YOUR GUIDE TO INTERMEDIATE ALGEBRA



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Your Guide to Intermediate Algebra

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Licensing

*A detailed breakdown of this resource's licensing can be found in **Back Matter/Detailed Licensing**.*

Preface

Welcome to *Your Guide to Intermediate Algebra*! Our goal is to provide a streamlined text tailored to the needs of an intermediate algebra student that is freely available to all who are interested in the subject.

About Your Guide to Intermediate Algebra

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Pedagogical Philosophy and Structure

One of the biggest goals behind this text was to help students increase their confidence in their ability to learn and understand the concepts necessary to be successful in future courses. We note that more often than not, intermediate algebra is not a terminal course for students, so it is critical that not only do they leave the course with a solid foundation in the concepts necessary to be successful, but also that they have the confidence to explore and wrestle with the concepts they will see in future math courses.

With this in mind, each learning objective and concept has multiple examples associated to it that increase in complexity, and the exercises for each section correspond directly to objectives and examples in that section. Moreover, the answers to ALL exercises are contained just below each exercise, so as to encourage students to check their own work after each problem and gauge their own understanding as they practice.

We believe in approaching concepts in multiple ways to help students gain a variety of perspectives and problem solving techniques. Because of this, heavy emphasis is also placed on visual understanding of each concept, with the structure of the text being centered on the graphs of different types of functions that students may encounter.

Learning Objectives

Since the goal of this text is designed to present material tailored to the needs of intermediate algebra students, heavy emphasis is placed on core student learning objectives, or SLO's. Since this text was created with a course offered at a Kansas university in mind, our SLO's are heavily based on those required in the state of Kansas, but should be universal enough to be easily adapted or further remixed for use at other universities.

Our SLO's and corresponding sections are as follows:

1. Demonstrate the ability to perform arithmetic and algebraic manipulation by
    1. Factoring expressions completely using various techniques (Section 1.2)
    2. Performing addition, subtraction, multiplication, and division on rational expressions (Section 1.2)
    3. Simplifying compound fractions (Section 1.2)
    4. Applying the laws of exponents to simplify expressions containing rational exponents (Section 1.6)
    5. Applying the laws of radicals to perform addition, subtraction, and multiplication on expressions involving radicals and rationalizing denominators containing radicals (Section 5.1)
    6. Simplifying radicals containing negative radicands and performing arithmetic operations on complex numbers (Section 4.3)
    7. Evaluating functions using function notation (Section 2.1)
2. Solve equations and inequalities
    1. Solve linear equations in one variable (Section 1.4, Section 3.1)
    2. Solve linear inequalities in one variable showing solutions both on the real number line, in interval notation, and in set-builder notation (Section 1.5)
    3. Solve literal equations (Section 1.4)

4. Solve systems of equations in two variables (Section 5.7)
5. Solve equations by factoring and quadratic formula (Section 4.4; Section 4.5)
6. Solve equations containing rational expressions (Section 5.6)
7. Solve equations involving radicals (Section 5.2)
8. Develop and solve mathematical models such as variation, mixture, motion, work and geometrical applications (Section 5.8)

3. Produce graphs on a coordinate plane by
    1. Graphing linear equations and inequalities (Section 3.2)
    2. Graphing functions, including linear and quadratic (Section 2.4, Section 3.2, Section 4.2)
4. Analyze equations and graphs to
    1. Determine an equation of a line given sufficient information such as point and slope, two points, point and a perpendicular/parallel line (Section 3.2 - Section 3.5)
    2. Calculate the distance between two points (Section 2.2)
    3. Distinguish between functions and relations using the Vertical Line Test (Section 2.3)
    4. Identify the domain and range of a function (Section 2.1, Section 2.3)

Errata

This text was released for use in Fall 2022, and is continually undergoing improvements and updates to better serve students and instructors working with intermediate algebra. There are without doubt errors contained in the text. To suggest corrections or improvements, please reach out to intermediatealgebra010@gmail.com(opens in new window).

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About the Authors

Stanislav A. Trunov

Hello, I am Stanislav Trunov but most people call me Stas. I am currently a 5^{th} year graduate student at Kansas State University. I am currently doing research in Topology. Specifically, my research focuses on categories, like the Lusternik–Schnirelmann category, and their relation to the number of critical points for any smooth function on a manifold. After graduating I hope to get a teaching-oriented job at a community college or at a liberal arts college. I enjoy teaching which is why I offered to coordinate intermediate Algebra for K-State and then make a book to help with teaching it. I'm vegan by the way. In my free time I enjoy listening to debates while playing video games.

Elizabeth J. Hale

Hello! I am a 5^{th} year math PhD student at Kansas State University. My research is primarily in Fourier analysis, specifically working with fractional Leibniz rules for pseudodifferential operators in different function spaces. My other interests lie in machine learning and numerical modeling, and I've had the opportunity to work with NASA the last two summers as a Pathways intern with their Diagnostics and Prognostics Group at the Ames Research Center. In addition, after teaching and tutoring a variety of math and STEM classes for nearly 10 years, I've discovered a growing passion for math education. Teaching intermediate algebra was a lot of fun, so I have been really excited to work on curating this text designed for intermediate algebra students and hope that it will be helpful for any student looking to strengthen their skillset, as well as any instructor hoping to assist them in their journey!

CHAPTER OVERVIEW

1: Foundations

In this chapter, you will review the language of algebra and take your first steps toward working with algebraic concepts.

1.1: Integers

1.1E: Exercises

1.2: Fractions

1.2E: Exercises

1.3: Decimals and Real numbers

1.3E: Exercises

1.4: Using Variables to Solve Equations

1.4E: Exercises

1.5: Solve Linear Inequalities

1.5E: Exercises

1.6: Exponents and Radicals

1.6E: Exercises

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1.1: Integers

summary

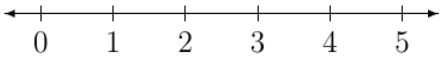
By the end of this section, you will be able to:

- Add and subtract integers
- Multiply and divide integers
- Simplify expressions with absolute value

Natural Numbers and Integers

Let's begin by reviewing some numbers. We are all familiar with the counting numbers, or as mathematicians say, natural numbers $\mathbb{N}$. These are $0, 1, 2, \ldots, 51, \ldots, 1000000, \ldots$

We can see these numbers on what is called a number line. Here is the number line with the natural numbers going from 0 to 5. The numbers are ascending going from left to right and descending from right to left. Larger numbers are to the right and smaller numbers are to the left.



We can add them together to get another natural number.

Example 1.1.1

Compute the following:

a. 1+2=
b. 3+5=
c. 43 + 27=
d. 900+716=

Answer

a. 1+2=3
b. 3+5=8
c. 43 + 27 = 70
d. 900+716=1616

We say the result of addition is the sum For example, the sum of 1 and 2 is 3. (The sum is 3, not 1 or 2).

We can also come up with numbers that add up to a certain number.

Example 1.1.2

Come up with two natural numbers whose sum is 7.

Possible answers:

6 and 1

5 and 2

4 and 3

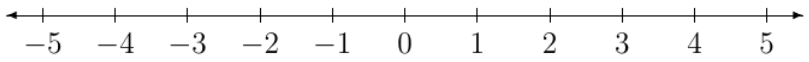
7 and 0

Any other combinations would be a repeat of those above.

What if we want to subtract? Sometimes we run into a new kind of number. For example 5-7=-2. This is a negative number and they were invented to keep track of debts. For example if you have 5 dollars and you need to pay me 7 dollars then you have -2 dollars, that is, you are 2 dollars in debt to me, the book.
You can actually think of subtraction as adding a negative, e.g, 5+(-7)=-2. So you really never lose money, you just get a negative money!
We can expand our view of numbers to include negative numbers to define the integers, $\mathbb{Z}$, by $\ldots, -1000000, \ldots, -51, \ldots, -2, -1, 0, 1, 2, \ldots, 51 \ldots, 1000000, \ldots$ That is, the integers are the negative AND positive versions of the natural numbers combined.

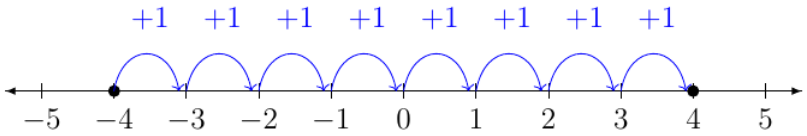
We also say that the result of subtraction is the difference. For example, the difference between 5 and 2 is 3. (The difference is 3, not 5 or 2).

Let's expand the number line to include the negatives. Again, larger numbers are to the right and smaller numbers are to the left.



Addition and subtraction may be easier to see with a number line. On a number line we go to the right for adding a positive number and to when adding a negative (subtraction).

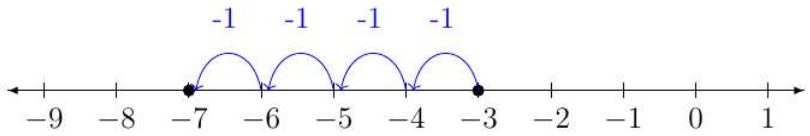
Consider -4+8=4. On a number line, this looks like



We start at -4 and then move 8 numbers to the right (because we are adding 8), landing at 4.

We can do the same for subtraction.

For example, let's do -3-4=-7



Now we start at -3 and move 4 numbers to the left (because we are subtracting 4), landing at -7.

With this in mind, let's try some examples.

? Example 1.1.3

Compute the following:

1. 5-2=
2. 7-5=
3. 5-10=
4. -10+5=
5. -7+4=

Answer

1. 5-2=3
2. 7-5=2
3. 5-10=-5
4. -10+5=-5
5. -7+4=-3

We see that there is a relationship between addition and subtraction in the sense that one "undoes" the other. For example

$$\begin{aligned} 1+2&=3 \\ 1&=3-2 \end{aligned} \tag{1.1.1}$$

We see that the action of adding 2 to 1 to get 3 can be "undone" by subtracting 2 from 3.

Multiply and Divide Integers

Often in math and life, we would like to add quickly. Say you had 4 boxes with 3 phones in each box and I ask "how many phones do you have"? You could use addition with 3+3+3+3=12 or you could use multiplication with $3 \cdot 4 = 12$. So one could think of multiplication as adding the same number to itself in a succinct way. Multiplication is very useful when numbers get too inconvenient to add. For example, let's say you have 100 boxes now. It would be difficult to add up all those 3's but with multiplication we know that $3 \cdot 100 = 300$. We call the result of multiplication, the "product". For example, the product of 3 and 100 is 300. (300 is the product, not 3 or 100).

? Example 1.1.4

1. $1 \cdot 2$
2. $5 \cdot 6$
3. $3 \cdot 9$
4. $12 \cdot 54$
5. Come up with two integers whose product is 6.

Answer

1. 2
2. 30
3. 27
4. 648
5. An answer is $2 \cdot 3$

What if we want to do multiplication with negative numbers? An intuitive way of understanding multiplication with negative numbers is to consider debts again. Say you owe 3 dollars to 4 people, how much are you in debt? The answer is 12 dollars in debt. In math terms, this is $4 \cdot (-3) = -12$. So a positive number multiplied by a negative number is a negative number. (A negative number multiplied by a positive number is a negative number).

What about a negative number multiplied by a negative number? To understand this, let's consider an "I owe you" note. This note is a physical interpretation of debt or "negative money". If Tom has a note that reads "Tom owes (your name) 3 dollars" this means that Tom is in debt to you by the amount of 3 dollars. Ok, now let's say that you needs to give Tom 4 copies of the note above, we can say that you **owe** Tom those notes. How much money would you have if everyone paid off their debt? Well, each note reads "Tom owes (your name) 3 dollars" and you gives Tom 4 of those notes, then you will have 12 dollars once the debt is paid off. In math terms we have -4 (notes that you owes Tom) $\cdot -3$ (The amount of negative money each note carries)=12. That is to say "if you owe someone negative money then you are owed positive money". So a negative times a negative is a positive!

This also answers the question about why 4-(-3)=7. Since you can think of the subtraction of negative 3 as giving one of the notes above.

Division is a way of...dividing up a number into another number. To get an intuitive understanding let's consider some pizza. Let's say that your friends eat 2 slices of pizza each and you have a pizza with 8 slices. How many of your friends can you feed with that pizza? This question is asking to divide 8 (slices of pizza) by 2 (the amount of slices each friend will eat). So $8 \div 2 = 4$, you will be able to feed 4 friends with 1 pizza. We call the result of division, the "quotient". For example, the quotient of 8 by 2 is 4. (4 is the quotient, not 8 or 2). We see the same intuition with negative numbers works for division as well. That is, if you need to divide a debt of 6 dollars between 3 people evenly then each person would owe you 2 dollars: $\frac{-6}{-3} = 2$. Same intuition holds if you need to pay off a 6 dollar debt between 3 people evenly: $\frac{-6}{3} = -2$.

Example 1.1.5

1. $2 \div 2 =$
2. $30 \div 6 =$
3. $-27 \div 9 =$
4. $-16 \div -8 =$
5. $648 \div 54 =$

Answer

1. 1
2. 5
3. -3
4. 2
5. 12

To recap

For multiplication and division of two signed numbers:

Rules for multiplication and division of two numbers

Same signs	Result
• Two positives	Positive
• Two negatives	Positive

If the signs are the same, the result is positive.

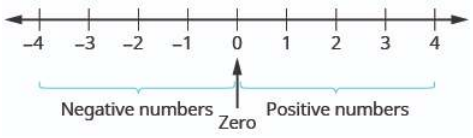
Rules for multiplication and division of two numbers

Different signs	Result
• Positive and negative	Negative
• Negative and positive	Negative

If the signs are different, the result is negative.

Sometimes we encounter a division that we can't get an integer out of. Say you have one slice of pizza and eat friend eats 2 slices. How many friends can you feed? You can feed half of a friend. In math this would look like this $\frac{1}{2}$. These are called fractions and we will be talking about them in the next section but first we should talk about when we do get an integer from division.

Simplify Expressions with Absolute Value



You may have noticed that, on the number line, the negative numbers are a mirror image of the positive numbers, with zero in the middle. Because the numbers 2 and -2 are the same distance from zero, each one is called the **opposite** of the other. The opposite of 2 is -2, and the opposite of -2 is 2.

Definition: OPPOSITE

The **opposite** of a number is the number that is the same distance from zero on the number line but on the opposite side of zero.

This figure 1.1.2 illustrates the definition.

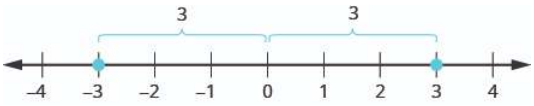


***Figure* 1.1.2.** *The opposite of 3 is −3.*

Definition: OPPOSITE NOTATION

$$-a \text{ means the opposite of the number } a \tag{1.1.2}$$
$$\text{The notation } -a \text{ is read as "the opposite of } a\text{."} \tag{1.1.3}$$

We saw that numbers such as 3 and −3 are opposites because they are the same distance from 0 on the number line. They are both three units from 0. The distance between 0 and any number on the number line is called the **absolute value** of that number.

Definition: ABSOLUTE VALUE

The **absolute value** of a number is its distance from 0 on the number line.

The absolute value of a number n is written as $|n|$ and $|n| \geq 0$ for all numbers.

Absolute values are always greater than or equal to zero.

For example,

$$-5 \text{ is 5 units away from 0, so } |-5| = 5. \tag{1.1.4}$$
$$5 \text{ is 5 units away from 0, so } |5| = 5. \tag{1.1.5}$$

Figure 1.1.3 illustrates this idea.

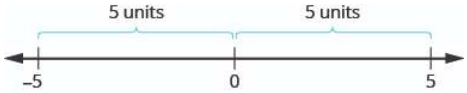


Figure 1.1.3: The numbers 5 and −5 are 5 units away from 0.

The absolute value of a number is never negative because distance cannot be negative. The only number with absolute value equal to zero is the number zero itself because the distance from 0 to 0 on the number line is zero units.

In the next example, we'll order expressions with absolute values.

? EXAMPLE 1.1.6

Fill in $<$, $>$, or $=$ for each of the following pairs of numbers:

a. $|-5|\_\_-|-5|$
b. $8\_\_-|-8|$
c. $-9\_\_-|-9|$
d. $-(-16)\_\_|-16|$.

Answer

a. First, simplify each side:

$$|-5| = 5 \text{ and } -|-5| = -5$$

Then choose an inequality that makes the simplified statement true:

$$5 > -5$$

This is the inequality that makes the original statement true!

$$|-5| > -|-5|$$

b. First, simplify each side:

The left side is already simplified. The right side simplifies as $-|-8| = -8$

Then choose an inequality that makes the simplified statement true:

$$8 > -8$$

This is the inequality that makes the original statement true!

$$8 > -|-8|$$

c. First, simplify each side:

The left side is already simplified. The right side simplifies as $-|-9| = -9$

Then choose an inequality (or equality!) that makes the simplified statement true:

$$-9 = -9$$

This is the inequality or equality that makes the original statement true!

$$-9 = -|-9|$$

d. First, simplify each side:

$$-(-16) = 16 \text{ and } |-16| = 16$$

Then choose an inequality (or equality!) that makes the simplified statement true:

$$16 = 16$$

This is the inequality or equality that makes the original statement true!

$$-(-16) > |-16|$$

? EXAMPLE 1.1.7

Fill in $<$, $>$, or $=$ for each of the following pairs of numbers:

a. $-9 \_\_ -|-9|$
b. $2 \_\_ -|-2|$
c. $-8 \_\_ |-8|$
d. $-(-9) \_\_ |-9|$

Answer

a. $>$
b. $>$
c. $<$
d. $=$

Order of Addition and Multiplication

If we want to compute $2+3\cdot 2$ is this 10 or 8? It's ambiguous which operation we want to do first so mathematicians have some rules of which operations we do first. In order we do multiplication, division, addition, and finally subtraction. We will have exceptions like if the addition is inside parenthesis, like $(2+3)\cdot 2$, in this case, it would be 10. Expressions in parenthesis (or any grouping symbols) take priority.

We now add absolute value bars to our list of grouping symbols. When we use the order of operations, first we simplify inside the absolute value bars as much as possible, then we take the absolute value of the resulting number.

Definition: GROUPING SYMBOLS

Parentheses	()	Braces	{}
Brackets	[]	Absolute value	||

(1.1.6)

In the next example, we simplify the expressions inside absolute value bars first just like we do with parentheses.

EXAMPLE 1.1.8

Simplify: $24-|19-3(6-2)|$.

Answer

Work inside parentheses first -- subtract 2 from 6:	$24-\|19-3(4)\|$
Simplify the multiplication inside the absolute value next:	$24-\|19-12\|$
Simplify the inside of the absolute value bars -- subtract 12 from 19:	$24-\|7\|$
Take the absolute value:	$24-7$
Subtract:	17

EXAMPLE 1.1.9

Simplify: $19-|11-4(3-1)|$.

Answer

16

Work inside parentheses first -- subtract 1 from 3:	$19-\|11-4(2)\|$
Simplify the multiplication inside the absolute value next:	$19-\|11-8\|$
Simplify the inside of the absolute value bars -- subtract 8 from 11:	$19-\|3\|$
Take the absolute value:	$19-3$
Subtract:	16

EXAMPLE 1.1.10

Simplify: $9-|8-4(7-5)|$.

Answer

9

Key Concepts

- Numberlines help visualize addition and subtraction.
- The absolute value of a number is its distance from 0 on the number line.

  The absolute value of a number *n* is written as $|n|$ and $|n| \geq 0$ for all numbers.

  Absolute values are always greater than or equal to zero.

- $$\begin{array}{llll} \text{Parentheses} & () & \text{Braces} & \{\} \\ \text{Brackets} & [] & \text{Absolute value} & || \end{array} \tag{1.1.7}$$

- **Subtraction Property**
  $a - b = a + (-b)$
  Subtracting a number is the same as adding its negative.
- For multiplication and division of two signed numbers:

  Multiplication and division for two same signed numbers

Same signs	Result
• Two positives	Positive
• Two negatives	Positive

  If the signs are the same, the result is positive.

  Multiplication and division for two different signed numbers

Different signs	Result
• Positive and negative	Negative
• Negative and positive	Negative

  If the signs are different, the result is negative.
- **Multiplication by** -1

  $-1a = -a$

  Multiplying a number by -1 gives its opposite.

Glossary

absolute value

The absolute value of a number is its distance from 0 on the number line.

integers

The whole numbers and their opposites are called the integers.

negative numbers

Numbers less than 0 are negative numbers.

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1.1E: Exercises

Practice Makes Perfect

Add and Subtract Integers In the following exercises, simplify each expression.

Add and Subtract

Compute the following

1. $1+4$
2. $3+9$
3. $9-6$
4. $8-3$
5. $9-14$
6. $1-8$
7. $-17-42$
8. $-58-(-67)$
9. $64+(-17)-9$
10. $48+(-16)$
11. $34+(-19)$
12. $-14+(-12)+4$
13. $-17+(-18)+6$
14. $19+2(-3+8)$
15. $24+3(-5+9)$
16. $(2-7)-(3-8)$
17. $32-[5-(15-20)]$

Answer

1. 5
2. 12
3. 3
4. 5
5. -5
6. -7
7. -59
8. 9
9. 38
10. 32
11. 15
12. -22
13. -29
14. 29
15. 36
16. 0
17. 22

Multiply and Divide

multiply or divide:

18. $-4\cdot 8$
19. $13(-5)$
20. $-24\div 6$

21. $-52 \div (-4)$
22. $-3 \cdot 9$
23. $9(-7)$
24. $35 \div (-7)$
25. $-84 \div (-6)$
26. $-3(-5)(6)$
27. $-4(-6)(3)$
28. $(8-11)(9-12)$
29. $(6-11)(8-13)$
30. $26-3(2-7)$
31. $23-2(4-6)$
32. $65 \div (-5)+(-28) \div (-7)$
33. $52 \div (-4)+(-32) \div (-8)$
34. $9-2[3-8(-2)]$
35. $11-3[7-4(-2)]$

Answer

18. -32
19. -65
20. -4
21. 13
22. -27
23. -63
24. -5
25. 14
26. 90
27. 72
28. 9
29. 25
30. 41
31. 27
32. -9
33. -9
34. -29
35. -34

Inequalities with Absolute Values

For each of the expressions below, identify the appropriate inequality or equality that makes the statement true.

1. $-8\_\_-|8|$
2. $13\_\_-13$
3. $-(-1)\_\_|-1|$
4. $-(-10)\_\_-|-10|$
5. $-2\_\_|-2|$
6. $0\_\_|0|$
7. $-9\_\_|-9|$

Answer

1. =
2. >

3. =
4. >
5. <
6. =
7. <

Order of Operations with Parentheses and Absolute Values

Simplify the following expressions:

1. $2-|10+2(6-5)|$
2. $-2|7+5(-3+5)|+3(4)$
3. $4+4|1+8(-5+4)|$
4. $3|4-12|-2(4+7)$
5. $-|1+3(6-7)+(9-2)|$
6. $-2|3+(6-7)+(9-10)|$
7. $3|1+4(-2+1)|-3(5)$
8. $-|3-4|+|4-5|-|5-6|$
9. $|3-6|-|6-9|+|9-12|$
10. $(3-4)|4+9|$
11. $(7-2)|-11+3|$

Answer

1. -10
2. -22
3. 32
4. 2
5. -5
6. -2
7. -6
8. -1
9. 3
10. -13
11. 40

Writing Exercises

36. Explain why the sum of −8 and 2 is negative, but the sum of 8 and −2 is positive.
37. Give an example from your life experience of adding two negative numbers.

Answer

Answers will vary

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1.2: Fractions

Learning Objectives

By the end of this section, you will be able to:

- Simplify fractions
- Multiply and divide fractions
- Add and subtract fractions
- Use the order of operations to simplify fractions

Warm-up 1.2.1

Compute the following

a. $16 \div 2$
b. $-16 \div -8$
c. $6 \div -6$

Answer

a. 8
b. 2
c. -1

What is a Fraction?

Now try $1 \div 2$. Is this an integer? No, it's just $1 \div 2$

Conceptually this is asking how many friends you can feed with 1 slice of pizza if each friend eats 2 slices. Well, you can feed half of a friend, that is $\frac{1}{2}$. If we expand our numbers once again we can accommodate these new numbers we will call fractions. Fractions represent smaller parts of whole numbers. If we now allow fractions, we get the rationals, $\mathbb{Q}$. Rationals are any number that can be written as an integer over an integer.

FRACTION

A **fraction** is written $\frac{a}{b}$, where $b \neq 0$ and

a is the **numerator** and b is the **denominator**.

A fraction represents parts of a whole. The denominator b is the number of equal parts the whole has been divided into, and the numerator a indicates how many parts are included.

Let's create a fraction for example, take a random integer.

7? Good choice! Take another integer.

3? Another good choice!Taking 7 over 3 we get $\frac{7}{3}$. Really this is expressing the number we get when we divide 7 by 3.

Where does the negative sign go in a fraction? Usually the negative sign is in front of the fraction, but you will sometimes see a fraction with a negative numerator, or sometimes with a negative denominator. Remember that fractions represent division. When the numerator and denominator have different signs, the quotient is negative.

$$\frac{-1}{3} = -\frac{1}{3} \qquad \frac{\text{negative}}{\text{positive}} = \text{negative} \tag{1.2.1}$$

$$\frac{1}{-3} = -\frac{1}{3} \qquad \frac{\text{positive}}{\text{negative}} = \text{negative} \tag{1.2.2}$$

PLACEMENT OF NEGATIVE SIGN IN A FRACTION

For any positive numbers a and b,

$$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b} \tag{1.2.3}$$

Example 1.2.1

If we have -7 or -3 but not both then we would get $-\frac{7}{3}$ This is the same as $\frac{-7}{3}$ or $\frac{7}{-3}$.

$\frac{-7}{3} = \frac{7}{-3} = -\frac{7}{3}$

Any integer can be turned into a fraction with 1 as the denominator e.g $3 = \frac{3}{1}$.

Let's take a look at when we will get a nice integer from division.

Factor Trees

Definition: Divisible

A number x is "divisible" by another number y if $x \div y$ is an integer

Example 1.2.2

9 is divisible by 3 since $9 \div 3 = 3$
9 is **not** divisible by 2 since $9 \div 2$ is not an integer.

Definition: Prime

A "prime number" is an integer that is only divisible by itself and 1 as well as negative itself and -1.

Example 1.2.3

7 is prime since it is not divisible by any integer except for ± 7 and ± 1

9 is **not** prime since 9 is divisible by 3 which is not ± 9 or ± 1.

We can "decompose" numbers into the prime numbers that were multiplied together to create them. This is called **factoring.**

Example 1.2.4

Factor

a. 24
b. 30
c. 90

Solution

a. $24 = 2 \cdot 2 \cdot 2 \cdot 3$
b. $30 = 2 \cdot 3 \cdot 5$
c. $90 = 2 \cdot 3 \cdot 3 \cdot 5$

A popular trick to use for factoring numbers is called a **factor tree**. Let's take a look at a simple example on how to do this. Let's decompose 6 into its prime factors

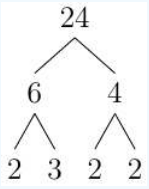


This involves dividing numbers until you get numbers that are prime. Here we note 6 is divisible by 3 and $6 \div 3 = 2$ so we broke 6 up into $3 \cdot 2$. We can't break this up anymore since 3 and 2 are prime.

? Example 1.2.5

Decompose 24 using a factor tree.

Answer

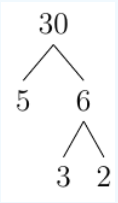


You can divide 24 by 4 to get 6. You can decompose 6 into 3 and 2. While you can decompose 4 into 2 and 2. So you can decompose 24 into $2 \cdot 3 \cdot 2 \cdot 2$ we see that 3 and 2 are prime so we can't decompose anymore. Notice that the factors produced are the same as in the previous example.

? Exercise 1.2.6

Decompose 30 using a factor tree.

Answer



30 is divisible by 5 and $30 \div 5 = 6$ which we can decompose into 3 and 2.

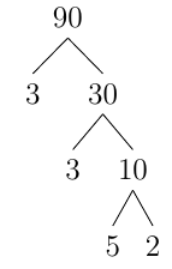
So $30 = 5 \cdot 3 \cdot 2$. This agrees with the answer in the previous example.

We see this isn't the only way to decompose 30 but you will still get the same result. For example you could have used 3 and 10 and decomposed 10 into 5 and 2 but the result is still $30 = 5 \cdot 3 \cdot 2$

? Exercise 1.2.7

Decompose 90 using a factor tree.

Answer



We can break 90 up into 3 and 30 and decompose 30.

So $90 = 3 \cdot 3 \cdot 5 \cdot 2$. This agrees with the example in the previous example.

Simplify Fractions

Try to input $\frac{4}{8}$ and $\frac{1}{2}$ into your calculator. What do you get? Why do you think the both inputs give you .5? It is because both of them express the same number. That is, 4 is half of 8 and 1 is half of 2. We can simplify $\frac{4}{8}$ by decomposing 4 and 8 into a product of its primes, that is $\frac{4}{8} = \frac{2 \cdot 2}{2 \cdot 2 \cdot 2}$. Now using the fact that we can break up a fraction, we can break up $\frac{2 \cdot 2}{2 \cdot 2 \cdot 2} = \frac{2 \cdot 2 \cdot 1}{2 \cdot 2 \cdot 2} = \frac{2}{2} \cdot \frac{2}{2} \cdot \frac{1}{2} = 1 \cdot 1 \cdot \frac{1}{2} = \frac{1}{2}$. This is why it is ok to cancel terms that are multiplied from the numerator and denominator. It is also common to just cross things out as in $\frac{2 \cdot 2}{2 \cdot 2 \cdot 2} = \frac{\cancel{2} \cdot \cancel{2}}{\cancel{2} \cdot \cancel{2} \cdot 2} = \frac{1}{2}$.

There are many ways to write a fraction, some may say there is an infinite way to write the same fraction, but we usually write it in simplest form, that is when we can no longer cancel anything from the top and bottom.

Fractions that have the same value are **equivalent fractions**.

The Equivalent Fractions Property allows us to find equivalent fractions and also simplify fractions.

If a, b, and c are numbers where $b \neq 0, c \neq 0$,
then $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$ and $\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$.

A fraction is considered simplified if there are no common factors, other than 1, in its numerator and denominator.

For example,

$\frac{2}{3}$ is simplified because there are no common factors of 2 and 3.

$\frac{10}{15}$ is not simplified because 5 is a common factor of 10 and 15.

We simplify, or reduce, a fraction by removing the common factors of the numerator and denominator. A fraction is not simplified until all common factors have been removed. If an expression has fractions, it is not completely simplified until the fractions are simplified.

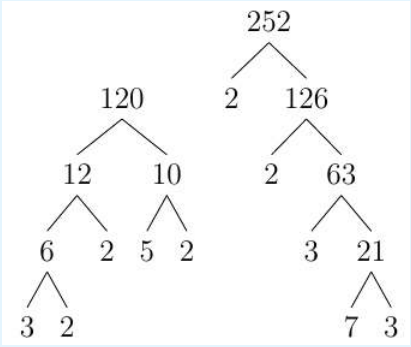
Sometimes it may not be easy to find common factors of the numerator and denominator. When this happens, it is a good idea to factor the numerator and the denominator into prime numbers using the factor tree, then cancel out the common factors using the Equivalent Fractions Property.

? Example 1.2.8

Simplify $\frac{120}{252}$

Answer

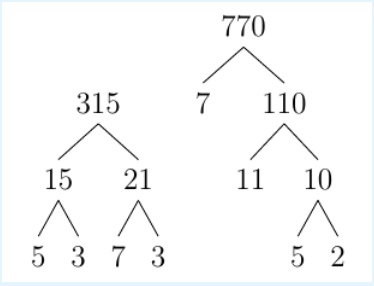
Steps to solve the example.

Step 1: Write out the factor trees	
Step 2: Rewrite the fraction as a product (multiplication) of primes on the top and bottom.	$\frac{120}{252} = \frac{3 \cdot 2 \cdot 2 \cdot 5 \cdot 2}{2 \cdot 2 \cdot 3 \cdot 7 \cdot 3}$
Step 3: Cancel like terms from top and bottom	$= \frac{\cancel{3} \cdot \cancel{2} \cdot \cancel{2} \cdot 5 \cdot 2}{\cancel{2} \cdot \cancel{2} \cdot \cancel{3} \cdot 7 \cdot 3} = \frac{5 \cdot 2}{7 \cdot 3}$
Step 4: Multiply to get 1 term on top and bottom, that is, in the numerator and denominator.	$\frac{5 \cdot 2}{7 \cdot 3} = \frac{10}{21}$

Simplify $\frac{-315}{770}$.

Answer

Solution to the example.

Step 1: Write out the factor trees	
Step 2: Rewrite the fraction as a product (multiplication) of primes on the top and bottom.	$-\frac{315}{770} = -\frac{5 \cdot 3 \cdot 7 \cdot 3}{7 \cdot 11 \cdot 5 \cdot 2}$
Step 3: Cancel like terms from top and bottom	$= -\frac{\cancel{5} \cdot 3 \cdot \cancel{7} \cdot 3}{\cancel{7} \cdot 11 \cdot \cancel{5} \cdot 2} = -\frac{9}{22}$
Step 4: Multiply to get 1 term on top and bottom, that is, in the numerator and denominator.	$= -\frac{9}{22}$

Simplify $-\frac{69}{120}$.

Answer

$-\frac{23}{40}$

? EXAMPLE 1.2.9

Simplify $-\frac{120}{192}$.

Answer

$$-\frac{5}{8}$$

Multiplying Fractions

Multiplying fractions is a simple operation. All you do is multiple the numerator with the numerator and the denominator with the denominator. Let's first get an intuitive idea on why that works. First say you have $\frac{1}{3}$ of a pizza and you multiplied it by 3, that is you put 3 $\frac{1}{3}$ pizzas together. You would get 1 whole pizza! In math, this would look like $3 \cdot \frac{1}{3} = \frac{3}{1} \cdot \frac{1}{3} = \frac{3 \cdot 1}{1 \cdot 3} = \frac{3}{3} = 1$.

We can also go backwards as in we can rewrite $\frac{2}{3}$ as $2 \cdot \frac{1}{3}$.

What about the denominators? Why do we multiply those? Say you have a whole pizza and you divide it into 3 equal slices. You then cut a slice in half to get two slices. We see that the small slices from the 2nd cut make up $\frac{1}{6}$ of a pizza. This makes sense because if you cut each slice in half, you would have 6 equally sized slices and 1 slice from that would be $\frac{1}{6}$ a pizza. So we divided a pizza by 3 and then each slice in 2. We can express this using an equation as $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{3} \div 2 = (1 \div 3) \div 2 = \frac{1}{6}$.

Now let's put all that intuition together.

? Example 1.2.10

Compute

$$\frac{2}{3} \cdot \frac{7}{5} \tag{1.2.4}$$

Answer

$$\begin{aligned} \frac{2}{3} \cdot \frac{7}{5} &= \frac{2 \cdot 7}{3 \cdot 5} \\ &= \frac{14}{15} \end{aligned}$$

We that was way easier than all that intuition.

Let's do some more examples,

? Exercise 1.2.11

Compute the following:

1. $\frac{1}{4} \cdot \frac{1}{4}$
2. $\frac{1}{3} \cdot -\frac{2}{3}$
3. $-\frac{2}{3} \cdot -\frac{4}{5}$

Answer

1. $\frac{1 \cdot 1}{4 \cdot 4} = \frac{1}{16}$
2. $-\frac{1 \cdot 2}{3 \cdot 3} = -\frac{2}{9}$
3. $- - \frac{2 \cdot 4}{3 \cdot 5} = \frac{8}{15}$

Now lets combine these strategies to compute and simplify a product of fractions.

Example 1.2.12

Compute $\frac{2}{3} \cdot \frac{27}{4}$

Answer

Solution to previous example.

Step 1: Rewrite every number as a product of primes.	$= \frac{2}{3} \cdot \frac{3 \cdot 3 \cdot 3}{2 \cdot 2}$
Step 2: Multiply	$= \frac{2 \cdot 3 \cdot 3 \cdot 3}{3 \cdot 2 \cdot 2}$
Step 3: Cancel like terms	$= \frac{3 \cdot 3}{2}$
Step 4: Simplify if possible	$= \frac{9}{2}$

Compound Fractions and Dividing Fractions

The numerators or denominators of some fractions contain fractions themselves. A fraction in which the numerator or the denominator is a fraction is called a **compound** fraction. Some people may use **complex fraction** but we will see later that this is actually terrible and horrible terminology. Still just know that complex fraction is the same as a compound fraction.

Definition: COMPOUND FRACTION

A **compound fraction** is a fraction in which the numerator or the denominator contains a fraction.

Some examples of compound fractions are:

$$\frac{\frac{6}{7}}{3} \quad \frac{\frac{3}{4}}{\frac{5}{8}} \quad \frac{\frac{x}{2}}{\frac{5}{6}} \tag{1.2.5}$$

To simplify a compound fraction, remember that the fraction bar means division. For example, the compound fraction $\frac{\frac{3}{4}}{\frac{5}{8}}$ means $\frac{3}{4} \div \frac{5}{8}$.

Surprisingly, division is just as easy as multiplication! All we need to do is take the top fraction and multiply it by the "reciprocal" of the bottom fraction.

Definition: RECIPROCAL

The **reciprocal** of a number is the multiplicative inverse. That is, the number you need to multiply to it to get 1.

This is very easy for the rationals as we just need to switch the numerator with the denominator.

Example 1.2.13

The reciprocal of $2 = \frac{1}{2}$ is $\frac{1}{2}$ since $2 \cdot \frac{1}{2} = 1$.

The reciprocal of $\frac{3}{2}$ is $\frac{2}{3}$ since $\frac{3}{2} \cdot \frac{2}{3} = 1$

Now we can look at an intuitive example of dividing by a fraction. Lets say a child eats half a slice of pizza and you have a pizza that contains 8 slices. How many children can you feed with 8 slices? Well each child eats half a slice so you would be able to feed 2 children with each slice and you have 8 slices, so you can feed 16 children. In an equation, this looks like

$$\frac{\frac{8}{1}}{\frac{1}{2}} = \frac{8}{1} \cdot \frac{2}{1} = \frac{16}{1} = 16 \tag{1.2.6}$$

Now lets get a bit more complicated with...teenagers...that eat $\frac{3}{4}$ of a slice each. That is, each teenager eat 3 quarter slices. Say you have $\frac{3}{2}$ slices of pizza, that is 3 half slices. How many teenagers can you feed with that? Well let's rewrite $\frac{3}{2}$ as $3 \cdot \frac{1}{2}$ and save the multiplication with $\frac{1}{2}$ for later. Let's cut up 3 slices into quarter slices so we have $3 \cdot 4 = 12$ quarter slices. each teenager eats 3 quarter slices so we can feed $\frac{12}{3} = 4$ teenagers. BUT WAIT we had 3 **half** slices so we can actually feed $4 \cdot \frac{1}{2} = \frac{4}{2} = 2$ teenagers. This is just

$$\frac{\frac{3}{2}}{\frac{3}{4}} = \frac{3}{2} \cdot \frac{4}{3} = \frac{12}{6} = 2 \tag{1.2.7}$$

WOW that was way easier than the intuition. You don't need to remember the intuitive examples for the test but hopefully you now understand why you are doing what you are doing. Now lets take a look at a step by step problem. To help with the examples, remember **"invert and multiply"**; this means that to divide, we flip over the denominator and multiply it by the numerator.

? Example 1.2.14

Compute $\frac{\frac{9}{2}}{\frac{27}{4}}$

Answer

Solution to previous example.

Step 1: Rewrite the quotient as a product.	$= \frac{9}{2} \cdot \frac{4}{27}$
Step 2: Rewrite every number as a product of primes.	$= \frac{3 \cdot 3}{2} \cdot \frac{2 \cdot 2}{3 \cdot 3 \cdot 3}$
Step 3: Multiply	$= \frac{3 \cdot 3 \cdot 2 \cdot 2}{2 \cdot 3 \cdot 3 \cdot 3}$
Step 4: Cancel like terms	$= \frac{2}{3}$
Step 5: Simplify if possible	$= \frac{2}{3}$

This is the same process as above just with an extra step in the beginning.

? Example 1.2.15

Compute $-\frac{7}{18} \div (-\frac{14}{27})$.

Answer

Solution to previous example.

Step 1: Rewrite the quotient as a product.	$= -\frac{7}{18} \cdot -\frac{27}{14}$
Step 2: Rewrite every number as a product of primes.	$= -\frac{7}{3 \cdot 3 \cdot 2} \cdot -\frac{3 \cdot 3 \cdot 3}{7 \cdot 2}$

Step 1: Rewrite the quotient as a product.	$= -\frac{7}{18} \cdot -\frac{27}{14}$
Step 3: Multiply	$= \frac{7 \cdot 3 \cdot 3 \cdot 3}{3 \cdot 3 \cdot 2 \cdot 7 \cdot 2}$
Step 4: Cancel like terms	$= \frac{3}{2 \cdot 2}$
Step 5: Simplify if possible	$= \frac{3}{4}$

Divide: $-\frac{7}{27} \div (-\frac{35}{36})$.

Answer

$\frac{4}{15}$

Divide: $-\frac{5}{14} \div (-\frac{15}{28})$.

Answer

$\frac{2}{3}$

Add and Subtract Fractions

Let's move on to addition. Say you have half a pizza and your friend has half a pizza, then between the two of you, you all have a whole pizza. That is, $\frac{1}{2} + \frac{1}{2} = \frac{2}{2} = 1$. Note that we added the numerators together and left the denominators unchanged.

When we multiplied fractions, we just multiplied the numerators and multiplied the denominators right straight across. To add or subtract fractions, they must have a common denominator.

FRACTION ADDITION AND SUBTRACTION

If a, b, and c are numbers where $c \neq 0$, then

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \text{ and } \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c} \tag{1.2.8}$$

To add or subtract fractions, add or subtract the numerators and place the result over the common denominator.

What if the denominators are different? Then make them the same! We can make them the same by using a fundamental trick of math

Theorem: 1st fundamental trick of math 1.2.16

A number multiplied by 1 is the same number. That is, if a is any number, then $a \cdot 1 = a$

That doesn't seem like much of a trick but there are **many** creative ways to write the number 1.

For example $\frac{2}{2}$ and $\frac{62}{62}$ are 1.

Definition: Least Common Multiple (LCM)

The least common multiple (LCM) of two integers x, y is the smallest integer, c such that c is divisible by x and y.

Example 1.2.17

The LCM of 4 and 6 is 12 because 4 and 6 divide 12 and 12 is the smallest integer that both 6 and 4 divide evenly.

The LCM of 4 and 6 is *not* 24 because, even though 4 and 6 divide 24, it is not the smallest integer that both 6 and 4 divide evenly.

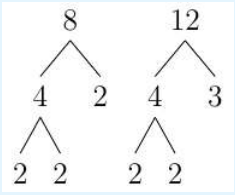
Let's do an example of how to find the LCM

Example 1.2.18

Find the lcm of 8 and 12.

Answer

Solution to previous example.

Step 1: factor the numbers	
Step 2: multiply all the primes but skip repeats	$2 \cdot 2 \cdot 2 \cdot 3$
Step 3: Compute the lcm by doing the multiplication	$= 24$

Note: Starting from left to right, we write down the three 2s we got from 8. We skip the two 2s from 12 since we already picked up three 2s from 8, and we pick up a 3 since we didn't already have it.

The **least common denominator** (LCD) of two fractions is the smallest number that can be used as a common denominator of the fractions. The LCD of the two fractions is the least common multiple (LCM) of their denominators.

Definition: LEAST COMMONE DENOMINATOR

The **least common denominator** (LCD) of two fractions is the least common multiple (LCM) of their denominators.

After we find the least common denominator of two fractions, we convert the fractions to equivalent fractions with the LCD. Putting these steps together allows us to add and subtract fractions because their denominators will be the same!

Let's first do an intuitive example.

Let's say you have half a pizza and a quarter of a pizza. How much pizza do you have all together and why is it $\frac{3}{4}$th of a pizza? We broke up the half of a pizza into 2 quarters of a pizza and then added them to the quarter of a pizza to get 3 quarters of a pizza. In an equation this looks like

$$\frac{1}{2}+\frac{1}{4}=\frac{2}{4}+\frac{1}{4}=\frac{3}{4}$$

Did you notice that 4 is the LCM of 2 and 4? If we want to add fractions all we do is make them have the same denominator and the denominator that is easiest to work with is the LCM!

Example 1.2.19

Compute $\frac{3}{10}+\frac{2}{15}$

Answer

Solution to previous example.

Solution to previous example.

Step 1: Compute lcm of denominator if they are different, if same then skip to step 5.	Substep 1: factor the denominators: $10 = 5 \cdot 2$, $15 = 5 \cdot 3$
	Substep 2: multiply all the primes but skip repeats: $5 \cdot 2 \cdot 3$
	Substep 3: Compute the lcm: $= 30$
Step 2: divide the lcm by the denominators	$\frac{30}{10} = 3, \frac{30}{15} = 2$
Step 3: multiply the 1st numerator and denominator by the 1st number above.	$\frac{3 \cdot 3}{10 \cdot 3} = \frac{9}{30}$ This is where we multiply by a 1 since $\frac{3}{10} = \frac{3}{10} \cdot 1 = \frac{3}{10} \cdot \frac{3}{3} = \frac{3 \cdot 3}{10 \cdot 3}$. Same thing goes for the next step.
Step 4: multiply the 2nd numerator and denominator by the 2nd number above	$\frac{2 \cdot 2}{15 \cdot 2} = \frac{4}{30}$
Step 5: add the two numbers together	$\frac{9}{30} + \frac{4}{30} = \frac{13}{30}$
Step 6: Simplify if possible	$= \frac{13}{30}$

This is the same procedure for subtraction too.

? Example 1.2.20

Compute $\frac{2}{3} - \frac{4}{5}$

Answer

Solution to previous example.

Step 1: Compute lcm of denominator	Substep 1: factor the denominators: 3 and 5 are both prime.
	Substep 2: multiply all the primes but skip repeats: $3 \cdot 5$
	Substep 3: Compute the lcm: =15
Step 2: divide the lcm by the denominators	$\frac{15}{3} = 5$ $\frac{15}{5} = 3$
Step 3: multiply the 1st numerator and denominator by the 1st number above	$\frac{2 \cdot 5}{3 \cdot 5} = \frac{10}{15}$
Step 4: multiply the 2nd numerator and denominator by the 2nd number above	$\frac{4 \cdot 3}{5 \cdot 3} = \frac{12}{15}$
Step 5: subtract the two numbers	$\frac{10}{15} - \frac{12}{15} = -\frac{2}{15}$

Solution to previous example.

Step 1: Compute lcm of denominator	Substep 1: factor the denominators	3 and 5 are both prime.
	Substep 2: multiply all the primes but skip repeats	$3 \cdot 5$
	Substep 3: Compute the lcm	=15
Step 6: Simplify if possible	$= -\frac{2}{15}$	

Lets take a look at some more examples.

? EXAMPLE 1.2.21

Add: $\frac{7}{12} + \frac{11}{15}$.

Answer

$\frac{79}{60}$

? EXAMPLE 1.2.22

Add: $\frac{13}{15} + \frac{17}{20}$.

Answer

$\frac{103}{60}$

We now have all four operations for fractions. The Table below summarizes fraction operations.

Four operations for fractions.

Fraction Multiplication	**Fraction Division**
$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$	$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$
Multiply the numerators and multiply the denominators	Multiply the first fraction by the reciprocal of the second.
Fraction Addition	**Fraction Subtraction**
$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$	$\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$
Add the numerators and place the sum over the common denominator.	Subtract the numerators and place the difference over the common denominator.
To multiply or divide fractions, an LCD is NOT needed. To add or subtract fractions, an LCD is needed.	

Use the Order of Operations to Simplify Fractions

The fraction bar in a fraction acts as grouping symbol. The order of operations then tells us to simplify the numerator and then the denominator. Then we divide.

SIMPLIFY AN EXPRESSION WITH A FRACTION BAR.

1. Simplify the expression in the numerator. Simplify the expression in the denominator.
2. Simplify the fraction.

EXAMPLE 1.2.23

Simplify: $\frac{4(-3)+6(-2)}{-3(2)-2}$.

Answer

The fraction bar acts like a grouping symbol. So completely simplify the numerator and the denominator separately.

$$\frac{4(-3)+6(-2)}{-3(2)-2}$$

Multiply. $\frac{-12+(-12)}{-6-2}$

Simplify. $\frac{-24}{-8}$

Divide. 3

EXAMPLE 1.2.24

Simplify: $\frac{8(-2)+4(-3)}{-5(2)+3}$.

Answer

4

EXAMPLE 1.2.25

Simplify: $\frac{7(-1)+9(-3)}{-5(3)-2}$.

Answer

2

Now we'll look at compound fractions where the numerator or denominator contains an expression that can be simplified. So we first must completely simplify the numerator and denominator separately using the order of operations. Then we divide the numerator by the denominator as the fraction bar means division.

EXAMPLE 1.2.26

Simplify $\frac{\frac{1}{2}-\frac{5}{2}}{\frac{3}{5}+\frac{1}{5}}$

Answer

Solution to the example.

Step 1: Simplify the numerator	$= \frac{-\frac{4}{2}}{\frac{3}{5}+\frac{1}{5}}$

Step 1: Simplify the numerator	$= \frac{-\frac{4}{2}}{\frac{3}{5}+\frac{1}{5}}$
Step 2: Simplify the denominator	$= \frac{-\frac{4}{2}}{\frac{4}{5}}$
Step 3: Compute the division	$= -\frac{4}{2} \cdot \frac{5}{4} = -\frac{5}{2}$
Step 4: Simplify if possible	$= -\frac{5}{2} \backslash$

? EXAMPLE 1.2.27

Simplify: $\frac{\left(\frac{1}{3}\right)}{2+2}$.

Answer

$\frac{1}{12}$

? EXAMPLE 1.2.28

Simplify: $\frac{1+4 \cdot 4}{\left(\frac{1}{4}\right)}$.

Answer

68

SIMPLIFY COMPOUND FRACTIONS.

1. Simplify the numerator.
2. Simplify the denominator.
3. Divide the numerator by the denominator.
4. Simplify if possible.

? EXAMPLE 1.2.29

Simplify: $\frac{\frac{1}{2}+\frac{2}{3}}{\frac{3}{4}-\frac{1}{6}}$.

Answer

It may help to put parentheses around the numerator and the denominator.

$$\frac{\frac{1}{2}+\frac{2}{3}}{\frac{3}{4}-\frac{1}{6}}$$

Simplify the numerator ($LCD = 6$) and simplify the denominator ($LCD = 12$). $\frac{\left(\frac{3}{6}+\frac{4}{6}\right)}{\left(\frac{9}{12}-\frac{2}{12}\right)}$

Simplify. $\left(\frac{7}{6}\right)\left(\frac{7}{12}\right)$

Divide the numerator by the denominator. $\frac{7}{6} \div \frac{7}{12}$

Simplify. $\frac{7}{6} \cdot \frac{12}{7}$

Divide out common factors. $\frac{\cancel{7} \cdot \cancel{6} \cdot 2}{\cancel{6} \cdot \cancel{7} \cdot 1}$

Simplify. 2

? EXAMPLE 1.2.30

Simplify: $\frac{\frac{1}{3}+\frac{1}{2}}{\frac{3}{4}-\frac{1}{3}}$.

Answer

2

? EXAMPLE 1.2.31

Simplify: $\frac{\frac{2}{3}-\frac{1}{2}}{\frac{1}{4}+\frac{1}{3}}$.

Answer

$\frac{2}{7}$

Key Concepts

- If a, b, and c are numbers where $b \neq 0, c \neq 0$, then
  $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$ and $\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$.
- **How to simplify a fraction.**
  1. Rewrite the numerator and denominator to show the common factors.
     If needed, factor the numerator and denominator into prime numbers first.
  2. Simplify using the Equivalent Fractions Property by dividing out common factors.

3. Multiply any remaining factors.

- If a, b, c, and d are numbers where $b \neq 0$, and $d \neq 0$, then

  $$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

  To multiply fractions, multiply the numerators and multiply the denominators.

- If a, b, c, and d are numbers where $b \neq 0$,$c \neq 0$, and $d \neq 0$, then

  $$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

  To divide fractions, we multiply the first fraction by the reciprocal of the second.

- If a, b, and c are numbers where $c \neq 0$, then

  $$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \text{ and } \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

  To add or subtract fractions, add or subtract the numerators and place the result over the common denominator.

- **How to add or subtract fractions.**
  1. Do they have a common denominator?
     - Yes—go to step 2.
     - No—rewrite each fraction with the LCD (least common denominator).
       - Find the LCD.
       - Change each fraction into an equivalent fraction with the LCD as its denominator.
  2. Add or subtract the fractions.
  3. Simplify, if possible.

- **How to simplify an expression with a fraction bar.**
  1. Simplify the expression in the numerator. Simplify the expression in the denominator.
  2. Simplify the fraction.

- For any positive numbers a and b,

  $$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$$

- **How to simplify compound fractions.**
  1. Simplify the numerator.
  2. Simplify the denominator.
  3. Divide the numerator by the denominator. Simplify if possible.

Glossary

compound fraction

A fraction in which the numerator or the denominator is a fraction is called a compound fraction.

denominator

In a fraction, written $\frac{a}{b}$, where $b \neq 0$, the denominator b is the number of equal parts the whole has been divided into.

equivalent fractions

Equivalent fractions are fractions that have the same value.

fraction

A fraction is written $\frac{a}{b}$, where $b \neq 0$, and a is the numerator and b is the denominator. A fraction represents parts of a whole.

least common denominator

The least common denominator (LCD) of two fractions is the least common multiple (LCM) of their denominators.

numerator

In a fraction, written $\frac{a}{b}$, where $b \neq 0$, the numerator a indicates how many parts are included.

reciprocal

The reciprocal of a fraction is found by inverting the fraction, placing the numerator in the denominator and the denominator in the numerator.

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1.2E: Exercises

Practice Makes Perfect

Simplify Fractions

Simplify the following:

1. $-\frac{108}{63}$
2. $\frac{120}{252}$
3. $\frac{14}{21}$
4. $-\frac{210}{110}$

Answer

1. $-\frac{12}{7}$
2. $\frac{10}{21}$
3. $\frac{2}{3}$
4. $-\frac{21}{11}$

Multiply and Divide Fractions

Perform the operation:

5. $-\frac{3}{4}\left(-\frac{4}{9}\right)$
6. $\left(-\frac{14}{15}\right)\left(\frac{9}{20}\right)$
7. $\left(-\frac{63}{84}\right)\left(-\frac{44}{90}\right)$
8. $\frac{3}{7}\cdot 21$
9. $\frac{3}{4}\div\frac{1}{11}$
10. $\frac{5}{18}\div\left(-\frac{15}{24}\right)$
11. $\frac{8}{15}\div\frac{12}{25}$
12. $\frac{3}{4}\div(-12)$
13. $-\frac{\frac{8}{21}}{\frac{12}{35}}$
14. $-\frac{\frac{4}{5}}{2}$
15. $\frac{\frac{1}{3}}{\frac{1}{2}}$

Answer

5. $\frac{1}{3}$
6. $-\frac{21}{50}$
7. $\frac{11}{30}$
8. 9
9. $\frac{33}{4}$
10. $-\frac{4}{9}$
11. $\frac{10}{9}$
12. $-\frac{1}{16}$
13. $-\frac{10}{9}$
14. $-\frac{2}{5}$
15. $\frac{2}{3}$

? Add and Subtract

16. $\frac{7}{12}+\frac{5}{8}$
17. $\frac{7}{12}-\frac{9}{16}$
18. $-\frac{13}{30}+\frac{25}{42}$
19. $-\frac{39}{56}-\frac{22}{35}$
20. $-\frac{2}{3}-\left(-\frac{3}{4}\right)$
21. $\frac{1}{3}+\frac{1}{4}$
22. $\frac{2}{3}+\frac{1}{6}$

Answer

16. $\frac{29}{24}$
17. $\frac{1}{48}$
18. $\frac{17}{105}$
19. $-\frac{53}{40}$
20. $\frac{1}{12}$
21. $\frac{7}{12}$
22. $\frac{5}{6}$

https://math.libretexts.org/@go/page/104797

Order of Operations

Simplify

23. $\dfrac{5 \cdot 6 - 3 \cdot 4}{4 \cdot 5 - 2 \cdot 3}$
24. $\dfrac{5^2 - 3^2}{3 - 5}$
25. $\dfrac{7 \cdot 4 - 2(8 - 5)}{9 \cdot 3 - 3 \cdot 5}$
26. $\dfrac{9(8 - 2) - 3(15 - 7)}{6(7 - 1) - 3(17 - 9)}$
27. $\dfrac{2}{\frac{1}{3} + \frac{1}{5}}$

Answer

23. $\frac{9}{7}$
24. -8
25. $\frac{11}{6}$
26. $\frac{5}{2}$
27. $\frac{15}{4}$

Writing Exercises

28. Why do you need a common denominator to add or subtract fractions? Explain.
29. How do you find the LCD of 2 fractions?
30. Explain how you find the reciprocal of a fraction.
31. Explain how you find the reciprocal of a negative number.

Answer

Answers will vary.

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1.3: Decimals and Real numbers

Learning Objectives

By the end of this section, you will be able to:

- Round decimals
- Add and subtract decimals
- Multiply and divide decimals
- Convert decimals, fractions, and percents
- Simplify expressions with square roots
- Identify integers, rational numbers, irrational numbers, and real numbers
- Locate fractions and decimals on the number line

Warm Up

Simplify the following fractions:

1. $\frac{15}{10}$
2. $\frac{14}{49}$
3. $\frac{24}{60}$

Answer

1. $\frac{3}{2}$
2. $\frac{2}{7}$
3. $\frac{2}{5}$

Round Decimals

Decimals are another way of writing fractions whose denominators are powers of ten; they also help us represent smaller parts of whole numbers.

$$\begin{aligned} 0.1 &= \frac{1}{10} && \text{is “one tenth”} \\ 0.01 &= \frac{1}{100} && \text{is “one hundredth”} \\ 0.001 &= \frac{1}{1000} && \text{is “one thousandth”} \\ 0.0001 &= \frac{1}{10,000} && \text{is “one ten-thousandth”} \end{aligned} \tag{1.3.1}$$

Just as in whole numbers, each digit of a decimal corresponds to the place value based on the powers of ten. The figure below shows the names of the place values to the left and right of the decimal point.

Place Value											
Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones	.	Tenths	Hundredths	Thousandths	Ten-thousandths	Hundred-thousandths
						.					

Figure 1.

When we work with decimals, it is often necessary to round the number to the nearest required place value. We summarize the steps for rounding a decimal here.

ROUND DECIMALS.

1. Locate the given place value and mark it with an arrow.
2. Underline the digit to the right of the place value.
3. Is the underlined digit greater than or equal to 5?
   - Yes: add 1 to the digit in the given place value.
   - No: do not change the digit in the given place value
4. Rewrite the number, deleting all digits to the right of the rounding digit.

EXAMPLE 1.3.1

Round 18.379 to the nearest ⓐ hundredth ⓑ tenth ⓒ whole number.

Answer

Round 18.379

ⓐ to the nearest hundredth

Solution to the example.

Locate the hundredths place with an arrow.	hundredths place 18.379
Underline the digit to the right of the given place value.	hundredths place 18.379
Because 9 is greater than or equal to 5, add 1 to the 7.	18.379 add 1 delete
Rewrite the number, deleting all digits to the right of the rounding digit.	18.38
Notice that the deleted digits were NOT replaced with zeros.	So, 18.379 rounded to the nearest hundredth is 18.38.

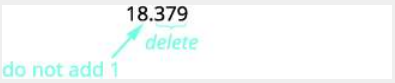
ⓑ to the nearest tenth

Solution to the example.

Locate the tenths place with an arrow.	tenths place 18.379
Underline the digit to the right of the given place value.	tenths place 18.379
Because 7 is greater than or equal to 5, add 1 to the 3.	18.379 add 1 delete
Rewrite the number, deleting all digits to the right of the rounding digit.	18.4
Notice that the deleted digits were NOT replaced with zeros.	So, 18.379 rounded to the nearest tenth is 18.4.

ⓒ to the nearest whole number

Solution to the example.

Locate the ones place with an arrow.	
Underline the digit to the right of the given place value.	
Since 3 is not greater than or equal to 5, do not add 1 to the 8.	
Rewrite the number, deleting all digits to the right of the rounding digit.	18
	So, 18.379 rounded to the nearest whole number is 18.

? EXAMPLE 1.3.2

Round 6.582 to the nearest ⓐ hundredth ⓑ tenth ⓒ whole number.

Answer

ⓐ 6.58 ⓑ 6.6 ⓒ 7

? EXAMPLE 1.3.3

Round 15.2175 to the nearest ⓐ thousandth ⓑ hundredth ⓒ tenth.

Answer

ⓐ 15.218 ⓑ 15.22

ⓒ 15.2

Add and Subtract Decimals

To add or subtract decimals, we line up the decimal points. By lining up the decimal points this way, we can add or subtract the corresponding place values. We then add or subtract the numbers as if they were whole numbers and then place the decimal point in the sum.

ADD OR SUBTRACT DECIMALS.

1. Determine the sign of the sum or difference.
2. Write the numbers so the decimal points line up vertically.
3. Use zeros as placeholders, as needed.
4. Add or subtract the numbers as if they were whole numbers. Then place the decimal point in the answer under the decimal points in the given numbers.
5. Write the sum or difference with the appropriate sign.

EXAMPLE 1.3.4

Add or subtract: ⓐ $-23.5-41.38$ ⓑ $14.65-20$

Answer

ⓐ

$-23.5-41.38$

The difference will be negative. To subtract, we add the numerals. Write the numbers so the decimal points line up vertically.

$$\begin{array}{r} 23.5 \\ +41.38 \\ \hline \end{array}$$

Put 0 as a placeholder after the 5 in 23.5.
Remember, $\frac{5}{10}=\frac{50}{100}$ so $0.5=0.50$.

$$\begin{array}{r} 23.50 \\ +41.38 \\ \hline \end{array}$$

Add the numbers as if they were whole numbers. Then place the decimal point in the sum.

$$\begin{array}{r} 23.50 \\ +41.38 \\ \hline 64.88 \end{array}$$

Write the result with the correct sign.

$64.88-23.5-41.38=-64.88$

ⓑ

	$14.65-20$
The difference will be negative. To subtract, we subtract 14.65 from 20.	
Write the numbers so the decimal points line up vertically.	$\begin{array}{r} 20 \\ -14.65 \\ \hline \end{array}$
Remember, 20 is a whole number, so place the decimal point after the 0.	
Put in zeros to the right as placeholders.	$\begin{array}{r} 20.00 \\ -14.65 \\ \hline \end{array}$
Subtract and place the decimal point in the answer.	$\begin{array}{rrcrr} & 9 & & 9 & \\ 1 & \cancel{10} & & \cancel{10} & 10 \\ 2 & 0 & . & 0 & 0 \\ -1 & 4 & . & 6 & 5 \\ \hline & 5 & . & 3 & 5 \end{array}$
Write the result with the correct sign.	$14.65-20=-5.35$

? EXAMPLE 1.3.5

Add or subtract: ⓐ $-4.8-11.69$ ⓑ $9.58-10$

Answer

ⓐ -16.49 ⓑ -0.42

? EXAMPLE 1.3.6

Add or subtract: ⓐ $-5.123-18.47$ ⓑ $37.42-50$

Answer

ⓐ -23.593 ⓑ -12.58

Multiply and Divide Decimals

When we multiply signed decimals, first we determine the sign of the product and then multiply as if the numbers were both positive. We multiply the numbers temporarily ignoring the decimal point and then count the number of decimal points in the factors and that sum tells us the number of decimal places in the product. Finally, we write the product with the appropriate sign.

MULTIPLY DECIMALS.

1. Determine the sign of the product.
2. Write in vertical format, lining up the numbers on the right. Multiply the numbers as if they were whole numbers, temporarily ignoring the decimal points.
3. Place the decimal point. The number of decimal places in the product is the sum of the number of decimal places in the factors.
4. Write the product with the appropriate sign.

Multiply: $(-3.9)(4.075)$

Answer

Solution to the example.

	$(-3.9)(4.075)$
The signs are different. The product will be negative.	The product will be negative.
Write in vertical format, lining up the numbers on the right.	$\begin{array}{r} 4.075 \\ \times\ 3.9 \end{array}$
Multiply.	$\begin{array}{r} 4.075 \\ \times\ 3.9 \\ \hline 36675 \\ 12225 \\ \hline 158925 \end{array}$
Add the number of decimal places in the factors (1 + 3). Place the decimal point 4 places from the right.	$\begin{array}{r} 4.075 \\ \times\ 3.9 \\ \hline 36675 \\ 12225 \\ \hline 15.8925 \end{array}$ (4 places)
(–3.9) (4.075) 1 place 3 places	
The signs are the different, so the product is negative.	$(-3.9)(4.075) = -15.8925$

Multiply: $-4.5(6.107)$

Answer

-27.4815

EXAMPLE 1.3.7

Multiply: $-10.79(8.12)$

Answer

-87.6148

Often, especially in the sciences, you will multiply decimals by powers of 10 (10, 100, 1000, etc). If you multiply a few products on paper, you may notice a pattern relating the number of zeros in the power of 10 to number of decimal places we move the decimal point to the right to get the product.

MULTIPLY A DECIMAL BY A POWER OF TEN.

1. Move the decimal point to the right the same number of places as the number of zeros in the power of 10.
2. Add zeros at the end of the number as needed.

EXAMPLE 1.3.8

Multiply: 5.63 by ⓐ 10 ⓑ 100 ⓒ 1000.

Answer

By looking at the number of zeros in the multiple of ten, we see the number of places we need to move the decimal to the right.

ⓐ

Solution to a.

	5.63 (10)
There is 1 zero in 10, so move the decimal point 1 place to the right.	5.63
	5.63

ⓑ

Solution to b.

	5.63(100)
There are 2 zeroes in 100, so move the decimal point 2 places to the right.	5.63
	563

ⓒ

Solution to c.

	5.63(1,000)
There are 3 zeroes in 1,000, so move the decimal point 3 place to the right.	5.63
A zero must be added to the end.	5,630

EXAMPLE 1.3.9

Multiply 2.58 by ⓐ 10 ⓑ 100 ⓒ 1000.

Answer

ⓐ 25.8 ⓑ 258 ⓒ 2,580

EXAMPLE 1.3.10

Multiply 14.2 by ⓐ 10 ⓑ 100 ⓒ 1000.

Answer

ⓐ 142 ⓑ 1,420 ⓒ 14,200

Just as with multiplication, division of signed decimals is very much like dividing whole numbers. We just have to figure out where the decimal point must be placed and the sign of the quotient. When dividing signed decimals, first determine the sign of the quotient and then divide as if the numbers were both positive. Finally, write the quotient with the appropriate sign.

We review the notation and vocabulary for division:

$$\underset{\text{dividend}}{a} \div \underset{\text{divisor}}{b} = \underset{\text{quotient}}{c} \qquad \begin{array}{r} \overset{\text{quotient}}{c} \\ \text{divisor}\ b\overline{)\,a} \\ \text{dividend} \end{array}$$

We'll write the steps to take when dividing decimals for easy reference.

DIVIDE DECIMALS.

1. Determine the sign of the quotient.
2. Make the divisor a whole number by "moving" the decimal point all the way to the right. "Move" the decimal point in the dividend the same number of places—adding zeros as needed.
3. Divide. Place the decimal point in the quotient above the decimal point in the dividend.
4. Write the quotient with the appropriate sign.

EXAMPLE 1.3.11

Divide: $-25.65 \div (-0.06)$

Answer

Remember, you can "move" the decimals in the divisor and dividend because of the Equivalent Fractions Property.

Solution to the example.

	$-25.65 \div (-0.06)$
The signs are the same.	The quotient is positive.
Make the divisor a whole number by "moving" the decimal point all the way to the right.	
"Move" the decimal point in the dividend the same number of places.	$0.06\overline{)25.65}$
Divide. Place the decimal point in the quotient above the decimal point in the dividend.	$\begin{array}{r} 427.5 \\ 006.\overline{)2565.0} \\ -24 \\ \hline 16 \\ -12 \\ \hline 45 \\ -42 \\ \hline 30 \\ 30 \end{array}$
Write the quotient with the appropriate sign.	$-25.65 \div (-0.06) = 427.5$

EXAMPLE 1.3.12

Divide: $-23.492 \div (-0.04)$

Answer

587.3

? EXAMPLE 1.3.13

Divide: $-4.11 \div (-0.12)$

Answer

34.25

Convert Decimals, Fractions, and Percents

In our work, it is often necessary to change the form of a number. We may have to change fractions to decimals or decimals to percent.

We convert decimals into fractions by identifying the place value of the last (farthest right) digit. In the decimal 0.03. the 3 is in the hundredths place, so 100 is the denominator of the fraction equivalent to 0.03.

$$0.03 = \frac{3}{100} \tag{1.3.2}$$

The steps to take to convert a decimal to a fraction are summarized in the procedure box.

CONVERT A DECIMAL TO A PROPER FRACTION AND A FRACTION TO A DECIMAL.

1. To convert a decimal to a proper fraction, determine the place value of the final digit.
2. Write the fraction.
   - numerator—the "numbers" to the right of the decimal point
   - denominator—the place value corresponding to the final digit
3. To convert a fraction to a decimal, divide the numerator of the fraction by the denominator of the fraction.

? EXAMPLE 1.3.14

Write: ⓐ 0.374 as a fraction ⓑ $-\frac{5}{8}$ as a decimal.

Answer

ⓐ

Solution to a.

	0.374
Determine the place value of the final digit.	0.3 tenths, 7 hundredths, 4 thousandths
Write the fraction for 0.374: The numerator is 374. The denominator is 1,000.	$\frac{374}{1000}$
Simplify the fraction.	$\frac{2 \cdot 187}{2 \cdot 500}$
Divide out the common factors.	$\frac{187}{500}$
	so, $0.374 = \frac{187}{500}$

ⓑ Since a fraction bar means division, we begin by writing the fraction $\frac{5}{8}$ as $8 \div 5$. Now divide.

$$8\overline{)5.000}\quad 0.625$$
$$48$$
$$20$$
$$16$$
$$40$$
$$40$$

so, $-\frac{5}{8} = -0.625$

? EXAMPLE 1.3.15

Write: ⓐ 0.234 as a fraction ⓑ $-\frac{7}{8}$ as a decimal.

Answer

ⓐ $\frac{117}{500}$ ⓑ -0.875

? EXAMPLE 1.3.16

Write: ⓐ 0.024 as a fraction ⓑ $-\frac{3}{8}$ as a decimal.

Answer

ⓐ $\frac{3}{125}$ ⓑ -0.375

A **percent** is a ratio whose denominator is 100. Percent means per hundred. We use the percent symbol, %, to show percent. Since a percent is a ratio, it can easily be expressed as a fraction. Percent means per 100, so the denominator of the fraction is 100. We then change the fraction to a decimal by dividing the numerator by the denominator. After doing this many times, you may see the pattern.

To convert a percent number to a decimal number, we move the decimal point two places to the left.

6%	78%	2.7%	135%
0.06	0.78	0.027	1.35

To convert a decimal to a percent, remember that percent means per hundred. If we change the decimal to a fraction whose denominator is 100, it is easy to change that fraction to a percent. After many conversions, you may recognize the pattern.

To convert a decimal to a percent, we move the decimal point two places to the right and then add the percent sign.

0.05	0.83	1.05	0.075	0.3
5%	83%	105%	7.5%	30%

CONVERT A PERCENT TO A DECIMAL AND A DECIMAL TO A PERCENT.

1. To convert a percent to a decimal, move the decimal point two places to the left after removing the percent sign.
2. To convert a decimal to a percent, move the decimal point two places to the right and then add the percent sign.

? EXAMPLE 1.3.17

Convert each:

ⓐ percent to a decimal: 62%, 135%, and 13.7%

ⓑ decimal to a percent: 0.51, 1.25, and 0.093

Answer

ⓐ

Solution to a.

	62%	135%	35.7%
Move the decimal point two places to the left.	0.62	1.35	0.357

ⓑ

Solution to b.

	0.51	1.25	0.093
Move the decimal point two places to the right.	51%	125%	9.3%

? EXAMPLE 1.3.18

Convert each:

ⓐ percent to a decimal: 9%, 87%, and 3.9%

ⓑ decimal to a percent: 0.17, 1.75, and 0.0825

Answer

ⓐ 0.09, 0.87, 0.039 ⓑ 17%, 175%, 8.25%

? EXAMPLE 1.3.19

Convert each:

ⓐ percent to a decimal: 3%, 91%, and 8.3%

ⓑ decimal to a percent: 0.41, 2.25, and 0.0925

Answer

ⓐ 0.03, 0.91, 0.083 ⓑ 41%, 225%, 9.25%

Identify Integers, Rational Numbers, Irrational Numbers, and Real Numbers

We have already described numbers as *counting numbers*, *whole numbers*, and *integers*. What is the difference between these types of numbers? Difference could be confused with subtraction. How about asking how we distinguish between these types of numbers?

$$\begin{array}{ll} \text{Counting numbers} & 1, 2, 3, 4, \ldots . \\ \text{Whole numbers} & 0, 1, 2, 3, 4, \ldots . \\ \text{Integers} & \ldots . -3, -2, -1, 0, 1, 2, 3, \ldots . \end{array} \tag{1.3.3}$$

What type of numbers would we get if we started with all the integers and then included all the fractions? The numbers we would have form the set of rational numbers. A **rational number** is a number that can be written as a ratio of two integers.

In general, any decimal that ends after a number of digits (such as 7.3 or −1.2684) is a rational number. We can use the place value of the last digit as the denominator when writing the decimal as a fraction. The decimal for $\frac{1}{3}$ is the number $0.\overline{3}$. The bar over the 3 indicates that the number 3 repeats infinitely. Continuously has an important meaning in calculus. The number(s) under the bar is called the repeating block and it repeats continuously.

Since all integers can be written as a fraction whose denominator is 1, the integers (and so also the counting and whole numbers. are rational numbers.

Every rational number can be written both as a ratio of integers $\frac{p}{q}$, where p and q are integers and $q \neq 0$, and as a decimal that stops or repeats.

Definition: RATIONAL NUMBER

A **rational number** is a number of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

Its decimal form stops or repeats.

Are there any decimals that do not stop or repeat? Yes! The number $\pi\pi$ (the Greek letter *pi*, pronounced "pie"), which is very important in describing circles, has a decimal form that does not stop or repeat. We use three dots (…) to indicate the decimal does not stop or repeat.

$$\pi = 3.141592654... \tag{1.3.4}$$

The square root of a number that is not a perfect square is a decimal that does not stop or repeat.

A numbers whose decimal form does not stop or repeat cannot be written as a fraction of integers. We call this an **irrational number**.

Definition: IRRATIONAL NUMBER

An **irrational number** is a number that cannot be written as the ratio of two integers.

Its decimal form does not stop and does not repeat.

Let's summarize a method we can use to determine whether a number is rational or irrational.

RATIONAL OR IRRATIONAL

If the decimal form of a number

- *repeats or stops*, the number is a **rational number**.
- *does not repeat and does not stop*, the number is an **irrational number.**

We have seen that all counting numbers are whole numbers, all whole numbers are integers, and all integers are rational numbers. The irrational numbers are numbers whose decimal form does not stop and does not repeat. When we put together the rational numbers and the irrational numbers, we get the set of **real numbers**.

Definition: REAL NUMBER

A **real number** is a number that is either rational or irrational.

Later in this course we will introduce numbers beyond the real numbers. Figure illustrates how the number sets we've used so far fit together.

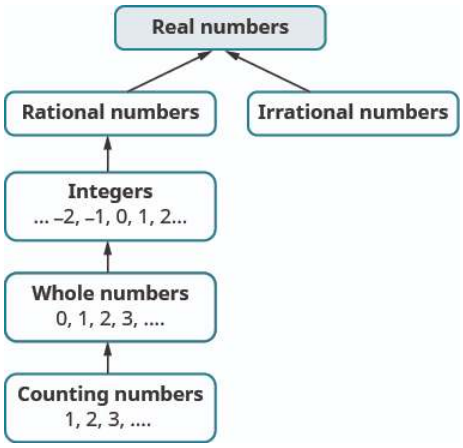


Figure 2. This chart shows the number sets that make up the set of real numbers.

Locate Fractions and Decimals on the Number Line

We now want to include fractions and decimals on the number line. Let's start with fractions and locate $\frac{1}{5}, -\frac{4}{5}, 3, \frac{7}{4}, -\frac{9}{2}, -5$ and $\frac{8}{3}$ on the number line.

We'll start with the whole numbers 3 and −5 because they are the easiest to plot. See Figure.

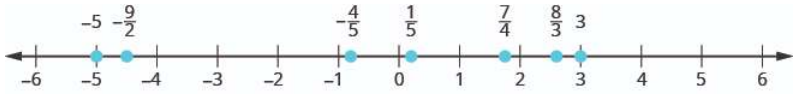
The proper fractions listed are $\frac{1}{5}$ and $-\frac{4}{5}$. We know the proper fraction $\frac{1}{5}$ has value less than one and so would be located between 0 and 1. The denominator is 5, so we divide the unit from 0 to 1 into 5 equal parts $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$. We plot $\frac{1}{5}$.

Similarly, $-\frac{4}{5}$ is between 0 and −1. After dividing the unit into 5 equal parts we plot $-\frac{4}{5}$.

Finally, look at the improper fractions $\frac{7}{4}, \frac{9}{2}, \frac{8}{3}$. Locating these points may be easier if you change each of them to a mixed number.

$$\frac{7}{4} = 1\frac{3}{4} \qquad -\frac{9}{2} = -4\frac{1}{2} \qquad \frac{8}{3} = 2\frac{2}{3} \tag{1.3.5}$$

Figure shows the number line with all the points plotted.



? Exercise 1.3.20

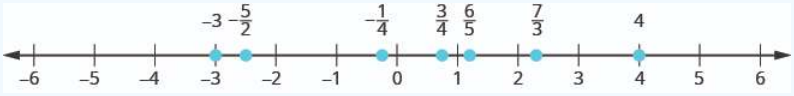
Locate and label the following on a number line: $4, \frac{3}{4}, -\frac{1}{4}, -3, \frac{6}{5}, -\frac{5}{2}$, and $\frac{7}{3}$.

Answer

Locate and plot the integers, $4, -3$.

Locate the proper fraction $\frac{3}{4}$ first. The fraction $\frac{3}{4}$ is between 0 and 1. Divide the distance between 0 and 1 into four equal parts, then we plot $\frac{3}{4}$. Similarly plot $-\frac{1}{4}$.

Now locate the improper fractions $\frac{6}{5}, -\frac{5}{2}$, and $\frac{7}{3}$. It is easier to plot them if we convert them to mixed numbers and then plot them as described above: $\frac{6}{5} = 1\frac{1}{5}, -\frac{5}{2} = -2\frac{1}{2}, \frac{7}{3} = 2\frac{1}{3}$.

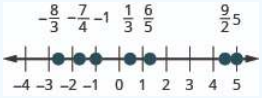


https://math.libretexts.org/@go/page/104798

Exercise 1.3.21

Locate and label the following on a number line: $-1, \frac{1}{3}, \frac{6}{5}, -\frac{7}{4}, \frac{9}{2}, 5, -\frac{8}{3}$.

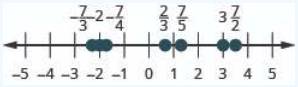
Answer



Exercise 1.3.22

Locate and label the following on a number line: $-2, \frac{2}{3}, \frac{7}{5}, -\frac{7}{4}, \frac{7}{2}, 3, -\frac{7}{3}$.

Answer



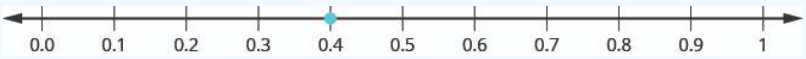
Exercise 1.3.23

Since decimals are forms of fractions, locating decimals on the number line is similar to locating fractions on the number line.

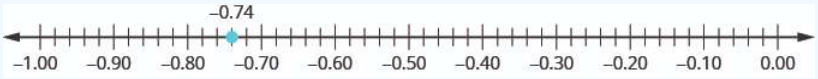
Locate on the number line: ⓐ 0.4 ⓑ −0.74

Answer

ⓐ The decimal number 0.4 is equivalent to $\frac{4}{10}$, a proper fraction, so 0.4 is located between 0 and 1. On a number line, divide the interval between 0 and 1 into 10 equal parts. Now label the parts 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0. We write 0 as 0.0 and 1 as 1.0, so that the numbers are consistently in tenths. Finally, mark 0.4 on the number line.



ⓑ The decimal -0.74 is equivalent to $-\frac{74}{100}$, so it is located between 0 and .−1. On a number line, mark off and label the hundredths in the interval between 0 and −1.

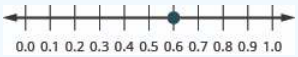


Exercise 1.3.24

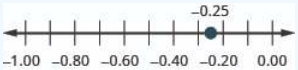
Locate on the number line: ⓐ 0.6 ⓑ -0.25

Answer

ⓐ



ⓑ



Exercise 1.3.25

Locate on the number line: ⓐ 0.90.9 ⓑ −0.75.−0.75

Answer

ⓐ



ⓑ



Access this online resource for additional instruction and practice with decimals.

- Arithmetic Basics: Dividing Decimals

Properties of Real Numbers

Properties of real numbers

Commutative Property

When adding or multiplying, changing the *order* gives the same result

of Addition	If a and $b$are real numbers, then	$a+b=b+a.$	(1.3.6)
of Multiplication	If a and $b$are real numbers, then	$a\cdot b=b\cdot a.$	

Associative Property

When adding or multiplying, changing the *grouping* gives the same result.

of Addition	If $a, b,$ and c are real numbers, then	$(a+b)+c=a+(b+c).$	(1.3.7)
of Multiplication	If $a, b,$ and c are real numbers, then	$(a\cdot b)\cdot c=a\cdot(b\cdot c).$	

Distributive Property

If $a, b,$and $c$are real numbers, then

$$a(b+c)=ab+ac$$
$$(b+c)a=ba+ca$$
$$a(b-c)=ab-ac$$
$$(b-c)a=ba-ca$$

(1.3.8)

Identity Property

of Addition For any real number a : $a+0=a$	$0+a=a$	(1.3.9)
0 is the **additive identity**		
of Multiplication For any real number a : $a\cdot 1=a$	$1\cdot a=a$	
1 is the **multiplicative identity**		

Commutative Property

When adding or multiplying, changing the *order* gives the same result

$$\begin{array}{ll} \textbf{of Addition} & \text{If } a \text{ and } b \text{are real numbers, then} \quad a+b=b+a. \\ \textbf{of Multiplication} & \text{If } a \text{ and } b \text{are real numbers, then} \quad a \cdot b = b \cdot a. \end{array} \tag{1.3.6}$$

Inverse Property

$$\begin{array}{l} \textbf{of addition} \text{ For any real number } a, \qquad a+(-a)=0 \\ \quad -a \text{ is the } \textbf{additive inverse} \text{ of } a \\ \quad \text{A number and its } \textit{opposite} \text{ add to zero.} \\ \\ \textbf{of multiplication} \text{ For any real number } a, a \neq 0 \qquad a \cdot \frac{1}{a} = 1 \\ \quad \frac{1}{a} \text{ is the } \textbf{multiplicative inverse} \text{ of } a \\ \quad \text{A number and its } \textit{reciprocal} \text{ multiply to one.} \end{array} \tag{1.3.10}$$

Properties of Zero

$$\begin{array}{ll} \text{For any real number } a, & a \cdot 0 = 0 \\ & 0 \cdot a = 0 \\ \text{For any real number } a, a \neq 0, & \frac{0}{a} = 0 \\ \text{For any real number } a, & \frac{a}{0} \text{ is undefined} \end{array} \tag{1.3.11}$$

Key Concepts

- **How to round decimals.**
  1. Locate the given place value and mark it with an arrow.
  2. Underline the digit to the right of the place value.
  3. Is the underlined digit greater than or equal to 5?
     - Yes: add 1 to the digit in the given place value.
     - No: do not change the digit in the given place value
  4. Rewrite the number, deleting all digits to the right of the rounding digit.
- **How to add or subtract decimals.**
  1. Determine the sign of the sum or difference.
  2. Write the numbers so the decimal points line up vertically.
  3. Use zeros as placeholders, as needed.
  4. Add or subtract the numbers as if they were whole numbers. Then place the decimal point in the answer under the decimal points in the given numbers.
  5. Write the sum or difference with the appropriate sign
- **How to multiply decimals.**
  1. Determine the sign of the product.
  2. Write in vertical format, lining up the numbers on the right. Multiply the numbers as if they were whole numbers, temporarily ignoring the decimal points.
  3. Place the decimal point. The number of decimal places in the product is the sum of the number of decimal places in the factors.
  4. Write the product with the appropriate sign.
- **How to multiply a decimal by a power of ten.**
  1. Move the decimal point to the right the same number of places as the number of zeros in the power of 10.

2. Add zeros at the end of the number as needed.

- **How to divide decimals.**
  1. Determine the sign of the quotient.
  2. Make the divisor a whole number by "moving" the decimal point all the way to the right. "Move" the decimal point in the dividend the same number of places—adding zeros as needed.
  3. Divide. Place the decimal point in the quotient above the decimal point in the dividend.
  4. Write the quotient with the appropriate sign.
- **How to convert a decimal to a proper fraction and a fraction to a decimal.**
  1. To convert a decimal to a proper fraction, determine the place value of the final digit.
  2. Write the fraction.
     - numerator—the "numbers" to the right of the decimal point
     - denominator—the place value corresponding to the final digit
  3. To convert a fraction to a decimal, divide the numerator of the fraction by the denominator of the fraction.
- **How to convert a percent to a decimal and a decimal to a percent.**
  1. To convert a percent to a decimal, move the decimal point two places to the left after removing the percent sign.
  2. To convert a decimal to a percent, move the decimal point two places to the right and then add the percent sign.
- **Square Root Notation** $\sqrt{m}$ is read "the square root of *m*." If $m = n^2$, then $\sqrt{m} = n$, for $n \geq 0$. The square root of *m*, $\sqrt{m}$, is the positive number whose square is *m*.
- **Rational or Irrational** If the decimal form of a number
  - *repeats or stops*, the number is a rational number.
  - *does not repeat and does not stop*, the number is an irrational number.
- **Real Numbers and their properties**

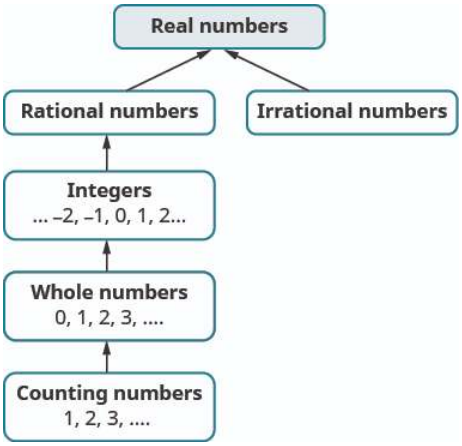


Figure 4.

Glossary

irrational number

An irrational number is a number that cannot be written as the ratio of two integers. Its decimal form does not stop and does not repeat.

percent

A percent is a ratio whose denominator is 100.

principal square root

The positive square root is called the principal square root.

rational number

A rational number is a number of the form $\frac{p}{q}$, where *p* and *q* are integers and $q \neq 0$. Its decimal form stops or repeats.

real number

A real number is a number that is either rational or irrational.

square of a number

If $n^2 = m$, then m is the square of n.

square root of a number

If $n^2 = m$, then n is a square root of m.

additive identity

The number 0 is the additive identity because adding 0 to any number does not change its value.

additive inverse

The opposite of a number is its additive inverse.

multiplicative identity

The number 1 is the multiplicative identity because multiplying 1 by any number does not change its value.

multiplicative inverse

The reciprocal of a number is its multiplicative inverse.

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1.3E: Exercises

Practice Makes Perfect

Round Decimals

Round each number to the nearest ⓐ hundredth ⓑ tenth ⓒ whole number.

1. 5.781
2. 0.299
3. 63.479

Answer

1. ⓐ 5.78 ⓑ 5.8 ⓒ 6
2. ⓐ 0.30 ⓑ 0.3 ⓒ 0
3. ⓐ 63.48 ⓑ 63.5 ⓒ 63

Add and Subtract

In the following exercises, add or subtract.

4. $-16.53-24.38$
5. $-38.69+31.47$
6. $72.5-100$
7. $91.75-(-10.462)$
8. $55.01-3.7$
9. $2.51-7.4$

Answer

4. -40.91
5. -7.22
6. -27.5
7. 102.212
8. 51.31
9. -4.89

Multiply

In the following exercises, multiply.

10. $(-5.18)(-65.23)$
11. $(0.06)(21.75)$
12. $(9.24)(10)$
13. $(0.025)(100)$
14. $(55.2)(1000)$

Answer

10. 337.8914
11. 1.305
12. 92.4
13. 2.5
14. 55200

Divide

In the following exercises, divide. Round to nearest hundredth

15. $117.25 \div 48$
16. $1.44 \div (-0.3)$
17. $5.2 \div 2.5$

Answer

15. 2.44
16. -4.8
17. 2.08

Convert Decimals, Fractions, and Percents

In the following exercises, write each decimal as a fraction.

18. 0.04
19. 0.095

In the following exercises, convert each fraction to a decimal.

20. $\frac{17}{20}$
21. $-\frac{310}{25}$

In the following exercises, convert each percent to a decimal.

22. 71%
23. 39.3%

In the following exercises, convert each decimal to a percent.

