

MODERN GEOMETRY



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Licensing

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Preface

Introduction

The course begins with a review of several of the most elementary results of traditional Euclidean Geometry; in fact, those of the beginning of Book 1 of *The Elements*, the Euclidean original. Since this work is about 2300 years old, rights to the original are not a problem. For millennia, mathematicians have been convinced that the original Greeks and generations of mathematicians who followed them believed that "uniqueness of parallels" could be proved from more fundamental "axioms" about geometry (lit., "earth measure") that were so obvious that they could be safely assumed without proof. Standard content for a course called "Modern Geometry" involves proving that this is not the case and studying what happens under a different assumption.

Understanding what modern mathematicians mean by "proof" dates from this era and this work, *The Elements*. Many of us are convinced that, of all the contributions of the ancient Greeks to modern human thought, their ideas of deductive logic, concretized with mathematical proof, is the most important.

There have been many books written and good textbooks are available that cover the material of this course. Unfortunately, books that go deeply enough to develop the subject well assume a year of study, a semester is not sufficient especially for students with weak backgrounds. The standard approach is to develop more results of advanced Euclidean geometry first and to eventually back up and go into hyperbolic geometry. In order to get where we need to get, we will not formally develop some of the advanced Euclidean results that are logically needed because we would never get to the "modern" part of the course's name. We will come back and develop, or at least introduce conceptually, some of these when we need them late in the course. Considerable compromise with mathematical formality will be sacrificed in order to meet this goal but not the idea of mathematical proof itself. In fact, the most important goal of the course is to solidify proving statements within the setting of Euclidean geometry. That the context may not be "Euclidean" can be very helpful in understanding the Euclidean setting itself. If you have never had any experience in proof-based Euclidean geometry, expect the course to be very difficult, maybe even too difficult for your success without exceptional effort on your part.

It will be assumed that you have a working knowledge of elementary logic; e.g., comfortable use of universal statements ("for every..." $\forall$) versus existential statements ("there exists..." $\exists$) where each ellipsis, "...", is itself a statement. That backs us up even further; what is a "statement"? It is a sentence (in our case English but any language will do) that has "truth value"; it makes sense to call it "true" or "false" as opposed to, say, questions or commands. These ideas and much more elementary Euclidean geometry could be taken for granted up to the last few decades but, regrettably, that is no longer the case. The traditional high school course, usually "sophomore geometry", gave a very different background than many students - even strong students with math-based career aspirations - have experienced. I deeply believe that such students are shortchanged by this "progress" but reality is what it is. As a concrete example, one of the best and most popular books in Projective Geometry (including this campus - using this same course name and number!) within recent history - is A. Seidenberg's *Lectures in Projective Geometry*, of the The University Series in Undergraduate Mathematics, 1962, of D. Van Nostrand Company, Inc. From the first paragraph of the Intro:

"The purpose of this book is to provide a text for an undergraduate course in Projective Geometry. The axiomatic approach to this subject is undoubtedly the proper one, but it seems desirable, if not necessary, to start from what the student knows, namely high school geometry, and proceed from there to new things. Accordingly, the first chapter is devoted to introducing some of the main topics in as naive a form as possible."

That "naïve a form as possible" of the first chapter should serve as a warning. I would defy more than two students in this class to read and understand it without some serious help; perhaps half of our semester would need to be devoted to get to the preparation UC Berkeley's Abraham Seidenberg had in mind for the beginning of the course. If you have not had a fairly traditional proof-based Euclidean geometry course, finding an older high school geometry text or one from some of the community colleges (many of which have not lost their way) to study through can be a very helpful **start** (as in NOW!) for the course. Another opportunity is the rather amazing online free Khan Academy that can offer this "review" that may be far more than review but necessary for success in this course and for an appropriate understanding of proofbased geometry specifically and proof-based mathematics more generally:

<https://www.khanacademy.org/math/geometry/>

This is a first year, high-school level course on Geometry (which is based on Euclid's elements). It revisits many of the basic geometrical concepts studied in earlier courses, but addresses them with more mathematical rigor. There is strong focus on proving theorems and results from basic postulates.

Proceeding (as in all of formal mathematics), a "statement" is a sentence that has an undefined "truth value" (hopefully, meaningful in the context) and new statements are constructed from existing ones in several ways. For a single statement, its *negation* means any reformulation of the statement such that if it was true, the new statement is false and inversely, if it was false, the new statement is true. Symbolically, if p is a statement, its negation is indicated by $\sim p$ or $\neg p$ or, less commonly, p' . Two or more statements can be compounded into new statements in several ways; the most obvious are *and* and *or* (formally, *conjunction* and *disjunction* that we will never use beyond this introduction along with their formal logic symbols, $\wedge$ and $\vee$). In the former case, the statement is true if and only if both (all, if more than two) statements are true and, in the latter, it is true if and only if at least one of them is true.

Back to universal and existential statements, their formal names will seldom be used and their formal symbols ($\forall$ and $\exists$) never used. Also understood - but this time often used - will be the idea that to prove that a universal statement false requires only to exhibit one *concrete* counterexample (one example of the statement that is demonstrably false) or, as is sometimes more enlightening, a set of examples of the statement with some common property that shows *all* (that is important!) of them to be false. For a nonmathematical example, consider the statement, "All dogs are white". Obviously it is false and single counterexample, exhibiting a single brown (or any color(s) other than white) dog proves that it is false as does the set of all brown dogs (since that set is nonempty). One less obvious fact - but entirely standard throughout mathematics - unless otherwise qualified, general statements are assumed to be universal. As a geometry example, "An equilateral triangle is equiangular" is assumed to mean a statement about all equilateral triangles, not just some one of them, otherwise it would have to say something like, "Some equilateral triangle is equilateral", or equivalently, "Some equilateral triangles are equilateral", since the plural adjective "some" in logic means the same as "one or more"; i.e., "at least one".

Already being used in the introductory paragraph, is a "working knowledge" of *conditional statements* (also known as *implications*); i.e., "If..., then..." statements where each "..." is a statement. Slightly more formally, "If p , then q ," where p (the *antecedent*) and q (the *consequent*) are statements. In formal logic, this is written " $p \Rightarrow q$ " (or " $p \rightarrow q$ ") and will seldom be used except for here where it makes it more convenient to review relevant terms. First off, " $p \Rightarrow q$ " is true except for one case, when p is true and q is false. For that reason, proving an implication is effected by assuming that p is true and proving that, with that assumption, q must also be true. On first acquaintance, this convention can seem to be counterintuitive but common sense logic does require that statements such as "If all dogs are white, then my dog is white" be considered to be true even if it is meaningless because, obviously, not all dogs are white but, in spite of the antecedent being clearly false, the statement must be taken to be true. If you are not convinced that that statement is meaningless, consider this slight modification, "If all dogs are white, then my dog is black." Again, the antecedent is false so the statement is taken to be true but, obviously, meaningless.

[Note: This is from the perspective of what is known as "propositional logic". This kind of ("quantified") statements are ordinarily considered from the perspective of what is known as "predicate logic."]

You might recall from logic that such an implication " $p \Rightarrow q$ " is logically equivalent (i.e., one statement is true if and only if the other statement is true) to the compound statement " $(\sim p) \vee q$ " which is true if either $\sim p$ is true (i.e., p is false) or q is true. That is, for the original implication to be true, either the antecedent p is false or the consequent q is true, hence the equivalence. If this does not seem familiar to you, it is easy to prove such statements using truth tables; i.e., directly confirm all possible truth values, in this case, four.

Back a few paragraphs, the word "inversely" was used although less careful but common usage might have said "conversely". The idea of the converse of an implication is critical in this course and throughout mathematics as are other permutations of the statements involved. Considering the general implication:

implication: $p \Rightarrow q$ The permutations are:

converse: $q \Rightarrow p$ That is, interchange the antecedent and consequent.

inverse: $\sim p \Rightarrow \sim q$ That is, negate the antecedent and also the consequent.

contrapositive: $\sim q \Rightarrow \sim p$ That is, both negate and interchange the antecedent and consequent.

Not a permutation of the original but a related statement that may or may not be true:

biconditional: $p \Leftrightarrow q$ That is, both the implication and its converse: $p \Rightarrow q$ and $q \Rightarrow p$.

You should know (it's easy to prove) that the original implication and its contrapositive are logically equivalent (one is true if and only if the other is true) as are its converse and its inverse (contrapositive implies that). Finally, the biconditional is true if and only if p and q are either both true or both false.

Of critical importance is the fact that the implication and its converse are not inherently logically equivalent (If my dog is white, then all dogs are white?) although it is often the case that implications in mathematics are biconditional; i.e., "if and only if" and abbreviated as "iff". For example, "A triangle is equilateral if and only if it is equiangular," or more succinctly, "A triangle is equilateral iff it is equiangular." At first glance, such a statement does not look like any kind of an implication but you need to equivalently and effortlessly think of it as, "If a triangle is equiangular then it is equilateral and conversely; i.e., if a triangle is equilateral then it is equiangular."

The "think of it as" might seem backwards but is not. If this is not clear, you need to more carefully examine the position of the "if" and the "only if" in the statement. Leave off the "and only if" to more easily see the idea. Symbolically, that would say " p if q " that is almost obviously equivalent to "if q , then p ". The "only if" is a little harder to understand at first but think about what " p only if q " must mean; it must mean that q follows from knowing p or, if p is true, then q must also be true. That is, if p , then q . Hence the equivalence.

Note

One important exception to "if" meaning "if and only if" is in definitions. It is entirely standard in mathematics to accept that in definitions, a stated implication means the biconditional.

The first 28 propositions of the original Book 1 of Euclid's Elements do not use "Euclid's 5th", known as "The Parallel Postulate", and those propositions are where we will start the course. We will use them as a review of geometry and of proof itself in the familiar geometric setting. The language is archaic but I like both the history and the geometry. From a modern perspective, the ancient Greek's introductory definitions are much too naïve and a few of the early propositions (SAS especially) need to be axioms or proved as consequences of unstated others but the presentation is surprisingly modern given the intervening millennia. The idea of trying so hard to avoid "the" Parallel Postulate (or any of the infinitely many statements that are logically equivalent) as long as possible is quite modern in its perspective. "As long as possible" is an exaggeration but the goal is evident and wasn't answered for a couple millennia until the development of hyperbolic geometry by the Bolyais (mostly son Johann) and Nikolai Lobachevski in the 1830 s.

Many other theorems that "feel like Euclidean geometry" can also be proved in "**neutral**" or "**absolute**" geometry (encompassing both Euclidean and hyperbolic geometry) that we will study in Chapter 2 such as "Hypotenuse-Leg", "construction of the incircle of a triangle", and "construction of the tangents from a point to a circle" along with many others that are provable in this context (no Parallel Postulate) before Chapter 3 where we specialize using the *negation* of the Parallel Postulate that (in conjunction with the other axioms of neutral geometry) is hyperbolic geometry.

Neither of these geometries is the "real world" in which we live; that is, on the surface of a sphere, a special case of elliptic geometry. Even there, many of the theorems hold with restrictions that figures not be "too big", whatever that might mean. This context can be helpful in understanding some results (so will occasionally be used herein) but are not to be considered as part of the negation of the Parallel Postulate because, given the *other* axioms, major differences arise. On a sphere, a "line" is a "great circle"; i.e., the plane of which contains the center of the sphere. Using a globe as a concrete model, all of the longitudinal lines are lines in the geometry but they all go through the North and South Poles so even the most fundamental, "two points determine a line", needs some kind of revision. As another example, consider three points in a row on the equator (a line in this geometry, other latitude circles are NOT lines) A, B, and C. Now think of "sliding" C all the way around the globe to be lined up as C , A, and B. Originally, B "felt like" it was between A and C. Now, it is A that has that feel with respect to C and B. The resolution? The concept of "**betweenness**" that is present and important in neutral geometry - does not exist in spherical geometry, at least not in the same unrestricted way. Still, thinking about spherical triangles and other geometric figures can be helpful in understanding validity and/or limitations of theorems in other geometry settings. When they fail to be true in spherical geometry, it helps us think more carefully about our proofs of the statements. Often the reasons explicit proofs fail are less than obvious. One enlightening example: On a sphere, any triangle - such as a triangle (ideally and perfectly) drawn on a poured concrete floor - has angle sum strictly *greater* than 180° !

From a modern perspective, the ancient Greek's introductory definitions are much too naïve and a few of the early propositions (SAS especially) need to be axioms or proved as consequences of unstated others but, generally speaking, the presentation is surprisingly modern given the intervening millennia. This is the content of Chapter 1. After consideration of some common additional theorems in neutral geometry in Chapter 2, we will start discussing some properties and proving some theorems that are unique to hyperbolic geometry in Chapter 3. To most knowledgeable people on first introduction, these can seem to be quite

counterintuitive, even wrong. Competence with formal deductive logic, i.e., proof, will make the results inevitable (from that starting point) but they still can have an air of make-believe. There is lots more to hyperbolic geometry than we will study but, again, we only have one term so this will only be a brief introduction with the primary goal being development of careful reading and proof in mathematics rather than content mastery of either hyperbolic geometry or advanced Euclidean geometry, both important topics worthy of study at this stage in your career.

The validity of the axioms of hyperbolic geometry is established by verifying "relative consistency" with Euclidean geometry; i.e., by establishing a model for this "new" geometry entirely within Euclidean geometry. That is, if anything is ever to be proved inconsistent in this new geometry, it would mean an inconsistency within the geometry that we all believe, Euclidean geometry. In reality, this is one long complicated Euclidean geometry theorem with lots of parts. To this end, a subset of the Euclidean plane is selected within which one redefines the constructs of "normal" geometry; i.e. giving new specifications of what to call points, lines, line segment measure, etc., and relationships among them. Within this setting, every axiom (postulate) of hyperbolic geometry must be proved as a Euclidean theorem; i.e., using any of the axioms and theorems of Euclidean geometry needed. That is, we are free to use conventional points, lines, etc., of the entire Euclidean plane, and all of the theorems among them, to prove each of the axioms of hyperbolic geometry as a theorem in the model. Many of the needed results in Euclidean geometry are not covered in introductory Euclidean geometry courses so we will need to introduce and study some of the properties of Euclidean geometry that are "well-known" but more advanced than high school geometry.

It should be noted that this consistency theorem is a trivial consequence of Complex Analysis (Linear Fractional Transformations and conformal mapping). However, most mathematics students don't have that level of mathematical sophistication at this stage (and many never will!). Moreover, much is to be gained from the review of elementary Euclidean geometry as well as new material learned in the process (inversions in a circle especially). Even worse, sometimes the ideas in Complex Analysis are not proved but refer to the student's knowledge Euclidean geometry. There is nothing wrong with this except that it can't be used both ways; circular reasoning is a no-no in mathematics!

After arguing that semi-formal deductive logic was the most important gift of the ancient Greeks to modern human thought, and that geometry is an effective and the most common (somewhat) formal introduction to that process, it might come as a surprise that *formal* axiomatic development will not be done in this course. It's even worse than that; the collection of axioms that we will use will never even be stated! The problem is that, in a genuinely formal development, even the underlying set theory and real analysis, would have to be axiomatically developed. Even taking naive set theory and properties of the real numbers for granted and focusing just on the geometry would still require a level of formality that would preclude getting much else done in the entire course. We will take the opposite extreme and only state the most critical of the axioms but how much formality will be expected is hard to quantify precisely. Hopefully, you will quickly develop a "feel" for it. In fact, you might have to "unlearn" some phony formalities; e.g., at this level and beyond, it is standard to ignore statements that are logically necessary but so obvious that their inclusion would be pedantic. Knowing what is "obvious" and what requires statements of verification can be tricky, of course; among other problems, it depends on the sophistication of both the writer and the target audience.

The terms *axiom* and *postulate* are taken to be interchangeable and mean statements that are accepted without proof. One of the "errors" of the Greeks was to assume that some statements are so obviously universally true as to always be true without proof and, therefore, a safe starting point. A more modern perspective is to leave the concept of "universal truth" out of the mix entirely. If we're in a setting where the statements are true - a so-called "model" of the geometry - what else must be true because it has already been proved? If not, that's fine too; we're just in some other setting, perhaps equally useful, maybe even more so. The same statements may or may not be true but, if they are, they require new proofs.

Axiom 1 - (For the rest of the course!) - Two points determine a line.

If this does not seem familiar to you, you are probably in the wrong course! However, really knowing what it means is very different. What are "points"? What are "lines"? What does "determine" mean?

Perhaps surprisingly, (except within any *model*) point and line will be taken as undefined terms restricted by certain axioms (that are also never formally stated). More surprisingly, perhaps, is that in any axiomatic system, there must be undefined terms or the structure is inherently circular. At the set level, "set" and "element of a set" are undefined. Doesn't a set mean a "collection of" or some such? Well, what does "collection of" mean? Etc., etc. That is, there is no place to stop so, over millennia, formal logicians have learned not to try. But, whatever they are, they behave in certain ways; i.e., according to statements called axioms. For set theory and the real or complex numbers, we will not even begin to try to say what they are axiomatically; we'll just use them at will (along with most mathematicians nearly all of the time).

Before we begin, there are some properties of a set of axioms that need to be understood.

Most fundamental is that they need to be **consistent**; i.e., that they have no inherent contradictions. In other words, it is critical to be sure that they cannot lead to statements that can both be proved and disproved; i.e. proved to be both true and false within the system. Such a statement would destroy the entire structure.

Proving that no such inherent contradictions *can* arise is a challenge in itself. In fact, in a very real sense, it is an impossible task! Entirely within a system itself, it is impossible to prove that the axioms are consistent. The standard proof of consistency of a set of axioms is to produce a "model"; a concrete realization of the situation (and even "concrete" is in the eye of the beholder!) in which all of the axioms can be proved as theorems. For example, if we have enough faith in the ordinary Cartesian (x, y) plane, it provides an example of a model for Euclidean geometry and, with that assumption, the axioms of Euclidean geometry must be consistent. For abstract systems, the most we can logically hope for is **relative consistency**; if we accept one system to be consistent, use that to prove that another must also be.

A less critical but desirable property is that the axioms be **independent**. This means that it is not possible to prove any of them using only the others. Failing to have this property would not destroy the system but it is definitely preferred. The fewer and more fundamental the axioms are, the "cleaner" the structure. We might have encountered the Pythagorean Theorem in 6th or 7th grade but it was not presented as the Pythagorean Axiom just because we couldn't yet understand a proof. The idea was that it was taken to be true without our proof but not without somebody's proof. Axioms are not just "pulled out of the air"; wise choice of them is a very sophisticated process and we won't begin to try. We won't even try to list them but some famous choices are easily available and will be referenced below.

A third property that a system might have is that the axioms be **categorical**. This is not always a desirable property since it implies that we are dealing in very specialized circumstances when, in mathematics, we often want to study properties that apply to many inherently different situations. For example, a "group" in mathematics is a specialized term that applies to lots of different structures that fit the group axioms but, if we add to them that the group has exactly seven elements, the set of axioms becomes categorical. In a very real sense, there is only one such group. That is, if the set of axioms is categorical, all "models" of the setting can be viewed as outwardly different forms of exactly the same thing. In the jargon, all models are "isomorphic." On the plus side, if a set of axioms are known to be categorical, a theorem proved in the context of one model must also be valid in all other models. As an example, if you are familiar with "analytic" proof versus "synthetic" proof in Euclidean geometry, knowing that the axioms are categorical assures us that either proof suffices. A theorem in any model is a theorem in the geometry. (See Chapt. 4.)

What is a *model* of the setting? Let's keep it in the terms of plane geometry. A **geometry** is an *abstract* set of elements called "points", certain abstract subsets of which that are called "lines" and perhaps other special sets assuming they satisfy certain confining properties called axioms. A **model** for this geometry is, in some sense, almost the opposite. It is a very concrete set, the elements of which are called points, concrete subsets of these points called lines and circles, etc., and all of the axioms of the geometry (i.e., statements assumed without proof in the geometry) are proved as *theorems* in the model. At that point, all of the theorems of the geometry become theorems in the model as well; i.e., they have already been proved.

Here are two famous sets of axioms that are readily available:

- Hilbert's Axioms for Euclidean Geometry:
- Birkhoff's Axioms for Euclidean Geometry:

Note: Of these formal sets of axioms (that we will not be using formally), only Birkhoff's (1959) includes what is known as a Ruler Postulate and a Protractor Postulate. However, Hilbert's (1899 as revised in 1902) do imply results consistent with them [Google: Desargues/Pappus/field and/or see Blumenthal, *A Modern View of Geometry*, the V. H. Freeman & Co, 1961] and were revised to include them by Saunders MacLane in 1923. These were not used by the Greeks (although some of the same ideas were present but expressed in terms of comparisons of line segments (for length), polygons (for area), and angles (for angle measure). We will include them but use them somewhat informally as follows:

Ruler Postulate: There is a 1-1 correspondence between points on a given line and the real numbers that behave in "nice ways"; i.e., given a distance and a direction, there is a unique point determined and vice versa. That is, once the line is "coordinatized," each point corresponds to a unique real number, and this association of line segments is consistent with real number arithmetic. For line segments, this distance between its endpoints is called the measure of the line segment but the word "measure" is misleading; there is no measuring involved since all measurement is approximate and these are taken to be infinitely accurate.

Protractor Postulate: There is a 1-1 correspondence between angles starting from a given ray and the real numbers of a closed interval $[0, 180]$ (or $[-90, 90]$ or $[0, \pi]$ or $[-\pi/2, \pi/2]$ or any other such including ones that might be reasonable but are almost never used such as $[0, 100]$) such that, given an angle size and a direction, there is a unique ray determined and vice versa. Once the semicircle is "coordinatized," each ray corresponds to a unique real number of the interval and, again, this association is consistent with the arithmetic of real numbers. This associated number is called the *measure* of the angle but, again, there is no measurement involved. Infinite precision is always assumed unless otherwise stated (almost never).

One last time, these ideas (especially their names) can be a little misleading; there is no actual *measuring* going on as that would be inherently approximate. These are theoretical abstractions only. Everything is assumed to be infinitely accurate which is never possible in measurement situations. Moreover, unless otherwise stated, *infinite accuracy will be expected* of you so calculators will be of little help except as a security blanket. Except for major exams, they will not be not prohibited but answers are expected to be precise; e.g., π is not 3.14 nor 3.14159, $2/7$ is not 0.2857, etc. Decimal answers are acceptable *only* if they are precise - decimal approximations will not receive full credit. *Simplified* ordinary fractions will suffice for the course and, in fact, are *preferred* whenever appropriate. Calculators, cell phones, and the like will be prohibited from the midterm and final because they have gotten too "smart"; i.e., they can record information and give you access to information that must be in your head to be able to think effectively.

How can we operate axiomatically without ever stating most of the axioms explicitly? Hopefully, you'll get the idea as we progress. One of the important ideas about which we will be kind of "sloppy" is that of "betweenness" of points on a line but it has to "be there" whether we worry about it or not. Another is continuity considerations. Say, for example, a line segment is determined by a point inside a circle and another outside of it. It should be obvious - and we will take it for granted - that there is a point of intersection of that segment and the circle. Formal axioms are needed but they will be ignored.

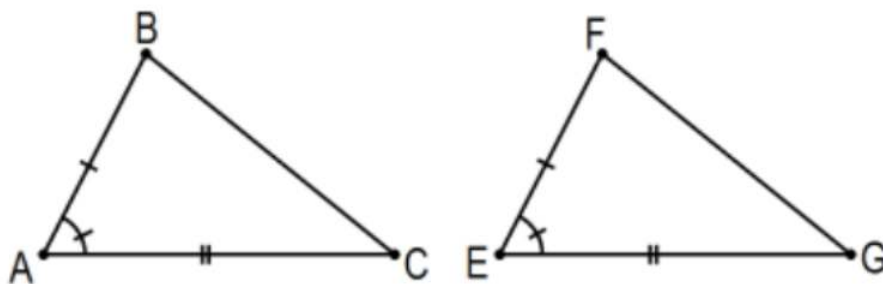
The same idea applies to definitions. They are *absolutely critical* and yet many of them will not be given in the course, at least not explicitly. If there is any doubt about what is true "*by definition*" about a named object of some kind, ask, and make careful note of it in your own personal glossary. Stating that a property of some geometric structure is true by definition when it is not part of the definition, but has been proved instead, is a serious error so be careful. If there is any ambiguity, also include the name of the word being referenced; e.g., "by definition of rectangle" instead of just "by definition". [What is the definition of rectangle? A quadrilateral with all angles being right angles. Nothing more. Anything else must be *proved*.]

Abbreviations are another place where care must be exercised in the sense that their "full-blown" forms need to be clearly in mind. That is, an abbreviation should not be used unless both the writer and the reader can expand it equivalently and effortlessly. A good example in geometry is SAS. Unless you have weak English or math skills, you can read it immediately as "Side-Angle-Side." That is *not* what is meant by "expanding it"; that is just the abbreviation. SAS is a *statement*; that is, it is a full (in our case, English) sentence that has truth value. Moreover, it is a *universal* statement; in this case, a statement about all triangles that meet the SAS conditions. It is common in geometry - and in mathematics settings more generally - to assume universality (i.e., "for every...") of statements that "sound like it". In general, an abbreviation should not be used unless you can expand it unambiguously on demand and not as some kind of verbatim, rote response but exactly the same mathematical content with as few mathematical symbols and/or figures as practicable and consistent with the expansion that any other knowledgeable practitioner would give it. In addition to being a universal statement, SAS is an example of a conditional statement or implication; i.e., it is an "if..., then..." statement as discussed earlier with the SAS only being an abbreviation of its hypothesis, not its conclusion, much less the entire implication.

SAS: If two triangles have two pairs of sides congruent and the angles included by these pairs of sides congruent, then the triangles are congruent.

Is it true? At least it makes sense to ask so it is a statement! Explaining its meaning to someone who does not know what it means, would it be clearer if supported by a good figure with the conditions properly indicated. That, however, should be a triviality for the person using the statement and for the person who is reading or hearing it; otherwise, the abbreviation should not be used at all. Their pictures would have different shaped triangles (but neither should appear to be specialized; i.e., not right triangles, not isosceles, etc.) and would probably be labeled differently but they would be communicating exactly the same idea. One thing that probably would be the same would be the use of "congruence marks", standard "tick mark" indicators that line segments or angles are congruent and a special mark for right angles. USE THEM! Unfortunately, my most powerful graphics software, *Cinderella*, does not include them so there will be times when you will need to add them in yourself. That should be easy to do from the context. Another point, you may be used to seeing such statements presented artificially such as:

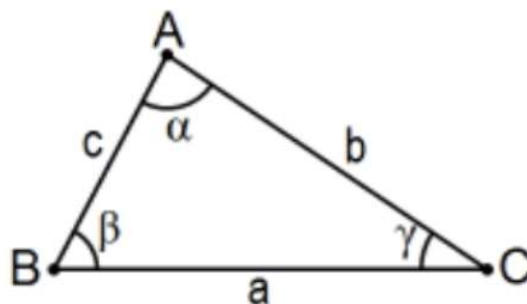
Given: $\overline{AB} \cong \overline{EF}$, $\overline{AC} \cong \overline{EG}$, and $\angle A \cong \angle E$ (or $\angle BAC \cong \angle FEG$)



Conclusion: $\triangle BAC \cong \triangle FEG$

Note: Another (and worse!) limitation of my software is that it is difficult and distorts line spacing to put mathematically appropriate bars over line segments ($\overline{AB}$), arrows over rays ($\overrightarrow{AB}$ or $\overleftarrow{BA}$), double-headed arrows over lines ($\longleftrightarrow{AB}$), and arcs over arcs ($\widehat{AB}$) when indicated by two points, usually uppercase English letters. USE THEM in your work consistently and correctly. Also be careful about "corresponding order". For example, in the above, we would have no reason to suspect that $\triangle ABC \cong \triangle FEG$ but $\triangle ABC \cong \triangle EFG$ is equivalent to the original. Note: As mentioned above, this is now a (universal) conditional statement or implication (i.e., if... then...), but often they are not so stated in mathematics. You should be able to effortlessly make the translation into an equivalent statement that does have that form and recognize universality when that is implied.

An artificial feature you might be used to is so-called "T-proof"; statements carefully matched with corresponding reasons why the statements are true in the context of the proof. If you feel more comfortable with such, feel free to use the structure but it is really just a pedagogical technique to convince beginning mathematics students that if a statement is made without a supporting reason (it could be understood but not explicitly stated), it is not a proof. True? Perhaps, but not proved.



Another detail that will be used, and you are encouraged to use as well, is standard notation, some of which is mandatory and some of which is only preferred. (That preference may or may not be standard outside of the US). Points will be indicated by uppercase English letters and the vertices of a polygon *must* be listed in sequential order (clockwise or counterclockwise).

Likewise, corresponding vertices of corresponding polygons (usually congruent or similar but not necessarily) must be listed in corresponding order. A variable written inside of an angle indicates the *measure* of the angle - i.e., its angle size - *not* a name for the angle itself so do NOT indicate an angle by using its *size* instead of its name, several angles may have the same size but, obviously, different names.

A convenient convention, but far from standard, is to use the corresponding lowercase Greek letter for the measure of an angle of the same English name. That is, $m(\angle A)$ will often be indicated by α , $m(\angle B)$ by β , $m(\angle C)$ by γ (because it is next in the Greek alphabet), etc., and to use lowercase English letters to indicate the measure of a side and will often be the same letter (but lowercase) of the opposite vertex. Geometer's Sketchpad automatically "fills in" angle measure indicators. Do NOT do that in your work; it is not standard usage. Another comment on usage, geometric entities such as angles, line segments, and polygons that are (informally speaking) the same size and shape are said to be **congruent**, as opposed to "equal", but measures (α , β , a , b etc.) are real numbers so, if they are the same, the appropriate word is "equal". This care with words (not shared by all geometers) comes from their underlying nature as sets of points. Sets are equal if and only if they have the same elements so consistency requires that, for example, $\angle A = \angle B$ would imply that the two angles (union of two rays with a common initial point) were *exactly* the same

two rays, not two angles of the same size that was probably the intent; instead express it symbolically as $\angle A \cong \angle B$ and read that as "angle A is congruent to angle B ".

To get a feel as to how this formal/informal proof is to work, below are items from a generic first quiz along with suggested solutions (that look very much like the items of Problem Set 1, hereafter PS 1). The instructions began simply with, "All items relate to the given isosceles triangle $\triangle ABC$." Implied, of course, is what *isosceles* means, even *triangle* itself. There are lots of such "familiar" terms that are used herein and throughout the course, terms such as "is perpendicular to", "ray", "angle bisector", ASA, and SSS. Hopefully, these ideas are not new, although they will be reviewed. One that we will use that is common but not universal is "cpctc", "corresponding parts of congruent triangles are congruent". Equivalently, one might say "by congruence of triangles" (or of congruent geometric figures more generally) but cpctc is so handy and so commonly used that I encourage its use. Again, make yourself an accurate glossary of terms about which you have any doubt (after asking if there is anything about which you are unsure).

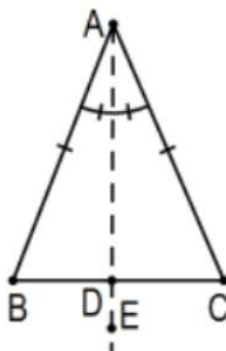
All items relate to the given isosceles triangle $\triangle ABC$ with base angles at B and C .



1. Indicate in the sketch all **given** information in the standard way. Equally importantly, indicate **nothing but** the given information.

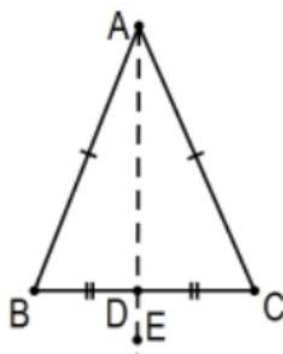
Solution: Add matching congruence marks, "tick-marks" to segments AB and AC .

2. Assume that point D is the point determined by the bisector of $\angle A$. Prove that $\angle B \cong \angle C$.



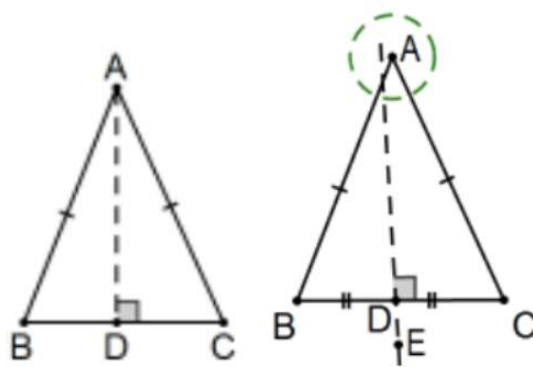
Solution: $AB \cong AC$ by definition of isosceles (or "given") and $\angle BAD \cong \angle CAD$ by definition of angle bisector (or "given"). Therefore, $\triangle BAD \cong \triangle CAD$ by SAS and $\angle B \cong \angle C$ by cpctc. [Note: What happened to $AD \cong AD$ needed for SAS? Formally, it is the reflexive property of congruence but don't bother. Isn't it obviously true?!]

3. Assume that point D is the midpoint of line segment BC . Prove that $\angle B \cong \angle C$.



Solution: $AB \cong AC$ by definition and $BD \cong CD$ by definition of midpoint (or "given"). Therefore, $\triangle BAD \cong \triangle CAD$ by SSS and $\angle B \cong \angle C$ by cpctc.

4. Assume that line DE is the perpendicular bisector of line segment BC. Don't prove anything but explain why an easy hypotenuse-leg "proof" that $\angle B \cong \angle C$ is *fallacious*.



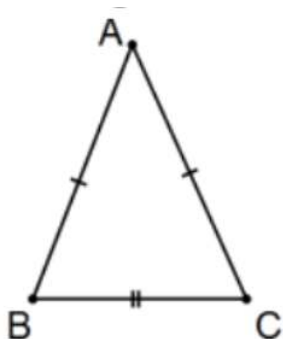
Answer: For all we know at this point, point A need not be on line DE.

(The circle at vertex A shows how the figure can be drawn to emphasize this.) That is, the "proof" assumes that $A \in DE$ without first proving that fact.

Valid Proof: Assume that D is the foot of the perpendicular from A to line BC (i.e., no tick-marks that $BD \cong CD$). Now, by Hyp-Leg, $\triangle ABD \cong \triangle ACD$ and $\angle B \cong \angle C$ by cpctc.

Note: The problem of #4 is a very important idea. Once an entity is determined (and the perpendicular bisector of a line segment completely determines the line) all other properties about it must be proved or the situation is "over-determined"; i.e., information is being assumed without proof and might possibly even be false. That's about as bad as it gets in "proof". In fact, it is better to say "I have no idea how to prove this."

5. Only using $\triangle ABC$ (that is, with no auxiliary points or lines), prove $\angle B \cong \angle C$.



Solution: $AB \cong AC$ by definition and $BC \cong CB$. Therefore, $\triangle BAC \cong \triangle CAB$ by SSS and $\angle B \cong \angle C$ by cpctc. [That is, the triangle is congruent to itself using a different correspondence.] Since $\angle BAC \cong \angle CAB$, $\triangle BAC \cong \triangle CAB$ by SAS as

well.

Note: The two proofs of #5 are a bit more subtle than they look. Congruence of geometric figures, in our interpretation, has been assumed to be in its "pure geometry" form; i.e., as an undefined term subject to some (albeit, unstated) axioms. For example, congruence is (among other things) an equivalence relation; i.e., it is reflexive, symmetric, and transitive. More explicitly:

- i. Figure F is congruent to Figure F,
- ii. If Figure F is congruent to Figure G, then Figure G is congruent to Figure F, and
- iii. If Figure F is congruent to Figure G and Figure G is congruent to Figure H, then Figure F is congruent to Figure H.

In the SSS proof, we assumed $BC \cong CB$ without proof and your instinct (perhaps past teaching?) would call that true by the reflexive property of congruence. It is *not*, however, because the orientation is reversed. A less "pure" form of geometry (e.g., including the Ruler and Protractor Postulates as we do) *defines* congruence, and not among specific geometric figures, but of the entire point set of the geometry. Formally, in this context, a congruence would be defined to be a one-to-one correspondence of the entire plane that preserves the distance (unsigned measure) between any two corresponding points and the size of any two corresponding angles. For example, $BC \cong CB$ because reflection in the perpendicular bisector of segment BC is a congruence that carries segment BC onto segment CB including the entire triangle in question onto itself but with a new correspondence. Similarly for the SAS proof, reflection in the line of the bisector of angle $\angle BAC$ is the needed congruence. Another congruence that you are probably familiar with is rotation about a single point and all compositions of these two. In Euclidean geometry, translation by a vector (a directed line segment) is another, sometimes called a "glide" in elementary books. Are there others? The answer is "no" but we will not prove that fact in this course.

Except for translations (that need *unique* parallels), these are also congruences in **neutral** (or **absolute**) geometry; that is, without reference to Euclid's 5th Postulate that gives us Euclidean geometry so these statements are true in both Euclidean and hyperbolic geometry even though we may be "thinking" Euclidean geometry only (and still have no idea what hyperbolic geometry might be!).

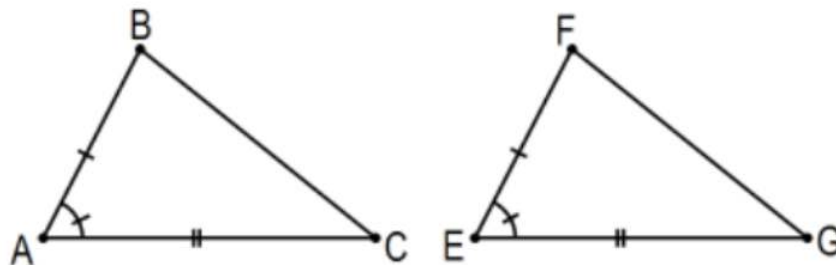
One last complexity that is often overlooked in introductory geometry classes is that, for independence of the axioms, the congruence statements (i.e., SAS, SSS, ASA, AAS, and Hypotenuse-Leg) are not "axioms", at least not most of them. Although the Greeks did not realize it, one of these (or something equivalent) must be taken as an axiom and the rest are theorems, conditional statements to be proved.

That last statement is not quite true but almost. For example, in Hilbert's axioms, congruence of segments and angles are described axiomatically and congruence of geometric figures (restricted to connected ones consisting of segments and angles) are then defined to be congruent if there is a one-to-one correspondence of their points such that all segments and angles correspond and are congruent. With this restriction, his basic axiom is not quite SAS; it includes congruence of the missing two angles but omits congruence of the missing side and proves the full SAS as a theorem. That is so close as to be almost trivial except for the idea of keeping the set of axioms at the absolute minimum needed. [See Hilbert's classic 1902 book, *Foundations of Geometry*, Open Court Publishing Co., Axiom IV, 6 and Theorem 10.]

We will take SAS as an axiom and use it to prove the others although, for example, sSS could be used instead. Although SSS will be stated independently, it really is, "If SAS, then SSS". In terms of formal logic (and ignoring the understood universal nature of the situation); i.e., "For every...", before the antecedent and again before the consequent in the implication, "If $(p \Rightarrow q)$, then $(r \Rightarrow s)$ ", or fully, " $(p \Rightarrow q) \Rightarrow (r \Rightarrow s)$ ". In such a situation, we simply assume the truth of the statement r and somehow prove that the statement s must also be true. In the process of proving s , it would be expected that somewhere along the line, maybe several times, the assumed truth of any occurrence of $p \Rightarrow q$ would be used.

In fact, it is more complicated than that. In reality, it says that "If all of the axioms of neutral geometry including SAS and previously proved theorems are true, then SSS, is also true." More generally, throughout formal mathematics, that is the idea: A large collection of "stuff" is used to prove some new conditional statement. *None* of that complicated antecedent is mentioned, only the antecedent of the new conditional statement that is to be proved. As mentioned above, we could just as well assume SSS as our axiom and use it to prove SAS. Focusing just on the congruence concepts, that makes the conditional statement "If SSS, then SAS" the converse of "If SAS, then SSS." In that case, were we to prove the statement, we would simply say, "Prove SAS." To do that, we would have to discard any statements we thought were true because they had been proved using SAS, and somehow build a proof using SSS to prove that the two triangles with the SAS hypothesis are, in fact, congruent. That is we start with two triangles that satisfy the hypotheses of SAS (NOT the hypothesis of SSS) and prove that the triangles are congruent. To do this, the setup could look exactly as the SAS that we looked at before:

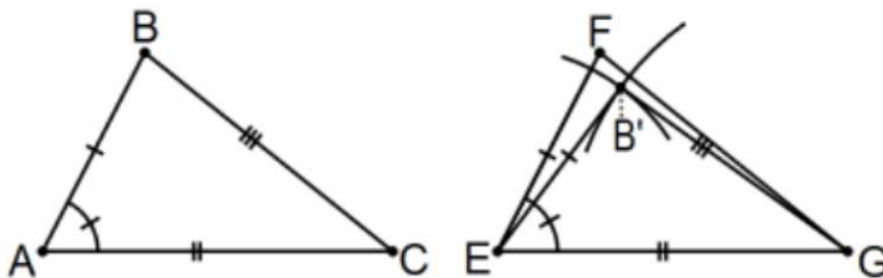
Given: $\overline{AB} \cong \overline{EF}$, $\overline{AC} \cong \overline{EG}$, and $\angle A \cong \angle E$ (or $\angle BAC \cong \angle FEG$)



Conclusion: $\triangle BAC \cong \triangle FEG$.

Where is the SSS? Whenever and wherever we have the conditions of SSS for two triangles (not these two(!), at least not yet), we are to conclude that those triangles are congruent. To prove the result, we need to somehow build triangles that do satisfy the conditions of SSS as a bridge to proving that these two are also congruent. How to do that may or may not be easy depending on one's experience and mathematical creativity. Whatever we do, however, we must be careful not to use any theorems that were proved using SAS or we have clear, albeit unintentional, circular reasoning. At this stage that is easy to do since we haven't proved any theorems except for the congruence of the base angles of isosceles triangles and we did offer a SSS proof for that fact (which we don't need anyway). To see how this could work, we introduce another familiar concept:

The allowed "tools" are an unmarked straightedge (a realization of Axiom 1) and a compass (a realization of "knowing" a circle given a point for a center and a line segment or its length for a radius). These are often called the "Euclidean tools" but (since they are artificial realizations of the abstract consequence of the axioms of neutral geometry) they are assumed to be available throughout the course, Euclidean geometry or not. New points, lines, etc., identified because of them will be said to exist "by construction".



Proof: Construct point B' as the appropriate intersection of $(E; AB)$, the circle with center at E of radius AB , and $(G; CB)$, the circle with center at G of radius CB . Now we have a new triangle with $\triangle EB'G \cong \triangle ABC$ by SSS since we have three pairs of congruent sides "by construction". Now $\angle B'EG \cong \angle BAC$ because corresponding parts of congruent triangles are congruent (cpctc). By hypothesis, $\angle BAC \cong \angle FEG$ so, by transitivity of congruence, $\angle B'EG \cong \angle FEG$. By the Protractor Postulate, ray EB' and ray EF are exactly the same ray. Moreover, $EB' \cong AB$ by construction and it was given that $AB \cong EF$ so, by transitivity, $EB' \cong EF$ so, by the Ruler Postulate, F and B' are exactly the same point ($F = B'$). Therefore, the original triangles are congruent because we already knew that they were congruent, we just have another name for one of the points. QED.

For more on the issue, see Ex 22 at the end of Problem Set 1 (hereafter indicated PS 1, #22).

CHAPTER OVERVIEW

Chapter 1: Euclid's Elements, Props 1-28 of Book 1

[1.1: Introduction to Euclid's Elements](#)

[1.2: Book 1 Definitions](#)

[1.3: Problem Set 1](#)

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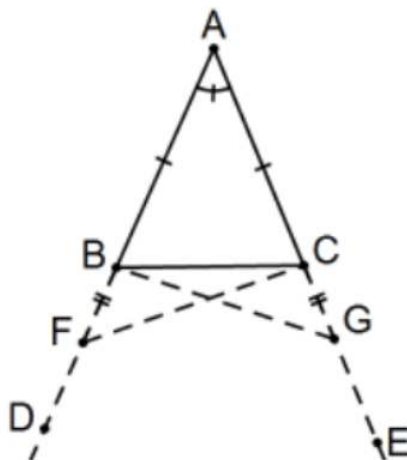
1.1: Introduction to Euclid's Elements

As an introduction, we'll cover only Propositions 1-28 of Book 1. This is an introduction to **neutral** or **absolute** geometry that subsumes both **Euclidean** and **hyperbolic** geometry. The given website is the Sir Thomas L. Heath translation and his very informative commentary; i.e., the (cheap!) Dover books in electronic form. Clicking on the proposition number brings up the statement and its historical proof. As an example, here is Proposition 5 the famous Pons Asinorum (Asses' Bridge), a double entendre from its shape and "honoring" those burros who cannot cross it (you get the idea). All the way back to the Greeks, proofbased geometry separated cognoscenti from commoner. Although used here, we will try to avoid identifying same-sized line segments or angles as "equal" because they are also sets of points and two sets are defined to be equal if they contain exactly the same elements. **Congruent** is the word we will ordinarily use or the term *equal in measure*, that theoretical "measure" of perfection that can never be achieved by measuring, but many great geometers have used the term "equal" in this context. Please **do not**.

Proposition 5

In isosceles triangles, the angles at the base are equal to one another, and, if the equal straight lines be produced further, the angles under the base will be equal to one another.

Proof: Let $\triangle ABC$ be an isosceles triangle having the side $\overline{AB}$ equal to the side $\overline{AC}$; and let the straight lines $\overleftrightarrow{BD}$, $\overleftrightarrow{CE}$ be produced further in a straight line with $\overline{AB}$, $\overline{AC}$. [Post. 2]



I say that the angle $\angle ABC$ is equal to the angle $\angle ACB$, and the angle $\angle CBD$ to the angle $\angle BCE$.

Let a point F be taken at random on BD ; from AE the greater [$AE > AD$] let AG be cut off equal to AF the less; and let the straight lines FC , GB be joined. [Post. 1]

Then, since AF is equal to AG and AB to AC , the two sides FA , AC are equal to the two sides GA , AB , respectively; and they contain a common angle, the angle FAG .

Therefore the base FC is equal to the base GB , and the triangle AFC is equal to the triangle AGB , and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend, that is, the angle ACF to the angle ABG , and the angle AFC to the angle AGB .

And, since the whole AF is equal to the whole AG , and in these AB is equal to AC , the remainder BF is equal to the remainder CG .

But FC was also proved equal to GB ; therefore the two sides BF , FC are equal to the two sides CG , GB respectively; and the angle BFC is equal to the angle CGB , while the base BC is common to them; therefore the triangle BFC is also equal to the triangle CGB , and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend; therefore the angle FBC is equal to the angle GCB , and the angle BCF to the angle CBG .

Accordingly, since the whole angle ABG was proved equal to the angle ACF , and in these the angle CBG is equal to the angle BCF , the remaining angle ABC is equal to the remaining angle ACB ; and they are at the base of the triangle ABC . But the angle FBC was also proved equal to the angle GCB ; and they are under the base.

QED.

[Quod Erat Demonstrandum, Latin for "that which was to be demonstrated."]

Note

QEF is another but less common, "Quod Erat Faciendum, "that which was to have been done". Are there easier proofs? Of course, we saw several in the Introduction! But we can't just start in anywhere. Why, for example, start with Prop 5? In part, because the beginning of the *Elements* is a little shaky and we'll only partly "fix it up". If we spent too much time on a modern approach, we wouldn't get beyond that.

