.

b. Find the rate at which the water is draining out of the tank after 10 minutes.

2. A mass attached to a vertical spring has position function given by $s(t) = 4sin(2t)$ where t is measured in seconds and s in inches.

a. Find the velocity at $t = 2$.

b. Find the acceleration at $t = 2$.

3. The motion of a spring that is subject to a frictional force or a damping force (such as a shock absorber in a car) is often modeled by the product of an exponential function and a sine or cosine function. Suppose that the equation of motion of a point on a spring is $s(t) = 2e^{-1.5t}sin(2\pi t)$, where s is measured in centimeters and t in seconds. Find the velocity after t seconds.

4.

Let $u(x)$ be a differentiable function. For each of the following functions, determine the derivative. Each response will involve u and/or u'.

1. $p(x) = e^{u(x)}$
2. $q(x) = u(e^x)$
3. $r(x) = \cot(u(x))$
4. $s(x) = u(\cot(x))$
5. $a(x) = u(x^4)$
6. $b(x) = u^4(x)$

5. If a spherical tank of radius 4 feet has h feet of water present in the tank, then the volume of water in the tank is given by the formula $V = \frac{\pi}{3}h^2(12 - h)$.

a. At what instantaneous rate is the volume of water in the tank changing with respect to the *height* of the water at the instant h=1? What are the units on this quantity?

b. Now suppose that the height of water in the tank is being regulated by an inflow and outflow (e.g., a faucet and a drain) so that the height of the water at time t is given by the rule $h(t) = sin(\pi t) + 1$, where t is measured in hours (and h is still measured in feet). At what rate is the height of the water changing with respect to time at the instant t =2?

c. Continuing under the assumptions in (b), at what instantaneous rate is the volume of water in the tank changing with respect to *time* at the instant t =2?

d. What are the main differences between the rates found in (a) and (c)? Include a discussion of the relevant units.

This page titled 6.4: The Chain Rule is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Matthew Boelkins, David Austin & Steven Schlicker (ScholarWorks @Grand Valley State University) via source content that was edited to the style and standards of the LibreTexts platform.

- **2.5: The Chain Rule** is licensed CC BY-SA 4.0. Original source: https://activecalculus.org/single.
- **2.5: The Chain Rule** by Matthew Boelkins, David Austin & Steven Schlicker is licensed CC BY-SA 4.0. Original source: https://activecalculus.org/single.

6.5: Derivatives of Functions Given Implicitely

Preview Video

This optional video from the textbook author will be helpful for completing the preview activity. The summary video and other supplemental videos for this section are posted at the end of the text.



Learning Objectives

In this section, we strive to understand the ideas generated by the following important questions:

- What does it mean to say that a curve is an implicit function of x, rather than an explicit function of x?
- How does implicit differentiation enable us to find a formula for dy/dx when y is an implicit function of x?
- In the context of an implicit curve, how can we use $\frac{dy}{dx}$ to answer important questions about the tangent line to the curve?

In all of our studies with derivatives to date, we have worked in a setting where we can express a formula for the function of interest explicitly in terms of x. But there are many interesting curves that are determined by an equation involving x and y for which it is impossible to solve for y in terms of x. Perhaps the simplest and most natural of all such curves are circles. Because of the circle's symmetry, for each x value strictly between the endpoints of the horizontal diameter, there are two corresponding y-values.

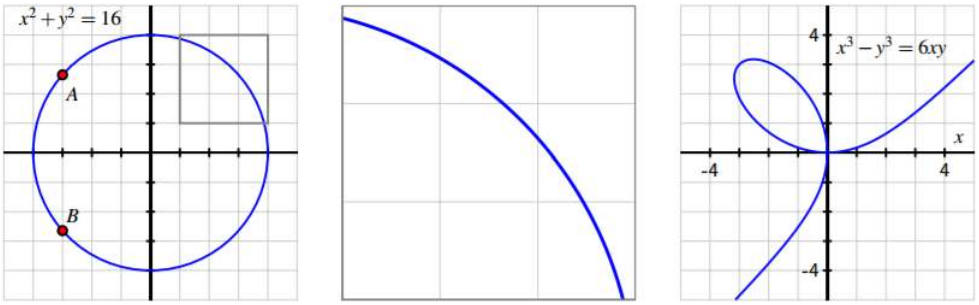


***Figure* 6.5.1:** *At left, the circle given by $x^2+y^2=16$. In the middle, the portion of the circle $x^2+y^2=16$ that has been highlighted in the box at left. And at right, the lemniscate given by $x^3-y^3=6xy$.*