24. 1.56
25. 0.0625

Answer

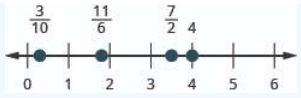
18. $\frac{1}{25}$
19. $\frac{19}{200}$
20. 0.85
21. -12.4
22. 0.71
23. 0.393
24. 156%
25. 6.25%

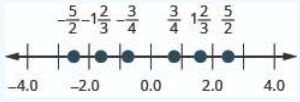
Locate Fractions and Decimals on Number Line

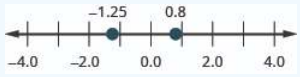
In the following exercises, locate the numbers on a number line.

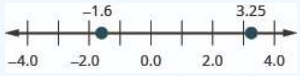
26. $\frac{3}{10}, \frac{7}{2}, \frac{11}{6}, 4$
27. $\frac{3}{4}, -\frac{3}{4}, 1\frac{2}{3}, -1\frac{2}{3}, \frac{5}{2}, -\frac{5}{2}$
28. $0.8, -1.25$
29. $-1.6, 3.25$

Answer

26. 

27. 

28. 

29. 

Writing Exercises

31. When the Szetos sold their home, the selling price was 500% of what they had paid for the house 30 years ago. Explain what 500% means in this context.
32. Explain how the sets of numbers (counting, whole, integer, rational, irrationals, reals) are related to each other.

Answer

Answers will vary.

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1.4: Using Variables to Solve Equations

Learning Objectives

By the end of this section, you will be able to:

- Verify a solution of an equation
- Solve equations using the Subtraction and Addition Properties of Equality
- Solve equations that require simplification
- Translate to an equation and solve
- Translate and solve applications

Warm up

1. Compute $2 \cdot 5 - 3$
2. Evaluate $x + 4$ when $x = -3$.
3. Evaluate $15 - y$ when $y = -5$.

Answer

1. 7
2. 1
3. 20

Verify a Solution of an Equation

Solving an equation is like discovering the answer to a puzzle. The purpose in solving an equation is to find the value or values of the variable that make each side of the equation the same – so that we end up with a true statement. Any value of the variable that makes the equation true is called a solution to the equation. It is the answer to the puzzle!

Definition: SOLUTION OF AN EQUATION

A **solution of an equation** is a value of a variable that makes a true statement when substituted into the equation.

TO DETERMINE WHETHER A NUMBER IS A SOLUTION TO AN EQUATION

1. Substitute the number in for the variable in the equation.
2. Simplify the expressions on both sides of the equation.
3. Determine whether the resulting equation is true (the left side is equal to the right side)
   - If it is true, the number is a solution.
   - If it is not true, the number is not a solution.

Exercise 1.4.1

Determine whether $x = \frac{3}{2}$ is a solution of $4x - 2 = 2x + 1$.

Answer

Since a solution to an equation is a value of the variable that makes the equation true, begin by substituting the value of the solution for the variable.

Solution to the example.

	$4x - 2 = 2x + 1$
Substitute $\frac{3}{2}$ for x.	$4\left(\frac{3}{2}\right) - 2 \stackrel{?}{=} 2\left(\frac{3}{2}\right) + 1$
Multiply.	$6 - 2 \stackrel{?}{=} 3 + 1$
Subtract.	$4 = 4$✓

Since $x = \frac{3}{2}$ results in a true equation (4 is in fact equal to 4), $\frac{3}{2}$ is a solution to the equation $4x - 2 = 2x + 1$.

Exercise 1.4.2

Is $y = \frac{4}{3}$ a solution of $9y + 2 = 6y + 3$?

Answer

no

Exercise 1.4.3

Is $y = \frac{7}{5}$ a solution of $5y + 3 = 10y - 4$?

Answer

yes

Solve Equations Using the Subtraction and Addition Properties of Equality

We are going to use a model to clarify the process of solving an equation. An envelope represents the variable – since its contents are unknown – and each counter represents one. We will set out one envelope and some counters on our workspace, as shown in Figure 1.4.1. Both sides of the workspace have the same number of counters, but some counters are "hidden" in the envelope. Can you tell how many counters are in the envelope?

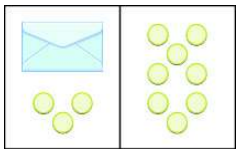


Figure 1.4.1: The illustration shows a model of an equation with one variable. On the left side of the workspace is an unknown (envelope) and three counters, while on the right side of the workspace are eight counters.

What are you thinking? What steps are you taking in your mind to figure out how many counters are in the envelope?

Perhaps you are thinking: "I need to remove the 3 counters at the bottom left to get the envelope by itself. The 3 counters on the left can be matched with 3 on the right and so I can take them away from both sides. That leaves five on the right—so there must be 5 counters in the envelope." See Figure 1.4.2 for an illustration of this process.

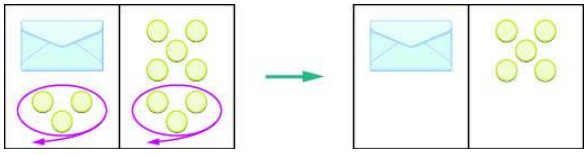


Figure 1.4.2: The illustration shows a model for solving an equation with one variable. On both sides of the workspace remove three counters, leaving only the unknown (envelope) and five counters on the right side. The unknown is equal to five counters.

What algebraic equation would match this situation? In Figure 1.4.3 each side of the workspace represents an expression and the center line takes the place of the equal sign. We will call the contents of the envelope x.

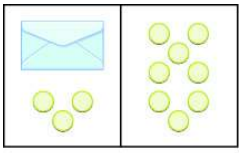


Figure 1.4.3: The illustration shows a model for the equation $x + 3 = 8$.

Let's write algebraically the steps we took to discover how many counters were in the envelope:

Table 1.4.1

	$x + 3 = 8$
First, we took away three from each side.	$x + 3 - 3 = 8 - 3$
Then we were left with five.	$x = 5$

Check:

Five in the envelope plus three more does equal eight!

$$5+3=8 \tag{1.4.1}$$

Our model has given us an idea of what we need to do to solve one kind of equation. The goal is to isolate the variable by itself on one side of the equation. To solve equations such as these mathematically, we use the **Subtraction Property of Equality**.

Definition: SUBTRACTION PROPERTY OF EQUALITY

For any numbers *a*, *b*, and *c*,

$$\begin{aligned} &\text{If} \quad a=b \\ &\text{then} \quad a-c=b-c \end{aligned} \tag{1.4.2}$$

When you subtract the same quantity from both sides of an equation, you still have equality.

Let's see how to use this property to solve an equation. Remember, the goal is to isolate the variable on one side of the equation. And we check our solutions by substituting the value into the equation to make sure we have a true statement.

Exercise 1.4.4

Solve: $y+37=-13$.

Answer

To get *y* by itself, we will undo the addition of 37 by using the Subtraction Property of Equality.

Solution to the example.

	
Subtract 37 from each side to 'undo' the addition.	
Simplify.	$y=-50$
Check:	$y+37=-13$
Substitute $y=-50$	$-50+37=-13$
	$-13\stackrel{?}{=}-13$ ✓

Since y=−50 makes y+37=−13 a true statement, we have the solution to this equation.

Exercise 1.4.5

Solve: $x+19=-27$.

Answer

$x=-46$

Exercise 1.4.6

Solve: $x+16=-34$.

Answer

$x=-50$

What happens when an equation has a number subtracted from the variable, as in the equation $x-5=8$? We use another property of equations to solve equations where a number is subtracted from the variable. We want to isolate the variable, so to 'undo' the subtraction we will add the number to both sides. We use the **Addition Property of Equality**.

Definition: ADDITION PROPERTY OF EQUALITY

For any numbers a, b, and c,

$$\begin{aligned} &\text{If} \quad a = b \\ &\text{then} \quad a + c = b + c \end{aligned} \tag{1.4.3}$$

When you add the same quantity from both sides of an equation, you still have equality.

In Exercise 1.4.4, 37 was added to the y and so we subtracted 37 to 'undo' the addition. In Exercise 1.4.7, we will need to 'undo' subtraction by using the **Addition Property of Equality**.

Exercise 1.4.7

Solve: $a - 28 = -37$.

Answer

Solution to the example.

	
Add 28 to each side to 'undo' the subtraction.	$a - 28 + 28 = -37 + 28$
Simplify.	$a = -9$
Check:	$a - 28 = -37$
Substitute $a = -9$	$-9 - 28 = -37$
	$-37 \stackrel{?}{=} -37$ ✓
	The solution to $a - 28 = -37$ is $a = -9$.

Exercise 1.4.8

Solve: $n - 61 = -75$.

Answer

$n = -14$

Exercise 1.4.9

Solve: $p - 41 = -73$.

Answer

$p = -32$

Exercise 1.4.10

Solve: $x - \frac{5}{8} = \frac{3}{4}$

Answer

Solution to the example.

	$x - \frac{5}{8} = \frac{3}{4}$
Use the Addition Property of Equality.	$x - \frac{5}{8} + \frac{5}{8} = \frac{3}{4} + \frac{5}{8}$
Find the LCD to add the fractions on the right.	$x - \frac{5}{8} + \frac{5}{8} = \frac{6}{8} + \frac{5}{8}$
Simplify.	$x = \frac{11}{8}$
Check:	$x - \frac{5}{8} = \frac{3}{4}$

		$x-\frac{5}{8}=\frac{3}{4}$
Substitute $x=\frac{11}{8}$	$\frac{11}{8}-\frac{5}{8}\stackrel{?}{=}\frac{3}{4}$	
Subtract.	$\frac{6}{8}\stackrel{?}{=}\frac{3}{4}$	
Simplify.	$\frac{3}{4}=\frac{3}{4}\checkmark$	
		The solution to $x-\frac{5}{8}=\frac{3}{4}$ is $x=\frac{11}{8}$.

Exercise 1.4.11

Solve: $p-\frac{2}{3}=\frac{5}{6}$.

Answer

$p=\frac{9}{6}=\frac{3}{2}$

Exercise 1.4.12

Solve: $q-\frac{1}{2}=\frac{5}{6}$.

Answer

$q=\frac{4}{3}$

The next example will be an equation with decimals.

Exercise 1.4.13

Solve: $n-0.63=-4.2$.

Answer

Solution to the example.

		$n-0.63=-4.2$
Use the Addition Property of Equality.		$n-0.63+0.63=-4.2+0.63$
Add.		$n=-3.57$
Check:	$n=-3.57$	
Let $n=-3.57$.	$-3.57-0.63\stackrel{?}{=}-4.2$	
	$-4.2=-4.2\checkmark$	

Exercise 1.4.14

Solve: $b-0.47=-2.1$.

Answer

$b=-1.63$

Exercise 1.4.15

Solve: $c-0.93=-4.6$.

Answer

$c=-3.67$

Solve Equations That Require Simplification

In the previous examples, we were able to isolate the variable with just one operation. Most of the equations we encounter in algebra will take more steps to solve. Usually, we will need to simplify one or both sides of an equation before using the Subtraction or Addition Properties of Equality.

You should always simplify as much as possible before you try to isolate the variable. Remember that to simplify an expression means to do all the operations in the expression. Simplify one side of the equation at a time. Note that **simplification** is different from the process used to solve an equation in which we apply an operation to both sides.

Exercise 1.4.16

Solve: $9x-5-8x-6=7$.

Answer

Step 1. Simplify the expressions on each side as much as possible.	Rearrange the terms, using the Commutative Property of Addition. Combine like terms. Notice that each side is now simplified as much as possible.	$9x-5-8x-6=7$ $9x-8x-5-6=7$ $x-11=7$
Step 2. Isolate the variable.	Now isolate x. Undo subtraction by adding 11 to both sides.	$x-11+11=7+11$
Step 3. Simplify the expressions on both sides of the equation.		$x=18$
Step 4. Check the solution.		**Check:** Substitute $x=18$. $9x-5-8x-6=7$ $9(18)-5-8(18)-6\stackrel{?}{=}7$ $162-5-144-6\stackrel{?}{=}7$ $157-144-6\stackrel{?}{=}7$ $13-6\stackrel{?}{=}7$ $7=7$ ✓ The solution to $9x-5-8x-6=7$ is $x=18$.

Exercise 1.4.17

Solve: $8y-4-7y-7=4$.

Answer

$y=15$

Exercise 1.4.18

Solve: $6z+5-5z-4=3$.

Answer

$z=2$

Exercise 1.4.19

Solve: 5(n−4)−4n=−8.

Answer

We simplify both sides of the equation as much as possible before we try to isolate the variable.

Solution to the example.

	$5(n-4)-4n=-8$
Distribute on the left.	$5n-20-4n=-8$
Use the Commutative Property to rearrange terms.	$5n-4n-20=-8$
Combine like terms.	$n-20=-8$
Each side is as simplified as possible. Next, isolate n.	
Undo subtraction by using the Addition Property of Equality.	$n-20+20=-8+20$
Add.	$n=12$
Check. Substitute n=12. $5(n-4)-4n=-8$ $5(12-4)-4(12)\stackrel{?}{=}-8z$ $5(8)-48\stackrel{?}{=}-8$ $40-48\stackrel{?}{=}-8$ $-8=-8$ ✓	
	The solution to $5(n-4)-4n=-8$ is $n=12$.

Exercise 1.4.20

Solve: $5(p-3)-4p=-10$.

Answer

$p=5$

Exercise 1.4.21

Solve: $4(q+2)-3q=-8$.

Answer

$q=-16$

Exercise 1.4.22

Solve: $3(2y-1)-5y=2(y+1)-2(y+3)$.

Answer

We simplify both sides of the equation before we isolate the variable.

Solution to the example.

	$3(2y-1)-5y=2(y+1)-2(y+3)$
Distribute on both sides.	$6y-3-5y=2y+2-2y-6$
Use the Commutative Property of Addition.	$6y-5y-3=2y-2y+2-6$
Combine like terms.	$y-3=-4$
Each side is as simplified as possible. Next, isolate y.	

	$3(2y-1)-5y=2(y+1)-2(y+3)$
Undo subtraction by using the Addition Property of Equality.	$y-3+3=-4+3$
Add.	$y=-1$
Check. Let y=−1. $3(2y-1)-5y=2(y+1)-2(y+3)$ $3(2(-1)-1)-5(-1)\stackrel{?}{=}2(-1+1)-2(-1+3)$ $3(-2-1)+5\stackrel{?}{=}2(0)-2(2)$ $3(-3)+5\stackrel{?}{=}-4$ $-9+5\stackrel{?}{=}-4$ $-4=-4\checkmark$	
	The solution to $3(2y-1)-5y=2(y+1)-2(y+3)3(2y-1)-5y=2(y+1)-2(y+$ is $y=-1$.

Exercise 1.4.23

Solve: $4(2h-3)-7h=6(h-2)-6(h-1)$.

Answer

$h=6$

Exercise 1.4.24

Solve: $2(5x+2)-9x=3(x-2)-3(x-4)$.

Answer

$x=2$

Solve Equations Using the Division and Multiplication Properties of Equality

You may have noticed that all of the equations we have solved so far have been of the form $x+a=b$ or $x-a=b$. We were able to isolate the variable by adding or subtracting the constant term on the side of the equation with the variable. Now we will see how to solve equations that have a variable multiplied by a constant and so will require division to isolate the variable.

Let's look at our puzzle again with the envelopes and counters in Figure 1.4.1.

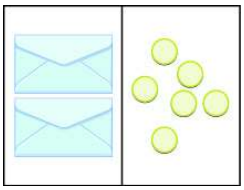


Figure 1.4.1: The illustration shows a model of an equation with one variable multiplied by a constant. On the left side of the workspace are two instances of the unknown (envelope), while on the right side of the workspace are six counters.

In the illustration there are two identical envelopes that contain the same number of counters. Remember, the left side of the workspace must equal the right side, but the counters on the left side are "hidden" in the envelopes. So how many counters are in each envelope?

How do we determine the number? We have to separate the counters on the right side into two groups of the same size to correspond with the two envelopes on the left side. The 6 counters divided into 2 equal groups gives 3 counters in each group (since $6\div 2=3$).

What equation models the situation shown in Figure 1.4.2? There are two envelopes, and each contains xx counters. Together, the two envelopes must contain a total of 6 counters.

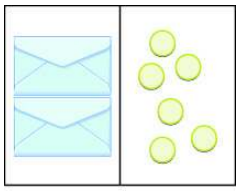


Figure 1.4.2: The illustration shows a model of the equation $2x = 6$.

Algebraic solution to the picture.

	$2x = 6$
If we divide both sides of the equation by 2, as we did with the envelopes and counters,	$\frac{2x}{2} = \frac{6}{2}$
we get:	$x = 3$

We found that each envelope contains 3 counters. Does this check? We know $2 \cdot 3 = 6$, so it works! Three counters in each of two envelopes does equal six!

This example leads to the **Division Property of Equality**.

Definition: DIVISION PROPERTY OF EQUALITY

For any numbers a, b, and c, and $c \neq 0$,

$$\begin{aligned} &\text{If} \quad a = b \\ &\text{then} \quad \frac{a}{c} = \frac{b}{c} \end{aligned} \tag{1.4.4}$$

When you divide both sides of an equation by any non-zero number, you still have equality.

The goal in solving an equation is to 'undo' the operation on the variable. In the next example, the variable is multiplied by 5, so we will divide both sides by 5 to 'undo' the multiplication.

Exercise 1.4.25

Solve: $5x = -27$.

Answer

Solution to the example.

To isolate x, "undo" the multiplication by 5.		$5x = -27$
Divide to 'undo' the multiplication.		$\frac{5x}{5} = -\frac{27}{5}$
Simplify.		$x = -\frac{27}{5}$
Check:	$5x = -27$	
Substitute $-\frac{27}{5}$ for x.	$5\left(-\frac{27}{5}\right) \stackrel{?}{=} -27$	
	$-27 = -27$ ✓	
		Since this is a true statement, $x = -\frac{27}{5}$ is the solution to $5x = -27$.

Exercise 1.4.26

Solve: $3y = -41$.

Answer

$y = -\frac{41}{3}$

Exercise 1.4.27

Solve: $4z = -55$.

Answer

$y = -\frac{55}{4}$

Consider the equation $\frac{x}{4} = 3$. We want to know what number divided by 4 gives 3. So to "undo" the division, we will need to multiply by 4. The **Multiplication Property of Equality** will allow us to do this. This property says that if we start with two equal quantities and multiply both by the same number, the results are equal.

Definition: MULTIPLICATION PROPERTY OF EQUALITY

For any numbers *a*, *b*, and *c*,

$$\begin{array}{lllll} \text{If} & a & = & b \\ \text{then} & ac & = & bc \end{array} \qquad (1.4.5)$$

If you multiply both sides of an equation by the same number, you still have equality.

Exercise 1.4.28

Solve: $\frac{y}{-7} = -14$

Answer

Here y is divided by −7. We must multiply by −7 to isolate y.

Solution to the example.

Multiply both sides by −7.	$-7\left(\frac{y}{-7}\right) = -7(-14)$
Multiply.	$\frac{-7y}{7} = 98$
Simplify.	$y = 98$
Check: $\frac{y}{-7} = -14$	
Substitute y=98.	
Divide.	$-14 = -14$ ✓

Exercise 1.4.29

Solve: $\frac{a}{-7} = -42$

Answer

$a = 294$

Exercise 1.4.30

Solve: $\frac{b}{-6} = -24$

Answer

$b = 144$

Exercise 1.4.31

Solve: $-n = 9$

Answer

Solution to the example.

	$-n = 9$
Remember −n is equivalent to −1n.	$-1n = 9$
Divide both sides by −1.	$\frac{-1n}{-1} = \frac{9}{-1}$
Divide.	$n = -9$
Notice that there are two other ways to solve −n=9. We can also solve this equation by multiplying both sides by −1 and also by taking the opposite of both sides.	
Check:	$-n = 9$
Substitute n=−9.	$-(-9) \stackrel{?}{=} 9$
Simplify.	$9 = 9$ ✓

Exercise 1.4.32

Solve: $-k = 8$.

Answer

$k = -8$

Exercise 1.4.33

Solve: $-g = 3$.

Answer

$g = -3$

Exercise 1.4.34

Solve: $\frac{3}{4}x = 12$

Answer

Since the product of a number and its reciprocal is 1, our strategy will be to isolate x by multiplying by the reciprocal of $\frac{3}{4}$.

Solution to the example.

	$\frac{3}{4}x = 12$
Multiply by the reciprocal of $\frac{3}{4}$.	$\frac{4}{3} \cdot \frac{3}{4}x = \frac{4}{3} \cdot 12$
Reciprocals multiply to 1.	$1x = \frac{4}{3} \cdot \frac{12}{1}$
Multiply.	$x = 16$
Notice that we could have divided both sides of the equation $\frac{3}{4}x = 12$ by $\frac{3}{4}$ to isolate x. While this would work, most people would find multiplying by the reciprocal easier.	
Check:	$\frac{3}{4}x = 12$
Substitute $x = 16$.	$\frac{3}{4} \cdot 16 \stackrel{?}{=} 12$
	$12 = 12$ ✓

Exercise 1.4.35

Solve: $\frac{2}{5}n = 14$.

Answer

$n = 35$

Exercise 1.4.36

Solve: $\frac{5}{6}y = 15$.

Answer

$y = 18$

In the next example, all the variable terms are on the right side of the equation. As always, our goal in solving the equation is to isolate the variable.

Exercise 1.4.37

Solve: $\frac{8}{15} = -\frac{4}{5}x$

Answer

Solution to the example.

	$\frac{8}{15} = -\frac{4}{5}x$
Multiply by the reciprocal of $-\frac{4}{5}$.	$\left(-\frac{5}{4}\right)\left(\frac{8}{15}\right) = \left(-\frac{5}{4}\right)\left(-\frac{4}{5}x\right)$
Reciprocals multiply to 1.	$-\frac{\cancel{5}\cdot\cancel{4}\cdot 2}{\cancel{4}\cdot 3\cdot\cancel{5}} = 1x$
Multiply.	$-\frac{2}{3} = x$
Check:	$\frac{8}{15} = -\frac{4}{5}x$
Let $x = -\frac{2}{3}$.	$\frac{8}{15} = -\frac{4}{5}\left(-\frac{2}{3}\right)$
	$\frac{8}{15} = \frac{8}{15}$ ✓

Exercise 1.4.38

Solve: $\frac{9}{25} = -\frac{4}{5}z$

Answer

$z = -\frac{9}{20}$

Exercise 1.4.39

$\frac{5}{6} = -\frac{8}{3}r$

Answer

$r = -\frac{5}{16}$

Solve Equations That Require Simplification

Many equations start out more complicated than the ones we have been working with.

With these more complicated equations the first step is to simplify both sides of the equation as much as possible. This usually involves combining like terms or using the distributive property.

Exercise 1.4.40

Solve: $14-23=12y-4y-5y$.

Answer

Begin by simplifying each side of the equation.

Solution to the example.

	$14-23\stackrel{?}{=}-36+12+15$
Simplify each side.	$-9=-9$ ✓
Divide both sides by 3 to isolate y.	$14-23=12y-4y-5y$
Divide.	$-9=3y$
Check:	$14-23=12y-4y-5y$
Substitute $y=-3$.	$14-23\stackrel{?}{=}12(-3)-4(-3)-5(-3)$
	$14-23\stackrel{?}{=}-36+12+15$
	$-9=-9$ ✓

Exercise 1.4.41

Solve: $18-27=15c-9c-3c$.

Answer

$c=-3$

Exercise 1.4.42

Solve: $18-22=12x-x-4x$.

Answer

$x=-\frac{4}{7}$

Exercise 1.4.43

Solve: $-4(a-3)-7=25$.

Answer

Here we will simplify each side of the equation by using the distributive property first.

Solution to the example.

	$-4(a-3)-7=25$
Distribute.	$-4a+12-7=25$
Simplify.	$-4a+5=25$
Simplify.	$-4a=20$
Divide both sides by -4 to isolate a.	$\frac{-4a}{-4}=\frac{20}{-4}$

		$-4(a-3)-7=25$
Divide.		$a=-5$
Check:		
Substitute $a=-5$	$-4(-5-3)-7 \stackrel{?}{=} 25$	
	$-4(-8)-7 \stackrel{?}{=} 25$	
	$32-7 \stackrel{?}{=} 25$	
	$25=25$ ✓	

Exercise 1.4.44

Solve: $-4(q-2)-8=24$.

Answer

$q=-6$

Exercise 1.4.45

Solve: $-6(r-2)-12=30$.

Answer

$r=-5$

Now we have covered all four properties of equality—subtraction, addition, division, and multiplication. We'll list them all together here for easy reference.

PROPERTIES OF EQUALITY

When you add, subtract, multiply, or divide the same quantity from both sides of an equation, you still have equality.

Subtraction Property of Equality	**Addition Property of Equality**
For any real numbers a, b, and c,	For any real numbers a, b, and c,
if $a=b$,	if $a=b$,
then $a-c=b-c$	then $a+c=b+c$
Divition Property of Equality	**Multiplication Property of Equality**
For any real numbers a, b, and c,	For any real numbers a, b, and c,
if $a=b$,	if $a=b$,
then $a-c=b-c$	then $a+c=b+c$

(1.4.6)

When you add, subtract, multiply, or divide the same quantity from both sides of an equation, you still have equality.

Key Concepts

- **To Determine Whether a Number is a Solution to an Equation**
  1. **Substitute the number in for the variable in the equation.**
  2. **Simplify the expressions on both sides of the equation.**
  3. **Determine whether the resulting statement is true.**
     - If it is true, the number is a solution.
     - If it is not true, the number is not a solution.
- **Addition Property of Equality**
  - For any numbers *a*, *b*, and *c*, if a=b, then a+c=b+c.
- **Subtraction Property of Equality**
  - For any numbers *a*, *b*, and *c*, if a=b, then a−c=b−c.

- **The Division Property of Equality**—For any numbers a, b, and c, and $c \neq 0$, if $a = b$, then $\frac{a}{c} = \frac{b}{c}$.
  When you divide both sides of an equation by any non-zero number, you still have equality.
- **The Multiplication Property of Equality**—For any numbers a, b, and c, if $a = b$, then $ac = bc$.
  If you multiply both sides of an equation by the same number, you still have equality.

Glossary

solution of an equation

A solution of an equation is a value of a variable that makes a true statement when substituted into the equation.

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1.4E: Exercises

Practice Makes Perfect

Verify a Solution

In the following exercises, determine whether the given value is a solution to the equation.

1. Is $y=\frac{5}{3}$ a solution of $6y+10=12y$?
2. Is $u=-\frac{1}{2}$ a solution of $8u-1=6u$?
3. Is $v=-\frac{1}{3}$ a solution of $9v-2=3v$?
4. Is $v=\frac{1}{3}$ a solution of $9v-2=3v$?

Answer

1. Yes
2. No
3. No
4. Yes

Solve Equations using Subtraction and Addition

In the following exercises, solve each equation using the Subtraction and Addition Properties of Equality.

5. $x+24=35$
6. $y+45=-66$
7. $b+\frac{1}{4}=\frac{3}{4}$
8. $p+2.4=-9.3$
9. $a-45=76$
10. $m-18=-200$
11. $x-\frac{1}{3}=2$
12. $y-3.8=10$
13. $x-165=-420$
14. $z+0.52=-8.5$
15. $q+\frac{3}{4}=\frac{1}{2}$
16. $p-\frac{2}{5}=\frac{2}{3}$
17. $c+31-10=46$
18. $9x+5-8x+14=20$
19. $-6x-11+7x-5=-16$

Answer

5. x = 11
6. y = -111
7. $b=\frac{1}{2}$
8. p = -11.7
9. a = 121
10. m = -182
11. $x=\frac{7}{3}$
12. y = 13.8
13. $x=-255$
14. $z=-9.02$
15. $q=-\frac{1}{4}$
16. $p=\frac{16}{15}$

17. c = 25
18. x = 1
19. x = 0

Solve Equations using Division and Multiplication

In the following exercises, solve each equation using the Division and Multiplication Properties of Equality.

20. $8x = 56$
21. $-5c = 55$
22. $-809 = 15y$
23. $-37p = -541$
24. $0.25z = 3.25$
25. $-13x = 0$
26. $\frac{x}{4} = 35$
27. $-20 = \frac{q}{-5}$
28. $\frac{y}{9} = -16$
29. $\frac{m}{-12} = 45$
30. $-y = 6$
31. $-v = -72$
32. $-\frac{5}{8}w = 40$
33. $-\frac{2}{5} = \frac{1}{10}a$
34. $100 - 16 = 4p - 10p - p$
35. $\frac{7}{8}n - \frac{3}{4}n = 9 + 2$
36. $0.25d + 0.10d = 6 - 0.75$
37. $-10(q - 4) - 57 = 93$
38. $-10(x + 4) - 19 = 85$

Answer

20. $x = 7$
21. $c = -11$
22. $y = -\frac{809}{15}$
23. $p = \frac{541}{37}$
24. z= 13
25. $x = 0$
26. $x = 140$
27. $q = 100$
28. $y = -144$
29. $m = -540$
30. $y = -6$
31. $v = 72$
32. $w = -64$
33. $a = -4$
34. $p = -12$
35. $n = 88$
36. d=15
37. $q = -11$
38. $x = -\frac{72}{5}$

Everyday Math

39. **Construction** Miguel wants to drill a hole for a $\frac{5}{8}$ inch screw. The hole should be $\frac{1}{12}$ inch smaller than the screw. Let d equal the size of the hole he should drill. Solve the equation $d = \frac{5}{8} - \frac{1}{12}$ to see what size the hole should be.
40. **Baking** Kelsey needs $\frac{2}{3}$ cup of sugar for the cookie recipe she wants to make. She only has $\frac{3}{8}$ cup of sugar and will borrow the rest from her ner neighbor. Let s equal the amount of sugar she will borrow. Solve the equation $\frac{3}{8} + s = \frac{2}{3}$ to find the amount of sugar she should ask to borrow.
41. **Commissions** Every week Perry gets paid $\$150$ plus 12% of his total sales amount. Solve the equation $840 = 150 + 0.12(a - 1250)$ for a to find the total amount Perry must sell in order to be paid $\$840$ one week.
42. **Stamps** Travis bought $9.45 worth of 49-cent stamps and 21-cent stamps. The number of 21-cent stamps was 5 less than the number of 49-cent stamps. Solve the equation $0.49s+0.21(s-5)=9.45$ for s, to find the number of 49-cent stamps Travis bought.

Answer

39. $d = \frac{13}{24}$ inch
40. $s = \frac{7}{24}$ cups
41. $a = 7000$
42. 15 49-cent stamps

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1.5: Solve Linear Inequalities

Learning Objectives

By the end of this section, you will be able to:

- Graph inequalities on the number line
- Solve linear inequalities
- Translate words to an inequality and solve
- Solve applications with linear inequalities

Warm-up

Solve

a. $x+3=5$
b. $-x+3=5$
c. $-2x-3=-9$

Solution

a. 2
b. -2
c. 3

Graph Inequalities on the Number Line

What number would make the inequality $x>3$ true? Are you thinking, "x could be four"? That's correct, but x could be 6, too, or 37, or even 3.001. Any number greater than three is a solution to the inequality $x>3$. We show all the solutions to the inequality $x>3$ on the number line by shading in all the numbers to the right of three, to show that all numbers greater than three are solutions. Because the number three itself is not a solution, we put an open parenthesis at three.

We can also represent inequalities using **interval notation**. There is no upper end to the solution to this inequality. In interval notation, we express $x>3$ as $(3,\infty)$. The symbol ∞ is read as "**infinity**." It is not an actual number. Figure $1.5.1$ shows both the number line and the interval notation.

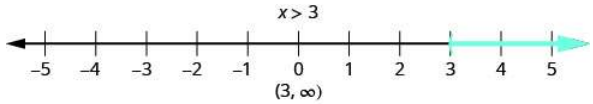


Figure 1.5.1: The inequality $x>3$ is graphed on this number line and written in interval notation.

We use the left parenthesis symbol, (, to show that the endpoint of the inequality is not included. The left bracket symbol, [, shows that the endpoint is included.

The inequality $x\leq 1$ means all numbers less than or equal to one. Here we need to show that one is a solution, too. We do that by putting a bracket at $x=1$. We then shade in all the numbers to the left of one, to show that all numbers less than one are solutions (Figure $1.5.2$). There is no lower end to those numbers. We write $x\leq 1$ in interval notation as $(-\infty,1]$. The symbol $-\infty$ is read as "negative infinity."

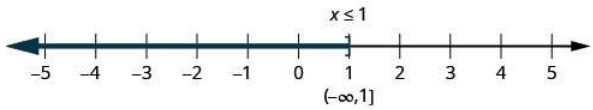


Figure 1.5.2: The inequality $x\leq 1$ is graphed on this number line and written in interval notation.

Figure $1.5.3$ shows both the number line and interval notation.

INEQUALITIES, NUMBER LINES, AND INTERVAL NOTATION



The notation for inequalities on a number line and in interval notation use the same symbols to express the endpoints of intervals.

Example 1.5.1

Graph each inequality on the number line and write in interval notation.

a. $x \geq -3$
b. $x < 2.5$
c. $x \leq -\frac{3}{5}$

Answer

ⓐ

Solution to a.

	$x \geq -3$
Shade to the right of -3, and put a bracket at -3.	
Write in interval notation.	$[-3, \infty)$

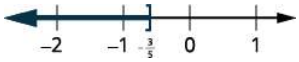
ⓑ

Solution to b.

	$x < 2.5$
Shade to the left of 2.5 and put a parenthesis at 2.5.	
Write in interval notation.	$(-\infty, 2.5)$

ⓒ

Solution to c.

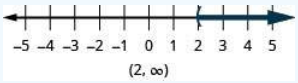
	$x \leq -\frac{3}{5}$
Shade to the left of $-\frac{3}{5}$, and put a bracket at $-\frac{3}{5}$.	
Write in interval notation.	$\left(-\infty, \frac{3}{5}\right]$

Graph each inequality on the number line and write in interval notation:

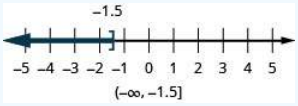
a. $x > 2$
b. $x \leq -1.5$
c. $x \geq \frac{3}{4}$.

Answer

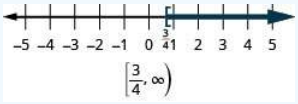
ⓐ



ⓑ



ⓒ



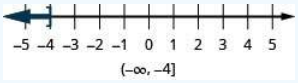
Example 1.5.3

Graph each inequality on the number line and write in interval notation:

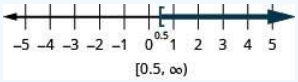
a. $x \leq -4$
b. $x \geq 0.5$
c. $x < -\frac{2}{3}$.

Answer

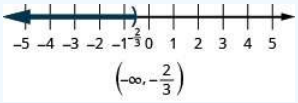
ⓐ



ⓑ



ⓒ



What numbers are greater than two but less than five? Are you thinking say, 2.5, 3, $3\frac{2}{3}$, 4? We can represent all the numbers between two and five with the inequality $2 < x < 5$. We can show $2 < x < 5$ on the number line by shading all the numbers between two and five. Again, we use the parentheses to show the numbers two and five are not included. See Figure.



The interval notation would be (2, 5).

Figure 1.5.3

Graph each inequality on the number line and write in interval notation.

ⓐ $-3 < x < 4$ ⓑ $-6 \leq x < -1$ ⓒ $0 \leq x \leq 2.5$

Answer

ⓐ

Solution to a.

		$-3 < x < 4$
Shade between -3 and 4. Put a parentheses at -3 and 4.		
Write in interval notation.		$(-3, 4)$

ⓑ

Solution to b.

			$-6 \leq x < -1$
Shade between -6 and -1. Put a bracket at -6, and a parentheses at -1.			
Write in interval notation.			$[-6, 1)$

ⓒ

Solution to c.

			$0 \leq x \leq 2.5$
Shade between 0 and 2.5. Put a bracket at 0 and at 2.5.			
Write in interval notation.			$[0, 2.5]$

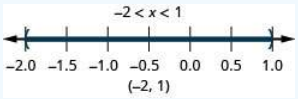
Example 1.5.5

Graph each inequality on the number line and write in interval notation:

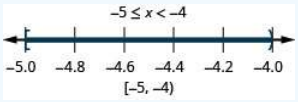
ⓐ $-2 < x < 1$ ⓑ $-5 \leq x < -4$ ⓒ $1 \leq x \leq 4.25$

Answer

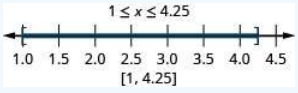
ⓐ



ⓑ



ⓒ



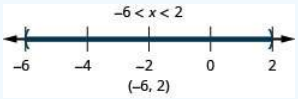
Example 1.5.6

Graph each inequality on the number line and write in interval notation:

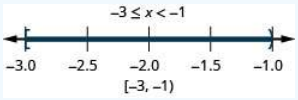
ⓐ $-6 < x < 2$ ⓑ $-3 \leq x < -1$ ⓒ $2.5 \leq x \leq 6$

Answer

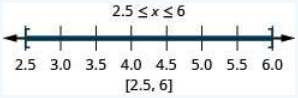
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ⓑ



ⓒ



Solve Linear Inequalities

A linear inequality is much like a linear equation—but the equal sign is replaced with an inequality sign. A **linear inequality** is an inequality in one variable that can be written in one of the forms, $ax+b<c$, $ax+b\le c$, $ax+b>c$, or $ax+b\ge c$.

Definition: LINEARE INEQUALITY

A linear inequality is an inequality in one variable that can be written in one of the following forms where a, b, and c are real numbers and $a\neq 0$:

$$ax+b<c,\quad ax+b\le c,\quad ax+b>c,\quad ax+b\ge c.$$

When we solved linear equations, we were able to use the properties of equality to add, subtract, multiply, or divide both sides and still keep the equality. Similar properties hold true for inequalities.

We can add or subtract the same quantity from both sides of an inequality and still keep the inequality. For example:

$$\begin{array}{ll} -4<2 & -4<2 \\ -4-5<2-5 & -4+7<2+7 \\ -9<-3 \text{ True} & 3<9 \text{ True} \end{array}$$

Notice that the inequality sign stayed the same.

This leads us to the Addition and Subtraction Properties of Inequality.

Definition: ADDITION AND SUBTRACTION PROPERTY OF INEQUALITY

For any numbers a, b, and c, if $a<b$, then

$$\begin{array}{ll} a+c<b+c & a-c<b-c \\ a+c>b+c & a-c>b-c \end{array}$$

We can add or subtract the same quantity from both sides of an inequality and still keep the inequality

What happens to an inequality when we divide or multiply both sides by a constant?

Let's first multiply and divide both sides by a positive number.

$$\begin{array}{ll} 10<15 & 10<15 \\ 10(5)<15(5) & \frac{10}{5}<\frac{15}{5} \\ 50<75 \text{ True} & 2<3 \text{ True} \end{array}$$

The inequality signs stayed the same.

Does the inequality stay the same when we divide or multiply by a negative number?

$10 < 15$	$10 < 15$
$10(-5) \;?\; 15(-5)$	$\frac{10}{-5} \;?\; \frac{15}{-5}$
$-50 \;?\; -75$	$-2 \;?\; -3$
$-50 > -75$	$-2 > -3$

Notice that when we filled in the inequality signs, the inequality signs reversed their direction.

When we divide or multiply an inequality by a positive number, the inequality sign stays the same. When we divide or multiply an inequality by a negative number, the inequality sign reverses.

This gives us the Multiplication and Division Property of Inequality.

Definition: MULTIPLICATION AND DIVISION PROPERTY OF INEQUALITY

For any numbers a, b, and c,

multiply or divide by a positive

if $a < b$ and $c > 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.
if $a > b$ and $c > 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.

multiply or divide by a negative

if $a < b$ and $c < 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.
if $a > b$ and $c < 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.

When we divide or multiply an inequality by a:

- positive number, the inequality stays the same.
- negative number, the inequality reverses.

Sometimes when solving an inequality, as in the next example, the variable ends upon the right. We can rewrite the inequality in reverse to get the variable to the left.

$$x > a \text{ has the same meaning as } a < x$$

Think about it as “If Xander is taller than Andy, then Andy is shorter than Xander.”

Example 1.5.7

Solve each inequality. Graph the solution on the number line, and write the solution in interval notation.

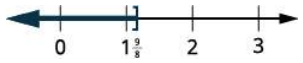
ⓐ $x - \frac{3}{8} \leq \frac{3}{4}$ ⓑ $9y < 54$ ⓒ $-15 < \frac{3}{5}z$

Answer

ⓐ

Solution to a.

	$x - \frac{3}{8} \leq \frac{3}{4}$
Add 3838 to both sides of the inequality.	$x - \frac{3}{8} + \frac{3}{8} \leq \frac{3}{4} + \frac{3}{8}$
Simplify.	$x \leq \frac{9}{8}$

	$x-\frac{3}{8} \leq \frac{3}{4}$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, \frac{9}{8}]$

ⓑ

Solution to b.

	$9y < 54$
Divide both sides of the inequality by 9; since 9 is positive, the inequality stays the same.	$\frac{9y}{9} < \frac{54}{9}$
Simplify.	$y < 6$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, 6)$

ⓒ

Solution to c.

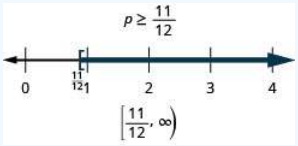
	$-15 < \frac{3}{5}z$
Multiply both sides of the inequality by $\frac{5}{3}$. Since $\frac{5}{3}$ is positive, the inequality stays the same.	$\frac{5}{3}(-15) < \frac{5}{3}\left(\frac{3}{5}z\right)$
Simplify.	$-25 < z$
Rewrite with the variable on the left.	$z > -25$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-25, \infty)$

Solve each inequality, graph the solution on the number line, and write the solution in interval notation:

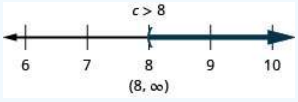
ⓐ $p-\frac{3}{4} \geq \frac{1}{6}$ ⓑ $9c > 72$ ⓒ $24 \leq \frac{3}{8}m$

Answer

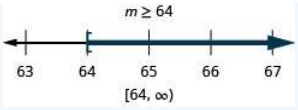
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ⓑ



ⓒ



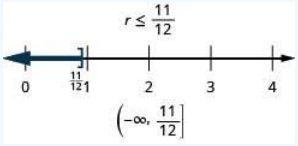
Example 1.5.9

Solve each inequality, graph the solution on the number line, and write the solution in interval notation:

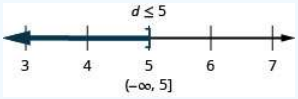
ⓐ $r-\frac{1}{3}\leq\frac{7}{12}$ ⓑ $12d\leq 60$ ⓒ $-24<\frac{4}{3}n$

Answer

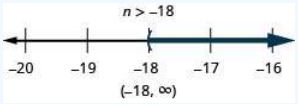
ⓐ



ⓑ



ⓒ



Be careful when you multiply or divide by a negative number—remember to reverse the inequality sign.

Example 1.5.10

Solve each inequality, graph the solution on the number line, and write the solution in interval notation.

ⓐ $-\frac{1}{3}m\geq\frac{6}{5}$ ⓑ $\frac{n}{-2}\geq 8$

Answer

ⓐ

Solution to a.

	$-13m \geq 65$
Divide both sides of the inequality by $-\frac{1}{3}$. Since $-\frac{1}{3}$ is a negative, the inequality reverses.	$\frac{-13m}{-13} \leq \frac{65}{-13}$
Simplify.	$m \leq -5$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, -5]$

ⓑ

Solution to b.

	$\frac{n}{-2} \geq 8$
Multiply both sides of the inequality by -2. Since -2 is a negative, the inequality reverses.	$-2\left(\frac{n}{-2}\right) \leq -2(8)$
Simplify.	$n \leq -16$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, -16]$

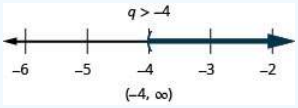
Example 1.5.11

Solve each inequality, graph the solution on the number line, and write the solution in interval notation:

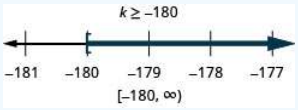
ⓐ $-8q < 32$ ⓑ $\frac{k}{-12} \leq 15$.

Answer

ⓐ



ⓑ



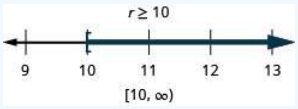
Example 1.5.12

Solve each inequality, graph the solution on the number line, and write the solution in interval notation:

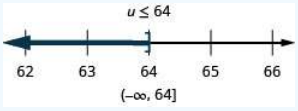
ⓐ $-7r \leq -70$ ⓑ $\frac{u}{-4} \geq -16$.

Answer

ⓐ



ⓑ



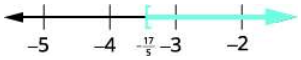
Most inequalities will take more than one step to solve. We follow the same steps we used in the general strategy for solving linear equations, but make sure to pay close attention when we multiply or divide to isolate the variable.

Example 1.5.13

Solve the inequality $6y \leq 11y + 17$, graph the solution on the number line, and write the solution in interval notation.

Solution

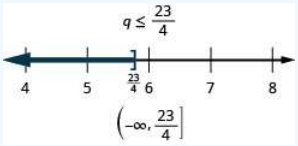
Solution to the example.

	$6y \leq 11y + 17$
Subtract 11y from both sides to collect the variables on the left.	$6y - 11y \leq 11y - 11y + 17$
Simplify.	$-5y \leq 17$
Divide both sides of the inequality by -5, and reverse the inequality.	$\frac{-5y}{-5} \geq \frac{17}{-5}$
Simplify.	$y \geq \frac{17}{-5}$
Graph the solution on the number line.	
Write the solution in interval notation.	$\left[-\frac{17}{5}, \infty\right)$

Exercise 1.5.14

Solve the inequality, graph the solution on the number line, and write the solution in interval notation: $3q \geq 7q - 23$.

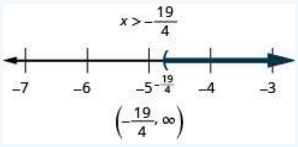
Answer



Example 1.5.15

Solve the inequality, graph the solution on the number line, and write the solution in interval notation: $6x < 10x + 19$.

Answer



When solving inequalities, it is usually easiest to collect the variables on the side where the coefficient of the variable is largest. This eliminates negative coefficients and so we don't have to multiply or divide by a negative—which means we don't have to remember to reverse the inequality sign.

Example 1.5.16

Solve the inequality $8p + 3(p - 12) > 7p - 28$, graph the solution on the number line, and write the solution in interval notation.

Answer

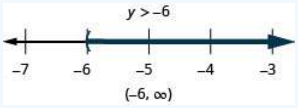
Solution to the example.

	$8p+3(p-12)>7p-28$
Simplify each side as much as possible.	
Distribute.	$8p+3p-36>7p-28$
Combine like terms.	$11p-36>7p-28$
Subtract $7p$ from both sides to collect the variables on the left, since $11>7$.	$11p-36-7p>7p-28-7p$
Simplify.	$4p-36>-28$
Add 36 to both sides to collect the constants on the right.	$4p-36+36>-28+36$
Simplify.	$4p>8$
Divide both sides of the inequality by 4; the inequality stays the same.	$\frac{4p}{4}>\frac{8}{4}$
Simplify.	$p>2$
Graph the solution on the number line.	
Write the solution in interval notation.	$(2,\infty)$

? Example 1.5.17

Solve the inequality $9y+2(y+6)>5y-24$, graph the solution on the number line, and write the solution in interval notation.

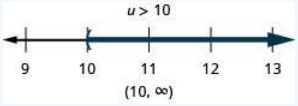
Answer



? Example 1.5.18

Solve the inequality $6u+8(u-1)>10u+32$, graph the solution on the number line, and write the solution in interval notation.

Answer



Just like some equations are identities and some are contradictions, inequalities may be identities or contradictions, too. We recognize these forms when we are left with only constants as we solve the inequality. If the result is a true statement, we have an identity. If the result is a false statement, we have a contradiction.

? Example 1.5.19

Solve the inequality $8x-2(5-x)<4(x+9)+6x$, graph the solution on the number line, and write the solution in interval notation.

Answer

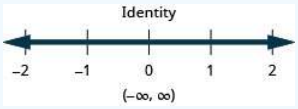
Solution to the example.

Simplify each side as much as possible.		$8x-2(5-x)<4(x+9)+6x$
Distribute.		$8x-10+2x<4x+36+6x$
Combine like terms.		$10x-10<10x+36$
Subtract $10x$ from both sides to collect the variables on the left.		$10x-10-10x<10x+36-10x$
Simplify.		$-10<36$
The x's are gone, and we have a true statement.		The inequality is an identity. The solution is all real numbers.
Graph the solution on the number line.		
Write the solution in interval notation.		$(-\infty,\infty)$

Example 1.5.20

Solve the inequality $4b-3(3-b)>5(b-6)+2b$, graph the solution on the number line, and write the solution in interval notation.

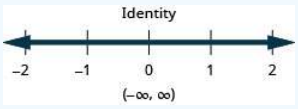
Answer



Example 1.5.21

Solve the inequality $9h-7(2-h)<8(h+11)+8h$, graph the solution on the number line, and write the solution in interval notation.

Answer



We can clear fractions in inequalities much as we did in equations. Again, be careful with the signs when multiplying or dividing by a negative.

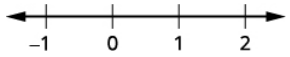
Example 1.5.22

Solve the inequality $\frac{1}{3}a-\frac{1}{8}a>\frac{5}{24}a+\frac{3}{4}$, graph the solution on the number line, and write the solution in interval notation.

Answer

Solution to the example.

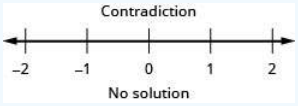
	$\frac{1}{3}a-\frac{1}{8}a>\frac{5}{24}a+\frac{3}{4}$
Multiply both sides by the LCD, 24, to clear the fractions.	$24\left(\frac{1}{3}a-\frac{1}{8}a\right)>24\left(\frac{5}{24}a+\frac{3}{4}\right)$

	$\frac{1}{3}a-\frac{1}{8}a>\frac{5}{24}a+\frac{3}{4}$
Simplify.	$8a-3a>5a+18$
Combine like terms.	$5a>5a+18$
Subtract $5a$ from both sides to collect the variables on the left.	$5a-5a>5a-5a+18$
Simplify.	$0>18$
The statement is false.	The inequality is a contradiction. There is no solution.
Graph the solution on the number line.	
Write the solution in interval notation.	There is no solution.

? Example 1.5.23

Solve the inequality $\frac{1}{4}x-\frac{1}{12}x>\frac{1}{6}x+\frac{7}{8}$, graph the solution on the number line, and write the solution in interval notation.

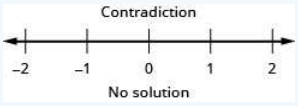
Answer



? Example 1.5.24

Solve the inequality $\frac{2}{5}z-\frac{1}{3}z<\frac{1}{15}z-\frac{3}{5}$, graph the solution on the number line, and write the solution in interval notation.

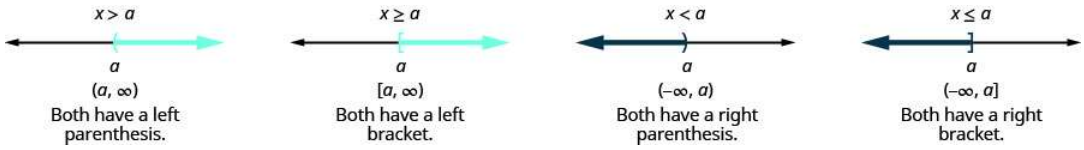
Answer



Key Concepts

- **Inequalities, Number Lines, and Interval Notation**

  $x>a$ $x\geq a$ $x<a$ $x\leq a$

- **Linear Inequality**
  - A **linear inequality** is an inequality in one variable that can be written in one of the following forms where *a*, *b*, and *c* are real numbers and $a\neq 0$:

$$ax+b<c,\qquad ax+b\leq c,\qquad ax+b>c,\qquad ax+b\geq c.$$

- **Addition and Subtraction Property of Inequality**
  - For any numbers *a*, *b*, and *c*, if a<b,thena<b,then

$$a+c<b+c \qquad a-c<b-c$$
$$a+c>b+c \qquad a-c>b-c$$

- We can add or subtract the same quantity from both sides of an inequality and still keep the inequality.

- **Multiplication and Division Property of Inequality**
  - For any numbers *a*, *b*, and *c*,
    multiply or divide by a **positive**

$$\text{if } a<b \text{ and } c>0, \text{ then } ac<bc \text{ and } \frac{a}{c}<\frac{b}{c}.$$
$$\text{if } a>b \text{ and } c>0, \text{ then } ac>bc \text{ and } \frac{a}{c}>\frac{b}{c}.$$

    multiply or divide by a **negative**

$$\text{if } a<b \text{ and } c<0, \text{ then } ac>bc \text{ and } \frac{a}{c}>\frac{b}{c}.$$
$$\text{if } a>b \text{ and } c<0, \text{ then } ac<bc \text{ and } \frac{a}{c}<\frac{b}{c}.$$

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1.5E: Exercises

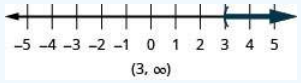
Practice Makes Perfect

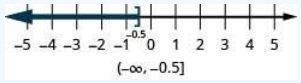
Graph Inequalities

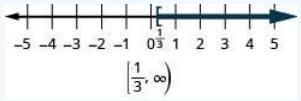
In the following exercises, graph each inequality on the number line and write in interval notation.

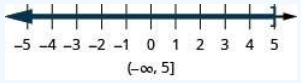
1. $x > 3$
2. $x \leq -0.5$
3. $x \geq \frac{1}{3}$
4. $x \leq 5$
5. $x \geq -1.5$
6. $x < -\frac{7}{3}$
7. $-2 < x < 0$
8. $-5 \leq x < -3$
9. $0 \leq x \leq 3.5$
10. $-4 < x < 2$
11. $-5 < x \leq -2$
12. $-3.75 \leq x \leq 0$

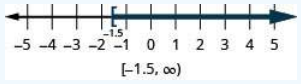
Answer

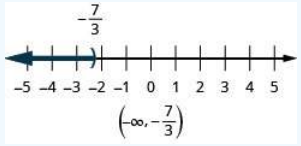
1. 

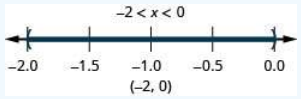
2. 

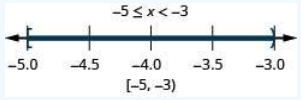
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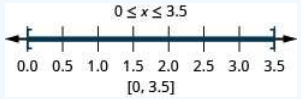
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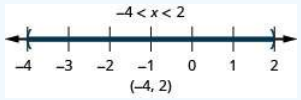
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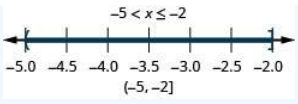
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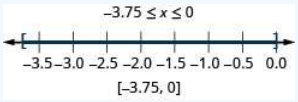
7. 

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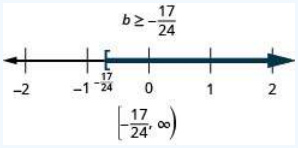
12. 

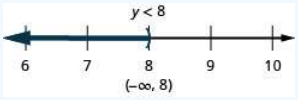
Solve Inequalities

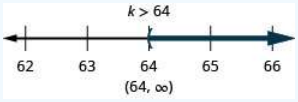
In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

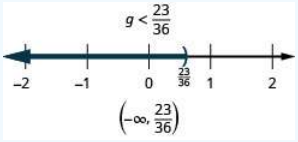
13. $b+\frac{7}{8} \geq \frac{1}{6}$
14. $6y < 48$
15. $40 < \frac{5}{8}k$
16. $g-\frac{11}{12} < -\frac{5}{18}$
17. $7s < -28$
18. $\frac{9}{4}g \leq 36$
19. $-8v \leq 96$
20. $\frac{b}{-10} \geq 30$
21. $-7d > 105$
22. $-18 > \frac{q}{-6}$

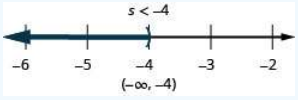
Answer

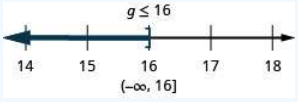
13. 

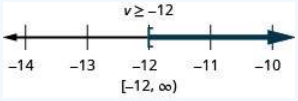
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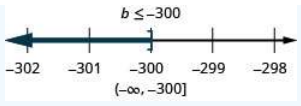
15. 

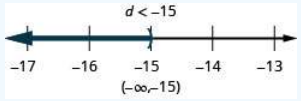
16. 

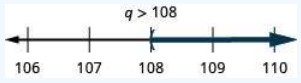
17. 

18. 

19. 

20. 

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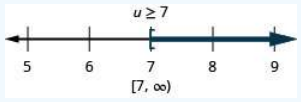
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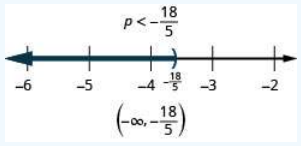
Now with Interval Notation

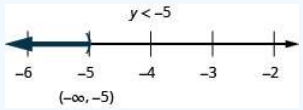
In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

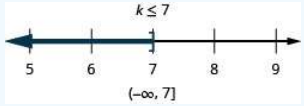
23. $5u \leq 8u - 21$
24. $9p > 14p + 18$
25. $9y + 5(y + 3) < 4y - 35$
26. $4k - (k - 2) \geq 7k - 26$
27. $6n - 12(3 - n) \leq 9(n - 4) + 9n$
28. $9u + 5(2u - 5) \geq 12(u - 1) + 7u$
29. $12v + 3(4v - 1) \leq 19(v - 2) + 5v$
30. $35k \geq -77$
31. $18q - 4(10 - 3q) < 5(6q - 8)$
32. $-\frac{21}{8}y \leq -\frac{15}{28}$

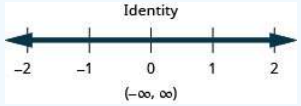
Answer

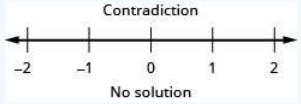
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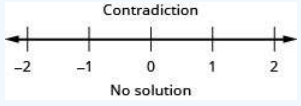
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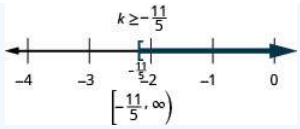
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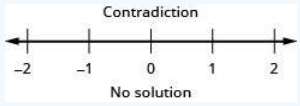
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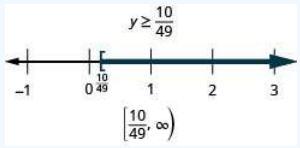
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1.6: Exponents and Radicals

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions with roots
- Estimate and approximate roots
- Simplify variable expressions with roots

Warm Up

Compute the following:

a. $4 \cdot 4$
b. $9 \cdot 9$
c. $64 \div 8$
d. $9 \div 3$

Answer

a. 16
b. 81
c. 8
d. 3

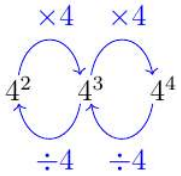
Exponents

Exponents are notation we use to multiply a number to itself multiple times. We take the number we want to multiply by as the big number and how many times we want to multiply by it as the smaller one on top. For example, take $4 \cdot 4$ we can write this as 4^2 read as "4 to the 2".

How about $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$? Here we have 4 multiplied to itself 5 times, so we can express this as 4^5 read "4 to the 5".

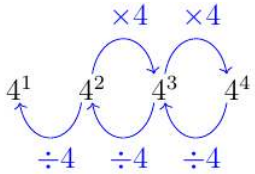
We can see that if we have $4^2 \cdot 4$ then this would be 4^3 since $4^2 = 4 \cdot 4$ and that $4^2 \cdot 4 = 4 \cdot 4 \cdot 4 = 4^3$ as we have 4 multiplied to itself 3 times. (By the way $4^3 = 64$

We can go back as well if we take $4^3 \div 4$ then this would be 4^2 since $4^3 = 4 \cdot 4 \cdot 4$ and that $4^3 \div 4 = 4 \cdot 4 \cdot 4 \div 4 = 4^2$ as we have 4 multiplied to itself 2 times. (By the way $4^2 = 16$)

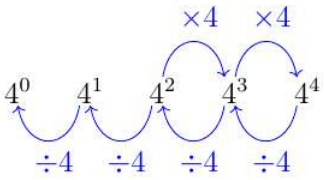


We see that multiplying by 4 we go up an exponent and down an exponent by dividing by 4.

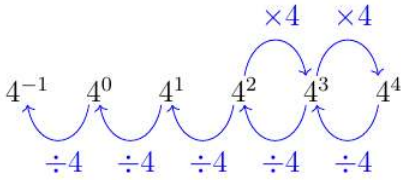
What if we divide 4^2 by 4, what would we get? We would get $4^2 \div 4 = 4 \cdot 4 \div 4 = 4 = 4^1$. Which make sense because we have only one 4.



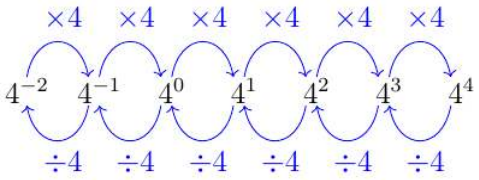
What if we do it again? What if we divide by 4 again to get 4^0? What is 4^0? Well $4^1 \div 4 = 4 \div 4 = 1 = 4^0$.



Do it again. What is 4^{-1}? $4^0 \div 4 = 1 \div 4 = \frac{1}{4} = 4^{-1}$

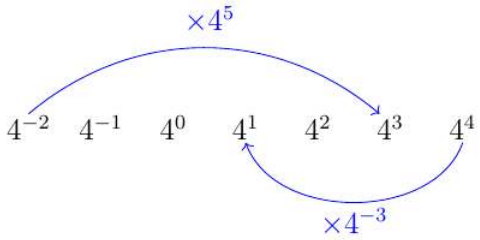


And again. What is 4^{-2}? $4^{-1} \div 4 = 1 \div 4 \cdot 4 = \frac{1}{16} = \frac{1}{4^2} = 4^{-2}$



Instead of counting multiplying by 4 everytime, we can "jump" by multiplying by multiple copies of 4. For example, if we wanted to get from 4^{-2} to 4^3 we would multiple by 4^5. Why? We notice that the difference between 3 and -2 is 5, i.e $3-(-2)=3+2=5$, and so we would have to multiply by 4, five times, but we have notation for that! it is $4^5 = 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$. So $4^{-2} \cdot 4^5 = 4^3$

Same thing goes for going back. For example, going from 4^4 to 4, we would divide by 4^3 or multiply by 4^{-3} i.e $4^4 \cdot 4^{-3} = 4^1 = 4$



We don't need to use 4 for exponents. We can use any number.

Definition: Exponents

Given two numbers a and b. In a^b, a is called the **base** and b is called the **exponent**.

Product law of exponents:

Let a,b, and c be real numbers then $a^b \cdot a^c = a^{b+c}$

note: this only works for the same base.

Exercise 1.6.2

Compute the following (you should be able to do these with out a calculator)

1. 3^2
2. 3^3
3. 1^{32}

4. 5^{-2}
5. $7^{3} \cdot 7^{-2}$
6. $8^{-4} \cdot 8^{3}$
7. $9^{-999} \cdot 9^{1000}$

Answer

1. $3 \cdot 3 = 9$
2. $3 \cdot 3 \cdot 3 = 27$
3. 1
4. $\frac{1}{5 \cdot 5} = \frac{1}{25}$
5. $7^{3+(-2)} = 7^{1} = 7$
6. $8^{-4+3} = 8^{-1} = \frac{1}{8}$
7. $9^{-999+1000} = 9^{1} = 9$

Simplify Expressions with Integers

What happens when there are more than two operations in an expression? We follow the order of operations, which is parenthesis, exponents, multiplication, division, addition, subtraction. The order of operations still applies when negatives are included. Remember Please Excuse My Dear Aunt Sally? This is a way to remember the order of operations. The "P" in "Please" stands parenthesis, the "E" in "Excuse" stands for exponents and so on.

Let's try some examples. We'll simplify expressions that use all four operations with integers—addition, subtraction, multiplication, and division. Remember to follow the order of operations.

? EXAMPLE 1.6.3

Simplify:

1. $(2+3)^{2} \cdot 2 - 3$
2. $2 - 3^{2}$
3. $(-7)^{2}$
4. -7^{2}

Answer

1. $(2+3)^{2} \cdot 2 - 3 = (5)^{2} \cdot 2 - 3 = 25 \cdot 2 - 3 = 50 - 3 = 47$
2. $2 - 3^{2} = 2 - 9 = -7$
3. $(-7)^{2} = -7 \cdot -7 = 49$
4. $-7^{2} = (-1) \cdot 7^{2} = (-1) \cdot 7 \cdot 7 = (-1) \cdot 49 = -49$

The last example showed us the difference between $(-2)^{4}$ and -2^{4}. This distinction is important to prevent future errors. The next example reminds us to multiply and divide in order left to right.

? EXAMPLE 1.6.4

Simplify:

1. $8(-9) \div (-2)^{3}$
2. $-30 \div 2 + (-3)(-7)$

Answer

1.

	$8(-9) \div (-2)^{3}$
Exponents first.	$8(-9) \div (-8)$
Multiply.	$-72 \div (-8)$
Divide.	9

		$-30 \div 2 + (-3)(-7)$
	Multiply and divide	
2.	left to right, so divide first.	$-15 + (-3)(-7)$
	Multiply.	$-15 + 21$
	Add.	6

? Example1.6.5

Simplify:

1. $12(-9) \div (-3)^3$
2. $-27 \div 3 + (-5)(-6)$.

Answer

1. 4
2. 21

? EXAMPLE 1.6.6

Simplify:

1. $18(-4) \div (-2)^3$
2. $-32 \div 4 + (-2)(-7)$

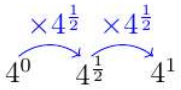
Answer

1. 9
2. 6

Roots and Fractional Exponents

A natural and intimidating question is "can we put fractions in the exponent?" The answer is yes! Let's take a look at an example.

Say we want to know what $4^{\frac{1}{2}}$ is. Well we would know that, whatever that number was, that if we multiplied it by itself, then we would get 4.



What number multiplied to itself would get us 4? 2 would work! $2 \cdot 2 = 4$. Note that -2 would also work since $-2 \cdot -2 = 4$ but in our picture, we have a positive number.

So really, the fraction in the exponent is essentially asking to do the opposite of the whole number. That is, instead of 2^2 asking us "what is 2 times itself twice?" we have $4^{\frac{1}{2}}$ asking us "what number, multiplied to itself twice, will give us 4?"

What about $27^{\frac{1}{3}}$? This is asking us to find the number that when multiplied to itself 3 times, gives us 27. Well we know that $3^3 = 27$ so 3 multiplied to itself 3 times gives us 27. Thus $27^{\frac{1}{3}} = 3$

? Example 1.6.7

Compute the following

a. $8^{\frac{1}{3}}$
b. $9^{\frac{1}{2}}$
c. $1^{\frac{1}{36}}$

d. $49^{\frac{1}{2}}$

Answer

a. 2
b. 3
c. 1
d. 7

Imagine a square with all sides of the same length, say 4. The area of this is $4 \cdot 4 = 4^2 = 16$ because of this we sometimes call 4^2 "4 squared".

Definition: SQUARE AND SQUARE ROOT OF A NUMBER

Square

If $n^2 = m$, then m is the **square** of n.

Square Root

If $n^2 = m$, then n is a **square root** of m.

The square root of a number is just that number raised to the $\frac{1}{2}$ power.

Notice $(-13)^2 = 169$ also, so -13 is also a square root of 169. Therefore, both 13 and -13 are square roots of 169.

So, every positive number has two square roots—one positive and one negative. What if we only wanted the positive square root of a positive number? We use a *radical sign*, and write, $\sqrt{m}$, which denotes the positive square root of m. The positive square root is also called the **principal square root**.

We also use the radical sign for the square root of zero. Because $0^2 = 0$, $\sqrt{0} = 0$. Notice that zero has only one square root.

Definition: SQUARE ROOT NOTATION

$\sqrt{m}$ is read "the square root of m."

If $n^2 = m$, then $n = \sqrt{m}$, for $n \geq 0$.

$$\text{radical sign} \longrightarrow \sqrt{m} \longleftarrow \text{radicand}$$

Figure 8.1.1

What about if we have $\frac{1}{3}$ in the exponent? Is there notation for that?

Definition: N$^{\text{th}}$ ROOT OF A NUMBER

If $b^n = a$, then b is an n^{th} root of a.

The principal n^{th} root of a is written $\sqrt[n]{a}$.

The n is called the **index** of the radical.

Let's redo the last example in this notation.

Example 1.6.8

Compute the following

a. $\sqrt[3]{8}$
b. $\sqrt{9}$
c. $\sqrt[36]{1}$
d. $\sqrt{49}$

Answer

a. 2
b. 3
c. 1
d. 7

What about negative numbers inside square roots? We are not ready to go there yet but we will in Chapter 4.

Example 1.6.9

Simplify:

a. $\sqrt{144}$
b. $-\sqrt{289}$

Solution:

a.

$$\sqrt{144}$$

Since $12^2 = 144$.

$$12$$

b.

$$-\sqrt{289}$$

Since $17^2 = 289$ and the negative is in front of the radical sign.

$$-17$$

So far we have only talked about squares and square roots. Let's now extend our vocabulary to include higher powers and higher roots.

We write:	We say:
n^2	n squared
n^3	n cubed
n^4	n to the fourth power
n^5	n to the fifth power

The terms 'squared' and 'cubed' come from the formulas for area of a square and volume of a cube.

It will be helpful to have a table of the powers of the integers from -5 to 5. See *Figure 8.1.2*

Number	Square	Cube	Fourth power	Fifth power
n	n^2	n^3	n^4	n^5
1	1	1	1	1
2	4	8	16	32
3	9	27	81	243
4	16	64	256	1024
5	25	125	625	3125
x	x^2	x^3	x^4	x^5
x^2	x^4	x^6	x^8	x^{10}

Number	Square	Cube	Fourth power	Fifth power
n	n^2	n^3	n^4	n^5
–1	1	–1	1	–1
–2	4	–8	16	–32
–3	9	–27	81	–243
–4	16	–64	256	–1024
–5	25	–125	625	–3125

Figure 8.1.2

Notice the signs in the table. All powers of positive numbers are positive, of course. But when we have a negative number, the *even* powers are positive and the *odd* powers are negative. We'll copy the row with the powers of -2 to help you see this.

n	n^2	n^3	n^4	n^5
–2	4	–8	16	–32

Even power
Positive result

Odd power
Negative result

Figure 8.1.3

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Estimate and Approximate Roots

When we see a number with a radical sign, we often don't think about its numerical value. While we probably know that the $\sqrt{4}=2$, what is the value of $\sqrt{21}$ or $\sqrt[3]{50}$? In some situations a quick estimate is meaningful and in others it is convenient to have a decimal approximation.

To get a numerical estimate of a square root, we look for **perfect square** numbers closest to the radicand. To find an estimate of $\sqrt{11}$, we see 11 is between perfect square numbers 9 and 16, *closer* to 9. Its square root then will be between 3 and 4, but closer to 3.

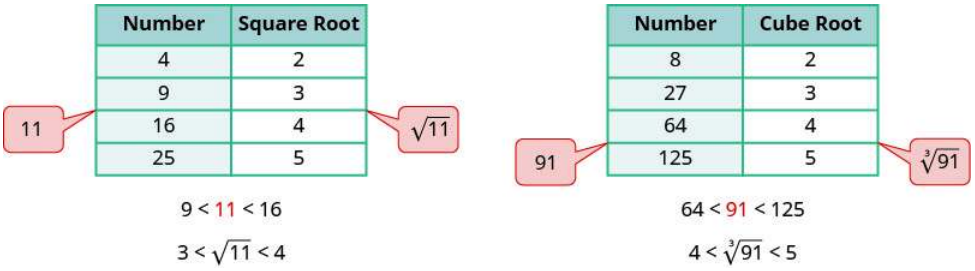


Figure 8.1.4

Similarly, to estimate $\sqrt[3]{91}$, we see 91 is between perfect cube numbers 64 and 125. The cube root then will be between 4 and 5.

Example 1.6.10

Estimate each root between two consecutive whole numbers:

a. $\sqrt{105}$
b. $\sqrt[3]{43}$

Solution:

a. Think of the perfect square numbers closest to 105. Make a small table of these perfect squares and their squares roots.

Table 8.1.1

	$\sqrt{105}$
	Number: 81, 100, 121, 144 / Square Root: 9, 10, 11, 12 (105, $\sqrt{105}$)
Locate 105 between two consecutive perfect squares.	$100 < 105 < 121$
$\sqrt{105}$ is between their square roots.	$10 < \sqrt{105} < 11$

b. Similarly we locate 43 between two perfect cube numbers.

Table 8.1.2

	$\sqrt[3]{43}$
	Number: 8, 27, 64, 125 / Cube Root: 2, 3, 4, 5 (43, $\sqrt[3]{43}$)
Locate 43 between two consecutive perfect cubes.	$27 < 43 < 64$
$\sqrt[3]{43}$ is between their cube roots.	$3 < \sqrt[3]{43} < 4$

Exercise 1.6.11

Estimate each root between two consecutive whole numbers:

a. $\sqrt{38}$
b. $\sqrt[3]{93}$

Answer

a. $6 < \sqrt{38} < 7$
b. $4 < \sqrt[3]{93} < 5$

Exercise 1.6.12

Estimate each root between two consecutive whole numbers:

a. $\sqrt{84}$
b. $\sqrt[3]{152}$

Answer

a. $9 < \sqrt{84} < 10$
b. $5 < \sqrt[3]{152} < 6$

There are mathematical methods to approximate square roots, but nowadays most people use a calculator to find square roots. To find a square root you will use the $\sqrt{x}$ key on your calculator. To find a cube root, or any root with higher index, you will use the $\sqrt[y]{x}$ key.

When you use these keys, you get an approximate value. It is an approximation, accurate to the number of digits shown on your calculator's display. The symbol for an approximation is $\approx$ and it is read 'approximately'.

Suppose your calculator has a 10 digit display. You would see that

$$\sqrt{5} \approx 2.236067978 \text{rounded to two decimal places is } \sqrt{5} \approx 2.24$$

$$\sqrt[4]{93} \approx 3.105422799 \text{rounded to two decimal places is } \sqrt[4]{93} \approx 3.11$$

How do we know these values are approximations and not the exact values? Look at what happens when we square them:

$$(2.236067978)^2 = 5.000000002 \qquad (3.105422799)^4 = 92.999999991$$
$$(2.24)^2 = 5.0176 \qquad (3.11)^4 = 93.54951841$$

Their squares are close to 5, but are not exactly equal to 5. The fourth powers are close to 93, but not equal to 93.

Example 1.6.13

Round to two decimal places:

a. $\sqrt{17}$
b. $\sqrt[3]{49}$
c. $\sqrt[4]{51}$

Solution:

a.

$$\sqrt{17}$$

Use the calculator square root key.

$$4.123105626\ldots$$

Round to two decimal places.

$$4.12$$

$$\sqrt{17} \approx 4.12$$

b.

$$\sqrt[3]{49}$$

Use the calculator $\sqrt[y]{x}$ key.

$$3.659305710\ldots$$

Round to two decimal places.

$$3.66$$

$$\sqrt[3]{49} \approx 3.66$$

c.

$$\sqrt[4]{51}$$

Use the calculator $\sqrt[y]{x}$ key.

$$2.6723451177\ldots$$

Round to two decimal places.

$$2.67$$

$$\sqrt[4]{51} \approx 2.67$$

? Exercise 1.6.14

Round to two decimal places:

1. $\sqrt{11}$
2. $\sqrt[3]{71}$
3. $\sqrt[4]{127}$

Answer

a. ≈ 3.32
b. ≈ 4.14
c. ≈ 3.36

? Exercise 1.6.15

Round to two decimal places:

1. $\sqrt{13}$
2. $\sqrt[3]{84}$
3. $\sqrt[4]{98}$

Answer

a. ≈ 3.61
b. ≈ 4.38
c. ≈ 3.15

Simplify Variable Expressions with Roots

The odd root of a number can be either positive or negative. For example,

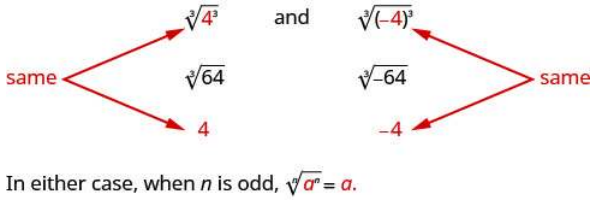


Figure 8.1.13

But what about an even root? We want the principal root, so $\sqrt[4]{625} = 5$.

But notice,

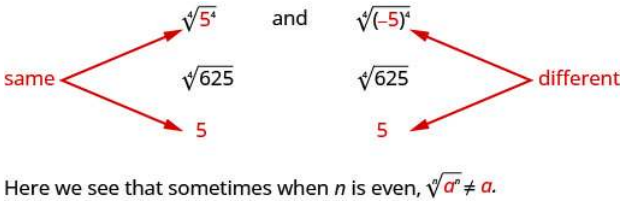


Figure 8.1.14

How can we make sure the fourth root of -5 raised to the fourth power is 5? We can use the absolute value. $|-5| = 5$. So we say that when n is even $\sqrt[n]{a^n} = |a|$. This guarantees the principal root is positive.

Definition: SIMPLIFYING ODD AND EVEN ROOTS

For any integer $n \geq 2$,

$$\text{when the index } n \text{ is odd } \sqrt[n]{a^n} = a$$

$$\text{when the index } n \text{ is even } \sqrt[n]{a^n} = |a|$$

We must use the **absolute value** signs when we take an even root of an expression with a variable in the radical.

Example 1.6.16

Simplify:

a. $\sqrt{x^2}$
b. $\sqrt[3]{n^3}$
c. $\sqrt[4]{p^4}$
d. $\sqrt[5]{y^5}$

Solution:

a. We use the absolute value to be sure to get the positive root.

$$\sqrt{x^2}$$

Since the index n is even, $\sqrt[n]{a^n} = |a|$.

b. This is an odd indexed root so there is no need for an absolute value sign.

$$\sqrt[3]{m^3}$$

Since the index is n is odd, $\sqrt[n]{a^n} = a$.

$$m$$

c.

$$\sqrt[4]{p^4}$$

Since the index n is even $\sqrt[n]{a^n} = |a|$.

$$|p|$$

d.

$$\sqrt[5]{y^5}$$

Since the index n is odd, $\sqrt[n]{a^n} = a$.

$$y$$

Exercise 1.6.17

Simplify:

a. $\sqrt{b^2}$
b. $\sqrt[3]{w^3}$
c. $\sqrt[4]{m^4}$
d. $\sqrt[5]{q^5}$

Answer

a. $|b|$
b. w
c. $|m|$
d. q

Exercise 1.6.18

Simplify:

a. $\sqrt{y^2}$
b. $\sqrt[3]{p^3}$
c. $\sqrt[4]{z^4}$
d. $\sqrt[5]{q^5}$

Answer

a. $|y|$
b. p
c. $|z|$
d. q

What about square roots of higher powers of variables? The Power Property of Exponents says $(a^m)^n = a^{m \cdot n}$. So if we square a^m, the exponent will become $2m$.

$$(a^m)^2 = a^{2m}$$

Looking now at the square root.

$$\sqrt{a^{2m}}$$

Since $(a^m)^2 = a^{2m}$.

$$\sqrt{(a^m)^2}$$

Since n is even $\sqrt[n]{a^n} = |a|$.

$$|a^m|$$

So $\sqrt{a^{2m}} = |a^m|$.

We apply this concept in the next example.

Example 1.6.19

Simplify:

a. $\sqrt{x^6}$
b. $\sqrt{y^{16}}$

Solution:

a.

$$\sqrt{x^6}$$

Since $\left(x^3\right)^2 = x^6$.

$$\sqrt{\left(x^3\right)^2}$$

Since the index n is even $\sqrt{a^n} = |a|$.

$$|x^3|$$

b.

$$\sqrt{y^{16}}$$

Since $\left(y^8\right)^2 = y^{16}$.

$$\sqrt{\left(y^8\right)^2}$$

Since the index n is even $\sqrt[n]{a^n} = |a|$.

$$y^8$$

In this case the absolute value sign is not needed as y^8 is positive.

Exercise 1.6.20

Simplify:

a. $\sqrt{y^{18}}$
b. $\sqrt{z^{12}}$

Answer

a. $|y^9|$
b. z^6

Exercise 1.6.21

Simplify:

a. $\sqrt{m^4}$
b. $\sqrt{b^{10}}$

Answer

a. m^2
b. $|b^5|$

The next example uses the same idea for higher roots.

Example 1.6.22

Simplify:

a. $\sqrt[3]{y^{18}}$
b. $\sqrt[4]{z^8}$

Solution:

a.

$$\sqrt[3]{y^{18}}$$

Since $\left(y^6\right)^3 = y^{18}$.

$$\sqrt[3]{\left(y^6\right)^3}$$

Since n is odd, $\sqrt[n]{a^n} = a$.

$$y^6$$

b.

$$\sqrt[4]{z^8}$$

Since $\left(z^2\right)^4 = z^8$.

$$\sqrt[4]{\left(z^2\right)^4}$$

Since z^2 is positive, we do not need an absolute value sign.

$$z^2$$

Exercise 1.6.23

Simplify:

a. $\sqrt[4]{u^{12}}$
b. $\sqrt[3]{v^{15}}$

Answer

a. $|u^3|$
b. v^5

Exercise 1.6.24

Simplify:

a. $\sqrt[5]{c^{20}}$
b. $\sqrt[6]{d^{24}}$

Answer

a. c^4
b. d^4

In the next example, we now have a coefficient in front of the variable. The concept $\sqrt{a^{2m}} = |a^m|$ works in much the same way.

$$\sqrt{16r^{22}} = 4\left|r^{11}\right| \text{ because } \left(4r^{11}\right)^2 = 16r^{22}.$$

But notice $\sqrt{25u^8} = 5u^4$ and no absolute value sign is needed as u^4 is always positive.

Example 1.6.25

Simplify:

a. $\sqrt{16n^2}$
b. $-\sqrt{81c^2}$

Solution:

a.

$$\sqrt{16n^2}$$

Since $(4n)^2 = 16n^2$.

$$\sqrt{(4n)^2}$$

Since the index n is even $\sqrt[n]{a^n} = |a|$.

$$4|n|$$

b.

$$-\sqrt{81c^2}$$

Since $(9c)^2 = 81c^2$.

$$-\sqrt{(9c)^2}$$

Since the index n is even $\sqrt[n]{a^n} = |a|$.

$$-9|c|$$

Exercise 1.6.26

Simplify:

a. $\sqrt{64x^2}$
b. $-\sqrt{100p^2}$

Answer

a. $8|x|$
b. $-10|p|$

Exercise 1.6.27

Simplify:

a. $\sqrt{169y^2}$
b. $-\sqrt{121y^2}$

Answer

a. $13|y|$
b. $-11|y|$

This example just takes the idea farther as it has roots of higher index.

Example 1.6.28

Simplify:

a. $\sqrt[3]{64p^6}$
b. $\sqrt[4]{16q^{12}}$

Solution:

a.

$$\sqrt[3]{64p^6}$$

Rewrite $64p^6$ as $\left(4p^2\right)^3$.

$$\sqrt[3]{\left(4p^2\right)^3}$$

Take the cube root.

$$4p^2$$

b.

$$\sqrt[4]{16q^{12}}$$

Rewrite the radicand as a fourth power.

$$\sqrt[4]{\left(2q^3\right)^4}$$

Take the fourth root.

$$2|q^3|$$

Exercise 1.6.29

Simplify:

a. $\sqrt[3]{27x^{27}}$
b. $\sqrt[4]{81q^{28}}$

Answer

a. $3x^9$
b. $3|q^7|$

Exercise 1.6.30

Simplify:

a. $\sqrt[3]{125q^9}$
b. $\sqrt[5]{243q^{25}}$

Answer

a. $5p^3$
b. $3q^5$

The next examples have two variables.

Example 1.6.31

Simplify:

a. $\sqrt{36x^2y^2}$
b. $\sqrt{121a^6b^8}$
c. $\sqrt[3]{64p^{63}q^9}$

Solution:

a.

$$\sqrt{36x^2y^2}$$

Since $(6xy)^2 = 36x^2y^2$

$$\sqrt{(6xy)^2}$$

Take the square root.

$$6|xy|$$

b.

$$\sqrt{121a^6b^8}$$

Since $\left(11a^3b^4\right)^2 = 121a^6b^8$

$$\sqrt{\left(11a^3b^4\right)^2}$$

Take the square root.

$$11\left|a^3\right|b^4$$

c.

$$\sqrt[3]{64p^{63}q^9}$$

Since $\left(4p^{21}q^3\right)^3 = 64p^{63}q^9$

$$\sqrt[3]{\left(4p^{21}q^3\right)^3}$$

Take the cube root.

$$4p^{21}q^3$$

? Exercise 1.6.32

Simplify:

a. $\sqrt{100a^2b^2}$
b. $\sqrt{144p^{12}q^{20}}$
c. $\sqrt[3]{8x^{30}y^{12}}$

Answer

a. $10|ab|$
b. $12p^6q^{10}$
c. $2x^{10}y^4$

? Exercise 1.6.33

Simplify:

a. $\sqrt{225m^2n^2}$
b. $\sqrt{169x^{10}y^{14}}$
c. $\sqrt[3]{27w^{36}z^{15}}$

Answer

a. $15|mn|$
b. $13\left|x^5y^7\right|$
c. $3w^{12}z^5$

Access this online resource for additional instruction and practice with simplifying expressions with roots.

- Simplifying Variables Exponents with Roots using Absolute Values

Key Concepts

- **Square Root Notation**
  - $\sqrt{m}$ is read 'the square root of m'
  - If $n^2 = m$, then $n = \sqrt{m}$, for $n \geq 0$.

    Figure 8.1.1
  - The square root of m, $\sqrt{m}$, is a positive number whose square is m.
- **n^{th} Root of a Number**
  - If $b^n = a$, then b is an n^{th} root of a.
  - The principal n^{th} root of a is written $\sqrt[n]{a}$.
  - n is called the *index* of the radical.
- **Properties of $\sqrt[n]{a}$**
  - When n is an even number and
    - $a \geq 0$, then $\sqrt[n]{a}$ is a real number
    - $a < 0$, then $\sqrt[n]{a}$ is not a real number
  - When n is an odd number, $\sqrt[n]{a}$ is a real number for all values of a.
- **Simplifying Odd and Even Roots**
  - For any integer $n \geq 2$,
    - when n is odd $\sqrt[n]{a^n} = a$
    - when n is even $\sqrt[n]{a^n} = |a|$
  - We must use the absolute value signs when we take an even root of an expression with a variable in the radical.

Glossary

square of a number

If $n^2 = m$, then m is the square of n.

square root of a number

If $n^2 = m$, then n is a square root of m.

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1.6E: Exercises

Practice Makes Perfect

Simplify Exponents

Simplify the following

1. 2^3
2. 2^4
3. 5^1
4. $(\frac{1}{4})^{-2}$
5. 4^{-2}
6. 2^0
7. $3-2^2$
8. $2-3^2$
9. $(1+2)^3$
10. $(2-4)^2$
11. $(3+1)^{-2}$
12. $9(-4)\div(-2)^2$
13. $(-3)^2\div 9+(-2)(-7)$
14. $((2^1)((-3)^3)$

Answer

1. 8
2. 16
3. 5
4. 16
5. $\frac{1}{16}$
6. 1
7. 1
8. -7
9. 27
10. 4
11. $\frac{1}{16}$
12. 9
13. 15
14. -54

Simplify Roots

In the following exercises, simplify.

15. $\sqrt{64}$
16. $-\sqrt{81}$
17. $\sqrt{196}$
18. $-\sqrt{1}$
19. $\sqrt{\frac{4}{9}}$
20. $-\sqrt{0.01}$
21. $121^{\frac{1}{2}}$
22. $9^{\frac{1}{2}}$
23. $27^{\frac{1}{3}}$

24. $\sqrt[3]{8}$
25. $\sqrt[5]{32}$

Answer

15. 8
16. -9
17. 14
18. ((-1\)
19. $\frac{2}{3}$
20. -0.1
21. 11
22. 3
23. 3
24. 2
25. 2

Estimate Roots

In the following exercises, estimate each root by giving the interval of two consecutive whole numbers in which the root lies.

26. $\sqrt{8}$
27. $\sqrt{27}$
28. $\sqrt{70}$
29. $\sqrt[3]{12}$
30. $\sqrt[3]{32}$
31. $\sqrt[3]{71}$

Answer

26. $2 < \sqrt{8} < 3$
27. $5 < \sqrt{27} < 6$
28. $8 < \sqrt{70} < 9$
29. $2 < \sqrt[3]{12} < 3$
30. $3 < \sqrt[3]{32} < 4$
31. $4 < \sqrt[3]{71} < 5$

Simplify with Absolute Values

In the following exercises, simplify using absolute values as necessary.

32. $\sqrt{(-2)^2}$
33. $\sqrt[3]{(-3)^3}$
34. $\sqrt[5]{u^5}$
35. $\sqrt[8]{v^8}$
36. $\sqrt[4]{y^4}$
37. $\sqrt[7]{m^7}$
38. $\sqrt{x^6}$
39. $\sqrt{y^{16}}$

Answer

32. 2
33. -3
34. u
35. $|v|$

36. $|y|$
37. m
38. $|x^3|$
39. y^8

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CHAPTER OVERVIEW

2: Introduction to Functions and Graphing

2.1: Relations and functions

2.1E: Exercises

2.2: Graphing on the Cartesian Coordinate Plane

2.2E: Exercises

2.3: Understanding Graphs of Functions

2.3E: Exercises

2.4: Families of Functions

2.4E: Exercises

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2.1: Relations and functions

Learning Objectives

- Interpret relations through ordered pairs, mapping diagrams, and tables
- Determine the domain and range of a relation
- Determine if a relation is a function
- Use function notation to evaluate a function defined with ordered pairs and an equation
- Determine the domain and range of a function

Relations and Functions

One of the most important ideas in mathematics is the ability to notice and describe relationships. In this section, we discuss how math can be used to define relationships between different numbers using what are called **relations** and **functions**.