Contributors

<http://aleph0.clarku.edu/~djoyce/java/elements/bookI/bookI.html> - Used with owner's permission. [Or just Google "Euclid Book 1" if this URL (and very useful site) becomes unavailable. For example: <http://www.perseus.tufts.edu/hopper/text?doc=Euc.+1>]

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1.2: Book 1 Definitions

1. A *point* is that which has no part. [Essentially meaningless as a definition. From a modern perspective, there must be some undefined terms. This is one.]
2. A *line* is breadthless length. [Again, this is no definition. It is another undefined term.]
3. The extremities of a line are points. [*Endpoints*. Formally, something about "betweenness".]
4. A straight line is a line which lies evenly with the points on itself. [Say what? Curves are "lines" here but what is "straight"? In Chapter 5, we will define it formally but it won't seem natural!]
5. A *surface* is that which has length and breadth only. [A region in the (plane) geometry.]
6. The extremities of a surface are lines. [Edges of a polygon or polyhedron are line *segments*.]
7. A plane surface is a surface which lies evenly with the straight lines on itself. [See the above.]
8. A plane *angle* is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line. [Modern? An *angle* is the union of two rays with a common initial point.]
9. And when the lines containing the angle are straight, the angle is called *rectilineal*. [As opposed to an angle determined by the intersection of two "smooth" curves; i.e. the intersection of their tangents at the point of intersection. We will need these in the "Poincaré disk model" for hyperbolic geometry.]
10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is *right*, and the straight line standing on the other is called a *perpendicular* to that on which it stands. [Modern? *Perpendicular* lines are lines that intersect to form congruent, adjacent angles. The angles they form are called right angles.]
11. An *obtuse* angle is an angle greater than a right angle. [A real definition.]
12. An *acute* angle is an angle less than a right angle. [Two in a row!]
13. A *boundary* is that which is an extremity of anything. [Your sarcasm here?]
14. A *figure* is that which is contained by any boundary or boundaries. [I.e., closed and bounded.]
15. A *circle* is a plane figure contained by one line [curve] such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another. [Another, "Say what?!"]
16. And the point is called the *center* of the circle. [Another real definition!]
17. A *diameter* of the circle is any straight line drawn through the center and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle. [Omit "circumference".]
18. A *semicircle* is the figure contained by the diameter and the circumference cut off by it. And the center of the semicircle is the same as that of the circle. [As in #17, "circumference" implies the circle itself, the set of points equidistant from a given point; circumference actually means the linear measure of the circle.]
19. *Rectilineal figures* are those which are contained by straight lines, *trilateral* figures being those contained by three, *quadrilateral* those contained by four, and *multilateral* those contained by more than four straight lines. [Real definitions but we'd say *polygon* and, in case of three, *triangle*, but *trilateral* is used as is *quadrangle* and *n*-gon if we wish to specify the number of sides, *n*.]
20. Of trilateral figures, an *equilateral triangle* is that which has its three sides equal, an *isosceles triangle* that which has two of its sides alone equal, and a *scalene triangle* that which has its three sides unequal.
21. Further, of trilateral figures, a right-angled triangle is that which has a right angle, an obtuse-angled triangle that which has an obtuse angle, and an acute angled triangle that which has its three angles acute. [For us, *right triangle*, *obtuse triangle*, and *acute triangle*.]
22. Of quadrilateral figures, a *square* is that which is both equilateral and right-angled; an *oblong* that which is right-angled but not [necessarily] equilateral; a *rhombus* that which is equilateral but not [necessarily] right-angled; and a *rhomboid* that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called *trapezia*. [Never used!]
23. *Parallel straight lines* are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction. [Do they exist? Yes (but not in elliptic geometry), but what does "infinite" mean? Formally, it is the Archimedean Property: Given two segments, enough copies of the shorter one exceeds the longer one (independent of their lengths.)]

Postulates. [*Axiom* and *postulate* are interchangeable terms from the modern perspective, statements accepted as true without proof. Not in some universal sense, only in the context under consideration]

Let the following be postulated:

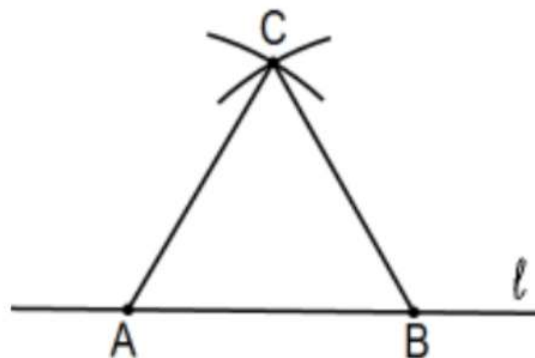
1. To draw a straight line from any point to any point. [Hereafter, **Axiom 1**: Two points determine...]
2. To produce a finite straight line continuously in a straight line. [Lines are "infinite" - Subsumed under the **Ruler Postulate**]
3. To describe a circle with any center and distance. [Axiomatically, a point and a radius determine a unique circle and circles behave as they should. 'Nuff said.]
4. That all right angles are equal to one another. [Subsumed under the **Protractor Postulate**.]
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles. [**Euclid's 5th Postulate**, traditional form. Far more common is **Playfair's** form: Given a line and a point not on the line, there is at most one line on the given point that is parallel to the given line. At most one? Maybe not even one?! No, but by proof, not by axiom].

Common Notions [From a modern perspective, these follow from set theory and properties of real numbers that can also be developed axiomatically. If we were to do so, we would never get done!]

1. Things which are equal to the same thing are also equal to one another. [Read: Congruence is transitive.]
2. If equals be added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

Proposition 1. On a given finite straight line to construct an equilateral triangle.

[Note the use of the infinitive as a command, "to construct". Think ancient Latin (or modern Spanish). Also "finite straight line" means line *segment*.]



Construction: Choose any 2 points on the line, say A and B. Let C be the intersection of circles (A; AB) and (B; AB), the circles centered at A and B both of radius AB (only *one* of the *two* intersections of these two circles to be more precise). The triangle determined by the 3 points, $\triangle ABC$ is equilateral.

Proof: By construction, all 3 sides are congruent, the definition of equilateral. [But how do we know the circles have to intersect? Can they "slip through" one another without intersecting? No, but there are geometries where they might. Without bothering to state them, we assume (as did the Greeks) that our axioms assure validity of such "continuity" considerations.]

Proposition 2. To place at a given point (as an extremity) a straight line equal to a given straight line

As stated, this really means "to construct"; i.e., use a compass to copy a line segment at any point on another line. It's immediate from the **Ruler Postulate**; exactly one point (with direction) for any given length.

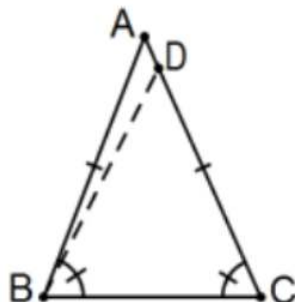
Proposition 3. Given two unequal straight lines, to cut off from the greater a straight line equal to the less. [This is also part of the **Ruler Postulate** so nothing to prove, at least from a modern viewpoint.]

Proposition 4. If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend. [That is, **Side-Angle-Side (SAS)** that we will assume as our triangle congruence axiom. It is proved in Book 1 but "superposition", the underlying assumption of the ancient Greek proof, is tantalizingly obvious so don't blame the Greeks too much for not realizing the need for an axiom; in our case, the **SAS Axiom**.]

Proposition 5. In isosceles triangles, the angles at the base are equal to one another, and, if the equal straight lines be produced further, the angles under the base will be equal to one another.

[See the Introduction and the introduction to this chapter and say "the base angles are *congruent*."]

Proposition 6. If in a triangle two angles be equal to one another, the sides which subtend the equal angles will also be equal to one another. [The *converse* of the "base angles of an isosceles triangle theorem"; as usually stated: In a triangle, sides opposite congruent angles are congruent.]



Given: $\triangle ABC$ with $\angle B \cong \angle C$ Prove: $AB \cong AC$.

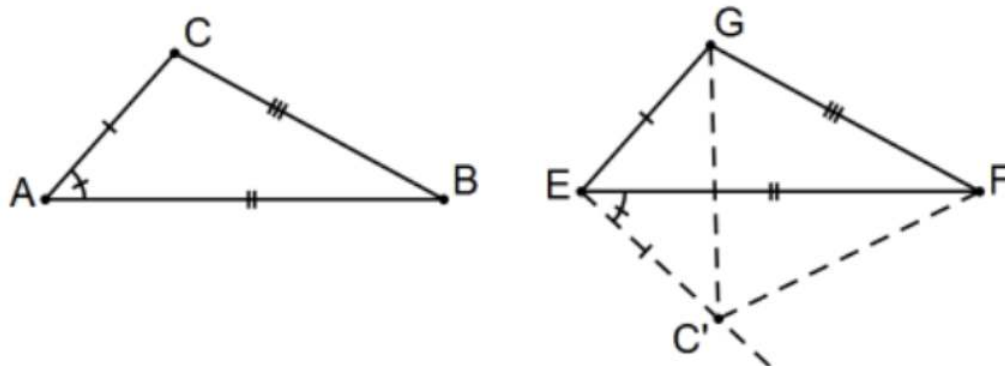
Proof: Take D to be the point along ray CA with $CD \cong BA$ (by the Ruler Postulate). Now $\triangle DCB \cong \triangle ABC$ by SAS so $\angle DBC \cong \angle ACB$ by cpctc and, by hypothesis and transitivity, $\angle DBC \cong \angle ABC$. By the Protractor Postulate, ray BD is the same ray as BA so, by the Ruler Postulate again, $D = A$ (exactly the same point!), and $CA \cong BA$ just by renaming CD as CA. QED.

Proposition 7. Given two straight lines constructed on a straight line (from its extremities) and meeting in a point, there cannot be constructed on the same straight line (from its extremities), and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each to that which has the same extremity with it. [This, together with **Prop 8**, is usually known as the **Side-Side-Side Congruence Theorem (SSS)**. Formally, if two triangles have three pairs of sides congruent, the triangles themselves are congruent.]

Proposition 8 (SSS). If two triangles have the two sides equal to two sides respectively, and have also the base equal to the base, they will also have the angles equal which are contained by the equal straight lines.

Given: $AB \cong EF$, $AC \cong EG$, $BC \cong FG$

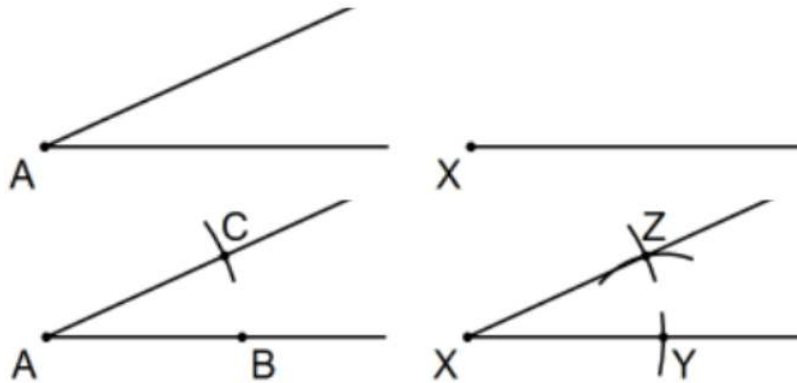
Prove: $\triangle ABC \cong \triangle EFG$



Proof: Copy $\angle A$ at E but on the opposite side of line EF than G (Protractor Postulate). Along that ray, let C' be the point such that $EC' \cong AC$ so, by the given hypothesis and transitivity, $\triangle EC'G$ is isosceles with base angles at C' and G . Moreover, $\triangle ABC \cong \triangle EFC'$ by SAS so $FC' \cong BC$ by cpctc so $\triangle FGC'$ is also isosceles, also with base angles at C' and G . Adding congruent angles, we have $\angle EC'F \cong \angle EGF$. Thus $\triangle EFC' \cong \triangle EFG$ by SAS and $\triangle ABC \cong \triangle EFG$ by transitivity. QED.

As long as it is done logically, there is nothing magic about following Euclid's Elements Book 1 in its original sequence and it is convenient to look at Prop 23 at this point:

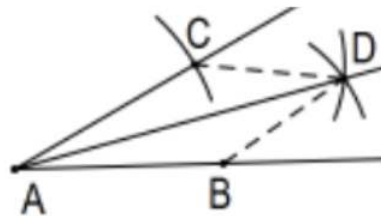
Proposition 23. On a given straight line and at a point on it to construct a rectilinear angle equal to a given rectilinear angle. [That is, copy an angle using straightedge and compass.]



Construction: To copy $\angle A$ at point X on the side given, choose any point $B \neq A$ on one of the rays of $\angle A$ and use that as the radius for circles centered at A and at X to establish points C and Y and an arc in the vicinity of (not yet determined Z). Construct the circle with center at Y of radius BC to establish point Z . Ray XZ is the desired angle at point X .

Proof: $\triangle ABC \cong \triangle XYZ$ by SSS so $\angle A \cong \angle X$ by cpctc. QED.

Proposition 9. To bisect a given rectilinear angle.

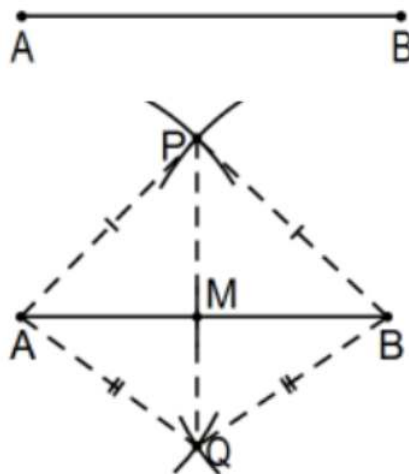


Construction: Choose a point, say B , on one of the rays of the angle and determine point C on the other ray by constructing the circle $(A; AB)$, the circle with center A of radius AB . With length $r > (1/2)m(BC)$, construct circles $(B; r)$ and $(C; r)$ and let D be their point of intersection (farthest from A , as pictured). Ray AD is the desired angle bisector. [Do NOT construct (or even indicate) segments BD and CD or any such dashed segments in these constructions; they are only indicated to clarify the proof.]

Proof: $\triangle ABD \cong \triangle ACD$ by SSS so that $\angle BAD \cong \angle CAD$ by cpctc and, by definition of angle bisector, we're done. [Why not until now? In fact, there is an SAS proof for this so that it could have been done before SSS. Can you see it?]

QED.

Proposition 10. To bisect a given finite straight line. [That is, construct the *midpoint* of a *line segment*.]



Construction: [Note: This does *not* say the *perpendicular* bisector, only the point that bisects; i.e., the midpoint. After the traditional construction, the perpendicular bisector of the segment to determine its midpoint, we will look at another that emphasizes this point. The problem is that some of you will say "perpendicular bisector" when you only mean one of "perpendicular" or "bisector". Words have meaning so use them correctly and meaningfully.]

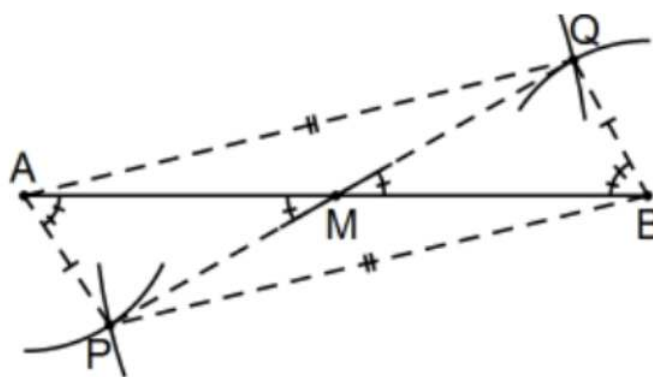
By means of circles, of the same radii, each greater than $(1/2)m(AB)$, construct two points, each equidistant from the endpoints; in this case, A and B are the endpoints and P and Q are the points, each equidistant from A and B . Finally, let M be the intersection of lines AB and PQ . [As in Prop 9, DON'T include the pictured dashed segments, only the *construction*. They are only there for clarity in the proof.]

Proof: By SSS, $\triangle APQ \cong \triangle BPQ$ so that $\angle APQ \cong \angle BPQ$ by cpctc. [Note: More clever would've been to say "From the proof of Prop 9, we have $\angle APQ \cong \angle BPQ$. Can you see it?"] In any case, we now have $\triangle APM \cong \triangle BPM$ by SAS so that $AM \cong BM$ by cpctc and M is the midpoint of AB . QED.

Note: A useful extension of the proof above is:

Theorem: A point is on the perpendicular bisector of a line segment if and only if it is equidistant from each of its endpoints (PS 1, #20).]

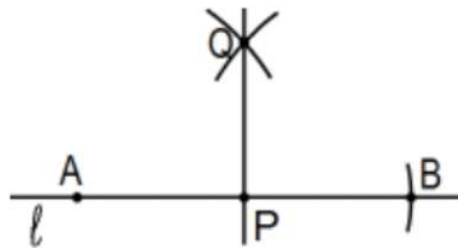
[As promised, here is a construction you probably would not have thought of that we can't prove yet but only because we don't yet have AAS or congruence of vertical angles. If we did...



Construction: Consider two pairs of circles with centers A and B of the same radii large enough to intersect at points P and Q . Let M be the intersection of lines AB and PQ ; then M is the desired point.

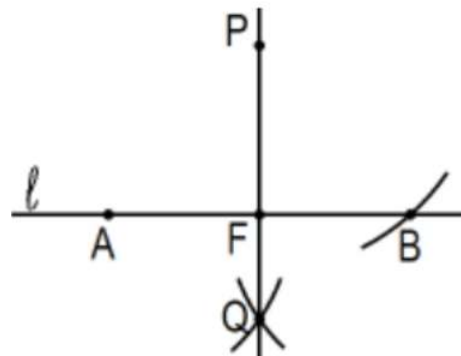
Proof: $\triangle ABQ \cong \triangle BAP$ by SSS so $\angle ABQ \cong \angle BAP$ by cpctc. But $\angle ABQ = \angle MBQ$ (i.e., exactly the same angle) and $\angle BAP = \angle MAP$ with $\angle AMP \cong \angle BMQ$ because they are vertical angles. Now $\triangle AMP \cong \triangle BMQ$ by AAS and, by cpctc, $AM \cong BM$ as desired. QED.]

Proposition 11. To draw a straight line at right angles to a given straight line from a given point on it. [Construct a perpendicular at a given point on a given line. Note: Do *not* use the other intersection of these two circles. Why not? Is PQ the *only* perpendicular to the line on the point P? Yes. Why?]



Proof: See PS 1, #9.

Proposition 12. To a given infinite straight line, from a given point which is not on it, to draw a perpendicular straight line. [Construct a perpendicular to a line from a point not on the line. Is it *the* perpendicular to the line on the point? Not yet; not until after Prop 16.]

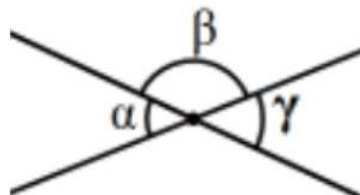


Proof: See PS 1, #8.

Proposition 13. If a straight line [ray] set up on a straight line make angles, it will make either two right angles or angles equal to two right angles. [Nothing to prove; it is axiomatic by the Protractor Postulate.]

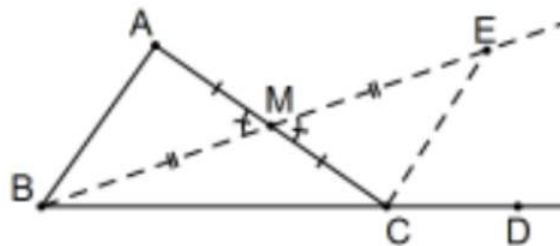
Proposition 14. If with any straight line, and at a point on it, two straight lines not lying on the same side make the adjacent angles equal to two right angles, the two straight lines will be in a straight line with one another. [Here either. It is also a consequence of the Protractor Postulate.]

Proposition 15. If two straight lines cut one another, they make the vertical angles equal to one another.

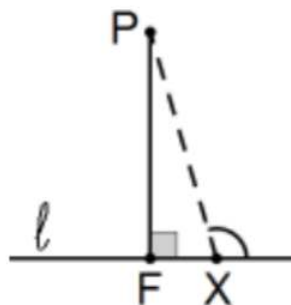


Proof: Viewed most simply: $\alpha + \beta = 180^\circ = \beta + \gamma$ implying $\alpha = \gamma$. QED.

Proposition 16. [Exterior Angle Theorem (neutral geometry form). KNOW IT!] In any triangle, if one of the sides be produced, the exterior angle is greater than either of the nonadjacent interior angles.



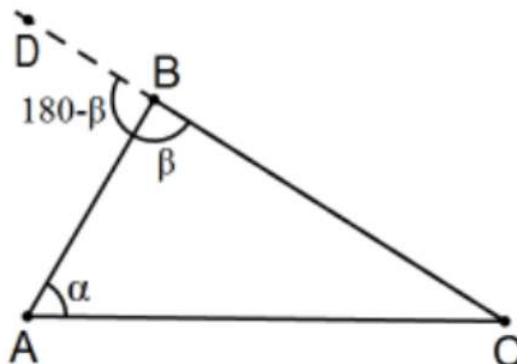
Proof: Let ray BM be the *median* on B [M is the midpoint of segment AC] and let E be along that ray such that $EM \cong BM$. Conclude that $\angle A \cong \angle ACE < \angle ACD$. Why does this prove that $\angle B < \angle ACD$? QED.



Corollary: The line of Prop 12 is unique. It is *the* line on the point that is perpendicular to the line.

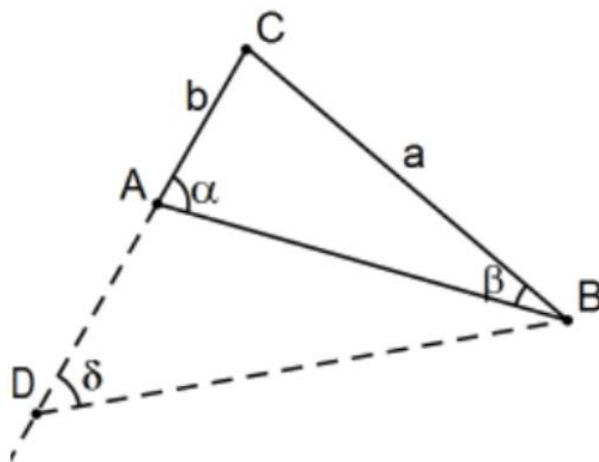
Other immediate corollaries are Props 17 and 27. Some others are only a step away.

Proposition 17. In any triangle, two angles taken together in any manner are less than two right angles.



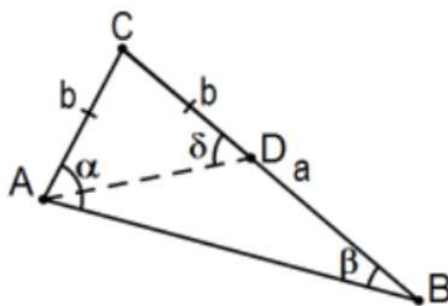
Proof: We focus on two of the angles, specifically those at A and B, of $\triangle ABC$. By supplementary angles, $m(\angle CBA) + m(\angle ABD) = \beta^\circ + (180 - \beta)^\circ$ where $\angle ABD$ is a nonadjacent exterior angle to $\angle A$ so, by Prop. 16, $m(\angle A) < m(\angle ABD)$ and $\alpha + \beta < (180^\circ - \beta) + \beta = 180^\circ$. Finally, $\alpha + \beta < 180^\circ$.

Proposition 18. In any triangle, the greater side subtends the greater angle. [Angles opposite greater sides are greater. Note: Be careful which is which.]



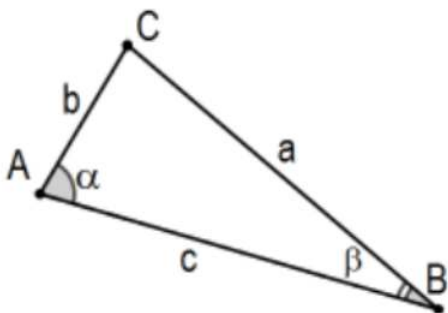
Proof 1: In $\triangle ABC$, assume $a > b$. Prove that $\alpha > \beta$.

Extend ray CA to point D such that $CD \cong CB$. Then $\angle D \cong \angle CBD$ (they are base angles of a triangle constructed to be isosceles) and point A is in the interior of $\angle CBD$ making $\angle CAB$ an exterior angle wrt (with respect to) $\angle ADB$ of $\triangle ADB$. By the Exterior Angle Theorem and the fact that $\angle A = \angle CAB$, we have that $\delta < \alpha$, and, since point A is interior to $\angle CBD$, $m(\angle B) (= \angle CBA) < m(\angle CBD)$. By transitivity, we have $m(\angle B) < m(\angle A)$ or $\alpha > \beta$. QED.



Proof 2: As above, in $\triangle ABC$, assume $a > b$. This time, let D be along ray CB such that $CD \cong CA$ and proceed similarly. QED.

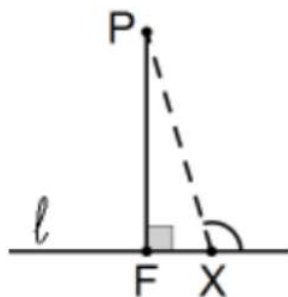
Proposition 19. In any triangle, the greater angle subtends the greater side. [The converse of #18; i.e., sides opposite greater angles are greater. Again, if..., then...?]



Proof: In $\triangle ABC$ pictured as in Prop.18 but assume that $\angle A > \angle B$. There are only three possibilities, $a < b$, $a = b$, or $a > b$. In the second case, the triangle would be isosceles and its base angles congruent, a contradiction to the hypothesis. In the first case, applying Prop 18, we would have that $\angle A < \angle B$; again, a contradiction. Thus $a > b$. QED.

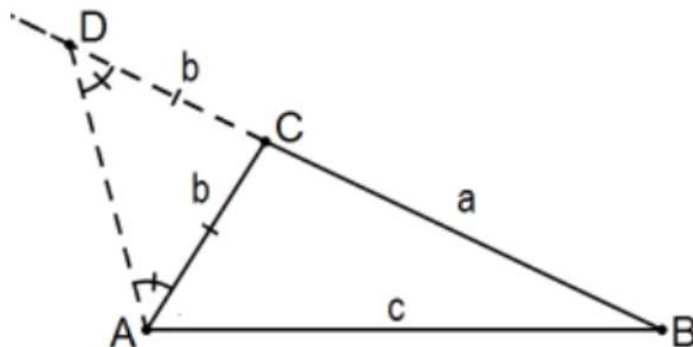
Note: Here, proof of the original statement made a proof of its converse to be very easy. Often that is the case (when the converse is true) but not necessarily. Sometimes a proof of the converse of statement can be quite different from using the validity of the original statement

Corollary: The shortest distance from a point to a line is the perpendicular distance.



Proof: Let F be the foot of the perpendicular on P and let X be any other point on the line. The greatest angle in the resulting triangle is... QED.

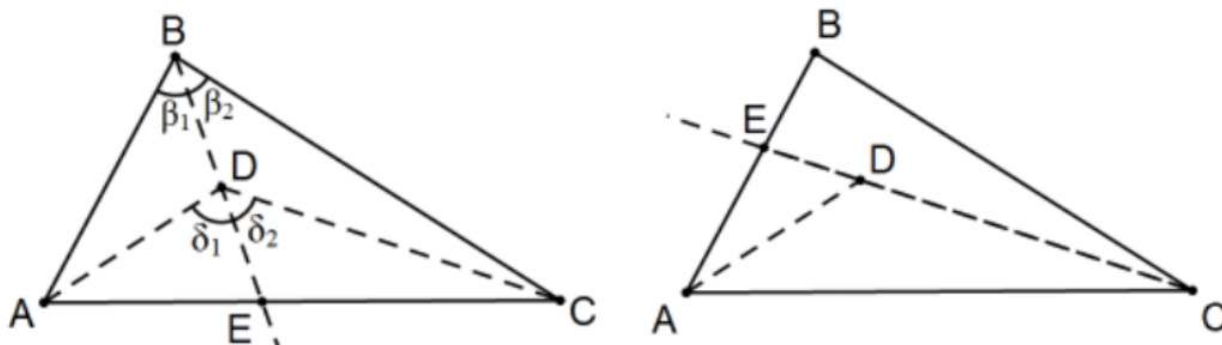
Proposition 20 [Triangle Inequality.] In any triangle, two sides taken together in any manner are greater than the remaining one. In other words, the shortest distance between any two points is (surprise, surprise) the length of the line segment that joins them.



Proof: In the picture, we prove that $a + b > c$. Let D be the point along ray BC of length b making $\triangle ACD$ to be an isosceles triangle so that $\angle D \cong \angle DAC$. It is clear that $\angle DAC < \angle DAB$ so that $\angle ADB < \angle DAB$. By Prop 19, $AB < BC + CD$. That is, $a + b > c$. QED.

Proposition 21. If on one of the sides of a triangle, from its extremities, there be constructed two straight lines meeting within the triangle, the straight lines so constructed will be less than the remaining two sides of the triangle, but will contain a greater angle.

Note: The archaic language is a little hard to understand but the idea is quite simple. In any triangle $\triangle ABC$, the triangle inequality says that the shortest distance between the two points A and C is the line segment that joins them. This proposition says that "going through" any interior point D of the triangle is shorter than following the sides of the triangle through B and that the angle determined is greater than the angle at that third point B .



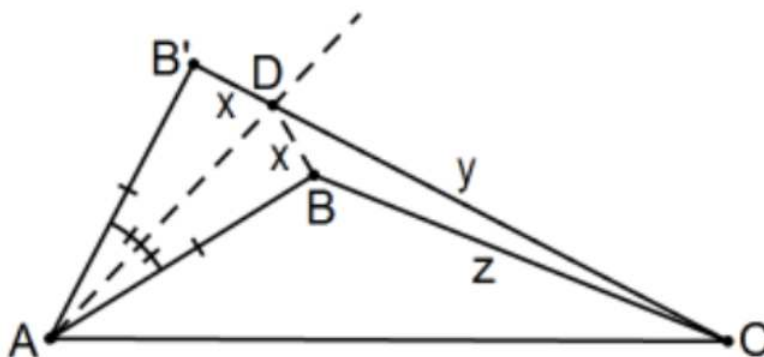
Proof: The angle situation is the easier of the two. Just consider ray BD and use the EAT at D by dividing the angle at B into two angles of size β_1 and β_2 using ray BD that also divides $\angle ADC$ into two angles of size δ_1 and δ_2 with both $\beta_1 < \delta_1$ and $\beta_2 < \delta_2$ by the exterior angle theorem and adding, $\beta = \beta_1 + \beta_2 < \delta_1 + \delta_2 = \delta$ so $\angle ABC < \angle ADC$. For the "shorter path" conclusion, extend ray CD to determine point E as pictured. The result follows from careful use of the Triangle Inequality: $AD < AE + ED$

so that $AD + DC < (AE + ED) + DC$ and reassociating, $AD + DC < AE + (ED + DC) = AE + EC$. But also $EC < EB + BC$ so that $AE + EC < AE + (EB + BC)$ and, since $AE + EB = AB$, we have that $AD + DC < AB + BC$.

Proposition 22. Out of three straight lines, which are equal to three given straight lines, to construct a triangle: thus it is necessary that two of the straight lines taken together in any manner should be greater than the remaining one. [The archaic language is awkward but the idea is simple.]

Proposition 23. Copying an angle using straightedge and compass was done immediately after Prop 8.

Proposition 24. If two triangles have the two sides equal to two sides respectively, but have the one of the angles contained by the equal straight lines greater than the other, they will also have the base greater than the base. [The **Hinge Theorem**, along with its converse Prop 25.]



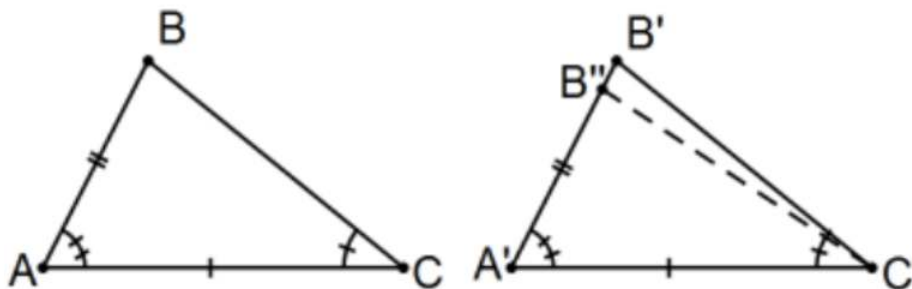
Proof: In the figure, $AB \cong AB'$ and $\angle BAC < \angle B'AC$.

Prove $BC < B'C$, or more simply, $x + y > z$.

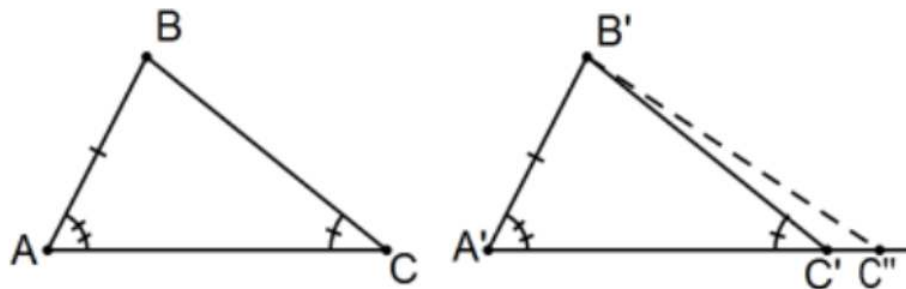
Bisect $\angle BAB'$ to determine point D and $\triangle BAD \cong \triangle B'AD$ by SAS so $BD \cong B'D$. But $BD + DC > BC$ so we're done. The picture could be misleading in that AD could be shorter than AB but the proof is the same. QED.

Proposition 25. If two triangles have the two sides equal to two sides respectively, but have the base greater than the base, they will also have the one of the angles contained by the equal straight lines greater than the other. [The converse of Prop 24 and easily follows from it.]

Proposition 26. If two triangles have the two angles equal to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, or that subtending one of the equal angles, they will also have the remaining sides equal to the remaining sides and the remaining angle to the remaining angle. [Both ASA and AAS. These are different theorems with different proofs so a mild structural error!]

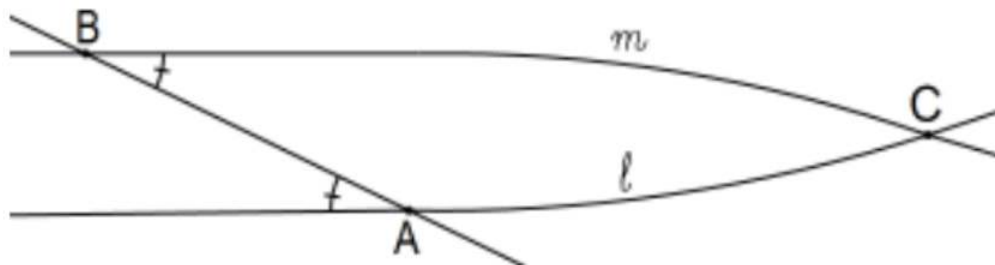


Proof of ASA: Assume $\angle A \cong \angle A'$, $\angle C \cong \angle C'$, and $AC \cong A'C'$. Let B'' be along ray $A'B'$ such that $AB \cong A'B''$. Then $\triangle ABC \cong \triangle A'B''C'$ by SAS so $\angle C \cong \angle A'C'B''$ by cpctc and, by transitivity, $\angle C' = \angle A'C'B''$; i.e., one and the same angle. Therefore, $B' = B''$ and the desired triangles are already known to be congruent. QED.



Proof of AAS: Assume $\angle A \cong \angle A'$, $\angle C \cong \angle C'$, and $AB \cong A'B'$. Let C'' be along ray $A'C'$ such that $AC \cong A'C''$. Then $\triangle ABC \cong \triangle A'B'C''$ by SAS so $\angle C \cong \angle C''$ by cpctc. If $C' \neq C''$, $\triangle B'C''C'$ would be a non-degenerate triangle and would therefore violate the Exterior Angle Theorem. Since that is impossible, $C' = C''$ and, as before, the triangles are already known to be congruent. QED.

Proposition 27. If a straight line falling on two straight lines make the alternate angles equal to one another, the straight lines will be parallel to one another. [With Prop 23, parallel lines exist!]



Proof: Suppose $\angle A \cong \angle B$ but the lines ℓ and m intersect at C as pictured. Then $\angle A$ would be an exterior angle not adjacent to interior angle $\angle B$ with respect to $\triangle ABC$. But by EAT, $\angle B < \angle A$ so that is impossible. That is, there is no such intersection on the right side. The lines can't intersect on the other side either for the same reason by interchanging which angle is interior and which is exterior. QED.

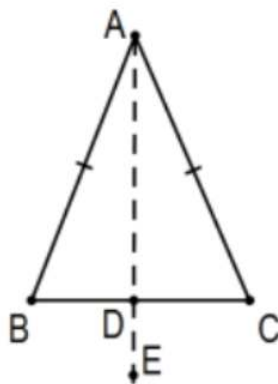
Proposition 28. If a straight line falling on two straight lines make the exterior angle equal to the interior and opposite angle on the same side, or the interior angles on the same side equal to two right angles, the straight lines will be parallel to one another. [Corollaries of Prop 27.]

Euclid's Elements, Book 1, Prop 29 is the converse of Prop 27 – 8 and that would mean we'd be in **Euclidean** geometry since (along with the rest of the axioms we are assuming) it is logically equivalent to Euclid's 5th Postulate and lots of variants some of which we'll look at eventually. For now, we are focusing on **neutral** or, by another common name, **absolute** geometry. Although there are many later theorems in Euclid's Elements (including in books that succeed Book 1) that are theorems of neutral geometry with identical proofs, the fact that Euclid went this far without using the 5th Postulate, including the limited form of the Exterior Angle Theorem, is persuasive indication that the Greeks were trying their best to prove it as a theorem instead of reluctantly accepting it as another axiom. For many centuries thereafter, very good geometers conducted extensive work in failed attempts to prove it. We will touch on a little of that in preliminary efforts of Saccheri and Lambert in the next chapter. Only the eventual acceptance of all of the other axioms along with the **negation** of Euclid's 5th Postulate as a legitimate geometry in its own right, hyperbolic geometry, resolved the millennia-old problem now known as modern geometry.

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1.3: Problem Set 1

Exs 1-6 relate to the pictured isosceles triangle with base angles at B and C. In each, indicate in a corresponding sketch all **given** information in the standard manner and, equally important, **nothing but** the given information. Then prove the base angles are congruent freely using, as appropriate, any of the standard congruence statements SAS, SSS, ASA, AAS, and Hypotenuse-Leg.



1. Assume that point D is the point determined by ray AE, the bisector of $\angle A$.
2. Assume that point D is the midpoint of line segment BC.
3. Assume that point D is determined by the perpendicular from (on) point A to line BC.
4. Explain why the picture is misleading if you were to assume that line DE is the perpendicular bisector of segment BC so that your ensuing proof would probably be flawed (i.e., not a proof at all).
5. Using only isosceles $\triangle ABC$ (that is, with no auxiliary points or lines) and SAS.
6. Using only isosceles $\triangle ABC$ (that is, with no auxiliary points or lines) and SSS.
7. Proposition 9 in Euclid's Elements, Book 1, and is translated as, "To bisect a given rectilinear angle." Construct the bisector of an angle and prove that it is valid.
8. Construct a perpendicular line to a line from (or "on") a point which is not on the line and prove your construction is valid.
9. Construct a perpendicular line to a line on a point which is on the line and prove your construction. [Note: In both Exs 8 and 9, the indefinite article "a" is deliberate but, and as remarked in the text for different reasons, both lines are unique so "the".]
10. Definitions are crucial to mathematics even though I generally will not ask you to quote any of the standard ones of elementary Euclidean geometry. For each, state the essence of its definition; i.e., only the information known about the geometric figure or situation by definition. Include a sketch of a representative of each situation and indicate in it only the information given from its definition.
 - i. isosceles triangle xii. complementary angles xxiii. rhombus
 - ii. right angle xiii. supplementary angles xxiv. bisector
 - iii. rectangle xiv. tangent line xxv. regular polygon
 - iv. scalene triangle xv. concentric circles xxvi. median
 - v. obtuse triangle xvi. trapezoid xxvii. altitude
 - vi. right triangle xvii. isosceles trapezoid xxviii. concurrent lines
 - vii. square xviii. radius xxix. skew lines
 - viii. parallel lines xix. diameter xxx. parallelogram
 - ix. vertical angles xx. tangent circles xxxi. Add your own!
 - x. perpendicular lines xxi. incircle of a polygon (need not exist)
 - xi. polygon xxii. circumcircle of polygon (need not exist)
11. State (in modern, coherent, mathematical English) and prove Euclid's Elements' Proposition 6, the converse of the base angles of an isosceles triangle theorem.

12. State (in modern, coherent, mathematical English) and prove Euclid's Elements' Prop. 18, "In any triangle the greater side subtends the greater angle." Complete the indicated second proof.
13. State (in modern, coherent mathematical English) and prove Euclid's Elements' Prop. 19, the converse of the result in Ex 12.
14. Consider the statement: All squares are rectangles.
 - a. By the inclusive definition (the usual in mathematics), the statement is true. State it as an implication; that is, change the statement to "If p , then q " form.
 - b. State its converse as a formal implication and restate the implication in casual (common use) form.
 - c. Is it true? Prove your assertion.
15. Prove that the shortest distance from a point to a line is along the perpendicular.
16. A line is defined to be tangent to a circle if they intersect in exactly 1 point. Prove that a line that intersects a circle is tangent to the circle if and only if it is perpendicular to the radius that it determines.
17. Once again, Ex 9 says "a perpendicular from a point to a line", not "*the* perpendicular from a point to a line. Prove that the definite article is correct and argue that on a *sphere* such a line need not be unique. [Hint: The North Pole and the Equator can be helpful here.] It is "*the*" under the axioms of neutral geometry but this is somewhat subtle, a consequence of the Exterior Angle Theorem, Prop.16.]
18. Construct the copying of a triangle and prove that the construction is valid. [Note: This is not "duplicating that triangle"; that is archaic usage that means to double its area.]
19. A **kite** is a quadrilateral with two pairs of congruent adjacent sides (the shape of a traditional kite).
 - i. Prove that the diagonals of a kite are perpendicular.
 - ii. State a reasonable (i.e., smooth, non-artificial sounding) form of the converse of 19*i*.
 - iii. Prove or disprove 19*ii*.
21. **Theorem:** A point is on the perpendicular bisector of a line segment if and only if it is equidistant from each of its endpoints.
Note: An alternate form for this theorem, especially in older books is, "The locus of all points equidistant from two points is the perpendicular bisector of the line segment that joins them."
22. Given three points on a circle, construct the center of the circle.

Note 1: This language means more than it says; it means to explain a construction procedure *and prove that your procedure is valid*.

Note 2: Do not construct perpendicular bisectors of all 3 of the determined chords. Why not?
23. Part of Proposition 26 (both ASA and AAS) is better thought of as "If SAS, then ASA." State its converse and prove it. [Hint: See the comments on SSS at the end of the Introduction.]
24. A quadrilateral with the property that its four angles make two pairs of successive congruent angles (they share a side) is an isosceles trapezoid. **Corollary:** The opposite sides of a rectangle are congruent. [Hint: Consider the perpendicular bisector of either base and include diagonals to make congruent triangles.]

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CHAPTER OVERVIEW

Chapter 2: More Elementary Neutral Geometry

[2.1: Elementary Neutral Geometry](#)

[2.2: Problem Set 2](#)

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2.1: Elementary Neutral Geometry

Most of the following results date from antiquity but, even if so, we will consistently use modern language and you should too. In addition, try to be careful with symbolism. It is too hard for me to do that in my word processor but always add distinguishing symbols as appropriate - Do as I say, not as I do! For example, the line segment $\overline{AB}$ should be written with a bar over it, ray $\overrightarrow{AB}$ with an arrow, the same ray could be indicated by writing $\overrightarrow{BA}$ with the arrow reversed (that my word processor can't do), and line $\overleftrightarrow{AB}$ with a double-headed arrow. Note that line segment $\overline{AB}$ is equal to line segment $\overline{BA}$ as sets. That is, they are one and the same set but rays $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are not the same set. Formally:

segment $\overline{AB} = \{X \mid X \in \overleftrightarrow{AB} \text{ with } X = A, X = B, \text{ or } X \text{ between } A \text{ and } B\}$, and ray $\overrightarrow{AB} = \{X \mid X \in \overleftrightarrow{AB} \text{ with } X = A, X = B, X \text{ between } A \text{ and } B, \text{ or } B \text{ between } A \text{ and } X\}$.

Note

Does "between" make sense? It had better! This is yet another example of our *semi-formality*.

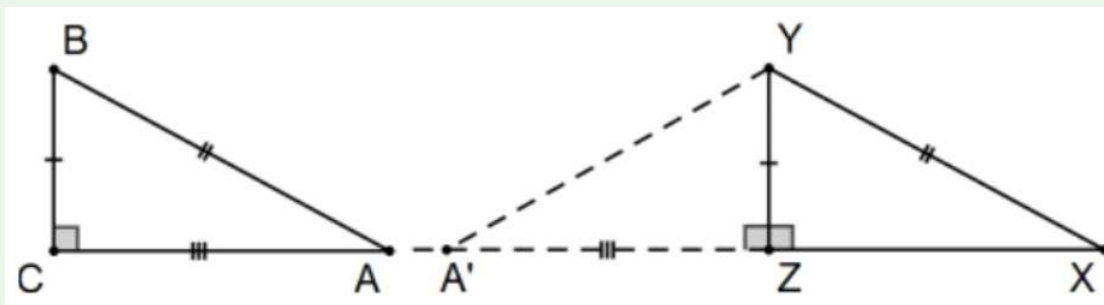
Using these conventions consistently "frees up" the use of AB with no symbolic modifiers to indicate the *measure* of line segment $\overline{AB}$; i.e., the *distance* from A to B , $m(\overline{AB})$, can be simplified as AB .

Although *congruence* is an undefined term in a strictly formal geometry, our axioms always have a concept of length of a line segment (the Ruler Postulate) and angle measure (the Protractor Postulate) and, with those, the idea we want can be (and is!) defined as a one-to-one correspondence that preserves the distance between any two points and the measure of any two corresponding angles. [Note: "Distance preserving" makes the "one-to-one" redundant.] Any set is congruent to itself in the abstract but to say $\overline{AB} \cong \overline{BA}$ is to assert more than its set equality; it implies such a distance-preserving correspondence *but in reverse order*. That is, A is "mapped to" B , the point $1/3$ of the way from A to B to the point $1/3$ of the way from B to A , etc.; i.e., all other points likewise. It is true although we won't worry about such things unless nagged but, for completeness sake, it is the reflection of the segment in its perpendicular bisector. Similarly, $\angle ABC \cong \angle CBA$ is true but *not* just because they are one and the same set; i.e., not because of the "reflexive property of congruence" (although you will see that in many elementary geometry books!). Again there is an underlying congruence, namely, the reflection of the angle in the line that bisects the angle.

In fact, it is even deeper than a distance and angle preserving correspondence between the two sets. To be formal about it, a congruence is defined to be a one-to-one, distance preserving, angle preserving correspondence from *the entire point set of the geometry* onto itself. That is, the assertion that two figures are congruent is an assertion that the entire point set can be matched to itself in a way that preserves all distances and angles as it "carries" one figure onto the other. In Euclidean plane geometry, a *translation* (glide or slide, in some elementary books), a *rotation* about a fixed point, and a *reflection* across a fixed line are all congruences, as well as compositions thereof. Are there others? Are these congruences of so-called "neutral" geometry as well? The answers are no, yes, and yes but we won't worry about it until Chapter 5 although some presentations are built around it. We still have not proved last of the congruence theorems:

Theorem: Hypotenuse-Leg

Two right triangles with hypotenuses congruent and one pair of legs congruent are congruent. Restated specifically:



Given: $\triangle ABC$ and $\triangle XYZ$ with $\angle C$ and $\angle Z$ right angles, $AB \cong XY$, and $BC \cong YZ$

Prove: $\triangle ABC \cong \triangle XYZ$

Proof: Take A' along ray XZ with $ZA' \cong CA$. Then $\triangle A'YZ \cong \triangle ABC$ by SAS so $A'Y \cong AB$ and, by transitivity, $A'Y \cong XY$ so $\triangle A'YX$ is isosceles. Then $\angle A' \cong \angle X$ but $\angle A' \cong \angle A$ by cpctc so $\triangle A'YZ \cong \triangle XYZ$ by AAS (in a couple of different ways). Finally, $\triangle ABC \cong \triangle XYZ$ by transitivity. QED.