For instance, in Figure 6.5.1, we have labeled $A=(-3,\sqrt{7})$ and $B=(-3,-\sqrt{7})$, and these points demonstrate that the circle fails the vertical line test. Hence, it is impossible to represent the circle through a single function of the form $y=f(x)$. At the same time, portions of the circle can be represented explicitly as a function of x, such as the highlighted arc that is magnified in the center of Figure 6.5.1. Moreover, it is evident that the circle is locally linear, so we ought to be able to find a tangent line to the curve at every point; thus, it makes sense to wonder if we can compute dy dx at any point on the circle, even though we cannot

write y explicitly as a function of x. Finally, we note that the righthand curve in Figure 6.5.1 is called a *lemniscate* and is just one of many fascinating possibilities for implicitly given curves.

In working with implicit functions, we will often be interested in finding an equation for dy dx that tells us the slope of the tangent line to the curve at a point (x, y). To do so, it will be necessary for us to work with y while thinking of y as a function of x, but without being able to write an explicit formula for y in terms of x. The following preview activity reminds us of some ways we can compute derivatives of functions in settings where the function's formula is not known. For instance, recall the earlier example

$$\frac{d}{dx}[e^{u(x)}] = e^{u(x)}u'(x). \tag{6.5.1}$$

Preview Activity 6.5.1

Let f be a differentiable function of x (whose formula is not known) and recall that $\frac{d}{dx}[f(x)]$ and $f'(x)$ are interchangeable notations. Determine each of the following derivatives of combinations of explicit functions of x, the unknown function f, and an arbitrary constant c.

a. $\frac{d}{dx}x^2 + f(x)$
b. $\frac{d}{dx}x^2 f(x)$
c. $\frac{d}{dx}c + x + f(x)^2$
d. $\frac{d}{dx}f(x^2)$
e. $\frac{d}{dx}[xf(x) + f(cx) + cf(x)]$

Implicit Differentiation

Because a circle is perhaps the simplest of all curves that cannot be represented explicitly as a single function of x, we begin our exploration of implicit differentiation with the example of the circle given by

$$x^2 + y^2 = 16. \tag{6.5.2}$$

It is visually apparent that this curve is locally linear, so it makes sense for us to want to find the slope of the tangent line to the curve at any point, and moreover to think that the curve is differentiable. The big question is: how do we find a formula for $\frac{dy}{dx}$, the slope of the tangent line to the circle at a given point on the circle? By viewing y as an implicit function of x, we essentially think of y as some function whose formula $f(x)$ is unknown, but which we can differentiate. Just as y represents an unknown formula, so too its derivative with respect to x, $\frac{dy}{dx}$, will be (at least temporarily) unknown.

Consider Equation 6.5.2 and view y as an unknown differentiable function of x. Differentiating both sides Equation 6.5.2 with respect to x, we have

$$\frac{d}{dx}\left[x^2 + y^2\right] = \frac{d}{dx}[16]. \tag{6.5.3}$$

On the right side of Equation 6.5.3, the derivative of the constant 16 is 0, and on the left we can apply the *sum rule*, so it follows that

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = 0. \tag{6.5.4}$$

Next, it is essential that we recognize the different roles being played by x and y. Since x is the independent variable, it is the variable with respect to which we are differentiating, and thus

$$\frac{d}{dx}[x^2] = 2x. \tag{6.5.5}$$

But y is the dependent variable and y is an implicit function of x. Thus, when we want to compute $\frac{d}{dx}[y^2]$ it is identical to the situation in Preview Activity 6.5.1 where we computed

$$\frac{d}{dx}[f(x)^2]. \tag{6.5.6}$$

In both situations, we have an unknown function being squared, and we seek the derivative of the result. This requires the chain rule, by which we find that

$$\frac{d}{dx}[y^2] = 2y^1\frac{dy}{dx}. \tag{6.5.7}$$

Therefore, continuing our work in differentiating both sides of Equation 6.5.2, we now have that

$$2x + 2y\frac{dy}{dx} = 0. \tag{6.5.8}$$

To find an expression for $\frac{dy}{dx}$, we solve for it in Equation 6.5.8. Subtracting $2x$ from both sides and dividing by $2y$,