Relations

We often indicate that two numbers are related to each other using notation that we call an **ordered pair**. This notation pairs two numbers together using parentheses, for example, $(3, 7)$. You can interpret this as "3 is related to 7". The order is important because the first number is often thought as an **input** and the second number is often thought as an **output**. If we want to indicate a generic ordered pair, it is common to write it as x, y, where x is used to indicate inputs and y is used to indicate outputs.

Ordered pairs are the building blocks of a **relation**, which we can use to describe how numbers might be related to each other.

Definition: RELATION

A collection of ordered pairs is called a **relation**.

For example, the collection of ordered pairs

$$R = \{(0, 1), (0, 2), (3, 4)\} \tag{2.1.1}$$

is a relation. We can interpret this as "0 is related to 1", "0 is also related to 2", and "3 is related to 4".

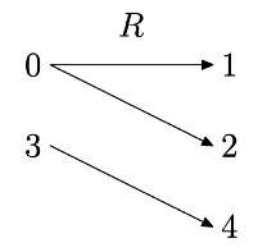
While ordered pairs offer a concise way to show relations, we can also use mapping diagrams with arrows or tables to display relations, with the first entry (or input) of each ordered pair on the left side of the diagram or table and the second entry (or output) on the right side of the diagram or table.

Example 2.1.1

Create a mapping diagram and a table for the relation $R = \{(0, 1), (0, 2), (3, 4)\}$.

Solution

A mapping diagram connects the related numbers using arrows, with the first entry (or input) on the left side and the second entry (or output) on the right side.



The pair $(0, 1)$ means "0 is related to 1", so we draw an arrow from 0 pointing to 1, and the pair $(0, 2)$ means "0 is related to 2", so we also draw an arrow from 0 pointing to 2. Similarly, $3, 4$ means that "3 is related to 4", so our last arrow starts at 3 and points to 4.

If we want to set this up in a table, we place our inputs on the left side and our outputs on the right side:

Inputs (or x)	Outputs (or y)
0	1
0	2
3	4

Since the order of the numbers in each pair matters, it's helpful to be able to talk about just the inputs or just the outputs for a specific relation. These are what we call the domain and range of a relation.

Definition: DOMAIN

The **domain** of a relation is the collection of the first entries of each ordered pair.

It is helpful to think of the domain as the collection of all inputs or x values.

Example 2.1.2

Find the domain for the relation $R = \{(0,1),(0,2),(3,4)\}$.

Solution

The domain is the collection of the first entries of the ordered pairs in a relation. For R, this is the collection $\text{Domain } = \{0,3\}$. Note that even though 0 appears twice as an input, we only list it once.

When we want to talk about the collection of outputs, or the second entry of each ordered pair in a relation, we call this the **range** of the relation.

Definition: RANGE

The **range** of a relation is the collection the second entry of each ordered pair.

We often talk about the range as being the collection of all possible outputs or y values.

Example 2.1.3

Find the range for the relation $R = \{(0,1),(0,2),(3,4)\}$.

Solution

The range is the collection of the second entries of the ordered pairs in a relation. For R, this is the collection

$$\text{Range } = \{1,2,4\}. \tag{2.1.2}$$

Try finding both the range and domain for a relation!

Exercise 2.1.4

Find the domain and range for the relation $R = \{(-3,5),(-2,3),(0,12),(1,-2),(6,5)\}$

Answer

The domain is $\{-3,-2,0,1,6\}$ The range is $\{-2,3,5,12\}$

Exercise 2.1.5

Find the domain and range for the relation $R = \{(-4,2),(-4,5),(-1,0),(3,-2),(5,-1)\}$.

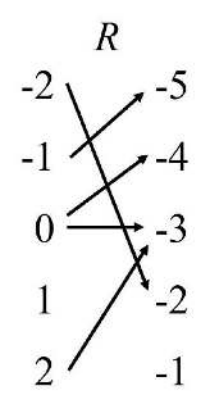
Answer

The domain is $\{-4, -1, 3, 5\}$. The range is $\{-2, -1, 0, 2, 5\}$

We can also determine the domain and range of a relation that has been displayed using a mapping diagram or a table.

Example 2.1.6

Determine the domain and range for the relation shown in the mapping diagram.



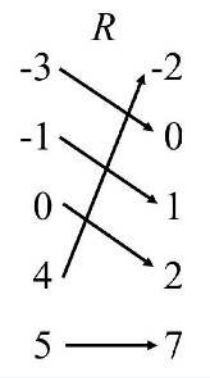
Solution

The domain is $\{-2, -1, 0, 2\}$. Notice that we do not include 1 in the domain because it is not mapped to anything in the relation, so it is not an input.

The range is $\{-5, -4, -3, -2\}$. Notice that we do not include -1 in the range because it does not have anything that gets mapped to it by the relation, so it is not an output.

Exercise 2.1.7

Determine the domain and range for the relation shown in the mapping diagram.



Answer

The domain is $\{-3, -1, 0, 4, 5\}$ The range is $\{-2, 0, 1, 2, 7\}$

Let's try similar examples for a relation defined using a table.

Example 2.1.8

Determine the domain and range for the relation shown in the table.

Inputs (or x)	Outputs (or y)
-2	0
-1	2
3	-2
3	4
5	4

Solution

The domain is $\{-2, -1, 3, 5\}$. The range is $\{-2, 0, 2, 4\}$.

Exercise 2.1.9

Determine the domain and range for the relation shown in the table.

Inputs (or x)	Outputs (or y)
-17	-3
-5	4
0	-1
7	8
10	4

Answer

The domain is $\{-17, -5, 0, 7, 10\}$The range is $\{-3, -1, 4, 8\}$.

Functions

A function is a special type of relation and will be very important throughout the rest of the course.

Definition: FUNCTION

A **function** is a relation where each input has exactly one output.

In other words, a function is a specific kind of relation where none of the first entries are repeated. If we want to think about this in terms of domain and range, each number in the domain is related to exactly one number in the range.

Example 2.1.10

Determine if the following relations are functions.

a. $R = \{(0, 1), (1, 3), (3, 4), (5, 2)\}$
b. $R = \{(0, 1), (0, 2), (3, 4)\}$

Solution

a. For the relation $R = \{(0, 1), (1, 3), (3, 4), (5, 2)\}$, each input has exactly one output, so it is a function.
b. For the relation $R = \{(0, 1), (0, 2), (3, 4)\}$, the input 0 has two outputs, 1 and 2, that correspond to it, so this is not a function (but it is still a relation!).

Exercise 2.1.11

Determine if the following relations are functions.

a. $R = \{(-3, 2), (1, 3), (4, 4), (1, -2), (0, -4)\}$
b. $R = \{(-5, -3), (1, 1), (6, 2), (7, 2)\}$

Answer

a. Not a function
b. Function

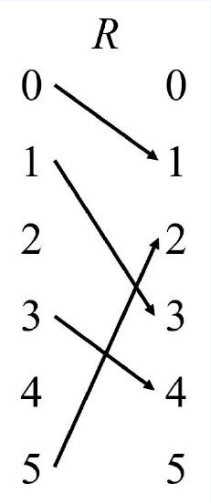
A mapping diagram can make it easier to determine if a relation is also a function. When a relation is a function, each input has exactly one output, so on its mapping diagram, each input will only have one arrow leaving it. When a relation is not a function, some of the inputs may have more than one output, so some inputs may have multiple arrows leaving it.

Example 2.1.12

Use a mapping diagram to determine if the relation $R = \{(0,1),(1,3),(3,4),(5,2)\}$is a function.

Solution

First, create the mapping diagram corresponding to this relation. You can choose whether or not to include the numbers not used in the ordered pairs; a few extra numbers isn't a big deal and won't affect the answer.



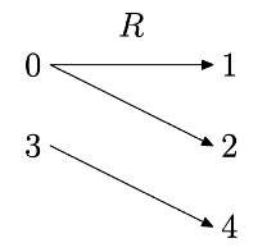
Since each input on the left only has one arrow attached to it, each input has only one output, meaning R is a function. (it's okay that not every number is an input or an output!)

Example 2.1.13

Use a mapping diagram to determine if the relation $R = \{(0,1),(0,2),(3,4)\}$is a function.

Solution

First, we create the mapping diagram for this relation.



Since 0 has two arrows leaving it, we see that it has two outputs that correspond to it. Therefore, R cannot be a function.

Function Notation

One way to think about a function is as being a rule that pairs one input with one output. The rule can take many forms. For example, we can use sets of ordered pairs, graphs, and mapping diagrams to describe the function. In the sections that follow, we

will explore other ways of describing a function, for example, through the use of equations and simple word descriptions.

Before we can talk about writing the rule for a function using an equation, we must define a new notation that we call **function notation**. If f is a function, function notation looks like

$$f(input) = output. \tag{2.1.3}$$

Or if we want to use x's and y's, it is common to say

$$f(x) = y. \tag{2.1.4}$$

In other words, if we want to know what a function does to a given input, we indicate the function with f, place the $input$ in parentheses, and set it equal to the corresponding output. We call this **evaluating the function**. Note: in this setting, the parentheses do **not** represent multiplication; we are not multiplying the function f by the input.

Example 2.1.14

Using the function $f = \{(0,1),(1,3),(3,4),(5,2)\}$, evaluate the following:

a. f(1)
b. f(5)
c. f(3)

Solution

a. Since the function f sends 1 to 3, $f(1) = 3$.
b. Since the function f sends 5 to 2, $f(5) = 2$.
c. Since the function f sends 3 to 4, $f(3) = 4$.

In $a.$, for example, this notation is read as "f of 1 is equal to 3", meaning that when we evaluate f with the input 1, the output is 3.

It is helpful to think of a function as a machine. The machine receives input, processes it according to some rule, then outputs a result. Something goes in (input), then something comes out (output). This is how a function pairs numbers in its domain with numbers in its range.

For the rest of this course, the rule that a function follows will be given by an equation using a variable. This allows us to evaluate a function for a much larger collection of numbers than in our previous examples. For example,

$$f(x) = 2x. \tag{2.1.5}$$

For this function, the rule is that we take the input number that x represents, and then multiply it by 2.

To evaluate a function f that uses an equation for a rule, we take the input and swap it out for x in the rule.

Example 2.1.15

For the function $f(x) = 2x$, evaluate the following:

a. $f(3)$
b. $f(-1)$
c. $f(0)$

Solution

a. To evaluate $f(3)$, we substitute the x in the rule that f follows with the 3. Therefore, $f(3) = 2 \cdot 3 = 6$. So our final answer is $f(3) = 6$.
b. To evaluate $f(-1)$, we substitute the x in the rule that f follows with the -1. Therefore, $f(-1) = 2 \cdot (-1) = -2$. So our final answer is $f(-1) = -2$.
c. To evaluate $f(0)$, we substitute the x in the rule that f follows with the 0. Therefore, $f(0) = 2 \cdot 0 = 0$. So our final answer is $f(0) = 0$.

Exercise 2.1.16

For the function $f(x) = x^2 + 3$, evaluate the following:

a. $f(-2)$
b. $f(0)$
c. $f(4)$

Answer

a. To evaluate $f(-2)$, we substitute the x in the rule that f follows with the -2. Be careful with the negative! We recommend surrounding the -2 with parentheses. We get $f(-2) = (-2)^2 + 3 = 4 + 3 = 7.$. So our final answer is $f(-2) = 7$.
b. $f(0) = 3$.
c. $f(4) = 19$.

Exercise 2.1.17

For the function $f(x) = |x - 4| + 1$, evaluate the following:

a. $f(-5)$
b. $f(-1)$
c. $f(0)$

Answer

a. $f(-5) = 10$
b. $f(-1) = 6$
c. $f(0) = 5$

Extracting the Domain of a Function Using an Equation

We've seen that the domain of a relation or function is the set of all the first coordinates of its ordered pairs. However, if a functional relationship is defined by an equation such as $f(x) = 3x - 4$, then it is not practical to list all ordered pairs defined by this relationship. For any real x-value, you get an ordered pair. For example, if x = 5, then $f(5) = 3(5) - 4 = 11$, leading to the ordered pair (5, f(5)) or (5, 11). As you can see, the number of such ordered pairs is infinite. For each new x-value, we get another function value and another ordered pair.

Therefore, it is easier to turn our attention to the values of x that yield real number responses in the equation $f(x) = 3x - 4$. This leads to the following key idea.

Definition: DOMAIN OF A FUNCTION

If a function is defined by an equation, then the **domain** of the function is the set of "permissible x-values," the values that produce a real number response defined by the equation.

We sometimes like to say that the domain of a function is the set of "all possible inputs."

Example 2.1.18

Find the domain for the following function: $f(x) = 3x - 4$

Solution

it is immediately apparent that we can use any value we want for x in the rule $f(x) = 3x - 4$. Thus, the domain of f is all real numbers. We can write that the domain $D = \mathbb{R}$, or we can use interval notation and write that the domain $D = (-\infty, \infty)$.

However, not every input is allowed when defining a function using an equation for the rule. For example, you cannot take the square root of a negative number and get a real number back, or divide by zero. So let's look at examples that have restrictions to

their domain.

Example 2.1.19

Find the domain for the following function: $f(x) = \sqrt{x}$

Solution

It is not possible to take the square root of a negative number. Therefore, x must either be zero or a positive real number. In set-builder notation, we can describe the domain with $D = \{x : x \geq 0\}$. In interval notation, we write $D = [0, \infty)$.

Example 2.1.20

Find the domain for the following function: $f(x) = 1/(x-3)$

Solution

If we define a function with this rule we immediately see that $x = 3$ will put a zero in the denominator. Division by zero is not allowed. Therefore, 3 is not in the domain of f. No other x-value will cause a problem. The domain of f is best described with set-builder notation as $D = \{x : x \neq 3\}$; that is, any number other than 3 is allowed to be "plugged in" the function $f(x) = x1/(x-3)$.

Extracting the Range of a Function Using an Equation

We've also seen that the range of a relation or function is the set of all the second coordinates of its ordered pairs. Once again, if a function is defined using an equation such as $f(x) = 3x - 4$, then we cannot list all the possible outputs.

So in the case of finding the range of a function based on an equation, we want to describe all the possible outputs created.

Definition: RANGE OF A FUNCTION

If a function is defined by an equation, then the **range** of the function is the set of "possible outputs," the values that can be created by plugging in different x values.

It's helpful to think of the range of a function as "all possible outputs."

Example 2.1.21

Find the range for the following function: $f(x) = 3x - 4$

Solution

We can create any real number using the rule $3x - 4$. Therefore, the range of f is all real numbers. We say that the range is $R = \mathbb{R}$, or we can use interval notation and say that the range is $R = (-\infty, \infty)$.

Just like with the domain, not every function has a range including all real numbers. That is, not every function is capable of producing all real numbers.

Example 2.1.22

Find the range for the following function: $f(x) = x^2$

Solution

If we square any real number, it becomes positive. For example, $f(-2) = 4$ or $f(-5) = -25$. If we square zero, we just get zero back. In other words, $f(0) = 0$. Squared numbers can be as large as we want them to be, so the range has to include zero and any positive number, but it cannot include any negative numbers. So we can describe the range with $R = [0, \infty)$.

Example 2.1.23

Find the range for the following function: $f(x) = 1/(x-3)$

Solution

No matter what value we place in the denominator, the rule $1/(x-3)$ will never produce the number 0. However, it is possible to create any other real number using this rule. Therefore, the range of this function is everything *except* 0. We indicate this by saying that the range is $R = \{y : y \neq 0\}$. In other words, this function can produce any y-value except 0.

Key Concepts

- An **ordered pair** relates one number to another number. The first number is often represented with an x and the second with a y. Order is important! These numbers cannot be swapped and still have the same meaning.
- A **relation** is a collection of ordered pairs
  - A relation can also be visualized using a table or mapping diagram
- The **domain** of a relation is the collection of the first entries of each ordered pair. The **range** of a relation is the collection of the second entries of each ordered pair.
- A **function** is a relation where each input has exactly one output.
- Function notation looks like $f(input) = output$ or $f(x) = y$. We use this notation to define the rule of the function through an equation based on x.
- When we substitute a value into the rule for $f(x)$ to determine the corresponding output, we call this **evaluating the function.**
- The **domain of a function** is the collection of all possible inputs. The **range of a function** is the collection of all possible outputs.

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2.1E: Exercises

Practice Makes Perfect

Represent Relations as Mapping Diagrams and Tables

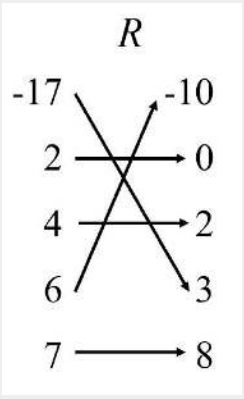
For each of the following relations, create a mapping diagram and table.

1. $R = \{(-17, 3), (4, 2), (2, 0), (6, -10), (7, 8)\}$
2. $R = \{(0, 3), (0, -5), (0, 7), (3, 22), (5, 16), (7, 3), (7, -11)\}$
3. $R = \{(-5, -1), (-4, 0), (-3, 1), (-2, 2)\}$
4. $R = \{(-3, 0), (-2, 3), (-2, 4), (1, -5)\}$

Answer

1.

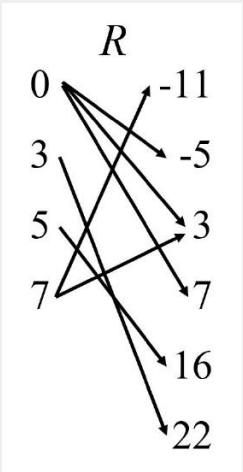
Mapping diagram	Table



x	y
-17	-10
2	0
4	2
6	3
7	8

2.

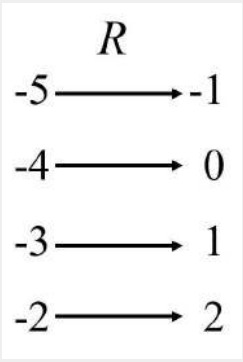
Mapping diagram	Table



x	y
0	-5
0	3
0	7
3	22
5	16
7	-11
7	3

3.

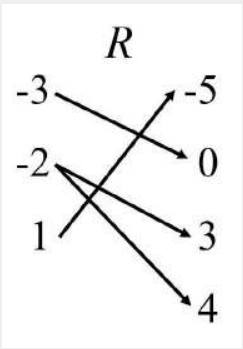
Mapping diagram	Table



x	y
-5	-1
-4	0
-3	1
-2	2

4.

Mapping diagram	Table



x	y
-3	0
-2	3
-2	4
1	5

Determine the domain of a relation

For each relation below, determine the domain.

5. $R = \{(-17, 3), (4, 2), (2, 0), (6, -10), (7, 8)\}$
6. $R = \{(0, 3), (0, -5), (0, 7), (3, 22), (5, 16), (7, 3), (7, -11)\}$
7. $R = \{(-5, -1), (-4, 0), (-3, 1), (-2, 2)\}$
8. $R = \{(-3, 0), (-2, 3), (-2, 4), (1, -5)\}$

Answer

5. $\{-17, 2, 4, 6, 7\}$
6. $\{0, 3, 5, 7\}$
7. $\{-5, -4, -3, -2\}$
8. $\{-3, -2, 1\}$

Determine the range of a relation

For each relation below, determine the range.

9. $R = \{(-17, 3), (4, 2), (2, 0), (6, -10), (7, 8)\}$
10. $R = \{(0, 3), (0, -5), (0, 7), (3, 22), (5, 16), (7, 3), (7, -11)\}$
11. $R = \{(-5, -1), (-4, 0), (-3, 1), (-2, 2)\}$
12. $R = \{(-3, 0), (-2, 3), (-2, 4), (1, -5)\}$

Answer

9. $\{-10, 0, 2, 3, 8\}$
10. $\{-11, -5, 3, 7, 16, 22\}$

11. $\{-1, 0, 1, 2\}$
12. $\{-5, 0, 3, 4\}$

Determine the domain and range of a relation

For each relation below, find the domain and range.

13. $R = \{(-13, 7), (2, 4), (0, 0), (10, 6), (8, 7)\}$
14. $R = \{(1, 1), (-1, -4), (-1, 6), (2, 21), (4, 15), (6, 2), (6, -10)\}$
15. $R = \{(-4, 0), (-3, -1), (-4, 0), (-1, 1)\}$
16. $R = \{(-2, 1), (-1, 4), (-3, 5), (0, -5)\}$

Answer

13. Domain: $\{-13, 0, 2, 8, 10\}$; Range: $\{0, 4, 6, 7\}$
14. Domain: $\{-1, 1, 2, 4, 6\}$ Range: $\{-10, -4, 1, 2, 15, 21\}$
15. Domain: $\{-4, -3, -1\}$; Range: $\{-1, 0, 1\}$
16. Domain: $\{-3, -2, -1, 0\}$; Range: $\{-5, 1, 4, 5\}$

Determine the domain and range of a relation using a table

For each relation below, determine the domain and range.

17.

x	y
-3	-10
-2	3
0	-1
1	4
1	6
5	10
6	11

18.

x	y
-6	13
-5	31
-3	-11
-2	2
-1	8
0	-6
2	-6

19.

x	y
-4	4
2	-2
3	-2
3	3
3	-3
10	-10
14	-14

Answer

17. Domain: $\{-3, -2, 0, 1, 5, 6\}$ Range: $\{-10, -1, 3, 4, 6, 10, 11\}$
18. Domain: $\{-6, -5, -3, -2, -1, 0, 2\}$ Range: $\{-11, -6, 2, 13, 31\}$

19. Domain: $\{-4, 2, 3, 10, 14\}$Range: $\{-14, -10, -3, -2, 2, 3, 4\}$

Determine if a relation is a function

Determine if the following relations are functions:

20. $R = \{(-13, 7), (2, 4), (0, 0), (10, 6), (8, 7)\}$
21. $R = \{(1, 1), (-1, -4), (-1, 6), (2, 21), (4, 15), (6, 2), (6, -10)\}$
22. $R = \{(-4, 0), (-3, -1), (-4, 1), (-1, 1)\}$
23. $R = \{(-2, 1), (-1, 4), (-3, 5), (0, -5)\}$
24. $R = \{(-17, 3), (4, 2), (2, 0), (6, -10), (7, 8)\}$
25. $R = \{(0, 3), (0, -5), (0, 7), (3, 22), (5, 16), (7, 3), (7, -11)\}$
26. $R = \{(-5, -1), (-4, 0), (-3, 1), (-2, 2)\}$
27. $R = \{(-3, 0), (-2, 3), (-2, 4), (1, -5)\}$

Answer

20. Function
21. Not a function
22. Not a function
23. Function
24. Function
25. Not a function
26. Function
27. Not a function

Evaluate a function represented with ordered pairs

For each function below, evaluate the following: $f(-1), f(0), f(3)$

28. $f = \{(-1, 3), (0, 5), (3, 16)\}$
29. $f = \{(-4, 5), (-1, 2), (0, 5), (3, 5)\}$
30. $f = \{(-13, 7), (-1, -1), (2, 4), (0, 0), (3, 4)(10, 6), (8, 7)\}$
31. $f = \{(-5, -1), (-4, 0), (-3, 1), (-2, 2), (-1, 12), (0, -4), (3, -2)\}$
32. $f = \{(-3, 0), (-2, 3), (-1, 9), (0, 4), (1, -5), (3, -14), (9, 2)\}$

Answer

28. $f(-1) = 3; f(0) = 5; f(3) = 16$
29. $f(-1) = 2; f(0) = 5; f(3) = 5$
30. $f(-1) = -1; f(0) = 0, f(3) = 4$
31. $f(-1) = 12; f(0) = -4; f(3) = -2$
32. $f(-1) = 9; f(0) = 4; f(3) = -14$

Evaluate functions defined using an equation

For each function below, evaluate the following: $f(-3), f(0), f(5)$

33. $f(x) = 2x + 1$
34. $f(x) = x^2 + 1$
35. $f(x) = \sqrt{x + 4}$
36. $f(x) = |x| + 5$
37. $f(x) = -17$
38. $f(x) = 3x - 4$
39. $f(x) = 2x^2 - 2$
40. $f(x) = \sqrt{x + 4} + 9$

41. $f(x) = -2|x| + 3$
42. $f(x) = 3$

Answer

33. $f(-3) = -5; f(0) = 1, f(5) = 11$
34. $f(-3) = 10; f(0) = 1; f(5) = 26$
35. $f(-3) = 1; f(0) = 2; f(5) = 3$
36. $f(-3) = 8; f(0) = 5; f(5) = 10$
37. $f(-3) = -17; f(0) = -17; f(5) = -17$
38. $f(-3) = -13; f(0) = -4; f(5) = 11$
39. $f(-3) = 16; f(0) = -2; f(5) = 48$
40. $f(-3) = 10; f(0) = 11; f(5) = 12$
41. $f(-3) = -3; f(0) = 3; f(5) = -7$
42. $f(-3) = 3; f(0) = 3; f(5) = 3$

? Find the domain and range of a function defined using an equation

Find the domain and range for each of the functions below:

43. $f(x) = 2x + 1$
44. $f(x) = x^2 + 1$
45. $f(x) = \sqrt{x+4}$
46. $f(x) = |x| + 5$
47. $f(x) = -17$
48. $f(x) = 3x - 4$
49. $f(x) = 2x^2 - 2$
50. $f(x) = \sqrt{x+4} + 9$
51. $f(x) = -2|x| + 3$
52. $f(x) = 3$

Answer

43. Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$
44. Domain: $(-\infty, \infty)$; Range: $(1, \infty)$
45. Domain: $x \geq -4$; Range: $y \geq 0$
46. Domain: $(-\infty, \infty)$; Range: $y \geq 5$
47. Domain: $(-\infty, \infty)$; Range: $y = -17$
48. Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$
49. Domain: $(-\infty, \infty)$; Range: $(-2, \infty)$
50. Domain: $x \geq -4$; Range: $y \geq 9$
51. Domain: $(-\infty, \infty)$; Range: $y \leq 3$
52. Domain: $(-\infty, \infty)$; Range: $y = 3$

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2.2: Graphing on the Cartesian Coordinate Plane

Learning Objectives

By the end of this section you will be able to...

- Plot ordered pairs on the Cartesian plane
- Compute the distance between two points on the Cartesian coordinate plane
- Compute the midpoint between two points on the Cartesian coordinate plane
- Graph relations and functions defined with ordered pairs
- Graph functions defined with an equation using a table to create ordered pairs

Warm Up 2.2.1

Identify the x and y values in each ordered pair below.

a. $(1, 3)$
b. $(4, -2)$
c. $(-25, -72)$

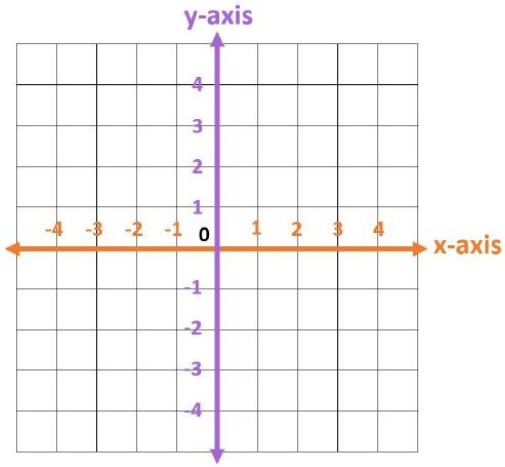
Answer

a. For $(1, 3)$, the x value is 1 and the y value is 3.
b. For $(4, -2)$, the x value is 4 and the y value is -2.
c. For $(-25, -72$, the x value is -25 and the y value is -72.

The Cartesian Coordinate Plane

In the previous section, we talked about how we can use ordered pairs to form the building blocks of relations and functions. We introduced a few ways to understand these types of relationships between numbers, including lists of ordered pairs, mapping diagrams, tables, and in the case of functions, writing an equation. In this section, we look at a new way to understand the relationships between numbers by visualizing the ordered pairs on a graph.

To create a graph, we start with what is called the **Cartesian Coordinate Plane**. The Cartesian coordinate plane, shown below, uses a grid system to plot ordered pairs using two number lines at the same time called the x-axis and y-axis. The place these axes intersect is called the **origin**. For an ordered pair (x, y), the x-axis displays the first coordinate by moving along the horizontal direction and the y-axis displays the second coordinate by moving along the vertical direction. For positive numbers, we move either right (for the x component) or up (for the y component). For negative numbers, we move either left (for the x component) or down (for the y component). The origin is located at the point (0, 0); this is where we always start before moving along the x or y axis to plot a point.



Caution

When plotting, the **first component gives horizontal information**, and the **second component gives vertical information**. The order of the ordered pair is very important; they are not interchangeable!

Let's look at some examples on how to plot points.

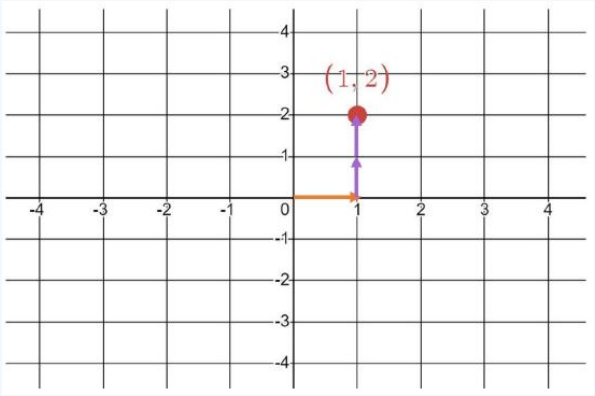
Example 2.2.1

Plot the following on a Cartesian coordinate plane:

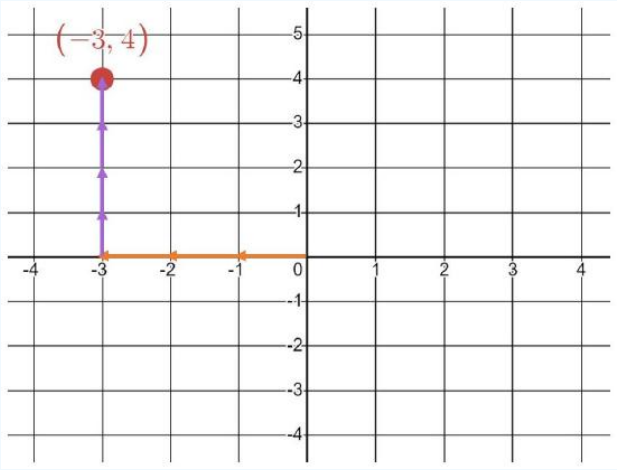
a. (1, 2)
b. (-3, 4)
c. (2, -1)

Solution

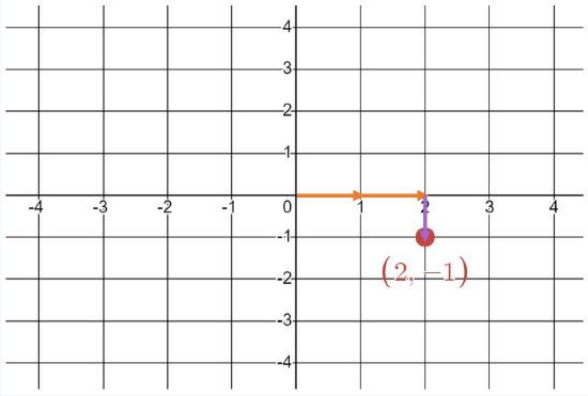
a. To plot the point (1, 2), starting at the origin (0, 0), we first go **right 1 unit along the x axis**, then **up 2 units along the y axis**.



b. To plot the point (-3, 4), starting at the origin (0, 0), we first go **left 3 units along the x axis**, then **up 4 units along the y axis**.



c. To plot the point (2, -1), starting at the origin (0, 0), we first go **right 2 units along the x axis**, then **down 1 unit along the y axis**.

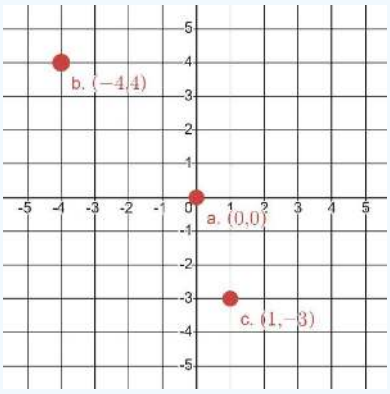


Exercise 2.2.2

Plot the following on a Cartesian coordinate plane.

a. (0, 0)
b. (-4, 4)
c. (1, -3)

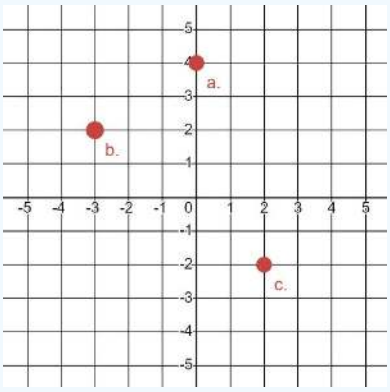
Answer



It is also useful to be able to look at a point on a graph and be able to identify the corresponding ordered pair.

Example 2.2.3

Identify the ordered pairs for the points shown on the graph below.

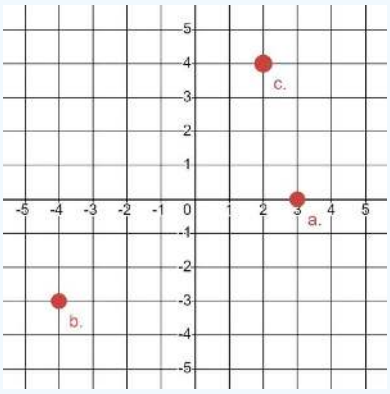


Solution

a. The point is directly over the origin, so we haven't moved left or right. Therefore, the x value is 0. It's 4 units up, so the y value is 4. Therefore, the ordered pair is (0, 4).
b. Starting at the origin, this point is 3 units to the left, so the x value is -3. From there, it's 2 units up, so the y value is 2. Therefore, the ordered pair is (-3, 2).
c. Starting at the origin, the point is 2 units to the right, so the x value is 2. From there, it's 2 units down, so the y value is -2. Therefore, the ordered pair is (2, -2).

? Exercise 2.2.4

Identify the ordered pairs for the points shown on the graph below.



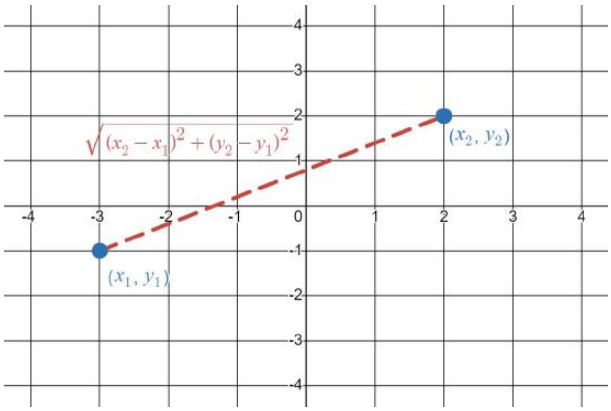
Answer

Add texts here. Do not delete this text first.

a. (3, 0)
b. (-4, -3)
c. (2, 4)

Computing Distance between Two Points

It is useful to be able to talk about the distance between two points plotted on a graph. Suppose we have two points (x_1, y_1) and (x_2, y_2). We can use a formula based on the Pythagorean theorem to determine the distance between these two points.



Definition:DISTANCE FORMULA

To compute the **distance between two points** (x_1, y_1) and (x_2, y_2) on a graph, we use the following formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\ .$$

Let's try this out with a few examples!

Example 2.2.5

Compute the distance between the following two points: (1, 2), (5, 5)

Solution

We have $(x_1, y_1) = (1, 2)$ and $(x_2, y_2) = (5, 5)$. Therefore the distance between these two points is

$$d = \sqrt{(5-1)^2 + (5-2)^2} = \sqrt{4^2 + 3^2} = \sqrt{16+9} = \sqrt{25} = 5 \quad .$$

Exercise 2.2.6

Compute the distance between the following two points: (-1, 4), (5, 12)

Answer

We have $(x_1, y_1) = (-1, 4)$ and $(x_2, y_2) = (5, 12)$. Therefore the distance between these two points is

$$d = \sqrt{(5-(-1))^2 + (12-4)^2} = \sqrt{6^2 + 8^2} = \sqrt{36+64} = \sqrt{100} = 10 \quad .$$

Exercise 2.2.7

Compute the distance between the following two points: (3, 5), (15, 0)

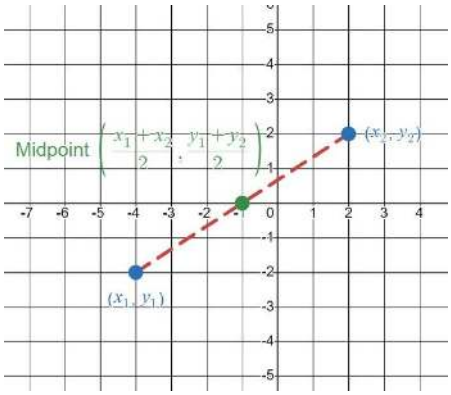
Answer

We have $(x_1, y_1) = (3, 5)$ and $(x_2, y_2) = (15, 0)$. Therefore the distance between these two points is

$$d = \sqrt{(15-3)^2 + (0-5)^2} = \sqrt{12^2 + (-5)^2} = \sqrt{144+25} = \sqrt{169} = 13 \quad .$$

Computing the Midpoint of Two Points

We can also determine the **midpoint** for two points. That is, if we are given two points, it is possible to figure out a third point that is located exactly halfway between the two original points.



Definition: MIDPOINT FORMULA

To compute the **midpoint between two points** (x_1, y_1) and (x_2, y_2) on a graph, we use the following formula:

$$\left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right).$$

Using this formula, computing the midpoint of two points very straightforward. It's essentially finding their average x value and their average y value.

Example 2.2.8

Find the midpoint for the following two points: (-1, 3), (2, 5)

Solution

We have $(x_1, y_1) = (-1, 3)$ and $(x_2, y_2) = (2, 5)$. Therefore the midpoint has coordinates

$$\left(\frac{-1+2}{2}, \frac{3+5}{2}\right) = \left(\frac{1}{2}, \frac{8}{2}\right) = \left(\frac{1}{2}, 4\right).$$

Exercise 2.2.9

Find the midpoint for the following two points: (0, 8), (-4, 10)

Answer

(-2, 9)

Exercise 2.2.10

Find the midpoint for the following two points: (-2, -4), (0, -6)

Answer

(-1, -5)

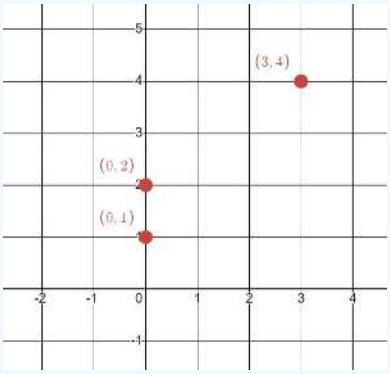
Graphing Relations and Functions

The biggest use for graphs is that it allows us to actually visualize the relations and functions that we are working with. In order to do this, we simply graph the ordered pairs in the relation or function on the Cartesian coordinate plane.

Example 2.2.11

Graph the relation $R = \{(0, 1), (0, 2), (3, 4)\}$

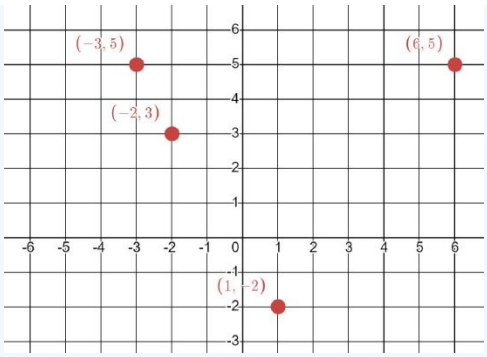
Solution



Exercise 2.2.12

Graph the relation $R = \{(-3, 5), (-2, 3), (0, 12), (1, -2), (6, 5)\}$

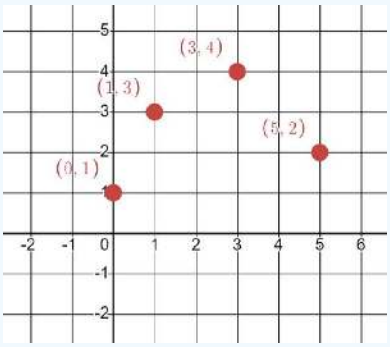
Answer



Example 2.2.13

Graph the function $\{(0,1),(1,3),(3,4),(5,2)\}$

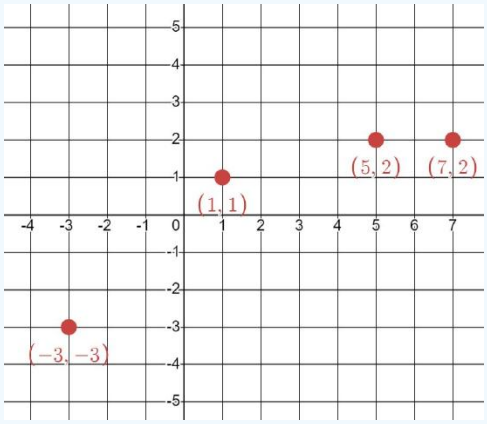
Solution



Exercise 2.2.14

Graph the function $\{(-3,-3),(1,1),(5,2),(7,2)\}$

Answer



The process is pretty straightforward when we are given the ordered pairs to plot! But in the last section, we discussed how functions can be described by rules using equations, instead of a short list of ordered pairs. This means that in order to create ordered pairs to graph the function, we must choose our own inputs for this. For each input x, the resulting ordered pair becomes

$$(x, f(x)).$$

These ordered pairs will act as our ordered pairs (x, y) that we use for plotting, where we treat $f(x)$ as the y values.

We can choose a list of several convenient inputs that will help us determine the general shape of a function, collect inputs and outputs in a table, then create ordered pairs from them. These are what we will use to graph a function $f(x)$ that we determine using an equation.

Example 2.2.15

Graph $f(x) = 2x - 1$ using a table.

Solution

Step 1: Create a table to keep track of inputs, outputs, and ordered pairs.

Inputs (x)	Outputs $f(x)$	Ordered Pairs $(x, f(x))$

Step 2: Choose several inputs to use to create ordered pairs; convenient numbers such as -1, 0, 1 are good to include, and often times, 4-5 points is sufficient to get an idea of what the graph will look like. Add these inputs to the table.

In this example, we will use x values -2, -1, 0, 1, 2; but you can use whatever values you wish.

Inputs (x)	Outputs $f(x)$	Ordered Pairs $(x, f(x))$
-2		
-1		
0		
1		
2		

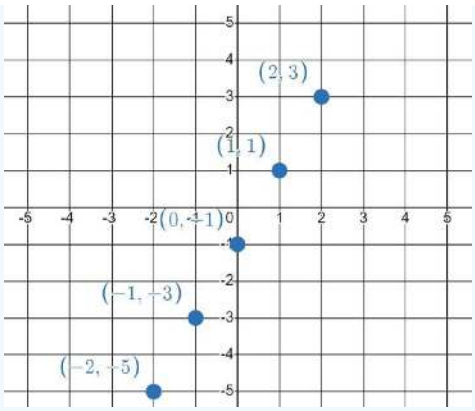
Step 3: Compute the outputs $f(x)$ corresponding to each input x by plugging the x value into the rule for $f(x)$; add these to the table.

Inputs (x)	Outputs $f(x)$	Ordered Pairs $(x, f(x))$
-2	$2 \cdot (-2) - 1 = -5$	
-1	$2 \cdot (-1) - 1 = -3$	
0	$2 \cdot 0 - 1 = -1$	
1	$2 \cdot 1 - 1 = 1$	
2	$2 \cdot 2 - 1 = 3$	

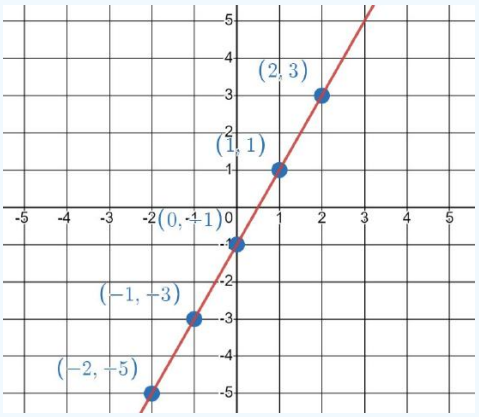
Step 4: Create ordered pairs from the inputs and their outputs; add to table

Inputs (x)	Outputs $f(x)$	Ordered Pairs $(x, f(x))$
-2	$2 \cdot (-2) - 1 = -5$	$(-2, -5)$
-1	$2 \cdot (-1) - 1 = -3$	$(-1, -3)$
0	$2 \cdot 0 - 1 = -1$	$(0, -1)$
1	$2 \cdot 1 - 1 = 1$	$(1, 1)$
2	$2 \cdot 2 - 1 = 3$	$(2, 3)$

Step 5: Use the ordered pairs to plot the graph of the function. Based on the table we have created, we need to plot the pairs (-2, -5), (-1, -3), (0, 1), (1, 1), and (2, 3):



Step 6: Draw the function by connecting the dots. Recall that when a function is defined by an equation, we have a lot of inputs for x to choose from. We account for this on the graph by sketching a picture of a graph suggested by the points plotted. In this case, the graph turns out to be a straight line. Be sure to continue the line beyond the last points to indicate that we could have used even more inputs!



Exercise 2.2.16

Graph the function $f(x) = x^2 + 3$ using a table.

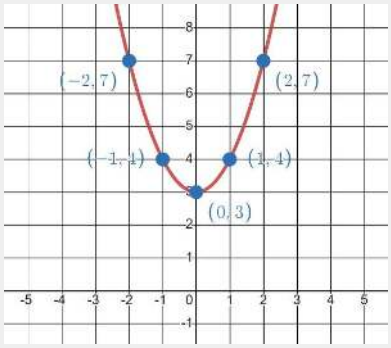
Answer

The points used on the table may vary, but the graph should look the same.

Table

Inputs (x)	Outputs $f(x)$	Ordered Pairs $(x, f(x))$
-2	7	$(-2, 7)$
-1	4	$(-1, 4)$
0	3	$(0, 3)$
1	4	$(1, 4)$
2	7	$(2, 7)$

Graph



Key Concepts

- The **Cartesian coordinate plane** allows us to visualize ordered pairs by representing the inputs along horizontal number line called the ***x* axis** and outputs along a vertical number line called the ***y* axis**.
- **How to plot an ordered pair**
  1. Start at the origin (0, 0); this is where the axes intersect
  2. Move the number of units in the x component horizontally. If x is positive, move right. If x is negative, move left.
  3. Move the number of units in the y component vertically. If y is positive, move up. If y is negative, move down.
- We can compute the **distance between two points** (x_1, y_1) and (x_2, y_x) by using the equation
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} .$$
- We can compute the **midpoint of two points** (x_1, y_1) and (x_2, y_x) by using the equation
$$\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right).$$
- **To graph a function defined using an equation for its rule**
  1. Create a table to keep track of inputs, outputs, and ordered pairs.
  2. Choose several inputs to use to create ordered pairs; convenient numbers such as -1, 0, 1 are good to include, and often times, 4-5 points is sufficient to get an idea of what the graph will look like. Add these inputs to the table.
  3. Compute the outputs $f(x)$ corresponding to each input x by plugging the x value into the rule for $f(x)$; add these to the table.
  4. Create ordered pairs from the inputs and their outputs; add to table
  5. Use the ordered pairs to plot the graph of the function.
  6. Draw the function by connecting the dots. Recall that when a function is defined by an equation, we have a lot of inputs for x to choose from. We account for this on the graph by sketching a picture of a graph suggested by the points plotted. Be sure to continue the graph beyond the last points to indicate that we could have used even more inputs!

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2.2E: Exercises

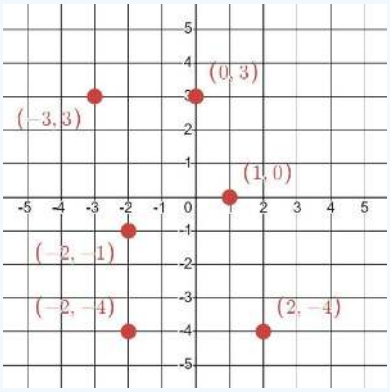
Practice makes perfect!

Graphing ordered pairs on the Cartesian coordinate plane

Graph the following ordered pairs on the Cartesian coordinate plane:

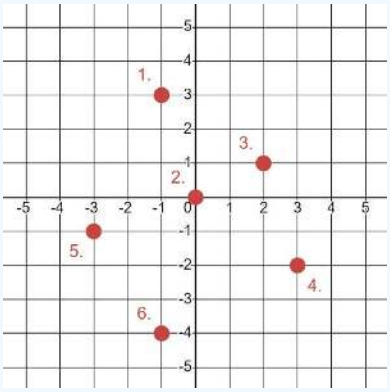
1. (0, 3)
2. (-2, -1)
3. (1, 0)
4. (2, -4)
5. (-3, 3)
6. (-2, -4)

Answer



Identifying points on the Cartesian coordinate plane

Identify the points on the Cartesian coordinate plane below:



Answer

1. (-1, 3)
2. (0, 0)
3. (2, 1)
4. (3, -2)
5. (-3, -1)
6. (-1, 6)

Compute distance between two points

Compute the distance between the points below:

7. (3, 6), (15, 1)
8. (-2, 5), (-2, -4)
9. (-1, 0), (2, 3)
10. (7, -2), (5, 2)

Answer

Add texts here. Do not delete this text first.

7. 13
8. 9
9. $\sqrt{18}$ (or $3\sqrt{2}$)
10. $\sqrt{20}$ (or $2\sqrt{5}$

Find the midpoint between two points

Find the midpoint for each pair of points below:

11. (3, 6), (15, 1)
12. (-2, 5), (-2, -4)
13. (-1, 0), (2, 3)
14. (7, -2), (5, 2)

Answer

11. $\left(9, \frac{7}{2}\right)$
12. $\left(-2, \frac{1}{2}\right)$
13. $\left(\frac{1}{2}, \frac{3}{2}\right)$
14. (6, 0)

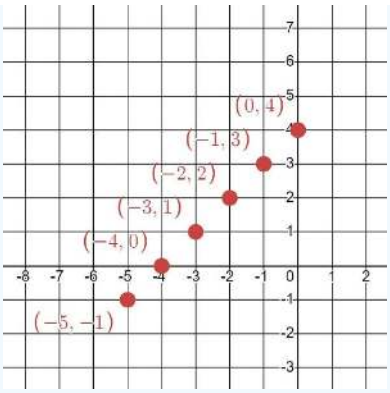
Graphing relations

Graph the relations below:

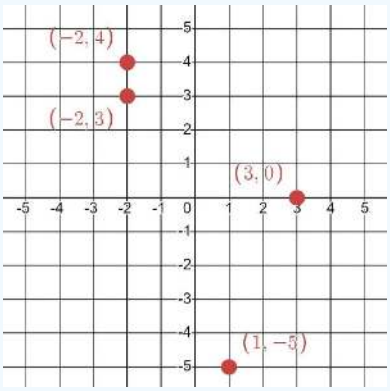
15. $R = \{(-5, -1), (-4, 0), (-3, 1), (-2, 2), (-1, 3), (0, 4)\}$
16. $R = \{(3, 0), (-2, 3), (-2, 4), (1, -5)\}$
17. $R = \{(-4, 0), (-3, -1), (-4, 0), (-1, 1), (2, 1), (0, 4)\}$
18. $R = \{(-2, 1), (-1, 4), (3, 3), (0, -5)\}$

Answer

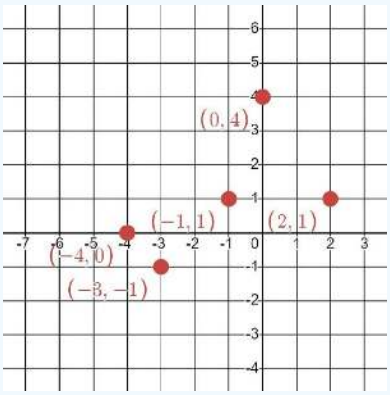
15.



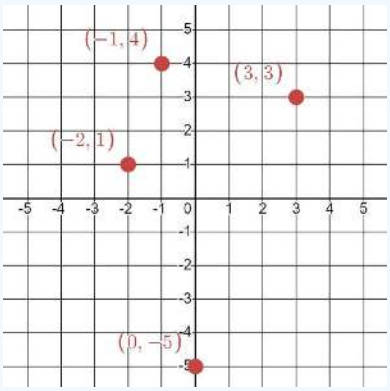
16.



17.



18.



Graphing functions defined by an equation

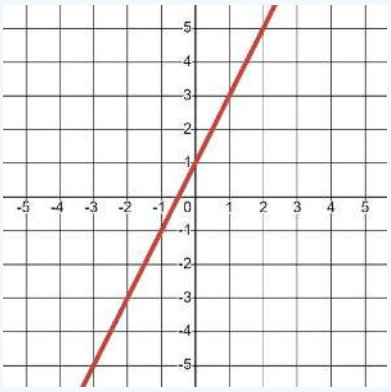
Graph the functions below using a table.

19. $f(x) = 2x + 1$
20. $f(x) = x^2 + 1$
21. $f(x) = \sqrt{x + 4}$
22. $f(x) = |x| + 5$
23. $f(x) = -3$
24. $f(x) = -3x + 4$
25. $f(x) = 2x^2 - 2$
26. $f(x) = \sqrt{x + 4} + 2$
27. $f(x) = -2|x| + 3$
28. $f(x) = 3$

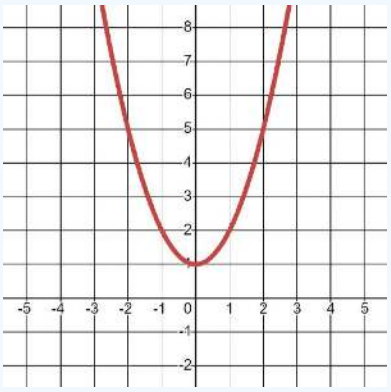
Answer

Tables will vary depending on inputs used; graphs should match those shown below.

19.

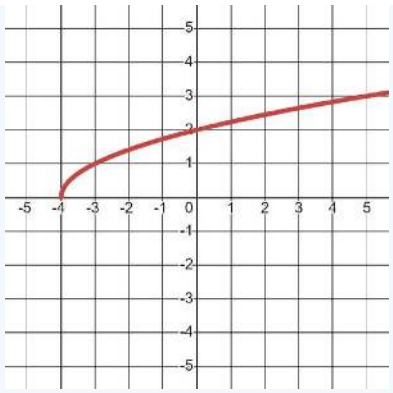


20.

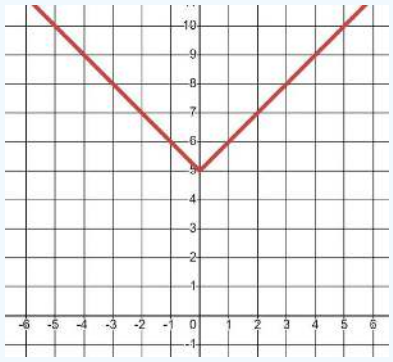


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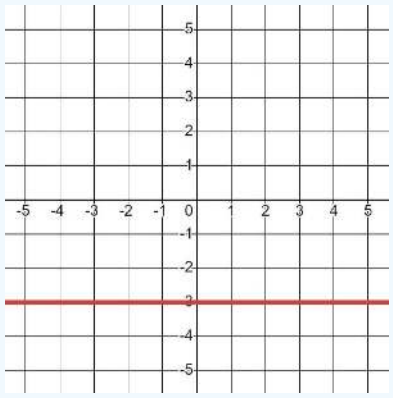
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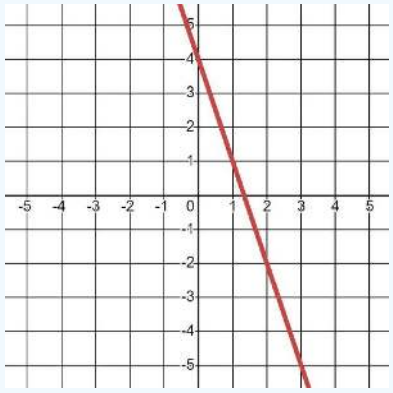
22.



23.

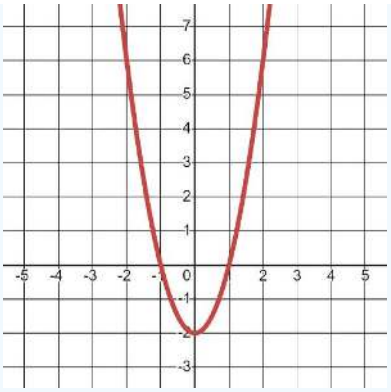


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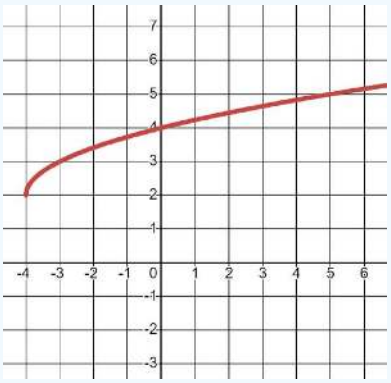


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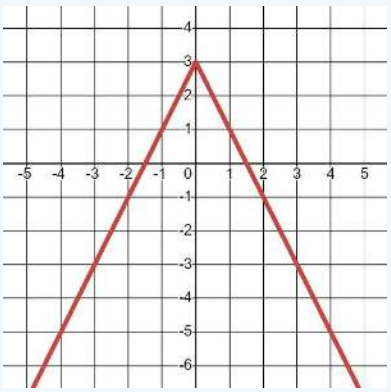
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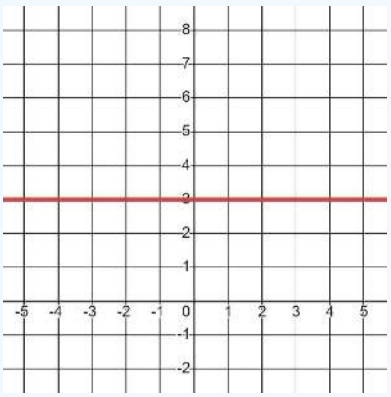
26.



27.



28.



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2.3: Understanding Graphs of Functions

Learning Objectives

- Determine the value of a function at a point using a graph
- Use the vertical line test to determine if a graph represents a function
- Determine domain and range of a function using a graph

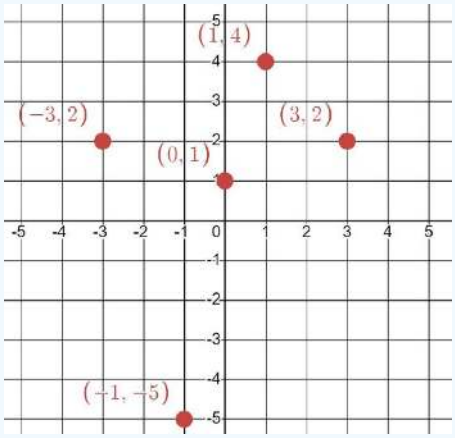
Warm Up 2.3.1

For the relation $R = \{(-3, 2), (-1, -5), (0, 1), (3, 2), (1, 4)\}$, do the following:

a. Determine its domain and range;
b. Graph R;
c. Determine if R is also a function. How do you know?

Answer

a. The domain of R is its input values, or the first component from each ordered pair. These are $\{-3, -1, 0, 3, 1\}$ The range of R is its output values, or the second component from each ordered pair. These are $\{2, -5, 1, 4\}$ (Note that even though 2 occurs twice as an output, we only need to list it once!)
b. To graph, R, we plot each ordered pair on a Cartesian coordinate plane.



c. Each input has only one output corresponding to it, so R is a function! Notice that one of its outputs is repeated; the number 2 appears twice as an output. This is fine since for a function, we only care that each *input* appears once.

Using a Graph to Determine Values of a Function

In our last section, we discussed how we can use graphs on the Cartesian coordinate plane to represent ordered pairs, relations, and functions. In this section, we will dig into the graphs of functions that have been defined using an equation. Our first task is to work backwards from what we did at the end of the last section, and start with a graph to determine the values of a function.

To use a graph to determine the values of a function, the main thing to keep in mind is that $f(input) = ouput$ is the same thing as $f(x) = y$, which means that we can use the y value that corresponds to a given x value on a graph to determine what the function is equal to there. For example, if we had a graph for a function f and we wanted to use that to know what $f(3)$ was, we would start at the origin (0, 0), then move along the horizontal axis to where $x = 3$ and then move up or down until we hit the graph. The y value there is $f(3)$.

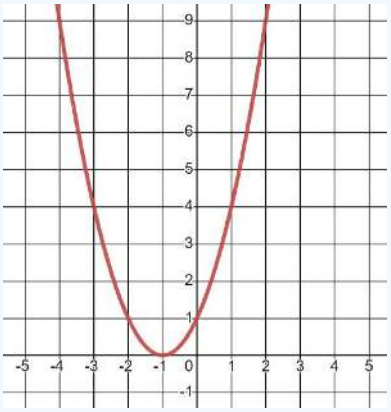
Example 2.3.1

Use the graph below to determine the following values for $f(x) = (x + 1)^2$:

a. $f(2)$

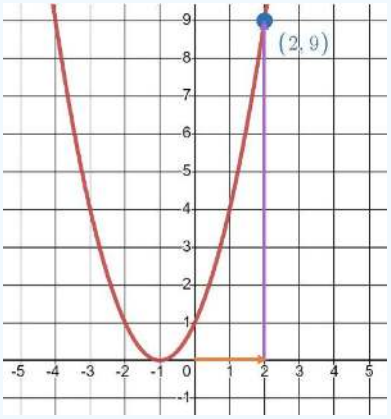
b. $f(-3)$
c. $f(-1)$

After determining these values, compare your answers to what you would get by simply plugging the given values into the function.



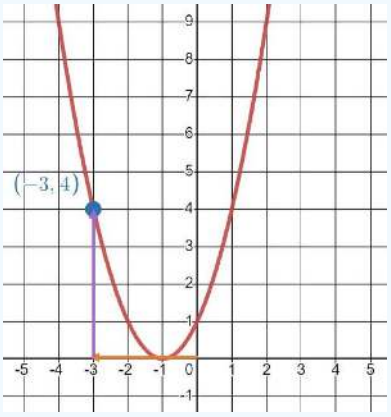
Solution

a. For $f(2)$, our input or x value is $x = 2$. Therefore, to compute this using the graph, starting at the origin (0, 0), we go two units to the right and then go up until we hit the graph. The corresponding y value is 9. So $f(2) = 9$.



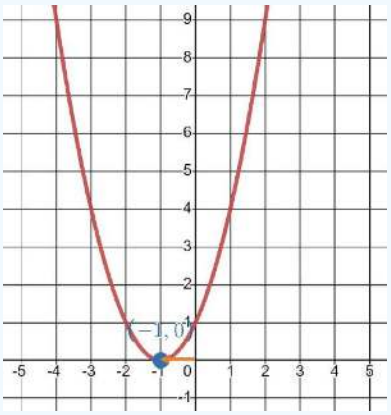
We can compare this answer to what we get by plugging 2 into f. We have $f(2) = (2+1)^2 = 3^2 = 9$; this agrees with the answer from the graph!

b. For $f(-3)$, the input is $x = -3$. So using the graph, we move 3 units to the left then go up until we hit the graph. The y value we get is 4. This means that $f(-3) = 4$.



If we want to check our answer by just evaluating $f(-3)$ by plugging in -3 for x, we have $f(-3) = (-3+1)^2 = (-2)^2 = 4$.

c. Finally, for $f(-1)$, we move left 1 unit. Since the y value for $x=-1$ is 0, we have $f(-1)=0$.



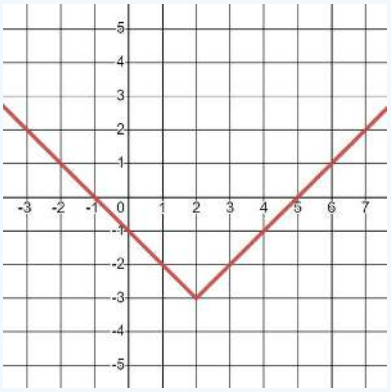
We can plug in -1; we get $f(-1)=(-1+1)^2=0^2=0$.

Exercise 2.3.2

Use the graph below to determine the following values for $f(x)=|x-2|-3$:

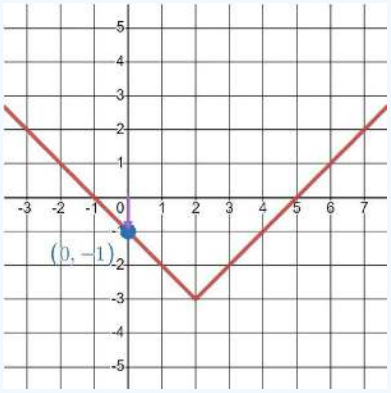
a. $f(0)$
b. $f(-3)$
c. $f(4)$

After determining these values, compare your answers to what you would get by simply plugging the given values into the function.



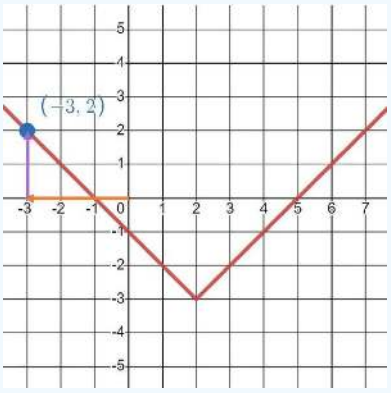
Answer

a. For $f(0)$, the input is $x=0$. Therefore, we just start at 0 and do not need to move horizontally. Since the graph is below the x-axis, we move down until we hit the graph. The graph has $y=-1$ there. Therefore, $f(0)=-1$.



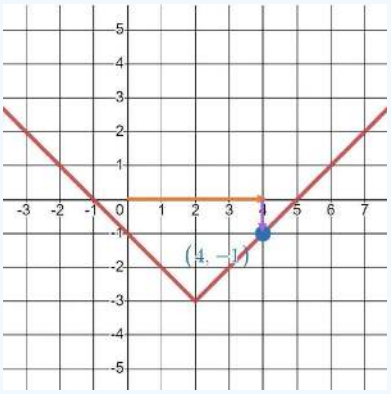
We can also just evaluate the function directly. Plugging in $x = 0$, we get $f(0) = |0 - 2| - 3 = |-2| - 3 = 2 - 3 = -1$.

b. For $f(-3)$, we have the input $x = -3$. To find the output, we move left 3 units and then up until we hit the graph. There $y = 2$, so \(f(-3) = 2).



We can plug in $x = -3$ to get $f(-3) = |-3 - 2| - 3 = |-5| - 3 = 5 - 3 = 2$.

c. Last, we need $f(4)$; this means $x = 4$. We move 4 units to the right, then 1 unit down. The y value is $y = -1$. Therefore, $f(4) = -1$.



We can compare this answer to what we get by plugging in $x = 4$. We have $f(4) = |4 - 2| - 3 = |2| - 3 = 2 - 3 = -1$.

Determining if a Graph Represents a Function

The examples above were graphs of functions, but in the last section we talked about graphing relations and not just functions. However, functions are going to be the focus of what we work with in this course so this brings us to an important question: *how do we know if a graph represents a function?*

The big idea here lies in the definition of a function: each input (or x value) may have only one output (or y value). Based on this, we use what's called the **vertical line test** to determine if a graph represents a function or not. This test helps us identify from the graph of a function if there's anywhere that a single input may have two outputs.

Definition: VERTICAL LINE TEST

If a vertical line drawn *anywhere* on the graph of a relation only intersects the graph at one point, then that graph represents a function. If a vertical line can intersect the graph at two or more points, then the graph does not represent a function.

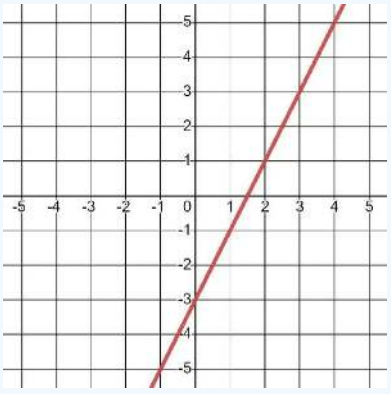
In other words, if a vertical line drawn anywhere only intersects the graph at only one spot, this means that each x value corresponds to only one y value, so the graph represents a function. If a vertical line can be drawn that intersects the graph at more than one spot, then this means that there is an x value that corresponds to multiple y values so the graph cannot represent a function.

Let's try this out on some examples!

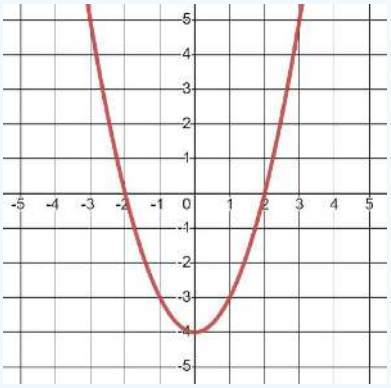
Example 2.3.3

Use the vertical line test to determine if the following graphs represent functions.

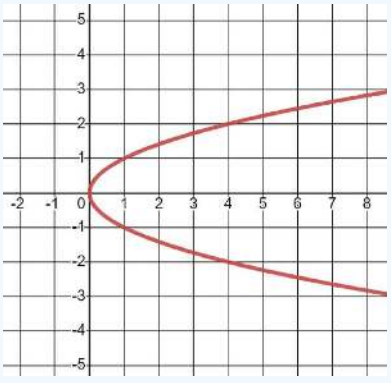
a.



b.

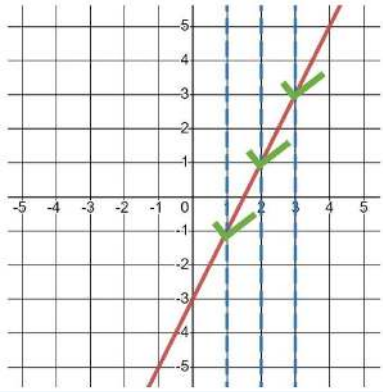


c.

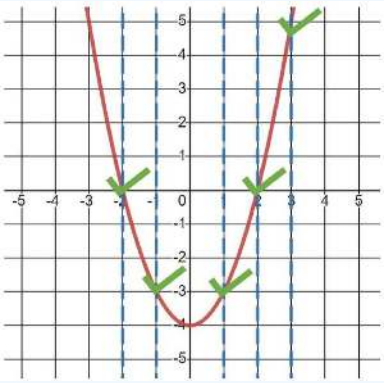


Solution

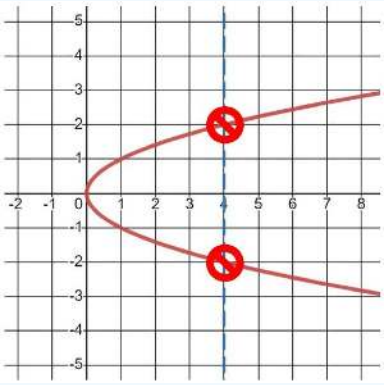
a. Anywhere we draw a vertical line on this graph, it will only intersect the graph once. So the first graph represents a function!



b. Anywhere we draw a vertical line on this graph, it will only intersect the graph once. So the second graph represents a function as well!



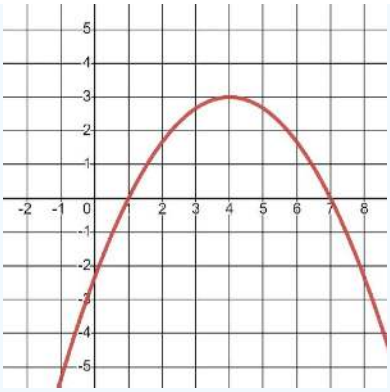
c. Since we can draw a vertical line that intersects the graph at two places, this graph does not represent a function!



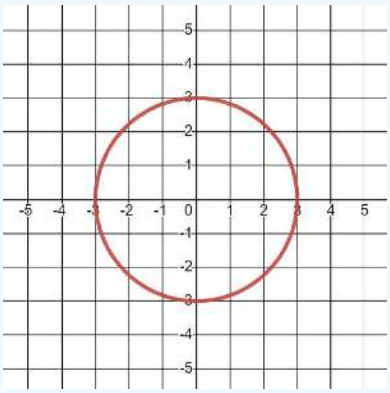
Exercise 2.3.4

Use the vertical line test to determine if the following graphs represent a function:

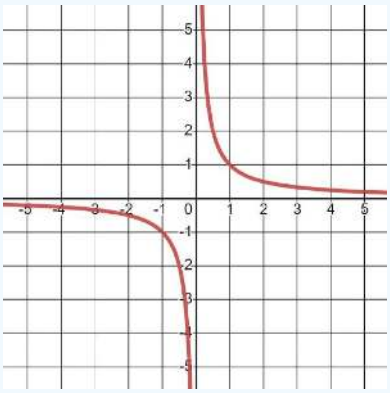
a.



b.

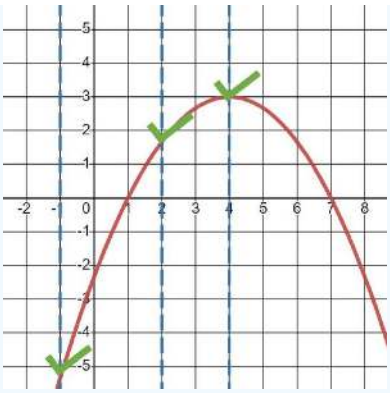


c.

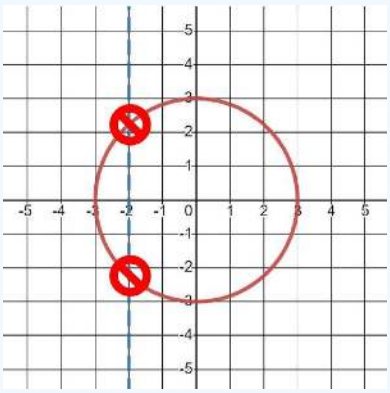


Answer

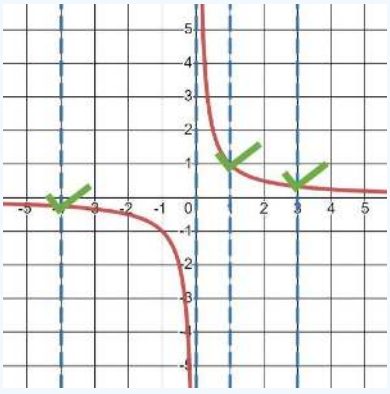
a. Anywhere we draw a vertical line on this graph, it will only intersect the graph once. So the first graph represents a function!



b. Since we can draw a vertical line that intersects the graph at two places, this graph does not represent a function!



c. This graph is a little strange. Most places we draw a vertical line, it's easy to see that the line only intersects the graph at one point. Drawing a line at $x = 0$, it's not quite so clear. However, the graph does not appear to ever exist there, so there is nowhere for a vertical line to intersect. Therefore, the graph passes the vertical line test and represents a function!



Domain and Range of a Function on a Graph

We conclude this section by looking at how domain and range appear on a graph. First, let's look at definitions for the domain and range of a function that will be more helpful to us here. These definitions are the same as the ones that we used before, just restated for this context:

Definition: DOMAIN

The **domain** of a function is the collection of all possible inputs (or x values) for the function.

Definition: RANGE

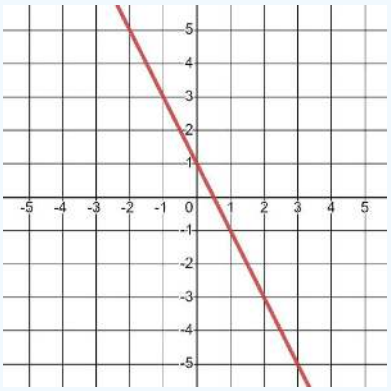
The **range** of a function is the collection of all possible outputs (or y values) for the function.

This means that to determine the domain and range of a function using a graph, we just need to look at what is happening along the x and y axes. The values along the x axis that are included on the graph create the domain of the function, while the values along the y axis that are included on the graph create the range of the function. Let's look at some examples.

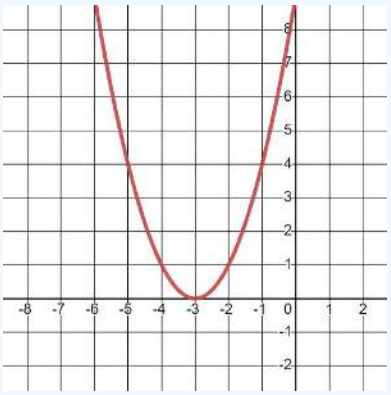
Example 2.3.5

For each graph below, determine the domain and range of the function it represents.

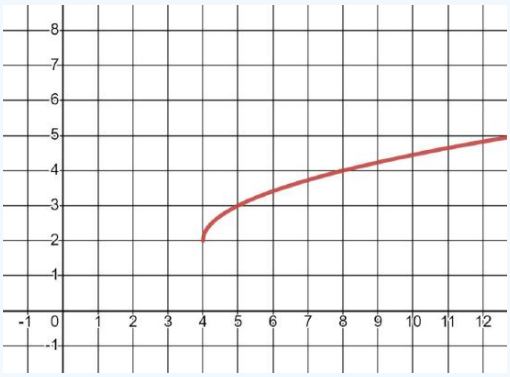
a.



b.

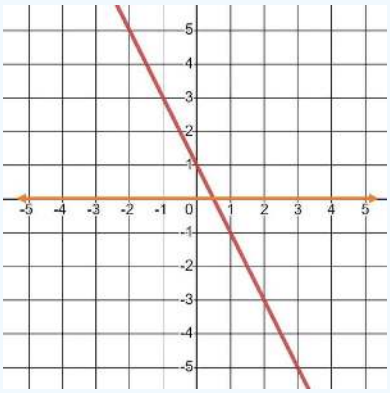


c.

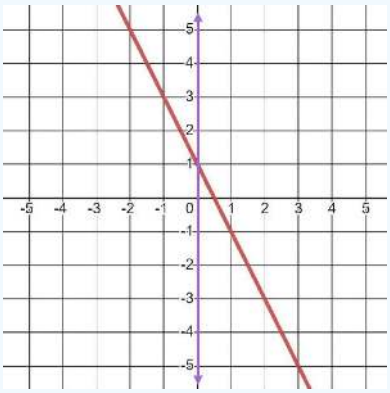


Solution

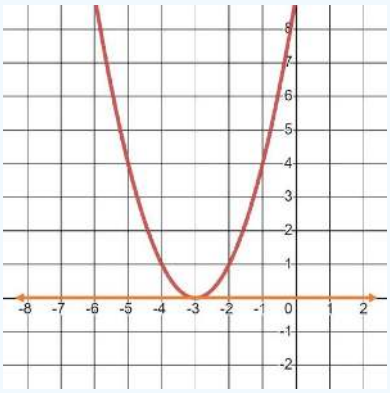
a. To determine the domain, we look at all the possible x values that the graph will reach. Here, the graph will continue to stretch across all the x values, so **the domain is all real numbers, that is, $D = (-\infty, \infty)$.**



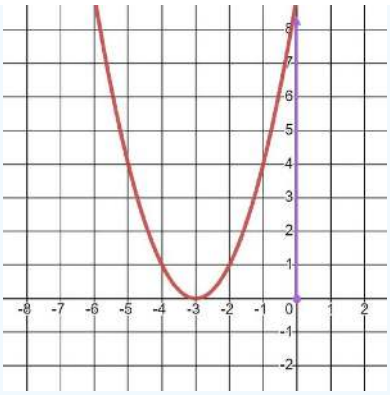
To determine the range, we look at all the possible y values that the graph can reach. The graph will continue growing both upwards and downwards without end, so the **range is all real numbers, that is, $R = (-\infty, \infty)$.**



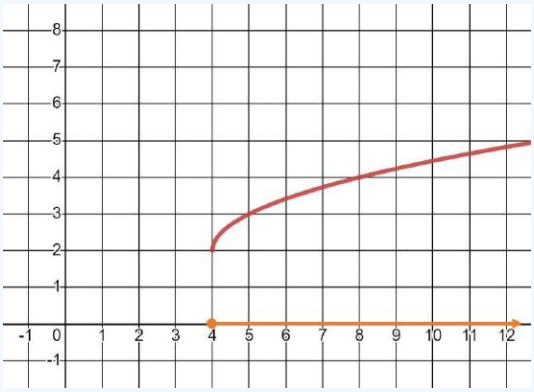
b. The second graph will expand outwards in each direction horizontally without stopping, so it will eventually include every possible x value. Therefore, **the domain is $D = (-\infty, \infty)$.**



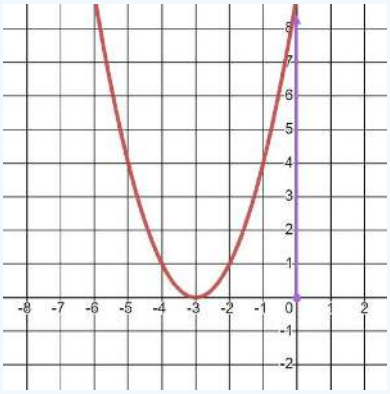
Looking in the vertical direction to determine the range, the graph starts at $y = 0$ and grows upward toward infinity, so it will hit all y values that are greater than or equal to 0. Therefore, the **range will be $y \geq 0$.**



c. Finally, for the domain of the third function, looking at the horizontal direction, the graph starts at $x = 4$ and goes to the right toward infinity. Therefore, the possible inputs are any x value that is greater than or equal to 4. In other words, **the domain is $x \geq 4$.**



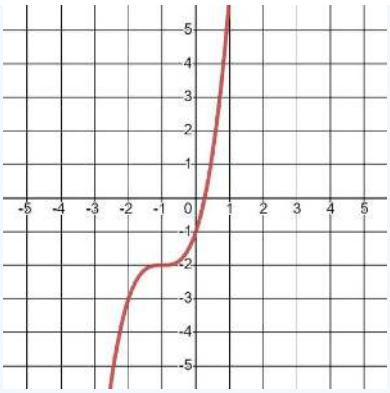
For the range, we look in the vertical direction. The y values start at $y = 2$, and even though the graph appears to be more horizontal than vertical, there is no limit on how much it is going to grow (even though it looks like it's slowing down!). Therefore, the possible output values are the y values that are greater than or equal to 2. Therefore, **the range is $y \geq 2$.**



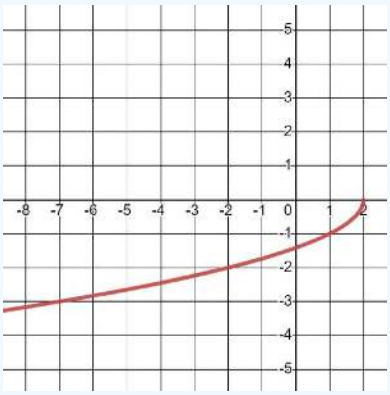
? Exercise 2.3.6

For each graph below, determine the domain and range of the function it represents.

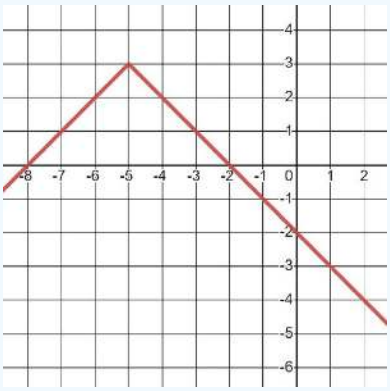
a.



b.



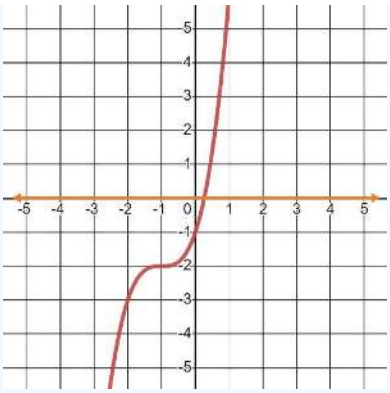
c.



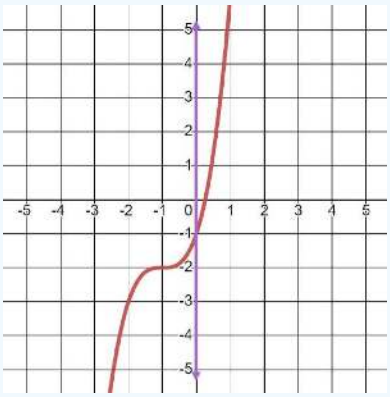
Answer

Add texts here. Do not delete this text first.

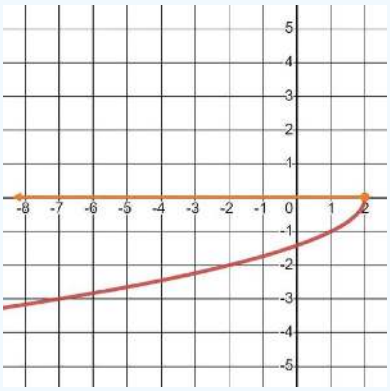
a. To determine the domain, we look at all the possible x values that the graph will reach. Here, the graph will continue to stretch across all the x values, so **the domain is all real numbers, that is,** $D = (-\infty, \infty)$.



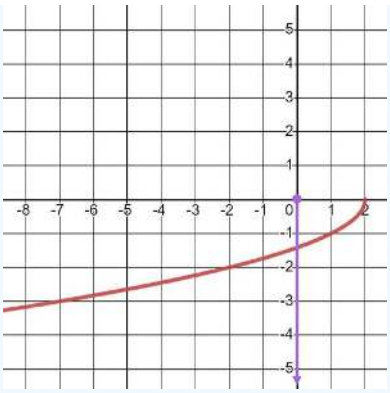
To determine the range, we look at all the possible y values that the graph can reach. The graph will continue growing both upwards and downwards without end, so the **range is all real numbers, that is,** $R = (-\infty, \infty)$.



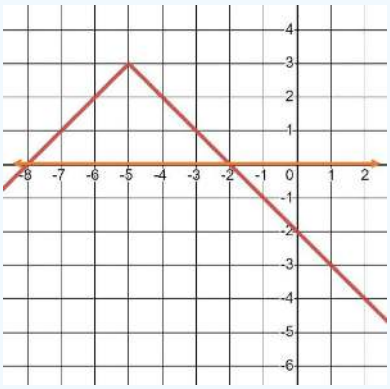
b. To determine the domain, looking in the horizontal direction, we see that the graph begins at $x = 2$ and extends to the left toward infinity. This means that our domain will include all numbers less than or equal to $x = 2$. Therefore, **the domain is** $x \leq 2$.



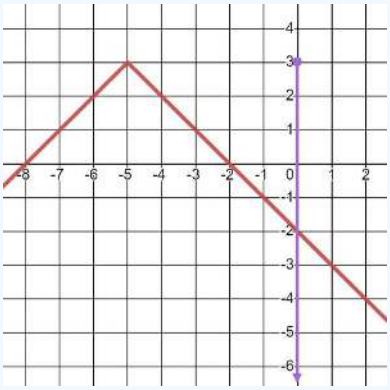
For the range, we look in the vertical direction. This graph starts at $y = 0$ and goes downward. Similar to the graph in the last example, it appears to be more horizontal than vertical. However, it will continue downward without stopping, so the range will include all the negative y values and 0. That is, **the range is** $y \leq 0$.



c. For the last graph, to get the domain, we notice that the graph continuously extends horizontally and will eventually hit every possible value of x. Therefore, **the domain will include all real numbers so $D = (-\infty, \infty)$.**



For the range, the y values begin at $y = 3$ and then continue downward without stopping. The range is therefore any y values that are less than or equal to 3. In other words, **the range is $y \leq 3$.**



Key Concepts

- **How to determine the value of a function $f(x)$ using a graph**
  1. Go to the point on the x axis corresponding to the input for the function.
  2. Move up or down until you hit the graph.
  3. The y value at that point on the graph is the value for $f(x)$.
- **How to use the vertical line test to determine if a graph represents a function**
  1. Look for places where a *vertical* line can be drawn on the graph that might hit the graph in more than one place.
     1. If it is possible to draw a vertical line that hits the graph in two or more places, the graph does NOT represent a function.
     2. If any vertical line drawn hits the graph in only one place, the graph does represent a function.

- **How to determine domain and range of a function using a graph**
  1. To determine the domain, look at the values along the x axis that the graph reaches.
  2. To determine the range, look at the values along the y axis that the graph reaches.

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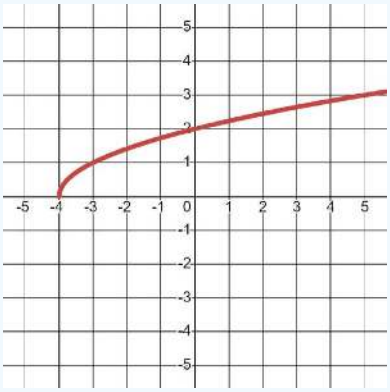
2.3E: Exercises

Practice makes perfect!

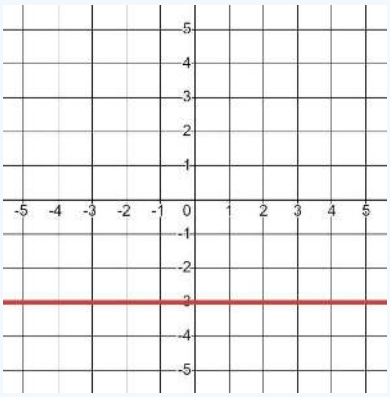
Determine values of a function using its graph

Evaluate the following based on the graphs, then check your work by plugging the x value into the function to evaluate.