Theorem: Construct the Center of a Circle

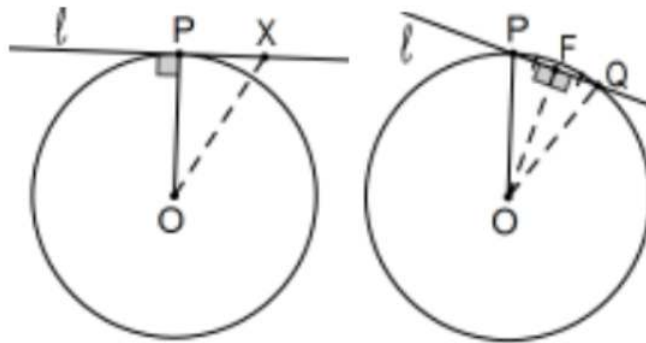
Select any three points on the given circle and construct the perpendicular bisectors of any two of the three chords they determine. These do intersect and their point of intersection is the center of the circle.

Proof: (PS 1, #21) Know it, its proof, and lots of other easy ones.

Note 1: Why not all three pairs? Hint: It is not wrong but makes the proof much harder. Why?

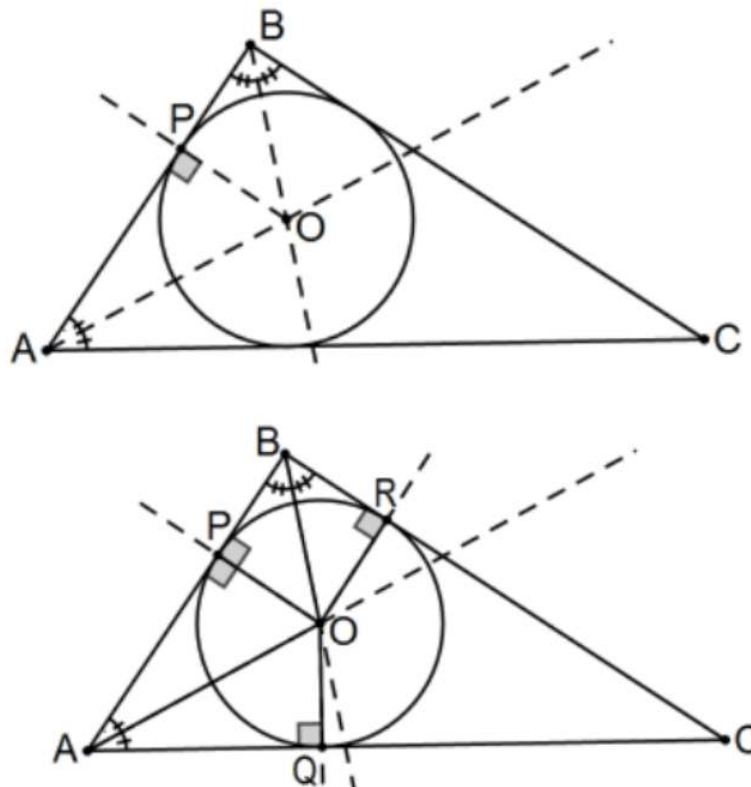
Note 2: This does **not** prove that any triangle **has** a *circumcircle* because it does not prove that the perpendicular bisectors of two of its sides intersect. If they do intersect, then all 3 vertices of the triangle are equidistant from that point of intersection and the set of all points equidistant from that point is a circle that contains the 3 vertices, hence the *circumcircle*, and the intersection point is its center, called the *circumcenter* of the triangle. Note that this point is equidistant from each pair of points on the circle and hence on its perpendicular bisector so all such perpendicular bisectors are concurrent. In our case, the three points were already on a circle so, obviously, its center must lie on each of the two perpendicular bisectors. Therefore they must intersect to identify the center. QED.

Theorem: Tangent iff Perpendicular to Radius: A line that intersects a circle at a point is tangent to the circle at that point iff it is perpendicular to the radius to that point of intersection.



Proof: First the "if", suppose P is on circle O and the given line ℓ is perpendicular to radius OP . Let X be any point on the line other than P . Then $\triangle OPX$ is a right triangle with hypotenuse $OX > OP$ that forces X to be outside the circle so only one point of intersection of line ℓ and circle O . By definition of tangent, the line is tangent to the circle at that point. Conversely, assume line ℓ is tangent to circle O and let F be the foot of the perpendicular from O to the line. If it were true that $F \neq P$, let Q be the point on the opposite side of F with the $FQ \cong FP$. Then $\triangle OFQ \cong \triangle OFP$, by SAS, so $OQ \cong OP$ and Q would be a second point of line ℓ on circle O and the line is not tangent since $Q \neq P$. This proves the inverse (if not perpendicular, then not tangent), not the converse as expected, but they are logically equivalent. QED.

Theorem: To Construct the Incircle of a Triangle:



Construction: Construct the bisectors of two of the angles of the triangle. The unspecified(!) axioms do guarantee that they intersect (since they are inside the closed and bounded triangle). That point is the *incenter* of the triangle. To determine the radius, construct the perpendicular to any one of its sides. With the center and the radius constructed, the circle is constructed.

Proof: Let Q and R be the perpendiculars to the other two sides where O was determined by the bisectors of angles A and B. $\triangle AOP \cong \triangle AOQ$ and $\triangle BOP \cong \triangle BOR$ by AAS so that $OP \cong OQ \cong OR$. That implies that P, Q, and R lie on the circle with center at O and radius OP. The perpendicularity of the radii with each of the sides implies that this circle is tangent to each of the sides, the **incircle**.

Corollary: The angle bisectors of a triangle are concurrent.

Proof: Just prove that the ray (already determined) from the third vertex C through the incenter O in the proof of the theorem above bisects that third angle. [Hint: Use Hypotenuse-Leg.] QED.

Note: This approach to proving that entities that share some common description share some other common description is important: Quit "constructing" as soon as the needed entity is determined and *prove everything else about it*. If not, the proof is made much more difficult by having to show that apparently different entities are really one and the same. That is, O was determined by bisecting two of the angles. Bisecting the third angle would (ostensibly) determine two more points that would have to be shown to be the same point. Avoid the problem by never "overdetermining" anything. Stop saying "let x be" when x is determined and prove everything appropriate about it. As another example from the construction, the second and third perpendiculars were not part of the actual *construction*, only part of the proof of validity.

Theorem: Angle Sum Theorem (neutral geometry form): The sum of the angles of a triangle is not greater than two right angles. [So for an n -gon, not greater than $180(n - 2)$.]

Proof: One nice proof is an extension of the previous proof of the Exterior Angle Theorem but first we consider some preliminary ideas.

[Note: We are no longer in our literal "real world"; i.e., life on a sphere. Here there are no parallel lines (any two lines intersect in antipodal points; i.e., the midpoint of the Euclidean segment they determine is the center of the sphere). Consider a triangle determined by two longitudinal line segments from the North Pole to the Equator and the segment of the Equator that they

determine. The angle sum of such a triangle is $90^\circ + 90^\circ + x^\circ$ where x° is the measure of the angle at the North Pole. What is wrong in the proof; something must be! Hint: Size matters on a sphere. What is "too big"? See PS 2, #27.]

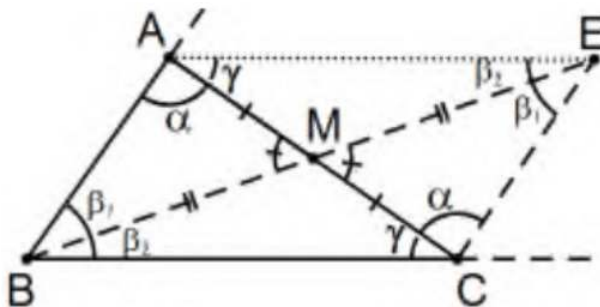
Definition: Two polygons are equivalent if one can be cut up into a finite number of polygons and the pieces rearranged to make the other; i.e., to fit perfectly. [A formal definition of this is tricky, so we won't bother!] Example: Snip a 2×8 rectangle in the middle and rearrange to obtain a 4×4 square.

Question? What is a rectangle? Are there any? Answer: Yes (in Euclidean geometry) or No (hyperbolic).

Note: *Area* hasn't been defined but, whatever it might be, it should be the same for equivalent figures!

Theorem: If two triangles are equivalent, then they have the same angle sum. More generally, if a triangle is equivalent to an n -gon the difference of their angle sums is $180(n - 3)$.

Proof: Handwaving. We're not supposed to do that in mathematics but a formal proof would get in the road and would not be helpful. The idea is that vertices introduced by the cutting will either be in the interior where the constituent angles add to 360° or along an edge where they add to 180° . As these pieces are rearranged, the perfect fit requires the same situation. A formal proof would be by some kind of tricky mathematical induction. Fortunately, we do not need the full power of this result to complete the proof of the Angle Sum Theorem; we can prove what we need as we go along:



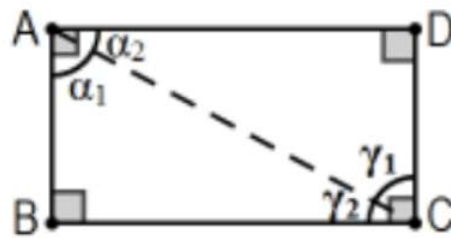
In the proof of the Exterior Angle Theorem, Prop 16, it is easy to see that $\triangle EBC$ is equivalent to the original $\triangle ABC$. In the process, $\angle B$ was divided into two angles and that each subangle is involved in this new triangle, $\angle EBC$ is one of those angles and $\angle EBA \cong \angle BEC$ is the other. Moreover, $\angle A$ is congruent to $\angle ECM$ that is a component part of $\angle BCE$ with the original $\angle C$ being the rest of $\angle BCE$. Therefore, the angle sum of this new $\triangle EBC$ is the same as the original $\triangle ABC$. If, as usual, we take $m(\angle B) = \beta$ and take the measures of the two subdivided angles as $\beta = \beta_1 + \beta_2$ as indicated, at least one of β_1 or β_2 must be less than or equal to $\beta/2$ because, if both were greater than $\beta/2$, their sum would exceed β , an impossibility. In our figure, the smaller appears to be β_2 , in which case, we would proceed with $\triangle EBC$ as pictured but it were to be β_1 , we would use $\triangle EBA$, also pictured and also equivalent to $\triangle ABC$ and where all of the same remarks apply.

Finally, we are ready to begin an indirect proof; i.e., assume this $\triangle ABC$ is one with angle sum greater than 180° . Express this "excess" by $\varepsilon > 0$. In other words, $\alpha + \beta + \gamma = 180 + \varepsilon$. The new triangle, $\triangle EBC$ has the same angle sum but with the new angle at B of measure less than or equal to $\beta/2 = \beta/2^1$.

Repeating the same construction (i.e., with the median of B through the midpoint of side CE , we obtain a triangle with the same angle sum but, this time, the measure of the angle at B is less than or equal to $\beta/2^2$, and repeating n times, $\beta/2^n$ where, for sufficiently large n , $\beta/2^n < \varepsilon$. So what? By Prop 17, the other two angles have a sum, say x , of less than 180° so adding in the angle at B , say y , we have a triangle with angle sum of $180 + \varepsilon = x + y$ with $x < 180$ and $y < \varepsilon$. Obviously, this cannot be so; there is no such triangle and the angle sum of any triangle does not exceed 180° . QED.

Corollary: If a line intersects two sides of a rectangle it divides it into two polygons with no "deficit"; i.e., their angle sums less can't be than they "should be". For example, if one of the polygons is a right triangle, then the other two angles are complementary or, if it's a quadrilateral, the two adjacent angles not known to be right angles are supplementary. The idea is that if either resulting n -gon has angle sum strictly less than $180^\circ(n - 2)$, then ... [But, again, are there any rectangles? Wait till the next chapter.]

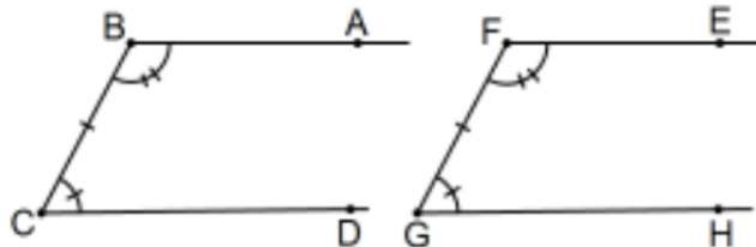
Corollary: The opposite sides of a rectangle are congruent. [Note: This was a corollary to PS 1, #23.] Moreover, it is not a theorem about *parallelograms* in general.]



Proof: Either diagonal of a rectangle divides it into two right triangles. There can be no "deficit", the angle sum of each must be 180° , since the two acute angles of each compose a right angle of the rectangle. Specifically with reference to the pictured rectangle, each vertex angle is a right angle so that $\alpha_1 + \alpha_2 = \gamma_1 + \gamma_2 = 90^\circ$ and, since each triangle has angle sum less than or equal 180° , $\alpha_1 + \gamma_2 \leq 90^\circ$ and $\alpha_2 + \gamma_1 \leq 90^\circ$. If either of these inequalities were strict (i.e., $<$), we would have $(\alpha_1 + \gamma_2) + (\alpha_2 + \gamma_1) < 180^\circ$ and, rearranging the terms, $180^\circ = (\alpha_1 + \alpha_2) + (\gamma_1 + \gamma_2) < 180^\circ$. This contradiction forces both $\alpha_1 + \gamma_2 = 90^\circ$ and $\alpha_2 + \gamma_1 = 90^\circ$. That is, $\alpha_1 = 90^\circ - \gamma_2 = \gamma_1$ and $\alpha_2 = 90^\circ - \gamma_1 = \gamma_2$. Either equation alone, along with right angles at B and D and the common diagonal, imply that $\triangle ACD \cong \triangle CAB$ by AAS or, using them both, by ASA. In either case, by cpctc, the opposite sides are congruent.

Theorem: ASA for Long Triangles. First off, there is no such thing; i.e., a "long triangle" is not a triangle at all. It is a convenient name for a figure that is the union of two rays that, together with the segment of their initial points, do not intersect. [Note: This name is not used universally in geometry.]

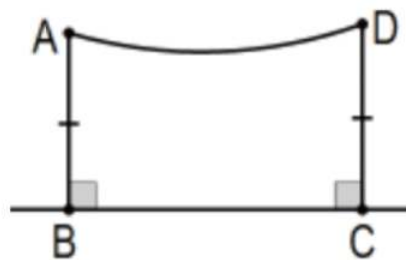
[Note: This was a *definition*; you need to know definitions even if never written down!] The *theorem*? Effectively, if ABCD and EFGH are geometric figures such that $\angle B \cong \angle F$, $\angle C \cong \angle G$, and segments $BC \cong FG$, then rays BA and CD intersect iff rays FE and GH intersect. [Equivalently, one pair does not intersect iff the other pair does not intersect; i.e., one is "long" iff the other is.]



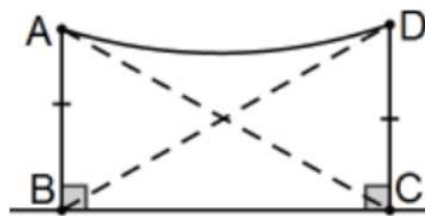
Proof: Suppose rays BA and CD do intersect, say at X, and take Y along GH with $GY \cong CX$. Then $\triangle BCX \cong \triangle FGY$ by SAS and $\angle CBX \cong \angle GFY$ by cpctc. But $\angle CBX = \angle CBA = \angle B$ (i.e., one and the same angle) so, by hypothesis and the Protractor Postulate, Y is also on ray FE, the point of intersection with ray GH. QED.

Note: These long triangles are "congruent"; i.e., there is a one-to-one correspondence such that all distances and angles are preserved) and this proof could be cleaned up to accommodate it. We won't need it.

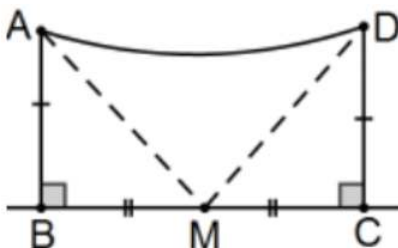
Definition: A *Saccheri quadrilateral* is a quadrilateral with one pair of opposite sides, called **legs**, that are congruent and perpendicular to (on the same side of) the line of its base. Its fourth side is called its **summit**. For example, Fig. ABCD is a Saccheri quadrilateral since lines AB and CD are perpendicular to line BC and segments $AB \cong DC$. In this case, these segments are the legs, segment BC is the base, and segment AD is the summit. Line AD is the summit line.



Theorem: The summit angles of a Saccheri quadrilateral are congruent (and therefore not obtuse).



Proof 1: Know several and be able to correctly finish any prompt I might give. Probably the most obvious proof uses the diagonals AC and BD as indicated. Obviously, $\triangle ABC \cong \triangle DCB$ by SAS so that $AC \cong DB$ and $\triangle ABD \cong \triangle DCA$ by SSS so that the original $\angle A \cong \angle D$ by cpctc. QED.

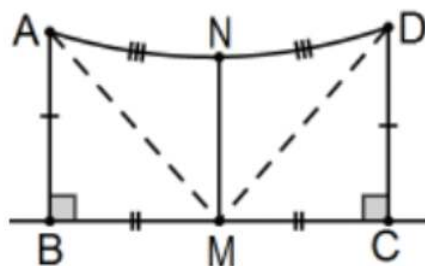


Proof 2: A less obvious but still easy proof is using exactly one auxiliary point, the midpoint M of the base BC as pictured. Obviously, $\triangle ABM \cong \triangle DCM$ by SAS so that $AM \cong DM$ and $\triangle AMD$ is isosceles so its base angles are congruent by previous theorem. Combined with the other pair of congruent angles at A and D (by cpctc), we have another proof that the summit angles are congruent. QED.

Note 1: As a matter of good form in a situation such as in that second proof, do say something like, "Let M be the midpoint of line segment BC." but don't say anything about "let" or "construct" anything else. At that point, segment AM, ray AM, and other such that may be needed already exist. Just use them.

Note 2: A Saccheri quadrilateral need not (at least not yet proved) be a rectangle. In Euclidean geometry, they turn out to be one and the same but in neutral geometry, the summit angles may or may not be right angles. What is a rectangle? [Hint: Clear your throat while saying "right" as if you were Olde Englisch.]

Theorem: In a Saccheri quadrilateral, these are the same line:



- i. The line of the midpoints of the base and the summit.
- ii. The perpendicular bisector of the base.
- iii. The perpendicular bisector of the summit.

Proof: See PS 2, #12-16. For i implies ii and iii:

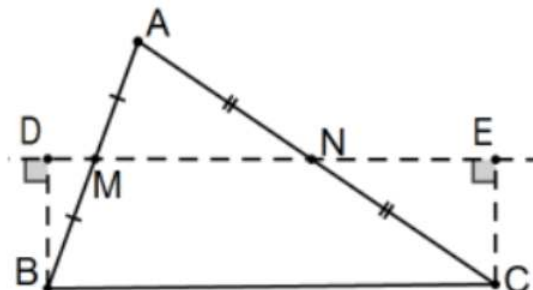
Let M be the midpoint of the base and N the midpoint of the summit of the Saccheri quadrilateral ABCD as pictured. Then $\triangle ABM \cong \triangle DCM$ by SAS so $AM \cong DM$ by cpctc and $\triangle AMN \cong \triangle DMN$ by SSS. Finally, the angles at N are both congruent (cpctc) and supplementary hence line MN is perpendicular to the summit and, by adding pairs of congruent angles, it is also perpendicular to the base. Since it was given that it bisected each (i.e., "let"), the line is the perpendicular bisector of each. QED

Note: The (lines of) the base and summit of a Saccheri quadrilateral share a common perpendicular. Therefore, among other things, they are parallel by the EAT. Another result of this proof is that it provides more proofs of the preceding theorem; the summit

angles of a Saccheri quadrilateral are congruent.

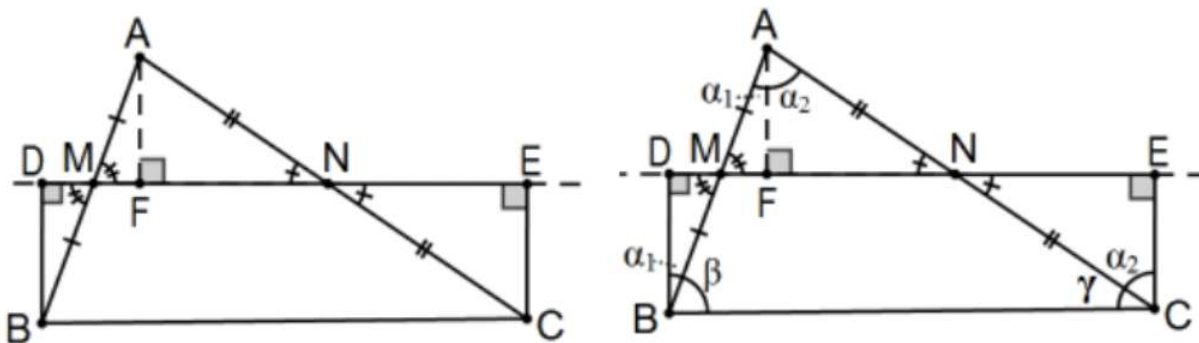
Theorem: [Associated Saccheri Quadrilateral] In a triangle, consider the line determined by the midpoints of two sides and the perpendiculars to that line from the two vertices on the same side of that line. The segments determined form a Saccheri quadrilateral that is equivalent to the original triangle with summit angles such that their sum is the sum of the interior angles of the triangle.

[What are its base, legs, and summit?]



Construction: Know it. In $\triangle ABC$, let M and N be the midpoints of two of its sides; in the pictured case, segments AB and AC , respectively. Let D and E be the feet of the lines perpendicular to line MN from the two vertices of $\triangle ABC$ on the same side of line MN ; in this case, vertices B and C . Then $BDEC$ is the desired Saccheri quadrilateral with summit BC .

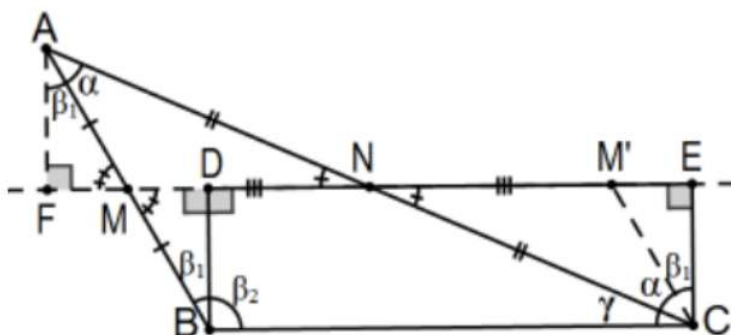
Note: This is the construction of only one of the three associated quadrilaterals of $\triangle ABC$; there is another with summit AB and a third with summit AC .



Proof: The most natural proof uses AAS by dropping the perpendicular from A to DE to determine F and conclude that $BD \cong AF \cong CE$ so that $BD \cong CE$, the only detail needed to conclude $BDEC$ is Saccheri and the equivalence of that quadrilateral with $\triangle ABC$ is obvious.

[Note: There is an alternate proof that uses SAS. See PS 2, #23.]

However, the proof is not complete because we haven't discussed angle sums. Moreover, even the rest of the proof is not complete because it implicitly assumes that F falls between M and N and that need not be the case. Continuing with that assumption, the angle sum conclusion is easy. Starting with the summit angles, we compute:



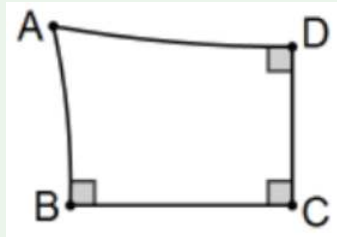
$(\alpha_1 + \beta) + (\gamma + \alpha_2) = (\alpha_1 + \alpha_2) + \beta + \gamma = \alpha + \beta + \gamma$ as required. Although the proof above is not valid if $\angle B$ is obtuse, the result it is still true. In the indicated figure, let M' be the point along ray NE such that segments $NM' \cong NM$. Now $\triangle ANM \cong \triangle CNM'$ by SAS so that $M'C \cong MA$ and, by supplementation, $\angle CM'E \cong \angle AMF$ so that $\triangle CM'E \cong \triangle AMF \cong \triangle BMD$ by AAS.

This time $\beta = \beta_1 + \beta_2$ and, again starting with the summit angles:

$$\beta_2 + (\gamma + \alpha + \beta_1) = \alpha + (\beta_1 + \beta_2) + \gamma = \alpha + \beta + \gamma \quad \text{QED.}$$

Definition: A Lambert Quadrilateral

A **Lambert quadrilateral** is a quadrilateral with three right angles. [In Fig. ABCD, angles B, C, and D are right angles. What is the size of $\angle A$?]



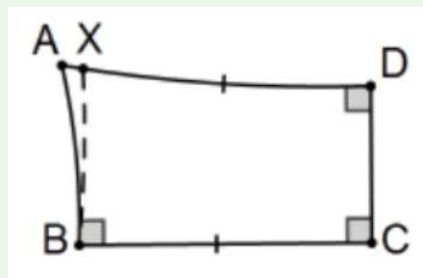
Theorem

In a Saccheri quadrilateral, the line determined by the midpoints of the base and the summit determine two congruent Lambert quadrilaterals.

Proof: It's done already except for the formal meaning of congruent polygons. [That is, a distance preserving, angle preserving transformation of the entire plane. Reflection in that line of midpoints provides the needed congruence function. QED.]

Theorem

Each side of the fourth angle of a Lambert quadrilateral is greater than or equal to (i.e., not shorter than) the side opposite it and its 4th angle is not obtuse.



Proof: Suppose ABCD is the Lambert quadrilateral with unknown angle A. Let X be along ray DA with segment DX congruent to segment CB . Then figure BCDX is a Saccheri quadrilateral with summit BX . $\angle XBC \leq \angle ABC$ so $DX \leq DA$ (why?). QED.

Theorem 2.1.1

The summit of a Saccheri quadrilateral is greater than or equal to its base and its summit angles are not obtuse.

Proof: These are immediate corollaries to the above (but start with a Saccheri quad, not a Lambert.)

These quadrilaterals (both "almost" rectangles) were studied extensively by their eponyms in an effort to prove Euclid's 5th Postulate. As is indicated in this last theorem, anything provable by one approach can be translated into a proof of the other. The trick is in being careful to start where you need to start. For example, in the above, start with a general Saccheri quadrilateral; do

not start with a Lambert quadrilateral. Another example is that the fourth angle of a Lambert quadrilateral is not obtuse. This time to prove it, start with a general Lambert quadrilateral and construct a useful Saccheri quadrilateral.

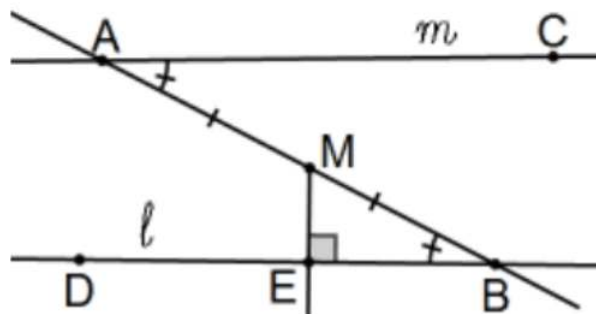
 Note

There are other theorems of neutral geometry that could be proved, especially ones that can be stated in this "greater than or equal to" form but will be stated only in their hyperbolic geometry "strict" form in the next chapter. We will break here and start looking at them from that perspective.

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2.2: Problem Set 2

Exs 1-6 relate to the pictured lines ℓ and m that share a transversal that forms a pair of congruent alternate interior angles, $\angle BAC \cong \angle ABD$. In each case, let M be the midpoint of segment AB and let E be the foot of the perpendicular from M to ℓ . Moreover, start by making a corresponding sketch all **given** information in the standard manner and, equally important, **nothing but** the given information. Then prove the resulting line EF is a common perpendicular of the two lines freely using any of the standard congruence statements SAS, SSS, ASA, AAS, or Hypotenuse-Leg as appropriate. [Assume that line AB is not perpendicular to line ℓ or we're already done!]



1. Let F be the intersection of line EM with m . [Assuming that they DO intersect!]
2. Let F be the foot of the perpendicular from M to m .
[Note: Don't forget to show that E , M , and F are collinear.]
3. Let F be the foot of the perpendicular from A to line EM .
4. Let F be the point along ray AC with $AF \cong BE$.
5. Let F be the point along ray EM with $MF \cong ME$.
6. Let F be the (correct) intersection of circles $(A; BE)$ and $(M; ME)$; i.e., the circles centered at A of radius BE and at M of radius ME .

Note: It is preferable to use fewer symbols but well-chosen words in stating mathematical results; it focuses more on the concepts instead of their specific presentation. Exs. 7-10 repeat four of Exs 1-6 exhibiting that philosophy.

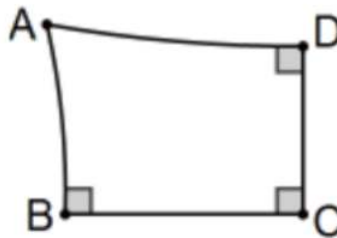
7. Use SAS to prove that if a transversal to two lines forms a pair of congruent alternate interior angles, then the lines have a common perpendicular.
8. Use ASA to prove that if a transversal to two lines forms a pair of congruent alternate interior angles, then the lines have a common perpendicular.
9. Use AAS to prove that if a transversal to two lines forms a pair of congruent alternate interior angles, then the lines have a common perpendicular.
10. Use SSS to prove that if a transversal to two lines forms a pair of congruent alternate interior angles, then the lines have a common perpendicular.
11. (In a sense, the converse of each of these 1-10) Prove that if a line is a common perpendicular to two other lines, any transversal to the lines that passes through the midpoint of the segment determined by the common perpendicular forms congruent alternate interior angles.

In Exs 12-16, assume $ABCD$ to be a Saccheri quadrilateral with summit AD and prove that line MN is the perpendicular bisector of both the base and the summit where the new figure is determined by the information given in each exercise. Include an appropriate, competent sketch for each with all of the *given* information indicated as well as auxiliary points and segments you use but do *not* indicate any information about them that has to be *proved*.



12. M and N are the midpoints of the base and summit, respectively, of $ABCD$.
13. M is the midpoint of the base of $ABCD$ and N is determined by the perpendicular at M . [Verbally, the perpendicular bisector of the base is also the perpendicular bisector of the summit.]
14. N is the midpoint of the summit of $ABCD$ and M is determined by the perpendicular at N . [Verbally, the perpendicular bisector of the summit is also the perpendicular bisector of the base.]
15. N is the midpoint of the summit of $ABCD$ and M is the foot of the perpendicular at N to the base.
16. M is the midpoint of the base of $ABCD$ and N is the foot of the perpendicular at M to the summit.

In Exs 17-21, assume $ABCD$ is a Lambert quadrilateral with fourth angle at A and prove that $ABEF$ is a Saccheri quadrilateral with summit line AD where the new figure is determined by the information given in each exercise. Include an appropriate, competent sketch for each with all of the given information indicated and **no other** information indicated as given.



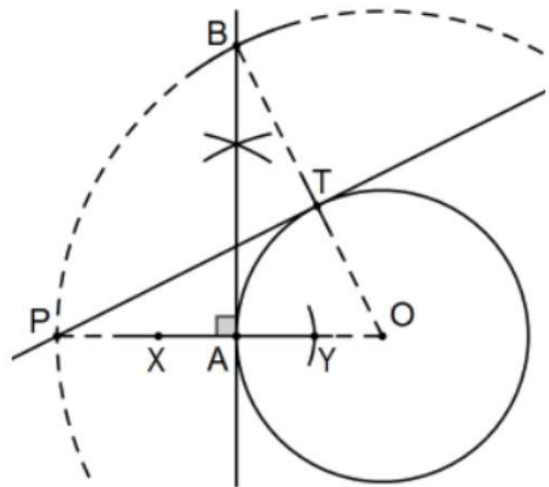
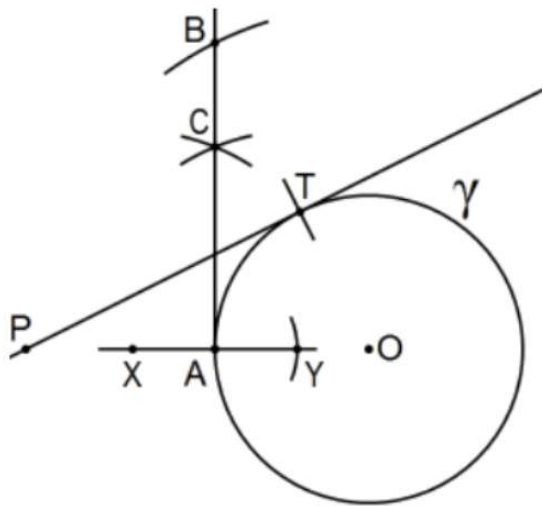
17. E is along ray BC with $CE \cong BC$ and F is along ray AD with $DF \cong AD$. [Informally, double each of a pair of opposite sides.]
18. E is along ray BC with $CE \cong BC$ and F is on the perpendicular ray at E with $EF \cong BA$.
19. F is along ray AD with $DF \cong AD$ and E is the foot of the perpendicular at F to line BC .
20. E is along ray BC with $CE \cong BC$ and F is the intersection of $(E; AB)$, the circle centered at E of radius AB and $(D; AD)$, the circle centered at D of radius AD . [Hint: Need to show TWO right angles.]
21. F is along ray AD with $DF \cong AD$ and E is the intersection of $(F; AB)$, the circle centered at F of radius AB , and $(C; BC)$, the circle centered at C of radius BC . [Hint: Need to show TWO right angles.]
22. E is along ray BC with $CE \cong BC$ and F is the intersection of the perpendicular at E with line AD .
23. Give an alternate proof of the fact the quadrilateral of any associated Saccheri quadrilateral of a triangle is, in fact, Saccheri as follows. Start as before but take F_1 along ray DM with $MF_1 \cong MD^M$ and F_2 along ray EN with $NF_2 \cong NE$. Prove that $F_1 = F_2 = F$, the unique perpendicular from A line MN and conclude, as before, that the resulting quadrilateral $BDEC$ is Saccheri with summit BC .
24. Prove that if the opposite sides of a quadrilateral are congruent, the figure is a parallelogram.
25. Prove that the opposite sides of a quadrilateral are congruent if and only if its diagonals bisect each other.
26. Prove that the perpendicular bisector of one side of triangle is perpendicular to the line which joins the midpoints of the other two sides. [Hint: Think of an associated Saccheri quadrilateral.]
27. Missing in the list of triangle congruency theorems SAS, SSS, ASA, and AAS is ASS. Beyond the lack of decorum, there is very good reason; the information is insufficient to guarantee congruence unless the angle is known to be a right angle in which case it is true but called "Hypotenuse-Leg". Show by explicit example that there exists two non-congruent triangles with the hypotheses of ASS satisfied.
28. Clearly the Exterior Angle Theorem does not hold on a sphere (e.g., two longitudinal lines through the North Pole and the Equator). Explain where the standard proof of the exterior angle goes wrong (as it must). Include a good sketch(es) to support

your answer. If you can, characterize (i.e., find necessary and sufficient conditions for) when it is valid. [Hint: To fail, the triangle has to be big in some sense. What is that sense and how big is too big?]

29. Prove that the following construction for the tangents to a circle from a point outside of a circle to the circle is valid: Let the circle be centered at O and let P be the point outside of the circle. Construct a (if any) tangent from P to the circle.

Construction: Assuming the center O has been constructed, let A be the intersection of segment OP with the circle and construct the perpendicular to line OP at A . Construct circle $(O; OP)$, the circle centered at O of radius OP and let B be an intersection of this line and this circle. Finally, let T be the intersection of BO with the original circle. Claim: Line PT is tangent to the original circle.

[Note: This construction is a neutral geometry construction but largely unknown to US geometry students in deference to a strictly Euclidean construction (see Chapt 4). It is a little amazing in that this construction comes earlier in Euclid's Elements, it is easier, and its proof is easier!]



30. Prove that the line PT is the only tangent on P to the circle on the same side of line of the OP as point B . [Hint: Let S be a point on the circle on the same side of line OP as T such that line PS is tangent to the circle and prove that $T = S$, hence exactly one tangent on each side of line OP .]

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CHAPTER OVERVIEW

Chapter 3: Introduction to Hyperbolic Geometry

[3.1: Hyperbolic Geometry](#)

[3.2: Problem Set 3](#)

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3.1: Hyperbolic Geometry

All of the rest of the axioms and definitions (that remain unspecified!) of neutral geometry remain in effect but in addition we add:

Hyperbolic Parallel Postulate (Local Form)

There exists a line and a point not on that line such that there are two lines on that point that are parallel to the original line.

It is always a good idea to hypothesize as little as possible but this axiom is sometimes stated as:

Hyperbolic Parallel Postulate (Global Form)

For any line and any point not on that line, there are two lines on that point that are parallel to the original line.

For more specificity, let ℓ be the line and let P be the point not on the line and let m and n be two lines on P that are parallel to ℓ .

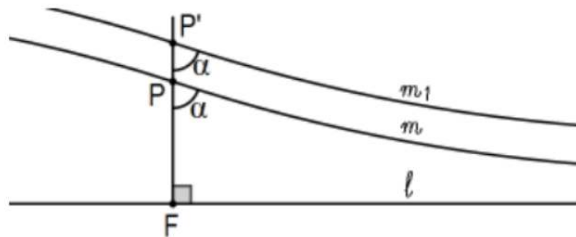
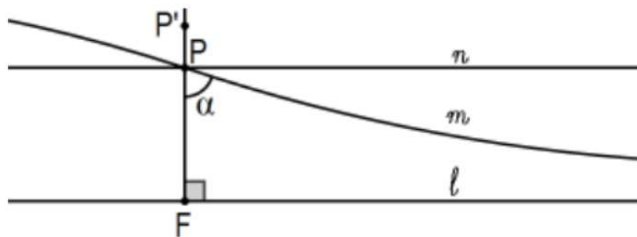


Theorem

These two axioms are equivalent; i.e., one is true if and only if the other is true.

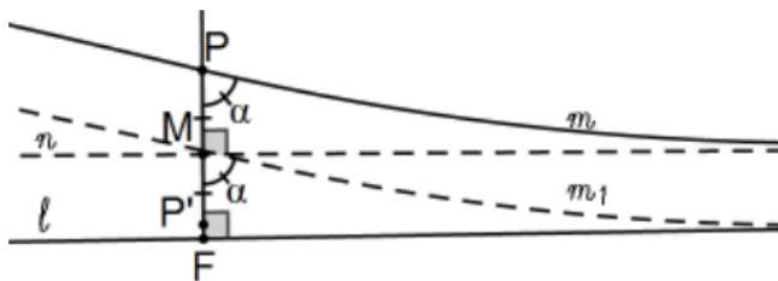
Proof: Obviously, the second implies the first so there is nothing to prove in this direction. Conversely, assume the local form is true, say line ℓ and point P with m and n distinct lines on P parallel to ℓ . First we prove a special case, multiple parallels at any point P' on the line on P perpendicular to ℓ , say PF , where F is the foot of the perpendicular from P .

Case 1: $P'F > PF$ [Easy.] Copy the angles at P determined by m and n with the perpendicular at P' .



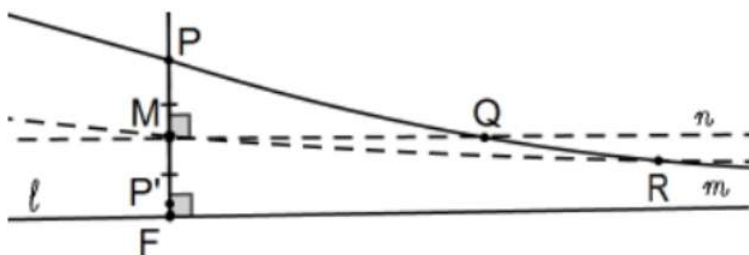
Case 2: $P'F < PF$ [Much harder.] Let m be a parallel to ℓ on P that is not perpendicular to PF . [We will not need a second parallel at P .] Focusing on the acute side, let M be the midpoint of segment PF , and let line n be the perpendicular to line PF at M . First we construct multiple parallels at M , not at P' and we already have one, namely n . Two cases arise, either m is parallel to n or it is not.

Case 2a: $m \parallel n$.



In this case, copy the angle at P with the perpendicular at M to obtain m_1 and use ASA for Long Triangles. Then m_1 and n_n are multiple parallels at M.

Case 2 b : $m \cap n \neq \emptyset$, say Q. Let R be along ray PQ beyond Q. Then line MR is another parallel to ℓ on M along with n . Why?



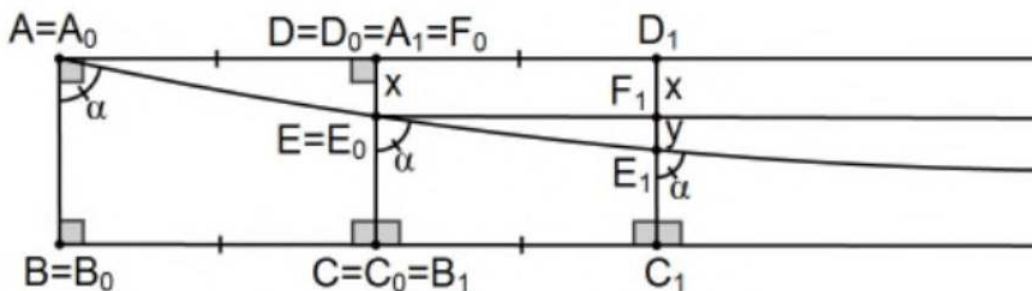
So either case leads to multiple parallels at M but we need multiple parallels at P' , not $M = M_1$. What to do? Repeat recursively until the new M_n is between F and P' (by the Archimedean property of length) and use Case 1.

General situation for any line and point not on it? Use ASA for Long Triangles from what has already been already done. QED.

Theorem

There is no rectangle (i.e., no quadrilateral with four right angles).

-[Note: In our "real world", the same result is true. No rectangles? What can that mean?! It means that we don't live on a plane; we live on a sphere and there are no rectangles on a sphere; in fact, there are no parallelograms or even trapezoids. Why not?]



Proof: Suppose for contradiction that $ABCD$ is a rectangle so (by definition) each of its angles are right angles and (by theorem) both pairs of opposite sides are congruent. Double the base and the summit; i.e., consider C_1 of ray BC with $C_1 \cong BC$ and D_1 of ray $ADDD_1 \cong AD$. Renaming $A = A_0$, $B = B_0$, $C = C_0 = B_1$ and $D = D_0 = A_1$ (to set up the natural mathematical induction that we will omit), we conclude that $A_0 B_0 C_0 D_0$ and $A_1 B_1 C_1 D_1$ are congruent Saccheri quadrilaterals so that the angles at C_1 and D_1 are also right angles and $A_1 B_1 C_1 D_1$ is a rectangle congruent to the original rectangle $ABCD$. Multiple parallels at A to line BC implies that there is a second parallel to line BC on point A that forms an angle of measure $\alpha < 90^\circ$ with the vertical line AB and let $E = E_0$ be the intersection of that line with CD , the opposite side of the original rectangle. Let E_1 be determined by that line with $C_1 D_1$. Now let F_1 be along ray $D_1 C_1$ with $D_1 F_1 \cong DE$ (segments of measure x). Then

$E_0 A_1 D_1 F_1$ and $E_0 B_1 C_1 F_1$ are Saccheri quadrilaterals with pairs of supplementary summit angles at E_0 and F_1 that are congruent so they are right angles so these Saccheri quadrilaterals are also rectangles. Therefore, $E_0 F_1 \cong A_1 D_1 \cong AD$. From the fact that $ABCD$ is a rectangle, the angle sum of quadrilateral $ABCE$ and $\triangle ADE$ must be exactly $360^\circ + 180^\circ$ (the supplementary angles at E) with neither exceeding those numbers. This implies $\angle BAE$ and $\angle AEC$ are supplementary and it follows that $\angle DAE \cong \angle F_1 E_0 E_1$. By ASA, $\triangle ADE \cong \triangle E_0 F_1 E_1$ so $F_1 E_1 \cong DE$ that $D_1 E_1 = x + y = 2(DE) = 2x$. Generalizing this process, we have $D_n E_n = (n+1)x = (n+1)DE$ for all n and, for sufficiently large n , $n x > AB$ (by the Archimedean property of length; i.e., enough copies of one segment, no matter how short, exceeds another, no matter how long, a consequence of the Ruler Postulate.) But that means that $D_n E_n > D_n C_n \cong AB$ forcing E_n to be on the other side of line BC and, by continuity, parallel lines AE and BC intersect somewhere between A and E_n , an obvious contradiction. The original hypothetical rectangle $ABCD$ did not exist. QED.

Corollary: The summit angles of a Saccheri quadrilateral are acute.

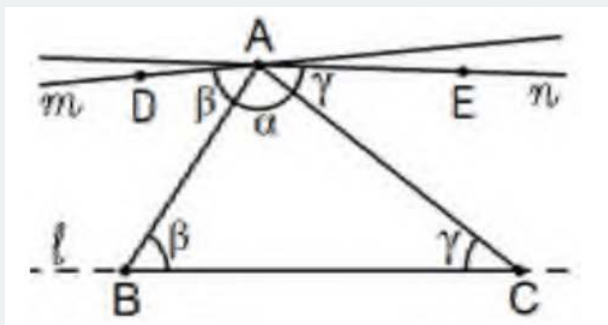
Corollary: The 4th angle of a Lambert quadrilateral is acute.

Corollary: The sum of the angles of a triangle is less than 180° (one straight angle).

Proof: Look at any of its associated Saccheri quadrilaterals. The sum of its summit angles ... QED.

Note

These consequences of the HPP (and lots of others) are actually equivalent to it.



For example, suppose instead of the HPP, we assumed that there is a triangle, say $\triangle ABC$, with angle sum $\alpha + \beta + \gamma < 180^\circ$.

At A on line AB , and on the opposite side, copy $\angle ABC$, say $\angle BAD$, and at A on AC , and on the opposite side, copy $\angle ACB$ to obtain $\angle CAE$. Look at lines AD , m , and AE , n . So? QED.

Corollary: Either side of the 4th angle of a Lambert quadrilateral is greater than the side opposite it.

Proof: From neutral geometry, we know that it is greater than or equal to the side opposite it. If they were "equal" (congruent), we would have a Saccheri quadrilateral with 4 right angles. QED.

Corollary: The summit of a Saccheri quadrilateral is greater than its base.

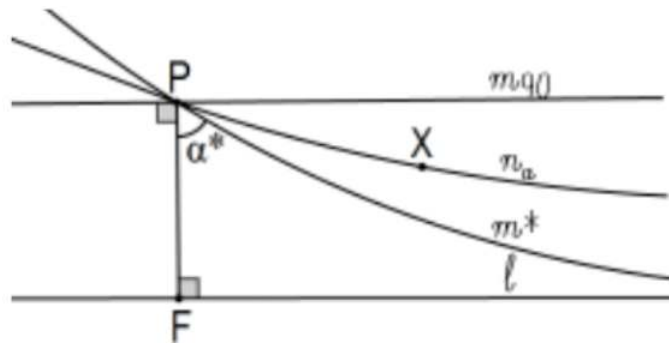
[Hint: Starting at the line of midpoints and "tipping your head 90° " either way yields a Lambert quadrilateral with a summit of the Saccheri quadrilateral as its 4th angle.]

Corollary: The line segment joining the midpoints of the base and the summit of a Saccheri quadrilateral is strictly shorter than any other line segment from its base line to its summit line.

Corollary: If two lines share a common perpendicular, that segment is shorter than any other line segment from one to the other. [That is, there are no "railroad tracks".]