$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}. \tag{6.5.9}$$

There are several important things to observe about the result that

$$\frac{d}{dx} = -\frac{x}{y}. \tag{6.5.10}$$

First, this expression for the derivative involves both x and y. It makes sense that this should be the case, since for each value of x between -4 and 4, there are two corresponding points on the circle, and the slope of the tangent line is different at each of these points. Second, this formula is entirely consistent with our understanding of circles. If we consider the radius from the origin to the point (a, b), the slope of this line segment is $m_r = ba$. The tangent line to the circle at (a, b) will be perpendicular to the radius, and thus have slope $m_t = -ab$, as shown in Figure 6.5.2. Finally, the slope of the tangent line is zero at $(0, 4)$ and $(0, -4)$, and is undefined at $(-4, 0)$ and $(4, 0)$; all of these values are consistent with Equation 6.5.10.

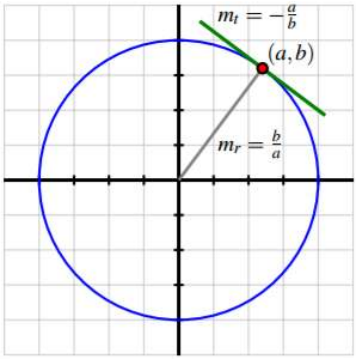


***Figure* 6.5.2**: *The circle given by $x^2 + y^2 = 16$ with point (a, b) on the circle and the tangent line at that point, with labeled slopes of the radial line, m_r, and tangent line, m_t.*

Essentially the idea of an implicit function is that it can be broken into pieces where each piece can be viewed as an explicit function of x, and the combination of those pieces constitutes the full implicit function. For the circle, we could choose to take the top half as one explicit function of x, and the bottom half as another.

We consider the following more complicated example to investigate and demonstrate some additional algebraic issues that arise in problems involving implicit differentiation.

Example 6.5.1

For the curve given implicitly by

$$x^3 + y^2 - 2xy = 2 \tag{6.5.11}$$

shown in Figure 6.5.3. Find the slope of the tangent line at $(-1, 1)$.

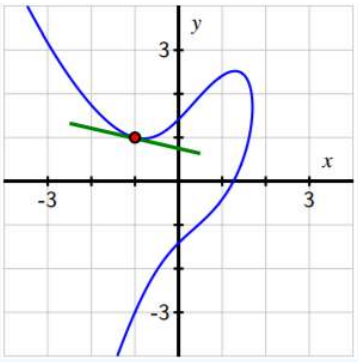


***Figure* 6.5.3**: *The curve* $x^3+y^2-2xy=2$.

Solution

We begin by differentiating the curve's equation implicitly. Taking the derivative of each side of Equation 6.5.11 with respect to x,

$$\frac{d}{dx}\left[x^3+y^2-2xy\right]=\frac{d}{dx}[2], \tag{6.5.12}$$

by the sum rule and the fact that the derivative of a constant is zero, we have

$$\frac{d}{dx}[x^3]+\frac{d}{dx}[y^2]-\frac{d}{dx}[2xy]=0. \tag{6.5.13}$$

For the three derivatives we now must execute, the first uses the simple power rule, the second requires the chain rule (since y is an implicit function of x), and the third necessitates the product rule (again since y is a function of x). Applying these rules, we now find that

$$3x^2+2y\frac{dy}{dx}-\left[2x\frac{dy}{dx}+2y\right]=0. \tag{6.5.14}$$

Remembering that our goal is to find an expression for $\frac{dy}{dx}$ so that we can determine the slope of a particular tangent line, we want to solve the preceding equation for $\frac{dy}{dx}$. To do so, we get all of the terms involving $\frac{dy}{dx}$ on one side of the equation and then factor. Expanding and then subtracting $3x^2-2y$ from both sides, it follows that

$$2y\frac{dy}{dx}-2x\frac{dy}{dx}=2y-3x^2. \tag{6.5.15}$$

Factoring the left side to isolate $\frac{dy}{dx}$, we have

$$\frac{dy}{dx}(2y-2x)=2y-3x^2. \tag{6.5.16}$$

Finally, we divide both sides by $(2y-2x)$ and conclude that

$$\frac{dy}{dx}=\frac{2y-3x^2}{2y-2x}. \tag{6.5.17}$$

Here again, the expression for $\frac{dy}{dx}$ depends on both x and y. To find the slope of the tangent line at (-1, 1), we substitute this point in the formula for $\frac{dy}{dx}$, using the notation

$$\left.\frac{dy}{dx}\right|_{(-1,1)}=\frac{2(1)-3(-1)^2}{2(1)-2(-1)}=-\frac{1}{4}. \tag{6.5.18}$$

This value matches our visual estimate of the slope of the tangent line shown in Figure 6.5.3.