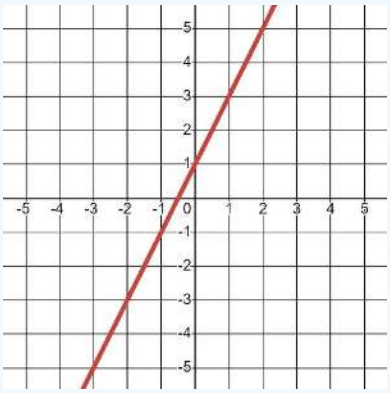
1. $f(x)=\sqrt{x+4}$; $f(-4), f(-3), f(0)$



2. $f(x)=-3$; $f(-3), f(0), f(4)$



3. $f(x)=2x+1$; $f(-2), f(0), f(2)$

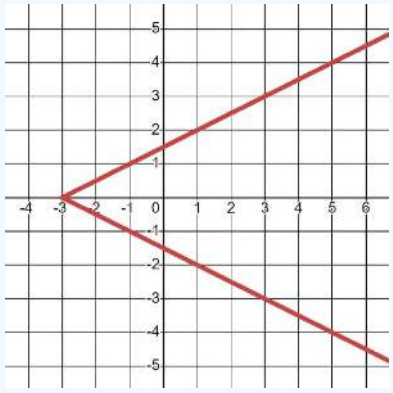


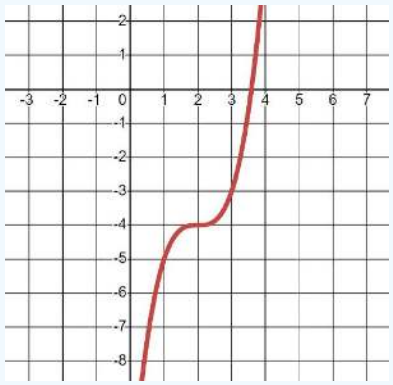
Answer

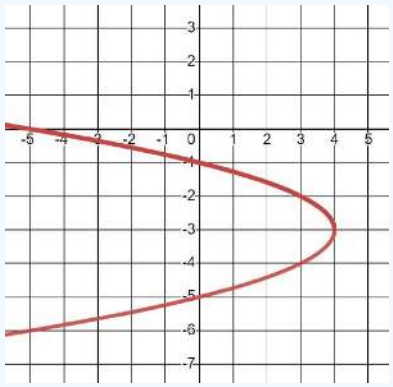
1. $f(-4)=0, f(-3)=1, f(0)=2$
2. $f(-3)=-3, f(0)=-3, f(4)=-3$
3. $f(-2)=-3, f(0)=1, f(2)=5$

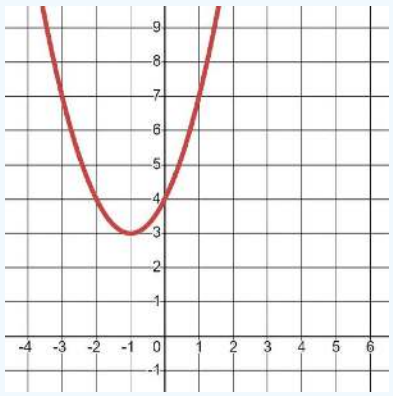
Determine if a graph represents a function

Use the vertical line test to determine if the following graphs represent a function:

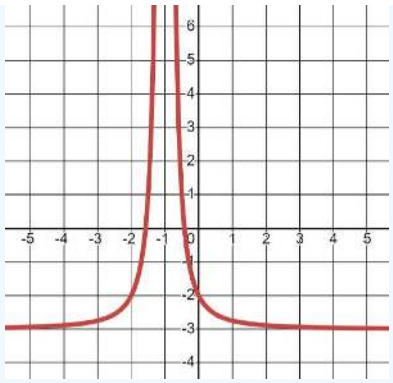
4. 

5. 

6. 

7. 

https://math.libretexts.org/@go/page/110609

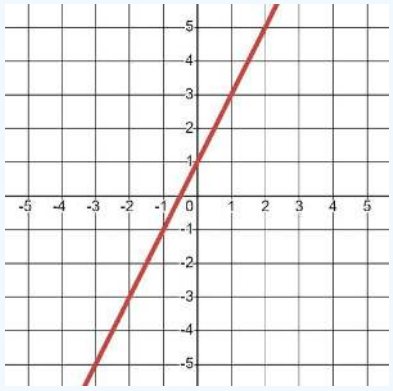
8. 

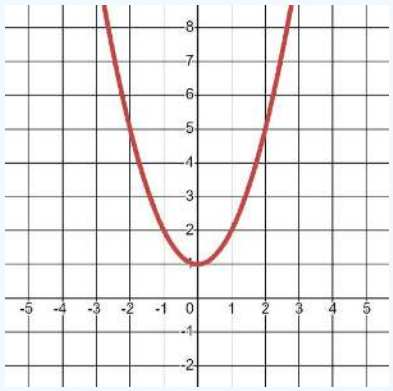
Answer

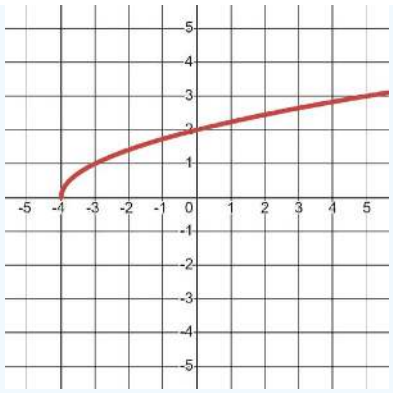
4. Not a function
5. Function
6. Not a function
7. Function
8. Function

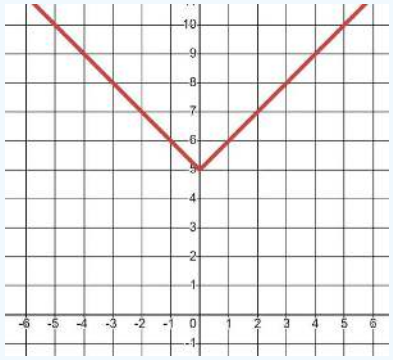
? Determine domain and range of a function using a graph

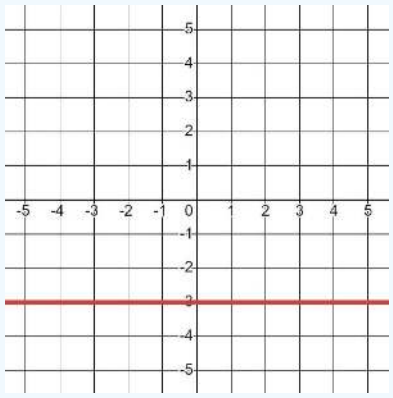
Determine the domain and range for each function represented by the graphs below:

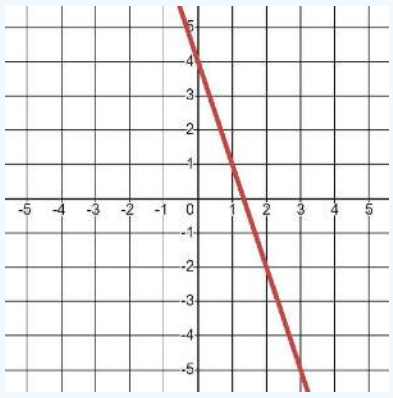
9. 

10. 

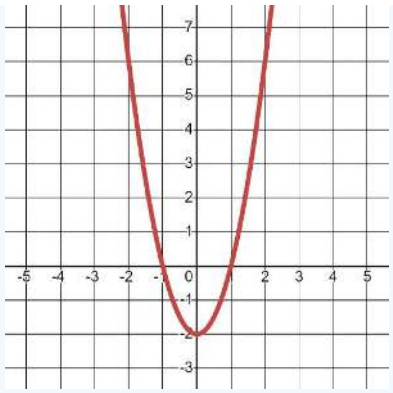
11. 

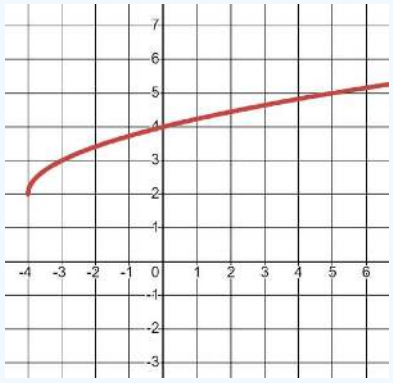
12. 

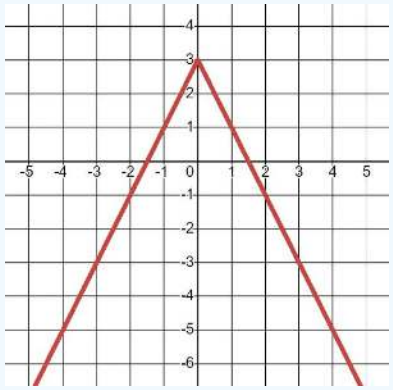
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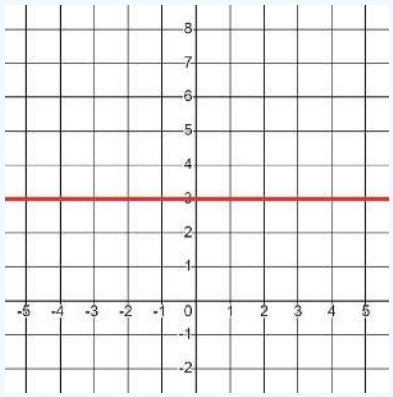
14. 

https://math.libretexts.org/@go/page/110609

15. 

16. 

17. 

18. 

Answer

9. Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$
10. Domain: $(-\infty, \infty)$; Range: $y \geq 1$
11. Domain: $x \geq -4$; Range: $y \geq 0$
12. Domain: $(-\infty, \infty)$; Range: $y \geq 5$

https://math.libretexts.org/@go/page/110609

13. Domain: $(-\infty, \infty)$; Range: $y = -3$
14. Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$
15. Domain: $(-\infty, \infty)$; Range: $y \geq 2$
16. Domain: $x \geq -4$; Range: $y \geq 2$
17. Domain: $(-\infty, \infty)$; Range: $y \leq 3$
18. Domain: $(-\infty, \infty)$; Range: $y = 3$

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2.4: Families of Functions

Learning Objectives

- Identify families of functions based on their rule
- Identify families of functions based on their graphs
- Match functions and their graphs based on their family

Families of Functions

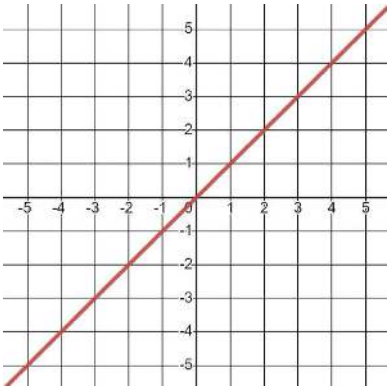
In the last few sections, we've studied functions and how we can represent them visually using a graph. So far, we've looked a wide variety of functions, and in some of the examples, you may have noticed that the resulting graphs looked really similar to each other. As it turns out, there are "families" of functions based on equations that result in specific shapes of graphs, where the only difference between them may be that they've been moved around or flipped over. In this section, we will identify the main families of functions and their graphs we are going to work with in this class. Of course, there are many more families of functions than we will talk about here, but these particular families will be the focus of our class!

Linear Functions

Our first family of functions is called linear functions. The "parent" function for this family is

$$f(x) = x.$$

As you may have guessed, these are the type of functions whose graphs are a straight line. The graph of $f(x) = x$ looks like



Graphs in this family may have different slants or be in a different location on the coordinate plane, but what they all have in common is their basic shape is a straight line. Their rules also all look similar, where the equation only has x which may be multiplied by a constant or have a constant added to it, but that's it. Let's plot some other examples of linear functions:

Example 2.4.1

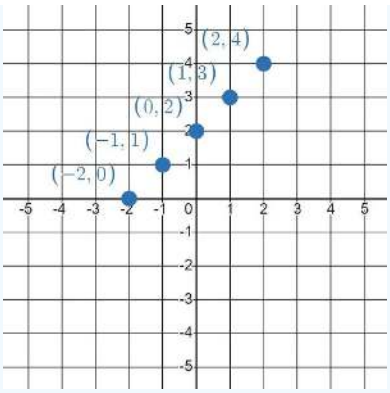
Graph the following linear function using a table: $f(x) = x + 2$

Solution

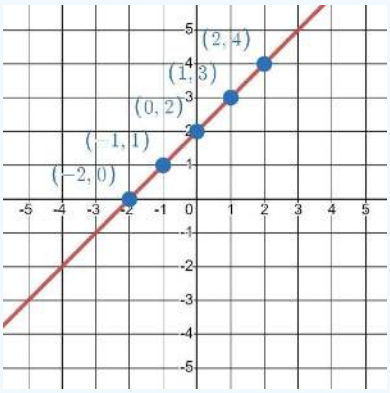
First, let's make a table of ordered pairs we can plot; five points should be sufficient to create the graph

x	$f(x)$	$(x, f(x))$
-2	$-2+2=0$	$(-2, 0)$
-1	$-1+2=1$	$(-1, 1)$
0	$0+2=2$	$(0, 2)$
1	$1+2=3$	$(1, 3)$
2	$2+2=4$	$(2, 4)$

Now let's plot the ordered pairs on a coordinate plane.



Finally, we add the line.



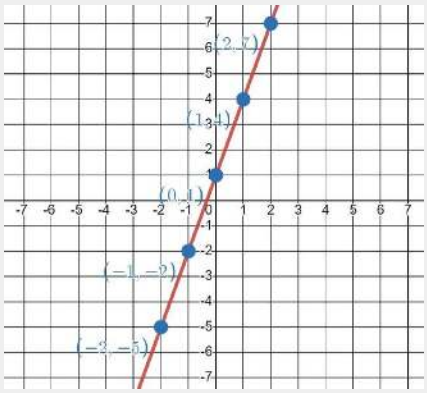
Example 2.4.2

Graph the following linear function using a table: $f(x) = 3x + 1$

Answer

Table	Graph

x	$f(x)$	$(x, f(x))$
-2	-5	$(-2, -5)$
-1	-2	$(-1, -2)$
0	1	$(0, 1)$
1	4	$(1, 4)$
2	7	$(2, 7)$



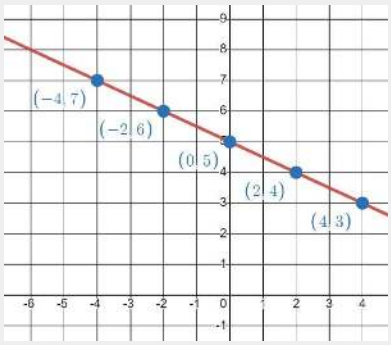
Exercise 2.4.3

Graph the following linear function using a table: $f(x) = -\frac{1}{2}x + 5$

Answer

Table	Graph

x	$f(x)$	$(x, f(x))$
-4	7	$(-4, 7)$
-2	6	$(-2, 6)$
0	5	$(0, 5)$
2	4	$(2, 4)$
4	3	$(4, 3)$



Other Linear Graphs

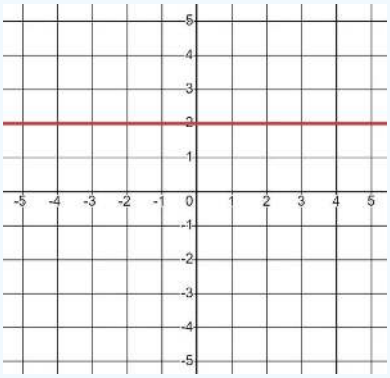
We should also mention two other types of equations that result in graphs that are straight lines. The first is the function $f(x) = a$, where a is any number. This means that regardless of what x is, the output is always the same. The result is a horizontal line at the height of a since the y components are all equal to a.

Example 2.4.4

Graph the function $f(x) = 2$.

Solution

The graph of $f(x) = 2$ is just a horizontal line at $y = 2$.



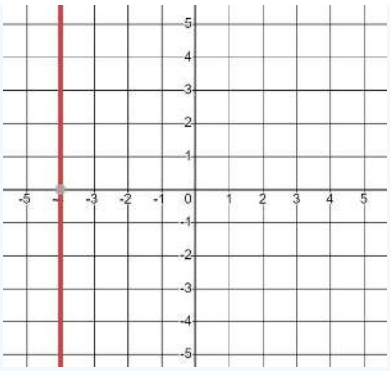
Another equation that results in a straight line is the equation $x = a$, where we only consider a single value of x and then allow y to be anything. This results in a vertical line at $x = a$. However, it is important to notice that a vertical line does NOT represent a function, as it would not pass the vertical line test (the one x value has infinitely many y values corresponding to it!).

Example 2.4.5

Graph the line $x = -4$

Solution

The graph of $x = -4$ is just a vertical line at $x = -4$.



Absolute value

Our next family of functions is those that look like linear functions, but incorporate an absolute value. We might recall that the absolute values return the positive version of whatever is inside of it. Intuitively, if we have a positive number then we don't do anything; if we have a negative number then we make it into a positive.

One way to think of this is as being a function that has two rules, one where we do nothing and one where we multiply negative values by a negative to make it a positive.

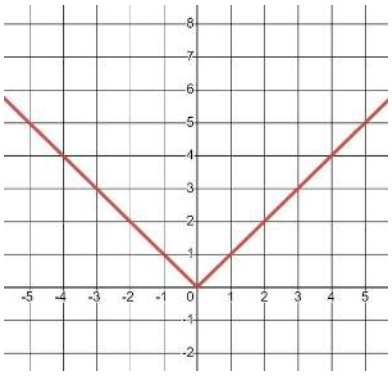
Definition: Absolute value function

$$f(x) = |x| = \begin{cases} -x & x \leq 0 \\ x & x \geq 0 \end{cases}$$

The "parent" function for this family of functions is

$$f(x) = |x|.$$

It has a graph similar to the linear graph, except it has a "v" shape due to the absolute value changing the sign on half of the graph.



All functions in this family will have graphs with this basic shape; however, they may be moved, flipped over (depending on how the rule has been changed, the graph may not always be positive!), or stretched. Let's look at some other examples of functions in this family.

Our first example has been moved around horizontally.

Example 2.4.6

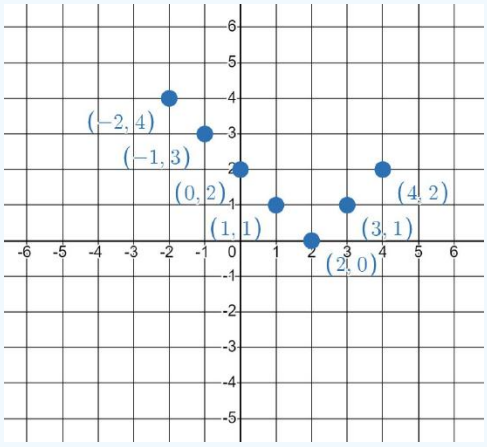
Graph the following absolute value function using a table: $f(x) = |x - 2|$

Solution

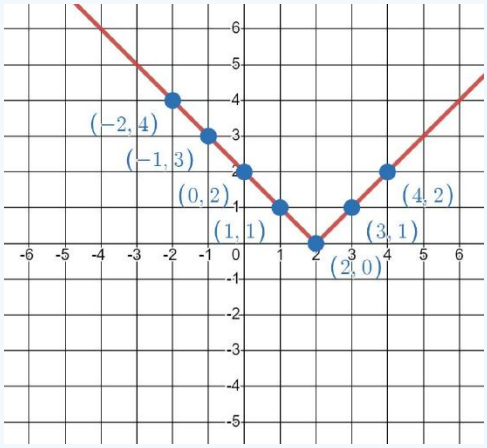
First, let's make a table to contain ordered pairs for plotting. We need to make sure include enough points to understand the shape of the graph. Since we expect the absolute value to have a "v" shape, we will use a couple more than we normally would.

x	$f(x)$	$(x, f(x))$
-2	$\lvert -2-2\rvert = \lvert -4\rvert = 4$	$(-2, 4)$
-1	$\lvert -1-2\rvert = \lvert -3\rvert = 3$	$(-1, 3)$
0	$\lvert 0-2\rvert = \lvert -2\rvert = 2$	$(0, 2)$
1	$\lvert 1-2\rvert = \lvert -1\rvert = 1$	$(1, 1)$
2	$\lvert 2-2\rvert = \lvert 0\rvert = 0$	$(2, 0)$
3	$\lvert 3-2\rvert = \lvert 1\rvert = 1$	$(3, 1)$
4	$\lvert 4-2\rvert = \lvert 2\rvert = 2$	$(4, 2)$

Let's plot the points.



Finally, we sketch the graph. We use both the dots and the knowledge that we expect a "v" shape to do this.



Our next example has been moved vertically.

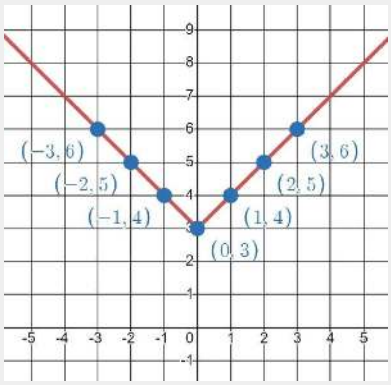
? Example 2.4.7

Graph the following absolute value function using a table: $f(x) = |x| + 3$

Answer

Table	Graph

x	$f(x)$	$(x, f(x))$
-3	$\lvert -3\rvert +3 = 3+3 = 6$	$(-3, 6)$
-2	$\lvert -2\rvert +3 = 2+3 = 5$	$(-2, 5)$
-1	$\lvert -1\rvert +3 = 1+3 = 4$	$(-1, 4)$
0	$\lvert 0\rvert +3 = 0+3 = 3$	$(0, 3)$
1	$\lvert 1\rvert +3 = 1+3 = 4$	$(1, 4)$
2	$\lvert 2\rvert +3 = 2+3 = 5$	$(2, 5)$
3	$\lvert 3\rvert +3 = 3+3 = 6$	$(3, 6)$



One more example! This time, our graph has been moved horizontally, vertically, and flipped over!

? Example 2.4.8

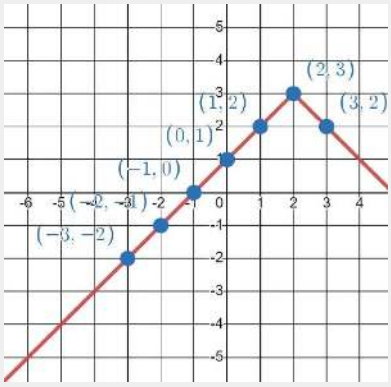
Graph the following absolute value function using a table: $f(x) = -|x-2|+3$

Answer

Table

x	$f(x)$	$(x, f(x))$
-3	$-\lvert -3-2\rvert +3 = -\lvert -5\rvert +3 = -(5)+3 = -2$	$(-3, -2)$
-2	$-\lvert -2-2\rvert +3 = -\lvert -4\rvert +3 = -(4)+3 = -1$	$(-2, -1)$
-1	$-\lvert -1-2\rvert +3 = -\lvert -3\rvert +3 = -(3)+3 = 0$	$(-1, 0)$
0	$-\lvert 0-2\rvert +3 = -\lvert -2\rvert +3 = -(2)+3 = 1$	$(0, 1)$
1	$-\lvert 1-2\rvert +3 = -\lvert -1\rvert +3 = -(1)+3 = 2$	$(1, 2)$
2	$-\lvert 2-2\rvert +3 = -\lvert 0\rvert +3 = -(0)+3 = 3$	$(2, 3)$
3	$-\lvert 3-2\rvert +3 = -\lvert 1\rvert +3 = -(1)+3 = 2$	$(3, 2)$

Graph

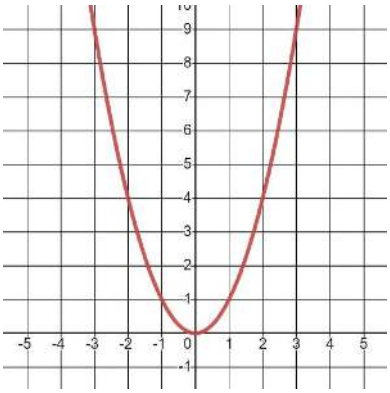


Quadratic Functions

Our third family of functions we want to look at are the quadratic functions. These functions are the ones where the largest exponent appearing in the rule is a 2. That is, x is allowed to be squared or appear as just x, but we can't have anything that looks like x^3, x^4, etc in the rule. The "parent" function for this family is

$$f(x) = x^2.$$

Similar to the absolute value function, this function has a graph that appears to have two branches reaching upward, since anything squared is positive, but it takes on more of a "U" shape since it curves smoothly around the base.



All quadratic functions have this same basic shape. However, just like the absolute value based functions, the graphs of quadratic functions may appear flipped over, stretched, or moved up and down. Let's check out some other examples of quadratic functions!

Our first example has been moved vertically.

Example 2.4.9

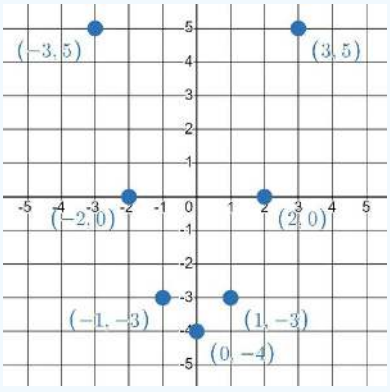
Graph the following quadratic function using a table: $f(x) = x^2 - 4$

Solution

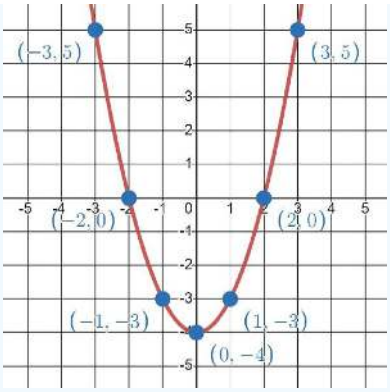
We start by making a table with several points to plot. Since we are expecting a "U" shape, we should use some extra points:

x	$f(x)$	$(x, f(x))$
-3	$(-3)^2 - 4 = 5$	$(-3, 5)$
-2	$(-2)^2 - 4 = 0$	$(-2, 0)$
-1	$(-1)^2 - 4 = -3$	$(-1, -3)$
0	$0^2 - 4 = -4$	$(0, -4)$
1	$1^2 - 4 = -3$	$(1, -3)$
2	$2^2 - 4 = 0$	$(2, 0)$
3	$3^2 - 4 = 5$	$(3, 5)$

Plot the points on a graph.



We connect the dots using a curved line, since quadratic functions all have a "U" shape, so the base of it should be rounded instead of being a corner.



Let's look at another example where the exponent 2 isn't directly attached to the x. This still counts as a quadratic function because the exponent affects the x even though it is outside of parentheses! We will look at this in more detail in Section 4.

In this example, the graph has been moved horizontally.

? Exercise 2.4.10

Graph the following quadratic function using a table: $f(x) = (x+2)^2$

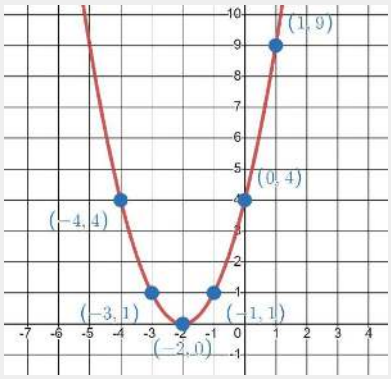
Answer

Notice that in this one, the points are a little "imbalanced", but since we know the general shape of the graph, we can be confident that the left side looks the same as the right side and we can complete the "U" shape in the sketch using that information.

Table

x	$f(x)$	$(x, f(x))$
-4	$(-4+2)^2 = (-2)^2 = 4$	$(-4, 4)$
-3	$(-3+2)^2 = (-1)^2 = 1$	$(-3, 1)$
-2	$(-2+2)^2 = 0^2 = 0$	$(-2, 0)$
-1	$(-1+2)^2 = 1^2 = 1$	$(-1, 1)$
0	$(0+2)^2 = 2^2 = 4$	$(0, 4)$
1	$(1+2)^2 = 3^2 = 9$	$(1, 9)$

Graph



Our next example is another format for a quadratic that includes the x^2, an x term, and a constant term. This graph has been moved horizontally, vertically, and flipped over!

? Exercise 2.4.11

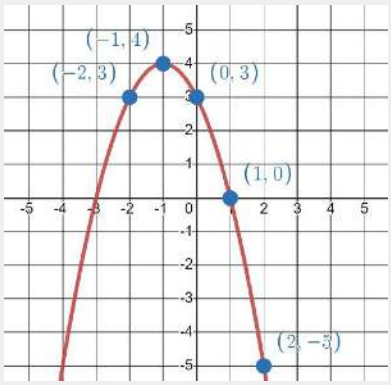
Graph the following quadratic function using a table: $f(x) = -x^2 - 2x + 3$

Answer

Table

Graph

x	$f(x)$	$(x, f(x))$
-2	$-(-2)^2-2(-2)+3=3$	$(-2,3)$
-1	$-(-1)^2-2(-1)+3=4$	$(-1,4)$
0	$-(0)^2-2(0)+3=3$	$(0,3)$
1	$-(1)^2-2(1)+3=0$	$(1,0)$
2	$-(2)^2-2(2)+3=-5$	$(2,-5)$



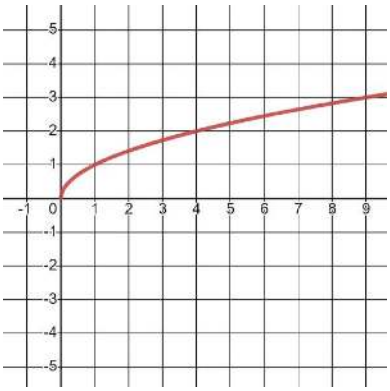
Square root.

Square root functions are closely related to quadratic functions, because as you may have already noticed, square roots "undo" squaring a number, and vice versa, squaring a number "undoes" a square root. As a result, the square root family of functions have graphs that somewhat resemble the quadratic graphs with two notable exceptions -- 1) they're sideways and 2) it's only half the graph.

The "parent" functions for the square root family is

$$f(x)=\sqrt{x}.$$

Notice how the graph looks like half of the quadratic graph that's been turned on its side. The basic shape is a "swoosh", but just like with the previous families, other functions in this family may be modified so that their graph is flipped over vertically or horizontally, moved around, or stretched.



There are three really important things about the square root functions to point out.

1. Remember that when we take the square root of a number, we get both a positive and a negative answer. However, when we are thinking about square root as a *function*, we only want it to have ONE output, not two. In order for a square root to be considered a function, we only consider the positive results from square roots.
2. Since every squared real number produces a positive number, going the opposite direction with a square root means we cannot take the square root of a negative number and get a real number back. That is, when we are looking at square roots as a function, we only take the square roots of positive numbers or 0.
3. Always make the first point you graph the place where you are taking the square root of 0 to make sure the "swoosh" starts in the right place.

When graphing these types of functions, it's pretty common to wind up with decimals for outputs because the inputs may not always result in taking "nice" square roots. If we want, we can choose "convenient" numbers to plug in that will always cause us to take square roots of these "nice" numbers (e.g., 0, 1, 4, 9, 16, etc), or we can just deal with the decimals. It's entirely up to you!

Our first example has been moved horizontally.

Example 2.4.12

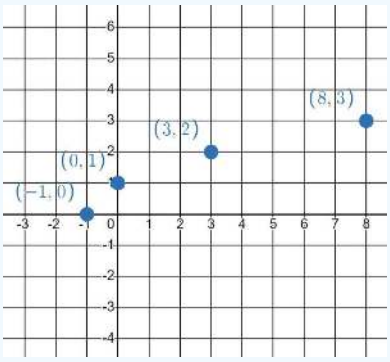
Graph the following square root function using a table: $f(x)=\sqrt{x+1}$

Solution

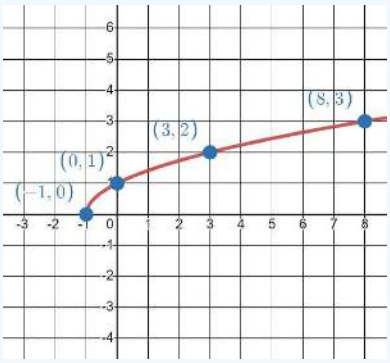
We first make the table with points to plot. Notice that since inside the square root is $x+1$, starting at $x=-1$ is fine because $-1+1=0$, which we are allowed to take the square root of! This will be the first point we want to include on the table. From here, we'd like to take square roots of the "nice" squared numbers 1, 4, and 9. So we want to plug in $x=0, 3$, and 8.

x	$f(x)$	$(x, f(x))$
-1	$\sqrt{-1+1}=\sqrt{0}=0$	$(-1, 0)$
0	$\sqrt{0+1}=\sqrt{1}=1$	$(0, 1)$
3	$\sqrt{3+1}=\sqrt{4}=2$	$(3, 2)$
8	$\sqrt{8+1}=\sqrt{9}=3$	$(8, 3)$

Now we plot the points.



We complete the graph by knowing that it starts at (-1, 0), and then must follow the "swoosh" shape to the right.



Let's look at a version of the graph that has been moved around both horizontally and vertically.

Example 2.4.13

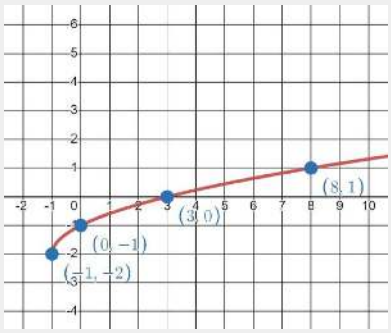
Graph $f(x)=\sqrt{x+1}-2$ by plotting points.

Answer

In regards to the table, notice that once again since inside the square root is $x+1$, starting at $x=-1$ is fine because $-1+1=0$, which we are allowed to take the square root of! This will be the first point we want to include on the table. From here, we'd like to take square roots of the "nice" squared numbers 1, 4, and 9. So we want to plug in $x=0, 3$, and 8.

Table	Graph

x	$f(x)$	$(x, f(x))$
-1	$\sqrt{-1+1}-2=\sqrt{0}-2=-2$	$(-1,-2)$
0	$\sqrt{0+1}-2=\sqrt{1}-2=-1$	$(0,-1)$
3	$\sqrt{3+1}-2=\sqrt{4}-2=0$	$(3,0)$
8	$\sqrt{8+1}-2=\sqrt{9}-2=1$	$(8,1)$



Finally, let's look at an example where the square root "swoosh" has been moved horizontally, vertically, and flipped over.

Exercise 2.4.14

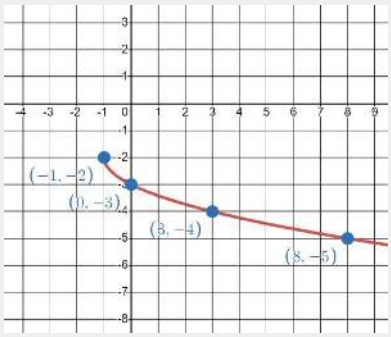
Graph the following square root function using a table: $f(x)=-\sqrt{x+1}-2$

Answer

Table

x	$f(x)$	$(x, f(x))$
-1	$-\sqrt{-1+1}-2=-\sqrt{0}-2=-2$	$(-1,-2)$
0	$-\sqrt{0+1}-2=-\sqrt{1}-2=-3$	$(0,-3)$
3	$-\sqrt{3+1}-2=-\sqrt{4}-2=-4$	$(3,-4)$
8	$-\sqrt{8+1}-2=-\sqrt{9}-2=-5$	$(8,-5)$

Graph

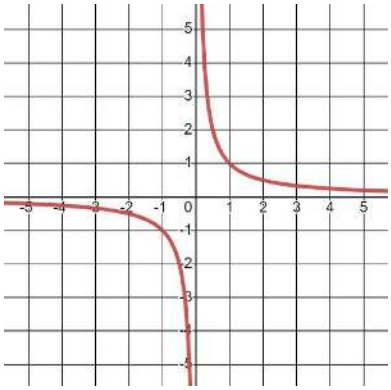


Quotient graph

Our last family of functions is the quotient function. Quotient is just the math way of saying division, and these functions get their name from dividing by the input x. The parent function for this family is

$$f(x)=\frac{1}{x}.$$

The graph of $f(x)=\frac{1}{x}$ is really distinct because unlike the previous functions, its graph has two pieces, and not just one.



The two pieces show up because if you'll remember, we cannot divide by $x=0$. This means there is a gap in the horizontal direction whenever 0 may appear in the denominator of one of these functions. There is also a gap in the vertical direction due to

the fact that nothing we divide by can ever cause the entire function to equal zero. These gaps have a special name:

Definition: Asymptote

A horizontal asymptote is the horizontal line that the graph approaches as $x \to \pm\infty$.

A vertical asymptote is the vertical line that the graph approaches as the denominator approaches 0.

Let's look at some other examples of functions that are in this family, starting with one that has been moved around vertically.

Example 2.4.15

Graph the following quotient function using a table: $f(x) = \frac{1}{x} + 4$

Solution

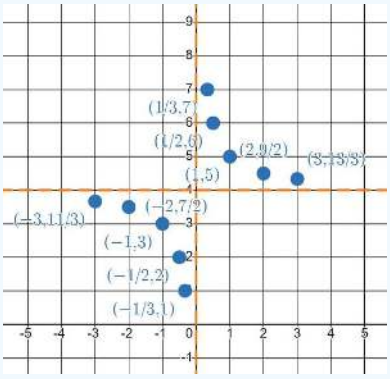
First, we make a table. With these kinds of functions, fractions will be a necessity to understand their general shape! Also, since there are two pieces to the graph, we want to make sure to use enough points to capture both pieces!

x	$f(x)$	$(x, f(x))$
-3	$\frac{1}{-3} + 4 = -\frac{1}{3} + 4 = \frac{11}{3}$	$(-3, \frac{11}{3})$
-2	$\frac{1}{-2} + 4 = -\frac{1}{2} + 4 = \frac{7}{2}$	$(-2, \frac{7}{2})$
-1	$\frac{1}{-1} + 4 = -1 + 4 = 3$	$(-1, 3)$
$-\frac{1}{2}$	$\frac{1}{-1/2} + 4 = -2 + 4 = 2$	$(-\frac{1}{2}, 2)$
$-\frac{1}{3}$	$\frac{1}{-1/3} + 4 = -3 + 4 = 1$	$(-\frac{1}{3}, 1)$
0	$??? + 4 = ???$	draw vertical gap line at $x = 0$
$\frac{1}{3}$	$\frac{1}{1/3} + 4 = 3 + 4 = 7$	$(\frac{1}{3}, 7)$
$\frac{1}{2}$	$\frac{1}{1/2} + 4 = 2 + 4 = 6$	$(\frac{1}{2}, 6)$
1	$\frac{1}{1} + 4 = 1 + 4 = 3$	$(1, 5)$
2	$\frac{1}{2} + 4 = \frac{1}{2} + 4 = \frac{9}{2}$	$(2, \frac{9}{2})$
3	$\frac{1}{3} + 4 = \frac{1}{3} + 4 = \frac{13}{3}$	$(3, \frac{13}{3})$

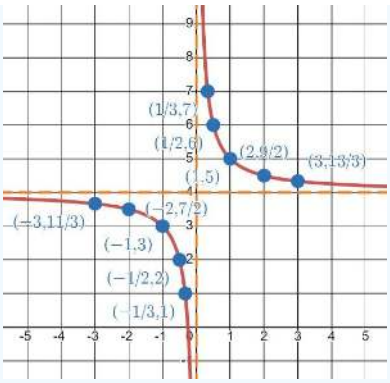
The vertical asymptote goes at the x value that make the denominator zero. In this example, the denomator is zero when $x = 0$, so we can add a dashed line at $x = 0$ to remind ourselves we don't want to graph there.

The horizontal asymptote goes at the y value that each part of the graph are approaching. In this example, each half of the graph will approach $y = 4$, so we can add a dashed line at $y = 4$ to remind ourselves that we don't want to graph there.

Now we plot the points and asymptotes.



When we add the lines, we have the two curves, where the curves follow the points AND asymptotes.



We can shift this graph too. Lets shift it in the same way as we did with $\sqrt{x}$.

Example 2.4.16

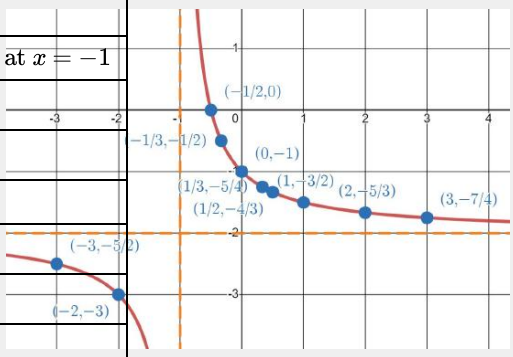
Graph $f(x)=\frac{1}{x+1}-2$ by plotting points.

Answer

Table

x	$f(x)$	$(x, f(x))$
-3	$\frac{1}{-3+1}-2=\frac{1}{-2}-2=\frac{-5}{2}$	$(-3, \frac{-5}{2})$
-2	$\frac{1}{-2+1}-2=\frac{1}{-1}-2=-3$	$(-2,-3)$
-1	$\frac{1}{-1+1}-2=???-2=???$	draw vertical gap line at $x=-1$
$-\frac{1}{2}$	$\frac{1}{-1/2+1}-2=\frac{1}{1/2}-2=0$	$(-\frac{1}{2}, 0)$
$-\frac{1}{3}$	$\frac{1}{-1/3+1}-2=\frac{3}{2}-2=-\frac{1}{2}$	$(-\frac{1}{3}, -\frac{1}{2})$
0	$\frac{1}{0+1}-2=1-2=-1$	$(0,-1)$
$\frac{1}{3}$	$\frac{1}{1/3+1}-2=\frac{3}{4}-2=-\frac{5}{4}$	$(\frac{1}{3}, -\frac{5}{4})$
$\frac{1}{2}$	$\frac{1}{1/2+1}-2=\frac{2}{3}-2=-\frac{4}{3}$	$(\frac{1}{2}, -\frac{4}{3})$
1	$\frac{1}{1+1}-2=\frac{1}{2}-2=-\frac{3}{2}$	$(1, -\frac{3}{2})$
2	$\frac{1}{2+1}-2=\frac{1}{2+1}-2=-\frac{5}{3}$	$(2, -\frac{5}{3})$
3	$\frac{1}{3+1}-2=\frac{1}{4}-2=-\frac{7}{4}$	$(3, -\frac{7}{4})$

Graph



Here we see that x=-1 is a vertical asymptote and y=-2 is a horizontal asymptote.

Identifying Graphs of Families of Functions

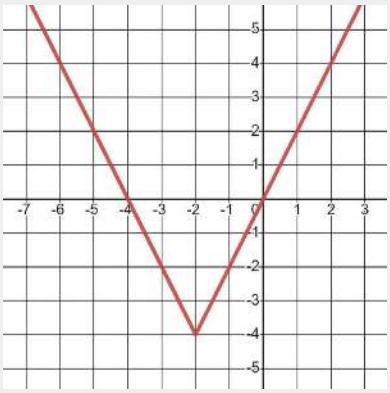
The main point of this section is to highlight the fact that graphs of certain functions always appear a certain way, dependent on the key feature present in the rule. Being able to identify which type of graph goes with which type of function can really help your intuition for what is happening in the long run in this class! So we conclude with some examples of how you can match a function to its graph and vice versa simply based on what type of function it is.

Example 2.4.17

Identify what family of functions each graph belongs to and then match it to the correct function.

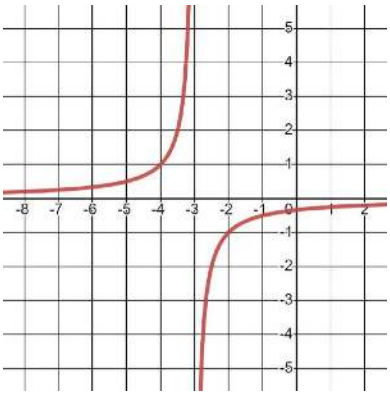
Graph	Function

a.



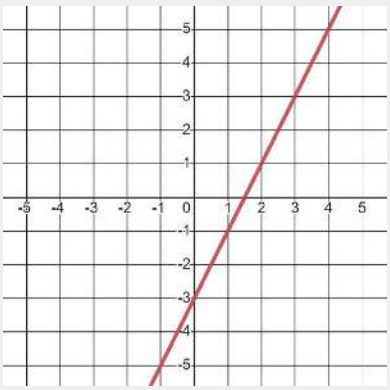
1. $f(x) = 2x - 3$

b.



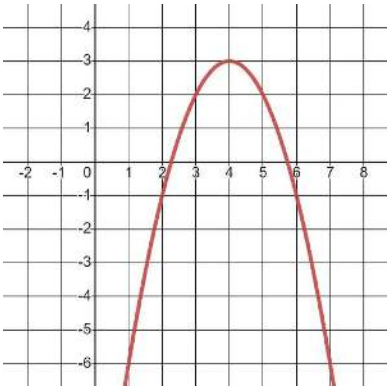
2. $f(x) = \frac{1}{x+3}$

c.



3. $f(x) = -(x-4)^2 + 3$

d.



4. $f(x) = 2|x-2| + 4$

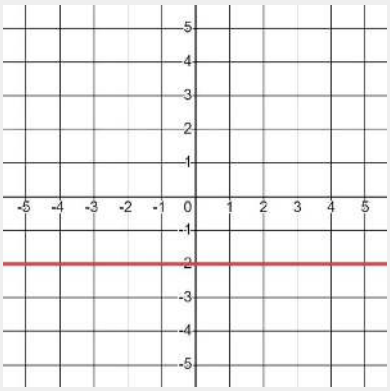
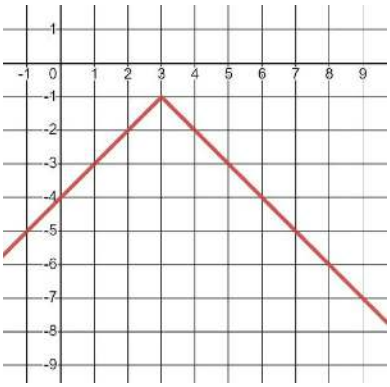
Solution

a. The first graph represents an absolute value function. Therefore, it must have come from the function $f(x) = 2|x-2|+4$, option (4).
b. The second graph represents a quotient function. Therefore, it must have come from the function $f(x) = \frac{1}{x+3}$, option (2).
c. The third graph represents a linear function. Therefore, it must have come from the function $f(x) = 2x-3$, option (1).
d. The final graph represents a quadratic function. Therefore, it must have come from the function $f(x) = -(x-4)^2+3$, option (3).

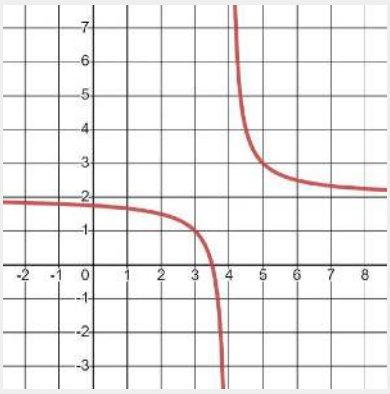
Try it out on your own!

Exercise 2.4.18

Identify what family of functions each graph belongs to and then match it to the correct function.

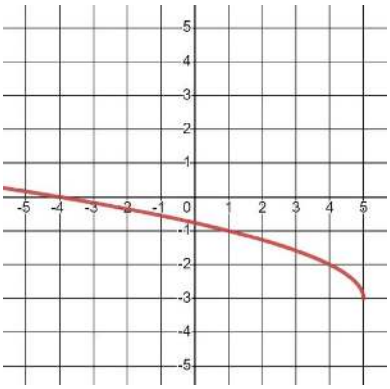
Graph	Function
a. 	1. $f(x) = -\|x-3\|-1$
b. 	2. $f(x) = -2$

c.



3. $f(x) = \frac{1}{x-4} + 2$

d.



4. $f(x) = \sqrt{-x+5} - 3$

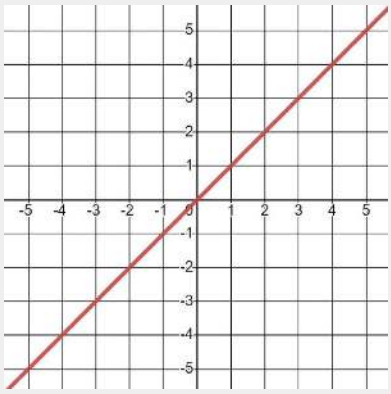
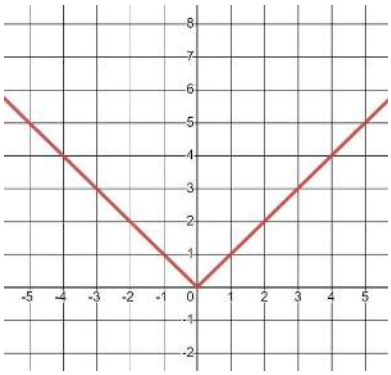
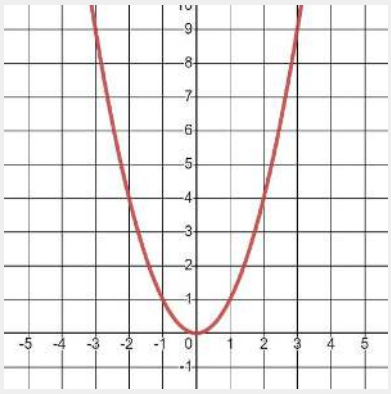
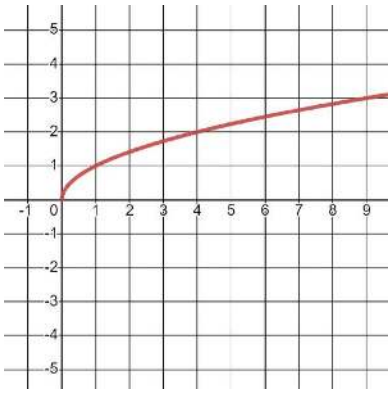
Answer

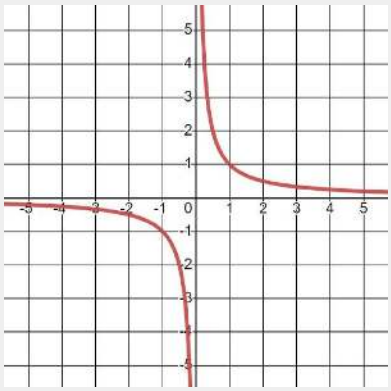
a. The first graph represents a linear function. Therefore, it must have come from the function $f(x) = -2$, option (2).
b. The second graph represents an absolute value function. Therefore, it must have come from the function $f(x) = -|x-3| - 1$, option (1).
c. The third graph represents a quotient function. Therefore, it must have come from the function $f(x) = \frac{1}{x-4} + 2$, option (3).
d. The final graph represents a square root function. Therefore, it must have come from the function $f(x) = \sqrt{-x+5} - 3$, option (4).

Key Concepts

Functions can be grouped into families according to the main features in their rules. These main features control how functions appear when graphed, and therefore functions of the same family all have graphs with the same basic shape.

Family	Parent Function	Basic Shape (graph of parent function)

Linear	$f(x) = x$	
Absolute Value	$f(x) = \|x\|$	
Quadratic	$f(x) = x^2$	
Square Root	$f(x) = \sqrt{x}$	

Quotient	$f(x) = \frac{1}{x}$	

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2.4E: Exercises

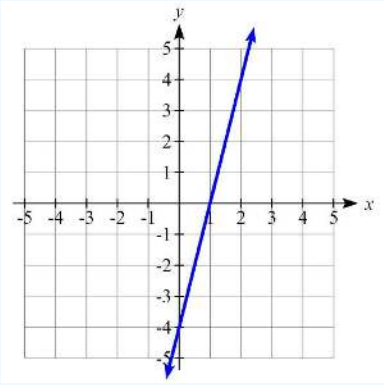
Practice makes perfect!

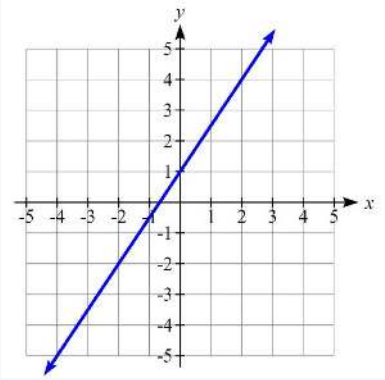
Graphs of linear functions

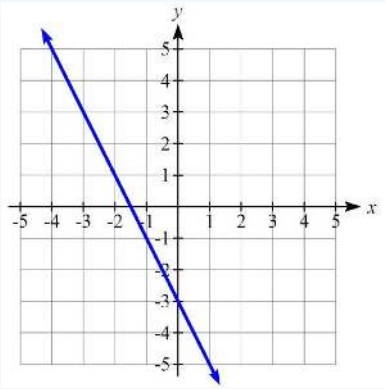
Graph the following linear functions:

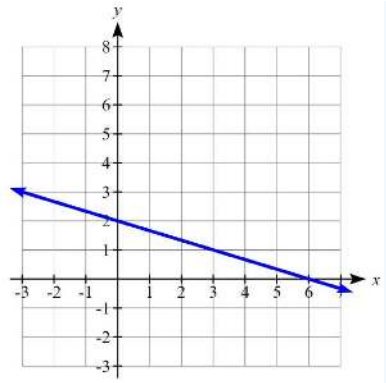
1. $f(x) = 4x - 4$
2. $f(x) = \frac{3}{2}x + 1$
3. $f(x) = -2x - 3$
4. $f(x) = -\frac{1}{3}x + 2$
5. $f(x) = 2$
6. $f(x) = -1$

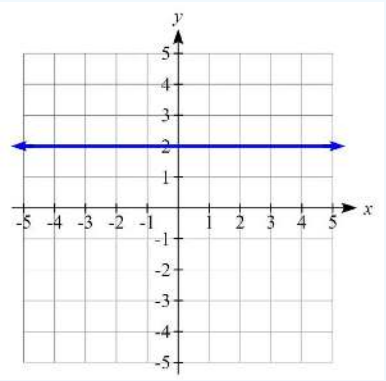
Answer

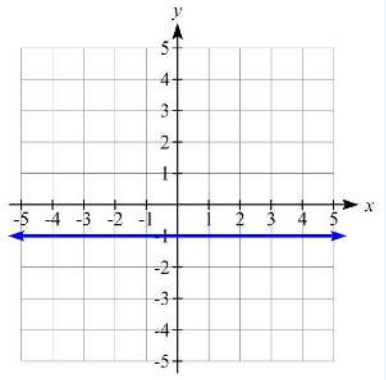
1. 

2. 

3. 

4. 

5. 

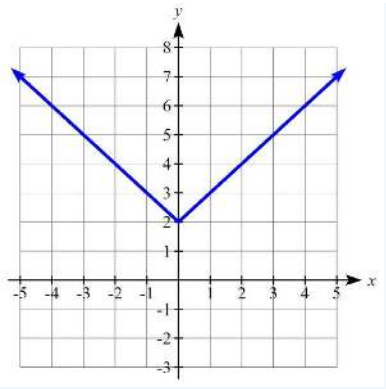
6. 

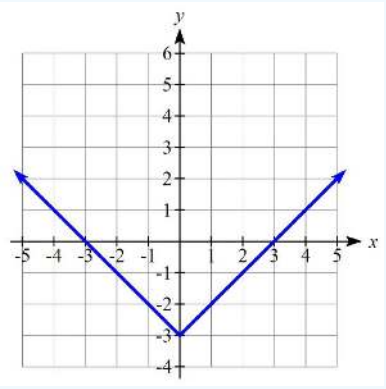
? Graphs of absolute value functions

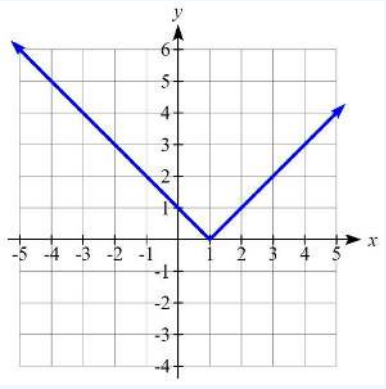
Graph the absolute value functions below:

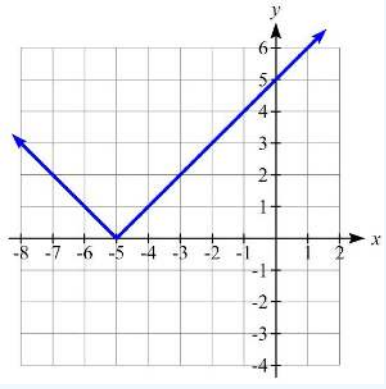
7. $f(x) = |x| + 2$
8. $f(x) = |x| - 3$
9. $f(x) = |x - 1|$
10. $f(x) = |x + 5|$
11. $f(x) = 2|x + 3| - 1$
12. $f(x) = -|x| + 4$

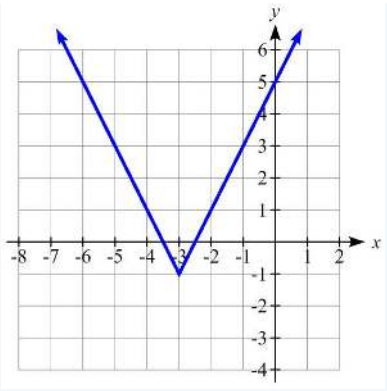
Answer

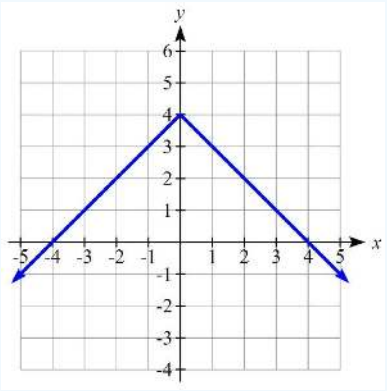
7. 

8. 

9. 

10. 

11. 

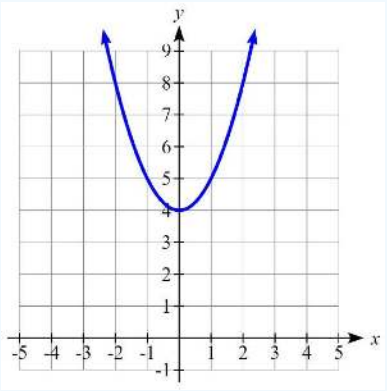
12. 

? Graphs of quadratic functions

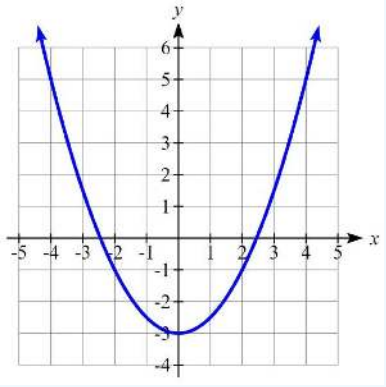
Graph the following quadratic functions

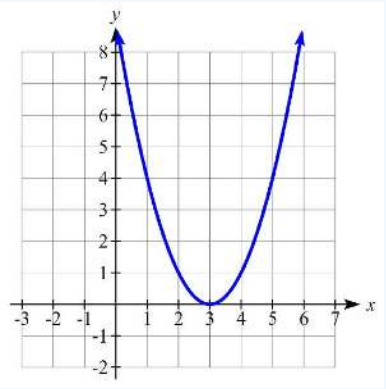
12. $f(x) = x^2 + 4$
13. $f(x) = \frac{1}{2}x^2 - 3$
14. $f(x) = (x-3)^2$
15. $f(x) = 2(x+1)^2$
16. $f(x) = (x-3)^2 + 4$
17. $f(x) = -(x+2)^2 + 5$

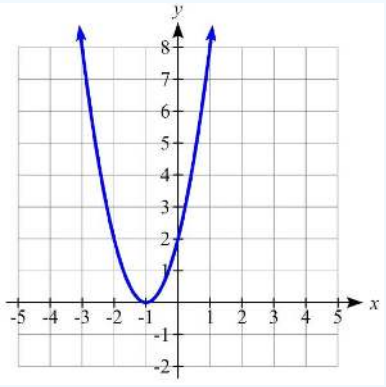
Answer

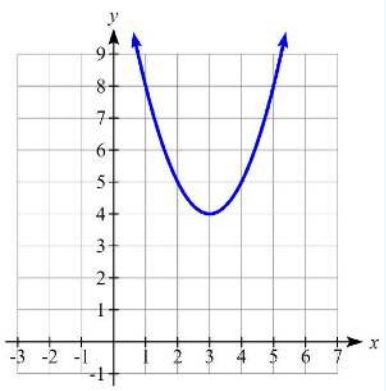
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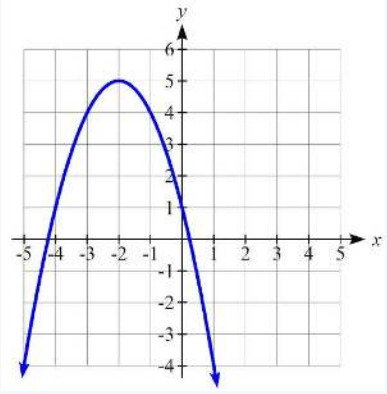
13. 

14. 

15. 

16. 

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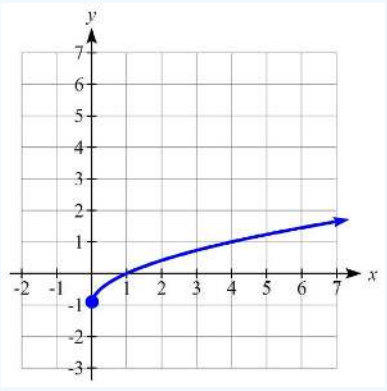
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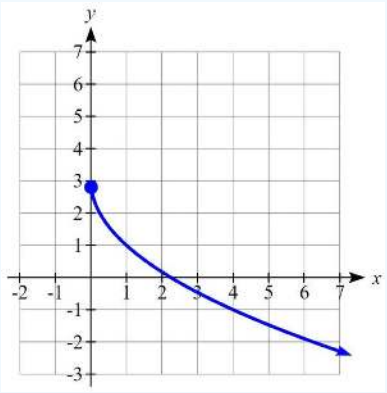
Graphs of square root functions

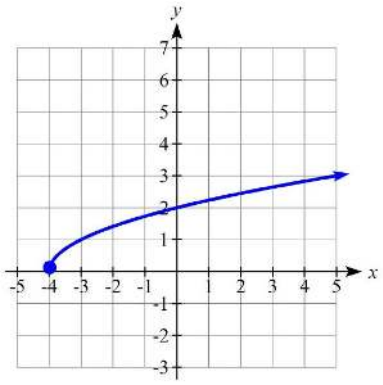
Graph the following square root functions:

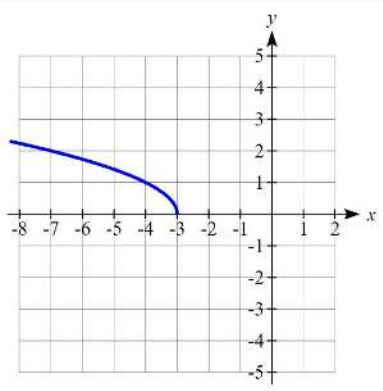
18. $f(x) = \sqrt{x} - 1$
19. $f(x) = -2\sqrt{x} + 3$
20. $f(x) = \sqrt{x+4}$
21. $f(x) = \sqrt{-x-3}$
22. $f(x) = 2\sqrt{x+1} - 2$
23. $f(x) = 3\sqrt{x-2} - 4$

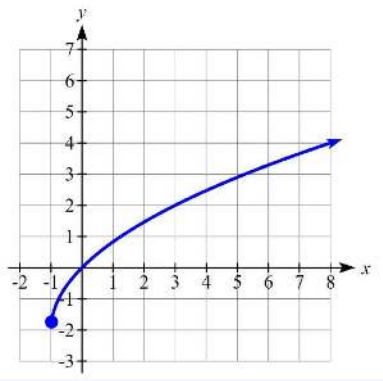
Answer

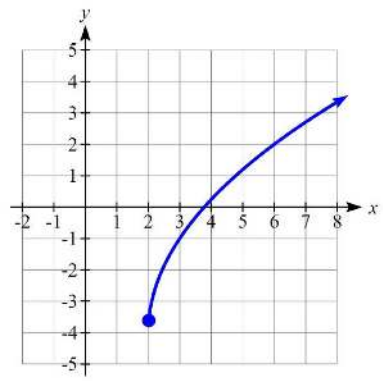
18. 

19. 

20. 

21. 

22. 

23. 

? Graphs of quotient functions

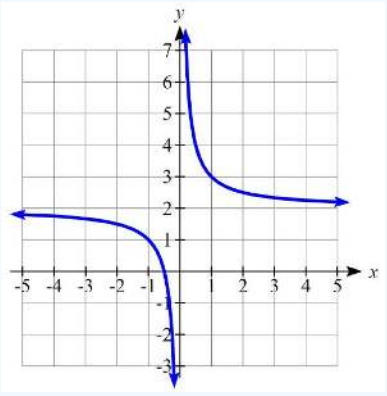
Graph the following quotient functions:

24. $f(x) = \frac{1}{x} + 2$

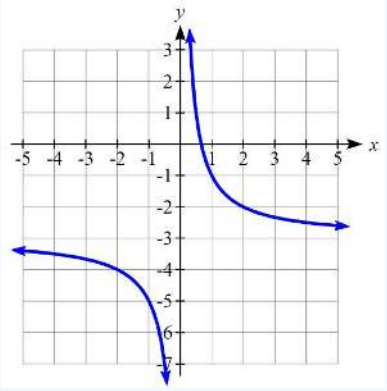
25. $f(x) = \frac{2}{x} - 3$
26. $f(x) = \frac{1}{x+4}$
27. $f(x) = -\frac{1}{x-1}$
28. $f(x) = \frac{1}{x+3} - 2$
29. $f(x) = \frac{3}{x-2} + 1$

Answer

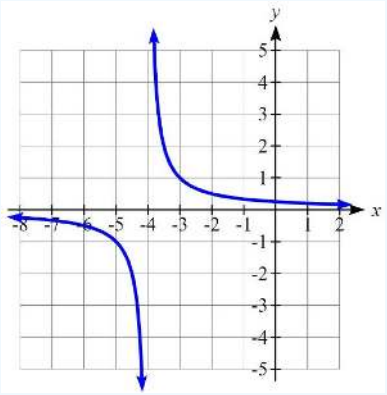
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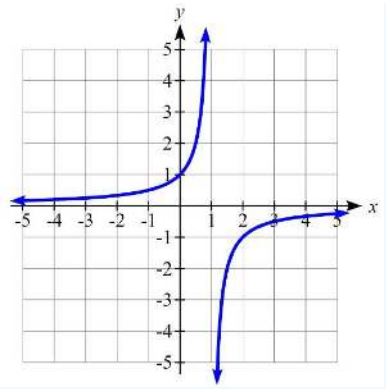
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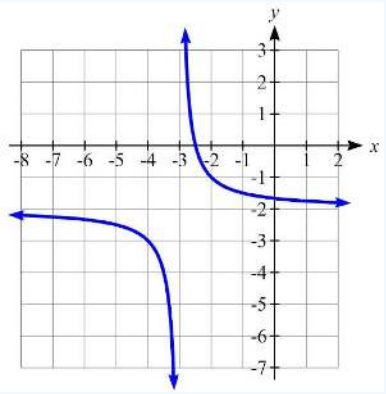


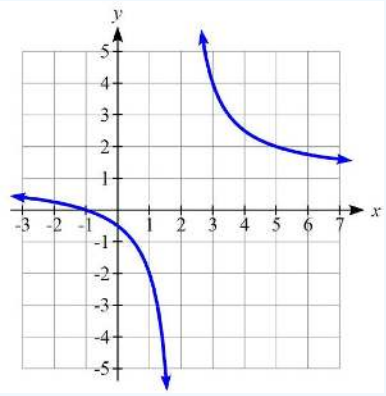
26.



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27. 

28. 

29. 

Classifying functions based on their rule

Classify the following functions according to their family:

30. $f(x) = x + 2$
31. $f(x) = x^2 - 4$
32. $f(x) = \frac{1}{3+x}$
33. $f(x) = x^2 - 3x + 4$
34. $f(x) = \sqrt{x+3} - 4$
35. $f(x) = -2|x - 3|$

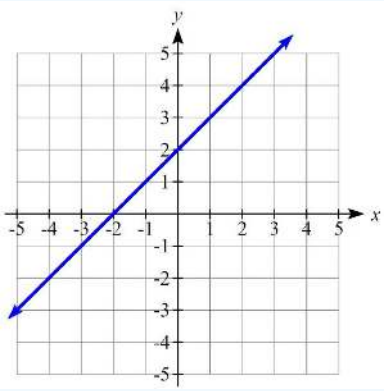
Answer

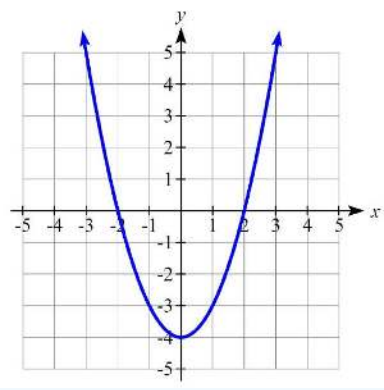
30. Linear
31. Quadratic
32. Quotient
33. Quadratic
34. Square root

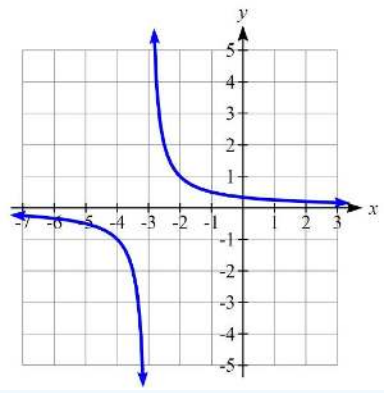
35. Absolute value

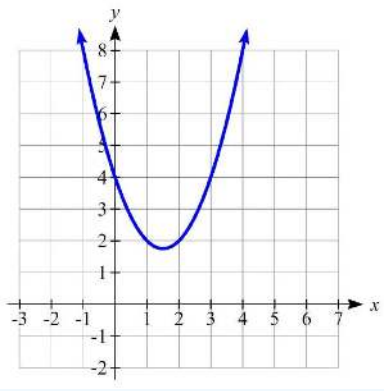
Classifying functions based on their graphs

Classify the functions represented by the graphs below by family:

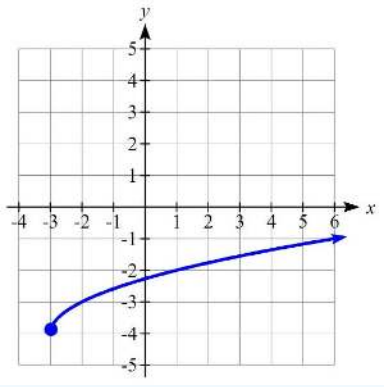
36. 

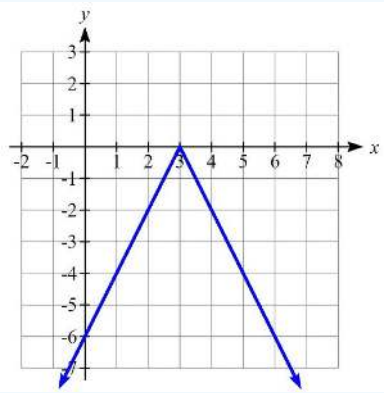
37. 

38. 

39. 

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40. 

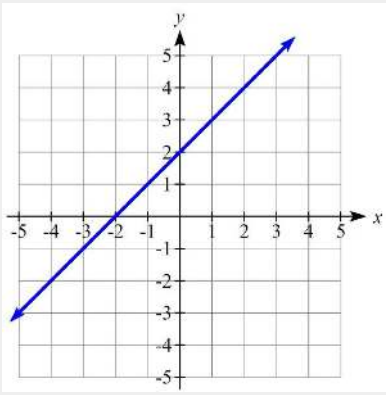
41. 

Answer

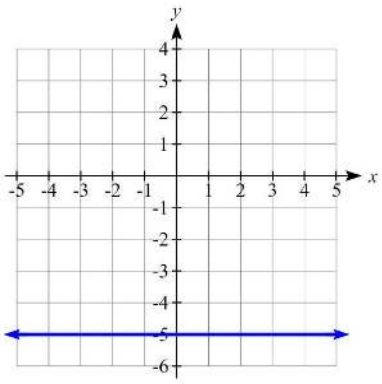
36. Linear
37. Quadratic
38. Quotient
39. Quadratic
40. Square root
41. Absolute value

? Identify graphs of functions based on "basic shape"

Match the graphs below to the function it might represent based on its family.

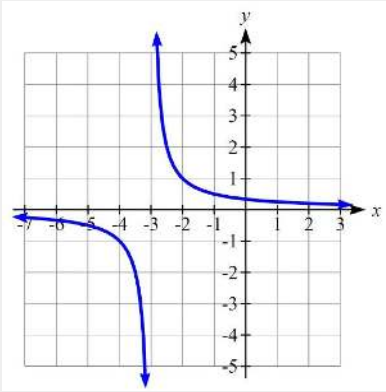
Graph	Function
42. 	a. $f(x) = x^2 - 3x + 4$

43.



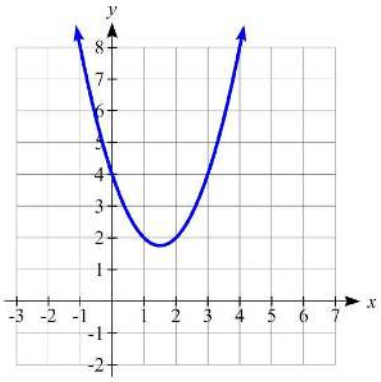
b. $f(x) = x + 2$

44.



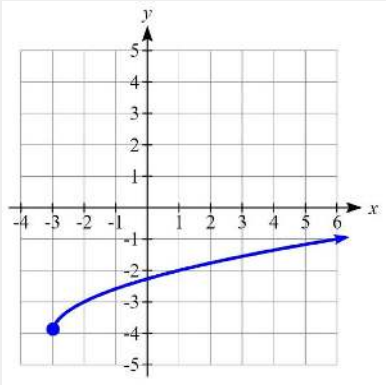
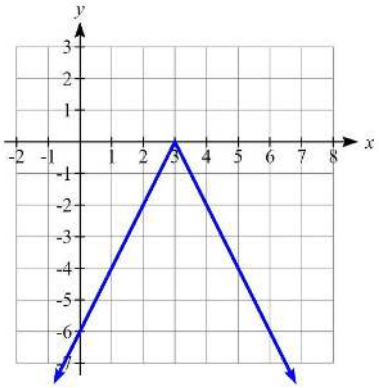
c. $f(x) = -5$

45.



d. $f(x) = \frac{1}{3+x}$

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46. 	e. $f(x) = -2\|x-3\|$
47. 	f. $f(x) = \sqrt{x+3} - 4$

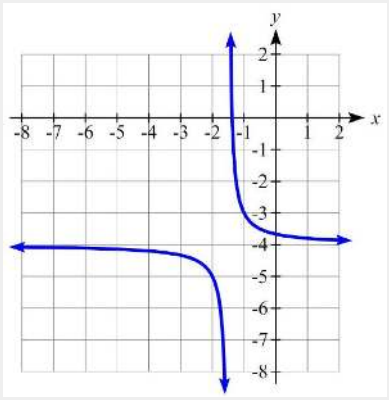
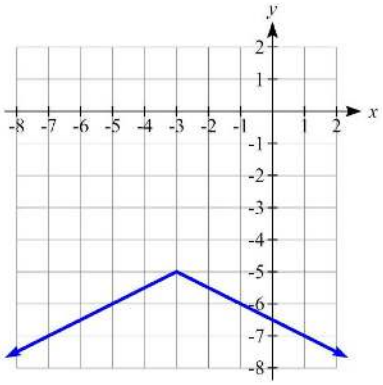
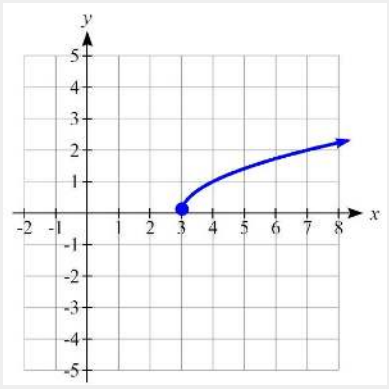
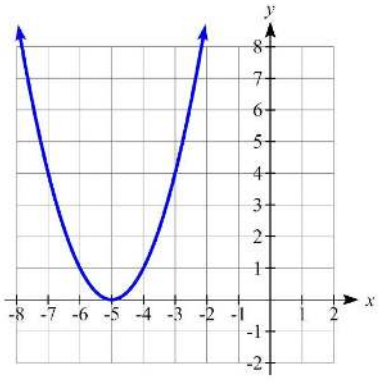
Answer

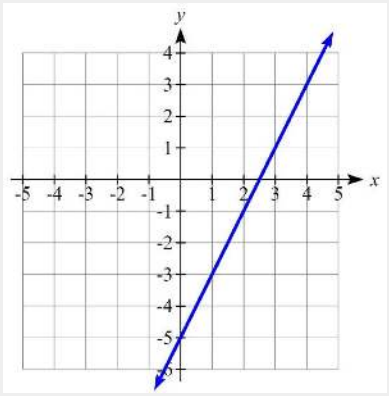
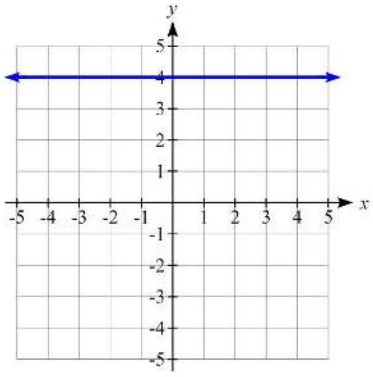
42. b
43. c
44. d
45. a
46. f
47. e

Identify graphs of functions based on their "basic shape"

Match each function below to its possible graph based on its family.

Function	Graph

48. $f(x) = \sqrt{x-3}$	a. 
49. $f(x) = 4$	b. 
50. $f(x) = (x+5)^2$	c. 
51. $f(x) = \frac{1}{2x+3} - 4$	d. 

52. $f(x) = \frac{1}{2}\|x+3\| - 5$	e. 
53. $f(x) = 2x - 5$	f. 

Answer

48. c
49. f
50. d
51. a
52. b
53. e

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CHAPTER OVERVIEW

3: Linear Equations and Graphs

In this chapter, you will explore linear equations, develop a strategy for solving them, and relate them to real-world situations.

3.1: Use a General Strategy to Solve Linear Equations

3.1E: Exercises

3.2: Graphing Linear Equations

3.2E: Exercises

3.3: Slope Intercept Form

3.3E: Exercises

3.4: Finding the Equation of a Line

3.4E: Exercises

3.5: Parallel and Perpendicular Lines

3.5E: Exercises

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3.1: Use a General Strategy to Solve Linear Equations

Learning Objectives

- Solve linear equations with integer coefficients
- Solve linear equations with fraction and decimal coefficients

Warm Up 3.1.1

a. Determine whether the values are solutions to the equation: $6x-2=4x+8$.

1. $x=5$
2. $x=2$

b. Solve for x: $3x-4+6x=22$

c. Find the LCD of $\frac{5}{6}$ and $\frac{1}{4}$.

Answer

a. For the equation $6x-2=4x+8$

1. $x=5$ is a solution
2. $x=2$ is not a solution

b. $x=2$

c. 12

Solve Linear Equations Using a General Strategy

In the first chapter, we did a quick introduction to the basics of solving equations. Before diving into the world of linear functions (as well as other kinds of functions) in depth, it is worth revisiting these skills and discussing a general strategy we can use to solve equations. This section will be a little shorter than usual because we've already covered (and used!) some of the necessary skills already; it's just time to bring them all together so we can solve more complicated problems!

There are many types of equations that we will learn to solve. In this section we will focus on a **linear equation.** You've already worked with linear equations, so we are just formalizing some information we already have.

Definition: LINEAR EQUATION

A **linear equation** is an equation in one variable that can be written, where a and b are real numbers and $a\neq 0$, as:

$$ax+b=0.$$

An easy way to think about linear equations is that these are the equations where the variable appears by itself, and has been multiplied by a constant or has a constant added to it, but there are no exponents attached to it, no absolute values, no square roots, etc. We will see later that these equations are closely related to the linear functions that we looked at in the last chapter.

To solve a linear equation it is a good idea to have an overall strategy that can be used to solve any linear equation. In the next example, we will give the steps of a general strategy for solving any linear equation. Simplifying each side of the equation as much as possible first makes the rest of the steps easier.

Example 3.1.1

Solve: $7(n-3)-8=-15$

Answer

Step 1. Simplify each side of the equation as much as possible.	Use the Distributive Property. Notice that each side of the equation is now simplified as much as possible.	$7(n-3)-8=-15$ $7n-21-8=-15$ $7n-29=-15$
Step 2. Collect all variable terms on one side of the equation.	Nothing to do—all n's are on the left side.	
Step 3. Collect constant terms on the other side of the equation.	To get constants only on the right, add 29 to each side. Simplify.	$7n-29+29=-15+29$ $7n=14$
Step 4. Make the coefficient of the variable term equal to 1.	Divide each side by 7. Simplify.	$\frac{7n}{7}=\frac{14}{7}$ $n=2$
Step 5. Check the solution.	Let $n=2$ Subtract.	**Check:** $7(n-3)-8=-15$ $7(2-3)-8\stackrel{?}{=}-15$ $7(-1)-8\stackrel{?}{=}-15$ $-7-8\stackrel{?}{=}-15$ $-15=-15$ ✓

Exercise 3.1.2

Solve: $2(m-4)+3=-1$.

Answer

$m=2$

Exercise 3.1.3

Solve: $5(a-3)+5=-10$.

Answer

$a=0$

These steps are summarized in the **General Strategy for Solving Linear Equations** below.

Strategy to Solve Linear Equations

1. Simplify each side of the equation as much as possible.
   Use the Distributive Property to remove any parentheses.
   Combine like terms.
2. Collect all the variable terms on one side of the equation.
   Use the Addition or Subtraction Property of Equality.
3. Collect all the constant terms on the other side of the equation.
   Use the Addition or Subtraction Property of Equality.
4. Make the coefficient of the variable term equal to 1.
   Use the Multiplication or Division Property of Equality.
   State the solution to the equation.
5. Check the solution.

Substitute the solution into the original equation to make sure the result is a true statement.

Exercise 3.1.4

Solve: $\frac{2}{3}(3m-6)=5-m$.

Answer

	$\frac{2}{3}(3m-6)=5-m$
Distribute.	$2m-4=5-m$
Add m to both sides to get the variables only on the left.	$2m+m-4=5-m+m$
Simplify.	$3m-4=5$
Add 4 to both sides to get constants only on the right.	$3m-4+4=5+4$
Simplify.	$3m=9$
Divide both sides by three.	$\frac{3m}{3}=\frac{9}{3}$
Simplify.	$m=3$

Check:	$\frac{2}{3}(3m-6)=5-m$
Let $m=3$.	$\frac{2}{3}(3\cdot 3-6)\stackrel{?}{=}5-3$
	$\frac{2}{3}(9-6)\stackrel{?}{=}2$
	$\frac{2}{3}(3)\stackrel{?}{=}2$
	$2=2\checkmark$

Exercise 3.1.5

Solve: $\frac{1}{3}(6u+3)=7-u$.

Answer

$u=2$

Exercise 3.1.6

Solve: $\frac{2}{3}(9x-12)=8+2x$.

Answer

$x=4$

We can solve equations by getting all the **variable** terms to either side of the **equal sign**. By collecting the variable terms on the side where the **coefficient** of the variable is larger, we avoid working with some negatives. This will be a good strategy when we solve inequalities later in this chapter. It also helps us prevent errors with negatives.

Exercise 3.1.7

Solve: $4(x-1)-2=5(2x+3)+6$.

Answer

	$4(x-1)-2=5(2x+3)+6$
Distribute.	$4x-4-2=10x+15+6$
Combine like terms.	$4x-6=10x+21$
Subtract $4x$ from each side to get the variables only on the right since $10>4$.	$4x-4x-6=10x-4x+21$
Simplify.	$-6=6x+21$
Subtract 21 from each side to get the constants on left.	$-6-21=6x+21-21$
Simplify.	$-27=6x$
Divide both sides by 6.	$\frac{-27}{6}=\frac{6x}{6}$
Simplify.	$-\frac{9}{2}=x$
Check:	$4(x-1)-2=5(2x+3)+6$ \)
Let $x=-92$.	$4\left(-\frac{9}{2}-1\right)-2\stackrel{?}{=}5\left(2\left(-\frac{9}{2}\right)+3\right)+6$
	$4\left(-\frac{11}{2}\right)-2\stackrel{?}{=}5(-9+3)+6$
	$-22-2\stackrel{?}{=}5(-6)+6$
	$-24\stackrel{?}{=}-30+6$
	$-24=-24$ ✓

Exercise 3.1.8

Solve: $6(p-3)-7=5(4p+3)-12$.

Answer

$p=-2$

Exercise 3.1.9

Solve: $8(q+1)-5=3(2q-4)-1$.

Answer

$q=-8$

Exercise 3.1.10

Solve: $10[3-8(2s-5)]=15(40-5s)$.

Answer

	$10[3-8(2s-5)]=15(40-5s)$
Simplify from the innermost parentheses first.	$10[3-16s+40]=15(40-5s)$

Combine like terms in the brackets.		$10[43-16s]=15(40-5s)$
Distribute.		$430-160s=600-75s$
Add $160s$ to both sides to get the $160s$ to both sides to get the variables to the right.		$430-160s+160s=600-75s+160s$
Simplify.		$430=600+85s$
Subtract 600 from both sides to get the constants to the left.		$430-600=600+85s-600$
Simplify.		$-170=85s$
Divide both sides by 85.		$\frac{-170}{85}=\frac{85s}{85}$
Simplify.		$-2=s$, so $s=-2$
Check:	$10[3-8(2s-5)]=15(40-5s)$	
Let $s=-2$.	$10[3-8(2(-2)-5)] \stackrel{?}{=} 15(40-5(-2))$	
	$10[3-8(-4-5)] \stackrel{?}{=} 15(40+10)$	
	$10[3-8(-9)] \stackrel{?}{=} 15(50)$	
	$10(3+72) \stackrel{?}{=} 750$	
	$10(75) \stackrel{?}{=} 750$	
	$750=750$ ✓	

Exercise 3.1.11

Solve: $6[4-2(7y-1)]=8(13-8y)$.

Answer

$y=-\frac{17}{5}$

Exercise 3.1.12

Solve: $12[1-5(4z-1)]=3(24+11z)$.

Answer

$z=0$

Solve Equations with Fraction or Decimal Coefficients

We could use the General Strategy to solve the next example. This method would work fine, but working with the fractions can get a bit tedious, and also increases the number of opportunities for making a mistake. So, we are going to show an alternate method to solve equations with fractions. This alternate method eliminates the fractions.

We will apply the Multiplication Property of Equality and multiply both sides of an equation by the **least common denominator** (LCD) of all the fractions in the equation. The result of this operation will be a new equation, equivalent to the first, but without fractions. This process is called *clearing* the equation of fractions.

To clear an equation of decimals, we think of all the decimals in their fraction form and then find the LCD of those denominators.

Exercise 3.1.13

Solve: $\frac{1}{12}x+\frac{5}{6}=\frac{3}{4}$.

Answer

Step 1. Find the least common denominator of *all* the fractions and decimals in the equation.	What is the LCD of $\frac{1}{12}$, $\frac{5}{6}$, and $\frac{3}{4}$?	$\frac{1}{12}x+\frac{5}{6}=\frac{3}{4}$ LCD = 12
Step 2. Multiply both sides of the equation by that LCD. This clears the fractions and decimals.	Multiply both sides of the equation by the LCD, 12. Use the Distributive Property. Simplify—and notice, no more fractions!	$12\left(\frac{1}{12}x+\frac{5}{6}\right)=12\left(\frac{3}{4}\right)$ $12\cdot\frac{1}{12}x+12\cdot\frac{5}{6}=12\cdot\frac{3}{4}$ $x+10=9$
Step 3. Solve using the General Strategy for Solving Linear Equations.	To isolate the variable term, subtract 10. Simplify.	$x+10-10=9-10$ $x=-1$ **Check:** $\frac{1}{12}x+\frac{5}{6}=\frac{3}{4}$ $\frac{1}{12}(-1)+\frac{5}{6}\stackrel{?}{=}\frac{3}{4}$ $-\frac{1}{12}+\frac{5}{6}\stackrel{?}{=}\frac{3}{4}$ $-\frac{1}{12}+\frac{10}{12}\stackrel{?}{=}\frac{9}{12}$ $\frac{9}{12}=\frac{9}{12}$ ✓

Exercise 3.1.14

Solve: $\frac{1}{4}x+\frac{1}{2}=\frac{5}{8}$.

Answer

$x=\frac{1}{2}$

Exercise 3.1.15

Solve: $\frac{1}{8}x+\frac{1}{2}=\frac{1}{4}$.

Answer

$x=-2$

Notice in the previous example, once we cleared the equation of fractions, the equation was like those we solved earlier in this chapter. We changed the problem to one we already knew how to solve. We then used the **General Strategy for Solving Linear Equations**.