In hyperbolic geometry, parallel lines do not necessarily have a common perpendicular. If we were being formally axiomatic, we would need to study the real numbers from that perspective as well (even the underlying set theory - we would never get done!) but one of the defining axioms we will use. It is called the "Real Number Completeness Axiom" and, in the form we need it, it guarantees that any nonempty subset of the real numbers that is bounded below has a greatest lower bound.

Limit (or Boundary) Parallel

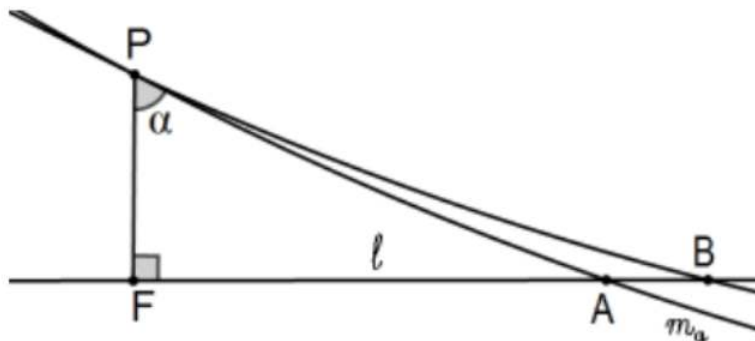


For a point P not on line l , there is a unique parallel line m^* (one in each direction) that minimizes the angle (on that side) made by lines on P with the perpendicular which are parallel to l . This is the right (pictured, or left) *limit* or *boundary parallel*.

[Note 1: Some authors use simply parallel with the rest of them being called hyperparallel.]

[Note 2: This is a theorem as well as a definition. To be mathematically "proper", the existence and uniqueness of such a line would need to be proved first and only then would its definition be meaningful. Proceeding backwards ...]

Proof: For any line on P , let F be the foot of the perpendicular on P and consider the angle made at P , $\angle FPX$, where $X \neq P$ is any other point on the line. For $\alpha = m(\angle FPX)$ (using, degrees, radians, whatever), we identify the line as m_α . Depending on the size of α , some lines m_α are parallel to l and some are not; i.e., some intersect l

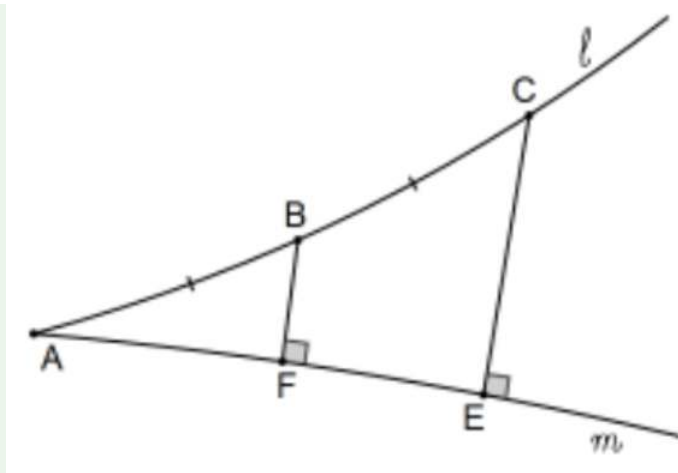


Let $\& = \{\alpha \mid m_\alpha \parallel l\}$; the angles determined by the parallel ones. $\&$ is nonempty. Why? $\&$ is "bounded below". Why? What is a lower bound greater than 0° ? Why is $\alpha = m(\angle FPA)$ in this figure a lower bound for $\&$? Why is there a greater lower bound? We define the **angle of parallelism** or **limit angle** or **boundary angle** or **critical angle** to be the greatest lower bound of $\&$. That is, $\alpha^* = glb(\&)$.

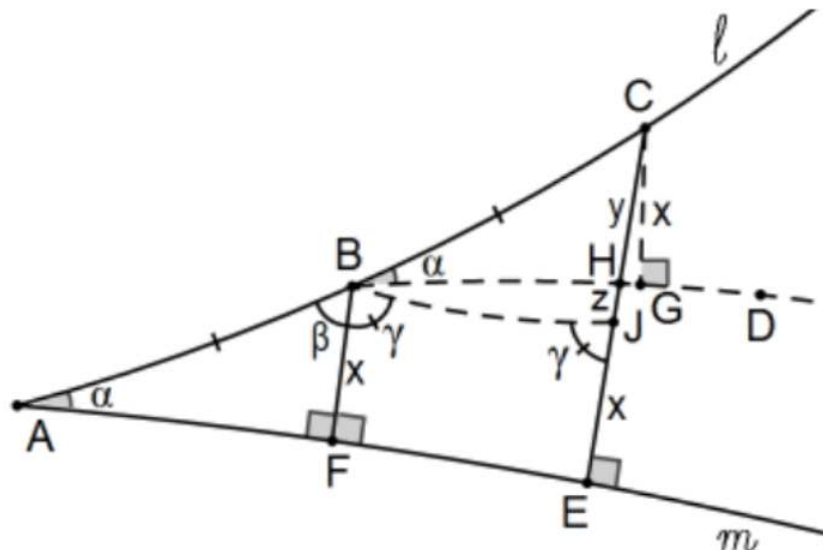
The proof concludes by confirming that m_{α^*} , the line on P that makes angle α^* with the perpendicular ray PF , is parallel to l so that α^* is actually in $\&$ and we have our parallel on P . This fact is immediate from the idea of the figure above; i.e., a line on P that intersects l , line PA determines an angle size (α) at P that is a lower bound for S but obviously not the *greatest* lower bound. QED.

Theorem 3.1.1

Intersecting lines diverge faster than proportionally. More specifically, the distance (BF in the figure) from a point on one side of an angle to the other side of the angle (F) is more than doubled (CF) if the distance from the vertex doubled. [Note: Another interpretation of this statement would be to double segments on each ray and prove that the third side of the determined triangle is more than doubled. This is also true although we don't need it except in the context of disproving the Pythagorean Theorem (in case $\angle A$ is a right angle). [See PS 3, #18; the same proof works for any $\angle A$.]

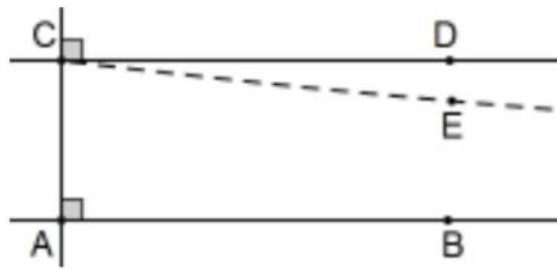


First off. What is meant by the distance from a point on one line to another line? As we have seen before, it is the length of the perpendicular segment from the point to the line. In the figure, lines l and m intersect at A , B is any point along one of them, C is such that $BC \cong AB$ (double segment AB), and F and E are determined by the perpendiculars to the other line from B and C . In Euclidean geometry, AA Similarity would imply that $CE = 2(BF)$. In hyperbolic geometry, we prove that $CE > 2(BF)$.



Proof: Copy $\angle A$ at B to obtain $\angle CBD$ and let G be the foot of the perpendicular from C to line BD so that $\triangle ABF \cong \triangle BCG$ by AAS and $CG \cong BF$ by cpctc. Let H be the intersection of CE with BD . Then CH is the hypotenuse of right triangle $\triangle CHG$. For clarity, let $x = m(BF)$ and let $y = m(CH)$. Now let J be along ray EC of length x so that $BFEJ$ is a Saccheri quadrilateral. Since the original $\angle A + \angle ABF$ is less than a right angle, $\alpha + \beta < 90^\circ$. This implies that E, J, H , and C are lined up correctly in the picture so $z > 0$ (why so?). With that, $x \leq y$, $2x < x + y + z$ so that $CE > 2(BF)$.

Corollary: If two lines share a common perpendicular, they diverge in both directions.



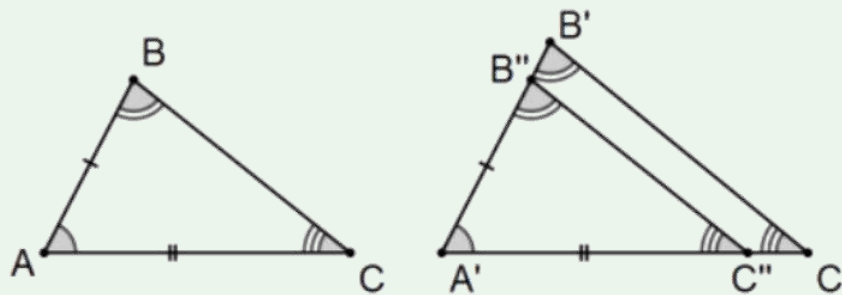
Proof: Suppose line AC is a common perpendicular to lines AB and CD and let CE be another parallel to line AB that forms an acute angle with the perpendicular AC . By the theorem just proved, lines CD and CE diverge. All the more so, CD and AB . By symmetry, the same applies if the acute angle is on the other side of AC . QED.

Note

This says more than that the lines are getting farther apart. That could be the case for lines behaving asymptotically. This says they are arbitrarily far apart; i.e., eventually exceeding any specified distance.

AAA Congruency Theorem

If two triangles...



Proof: Copy one onto the other using SAS Congruency and look at the resulting quadrilateral. The sum of its angles must be 360° but there is no such. QED.

Note: The same idea holds on a sphere; i.e., in our "real world", two triangles with all three angles pairwise congruent are congruent triangles. The proof uses "surplus" to replace "deficit" from 180° .

There are other theorems that can be proved in much the same way. For example:

Theorem: Summit Theorem: If two Saccheri quadrilaterals have congruent summits and congruent summit angles, then they are congruent.

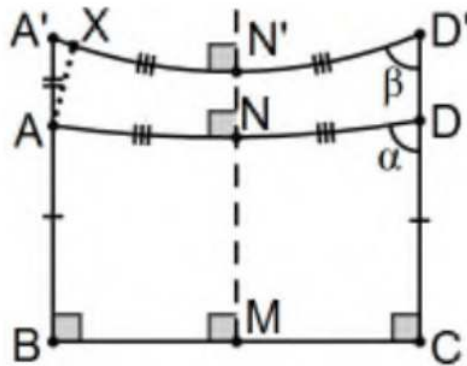
Proof: Copy one onto the other from the summit downward. [PS 3, #19] QED.

Theorem: If two Saccheri quadrilaterals have congruent bases and congruent summit angles, they are congruent.

Proof: PS 3, #20. QED.

What if some parts of a Saccheri quadrilateral are changed while others are held fixed?

Theorem: If the legs of a Saccheri quadrilateral are extended, then the summit is increased and the summit angles are decreased. [By implication, the base is unchanged here.]

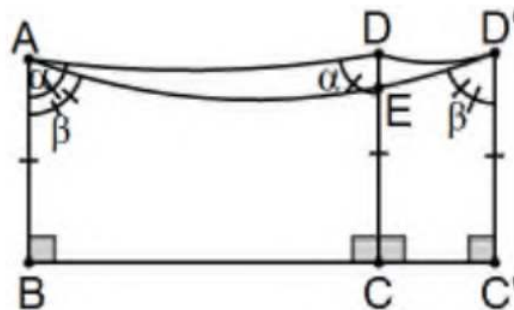


Proof: Consider Saccheri Quadrilateral $ABCD$ with rays BA and CD extended the same distance; i.e., $AA' \cong DD'$ so that $A'BCD'$ is still Saccheri.

Let $\alpha = m(\angle A)$ and $\beta = m(\angle A')$. Then $2\beta + 2(180 - \alpha) < 360^\circ$ so that $\beta < \alpha$ and the summit angles are strictly decreased as the theorem asserts. Why must AD be less than $A'D'$? Because moving symmetrically away from the perpendicular of two lines that share a common perpendicular, widens the separation. We already know they diverge, but this says that they are consistently diverging with increasing distance from the common perpendicular. Let line MN be the perpendicular bisector of segment BC . We know that this line bisects and is perpendicular to both summits so that, by letting X be along ray $N'A'$ with $N'X \cong NA$, $ANN'X$ is a Saccheri quadrilateral with $\angle NAX$ acute but $\angle NAA'$ obtuse. Therefore, the picture is drawn correctly; i.e., X is between N' and A' . QED.

The next is a result similar to the one above but with a very different proof:

Theorem: If the base of a Saccheri quadrilateral is extended, then the summit is increased and the summit angles are decreased. [By implication, the length of the legs is unchanged.]



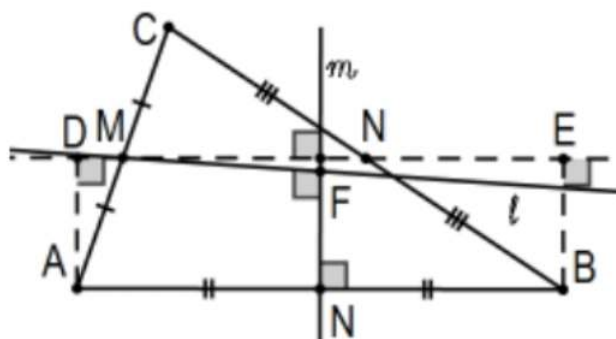
Proof: First note that there are three Saccheri quadrilaterals involved, the two we are comparing and the extension itself. This proof rests on noting that two adjacent angles (in the figure, $\angle ADC$ and $\angle D'DC$) are each acute so together less than one straight angle so that $\triangle ADD'$ is a genuine triangle that forces E to be correctly positioned (not outside of D along ray CD). That implies that summit $\angle BBD'$ is less than the original summit $\angle BAD$ verifying the second conclusion of the theorem. The greatest angle of $\triangle ADD'$ is $\angle D$ so the greatest side is AD' and the summit of the new quadrilateral is greater than the old summit, AD . QED.

Note: Recall that the summit of a Saccheri quadrilateral is parallel to the base but, by "stretching the base," there is always another line parallel to the base that makes a smaller angle with the leg of the Saccheri quadrilateral from that point (i.e., just stretch the base a little more.) This confirms that the angle formed by the summit of a Saccheri quadrilateral is never the critical angle with a leg as the perpendiculars to its base.

Theorem: If two triangles have the same angle sum, then they are equivalent. [Converse of...]

Proof: In summary, we first prove the case where the triangles have one side of common length and then cleverly force that case using the following (neutral geometry) lemma.

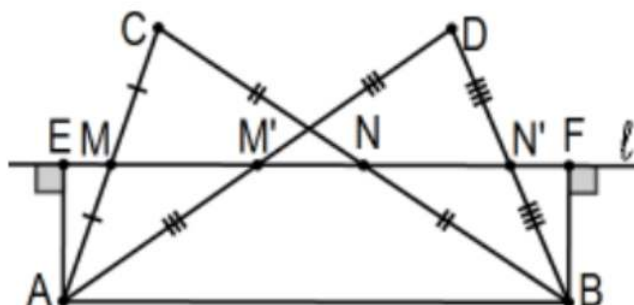
Lemma: If a line that bisects one side of a triangle is perpendicular to the perpendicular bisector of a second side, then it bisects the third side.



Proof: In given triangle $\triangle ABC$, let M be the midpoint of AC , let m be the perpendicular bisector of AB , and let F be the foot of the line ℓ on M perpendicular to m . Let N be the midpoint of the third side BC and consider the associated Saccheri quadrilateral $ADEB$. Its summit is AB and we know that the perpendicular bisector of the summit is perpendicular to its base line MN . Thus $\ell = MN$. Why? QED.

Case 1: The two triangles have one pair of sides of the same length.

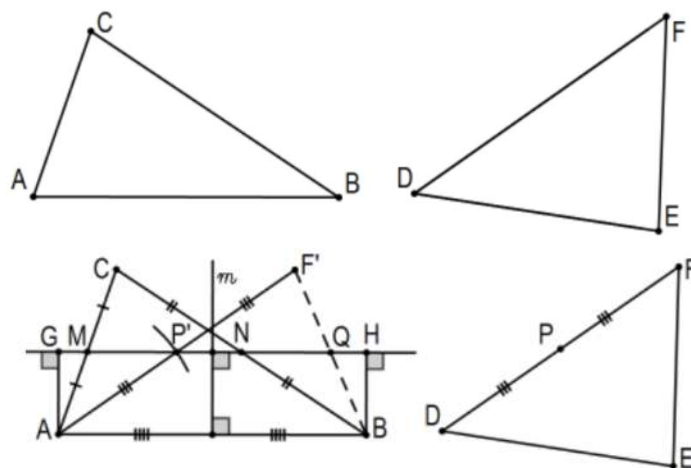
Without loss of generality, we may assume that one side is superimposed on the other as pictured here (copy one triangle onto the other if it is not already so positioned).



Proof: In the given figure, assume that $\triangle ABC$ and $\triangle ABD$ have the same angle sum and consider their associated Saccheri quadrilaterals on the common side AB . By previous theorem, their summit angles have the same sum as the triangles themselves so the two Saccheri quadrilaterals have the same summit AB and congruent summit angles so, by the Summit Theorem, they are the same so that $\triangle ABC \sim AEFB \sim \triangle ABD$.

Note: Even in this case, producing the concrete equivalence is harder than it looks. Remember that the equivalence $\triangle ABC$ of $AEFB$ is shown by dropping the perpendicular from C to line MN and similarly for $\triangle ABD$ from D to line $M'N'$, the same line ℓ . The idea is to keep track of both sets of cuts as the triangles get cut twice, first to rearrange $\triangle ABC$ to obtain $AEFB$ and then rearrange again to obtain $\triangle ABD$. It sounds easy but...

Case 2: The general case. The idea is to reduce Case 2 to Case 1. If the triangles are not congruent equilateral triangles, there is one side of one of them greater than some side of the other or, by SSS, they would be congruent.



In $\triangle ABC$ and $\triangle DEF$, assume only that they have the same angle sum and $AC < DF$. (No vertex-by-vertex correspondence is necessary nor implied.) Consider the associated Saccheri quadrilateral for $\triangle ABC$ with summit on either of the other two sides; we choose AB so that $AGHB$ is its associated Saccheri quadrilateral with base on line MN , the line of midpoints. Let P be the midpoint of segment DF and let P' be either intersection of circle $(A; DP)$ (i.e., center at A and radius DP), and line MN . [Note that $(1/2)AC < (1/2)DF$ so they do indeed intersect]. Double segment AP' along ray AP' to determine F' so, by construction, $AF' \cong DF$. Let Q be the intersection of segment BF' and line MN , and let m be the line of midpoints of the base and summit of Saccheri quadrilateral $AGHB$; i.e., the common perpendicular. Considering $\triangle ABF'$, note that P' is the midpoint of side AF' and m is the perpendicular bisector of side AB so, since line MN is a perpendicular from the midpoint of one side of a triangle to the perpendicular bisector of a second side, its intersection with the third side is the midpoint of that side so that Q is the midpoint of the third side BF' (by the Lemma). Thus line MN is not only a line of midpoints of $\triangle ABC$, it is a line of midpoints of $\triangle ABF'$ as well so that $AGHB$ is also its associated Saccheri quadrilateral with summit AB and the sum of the summit angles is the sum of the angles of both $\triangle ABC$ and $\triangle ABF'$. Now $\triangle ABF'$ and $\triangle DEF$ satisfy the conditions of Case 1 because $\triangle ABF'$ has the same angle sum and shares a side length with $\triangle DEF$ ($AF' \cong DF$). Thus $\triangle ABC \sim \triangle ABF' \sim \triangle DEF$ QED.

Note

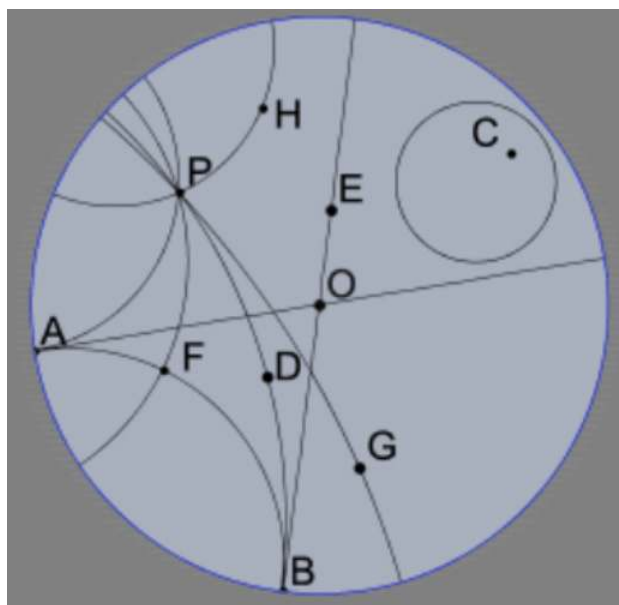
In this case, producing the concrete equivalence is much, much harder. Why is that? The note after Case 1 describes how to effect this for $\triangle ABC \sim \triangle ABF'$ and also for $\triangle ABF' \sim \triangle DEF$ but, tedious as this is, it still appears to be misleadingly easy. The difficulty is that the Saccheri quadrilateral for the first equivalence is based on the common summit AB but for the second it is the Saccheri quadrilateral of $\triangle ABF'$ with summit AF' so different (but, of course, equivalent) Saccheri quadrilaterals. Keeping track of both sets of cuts on the intermediate $\triangle ABF'$ and how they are to be rearranged to go directly from $\triangle ABC$ to $\triangle DEF$ is closer to a 1,000 -piece jigsaw puzzle than mathematics!

If we haven't already done so by the time we get to this point in the course, it is time to start looking at a model of this geometry to make sure that we are not building a pretty castle in the sky. There are several standard ones but the only one we will work with is the Poincaré disk model for hyperbolic geometry:

$\mathcal{P}$, the set of points, is the set of interior points of a fixed Euclidean circle γ centered at O

$\mathbb{C}$, the set of circles, is the set of Euclidean circles that lie entirely inside the defining circle γ .

Note 2: Lines determined by orthogonal circles always "curve away" from the center of the defining circle. This is easy to see by noticing that the triangle formed by the two circle centers and either point of intersection is a right triangle (by orthogonality) with the segment of centers being the hypotenuse so longer than the defining circle's radius so the second center must be outside of the defining circle.



In this figure, note the multiple parallels on P to lines AB and AO . That is, line PH is obviously parallel to lines AB and AO but so is PA since A , the point of Euclidean tangency, is not a point in the Poincaré plane. PA is a boundary parallel to them. Similarly, line PD is the other boundary parallel on P to line AB and PG is another nonboundary parallel on P to line AB . The foot of the perpendicular from P to line AB is F . For reasons not yet proved, $\angle APF \cong \angle BPF$. There is much to do to formally prove that this really is a model for hyperbolic geometry but it is. In essence, this fact is a theorem, that hyperbolic geometry is "just as good" as Euclidean geometry. That is, if an inconsistency were to ever arise in hyperbolic geometry, it could be translated back into an inconsistency in our trusted Euclidean geometry. Formally, hyperbolic geometry is "relatively consistent" with Euclidean geometry; that is, if the axioms of Euclidean geometry are consistent, so are those of hyperbolic geometry. The converse is also true but we won't be looking at that fact.

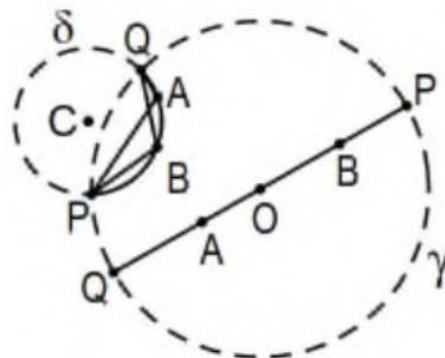
Moreover, recalling and/or learning enough Euclidean geometry to understand the essence of this proof will take up most of the last part of the course. There is a sophisticated approach that nails the result almost as an afterthought; the conformal mapping of linear fractional transformations in complex analysis:

<http://www.mathpages.com/home/kmath464/kmath464.htm>

In general, $f(z) = (az + b)/(cz + d)$ but, in our case, with $a = d = 0$, $b = R^2$, and $c = 1$ followed by a reflection in the real axis; i.e., conjugation. QED? No, since complex analysis will not be assumed. (Aren't you glad?!) The ideas behind a strictly geometric proof require lots of Euclidean geometry, some of which you have seen but much of which you have not. For example, measures of angles in this model are pretty obvious. As in calculus, they are just the angles between their tangents at the points of intersection. Distance is new.

Poincaré distance between two points (length of their Poincaré segment) is the absolute value of natural logarithm of the (Euclidean) cross ratio of the two points and the two points that the Euclidean circle of their line determines.

$$d_p(A, B) = |\ln(A, B; P, Q)|$$



Absolute value and natural logarithms are familiar. For any points A, B, C , and D , where $A = B$ is possible but all others are distinct, the cross ratio of A, B, C , and D (in this order) is:

$$(A, B; C, D) = \frac{AC/CB}{AD/DB} = \frac{(AC)(DB)}{(AD)(CB)}$$

Note

In some applications, these measures of Euclidean line segments are signed measures.

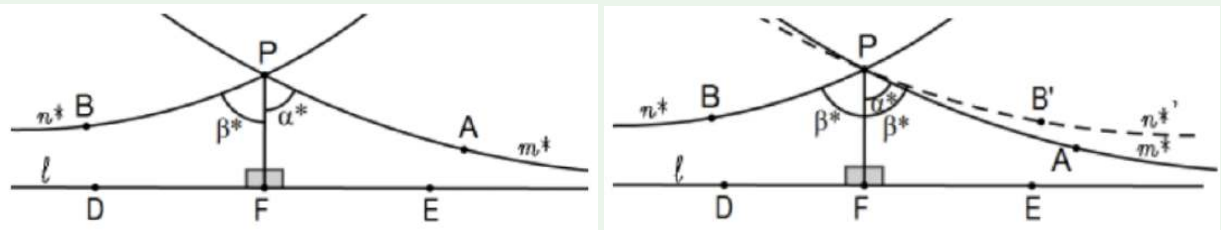
But what are P and Q ? These are easy assuming you believe Axiom 1 to be true in the Poincaré disk model for hyperbolic geometry; i.e. two points determine a line. Assuming that as fact (i.e., that two points and orthogonality with the defining circle determine a unique Euclidean circle or diameter (both pictured here), these are the two points of intersection of that circle or line with the defining circle. You might object, asserting that those points are not interior! You would be right as you were being wrong. They are perfectly good *Euclidean* points and everything in the definition and in the proofs are in Euclidean geometry; in fact, that's the idea. The Poincaré model for hyperbolic geometry is built entirely within Euclidean geometry with Euclidean lines and circles and we have the entire Euclidean plane in which to work. If signed measures are being used, assign a positive direction on

each line and, for points A and B on a line, if AB (the measure of line segment AB) is positive, then BA has the same absolute value but is negative.

Before studying the Poincaré disk model more carefully, we will develop some more ideas unique to hyperbolic geometry. That is not quite true. The next three theorems are true in Euclidian geometry; it's just that they are uninteresting in that case since that angle in question is always just a right angle. Getting used to it not necessarily being a right angle can be a challenge to some students.

Theorem

The left and right angles of parallelism (critical angle) critical angle at a defining point P to a defining line l are the same so, hereafter, "the angle of parallelism"; i.e., unqualified as to left or right.

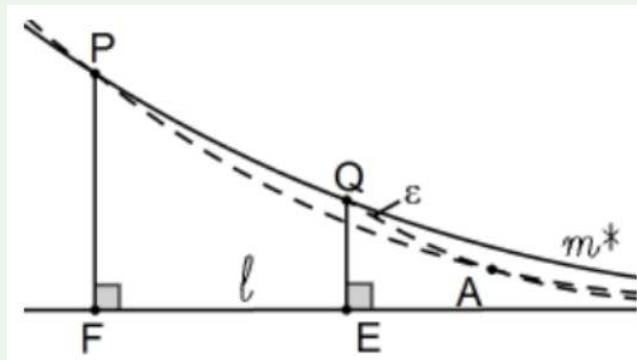


Proof: Let $n = PB$ be the parallel to line l determined by the left critical angle of size β^* with the perpendicular at P , PF , and let $m^* = PA$ be determined by the right critical angle of size α^* with the same perpendicular PF . Copy $\angle FPB$ at P on the other side as indicated to establish $\angle FPB'$ and line n^* . That line must be parallel by ASA for Long Triangles and therefore $\alpha^* \leq \beta^*$ since is α^* a lower bound for all parallels to l on the right at P . However, there was nothing special about starting with copying $\angle FPB$. Copying $\angle FPA$ on the other side implies that $\beta^* \leq \alpha^*$. In other words, both $\alpha^* \leq \beta^*$ and $\beta^* \leq \alpha^*$ so we have equality. QED.

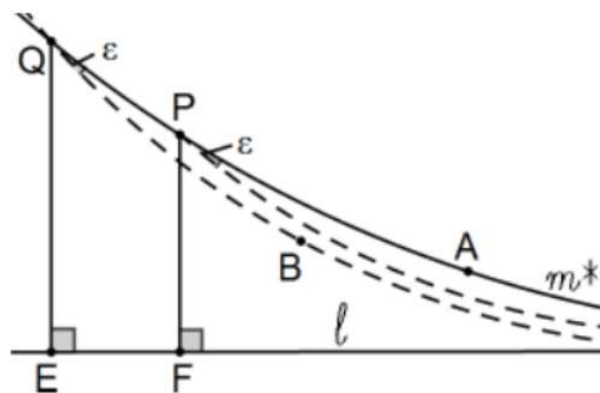
More surprising, perhaps, is that this property does not depend on the point P . That might be confusing but simply means that the boundary parallel at other points along the same line is exactly the same line:

Theorem 3.1.1

A limit (boundary) parallel m^* is a boundary parallel to the a line l at each of its points, not just its defining point P . [Note: This implies that it should *not* carry α^* as part of its name.]



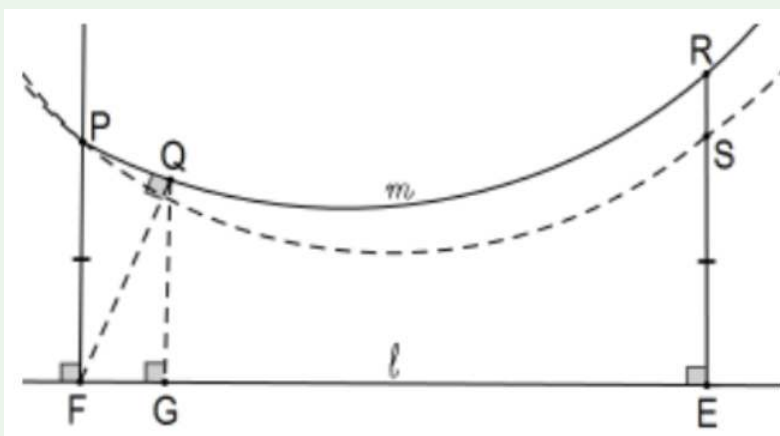
Proof: Case 1: Q on m^* on the acute side of P . In this case, drop the perpendicular to l , say E , and consider any ray from Q in the interior of the angle determined with the boundary parallel, say $\epsilon > 0$. Let A be any point on this ray and consider ray PA . This ray must intersect l . Why? Therefore so must ray QA . Why? This new angle of parallelism at Q is strictly greater than that at P . Why? [PS 3, #13.]



Case 2: Q on $m^* = PA$ on the obtuse side of P . Drop the perpendicular to obtain E , and again consider any ray from Q in the interior of $\angle EQP$, say ray QB , and the angle it determines with the boundary parallel, $m(\angle PQB) = \varepsilon > 0$. Copy that angle as a corresponding angle at P forming an angle with ray PF of size $\alpha^* - \varepsilon < \alpha^*$, the angle of parallelism, so this ray must intersect the original line l so that the parallel ray QB must intersect it as well. Since ray QB was arbitrary, the angle with the perpendicular is the angle of parallelism at Q . Consistent with Case 1, this new angle of parallelism at Q is smaller than $\angle FPA$. QED.

Theorem

A limit (boundary) parallel is bounded by the distance from the distance from the defining point to the defining line.



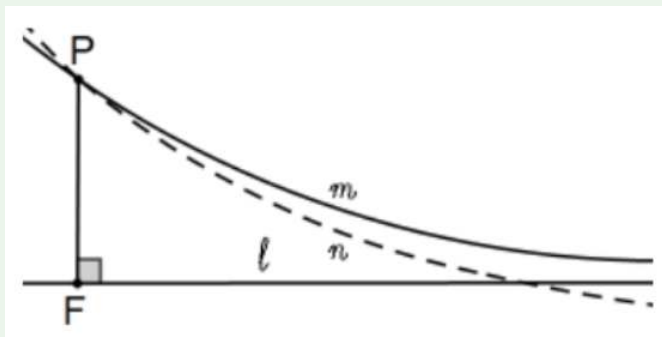
Note

Obviously, we're only looking on the side of the acute angle with the perpendicular. Throughout, let l be the line, let P be the point not on l , let F be the foot of the perpendicular from P to l , and let m be any line on P that forms an acute angle with the perpendicular, parallel or not. First note that (for some distance) the distance to l from a point on m is strictly less than the distance from P to l . To see this, let Q be the foot of the perpendicular from F back to the other line m and then let G be the foot of the perpendicular back to l . Considering the determined right triangles and their hypotenuses, clearly $QG < QF < PF$. We can repeat this ad infinitum but that does not mean that all segments are always shorter; e.g. consider any Saccheri quadrilateral and the line of midpoints of its base and summit.

Proof: Assuming only that such an m on P is not bounded, we show that m is not a boundary parallel to l ; i.e., the contrapositive. Assuming this m is not bounded is tantamount to assuming a point on it, say R , such that the distance to l is greater than the distance at P . That is, if E is the foot of the perpendicular, then $RE > PF$. By the Ruler Postulate, there is a point S along the ray ER such that $ES \cong FP$. But then $PFES$ is a Saccheri quadrilateral with summit PS ; i.e., a parallel line on P that makes a smaller angle than $\angle FPR$ so that angle was not a lower bound for all angles with the perpendicular at P that determine parallel lines. That is, m is not a boundary parallel to l . QED.

Theorem (Converse)

If two parallel lines are bounded (in one direction), then they are boundary parallels to each other.



Proof: If two lines intersect, we know they diverge in both directions. Suppose ℓ and m are bounded in the direction indicated in the figure, P is a point on m , and F is the foot of the perpendicular from P to ℓ . Finally, let n be any line on P that forms an angle with the perpendicular inside the angle formed by m . It suffices to show that n intersects ℓ . Since m and n intersect at P , they diverge so the boundedness of m to ℓ forces the desired result. QED.

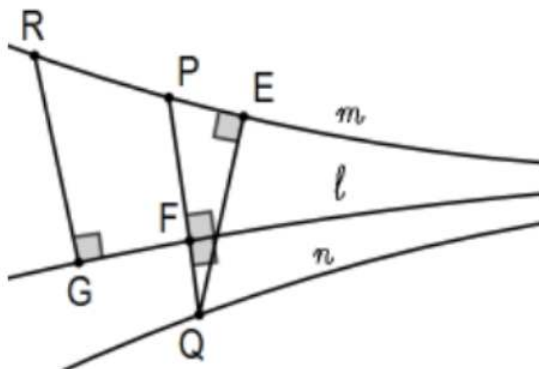
Theorem 3.1.1

Assuming that a limit parallel is defined to be a limit parallel to itself, limit parallelism is an equivalence relation (in the same direction of boundedness, of course).

Proof:

Reflexivity: True by definition.

Symmetry: We already know that a limit (boundary) parallel m is bounded to the defining line ℓ independent of the point that determined it and that parallels that are not bounded (in the acute direction) share a common perpendicular so diverge in both directions. It suffices to know that the defining line ℓ is bounded to that limit (boundary) parallel m . Suppose it were not so. Then it would be unbounded in that direction and, as before, determine a Saccheri quadrilateral with summit and base that would diverge in both directions; a contradiction.

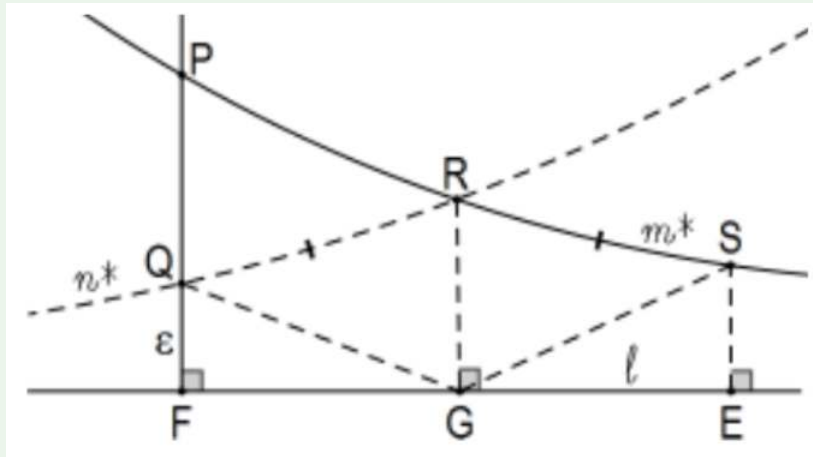


Transitivity: Assume m is a limit parallel to ℓ and n is a limit parallel to ℓ . In case m or n is between ℓ and the other, that parallel is "squeezed" between two that are bounded toward each other so the only case of interest is when ℓ is between m and n . In the figure, assume that is the case with $P \in m$. Let F be the foot of the perpendicular to ℓ and let Q be the intersection of line PF with n . That PF intersects n at all is not as obvious as it might look for a very good reason; it need not be true! That is what line segment RG is doing in the figure; it looks irrelevant. If ray RG fails to intersect n , choose any point $Q \in n$, say F again, and choose between G and F depending on which one is further along in the direction of boundedness, as F is in figure. Then ray QF must intersect m , say P , due to the bounding to ℓ . Dropping the perpendicular

from Q to m , we have a right triangle $\triangle QEP$ with PQ the hypotenuse so $QE < PF + FQ$. This argument can be applied to any point to the right of Q so n is bounded toward m by line segment PQ . Hence, by the preceding, boundary to each other.

Theorem

Boundary parallels are asymptotic.



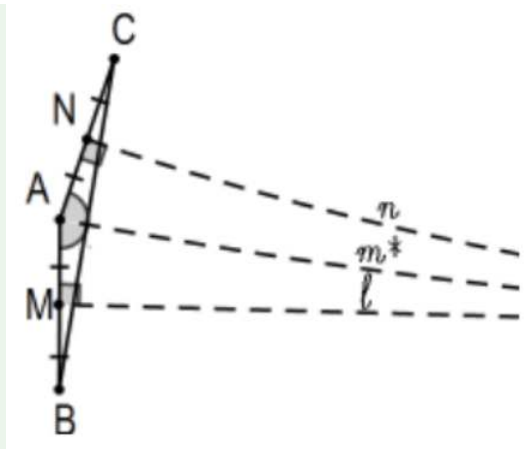
Proof: Let m_u^* be a boundary (limit) parallel to line ℓ .

We prove that m^* is eventually bounded (in the acute angle direction) by any distance $\varepsilon > 0$. Let $\varepsilon > 0$ be arbitrary and let P be a point of m^* and F is the foot of the perpendicular. Let Q be a point along ray FP with $m(FQ) = FQ = \varepsilon$. By the boundedness result, there is nothing to prove if $\varepsilon \geq FP$ so assume that $\varepsilon < FP$; i.e., Q is between F and P as pictured and let n^* be the limit parallel to ℓ at Q in the opposite direction. That is, if the acute angle at P is to the right, then the acute angle at Q is to the left. Since m^* is bounded toward line ℓ and n^* is unbounded in that direction, they must intersect; say at R . Then duplicate segment QR along ray PR to obtain S ; i.e., $RS \cong RQ$. Claim: m^* is bounded by ε to the right of S . To prove this, let E be the foot of the perpendicular on S and prove that $ES \cong FQ$ so that m^* is bounded to the right of SE by ε . The approach? Let G be the foot of the perpendicular on R and prove that $\triangle QFG \cong \triangle SEG$. [It's not hard but it is a bit tricky. Why is $\triangle QRG \cong \triangle SRG$?] QED.

In summary, two lines are boundary parallels if and only if they do not share a common perpendicular if and only if they are bounded to each other if and only if they are asymptotic to each other. To see these theorems in action, look back at the figure associated with the presentation of the Poincaré disk model for hyperbolic geometry, often simply the Poincaré model although he gave another model for hyperbolic geometry. [That model uses an open half-plane instead of a disk for the point-set. The upper (open) half of the xy -plane is the point-set, lines are vertical (open) rays and (open) semicircles orthogonal to the X -axis, and circles are Euclidian circles that lie entirely in the point-set. For your own amusement, identify parallels that are not boundary parallels, boundary parallels, and left and right angles of parallelism.]

Theorem

There exists triangles that cannot be circumscribed.



Proof: That is, produce one. We show one way to produce examples:

Starting with any line ℓ and point A not on ℓ , let m^* be the boundary parallel on A to ℓ , let M be the foot of the perpendicular from A to ℓ , and let B be the point along ray AM with $MB \cong MA$. Now copy the critical angle at A on the opposite side of m^* and take C along the determined ray with $\angle CAB$. Then $\triangle ABC$ is a triangle that cannot be circumscribed. Why? [See PS3, #16.]

Instead of proving deeper results of hyperbolic geometry, the last two chapters will be studying the geometry obtained from assuming the more familiar parallel postulate instead of the hyperbolic one. That is, we will be studying Euclidean geometry with a primary goal of outlining a proof that the Poincaré disk model is genuinely a model for hyperbolic geometry built entirely within Euclidean geometry. By implication, if hyperbolic geometry had any inherent inconsistencies, they could be carried back to exhibit inconsistencies within Euclidean geometry. In the vernacular, hyperbolic geometry is relatively consistent with Euclidean geometry. That is, if we threw out hyperbolic geometry, we would have to reject Euclidean geometry as well. Most of us don't want to do that.

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3.2: Problem Set 3

Unless otherwise stated, assume all of the axioms of neutral geometry (albeit unstated) and the global form of the Hyperbolic Parallel Postulate (HPP) as well; i.e., assume hyperbolic geometry.

1. Prove that the sum of the interior angles of any triangle is less than two right angles (180°).
2. Don't assume the HPP but do assume that there is one triangle for which the sum of the interior angles is less than 180° . Prove the local form of the HPP.
3. Don't assume the HPP but do assume that there is one pair of parallel lines that share a transversal that forms a pair of non-congruent alternate interior angles. Prove the local form of the HPP.
4. Prove that the summit of a Saccheri quadrilateral is never a boundary (limit) parallel to its base.
5. Prove that a side of a Lambert quadrilateral is never a boundary (limit) parallel to its opposite side.
6. Prove that two lines do not share more than one common perpendicular (i.e., no "railroad track ties").

In Exs 7-10, consider the Poincaré model of hyperbolic geometry in Euclidean geometry determined by the circle of radius 2 centered at the origin $O(0, 0)$ of the Cartesian plane, $\gamma = \{(x, y) \mid x^2 + y^2 < 2^2\}$

7. Find the equation for the circle δ that determines the right boundary (limit) parallel to the x-axis on the point $P(1, 1)$ and accurately sketch the situation.

[Hint: The center of this circle must lie on the line $x = 2$. Why? Recall the equation of the circle with center $C(h, k)$ and radius r : $\delta = \{(x, y) \mid (x - h)^2 + (y - k)^2 = r^2\}$.]

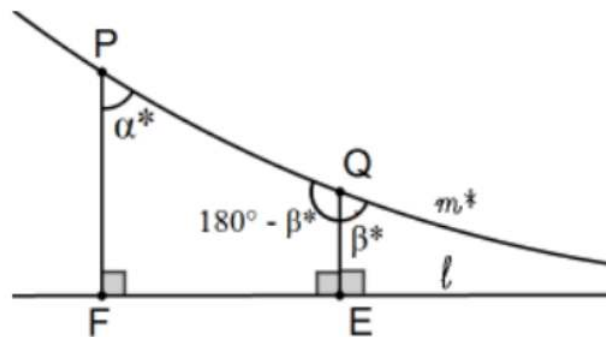
9. Redo Ex 7 for the left boundary parallel at the same point $P(1, 1)$.
10. Make an accurate sketch of the situation in Exs 7 and 8 and "sketch in" the Poincaré line on $P(1, 1)$ that is perpendicular to the x-axis.
11. Find the equation for the Euclidean circle δ of the Poincaré line in Ex 9. [Hint: For orthogonality with the x-axis, its center C will have to be on the x-axis, say $(c, 0)$, so $(x - c)^2 + y^2 = r^2$ is the equation and it passes through $P(1, 1)$. The needed additional condition is orthogonality with the defining circle. This can easily be done by using the Pythagorean Theorem with the radii and distance between the centers, $|c|$]
12. Consider the Poincaré model of hyperbolic geometry in Euclidean geometry determined by the circle of radius 4 centered at the origin $O(0, 0)$ of the Cartesian plane, $\gamma = \{(x, y) \mid x^2 + y^2 < 4^2\}$. Find the equation for the Euclidean circle δ of the Poincaré line determined by the points $A(1, 2)$ and $B(3, 0)$.

[Hint: For its equation, $\delta = \{(x, y) \mid (x - h)^2 + (y - k)^2 = r^2\}$, orthogonality with the defining circle can be involved using Pythagorean theorem: $4^2 + r^2 = h^2 + k^2$ so that $r^2 = h^2 + k^2 - 4^2$. The perpendicular bisector of Euclidean segment AB has slope 1, passes through the midpoint, $M(2, 1)$, and (h, k) .] 12. The idea of Ex 11 can be generalized to provide an analytic proof (See Chapt. 4) of Axiom 1 as a Euclidean geometry theorem as one step in the proof that the Poincaré model really is a model for hyperbolic geometry entirely within Euclidean geometry (establishing "relative consistency" of the hyperbolic geometry axioms (see Chapter 5 for a trivial synthetic proof!). Consider the Poincaré model determined by the circle of radius R centered at the origin $O(0, 0)$ of the Cartesian plane, $\gamma = \{(x, y) \mid x^2 + y^2 < R^2\}$.

Find the equation for the Euclidean circle δ of the Poincaré line determined by two general Poincaré points $A(a_x, a_y)$ and $B(b_x, b_y)$ where A, B , and O are not collinear.

That is, if $(x - h)^2 + (y - k)^2 = r^2$ is its equation, express h, k , and r uniquely in terms of R and the coordinates of A and B . Orthogonality with the defining circle can be involved using the Pythagorean theorem:

$R^2 + r^2 = h^2 + k^2$ so that $r^2 = h^2 + k^2 - R^2$. The perpendicular bisector of Euclidean segment AB has to pass through its midpoint M and the center of the circle $C(h, k)$. Three cases occur with the most general being that Euclidean line AB is neither vertical nor horizontal. In that case, calculate the equation of the perpendicular bisector, $y = mx + b$ with m being the negative reciprocal of the slope of line AB and using M to calculate b . Since C is on that line, we have $k = mh + b$ and, substituting in the coordinates of either A or B that must lie on the circle, we will have $(x - h)^2 + (y - k)^2 = r^2 = h^2 + k^2 - R^2$ entirely in terms of h (and the given coefficients). Luckily, the terms involving h^2 cancel out and we can solve uniquely for h , then $k = mh + b$, and finally $r^2 = h^2 + k^2 - R^2$. That is, there is one and only one such circle. The other two cases use same idea but are easier.



13. Prove that angles of parallelism strictly increase along boundary parallels in the direction of boundedness.

[Hint: The given figure says it all: $\alpha^* + (180 - \beta^*) < 180^\circ$ because PQEF is a quadrilateral.]

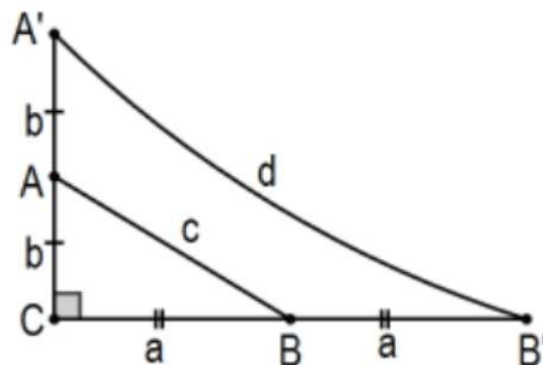
A polygon can be *circumscribed* if there is a circle, called the *circumcircle*, that contains every vertex of the polygon. It should be obvious that a polygon can be circumscribed iff there is a point equidistant from each of its vertices.