Example 6.5.1 shows that it is possible when differentiating implicitly to have multiple terms involving $\frac{dy}{dx}$. Regardless of the particular curve involved, our approach will be 145 similar each time. After differentiating, we expand so that each side of the

equation is a sum of terms, some of which involve $\frac{dy}{dx}$. Next, addition and subtraction are used to get all terms involving dy dx on one side of the equation, with all remaining terms on the other. Finally, we factor to get a single instance of $\frac{dy}{dx}$, and then divide to solve for $\frac{dy}{dx}$.

Note, too, that since $\frac{dy}{dx}$ is often a function of both x and y, we use the notation

$$\left.\frac{dy}{dx}\right|_{(a,b)} \tag{6.5.19}$$

to denote the evaluation of $\frac{dy}{dx}$ at the point (a, b). This is analogous to writing $f'(a)$ when f' depends on a single variable.

Finally, there is a big difference between writing $\frac{d}{dx}$ and $\frac{dy}{dx}$. For example,

$$\frac{d}{dx}[x^2+y^2] \tag{6.5.20}$$

gives an instruction to take the derivative with respect to x of the quantity x^2+y^2, presumably where y is a function of x. On the other hand,

$$\frac{dy}{dx}(x^2+y^2) \tag{6.5.21}$$

means the product of the derivative of y with respect to x with the quantity x^2+y^2. Understanding this notational subtlety is essential. The following activities present opportunities to explore several different problems involving implicit differentiation.

Activity 6.5.1

Consider the curve defined by the equation $x(y)=y^5-5y^3+4y$, whose graph is pictured in Figure 6.5.4.

a. Explain why it is not possible to express y as an explicit function of x.
b. Use implicit differentiation to find a formula for $\frac{dy}{dx}$.
c. Use your result from part (b) to find an equation of the line tangent to the graph of $x(y)$ at the point (0, 1).
d. Use your result from part (b) to determine all of the points at which the graph of $x(y)$ has a vertical tangent line.

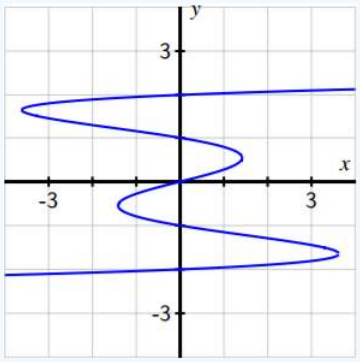


Figure 6.5.4: *The curve x = y 5 - 5y 3 + 4y.*

Two natural questions to ask about any curve involve where the tangent line can be vertical or horizontal. To be horizontal, the slope of the tangent line must be zero, while to be vertical, the slope must be undefined. It is typically the case when differentiating implicitly that the formula for dy dx is expressed as a quotient of functions of x and y, say

$$\frac{dy}{dx}=\frac{p(x,y)}{q(x,y)}. \tag{6.5.22}$$

Thus, we observe that the tangent line will be horizontal precisely when the numerator is zero and the denominator is nonzero, making the slope of the tangent line zero. Similarly, the tangent line will be vertical whenever $q(x,y)=0$ and $p(x,y)\neq 0$, making the slope undefined. If both x and y are involved in an equation such as $p(x,y)=0$, we try to solve for one of them in terms of the other, and then use the resulting condition in the original equation that defines the curve to find an equation in a single variable that

we can solve to determine the point(s) that lie on the curve at which the condition holds. It is not always possible to execute the desired algebra due to the possibly complicated combinations of functions that often arise.