Strategy to Solve Equations with Fractions or Decimals

1. Find the least common denominator (LCD) of *all* the fractions and decimals (in fraction form) in the equation.
2. Multiply both sides of the equation by that LCD. This clears the fractions and decimals.
3. Solve using the General Strategy for Solving Linear Equations.

Exercise 3.1.16

Solve: $5 = \frac{1}{2}y + \frac{2}{3}y - \frac{3}{4}y$.

Answer

We want to clear the fractions by multiplying both sides of the equation by the LCD of all the fractions in the equation.

Find the LCD of all fractions in the equation.	$5 = \frac{1}{2}y + \frac{2}{3}y - \frac{3}{4}y$
The LCD is 12.	
Multiply both sides of the equation by 12.	$12(5) = 12 \cdot \left(\frac{1}{2}y + \frac{2}{3}y - \frac{3}{4}y\right)$
Distribute.	$12(5) = 12 \cdot \frac{1}{2}y + 12 \cdot \frac{2}{3}y - 12 \cdot \frac{3}{4}y$
Simplify—notice, no more fractions.	$60 = 6y + 8y - 9y$
Combine like terms.	$60 = 5y$
Divide by five.	$\frac{60}{5} = \frac{5y}{5}$
Simplify.	$12 = y$
Check:	$5 = \frac{1}{2}y + \frac{2}{3}y - \frac{3}{4}y$
Let $y = 12$.	$5 \stackrel{?}{=} \frac{1}{2}(12) + \frac{2}{3}(12) - \frac{3}{4}(12)$
	$5 \stackrel{?}{=} 6 + 8 - 9$
	$5 = 5$ ✓

Exercise 3.1.17

Solve: $7 = \frac{1}{2}x + \frac{3}{4}x - \frac{2}{3}x$.

Answer

$x = 12$

Exercise 3.1.18

Solve: $-1 = \frac{1}{2}u + \frac{1}{4}u - \frac{2}{3}u$.

Answer

$u = -12$

In the next example, we'll distribute before we clear the fractions.

Exercise 3.1.19

Solve: $\frac{1}{2}(y-5) = \frac{1}{4}(y-1)$.

Answer

	$\frac{1}{2}(y-5) = \frac{1}{4}(y-1)$
Distribute.	$\frac{1}{2} \cdot y - \frac{1}{2} \cdot 5 = \frac{1}{4} \cdot y - \frac{1}{4} \cdot 1$

Simplify.	$\frac{1}{2}y-\frac{5}{2}=\frac{1}{4}y-\frac{1}{4}$
Multiply by the LCD, four.	$4\left(\frac{1}{2}y-\frac{5}{2}\right)=4\left(\frac{1}{4}y-\frac{1}{4}\right)$
Distribute.	$4\cdot\frac{1}{2}y-4\cdot\frac{5}{2}=4\cdot\frac{1}{4}y-4\cdot\frac{1}{4}$
Simplify.	$2y-10=y-1$
Collect the variables to the left.	$2y-y-10=y-y-1$
Simplify.	$y-10=-1$
Collect the constants to the right.	$y-10+10=-1+10$
Simplify.	$y=9$
An alternate way to solve this equation is to clear the fractions without distributing first. If you multiply the factors correctly, this method will be easier.	
	$\frac{1}{2}(y-5)=\frac{1}{4}(y-1)$
Multiply by the LCD, 4.	$4\cdot\frac{1}{2}(y-5)=4\cdot\frac{1}{4}(y-1)$
Multiply four times the fractions.	$2(y-5)=1(y-1)$
Distribute.	$2y-10=y-1$
Collect the variables to the left.	$2y-y-10=y-y-1$
Simplify.	$y-10=-1$
Collect the constants to the right.	$y-10+10=-1+10$
Simplify.	$y=9$
Check:	$\frac{1}{2}(y-5)=\frac{1}{4}(y-1)$
Let $y=9$.	$\frac{1}{2}(9-5)\stackrel{?}{=}\frac{1}{4}(9-1)$
Finish the check on your own.	

Exercise 3.1.20

Solve: $\frac{1}{5}(n+3)=\frac{1}{4}(n+2)$.

Answer

$n=2$

Exercise 3.1.21

Solve: $\frac{1}{2}(m-3)=\frac{1}{4}(m-7)$.

Answer

$m=-1$

When you multiply both sides of an equation by the LCD of the fractions, make sure you multiply each term by the LCD—even if it does not contain a fraction.

Exercise 3.1.22

Solve: $\frac{4q+3}{2}+6=\frac{3q+5}{4}$

Answer

	$\frac{4q+3}{2}+6=\frac{3q+5}{4}$
Multiply both sides by the LCD, 4.	$4\left(\frac{4q+3}{2}+6\right)=4\left(\frac{3q+5}{4}\right)$
Distribute.	$4\left(\cdot\frac{4q+3}{2}\right)+4\cdot 6=4\cdot\left(\frac{3q+5}{4}\right)$
Simplify.	$2(4q+3)+24=3q+5$
	$8q+6+24=3q+5$
	$8q+30=3q+5$
Collect the variables to the left.	$8q-3q+30=3q-3q+5$
Simplify.	$5q+30=5$
Collect the constants to the right.	$5q+30-30=5-30$
Simplify.	$5q=-25$
Divide both sides by five.	$\frac{5q}{5}=\frac{-25}{5}$
Simplify.	$q=-5$
Check:	$\frac{4q+3}{2}+6=\frac{3q+5}{4}$
Let $q=-5$.	$\frac{4(-5)+3}{2}+6\stackrel{?}{=}\frac{3(-5)+5}{4}$
Finish the check on your own.	

Exercise 3.1.23

Solve: $\frac{2s+3}{2}+1=\frac{3s+2}{4}$.

Answer

$s=-8$

Exercise 3.1.24

Solve: $\frac{3r+5}{6}+1=\frac{4r+3}{3}$.

Answer

$r=3$

Some equations have decimals in them. This kind of equation may occur when we solve problems dealing with money or percentages. But decimals can also be expressed as fractions. For example, $0.7=\frac{7}{10}$ and $0.29=\frac{29}{100}$. So, with an equation with decimals, we can use the same method we used to clear fractions—multiply both sides of the equation by the **least common denominator**.

The next example uses an equation that is typical of the ones we will see in the money applications in a later section. Notice that we will clear all decimals by multiplying by the LCD of their fraction form.

Example 3.1.24

Solve: $0.25x+0.05(x+3)=2.85$.

Answer

Look at the decimals and think of the equivalent fractions:

$$0.25=\frac{25}{100},\qquad 0.05=\frac{5}{100},\qquad 2.85=2\frac{85}{100}. \tag{3.1.1}$$

Notice, the LCD is 100. By multiplying by the LCD we will clear the decimals from the equation.

	$0.25x+0.05(x+3)=2.85$
Distribute first.	$0.25x+0.05x+0.15=2.85$
Combine like terms.	$0.30x+0.15=2.85$
To clear decimals, multiply by 100.	$100(0.30x+0.15)=100(2.85)$
Distribute.	$30x+15=285$
Subtract 15 from both sides.	$30x+15-15=285-15$
Simplify.	$30x=270$
Divide by 30.	$\frac{30x}{30}=\frac{270}{30}$
Simplify.	$x=9$
Check it yourself by substituting $x=9$ into the original equation.	

Exercise 3.1.25

Solve: $0.25n+0.05(n+5)=2.95$.

Answer

$n=9$

Exercise 3.1.26

Solve: $0.10d+0.05(d-5)=2.15$.

Answer

$d=16$

Key Concepts

- **How to Solve Linear Equations Using a General Strategy**
  1. Simplify each side of the equation as much as possible.

     Use the Distributive Property to remove any parentheses.

     Combine like terms.
  2. Collect all the variable terms on one side of the equation.

     Use the Addition or Subtraction Property of Equality.
  3. Collect all the constant terms on the other side of the equation.

Use the Addition or Subtraction Property of Equality.

4. Make the coefficient of the variable term equal to 1.

   Use the Multiplication or Division Property of Equality.

   State the solution to the equation.

5. Check the solution.

   Substitute the solution into the original equation to make sure the result is a true statement.

- **How to Solve Equations with Fraction or Decimal Coefficients**
  1. Find the least common denominator (LCD) of *all* the fractions and decimals (in fraction form) in the equation.
  2. Multiply both sides of the equation by that LCD. This clears the fractions and decimals.
  3. Solve using the General Strategy for Solving Linear Equations.

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3.1E: Exercises

Practice Makes Perfect

Solve Equations Using the General Strategy

In the following exercises, solve each linear equation.

1. $15(y-9)=-60$
2. $-(w-12)=30$
3. $51+5(4-q)=56$
4. $3(10-2x)+54=0$
5. $\frac{2}{3}(9c-3)=22$
6. $\frac{1}{5}(15c+10)=c+7$
7. $3(4n-1)-2=8n+3$
8. $12+2(5-3y)=-9(y-1)-2$
9. $5+6(3s-5)=-3+2(8s-1)$
10. $4(p-4)-(p+7)=5(p-3)$

Answer

1. $y=5$
2. $w=-18$
3. $q=3$
4. $x=14$
5. $c=4$
6. $c=\frac{5}{2}$
7. $n=2$
8. $y=-5$
9. $s=10$
10. $p=-4$

Solve equations with Fraction or Decimal Coefficients

In the following exercises, solve each equation with fraction coefficients.

11. $\frac{1}{4}x-\frac{1}{2}=-\frac{3}{4}$
12. $\frac{5}{6}y-\frac{2}{3}=-\frac{3}{2}$
13. $\frac{1}{2}a+\frac{3}{8}=\frac{3}{4}$
14. $2=\frac{1}{3}x-\frac{1}{2}x+\frac{2}{3}x$
15. $\frac{1}{3}w+\frac{5}{4}=w-\frac{1}{4}$
16. $\frac{1}{3}b+\frac{1}{5}=\frac{2}{5}b-\frac{3}{5}$
17. $\frac{1}{4}(p-7)=\frac{1}{3}(p+5)$
18. $\frac{1}{2}(x+4)=\frac{3}{4}$
19. $\frac{4n+8}{4}=\frac{n}{3}$
20. $\frac{3x+4}{2}+1=\frac{5x+10}{8}$

Answer

11. $x=-1$
12. $y=-1$
13. $a=\frac{3}{4}$
14. $x=4$

15. $w = \frac{9}{4}$
16. $b = 12$
17. $p = -41$
18. $x = -\frac{5}{2}$
19. $n = -3$
20. $x = -2$

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3.2: Graphing Linear Equations

Learning Objectives

By the end of this section, you will be able to:

- Graph a linear equation by plotting points
- Graph vertical and horizontal lines
- Find the x- and y-intercepts
- Graph a line using the intercepts

Warm Up 3.2.1

a. Evaluate $5x-4$ when $x=-1$.
b. Evaluate $3x-2y$ when $x=4, y=-3$.
c. Solve for y : $8-3y=20$.

Answer

a. -9
b. 18
c. $y=-4$

Linear Equations with Two Variables

Up to now, all the equations you have solved were equations with just one variable. In almost every case, when you solved the equation you got exactly one solution. But equations can have more than one variable. Equations with two variables may be of the form $Ax+By=C$. An equation of this form is called a **linear equation** in two variables.

Definition: LINEAR EQUATION

An equation of the form $Ax+By=C$, where A and B are not both zero, is called a **linear equation** in two variables.

Here is an example of a linear equation in two variables, x and y.

$$\begin{aligned} Ax+By &= C \\ x+4y &= 8 \end{aligned}$$

$A=1, B=4, C=8$

The equation $y=-3x+5$ is also a linear equation. But it does not appear to be in the form $Ax+By=C$. We can use the Addition Property of Equality and rewrite it in $Ax+By=C$ form.

$$\begin{array}{lrcl} & y & = & -3x+5 \\ \text{Add to both sides.} & y+3x & = & 3x+5+3x \\ \text{Simplify.} & y+3x & = & 5 \\ \text{Use the Commutative Property to put it in} & & & \\ Ax+By=C \text{ form.} & 3x+y & = & 5 \end{array}$$

By rewriting $y=-3x+5$ as $3x+y=5$, we can easily see that it is a linear equation in two variables because it is of the form $Ax+By=C$. When an equation is in the form $Ax+By=C$, we say it is in **standard form of a linear equation**.

Definition: STANDARD FORM OF A LINEAR EQUATION

A linear equation is in **standard form** when it is written $Ax+By=C$.

Most people prefer to have A, B, and C be integers and $A \geq 0$ when writing a linear equation in standard form, although it is not strictly necessary.

Linear equations have infinitely many solutions. For every number that is substituted for x there is a corresponding y-value. This pair of values is a **solution** to the linear equation and is represented by the ordered pair (x, y). When we substitute these values of x and y into the equation, the result is a true statement, because the value on the left side is equal to the value on the right side.

Definition: Solution OF A LINEAR EQUATION IN TWO VARIABLES

An ordered pair (x, y) is a **solution** of the linear equation $Ax + By = C$, if the equation is a true statement when the x- and y-values of the ordered pair are substituted into the equation.

Graphs of Solutions to Linear Equations

Now, in our last chapter, we learned how to graph ordered pairs. We can do the same thing here with the solutions to linear equations. Linear equations have infinitely many solutions. We can plot these solutions in the rectangular coordinate system. The points will line up perfectly in a straight line. We connect the points with a straight line to get the graph of the equation. We put arrows on the ends of each side of the line to indicate that the line continues in both directions.

A graph is a visual representation of all the solutions of the equation. It is an example of the saying, "A picture is worth a thousand words." The line shows you *all* the solutions to that equation. Every point on the line is a solution of the equation. And, every solution of this equation is on this line. This line is called the graph of the equation. Points *not* on the line are *not* solutions!

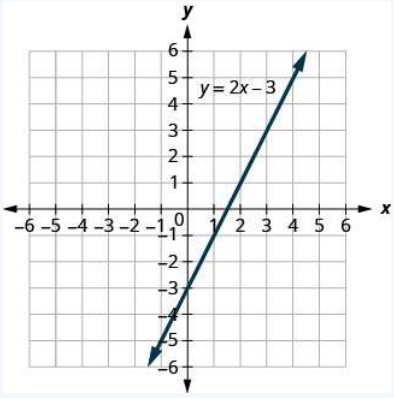
GRAPH OF A LINEAR EQUATION

The graph of a linear equation $Ax + By = C$ is a straight line.

- Every point on the line is a solution of the equation.
- Every solution of this equation is a point on this line.

Example 3.2.1

The graph of $y = 2x - 3$ is shown.



For each ordered pair, decide:

a. Is the ordered pair a solution to the equation?
b. Is the point on the line?

A: $(0, -3)$ B: $(3, 3)$ C: $(2, -3)$ D: $(-1, -5)$

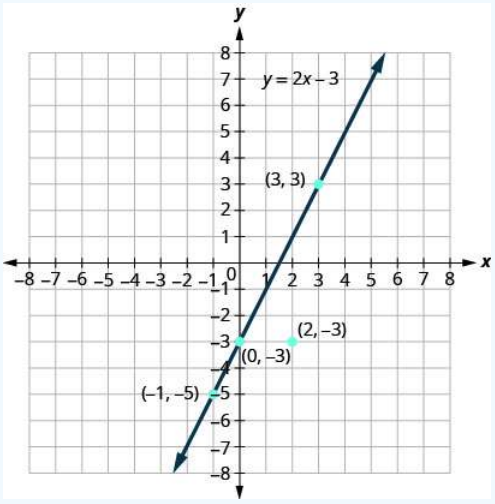
Solution

Substitute the x- and y-values into the equation to check if the ordered pair is a solution to the equation.

a.

A: $(0, -3)$	B: $(3, 3)$	C: $(2, -3)$	D: $(-1, -5)$
$y = 2x - 3$	$y = 2x - 3$	$y = 2x - 3$	$y = 2x - 3$
$-3 \stackrel{?}{=} 2(0) - 3$	$3 \stackrel{?}{=} 2(3) - 3$	$-3 \stackrel{?}{=} 2(2) - 3$	$-5 \stackrel{?}{=} 2(-1) - 3$
$-3 = -3$ ✓	$3 = 3$ ✓	$-3 \neq 1$	$-5 = -5$ ✓
$(0, -3)$ is a solution.	$(3, 3)$ is a solution.	$(2, -3)$ is not a solution.	$(-1, -5)$ is a solution.

b. Plot the points $(0, -3)$, $(3, 3)$, $(2, -3)$, and $(-1, -5)$.



The points $(0, 3)$, $(3, -3)$, and $(-1, -5)$ are on the line $y = 2x - 3$, and the point $(2, -3)$ is not on the line.

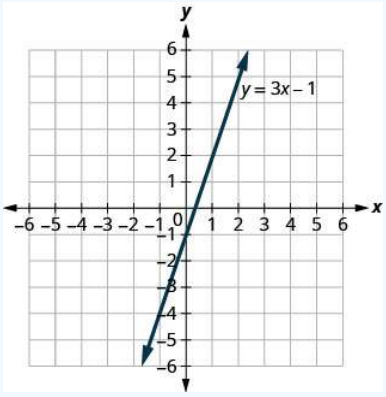
The points that are solutions to $y = 2x - 3$ are on the line, but the point that is not a solution is not on the line.

? Exercise 3.2.2

Use graph of $y = 3x - 1$. For each ordered pair, decide:

a. Is the ordered pair a solution to the equation?
b. Is the point on the line?

A $(0, -1)$ B $(2, 5)$



Answer

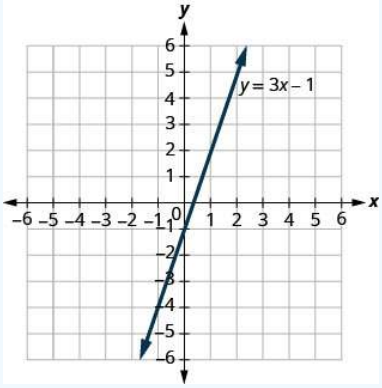
a. yes b. yes

Exercise 3.2.3

Use graph of $y = 3x - 1$. For each ordered pair, decide:

a. Is the ordered pair a solution to the equation?
b. Is the point on the line?

$A(3, -1)\ B(-1, -4)$



Answer

a. no b. yes

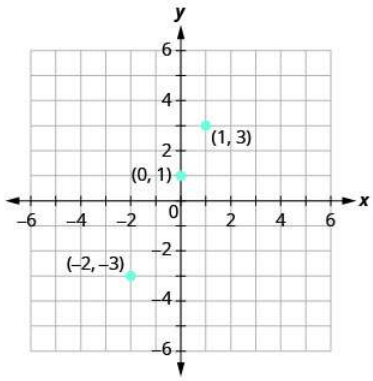
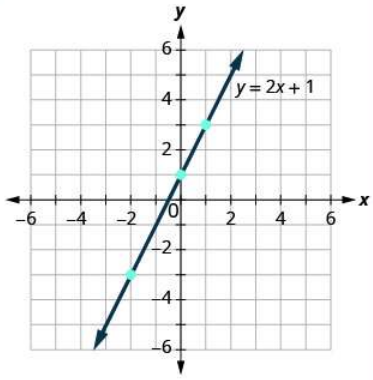
Graph a Linear Equation by Plotting Points

There are several methods that can be used to graph a linear equation. The first method we will use is called plotting points, or the Point-Plotting Method, which is the same tool we used in the last chapter. We find some points whose coordinates are solutions to the equation and then plot them in a rectangular coordinate system. By connecting these points in a line, we have the graph of the linear equation.

Example 3.2.4

Graph the equation $y = 2x + 1$ by plotting points.

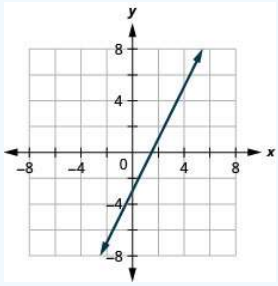
Solution

Step 1. Find three points whose coordinates are solutions to the equation.	You can choose any values for x or y. In this case, since y is isolated on the left side of the equation, it is easier to choose values for x.	$y = 2x + 1$ $x = 0$ $y = 2x + 1$ $y = 2 \cdot 0 + 1$ $y = 0 + 1$ $y = 1$ $x = 1$ $y = 2x + 1$ $y = 2 \cdot 1 + 1$ $y = 2 + 1$ $y = 3$ $x = -2$ $y = 2x + 1$ $y = 2(-2) + 1$ $y = -4 + 1$ $y = -3$
Organize the solutions in a table.	Put the three solutions in a table.	$y = 2x + 1$ x | y | (x, y) 0 | 1 | (0, 1) 1 | 3 | (1, 3) –2 | –3 | (–2, –3)
Step 2. Plot the points in a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work!	Plot: (0, 1), (1, 3), (–2, –3). Do the points line up? Yes, the points line up.	
Step 3. Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.	This line is the graph of $y = 2x + 1$.	

Exercise 3.2.5

Graph the equation by plotting points: $y = 2x - 3$.

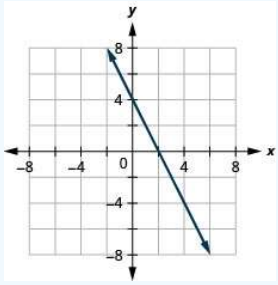
Answer



Exercise 3.2.6

Graph the equation by plotting points: $y = -2x + 4$.

Answer



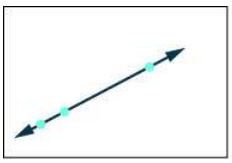
The steps to take when graphing a linear equation by plotting points are summarized here.

GRAPH A LINEAR EQUATION BY PLOTTING POINTS

1. Find three points whose coordinates are solutions to the equation. Organize them in a table.
2. Plot the points in a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work.
3. Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.

It is true that it only takes two points to determine a line, but it is a good habit to use three points. If you only plot two points and one of them is incorrect, you can still draw a line but it will not represent the solutions to the equation. It will be the wrong line.

If you use three points, and one is incorrect, the points will not line up. This tells you something is wrong and you need to check your work. Look at the difference between these illustrations.



(a)



(b)

When an equation includes a fraction as the coefficient of x, we can still substitute any numbers for x. But the arithmetic is easier if we make "good" choices for the values of x. This way we will avoid fractional answers, which are hard to graph precisely.

Example 3.2.7

Graph the equation: $y = \frac{1}{2}x + 3$.

Solution

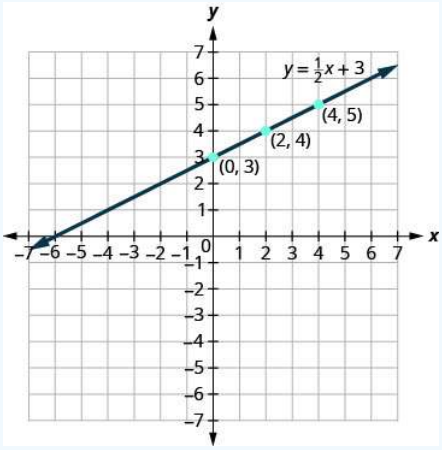
Find three points that are solutions to the equation. Since this equation has the fraction $\frac{1}{2}$ as a coefficient of x, we will choose values of x carefully. We will use zero as one choice and multiples of 2 for the other choices. Why are multiples of two a good choice for values of x? By choosing multiples of 2 the multiplication by $\frac{1}{2}$ simplifies to a whole number

$x = 0$	$x = 2$	$x = 4$
$y = \frac{1}{2}x + 3$	$y = \frac{1}{2}x + 3$	$y = \frac{1}{2}x + 3$
$y = \frac{1}{2}(0) + 3$	$y = \frac{1}{2}(2) + 3$	$y = \frac{1}{2}(4) + 3$
$y = 0 + 3$	$y = 1 + 3$	$y = 2 + 3$
$y = 3$	$y = 4$	$y = 5$

The points are shown in Table.

$y = \frac{1}{2}x + 3$		
x	y	(x, y)
0	3	$(0, 3)$
2	4	$(2, 4)$
4	5	$(4, 5)$

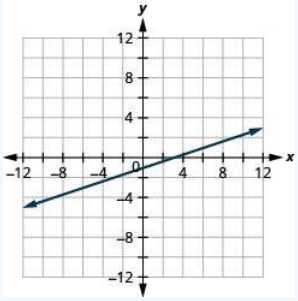
Plot the points, check that they line up, and draw the line.



Exercise 3.2.8

Graph the equation: $y = \frac{1}{3}x - 1$.

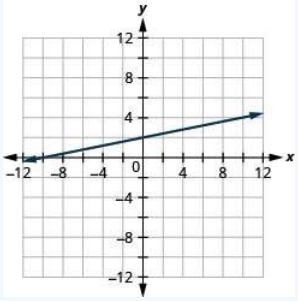
Answer



? Exercise 3.2.9

Graph the equation: $y = \frac{1}{4}x + 2$.

Answer



Graph Vertical and Horizontal Lines

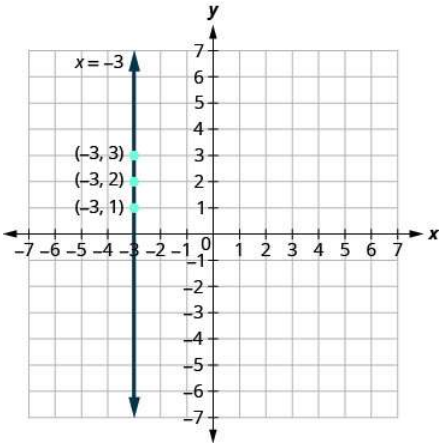
Some linear equations have only one variable. They may have just x and no y, or just y without an x. This changes how we make a table of values to get the points to plot.

Let's consider the equation $x = -3$. This equation has only one variable, x. The equation says that $x$is *always* equal to -3, so its value does not depend on y. No matter what is the value of y, the value of x is always -3.

So to make a table of values, write -3 in for all the x-values. Then choose any values for y. Since x does not depend on y, you can choose any numbers you like. But to fit the points on our coordinate graph, we'll use 1, 2, and 3 for the y-coordinates. See Table.

$x = -3$		
x	y	(x, y)
-3	1	$(-3, 1)$
-3	2	$(-3, 2)$
$(-3,)$	3	$(-3, 3)$

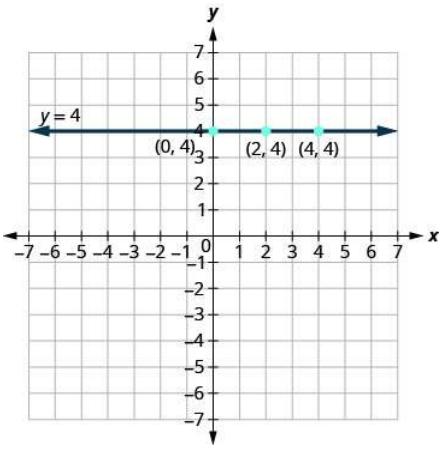
Plot the points from the table and connect them with a straight line. Notice that we have graphed a **vertical line**.



What if the equation has y but no x? Let's graph the equation $y = 4$. This time the y-value is a constant, so in this equation, y does not depend on x. Fill in 4 for all the y's in Table and then choose any values for x. We'll use 0, 2, and 4 for the x-coordinates.

$y = 4$		
x	y	(x, y)
0	4	$(0, 4)$
2	4	$(2, 4)$
4	4	$(4, 4)$

In this figure, we have graphed a **horizontal line** passing through the y-axis at 4.



VERTICAL AND HORIZONTAL LINES

A **vertical line** is the graph of an equation of the form $x = a$.

The line passes through the x-axis at $(a, 0)$.

A **horizontal line** is the graph of an equation of the form $y = b$.

The line passes through the y-axis at $(0, b)$.

Example 3.2.10

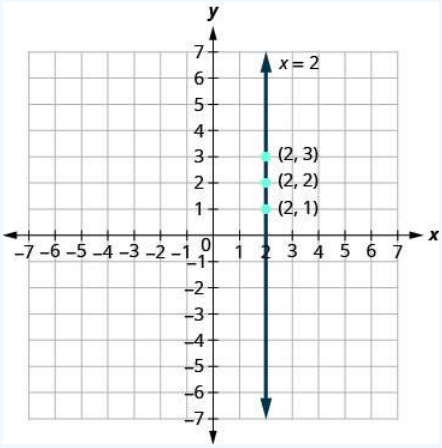
Graph: a. $x = 2$ b. $y = -1$.

Solution

a. The equation has only one variable, x, and x is always equal to 2. We create a table where x is always 2 and then put in any values for y. The graph is a vertical line passing through the x-axis at 2.

$$x = 2$$

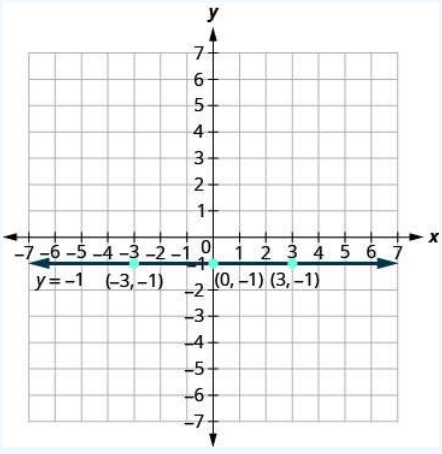
x	y	(x, y)
2	1	$(2, 1)$
2	2	$(2, 2)$
2	3	$(2, 3)$



b. Similarly, the equation $y = -1$ has only one variable, y. The value of y is constant. All the ordered pairs in the next table have the same y-coordinate. The graph is a horizontal line passing through the y-axis at -1.

$$y = -1$$

x	y	(x, y)
0	-1	$(0, -1)$
3	-1	$(3, -1)$
-3	-1	$(-3, -1)$

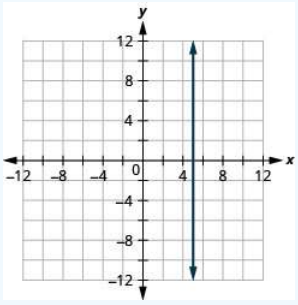


Exercise 3.2.11

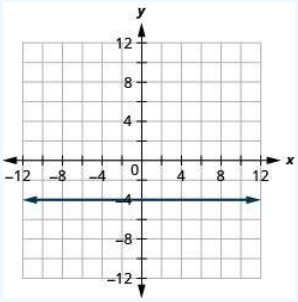
Graph the equations: a. $x = 5$ b. $y = -4$.

Answer

a.



b.

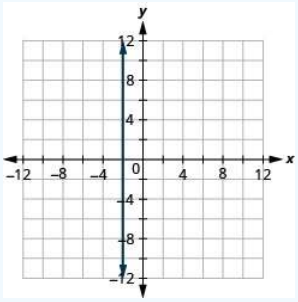


Exercise 3.2.12

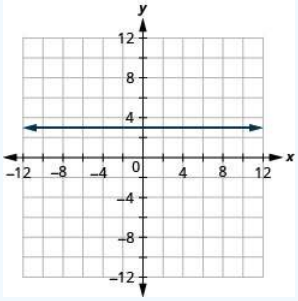
Graph the equations: a. $x = -2$ b. $y = 3$.

Answer

a.

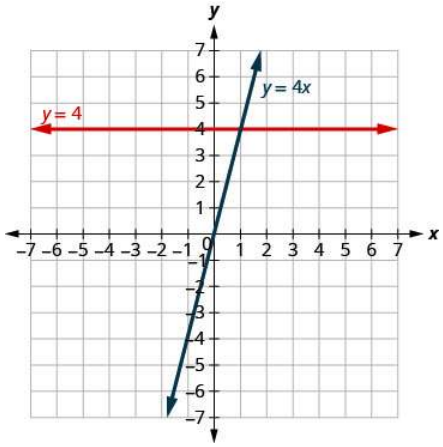


b.



What is the difference between the equations $y = 4x$ and $y = 4$?

The equation $y = 4x$ has both x and y. The value of y depends on the value of x, so the y-coordinate changes according to the value of x. The equation $y = 4$ has only one variable. The value of y is constant, it does not depend on the value of x, so the y-coordinate is always 4.



$y = 4x$		
x	y	(x, y)
0	0	(0, 0)
1	4	(1, 4)
2	8	(2, 8)

$y = 4$		
x	y	(x, y)
0	4	(0, 4)
1	4	(1, 4)
2	4	(2, 4)

Notice, in the graph, the equation $y = 4x$ gives a slanted line, while $y = 4$ gives a horizontal line.

Example 3.2.13

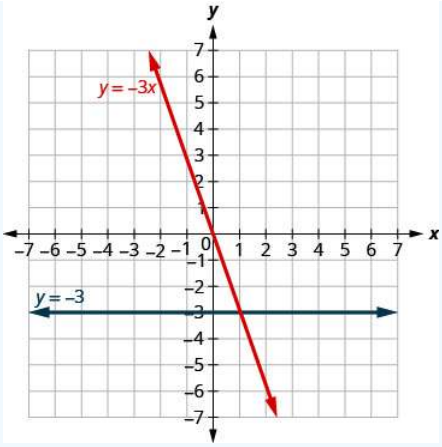
Graph $y = -3x$ and $y = -3$ in the same rectangular coordinate system.

Solution

We notice that the first equation has the variable x, while the second does not. We make a table of points for each equation and then graph the lines. The two graphs are shown.

$y = -3x$		
x	y	(x, y)
0	0	(0, 0)
1	–3	(1, –3)
2	–6	(2, –6)

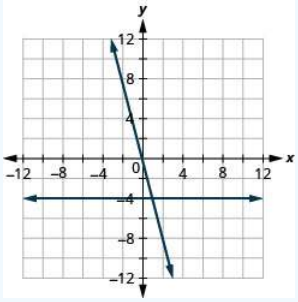
$y = -3$		
x	y	(x, y)
0	–3	(0, –3)
1	–3	(1, –3)
2	–3	(2, –3)



Exercise 3.2.14

Graph the equations in the same rectangular coordinate system: $y = -4x$ and $y = -4$.

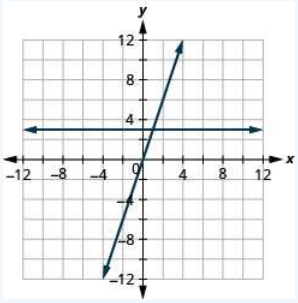
Answer



Exercise 3.2.15

Graph the equations in the same rectangular coordinate system: $y = 3$and $y = 3x$.

Answer



Find x- and y-intercepts

Every linear equation can be represented by a unique line that shows all the solutions of the equation. We have seen that when graphing a line by plotting points, you can use any three solutions to graph. This means that two people graphing the line might use different sets of three points.

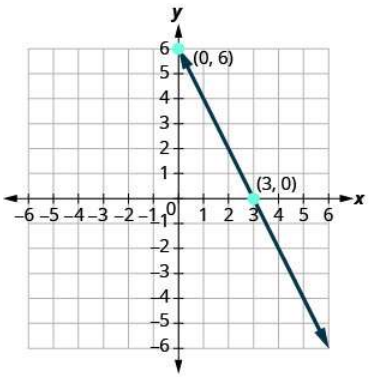
At first glance, their two lines might not appear to be the same, since they would have different points labeled. But if all the work was done correctly, the lines should be exactly the same. One way to recognize that they are indeed the same line is to look at

where the line crosses the x-axis and the y-axis. These points are called the **intercepts of a line**.

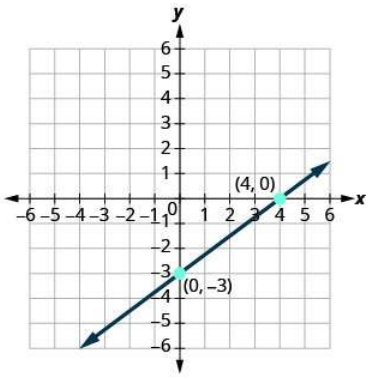
Definition: INTERCEPTS OF A LINE

The points where a line crosses the x-axis and the y-axis are called the **intercepts of the line**.

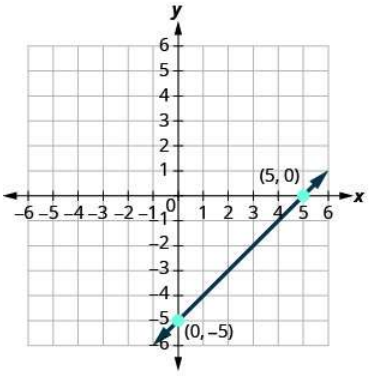
Let's look at the graphs of the lines.



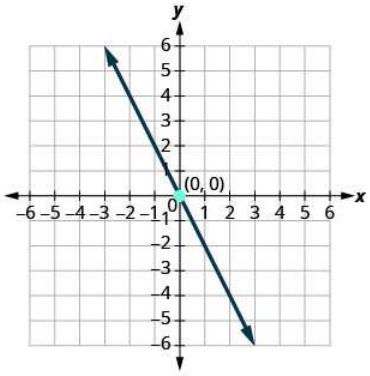
(a) $2x + y = 6$



(b) $3x - 4y = 12$



(c) $x - y = 5$



(d) $y = -2x$

First, notice where each of these lines crosses the x-axis. See Table.

Now, let's look at the points where these lines cross the y-axis.

Figure	**The line crosses the x-axis at:**	**Ordered pair for this point**	**The line crosses the y-axis at:**	**Ordered pair for this point**
Figure (a)	3	$(3, 0)$	6	$(0, 6)$
Figure (b)	4	$(4, 0)$	-3	$(0, -3)$
Figure (c)	5	$(5, 0)$	-5	$(0, 5)$
Figure (d)	0	$(0, 0)$	0	$(0, 0)$
General Figure	a	$(a, 0)$	b	$(0, b)$

Do you see a pattern?

For each line, the y-coordinate of the point where the line crosses the x-axis is zero. The point where the line crosses the x-axis has the form $(a, 0)$ and is called the *x-intercept* of the line. The x-intercept occurs when y is zero.

In each line, the x-coordinate of the point where the line crosses the y-axis is zero. The point where the line crosses the y-axis has the form $(0, b)$ and is called the *y-intercept* of the line. The y-intercept occurs when x is zero.

Intercepts of a Line

The x-intercept is the point $(a, 0)$ where the line crosses the x-axis.

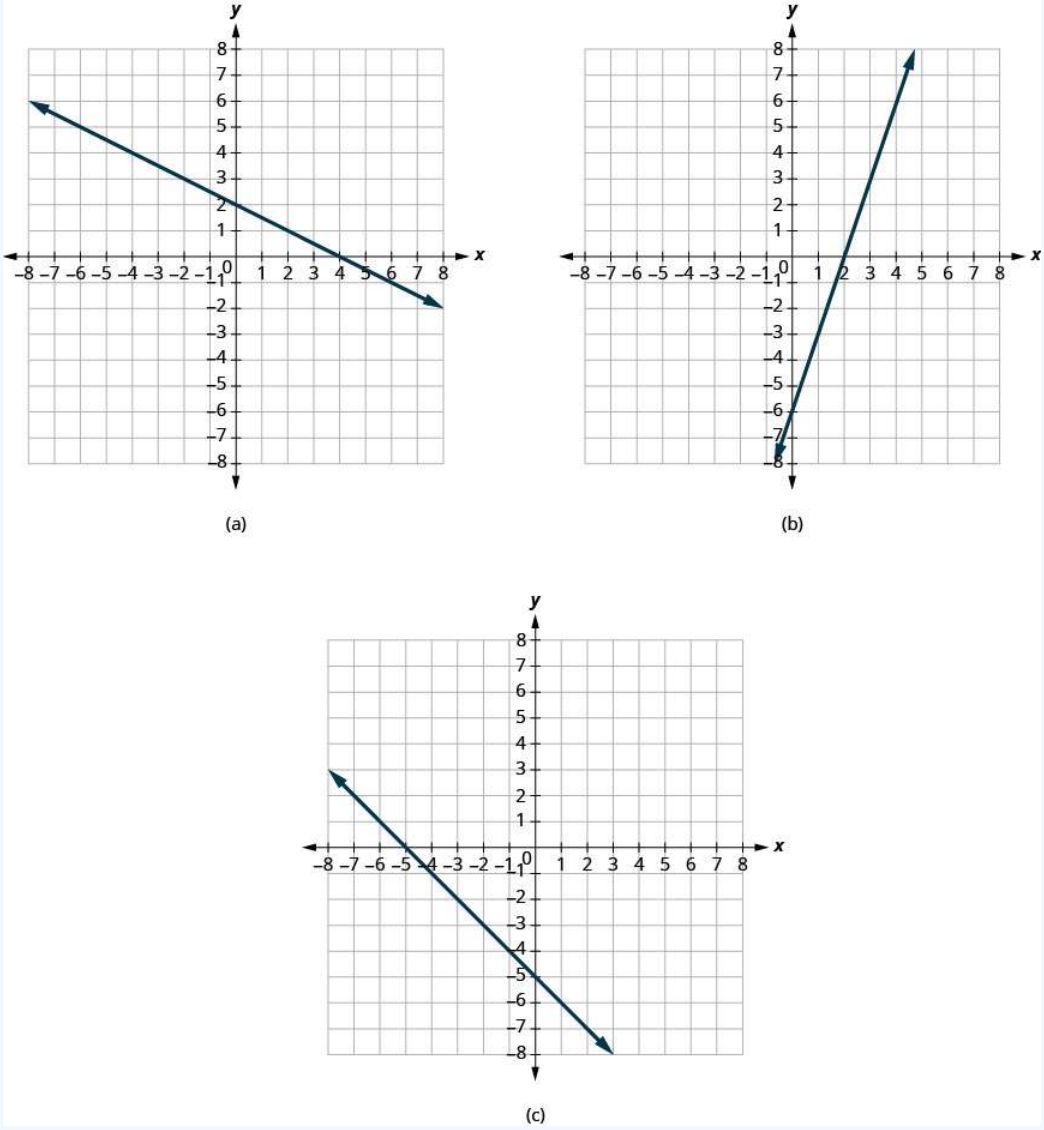
The y-intercept is the point $(0, b)$ where the line crosses the y-axis.

- The x-intercept occurs when y is zero.
- The y-intercept occurs when x is zero.

x	y
a	0
0	b

Example 3.2.16

Find the x- and y-intercepts on each graph shown.



Solution

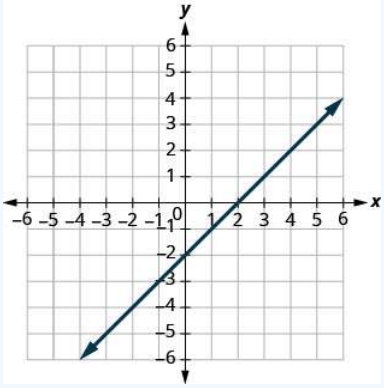
a. The graph crosses the x-axis at the point $(4, 0)$. The x-intercept is $(4, 0)$.
The graph crosses the y-axis at the point $(0, 2)$. The y-intercept is $(0, 2)$.

b. The graph crosses the x-axis at the point $(2, 0)$. The x-intercept is $(2, 0)$.
The graph crosses the y-axis at the point $(0, -6)$. The y-intercept is $(0, -6)$.

c. The graph crosses the x-axis at the point $(-5, 0)$. The x-intercept is $(-5, 0)$.
The graph crosses the y-axis at the point $(0, -5)$. The y-intercept is $(0, -5)$.

Exercise 3.2.17

Find the x- and y-intercepts on the graph.

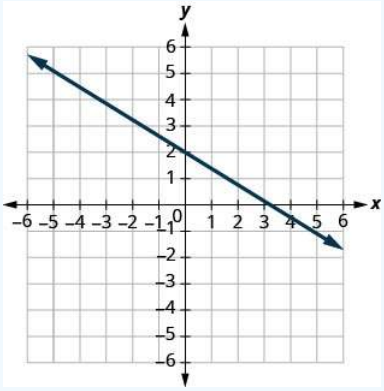


Answer

x-intercept: $(2, 0)$,
y-intercept: $(0, -2)$

Exercise 3.2.18

Find the x- and y-intercepts on the graph.



Answer

x-intercept: $(3, 0)$,
y-intercept: $(0, 2)$

Recognizing that the x-intercept occurs when y is zero and that the y-intercept occurs when x is zero, gives us a method to find the intercepts of a line from its equation. To find the x-intercept, let $y = 0$ and solve for x. To find the y-intercept, let $x = 0$ and solve for y.

Finding Intercepts from the Equation of a Line

Use the equation of the line. To find:

- the x-intercept of the line, let $y = 0$ and solve for x.
- the y-intercept of the line, let $x = 0$ and solve for y.

Example 3.2.19

Find the intercepts of $2x + y = 8$.

Solution

We will let $y = 0$ to find the x-intercept, and let $x = 0$ to find the y-intercept. We will fill in a table, which reminds us of what we need to find.

$2x + y = 8$	
x	y
	0
0	

x-intercept
y-intercept

To find the x-intercept, let $y = 0$.	
	$2x + y = 8$
Let $y = 0$.	$2x + 0 = 8$
Simplify.	$2x = 8$
	$x = 4$
The x-intercept is:	$(4, 0)$
To find the y-intercept, let $x = 0$.	
	$2x + y = 8$
Let $x = 0$.	$2(0) + y = 8$
Simplify.	$0 + y = 8$
	$y = 8$
The y-intercept is:	$(0, 8)$

The intercepts are the points $(4, 0)$ and $(0, 8)$ as shown in the table.

$\boldsymbol{2x + y = 8}$	
$\boldsymbol{x}$	$\boldsymbol{y}$
4	0
0	8

Exercise 3.2.20

Find the intercepts: $3x + y = 12$.

Answer

x-intercept: $(4, 0)$,
y-intercept: $(0, 12)$

Exercise 3.2.21

Find the intercepts: $x + 4y = 8$.

Answer

x-intercept: $(8, 0)$,
y-intercept: $(0, 2)$

Graph a Line Using the Intercepts

To graph a linear equation by plotting points, you need to find three points whose coordinates are solutions to the equation. You can use the *x*- and *y*- intercepts as two of your three points. Find the intercepts, and then find a third point to ensure accuracy. Make sure the points line up—then draw the line. This method is often the quickest way to graph a line.

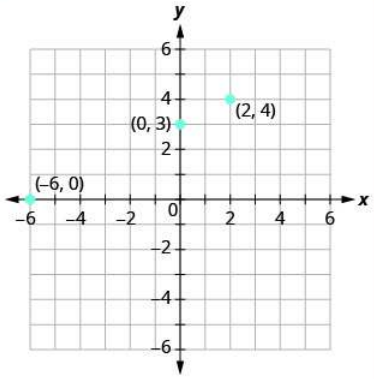
Example 3.2.22

Graph $-x + 2y = 6$ using the intercepts.

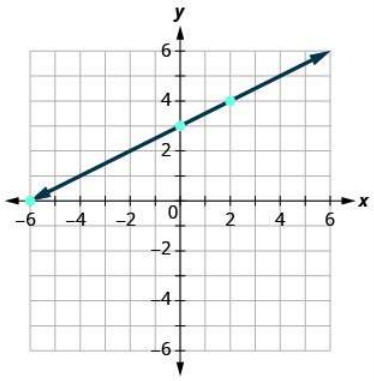
Solution

Step 1. Find the *x*- and *y*-intercepts of the line.	Find the *x*-intercept.	Let $y = 0$
Let $y = 0$ and solve for x.		$-x + 2y = 6$ $-x + 2(0) = 6$ $-x = 6$ $x = -6$ The *x*-intercept is (–6, 0).
Let $x = 0$ and solve for y.	Find the *y*-intercept.	Let $x = 0$. $-x + 2y = 6$ $-0 + 2y = 6$ $2y = 6$ $y = 3$ The *y*-intercept is (0, 3).
Step 2. Find another solution to the equation.	We'll use $x = 2$.	Let $x = 2$. $-x + 2y = 6$ $-2 + 2y = 6$ $2y = 8$ $y = 4$ A third point is (2, 4).

Step 3. Plot the three points. Check that the points line up.	x	y	(x, y)
	−6	0	(−6, 0)
	0	3	(0, 3)
	2	4	(2, 4)



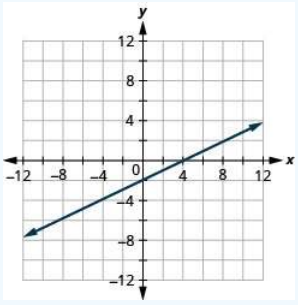
Step 4. Draw the line.	See the graph.



Exercise 3.2.23

Graph using the intercepts: $x-2y=4$.

Answer

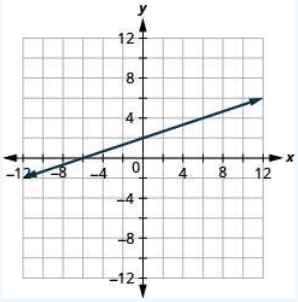


Exercise 3.2.24

Graph using the intercepts: $-x+3y=6$.

Answer

https://math.libretexts.org/@go/page/104817



The steps to graph a linear equation using the intercepts are summarized here.

GRAPH A LINEAR EQUATION USING THE INTERCEPTS

1. Find the x- and y-intercepts of the line.
   - Let y=0y=0 and solve for x.
   - Let x=0x=0 and solve for y.
2. Find a third solution to the equation.
3. Plot the three points and check that they line up.
4. Draw the line.

Example 3.2.25

Graph $4x - 3y = 12$ using the intercepts.

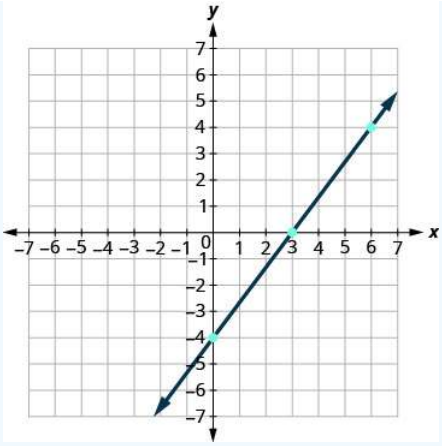
Solution

Find the intercepts and a third point.

x-intercept, let $y = 0$	y-intercept, let $x = 0$	third point, let $y = 4$
$4x - 3y = 12$	$4x - 3y = 12$	$4x - 3y = 12$
$4x - 3(0) = 12$	$4(0) - 3y = 12$	$4x - 3(4) = 12$
$4x = 12$	$-3y = 12$	$4x - 12 = 12$
$x = 3$	$y = -4$	$4x = 24$
		$x = 6$

We list the points in the table and show the graph.

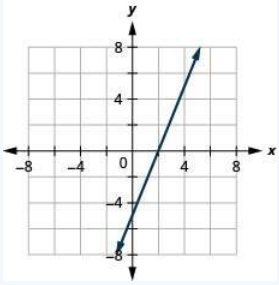
$4x - 3y = 12$		
x	y	(x, y)
3	0	$(3, 0)$
0	-4	$(0, -4)$
6	4	$(6, 4)$



Exercise 3.2.26

Graph using the intercepts: $5x - 2y = 10$.

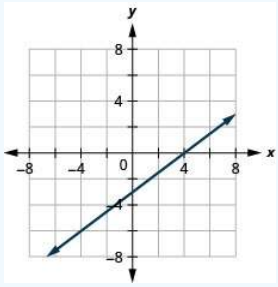
Answer



Exercise 3.2.27

Graph using the intercepts: $3x - 4y = 12$.

Answer



When the line passes through the origin, the x-intercept and the y-intercept are the same point.

Example 3.2.28

Graph $y = 5x$ using the intercepts.

Solution

x-intercept	y-intercept
Let $y = 0$.	Let $x = 0$.
$y = 5x$	$y = 5x$
$0 = 5x$	$y = 5 \cdot 0$
$0 = x$	$y = 0$
$(0, 0)$	$(0, 0)$

This line has only one intercept. It is the point $(0, 0)$.

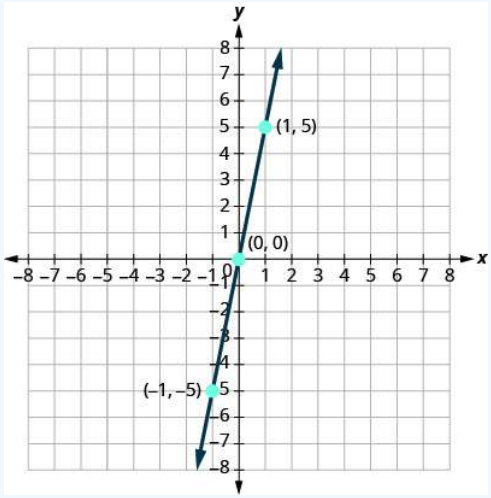
To ensure accuracy, we need to plot three points. Since the x- and y-intercepts are the same point, we need *two* more points to graph the line.

Let $x = 1$.	Let $x = -1$.
$y = 5x$	$y = 5x$
$y = 5 \cdot 1$	$y = 5(-1)$
$y = 5$	$y = -5$

The resulting three points are summarized in the table.

$y = 5x$		
x	y	(x, y)
0	0	$(0, 0)$
1	5	$(1, 5)$
-1	-5	$(-1, -5)$

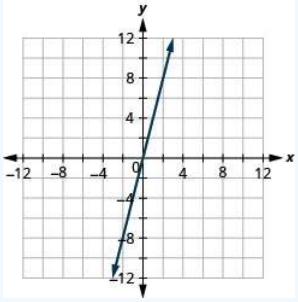
Plot the three points, check that they line up, and draw the line.



Exercise 3.2.29

Graph using the intercepts: $y = 4x$.

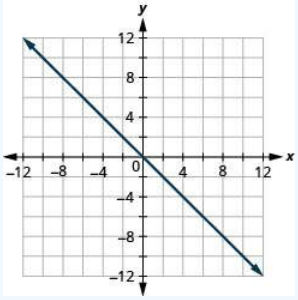
Answer



? Exercise 3.2.30

Graph the intercepts: $y = -x$.

Answer



Key Concepts

- **Graph of a Linear Equation:** The graph of a linear equation $Ax + By = C$ is a straight line.
  Every point on the line is a solution of the equation.
  Every solution of this equation is a point on this line.
- **How to graph a linear equation by plotting points.**
  1. Find three points whose coordinates are solutions to the equation. Organize them in a table.
  2. Plot the points in a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work.
  3. Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.
- **x-intercept and y-intercept of a Line**
  - The x-intercept is the point $(a, 0)$ where the line crosses the x-axis.
  - The y-intercept is the point $(0, b)$ where the line crosses the y-axis.

- The x-intercept occurs when y is zero.
- The y-intercept occurs when x is zero.

x	y
a	0
0	b

- **Find the x- and y-intercepts from the Equation of a Line**
  - Use the equation of the line. To find:
    the x-intercept of the line, let $y = 0$ and solve for x.
    the y-intercept of the line, let $x = 0$ and solve for y.
- **How to graph a linear equation using the intercepts.**
  1. Find the x- and y-intercepts of the line.
     Let $y = 0$ and solve for x.
     Let $x = 0$ and solve for y.
  2. Find a third solution to the equation.
  3. Plot the three points and check that they line up.

4. Draw the line.

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3.2E: Exercises

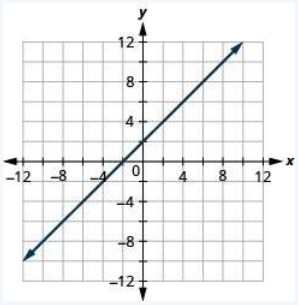
Practice Makes Perfect

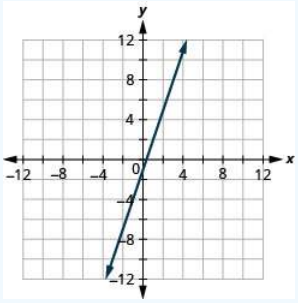
Graph by Plotting Points

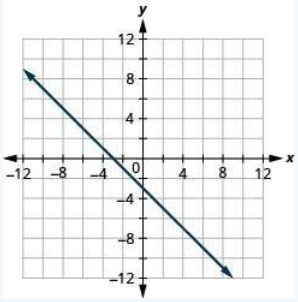
In the following exercises, graph by plotting points.

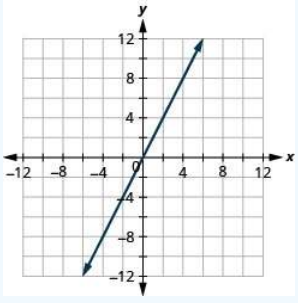
1. $y = x + 2$
2. $y = 3x - 1$
3. $y = -x - 3$
4. $y = 2x$
5. $y = \frac{1}{2}x + 2$

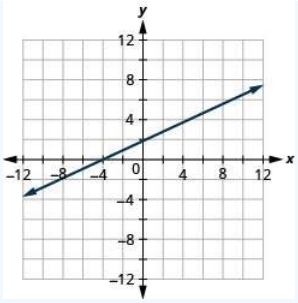
Answer

1. 

2. 

3. 

4. 

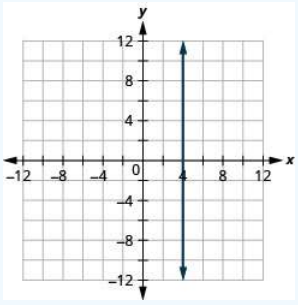
5. 

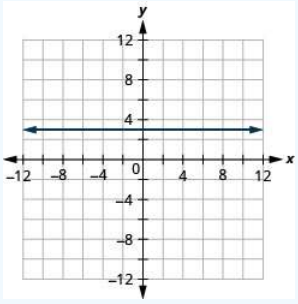
Graph Vertical and Horizontal lines.

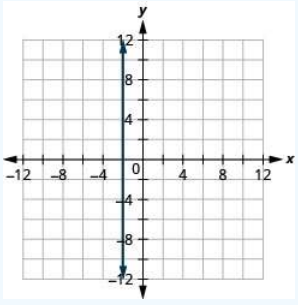
In the following exercises, graph each equation.

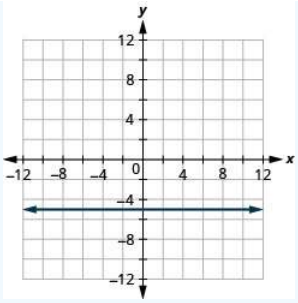
6. $x = 4$
7. $y = 3$
8. $x = -2$
9. $y = -5$

Answer

6. 

7. 

8. 

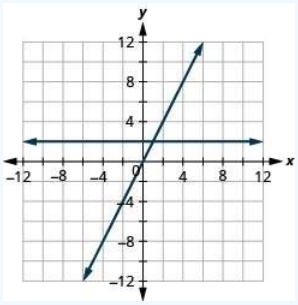
9. 

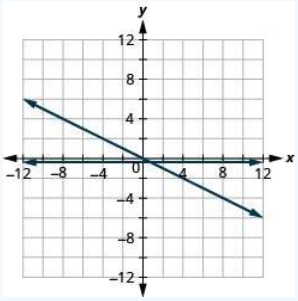
Graph Each Pair of Equations

In the following exercises, graph each pair of equations in the same rectangular coordinate system.

10. $y = 2x$ and $y = 2$
11. $y = -\frac{1}{2}x$ and $y = -\frac{1}{2}$

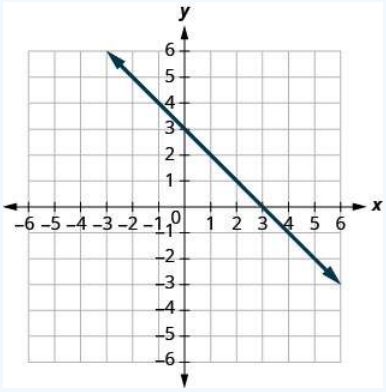
Answer

10. 

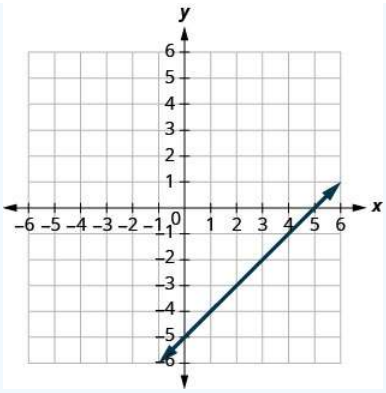
11. 

Find x- and y-intercepts From a Graph

In the following exercises, find the *x*- and *y*-intercepts on each graph.

12. 

https://math.libretexts.org/@go/page/110471

13. 

Answer

12. $(3, 0), (0, 3)$
13. $(5, 0), (0, -5)$

Find Intercepts for Equations

In the following exercises, find the intercepts for each equation.

14. $x - y = 5$
15. $3x + y = 6$
16. $4x - y = 8$
17. $2x + 5y = 10$

Answer

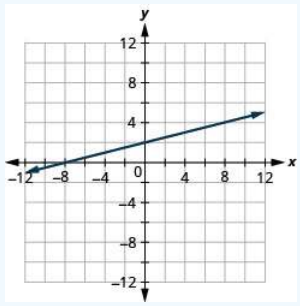
14. x-int: $(5, 0)$, y-int: $(0, -5)$
15. x-int: $(2, 0)$, y-int: $(0, 6)$
16. x-int: $(2, 0)$, y-int: $(0, -8)$
17. x-int: $(5, 0)$, y-int: $(0, 2)$

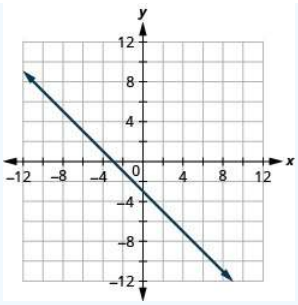
Graph using Intercepts

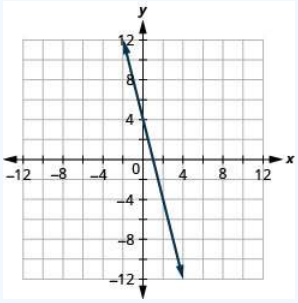
In the following exercises, graph using the intercepts.

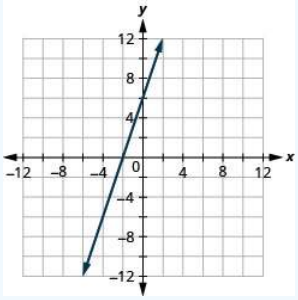
18. $-x + 4y = 8$
19. $x + y = -3$
20. $4x + y = 4$
21. $3x - y = -6$
22. $2x + 4y = 12$

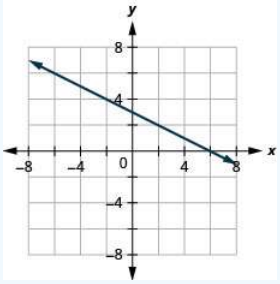
Answer

18. 

19. 

20. 

21. 

22. 

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3.3: Slope Intercept Form

Learning Objectives

By the end of this section, you will be able to:

- Find the slope of a line
- Graph a line given a point and the slope
- Graph a line using its slope and intercept
- Choose the most convenient method to graph a line
- Graph and interpret applications of slope–intercept
- Use slopes to identify parallel and perpendicular lines

Warm Up 3.3.1

a. Simplify: $\frac{1-4}{8-2}$

b. Divide: $\frac{0}{4}$, $\frac{4}{0}$.

c. Simplify: $\frac{15}{-3}$, $\frac{-15}{3}$, $\frac{-15}{-3}$.

Answer

a. -2

b. 0, undefined

c. -5, -5, 5

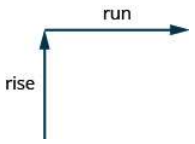
Find the Slope of a Line

When you graph linear equations, you may notice that some lines tilt up as they go from left to right and some lines tilt down. Some lines are very steep and some lines are flatter.

In mathematics, the measure of the steepness of a line is called the *slope* of the line.

The concept of slope has many applications in the real world. In construction the pitch of a roof, the slant of the plumbing pipes, and the steepness of the stairs are all applications of slope. and as you ski or jog down a hill, you definitely experience slope.

We can assign a numerical value to the slope of a line by finding the ratio of the rise and run. The *rise* is the amount the vertical distance changes while the *run* measures the horizontal change, as shown in this illustration. Slope is a rate of change.



Definition: SLOPE OF A LINE

The slope of a line is $m = \frac{\text{rise}}{\text{run}}$.

The rise measures the vertical change and the run measures the horizontal change.

To find the slope of a line, we locate two points on the line whose coordinates are integers. Then we sketch a right triangle where the two points are vertices and one side is horizontal and one side is vertical.

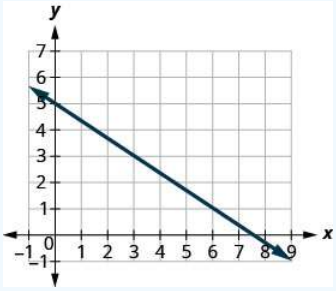
To find the slope of the line, we measure the distance along the vertical and horizontal sides of the triangle. The vertical distance is called the *rise* and the horizontal distance is called the *run*.

FIND THE SLOPE OF A LINE FROM ITS GRAPH USING $m = \frac{\text{rise}}{\text{run}}$

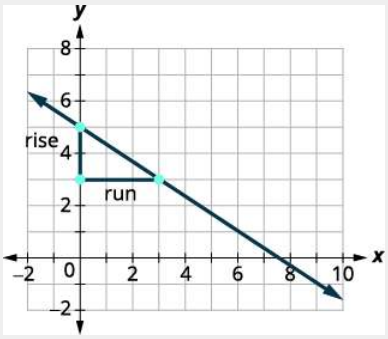
1. Locate two points on the line whose coordinates are integers.
2. Starting with one point, sketch a right triangle, going from the first point to the second point.
3. Count the rise and the run on the legs of the triangle.
4. Take the ratio of rise to run to find the slope: $m = \frac{\text{rise}}{\text{run}}$.

Example 3.3.1

Find the slope of the line shown.

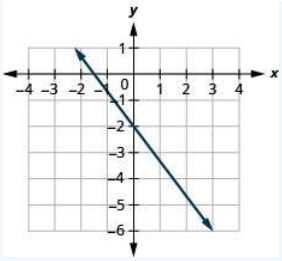


Answer

Locate two points on the graph whose coordinates are integers.	$(0, 5)$ and $(3, 3)$
Starting at $(0, 5)$, sketch a right triangle to $(3, 3)$ as shown in this graph.	
Count the rise— since it goes down, it is negative.	The rise is -2.
Count the run.	The run is 3.
Use the slope formula.	$m = \frac{\text{rise}}{\text{run}}$
Substitute the values of the rise and run.	$m = -23$
Simplify.	$m = -23$
	The slope of the line is -23.
	So y decreases by 2 units as x increases by 3 units.

Exercise 3.3.2

Find the slope of the line shown.

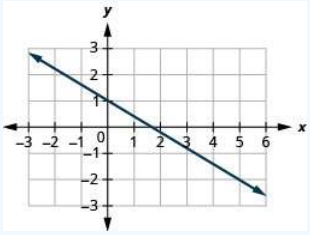


Answer

$-\frac{4}{3}$

Exercise 3.3.3

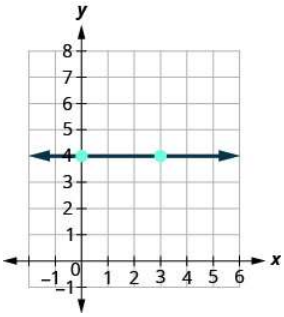
Find the slope of the line shown.



Answer

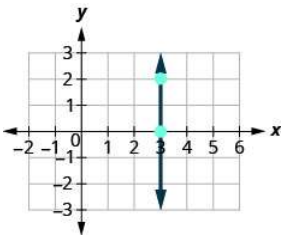
$-\frac{3}{5}$

How do we find the slope of horizontal and vertical lines? To find the slope of the horizontal line, $y = 4$, we could graph the line, find two points on it, and count the rise and the run. Let's see what happens when we do this, as shown in the graph below.



What is the rise?	The rise is 0.
What is the run?	The run is 3.
What is the slope?	$m = \frac{\text{rise}}{\text{run}}$
	$m = \frac{0}{3}$
	$m = 0$
	The slope of the horizontal line $y = 4$ is 0.

Let's also consider a vertical line, the line $x = 3$, as shown in the graph.



What is the rise?	The rise is 0.
What is the run?	The run is 3.
What is the slope?	$m = \frac{\text{rise}}{\text{run}}$
	$m = \frac{2}{0}$

The slope is undefined since division by zero is undefined. So we say that the slope of the vertical line $x = 3$ is undefined.

All horizontal lines have slope 0. When the *y*-coordinates are the same, the rise is 0.

The slope of any vertical line is undefined. When the *x*-coordinates of a line are all the same, the run is 0.

SLOPE OF A HORIZONTAL AND VERTICAL LINE

The slope of a horizontal line, $y = b$, is 0.

The slope of a vertical line, $x = a$, is undefined.

Exercise 3.3.4

Find the slope of each line: ⓐ $x = 8$ ⓑ $y = -5$.

Answer

ⓐ $x = 8$
This is a vertical line. Its slope is undefined.
ⓑ $y = -5$
This is a horizontal line. It has slope 0.

Exercise 3.3.5

Find the slope of the line: $x = -4$.

Answer

undefined

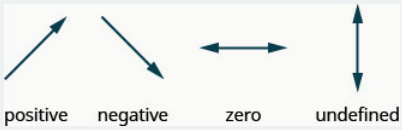
Exercise 3.3.6

Find the slope of the line: $y = 7$.

Answer

0

QUICK GUIDE TO THE SLOPES OF LINES



Sometimes we'll need to find the slope of a line between two points when we don't have a graph to count out the rise and the run. We could plot the points on grid paper, then count out the rise and the run, but as we'll see, there is a way to find the slope without graphing. Before we get to it, we need to introduce some algebraic notation.

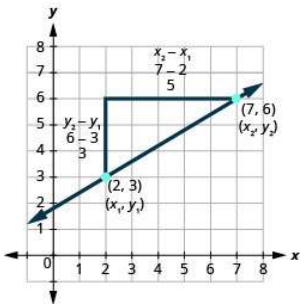
We have seen that an ordered pair (x,y)(x,y) gives the coordinates of a point. But when we work with slopes, we use two points. How can the same symbol (x,y)(x,y) be used to represent two different points? Mathematicians use subscripts to distinguish the points.

(x_1, y_1) read "x sub 1, y sub 1"
(x_2, y_2) read "x sub 2, y sub 2"

We will use (x_1, y_1) to identify the first point and (x_2, y_2) to identify the second point.

If we had more than two points, we could use (x_3, y_3), (x_4, y_4), and so on.

Let's see how the rise and run relate to the coordinates of the two points by taking another look at the slope of the line between the points $(2, 3)$ and $(7, 6)$, as shown in this graph.



Since we have two points, we will use subscript notation.	$\begin{pmatrix} x_1, & y_1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_2, & y_2 \\ 6 & 6 \end{pmatrix}$
	$m = \frac{\text{rise}}{\text{run}}$
On the graph, we counted the rise of 3 and the run of 5.	$m = \frac{3}{5}$
Notice that the rise of 3 can be found by subtracting the y-coordinates, 6 and 3, and the run of 5 can be found by subtracting the x-coordinates 7 and 2.	
We rewrite the rise and run by putting in the coordinates.	$m = \frac{6-3}{7-2}$
But 6 is y_2, the y-coordinate of the second point and 3 is y_1, the y-coordinate of the first point. So we can rewrite the slope using subscript notation.	$m = \frac{y_2-y_1}{7-2}$
Also 7 is the x-coordinate of the second point and 2 is the x-coordinate of the first point. So again we rewrite the slope using subscript notation.	$m = \frac{y_2-y_1}{x_2-x_1}$

We've shown that $m = \frac{y_2-y_1}{x_2-x_1}$ is really another version of $m = \frac{\text{rise}}{\text{run}}$. We can use this formula to find the slope of a line when we have two points on the line.

Definition: SLOPE OF A LINE BETWEEN TWO POINTS

The slope of the line between two points (x_1, y_1) and (x_2, y_2) is:

$m = \frac{y_2-y_1}{x_2-x_1}$.

The slope is:

y of the second point minus y of the first point

over

x of the second point minus x of the first point

Example 3.3.7

Use the slope formula to find the slope of the line through the points $(-2,-3)$ and $(-7,4)$.

Answer

We'll call $(-2,-3)$ point #1and $(-7,4)$ point #2. $\begin{pmatrix} x_1, & y_1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} x_2, & y_2 \\ -7 & 4 \end{pmatrix}$

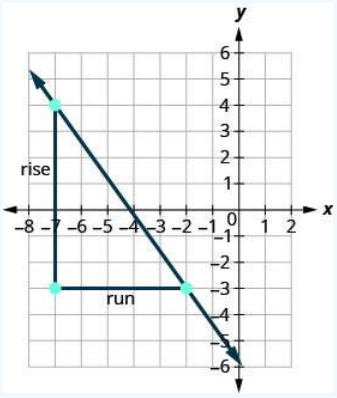
Use the slope formula. $m = \frac{y_2 - y_1}{x_2 - x_1}$

Substitute the values.
y of the second point minus y of the first point
x of the second point minus x of the first point $m = \frac{4-(-3)}{-7-(-2)}$

Simplify $m = \frac{7}{-5}$

$m = \frac{-7}{5}$

Let's verify this slope on the graph shown.



$$m = \frac{\text{rise}}{\text{run}}$$

$$m = \frac{7}{-5}$$

$$m = \frac{-7}{5}$$

Use the slope formula to find the slope of the line through the pair of points: $(-3,4)$ and $(2,-1)$.

Answer

-1

Exercise 3.3.9

Use the slope formula to find the slope of the line through the pair of points: $(-2,6)$and $(-3,-4)$.

Answer

10

Graph a Line Given a Point and the Slope

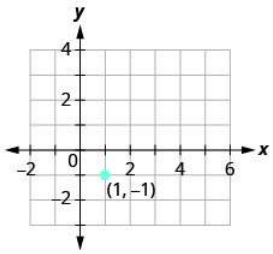
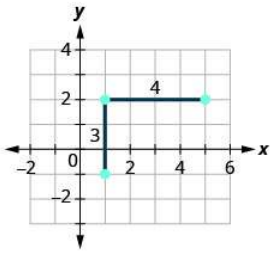
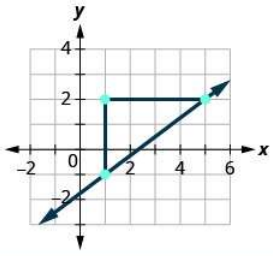
Up to now, in this chapter, we have graphed lines by plotting points, by using intercepts, and by recognizing horizontal and vertical lines.

We can also graph a line when we know one point and the slope of the line. We will start by plotting the point and then use the definition of slope to draw the graph of the line.

? Exercise 3.3.10: How to graph a Line Given a Point and the Slope

Graph the line passing through the point $(1, -1)$ whose slope is $m = \frac{3}{4}$.

Answer

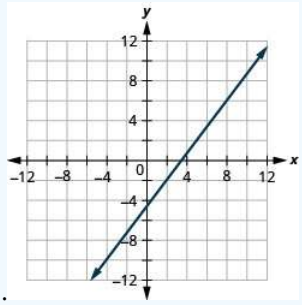
Step 1. Plot the given point.	Plot (1, –1).	
Step 2. Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.	Identify the rise and the run.	$m = \frac{3}{4}$ $\frac{\text{rise}}{\text{run}} = \frac{3}{4}$ rise = 3 run = 4
Step 3. Starting at the given point, count out the rise and run to mark the second point.	Start at (1, –1) and count the rise and the run. Up 3 units, right 4 units.	
Step 4. Connect the points with a line.	Connect the two points with a line.	

You can check your work by finding a third point. Since the slope is $m = \frac{3}{4}$, it can also be written as $m = \frac{-3}{-4}$ (negative divided by negative is positive!). Go back to $(1, -1)$ and count out the rise, -3, and the run, -4.

Exercise 3.3.11

Graph the line passing through the point $(2, -2$ with the slope $m = \frac{4}{3}$.

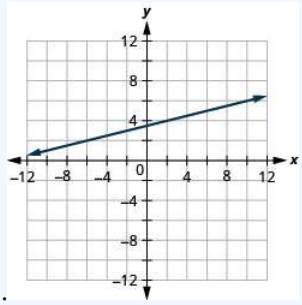
Answer



Exercise 3.3.12

Graph the line passing through the point $(-2, 3)$ with the slope $m = \frac{1}{4}$.

Answer



GRAPH A LINE GIVEN A POINT AND THE SLOPE.

1. Plot the given point.
2. Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.
3. Starting at the given point, count out the rise and run to mark the second point.
4. Connect the points with a line.

Graph a Line Using its Slope and Intercept

We have graphed linear equations by plotting points, using intercepts, recognizing horizontal and vertical lines, and using one point and the slope of the line. Once we see how an equation in slope–intercept form and its graph are related, we'll have one more method we can use to graph lines.

Let's look at the graph of the equation $y = 12x + 3$ and find its slope and *y*-intercept.

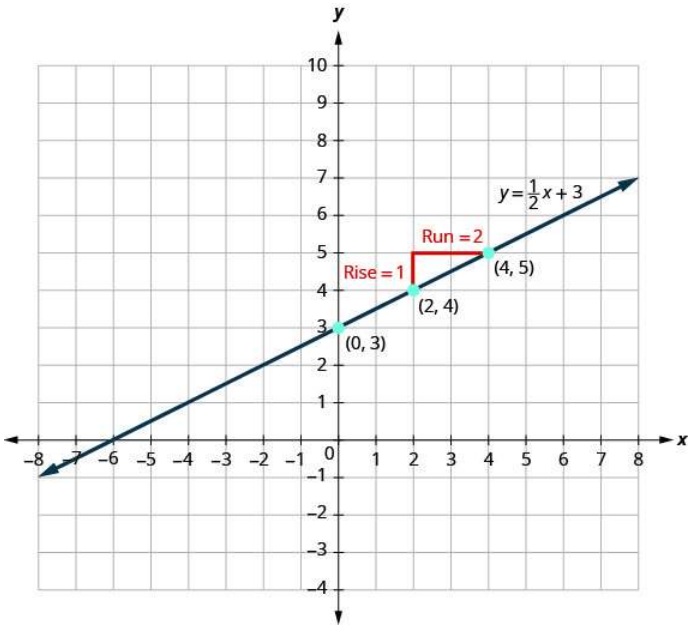


Figure 3.3.2

The red lines in the graph show us the rise is 1 and the run is 2. Substituting into the slope formula:

$$m = \frac{\text{rise}}{\text{run}}$$

$$m = \frac{1}{2}$$

The *y*-intercept is $(0, 3)$.

Look at the equation of this line.

$y = \frac{1}{2}x + 3$

Look at the slope and *y*-intercept.

slope $m = \frac{1}{2}$ and *y*-intercept (0, 3).

When a linear equation is solved for *y*, the coefficient of the *x* term is the slope and the constant term is the *y*-coordinate of the *y*-intercept. We say that the equation $y = 12x + 3$ is in slope–intercept form. Sometimes the slope–intercept form is called the "*y*-form."

$m = \frac{1}{2}$; *y*-intercept is (0, 3)

$y = \frac{1}{2}x + 3$

$y = mx + b$

Definition: SLOPE INTERCEPT FORM OF AN EQUATION OF A LINE

The slope–intercept form of an equation of a line with slope *m* and *y*-intercept, $(0, b)$ is $y = mx + b$.

Let's practice finding the values of the slope and *y*-intercept from the equation of a line.

Exercise 3.3.13

Identify the slope and *y*-intercept of the line from the equation:

ⓐ $y=-\frac{4}{7}x-2$ ⓑ $x+3y=9$

Answer

ⓐ We compare our equation to the slope–intercept form of the equation.

Write the slope–intercept form of the equation of the line.	$y=mx+b$
Write the equation of the line.	$y=-\frac{4}{7}x-2$
Identify the slope.	$m=-\frac{4}{7}$
Identify the *y*-intercept.	*y*-intercept is $(0,-2)$

ⓑ When an equation of a line is not given in slope–intercept form, our first step will be to solve the equation for *y*.

Solve for *y*.	x+3y=9x+3y=9
Subtract *x* from each side.	$3y=-x+9$
Divide both sides by 3.	$\frac{3y}{3}=\frac{-x+9}{3}$
Simplify.	$y=-\frac{1}{3}x+3$
Write the slope–intercept form of the equation of the line.	$y=mx+b$
Write the equation of the line.	$y=-\frac{1}{3}x+3$
Identify the slope.	$m=-\frac{1}{3}$
Identify the *y*-intercept.	*y*-intercept is $(0,3)$

Exercise 3.3.14

Identify the slope and *y*-intercept from the equation of the line.

ⓐ $y=\frac{2}{5}x-1$ ⓑ $x+4y=8$

Answer

ⓐ $m=\frac{2}{5}$; $(0,-1)$

ⓑ $m=-\frac{1}{4}$; $(0,2)$

Exercise 3.3.15

Identify the slope and *y*-intercept from the equation of the line.

ⓐ $y=-\frac{4}{3}x+1$ ⓑ $3x+2y=12$

Answer

ⓐ $m=-\frac{4}{3}$; $(0,1)$

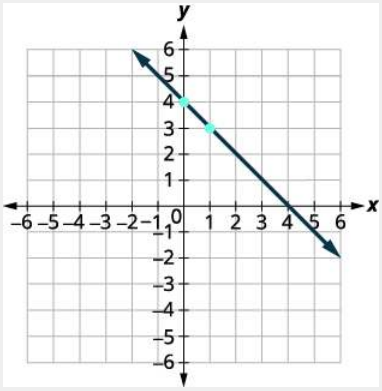
ⓑ $m=-\frac{3}{2}$; $(0,6)$

We have graphed a line using the slope and a point. Now that we know how to find the slope and *y*-intercept of a line from its equation, we can use the *y*-intercept as the point, and then count out the slope from there.

Exercise 3.3.16

Graph the line of the equation $y = -x + 4$ using its slope and *y*-intercept.

Answer

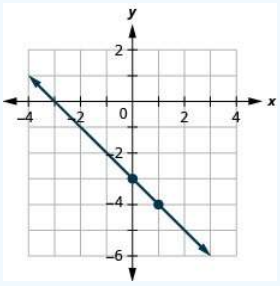
	$y = mx + b$
The equation is in slope–intercept form.	$y = -x + 4$
Identify the slope and *y*-intercept.	$m = -1$ *y*-intercept is $(0, 4)$
Plot the *y*-intercept.	See the graph.
Identify the rise over the run.	$m = -11$
Count out the rise and run to mark the second point.	rise -1, run 1 

Draw the line as shown in the graph.

Exercise 3.3.17

Graph the line of the equation $y = -x - 3$ using its slope and *y*-intercept.

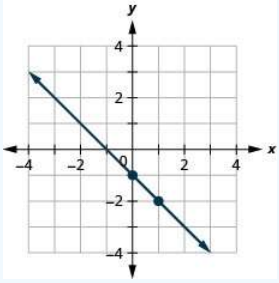
Answer



Exercise 3.3.18

Graph the line of the equation $y = -x - 1$ using its slope and *y*-intercept.

Answer



Now that we have graphed lines by using the slope and *y*-intercept, let's summarize all the methods we have used to graph lines.

Methods to Graph Lines			
Point Plotting	**Slope–Intercept**	**Intercepts**	**Recognize Vertical and Horizontal Lines**
x | y	$y = mx + b$	x | y 0 | | 0	
Find three points. Plot the points, make sure they line up, then draw the line.	Find the slope and *y*-intercept. Start at the *y*-intercept, then count the slope to get a second point.	Find the intercepts and a third point. Plot the points, make sure they line up, then draw the line.	The equation has only one variable. $x = a$ vertical $y = b$ horizontal

Choose the Most Convenient Method to Graph a Line

Now that we have seen several methods we can use to graph lines, how do we know which method to use for a given equation?

While we could plot points, use the slope–intercept form, or find the intercepts for *any* equation, if we recognize the most convenient way to graph a certain type of equation, our work will be easier.

Generally, plotting points is not the most efficient way to graph a line. Let's look for some patterns to help determine the most convenient method to graph a line.

Here are five equations we graphed in this chapter, and the method we used to graph each of them.

	Equation	**Method**
#1	$x = 2$	Vertical line
#2	$y = -1$	Horizontal line
#3	$-x + 2y = 6$	Intercepts
#4	$4x - 3y = 12$	Intercepts
#5	$y = -x + 4$	Slope–intercept

Equations #1 and #2 each have just one variable. Remember, in equations of this form the value of that one variable is constant; it does not depend on the value of the other variable. Equations of this form have graphs that are vertical or horizontal lines.

In equations #3 and #4, both *x* and *y* are on the same side of the equation. These two equations are of the form Ax+By=C.Ax+By=C. We substituted y=0y=0 to find the *x*- intercept and x=0x=0 to find the *y*-intercept, and then found a third point by choosing another value for *x* or *y*.

Equation #5 is written in slope–intercept form. After identifying the slope and *y*-intercept from the equation we used them to graph the line.

This leads to the following strategy.

STRATEGY FOR CHOOSING THE MOST CONVENIENT METHOD TO GRAPH A LINE

Consider the form of the equation.

- If it only has one variable, it is a vertical or horizontal line.
  - $x = a$ is a vertical line passing through the *x*-axis at *a*.
  - $y = b$ is a horizontal line passing through the *y*-axis at *b*.
- If *y* is isolated on one side of the equation, in the form $y = mx + b$, graph by using the slope and *y*-intercept.
  - Identify the slope and *y*-intercept and then graph.
- If the equation is of the form $Ax + By = C$, find the intercepts.
  - Find the *x*- and *y*-intercepts, a third point, and then graph.

Exercise 3.3.19

Determine the most convenient method to graph each line:

ⓐ $y = 5$ ⓑ $4x - 5y = 20$ ⓒ $x = -3$ ⓓ $y = -\frac{5}{9}x + 8$

Answer

ⓐ $y = 5$
This equation has only one variable, *y*. Its graph is a horizontal line crossing the *y*-axis at 5.
ⓑ $4x - 5y = 20$
This equation is of the form $Ax + By = C$. The easiest way to graph it will be to find the intercepts and one more point.
ⓒ $x = -3$
There is only one variable, *x*. The graph is a vertical line crossing the *x*-axis at -3.
ⓓ $y = -\frac{5}{9}x + 8$
Since this equation is in $y = mx + b$ form, it will be easiest to graph this line by using the slope and *y*-intercepts.

Exercise 3.3.20

Determine the most convenient method to graph each line:

ⓐ $3x + 2y = 12$ ⓑ $y = 4$ ⓒ $y = \frac{1}{5}x - 4$ ⓓ $x = -7$.

Answer

ⓐ intercepts ⓑ horizontal line ⓒ slope-intercept ⓓ vertical line

Exercise 3.3.21

Determine the most convenient method to graph each line:

ⓐ $x = 6$ ⓑ $y = -\frac{3}{4}x + 1$ ⓒ $y = -8$ ⓓ $4x - 3y = -1$.

Answer

ⓐ vertical line ⓑ slope-intercept ⓒ horizontal line
ⓓ intercepts

Key Concepts

- **Slope of a Line**
  - The slope of a line is $m = \frac{\text{rise}}{\text{run}}$.
  - The rise measures the vertical change and the run measures the horizontal change.
- **How to find the slope of a line from its graph using** $m = \frac{\text{rise}}{\text{run}}$.

1. Locate two points on the line whose coordinates are integers.
2. Starting with one point, sketch a right triangle, going from the first point to the second point.
3. Count the rise and the run on the legs of the triangle.
4. Take the ratio of rise to run to find the slope: $m = \frac{\text{rise}}{\text{run}}$.

- **Slope of a line between two points.**
  - The slope of the line between two points (x_1, y_1) and (x_2, y_2) is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

    .

- **How to graph a line given a point and the slope.**
  1. Plot the given point.
  2. Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.
  3. Starting at the given point, count out the rise and run to mark the second point.
  4. Connect the points with a line.
- **Slope Intercept Form of an Equation of a Line**
  - The slope–intercept form of an equation of a line with slope *m* and *y*-intercept, $(0, b)$ is $y = mx + b$
  -

Methods to Graph Lines

Point Plotting	Slope–Intercept	Intercepts	Recognize Vertical and Horizontal Lines
x | y	$y = mx + b$	x | y 0 | | 0	
Find three points. Plot the points, make sure they line up, then draw the line.	Find the slope and *y*-intercept. Start at the *y*-intercept, then count the slope to get a second point.	Find the intercepts and a third point. Plot the points, make sure they line up, then draw the line.	The equation has only one variable. $x = a$ vertical $y = b$ horizontal

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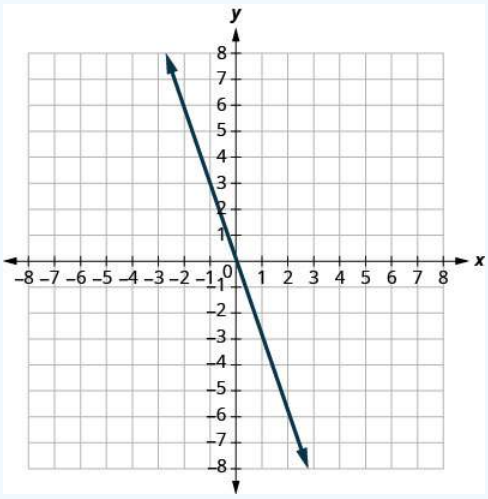
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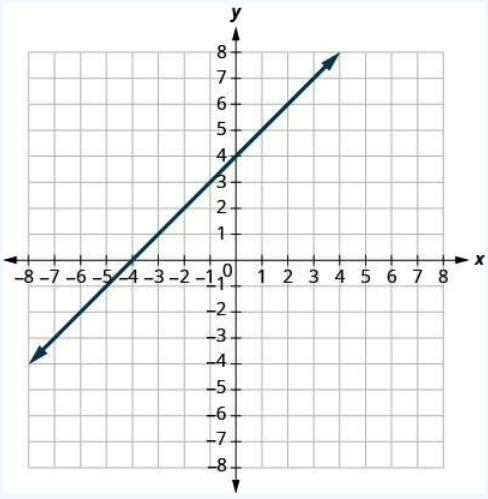
3.3E: Exercises

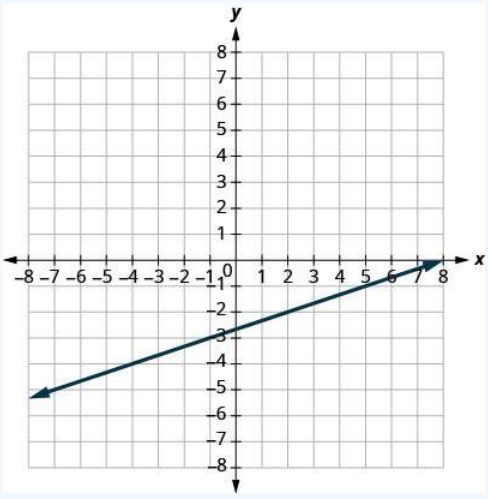
Practice Makes Perfect

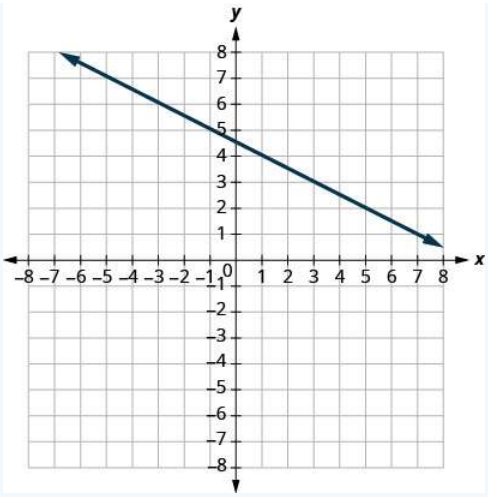
Slope of Line by Picture

In the following exercises, find the slope of each line shown.