14. Prove that a Lambert quadrilateral can never be circumscribed. [Hint: Assume that there is a circumcircle and derive a contradiction; i.e., use proof by contradiction. Note: Do not assume that the intersection of the diagonals is the center of the assumed circumcircle but you may use the fact that it would have to have a center.]

15. Construct the circumcircle of a Saccheri quadrilateral. [Hint: Construct the perpendicular bisectors of a leg and the base and prove that the intersection is equidistant from each of the vertices, hence the center of the circumcircle.]

16. Complete the proof that the triangles of the last theorem of the chapter are genuine triangles and that they cannot be circumscribed. [Why does the presentation fail in Euclidean geometry?]

17. Show by a careful sketch in the Poincaré plane that not all triangles can be circumscribed and argue why it cannot be circumscribed. A Poincaré argument? A Euclidean argument?



18. Prove that the Pythagorean Theorem fails by showing that if $\triangle ABC$ is a right triangle with $a^2 + b^2 = c^2$, then a right triangle with legs of $2a$ and $2b$ does not satisfy the Pythagorean equation.

[Hint: Consider the associated Saccheri quadrilateral with summit $A'B'$.]

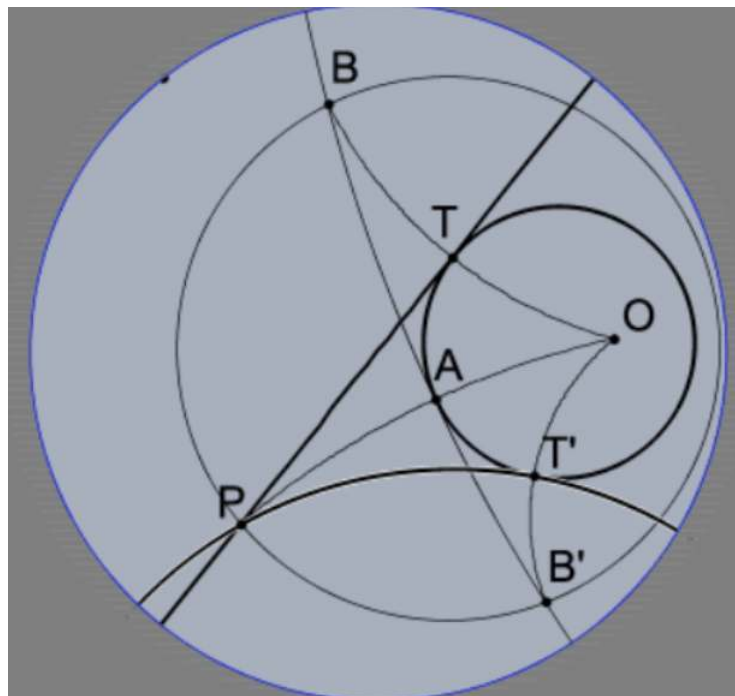
19. Prove the Summit Theorem.

20. **Theorem:** If two Saccheri quadrilaterals have congruent bases and congruent summits, then they are congruent. [That is, prove it.]

In Ex 21-24, assume that A and B are points inside the fixed Euclidian circle γ that defines the Poincaré disk model for hyperbolic geometry, δ is the orthogonal circle determined by A and B , and P and Q are the points of intersection of the circles γ and δ . Recall that Poincaré distance between points A and B (the measure of Poincaré line segment AB) is:

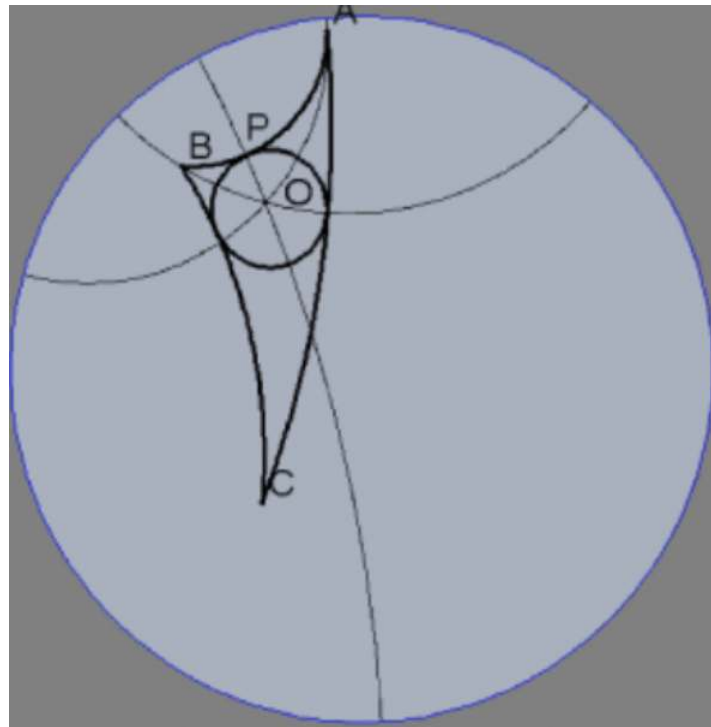
$$m_p(AB) = d_p(A, B) = \left| \ln(A, B; P, Q) \right| = \left| \ln \left(\frac{AP/PB}{AQ/QB} \right) \right| = \left| \ln \left(\frac{(AP)(QB)}{(AQ)(PB)} \right) \right|.$$

[Note 2: The full notion of "betweenness" requires that, for B to be between A and C, this equation holds without the assumption that the points be collinear. This is true for Poincaré measures of line segments but the proof is much more difficult. See Chapter 5 , concluding with PS 5, #8.]



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construction of these tangents in Euclidean geometry can be carried out with a Poincaré compass and straightedge in the Poincaré disk model but the result is not correct. In other words, proofs matter!]

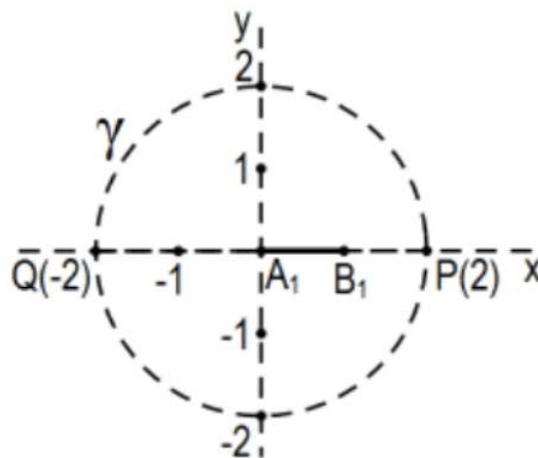


26. Assuming that the Poincaré disk model really is a model for hyperbolic geometry, recall the construction for the incircle of a triangle from Chapter 2 (neutral geometry). Explain the indicated construction of the incircle of the given triangle in the pictured Poincaré disk model and prove that it is valid.

[Note: As with Ex. 25, the pictured construction uses a Poincaré compass and straightedge but it is possible to construct the incircle of a Poincaré triangle with Euclidean compass and straightedge but it is exceedingly tedious.]

27a. The segment $A_1 B_1 = [0, 1] = \{x \in \mathbf{R} \mid 0 \leq x \leq 1\}$ has Euclidean measure 1 (surprise, surprise). Confirm that its Poincaré measure is $\ln(3)$ in the Poincaré model determined by $\gamma = (O; 2)$ by explaining this computation:

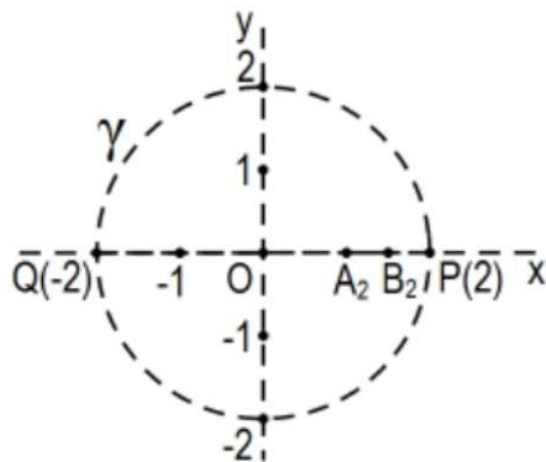
$$m_f([0, 1]) = \left| \ln \left(\frac{AP/PB}{AQ/QB} \right) \right| = \left| \ln \left(\frac{2/1}{2/3} \right) \right| = \left| \ln \left(\frac{3}{1} \right) \right| = |\ln(3)| = \ln(3)$$



27b. Do the same for segment $A_2 B_2 = [1, 3/2]$; that is, calculate both its Euclidean and Poincaré measures.

27c. Do the same for segment $A_3 B_3 = [3/2, 7/4]$.

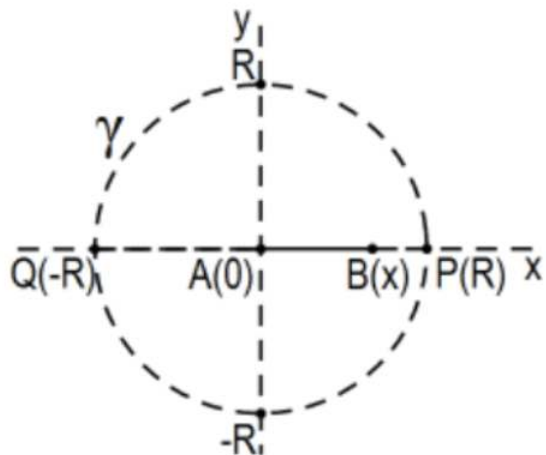
27d. Do the same for segment $A_n B_n = [(2^{n-1} - 1) / 2^{n-2}, (2^n - 1) / 2^{n-1}]$.



What is the limit as $n \rightarrow \infty$?

Note: Segments of the type $[0, 1]$ in #27a above are especially easy to compute:

Let $[0, x]$ be the Euclidean segment along the x -axis that is also a segment in the Poincaré model defined by circle $\gamma = (O; R)$, segment AB in the figure, $0 < x < R$. Clearly, its Euclidean measure is just x . For its Poincaré measure: [Here we take account of direction as well as positive Euclidean length.]



$$m_f([0, x]) = \left| \ln \left(\frac{AP/PB}{AQ/QB} \right) \right| = \left| \ln \left(\frac{R/(x-R)}{-R/(R+x)} \right) \right| = \left| \ln \left(\frac{R+x}{R-x} \right) \right| = \ln \left(\frac{R+x}{R-x} \right)$$

Note: The last expression makes it clear that, as x approaches R , the length of segment $[A(0), B(x)]$ becomes infinite (as it must). In Chapter 5 (PS 5, #5), we will see that by an easy isometric (congruence) transformation, the Poincaré measure of any Poincaré segment in a Poincaré model in the x, y -coordinate (Cartesian) plane can be computed in this convenient manner, no simultaneous solution of two quadratic equations in two variables (to find the coordinates of the defining points P and Q , and no Euclidean distance formula computations required.

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CHAPTER OVERVIEW

Chapter 4: Elementary Euclidean Geometry

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[4.2: Analytic Geometry Proof](#)

[4.3: Theorems of Ceva and Menelaus](#)

[4.4: Problem Set 4](#)

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4.1: Euclidean Geometry

All of the rest of the axioms and definitions (that remain unspecified!) of neutral geometry remain in effect but in addition we add:

Euclidean Parallel Postulate (Local Form)

There exists a line and a point not on that line such that there is at most one line on that point that is parallel to the original line.

[Note: Recall from neutral geometry that we always have at least one parallel so now exactly one parallel.]

Euclidean Parallel Postulate (Global Form)

For any line and any point not on that line, there is at most one line on that point that is parallel to the original line.

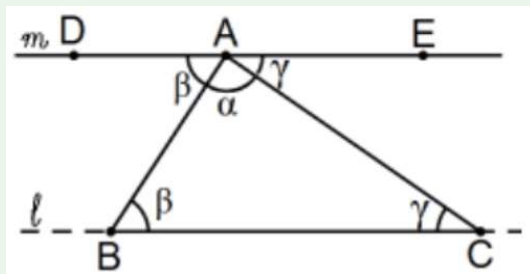
Theorem

These two axioms are equivalent.

Proof: Obviously, Global implies Local. Conversely, assume Local is true. Then, since the "no parallels" case does not exist (we're in neutral geometry), the negation of Global would imply some line and some point not on that line at which there were multiple parallels. But then the global form of the Hyperbolic Parallel Postulate would imply that the given local hyperbolic situation must be true, the needed contradiction. QED.

Angle Sum Theorem (Euclidean geometry form)

The sum of the angles of a triangle is equal to two right angles. [So for an n -gon, exactly $180(n - 2)$.]



Proof: Consider any triangle, say $\triangle ABC$. At A on AB, and on the opposite side, copy $\angle ABC$, say $\angle DAB$, and at A on AC, and on the opposite side, copy $\angle ACB$ to obtain $\angle EAC$. Why is line $DA = AE$ so that $\alpha + \beta + \gamma = 180^\circ$?

Alternate Interior Angle Theorem

If two parallel lines are cut by a transversal, they form congruent alternate interior angles with the transversal [and alternate exterior angles and corresponding angles].

Proof: PS 4, #2. Note: This gives another easy proof of the Angle Sum Theorem - start with the parallel.

Definition

The *area* of square of length 1 unit is 1 *square unit* and, by extension, the area of any $m \times n$ rectangle is mn square units. The area of a finite, bounded, simple (no overlaps), region (in the plane) is the greatest lower bound of the sums of the areas of all rectangles that can be inscribed inside the region with no overlap of their interiors provided that number is equal to the least upper bound of the sums of the areas of all rectangles that contain the region with no overlap of their interiors.

[Note: To prove this definition is meaningful requires more real analysis than we are going to assume.]

Theorem 4.1.1

Two triangles are equivalent if and only if they have the same area. More generally, a triangle is equivalent to an n -gon iff they have the same area.

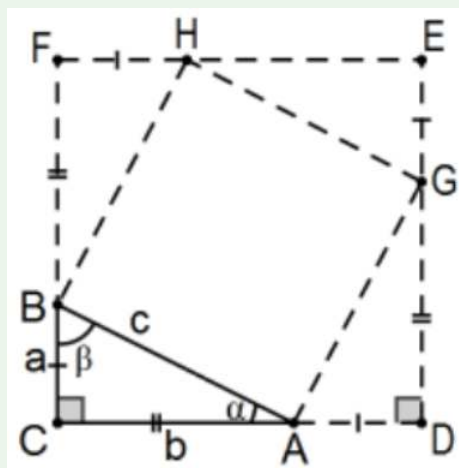
Proof: It should not be surprising that area would be preserved under equivalence but why not in hyperbolic geometry? That said, one proof (for triangles only) is to modify the corresponding theorem in hyperbolic geometry by replacing every occurrence of "angle sum" by "area" (PS 4, #28). Do less sophisticated proofs exist? Absolutely. For other polygons, induct on the number of sides (PS 4, #33). (This is *not* true for angle sum in hyperbolic geometry.) QED.

Theorem

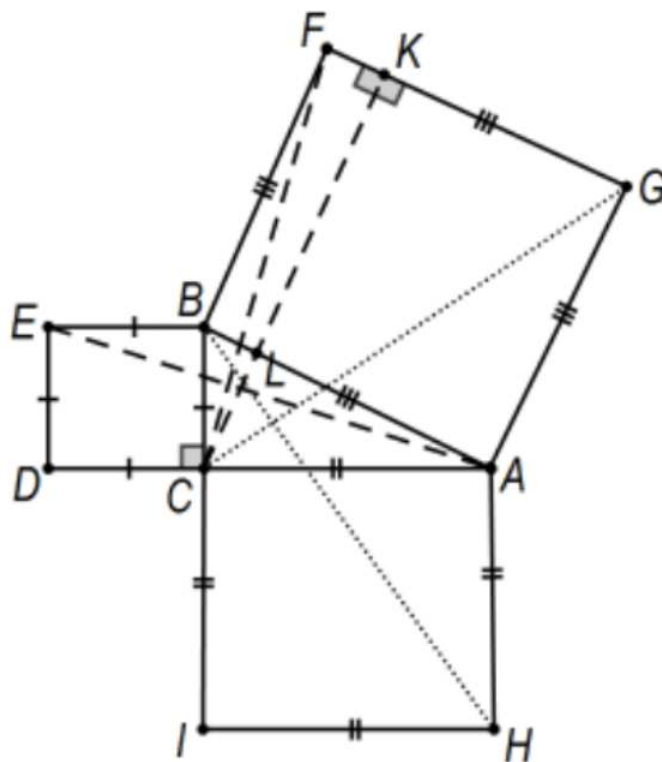
Area formulas of ordinary polygons.

Pythagorean Theorem

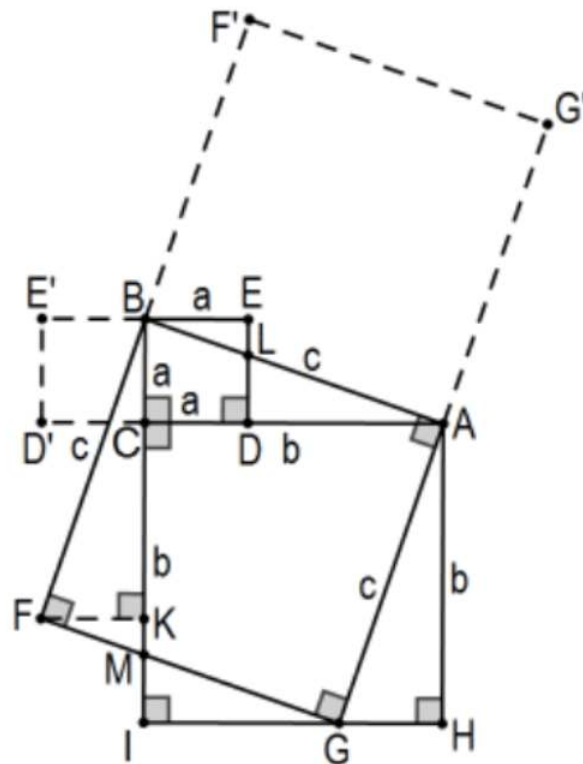
If $\triangle ABC$ with sides a, b, c is a right triangle with right angle at C , then $a^2 + b^2 = c^2$. That is, the sum of the areas of the squares on its two legs is equal to the area of the square on its hypotenuse.



Proof 1: There are many proofs of this result; some quite different from others. Among the easiest is to build a square of sides $a + b$ around the triangle and use a little algebra. Adding the areas of the pieces we have $(a + b)^2 = 4((1/2)ab) + c^2$ that simplifies to: $a^2 + b^2 = c^2$. The details are left as PS4, #19. QED.



Proof 2: This is the traditional proof from Euclid's Elements in Book 1, Prop 47 and every prospective high school geometry teacher should be familiar with it. With a little more work, it can be adjusted to prove the squares are equivalent; i.e., the two smaller squares can be "cut up" into a finite number of pieces and rearranged into the larger. In the right triangle of the figure, $\triangle ABC$, with squares on each side, assume auxiliary lines are as they appear with CK determined by the line on C perpendicular to the hypotenuse AB of the original right triangle. The area of $\triangle EBA$ is half the area of the small square BCDE (base BE and height BC) and, by SAS, $\triangle EBA \cong \triangle CBF$ (rotate by 90° about point B). For the second triangle, use base BF with height BL. Thus square BCDE has the same area as rectangle BFKL. By symmetry, (segments BH and CG) square ACIH and rectangle AGKL have the same area. QED.



Proof 3: Much easier is a proof that directly proves that the two smaller squares are equivalent to the larger one. In this one, fold the square on one leg to overlap the triangle and fold the square on the hypotenuse across both the triangle and the adjacent squares. Note that D' , E' , F' , and G' are not part of the proof; they are only there to help visualize the situation in its traditional form. Amazingly, this requires only one auxiliary line, FK , where K is the perpendicular on F to line BC . The idea is to show that the square on BC (a^2) together with the square on AC (b^2) is equivalent to the square on AB (c^2). Since there is so much overlap, the equivalence only requires confirming that $\triangle AGH \cong \triangle BFK$, $\triangle BLE \cong \triangle FMK$, and $\triangle ALD \cong \triangle GMI$ and that G is collinear with H and I . The details are left as PS 4, #20. QED.

Other proofs are given in in PS 4, #21-24.

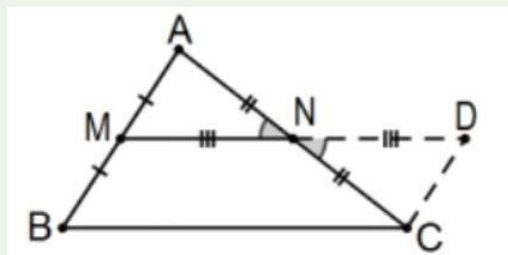
Converse of the Pythagorean Theorem

If $\triangle ABC$ is such that $a^2 + b^2 = c^2$, then it is a right triangle with right angle at C .

Proof: Construct a right triangle with legs of size a and b . By the Pythagorean Theorem, the third side is c . Then by SSS...? QED.

Midpoints Theorem

The segment joining the midpoints of two sides of a triangle is half the length of the third side and parallel to it.



Proof: Let M and N be the midpoints of segments AB and AC of $\triangle ABC$ and double segment MN to determine D as indicated. With the vertical angles at N we have $\triangle CND \cong \triangle ANM$ by SAS. Then $AM \cong CD$ and $\angle DCN \cong \angle A$ by cpctc and we have

congruent alternate interior angles so lines $AM = BM$ and CD are parallel. With $AM \cong BM$ so that $BM \cong CD$, we have quadrilateral $BCDM$ with a pair of opposite sides that are both parallel and congruent. Since we are in Euclidean geometry, the quadrilateral is a parallelogram (PS 4, #15) so its opposite sides are congruent (PS 4, #12). That is, $MD \cong BC$ and, since $MN \cong ND$ by construction, N is the midpoint of segment MD known to be congruent to BC . Thus we have segment MN parallel to and half of segment BC as required. QED.

Definition: Term

Two polygons are *similar* (indicated “ $\sim$ ”), and the correspondence a *similarity*, if (so iff) there is a proportional correspondence between their sides that preserves its angles; i.e., a 1-1 correspondence of the sides of one to the other with a positive real number r , the *common ratio*, such that the length of the second is the length of the first multiplied by r and the angle between two corresponding sides is unchanged.

For example, $\triangle MAN \sim \triangle BAC$ with $r = 2$ in the last theorem since the other two pairs of angles are congruent because the lines are parallel. Even more trivially, congruent figures are similar with common ratio $r = 1$.

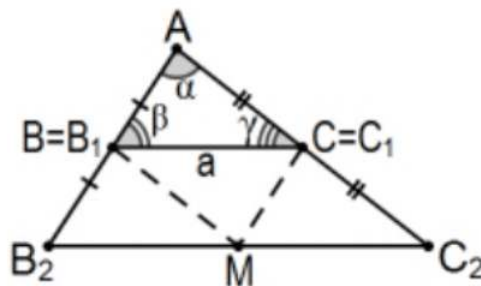
Theorem: SAS Similarity

If two triangles are such that the ratios between two of its sides are the same and the angles between those two pairs of sides are congruent, then the triangles are similar.orem text

Proof: First off, this looks like our proportion is wrong because this speaks of the ratio of two sides of the same triangle instead of the presence of a constant of proportionality for the correspondence (image side length to original side length). This fact is due to the property of ordinary fractions:

$a/b = x/y$ iff $a/x = b/y$ iff their reciprocals agree; i.e., $x/a = y/b$.

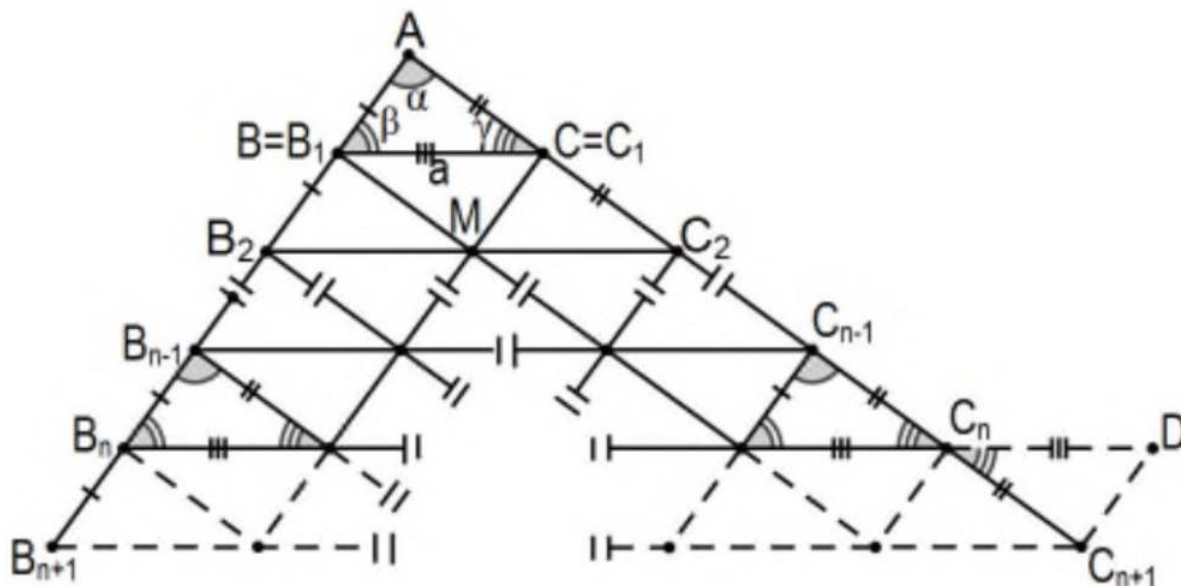
Here, think of x as the length of the image of the side of length a , and y likewise for b , and under the correspondence that, we hope, turns out to be a similarity correspondence. Let r be that common ratio; i.e., $r = x/a$ or $x = ra$ and $y = rb$. To be proved is that, for any $r > 0$, the other 2 angles are also congruent and the third sides are in the same ratio r . The proof is based on induction on $r = n$ as natural numbers and simple fractions $r = 1/n$ with the new triangle with two sides and included angle superimposed on the original $\triangle ABC$ (by SAS for congruent triangles) with the two sides of length $r(AB)$ and $r(AC)$.



Case $r = 1$: This is just SAS Congruence as mentioned above.

Case $r = 2$: This is just an immediate consequence of the last theorem. In order to prepare for a proof by Mathematical Induction, we rename the vertices as pictured, with midpoints $B = B_1$ and $C = C_1$ making lengths $AB_2 = 2AB$ and $AC_2 = 2AC$. By the last theorem, the length of $B_2C_2 = 2B_1C_1$ and line B_2C_2 parallel to line B_1C_1 confirming $\angle AB_1C_1 \cong \angle AB_2C_2$ and $\angle AC_1B_1 \cong \angle AC_2B_2$. Letting M_2 be the midpoint of segment B_2C_2 determines 3 new triangles all congruent to the original $\triangle ABC$, $4 = 2^2$ congruent triangles.

Case $r = 1/2$: This time, interchange the initial triangle $B = B_2$ and $C = C_2$ making lengths $AB_1 = (1/2)AB$ and $AC_1 = (1/2)AC$, the midpoint situation of the motivating theorem above.



Case $r = n$: For n any positive integer, induct on n beyond the case of $r = 2$ as follows:

Assume the result (similarity with common factor $r = n$ with n^2 internal congruent triangles) is valid for $r = 2, 3, \dots, n$ and consider any triangle $\triangle AB_{n+1}C_{n+1}$ with length $AB_{n+1} = (n+1)AB$, length $AC_{n+1} = (n+1)AC$, and, for convenience, the same $\angle A$. By the Ruler Postulate, identify n points to determine $n+1$ congruent segments along each segment AB_{n+1} and AC_{n+1} by subscript number and apply the inductive hypothesis to $\triangle AB_nC_n$. Following the proof of the preceding theorem, extend segment B_nC_n , of length nBC by the inductive hypothesis, by a segment of length BC to determine point D and consider $\triangle C_{n+1}DC_n$. This triangle is congruent to the original, $\triangle C_{n+1}C_nD \cong \triangle ACB$ by SAS. By cpct of segments and angles and transitivity, $DC_{n+1} \cong B_nB_{n+1}$ and their lines are parallel so quadrilateral $B_nB_{n+1}C_{n+1}D$ has a pair of opposite sides that are both congruent and parallel so it is a parallelogram (PS 4, #15) and its other pair of opposite sides are also congruent and parallel (PS4, #12). Since B_nD is of length $nBC + BC = (n+1)BC$, the needed side $B_{n+1}C_{n+1}$ is also along with the appropriate congruence of the angles at B_{n+1} and C_{n+1} confirming similarity with $\triangle ABC$. Finally, counting the number of the internal congruent triangles, we have the original n^2 plus $n+1$ new ones oriented with $\triangle ABC$ and n between them oriented upside-down: $n^2 + (n+1) + n = n^2 + 2n + 1 = (n+1)^2$ and the inductive step is complete. (For better understanding, carefully follow the proof with $n = 3$; the ideas are exactly the same.)

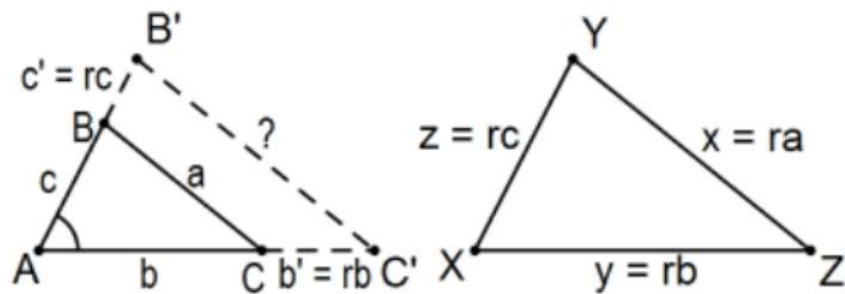
Case $r = 1/n$: For any positive integer n and two triangles that satisfy the SAS for Similarity conditions with common ratio $r = 1/n$ (one pair of congruent angles and there including pairs of sides in the same $1/n$ ratio), divide each of the including sides of the larger triangle, say $\triangle ABC$ into n congruent segments (Ruler Postulate) and index them as with the proof of any natural number $r = n$; that is $B_n = B$ and $C_n = C$ and each point of those subdivisions as B_i and C_i along the way. Then by SAS Congruency, the smaller of the triangles is congruent to $\triangle AB_1C_1$ starting from which we can recover similarity with the original $\triangle ABC$ by the case of $r = n$.

Case $r = m/n$: For any positive rational number: In this case, start with either the numerator or the denominator and apply the appropriate cases in sequence.

Case r : For any positive real number r : If r is rational, it is already done so assume r is irrational. Consider any sequence of positive rational numbers that converges to r and use continuity. *QED*

Theorem: SSS Similarity

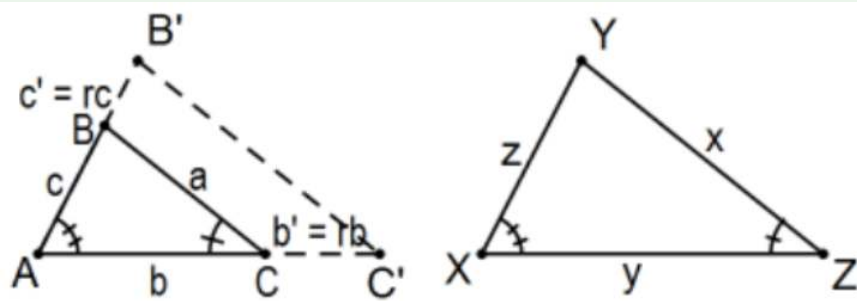
If two triangles are such that the ratio between all three pairs of their corresponding sides is the same, then the triangles are similar.



Proof: Here the constant of proportionality for the correspondence is that common ratio r . It would be possible to start from scratch as in the proof of SAS Similarity but it's not necessary. Choose two of the sides of the first triangle and consider the triangle formed by the angle between them but with the two sides r times as long. By SAS Similarity, the third side of that triangle is r times the original third side, ra . But then, by SSS Congruence,... QED.

Theorem: AA Similarity

If two triangles are such that two of its angles are congruent, then the triangles are similar.

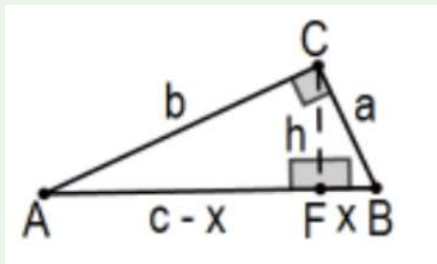


Proof: Where is the constant of proportionality for the correspondence? It is the ratio r of the sides included by the two pairs of congruent angles, $r = y/b$.

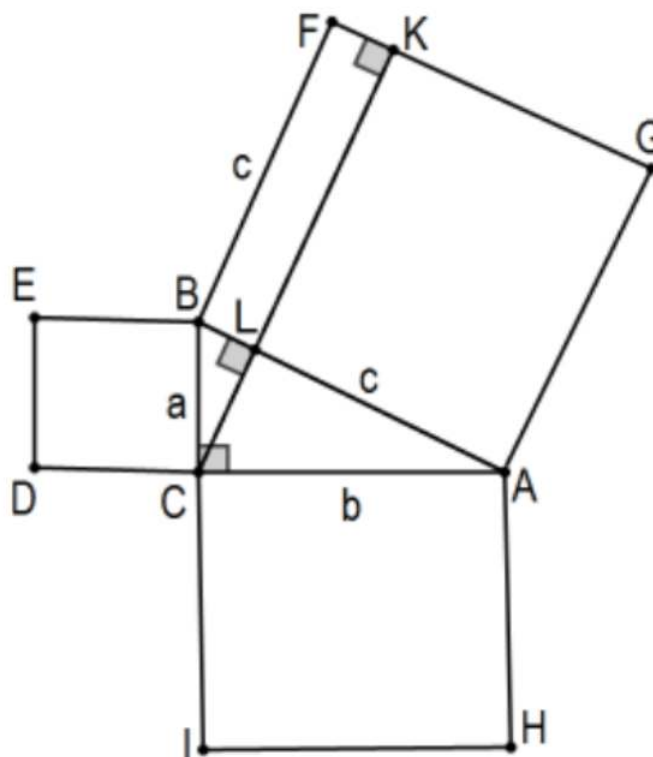
Since there is only one pair of sides under consideration, "they" all agree and proceed as with SSS as above. More explicitly, suppose $\triangle ABC$ and $\triangle XYZ$ are such that $\angle A \cong \angle X$ and $\angle C \cong \angle Z$ as indicated. Multiplying the 2 sides of $\angle A$ by the same factor r , $r > 0$, we have similar triangles $\triangle ABC \sim \triangle AB'C'$ by SAS similarity. By definition of similarity, $\angle C' \cong \angle C$ so, by transitivity, $\angle C' \cong \angle Z$ and we have $\triangle AB'C' \cong \triangle XYZ$ by ASA congruence. Finally, $\triangle ABC \sim \triangle XYZ$ by transitivity.

Theorem

The altitude from the right angle of a right triangle divides the triangle into two triangles each similar to the original.



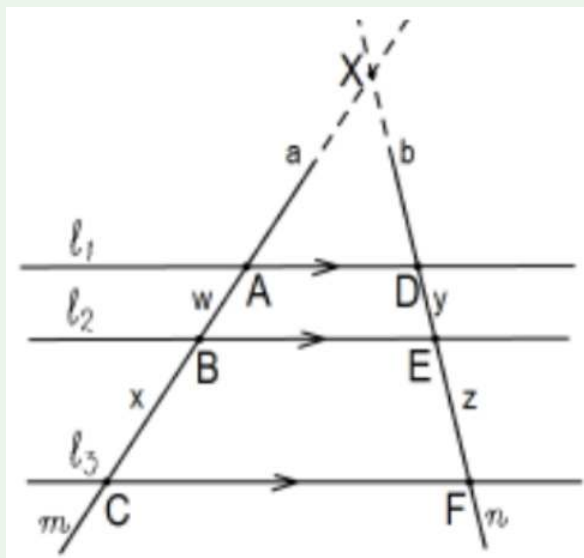
Proof: The acute angles of a right triangle are complementary so the result is immediate by AA Similarity. QED.



Proof 4: This fact gives the heart of yet another proof of the Pythagorean Theorem; nice in that it has the traditional picture but with a much easier proof. Starting as before with right $\triangle ABC$ with a square on each side, take the line on the right angle vertex C perpendicular to the hypotenuse so that CL is the altitude from the right angle of right $\triangle ABC$. That implies that $BL/a = a/c$ so that $c(BL) = a^2$. That is, the area of square $BCDE$ is the area of rectangle $BFKL$. Similarly b^2 is the area of rectangle $AGKL$. QED. (see PS 4, #21)

Parallel lines cut off proportional segments

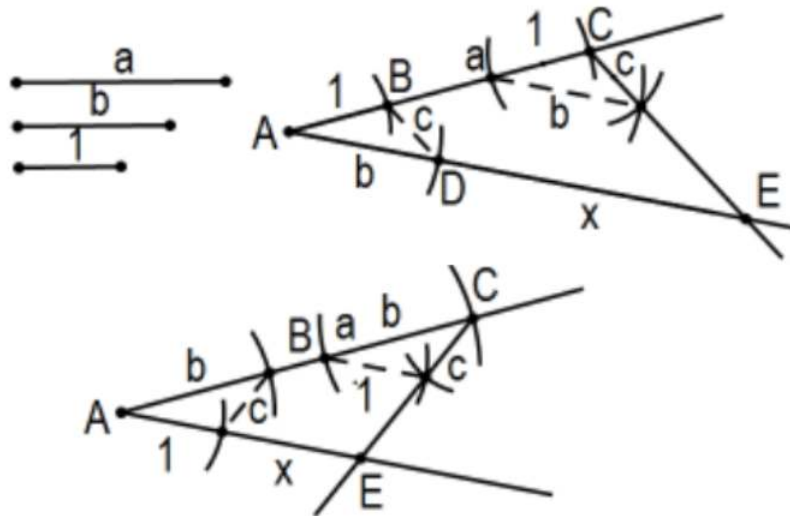
[More explicitly, the lengths of the segments of 2 transversals to 3 (or more) parallel lines are proportional (have the same ratios)].



Proof: In the figure, the 3 lines ℓ_1, ℓ_2 , and ℓ_3 are parallel lines cut by transversals m and n . We need to show that the determined segments are proportional; i.e., show that $w/x = y/z$.

If the transversals m and n are parallel, the determined quadrilaterals are parallelograms so $w = y$ and $x = z$ and trivial equality of the ratios so assume they intersect. Let X be the intersection of m and n . That determines 3 triangles, all similar by AA Similarity. The equality of the desired ratios follows from straightforward application of the similarity ratios of these triangles. The details are left as PS 4, #18.

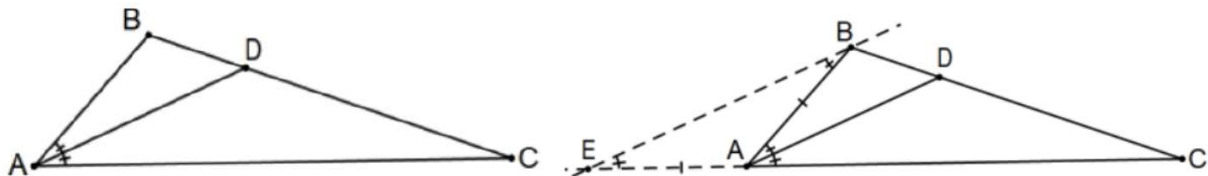
Construct: The product and the quotient of two line segments.



More specifically, given line segments of lengths a , b , and unit length 1 , construct segments of length ab and a/b . and D as indicated in the figure determines $\triangle ABD$. Copy $\angle B$ at point C as indicated to determine point E on the other ray. Then x , the length of segment DE , is the required length, $x = ab$. For a/b , it's the same construction except start with the b followed by the a on one ray and the 1 on the other ray.

Proof: By construction, $\angle ABD \cong \angle ACE$ and congruent corresponding angles implies that lines CE and BD are parallel. Since parallel lines cut off proportional segments, $1/b = a/x$. Therefore $x = ab$.

Theorem: The bisector of an angle of a triangle divides the opposite side proportionately with the sides of the angle.

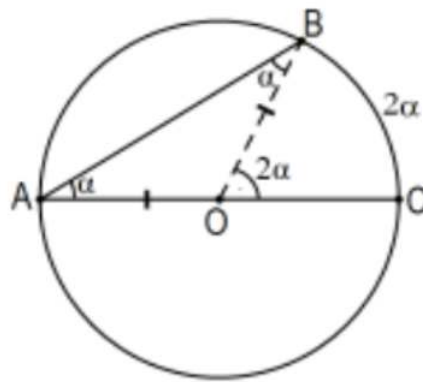


That is, in the associated figure, if ray AD is the bisector of $\angle A$, then $AB/AC = DB/DC$.

Proof: Let E be the intersection of line AC with the line on B parallel to line AD and consider the triangle created: $\triangle DCA \sim \triangle BCE$. QED.

Corollary: The bisector of an angle of a triangle is a median if and only if the triangle is isosceles.

Theorem: Inscribed Angle Theorem: An angle inscribed in a circle has measure half the measure of its intercepted (subtended) arc; i.e., half that of its corresponding central angle.

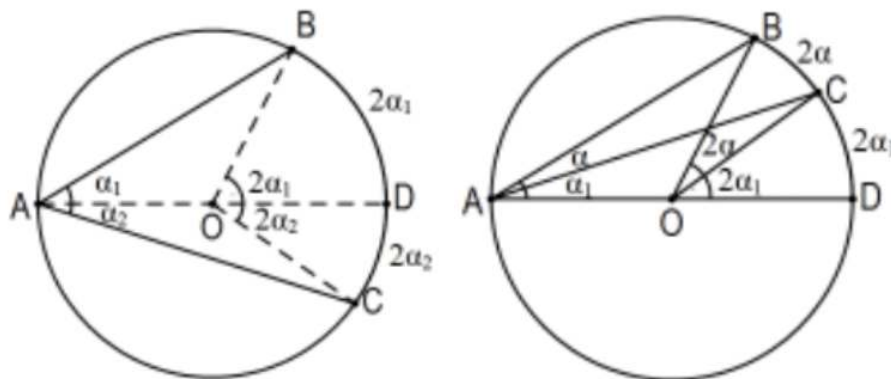


Proof: In all cases, vertex A is on the circle with center at O .

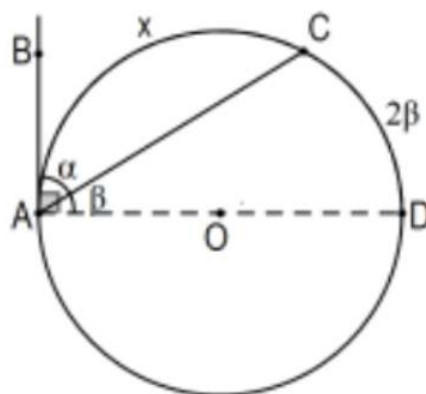
Case 1: One side of the angle is a diameter. In this case, the central angle is exactly the exterior angle of the isosceles triangle determined by the inscribed angle. Explicitly, since OA and OB are radii of the same circle, $\triangle AOB$ is isosceles and its base angles (at A and B) are congruent so that their sum is twice $\alpha = m(\angle A)$. By the Euclidean form of the EAT, $2\alpha = 2m(\angle A) = m(\angle BOC)$. Since arc measure is defined to be the measure of the central angle that subtends it, the assertion is proved.

Case 2: The center of the circle is interior to the angle. Include diameter OD : $2\alpha = 2(\alpha_1 + \alpha_2) = 2\alpha_1 + 2\alpha_2$.

Case 3: The center of the circle is exterior to the angle.



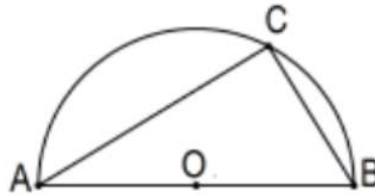
Note that $\angle BAC = \angle BAD - \angle CAD$. This implies that $2\alpha = 2((\alpha + \alpha_1) - \alpha_1) = 2(\alpha + \alpha_1) - 2\alpha_1$.



Case 4: One side of the angle is tangent to the circle. Assume line AB is tangent to the circle at A , and that $\angle BAC$ is acute. Consider $\angle CAD$, the complementary angle with the diameter AD determined by the radius OA since, by previous theorem, lines AB and AD are perpendicular. Obviously, the measure of arc AC is $x = 180^\circ - m(\text{arc } CD)$ so, by Case 1,

$x = 180^\circ - 2\beta = 2(90^\circ - \beta) = 2\alpha$ and we're done. If $\angle BAC$ is obtuse, its supplement is acute so use the acute case and subtract from 360° .

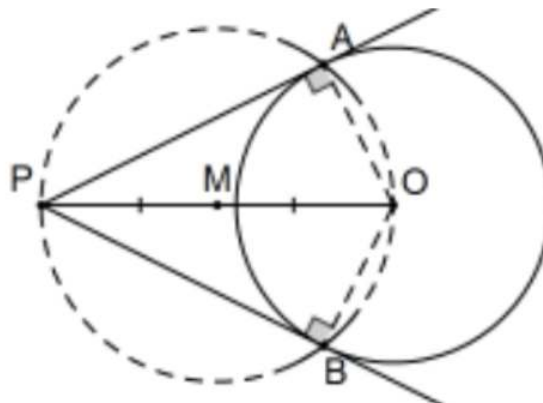
Corollary: A triangle inscribed in a semicircle (i.e., one side is a diameter) is a right triangle.



Historically, this fact is known as the Theorem of Thales. Thales was a philosopher mathematician who lived approximately 300 years prior to Euclid and has been called the first philosopher of the Greek tradition.

Corollary: Construct the tangents from a point outside a circle to the circle.

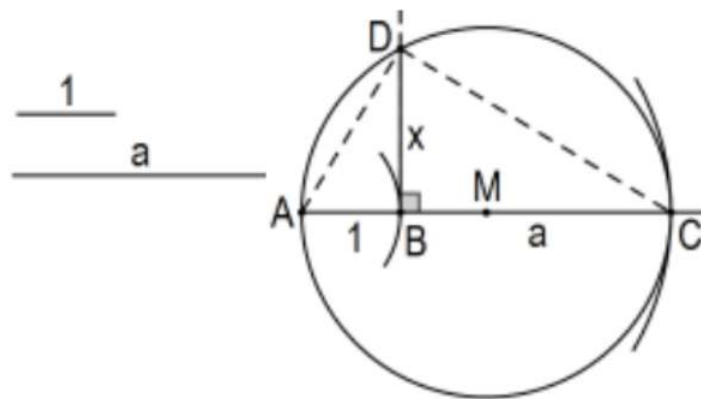
[Note: This was PS 2, #25, in neutral geometry. Why do it again? This construction is the one that is usually taught so is much better known than the earlier one. This situation is a bit strange since both constructions are in the original Euclid, Book III, the neutral geometry one is easier to prove, and is given earlier (Prop. 17 versus Prop. 31). It does make a nice example, however, of a situation where changing the axioms leaves the proposition true (the ability to construct the tangents) but the Euclidean form given here is not valid in neutral geometry. The steps of the construction can be done but, not only does the proof fail, it never produces the actual tangents!]



Construction: Bisect the segment determined by the point and the center of the circle and consider the circle centered at that point with radius determined by either of those points, (M; MO). Its intersections with the circle are the points of tangency. More explicitly, (if the center of the circle is not given) construct the center of the circle, O, the midpoint M of line segment PO, and the circle with center at M and radius $MP = MO$. Letting A and B be the points of intersection of this circle with the original circle, the lines PA and PB are the desired tangents.