Exercise 6.5.2

Consider the curve defined by the equation

$$y(y^2-1)(y-2)=x(x-1)(x-2), \tag{6.5.23}$$

whose graph is pictured in Figure 6.5.5. Through implicit differentiation, it can be shown that

$$\frac{dy}{dx}=\frac{(x-1)(x-2)+x(x-2)+x(x-1)}{(y^2-1)(y-2)+2y^2(y-2)+y(y^2-1)}. \tag{6.5.24}$$

Use this fact to answer each of the following questions.

a. Determine all points (x,y) at which the tangent line to the curve is horizontal. (Use technology appropriately to find the needed zeros of the relevant polynomial function.)
b. Determine all points (x,y) at which the tangent line is vertical. (Use technology appropriately to find the needed zeros of the relevant polynomial function.)
c. Find the equation of the tangent line to the curve at one of the points where $x=1$.

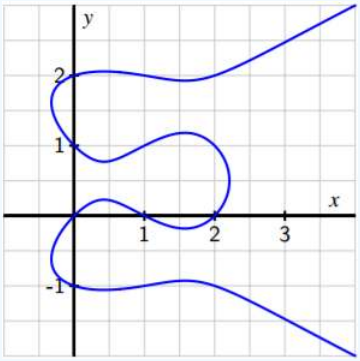


Figure 6.5.5: *The curve* $y(y^2-1)(y-2)=x(x-1)(x-2)$.

The closing activity in this section offers more opportunities to practice implicit differentiation.

Activity 6.5.3

For each of the following curves, use implicit differentiation to find dy/dx and determine the equation of the tangent line at the given point.

a. $x^3-y^3=6xy$ at $(-3,3)$
b. $\sin(y)+y=x^3+x$ at $(0,0)$
c. $3xe^{-xy}=y^2$ at $(0.619061,1)$

Summary

In this section, we encountered the following important ideas:

- When we have an equation involving x and y where y cannot be solved for explicitly in terms of x, but where portions of the curve can be thought of as being generated by explicit functions of x, we say that y is an implicit function of x. A good example of such a curve is the unit circle.
- In the process of implicit differentiation, we take the equation that generates an implicitly given curve and differentiate both sides with respect to x while treating y as a function of x. In so doing, the chain rule leads $\frac{dy}{dx}$ to arise, and then we may subsequently solve for $\frac{dy}{dx}$ using algebra.

- While $\frac{dy}{dx}$ may now involve both the variables x and y, $\frac{dy}{dx}$ still measures the slope of the tangent line to the curve, and thus this derivative may be used to decide when the tangent line is horizontal ($\frac{dy}{dx} = 0$) or vertical ($\frac{dy}{dx}$ is undefined), or to find the equation of the tangent line at a particular point on the curve.

Supplemental videos





Homework Exercises 6.5

1. Find $\frac{dy}{dx}$ for the implicit function:

$$x^2y - x - 5y - 11 = 0$$

2. Find the slope of the tangent line to the implicit function of *x* at the point (1, 5). Use Desmos to sketch a graph of the curve and the tangent line to show that your answer makes sense.

$$x^3 + xy + y^2 = 31$$

3. Find the equation of the tangent line to the implicit function of *x* at the point (4, 1).

$$3xy^3 + xy = 16$$

4a. Find $\frac{dy}{dx}$ for the implicit function:

$$2y^3+y^2-y^5=x^4-2x^3+x^2$$

b. Find where $\frac{dy}{dx}=0$ to find the *x*-coordinates where there are horizontal tangent lines. You should be able to solve for the zeros by factoring the numerator.

c. Use Desmos to sketch a graph of the curve to show there are nine points where there are horizontal tangent lines.

5a. For the curve given by $\sin(x+y)+\cos(x-y)=1$, find $\frac{dy}{dx}$.

b. Use Desmos to graph this implicit function. (Isn't it interesting??) Then zoom in near the point $(\frac{\pi}{2},\frac{\pi}{2})$. Give a rough estimate of $\left.\frac{dy}{dx}\right|_{(\frac{\pi}{2},\frac{\pi}{2})}$.

c. Find the exact value of the slope using your derivative function and knowledge of the unit circle. Then find the equation of the tangent line and graph this along with the implicit function on desmos.

This page titled 6.5: Derivatives of Functions Given Implicitely is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Matthew Boelkins, David Austin & Steven Schlicker (ScholarWorks @Grand Valley State University) via source content that was edited to the style and standards of the LibreTexts platform.