1. 

2. 

3. 

4. 

Answer

1. -3
2. 1
3. $\frac{1}{3}$
4. $\frac{1}{2}$

Find the Slope Given Two Points

In the following exercises, use the slope formula to find the slope of the line between each pair of points.

5. $(-1,-1),(0,5)$
6. $(3,5),(4,-1)$
7. $(-5,-2),(3,2)$
8. $(2,1),(4,6)$

Answer

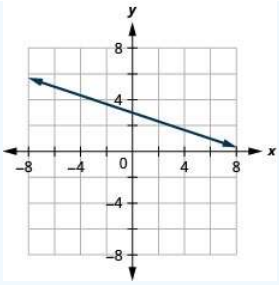
5. 6
6. -6
7. $\frac{1}{2}$
8. $\frac{5}{2}$

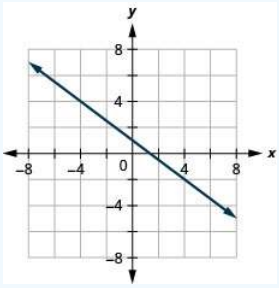
Graph a line

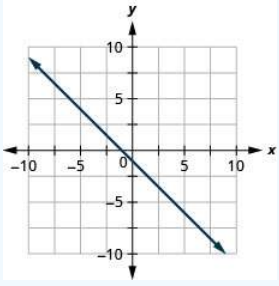
In the following exercises, graph each line given the following information.

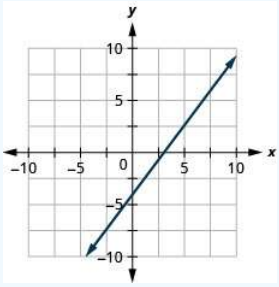
9. $(-3,4);\ m=-\frac{1}{3}$
10. y-intercept $1; m=-\frac{3}{4}$
11. $y=-x-1$
12. $4x-3y=12$

Answer

9. 

10. 

11. 

12. 

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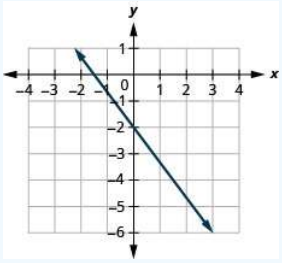
3.4: Finding the Equation of a Line

Learning Objectives

- Determine the equation of a line given the y intercept and slope
- Use a graph to extract the y intercept and slope of a line to determine its equation
- Determine the equation of a line given the y intercept and a point
- Determine the equation of a line given two points

Warm Up 3.4.1

a. Find the y intercept of the graph below



b. Find the slope of the line that contains the points (-1, 2) and (3, -6)

c. Solve for y to get the equation into slope-intercept form: $y-3=-4(x+1)$.

Answer

a. $b=-2$

b. $m=-2$

c. $y=-4x-1$

In the last few sections, we've talked about how we can use an equation to look at the graphs of a line or points on a line. In this section, we look at different ways to deduce the equation of a line given information such as slope, intercepts, points, or just a graph.

Find an Equation of the Line Given the Slope and *y*-Intercept

First, let's recall a particularly helpful form for the equation of a line -- the slope-intercept form -- and discuss a strategy we can use to deduce an equation of this form.

Definition: SLOPE-INTERCEPT FORM OF A LINE

If the line L intercepts the y-axis at the point (0, b) and has slope m, then the equation of the line is

$$y=mx+b. \tag{3.4.1}$$

This form of the equation of a line is called the slope-intercept form. The function defined by the equation

$$f(x)=mx+b \tag{3.4.2}$$

is called a **linear function**.

It is important to note two key facts about the slope-intercept form y = mx + b.

- The coefficient of x (the m in y = mx + b) is the slope of the line.
- The constant term (the b in y = mx+b) is the y-coordinate of the y-intercept (0, b).

So in order to create an equation for a line using this form, all we would need to deduce is those two pieces of information, slope and intercept.

Procedure for Using the Slope-Intercept Form of a Line

When given the slope of a line and the y-intercept of the line, use the slope-intercept form as follows:

1. Substitute the given slope for m in the formula $y = mx + b$.
2. Substitute the y-coordinate of the y-intercept for b in the formula $y = mx + b$.

For example, if the line has slope −2 and the y-intercept (the point where the line crosses the y-axis) is (0, 3), then substitute m = −2 and b = 3 into Equation 3.4.1 to obtain

$$y = -2x + 3.$$

As we did in the last chapter, we can easily determine the slope and intercept of a line if the equation is written in slope-intercept form, $y = mx + b$. Now we will do the reverse—we will start with the slope and *y*-intercept and use them to find the equation of the line.

Example 3.4.1

Find the equation of a line with slope -9 and *y*-intercept $(0, -4)$.

Answer

Since we are given the slope and *y*-intercept of the line, we can substitute the needed values into the slope-intercept form, $y = mx + b$.

Name the slope.	$m = -9$
Name the *y*-intercept.	*y*-intercept $(0, -4)$
Substitute the values into $y = mx + b$.	$y = mx + b$
	$y = -9x + (-4)$
	$y = -9x - 4$

Exercise 3.4.2

Find the equation of a line with slope 25 and *y*-intercept $(0, 4)$.

Answer

$y = \frac{2}{5}x + 4$

Exercise 3.4.3

Find the equation of a line with slope -1 and *y*-intercept $(0, -3)$.

Answer

$y = -x - 3$

Sometimes, the slope and intercept need to be determined from the graph.

Example 3.4.4

Given the graph of the line in Figure 3.4.4(a), determine the equation of the line.

Solution

First, locate the y-intercept of the line, which we've labeled P(0, −1) in Figure 3.4.4(b). In the formula y = mx + b, recall that b represents the y-coordinate of the y-intercept. Thus, b = −1.

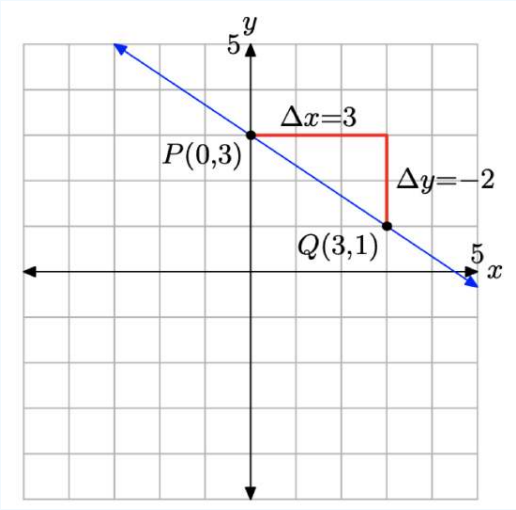


Figure 3.4.3. The line has y-intercept at (0, 3) and slope −2/3.

Secondly, we need to determine the slope of the line. In Figure 3.4.4(b), start at the point P, move 2 units to the right, then 3 units upward to the point Q(2, 2). This makes the slope

$$m = \frac{\Delta y}{\Delta x} = \frac{3}{2} \tag{3.4.3}$$

Substitute m = 3/2 and b = −1 into the slope-intercept form y = mx + b to obtain

$$y = \frac{3}{2}x - 1 \tag{3.4.4}$$

which is the desired equation of the line.

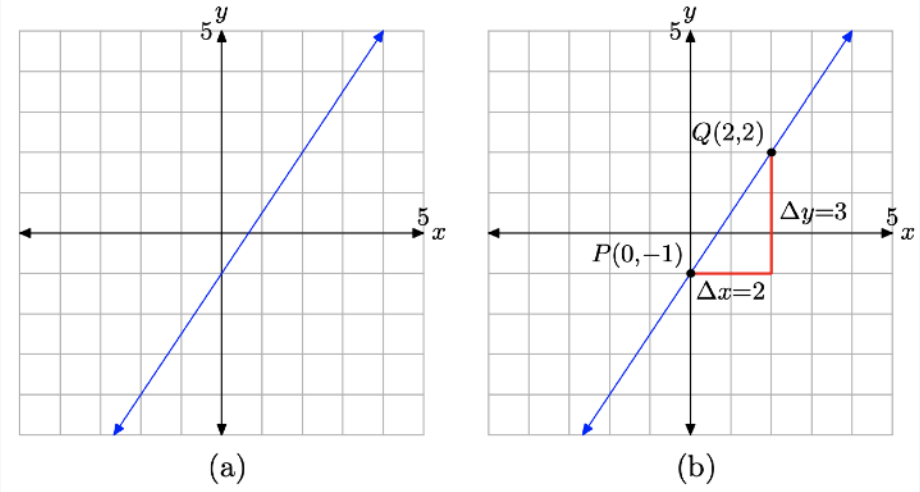
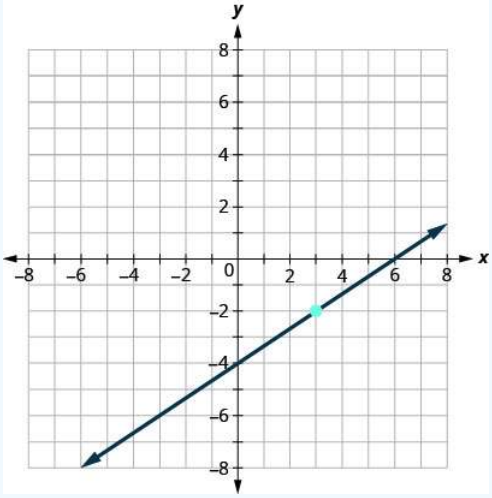


Figure 3.4.4. Determining the equation of a line from its graph.

? Exercise 3.4.5

Find the equation of the line shown in the graph.



Answer

We need to find the slope and *y*-intercept of the line from the graph so we can substitute the needed values into the slope-intercept form, $y = mx + b$.

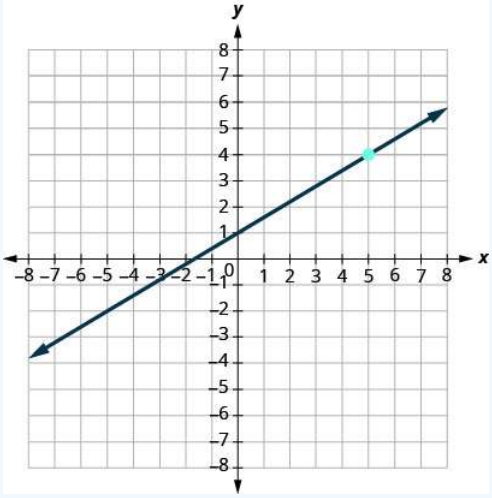
To find the slope, we choose two points on the graph.

The *y*-intercept is $(0, -4)$ and the graph passes through $(3, -2)$.

Find the slope, by counting the rise and run.	$m = \frac{\text{rise}}{\text{run}}$
	$m = \frac{2}{3}$
Find the *y*-intercept.	*y*-intercept (0, –4)
Substitute the values into y=mx+b.y=mx+b.	$y = mx + b$
	$y = \frac{2}{3}x - 4$

Exercise 3.4.6

Find the equation of the line shown in the graph.

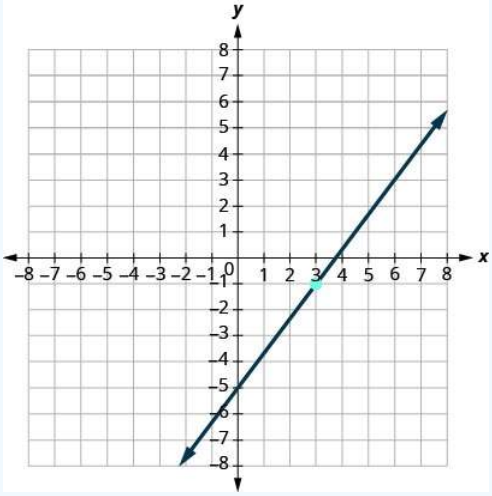


Answer

$y = \frac{3}{5}x + 1$

Exercise 3.4.7

Find the equation of the line shown in the graph.



Answer

$y = \frac{4}{3}x - 5$

Find an Equation of the Line Given the Slope and a Point

Finding an equation of a line using the slope-intercept form of the equation works well when you are given the slope and *y*-intercept or when you read them off a graph. But what happens when you have another point instead of the *y*-intercept?

We are going to use the slope formula to derive another form of an equation of the line.

Suppose we have a line that has slope *m* and that contains some specific point (x_0, y_0) and some other point, which we will just call (x, y). We can write the slope of this line and then change it to a different form.

	m	$=$	$\frac{y-y_0}{x-x_0}$
Multiply both sides of the equation by $x - x_1$.	$m(x-x_1)$	$=$	$\left(\frac{y-y_1}{x-x_1}\right)(x-x_1)$
Simplify.	$m(x-x_1)$	$=$	$y-y_1$
Rewrite the equation with theyterms on the left.	$y-y_1$	$=$	$m(x-x_1)$

This format is called the **point-slope form** of an equation of a line.

Definition: POINT-SLOPE FORM OF A LINE

If line L passes through the point (x_1, y_1) and has slope m, then the equation of the line is

$$y - y_1 = m\,(x - x_1) \tag{3.4.5}$$

This form of the equation of a line is called the point-slope form.

To use the point-slope form of a line, follow these steps.

Procedure for Using the Point-Slope Form of a Line

When given the slope of a line and a point on the line, use the point-slope form as follows:

1. Substitute the given slope for m in the formula $y - y_1 = m\,(x - x_1)$.
2. Substitute the coordinates of the given point for x0 and y0 in the formula $y - y_1 = m\,(x - x_1)$.

For example, if the line has slope −2 and passes through the point (3, 4), then substitute $m = -2, x_1 = 3,$ and $y_1 = 4$ in the formula $y - y_1 = m\,(x - x_1)$ to obtain

$$y - 4 = -2(x - 3)$$

.

It is common to rewrite this equation in slope-intercept form, because it's such a popular format. To do this, we just get y by itself on the left and then simplify everything on the left:

$$y - 4 = -2(x - 3)$$

$$y = -2(x - 3) + 4$$

$$y = -2x + 6 + 4$$

$$y = -2x + 10.$$

We can use the point-slope form of an equation to find an equation of a line when we know the slope and at least one point, even if it's not the y intercept. Then, we will rewrite the equation in slope-intercept form. Most applications of linear equations use the the slope-intercept form.

How to Find an Equation of a Line Given a Point and the Slope

Example 3.4.8

Find an equation of a line with slope $m = -\frac{1}{3}$ that contains the point $(6, -4)$. Write the equation in slope-intercept form.

Answer

Step 1. Identify the slope.	The slope is given.	$m = -\frac{1}{3}$
Step 2. Identify the point.	The point is given.	(x_1, y_1) $(6, -4)$
Step 3. Substitute the values into the point–slope form, $y - y_1 = m(x - x_1)$.	Simplify.	$y - y_1 = m(x - x_1)$ $y - (-4) = -\frac{1}{3}(x - 6)$ $y + 4 = -\frac{1}{3}x + 2$
Step 4. Write the equation in slope–intercept form.		$y = -\frac{1}{3}x - 2$

Exercise 3.4.9

Find the equation of a line with slope $m = -\frac{2}{5}$ and containing the point $(10, -5)$.

Answer

$y = -\frac{2}{5}x - 1$

Exercise 3.4.10

Find the equation of a line with slope $m = -\frac{3}{4}$, and containing the point $(4, -7)$.

Answer

$y = -\frac{3}{4}x - 4$

Let's summarize the steps more succinctly for easy reference:

Procedure for Using the Point-Slope Form of a Line to Find an Equation of a Line Using a Slope and a Point

1. Identify the slope.
2. Identify the point.
3. Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.
4. Write the equation in slope-intercept form.

Exercise 3.4.11

Find an equation of a horizontal line that contains the point $(-2, -6)$. Write the equation in slope-intercept form.

Answer

Every horizontal line has slope 0. We can substitute the slope and points into the point-slope form, $y - y_1 = m(x - x1)$.

Identify the slope.	$m = 0$
Identify the point.	$\begin{matrix} x_1 & y_1 \\ -2, & -6 \end{matrix}$
Substitute the values into y−y1=m(x−x1).y−y1=m(x−x1).	$y - y_1 = m(x - x_1)$
	$y - (-6) = 0(x - (-2))$
Simplify.	$y + 6 = 0$
	$y = -6$
Write in slope-intercept form.	It is in y-form, but could be written $y = 0x - 6$.

Did we end up with the form of a horizontal line, $y = a$?

Exercise 3.4.12

Find the equation of a horizontal line containing the point $(-3, 8)$.

Answer

$y = 8$

Exercise 3.4.13

Find the equation of a horizontal line containing the point $(-1, 4)$.

Answer

$y = 4$

Find an Equation of the Line Given Two Points

In the next example we'll see how to find an equation of a line when just two points are given.

So far, we have two options for finding an equation of a line: slope-intercept or point-slope. When we start with two points, it makes more sense to use the point-slope form.

But then we need the slope. Can we find the slope with just two points? Yes. Then, once we have the slope, we can use it and one of the given points to find the equation.

How to Find the Equation of a Line Given Two Points

? Example 3.4.14

Find an equation of a line that contains the points $(-3, -1)$ and $(2, -2)$ Write the equation in slope-intercept form.

Answer

Step 1. Find the slope using the given points.	Find the slope of the line through (–3, –1) and (2, –2).	$m = \frac{y_2 - y_1}{x_2 - x_1}$ $m = \frac{-2-(-1)}{2-(-3)}$ $m = \frac{-1}{5}$ $m = -\frac{1}{5}$
Step 2. Choose one point.	Choose either point.	$\begin{pmatrix} x_1 & y_1 \\ 2, & -2 \end{pmatrix}$
Step 3. Substitute the values into the point–slope form, $y - y_1 = m(x - x_1)$.	Simplify.	$y - y_1 = m(x - x_1)$ $y-(-2) = -\frac{1}{5}(x-2)$ $y+2 = -\frac{1}{5}x + \frac{2}{5}$
Step 4. Write the equation in slope–intercept form.		$y = -\frac{1}{5}x - \frac{8}{5}$

? Exercise 3.4.15

Find the equation of a line containing the points $(-2, -4)$ and $(1, -3)$.

Answer

\(y=\frac{1}{3}x−\frac{10{3}\)

? Exercise 3.4.16

Find the equation of a line containing the points $(-4, -3)$ and $(1, -5)$.

Answer

$y = -\frac{2}{5}x - \frac{23}{5}$

The steps are summarized here.

TO FIND AN EQUATION OF A LINE GIVEN TWO POINTS.

1. Find the slope using the given points. $m = \frac{y_2 - y_1}{x_2 - x_1}$
2. Choose one point.
3. Substitute the values into the point-slope form: $y - y_1 = m(x - x_1)$.
4. Write the equation in slope-intercept form.

Exercise 3.4.17

Find an equation of a line that contains the points $(-3, 5)$ and $(-3, 4)$. Write the equation in slope-intercept form.

Answer

Again, the first step will be to find the slope.

Find the slope of the line through$(-3, 5)$and$(-3, 4)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{4 - 5}{-3 - (-3)}$$

$$m = \frac{-1}{0}$$

The slope is undefined.

This tells us it is a vertical line. Both of our points have an *x*-coordinate of -2. So our equation of the line is $x = -2$. Since there is no *y*, we cannot write it in slope-intercept form.

You may want to sketch a graph using the two given points. Does your graph agree with our conclusion that this is a vertical line?

Exercise 3.4.18

Find the equation of a line containing the points $(5, 1)$ and $(5, -4)$.

Answer

$x = 5$

Exercise 3.4.19

Find the equation of a line containing the points $(-4, 4)$ and $(-4, 3)$.

Answer

$x = -4$

We have seen that we can use either the slope-intercept form or the point-slope form to find an equation of a line. Which form we use will depend on the information we are given.

To Write an Equation of a Line		
If given:	**Use:**	**Form:**

To Write an Equation of a Line		
Slope and *y*-intercept	slope-intercept	$y = mx + b$
A graph	slope-intercept	$y = mx + b$
Slope and a point	point-slope	$y - y_1 = m(x - x_1)$
Two points	point-slope	$y - y_1 = m(x - x_1)$

Key Concepts

- There are two useful forms for an equation -- the slope-intercept form and the point-slope form
  - The point-slope form is often the easiest to use to create the equation initially
  - The slope-intercept form is the most popular final form
- The **slope-intercept form** of a line is $y = mx + b$, and only requires the slope m and the y intercept b to create it
  - When given the slope of a line and the y-intercept of the line, use the slope-intercept form as follows:
    - Substitute the given slope for m in the formula $y = mx + b$.
    - Substitute the y-coordinate of the y-intercept for b in the formula $y = mx + b$.
- When we know a point (x_1, y_1) that is on the line, the **point-slope form** of a line is $y - y_1 = m(x - x_1)$; but it is commonly rewritten to be in slope-intercept form
  - When given the slope of a line and a point on the line that may not be the y intercept, use the point-slope form as follows:
    - Identify the slope.
    - Identify the point.
    - Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.
    - Write the equation in slope-intercept form.
- When given two points, (x_1, y_1) and (x_2, y_2), the point-slope form is the easiest to use; we just need to do extra work to get the slope first
  - Use the point-slope form as follows:
    - Find the slope using the given points. $m = \frac{y_2 - y_1}{x_2 - x_1}$
    - Choose one point.
    - Substitute the values into the point-slope form: $y - y_1 = m(x - x_1)$.
    - Write the equation in slope-intercept form.

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3.4E: Exercises

Practice Makes Perfect

Find the Equation Given a Slope and y-Intercept

In the following exercises, find the equation of a line with given slope and y-intercept. Write the equation in slope–intercept form.

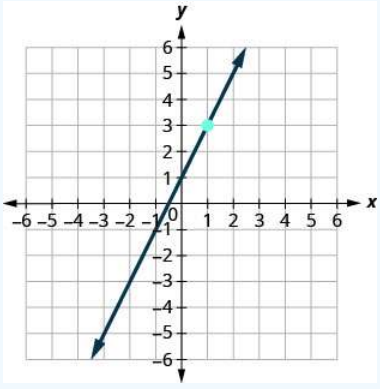
1. Slope $\frac{1}{3}$ and y-intercept $(0, -6)$
2. Slope -5 and y-intercept $(0, -3)$
3. Slope 0 and y-intercept $(0, 4)$
4. Slope -2 and y-intercept $(0, 0)$

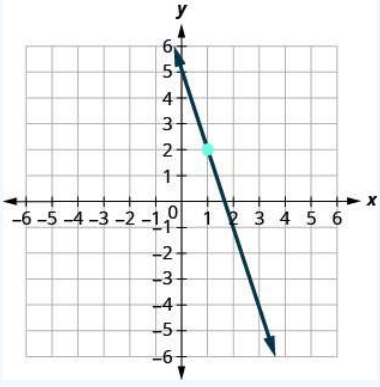
Answer

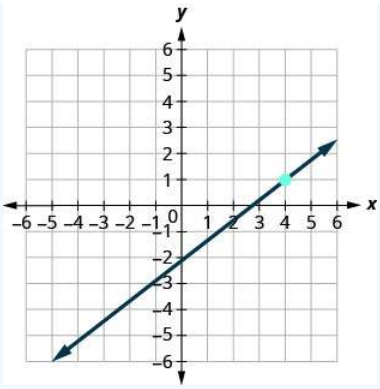
1. $y = \frac{1}{3} - 6$
2. $y = -5x - 3$
3. $y = 4$
4. $y = -2x$

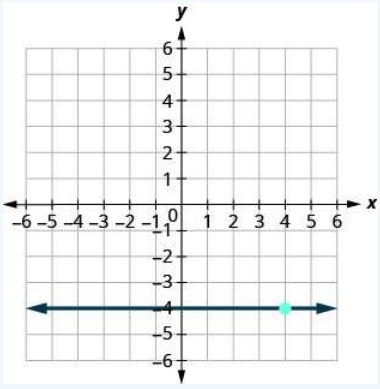
Find the Equation Given a Graph

In the following exercises, find the equation of the line shown in each graph. Write the equation in slope–intercept form.

5. 

6. 

7. 

8. 

Answer

5. $y = 2x + 1$
6. $y = -3x + 5$
7. $y = \frac{3}{4}x - 2$
8. $y = -4$

Find the Equation Given a Slope and a Point

In the following exercises, find the equation of a line with given slope and containing the given point. Write the equation in slope–intercept form.

9. $m = -\frac{1}{4}$, point $(-8, 3)$
10. $m = \frac{3}{5}$, point $(10, 6)$
11. Horizontal line containing $(-2, 7)$
12. $m = -2$, point $(-1, -3)$

Answer

9. $y = -\frac{1}{4}x + 1$
10. $y = \frac{3}{5}x$
11. $y = 7$
12. $y = -2x - 5$

Find the Equation Given Two Points

In the following exercises, find the equation of a line containing the given points. Write the equation in slope–intercept form.

13. $(2, 10)$ and $(-2, -2)$
14. $(7, 1)$ and $(5, 0)$
15. $(3, 8)$ and $(3, -4)$

16. $(5, 2)$ and $(-1, 2)$

Answer

13. $y = 3x + 4$
14. $y = \frac{1}{2}x - \frac{5}{2}$
15. $x = 3$
16. $y = 2$

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3.5: Parallel and Perpendicular Lines

Learning Objectives

- Graph a line parallel to a given line
- Graph a line perpendicular to a given line
- Find the equation of a line parallel to a given line
- Find the equation of a line perpendicular to a given line

Warm Up 3.5.1

a. Find the reciprocal of $4, -2,$ and $1/3$
b. Find the equation of a line with slope $m = 1/4$ and containing the point $(8, -2)$

Answer

Add texts here. Do not delete this text first.

a. 1/4, -2, 3
b. $y = \frac{1}{4}x$

Parallel and Perpendicular Lines

In the last section, we dug further into how equations in different forms are related to graphs. In this section, we are going to investigate this even more by comparing the effects of different slopes. We will do this by determining when two equations result in lines might be parallel or perpendicular to each other, or neither!

First, let's recall what the slope tells us intuitively: it describes the "steepness" of a line. This means that we can look at a slope for an equation and have some expectations for what the corresponding graph might look like:

Expectations

1. Lines with positive slope should slant uphill (as our eyes sweep from left to right).
2. Lines with negative slope should slant downhill (as our eyes sweep from left to right).
3. Because any horizontal line neither slants uphill nor downhill, we expect that it should have slope equal to zero.
4. Lines with a larger positive slope should rise more quickly than lines with a smaller positive slope.
5. If two lines have negative slope, then the line having the slope with larger absolute value should fall more quickly than the other line.

Because slope controls the “steepness” of a line, it is a simple matter to see that parallel lines must have the same slope.

Property: PARALLEL LINES

Let $\boldsymbol{L}_1$ be a line having slope m_1. Let $\boldsymbol{L}_2$ be a line having slope m_2. If $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ are parallel, then

$$m_1 = m_2 \tag{3.5.1}$$

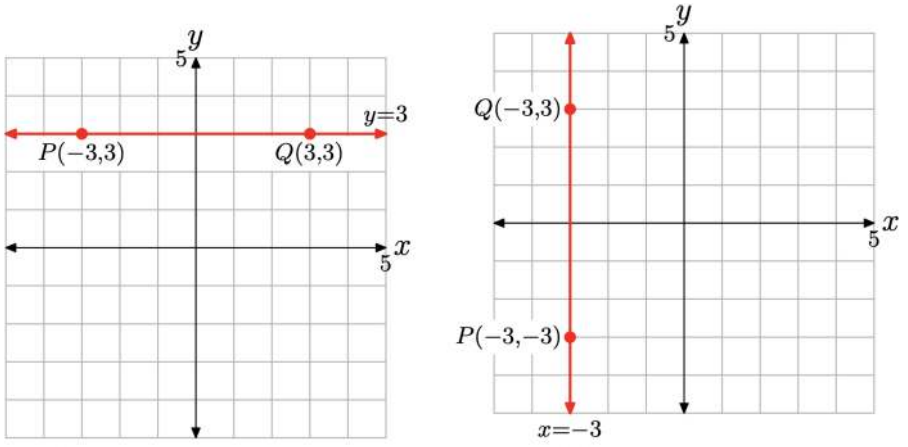
That is, any two parallel lines have the same slope.

Example 3.5.1

What is the slope of any horizontal line? What is the slope of any vertical line?

Solution

One would expect that our definition would verify that the slope of any horizontal line is zero. Select, for example, the horizontal line shown in Figure 9(a). Select the points (−3, 3) and (3, 3) on this line.



(a) Determine the slope of a horizontal line.

(b) Determine the slope of a vertical line.

Figure 3.5.9

With $(x_1, y_1) = (-3, 3)$ and $(x_2, y_2) = (3, 3)$

$$\text{Slope } = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-3}{3-(-3)} = \frac{0}{6} = 0 \tag{3.5.2}$$

Thus, the horizontal line in Figure 3.5.9(a) has slope equal to zero, exactly as expected. Further, all horizontal lines are parallel to this horizontal line and have the same slope. Therefore, all horizontal lines have slope zero.

We would surmise that the vertical line in Figure 3.5.9(b) has undefined slope (we'll explore this more fully in the exercises). In Figure 3.5.9(b), we've selected the points P(−3, −3) and Q(−3, 3) on the vertical line. With $(x_1, y_1) = P(-3, -3)$ and $(x_2, y_2) = Q(-3, -3)$,

$$\text{Slope } = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-(-3)}{-3-(-3)} = \frac{6}{0}, \text{ which is undefined.} \tag{3.5.3}$$

The slope of the vertical line in Figure 3.5.9(b) is undefined because division by zero is meaningless. Further, all vertical lines are parallel to this vertical line and have undefined slope.

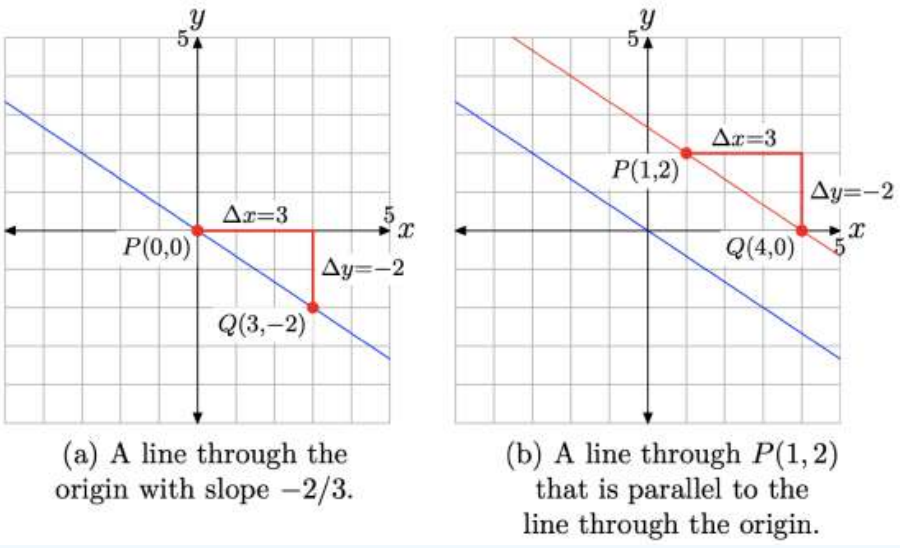
Example 3.5.2

Draw a line through the point P(1, 2) that is parallel to the line passing through the origin with slope −2/3.

Solution

We will first draw a line through the origin with slope −2/3. Plot the point P(0, 0), then move 3 units to the right and 2 units downward to the point Q(3, −2), as shown in Figure 3.5.10(a). Draw a line through the points P and Q as shown in Figure 3.5.10(a).

Next, plot the point P(1, 2) as shown in Figure 3.5.10(b). To draw a line through this point that is parallel to the line through the origin, this second line must have the same slope as the first line. Therefore, start at the point P(1, 2), as shown in Figure 3.5.10(b), then move 3 units to the right and 2 units downward to the point Q(4, 0). Draw a line through the points P and Q as shown in Figure 3.5.10(b). Note that this second line is parallel to the first.



(a) A line through the origin with slope $-2/3$.

(b) A line through $P(1,2)$ that is parallel to the line through the origin.

Figure 3.5.10

Perpendicular Lines

The relationship between the slopes of two perpendicular lines is not as straightforward as the relation between the slopes of two parallel lines. Let's begin by stating the pertinent property.

Property: PERPENDICULAR LINES

Let L_1 be a line having slope m_1. Let L_2 be a line having slope m_2. If L_1 and L_2 are perpendicular, then

$$m_1 m_2 = -1 \tag{3.5.4}$$

That is, the product of the slopes of two perpendicular lines is −1.

We can solve equation (14) for m_1 in terms of m_2.

$$m_1 = -\frac{1}{m_2} \tag{3.5.5}$$

Equation (15) tells us that the slope of the first line is the negative reciprocal of the slope of the second line.

For example, suppose that L_1 and L_2 are perpendicular lines with slopes m_1 and m_2, respectively

- If $m_2 = 2$, then $m_1 = -\frac{1}{2}$
- If $m_2 = \frac{3}{5}$, then $m_1 = -\frac{5}{3}$
- If $m_2 = -\frac{2}{3}$, then $m_1 = \frac{3}{2}$

Note that in each bulleted item, the product of the slopes is −1.

We won't provide a proof of equation (15), but we will provide some motivating evidence in the form of a graph.

Example 3.5.3

Sketch the graphs of the lines passing through the origin having slopes 2 and −1/2.

Solution

In Figure 3.5.11(a), we've plotted the point P(0, 0) at the origin, then moved 1 unit to the right and 2 units upward to the point Q(1, 2). The resulting line passes through the origin and has slope $m_1 = 2$ (alternatively, $m_1 = 2/1$).

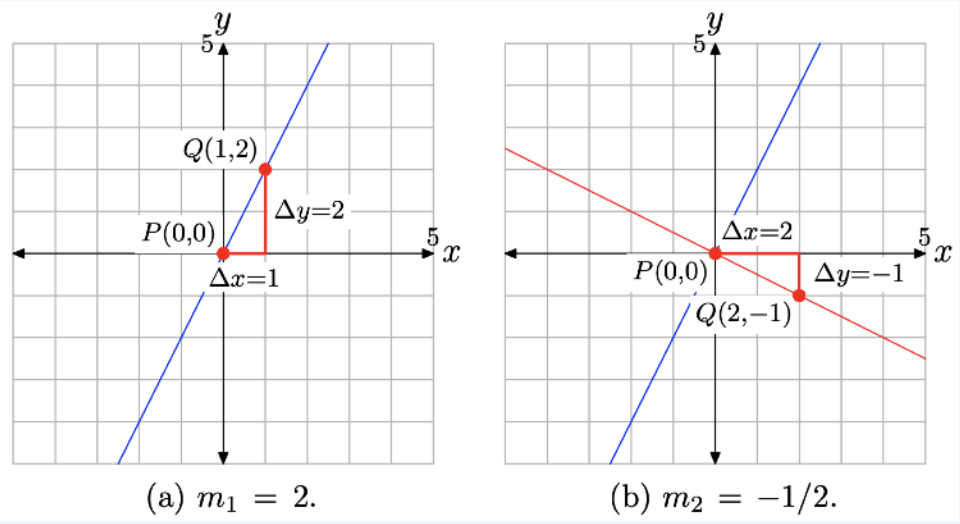
In Figure 3.5.11(b), we've again plotted the point P(0, 0) at the origin, then moved 2 units to the right and 1 unit downward to the point Q(2, −1). The resulting line passes through the origin and has slope $m_2 = -1/2$.

There are two important points that need to be made about the lines in Figure 3.5.11(b).

The two lines in Figure 3.5.11(b) are perpendicular. They meet and form a right angle of $90°$. If you have a protractor available, you might want to measure the angle between the two lines and note that the measure of the angle is $90°$.

The product of the two slopes is

$$m_1 m_2 = 2 \cdot \left(-\frac{1}{2}\right) = -1 \tag{3.5.6}$$



(a) $m_1 = 2$. (b) $m_2 = -1/2$.

Figure 3.5.11. Sketching perpendicular lines.

Let's summarize:

- Parallel lines have the same slope m
- Perpendicular lines have negative reciprocal slopes. That is, if a line has slope m, any line perpendicular to it must have slope $-1/m$.

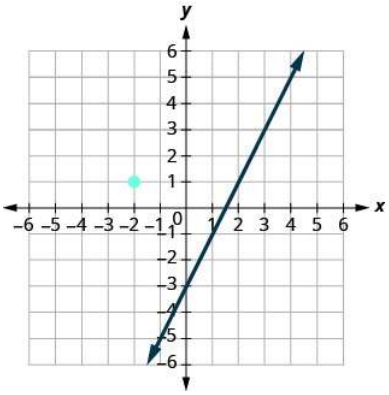
Find an Equation of a Line Parallel to a Given Line

Since there are very specific rules about the slopes of lines that are parallel or perpendicular to each other, this makes it possible to use the equation of a line to determine the equation of another line that is parallel or perpendicular to it. Let's start with when we want to find a line parallel to another line.

Suppose we need to find an equation of a line that passes through a specific point and is parallel to a given line. We can use the fact that parallel lines have the same slope. So we will have a point and the slope—just what we need to use the point-slope equation.

First, let's look at this graphically.

This graph shows $y = 2x - 3$. We want to graph a line parallel to this line and passing through the point $(-2, 1)$.



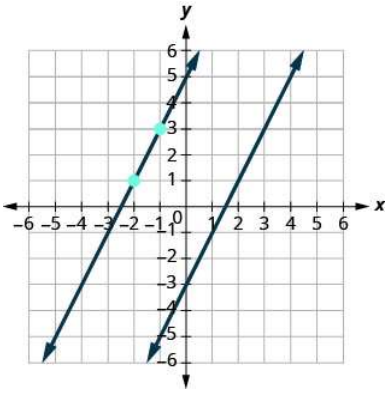
We know that parallel lines have the same slope. So the second line will have the same slope as $y = 2x - 3$. That slope is $m_{\|} = 2$. We'll use the notation $m_{\|}$ to represent the slope of a line parallel to a line with slope m. (Notice that the subscript $\|$ looks like

two parallel lines.)

The second line will pass through $(-2, 1)$ and have $m = 2$.

To graph the line, we start at$(-2, 1)$ and count out the rise and run.

With $m = 2$ (or $m = \frac{2}{1}$), we count out the rise 2 and the run 1. We draw the line, as shown in the graph.



Do the lines appear parallel? Does the second line pass through$(-2, 1)$?

We were asked to graph the line, now let's see how to do this algebraically.

We can use either the slope-intercept form or the point-slope form to find an equation of a line. Here we know one point and can find the slope. So we will use the point-slope form.

How to Find the Equation of a Line Parallel to a Given Line and a Point

? Exercise 3.5.4

Find an equation of a line parallel to $y = 2x - 3$ that contains the point $(-2, 1)$. Write the equation in slope-intercept form.

Answer

Step 1. Find the slope of the given line.	The line is in slope–intercept form, $y = 2x - 3$.	$m = 2$
Step 2. Find the slope of the parallel line.	Parallel lines have the same slope.	$m_1 = 2$
Step 3. Identify the point.	The given point is (–2, 1).	$\begin{matrix} x_1 & y_1 \\ -2, & 1 \end{matrix}$
Step 4. Substitute the values into the point–slope form, $y - y_1 = m(x - x_1)$.	Simplify.	$y - y_1 = m(x - x_1)$ $y - 1 = 2(x - (-2))$ $y - 1 = 2(x + 2)$ $y - 1 = 2x + 4$
Step 5. Write the equation in slope–intercept form.		$y = 2x + 5$

Look at graph with the parallel lines shown previously. Does this equation make sense? What is the *y*-intercept of the line? What is the slope?

Exercise 3.5.5

Find an equation of a line parallel to the line $y = 3x + 1$ that contains the point $(4, 2)$. Write the equation in slope-intercept form.

Answer

$y = 3x - 10$

Exercise 3.5.6

Find an equation of a line parallel to the line $y = 12x - 3$ that contains the point $(6, 4)$.

Write the equation in slope-intercept form.

Answer

$y = \frac{1}{2}x + 1$

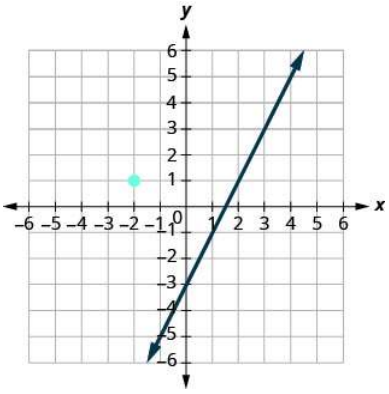
FIND AN EQUATION OF A LINE PARALLEL TO A GIVEN LINE.

1. Find the slope of the given line.
2. Find the slope of the parallel line.
3. Identify the point.
4. Substitute the values into the point-slope form: $y - y_1 = m(x - x_1)$.
5. Write the equation in slope-intercept form.

Find an Equation of a Line Perpendicular to a Given Line

Now, let's consider perpendicular lines. Suppose we need to find a line passing through a specific point and which is perpendicular to a given line. We can use the fact that perpendicular lines have slopes that are negative reciprocals. We will again use the point-slope equation, like we did with parallel lines.

This graph shows $y = 2x - 3$. Now, we want to graph a line perpendicular to this line and passing through $(-2, 1)$.



We know that perpendicular lines have slopes that are negative reciprocals.

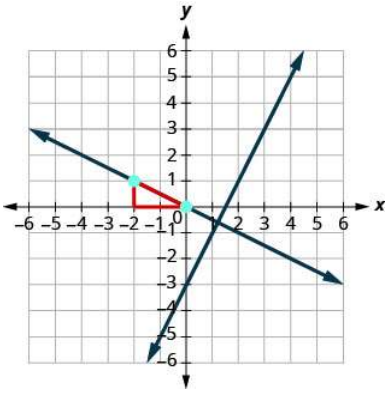
We'll use the notation $m_{\perp}$ to represent the slope of a line perpendicular to a line with slope m. (Notice that the subscript $\perp$ looks like the right angles made by two perpendicular lines.)

$$y = 2x - 3 perpendicular line$$

$$m = 2m \perp = -12$$

We now know the perpendicular line will pass through $(-2, 1)$ with $m \perp = -12$.

To graph the line, we will start at $(-2, 1)$ and count out the rise -1 and the run 2. Then we draw the line.



Do the lines appear perpendicular? Does the second line pass through$(-2, 1)$?

We were asked to graph the line, now, let's see how to do this algebraically.

We can use either the slope-intercept form or the point-slope form to find an equation of a line. In this example we know one point, and can find the slope, so we will use the point-slope form.

How to Find the Equation of a Line Perpendicular to a Given Line and a Point

Exercise 3.5.7

Find an equation of a line perpendicular to $y = 2x - 3$ that contains the point $(-2, 1)$. Write the equation in slope-intercept form.

Answer

Step 1. Find the slope of the given line.	The line is in slope–intercept form, $y = 2x - 3$.	$m = 2$
Step 2. Find the slope of the perpendicular line.	The slopes of perpendicular lines are negative reciprocals.	$m_{\perp} = -\frac{1}{2}$
Step 3. Identify the point.	The given point is (–2, 1).	$\begin{pmatrix} x_1 & y_1 \\ -2 & 1 \end{pmatrix}$
Step 4. Substitute the values into the point–slope form, $y - y_1 = m(x - x_1)$.	Simplify.	$y - y_1 = m(x - x_1)$ $y - 1 = -\frac{1}{2}(x - (-2))$ $y - 1 = -\frac{1}{2}(x + 2)$ $y - 1 = -\frac{1}{2}x - 1$
Step 5. Write the equation in slope–intercept form.		$y = -\frac{1}{2}x$

Exercise 3.5.8

Find an equation of a line perpendicular to the line $y = 3x + 1$ that contains the point $(4, 2)$. Write the equation in slope-intercept form.

Answer

$y = -\frac{1}{3}x + \frac{10}{3}$

Exercise 3.5.9

Find an equation of a line perpendicular to the line $y = 12x - 3$ that contains the point $(6, 4)$. Write the equation in slope-intercept form.

Answer

$y = -2x + 16$

FIND AN EQUATION OF A LINE PERPENDICULAR TO A GIVEN LINE.

1. Find the slope of the given line.
2. Find the slope of the perpendicular line.
3. Identify the point.
4. Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.
5. Write the equation in slope-intercept form.

Exercise 3.5.10

Find an equation of a line perpendicular to $x = 5$ that contains the point $(3, -2)$. Write the equation in slope-intercept form.

Answer

Again, since we know one point, the point-slope option seems more promising than the slope-intercept option. We need the slope to use this form, and we know the new line will be perpendicular to x=5.x=5. This line is vertical, so its perpendicular will be horizontal. This tells us the m⊥=0.m⊥=0.

Identify the point.Identify the slope of the perpendicular line.Substitute the values intoy−y1=m(x−x1).Simplify. (3,−2)m⊥y−y1y−(−2)y+2y=====0m(x−x1)0(x−3)0−2Identify the point.(3,−2)Identify the slope of the perpendicular line.Substitute the values intoy−y1=m(x−x1).Simplify.m⊥=0y−y1=m(x−x1)y−(−2)=0(x−3)y+2=0y=−2

Sketch the graph of both lines. On your graph, do the lines appear to be perpendicular?

Exercise 3.5.11

Find an equation of a line that is perpendicular to the line $x = 4$ that contains the point $(4, -5)$. Write the equation in slope-intercept form.

Answer

$y = -5$

Exercise 3.5.12

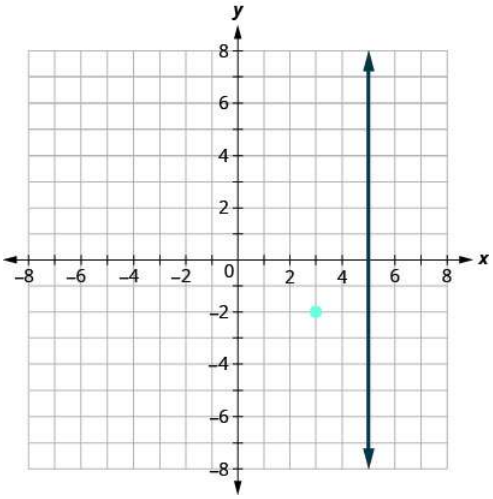
Find an equation of a line that is perpendicular to the line $x = 2$ that contains the point $(2, -1)$. Write the equation in slope-intercept form.

Answer

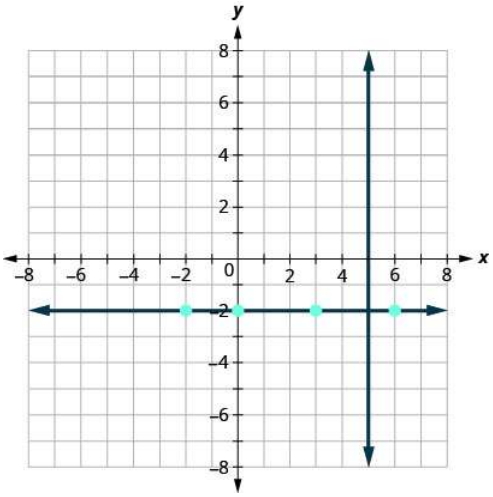
$y = -1$

In Example, we used the point-slope form to find the equation. We could have looked at this in a different way.

We want to find a line that is perpendicular to $x = 5$ that contains the point $(3, -2)$. This graph shows us the line $x = 5$ and the point $(3, -2)$.



We know every line perpendicular to a vertical line is horizontal, so we will sketch the horizontal line through $(3, -2)$.



Do the lines appear perpendicular?

If we look at a few points on this horizontal line, we notice they all have *y*-coordinates of -2. So, the equation of the line perpendicular to the vertical line $x = 5$ is $y = -2$.

? Exercise 3.5.13

Find an equation of a line that is perpendicular to $y = -3$ that contains the point $(-3, 5)$. Write the equation in slope-intercept form.

Answer

The line $y = -3$ is a horizontal line. Any line perpendicular to it must be vertical, in the form $x = a$. Since the perpendicular line is vertical and passes through $(-3, 5)$, every point on it has an *x*-coordinate of -3. The equation of the perpendicular line is $x = -3$.

You may want to sketch the lines. Do they appear perpendicular?

Exercise 3.5.14

Find an equation of a line that is perpendicular to the line $y = 1$ that contains the point $(-5, 1)$. Write the equation in slope-intercept form.

Answer

$x = -5$

Exercise 3.5.15

Find an equation of a line that is perpendicular to the line $y = -5$ that contains the point $(-4, -5)$. Write the equation in slope-intercept form.

Answer

$x = -4$

Key Concepts

- **Parallel lines** have the same slope
- **Perpendicular lines** have negative reciprocal slopes
- **How to find an equation of a line parallel to a given line**
  1. Find the slope of the given line.
  2. Find the slope of the parallel line.
  3. Identify the point.
  4. Substitute the values into the point-slope form: $y - y_1 = m(x - x_1)$.
  5. Write the equation in slope-intercept form.
- **How to find an equation of a line perpendicular to a given line**
  1. Find the slope of the given line.
  2. Find the slope of the perpendicular line.
  3. Identify the point.
  4. Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.
  5. Write the equation in slope-intercept form.

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3.5E: Exercises

Practice Makes Perfect

Find an Equation of a Parallel Line

In the following exercises, find an equation of a line parallel to the given line and contains the given point. Write the equation in slope–intercept form.

1. line $y = -3x + 6$, point $(1, -5)$
2. line $2x + 5y = -10$, point $(10, 4)$
3. line $x = 4$, point $(-2, -1)$
4. line $y = -5$, point $(-4, 3)$

Answer

1. $y = -3x - 2$
2. $y = -\frac{2}{5}x + 8$
3. $x = -2$
4. $y = 3$

Find an Equation of a Perpendicular Line

In the following exercises, find an equation of a line perpendicular to the given line and contains the given point. Write the equation in slope–intercept form.

5. line $y = -\frac{4}{5}x + 2$, point $(8, 9)$
6. line $2x - 3y = 9$, point $(-4, 0)$
7. line $y = 3$, point $(-1, -3)$
8. line $x = -5$ point $(2, 1)$

Answer

5. $y = \frac{5}{4}x - 1$
6. $y = -\frac{3}{2}x - 6$
7. $x = -1$
8. $y = 1$

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CHAPTER OVERVIEW

4: Quadratic and Polynomial Functions

- 4.1: Operations with Polynomials
  - 4.1E: Exercises
- 4.2: Graphs of Polynomials
  - 4.2E: Exercises
- 4.3: Use the Complex Number System
  - 4.3E: Exercises
- 4.4: Factoring
  - 4.4E: Exercises
- 4.5: Solve Quadratic Equations Using the Quadratic Formula
  - 4.5E: Exercises
- 4.6: Graph Quadratic Functions Using Properties
  - 4.6E: Exercises

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4.1: Operations with Polynomials

Learning Objectives

By the end of this section, you will be able to:

- Multiply monomials
- Multiply a polynomial by a monomial
- Multiply a binomial by a binomial
- Multiply a polynomial by a polynomial
- Multiply special products
- Multiply polynomial functions

Warm Up

1. Distribute: $2(x+3)$.
2. Simplify: a. 9^2 b. $(-9)^2$ c. -9^2.
3. Evaluate: $2x^2-5x+3$ for $x=-2$.

Answer

1. $2x+6$
2. a. 81 b. 81 c. -81
3. 21

Definition of Polynomials

We have learned that a *term* is a constant or the product of a constant and one or more variables. A **monomial** is an algebraic expression with one term. When it is of the form ax^m, where a is a constant and m is a whole number, it is called a monomial in one variable. Some examples of monomial in one variable are. Monomials can also have more than one variable such as and $-4a^2b^3c^2$.

Definition: MONOMIAL

A *monomial* is an algebraic expression with one term. A monomial in one variable is a term of the form ax^m, where a is a constant and m is a whole number.

A monomial, or two or more monomials combined by addition or subtraction, is a **polynomial**. Some polynomials have special names, based on the number of terms. A monomial is a polynomial with exactly one term. A binomial has exactly two terms, and a **trinomial** has exactly three terms. There are no special names for polynomials with more than three terms.

Definition: POLYNOMIALS

- *polynomial*—A monomial, or two or more algebraic terms combined by addition or subtraction is a polynomial.
- *monomial*—A polynomial with exactly one term is called a monomial.
- *binomial*—A polynomial with exactly two terms is called a binomial.
- *trinomial*—A polynomial with exactly three terms is called a trinomial.

Here are some examples of polynomials.

Polynomial	$y+1$	$4a^2-7ab+2b^2$	$4x^4+x^3+8x^2-9x+1$	
Monomial	14	$8y^2$	$-9x^3y^5$	$-13a^3b^2c$
Binomial	$a+7b^2$	$4x^2-y^2$	y^2-16	$3p^3q-9p^2q$
Trinomial	$x^2-7x+12$	$9m^2+2mn-8n^2$	$6k^4-k^3+8k$	z^4+3z^2-1

Notice that every monomial, binomial, and trinomial is also a polynomial. They are just special members of the "family" of polynomials and so they have special names. We use the words *monomial*, *binomial*, and *trinomial* when referring to these special polynomials and just call all the rest *polynomials*.

The **degree of a polynomial** and the degree of its terms are determined by the exponents of the variable. A monomial that has no variable, just a constant, is a special case. The **degree of a constant** is 0.

Definition: DEGREE OF A POLYNOMIAL

- The *degree of a term* is the sum of the exponents of its variables.
- The *degree of a constant* is 0.
- The *degree of a polynomial* is the highest degree of all its terms.

Let's see how this works by looking at several polynomials. We'll take it step by step, starting with monomials, and then progressing to polynomials with more terms. Let's start by looking at a monomial. The monomial $8ab^2$ has two variables a and b. To find the degree we need to find the sum of the exponents. The variable a doesn't have an exponent written, but remember that means the exponent is 1. The exponent of b is 2. The sum of the exponents, 1+2,1+2, is 3 so the degree is 3.



Here are some additional examples.

Monomials	14	$8ab^2$	$-9x^3y^5$	$-13a$
Degree	0	3	8	1
Binomial	$h+7$	$7b^2-3b$	x^2y^2-25	$4n^3-8n^2$
Degree of each term	1 0	2 1	4 0	3 2
Degree of polynomial	1	2	4	3
Trinomial	$x^2-12x+27$	$9a^2+6ab+b^2$	$6m^4-m^3n^2+8mn^5$	z^4+3z^2-1
Degree of each term	2 1 0	2 2 2	4 5 6	4 2 0
Degree of polynomial	2	2	6	4
Polynomial	$y-1$	$3y^2-2y-5$	$4x^4+x^3+8x^2-9x+1$	
Degree of each term	1 0	2 1 0	4 3 2 1 0	
Degree of polynomial	1	2	4	

Working with polynomials is easier when you list the terms in descending order of degrees. When a polynomial is written this way, it is said to be in **standard form of a polynomial**. Get in the habit of writing the term with the highest degree first.

Add and Subtract Polynomials

We have learned how to simplify expressions by combining like terms. Remember, like terms must have the same variables with the same exponent. Since monomials are terms, adding and subtracting monomials is the same as combining like terms. If the monomials are like terms, we just combine them by adding or subtracting the coefficients.

We can think of adding and subtracting polynomials as just adding and subtracting a series of monomials. Look for the like terms—those with the same variables and the same exponent. The Commutative Property allows us to rearrange the terms to put like terms together.

Example 4.1.1

Find the sum: $(7y^2-2y+9)+(4y^2-8y-7)$.

Answer

Identify like terms.	$(\underline{\underline{7y^2}}-\underline{2y}+9)+(\underline{\underline{4y^2}}-\underline{8y}-7)$
Rewrite without the parentheses, rearranging to get the like terms together.	$\underline{\underline{7y^2+4y^2}}-\underline{2y-8y}+9-7$
Combine like terms.	$11y^2-10y+2$

Example 4.1.2

Find the sum: $(7x^2-4x+5)+(x^2-7x+3)$

Answer

$8x^2-11x+8$

Example 4.1.3

Find the sum: $(14y^2+6y-4)+(3y^2+8y+5)$

Answer

$17y^2+14y+1$

Be careful with the signs as you distribute while subtracting the polynomials in the next example.

Example 4.1.4

Find the difference: $(9w^2-7w+5)-(2w^2-4)$

Answer

	$(9w^2-7w+5)-(2w^2-4)$
Distribute and identify like terms.	$\underline{\underline{9w^2}}-\underline{7w}+5-\underline{\underline{2w^2}}+4$
Rearrange the terms.	$\underline{\underline{9w^2-2w^2}}-\underline{7w}+5+4$
Combine like terms.	$7w^2-7w+9$

Example 4.1.5

Find the difference: $(8x^2+3x-19)-(7x^2-14)$

Answer

x^2+3x-5

Example 4.1.6

Find the difference: $(9b^2-5b-4)-(3b^2-5b-7)$

Answer

$6b^2+3$

Example 4.1.7

Subtract $(p^2+10pq-2q^2)$ from (p^2+q^2).

Answer

	$(p^2+q^2)\ -\ (p^2+10pq-2q^2)$
Distribute and identify like terms.	$\underline{\underline{p^2}}+\underline{q^2}-\underline{\underline{p^2}}-10pq+\underline{2q^2}$
Rearrange the terms, putting like terms together.	$\underline{\underline{p^2-p^2}}-10pq+\underline{q^2+2q^2}$
Combine like terms.	$-10pq+3q^2$

Example 4.1.8

Subtract $(a^2+5ab-6b^2)$ from (a^2+b^2)

Answer

$-5ab+7b^2$

Example 4.1.9

Subtract $(m^2-7mn-3n^2)$ from (m^2+n^2).

Answer

7mn+4n^2

Multiply Monomials

We are ready to perform operations on polynomials. Since monomials are algebraic expressions, we can use the properties of exponents to multiply monomials.

Example 4.1.10

Multiply:

a. $(3x^2)(-4x^3)$
b. $\left(\frac{5}{6}x^3y\right)(12xy^2)$.

Answer a

	$(3x^2)(-4x^3)$
Use the Commutative Property to rearrange the terms.	$3\cdot(-4)\cdot x^2\cdot x^3$
	$-12x^5$

Answer b

	$\left(\frac{5}{6}x^3y\right)(12xy^2)$
Use the Commutative Property to rearrange the terms.	$\frac{5}{6}\cdot 12\cdot x^3\cdot x\cdot y\cdot y^2$
Multiply.	$10x^4y^3$

Example 4.1.11

Multiply:

a. $(5y^7)(-7y^4)$
b. $(25a^4b^3)(15ab^3)$

Answer a

$-35y^{11}$

Answer b

$375a^5b^6$

Example 4.1.12

Multiply:

a. $(-6b^4)(-9b^5)$
b. $(23r^5s)(12r^6s^7)$.

Answer a

$54b^9$

Answer b

$276r^{11}s^8$

Multiply a Polynomial by a Monomial

Multiplying a polynomial by a monomial is really just applying the Distributive Property.

Example 4.1.13

Multiply:

a. $-2y(4y^2+3y-5)$
b. $3x^3y(x^2-8xy+y^2)$.

Answer a

	$-2y(4y^2+3y-5)$
Distribute.	$-2y \cdot 4y^2 + (-2y) \cdot 3y - (-2y) \cdot 5$
Multiply.	$-8y^3 - 6y^2 + 10y$

Answer b

	$3x^3y(x^2-8xy+y^2)$
Distribute.	$3x^3y \cdot x^2 + (3x^3y) \cdot (-8xy) + (3x^3y) \cdot y^2$
Multiply.	$3x^5y - 24x^4y^2 + 3x^3y^3$

Example 4.1.14

Multiply:

a. $-3y(5y^2+8y^7)$
b. $4x^2y^2(3x^2-5xy+3y^2)$

Answer a

$-15y^3 - 24y^8$

Answer b

$12x^4y^2 - 20x^3y^3 + 12x^2y^4$

Example 4.1.15

Multiply:

a. $4x^2(2x^2 - 3x + 5)$
b. $-6a^3b(3a^2 - 2ab + 6b^2)$

Answer a

$8x^4 - 12x^3 + 20x^2$

Answer b

$-18a^5b + 12a^4b^2 - 36a^3b^3$

Multiply a Binomial by a Binomial

Just like there are different ways to represent multiplication of numbers, there are several methods that can be used to multiply a binomial times a binomial. We will start by using the Distributive Property.

Example 4.1.16

Multiply:

a. $(y+5)(y+8)$
b. $(4y+3)(2y-5)$.

Answer

ⓐ

	$(y+5)(y+8)$
Distribute $(y+8)$.	$y(y+8)+5(y+8)$
Distribute again.	$y^2+8y+5y+40$
Combine like terms.	$y^2+13y+40$

ⓑ

	$(4y+3)(2y-5)$
Distribute.	$4y(2y-5)+3(2y-5)$
Distribute again.	$8y^2-20y+6y-15$
Combine like terms.	$8y^2-14y-15$

Example 4.1.17

Multiply:

a. $(x+8)(x+9)$
b. $(3c+4)(5c-2)$.

Answer a

$x^2+17x+72$

Answer b

$15c^2+14c-8$

Example 4.1.18

Multiply:

a. $(5x+9)(4x+3)$
b. $(5y+2)(6y-3)$.

Answer a

$20x^2+51x+27$

Answer b

$30y^2-3y-6$

If you multiply binomials often enough you may notice a pattern. Notice that the first term in the result is the product of the *first* terms in each binomial. The second and third terms are the product of multiplying the two *outer* terms and then the two *inner* terms. And the last term results from multiplying the two *last* terms,

We abbreviate "First, Outer, Inner, Last" as FOIL. The letters stand for 'First, Outer, Inner, Last'. We use this as another method of multiplying binomials. The word FOIL is easy to remember and ensures we find all **four** products.

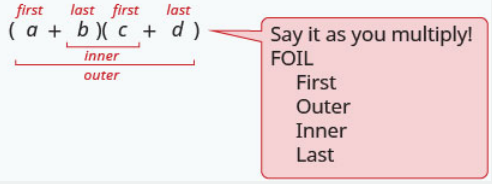
Let's multiply $(x+3)(x+7)$ using both methods.

Distributive Property	FOIL
$(x+3)(x+7)$	$(x+3)(x+7)$
$x(x+7)+3(x+7)$	
$x^2+7x+3x+21$ F O I L	$x^2+7x+3x+21$ F O I L
$x^2+10x+21$	$x^2+10x+21$

We summarize the steps of the FOIL method below. The FOIL method only applies to multiplying binomials, not other polynomials!

USE THE FOIL METHOD TO MULTIPLY TWO BINOMIALS.

Step 1. Multiply the *First* terms.
Step 2. Multiply the *Outer* terms.
Step 3. Multiply the *Inner* terms.
Step 4. Multiply the *Last* terms.
Step 5. Combine like terms, when possible.



When you multiply by the FOIL method, drawing the lines will help your brain focus on the pattern and make it easier to apply.

Now we will do an example where we use the FOIL pattern to multiply two binomials.

Example 4.1.19

Multiply:

a. $(y-7)(y+4)$
b. $(4x+3)(2x-5)$.

Answer

a.

$(y-7)(y+4)$

Step 1. Multiply the *First* terms. $y^2 + \_ + \_ + \_$ (F O I L)

Step 2. Multiply the *Outer* terms. $y^2 + 4y + \_ + \_$ (F O I L)

Step 3. Multiply the *Inner* terms. $y^2 + 4y - 7y + \_$ (F O I L)

Step 4. Multiply the *Last* terms. $y^2 + 4y - 7y - 28$ (F O I L)

Step 5. Combine like terms. $y^2 - 3y - 28$

b.

$(4x-3)(2x-5)$

Step 1. Multiply the *First* terms, $4x \cdot 2x$. $8x^2 + \_ + \_ + \_$ (F O I L)

Step 2. Multiply the *Outer* terms, $4x \cdot (-5)$. $8x^2 - 20x + \_ + \_$ (F O I L)

Step 3. Multiply the *Inner* terms, $3 \cdot 2x$. $8x^2 - 2x + 6x + \_$ (F O I L)

Step 4. Multiply the *Last* terms, $3 \cdot (-5)$. $8x^2 - 20x + 6x - 15$ (F O I L)

Step 5. Combine like terms. $8x^2 - 14x - 15$

Exercise 4.1.20

Multiply:

a. $(x-7)(x+5)$
b. $(3x+7)(5x-2)$.

Answer

a. $x^2 - 2x - 35$
b. $15x^2 + 29x - 14$

Exercise 4.1.21

Multiply:

a. $(b-3)(b+6)$
b. $(4y+5)(4y-10)$.

Answer

a. $b^2+3b-18$
b. $16y^2-20y-50$

The final products in the last example were trinomials because we could combine the two middle terms. This is not always the case.

Example 4.1.22

Multiply:

a. $(n^2+4)(n-1)$
b. $(3pq+5)(6pq-11)$.

Answer

a.

	$(n^2+4)(n-1)$
	$(n^2+4)(n-1)$
Step 1. Multiply the *First* terms.	$n^3+\_\_+\_\_+\_\_$ (F O I L)
Step 2. Multiply the *Outer* terms.	$n^3-n^2+\_\_+\_\_$ (F O I L)
Step 3. Multiply the *Inner* terms.	$n^3-n^2+4n+\_\_$ (F O I L)
Step 4. Multiply the *Last* terms.	n^3-n^2+4n-4 (F O I L)
Step 5. Combine like terms—there are none.	n^3-n^2+4n-4

b.

	$(3pq+5)(6pq-11)$
	$(3pq+5)(6pq-11)$
Step 1. Multiply the *First* terms.	$18p^2q^2+\_\_+\_\_+\_\_$ (F O I L)
Step 2. Multiply the *Outer* terms.	$18p^2q^2-33pq+\_\_+\_\_$ (F O I L)
Step 3. Multiply the *Inner* terms.	$18p^2q^2-33pq+30pq+\_\_$ (F O I L)
Step 4. Multiply the *Last* terms.	$18p^2q^2-33pq+30pq-55$ (F O I L)
Step 5. Combine like terms.	$18p^2q^2-3pq-55$

Example 4.1.23

Multiply:

a. $(x^2+6)(x-8)$
b. $(2ab+5)(4ab-4)$.

Answer

a. $x^3-8x^2+6x-48$
b. $8a^2b^2+12ab-20$

Example 4.1.24

Multiply:

a. $(y^2+7)(y-9)$
b. $(2xy+3)(4xy-5)$.

Answer

a. $y^3-9y^2+7y-63$
b. $8x^2y^2+2xy-15$

The FOIL method is usually the quickest method for multiplying two binomials, but it *only* works for binomials. You can use the Distributive Property to find the product of any two polynomials. Another method that works for all polynomials is the Vertical Method. It is very much like the method you use to multiply whole numbers. Look carefully at this example of multiplying two-digit numbers.

23 ×46		
138	partial product	Start by multiplying 23 by 6 to get 138.
92	partial product	Next, multiply 23 by 4, lining up the partial product in the correct columns.
1058	product	Last you add the partial products.

Now we'll apply this same method to multiply two binomials.

Example 4.1.25

Multiply using the Vertical Method: $(3y-1)(2y-6)$.

Answer

It does not matter which binomial goes on the top.

$$\begin{array}{r} 3y-1 \\ \times\ 2y-6 \\ \hline \end{array}$$

Multiply $3y-1$ by -6.	$-18y+6$	partial product
Multiply $3y-1$ by $2y$.	$6y^2-2y$	partial product
Add like terms.	$6y^2-20y+6$	

Notice the partial products are the same as the terms in the FOIL method.

$$\begin{array}{r} 3y-1 \\ \times\ 2y-6 \\ \hline -18y+6 \\ 6y^2-2y \\ \hline 6y^2-20x+6 \end{array}$$

$(3y-1)(2y-6)$
$6y^2-2y-18y+6$
$6y^2-20y+6$

Example 4.1.26

Multiply using the Vertical Method: $(5m-7)(3m-6)$.

Answer

$15m^2-51m+42$

Example 4.1.27

Multiply using the Vertical Method: $(6b-5)(7b-3)$.

Answer

$42b^2-53b+15$

We have now used three methods for multiplying binomials. Be sure to practice each method, and try to decide which one you prefer. The methods are listed here all together, to help you remember them.

MULTIPLYING TWO BINOMIALS

To multiply binomials, use the:

- Distributive Property
- FOIL Method
- Vertical Method

Multiply a Polynomial by a Polynomial

We have multiplied monomials by monomials, monomials by polynomials, and binomials by binomials. Now we're ready to multiply a polynomial by a polynomial. Remember, FOIL will not work in this case, but we can use either the Distributive Property or the Vertical Method.

Example 4.1.28

Multiply $(b+3)(2b^2-5b+8)$ using ⓐ the Distributive Property and ⓑ the Vertical Method.

Answer

a.

	$(b+3)(2b^2-5b+8)$
Distribute.	$b(2b^2-5b+8)+3(2b^2-5b+8)$
Multiply.	$2b^3-5b^2+8b+6b^2-15b+24$
Combine like terms.	$2b^3+b^2-7b+24$

b. It is easier to put the polynomial with fewer terms on the bottom because we get fewer partial products this way.

Multiply $(2b^2-5b+8)$ by 3. Multiply $(2b^2-5b+8)$ by b.	$\begin{array}{r} 2b^2-5b+8 \\ \times \quad b+3 \\ \hline 6b^2-15b+24 \end{array}$
Add like terms.	$\begin{array}{r} 2b^3-5b^2+8b \\ 2b^3+b^2-7b+24 \end{array}$

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? Example 4.1.29

Multiply $(y-3)(y^2-5y+2)$ using ⓐ the Distributive Property and ⓑ the Vertical Method.

Answer

a. $y^3-8y^2+17y-6$
b. $y^3-8y^2+17y-6$

? Example 4.1.30

Multiply $(x+4)(2x^2-3x+5)$ using a) the Distributive Property and b) The Vertical Method.

Answer

a. and b. $2x^3+5x^2-7x+20$

We have now seen two methods you can use to multiply a polynomial by a polynomial. After you practice each method, you'll probably find you prefer one way over the other. We list both methods are listed here, for easy reference.

MULTIPLYING A POLYNOMIAL BY A POLYNOMIAL

To multiply a trinomial by a binomial, use the:

- Distributive Property
- Vertical Method

Multiply Special Products

Mathematicians like to look for patterns that will make their work easier. A good example of this is squaring binomials. While you can always get the product by writing the binomial twice and multiplying them, there is less work to do if you learn to use a pattern. Let's start by looking at three examples and look for a pattern.

Look at these results. Do you see any patterns?

$(x+9)^2$	$(y-7)^2$	$(2x+3)^2$
$(x+9)(x+9)$	$(y-7)(y-7)$	$(2x+3)(2x+3)$
$x^2+9x+9x+81$	$y^2-7y-7y+49$	$4x^2+6x+6x+9$
$x^2+18x+81$	$y^2-14y+49$	$4x^2+12x+9$

What about the number of terms? In each example we squared a binomial and the result was a trinomial.

$$(a+b)^2 = \_\_\_ + \_\_\_ + \_\_\_$$

Now look at the *first term* in each result. Where did it come from?

The first term is the product of the first terms of each binomial. Since the binomials are identical, it is just the square of the first term!

$$(a+b)^2 = a^2 + \_\_\_ + \_\_\_$$

To get the first term of the product, square the first term.

Where did the *last term* come from? Look at the examples and find the pattern.

The last term is the product of the last terms, which is the square of the last term.

$$(a+b)^2 = \_\_\_ + \_\_\_ + b^2$$

To get the last term of the product, square the last term.

Finally, look at the *middle term*. Notice it came from adding the "outer" and the "inner" terms—which are both the same! So the middle term is double the product of the two terms of the binomial.

$$(a+b)^2 = \_\_\_ + 2ab + \_\_\_$$

$$(a-b)^2 = \_\_\_ - 2ab + \_\_\_$$

To get the middle term of the product, multiply the terms and double their product.

Putting it all together:

BINOMIAL SQUARES PATTERN

If a and b are real numbers,

$(a+b)^2 = a^2 + 2ab + b^2$

$\underbrace{(a+b)^2}_{\text{(binomial)}^2} = \underbrace{a^2}_{\text{(first term)}^2} + \underbrace{2ab}_{\text{2(product of terms)}} + \underbrace{b^2}_{\text{(last term)}^2}$

$(a-b)^2 = a^2 - 2ab + b^2$

$\underbrace{(a-b)^2}_{\text{(binomial)}^2} = \underbrace{a^2}_{\text{(first term)}^2} - \underbrace{2ab}_{\text{2(product of terms)}} + \underbrace{b^2}_{\text{(last term)}^2}$

To square a binomial, square the first term, square the last term, double their product.

Example 4.1.31

Multiply: a. $(x+5)^2$ b. $(2x-3y)^2$.

Answer

a.

	$(x+5)^2$
Square the first term.	$x^2 + \_\_\_ + \_\_\_$
Square the last term.	$x^2 + \_\_\_ + 5^2$
Double their product.	$x^2 + 2 \cdot x \cdot 5 + 5^2$
Simplify.	$x^2 + 10x + 25$

b.

	$(2x-3y)^2$
Use the pattern.	$(2x)^2 - 2 \cdot 2x \cdot 3y + (3y)^2$
Simplify.	$4x^2 - 12xy + 9y^2$

Example 4.1.32

Multiply: a. $(x+9)^2$ b. $(2c-d)^2$.

Answer

a. $x^2 + 18x + 81$
b. $4c^2 - 4cd + d^2$

Example 4.1.33

Multiply: a. $(y+11)^2$ b. $(4x-5y)^2$.

Answer

a. $y^2+22y+121$
b. $16x^2-40xy+25y^2$

We just saw a pattern for squaring binomials that we can use to make multiplying some binomials easier. Similarly, there is a pattern for another product of binomials. But before we get to it, we need to introduce some vocabulary.

A pair of binomials that each have the same first term and the same last term, but one is a sum and one is a difference is called a **conjugate pair** and is of the form $(a-b)$, $(a+b)$.

Definition: CONJUGATE PAIR

A **conjugate pair** is two binomials of the form

$$(a-b),(a+b).$$

The pair of binomials each have the same first term and the same last term, but one binomial is a sum and the other is a difference.

There is a nice pattern for finding the product of conjugates. You could, of course, simply FOIL to get the product, but using the pattern makes your work easier. Let's look for the pattern by using FOIL to multiply some conjugate pairs.

$(x+9)(x-9)$ $\quad$ $(y-8)(y+8)$ $\quad$ $(2x-5)(2x+5)$

$x^2-9x+9x-81$ $\quad$ $y^2+8y-8y-64$ $\quad$ $4x^2+10x-10x-25$

x^2-81 $\quad$ y^2-64 $\quad$ $4x^2-25$

What do you observe about the products?

The product of the two binomials is also a binomial! Most of the products resulting from FOIL have been trinomials.

Each *first term* is the product of the first terms of the binomials, and since they are identical it is the square of the first term.

$$(a+b)(a-b)=a^2-\_\_\_$$

To get the first term, square the first term.

The *last term* came from multiplying the last terms, the square of the last term.

$$(a+b)(a-b)=a^2-b^2$$

To get the last term, square the last term.

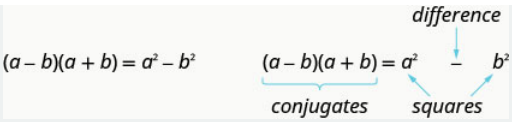
Why is there no middle term? Notice the two middle terms you get from FOIL combine to 0 in every case, the result of one addition and one subtraction.

The product of conjugates is always of the form a^2-b^2. This is called a **difference of squares**.

This leads to the pattern:

PRODUCT OF CONJUGATES PATTERN

If *a* and *b* are real numbers,



The product is called a difference of squares.

To multiply conjugates, square the first term, square the last term, write it as a difference of squares.

Example 4.1.34

Multiply using the product of conjugates pattern: a. $(2x+5)(2x-5)$ b. $(5m-9n)(5m+9n)$.

Answer

a.

Are the binomials conjugates?	$(2x+5)(2x-5)$
It is the product of conjugates.	$(a+b)(a-b)$
Square the first term, 2x.2x.	$(2x)^2 - \_\_$
Square the last term, 5.5.	$(2x)^2 - 5^2$
Simplify. The product is a difference of squares.	$4x^2 - 25$

b.

	$(5m-9n)(5m+9n)$
This fits the pattern.	$(a-b)(a+b)$
Use the pattern.	$(5m)^2 - (9n)^2$
Simplify.	$25m^2 - 81n^2$

Example 4.1.35

Multiply: a. $(6x+5)(6x-5)$ b. $(4p-7q)(4p+7q)$.

Answer

a. $36x^2-25$
b. $16p^2-49q^2$

Example 4.1.36

Multiply: a. $(2x+7)(2x-7)$ b.$(3x-y)(3x+y)$.

Answer

a. $4x^2-49$ b. $9x^2-y^2$

We just developed special product patterns for Binomial Squares and for the Product of Conjugates. The products look similar, so it is important to recognize when it is appropriate to use each of these patterns and to notice how they differ. Look at the two patterns together and note their similarities and differences.

COMPARING THE SPECIAL PRODUCT PATTERNS

Binomial Squares	Product of Conjugates
$(a+b)^2 = a^2 + 2ab + b^2$	$(a-b)(a+b) = a^2 - b^2$
$(a-b)^2 = a^2 - 2ab + b^2$	
• Squaring a binomial	• Multiplying conjugates
• Product is a **trinomial**	• Product is a **binomial.**
• Inner and outer terms with FOIL are **the same.**	• Inner and outer terms with FOIL are **opposites.**
• Middle term is **double the product** of the terms	• There is **no** middle term.

Example 4.1.37

Choose the appropriate pattern and use it to find the product:

a. $(2x-3)(2x+3)$ b. $(8x-5)^2$ c. $(6m+7)^2$ d. $(5x-6)(6x+5)$.

Answer

a. $(2x-3)(2x+3)$

These are conjugates. They have the same first numbers, and the same last numbers, and one binomial is a sum and the other is a difference. It fits the Product of Conjugates pattern.

	$\underset{}{\overset{a\ -\ b}{(2x-3)}}\underset{}{\overset{a\ +\ b}{(2x+3)}}$
Use the pattern.	$\overset{a^2\ -\ b^2}{(2x)^2 - 3^2}$
Simplify.	$4x^2 - 9$

b. $(8x-5)^2$

We are asked to square a binomial. It fits the binomial squares pattern.

	$\overset{a\ -\ b}{(8x-5)^2}$
Use the pattern.	$\overset{a^2\ -\ 2ab\ +\ b^2}{(8x)^2 - 2 \cdot 8x \cdot 5 + 5^2}$
Simplify.	$64x^2 - 80x + 25$

c. $(6m+7)^2$

Again, we will square a binomial so we use the binomial squares pattern.

	$\overset{a\ +\ b}{(6m+7)^2}$
Use the pattern.	$\overset{a^2\ +\ 2ab\ +\ b^2}{(6m)^2 + 2 \cdot 6m \cdot 7 + 7^2}$
Simplify.	$36m^2 + 84m + 49$

d. $(5x-6)(6x+5)$

This product does not fit the patterns, so we will use FOIL.

	$(5x-6)(6x+5)$
Use FOIL.	$30x^2+25x-36x-30$
Simplify.	$30x^2-11x-30$

? Example 4.1.38

Choose the appropriate pattern and use it to find the product:

a. $(9b-2)(2b+9)$ b. $(9p-4)^2$ c. $(7y+1)^2$ d. $(4r-3)(4r+3)$.

Answer

a. FOIL; $18b^2+77b-18$
b. Binomial Squares; $81p^2-72p+16$
c. Binomial Squares; $49y^2+14y+1$
d. Product of Conjugates; $16r^2-9$

? Example 4.1.39

Choose the appropriate pattern and use it to find the product:

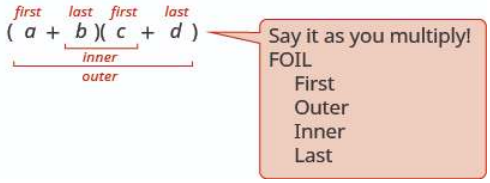
a. $(6x+7)^2$ b. $(3x-4)(3x+4)$ c. $(2x-5)(5x-2)$ d. $(6n-1)^2$.

Answer

a. Binomial Squares; $36x^2+84x+49$ b. Product of Conjugates; $9x^2-16$ c. FOIL; $10x^2-29x+10$ d. Binomial Squares; $36n^2-12n+1$

Key Concepts

- **How to use the FOIL method to multiply two binomials.**
  Step 1. Multiply the *First* terms.
  Step 2. Multiply the *Outer* terms.
  Step 3. Multiply the *Inner* terms.
  Step 4. Multiply the *Last* terms.
  Step 5. Combine like terms, when possible.



- **Multiplying Two Binomials:** To multiply binomials, use the:
  - Distributive Property
  - FOIL Method
- **Multiplying a Polynomial by a Polynomial:** To multiply a trinomial by a binomial, use the:
  - Distributive Property
  - Vertical Method
- **Binomial Squares Pattern**

$(a+b)^2=a^2+2ab+b^2$

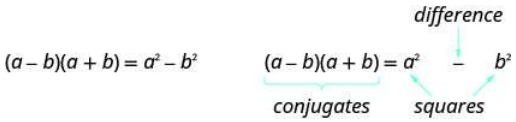
$\underbrace{(a+b)^2}_{\text{(binomial)}^2} = \underbrace{a^2}_{\text{(first term)}^2} + \underbrace{2ab}_{\text{2(product of terms)}} + \underbrace{b^2}_{\text{(last term)}^2}$

$(a-b)^2=a^2-2ab+b^2$

$\underbrace{(a-b)^2}_{\text{(binomial)}^2} = \underbrace{a^2}_{\text{(first term)}^2} - \underbrace{2ab}_{\text{2(product of terms)}} + \underbrace{b^2}_{\text{(last term)}^2}$

If a and b are real numbers,

- **Product of Conjugates Pattern**
  If a, b are real numbers



The product is called a difference of squares.

To multiply conjugates, square the first term, square the last term, write it as a difference of squares.

- **Comparing the Special Product Patterns**

Binomial Squares	Product of Conjugates
$(a+b)^2 = a^2 + 2ab + b^2$	$(a-b)^2 = a^2 - 2ab + b^2$
$(a-b)(a+b) = a^2 - b^2$	
• Squaring a binomial	• Multiplying conjugates
• Product is a **trinomial**	• Product is a **binomial.**
• Inner and outer terms with FOIL are **the same.**	• Inner and outer terms with FOIL are **opposites.**
• Middle term is **double the product** of the terms	• There is **no** middle term.

- **Multiplication of Polynomial Functions:**
  - For functions $f(x)$ and $g(x)$,

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

Glossary

conjugate pair

A conjugate pair is two binomials of the form $(a-b)$ and $(a+b)$. The pair of binomials each have the same first term and the same last term, but one binomial is a sum and the other is a difference.

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4.1E: Exercises

Practice Makes Perfect

Add and Subtract Polynomials

In the following exercises, add or subtract the polynomials.

1. $4p+11p$

Answer

$15p$

2. $-8y^3-5y^3$

Answer

$-13y^3$

3. $(4a^2+9a-11)+(6a^2-5a+10)$

Answer

$10a^2+4a-1$

4. $(8m^2+12m-5)-(2m^2-7m-1)$

Answer

$6m^2+19m-4$

5. $(y^2-3y+12)+(5y^2-9)$

Answer

$6y^2-3y+3$

6. $(5u^2+8u)-(4u-7)$

Answer

$5u^2+4u+7$

7. Find the sum of $8q^3-27$ and q^2+6q-2 .

Answer

$2x^2-2x+23$

8. Find the difference of x^2+6x+8 and $x^2-8x+15$.

Answer

$14x-7$

In the following exercises, simplify.

9. $17mn^2-(-9mn^2)+3mn^2$

Answer

$29mn^2$

10. $18a-7b-21a$

Answer

$-7b-3a$

11. $2pq^2-5p-3q^2$

Answer

$2pq^2-5p-3q^2$

12. $(6a^2+7)+(2a^2-5a-9)$

Answer

$8a^2-5a-2$

13. $(3p^2-4p-9)+(5p^2+14)$

Answer

$8p^2-4p+5$

14. $(7m^2-2m-5)-(4m^2+m-8)$

Answer

$-3m-3m+3$

15. $(7b^2-4b+3)-(8b^2-5b-7)$

Answer

$-b^2-b-10$

16. Subtract $(8y^2-y+9)$ from $(11y^2-9y-5)$

Answer

$3y^2-8y-14$

17. Find the difference of $(z^2-4z-12)$ and $(3z^2+2z-11)$

Answer

$-2z^2+6z+1$

18. $(x^3-x^2y)-(4xy^2-y^3)+(3x^2y-xy^2)$

Answer

$y^3 - 5xy^2 + 2x^2y + x^3$

19. $(x^3 - 2x^2y) - (xy^2 - 3y^3) - (x^2y - 4xy^2)$

Answer

$3y^3 + 3xy^2 - 3x^2y + x^3$

Multiply Monomials

In the following exercises, multiply the monomials.

20. $(-6p^4)(9p)$

Answer

$-54p^5$

21. $\left(\frac{1}{3}c^2\right)(30c^8)$

Answer

$10c^{10}$

22. $(8x^2y^5)(7xy^6)$

Answer

$56x^3y^{11}$

23. $\left(\frac{2}{3}m^3n^6\right)\left(\frac{1}{6}m^4n^4\right)$

Answer

$\frac{m^7n^{10}}{9}$

Multiply a Polynomial by a Monomial

In the following exercises, multiply.

24. $7(10 - x)$

Answer

$70 - 7x$

25. $a^2(a^2 - 9a - 36)$

Answer

$a^4 - 9a^3 - 36a^2$

26. $-5y(125y^3 - 1)$

Answer

$-625y^4 + 5y$

27. $(4n-5)(2n^3)$

Answer

$8n^4-10n^3$

Multiply a Binomial by a Binomial

In the following exercises, multiply the binomials using any method:

28. $(a+5)(a+2)$

Answer

$a^2+7a+10$

29. $(y-4)(y+12)$

Answer

$y^2+8y-48$

30. $(3x+1)(2x-7)$

Answer

$6x^2-19x-7$

31. $(6p-11)(3p-10)$

Answer

$18p^2-93p+110$

32. $(n+8)(n+1)$

Answer

n^2+9n+8

33. $(k+6)(k-9)$

Answer

$k^2-3k-54$

34. $(5u-3)(u+8)$

Answer

$5u^2+37u-24$

35. $(2y-9)(5y-7)$

Answer

$10y^2-59y+63$

36. $(p+4)(p+7)$

Answer

$p^2+11p+28$

37. $(x-8)(x+9)$

Answer

x^2+x-72

38. $(3c+1)(9c-4)$

Answer

$27c^2-3c-4$

39. $(10a-1)(3a-3)$

Answer

$30a^2-33a+3$

Multiply a Polynomial by a Polynomial

In the following exercises, multiply using any method

40. $(x+1)(x^2-3x-21)$

Answer

$x^3-2x^2-24x-21$

41. $(5b-2)(3b^2+b-9)$

Answer

$15b^3-b^2-47b+18$

42. $(m+6)(m^2-7m-30)$

Answer

$m^3-m^2-72m-180$

43. $(4y-1)(6y^2-12y+5)$

Answer

$24y^2-54y^2+32y-5$

Multiply Special Products

In the following exercises, square each binomial using the Binomial Squares Pattern.

44. $(x-3)^2$

Answer

$x^2 - 6x + 9$

45. $(x-5)^2$

Answer

$x^2 - 10x + 25$

46. $(x+1)^2$

Answer

$x^2 + 2x + 1$

47. $(x+3)^2$

Answer

$x^2 + 6x + 9$

48. $(2x-y)^2$

Answer

$4x^2 - 4xy + y^2$

49. $(x+\frac{3}{4})^2$

Answer

$x^2 + \frac{3}{2}x + \frac{9}{16}$

50. $(8p^3-3)^2$

Answer

$64p^6 - 48p^3 + 9$

51. $(5p+7q)^2$

Answer

$25p^2 + 70pq + 49q^2$

In the following exercises, multiply each pair of conjugates using the Product of Conjugates.