Proof: This is immediate from the theorem on triangles inscribed in semicircles. QED.

Theorem: Construct a line segment of length the square root of the length of a segment:



Solution: Given segments of unit length and length a , construct M , the midpoint of a segment of length $1 + a$, ($AB + BC = AC$ in the figure), and the circle centered at M of radius MC . Finally, construct the perpendicular at point B to determine point D . Then segment BD is the required segment.

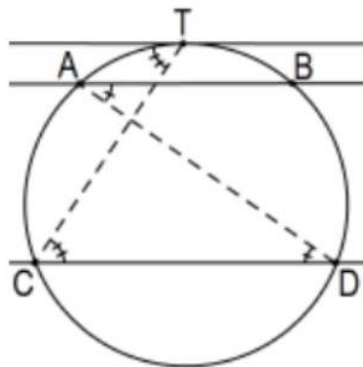
Proof: This is a triangle inscribed in a semicircle so $\triangle ACD$ is a right triangle and BD is the altitude from the right angle so $\triangle ADB \sim \triangle DCB$ and we have $x/1 = a/x$ so that $x^2 = a$ or $x = \sqrt{a}$. QED.

Theorem: Construct the geometric mean of the lengths of two line segments:

That is, given segments of lengths a and b , solve: $x^2 = ab$ geometrically.

Solution: Start with $m(AB) = a$ and $m(BC) = b$ in the figure above. QED.

Construct all positive solutions of a quadratic equation: That is, if a , b , and c are constructible and the equation has real solutions, solve $ax^2 + bx + c = 0$ geometrically.

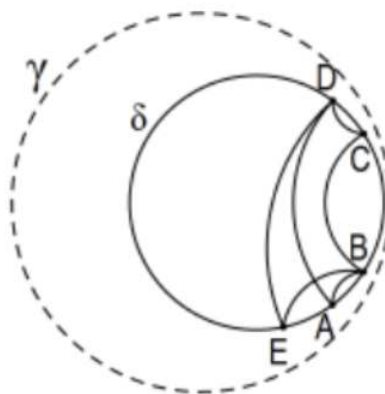


Solution: For $ax^2 + bx + c = 0$, $a \neq 0$, recall that the quadratic formula gives its solutions:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Each step of the algorithm is constructible. QED.

Theorem: Parallel lines intersecting a circle determine congruent arcs.

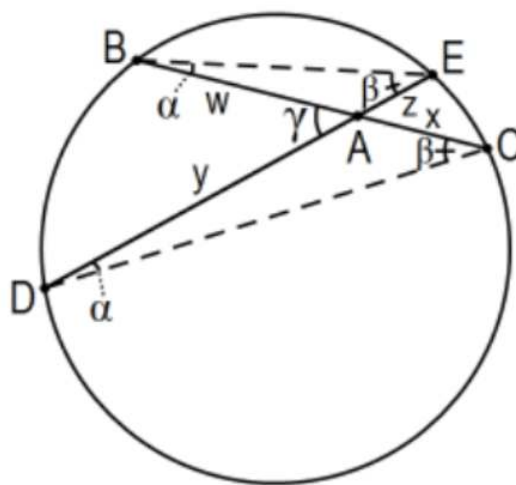


Proof: This is immediate from the Inscribed Angle Theorem.

Note: There is a problem in that we have never discussed what congruent arcs even means; everything has been in terms of breaking a figure into congruent triangles which is impossible for arcs of circles. Obviously, congruent circles are those with congruent radii. For congruent arcs, we mean arcs of congruent circles that determine congruent central angles or equivalently (by SAS congruence of triangles) congruent chords.

[Note: This is NOT true in hyperbolic geometry: (Poincaré arc: $DC \cong AB < EB$).]

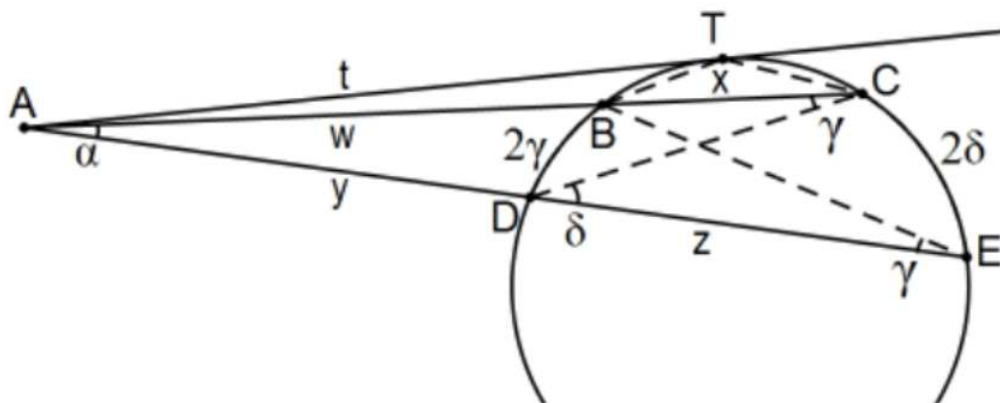
Theorem: The Chord Theorem: For a point inside a circle, the product of the lengths of the segments determined on any chord through it is constant (i.e., $wx = yz$ in the figure), and the measure of an angle determined by two chords through the point is half the sum of its intercepted (subtended) arcs.



Proof: By the Inscribed Angle Theorem, $\angle E \cong \angle C$ and $\angle B \cong \angle D$ so, by AA Similarity, $\triangle ABE \sim \triangle ADC$ so $AB/AD = AE/AC$ or $w/y = z/x$. Finally, $wx = yz$. For the angles, $m(\angle BAD) = m(\angle ABE) + m(\angle AEB)$ by the Euclidean form of the EAT and use the Inscribed Angle Theorem to express $m(\angle ABE)$ and $m(\angle AEB)$ in terms of the arc measures of their subtended arcs. QED.

Theorem: The Secant Theorem

For a point outside of a circle, the product of the lengths of the external segment times its entire secant (external segment plus the chord) is constant and in the tangent case that constant is just the square of the length of the segment. The measure of the angle at the point determined by two secants is half the difference of the subtended arcs. That is and as pictured, including the tangent case, $w(w + x) = y(y + z) = t^2$.

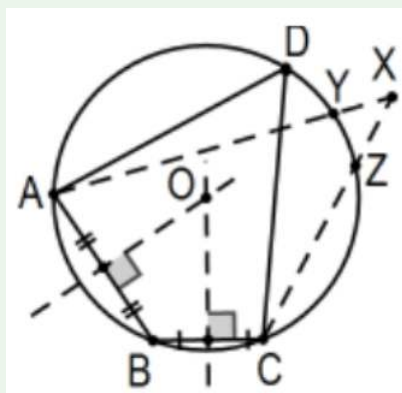


Proof: The angle at A is in common and, by the Inscribed Angle Theorem, $\angle C \cong \angle E$ so we have $\triangle ABE \sim \triangle ADC$ by AA Similarity. For the constant product conclusion, $AB/AE = AD/AC$ and cross multiply. For the angles, use the Euclidian form of the ETA, $m(\angle CDE) = m(\angle DAC) + m(\angle DCA)$, or $\delta = \alpha + \gamma$ or $\alpha = \delta - \gamma$. Now by the Inscribed Angle Theorem with arcs CE and BD , we have $\delta = (1/2)(m(CE) - m(BD))$. For the tangent case, see PS4, #7 QED.

Back in Chapter 1, we learned how to construct the center of a circle in neutral geometry; namely, choose any three points of the circle and the perpendicular bisectors of any two chords determined intersect to determine the center of the circle. This construction was familiar to many of you as a construction for the circumcircle of a triangle. That is, construct the perpendicular bisectors of two of the sides of the triangle to determine the circumcircle and the segment from that point to any of its vertices is its radius. We saw in Chapter 3 that it is not valid in neutral geometry because it is false in hyperbolic geometry; i.e., we know that some triangles have no circumcircle because sometimes those perpendicular bisectors do not intersect. In the Euclidean case, failure to intersect would imply that the two sides chosen were part the same line so that the triangle was not a triangle but a so-called "degenerate" triangle; i.e., three collinear points. Polygons with more than three sides, however, are not so clear; usually a polygon has no circumcircle but there is a surprising characterization for quadrilaterals:

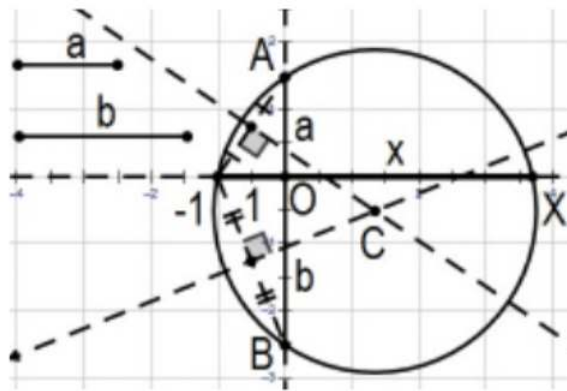
Theorem

A quadrilateral has a circumcircle if and only if each pair of opposite angles has sum of 180° .



Proof: The inscribed angle theorem immediately implies that if a quadrilateral can be inscribed in a circle, opposing angles must add to 180° . Conversely, suppose we have quadrilateral with opposite angles that add to 180° . Use any three of the vertices to construct the circumcircle of that triangle, the triangle those three points determine and prove that the fourth vertex must lie on it as well. In essence, the chord theorem (if the point were inside) and the secant theorem (if the point were outside) would imply that the angle sum would be wrong. The details are left as Ex 30.

Construct: The product and the quotient of two line segments using the chord theorem:



An alternate construction for the product (and for the quotient) of the lengths of two line segments of given lengths is by starting with any two intersecting lines as the lines of the their eventual chords with the constructed sum of the segments from the intersection of the lines as one of the chords and a segment of unit length from that point as one segment of the other chord. That determines 3 points of the desired circle and use them to construct the determined circle. The length of the segment from the point of intersection to the circle is the desired length (PS 4, #31). [Note: Using the x and y axes as the chord lines and the lengths as pictured, the product can be directly "read off" the positive x -axis ($\sim (3/2)(5/2) = 15/4$).]

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4.2: Analytic Geometry Proof

We have been doing what is known as "synthetic" proof, proving theorems directly from the axioms entirely in the abstract (in spite of the somewhat unorthodox approach of acknowledging that we never formally state these axioms instead of the usual, lying about it). The pictures we draw are helpful but meaningless unless we accept the idealized plane of a blackboard or sheet of paper (and associated points, lines, and the like) as a model for the abstract geometry that is intended to be strictly conceptual. Another model that we have been trained to believe is exactly the same thing is the Cartesian plane. In fact, this is an entirely separate model that has nothing in common with our usual plane except for our conviction that they are one and the same. In fact, they are very different. In another sense, however, they are the same; that is the nature of the axioms for Euclidean geometry being "categorical" introduced in the Introduction. The idea is that a proof in one model of Euclidean geometry can be identified completely (what are points, lines, etc.) in any other model or in the abstract "model-free" situation and the proof will be equally valid. That is, a Cartesian plane proof really is a valid proof. Although some of the full geometry (especially in n -dimensional Euclidean space) are better handled with vector notation (angle congruence for example), we are trying to present the situation at the lowest level possible for the most basic understanding.

Recall that when introducing the Poincaré disk model for hyperbolic geometry, we started off with the model being a triple, $\mathcal{L} = (\theta, L, C)$, for the set of points of the geometry, the subsets to be called lines, and the subsets to be called circles. We were "fudging" a little in that the concept for segment length (distance between 2 points needed to prove the Ruler Postulate) meant that "circles" already had a definition as the set of all points equidistant from some point called its center. The problem was that we had not developed much feeling for Poincaré distance (still have not, truth be told!). In this "new" situation, Euclidean distance is not a problem, the traditional distance formula, so $l = (0, L)$ where:

$$p = \{(x, y) \mid x, y \text{ the set of all ordered pairs, } \mathbf{R} \times \mathbf{R} = \mathbf{R}^2\}$$

$$\mathcal{L} = \{l \mid l = \{(x, y) \mid ax + by = c\} \text{ for some } a, b, c \in \mathbf{R}, a^2 + b^2 > 0\} \text{ (i.e., not both } a = 0 \text{ and } b = 0).$$

That is, a line is just the set of all solutions to any "linear equation" (even the words we use are tilting the pinball table!). In this Cartesian product setting, we are so used to identifying each ordered pair as a unique point in the coordinate plane that we have to force ourselves to think that its being nothing more than a left parenthesis "(" followed by two real numbers separated by a comma and a right parenthesis. Even Euclidean 3-space makes the identification obvious albeit harder to sketch. To get a better idea, think of a point in $\mathbf{R}^n$ for $n \geq 4$ and see if the identification remains as clear. To confirm that it is a model for Euclidean plane geometry when $n = 2$, we would have to prove all of the axioms as theorems but, in fact, some of them we've been doing ever since Algebra 1! For example, proof of Axiom 1 is nothing more than finding the (being more careful, "an") equation that is satisfied by two given pairs of ordered pairs and confirming that any linear equation that is satisfied by these two pairs has exactly the same set of solutions (same solution set). That is just a standard Algebra I problem if done in terms of general coefficients $P_1(x_1, y_1) \neq P_2(x_2, y_2)$ taking into account the special cases of $x_1 = x_2$ exclusive or $y_1 = y_2$. However, considering this model for Euclidean geometry in such a formal way would defeat the purpose. The whole idea is to use our close identification of the Cartesian plane (coordinate axes on the conventional plane) to give an alternate, sometimes simpler, proof.

The distance between two points is just the Pythagorean theorem-based distance formula. Angles are a little trickier but usually don't show up in the kinds of proofs that are easily constructed "analytically". For lines that are neither parallel to the x -axis or to the y -axis, it is easy to prove that distinct lines are parallel (no points in common) if and only if they have the same slope as long as we extend parallel to include the equivalence property "reflexivity" idea that a line be defined to be parallel to itself. Similarly, it is easy to prove that lines are perpendicular if and only if their slopes are negative reciprocals of each other (product is -1) along with the special case of parallels to the coordinate axes. Note: This is not a definition perpendicular already has defined meaning - this is a theorem. The measure of angles is most easily done with vector notation. We won't need it but for completeness sake:

Definition: The measure of the angle

The measure of the angle between two vectors u and v at the origin is $\alpha = \cos^{-1}(u \cdot v / \|u\| \|v\|)$, the size of an angle with cosine being the dot product of the vectors divided by the product of their lengths so for general $\angle BAC$ with $B(x_B, y_B)$, $A(x_A, y_A)$, and $C(x_C, y_C)$:

$$m(\angle BAC) = \cos^{-1} \left(\frac{(x_B - x_A)(x_C - x_A) + (y_B - x_A)(y_C - x_A)}{\left(\sqrt{(x_B - x_A)^2 + (y_B - x_A)^2} \right) \left(\sqrt{(x_C - x_A)^2 + (y_C - x_A)^2} \right)} \right)$$

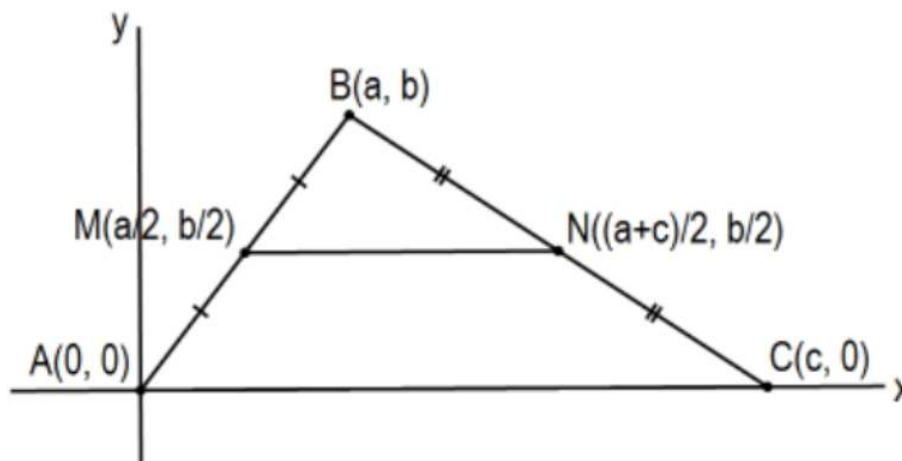
This was expressed in terms of the plane ($\mathbf{R}^n$ where $n = 2$) but the same vector approach is valid for any n . That is, the same vector formula for the measure of an angle for $\mathbf{R}^n$ for any n .

Instead of more of the details, here is an example of a theorem proved synthetically now using an analytic proof. Examples of other theorems are given in PS 4, #27 and #28.

Theorem

The line determined by the midpoints of two sides of a triangle is parallel to the third side and the length of its segment is half of that third side.

Proof: In order to minimize algebraic complexity, it is very helpful to coordinate the plane in such a way as to make the algebraic arithmetic as easy as possible being careful, of course, to be completely general in the assignment. A common simplification is with one side of a figure being studied along the x -axis and an important point $(0, 0)$. In this case, a reasonable place to start would be for our general triangle to be $\triangle ABC$ with ordered pair vertices $A(0, 0)$, $B(a, b)$, and $C(c, 0)$ with $b \neq 0$ and $c \neq 0$ as pictured. [Even easier would have been $A(a, 0)$, $B(0, b)$, and $C(c, 0)$ with $b \neq 0$ and $a \neq c$.] Of course, this assumes that calculation of a segment's midpoint is known but writing an equation for line MN ($y = b/2$) to confirm that it is parallel to the x -axis ($y = 0$) is a triviality as is calculating the length of segment MN and confirming that it is half the length of segment AC . QED.



Note: This is so simple that it almost feels like we are cheating but it is a perfectly valid proof assuming the fact that the axioms for Euclidean geometry are categorical and that this is a model for Euclidean geometry.

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4.3: Theorems of Ceva and Menelaus

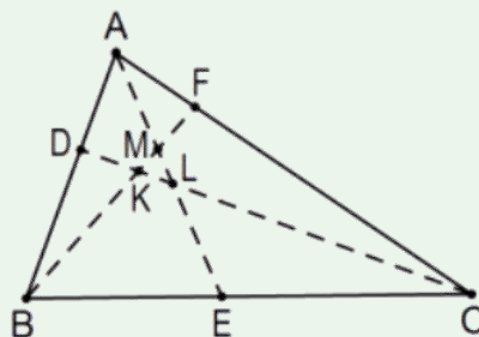
Cevian (from the 17th century Italian mathematician Giovanni Ceva (cheh'va)) is a line of a triangle from a vertex to a (non-vertex) point of the line of the side opposite. As examples, the medians of a triangle, its angle bisectors, and its altitudes are all Cevians, but they need not be anything so special. Three of them together, however, do lead to a surprising and powerful result. The theorem was proved by Ceva but it was also proved much earlier by Al-Mu'taman ibn Hüd, an eleventh-century king of Zaragoza, Spain.

Note

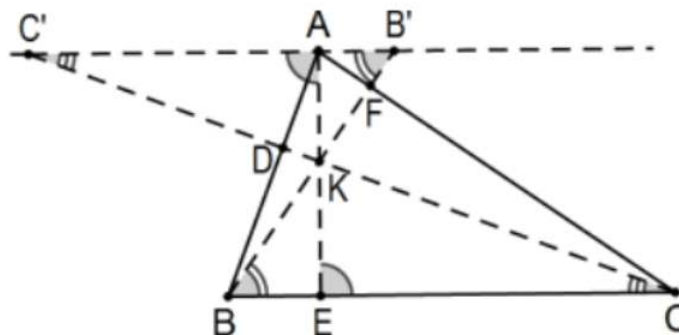
To this point we have used positive measures only but this theorem uses negative measures as well. It doesn't matter which direction on a line we call positive but, once that decision is made, all other measures on that line are consistent that and reversing the endpoints of a line segment reverses its sign.

Theorem (Ceva's Theorem)

Cevians from each vertex are concurrent if and only if the product of the signed ratios they determine on each side line is 1. That is, in the figure, the Cevians AE, BF, and CD are concurrent if and only if $(AD/DB)(BE/EC)(CF/FA) = 1$.



Proof: Whether or not the Cevians are concurrent, consider the line on A parallel to line BC and let B' be its intersection with line BF and C' its intersection with CD. Even without common intersection of the Cevians, congruent alternate interior angles and vertical angles verify five pairs of similar triangles; D and F imply that $\triangle AC'D \sim \triangle BCD$ and $\triangle AB'F \sim \triangle CBF$ and the three with the intersections of each pair of Cevians, AE with BF, AE with CD, and BF with CD. Arbitrarily, we take directions on each line as positive if vertex to vertex or vertex to Cevian is alphabetically increasing and A toward B' along the remaining.



These lead to many pairs of equal ratios including:

1. $(AD/BD = -AC'/-BC) \Rightarrow (AD/DB = AC/CB)$ (worrying about signed measures) and
2. $CF/FA = CB/B'A$ (the signs of the measures already agree). Suppose these three Cevians do intersect at a single point, say K. Then $\triangle AB'K \sim \triangle EBK$ and $\triangle AC'K \sim \triangle ECK$ so that:
 $(AB'/EB = AK/EK = -AC'/-EC) \Rightarrow (EB/AB' = EC/AC')$ (reciprocals of first and last) and

3. $(EB/EC = AB'/AC') \Rightarrow (BE/EC = B'A/AC')$ (signs again and also "interchanging the means"). Multiplying equations (1), (3), and (2), respectively, we have:

$$(AD/DB)(BE/EC)(CF/FA) = (AC/CB) (B'A/AC') (CB/B'A) = 1$$

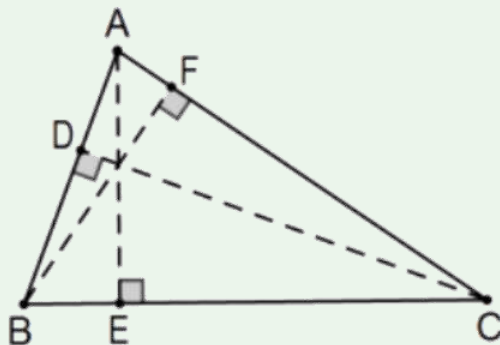
The converse is that, if the equation holds, then AE, BF, CD are concurrent. To prove this, assume that K is the point of intersection of only two of the Cevians, BF and CD , and let E' be the intersection of line AK with line BC . Then, from the part of the theorem just proved, $(AD/DB) (BE'/E'C) (CF/FA) = 1$ and, by the hypothesis, we have that $(AD/DB) (BE/EC) (CF/FA) = 1$. Simple algebra yields $BE'/E'C = BE/EC$ and, adding 1 to each side and adding the resulting mixed fractions, we have $(BE' + E'C)/E'C = (BE + EC)/EC$. Noticing that E' is between B and C and so is E , the numerators both simplify to BC . That is, $BC/E'C = BC/EC$. From this it follows immediately that $E'C = EC$ and, from their collinearity and the Ruler Postulate, $E' = E$ which proves that the Cevians are concurrent. QED.

Is the "signed" detail necessary? Is it possible to have the product of these ratios to be -1 ? Not only is the answer "yes", the result is an important theorem in its own right. There are some nice consequences of Ceva's Theorem. A couple of these we have already seen but are almost trivial consequences of this theorem. Truly trivial is that the medians of a triangle are concurrent. PS 4, #26 is a nontrivial proof but also gives more information.

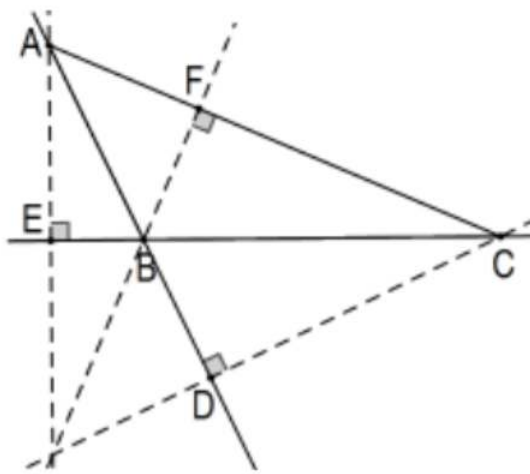
Slightly more involved is another proof that the angle bisectors of a triangle are concurrent. This time, the proportionality theorem for angle bisectors provides the answer (See PS 4, #33). More complicated, but indicative of the power of the theorem:

Theorem

The altitudes of a triangle are concurrent (the point of intersection is called the **orthocenter**).



Proof: Consider acute $\triangle ABC$ and let E, F , and D be the feet of the altitudes from vertices A, B , and C , respectively. By AA Similarity: $\triangle CAD \sim \triangle BAF$ ($\angle A$ in common) $\triangle ABE \sim \triangle CBD$ ($\angle B$ in common) $\triangle BCF \sim \triangle ACE$ ($\angle C$ in common)



As before, assigning positive sign from A to B, B to C , and A to C :

$$[AD/CA = AF/BA] \Rightarrow [(AD/CA)(BA/AF) = 1]$$

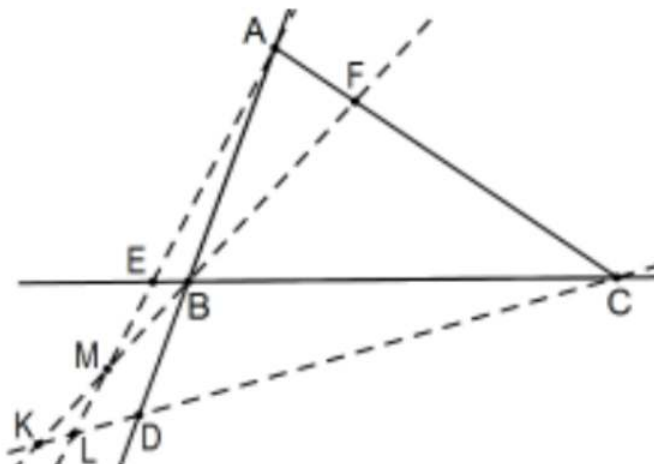
$$[BE/AB = -BD/-CB] \Rightarrow [(BE/AB)(CB/BD) = 1]$$

$$[CF/BC = -CE/-AC] \Rightarrow [(CF/BC)(AC/CE) = 1]$$

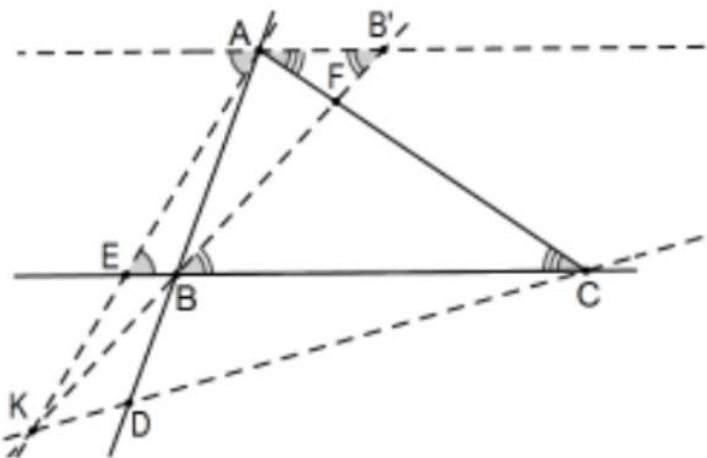
Multiplying these three yields the equation:

$$\begin{aligned} 1 &= [(AD/CA)(BA/AF)][(BE/AB)(CB/BD)][(CF/BC)(AC/CE)] \\ &= [(AD \cdot BE \cdot CF)(BA \cdot CB \cdot AC)] / [(BD \cdot CE \cdot AF)(CA \cdot AB \cdot BC)] \\ &= [(AD \cdot BE \cdot CF)(BA \cdot CB \cdot AC)] / [(DB \cdot EC \cdot FA)(AC \cdot BA \cdot CB)] \\ &= [(AD/DB)(BE/EC)(CF/FA)][(BA/BA)(CB/CB)(AC/AC)] \\ &= (AD/DB)(BE/EC)(CF/FA) \end{aligned}$$

By Ceva's Theorem, lines AE, BF, and CD are concurrent.



However, as the second figure implies, we do need to extend the result to obtuse triangles. Looking back at the similar triangles used in the acute case, the indicated triangles are still similar but there is a twist, the one that uses common $\angle B$ uses vertical angles instead of the common angle. Thus the ratios themselves, still hold and, checking, so do the signs of the ratios. Because of that fact, the same proof for congruence of the altitudes still works provided Ceva's Theorem also holds for obtuse triangles. Fortunately, it does. If two of the Cevian points are outside of their corresponding sides of the triangle, Ceva's Theorem still holds:



In the figure, D is outside of segment AB and E is outside of segment BC. Assuming the direction assignments as before, this time DB and BE are negative while F is still between A and C so CF is of the same sign. Then, whether the lines intersect in a single point or not, $(AD/DB)(BE/EC)(CF/FA)$ is positive. That fact could simplify the mechanics of the proof a little in that we

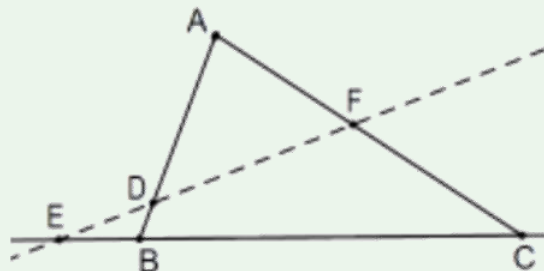
could ignore their signs and consider all lengths as positive, an observation that would have meant we would not have had to worry about two reversals at a time in the internal case. Proceeding exactly as before, consider the parallel on A to the opposite side line BC with B' and C' (C' is off the page to the right) as before and positive direction from A toward B' so that, this time, AC' is positive. All of the identified triangles are still similar (common vertex at D and K instead of vertical angles) so that $\triangle AC'D \sim \triangle BCD$ and $\triangle AB'F \sim \triangle CBF$ and therefore:

1. $(AD/BD = AC'/BC) \Rightarrow (AD/DB = AC/CB)$ (worrying about signed measures) and
2. $(AF/CF = AB'/CB) \Rightarrow (CF/FA = CB/B'A)$ (taking reciprocals and reverse two signed measures). Suppose these three Cevians do intersect at a single point, say K . Then $\triangle AB'K \sim \triangle EBK$ and $\triangle AC'K \sim \triangle ECK$ so that $(AB'/EB = AK/EK = AC'/EC) \Rightarrow (EB/AB' = EC/AC')$ and
3. $(EB/EC = AB'/AC') \Rightarrow (BE/EC = B'A/AC')$ ("interchanging means" and signs again). These are exactly the same three equations we had before so that, if the Cevians do intersect, then the product of the ratios is 1. Conversely, if we assume that K is the point of intersection of the two Cevians BF and CD and let E' be the intersection of line AK with line BC , we can proceed exactly as before. QED

Since Ceva's Theorem is relatively new, it is somewhat surprising to find out that the case of -1 dates from antiquity with its discovery and first proof unknown. The theorem is called **Menelaus' Theorem** but Menelaus was a figure from Greek mythology, not one of the great Greek geometers. As with Ceva's Theorem, there are many proofs of this famous theorem.

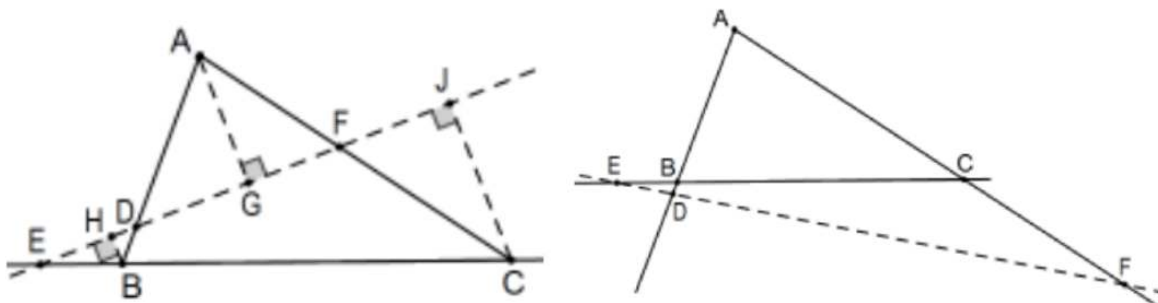
Theorem (Menelaus' Theorem)

Three points, one non-vertex point from each side line of a triangle, are collinear if and only if the product of the signed ratios they determine on all three side lines is -1 . That is, in the figure, D, E , and F are collinear if and only if $(AD/DB)(BE/EC)(CF/FA) = -1$.



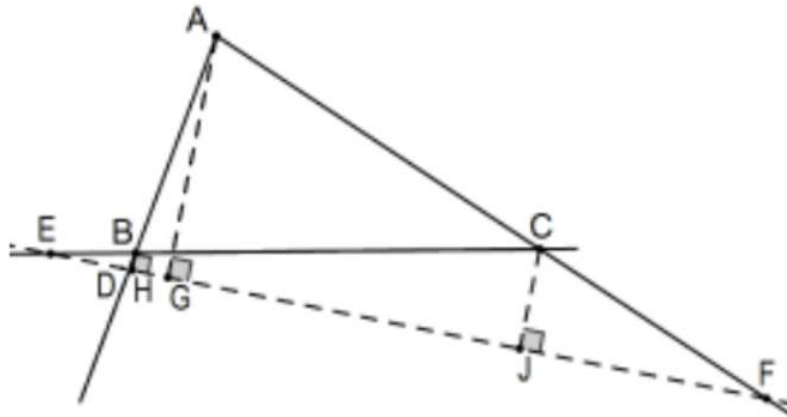
Proof: If you haven't already noticed, these ratios are positive if and only if the point is within the interior of its side (numerator and denominator positive or negative together). Moreover, if one of these is on the triangle itself, two must be. That is, if the three points are collinear, the line they determine intersects the interior of the sides of triangle in either two or no points and both of these are possible as the figures indicate.

That is to say, if the points are collinear, either one ratio is negative or all three are negative and, in either case, the product of ratios is negative so we do not really need to worry about signed measures; the product will be negative.



To start the proof of the case of two intersections, assume D, E , and F are collinear and drop the perpendicular from each vertex. AA Similarity yields several similar triangles (E in common and vertical angles at D and F in the two-side case and, in the zero-side case, common angles at E and F and vertical angles at D):

$[(AD/AG = BD/BH)] \Rightarrow [(AD/BD = AG/BH)]$ (interchange means) $[(BE/BH = CE/CJ)] \Rightarrow [(BE/CE = BH/CJ)]$
(interchange means) $[(CF/CJ = AF/AG)] \Rightarrow [(CF/AF = CJ/AG)]$ (interchange means) From these three we have:
 $(AD/DB)(BE/EC)(CF/FA) = (AG/HB)(BH/JC)(CJ/GA) = (AG/GA)(BH/HB)(CJ/JC) = (-1)^3 = -1$
and Menelaus' Theorem is proved in one direction.



Proof of the converse follows from this result just as with Ceva. That is, assume this product is -1 and let E' be the intersection of line DF with line BC. By the theorem just proved, we have:

$$(AD/DB)(BE/EC)(CF/FA) = (AD/DB)(BE'/E'C)(CF/FA)$$

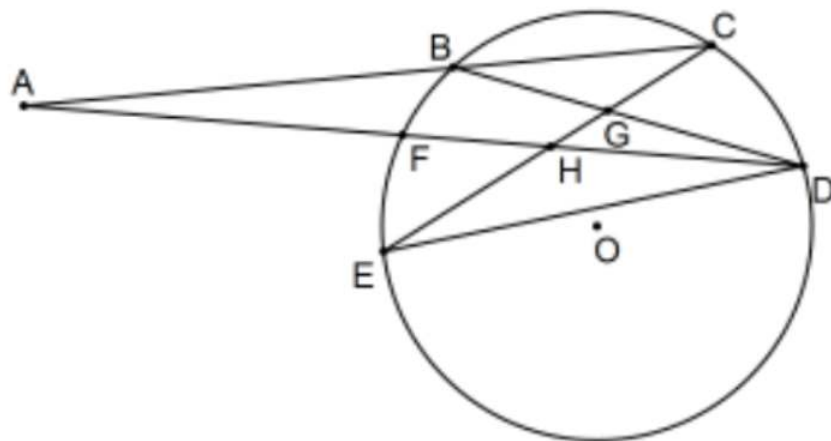
and the fact that $E = E'$ follows exactly as before. QED.

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4.4: Problem Set 4

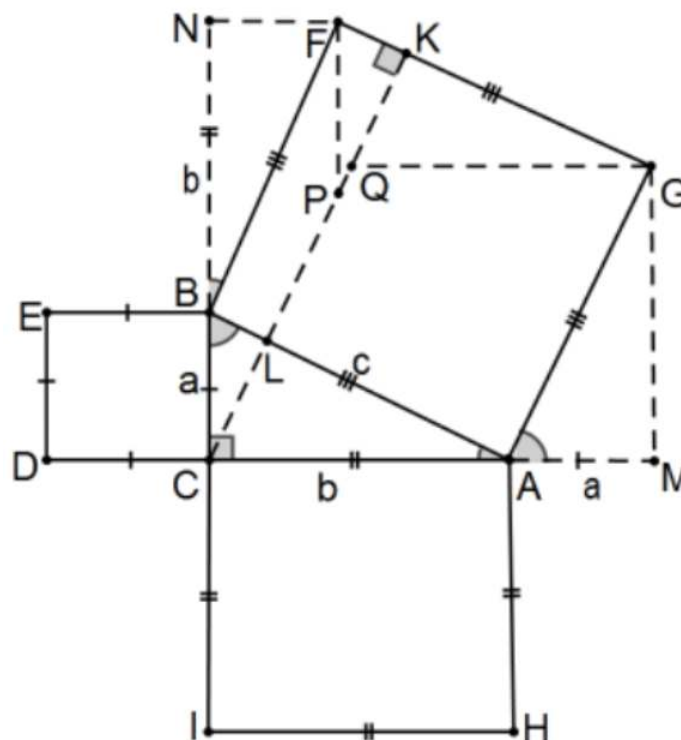
Unless otherwise stated, assume all of the axioms of neutral geometry (albeit unstated) and the global form of the Euclidean Parallel Postulate as well.

1. Prove that the sum of the interior angles of any triangle is two right angles (180°).
2. Prove that if two parallel lines are cut by a transversal, they form congruent alternate interior angles with the transversal.
3. Prove that a Saccheri or Lambert quadrilateral is a rectangle.
4. Construct and prove the standard Euclidean construction of the tangents from a point to a circle.
5. Prove all four cases of the Inscribed Angle Theorem. That is:
 - a. The center of the circle on one ray.
 - b. The center of the circle in the interior of the angle.
 - c. The center of the circle outside the angle.
 - d. The limit case; i.e, the case where one ray of the angle is not a chord but is tangent to the circle.
6. Prove the Chord Theorem.
7. Prove the Tangent Case of the Secant Theorem.
8. Assume in the sketch that all points and lines are as indicated along with the apparent circle with its center at O and assume $AB = 9$, $AF = 8$, $BC = 6$, $GB = 4$, $GC = 7/2$, and $GD = 6$. Find each and (briefly) explain your answer.

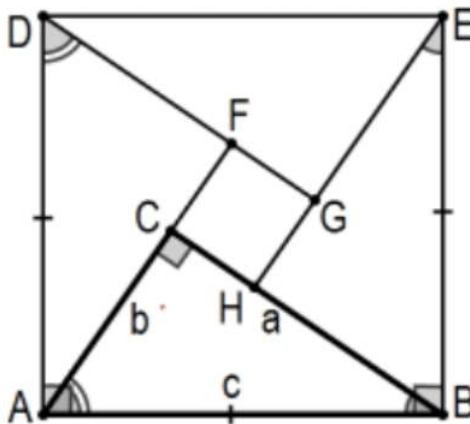


- a. $AC =$
 - b. $AD =$
 - c. $ED =$
 - d. $FD =$
 - e. $EG =$
9. As #8 but assume $m(\angle A) = 5^\circ$, $m(\angle E) = 30^\circ$, and $AB \parallel DE$. Find each and (briefly) explain why. [Note: NEW DATA!]
 - a. $m(\angle C) =$
 - b. $m(\angle CBD) =$
 - c. $m(\angle BDA) =$
 - d. $m(\angle FDE) =$
 - e. $m(\angle BGE) =$
 - f. Prove that the data of #9 is inconsistent with the data of #8.

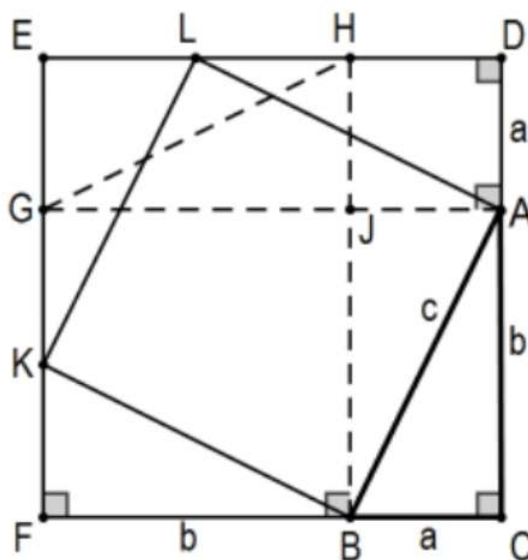
10. Prove that, in hyperbolic geometry, an inscribed angle is less than half its subtended arc (determined central angle) if the center of the circle is on or in the interior of the angle. The proof fails (may or may not be true) if the center is outside of the angle.
 11. Conclude from Ex 10 that, in hyperbolic geometry, a triangle inscribed in a semicircle is never a right triangle. Don't forget to confirm all 3 angles.
 12. Prove that a quadrilateral is a parallelogram iff its opposite sides are congruent.
 13. Prove that a quadrilateral is a parallelogram iff its opposite angles are congruent.
 14. Prove that a quadrilateral is a parallelogram iff its diagonals bisect each other.
 15. Prove that a quadrilateral is a parallelogram iff it has one pair of opposite sides that are both parallel and congruent.
 16. Prove that if a line segment joins the midpoints of two sides of a triangle, then it is parallel to and half of its third side.
- [Note: Although a much more sophisticated approach, this is immediate from a careful look at the determined associated quadrilateral.]
17. Given the unit line segment (e.g., the one in #18), construct segments of the lengths of all positive solutions to the equation: $x^2 - 2x - 1 = 0$.
 18. Prove that parallel lines cut off proportional segments. [More explicitly, the lengths of the segments of 2 transversals to 3 (or more) parallel lines are proportional (have the same ratios)].
 19. Prove the algebraic Proof 1 of the Pythagorean Theorem as outlined in the text (there are others). If there are misconceptions as to what details must be proved, they are that the resulting figure is indeed a square of side length $a + b$, that the four corners are congruent triangles and that the figure in the middle is indeed a square of side length c . "Obvious" may be tempting but it is not proof. That only works for professors and textbook authors.
 20. Prove the "equivalent triangle" Proof 3 of the Pythagorean Theorem as outlined in the text.
 21. Prove the "similar triangle" Proof 4 of the Pythagorean Theorem as outlined in the text.



22. Prove this alternate proof of the Pythagorean Theorem indicated that avoids areas of triangles but instead uses the parallelograms BCPF and ACQG. Explicitly, extend ray CB to determine N with BN of length b and M along ray CA with AM of length a and K is the foot of the perpendicular on C to line FG . Finally, let P and Q be determined by the intersections with line CK and the lines parallel to BC and AC .



23. Prove Bhāshkara's algebraic proof (India, b. A.D. 1114) of the Pythagorean Theorem as pictured here. ("Behold!" was his entire proof!) As usual, $\triangle ABC$ is a right triangle with right angle at C with $a \geq b$. Let segment AD be the perpendicular at A of length c and, similarly, let segment BE be the perpendicular at B , also of length c to establish quadrilateral $ADEB$. Copy the original $\angle A$ at D as indicated and the original $\angle B$ at E to establish quadrilateral $CFGH$. Prove the Pythagorean Theorem by proving that $ADEB$ is a square with area c^2 , $CFGH$ is a square with area $(a - b)^2$, and the four triangles are all congruent with area $(1/2)ab$. Conclude: $c^2 = a^2 + b^2$. [Note: What is the serious logical error in Khan's delightful presentation? <https://www.khanacademy.org/>, register, and enter Bhashkar in "Search".]

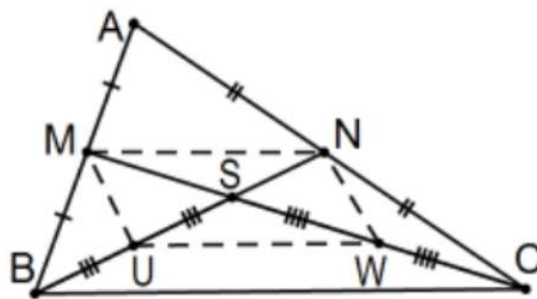


24. Here is another "Behold!" proof but I don't know who came up with it. This time, start with the right triangle $\triangle ABC$ and extend it, equivalent to #19, to obtain the larger square as indicated by the lengths and perpendicular indicators. The resulting square consists of the 4 corners plus a square of side length c . Pairing the corner pieces and rearranging them as indicated by the dashed lines, we have the "left over" region as a square of side length a and another square of side length b ! Complete the details of a valid proof.

25. Consider the following problem: The hypotenuse of a right triangle is 10 inches and the corresponding altitude of the triangle is 6 inches. Find the area of the triangle.

- Explain why $A = (1/2)bh = (1/2)(10)(6) = 30$ is wrong.
- Find and verify the range of possible altitudes to a hypotenuse of 10 in a right triangle.

26. Prove that the medians of a triangle are concurrent and intersect at a point $2/3$ of the distance from any vertex to its opposite side. This point is called the **centroid** of the triangle and is its center of gravity (i.e., it would balance on that point if it were a plate of uniform thickness and density).



[Hint: With the given sketch, use the results of Exs. 14, 15 to prove that S is $2/3$ of the way from C to M and from B to N . To prove the concurrence, let T be the intersection of the third median with one of these two and consider how they intersect from what has already been proved; i.e., use that to prove that $T = S$.]

Note: The intent was a so-called "synthetic" proof (as indicated) but, in Euclidean geometry, so-called "analytic" proofs are also available for more advanced theorems after similar triangles and the Pythagorean Theorem (needed for the distance formula) have been proved and assuming that the axioms of Euclidean geometry are "categorical". That is, proving the theorem in one model is tantamount to proving the theorem in all models. Although any (general) assignment of coordinates will suffice, the algebraic arithmetic (and hence, clarity) is simplified immensely by wisely choosing the assignment of coordinates to the points needed in the proof.

27. Give an analytic proof of the result of Ex. 26.

[Hint: A good way to start to let one vertex be at the origin and one side be along the x -axis so that the general triangle $\triangle ABC$ has coordinates $A = (0, 0)$, $B = (b, 0)$, and $C = (c, d)$.]

28. Prove the **Parallelogram Law**: In a parallelogram, the sum of the squares of the diagonals is equal to the sum of the squares of all four sides.

[Note: The easiest (non-vector) proof I know is analytic. Let $ABCD$ be the parallelogram. Coordinatize with the origin at $A = (0, 0)$, $B = (b, 0)$, $D = (c, d)$ and confirm that $C = (b + c, d)$. Now calculate the length of each of the segments involved.]

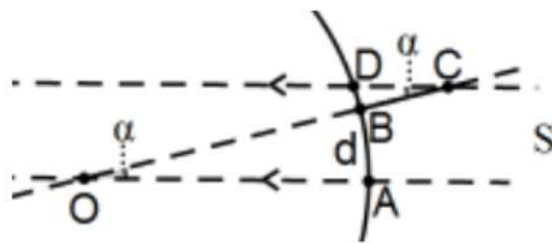
29. Complete the proof that two triangles of the same area are equivalent by modifying the corresponding theorem about triangles of the same angle sum in hyperbolic geometry.

[Note: This is not the traditional proof of the fact that has been known from antiquity but it is natural from our presentation.]