- **2.7: Derivatives of Functions Given Implicitely** is licensed CC BY-SA 4.0. Original source: https://activecalculus.org/single.

Index

A

absolute value equations
1.10: Other Types of Equations
adjacent side
5.2: Right Triangle Trigonometry
algebraic expression
1.1: Real Numbers - Algebra Essentials
amplitude
5.5: Graphs of the Sine and Cosine Functions
angle
5.1: Angles
angle of depression
5.2: Right Triangle Trigonometry
angle of elevation
5.2: Right Triangle Trigonometry
angular speed
5.1: Angles
arc length
5.1: Angles
area of a sector
5.1: Angles
associative property of addition
1.1: Real Numbers - Algebra Essentials
associative property of multiplication
1.1: Real Numbers - Algebra Essentials
average rate of change
2.3: Rates of Change and Behavior of Graphs

B

base
1.1: Real Numbers - Algebra Essentials

C

chain rule
6.4: The Chain Rule
commutative property of addition
1.1: Real Numbers - Algebra Essentials
completing the square
1.9: Quadratic Equations
Complex rational expressions
1.6: Rational Expressions
concavity
4.2: The Second Derivative
conditional equation
1.8: Linear Equations in One Variable
constant
1.1: Real Numbers - Algebra Essentials
constant multiple rule
4.4: Elementary Derivative Rules
continuity
3.5: Limits, Continuity, and Differentiability
cosecant
5.4: The Other Trigonometric Functions
Cosine Function
5.3: Unit Circle
5.5: Graphs of the Sine and Cosine Functions
cotangent
5.4: The Other Trigonometric Functions
coterminal angles
5.1: Angles

D

degree
5.1: Angles
degree of a polynomial
1.4: Polynomials
dependent variable
2.1: Functions and Function Notation
Derivative of cosine function
6.1: The Sine and Cosine Function
Derivative of sine function
6.1: The Sine and Cosine Function
Difference of Cubes
1.5: Factoring Polynomials
Difference of Squares
1.4: Polynomials
1.5: Factoring Polynomials
Differentiability (two variables)
3.5: Limits, Continuity, and Differentiability
discriminant
1.9: Quadratic Equations
distance formula
1.7: The Rectangular Coordinate Systems and Graphs
distributive property
1.1: Real Numbers - Algebra Essentials
domain
2.1: Functions and Function Notation

E

equation
1.1: Real Numbers - Algebra Essentials
equation in two variables
1.7: The Rectangular Coordinate Systems and Graphs
even function
2.4: Transformation of Functions
exponent
1.1: Real Numbers - Algebra Essentials
Exponential Notation
1.1: Real Numbers - Algebra Essentials
1.2: Exponents and Scientific Notation
extrema
2.3: Rates of Change and Behavior of Graphs

F

FOIL
1.4: Polynomials
formula
1.1: Real Numbers - Algebra Essentials
function
2.1: Functions and Function Notation

G

graph in two variables
1.7: The Rectangular Coordinate Systems and Graphs
greatest common factor
1.5: Factoring Polynomials
grouping method
1.9: Quadratic Equations

H

horizontal compression
2.4: Transformation of Functions
horizontal line
1.8: Linear Equations in One Variable
horizontal line test
2.1: Functions and Function Notation
horizontal reflection
2.4: Transformation of Functions
horizontal shift
2.4: Transformation of Functions
5.5: Graphs of the Sine and Cosine Functions
horizontal stretch
2.4: Transformation of Functions
Hypotenuse
5.2: Right Triangle Trigonometry

I

identities
5.4: The Other Trigonometric Functions
identity equation
1.8: Linear Equations in One Variable
identity property of addition
1.1: Real Numbers - Algebra Essentials
identity property of multiplication
1.1: Real Numbers - Algebra Essentials
implicit differentiation
6.5: Derivatives of Functions Given Implicitely
inconsistent equation
1.8: Linear Equations in One Variable
independent variable
2.1: Functions and Function Notation
initial side
5.1: Angles
inner function
6.4: The Chain Rule
input
2.1: Functions and Function Notation
integers
1.1: Real Numbers - Algebra Essentials
intercepts
1.7: The Rectangular Coordinate Systems and Graphs
Interval Notation
2.2: Domain and Range
inverse property of addition
1.1: Real Numbers - Algebra Essentials
inverse property of multiplication
1.1: Real Numbers - Algebra Essentials
irrational numbers
1.1: Real Numbers - Algebra Essentials