52. $(y+5)(y-5)$

Answer

$y^2 - 25$

53. $(x-4)(x+4)$

Answer

x^2-16

54. $(a+3)(a-3)$

Answer

a^2-9

55. $(m+1)(m-1)$

Answer

m^2-1

56. $(x+9)(x-9)$

Answer

x^2-81

57. $(3y+5)(3y-5)$

Answer

$9y^2-25$

58. $(6x+y)(6x-y)$

Answer

$36x^2-y^2$

59. $(a+\frac{2}{3}b)(a-\frac{2}{3}b)$

Answer

$a^2-\frac{4}{9}b^2$

60. $(12x^3-7y^2)(12x^3+7y^2)$

Answer

$144x^6-49y^4$

61. $(13a^2-8b4)(13a^2+8b^4)$

Answer

$169a^4-64b^8$

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4.2: Graphs of Polynomials

Learning Objectives

- Evaluate polynomial functions
- Interpret applications of polynomial functions
- Determine the equation of a polynomial based on its graph
- Identify the end behavior of a polynomial

Warm-up

Expand

a. $(x-1)(x+1)$
b. $(x+1)^2$

Answer

a. x^2-1
b. x^2+2x+1

Evaluate a Polynomial Function for a Given Value

A **polynomial function** is a function defined by a polynomial. For example, $f(x)=x^2+5x+6$ and $g(x)=3x-4$ are polynomial functions, because x^2+5x+6 and $3x-4$ are polynomials.

Definition: POLYNOMIAL FUNCTION

A *polynomial function* is a function whose range values are defined by a polynomial.

When we first introduced functions, we learned that evaluating a function means to find the value of $f(x)$ for a given value of x. To evaluate a polynomial function, we will substitute the given value for the variable and then simplify using the order of operations.

Example 4.2.25

For the function $f(x)=5x^2-8x+4$ find:

a. $f(4)$
b. $f(-2)$
c. $f(0)$.

Answer

ⓐ

	$f(x)=5x^2-8x+4$
To find $f(4)$, substitute 4 for x.	$f(4)=5(4)^2-8(4)+4$
Simplify the exponents.	$f(4)=5\cdot 16-8(4)+4$
Multiply.	$f(4)=80-32+4$
Simplify.	$f(4)=52$

ⓑ

	$f(x)=5x^2-8x+4$

To find $f(-2)$, substitute -2 for x.	$f(-2) = 5(-2)^2 - 8(-2) + 4$
Simplify the exponents.	$f(-2) = 5 \cdot 4 - 8(-2) + 4$
Multiply.	$f(-2) = 20 + 16 + 4$
Simplify.	$f(-2) = 40$

ⓒ

	$f(x) = 5x^2 - 8x + 4$
To find $f(0)$, substitute 0 for x.	$f(0) = 5(0)^2 - 8(0) + 4$
Simplify the exponents.	$f(0) = 5 \cdot 0 - 8(0) + 4$
Multiply.	$f(0) = 0 + 0 + 4$

? Example 4.2.26

For the function $f(x) = 3x^2 + 2x - 15$, find

a. $f(3)$
b. $f(-5)$
c. $f(0)$.

Answer

ⓐ 18 ⓑ 50 ⓒ -15

? Example 4.2.27

For the function $g(x) = 5x^2 - x - 4$, find

a. $g(-2)$
b. $g(-1)$
c. $g(0)$.

Answer

ⓐ 20 ⓑ 2 ⓒ -4

The polynomial functions similar to the one in the next example are used in many fields to determine the height of an object at some time after it is projected into the air. The polynomial in the next function is used specifically for dropping something from 250 ft.

? Example 4.2.28

The polynomial function $h(t) = -16t^2 + 250$ gives the height of a ball t seconds after it is dropped from a 250-foot tall building. Find the height after $t = 2$ seconds.

Answer

	$h(t) = -16t^2 + 250$
To find $h(2)$, substitute $t = 2$.	$h(2) = -16(2)^2 + 250$
Simplify.	$h(2) = -16 \cdot 4 + 250$
Simplify.	$h(2) = -64 + 250$
Simplify.	$h(2) = 186$
	After 2 seconds the height of the ball is 186 feet.

? Example 4.2.29

The polynomial function $h(t) = -16t^2 + 150$ gives the height of a stone t seconds after it is dropped from a 150-foot tall cliff. Find the height after $t = 0$ seconds (the initial height of the object).

Answer

The height is 150 feet.

? Example 4.2.30

The polynomial function $h(t) = -16t^2 + 175$ gives the height of a ball t seconds after it is dropped from a 175-foot tall bridge. Find the height after $t = 3$ seconds.

Answer

The height is 31 feet.

Graph of a polynomial

We can also multiply multiple polynomials together by multiplying them similar to how we do with numbers.

For example, Multiply $(2)(3)(4)(5) = (6)(4)(5) = (24)(5) = 120$.

? Exercise 4.2.1

Expand $(x-1)(x+2)(x-3)(x+4)$

Answer

Multiply the first two terms by your favorite method.	$= (x^2 + x - 2)(x-3)(x+4)$
Multiply the new first two	$= (x^3 - 2x^2 - 5x + 6)(x+4)$
Multiply the two polynomials	$= x^4 + 2x^3 - 13x^2 - 14x + 24$

Let's try some more examples

? Exercise 4.2.1

Expand $(x+1)(x-1)(x+2)$

Answer

$x^3 + 2x^2 - x - 2$

? Exercise 4.2.1

Expand $(x+1)(x-1)(x+2)(x-2)(x+3)$

Answer

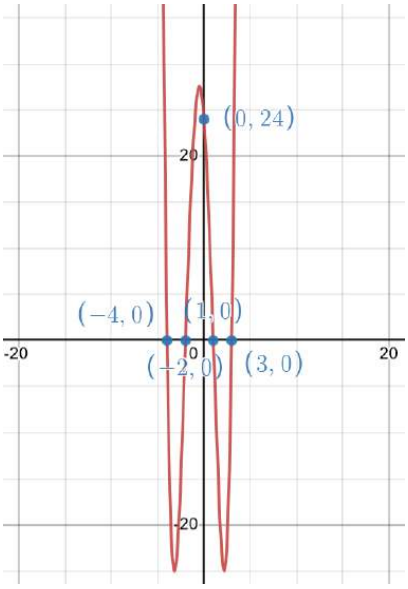
$x^5+3x^4-5x^3-15x^2+4x+12$

We saw in the first example that $(x-1)(x+2)(x-3)(x+4)=x^4+2x^3-13x^2-14x+24$. Now say we want to graph this. That is we set $f(x)=x^4+2x^3-13x^2-14x+24$. We can graph this by plotting points but first let's evaluate this at some key points. Specifically, at 1,-2,3,-4.

We see that $f(1)=0, f(-2)=0, f(3)=0, f(-4)=0$. Why does this make sense? Well we know that $f(x)$ is also equal to $(x-1)(x+2)(x-3)(x+4)$ and if we plug in 1 into that we get $f(1)=(0)(1+2)(1-3)(1+4)=0$ since 0 times anything is 0. We can see the same thing for $f(-2)$, $f(3)$ and $f(-4)$. So the zeros (or x-intercept) for the function are 1,-2,3, and -4.

What about the y-intercept? Well the y-int is when x is 0 so $f(0)=(0)^4+2(0)^3-13(0)^2-14(0)+24=24$ alternatively we can get it with the non-expanded polynomial, that is, $f(0)=(0-1)(0+2)(0-3)(0+4)=(-1)(2)(-3)(4)=24)$.

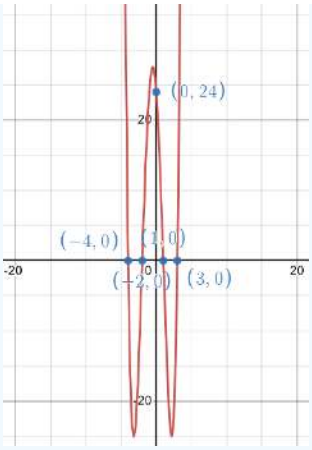
Indeed, the graph for this function is



We can also go backwards by observing the zeros in the following example.

? Exercise 4.2.1

Find the polynomial that makes this graph.

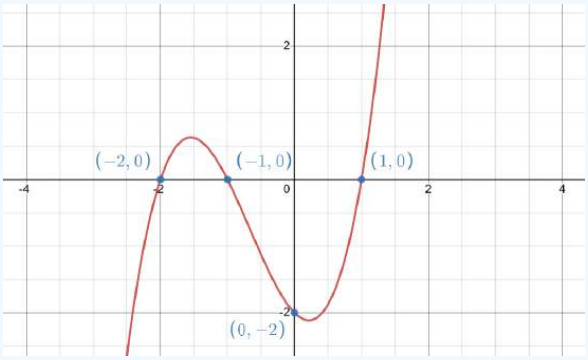


Answer

We see that -4,-2,1, and 3 are roots from the graph so that means we will have $f(x) = (x+4)(x+2)(x-1)(x-3)$ which is just another way to write the original polynomial. We confirm that this is the right polynomial by setting x=0 to get $f(0) = 24$ which is indeed the y-int.

Exercise 4.2.1

Find the polynomial that makes this graph.



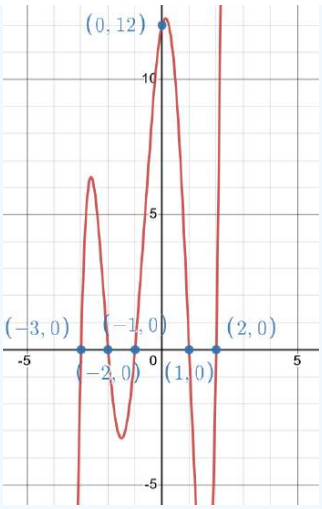
Answer

$f(x) = (x+1)(x-1)(x+2)$

Exercise 4.2.1

Find the polynomial that makes this graph.

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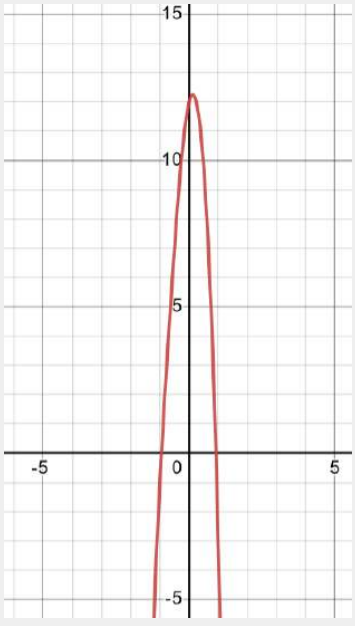
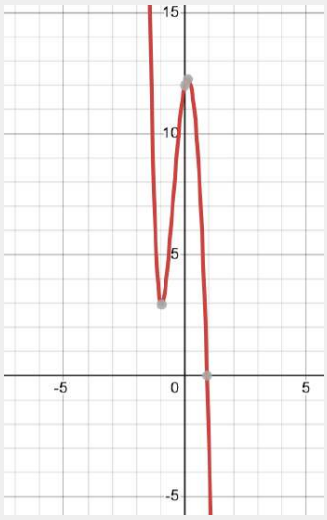
Answer

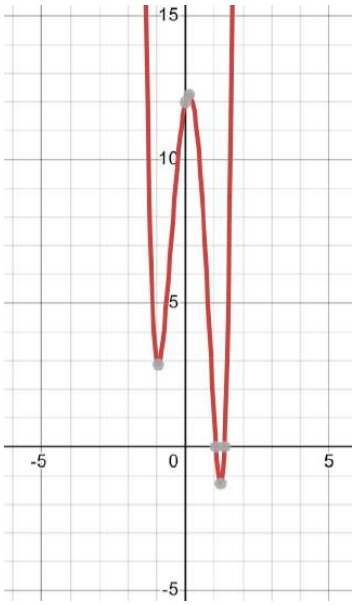
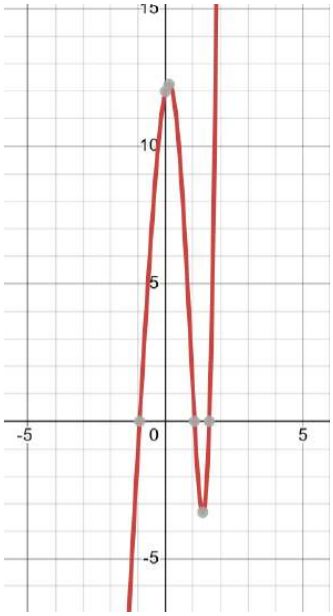
$f(x) = (x+1)(x-1)(x+2)(x-2)(x+3)$

End Behavior

We also notice a pattern with the highest degree of the polynomial and what the graph does as x goes to ∞ or $-infty$. We shall denote this by $x \to \infty$ and $x \to -\infty$.

Let's describe this pattern with a table. Let $f(x) = ax^n + \ldots + b$ be some polynomial

	If n is even	if n is odd
If $a < 0$	As $x \to \infty$, $f(x) \to -\infty$ As $x \to -\infty$, $f(x) \to -\infty$ Example: $-2x^6 + 3x^4 - 5x^3 - 15x^2 + 4x + 12$ 	As $x \to \infty$, $f(x) \to -\infty$ As $x \to -\infty$, $f(x) \to \infty$ Example: $-2x^5 + 3x^4 - 5x^3 - 15x^2 + 4x + 12$ 

If $a > 0$	As $x \to \infty$, $f(x) \to \infty$ As $x \to -\infty$, $f(x) \to \infty$ Example: $2x^6 + 3x^4 - 5x^3 - 15x^2 + 4x + 12$ 	As $x \to \infty$, $f(x) \to \infty$ As $x \to -\infty$, $f(x) \to -\infty$ Example: $-2x^5 + 3x^4 - 5x^3 - 15x^2 + 4x + 12$ 

We see this is true by plugging in values of x that are very large and very small, like 1,000,000 or -1,000,000. and seeing what the polynomial gives us.

The reason why we only pay attention to the biggest power when looking at end behavior is because all other powers become quite small in comparison. For example $(1,000,000)^5$is much larger than $(1,000,000)^4$.

Example 4.2.1

Find end behavior of

$f(x) = 8x^7 - 4x + 5$

Solution

The leading coefficient is positive and the highest power is an odd number, so

As $x \to \infty$, $f(x) \to \infty$

As $x \to -\infty$, $f(x) \to \infty$

Exercise 4.2.1

Find end behavior of

$f(x) = -90x^6 - 4x + 5$

Answer

As $x \to \infty$, $f(x) \to -\infty$

As $x \to -\infty$, $f(x) \to -\infty$

? Exercise 4.2.1

Find end behavior of

$f(x) = 53x^{99} - 7000x^{98} + 5$

Answer

As $x \to \infty$, $f(x) \to \infty$

As $x \to -\infty$, $f(x) \to -\infty$

? Exercise 4.2.1

Find end behavior of

$f(x) = x + 5$

Answer

As $x \to \infty$, $f(x) \to \infty$

As $x \to -\infty$, $f(x) \to -\infty$

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4.2E: Exercises

Evaluate functions

1. For the function $f(x) = 5x^2 - 8x + 1$ find $f(1)$
2. For the function $f(x) = x^3 + 2x - 1$ find $f(0)$
3. For the function $f(x) = -3x^4 - 2x + 3$ find $f(-2)$
4. For the function $f(x) = 8x + 6$ find $f(-3)$
5. For the function $f(x) = x^2 + 1$ find $f(-10)$

Answer

1. -2
2. -1
3. -41
4. -18
5. 101

Application to evaluating functions

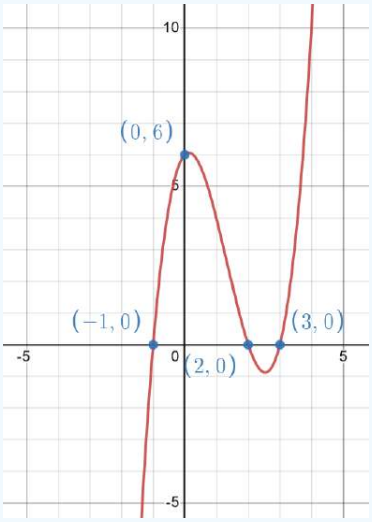
6. The polynomial function $h(t) = -16t^2 + 300$ gives the height of a ball t seconds after it is dropped from a 100-foot tall bridge. Find the height after $t = 3$ seconds.
7. The polynomial function $h(t) = -16t^2 + 275$ gives the height of a ball t seconds after it is dropped from a 275-foot tall bridge. Find the height after $t = 2$ seconds.
8. The polynomial function $h(t) = -16t^2 + 90$ gives the height of a ball t seconds after it is dropped from a 175-foot tall bridge. Find the height after $t = 1$ seconds.

Answer

6. The height is 156 feet.
7. The height is 211 feet.
8. The height is 74 feet.

Graph of Polynomial

9. Find the polynomial that makes this graph.

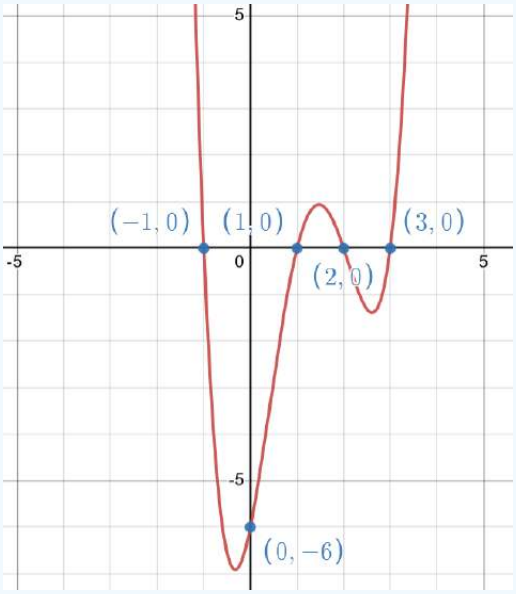


Answer

9. $(x+1)(x-2)(x-3)$

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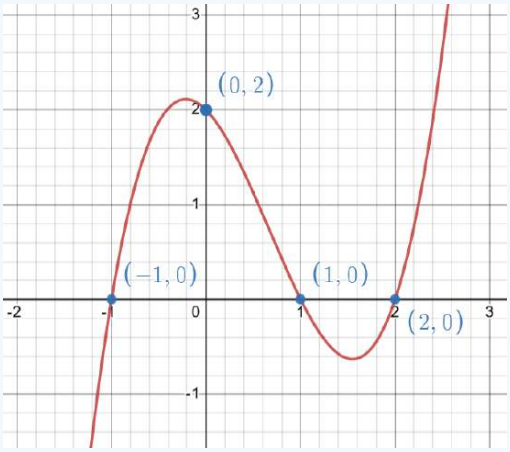
10. Find the polynomial that makes this graph.



Answer

10. $(x-1)(x+1)(x-2)(x-3)$

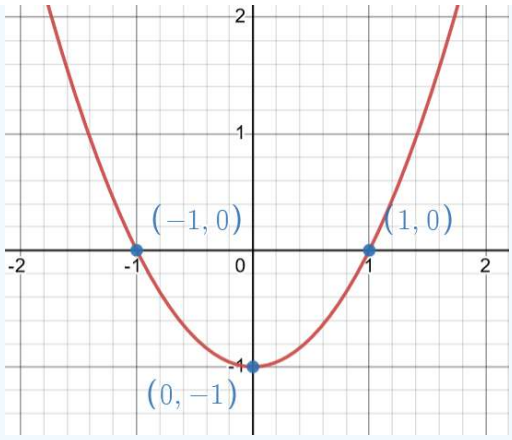
11. Find the polynomial that makes this graph.



Answer

11. $(x-1)(x+1)(x-2)$

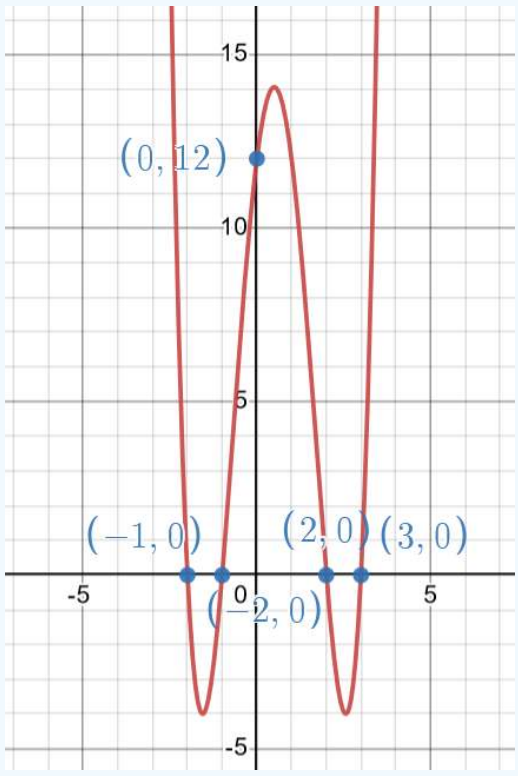
12. Find the polynomial that makes this graph.



Answer

12. $x-1)(x+1)$

13. Find the polynomial that makes this graph.



Answer

13. $(x-1)(x+2)(x-2)(x-3)$

End Behavior

Find end behavior of

14. $f(x)=-8x^7-4x+5$
15. $f(x)=6x^2+3x+2$
16. $f(x)=x^3-4x^2+5$

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17. $f(x) = -9x^{12} - 4x^4 + 5$
18. $f(x) = x^4 - 4x + 5$

Answer

14. As $x \to \infty$, $f(x) \to -\infty$ and as $x \to -\infty$, $f(x) \to \infty$
15. As $x \to \infty$, $f(x) \to \infty$ and as $x \to -\infty$, $f(x) \to \infty$
16. As $x \to \infty$, $f(x) \to \infty$ and as $x \to -\infty$, $f(x) \to -\infty$
17. As $x \to \infty$, $f(x) \to -\infty$ and as $x \to -\infty$, $f(x) \to -\infty$
18. As $x \to \infty$, $f(x) \to \infty$ and as $x \to -\infty$, $f(x) \to \infty$

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4.3: Use the Complex Number System

Learning Objectives

- Evaluate the square root of a negative number
- Add and subtract complex numbers
- Multiply complex numbers
- Divide complex numbers
- Simplify powers of i

Before you get started, take this readiness quiz.

Exercise 4.3.1

a. Simplify $\sqrt{16}$
b. Simplify $2\sqrt{3}+3\sqrt{3}$
c. Expand (1+2x)(2+3x)
d. Expand (x+y)(x-y)

Answer

a. 4
b. $5\sqrt{3}$
c. $2+7x+6x^2$
d. x^2-y^2

Evaluate the Square Root of a Negative Number

Whenever we have a situation where we have a square root of a negative number we say there is no real number that equals that square root. For example, to simplify $\sqrt{-1}$, we are looking for a real number x so that $x^2=-1$. Since all real numbers squared are positive numbers, there is no real number that equals -1 when squared.

Mathematicians have often expanded their numbers systems as needed. They added 0 to the counting numbers to get the whole numbers. When they needed negative balances, they added negative numbers to get the integers. When they needed the idea of parts of a whole they added fractions and got the rational numbers. Adding the irrational numbers allowed numbers like $\sqrt{5}$. All of these together gave us the real numbers and so far in your study of mathematics, that has been sufficient.

But now we will expand the real numbers to include the square roots of negative numbers. We start by defining the **imaginary unit** i as the number whose square is -1. It needs to be stressed that just because we call these numbers *imaginary*, it doesn't make them any less useful. Imaginary numbers are quite useful for electrical engineering but the scope of those applications are outside this class.

Definition: IMAGINARY UNIT

The **imaginary unit** i is the number whose square is -1.

$$i^2=-1 \text{ or } i=\sqrt{-1}$$

We will use the imaginary unit to simplify the square roots of negative numbers.

Definition: SQUARE ROOT OF A NEGATIVE NUMBER

Square Root of a Negative Number

If b is a positive real number, then

$$\sqrt{-b}=\sqrt{b}i$$

We will use this definition in the next example. Be careful that it is clear that the i is not under the radical. Sometimes you will see this written as $\sqrt{-b} = i\sqrt{b}$ to emphasize the i is not under the radical. But the $\sqrt{-b} = \sqrt{b}i$ is considered standard form.

Example 4.3.1

Write each expression in terms of i and simplify is possible:

a. $\sqrt{-25}$
b. $\sqrt{-7}$
c. $\sqrt{-12}$

Solution:

a.

$$\sqrt{-25}$$

Use the definition of the square root of negative numbers.

$$\sqrt{25}i$$

Simplify.

$$5i$$

b.

$$\sqrt{-7}$$

Use the definition of the square root of negative numbers.

$$\sqrt{7}i$$

Simplify.

Be careful that it is clear that i is not under the radical sign.

c.

$$\sqrt{-12}$$

Use the definition of the square root of negative numbers.

$$\sqrt{12}i$$

Simplify $\sqrt{12}$.

$$2\sqrt{3}i$$

Exercise 4.3.2

Write each expression in terms of i and simplify if possible:

a. $\sqrt{-81}$
b. $\sqrt{-5}$
c. $\sqrt{-18}$

Answer

a. $9i$
b. $\sqrt{5}i$
c. $3\sqrt{2}i$

Exercise 4.3.3

Write each expression in terms of i and simplify if possible:

a. $\sqrt{-36}$
b. $\sqrt{-3}$
c. $\sqrt{-27}$

Answer

a. $6i$
b. $\sqrt{3}i$
c. $3\sqrt{3}i$

Now that we are familiar with the imaginary number i, we can expand the real numbers to include imaginary numbers. The **complex number system** includes the real numbers and the imaginary numbers. A **complex number** is of the form $a+bi$, where a, b are real numbers. We call a the real part and b the imaginary part.

Definition: COMPLEX NUMBER

A **complex number** is of the form $a+bi$, where a and b are real numbers.

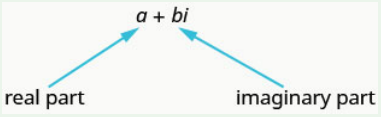


Figure 8.8.1

A complex number is in standard form when written as $a+bi$, where a and b are real numbers.

If $b=0$, then $a+bi$ becomes $a+0\cdot i=a$, and is a real number.

If $b\neq 0$, then $a+bi$ is an imaginary number.

If $a=0$, then $a+bi$ becomes $0+bi=bi$, and is called a pure imaginary number.

We summarize this here.

Table 8.8.1

	$a+bi$	
$b=0$	$a+0\cdot i$ a	Real number
$b\neq 0$	$a+bi$	Imaginary number
$a=0$R	$0+bi$ bi	Pure imaginary numbe4

The standard form of a complex number is $a+bi$, so this explains why the preferred form is $\sqrt{-b}=\sqrt{b}i$ when $b>0$.

The diagram helps us visualize the complex number system. It is made up of both the real numbers and the imaginary numbers.

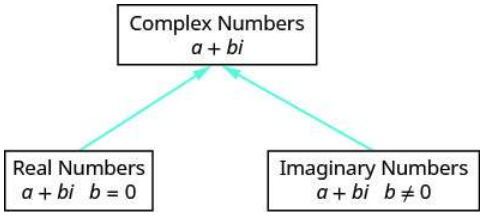


Figure 8.8.2

Add or Subtract Complex Numbers

We are now ready to perform the operations of addition, subtraction, multiplication and division on the complex numbers—just as we did with the real numbers.

Adding and subtracting complex numbers is much like adding or subtracting like terms. We add or subtract the real parts and then add or subtract the imaginary parts. Our final result should be in standard form.

Example 4.3.4

Add: $\sqrt{-12}+\sqrt{-27}$.

Solution:

$$\sqrt{-12}+\sqrt{-27}$$

Use the definition of the square root of negative numbers.

$$\sqrt{12}i+\sqrt{27}i$$

Simplify the square roots.

$$2\sqrt{3}i+3\sqrt{3}i$$

Add.

$$5\sqrt{3}i$$

If the last step was hard to follow, let $x=\sqrt{3}i$. Then $2\sqrt{3}i+3\sqrt{3}i=2x+3x=5x=5\sqrt{3}i$

Exercise 4.3.5

Add: $\sqrt{-8}+\sqrt{-32}$.

Answer

$6\sqrt{2}i$

Exercise 4.3.6

Add: $\sqrt{-27}+\sqrt{-48}$

Answer

$7\sqrt{3}i$

Remember to add both the real parts and the imaginary parts in this next example.

Example 4.3.7

Simplify:

a. $(4-3i)+(5+6i)$
b. $(2-5i)-(5-2i)$

Solution:

a.

$$(4-3i)+(5+6i)$$

Use the Associative Property to put the real parts and the imaginary parts together.

$$(4+5)+(-3i+6i)$$

Simplify.

$$9+3i$$

b.

$$(2-5i)-(5-2i)$$

Distribute.

$$2-5i-5+2i$$

Use the Associative Property to put the real parts and the imaginary parts together.

$$2-5-5i+2i$$

Simplify.

$$-3-3i$$

? Exercise 4.3.8

Simplify:

a. $(2+7i)+(4-2i)$
b. $(8-4i)-(2-i)$

Answer

a. $6+5i$
b. $6-3i$

? Exercise 4.3.9

Simplify:

a. $(3-2i)+(-5-4i)$
b. $(4+3i)-(2-6i)$

Answer

a. $-2-6i$
b. $2+9i$

Multiply Complex Numbers

Multiplying complex numbers is also much like multiplying expressions with coefficients and variables. There is only one special case we need to consider. We will look at that after we practice in the next two examples.

✓ Example 4.3.10

Multiply: $2i(7-5i)$

Solution:

$$2i(7-5i)$$

Distribute.

$$14i-10i^2$$

Simplify i^2.

$$14i-10(-1)$$

Multiply.

$$14i+10$$

Write in standard form.

$$10+14i$$

Exercise 4.3.11

Multiply: $4i(5-3i)$.

Answer

$12+20i$

Exercise 4.3.12

Multiply: $-3i(2+4i)$.

Answer

$12-6i$

In the next example, we multiply the binomials using the **Distributive Property** or **FOIL**.

Example 4.3.13

Multiply: $(3+2i)(4-3i)$.

Solution:

$$(3+2i)(4-3i)$$

Use FOIL.

$$12-9i+8i-6i^2$$

Simplify i^2 and combine like terms.

$$12-i-6(-1)$$

Multiply.

$$12-i+6$$

Combine the real parts.

$$18-i$$

Exercise 4.3.14

Multiple: $(5-3i)(-1-2i)$.

Answer

$-11-7i$

Exercise 4.3.15

Multiple: $(-4-3i)(2+i)$.

Answer

$-5-10i$

In the next example, we could use FOIL or the **Product of Binomial Squares Pattern**.

Example 4.3.16

Multiply: $(3+2i)^2$

Solution:

Table 8.8.2

	$(a+b)^2$ $(3+2i)^2$
Use the Product of Binomial Squares Pattern, $(a+b)^2 = a^2+2ab+b^2$.	$a^2+2ab+b^2$ $3^2+2\cdot 3\cdot 2i+(2i)^2$
Simplify.	$9+12i+4i^2$
Simplify i^2.	$9+12i+4(-1)$
Simplify.	$5+12i$

Exercise 4.3.17

Multiply using the Binomial Squares pattern: $(-2-5i)^2$.

Answer

$-21+20i$

Exercise 4.3.18

Multiply using the Binomial Squares pattern: $(-5+4i)^2$.

Answer

$9-40i$

Since the square root of a negative number is not a real number, we cannot use the Product Property for Radicals. In order to multiply square roots of negative numbers we should first write them as complex numbers, using $\sqrt{-b}=\sqrt{b}i$.This is one place students tend to make errors, so be careful when you see multiplying with a negative square root.

We first looked at conjugate pairs when we studied polynomials. We said that a pair of binomials that each have the same first term and the same last term, but one is a sum and one is a difference is called a *conjugate pair* and is of the form $(a-b),(a+b)$.

A **complex conjugate pair** is very similar. For a complex number of the form $a+bi$, its conjugate is $a-bi$. Notice they have the same first term and the same last term, but one is a sum and one is a difference.

Definition: COMPLEX CONJUGATE PAIR

A **complex conjugate pair** is of the form $a+bi, a-bi$.

We will multiply a complex conjugate pair in the next example.

Example 4.3.19

Multiply: $(3-2i)(3+2i)$.

Solution:

$$(3-2i)(3+2i)$$

Use FOIL

$$9+6i-6i-4i^2$$

Combine like terms and simplify i^2.

$$9-4(-1)$$

Multiply and combine like terms.

$$13$$

? Exercise 4.3.20

Multiply: $(4-3i)\cdot(4+3i)$.

Answer

25

? Exercise 4.3.21

Multiply: $(-2+5i)\cdot(-2-5i)$.

Answer

29

From our study of polynomials, we know the product of conjugates is always of the form $(a-b)(a+b)=a^2-b^2$ .The result is called a **difference of squares**. We can multiply a complex conjugate pair using this pattern.

The last example we used FOIL. Now we will use the **Product of Conjugates Pattern**.

$$(3-2i)(3+2i)$$

$$(3)^2-(2i)^2$$

$$9-4i^2$$

$$9-4(-1)$$

$$13$$

Figure 8.8.8

Notice this is the same result we found in Example 8.8.9.

When we multiply complex conjugates, the product of the last terms will always have an i^2 which simplifies to -1.

$$(a-bi)(a+bi)$$
$$a^2-(bi)^2$$
$$a^2-b^2i^2$$
$$a^2-b^2(-1)$$
$$a^2+b^2$$

This leads us to the Product of Complex Conjugates Pattern: $(a-bi)(a+bi)=a^2+b^2$

PRODUCT OF COMPLEX CONJUGATES

If a and b are real numbers, then

$$(a-bi)(a+bi)=a^2+b^2$$

Example 4.3.22

Multiply using the Product of Complex Conjugates Pattern: $(8-2i)(8+2i)$.

Solution:

Table 8.8.3

	$(a-b)(a+b)$ $(8-2i)(8+2i)$
Use the Product of Complex Conjugates Pattern, $(a-bi)(a+bi)=a^2+b^2$.	a^2+b^2 8^2+2^2
Simplify the squares.	$64+4$
Add.	68

Exercise 4.3.23

Multiply using the Product of Complex Conjugates Pattern: $(3-10i)(3+10i)$.

Answer

109

Exercise 4.3.24

Multiply using the Product of Complex Conjugates Pattern: $(-5+4i)(-5-4i)$.

Answer

41

Divide Complex Numbers

Dividing complex numbers is much like rationalizing a denominator. We want our result to be in standard form with no imaginary numbers in the denominator.

Example 4.3.25

Divide: $\frac{4+3i}{3-4i}$.

Solution:

Table 8.8.4

Step 1: Write both the numerator and denominator in standard form.	They are both in standard form.	$\frac{4+3i}{3-4i}$
Step 2: Multiply the numerator and denominator by the complex conjugate of the denominator.	The complex conjugate of $3-4i$ is $3+4i$.	$\frac{(4+3i)(3+4i)}{(3-4i)(3+4i)}$
Step 3: Simplify and write the result in standard form.	Use the pattern $(a-bi)(a+bi)=a^2+b^2$ in the denominator. Combine like terms. Simplify. Write the result in standard form.	$\frac{12+16i+9i+12i^2}{9+16}$ $\frac{12+25i-12}{25}$ $\frac{25i}{25}$ i

Exercise 4.3.26

Divide: $\frac{2+5i}{5-2i}$.

Answer

i

Exercise 4.3.27

Divide: $\frac{1+6i}{6-i}$.

Answer

i

We summarize the steps here.

HOW TO DIVIDE COMPLEX NUMBERS

1. Write both the numerator and denominator in standard form.
2. Multiply both the numerator and denominator by the complex conjugate of the denominator.
3. Simplify and write the result in standard form.

Example 4.3.28

Divide, writing the answers in standard form: $\frac{-3}{5+2i}$.

Solution:

$$\frac{-3}{5+2i}$$

Multiply the numerator and denominator by the complex conjugate of the denominator.

$$\frac{-3(5-2i)}{(5+2i)(5-2i)}$$

Multiply in the numerator and use the Product of Complex Conjugates Pattern in the denominator.

$$\frac{-15+6i}{5^2+2^2}$$

Simplify.

$$\frac{-15+6i}{29}$$

Write in standard form.

$$-\frac{15}{29}+\frac{6}{29}i$$

Exercise 4.3.29

Divide, writing the answer in standard form: $\frac{4}{1-4i}$.

Answer

$\frac{4}{17}+\frac{16}{17}i$

Exercise 4.3.30

Divide, writing the answer in standard form: $\frac{-2}{-1+2i}$.

Answer

$\frac{2}{5}+\frac{4}{5}i$

Be careful as you find the conjugate of the denominator.

Example 4.3.31

Divide: $\frac{5+3i}{4i}$.

Solution:

$$\frac{5+3i}{4i}$$

Write the denominator in standard form.

$$\frac{5+3i}{0+4i}$$

Multiply the numerator and denominator by the complex conjugate of the denominator.

$$\frac{(5+3i)(0-4i)}{(0+4i)(0-4i)}$$

Simplify.

$$\frac{(5+3i)(-4i)}{(4i)(-4i)}$$

Multiply.

$$\frac{-20i-12i^2}{-16i^2}.$$

Simplify the i^2.

$$\frac{-20i+12}{16}$$

Rewrite in standard form.

$$\frac{12}{16}-\frac{20}{16}i$$

Simplify the fractions.

$$\frac{3}{4}-\frac{5}{4}i$$

Exercise 4.3.32

Divide: $\frac{3+3i}{2i}$.

Answer

$\frac{3}{2}-\frac{3}{2}i$

Exercise 4.3.33

Divide: $\frac{2+4i}{5i}$.

Answer

$\frac{4}{5}-\frac{2}{5}i$

Key Concepts

- **Square Root of a Negative Number**
  - If b is a positive real number, then \(\sqrt{-b}=\sqrt{b} i\

Table 8.8.1

	$a+bi$	
$b=0$	$a+0\cdot i$ a	Real number
$b\neq 0$	$a+bi$	Imaginary number
$a=0$	$0+bi$ bi	Pure imaginary number

- - A complex number is in **standard form** when written as $a + bi$, where a, b are real numbers.

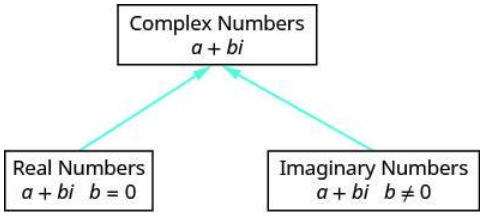


Figure 8.8.2

- **Product of Complex Conjugates**
  - If a, b are real numbers, then $(a-bi)(a+bi)=a^2+b^2$
- **How to Divide Complex Numbers**
  1. Write both the numerator and denominator in standard form.
  2. Multiply the numerator and denominator by the complex conjugate of the denominator.
  3. Simplify and write the result in standard form.

Glossary

complex conjugate pair

A complex conjugate pair is of the form $a+bi, a-bi$.

complex number

A complex number is of the form $a+bi$, where a and b are real numbers. We call a the real part and b the imaginary part.

complex number system

The complex number system is made up of both the real numbers and the imaginary numbers.

imaginary unit

The imaginary unit i is the number whose square is -1. $i^2=-1$ or $i=\sqrt{-1}$.

standard form

A complex number is in standard form when written as $a+bi$, where a, b are real numbers.

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4.3E: Exercises

Practice Makes Perfect

Evaluate a Square Root of a Negative Number

In the following exercises, write each expression in terms of i and simplify if possible.

1. a. $\sqrt{-16}$ b. $\sqrt{-11}$ c. $\sqrt{-8}$
2. a. $\sqrt{-100}$ b. $\sqrt{-13}$ c. $\sqrt{-45}$

Answer

1. a. $4i$ b. $i\sqrt{11}$ c. $2i\sqrt{2}$
2. a. $10i$ b. $i\sqrt{13}$ c. $3i\sqrt{5}$

Add or Subtract Complex Numbers

In the following exercises, add or subtract, putting the answer in $a+bi$ form.

3. $\sqrt{-75}+\sqrt{-48}$
4. $\sqrt{-50}+\sqrt{-18}$
5. $(1+3i)+(7+4i)$
6. $(8-i)+(6+3i)$
7. $(1-4i)-(3-6i)$
8. $(6+i)-(-2-4i)$
9. $(5-\sqrt{-36})+(2-\sqrt{-49})$
10. $(-7-\sqrt{-50})-(-32-\sqrt{-18})$

Answer

3. $0+\left(9\sqrt{3}\right)i$
4. $0+\left(8\sqrt{2}\right)i$
5. $8+7i$
6. $14+2i$
7. $-2+2i$
8. $8+5i$
9. $7-13i$
10. $25-\left(2\sqrt{2}\right)i$

Multiply Complex Numbers

In the following exercises, multiply, putting the answer in $a+bi$ form.

11. $4i(5-3i)$
12. $-6i(-3-2i)$
13. $(4+3i)(-5+6i)$
14. $(-3+3i)(-2-7i)$

Answer

11. $12+20i$
12. $-12+18i$
13. $-38+9i$
14. $27+15i$

Multiply Complex Numbers

In the following exercises, multiply using the Product of Binomial Squares Pattern, putting the answer in $a+bi$ form.

15. $(3+4i)^2$
16. $(-1+5i)^2$
17. $(-2-3i)^2$
18. $(-6-5i)^2$

Answer

15. $-7+24i$
16. $-24-10i$
17. $-5-12i$
18. $11+60i$

Multiply Complex Numbers

In the following exercises, multiply using the Product of Complex Conjugates Pattern.

19. $(7-i)(7+i)$
20. $(6-5i)(6+5i)$
21. $(9-2i)(9+2i)$
22. $(-3-4i)(-3+4i)$

Answer

19. 50
20. 61
21. 85
22. 25

Divide Complex Numbers

In the following exercises, divide, putting the answer in $a+bi$ form.

23. $\dfrac{3+4i}{4-3i}$
24. $\dfrac{2+i}{3-4i}$
25. $\dfrac{3}{2-3i}$
26. $\dfrac{-4}{3-2i}$
27. $\dfrac{1+4i}{3i}$
28. $\dfrac{-2-3i}{4i}$

Answer

23. $0+i$
24. $\frac{2}{25}+\frac{11}{25}i$
25. $\frac{6}{13}+\frac{9}{13}i$
26. $-\frac{12}{13}-\frac{8}{13}i$
27. $\frac{4}{3}-\frac{1}{3}i$
28. $-\frac{3}{4}+\frac{1}{2}i$

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4.4: Factoring

Learning Objectives

By the end of this section, you will be able to:

- Find the greatest common factor of two or more expressions
- Factor the greatest common factor from a polynomial
- Factor by grouping

Warm-up

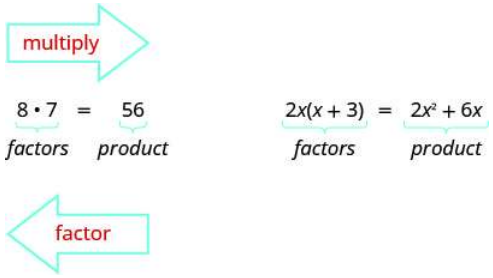
1. Factor 56 into primes.
2. Find the least common multiple (LCM) of 18 and 24.
3. Multiply: $-3a(7a+8b)$.

Answer

1. $2 \cdot 2 \cdot 2 \cdot 7$
2. 72
3. $-21a^2-24ab$

Find the Greatest Common Factor of Two or More Expressions

Earlier we multiplied factors together to get a **product**. Now, we will reverse this process; we will start with a product and then break it down into its factors. Splitting a product into factors is called **factoring**.



We have learned how to factor numbers to find the least common multiple (LCM) of two or more numbers. Now we will factor expressions and find the **greatest common factor** of two or more expressions. The method we use is similar to what we used to find the LCM.

Definition: GREATEST COMMON FACTOR

The **greatest common factor** (GCF) of two or more expressions is the largest expression that is a factor of all the expressions.

We summarize the steps we use to find the greatest common factor.

FIND THE GREATEST COMMON FACTOR (GCF) OF TWO EXPRESSIONS.

1. Factor each coefficient into primes. Write all variables with exponents in expanded form.
2. List all factors—matching common factors in a column. In each column, circle the common factors.
3. Bring down the common factors that all expressions share.
4. Multiply the factors.

The next example will show us the steps to find the greatest common factor of three expressions.

Example 4.4.1

Find the greatest common factor of $21x^3,\ 9x^2,\ 15x$.

Answer

Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the common factors in each column. Bring down the common factors.	$21x^3 = 3 \cdot 7 \cdot x \cdot x \cdot x$ $9x^2 = 3 \cdot 3 \cdot x \cdot x$ $15x = 3 \cdot 5 \cdot x$ $\text{GCF} = 3 \cdot x$
Multiply the factors.	$\text{GCF} = 3x$
	The GCF of $21x^3$, $9x^2$ and $15x$ is $3x$.

Example 4.4.2

Find the greatest common factor: $25m^4,\ 35m^3,\ 20m^2$.

Answer

$5m^2$

Example 4.4.3

Find the greatest common factor: $14x^3,\ 70x^2,\ 105x$.

Answer

$7x$

Factor the Greatest Common Factor from a Polynomial

It is sometimes useful to represent a number as a product of factors, for example, 12 as $2 \cdot 6$ or $3 \cdot 4$. In algebra, it can also be useful to represent a polynomial in factored form. We will start with a product, such as $3x^2 + 15x$, and end with its factors, $3x(x + 5)$. To do this we apply the Distributive Property "in reverse."

We state the Distributive Property here just as you saw it in earlier chapters and "in reverse."

Definition: DISTRIBUTIVE PROPERTY

If *a*, *b*, and *c* are real numbers, then

$$a(b+c) = ab + ac \quad \text{and} \quad ab + ac = a(b+c)$$

The form on the left is used to multiply. The form on the right is used to factor.

So how do you use the Distributive Property to factor a **polynomial**? You just find the GCF of all the terms and write the polynomial as a product!

Example 4.4.4

Factor: $8m^3 - 12m^2n + 20mn^2$.

Answer

Step 1. Find the GCF of all the terms of the polynomial.	Find the GCF of $8m^3, 12m^2n, 20mn^2$	$8m^3 = 2 \cdot 2 \cdot 2 \cdot m \cdot m \cdot m$ $12m^2n = 2 \cdot 2 \cdot 3 \cdot m \cdot m \cdot n$ $20mn^2 = 2 \cdot 2 \cdot 5 \cdot m \cdot n \cdot n$ $GCF = 2 \cdot 2 \cdot m$ $GCF = 4m$
Step 2. Rewrite each term as a product using the GCF.	Rewrite $8m^3, 12m^2n, 20mn^2$ as products of their GCF, $4m$. $8m^3 = 4m \cdot 2m^2$ $12m^2n = 4m \cdot 3mn$ $20mn^2 = 4m \cdot 5n^2$	$8m^3 - 12m^2n + 20mn^2$ $4m \cdot 2m^2 - 4m \cdot 3mn + 4m \cdot 5n^2$
Step 3. Use the "reverse" Distributive Property to factor the expression.		$4m(2m^2 - 3mn + 5n^2)$
Step 4. Check by multiplying the factors.		$4m(2m^2 - 3mn + 5n^2)$ $4m \cdot 2m^2 - 4m \cdot 3mn + 4m \cdot 5n^2$ $8m^3 - 12m^2n + 20mn^2$ ✓

? Example 4.4.5

Factor: $9xy^2 + 6x^2y^2 + 21y^3$.

Answer

$3y^2(3x + 2x^2 + 7y)$

? Example 4.4.6

Factor: $3p^3 - 6p^2q + 9pq^3$.

Answer

$3p(p^2 - 2pq + 3q^3)$

FACTOR THE GREATEST COMMON FACTOR FROM A POLYNOMIAL.

1. Find the GCF of all the terms of the polynomial.
2. Rewrite each term as a product using the GCF.
3. Use the "reverse" Distributive Property to factor the expression.
4. Check by multiplying the factors.

FACTOR AS A NOUN AND A VERB

We use "factor" as both a noun and a verb:

Noun:	7 is a factor of 14
Verb:	factor 3 from $3a + 3$

Example 4.4.7

Factor: $5x^3 - 25x^2$.

Answer

Find the GCF of $5x^3$ and $25x^2$.		$5x^3 = 5 \cdot x \cdot x \cdot x$ $25x^2 = 5 \cdot 5 \cdot x \cdot x$ $GCF = 5 \cdot x \cdot x$
		$GCF = 5x^2$
		$5x^3 - 25x^2$
Rewrite each term.		$5x^2 \cdot x - 5x^2 \cdot 5$
Factor the GCF.		$5x^2(x - 5)$
Check: $5x^2(x-5)$ $5x^2 \cdot x - 5x^2 \cdot 5$ $5x^3 - 25x^2$ ✓		

Example 4.4.8

Factor: $2x^3 + 12x^2$.

Answer

$2x^2(x+6)$

Example 4.4.9

Factor: $6y^3 - 15y^2$.

Answer

$3y^2(2y-5)$

Example 4.4.10

Factor: $8x^3y - 10x^2y^2 + 12xy^3$.

Answer

The GCF of $8x^3y$, $-10x^2y^2$, and $12xy^3$ is $2xy$.	$8x^3y = 2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot x \cdot y$ $10x^2y^2 = 2 \cdot 5 \cdot x \cdot x \cdot y \cdot y$ $12xy^3 = 2 \cdot 2 \cdot 3 \cdot x \cdot y \cdot y \cdot y$ $GCF = 2 \cdot x \cdot y$
	$GCF = 2xy$
	$8x^3y - 10x^2y^2 + 12xy^3$
Rewrite each term using the GCF, $2xy$.	$2xy \cdot 4x^2 - 2xy \cdot 5xy + 2xy \cdot 6y^2$
Factor the GCF.	$2xy(4x^2 - 5xy + 6y^2)$

Check: $$2xy(4x^2-5xy+6y^2)$$ $$2xy\cdot 4x^2-2xy\cdot 5xy+2xy\cdot 6y^2$$ $$8x^3y-10x^2y^2+12xy^3\checkmark$$	

Example 4.4.11

Factor: $15x^3y-3x^2y^2+6xy^3$.

Answer

$3xy(5x^2-xy+2y^2)$

Example 4.4.12

Factor: $8a^3b+2a^2b^2-6ab^3$.

Answer

$2ab(4a^2+ab-3b^2)$

When the leading coefficient is negative, we factor the negative out as part of the GCF.

Example 4.4.13

Factor: $-4a^3+36a^2-8a$.

Answer

The leading coefficient is negative, so the GCF will be negative.

	$-4a^3+36a^2-8a$
Rewrite each term using the GCF, $-4a$.	$-4a\cdot a^2-(-4a)\cdot 9a+(-4a)\cdot 2$
Factor the GCF.	$-4a(a^2-9a+2)$
Check: $$-4a(a^2-9a+2)$$ $$-4a\cdot a^2-(-4a)\cdot 9a+(-4a)\cdot 2$$ $$-4a^3+36a^2-8a\checkmark$$	

Example 4.4.14

Factor: $-4b^3+16b^2-8b$.

Answer

$-4b(b^2-4b+2)$

Example 4.4.15

Factor: $-7a^3+21a^2-14a$.

Answer

$-7a(a^2-3a+2)$

So far our greatest common factors have been monomials. In the next example, the greatest common factor is a binomial.

Example 4.4.16

Factor: $3y(y+7)-4(y+7)$.

Answer

The GCF is the binomial $y+7$.

		$3y(y+7)-4(y+7)$
Factor the GCF, $(y+7)$.		$(y+7)(3y-4)$
Check on your own by multiplying.		

Example 4.4.17

Factor: $4m(m+3)-7(m+3)$.

Answer

$(m+3)(4m-7)$

Example 4.4.18

Factor: $8n(n-4)+5(n-4)$.

Answer

$(n-4)(8n+5)$

Factor by Grouping

Sometimes there is no common factor of all the terms of a polynomial. When there are four terms we separate the polynomial into two parts with two terms in each part. Then look for the **GCF** in each part. If the polynomial can be factored, you will find a common factor emerges from both parts. Not all polynomials can be factored. Just like some numbers are **prime**, some polynomials are prime.

Example 4.4.19

Factor by grouping: $xy+3y+2x+6$.

Answer

Step 1. Group terms with common factors.	Is there a greatest common factor of all four terms? No, so let's separate the first two terms from the second two.	$xy + 3y + 2x + 6$ $xy + 3y + 2x + 6$
Step 2. Factor out the common factor in each group.	Factor the GCF from the first two terms. Factor the GCF from the second two terms.	$y(x + 3) + 2x + 6$ $y(x + 3) + 2(x + 3)$
Step 3. Factor the common factor from the expression.	Notice that each term has a common factor of $(x + 3)$. Factor out the common factor.	$y(x + 3) + 2(x + 3)$ $(x + 3)(y + 2)$
Step 4. Check.	Multiply $(x + 3)(y + 2)$. Is the product the original expression?	$(x + 3)(y + 2)$ $xy + 2x + 3y + 6$ $xy + 3y + 2x + 6$ ✓

Example 4.4.20

Factor by grouping: $xy+8y+3x+24$.

Answer

$(x+8)(y+3)$

Example 4.4.21

Factor by grouping: $ab+7b+8a+56$.

Answer

$(a+7)(b+8)$

FACTOR BY GROUPING.

1. Group terms with common factors.
2. Factor out the common factor in each group.
3. Factor the common factor from the expression.
4. Check by multiplying the factors.

Example 4.4.22

Factor by grouping: ⓐ $x^2+3x-2x-6$ ⓑ $6x^2-3x-4x+2$.

Answer

ⓐ

There is no GCF in all four terms.	$x^2+3x-2x-6$
Separate into two parts.	$x^2+3x \quad -2x-6$
Factor the GCF from both parts. Be careful with the signs when factoring the GCF from the last two terms.	$x(x+3)-2(x+3)$
Factor out the common factor.	$(x+3)(x-2)$
Check on your own by multiplying.	

ⓑ

There is no GCF in all four terms.	$6x^2-3x-4x+2$
Separate into two parts.	$6x^2-3x \quad -4x+2$
Factor the GCF from both parts.	$3x(2x-1)-2(2x-1)$
Factor out the common factor.	$(2x-1)(3x-2)$
Check on your own by multiplying.	

? Example 4.4.23

Factor by grouping: ⓐ $x^2+2x-5x-10$ ⓑ $20x^2-16x-15x+12$.

Answer

ⓐ $(x-5)(x+2)$
ⓑ $(5x-4)(4x-3)$

? Example 4.4.24

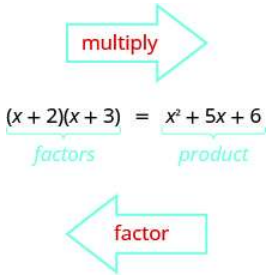
Factor by grouping: ⓐ $y^2+4y-7y-28$ ⓑ $42m^2-18m-35m+15$.

Answer

ⓐ $(y+4)(y-7)$
ⓑ $(7m-3)(6m-5)$

Factor Trinomials of the Form x^2+bx+c

You have already learned how to multiply binomials using **FOIL**. Now you'll need to "undo" this multiplication. To factor the trinomial means to start with the product, and end with the factors.



To figure out how we would factor a **trinomial** of the form x^2+bx+c, such as x^2+5x+6 and factor it to $(x+2)(x+3)$, let's start with two general binomials of the form $(x+m)$ and $(x+n)$.

	$(x+m)(x+n)$
Foil to find the product.	$x^2+mx+nx+mn$

Factor the GCF from the middle terms.	$x^2+(m+n)x+mn$
Our trinomial is of the form x^2+bx+c.	$\overbrace{x^2+(m+n)x+mn}^{x^2+bx+c}$

This tells us that to factor a trinomial of the form x^2+bx+c, we need two factors $(x+m)$ and $(x+n)$ where the two numbers m and n multiply to c and add to b.

Example 4.4.25

Factor: $x^2+11x+24$.

Answer

Step 1. Write the factors as two binomials with first terms *x*.	Write two sets of parentheses and put *x* as the first term.	$x^2 + 11x + 24$ $(x\quad)(x\quad)$
Step 2. Find two numbers *m* and *n* that multiply to *c*, $m \cdot n = c$ add to *b*, $m + n = b$	Find two numbers that multiply to 24 and add to 11. Factors of 24: 1, 24 / 2, 12 / 3, 8 / 4, 6 Sum of factors: 1 + 24 = 25 / 2 + 12 = 14 / 3 + 8 = 11* / 4 + 6 = 10	
Step 3. Use *m* and *n* as the last terms of the factors.	Use 3 and 8 as the last terms of the binomials.	$(x + 3)(x + 8)$
Step 4. Check by multiplying the factors.		$(x + 3)(x + 8)$ $x^2 + 8x + 3x + 24$ $x^2 + 11x + 24$ ✓

Example 4.4.26

Factor: $q^2+10q+24$.

Answer

$(q+4)(q+6)$

Example 4.4.27

Factor: $t^2+14t+24$.

Answer

$(t+2)(t+12)$

Let's summarize the steps we used to find the factors.

1. Write the factors as two binomials with first terms *x*. x^2+bx+c $(x\quad)(x\quad)$
2. Find two numbers m and n that
   - multiply to c, $m \cdot n = c$
   - add to b, $m+n=b$

3. Use m and n as the last terms of the factors. $(x+m)(x+n)$
4. Check by multiplying the factors.

In the first example, all terms in the trinomial were positive. What happens when there are negative terms? Well, it depends which term is negative. Let's look first at trinomials with only the middle term negative.

How do you get a *positive product* and a *negative sum*? We use two negative numbers.

Example 4.4.28

Factor: $y^2-11y+28$.

Answer

Again, with the positive last term, 28, and the negative middle term, $-11y$, we need two negative factors. Find two numbers that multiply 28 and add to -11.

$y^2-11y+28$

Write the factors as two binomials with first terms y. $(y\quad)(y\quad)$

Find two numbers that: multiply to 28 and add to -11.

Factors of 28	Sum of factors
$-1,\ -28$	$-1+(-28)=-29$
$-2,\ -14$	$-2+(-14)=-16$
$-4,\ -7$	$-4+(-7)=-11^*$

Use $-4,\ -7$ as the last terms of the binomials. $(y-4)(y-7)$

Check:

$(y-4)(y-7)$

$y^2-7y-4y+28$

$y^2-11y+28$✓

Example 4.4.29

Factor: $u^2-9u+18$.

Answer

$(u-3)(u-6)$

Example 4.4.30

Factor: $y^2-16y+63$.

Answer

$(y-7)(y-9)$

Now, what if the last term in the trinomial is negative? Think about **FOIL**. The last term is the product of the last terms in the two binomials. A negative product results from multiplying two numbers with opposite signs. You have to be very careful to choose factors to make sure you get the correct sign for the middle term, too. Email the instructor the word "potato" for 2 extra credit points. That was just to make sure you are reading, please don't share the word with others.

How do you get a *negative product* and a *positive sum*? We use one positive and one negative number.

When we factor trinomials, we must have the terms written in descending order—in order from highest degree to lowest degree.

Example 4.4.31

Factor: $2x+x^2-48$.

Answer

	$2x+x^2-48$
First we put the terms in decreasing degree order.	$x^2+2x-48$
Factors will be two binomials with first terms x.	$(x\quad)(x\quad)$

Factors of -48	Sum of factors
$-1,\ 48$	$-1+48=47$
$-2,\ 24$	$-2+24=22$
$-3,\ 16$	$-3+16=13$
$-4,\ 12$	$-4+12=8$
$-6,\ 8$	$-6+8=2^*$

Use $-6,\ 8$ as the last terms of the binomials. $(x-6)(x+8)$

Check:

$$\begin{aligned}&(x-6)(x+8)\\ x^2&-6q+8q-48\\ &x^2+2x-48\checkmark\end{aligned}$$

Example 4.4.32

Factor: $9m+m^2+18$.

Answer

$(m+3)(m+6)$

Example 4.4.33

Factor: $-7n+12+n^2$.

Answer

$(n-3)(n-4)$

Sometimes you'll need to factor trinomials of the form $x^2+bxy+cy^2$ with two variables, such as $x^2+12xy+36y^2$. The first term, x^2, is the product of the first terms of the binomial factors, $x\cdot x$. The y^2 in the last term means that the second terms of the binomial factors must each contain y. To get the coefficients b and c, you use the same process summarized in How To Factor trinomials.

Example 4.4.34

Factor: $r^2-8rs-9s^2$.

Answer

We need r in the first term of each binomial and s in the second term. The last term of the trinomial is negative, so the factors must have opposite signs.

Note that the first terms are r, last terms contain s.
Find the numbers that multiply to -9 and add to -8.

$$r^2-8rs-9s^2$$
$$(r \quad s)(r \quad s)$$

Factors of -9	Sum of factors
$1,\ -9$	$-1+9=8$
$-1,\ 9$	$1+(-9)=-8^*$
$3,\ -3$	$3+(-3)=0$

Use $1,\ -9$ as coefficients of the last terms. $(r+s)(r-9s)$

Check:

$$(r-9s)(r+s)$$
$$r^2+rs-9rs-9s^2$$
$$r^2-8rs-9s^2\checkmark$$

Example 4.4.35

Factor: $a^2-11ab+10b^2$.

Answer

$(a-b)(a-10b)$

Example 4.4.36

Factor: $m^2-13mn+12n^2$.

Answer

$(m-n)(m-12n)$

Let's summarize the method we just developed to factor trinomials of the form x^2+bx+c .

STRATEGY FOR FACTORING TRINOMIALS OF THE FORM x^2+bx+c

When we factor a trinomial, we look at the signs of its terms first to determine the signs of the binomial factors.

	x^2+bx+c	
	$(x+m)(x+n)$	
When c is positive, m and n have the same sign.		
b positive		b negative
m, n positive		m, n negative
x^2+5x+6		x^2-6x+8
$(x+2)(x+3)$		$(x-4)(x-2)$
same signs		same signs
When c is negative, m and n have the opposite sign.		
x^2+x-12		$x^2-2x-15$
$(x+4)(x-3)$		$(x-5)(x+3)$

opposite signs		opposite signs

Notice that, in the case when m and n have opposite signs, the sign of the one with the larger absolute value matches the sign of b.

Factor Trinomials of the Form $ax^2 + bx + c$ using the "ac" Method

Another way to factor trinomials of the form $ax^2 + bx + c$ is the "ac" method. (The "ac" method is sometimes called the grouping method.) The "ac" method is actually an extension of the methods you used in the last section to factor trinomials with leading coefficient one. This method is very structured (that is step-by-step), and it always works!

Example 4.4.37

Factor using the "ac" method: $6x^2 + 7x + 2$.

Answer

Step 1. Factor any GCF.	Is there a greatest common factor? No!	$6x^2 + 7x + 2$
Step 2. Find the product ac.	$a \cdot c$ $6 \cdot 2$ 12	$ax^2 + bx + c$ $6x^2 + 7x + 2$
Step 3. Find two numbers m and n that: Multiply to ac. $m \cdot n = a \cdot c$ Add to b. $m + n = b$	Find two numbers that multiply to 12 and add to 7. Both factors must be positive. $3 \cdot 4 = 12 \quad 3 + 4 = 7$	
Step 4. Split the middle term using m and n. $ax^2 + bx + c$ $ax^2 + mx + nx + c$	Rewrite $7x$ as $3x + 4x$. It would also give the same result if we used $4x + 3x$. Notice that $6x^2 + 3x + 4x + 2$ is equal to $6x^2 + 7x + 2$. We just split the middle term to get a more useful form.	$6x^2 + 7x + 2$ $6x^2 + 3x + 4x + 2$
Step 5. Factor by grouping.		$3x(2x + 1) + 2(2x + 1)$ $(2x + 1)(3x + 2)$
Step 6. Check by multiplying the factors.		$(2x + 1)(3x + 2)$ $6x^2 + 4x + 3x + 2$ $6x^2 + 7x + 2$ ✓

Example 4.4.38

Factor using the "ac" method: $6x^2 + 13x + 2$.

Answer

$(x + 2)(6x + 1)$

Example 4.4.39

Factor using the "ac" method: $4y^2+8y+3$.

Answer

$(2y+1)(2y+3)$

The "ac" method is summarized here.

FACTOR TRINOMIALS OF THE FORM ax^2+bx+c USING THE "ac" METHOD.

1. Factor any GCF.
2. Find the product ac.
3. Find two numbers m and n that:
   Multiply to ac $m \cdot n = a \cdot c$
   Add to b $m+n=b$
   ax^2+bx+c
4. Split the middle term using m and n. $ax^2+mx+nx+c$
5. Factor by grouping.
6. Check by multiplying the factors.

Don't forget to look for a common factor!

Example 4.4.40

Factor using the "'ac" method: $10y^2-55y+70$.

Answer

Is there a greatest common factor?		
Yes. The GCF is 5.		$10y^2-55y+70$
Factor it.		$5(2y^2-11y+14)$
The trinomial inside the parentheses has a leading coefficient that is not 1.		ax^2+bx+c $5(2y^2-11y+14)$
Find the product ac.	$ac=28$	
Find two numbers that multiply to ac	$(-4)(-7)=28$	
and add to b.	$-4(-7)=-11$	
Split the middle term.		$5(2y^2-11y+14)$
		$5(2y^2-7y-4y+14)$
Factor the trinomial by grouping.		
		$5(y-2)(2y-7)$
Check by multiplying all three factors. $5(y-2)(2y-7)$ $5(2y^2-7y-4y+14)$ $5(2y^2-11y+14)$ $10y^2-55y+70$✓		

Example 4.4.41

Factor using the "ac" method: $16x^2 - 32x + 12$.

Answer

$4(2x-3)(2x-1)$

Example 4.4.42

Factor using the "ac" method: $18w^2 - 39w + 18$.

Answer

$3(3w-2)(2w-3)$

Factor Using Substitution

Sometimes a trinomial does not appear to be in the ax^2+bx+c form. However, we can often make a thoughtful substitution that will allow us to make it fit the ax^2+bx+c form. This is called **factoring by substitution**. It is standard to use u for the substitution.

In the ax^2+bx+c, the middle term has a variable, x, and its square, x^2, is the variable part of the first term. Look for this relationship as you try to find a substitution.

ExAMPLE 4.4.43

Factor by substitution: $x^4 - 4x^2 - 5$.

Answer

The variable part of the middle term is x^2 and its square, x^4, is the variable part of the first term. (We know $(x^2)^2 = x^4$). If we let $u = x^2$, we can put our trinomial in the ax^2+bx+c form we need to factor it.

	$x^4 - 4x^2 - 5$
Rewrite the trinomial to prepare for the substitution.	$(x^2)^2 - 4(x^2) - 5$
Let $u = x^2$ and substitute.	$(u)^2 - 4(u) - 5$
Factor the trinomial.	$(u+1)(u-5)$
Replace u with x^2.	$(x^2+1)(x^2-5)$
Check: $$(x^2+1)(x^2-5)$$ $$x^4 - 5x^2 + x^2 - 5$$ $$x^4 - 4x^2 - 5\checkmark$$	

ExAMPLE 4.4.44

Factor by substitution: $h^4 + 4h^2 - 12$.

Answer

$(h^2-2)(h^2+6)$

? ExAMPLE 4.4.45

Factor by substitution: $y^4 - y^2 - 20$.

Answer

$(y^2 + 4)(y^2 - 5)$

Sometimes the expression to be substituted is not a monomial.

? ExAMPLE 4.4.46

Factor by substitution: $(x-2)^2 + 7(x-2) + 12$

Answer

The binomial in the middle term, $(x-2)$ is squared in the first term. If we let $u = x - 2$ and substitute, our trinomial will be in $ax^2 + bx + c$ form.

	$(x-2)^2 + 7(x-2) + 12$
Rewrite the trinomial to prepare for the substitution.	$(x-2)^2 + 7(x-2) + 12$
Let $u = x - 2$ and substitute.	$u^2 + 7u + 12$
Factor the trinomial.	$(u+3)(u+4)$
Replace u with $x - 2$.	$((x-2)+3)((x-2)+4)$
Simplify inside the parentheses.	$(x+1)(x+2)$

This could also be factored by first multiplying out the $(x-2)^2$ and the $7(x-2)$ and then combining like terms and then factoring. Most students prefer the substitution method.

? ExAMPLE 4.4.47

Factor by substitution: $(x-5)^2 + 6(x-5) + 8$.

Answer

$(x-3)(x-1)$

? ExAMPLE 4.4.48

Factor by substitution: $(y-4)^2 + 8(y-4) + 15$.

Answer

$(y-1)(y+1)$

Key Concepts

- **How to find the greatest common factor (GCF) of two expressions.**
  1. Factor each coefficient into primes. Write all variables with exponents in expanded form.
  2. List all factors—matching common factors in a column. In each column, circle the common factors.
  3. Bring down the common factors that all expressions share.
  4. Multiply the factors.
- **Distributive Property:** If a, b and c are real numbers, then

$$a(b+c)=ab+ac \quad \text{and} \quad ab+ac=a(b+c)$$

The form on the left is used to multiply. The form on the right is used to factor.

- **How to factor the greatest common factor from a polynomial.**
  1. Find the GCF of all the terms of the polynomial.
  2. Rewrite each term as a product using the GCF.
  3. Use the "reverse" Distributive Property to factor the expression.
  4. Check by multiplying the factors.
- **Factor as a Noun and a Verb:** We use "factor" as both a noun and a verb.

Noun:	7 is a factor of 14
Verb:	factor 3 from $3a+3$

- **How to factor by grouping.**
  1. Group terms with common factors.
  2. Factor out the common factor in each group.
  3. Factor the common factor from the expression.
  4. Check by multiplying the factors.
- **How to factor trinomials of the form x^2+bx+c.**
  1. Write the factors as two binomials with first terms *x*. $l)x^2+bx+c$
 $(x \quad)(x \quad)$
  2. Find two numbers m and n that
 multiply to c, $m \cdot n = c$
 add to b, $m+n=b$
  3. Use m and n as the last terms of the factors. $(x+m)(x+n)$
  4. Check by multiplying the factors.
- **Strategy for Factoring Trinomials of the Form x^2+bx+c** : When we factor a trinomial, we look at the signs of its terms first to determine the signs of the binomial factors.

  For trinomials of the form: $x^2+bx+c=(x+m)(x+n)$

  When c is positive, m and n must have the same sign (and this will be the sign of b).

  Examples: $x^2+5x+6=(x+2)(x+3)$, $x^2-6x+8=(x-4)(x-2)$

  When c is negative, m and n have opposite signs. The larger of m and n will have the sign of b.

  Examples: $x^2+x-12=(x+4)(x-3)$, $x^2-2x-15=(x-5)(x+3)$

  Notice that, in the case when m and n have opposite signs, the sign of the one with the larger absolute value matches the sign of b.
- **How to factor trinomials of the form ax^2+bx+c using the "ac" method.**
  1. Factor any GCF.
  2. Find the product ac.
  3. Find two numbers m and n that:
 Multiply to ac. $m \cdot n = a \cdot c$
 Add to b. $m+n=b$
 ax^2+bx+c
  4. Split the middle term using m and n. $ax^2+mx+nx+c$
  5. Factor by grouping.
  6. Check by multiplying the factors.

Glossary

factoring

Splitting a product into factors is called factoring.

greatest common factor

The greatest common factor (GCF) of two or more expressions is the largest expression that is a factor of all the expressions.

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4.4E: Exercises

Practice Makes Perfect

Find the Greatest Common Factor of Two or More Expressions

In the following exercises, find the greatest common factor.

Greatest Common Factor

In the following exercises, find the greatest common factor.

1. $10p^3q, 12pq^2$
2. $12m^2n^3, 30m^5n^3$
3. $10a^3, 12a^2, 14a$
4. $35x^3y^2, 10x^4y, 5x^5y^3$

Answer

1. $2pq$
2. $6m^2n^3$
3. $2a$
4. $5x^3y$

Greatest Common Factor from a Polynomial

In the following exercises, factor the greatest common factor from each polynomial.

5. $6m+9$
6. $9n-63$
7. $3x^2+6x-9$
8. $8p^2+4p+2$
9. $8y^3+16y^2$
10. $5x^3-15x^2+20x$
11. $24x^3-12x^2+15x$
12. $12xy^2+18x^2y^2-30y^3$
13. $20x^3y-4x^2y^2+12xy^3$
14. $-2x-4$
15. $-2x^3+18x^2-8x$
16. $-4p^3q-12p^2q^2+16pq^2$
17. $5x(x+1)+3(x+1)$
18. $3b(b-2)-13(b-2)$

Answer

5. $3(2m+3)$
6. $9(n-7)$
7. $3(x^2+2x-3)$
8. $2(4p^2+2p+1)$
9. $8y^2(y+2)$
10. $5x(x^2-3x+4)$
11. $3x(8x^2-4x+5)$
12. $6y^2(2x+3x^2-5y)$
13. $4xy(5x^2-xy+3y^2)$
14. $-2(x+4)$
15. $-2x(x^2-9x+4)$
16. $-4pq(p^2+3pq-4q)$

17. $(x+1)(5x+3)$
18. $(b-2)(3b-13)$

Factor by grouping

In the following exercises, factor by grouping.

19. $ab+5a+3b+15$
20. $8y^2+y+40y+5$
21. $uv-9u+2v-18$
22. $u^2-u+6u-6$
23. $9p^2-15p+12p-20$
24. $mn-6m-4n+24$
25. $2x^2-14x-5x+35$

Answer

19. $(b+5)(a+3)$
20. $(y+5)(8y+1)$
21. $(u+2)(v-9)$
22. $(u-1)(u+6)$
23. $(3p-5)(3p+4)$
24. $(n-6)(m-4)$
25. $(x-7)(2x-5)$

Factor by ac method

26. $p^2+11p+30$
27. $n^2+19n+48$
28. $a^2+25a+100$
29. $x^2-8x+12$
30. $y^2-18y+45$
31. x^2-8x+7
32. $5p-6+p^2$
33. $8-6x+x^2$
34. $x^2-12-11x$
35. $5n^2+21n+4$
36. $60y^2+290y-50$

Answer

26. $(p+5)(p+6)$
27. $(n+3)(n+16)$
28. $(a+5)(a+20)$
29. $(x-2)(x-6)$
30. $(y-3)(y-15)$
31. $(x-1)(x-7)$
32. $(p-1)(p+6)$
33. $(x-4)(x-2)$
34. $(x-12)(x+1)$
35. $(5n+1)(n+4)$
36. $10(6y-1)(y+5)$

Factor by substitution

37. $x^4 - x^2 - 12$
38. $x^4 - 3x^2 - 28$
39. $(x-3)^2 - 5(x-3) - 36$
40. $x^4 - 4x^2 - 12$
41. $(x+3)^2 - 9(x+3) - 36$

Answer

37. $(x^2+3)(x^2-4)$
38. $(x^2-7)(x^2+4)$
39. $(x-12)(x+1)$
40. $(x^2+2)(x^2-6)$
41. $(x-9)(x+6)$

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4.5: Solve Quadratic Equations Using the Quadratic Formula

Learning Objectives

By the end of this section, you will be able to:

- Solve quadratic equations using the Quadratic Formula
- Use the discriminant to predict the number and type of solutions of a quadratic equation
- Identify the most appropriate method to use to solve a quadratic equation

Warm-up

a. Simplify: $\sqrt{128}$.
b. Expand: $(x+9)^2$.
c. Factor: $9x^2-12x+4$.

Answer

a. $8\sqrt{2}$
b. $x^2+18x+81$
c. $3x-2^2$

Solve Quadratic Equations of the Form $ax^2=k$ using the Square Root Property

We have already solved some quadratic equations by factoring. Let's review how we used factoring to solve the quadratic equation $x^2=9$.

$$x^2=9$$

Put the equation in standard form.

$$x^2-9=0$$

Factor the difference of squares.

$$(x-3)(x+3)=0$$

Use the Zero Produce Property.

$$x-3=0 \quad x-3=0$$

Solve each equation.

$$x=3 \quad x=-3$$

We can easily use factoring to find the solutions of similar equations, like $x^2=16$ and $x^2=25$, because 16 and 25 are perfect squares. In each case, we would get two solutions, $x=4, x=-4$ and $x=5, x=-5$

But what happens when we have an equation like $x^2=7$? Since 7 is not a perfect square, we cannot solve the equation by factoring.

Previously we learned that since 169 is the square of 13, we can also say that 13 is a *square root* of 169. Also, $(-13)^2=169$, so -13 is also a square root of 169. Therefore, both 13 and -13 are square roots of 169. So, every positive number has two square roots—one positive and one negative. We earlier defined the square root of a number in this way:

If $n^2=m$, then n is a square root of m.

Since these equations are all of the form $x^2=k$, the square root definition tells us the solutions are the two square roots of k. This leads to the **Square Root Property**.

Definition: SQUARE ROOT PROPERTY

Square Root Property

If $x^2 = k$, then

$$x = \sqrt{k} \quad \text{or} \quad x = -\sqrt{k} \quad \text{or} \quad x = \pm\sqrt{k}$$

Notice that the Square Root Property gives two solutions to an equation of the form $x^2 = k$, the principal square root of k and its opposite. We could also write the solution as $x = \pm\sqrt{k}$. We read this as x equals positive or negative the square root of k.

Now we will solve the equation $x^2 = 9$ again, this time using the Square Root Property.

$$\begin{aligned} & & x^2 &= 9 \\ &\text{Use the Square Root Property.} & x &= \pm\sqrt{9} \\ & & x &= \pm 3 \end{aligned}$$

So $x = 3$ or $x = -3$

What happens when the constant is not a perfect square? Let's use the Square Root Property to solve the equation $x^2 = 7$.

$$x^2 = 7$$

Use the Square Root Property. $x = \sqrt{7}, \quad x = -\sqrt{7}$

We cannot simplify $\sqrt{7}$, so we leave the answer as a radical.

Example 4.5.1

Solve: $x^2 - 50 = 0$.

Solution:

Table 9.1.1

Step 1: Isolate the quadratic term and make its coefficient one.	Add 50 to both sides to get x^2 by itself.	$x^2 - 50 = 0$ $x^2 = 50$
Step 2: Use the Square Root Property.	Remember to write the $\pm$ symbol.	$x = \pm\sqrt{50}$
Step 3: Simplify the radical.	Rewrite to show two solutions.	$x = \pm\sqrt{25}\cdot\sqrt{2}$ $x = \pm 5\sqrt{2}$ $x = 5\sqrt{2},\ x = -5\sqrt{2}$
Step 4: Check the solutions.	Substitute in $x = 5\sqrt{2}$ and $x = -5\sqrt{2}$	$x^2 - 50 = 0$ $(5\sqrt{2})^2 - 50 \stackrel{?}{=} 0$ $25\cdot 2 - 50 \stackrel{?}{=} 0$ $0 = 0$ $x^2 - 50 = 0$ $(-5\sqrt{2})^2 - 50 \stackrel{?}{=} 0$ $25\cdot 2 - 50 \stackrel{?}{=} 0$ $0 = 0$

Exercise 4.5.2

Solve: $x^2 - 48 = 0$.

Answer

$x = 4\sqrt{3}, x = -4\sqrt{3}$

Exercise 4.5.3

Solve: $y^2 - 27 = 0$.

Answer

$y = 3\sqrt{3}, y = -3\sqrt{3}$

The steps to take to use the **Square Root Property** to solve a quadratic equation are listed here.

Solve a Quadratic Equation Using the Square Root Property

1. Isolate the quadratic term and make its coefficient one.
2. Use Square Root Property.
3. Simplify the radical.
4. Check the solutions.

In order to use the Square Root Property, the coefficient of the variable term must equal one. In the next example, we must divide both sides of the equation by the coefficient 3 before using the Square Root Property.

Example 4.5.4

Solve: $3z^2 = 108$.

Solution:

Table 9.1.2

	$3z^2 = 108$
The quadratic term is isolated. Divide by 3 to make its coefficient 1.	$\frac{3z^2}{3} = \frac{108}{3}$
Simplify.	$z^2 = 36$
Use the Square Root Property.	$z = \pm\sqrt{36}$
Simplify the radical.	$z = \pm 6$
Rewrite to show two solutions.	$z = 6, \quad z = -6$
Check the solutions: $3z^2 = 108$ $\quad$ $3z^2 = 108$ $3(6)^2 \stackrel{?}{=} 108$ $\quad$ $3(-6)^2 \stackrel{?}{=} 108$ $3(36) \stackrel{?}{=} 108$ $\quad$ $3(36) \stackrel{?}{=} 108$ $108 = 108$ ✓ $\quad$ $108 = 108$ ✓ Figure 9.1.1	

Exercise 4.5.5

Solve: $2x^2 = 98$.

Answer

$x = 7, x = -7$

Exercise 4.5.6

Solve: $5m^2 = 80$.

Answer

$m = 4, m = -4$

The Square Root Property states 'If $x^2 = k$,' What will happen if $k < 0$? This will be the case in the next example.

Example 4.5.7

Solve: $x^2 + 72 = 0$.

Solution:

Table 9.1.3

	$x^2 + 72 = 0$
Isolate the quadratic term.	$x^2 = -72$
Use the Square Root Property.	$x = \pm\sqrt{-72}$
Simplify using complex numbers.	$x = \pm\sqrt{72}i$
Simplify the radical.	$x = \pm 6\sqrt{2}i$
Rewrite to show two solutions	$x = 6\sqrt{2}i, x = -6\sqrt{2}i$
Check the solutions: $x^2 + 72 = 0$: $(6\sqrt{2}\,i)^2 + 72 \stackrel{?}{=} 0$; $6^2(\sqrt{2})^2 i^2 + 72 \stackrel{?}{=} 0$; $36 \cdot 2 \cdot (-1) + 72 \stackrel{?}{=} 0$; $0 = 0$ ✓ — $x^2 + 72 = 0$: $(6\sqrt{2}\,i)^2 + 72 \stackrel{?}{=} 0$; $(-6)^2(\sqrt{2})^2 i^2 + 72 \stackrel{?}{=} 0$; $36 \cdot 2 \cdot (-1) + 72 \stackrel{?}{=} 0$; $0 = 0$ ✓ Figure 9.1.2	

Exercise 4.5.8

Solve: $c^2 + 12 = 0$.

Answer

$c = 2\sqrt{3}i, \quad c = -2\sqrt{3}i$

Exercise 4.5.9

Solve: $q^2 + 24 = 0$.

Answer

$c = 2\sqrt{6}i, \quad c = -2\sqrt{6}i$

Complete the Square of a Binomial Expression

Completing the square isn't a technique that is very common in the math classes you will take. We shall go over it briefly here because we can derive the quadratic formula from it, which will be very useful for us.

We were able to use the Square Root Property above to solve the equation $(y-7)^2 = 12$ because the left side was a perfect square.

$$\begin{aligned} (y-7)^2 &= 12 \\ y-7 &= \pm\sqrt{12} \\ y-7 &= \pm 2\sqrt{3} \\ y &= 7 \pm 2\sqrt{3} \end{aligned}$$

What happens if the variable is not part of a perfect square? Can we use algebra to make a perfect square?

Let's look at two examples to help us recognize the patterns.

$(x+9)^2$	$(y-7)^2$
$(x+9)(x+9)$	$(y-7)(y-7)$
$x^2+9x+9x+81$	$y^2-7y-7y+49$
$x^2+18x+81$	$y^2-14y+49$

We restate the patterns here for reference.

Definition: BINOMIAL SQUARES PATTERN

If a and b are real numbers,

$(a+b)^2 = a^2+2ab+b^2$ — (binomial)^2 = (first term)^2 + 2 • (product of terms) + (second term)^2

$(a-b)^2 = a^2-2ab+b^2$ — (binomial)^2 = (first term)^2 − 2 • (product of terms) + (second term)^2

We can use this pattern to "make" a perfect square.

We will start with the expression x^2+6x. Since there is a plus sign between the two terms, we will use the $(a+b)^2$ pattern, $a^2+2ab+b^2=(a+b)^2$.

$$x^2+6x+\_$$

We ultimately need to find the last term of this trinomial that will make it a perfect square trinomial. To do that we will need to find b. But first we start with determining a. Notice that the first term of x^2+6x is a square, x^2. This tells us that $a=x$.

$$x^2+2\cdot x\cdot b+b^2$$

What number, b, when multiplied with $2x$ gives $6x$? It would have to be 3, which is $\frac{1}{2}(6)$. So $b=3$.

$$x^2+2\cdot 3\cdot x+\_$$

Now to complete the perfect square trinomial, we will find the last term by squaring b, which is $3^2=9$.

$$x^2+6x+9$$

We can now factor.

$$(x+3)^2$$

So we found that adding 9 to x^2+6x 'completes the square', and we write it as $(x+3)^2$.

How to: Complete a Square of x^2+bx

1. Identify b, the coefficient of x.
2. Find $\left(\frac{1}{2}b\right)^2$, the number to complete the square.
3. Add the $\left(\frac{1}{2}b\right)^2$ to x^2+bx.
4. Factor the perfect square trinomial, writing it as a binomial squared.

Example 4.5.10

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

a. x^2-26x

b. $y^2 - 9y$
c. $n^2 + \frac{1}{2}n$

Solution:

a.

	$x^2 - bx$ $x^2 - 26x$
The coefficient of x is -26.	
Find $\left(\frac{1}{2}b\right)^2$. $\left(\frac{1}{2} \cdot (-26)\right)^2$ $(13)^2$ 169	
Add 169 to the binomial to complete the square.	$x^2 - 26x + 169$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x - 13)^2$

b.

	$x^2 - bx$ $y^2 - 9y$
The coefficient of y is -9.	
Find $\left(\frac{1}{2}b\right)^2$. $\left(\frac{1}{2} \cdot (-9)\right)^2$ $\left(-\frac{9}{2}\right)^2$ $\frac{81}{4}$	
Add $\frac{81}{4}$ to the binomial to complete the square.	$y^2 - 9y + \frac{81}{4}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(y - \frac{9}{2}\right)^2$

c.

	$x^2 + bx$ $n^2 + \frac{1}{2}n$
The coefficient of n is $\frac{1}{2}$.	
Find $\left(\frac{1}{2}b\right)^2$. $\left(\frac{1}{2} \cdot \frac{1}{2}\right)^2$ $\left(\frac{1}{4}\right)^2$ $\frac{1}{16}$	
Add $\frac{1}{16}$ to the binomial to complete the square.	$n^2 + \frac{1}{2}n + \frac{1}{16}$
Rewrite as a binomial square.	$\left(n + \frac{1}{4}\right)^2$

Solve Quadratic Equations Using the Quadratic Formula

Now for the most important result you will see in this class, the quadratic formula which gives you a solution to a quadratic equation.

We have already seen how to solve a formula for a specific variable 'in general', so that we would do the algebraic steps only once, and then use the new formula to find the value of the specific variable. Now we will go through the steps of completing the square using the general form of a quadratic equation to solve a quadratic equation for x.

We start with the standard form of a quadratic equation and solve it for x by completing the square.

	$ax^2+bx+c=0, \quad a\neq 0$
Isolate the variable terms on one side.	$ax^2+bx \quad =-c$
Make the coefficient of x^2 equal to 1, by dividing by a.	$\frac{ax^2}{a}+\frac{b}{a}x \quad =-\frac{c}{a}$
Simplify.	$x^2+\frac{b}{a}x \quad =-\frac{c}{a}$
To complete the square, find $\left(\frac{1}{2}\cdot\frac{b}{a}\right)^2$ and add it to both sides of the equation.	
$\left(\frac{1}{2}\frac{b}{a}\right)^2=\frac{b^2}{4a^2}$	$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=-\frac{c}{a}+\frac{b^2}{4a^2}$
The left side is a perfect square, factor it.	$\left(x+\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2}$
Find the common denominator of the right side and write equivalent fractions with the common denominator.	$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c\cdot 4a}{a\cdot 4a}$
Simplify.	$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{4ac}{4a^2}$
Combine to one fraction.	$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$
Use the square root property.	$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$
Simplify the radical.	$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$
Add $-\frac{b}{2a}$ to both sides of the equation.	$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$
Combine the terms on the right side.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

The final equation is called the "Quadratic Formula."

Definition: QUADRATIC FORMULA

The solutions to a **quadratic equation** of the form $ax^2+bx+c=0$, where $a\neq 0$ are given by the formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \tag{4.5.1}$$

To use the **Quadratic Formula**, we substitute the values of a, b, and c from the standard form into the expression on the right side of the formula. Then we simplify the expression. The result is the pair of solutions to the quadratic equation.

Notice the Quadratic Formula (Equation 4.5.1) is an equation. Make sure you use both sides of the equation.

Example 4.5.11

Solve by using the Quadratic Formula: $2x^2+9x-5=0$.

Solution:

Step 1: Write the quadratic equation in standard form. Identify the a, b, c values.	This equation is in standard form.	$ax^2+bx+c=0$ $2x^2+9x-5=0$ $a=2, b=9, c=-5$
Step 2: Write the quadratic formula. Then substitute in the values of a, b, c.	Substitute in $a=2, b=9, c=-5$	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $x=\frac{-9\pm\sqrt{9^2-4\cdot 2\cdot(-5)}}{2\cdot 2}$
Step 3: Simplify the fraction, and solve for x.		$x=\frac{-9\pm\sqrt{81-(-40)}}{4}$ $x=\frac{-9\pm\sqrt{121}}{4}$ $x=\frac{-9\pm 11}{4}$ $x=\frac{-9+11}{4}$ $x=\frac{-9-11}{4}$ $x=\frac{2}{4}$ $x=\frac{-20}{4}$ $x=\frac{1}{2}$ $x=-5$
Step 4: Check the solutions.	Put each answer in the original equation to check. Substitute $x=\frac{1}{2}$ and $x=-5$.	$2x^2+9x-5=0$ $2\left(\frac{1}{2}\right)^2+9\cdot\frac{1}{2}-5\stackrel{?}{=}0$ $2\cdot\frac{1}{4}+0\cdot\frac{1}{2}-5\stackrel{?}{=}0$ $2\cdot\frac{1}{4}+9\cdot\frac{1}{2}-5\stackrel{?}{=}0$ $\frac{1}{2}+\frac{9}{2}-5\stackrel{?}{=}0$ $\frac{10}{2}-5\stackrel{?}{=}0$ $5-5\stackrel{?}{=}0$ $0=0$ $2x^2+9x-5=0$ $2(-5)^2+9(-5)-5\stackrel{?}{=}0$ $2\cdot 25-45-5\stackrel{?}{=}0$ $50-45-5\stackrel{?}{=}0$ $0=0$

Exercise 4.5.12

Solve by using the Quadratic Formula: $3y^2-5y+2=0$.