30. Complete the proof that a quadrilateral has a circumcircle if and only if each pair of opposite angles are supplementary (has sum of 180°). [Hint: The figure in the text is helpful. Secant theorem? Chord theorem?]

31. Construct the product of the lengths of two line segments of given lengths by starting with any two intersecting lines as the lines of the their eventual chords with the constructed sum of the segments from the intersection of the lines as one of the chords and a segment of unit length from that point as one segment of the other chord. That determines 3 points of the desired circle and use them to construct the determined circle. The length of the segment from the point of intersection to the circle is the desired length. Use the Chord Theorem to prove the construction is valid. Finally, make the appropriate adjustments to construct the quotient of the lengths (the proof is essentially the same).

32. **Greek Diameter of the Earth** The ancient Greeks not only knew the earth was a sphere, they knew its size with reasonable accuracy. In 240BC, Eratosthenes estimated the diameter of the earth by measuring the angle formed at the top of a vertical pole that subtended its shadow at a point a known distance directly north of another point when the sun was directly overhead of the first point where a pole would have no visible shadow (he used the sun shining on the bottom of a well). There are a couple of approaches using this idea to approximate the circumference (equivalently, the diameter or radius) of the earth and both begin with the same setup described as follows.



Only the northern point is needed if it is at a known distance directly north of the great circle determined by a point with sun directly overhead at a given time on a given day. Suppose the accompanying figure is the cross-section of the earth determined by the plane of O, S and B; respectively, the center of the earth, the center of the sun (too far away on the right to be visible), and a point a known distance d directly north of the point A at which the sun is directly overhead (for simplicity, on the equator, at noon on an equinox). Both ideas assume that the sun is so far away that we can consider the sun's rays to be parallel instead of radiating from the point source S . Think of the line $SO = AO$ as a light ray from the sun and the other line is the parallel ray through C , the top of a vertical pole erected at B . Let D be the point where that ray strikes the earth. Since line OC is a transversal to two parallel lines, $\angle AOB \cong \angle BCD$ because they are alternate interior angles formed by a transversal to parallel lines and let $\alpha = m(\angle AOB) = m(\angle BCD)$.

32a. Eratosthenes used the simple observation that the central angle measure α° compared with the full circle 360° is proportional to the arc length d compared with the Earth's (the circle's) circumference c . Symbolically, we have $\alpha/360 = d/c$ so that $c = 360 d/\alpha$. Following Eratosthenes, approximate the circumference of the earth for $d \sim 575\text{mi}$ and $\alpha \sim 7.2^\circ$ (it is hard to measure angle α so accurately).

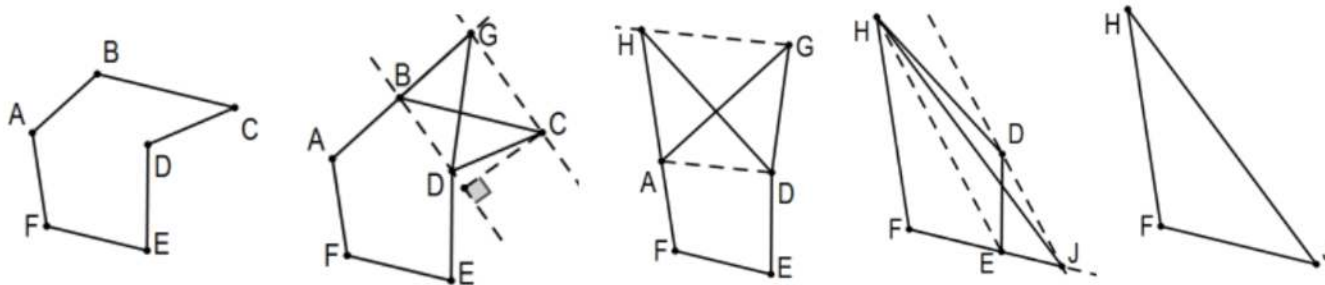
32b. Assuming point B is not too far from A (relatively small α), it is reasonable to consider the earth to be flat (it certainly feels flat!) and we replace arc AD by the common perpendicular to the two parallel lines through point B yielding points A' , D' , and, with Earth's radius r , two similar, $r/BA' = BC/BD'$ from which we have the obvious approximation: $r/m(\text{arc}(BA)) \sim BC/m(\text{arc}(BD))$ and finally:

$r \sim (m(\text{arc}(BA))) (BC/m(\text{arc}(BD)))$ with everything on the right easily measured (easy to measure $m(\text{arc}(BD)) \sim 11/2\text{ft}$ for pole length $BC = 10\text{ft}$. Sketch the situation and approximate the radius of the earth r and its circumference C .

33. Use Ceva's Theorem to prove that the medians of a triangle are concurrent. [Hint: Absolutely trivial!]

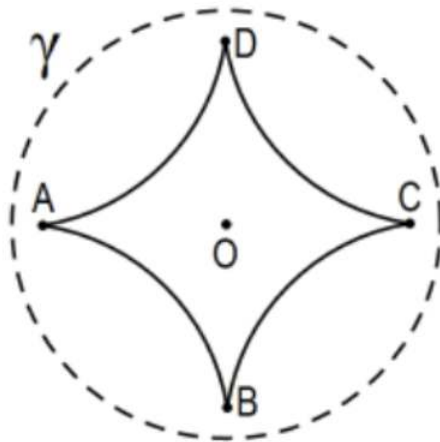
34. Use Ceva's Theorem to prove that the angle bisectors of a triangle are concurrent. [Hint: Use the theorem that an angle bisector of a triangle divides the opposite side proportionally with the sides of the angle.]

35. In Euclidean geometry, two polygons are equal in area if and only if they are equivalent. [The formal proof is by induction on n , the number sides in the polygon, as being equivalent to some triangle, so then the triangle case completes the solution. That is, for two polygons of number of sides m and n of the same area, reduce them both to triangles of the same area, use the fact that these two are equivalent and then, by symmetry and transitivity, conclude that the original polygons are equivalent. Rather than formally proving this, explain the reduction of this hexagon $ABCDEF$ to the triangle FHJ by, one-by-one, replacing a vertex with an edge while maintaining the same area because of equivalence (i.e., "snipping off" and moving congruent triangles.)]



Although angle sum replaces area for triangles in the corresponding theorem in hyperbolic geometry, the natural generalization to polygons is not true. The idea looks like it might work by "cutting off" a triangle to reduce the number of vertices by using the line of midpoints and Saccheri quadrilaterals. That is, in the first step above, instead of the parallel on C to line BD (that is not unique),

start with the intersection of line AB and the line of midpoints of segments BC and DC and double the segment from B to the point of intersection to establish point G thereby eliminating vertex B . In the Euclidean case, that approach determines exactly the same point G . Eliminating a vertex with an equivalent polygon reduces the angle sum by exactly 180° , the importance of the construction. Problem? There may not be 180° "to spare". To see why this is so, consider the regular quadrilateral $ABCD$ pictured in this Poincaré model, the angle sum is much less than 180° (each angle is less than 45°) so no vertex can be removed to obtain an equivalent triangle because it would have to have angle sum exactly 180° less but this is impossible.



36. What goes wrong with the Euclidean proof in the hyperbolic situation?

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CHAPTER OVERVIEW

Chapter 5: Advanced Euclidean Geometry

[5.1: Advanced Euclidean Geometry](#)

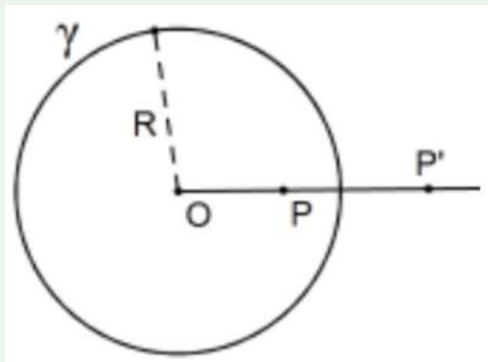
[5.2: Problem Set 5](#)

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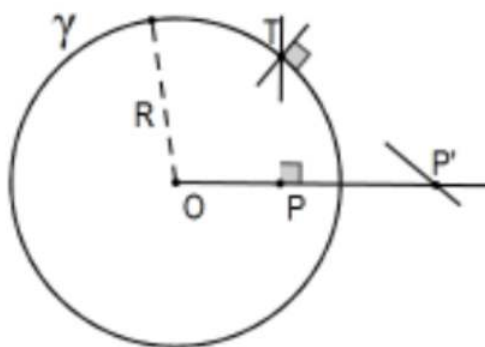
5.1: Advanced Euclidean Geometry

Definition: Inversion of a Point

The *inversion* of a point $P \neq O$ in a circle γ with center O and radius R is the point P' on ray OP such that $(OP)(OP') = R^2$.



Note: Sometimes we are just thinking of the inversion of particular point P and at other times we'll be thinking of the inversion of the entire "punctured plane". Why punctured?



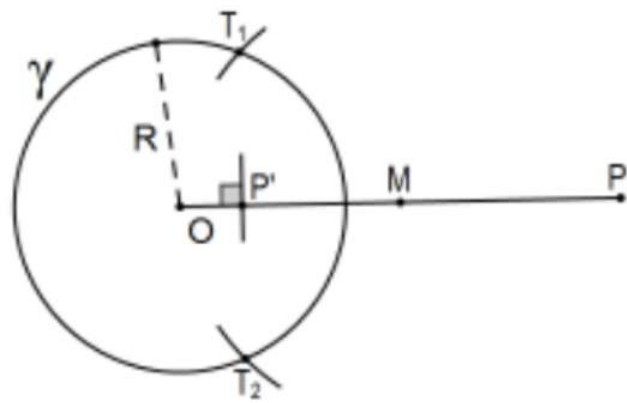
✓ Example 5.1.1

Construct the inversion of a given point $P \neq O$ in a given circle γ .

- *Case 1:* P inside the circle.
- *Case 2:* P on the circle.
- *Case 3:* P outside the circle.

Solution

For Case 1, let T be (either point) determined by the perpendicular to ray OP at P . Then P' is the intersection of ray OP with the tangent at T , the perpendicular to the radius to T .



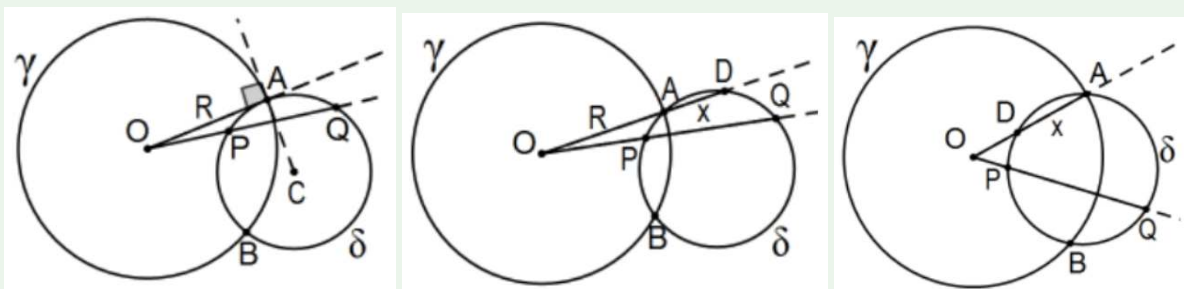
Since line TP is the altitude from the right angle of right triangle $\triangle OP'T$, $\triangle TOP \sim \triangle OP'T$ so $OP/OT = OT/OP'$ or $OP/R = R/OP'$ and $(OP)(OP') = R^2$ as desired.

For Case 2, $P = P'$ so the construction instructions are "do nothing"!

For Case 3, let T_1 and T_2 be the points that determine the tangent lines from P . Then P' is the intersection of ray OP with line T_1T_2 . (Why is this line perpendicular to line OP' ?) The proof is exactly that of Case 1 with the roles of P and P' interchanged. QED.

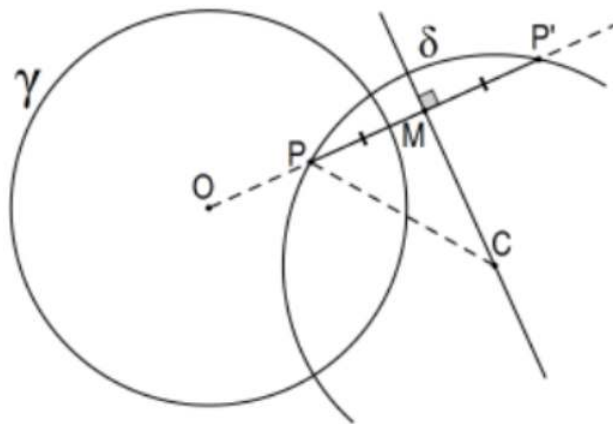
Fundamental Theorem of Orthogonal Circles

If two circles γ and δ intersect at points A and B with one of the circles centered at O and P and Q are points of the other circle determined by a ray that originates at O , then the circles are orthogonal if and only if P and Q are inverses of each other in the circle γ centered at O .



Proof: Consider ray OA . If the circles are orthogonal, then the tangent case of the secant theorem states the conclusion; i.e., $OP(OQ) = (OA)^2 = R^2$.

Conversely, assume P and Q are inverses in circle γ ; i.e., $OP(OQ) = (OA)^2 = R^2$. If the circles were not orthogonal, ray OA intersects the circle at a second point, say D , and there would be a little piece of chord, $AD = x$ with $OA = R$ being either the internal segment of the secant of length $R + x$ (as pictured first) or the entire secant with the external segment of length $R - x$ (second picture). Either way, the hypothesis forces x to be 0 and OA is a tangent from O . More explicitly, we have $R(R \pm x) = R^2$. That is, $R \pm x = R$ so that $x = 0$ and we have orthogonality.



Corollary: If a point on one circle (think P on δ) is in the interior of another circle (think γ), the circles are orthogonal if and only if the center of the first circle lies on the perpendicular bisector of the segment that joins the point with its inversion in the second circle.

Theorem: Axiom 1 for the Poincaré model.

Proof: Let γ be the defining circle and let A and B be distinct points of the plane not collinear with O . Then A' and B' are also on the desired Euclidean circle. That makes four noncollinear points, A , B , A' , and B' , any three of which completely determines the unique Euclidean circle that determines the open arc that is the desired unique Poincaré line. Moreover, the standard construction for that circle is a Euclidean construction of the Poincaré line. That is, the following is a triviality:

Corollary: Construct the Poincaré line determined by two points.

Beyond that, these additional two Euclidean points are very helpful in "eyeball" sketching the Poincaré line determined by the two points, the line on a point perpendicular to a line, the boundary parallels to a line on a point, etc. Don't formally construct these points (although some of these are now easy formal Euclidean constructions as well; see Problem Set 5) but, if you need to, use a straightedge (edge of a book, twicfolded piece of paper, or some such) and your knowledge of orthogonal circles to sketch the Poincaré line or lines required for the problem under consideration.

[Note: This is Step 1 of a very complicated proof, that *all* of the axioms of hyperbolic geometry (stated herein or not) can be proved as *theorems* of Euclidean geometry about the Poincaré model from which we can conclude that Poincaré geometry is relatively consistent with Euclidean geometry.]

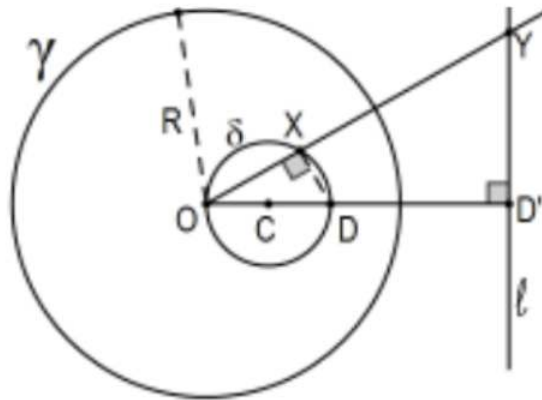
Theorem: In the inversion of the entire punctured plane in a fixed circle, circles and lines get mapped to lines and circles. More explicitly let γ be the fixed circle with center at O and radius of length R . Then for any Euclidean figure φ that does not contain O , let φ' indicate the inversion of all of its points in γ and, if φ does contain O , let φ^* be the same Euclidean figure but excluding O , and let δ be any circle and ℓ be any line in the Euclidean plane. Then:

- i. If $O \in \delta$, then the inversion of δ^* is ℓ , the line perpendicular to the ray OC at D' where C is the center of δ and D is determined by the diameter of δ determined by the ray OC .
- ii. If $O \notin \delta$, then the inversion of ℓ is the "punctured" circle δ^* where F is the foot of the perpendicular from O to ℓ and OF' is the diameter of δ (i.e., the inversion of the inversion in i).
- iii. If $O \in \ell$, then the inversion of ℓ^* is ℓ^* (the punctured line turned inside-out).
- iv. If $O \notin \delta$, let C be its center and segment AB be its diameter determined by the line OC . Then the inversion of δ is the circle δ' with collinear diameter $A'B'$ with two cases to consider:
 - a. If $O \notin AB$, then $A'B'$ is a subset of ray AB (the case pictured), or
 - b. If $O \in AB$, then $O \in A'B'$ (because A and A' are on one side of O and B and B' are on the other).

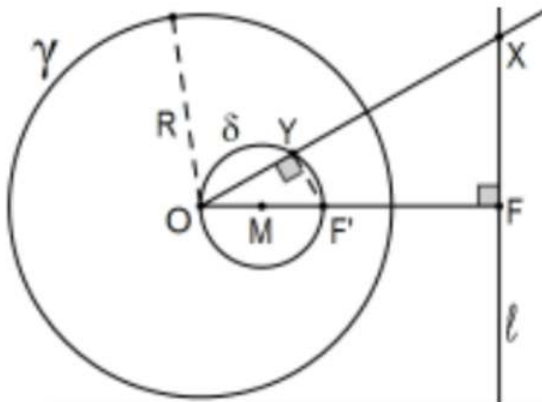
We are especially interested in reflections of the Poincaré plane in a Poincaré line but the theorem speaks of the full Euclidean plane. Reflections in a line and rotations about any point are provable as congruences in neutral geometry, hence in hyperbolic geometry. The problem is that we do not yet know that the Poincaré disk model even is a model for hyperbolic geometry; to assume

such would be assuming what we are trying to prove! (The special cases of a reflection across a diameter and a rotation about the center are obvious but not the others).

These proofs are easy but require some insight,

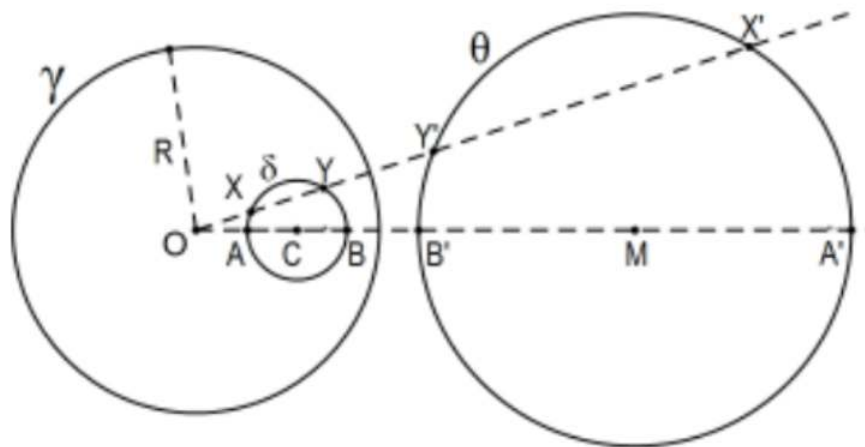


Case i : Let $X \in \delta^*$ and let $Y = f(X)$ be the intersection of ray OX with ℓ , the line perpendicular to ray OC' at D' . By AA Similarity, $\triangle OD'Y$ is similar to $\triangle OXD$ but so is $\triangle OD'X'$ so that $Y = X'$ and we have $\ell = \delta^{*'} as claimed.$



Case ii: This is exactly the same idea only this time we start with F , the foot of the perpendicular from O to ℓ and let F' be its inversion in γ . Let M be the midpoint of segment OF' . Let circle $\delta^* = (M; MO)$; i.e., the circle with center at M of radius $MO = MF'$. Let $X \in \ell$ let $Y = f(X)$ be the point of intersection of ray OX with δ^* . Since OF' is a diameter, $\triangle OYF$ is a right triangle so, by AA Similarity, $\triangle OYF' \sim \triangle OFX$. As before, Y is the inversion of X ; i.e., $Y = X'$ and we have $\ell' = \delta^*$ as claimed.

Case iii: If you think about it, it's obvious.



Case iv a: The construction is easy; the figure almost says it all. For circle δ , let A and B be determined by line OC , the line of centers of the two circles and A' and B' be their inverses with M the midpoint of segment $B'A'$. Then $\delta' = \theta$, the circle $(M; MA')$, with center at M and radius MA' . The tricky part of the proof is the idea is that the inversion of δ is a dilation (contraction) of itself from O but only as a set, not "point-wise". That is, there exists some constant k such that every point of δ' is determined by a point of δ on the ray it determines from O and k times its distance from O (dilation for $k > 1$, contraction for $0 < k < 1$). Dilations preserve circles because they preserve shapes; the ratio of any pair of distances is the same because all the factors of k cancel out. The tricky part is that the image is "turned inside out" as opposed to a "point-wise" dilation of the Euclidean plane where every point X of the plane would be mapped to the point $k(OX)$ from O along the same ray. That is not true under inversion but, as a set of points, any circle is carried onto another circle, a dilation of itself. For any $X \in \delta$, let Y be the other point determined by the ray OX (if tangent, $Y = X$). Now:

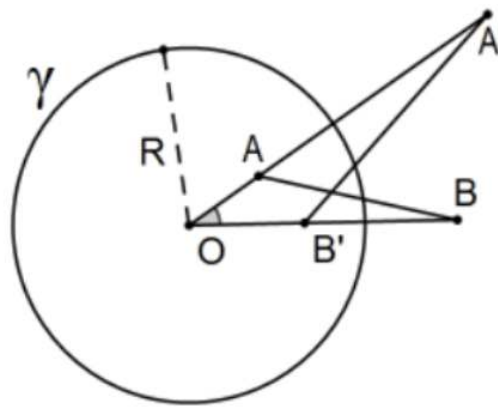
$$O' = \frac{R^2}{OY} = \left(\frac{OX}{OX} \right) \frac{R^2}{OY} = \frac{R^2}{(OX)(OY)} (OX) = \frac{R^2}{(OA)(OB)} (OX)$$

Where the first equality is the definition of inversion and the last one is use of the Secant Theorem. That is, the constant we needed is $k = R^2 / (OA)(OB)$. The point is that this is a constant; i.e. it only depends on the size of the circle and its distance from O , not the choice of point on the circle.

[Note: Many books take the constant to be $k = R^2 / (OT)^2 = (R/OT)^2$ where T is one of the tangents to the circle δ from the point O , the same number from the tangent case of the secant theorem. The problem is that it makes Case iv b more of a special case than it needs to be.]

Case iv b: This is the same except that the final equality holds by the Chord Theorem instead of by the Secant Theorem. This verifies the result (but with a twist; see PS 5, #10).

Lemma: For two points A and B not both collinear with the center O of circle γ , and A' and B' the inverses of A and B in the circle, $\triangle OA'B' \sim \triangle OBA$; i.e., the triangles are similar but the correspondence (so the other two corresponding angles) is interchanged.



Proof: Since $\angle O$ is in common, by SAS Similarity, we only need to check the ratio of its two sides. Letting R be the radius of the circle:

$$\frac{OA'}{OB} = \frac{R^2/OA}{OB} = \frac{R^2}{(OA)(OB)} = \frac{OB'}{OA} \quad \text{so} \quad \frac{OA'}{OB} = \frac{OB'}{OA}$$

Theorem: Inversion in a circle preserves cross ratio. That is, if A, B, C , and D are four points inverted in a circle, then $(A, B; C, D) = (A', B'; C', D')$

Proof: This important fact is simply an application of the lemma applied to the defining ratio of cross ratio. More explicitly, assuming the circle is centered at O , use the lemma on every defining fraction and simplify. The numerator fraction:

$$A'C' = \left(\frac{R^2}{(OA)(OC)} \right) AC \quad \text{and} \quad C'B' = \left(\frac{R^2}{(OC)(OB)} \right) CB \quad \text{so that} \quad \frac{A'C'}{C'B'} = \left(\frac{OB}{OA} \right) \left(\frac{AC}{CB} \right)$$

The denominator fraction

$$A'D' = \left(\frac{R^2}{(OA)(OD)} \right) AD \quad \text{and} \quad D'B' = \left(\frac{R^2}{(OD)(OB)} \right) DB \quad \text{so that} \quad \frac{A'D'}{D'B'} = \left(\frac{OB}{OA} \right) \left(\frac{AD}{DB} \right)$$

These have the same multiplier (OB/OA) so they cancel and the cross ratio is preserved. QED.

Theorem: Inversion in a circle preserves angles formed by intersections of circles and lines.

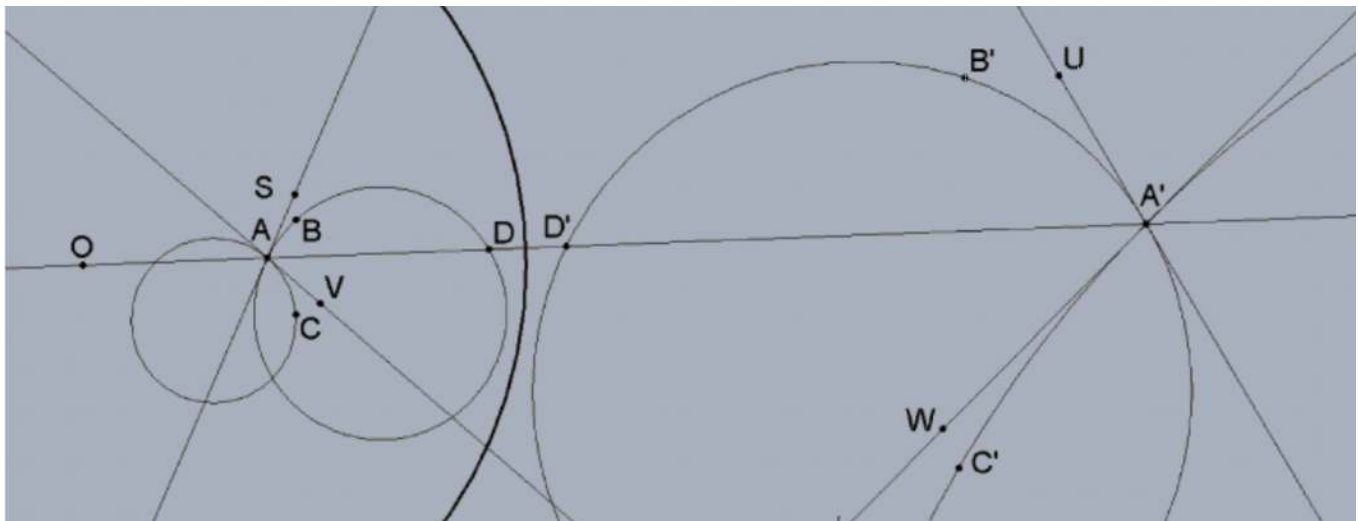
[Note: This "circles and lines" is in the sense of the figures themselves - inversion of a circle may be a line or vice versa.]

Proof: This is a trivial consequence of elementary complex analysis, an analytic function (so any linear fractional transformation) is conformal (so angle preserving) at any point where it has a nonzero derivative. This is not quite inversion in a circle but, with proper choice of coefficients and followed by a reflection in the x -axis (conjugation), it is. This power, of course, is unavailable to us in a strictly Euclidean geometry setting so here is a synthetic geometry proof.

A number of cases must be considered, a conventional angle - the union of two rays (with a common initial point), the arc of a circle and a ray, and the union of arcs of two circles. Furthermore, the situation is different depending on whether the circle of a ray or the line of a ray of an angle contains the center of the circle of inversion - this latter situation is convenient enough to use to force it to simplify the other situations. For example, consider an angle determined by two circles that do not contain the center of the circle of inversion - we need to prove that the angle between their inversion circles is unchanged; i.e., the angle formed by their tangents at the point of intersection is the same after the inversion. Note that these are NEW tangent lines, not inversions of each other - the inversion of a line that does not contain the center of the circle of inversion, is not even a line - it's a circle that does contain the center. For $\angle A = \angle BAC$ for arcs AB and AC . We need to prove that the angles between the tangents to the circles at A and A' are congruent; i.e., we need to prove that conventional angles $\angle SAV \cong \angle UA'W$ where S, U, V , and W are any points along the rays determined by the tangent rays to the circles at A and A' as indicated.



It is sufficient to "split the angle" $\angle A = \angle BAC$ by line $OA = OA'$ (that contains both angle vertices) as follows: Letting D be the other intersection of ray OA with the circle of arc AB , we can consider the original angle as two angles of the same measure sum $\alpha = \alpha_1 + \alpha_2$ where these are the measures of $\angle BAO$ and $\angle DAO$, respectively. The other special case, but handled similarly, is for $\alpha = \alpha_1 - \alpha_2$.

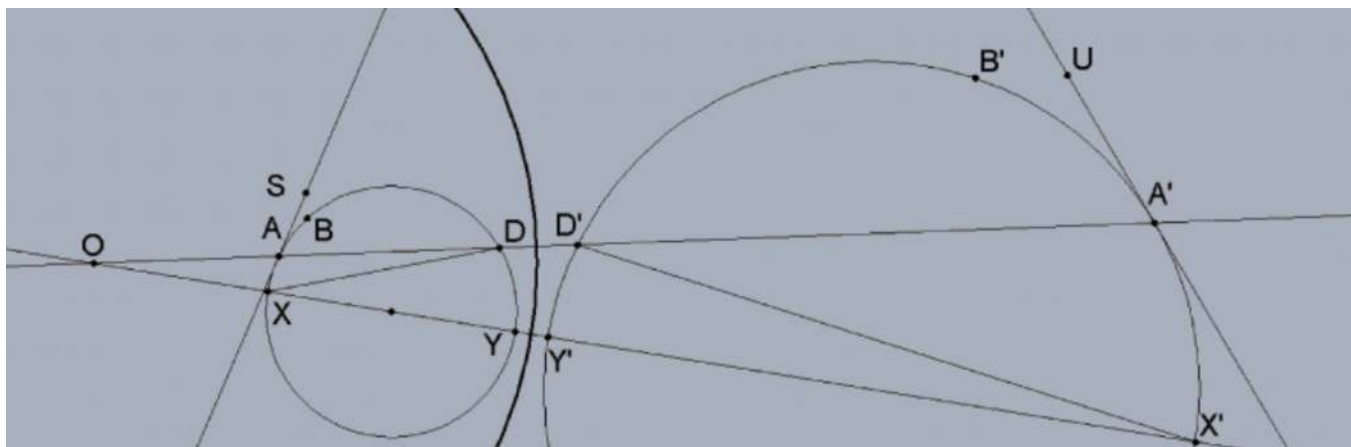


In other words, and without loss of generality, we can assume that one of the rays of the angle at A is on line OA (so we can ignore the portion of the angle below line OA). Note that in this restricted case, the inversion of the angle ray AD is the other angle ray $A'D'$ since the inversion of a (punctured) line that contains the center of the circle of inversion is the same line "turned inside out". This idea of assuming that one side of the angle is on the line determined by the center of inversion and the vertex of the angle (O and A , respectively) greatly reduces the number of cases needed to prove the theorem because each part of the divided angle can be considered separately and we don't have to worry about angles with one side contained in a line through the center of inversion (all these are of that type!):

Case 1: An arc of a circle to an arc of a circle (as pictured below).

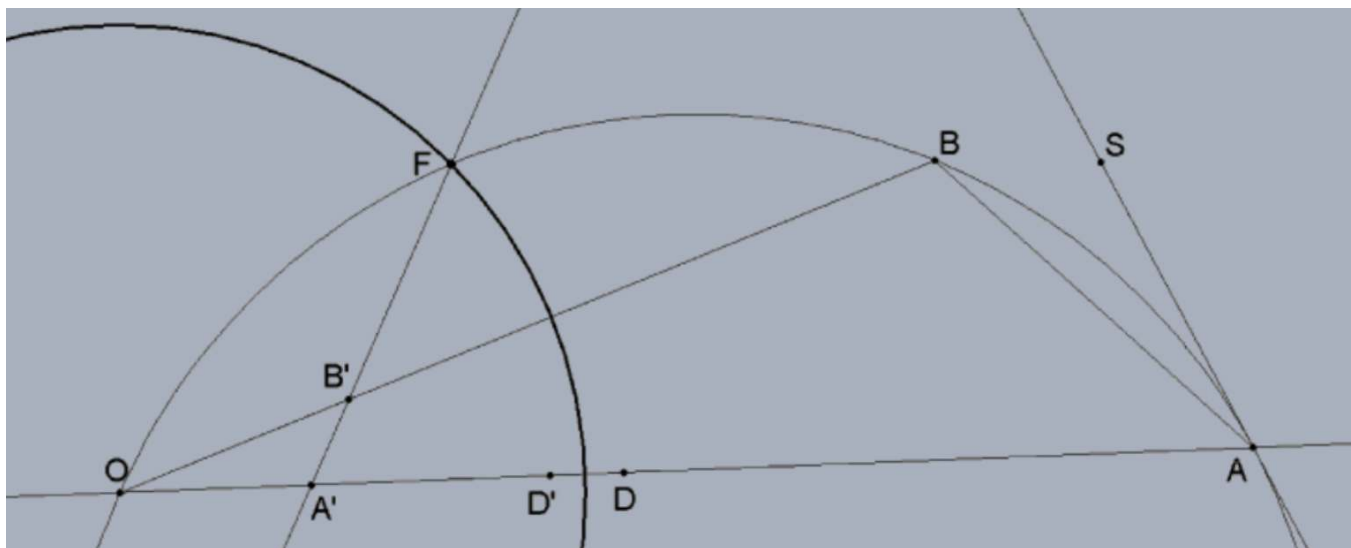
Case 2: An arc of a circle that contains the center of inversion to a ray.

Case 3: A ray within a line that does not contain the center of inversion to an arc of a circle that does.



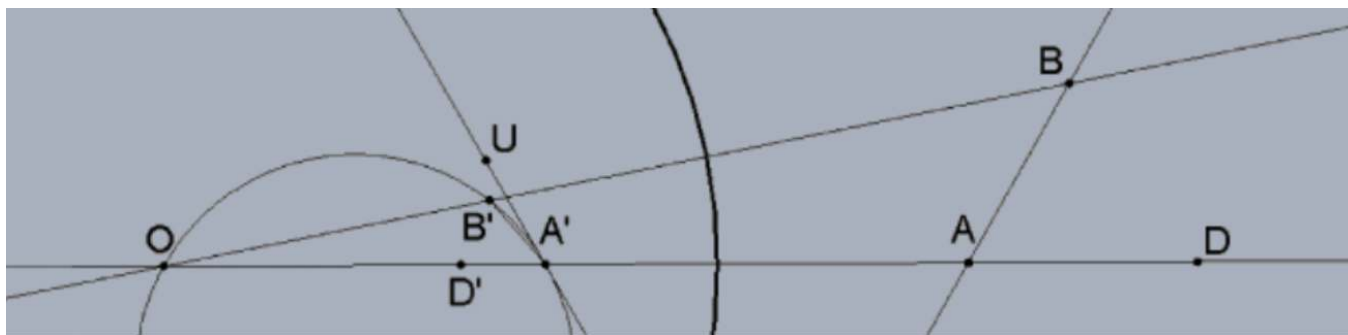
To that end, let X and Y be the diametric intersections of the circle of arc AB determined by the center of inversion O and the center of the circle of arc AB with inversions as indicated X' and Y' . We will identify these circles by these diameters (XY) and $(X'Y')$. By previous theorem, $\triangle ODX \cong \triangle OX'D'$ so that $\angle ODX \cong \angle OX'D'$. By the inscribed angle theorem (and the fact $\angle ODX = \angle ADX$) that $m(\angle ADX) = (1/2)(AX)$ where AX is the subtended arc of circle (XY) and $m(\angle Y'X'D') = (1/2)(Y'D')$. Likewise, $\angle OXD \cong \angle OD'X'$ so, by supplementation, $\angle DXY \cong \angle X'D'A'$ and arc DY of circle (XY) has the same measure as arc $A'X'$ of circle $(X'Y')$. In each circle, the three arcs make a semicircle: $XA + AD + DY$ of circle (XY) and $X'A' + A'D' + D'Y'$ of circle $(X'Y')$ with two of the three arcs being equal in measure. Since together they make semicircles (180 if using degree measure), we conclude that arc AD of circle (XY) has the same measure as arc $A'D'$ of circle $(X'Y')$. By the tangent case of the inscribed angle theorem, $\alpha_1 = m(\angle SAD) = (1/2)(AD) = (1/2)(A'D') = m(\angle U'D')$.

Case 2: An arc of a circle that contains the center of inversion to a ray.



In this case, the inversion of $\angle BAD$ where AB is an arc of a circle that contains O is the genuine angle $\angle B'A'D'$. By previous theorem, $\triangle OBA \cong \triangle OA'B'$ so that $\angle OAB \cong \angle OB'A'$. By the inscribed angle theorem, $m(\angle OAB) = (1/2)(OB)$ where OB is the subtended arc of the the original circle. Moreover, the rest of arc OA , arc BA , is subtended by $\angle O$ and the entire arc gives the measure of the tangent angle $m(\angle SAD) = (1/2)(OA)$. Now consider triangle $\triangle OA'B'$ and its external angle at A' . By the Euclidean form of the Exterior Angle Theorem, $\angle B'A'D'$ is the sum of the two non-adjacent interior angles, the angle at O and $\angle OB'A' \cong \angle A$. That is, the inversion angle at A' agrees with the tangent angle at A as needed.

Case 3: A ray within a line that does not contain the center of inversion to an arc of a circle that does. (This is the reverse of Case 2 with very similar proof.)



In this case, the inversion of actual angle $\angle BAD$ is $\angle B'A'D'$ where $A'B'$ an arc of a circle, the inversion of line AB . By previous theorem, $\triangle OBA \approx \triangle OA'B'$ so that $\angle OBA \cong \angle OA'B'$, by the inscribed angle theorem, $m(\angle OA'B') = (1/2)(\widehat{OB'})$ where $\widehat{OB'}$ is the subtended arc of the inversion circle. Moreover, the rest of arc OA' is arc $B'A'$ that is subtended by $\angle O$ and the entire arc gives the measure of the tangent angle $m(\angle UA'D') = (1/2)(\widehat{OA'})$. Now consider triangle $\triangle OAB$ and its external angle at A . By the Euclidean form of the Exterior Angle Theorem, our original $\angle BAD$ is the sum of the two non-adjacent interior angles, exactly the angle at O and $\angle OA'B' \cong \angle B$. That is, the original angle at A agrees with the tangent angle at A' .

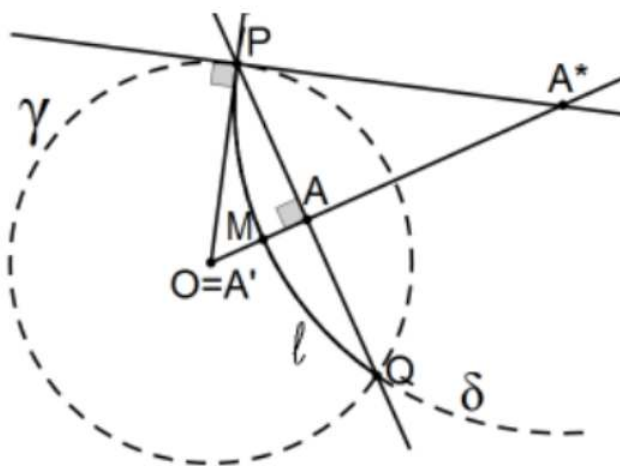
Does the situation change if ray AD is reversed? Slightly, and worthy of comment, but not much. The same argument together with supplementation confirms the same result. QED.

Theorem: Reflection in a Poincaré line (i.e., inversion in its Euclidean circle) is a congruence of the Poincaré plane.

Proof: We've done some hand-waving but the pieces are all in place. That is, angles are preserved and Poincaré distance between two points (i.e., lengths of Poincaré line segments) is preserved (at the cross ratio level). All that remains is seeing why such an inversion carries the defining disk back onto itself. It is obvious when you think about it. Explicitly, each point of the "inside" of the defining circle is inverted to its corresponding point on the "outside" and vice versa including the boundary points of the defining circle to corresponding boundary points and all points between are reversed in a similar manner. QED.

Theorem: Given any point of the Poincaré disk, there is a unique Euclidean constructible Poincaré line (i.e., Euclidean circle) such that reflection in that line carries the point onto the center of the disk.

[Note: This is an especially easy case of constructing the unique Poincaré line such that (Poincaré) reflection in that line carries any given point onto any other given point. General: PS 5, #16.]

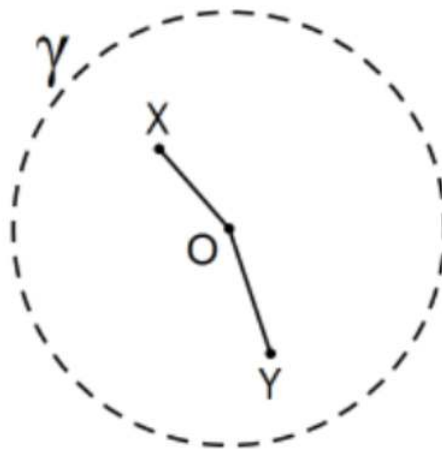


Construction: If this is to be a model for hyperbolic geometry, that there must be such a line is a consequence of neutral geometry (even though we didn't prove it) since reflection in the perpendicular bisector of the line segment joining any two points in neutral geometry is the unique line with that property. However, we don't yet know that this is a model so that fact is not available to us. That fact does, however, help us know how to proceed. Amazingly, if A is the point and O is the center of γ , the defining circle, the standard construction of its inversion in the circle, gives us everything! This time, however, we will identify the inversion of A in this circle by A^* in order to reserve A' for the inversion of A in the new circle yet to be located.

Letting P be the intersection of the perpendicular to the ray OA at A with γ , recall that A^* is determined by the perpendicular at P to the radius OP . Then the circle $(A^*; A^*P) = \delta$ determines the desired Poincaré line, l . Emphasizing this conclusion, the needed circle is centered at the inversion of A in the defining circle with radius already determined in its construction!

Proof: We need to prove that $(A^*A)(A^*O) = (A^*P)^2$ and that this Euclidean circle determines a Poincaré line. Orthogonality is a triviality (by construction!) and, since PA is the altitude from the right angle of $\triangle OAA^*P$, $\triangle OAA^*P \sim \triangle PA^*A$ from which $(A^*O)/(A^*P) = (A^*P)/(A^*A)$ or $(A^*A)(A^*O) = (A^*P)^2$ so that O is the inversion of A in this orthogonal circle centered at A^* ; i.e., it is the reflection of A in that Poincaré line. Why is there no other such circle? Inversion in a circle (reflection in a Poincaré line) preserves cross ratio so that the point of intersection, M in the figure, must be the Poincaré midpoint of segment AO . Since the center of any orthogonal circle that would carry A to O must lie on ray OA and the reflection would have to preserve M (i.e., not move it), this circle would have to be perpendicular to line OA as well as orthogonal to the defining circle forcing it to be the same line; i.e., uniqueness. Note that this l determines the midpoint M of segment AO and is perpendicular to its line so it is the perpendicular bisector of segment AO . QED.

Lemma: One point of the Poincaré disk model is closer to the center of the defining circle than another point using Euclidean measure if and only if it is closer using Poincaré measure.



Proof: Let X and Y be the points with X closer to O than Y using Euclidean measure as indicated. Rotate each segment and its diameter so that the segments are in the same direction along the same diameter. The segment lengths, whether Euclidean or Poincaré, remain unchanged but now they determine the same intersections with the defining circle and the Euclidean measures of segments OX and OY can be viewed as real numbers on the Euclidean number line: with $-R < 0 < x < y < R$.

Comparing their cross ratios:

$$(O, X; R, -R) = (R/(R-x))/(R/(R+x)) = (R+x)/(R-x) \quad \text{and} \quad \text{likewise} \\ (O, Y; R, -R) = (R/(R-y))/(R/(R+y)) = (R+y)/(R-y)$$

Although the numerators and denominators are all positive, comparison of fractions can be tricky: $R+x$ is greater than $R+y$ but $R-x$ is also greater than $R-y$ and a larger number divided by a larger number can be smaller.

Working in reverse from what we seek: $(R+x)/(R-x) < (R+y)/(R-y)$ iff $(R+x)/(R-y) < (R+y)/(R-x)$ iff $Rx - Ry < Ry - Rx$ iff $2Rx < 2Ry$ iff $x < y$. Since $x < y$, these cross ratios satisfy the same inequality as the distances of the rotated points. However, cross ratio was not the point, only their Poincaré length; i.e., $|\ln(O, X; R, -R)|$ and $|\ln(O, Y; R, -R)|$. This follows from the fact that, because of the alignment of the points, we know the values: $R-x < R+x$ so their quotient is greater than 1 so positive logarithm and similarly for Y . Since the natural logarithm function is strictly increasing:

$|\ln(O, X; R, -R)| = \ln(O, X; R, -R) < \ln(O, Y; R, -R) = |\ln(O, Y; R, -R)|$. That is, $d_p(O, X) < d_p(O, Y)$. Instead of the converse, it is easier to use this result to prove the inverse; in essence, what happens if $x = y$ or if $x > y$. In case $x = y$, direct (trivial) computation confirms that $d_p(O, X) = d_p(O, Y)$ and in case $x > y$, we conclude $d_p(O, X) > d_p(O, Y)$ by interchanging the roles of x and y in the preceding. QED.

Theorem: Poincaré line segments are congruent if and only if the reflection of an endpoint of each of them to the center of the defining circle (inversion in orthogonal circles) are congruent Euclidean segments.

Proof: This is an immediate consequence of the preceding Lemma and the fact that inversion in an orthogonal circle preserves cross ratio so that reflection in a Poincaré line preserves Poincaré measure. That is, two Poincaré segments are congruent if and only if their reflected images are congruent Poincaré segments and, since one endpoint of each reflected segment is the defining center, they are also Euclidean segments that are congruent Euclidean segments if and only if they are congruent Poincaré segments. QED.

Theorem: A Poincaré circle is (as a set) a Euclidean circle that stays entirely inside the Poincaré disc and its Poincaré center is collinear with the center of the defining circle and its Euclidean center.

Question: Wasn't this the definition of Poincaré circle back in Chapter 3? Yes, but we "cheated". In neutral geometry (in which hyperbolic geometry is subsumed), the concept "circle" already has a standard definition so this can only be a model for hyperbolic geometry if the original "definition" is consistent with the real definition. We can prove that fact now that more sophistication has been developed.

Proof: Suppose $C \in P$ and $r > 0$. By the (real) definition of circle, the Poincaré circle with center C and radius r is $(C; r)_P = \{X \in P \mid d_P(C, X) = r\}$. To prove that this set is a Euclidean circle, consider two cases:

- i. $C = O$, the center of the defining circle, or
- ii. $C \neq O$.

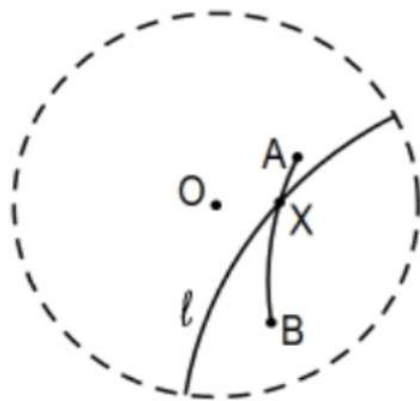
In the first case, Poincaré segments OX for all $X \in P$ are also Euclidean segments and line segments from O are congruent as Poincaré segments if and only if they are congruent as Euclidean segments by the preceding Lemma. This implies that set $(O; r)_P = \{X \in P \mid d_P(O, X) = r\} = \{X \in P \mid d(O, X) = r_E\}$, the Euclidean measure of the segment for some fixed real number r_E . This is exactly the definition of the Euclidean circle $(O; r_E)$ so that the original set of points, $(O; r)_P$, is a Euclidean circle.