L

least common denominator
1.8: Linear Equations in One Variable
lemniscate
6.5: Derivatives of Functions Given Implicitely
limit
3.5: Limits, Continuity, and Differentiability
linear speed
5.1: Angles
local extrema
2.3: Rates of Change and Behavior of Graphs

local maximum
2.3: Rates of Change and Behavior of Graphs
local minimum
2.3: Rates of Change and Behavior of Graphs

M
maxima
2.3: Rates of Change and Behavior of Graphs
Mean Value Theorem
3.6: The Mean Value Theorem
measure of an angle
5.1: Angles
midline
5.5: Graphs of the Sine and Cosine Functions
Midpoint Formula
1.7: The Rectangular Coordinate Systems and Graphs
minima
2.3: Rates of Change and Behavior of Graphs

N
natural numbers
1.1: Real Numbers - Algebra Essentials
negative angle
5.1: Angles
negative numbers
1.1: Real Numbers - Algebra Essentials
Negative rule
1.2: Exponents and Scientific Notation

O
odd function
2.4: Transformation of Functions
opposite side
5.2: Right Triangle Trigonometry
order of operations
1.1: Real Numbers - Algebra Essentials
outer function
6.4: The Chain Rule
output
2.1: Functions and Function Notation

P
perfect square trinomial
1.5: Factoring Polynomials
Perfect square trinomials
1.4: Polynomials
period
5.4: The Other Trigonometric Functions
5.5: Graphs of the Sine and Cosine Functions
periodic functions
5.5: Graphs of the Sine and Cosine Functions
piecewise function
2.2: Domain and Range
polynomial
1.4: Polynomials
positive angle
5.1: Angles
Power of a product rule
1.2: Exponents and Scientific Notation
Power of a quotient rule
1.2: Exponents and Scientific Notation
power rule
1.2: Exponents and Scientific Notation
4.4: Elementary Derivative Rules
principal square root
1.3: Radicals and Rational Expressions
product rule
1.2: Exponents and Scientific Notation
properties of exponents
1.2: Exponents and Scientific Notation
Pythagorean Identity
5.3: Unit Circle
Pythagorean Theorem
1.7: The Rectangular Coordinate Systems and Graphs
1.9: Quadratic Equations

Q
quadrantal angle
5.1: Angles
quadratic equations
1.9: Quadratic Equations
quadratic formula
1.9: Quadratic Equations
quotient rule
1.2: Exponents and Scientific Notation
6.2: The Product and Quotient Rules
Quotient Rules
6.2: The Product and Quotient Rules

R
radian
5.1: Angles
radian measure
5.1: Angles
radical expressions
1.3: Radicals and Rational Expressions
radicals
1.3: Radicals and Rational Expressions
range
2.1: Functions and Function Notation
rate of change
2.3: Rates of Change and Behavior of Graphs
Rational Exponents
1.3: Radicals and Rational Expressions
1.10: Other Types of Equations
Rational expressions
1.6: Rational Expressions
rational numbers
1.1: Real Numbers - Algebra Essentials
ray
5.1: Angles
real number line
1.1: Real Numbers - Algebra Essentials
real numbers
1.1: Real Numbers - Algebra Essentials
reference angle
5.1: Angles
relation
2.1: Functions and Function Notation
Rolle’s Theorem
3.6: The Mean Value Theorem

S
scientific notation
1.2: Exponents and Scientific Notation
secant
5.4: The Other Trigonometric Functions
Second Derivative
4.2: The Second Derivative
sine function
5.3: Unit Circle
sinusoidal function
5.5: Graphs of the Sine and Cosine Functions
slope
1.8: Linear Equations in One Variable
standard position
5.1: Angles
Sum of Cubes
1.5: Factoring Polynomials
Sum Rule
4.4: Elementary Derivative Rules

T
tangent
5.4: The Other Trigonometric Functions
tangent line approximation
4.3: The Tangent Line Approximation
terminal side
5.1: Angles
the grouping method
1.9: Quadratic Equations
the quadratic formula
1.9: Quadratic Equations