In the second case, reflect O in the unique Poincaré line ℓ (inversion in the appropriate orthogonal circle) that carries C to O . Since reflection in ℓ preserves distance, $((C; r)_P)' = \{X' \in P \mid d_P(C', X') = r\}$.

Since the X' of this set contains all of the elements of P , this set is exactly $\{X \in P \mid d_P(O, X) = r\}$ and this set is exactly the definition of the Poincaré circle $(O; r)_P$ that is, by case i, a Euclidean circle. Reflecting in ℓ once again, the inverted set is inverted back onto the original set and, since the inversion of a circle in a circle is a circle or a line with a line being impossible here (the set is contained inside the defining circle so it cannot contain the center of the orthogonal circle of inversion) the original set is a Euclidean circle.

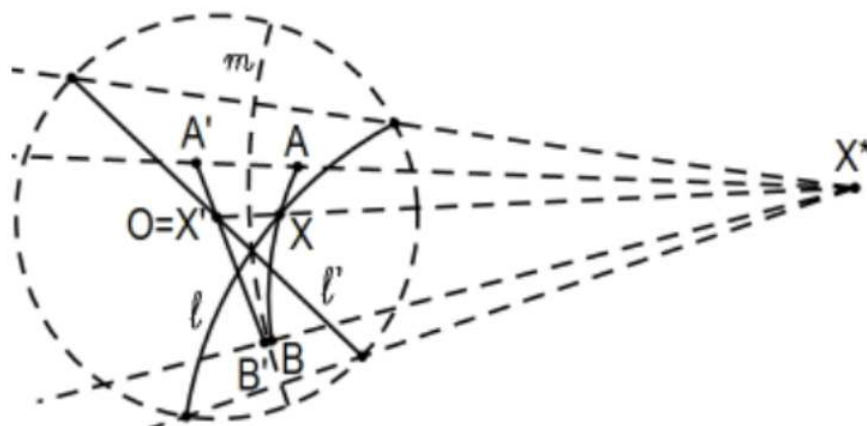
Conversely, suppose δ is a Euclidean circle that is contained in the interior of (that is not centered at O , the center of the defining circle). Construct its Euclidean center C_E by the standard construction and let A and B be the points of intersection of ray OC_E with δ so that segment AB is a diameter of the circle and also a Poincaré segment since it is collinear with O . Choose a 3rd point $C \in \delta$ and construct the perpendicular bisector of the Poincaré segment AC (or B and C) (see PS5, #17), identify the intersection with AB as C_P that is Poincaré equidistant from A and C (or B and C), and let $r = d_P(C_P, C)$. The Euclidean reflection of the plane in the line will $OC_E = OC_P$ is also a Poincaré reflection in the same line (Euclidean open diameter) identifies another point $D \in \delta$ for which C_P is also Poincaré equidistant from A (by symmetry of the situation with respect to diameter line OC_E , see X and Y in the proof of #16). Now reflect ρ in the unique Poincaré line ℓ (inversion in the appropriate orthogonal circle) that carries C_P to O . Since reflection in ℓ preserves distance, O is the same Poincaré distance r from A' , C' , and D' on δ' , the inversion of δ , that must be a Euclidean circle since it is the inversion of a Euclidean circle. Since OA' , OC' , and OD' are Poincaré' congruent segments, they are also Euclidean congruent segments so A' , C' , and D' are 3 distinct points on this Euclidean circle that are equidistant from $O = (C_P)'$ so it is the center of circle δ' . Reflecting in ℓ once again, we have that $\delta = \delta''$ is not only a Euclidean circle, it is the Poincaré circle $(C_P; r)_P$. QED.

Theorem: Poincaré lines are straight.



Proof: This tongue-in-cheek statement is an important theorem and is intended to point out the nebulous nature of "straightness". Our experience is so strong that we think we know until asked to describe what it means to be "straight" in mathematically defensible language. In formal terms, we do not try. Instead, the subsets we call lines have certain definitional properties that are never discussed in introductory courses (sometimes never!) and impose an axiom (Axiom 1) that two points determine exactly one of those sets. The actual definition of is any subset of the set of points (the plane) that separates the set into three mutually disjoint convex subsets, the set itself and two subsets (adjust analogously for "flatness" of a plane in space). This, of course, presupposes a notion of "convex" that itself depends on the notion of betweenness. A set is **convex** if, for any two points in the set, the set of all of the points between them is also in the set. In the language of geometry, if any two points are in the set, so is the line segment that they determine. That a Poincaré line, say l , is itself convex is a consequence of an earlier result, PS3, #24. The idea is that the concept of distance in neutral geometry must satisfy the strict form of the triangle inequality; i.e., for three distinct points A , B , and C , point B is between A and C iff $d_p(A, B) + d_p(B, C) = d_p(A, C)$. That exercise only verified that the condition is satisfied for three *collinear* points in a Poincaré disk model. Any 3 (distinct) points is done below.

For the other two sets determined by a Poincaré line, we prove the contrapositive. That is, if Poincaré line segment AB intersects a given line l where neither A nor B is in l (as pictured above), then the points are on opposite sides of l . To that end, in the following Poincaré disk model, suppose X is the intersection of segment AB and line l . The figure seems to imply the desired result but that



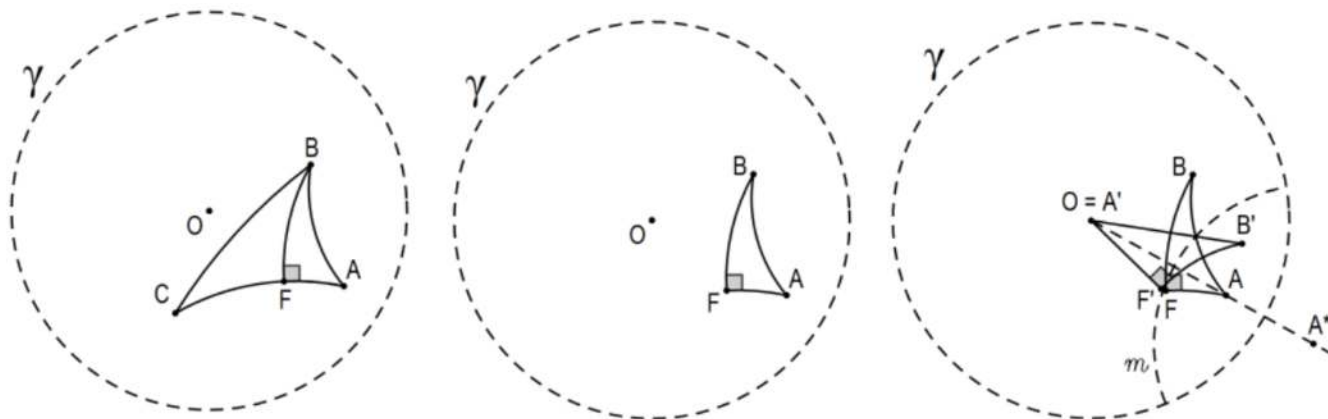
fact is a consequence of the proof; it is not a given. Let m be the Poincaré line in which X is reflected onto O ; i.e., $O = X'$ where this reflection means the inversion in the circle of which m is an arc that is orthogonal to the defining circle. The reflected lines l' and $A'B'$ are Euclidean lines and the Poincaré plane is divided into subsets of the two half-planes of the Euclidean plane determined by the Euclidean line of segment l' . Since $A'B'$ is also a Euclidean segment with point $O = X'$ in the interior of the segment, "straightness" of Euclidean lines (i.e., convexity of both half planes) implies that A' and B' lie on opposite sides of the Euclidean line l' . By properties of inversion in a circle, A and B must lie on opposite sides of l .

There is another technicality that we should not overlook (along with many others that we will!): We need a nonnegative real valued distance function d on a set S to satisfy the triangle inequality property; for any three elements A , B , and C , it must be true that $d_p(A, B) + d_p(B, C) \geq d_p(A, C)$. We need to check this condition for Poincaré distance. We already have this result for

collinear points but we need it to be true in general. It is tempting to claim that, for non-collinear points, this is an immediate consequence of Proposition 20 of neutral geometry but that is rather complicated circular reasoning. The truth is that in assuming measure-based formal geometry (i.e. accepting Ruler and Protractor Postulates), that proposition was only proved herein for historical reasons. The fact is that the generalized inequality had to be true axiomatically so to be a valid model it has to be proved for the Poincaré plane (or any other model of a neutral geometry axiomatic system, hyperbolic or not).

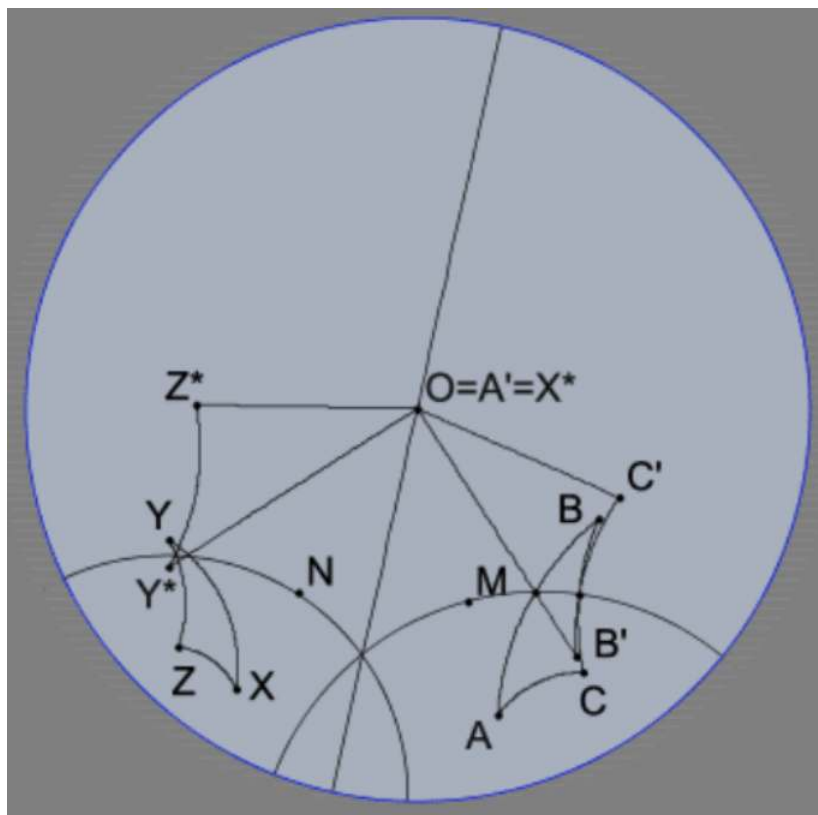
Theorem: For three non-collinear points A, B , and C in a Poincaré disk plane, $d_p(A, B) + d_p(B, C) > d_p(A, C)$

Proof: Consider Poincaré $\triangle ABC$. For the following reason, it suffices to prove only the special case, that the hypotenuse of a right triangle is greater than either of its legs. By the construction of Ex 10, we can construct the perpendicular from B to line AC , say F for the foot of the perpendicular. Assuming the right triangle case has been proved (the second figure), $d_p(A, B) > d_p(A, F)$ and $d_p(B, C) > d_p(F, C)$ so that $d_p(A, B) + d_p(B, C) > d_p(A, F) + d_p(F, C) = d_p(A, C)$ in the case that F falls between A and C and $d_p(A, F) + d_p(F, C) > d_p(A, C)$ if F falls outside Poincaré segment AC



To prove $d_p(A, B) > d_p(A, F)$ (and, by symmetry, $d_p(B, C) > d_p(F, C)$), (Poincaré) reflect A onto O in m (the third figure) the perpendicular bisector of Poincaré line segment AO (i.e., inversion in the appropriate Euclidean circle). This determines Poincaré $\triangle OB'F' \cong \triangle ABF$ so $AB \cong OB$, $AF \cong OF'$, and $\angle OF'B' \cong \angle AFB$ so that $\angle OF'B'$ is a right angle. The desired result follows from PS5, #8.

Theorem: SAS holds for the Poincaré model.



Proof: Suppose $\triangle ABC$ and $\triangle XYZ$ are Poincaré triangles that satisfy SAS as Poincaré triangles. Specifically, suppose that, with respect to Poincaré measure, $AB \cong XY$, $AC \cong XZ$, and that their included angles are congruent, $\angle A \cong \angle X$. Reflecting A to O (A') and X to O (X^*), the reflections in the (Poincaré) perpendicular bisectors of segments AO and XO respectively; i.e., the indicated lines through their (Poincaré) midpoints, M and N . The reflections of the triangles preserves their Poincaré lengths and angle measures so that $\triangle ABC \cong \triangle A'B'C'$ and $\triangle XYZ \cong \triangle X^*Y^*Z^*$ (with $O = A' = X^*$) so these new triangles satisfy the same Poincaré conditions as do the originals; i.e, $OB' \cong OY^*$ and $OC' \cong OZ^*$ (as Poincaré segments) and their included angles at O are also congruent.

By the preceding theorem, however, these transformed segments (since they are radii of the defining circle) are also congruent as Euclidean segments so that the Euclidean triangles they determine are congruent, $\triangle OB'C' \cong \triangle OY^*Z^*$, by the SAS Axiom. Now either a flip (inversion) in a diameter (as pictured) or a rotation around O carries one *Euclidean* triangle $\triangle OB'C'$ onto the other, $\triangle OX^*Y^*$. This last inversion is a Poincaré congruence as well as Euclidean since the Poincaré lengths and angles are preserved for the two sides from O and their included angle leaving only their third (Poincaré) side and other two angles. These must coincide as well because (by Axiom 1) these other two points determine a unique Poincaré line which determines the Poincaré side in question. Similar remarks apply to the angles these arcs determine with the two radial sides. By symmetry and transitivity (i.e., composition), the original $\triangle ABC$ and $\triangle XYZ$ are congruent as Poincaré triangles. QED.

Theorem: The Poincaré model is a model for hyperbolic geometry.

Proof: We would first have to make precise all of the rest of those unspecified axioms and then prove them. No thanks.



M. C. Escher was an artist who expressed his intrigue with unusual perspectives that offers insight into inversions in circles and the Poincaré disk model for hyperbolic geometry. One of his more famous is this *Self Portrait in a Spherical Mirror*. It is a three-dimensional analog of the two-dimensional situation that we have been studying, inversion of the Euclidean plane in a circle. As is always the case with a mirror, reflection in a mirror is not quite the mathematical equivalent because it only an illusion. A perfect mirror does actually reflect each point in 3-space to the point on the opposite side such that the mirror is the plane that is the perpendicular bisector of the segment determined by the point and its image; it just looks that way. Obviously, points on the reverse side of the mirror are not part of the scene at all. In the spherical case of this artwork, points outside of the sphere appear to be inside (although obviously they are not) and points inside are not part of the scene at all so the situation is not a transformation all of the is "punctured" three-dimensional Euclidean space but is still highly suggestive of it. The plane situation we have been studying would be that of each plane through the center of the sphere and the inversion of the region outside of the circle of intersection in that circle. [[www.freakingnews.com/ Escher-Hand-with-Reflecting-Sphere-in-Color-Pics-50737.asp](http://www.freakingnews.com/Escher-Hand-with-Reflecting-Sphere-in-Color-Pics-50737.asp) - Used without permission.]



M.C. Escher's "Circle limit IV".

All M.C. Escher works

Cordon Art - Baarn - Holland

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H.S.M. Coxeter's tessellation of the
Poincaré plane with congruent 30-45-90 triangles
(or isosceles 60 – 45 – 45 triangles)

mathaware.org/mam/03/essay1.html

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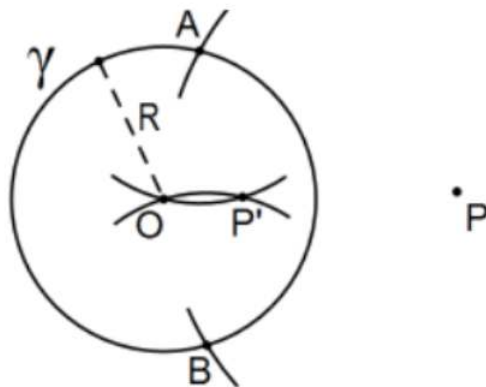
Finally, why "Hyperbolic" Geometry and what is its connection with the Poincaré model? From this cover of a wonderful book on the history of Hyperbolic Geometry, identify corresponding points of an elliptic hyperboloid using the Poincaré model as a guide: Sources of Hyperbolic Geometry, John Stillwell. A co-publication of the AMS and the London Mathematical Society, History of Math, Vol: 10, 1996. Cover picture reproduced with permission of Dr. Konrad Polthier.



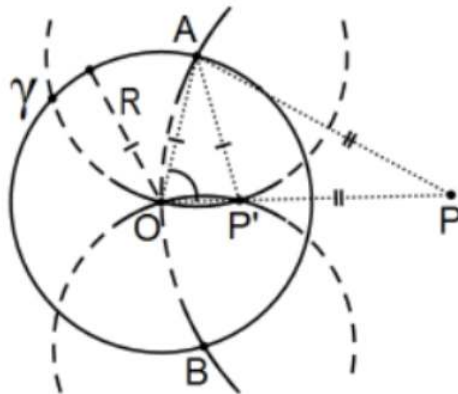
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5.2: Problem Set 5

Unless otherwise stated, assume all of the axioms of neutral geometry (albeit unstated) and the global form of the Euclidean Parallel Postulate as well.



1. Prove that the following alternate construction of the inversion of a given point outside of a given circle in the circle is valid. Don't forget to confirm that P' is on ray OP .



Construction: Given the circle $\gamma = (O; R)$ (i.e., centered at O of radius R) and P outside of the circle, let A and B be determined by circle $(P; OP)$ and let P' be determined by the other intersection of circles $(A; AO)$ and $(B; BO)$. Then P' is the desired inversion of P in the given circle γ .

[Hint: First show that O , P , and P' are collinear (think of segment AB).]

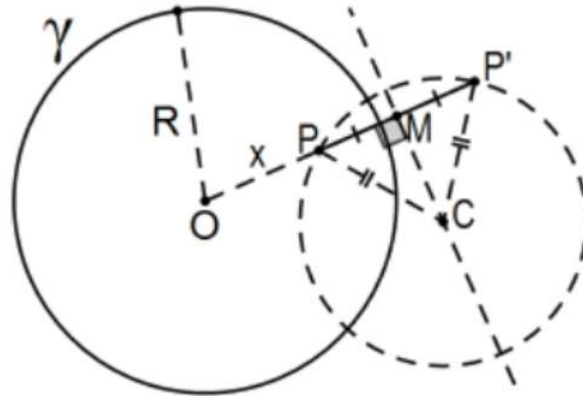
Why is this not a theorem in neutral geometry? That is, all the steps of this construction can be made in neutral geometry so why restrict to Euclidean?

Note: The dotted line segments in the second sketch construction were not part of the construction; they are only there to emphasize their existence part of the proof. The significance of this is that this construction was made using compass only. An amazing theorem of Euclidean geometry (Mohr-Mascheroni) is that every possible straightedge and compass construction can be done without the straightedge! That is, although two points determine a line, the line itself need not be a part of the next stage of the construction using some alternate procedure. To get an idea of what this means (and how tricky things are), consider trying to construct the intersection of two lines, each given by only two points for each line without actually drawing the lines. The solution involves the fact that the inversion of the intersecting lines in an appropriate circle are intersecting circles with one intersection at the center of the circle of inversion and the other being the inversion of the intersection of the original lines that can be constructed because all of these can be done with only a compass. Finally, the inversion of that point of intersection of the circles is the desired intersection.

The process just described requires the ability to do inversions of all points, not just those outside of the circle as with Ex 1. Here is how that can be done: Exactly the same proof works if P is inside the circle but more than halfway "out"; that is, $OP > R/2$. With a slight modification, this works for any $P \neq O$. The idea rests on the fact that there is a trivial circles only construction to double a

line segment and, by extension, any number of times the segment along the same line; i.e., collinear with the original 2 points. The idea is to use the compass at one of the points, construct the circle with that segment as its radius, and then "walk halfway around the circle with the same radius" as with the trivial construction of a regular hexagon. That point, say P_2 , is diametrically opposed to O through P and hence collinear with them doubling of length of segment OP . Repeating the process, gives any number n of equally spaced collinear points. Use of the inversion technique rests on proving that if P_n is the point such that $OP_n = n(OP)$, then inverting P_n to obtain P'_n , then $n(OOP'_n) = P'$, the inversion of the original P . In the algebra, the proof rests on the fact that if $0 < x < R$, then $0 < (R-x)^2$.

2. Almost by definition, if two circles are orthogonal, the center of either is exterior to the other (see Note 2 immediately following the description of Poincaré lines in Chapter 3). By the Fundamental Theorem of Orthogonal Circles, we can say much more. Prove that two intersecting circles are orthogonal if and only if a point P on one of them that is in the interior of the other has its center on the perpendicular bisector of the line segment PP' where P' is the inversion of P in the circle containing P in its interior.



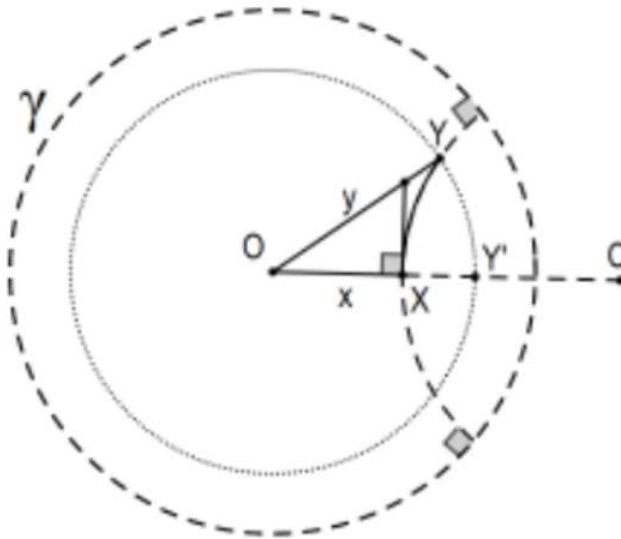
Note: The proof of #2 is very easy but it is interesting to calculate exactly where that perpendicular bisector is in relation to the position of P and the radius of the circle. That is, calculate the position of the midpoint of the line segment as a function of $x = m(OP)$ and R in this figure, the center of the circle (orthogonal) with diameter PP' .

- Let A and B be two points in the interior of circle γ centered at O that are not collinear with O in the Poincaré model determined by Euclidean circle γ . Construct Poincaré line segment AB .
- Starting with the result of #3, construct the isometric inversion (i.e., congruence) of the Poincaré model onto itself that carries A to O ; i.e., $A' = O$ and the Poincaré line segment $A'B' = OB'$, the (Poincaré congruent) image of segment AB thus determined.
- Starting with the result of #4, prove that the Poincaré length of segment AB is simply $\ln\left(\frac{R+x}{R-x}\right)$ where x is the Euclidean length of segment OB' .

Note: Approximate measures can be determined by direct measurement (and this final computation) but in theoretical geometry, direct measurement "doesn't count." All angle, line segment, area, and volume "measures" are not really measures at all; they are infinitely accurate numerical values - an impossibility with direct measurement. Use of this in the x, y -plane (given radius R and given the coordinates of interior points A and B) can yield the theoretically precise value of x as a function of the coordinates of A and B so the theoretically precise value of the Poincaré length of the determined segment AB . This is developed in the Solutions Manual along with a numerical example.

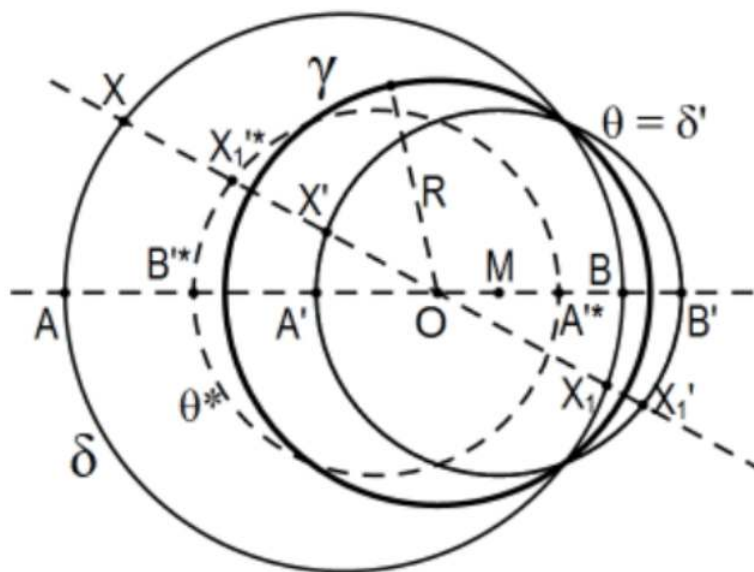
6. Continuing with #4, let $C \neq O$ be a third point in the interior of γ that is not collinear (in the Poincaré sense) with A and B . Sketch in (or carefully construct) Poincaré segments AC and BC to obtain Poincaré triangle $\triangle ABC$ and complete the congruence (isometric = distance preserving and angle preserving) transformation of the Poincaré model onto itself that carries A to O ; i.e., $A' = O$ and carefully sketch $\triangle A'B'C'$ the image of triangle $\triangle ABC$ thus determined. [Note: This is the basic construction in the proof of SAS for the Poincaré disk model.]

- Prove that, in the Cartesian plane, the circle of radius 2 centered at the origin and the circle of radius $3/2$ centered at $(5/2, 0)$ are orthogonal. Know at least three different solutions but the easiest is that $(1)(4) = 2^2$. Why is that fact sufficient?
- Complete the proof of the triangle inequality of Poincaré measure in the non-collinear case (strict inequality) by completing the proof indicated in this figure that the hypotenuse of a Poincaré right triangle with one acute angle vertex at the center of the



Let σ be the Poincaré disk model centered at O where the model of the Euclidean ε is the r, θ plane: $f: \otimes \rightarrow \varepsilon$ via $f(r, \theta) = (d_f(O, (r, \theta)), \theta)$ is a homomorphism; i.e., topologically, exactly the same.

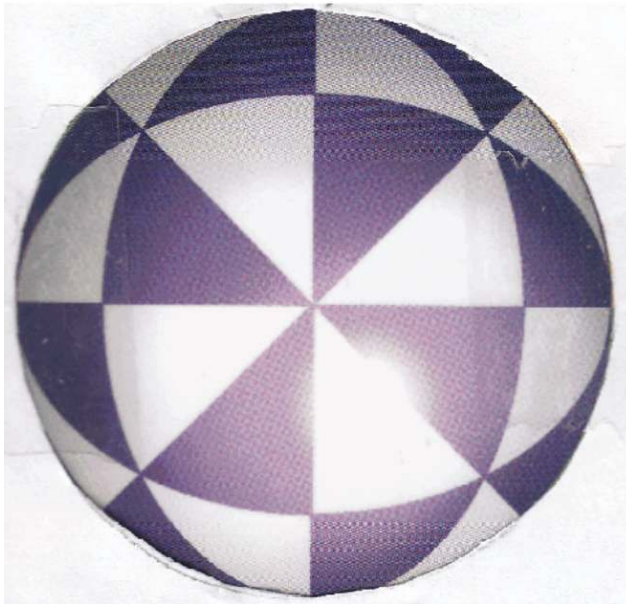
10. Prove Case iv b of the "Lines and Circles Theorem".



The idea is that the equation of Part a is still valid but the resulting inversion is not a dilation in the usual sense but it is close. If one extends dilation to include negative multipliers; i.e., opposite vectors, the proof is exactly the same. For the usual idea of dilation, take the opposite of each, X^{*} , the dashed circle instead of the inversion itself. In fact, using "negative length" when the direction

of a segment along a line is reversed, the fact that O is between A and B implies the change of sign so the negativity of the multiplier $R^2/(OA)(OB)$.

11. Analogous to tessellations of the Poincaré disk with infinitely many congruent triangles, a sphere (our real-world) can easily be tessellated with finitely many ones. As with hyperbolic geometry, there is an AAA congruency theorem in spherical geometry (if triangles are not too big). Probably the easiest is to consider the northern hemisphere (equator to the north pole) and, at the north pole, consider 4 longitudinal lines at 90° intervals $-0^\circ, 90^\circ, 180^\circ$, and 270° . The result is eight $90^\circ - 90^\circ - 90^\circ$ congruent triangles covering the entire sphere. Here are some others. Find the size of the vertex angles of each single-colored triangle. [Hint: Look at all of the vertices that come together to make 360° .]



Geometries, A.B. Sossinsky, AMS Student Mathematical Library No. 64, 2012. From the cover and used without permission.

Euclidean constructions of Poincaré Disk points and lines

It should not come as any great surprise that any constructible geometric figure in neutral geometry (i.e., with abstractions of a straightedge and compass) can be constructed in the Poincaré disk model for hyperbolic geometry using strictly Euclidean constructions. Some of these we have already done and some are given here complete with proof to help get the "feel" of how this works. Finally, there are some with at most hints as how to begin. In the following, assume the Poincaré plane is θ determined by circle γ centered at O . Prerequisite Euclidean constructions are assumed so the details will be omitted unless they seem particularly helpful or needed:

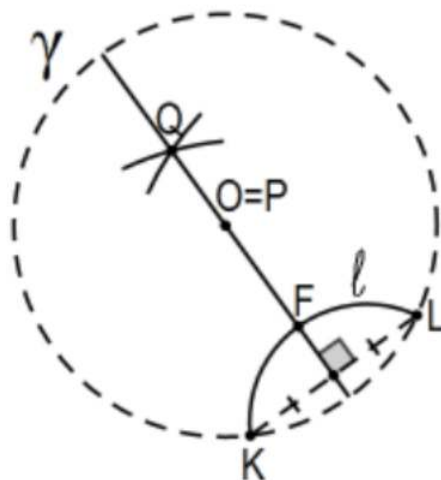
The circle determined by three noncollinear points.

The tangent to a circle at a point on it.

The tangents to a circle from a point outside of it.

The inversion in a circle of a point inside or outside the circle.

12. The Poincaré line perpendicular to a Poincaré line ℓ on a point P (whether or not $P \in \ell$).



Two cases occur:

Let K and L be the intersections of the Euclidean circle of the Poincaré line ℓ with the defining circle γ with center at O .

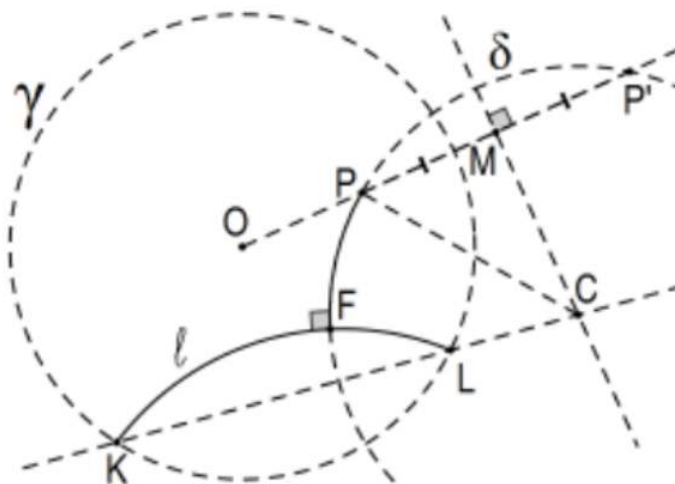
Case 1: P is equidistant from K and L , or

Case 2: P is not equidistant from K and L .

Case 1: P is equidistant from K and L and so is O (radii of the defining circle) so line OP is the perpendicular bisector of segment KL unless $P = O$ (figure) in which case construct Q , another point equidistant from K and L . Then line $PQ = OQ$ is the perpendicular bisector of segment KL . Line $PQ = OQ$ is the desired Poincaré perpendicular to line ℓ .

Proof: Let F be the intersection of this line with ℓ . We need to know that line PQ is the Poincaré perpendicular to Poincaré line ℓ but (by the definition of Poincaré angle measure) that is if and only if it is perpendicular to the Euclidean tangent at point F . However, we get this "for free" because segment KL is also a chord of the Euclidean circle that determines Poincaré line ℓ so that its center, say C , is equidistant from K and L so C also lies on the line OP . Therefore the Euclidean perpendicular at F is perpendicular to the radius FC and, by the standard theorem, this perpendicular line is the Euclidean tangent to that circle.

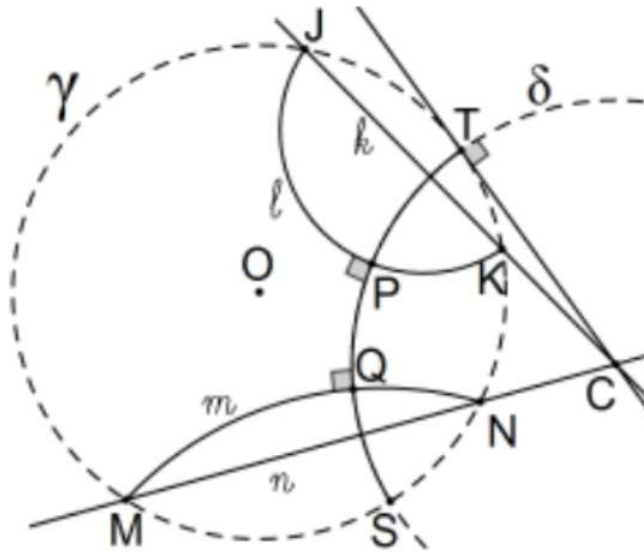
Case 2: Let P' be the inversion of P in γ and let C be determined as the intersection of the perpendicular bisector of segment PP' with line KL since they must intersect. Then the desired Poincaré line is the open arc of the indicated circle $\delta = (C; CP)$. In case $P \in \ell$, the situation is exactly the same and we would conclude that $F = P$.



Proof: Orthogonality of δ with γ is immediate from the Fundamental Theorem of Orthogonal Circles since ray OP also contains P' , the inversion of P in γ . By definition of Poincaré line, the defining circle of Poincaré line ℓ is also orthogonal to γ and ray CL also contains K . Again by the Fundamental Theorem of Orthogonal Circles, the circle of Poincaré line ℓ is also orthogonal to δ . That is,

the Poincaré line on P determined by orthogonal (with γ) circle δ is the desired Poincaré perpendicular to ℓ and the indicated point F is the foot F of the Poincaré perpendicular. Amazingly, we have everything we need almost for free!

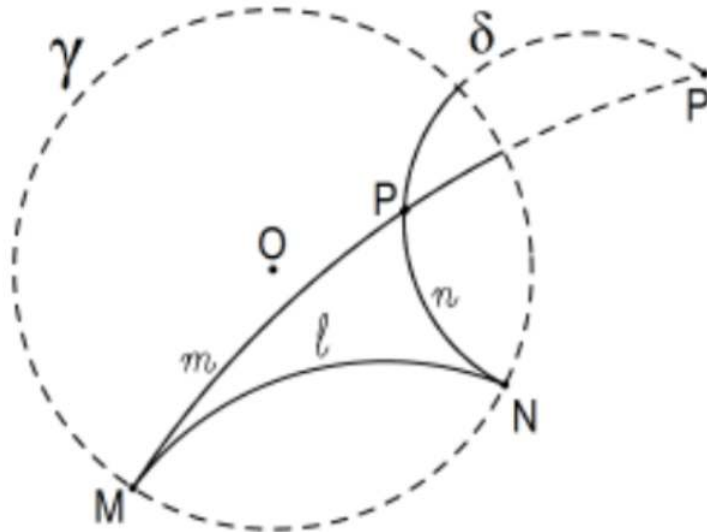
13. The common perpendicular to two non-boundary parallel lines:



Let l and m be Poincaré lines and k and n be the Euclidean lines of the chords of the Euclidean circles (or line) determined by l and m , respectively, with γ . If k and n do not intersect, that is, they are Euclidean parallel, the line is the diameter of common perpendicular bisectors of the chords. If k and n intersect, say C , this is the center of the desired circle and the point of tangency from it to any of the Euclidean circle arcs γ , l , or m , e.g., T , determines the radius so $\delta = (C; CT)$ determines the desired Poincaré line PQ .

Proof: Since Euclidean line CT is tangent to γ , the new circle is orthogonal so that a Poincaré line is determined. Orthogonality with l and m is immediate as in Ex. 12.

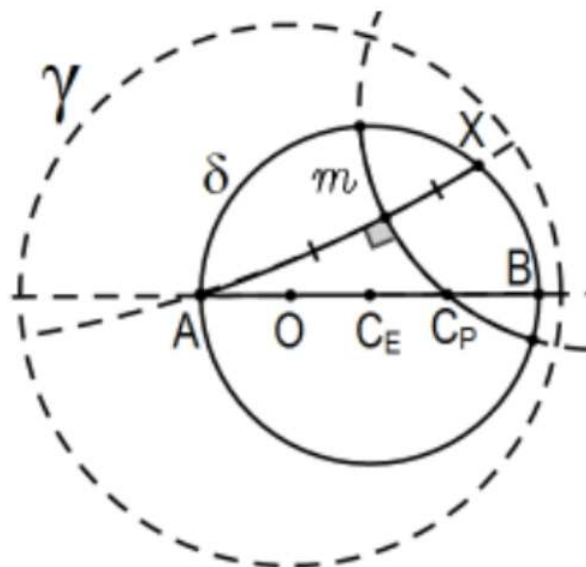
14. The boundary parallels to line l on point p not on l :



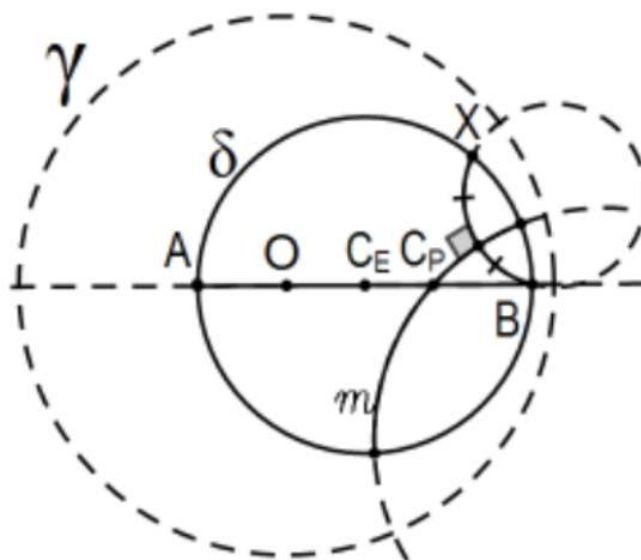
Case 1: If $P = O$, the desired lines are the obvious diameters.

Case 2: $P \neq O$, let P' be the inversion P in γ to obtain P' .

Let M , and N be the points determined by the extension (Euclidean circle or line) of l with γ . The Euclidean circles (i.e., Poincaré lines) are the ones determined by P, P' , and M or N .



Construction: Let O be the center of the defining circle γ and let C_E be the Euclidean center of the Poincaré circle. If $O = C_E$, we are done so assume they determine a line and let A and B be its intersections with the circle in question. Let X be any other point of circle. Construct the Poincaré line determined by A and X and the perpendicular bisector m of the Poincaré segment AX (Ex. 15). The intersection of m and diameter AB , C_P , is the Poincaré center of the circle. The second construction is the same but using BX instead of AX . As described in the proof below, there is a practical advantage when the points A and B are so positioned.

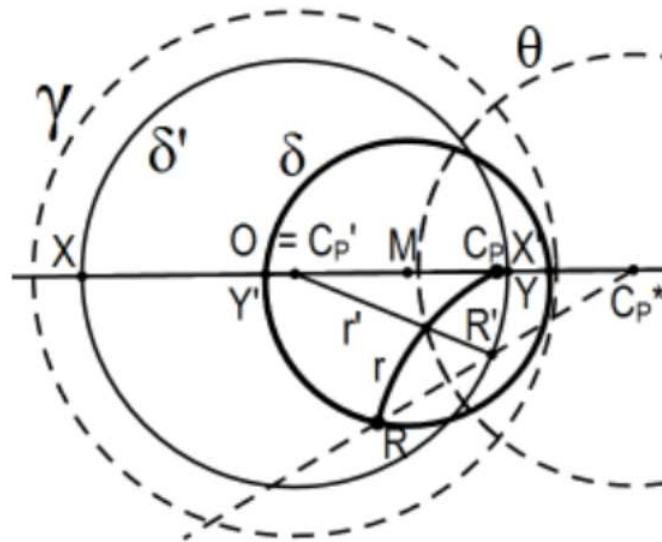


Proof: The fact that the Poincaré center is on the diameter of Euclidean centers will be a consequence of the proof and is based on carefully choosing the 3 points for 2 Poincaré chords of which the Poincaré perpendicular bisectors are to intersect to determine the center by neutral geometry considerations. The idea is to use X and A (or B) to determine one of the chords and Euclidean segment OX (or AX or BX) to swing an arc to establish Y as the Euclidean reflection of X in the line of centers. Entirely by symmetry, the Poincaré perpendicular bisector of Poincaré chord AY (or BY) must intersect at the same point, C_P .

Note 1: The construction of the Poincaré line AX is particularly messy because it is so close to O that the center of the Euclidean circle of the Poincaré line is far off the page so difficult to physically construct with Euclidean tools or even to really see (in a practical sense, not theoretical) the location of its center. Choosing BX instead makes it much clearer.

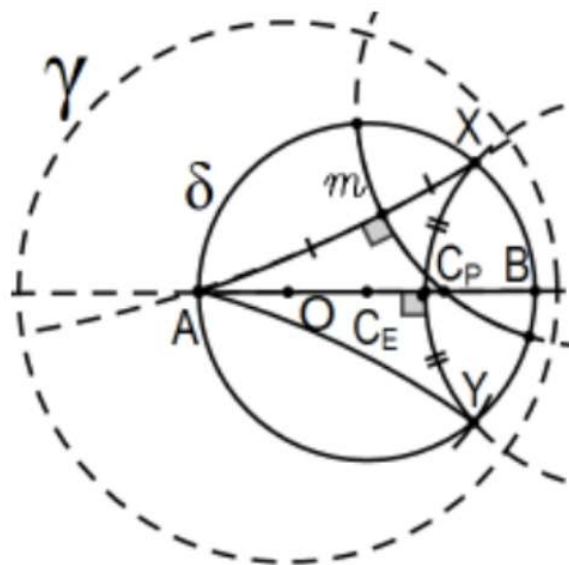
Note 2: This construction can be used to construct the Poincaré midpoint of a line segment AB when A and B are collinear with O . The idea is to construct its Euclidean midpoint and the Euclidean circle with diameter AB . Finally, choose any other point X on the circle, and follow the same construction. Then C_P is the Poincaré midpoint of AB .

18. The Circle given its Poincaré Center and Radius:



Construction: Let O be the center of the defining circle γ and let C_p be the Poincaré center and Poincaré segment $C_pR = r$ be the radius of the Poincaré circle determined $\delta = (C_p; C_pR)$ to be constructed using Euclidean tools, all indicated in bold. First, construct θ , inversion in which carries C_p to O , $O = C_p'$. It also carries R to R' that must lie on δ' , the inversion of δ in θ . construction of $\delta' = (O; OR') = (O; r')$ is a triviality and, letting X and Y be the intersections of δ' , we have a diameter, the inversion of which in θ , X' and Y' , must be a diameter of the desired circle δ . We now have two ways to finish the construction, either use the 3 points of δ knowing that it is a Euclidean circle X' , Y' , and R , or construct the Euclidean midpoint M of segment $X'Y'$ and we have $\delta = (M, M') = (M, M')$

19. Theorem: A set of Poincaré points is a Poincaré circle if and only if it is a Euclidean circle that lies entirely inside the defining circle.



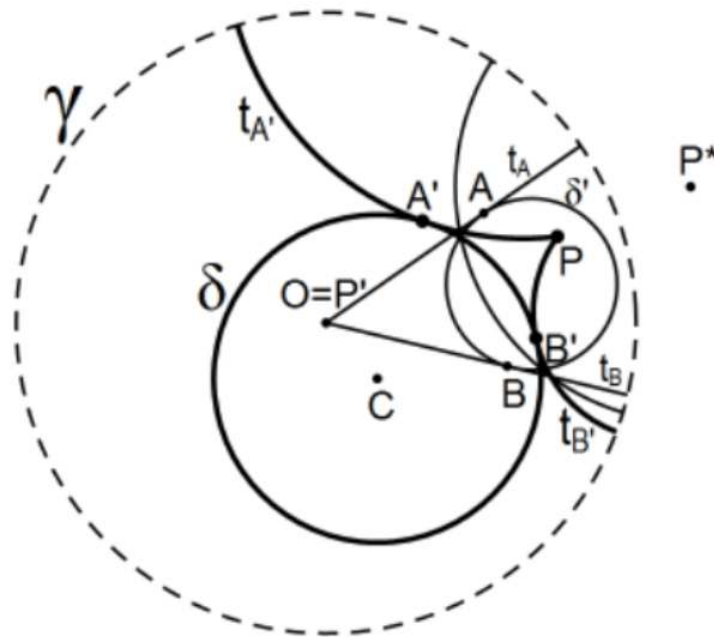
20. The Poincaré Bisector of a Poincaré Angle:

Here are two solutions. Reflect the vertex to the center and follow a point on each ray... Or construct the Euclidean tangents to each Poincaré line at the vertex and bisect the resulting Euclidean angle. Finally, construct the orthogonal circle through that point and tangent to the angle bisector.

Note: Once you get a feel for how this goes, more complicated ones are not so hard but even copying a line segment or an angle is more involved than you might think but are absolutely critical to the proof that the Poincaré Disk is a model for hyperbolic geometry. Here is an example:

21. The Tangents to a Poincaré Circle P outside the Circle:

The most obvious solution is to follow the neutral geometry construction (PS 2, #25 as pictured in PS 3, #25) since each of the needed Euclidean constructions has now been presented. Another approach is the idea of inverting a critical point to O , doing the work, and then inverting back, a common technique that has been used before (e.g., SAS).



The idea is to choose the critical point to map to the center of the defining circle by inversion in an appropriate Poincaré line. Let the circle be the one with Poincaré center C (C is not used in the Euclidean construction - see details below) and P be the point from which the tangents are to be constructed and the appropriate Poincaré line is the one centered at the inversion of P in the defining circle, say P^* (not pictured here) so as not to confuse it with inversions in this new circle) with radius determined by the Euclidean *perpendicular at P to ray OP where O is the center of the defining circle. This circle determines, the arc through M , the Poincaré perpendicular bisector of line segment OP , the line that reflects P to O . Reflecting the circle in this line (that is, inversion in this Euclidean circle), its reflection will be another circle but, this time, the point in question is the center of the defining circle so that its Euclidean tangents are also its Poincaré tangents, points A and B in the figure. Inverting A and B in the same line we obtain the points of tangency, A' and B' . Using P^* , we have three points to construct each of the needed orthogonal Euclidean circles; i.e., each of the circles determined by $P'A'$, and P^* and P, B' , and P^* . These arcs are the desired tangents since they preserve orthogonality with the Poincaré radii at A and B .

Actually doing this without the assistance of Cinderella, even with the assistance of Geometer's Sketchpad, gets kind of messy: As the last exercise in Chapter 4, we start with α , a circle that contains O , the center of the defining circle just to show how the center gets inverted outside as P gets inverted to O . How to do that? Let C be its Euclidean center (instead of its Poincaré center in the picture above) and consider points X and Y (not pictured) determined by the ray P^*C , a Euclidean diameter of the circle. Invert X and Y in the inverting circle to obtain X' and Y' that must be a diameter of the inverted circle. Now the Euclidean midpoint M of segment $X'Y'$ with either point for the radius determines the complete inverted circle α' . The Poincaré tangents from $O = P'$ to this new circle are exactly the same as the Euclidean ones so determine the Euclidean tangent points A and B of the *inverted circle*. These are also the Poincaré tangent points so invert them back to the original circle A' and B' . The Euclidean circles determined by P, A' , and P^* and P, B' , and P^* determine the desired Poincaré tangent lines.

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