U
unit circle
5.3: Unit Circle

V
variable
1.1: Real Numbers - Algebra Essentials
vertex
5.1: Angles
vertical compression
2.4: Transformation of Functions
vertical line
1.8: Linear Equations in One Variable
vertical line test
2.1: Functions and Function Notation
vertical reflection
2.4: Transformation of Functions
vertical shift
2.4: Transformation of Functions
vertical stretch
2.4: Transformation of Functions

W
whole numbers
1.1: Real Numbers - Algebra Essentials

Z
Zero exponent rule
1.2: Exponents and Scientific Notation

Glossary

Sample Word 1 | Sample Definition 1

Detailed Licensing

Overview

Title: Math 125: Hope College

Webpages: 50

Applicable Restrictions: Noncommercial

All licenses found:

- CC BY 4.0: 38% (19 pages)
- Undeclared: 32% (16 pages)
- CC BY-SA 4.0: 28% (14 pages)
- CC BY-NC-SA 4.0: 2% (1 page)

By Page

- Math 125: Hope College - *Undeclared*
  - Front Matter - *Undeclared*
    - TitlePage - *Undeclared*
    - InfoPage - *Undeclared*
    - Table of Contents - *Undeclared*
    - Licensing - *Undeclared*
  - 1: Algebra Essentials - *Undeclared*
    - 1.1: Real Numbers - Algebra Essentials - *CC BY 4.0*
    - 1.2: Exponents and Scientific Notation - *CC BY 4.0*
    - 1.3: Radicals and Rational Expressions - *CC BY 4.0*
    - 1.4: Polynomials - *CC BY 4.0*
    - 1.5: Factoring Polynomials - *CC BY 4.0*
    - 1.6: Rational Expressions - *CC BY 4.0*
    - 1.7: The Rectangular Coordinate Systems and Graphs - *CC BY 4.0*
    - 1.8: Linear Equations in One Variable - *CC BY 4.0*
    - 1.9: Quadratic Equations - *CC BY 4.0*
    - 1.10: Other Types of Equations - *CC BY 4.0*
  - 2: Function Essentials - *Undeclared*
    - 2.1: Functions and Function Notation - *CC BY 4.0*
    - 2.2: Domain and Range - *CC BY 4.0*
    - 2.3: Rates of Change and Behavior of Graphs - *CC BY 4.0*
    - 2.4: Transformation of Functions - *CC BY 4.0*
  - 3: Derivative Essentials - *Undeclared*
    - 3.1: How do we Measure Velocity? - *CC BY-SA 4.0*
    - 3.2: The Notion of Limit - *CC BY-SA 4.0*
    - 3.3: The Derivative of a Function at a Point - *CC BY-SA 4.0*
    - 3.4: The Derivative Function - *CC BY-SA 4.0*
    - 3.5: Limits, Continuity, and Differentiability - *CC BY-SA 4.0*
    - 3.6: The Mean Value Theorem - *CC BY-NC-SA 4.0*
  - 4: Applications of the Derivative - *Undeclared*
    - 4.1: Interpretating, Estimating, and Using the Derivative - *CC BY-SA 4.0*
    - 4.2: The Second Derivative - *CC BY-SA 4.0*
    - 4.3: The Tangent Line Approximation - *CC BY-SA 4.0*
    - 4.4: Elementary Derivative Rules - *CC BY-SA 4.0*
  - 5: Trigonometry Essentials - *Undeclared*
    - 5.1: Angles - *CC BY 4.0*
    - 5.2: Right Triangle Trigonometry - *CC BY 4.0*
    - 5.3: Unit Circle - *CC BY 4.0*
    - 5.4: The Other Trigonometric Functions - *CC BY 4.0*
    - 5.5: Graphs of the Sine and Cosine Functions - *CC BY 4.0*
  - 6: Differentiation Rules and Applications - *Undeclared*
    - 6.1: The Sine and Cosine Function - *CC BY-SA 4.0*
    - 6.2: The Product and Quotient Rules - *CC BY-SA 4.0*
    - 6.3: Derivatives of Other Trigonometric Functions - *CC BY-SA 4.0*
    - 6.4: The Chain Rule - *CC BY-SA 4.0*
    - 6.5: Derivatives of Functions Given Implicitely - *CC BY-SA 4.0*
  - Back Matter - *Undeclared*
    - Index - *Undeclared*
    - Glossary - *Undeclared*
    - Detailed Licensing - *Undeclared*