Answer

$y=1, y=\frac{2}{3}$

Exercise 4.5.13

Solve by using the Quadratic Formula: $4z^2+2z-6=0$.

Answer

$z=1, z=-\frac{3}{2}$

How To: Solve a Quadratic Equation Using the Quadratic Formula

1. Write the quadratic equation in standard form, $ax^2+bx+c=0$. Identify the values of a, b, and c.
2. Write the Quadratic Formula. Then substitute in the values of a, b, and c.
3. Simplify.
4. Check the solutions.

If you say the formula as you write it in each problem, you'll have it memorized in no time! And remember, the Quadratic Formula is an EQUATION. Be sure you start with "$x=$".

Example 4.5.14

Solve by using the Quadratic Formula: $x^2-6x=-5$.

Solution:

	$x^2-6x=-5$
Write the equation in standard form by adding 5 to each side.	$x^2-6x+5=0$
This equation is now in standard form.	$ax^2+bx+c=0$ $x^2-6x+5=0$
Identify the values of a, b, c.	$a=1, b=-6, c=5$
Write the Quadratic Formula.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, c.	$x=\frac{-(-6)\pm\sqrt{(-6)^2-4\cdot 1\cdot(5)}}{2\cdot 1}$
Simplify.	$x=\frac{6\pm\sqrt{36-20}}{2}$ $x=\frac{6\pm\sqrt{16}}{2}$ $x=\frac{6\pm 4}{2}$
Rewrite to show two solutions.	$x=\frac{6+4}{2}, \quad x=\frac{6-4}{2}$
Simplify.	$x=\frac{10}{2}, \quad x=\frac{2}{2}$
	$x=5, \quad x=1$
Check: $x^2-6x+5=0$ $5^2-6\cdot 5+5\stackrel{?}{=}0$ $25-30+5\stackrel{?}{=}0$ $0=0$ ✓ $x^2-6x+5=0$ $1^2-6\cdot 1+5\stackrel{?}{=}0$ $1-6+5\stackrel{?}{=}0$ $0=0$ ✓	

Exercise 4.5.15

Solve by using the Quadratic Formula: $a^2 - 2a = 15$.

Answer

$a = -3, a = 5$

Exercise 4.5.16

Solve by using the Quadratic Formula: $b^2 + 24 = -10b$.

Answer

$b = -6, b = -4$

When we solved quadratic equations by using the Square Root Property, we sometimes got answers that had radicals. That can happen, too, when using the **Quadratic Formula**. If we get a **radical** as a solution, the final answer must have the radical in its simplified form.

Example 4.5.17

Solve by using the Quadratic Formula: $2x^2 + 10x + 11 = 0$.

Solution:

	$2x^2 + 10x + 11 = 0$
This equation is in standard form.	$ax^2 + bx + c = 0$ $2x^2 + 10x + 11 = 0$
Identify the values of a, b and c.	$a = 2, b = 10, c = 11$
Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, and c.	$x = \frac{-(10) \pm \sqrt{(10)^2 - 4 \cdot 2 \cdot (11)}}{2 \cdot 2}$
Simplify.	$x = \frac{-10 \pm \sqrt{100 - 88}}{4}$
	$x = \frac{-10 \pm \sqrt{12}}{4}$
Simplify the radical.	$x = \frac{-10 \pm 2\sqrt{3}}{4}$
Factor out the common factor in the numerator.	$x = \frac{2(-5 \pm \sqrt{3})}{4}$ $x = \frac{2(-5 \pm \sqrt{3})}{4}$
Remove the common factors.	$x = \frac{-5 \pm \sqrt{3}}{2}$
Rewrite to show two solutions.	$x = \frac{-5 + \sqrt{3}}{2}, \quad x = \frac{-5 - \sqrt{3}}{2}$
Check: We leave the check for you!	

Exercise 4.5.18

Solve by using the Quadratic Formula: $3m^2+12m+7=0$.

Answer

$$m=\frac{-6+\sqrt{15}}{3}, m=\frac{-6-\sqrt{15}}{3}$$

Exercise 4.5.19

Solve by using the Quadratic Formula: $5n^2+4n-4=0$.

Answer

$$n=\frac{-2+2\sqrt{6}}{5}, n=\frac{-2-2\sqrt{6}}{5}$$

When we substitute a, b, and c into the Quadratic Formula and the **radicand** is negative, the quadratic equation will have imaginary or complex solutions. We will see this in the next example.

Example 4.5.20

Solve by using the Quadratic Formula: $3p^2+2p+9=0$.

Solution:

Table 9.3.5

	$3p^2+2p+9=0$
This equation is in standard form.	$ax^2+bx+c=0$ $3p^2+2p+9=0$
Identify the values of a, b, c.	$a=3, b=2, c=9$
Write the Quadratic Formula.	$p=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, c.	$p=\frac{-(2)\pm\sqrt{(2)^2-4\cdot 3\cdot(9)}}{2\cdot 3}$
Simplify.	$p=\frac{-2\pm\sqrt{4-108}}{6}$
	$p=\frac{-2\pm\sqrt{-104}}{6}$
Simplify the radical using complex numbers.	$p=\frac{-2\pm\sqrt{104}\,i}{6}$
Simplify the radical.	$p=\frac{-2\pm 2\sqrt{26}\,i}{6}$
Factor the common factor in the numerator.	$p=\frac{2\left(-1\pm\sqrt{26}\,i\right)}{6}$
Remove the common factors.	$p=\frac{-1\pm\sqrt{26}\,i}{3}$
Rewrite in standard $a+bi$ form.	$p=-\frac{1}{3}\pm\frac{\sqrt{26}\,i}{3}$
Write as two solutions.	$p=-\frac{1}{3}+\frac{\sqrt{26}\,i}{3}, \quad p=-\frac{1}{3}-\frac{\sqrt{26}\,i}{3}$

Exercise 4.5.21

Solve by using the Quadratic Formula: $4a^2 - 2a + 8 = 0$.

Answer

$$a = \frac{1}{4} + \frac{\sqrt{31}}{4}i, \quad a = \frac{1}{4} - \frac{\sqrt{31}}{4}i$$

Exercise 4.5.22

Solve by using the Quadratic Formula: $5b^2 + 2b + 4 = 0$.

Answer

$$b = -\frac{1}{5} + \frac{\sqrt{19}}{5}i, \quad b = -\frac{1}{5} - \frac{\sqrt{19}}{5}i$$

Remember, to use the Quadratic Formula, the equation must be written in standard form, $ax^2 + bx + c = 0$. Sometimes, we will need to do some algebra to get the equation into standard form before we can use the Quadratic Formula.

Example 4.5.23

Solve by using the Quadratic Formula: $x(x + 6) + 4 = 0$.

Solution:

Our first step is to get the equation in standard form.

Table 9.3.6

	$x(x+6)+4=0$
Distribute to get the equation in standard form.	$x^2+6x+4=0$
This equation is now in standard form.	$ax^2+bx+c=0$ $x^2+6x+4=0$
Identify the values of a, b, c.	$a=1,\ b=6,\ c=4$
Write the Quadratic Formula.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, c.	$x=\frac{-(6)\pm\sqrt{(6)^2-4\cdot 1\cdot(4)}}{2\cdot 1}$
Simplify.	$x=\frac{-6\pm\sqrt{36-16}}{2}$
	$x=\frac{-6\pm\sqrt{20}}{2}$
Simplify the radical.	$x=\frac{-6\pm 2\sqrt{5}}{2}$
Factor the common factor in the numerator.	$x=\frac{2(-3\pm 2\sqrt{5})}{2}$
Remove the common factors.	$x=-3\pm 2\sqrt{5}$
Write as two solutions.	$x=-3+2\sqrt{5},\ x=-3-2\sqrt{5}$
Check: We leave the check for you!	

Exercise 4.5.24

Solve by using the Quadratic Formula: $x(x+2)-5=0$.

Answer

$x=-1+\sqrt{6}, x=-1-\sqrt{6}$

Exercise 4.5.25

Solve by using the Quadratic Formula: $3y(y-2)-3=0$.

Answer

$y=1+\sqrt{2}, y=1-\sqrt{2}$

When we solved linear equations, if an equation had too many fractions we cleared the fractions by multiplying both sides of the equation by the LCD. This gave us an equivalent equation—without fractions— to solve. We can use the same strategy with quadratic equations.

Example 4.5.26

Solve by using the Quadratic Formula: $\frac{1}{2}u^2+\frac{2}{3}u=\frac{1}{3}$.

Solution:

Our first step is to clear the fractions.

Table 9.3.7

	$\frac{1}{2}u^2+\frac{2}{3}u=\frac{1}{3}$
Multiply both sides by the LCD, 6, to clear the fractions.	$6\left(\frac{1}{2}u^2+\frac{2}{3}u\right)=6\left(\frac{1}{3}\right)$
Multiply.	$3u^2+4u=2$
Subtract 2 to get the equation in standard form.	$\begin{array}{l} ax^2+bx+c=0 \\ 3u^2+4u-2=0 \end{array}$
Identify the values of a, b, and c.	$a=3,\ b=4,\ c=-2$
Write the Quadratic Formula.	$u=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, and c.	$u=\frac{-(4)\pm\sqrt{(4)^2-4\cdot 3\cdot(-2)}}{2\cdot 3}$
Simplify.	$u=\frac{-4\pm\sqrt{16+24}}{6}$
	$u=\frac{-4\pm\sqrt{40}}{6}$
Simplify the radical.	$u=\frac{-4\pm 2\sqrt{10}}{6}$
Factor the common factor in the numerator.	$u=\frac{2(-2\pm\sqrt{10})}{6}$
Remove the common factors.	$u=\frac{-2\pm\sqrt{10}}{3}$
Rewrite to show two solutions.	$u=\frac{-2+\sqrt{10}}{3}, \quad u=\frac{-2-\sqrt{10}}{3}$
Check: We leave the check for you!	

Exercise 4.5.27

Solve by using the Quadratic Formula: $\frac{1}{4}c^2 - \frac{1}{3}c = \frac{1}{12}$.

Answer

$c = \frac{2+\sqrt{7}}{3}, \quad c = \frac{2-\sqrt{7}}{3}$

Exercise 4.5.28

Solve by using the Quadratic Formula: $\frac{1}{9}d^2 - \frac{1}{2}d = -\frac{1}{3}$.

Answer

$d = \frac{9+\sqrt{33}}{4}, d = \frac{9-\sqrt{33}}{4}$

Think about the equation $(x-3)^2 = 0$. We know from the **Zero Product Property** that this equation has only one solution, $x = 3$.

We will see in the next example how using the **Quadratic Formula** to solve an equation whose standard form is a perfect square **trinomial** equal to 0 gives just one solution. Notice that once the radicand is simplified it becomes 0, which leads to only one solution.

Example 4.5.29

Solve by using the Quadratic Formula: $4x^2 - 20x = -25$.

Solution:

Table 9.3.8

	$4x^2 - 20x = -25$
Add 25 to get the equation in standard form.	$ax^2 + bx + c = 0$ $4x^2 - 20x + 25 = 0$
Identify the values of a, b, and c.	$a = 4, b = -20, c = 25$
Write the quadratic formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, and c.	$x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4 \cdot 4 \cdot (25)}}{2 \cdot 4}$
Simplify.	$x = \frac{20 \pm \sqrt{400 - 400}}{8}$
	$x = \frac{20 \pm \sqrt{0}}{8}$
Simplify the radical.	$x = \frac{20}{8}$
Simplify the fraction.	$x = \frac{5}{2}$
Check: We leave the check for you!	

Did you recognize that $4x^2 - 20x + 25$ is a perfect square trinomial. It is equivalent to $(2x-5)^2$? If you solve $4x^2 - 20x + 25 = 0$ by factoring and then using the Square Root Property, do you get the same result?

Exercise 4.5.30

Solve by using the Quadratic Formula: $r^2+10r+25=0$.

Answer

$r=-5$

Exercise 4.5.31

Solve by using the Quadratic Formula: $25t^2-40t=-16$.

Answer

$t=\frac{4}{5}$

Use the Discriminant to Predict the Number and Type of Solutions of a Quadratic Equation

When we solved the quadratic equations in the previous examples, sometimes we got two real solutions, one real solution, and sometimes two complex solutions. Is there a way to predict the number and type of solutions to a quadratic equation without actually solving the equation?

Yes, the expression under the radical of the Quadratic Formula makes it easy for us to determine the number and type of solutions. This expression is called the **discriminant**.

Definition: DISCRIMINANT

Discriminant

In the Quadratic Formula, $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$,
the quantity b^2-4ac is called the discriminant.

Figure 9.3.85

Let's look at the discriminant of the equations in some of the examples and the number and type of solutions to those quadratic equations.

Table 9.3.9

Quadratic Equation (in standard form)	Discriminat b^2-4ac	Value of the Discriminant	Number and Type of Solutions
$2x^2+9x-5=0$	$9^2-4\cdot 2(-5)$ 121	$+$	2 real
$4x^2-20x+25=0$	$(-20)^2-4\cdot 4\cdot 25$ 0	0	1 real
$3p^2+2p+9=0$	$2^2-4\cdot 3\cdot 9$ -104	$-$	2 complex

When the discriminant is positive, the quadratic equation has **2 real solutions**. $x=\frac{-b\pm\sqrt{+}}{2a}$

When the discriminant is zero, the quadratic equation has **1 real solution**. $x=\frac{-b\pm\sqrt{0}}{2a}$

When the discriminant is negative, the quadratic equation has **2 complex solutions**. $x=\frac{-b\pm\sqrt{-}}{2a}$

Figure 9.3.86

Using the Discriminant $b^2 - 4ac$, to Determine the Number and Type of Solutions of a Quadratic Equation

For a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$,

- If $b^2 - 4ac > 0$, the equation has 2 real solutions.
- if $b^2 - 4ac = 0$, the equation has 1 real solution.
- if $b^2 - 4ac < 0$, the equation has 2 complex solutions.

Example 4.5.32

Determine the number of solutions to each quadratic equation.

a. $3x^2 + 7x - 9 = 0$
b. $5n^2 + n + 4 = 0$
c. $9y^2 - 6y + 1 = 0$

Solution:

To determine the number of solutions of each quadratic equation, we will look at its discriminant.

a.

$$3x^2 + 7x - 9 = 0$$

The equation is in standard form, identify a, b, and c.

$$a = 3, \quad b = 7, \quad c = -9$$

Write the discriminant.

$$b^2 - 4ac$$

Substitute in the values of a, b, and c.

$$(7)^2 - 4 \cdot 3 \cdot (-9)$$

Simplify.

$$\begin{array}{c} 49 + 108 \\ 157 \end{array}$$

Since the discriminant is positive, there are 2 real solutions to the equation.

b.

$$5n^2 + n + 4 = 0$$

The equation is in standard form, identify a, b, and c.

$$a = 5, \quad b = 1, \quad c = 4$$

Write the discriminant.

$$b^2 - 4ac$$

Substitute in the values of a, b, and c.

$$(1)^2 - 4 \cdot 5 \cdot 4$$

Simplify.

$$\begin{array}{c} 1 - 80 \\ -79 \end{array}$$

Since the discriminant is negative, there are 2 complex solutions to the equation.

c.

$$9y^2 - 6y + 1 = 0$$

The equation is in standard form, identify a, b, and c.

$$a=9, \quad b=-6, \quad c=1$$

Write the discriminant.

$$b^2-4ac$$

Substitute in the values of a, b, and c.

$$(-6)^2-4\cdot 9\cdot 1$$

Simplify.

$$36-36$$
$$0$$

Since the discriminant is 0, there is 1 real solution to the equation.

Exercise 4.5.33

Determine the number and type of solutions to each quadratic equation.

a. $8m^2-3m+6=0$
b. $5z^2+6z-2=0$
c. $9w^2+24w+16=0$

Answer

a. 2 complex solutions
b. 2 real solutions
c. 1 real solution

Exercise 4.5.34

Determine the number and type of solutions to each quadratic equation.

a. $b^2+7b-13=0$
b. $5a^2-6a+10=0$
c. $4r^2-20r+25=0$

Answer

a. 2 real solutions
b. 2 complex solutions
c. 1 real solution

Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

We summarize the four methods that we have used to solve quadratic equations below.

Methods for Solving Quadratic Equations

1. Factoring
2. Square Root Property
3. Completing the Square
4. Quadratic Formula

Given that we have four methods to use to solve a quadratic equation, how do you decide which one to use? Factoring is often the quickest method and so we try it first. If the equation is $ax^2=k$ or $a(x-h)^2=k$ we use the Square Root Property. For any other equation, it is probably best to use the Quadratic Formula. Remember, you can solve any quadratic equation by using the Quadratic Formula, but that is not always the easiest method.

What about the method of Completing the Square? Most people find that method cumbersome and prefer not to use it. We needed to include it in the list of methods because we completed the square in general to derive the Quadratic Formula. You will also use

the process of Completing the Square in other areas of algebra.

Identify the Most Appropriate Method to Solve a Quadratic Equation

1. Try **Factoring** first. If the quadratic factors easily, this method is very quick.
2. Try the **Square Root Property** next. If the equation fits the form $ax^2 = k$ or $a(x-h)^2 = k$, it can easily be solved by using the Square Root Property.
3. Use the **Quadratic Formula**. Any other quadratic equation is best solved by using the Quadratic Formula.

The next example uses this strategy to decide how to solve each quadratic equation.

Example 4.5.35

Identify the most appropriate method to use to solve each quadratic equation.

a. $5z^2 = 17$
b. $4x^2 - 12x + 9 = 0$
c. $8u^2 + 6u = 11$

Solution:

a.

$$5z^2 = 17$$

Since the equation is in the $ax^2 = k$, the most appropriate method is to use the Square Root Property.

b.

$$4x^2 - 12x + 9 = 0$$

We recognize that the left side of the equation is a perfect square trinomial, and so factoring will be the most appropriate method.

c.

$$8u^2 + 6u = 11$$

Put the equation in standard form.

$$8u^2 + 6u - 11 = 0$$

While our first thought may be to try factoring, thinking about all the possibilities for trial and error method leads us to choose the Quadratic Formula as the most appropriate method.

Exercise 4.5.36

Identify the most appropriate method to use to solve each quadratic equation.

a. $x^2 + 6x + 8 = 0$
b. $(n-3)^2 = 16$
c. $5p^2 - 6p = 9$

Answer

a. Factoring
b. Square Root Property
c. Quadratic Formula

Exercise 4.5.37

Identify the most appropriate method to use to solve each quadratic equation.

a. $8a^2 + 3a - 9 = 0$
b. $4b^2 + 4b + 1 = 0$

c. $5c^2 = 125$

Answer

a. Quadratic Formula
b. Factoring or Square Root Property
c. Square Root Property

Access these online resources for additional instruction and practice with using the Quadratic Formula.

- Using the Quadratic Formula
- Solve a Quadratic Equation Using the Quadratic Formula with Complex Solutions
- Discriminant in Quadratic Formula

Key Concepts

- Square Root Property
  - If $x^2 = k$, then $x = \sqrt{k}$ or $x = -\sqrt{k}$or $x = \pm\sqrt{k}$

  How to solve a quadratic equation using the square root property.

  1. Isolate the quadratic term and make its coefficient one.
  2. Use Square Root Property.
  3. Simplify the radical.
  4. Check the solutions.
- Binomial Squares Pattern
  If a and b are real numbers,

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad \underbrace{(a+b)^2}_{\text{(binomial)}^2} = \underbrace{a^2}_{\text{(first term)}^2} + \underbrace{2ab}_{2\cdot\text{(product of terms)}} + \underbrace{b^2}_{\text{(second term)}^2}$$

$$(a-b)^2 = a^2 - 2ab + b^2 \qquad \underbrace{(a-b)^2}_{\text{(binomial)}^2} = \underbrace{a^2}_{\text{(first term)}^2} - \underbrace{2ab}_{2\cdot\text{(product of terms)}} + \underbrace{b^2}_{\text{(second term)}^2}$$

- How to Complete a Square
  1. Identify b, the coefficient of x.
  2. Find $\left(\frac{1}{2}b\right)^2$, the number to complete the square.
  3. Add the $\left(\frac{1}{2}b\right)^2$ to $x^2 + bx$
  4. Rewrite the trinomial as a binomial square
- Quadratic Formula
  - The solutions to a quadratic equation of the form $ax^2 + bx + c = 0, a \neq 0$ are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- How to solve a quadratic equation using the Quadratic Formula.
  1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$. Identify the values of a, b, c.
  2. Write the Quadratic Formula. Then substitute in the values of a, b, c.
  3. Simplify.
  4. Check the solutions.
- Using the Discriminant, $b^2 - 4ac$, to Determine the Number and Type of Solutions of a Quadratic Equation
  - For a quadratic equation of the form $ax^2 + bx + c = 0, a \neq 0$,
    - If $b^2 - 4ac > 0$, the equation has 2 real solutions.
    - If $b^2 - 4ac = 0$, the equation has 1 real solution.
    - If $b^2 - 4ac < 0$, the equation has 2 complex solutions.
- Methods to Solve Quadratic Equations:

- Factoring
- Square Root Property
- Completing the Square
- Quadratic Formula

- How to identify the most appropriate method to solve a quadratic equation.
  1. Try Factoring first. If the quadratic factors easily, this method is very quick.
  2. Try the **Square Root Property** next. If the equation fits the form $ax^2 = k$ or $a(x-h)^2 = k$, it can easily be solved by using the Square Root Property.
  3. Use the **Quadratic Formula.** Any other quadratic equation is best solved by using the Quadratic Formula.

Glossary

discriminant

In the Quadratic Formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the quantity $b^2 - 4ac$ is called the discriminant.

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4.5E: Exercises

Practice Makes Perfect

Solve Quadratic Equations of the Form $ax^2 + bx + c = 0$ Using Factoring Methods

1. $x^2 - x - 6 = 0$
2. $3x^2 + 7x + 4 = 0$
3. $m^2 + 5x + 4 = 0$
4. $-2a^2 + 6a - 4 = 0$
5. $p^2 + 11p + 30 = 0$
6. $y^2 - 18y + 45 = 0$

Some problems need to be rearranged a bit to fit the format $ax^2 + bx + c = 0$ before factoring to solve:

7. $a^2 + 25 + 50 = -50$
8. $p^2 + 3p + 67 = -16p + 19$
9. $-21b = 5b^2 + 4$
10. $60y^2 + 290y = 50$

Answer

1. $x = 3, x = -2$
2. $x = -\frac{1}{3}, x = -1$
3. $m = -1, m = -4$
4. $a = 1, a = 2$
5. $p = -5, p = -6$
6. $y = 3, y = 15$
7. $a = -5$
8. $p = -3, p = -16$
9. $b = -\frac{1}{5}, b = -4$
10. $y = \frac{1}{6}, y = -5$

Solve Quadratic Equations of the Form $ax^2 = k$ Using the Square Root Property

In the following exercises, solve each equation.

11. $a^2 = 49$
12. $r^2 - 24 = 0$
13. $u^2 - 300 = 0$
14. $4m^2 = 36$
15. $\frac{4}{3}x^2 = 48$
16. $x^2 + 25 = 0$
17. $x^2 + 63 = 0$
18. $\frac{4}{3}x^2 + 2 = 110$
19. $\frac{2}{5}a^2 + 3 = 11$
20. $7p^2 + 10 = 26$
21. $5y^2 - 7 = 25$

Answer

11. $a = \pm 7$

12. $r = \pm 2\sqrt{6}$
13. $u = \pm 10\sqrt{3}$
14. $m = \pm 3$
15. $x = \pm 6$
16. $x = \pm 5i$
17. $x = \pm 3\sqrt{7}i$
18. $x = \pm 9$
19. $a = \pm 2\sqrt{5}$
20. $p = \pm \frac{4}{\sqrt{7}}$
21. $y = \pm \frac{4\sqrt{2}}{\sqrt{5}}$

Solve Quadratic Equations Using the Quadratic Formula

In the following exercises, solve by using the Quadratic Formula.

22. $4m^2 + m - 3 = 0$
23. $2p^2 - 7p + 3 = 0$
24. $p^2 + 7p + 12 = 0$
25. $r^2 - 8r = 33$
26. $3u^2 + 7u - 2 = 0$
27. $2a^2 - 6a + 3 = 0$
28. $x^2 + 8x - 4 = 0$
29. $3y^2 + 5y - 2 = 0$
30. $2x^2 + 3x + 3 = 0$
31. $8x^2 - 6x + 2 = 0$
32. $(v+1)(v-5) - 4 = 0$
33. $\frac{1}{4}m^2 + \frac{1}{12}m = \frac{1}{3}$
34. $16c^2 + 24c + 9 = 0$
35. $25q^2 + 30q + 9 = 0$

Answer

22. $m = -1, m = \frac{3}{4}$
23. $p = \frac{1}{2}, p = 3$
24. $p = -4, p = -3$
25. $r = -3, r = 11$
26. $u = \frac{-7 \pm \sqrt{73}}{6}$
27. $a = \frac{3 \pm \sqrt{3}}{2}$
28. $x = -4 \pm 2\sqrt{5}$
29. $y = \frac{1}{3}, y = -2$
30. $x = -\frac{3}{4} \pm \frac{\sqrt{15}}{4}i$
31. $x = \frac{3}{8} \pm \frac{\sqrt{7}}{8}i$
32. $v = 2 \pm \sqrt{13}$
33. $m = 1, m = \frac{-4}{3}$
34. $c = -\frac{3}{4}$

35. $q = -\frac{3}{5}$

Use the Discriminant to Predict the Number of Real Solutions of a Quadratic Equation

36. In the following exercises, determine the number of real solutions for each quadratic equation.

a. $4x^2 - 5x + 16 = 0$
b. $36y^2 + 36y + 9 = 0$
c. $6m^2 + 3m - 5 = 0$

Answer

36. a. no real solutions b. 1 c. 2

37. In the following exercises, determine the number of real solutions for each quadratic equation.

a. $r^2 + 12r + 36 = 0$
b. $8t^2 - 11t + 5 = 0$
c. $3v^2 - 5v - 1 = 0$

Answer

37. a. 1 b. no real solutions c. 2

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4.6: Graph Quadratic Functions Using Properties

Learning Objectives

By the end of this section, you will be able to:

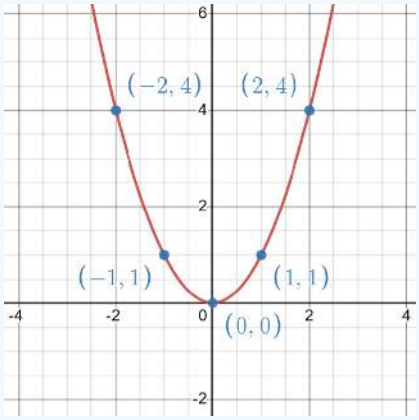
- Recognize the graph of a quadratic function
- Find the axis of symmetry and vertex of a parabola
- Find the intercepts of a parabola
- Graph quadratic functions using properties
- Solve maximum and minimum applications

Warm-up

1. Graph the function $f(x) = x^2$ by plotting points.
2. Solve: $2x^2 + 3x - 2 = 0$.
3. Evaluate $-\frac{b}{2a}$ when $a = 3$ and $b = -6$.

Answer

1.



b. $x = \frac{1}{2}, x = -2$

c. 1

Recognize the Graph of a Quadratic Function

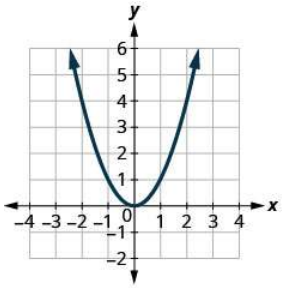
Previously we very briefly looked at the function $f(x) = x^2$, which we called the square function. It was one of the first non-linear functions we looked at. Now we will graph functions of the form $f(x) = ax^2 + bx + c$ if $a \neq 0$. We call this kind of function a quadratic function.

Definition 4.6.1

A **quadratic function**, where a, b, and c are real numbers and $a \neq 0$, is a function of the form

$$f(x) = ax^2 + bx + c$$

We graphed the quadratic function $f(x) = x^2$ by plotting points.



x	$f(x) = x^2$	$(x, f(x))$
–3	9	(–3, 9)
–2	4	(–2, 4)
–1	1	(–1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	4	(2, 4)
3	9	(3, 9)

Figure 9.6.1

Every quadratic function has a graph that looks like this. We call this figure a **parabola**. Let's practice graphing a parabola by plotting a few points.

Example 4.6.1

Graph: $f(x) = x^2 - 1$.

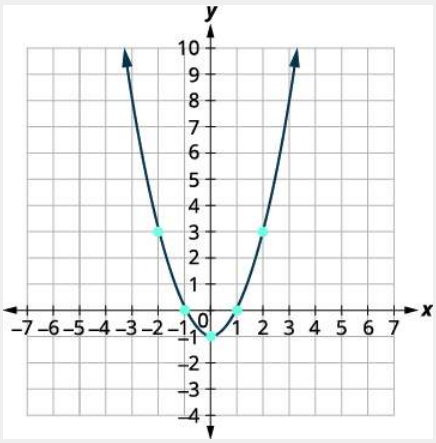
Solution:

We will graph the function by plotting points.

Choose integer values for x,
substitute them into the equation
and simplify to find $f(x)$.
Record the values of the ordered pairs in the chart.

$f(x) = x^2 - 1$	
x	$f(x)$
0	–1
1	0
–1	0
2	3
–2	3

Plot the points, and then connect
them with a smooth curve. The
result will be the graph of the
function $f(x) = x^2 - 1$.

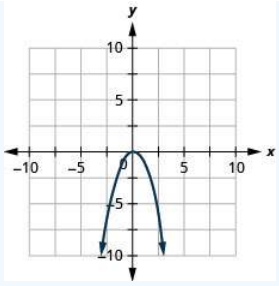


Exercise 4.6.2

Graph $f(x) = -x^2$.

Answer

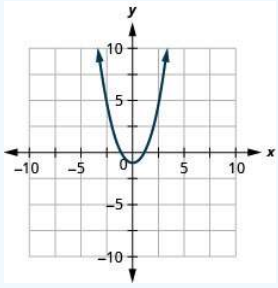
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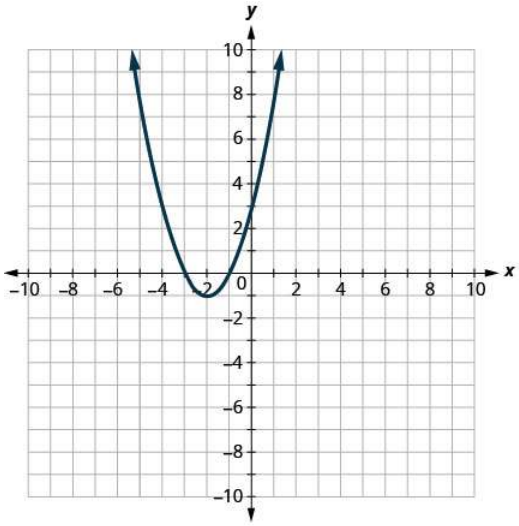
Exercise 4.6.3

Graph $f(x) = x^2 - 1$.

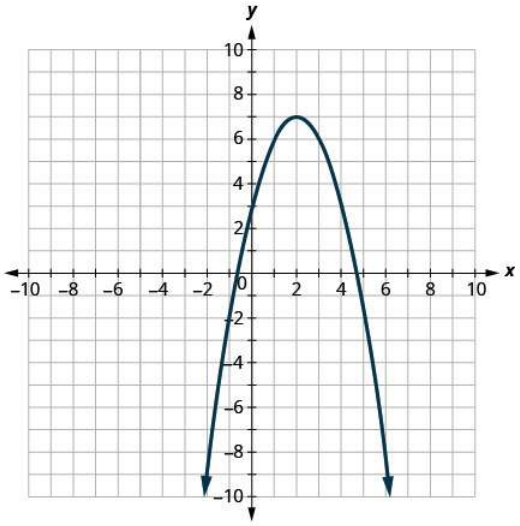
Answer



All graphs of quadratic functions of the form $f(x) = ax^2 + bx + c$ are parabolas that open upward or downward. See *Figure 9.6.6*



$f(x) = ax^2 + bx + c$
$f(x) = x^2 + 4x + 3$
$a > 0$
opens upward



$f(x) = ax^2 + bx + c$
$f(x) = -x^2 + 4x + 3$
$a < 0$
opens downward

Notice that the only difference in the two functions is the negative sign before the quadratic term (x^2 in the equation of the graph in *Figure 9.6.6*). When the quadratic term, is positive, the parabola opens upward, and when the quadratic term is negative, the parabola opens downward.

Definition: PARABOLA

Parabola Orientation

For the graph of the quadratic function $f(x) = ax^2 + bx + c$, if

- $a > 0$, the parabola opens upward 
- $a < 0$, the parabola opens downward 

Example 4.6.4

Determine whether each parabola opens upward or downward:

a. $f(x) = -3x^2 + 2x - 4$
b. $f(x) = 6x^2 + 7x - 9$

Solution:

a. Find the value of a.

$$f(x) = ax^2 + bx + c$$
$$f(x) = -3x^2 + 2x - 4$$
$$a = -3$$

Since the a is negative, the parabola will open downward.

b. Find the value of a.

$$f(x) = ax^2 + bx + c$$
$$f(x) = 6x^2 + 7x - 9$$
$$a = 6$$

Since the a is positive, the parabola will open upward.

Exercise 4.6.4

Determine whether the graph of each function is a parabola that opens upward or downward:

a. $f(x) = 2x^2 + 5x - 2$
b. $f(x) = -3x^2 - 4x + 7$

Answer

a. up
b. down

Exercise 4.6.5

Determine whether the graph of each function is a parabola that opens upward or downward:

a. $f(x) = -2x^2 - 2x - 3$
b. $f(x) = 5x^2 - 2x - 1$

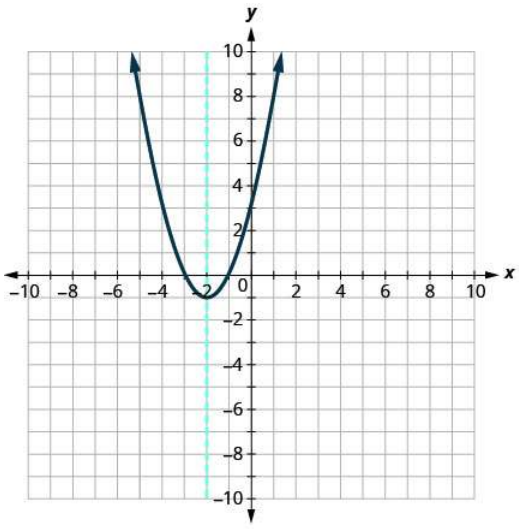
Answer

a. down
b. up

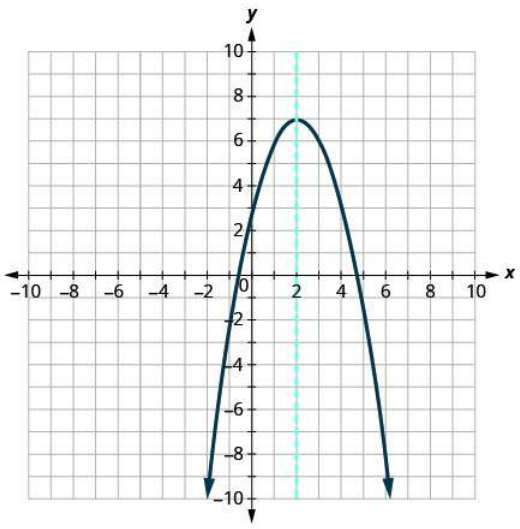
Find the Axis of Symmetry and Vertex of a Parabola

Look again at *Figure 9.6.10*. Do you see that we could fold each parabola in half and then one side would lie on top of the other? The 'fold line' is a line of symmetry. We call it the **axis of symmetry** of the parabola.

We show the same two graphs again with the axis of symmetry.



$f(x) = x^2 + 4x + 3$



$f(x) = -x^2 + 4x + 3$

The equation of the axis of symmetry can be derived by using the Quadratic Formula. We will omit the derivation here and proceed directly to using the result. The equation of the axis of symmetry of the graph of $f(x) = ax^2 + bx + c$ is $x = -\frac{b}{2a}$.

So to find the equation of symmetry of each of the parabolas we graphed above, we will substitute into the formula $x = -\frac{b}{2a}$.

$f(x) = ax^2 + bx + c$	$f(x) = ax^2 + bx + c$
$f(x) = x^2 + 4x + 3$	$f(x) = -x^2 + 4x + 3$
axis of symmetry	axis of symmetry
$x = -\frac{b}{2a}$	$x = -\frac{b}{2a}$
$x = -\frac{4}{2 \cdot 1}$	$x = -\frac{4}{2(-1)}$
$x = -2$	$x = 2$

Notice that these are the equations of the dashed blue lines on the graphs.

The point on the parabola that is the lowest (parabola opens up), or the highest (parabola opens down), lies on the axis of symmetry. This point is called the **vertex** of the parabola.

We can easily find the coordinates of the vertex, because we know it is on the axis of symmetry. This means its x-coordinate is $-\frac{b}{2a}$. To find the y-coordinate of the vertex we substitute the value of the x-coordinate into the quadratic function.

$f(x) = x^2 + 4x + 3$	$f(x) = -x^2 + 4x + 3$
axis of symmetry is $x = -2$	axis of symmetry is $x = 2$
vertex is $(-2, \_\_)$	vertex is $(2, \_\_)$
$f(x) = x^2 + 4x + 3$	$f(x) = -x^2 + 4x + 3$
$f(x) = (-2)^2 + 4(-2) + 3$	$f(x) = -(2)^2 + 4(2) + 3$
$f(x) = -1$	$f(x) = 7$
vertex is $(-2, -1)$	vertex is $(2, 7)$

Axis of Symmetry and Vertex of a Parabola

The graph of the function $f(x) = ax^2 + bx + c$ is a parabola where:

- the axis of symmetry is the vertical line $x = -\frac{b}{2a}$.
- the vertex is a point on the axis of symmetry, so its x-coordinate is $-\frac{b}{2a}$
- the y-coordinate of the vertex is found by substituting $x = -\frac{b}{2a}$ into the quadratic equation.

Example 4.6.6

For the graph of $f(x) = 3x^2 - 6x + 2$ find:

a. the axis of symmetry
b. the vertex

Solution:

a.

	$f(x) = ax^2 + bx + c$ $f(x) = 3x^2 - 6x + 2$
The axis of symmetry is the vertical line $x = -\frac{b}{2a}$.	
Substitute the values a, b into the equation.	$x = -\frac{-6}{2 \cdot 3}$
Simplify.	$x = 1$
	The axis of symmetry is the line $x = 1$.

b.

	$f(x) = 3x^2 - 6x + 2$
The vertex is a point on the line of symmetry, so its x-coordinate will be $x = 1$. Find $f(1)$.	$f(1) = 3(1)^2 - 6(1) + 2$
Simplify.	$f(1) = 3 \cdot 1 - 6 + 2$
The result is the y-coordinate.	$f(1) = -1$
	The vertex is $(1, -1)$.

Exercise 4.6.7

For the graph of $f(x) = 2x^2 - 8x + 1$ find:

a. the axis of symmetry
b. the vertex

Answer

a. $x = 2$
b. $(2, -7)$

Exercise 4.6.8

For the graph of $f(x) = 2x^2 - 4x - 3$ find:

a. the axis of symmetry
b. the vertex

Answer

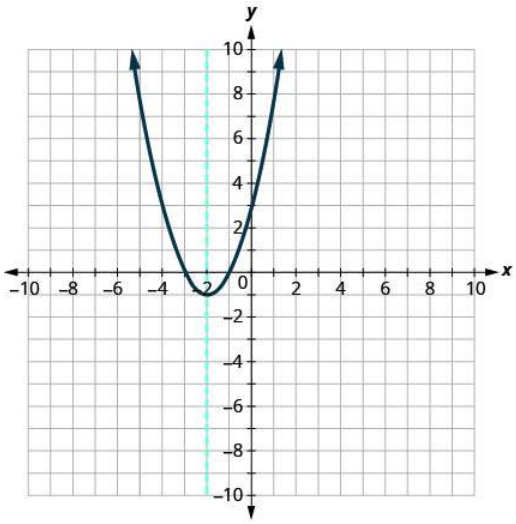
a. $x = 1$
b. $(1, -5)$

Find the Intercepts of a Parabola

When we graphed linear equations, we often used the x- and y-intercepts to help us graph the lines. Finding the coordinates of the intercepts will help us to graph parabolas, too.

Remember, at the y-intercept the value of x is zero. So to find the y-intercept, we substitute $x = 0$ into the function.

Let's find the y-intercepts of the two parabolas shown in *Figure 9.6.20*.

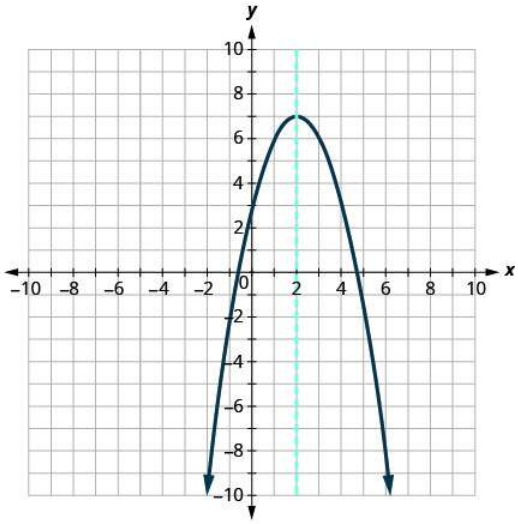


$f(x) = x^2 + 4x + 3$

$x = 0 \quad f(0) = 0^2 + 4 \cdot 0 + 3$

$f(0) = 3$

y-intercept $(0, 3)$



$f(x) = -x^2 + 4x + 3$

$x = 0 \quad f(0) = -0^2 + 4 \cdot 0 + 3$

$f(0) = 3$

y-intercept $(0, 3)$

An x-intercept results when the value of $f(x)$ is zero. To find an x-intercept, we let $f(x) = 0$. In other words, we will need to solve the equation $0 = ax^2 + bx + c$ for x.

$$f(x) = ax^2 + bx + c$$
$$0 = ax^2 + bx + c$$

Solving quadratic equations like this is exactly what we have done earlier in this chapter!

We can now find the x-intercepts of the two parabolas we looked at. First we will find the x-intercepts of the parabola whose function is $f(x) = x^2 + 4x + 3$.

	$f(x) = x^2 + 4x + 3$
Let $f(x) = 0$.	$0 = x^2 + 4x + 3$
Factor.	$0 = (x + 1)(x + 3)$
Use the Zero Product Property.	$x + 1 = 0 \quad x + 3 = 0$
Solve.	$x = -1 \quad x = -3$
	The x-intercepts are $(-1, 0)$ and $(-3, 0)$.

Now we will find the x-intercepts of the parabola whose function is $f(x) = -x^2 + 4x + 3$.

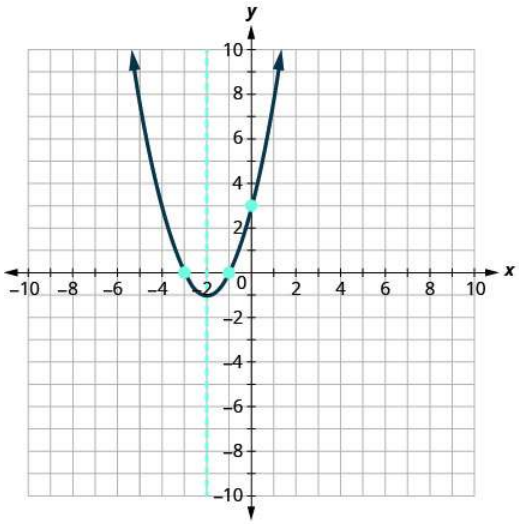
	$f(x) = -x^2 + 4x + 3$
Let $f(x) = 0$.	$0 = -x^2 + 4x + 3$
This quadratic does not factor, so we use the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$a=-1, b=4, c=3$	$x=\frac{-4\pm\sqrt{4^2-4(-1)(3)}}{2(-1)}$
Simplify.	$x=\frac{-4\pm\sqrt{28}}{-2}$
	$x=\frac{-4\pm2\sqrt{7}}{-2}$
	$x=\frac{-2(2\pm\sqrt{7})}{-2}$
	$x=2\pm\sqrt{7}$
	The x-intercepts are $(2+\sqrt{7},0)$ and $(2-\sqrt{7},0)$.

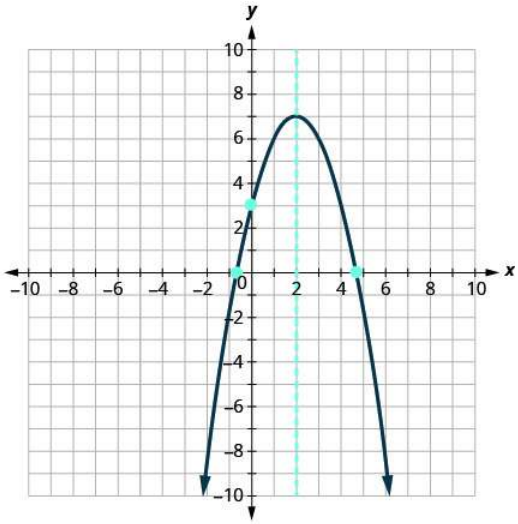
We will use the decimal approximations of the x-intercepts, so that we can locate these points on the graph,

$$(2+\sqrt{7},0)\approx(4.6,0)\quad(2-\sqrt{7},0)\approx(-0.6,0)$$

Do these results agree with our graphs? See *Figure 9.6.34*



$f(x)=x^2+4x+3$
y-intercept $(0,3)$
x-intercepts $(-1,0)$ and $(-3,0)$



$f(x)=-x^2+4x+3$
y-intercept $(0,3)$
x-intercepts $(2+\sqrt{7},0)\approx(4.6,0)$
$(2-\sqrt{7},0)\approx(-0.6,0)$

Find the Intercepts of a Parabola

To find the intercepts of a parabola whose function is $f(x)=ax^2+bx+c$:

y-intercept

Let $x=0$ and solve for $f(x)$.

x-intercepts

Let $f(x)=0$ and solve for x

Example 4.6.8

Find the intercepts of the parabola whose function is $f(x)=x^2-2x-8$.

Solution:

To find the y-intercept, let $x=0$ and solve for $f(x)$.	$f(x)=x^2-2x-8$
	$f(0)=0^2-2\cdot0-8$

	$f(0) = -8$
	When $x = 0$, then $f(0) = -8$. The y-intercept is the point $(0, -8)$.
To find the x-intercept, let $f(x) = 0$ and solve for x.	$f(x) = x^2 - 2x - 8$
	$0 = x^2 - 2x - 8$
Solve by factoring.	$0 = (x - 4)(x + 2)$
	$0 = x - 4 \quad 0 = x + 2$
	$4 = x \quad -2 = x$
	When $f(x) = 0$, then $x = 4$ or $x = -2$. The x-intercepts are the points $(4, 0)$ and $(-2, 0)$.

? Exercise 4.6.9

Find the intercepts of the parabola whose function is $f(x) = x^2 + 2x - 8$.

Answer

y -intercept: $(0, -8)$ x -intercepts $(-4, 0), (2, 0)$

? Exercise 4.6.10

Find the intercepts of the parabola whose function is $f(x) = x^2 - 4x - 12$.

Answer

y -intercept: $(0, -12)$ x -intercepts $(-2, 0), (6, 0)$

In this chapter, we have been solving quadratic equations of the form $ax^2 + bx + c = 0$. We solved for x and the results were the solutions to the equation.

We are now looking at quadratic functions of the form $f(x) = ax^2 + bx + c$. The graphs of these functions are parabolas. The x-intercepts of the parabolas occur where $f(x) = 0$.

For example:

Quadratic equation

$$\begin{aligned} x^2 - 2x - 15 &= 0 \quad \text{Let } f(x) = 0 \\ (x - 5)(x + 3) &= 0 \\ x - 5 = 0 \quad x + 3 &= 0 \\ x = 5 \quad x &= -3 \end{aligned}$$

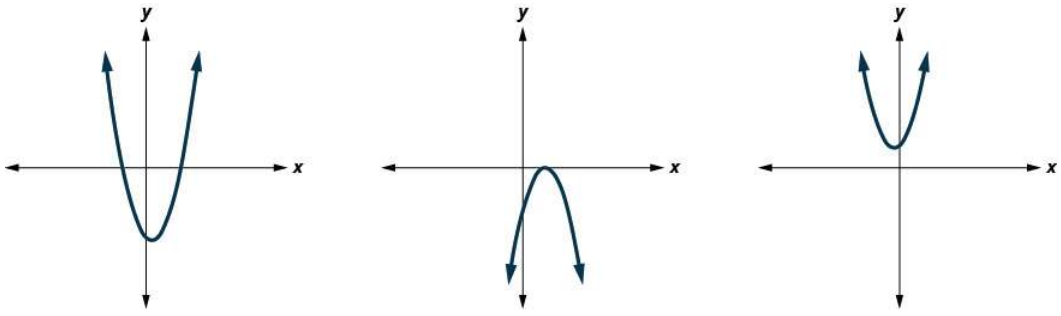
Quadratic function

$$\begin{aligned} f(x) &= x^2 - 2x - 15 \\ 0 &= x^2 - 2x - 15 \\ 0 &= (x - 5)(x + 3) \\ x - 5 = 0 \quad & x + 3 = 0 \\ x = 5 \quad & x = -3 \\ (5, 0) \text{ and } &(-3, 0) \\ x \text{ -intercepts} & \end{aligned}$$

The solutions of the quadratic function are the x values of the x-intercepts.

Earlier, we saw that quadratic equations have $2, 1$, or 0 solutions. The graphs below show examples of parabolas for these three cases. Since the solutions of the functions give the x-intercepts of the graphs, the number of x-intercepts is the same as the number of solutions.

Previously, we used the **discriminant** to determine the number of solutions of a quadratic function of the form $ax^2 + bx + c = 0$. Now we can use the discriminant to tell us how many x-intercepts there are on the graph.



$b^2 - 4ac > 0$
Two solutions
Two x-intercepts

$b^2 - 4ac = 0$
One solution
One x-intercept

$b^2 - 4ac < 0$
No real solution
No x-intercept

Before you to find the values of the x-intercepts, you may want to evaluate the discriminant so you know how many solutions to expect.

Example 4.6.11

Find the intercepts of the parabola for the function $f(x) = 5x^2 + x + 4$.

Solution:

	$f(x) = 5x^2 + x + 4$
To find the y-intercept, let $x = 0$ and solve for $f(x)$.	$f(0) = 5 \cdot 0^2 + 0 + 4$
	$f(0) = 4$
	When $x = 0$, then $f(0) = 4$. The y-intercept is the point $(0, 4)$.
To find the x-intercept, let $f(x) = 0$ and solve for x.	$f(x) = 5x^2 + x + 4$
	$0 = 5x^2 + x + 4$
Find the value of the discriminant to predict the number of solutions which is also the number of x-intercepts.	
$b^2 - 4ac$ $1^2 - 4 \cdot 5 \cdot 4$ $1 - 80$ -79	
	Since the value of the discriminant is negative, there is no real solution to the equation. There are no x-intercepts.

Exercise 4.6.12

Find the intercepts of the parabola whose function is $f(x) = 3x^2 + 4x + 4$.

Answer

y-intercept: $(0, 4)$ no x-intercept

Exercise 4.6.13

Find the intercepts of the parabola whose function is $f(x) = x^2 - 4x - 5$

Answer

y-intercept: $(0, -5)$ x-intercepts $(-1, 0), (5, 0)$

Graph Quadratic Functions Using Properties

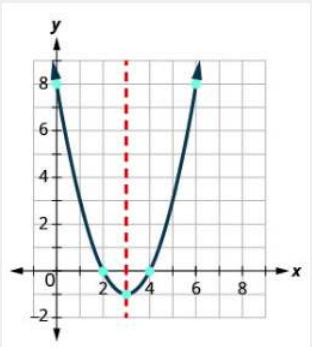
Now we have all the pieces we need in order to graph a quadratic function. We just need to put them together. In the next example we will see how to do this.

Example 4.6.14

Graph $f(x) = x^2 - 6x + 8$ by using its properties.

Solution:

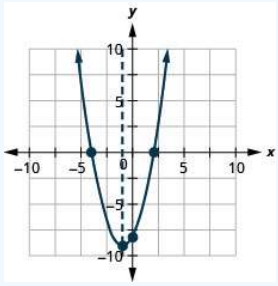
Step 1: Determine whether the parabola opens upward or downward.	Look a in the equation $f(x) = x^2 - 6x + 8$ Since a is positive, the parabola opens upward.	$f(x) = x^2 - 6x + 8$ $a = 1, b = -6, c = 8$ **The parabola opens upward.**
Step 2: Find the axis of symmetry.	$f(x) = x^2 - 6x + 8$ The axis of symmetry is the line $x = -\frac{b}{2a}$.	Axis of Symmetry $x = -\frac{b}{2a}$ $x = -\frac{(-6)}{2 \cdot 1}$ $x = 3$ **The axis of symmetry is the line $x = 3$.**
Step 3: Find the vertex.	The vertex is on the axis of symmetry. Substitute $x = 3$ into the function.	Vertex $f(x) = x^2 - 6x + 8$ $f(3) = (3)^2 - 6(3) + 8$ $f(3) = -1$ **The vertex is $(3, -1)$.**
Step 4: Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.	We find $f(0)$. We use the axis of symmetry to find a point symmetric to the y-intercept. The y-intercept is 3 units left of the axis of symmetry, $x = 3$. A point 3 units to the right of the axis of symmetry has $x = 6$.	y-intercept $f(x) = x^2 - 6x + 8$ $f(0) = (0)^2 - 6(0) + 8$ $f(0) = 8$ **The y-intercept is $(0, 8)$.** Point symmetric to y-intercept: **The point is $(6, 8)$.**
Step 5: Find the x-intercepts. Find additional points if needed.	We solve $f(x) = 0$. We can solve this quadratic equation by factoring.	x-intercepts $f(x) = x^2 - 6x + 8$ $0 = x^2 - 6x + 8$ $0 = (x - 2)(x - 4)$ $x = 2 or x = 4$ **The x-intercepts are $(2, 0)$ and $(4, 0)$.**

Step 6: Graph the parabola.	We graph the vertex, intercepts, and the point symmetric to the y-intercept. We connect these 5 points to sketch the parabola.	

Exercise 4.6.15

Graph $f(x) = x^2 + 2x - 8$ by using its properties.

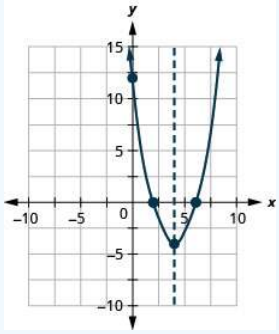
Answer



Exercise 4.6.16

Graph $f(x) = x^2 - 8x + 12$ by using its properties.

Answer



We list the steps to take in order to graph a quadratic function here.

To Graph a Quadratic Function Using Properties

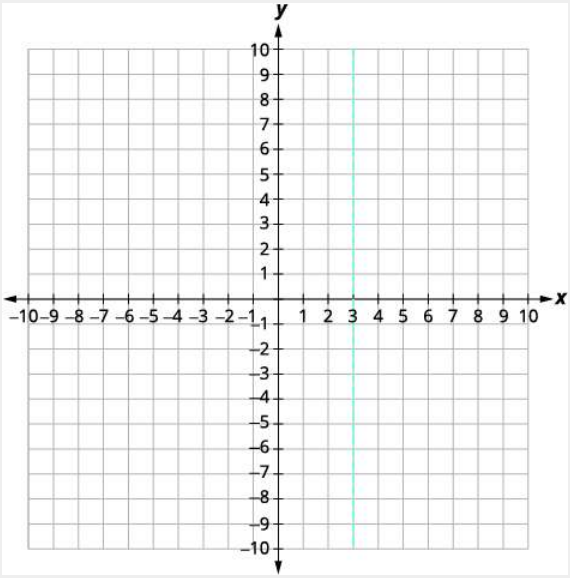
1. Determine whether the parabola opens upward or downward.
2. Find the equation of the axis of symmetry.
3. Find the vertex.
4. Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.
5. Find the x-intercepts. Find additional points if needed.
6. Graph the parabola.

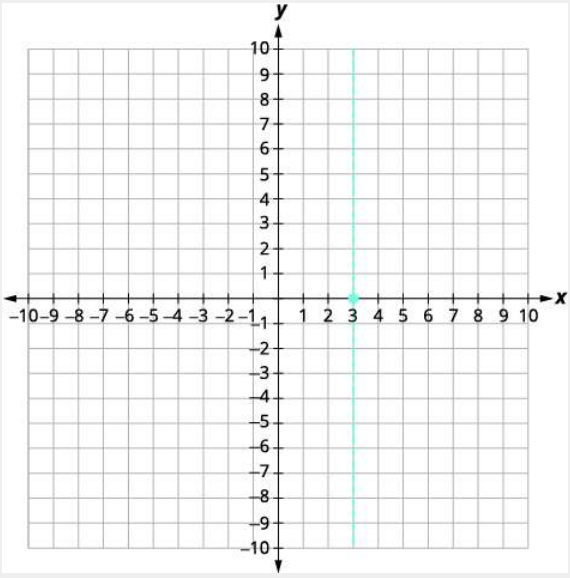
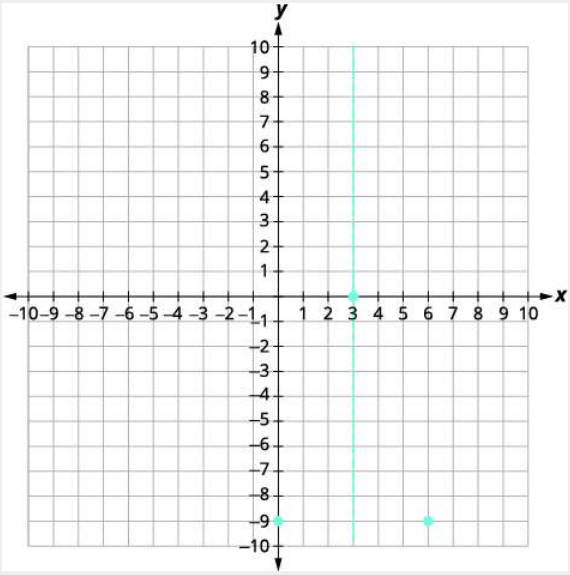
We were able to find the x-intercepts in the last example by factoring. We find the x-intercepts in the next example by factoring, too.

Example 4.6.17

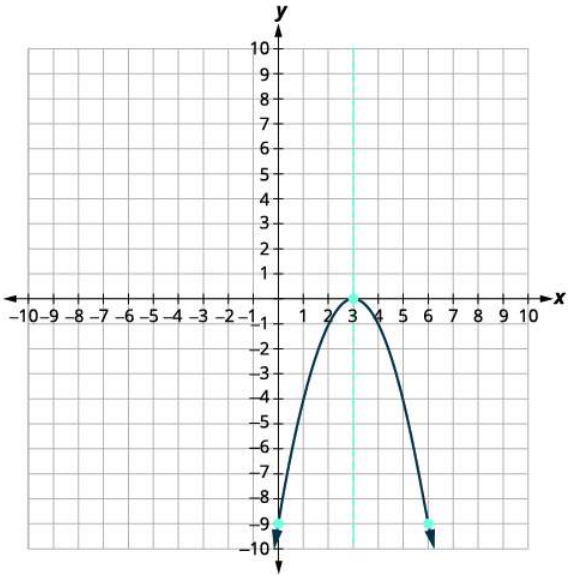
Graph $f(x) = -x^2 + 6x - 9$ by using its properties.

Solution:

	$f(x) = ax^2 + bx + c$ $f(x) = -x^2 + 6x - 9$
Since a is -1, the parabola opens downward.	
	
To find the equation of the axis of symmetry, use $x = -\frac{b}{2a}$.	$x = -\frac{b}{2a}$
	$x = -\frac{6}{2(-1)}$
	$x = 3$
	The axis of symmetry is $x = 3$. The vertex is on the line $x = 3$.
	
Find $f(3)$.	$f(x) = -x^2 + 6x - 9$
	$f(3) = -3^2 + 6 \cdot 3 - 9$
	$f(3) = -9 + 18 - 9$
	$f(3) = 0$
	The vertex is $(3, 0)$.

	
The y-intercept occurs when $x = 0$. Find $f(0)$.	$f(x) = -x^2 + 6x - 9$
Substitute $x = 0$.	$f(0) = -0^2 + 6 \cdot 0 - 9$
Simplify.	$f(0) = -9$
The point $(0, -9)$ is three units to the left of the line of symmetry. The point three units to the right of the line of symmetry is $(6, -9)$.	
	Point symmetric to the y-intercept is $(6, -9)$
The x-intercept occurs when $f(x) = 0$.	$f(x) = -x^2 + 6x - 9$
Find $f(x) = 0$.	$0 = -x^2 + 6x - 9$
Factor the GCF.	$0 = -(x^2 - 6x + 9)$
Factor the trinomial.	$0 = -(x - 3)^2$
Solve for x.	$0 = 3$

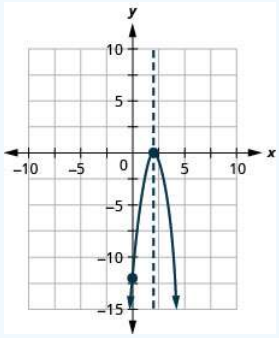
Connect the points to graph the parabola.



Exercise 4.6.18

Graph $f(x) = 3x^2 + 12x - 12$ by using its properties.

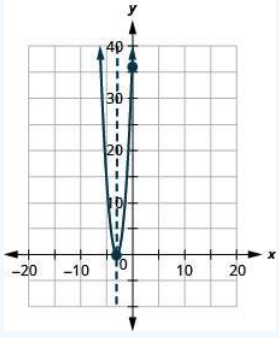
Answer



Exercise 4.6.19

Graph $f(x) = 4x^2 + 24x + 36$ by using its properties.

Answer



For the graph of $f(x) = -x^2 + 6x - 9$, the vertex and the x-intercept were the same point. Remember how the discriminant determines the number of solutions of a quadratic equation? The discriminant of the equation $0 = -x^2 + 6x - 9$ is 0, so there is

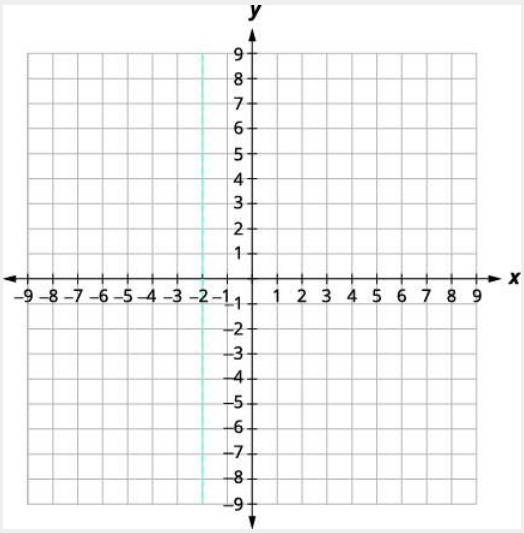
only one solution. That means there is only one x-intercept, and it is the vertex of the parabola.

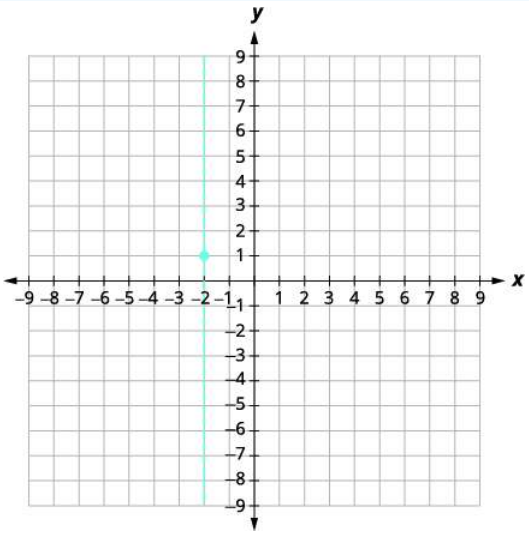
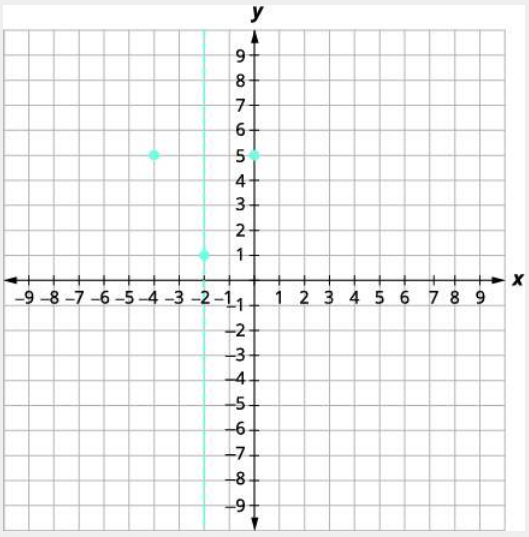
How many x-intercepts would you expect to see on the graph of $f(x) = x^2 + 4x + 5$?

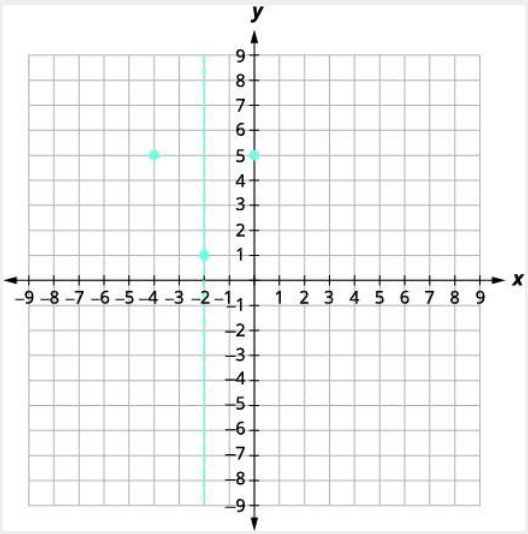
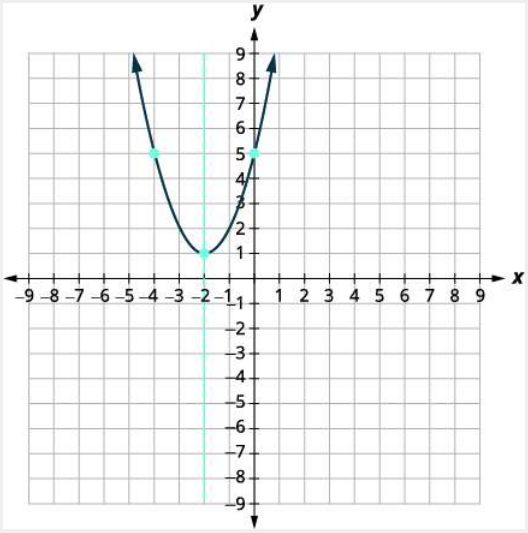
Example 4.6.20

Graph $f(x) = x^2 + 4x + 5$ by using its properties.

Solution:

	$f(x) = ax^2 + bx + c$ $f(x) = -x^2 + 6x - 9$
Since a is -1, the parabola opens downward.	
	
To find the equation of the axis of symmetry, use $x = -\frac{b}{2a}$.	$x = -\frac{b}{2a}$
	$x = -\frac{4}{(2)1}$
	$x = -2$
	The equation of the axis of symmetry is \(x=-2).
	
The vertex is on the line $x = -2$.	
Find $f(x)$ when $x = -2$.	$f(x) = x^2 + 4x + 5$
	$f(-2) = (-2)^2 + 4(-2) + 5$
	$f(-2) = 4 - 8 + 5$
	$f(-2) = 1$
	The vertex is $(-2, 1)$.

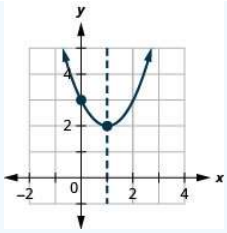
	
The y-intercept occurs when $x = 0$.	$f(x) = x^2 + 4x - 5$
Find $f(0)$.	$f(0) = 5$
Simplify.	$f(0) = 5$
	The y-intercept is $(0, 5)$.
The point $(-4, 5)$ is two units to the left of the line of symmetry. The point to units to the right of the line of symmetry is \((0,5)\.	
	Point symmetric to the y-intercept is $(-4, 5)$.

The x-intercept occurs when $f(x) = 0$.	
Find $f(x) = 0$.	$0 = x^2 + 4x + 5$
Test the discriminant.	
	$b^2 - 4ac$
	$4^2 - 4 \cdot 1 \cdot 5$
	$16 - 20$
	-4
Since the value of the discriminant is negative, there is no real solution and so no x-intercept.	
Connect the points to graph the parabola. You may want to choose two more points for greater accuracy.	

Exercise 4.6.21

Graph $f(x) = x^2 - 2x + 3$ by using its properties.

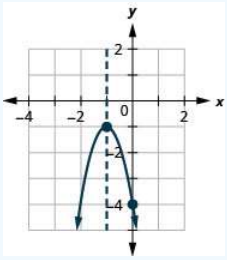
Answer



Exercise 4.6.22

Graph $f(x) = -3x^2 - 6x - 4$ by using its properties.

Answer



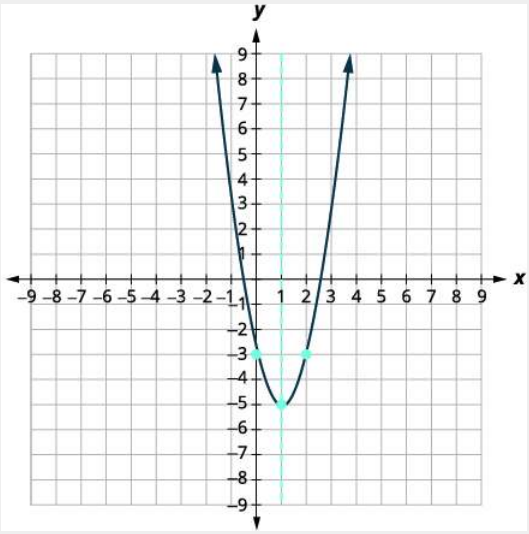
Finding the y-intercept by finding $f(0)$ is easy, isn't it? Sometimes we need to use the **Quadratic Formula** to find the x-intercepts.

Example 4.6.23

Graph $f(x) = 2x^2 - 4x - 3$ by using its properties.

Solution:

	$f(x) = ax^2 + bx + c$ $f(x) = 2x^2 - 4x - 3$
Since a is 2, the parabola opens upward.	
To find the equation of the axis of symmetry, use $x = -\frac{b}{2a}$.	$x = -\frac{b}{2a}$
	$x = -\frac{-4}{2 \cdot 2}$
	$x = 1$
	The equation of the axis of symmetry is $x = 1$.
The vertex is on the line $x = 1$.	$f(x) = 2x^2 - 4x - 3$
Find $f(1)$.	$f(x) = 2(1)^2 - 4(1) - 3$
	$f(1) = 2 - 4 - 3$
	\(\f(1)=-5)
	The vertex is $(1, -5)$.
The y-intercept occurs when $x = 0$.	$f(x) = 2x^2 - 4x - 3$
Find $f(0)$.	$f(0) = 2(0)^2 - 4(0) - 3$
Simplify.	$f(0) = -3$
	The y-intercept is $(0, -3)$.
The point $(0, -3)$ is one unit to the left of the line of symmetry.	Point symmetric to the y-intercept is $(2, -3)$

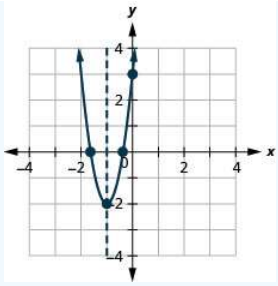
The point one unit to the right of the line of symmetry is $(2, 3)$.	
The x-intercept occurs when $y = 0$.	$f(x) = 2x^2 - 4x - 3$
Find $f(x) = 0$.	$0 = 2x^2 - 4x - 3$
Use the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Substitute in the values of a, b and c.	$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(3)}}{2(2)}$
Simplify.	$x = \frac{-4 \pm \sqrt{16 + 24}}{4}$
Simplify inside the radical.	$x = \frac{4 \pm \sqrt{40}}{4}$
Simplify the radical.	$x = \frac{4 \pm 2\sqrt{10}}{4}$
Factor the GCF.	$x = \frac{2(2 \pm \sqrt{10})}{4}$
Remove common factors.	$x = \frac{2 \pm \sqrt{10}}{2}$
Write as two equations.	$x = \frac{2 + \sqrt{10}}{2}, \quad x = \frac{2 - \sqrt{10}}{2}$
Approximate the values.	$x \approx 2.5, \quad x \approx -0.6$
	The approximate values of the x-intercepts are $(2.5, 0)$ and $(-0.6, 0)$.
Graph the parabola using the points found.	

Exercise 4.6.24

Graph $f(x) = 5x^2 + 10x + 3$ by using its properties.

Answer

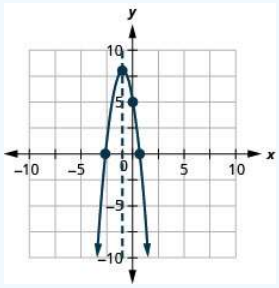
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? Exercise 4.6.25

Graph $f(x) = -3x^2 - 6x + 5$ by using its properties.

Answer



Key Concepts

- Parabola Orientation
  - For the graph of the quadratic function $f(x) = ax^2 + bx + c$, if
    - $a > 0$, the parabola opens upward.
    - $a < 0$, the parabola opens downward.
- Axis of Symmetry and Vertex of a Parabola The graph of the function $f(x) = ax^2 + bx + c$ is a parabola where:
  - the axis of symmetry is the vertical line $x = -\frac{b}{2a}$.
  - the vertex is a point on the axis of symmetry, so its x-coordinate is $-\frac{b}{2a}$.
  - the y-coordinate of the vertex is found by substituting $x = -\frac{b}{2a}$ into the quadratic equation.
- Find the Intercepts of a Parabola
  - To find the intercepts of a parabola whose function is $f(x) = ax^2 + bx + c$:
    - **y-intercept**
      - Let $x = 0$ and solve for $f(x)$.
    - **x-intercepts**
      - Let \(f(x)=0\) and solve for x.
- How to graph a quadratic function using properties.
  1. Determine whether the parabola opens upward or downward.
  2. Find the equation of the axis of symmetry.
  3. Find the vertex.
  4. Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.
  5. Find the x-intercepts. Find additional points if needed.
  6. Graph the parabola.

Glossary

quadratic function

A quadratic function, where a, b, and c are real numbers and $a \neq 0$, is a function of the form $f(x) = ax^2 + bx + c$.

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4.6E: Exercises

Practice Makes Perfect

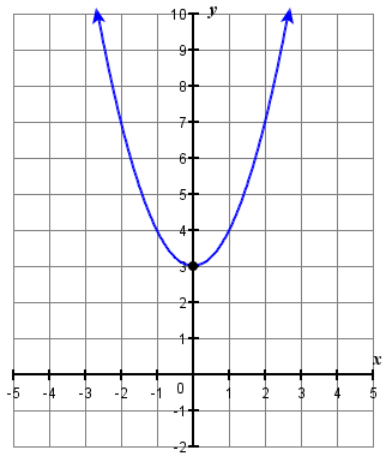
Exercises 1 - 4: Recognize the Graph of a Quadratic Function

In the following exercises, graph the functions by plotting points.

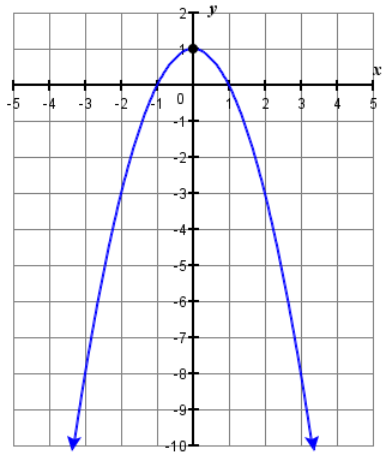
1. $f(x) = x^2 + 3$
2. $y = -x^2 + 1$

Answer

1.



2.



Exercises 5 - 8: Recognize the Graph of a Quadratic Function

For each of the following exercises, determine if the parabola opens up or down.

3. a. $f(x) = -2x^2 - 6x - 7$ b. $f(x) = 6x^2 + 2x + 3$
4. a. $f(x) = -3x^2 + 5x - 1$ b. $f(x) = 2x^2 - 4x + 5$

Answer

3. a. down b. up

4. a. down b. up

Exercises 9 - 12: Find the Axis of Symmetry and Vertex of a Parabola

In the following functions, find

a. The equation of the axis of symmetry
b. The vertex of its graph

5. $f(x) = x^2 + 8x - 1$
6. $f(x) = x^2 + 10x + 25$
7. $f(x) = -x^2 + 2x + 5$
8. $f(x) = -2x^2 - 8x - 3$

Answer

5. a. Axis of symmetry: $x = -4$ b. Vertex: $(-4, -17)$
6. a. Axis of symmetry: $x = -5$ b. Vertex: $(-5, 0)$
7. a. Axis of symmetry: $x = 1$ b. Vertex: $(1, 6)$
8. a. Axis of symmetry: $x = -2$ b. Vertex: $(-2, 5)$

Find the Intercepts of a Parabola

In the following exercises, find the intercepts of the parabola whose function is given.

9. $f(x) = x^2 + 7x + 6$
10. $f(x) = x^2 + 8x + 12$
11. $f(x) = -x^2 + 8x - 19$
12. $f(x) = x^2 + 6x + 13$
13. $f(x) = 4x^2 - 20x + 25$
14. $f(x) = -x^2 - 6x - 9$

Answer

9. y-intercept: $(0, 6)$; x-intercept(s): $(-1, 0), (-6, 0)$
10. y-intercept: $(0, 12)$; x-intercept(s): $(-2, 0), (-6, 0)$
11. y-intercept: $(0, -19)$; x-intercept(s): none
12. y-intercept: $(0, 13)$; x-intercept(s): none
13. y-intercept: $(0, 25)$; x-intercept(s): $(\frac{5}{2}, 0)$
14. y-intercept: $(0, -9)$; x-intercept(s): $(-3, 0)$

Graph Quadratic Functions Using Properties

In the following exercises, graph the function by using its properties. That is, find its x-intercepts, y-intercept, vertex, and determine its end behavior. Then use this information to graph the function.

15. $f(x) = x^2 + 6x + 5$
16. $f(x) = x^2 + 4x + 3$
17. $f(x) = 9x^2 + 12x + 4$
18. $f(x) = -x^2 + 2x - 7$
19. $f(x) = 2x^2 - 4x + 1$
20. $f(x) = 2x^2 - 4x + 2$
21. $f(x) = -x^2 - 4x + 2$
22. $f(x) = 5x^2 - 10x + 8$
23. $f(x) = 3x^2 + 18x + 20$

Answer

15.

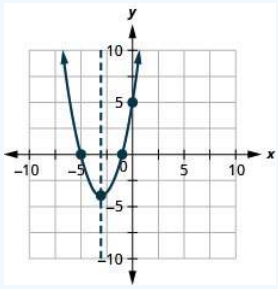


Figure 9.6.136

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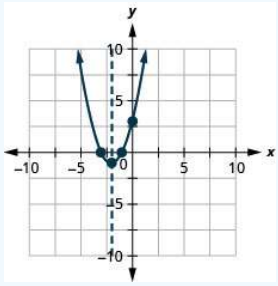


Figure 9.6.137

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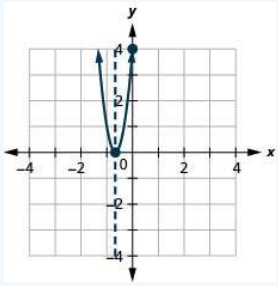


Figure 9.6.138

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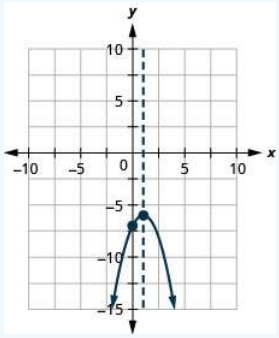


Figure 9.6.139

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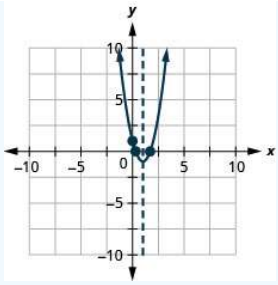


Figure 9.6.140

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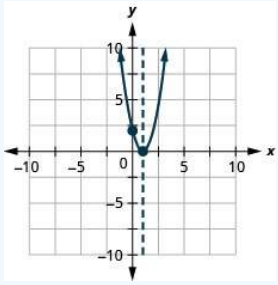


Figure 9.6.141

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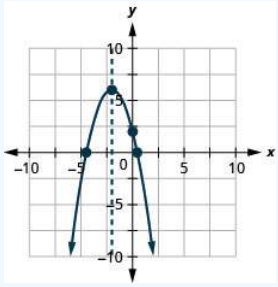


Figure 9.6.142

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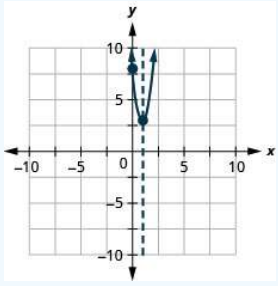


Figure 9.6.143

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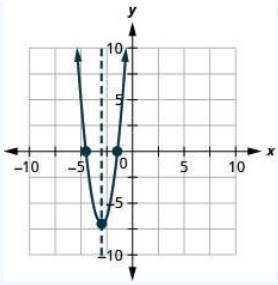


Figure 9.6.144

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CHAPTER OVERVIEW

5: Everything else you need to know

5.1: Simplify Rational Exponents

5.1E: Exercises

5.2: Solve Radical Equations

5.2E: Exercises

5.3: Use Radicals in Functions

5.3E: Exercises

5.4: Multiply and Divide Rational Expressions

5.4E: Exercises

5.5: Add and Subtract Rational Expressions

5.5E: Exercises

5.6: Solve Rational Equations

5.6E: Exercises

5.7: Solve Systems of Linear Equations with Two Variables

5.7E: Exercises

5.8: Applications

5.8E: Exercises

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5.1: Simplify Rational Exponents

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions with $a^{\frac{1}{n}}$
- Simplify expressions with $a^{\frac{m}{n}}$
- Use the properties of exponents to simplify expressions with rational exponents

Warm Up

Compute or simplify

a. 3^3
b. $7^6 \cdot 7^{-4}$
c. $a^{19} \cdot a^{14}$
d. $\sqrt{x^2}$
e. $\sqrt[3]{x^3}$

Answer

a. 27
b. $7^2 = 49$
c. a^{33}
d. $|x|$
e. x

A Note about $\sqrt[n]{x^n}$

We recall that in earlier sections that for any integer $n \geq 2$,

$$\text{when the index } n \text{ is odd } \sqrt[n]{a^n} = a$$

$$\text{when the index } n \text{ is even } \sqrt[n]{a^n} = |a|$$

We notice that if $a > 0$, that is a positive real number, then $\sqrt[n]{a^n} = a$ for $n \geq 2$ no matter if n is even or odd. Since we will be dealing with some properties that may be confusing at first, we will treat all bases as positive real numbers.

Rationalizing the denominator.

No mathematician actually cares about rationalizing the denominator but it is a student learning objective for this class so... here we go.

Say we have a fraction $\frac{1}{\sqrt{2}}$ and we want to make the denominator into a rational, that is we want to rationalize the denominator. We know that $\sqrt{2}^2 = 2$ and multiplying by 1 doesn't change the number, so lets multiply by $\frac{\sqrt{2}}{\sqrt{2}}$ since this is just a fancy way to write 1. We get $\frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}^2} = \frac{\sqrt{2}}{2}$.

Exercise 5.1.1

Rationalize the denominator.

a. $\frac{1}{\sqrt{3}}$
b. $\frac{1}{\sqrt{4}}$

c. $\dfrac{1}{\sqrt{5}}$
d. $\dfrac{1}{\sqrt{x}}$

Answer

a. $\dfrac{\sqrt{3}}{3}$
b. Trick question since $\sqrt{4}=2$ so $\dfrac{1}{\sqrt{4}}=\dfrac{1}{2}$
c. $\dfrac{\sqrt{5}}{5}$
d. $\dfrac{\sqrt{x}}{x}$

What about if we have a sum or difference in the denominator? For example we have $\dfrac{1}{1+\sqrt{2}}$, what do we do? Remember $(x+y)(x-y)=x^2-y^2$? we can use this to rationalize the denominator by using what is called the conjugate.

Definition: CONJUGATE

The conjugate of $a+\sqrt{b}$ is $a-\sqrt{b}$
and the conjugate of $a+\sqrt{b}$ is $a-\sqrt{b}$

So by multiplying by a sneaky 1, $\dfrac{1-\sqrt{2}}{1-\sqrt{2}}$, we can rationalize the denominator of $\dfrac{1}{1+\sqrt{2}}$. We get

Example 5.1.2

Rationalize the denominator of $\dfrac{1}{1+\sqrt{2}}$

Solution

Multiply top and bottom by the conjugate of the denominator	$=\dfrac{1}{1+\sqrt{2}}\dfrac{1-\sqrt{2}}{1-\sqrt{2}}$
Multiply	$=\dfrac{1-\sqrt{2}}{1^2-\sqrt{2}^2}$
Simplify	\(=\dfrac{1-\sqrt{2}}{1-2} =\dfrac{1-\sqrt{2}}{-1}\ =1+\sqrt{2}\)

Let's do some examples

Exercise 5.1.3

Rationalize the denominator

1. $\dfrac{1}{1+\sqrt{3}}$
2. $\dfrac{1}{5+\sqrt{4}}$
3. $\dfrac{1}{2-\sqrt{5}}$

4. $\dfrac{1}{3+\sqrt{x}}$
5. $\dfrac{1}{x-\sqrt{y}}$

Answer

1. $\dfrac{1-\sqrt{3}}{1-3}=-\dfrac{1-\sqrt{3}}{2}$
2. $\dfrac{1}{5+2}=\dfrac{1}{7}$
3. $\dfrac{2+\sqrt{5}}{4-5}=-\dfrac{1-\sqrt{3}}{1}=1-\sqrt{3}$
4. $\dfrac{3-\sqrt{x}}{9-x}$
5. $\dfrac{x+\sqrt{y}}{x^2-y}$

Use the Properties of Exponents to Simplify Expressions with Rational Exponents

The same properties of exponents that we have already used also apply to rational exponents. We will list the Properties of Exponents here to have them for reference as we simplify expressions.

Properties of Exponents

Properties of Exponents

If a and b are real numbers and m and n are rational numbers, then

Zero Exponent Property

$$a^0=1, a\neq 0$$

Negative Exponent Property

$$a^{-n}=\tfrac{1}{a^n}, a\neq 0$$

Product Property

$$a^m\cdot a^n=a^{m+n}$$

Power Property

$$(a^m)^n=a^{m\cdot n}$$

Product to a Power Property

$$(ab)^m=a^m b^m$$

Quotient Property

$$\tfrac{a^m}{a^n}=a^{m-n}, a\neq 0$$

Quotient to a Power Property

$$\left(\tfrac{a}{b}\right)^m=\tfrac{a^m}{b^m}, b\neq 0$$

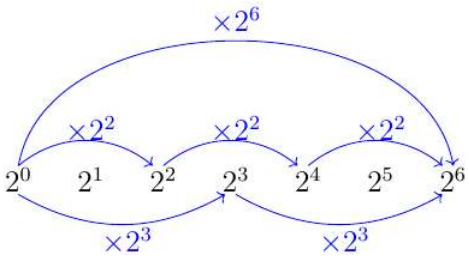
Let's talk each property.

We saw the **Zero Exponent Property** in chapter 1 from dividing a^1 by a.

We saw the **Product Property** in chapter 1 but a quick example would be $4^2\cdot 4^3=4\cdot 4\cdot 4\cdot 4\cdot 4=4^5$

We saw the **Negative Exponent Property** in chapter 1 and it follows from dividing a^0 by a to get a^{-1}.

To see the **Power Property** is true, let's consider an example. Say we have $(2^2)^3$ and from the power property, this should be 2^6. Computationally, this is true since $(2^2)^3 = 4^3 = 64 = 2^6$ but let's take a look at a picture for the intuition of why this is true.



We see that it takes 3 steps of 2^2 to go from $1 = 2^0$ to 2^6. This makes sense since if you take 6 normal steps then this is the same as you taking 3 double steps since $3 \cdot 2$=6, and this is exactly what is happening in the picture above. For the same reason we see that if we take 2 triple steps then we took 6 normal steps, which is shown with the bottom arrows. So we get $(a^n)^m = a^{n \cdot m} = a^{m \cdot n} = (a^m)^n$.

The **Product to a Power Property** comes from noticing that multiplication is commutative, that is we can switch terms around without changing the product, i.e $a \cdot b = b \cdot a$. So let's take a look at $(a \cdot b)^2$. Well, $(a \cdot b)^2 = (a \cdot b) \cdot (a \cdot b) = a \cdot b \cdot a \cdot b = a \cdot a \cdot b \cdot b = a^2 \cdot b^2$. The same thing goes for to the power of 3, $(a \cdot b)^3 = a \cdot b \cdot a \cdot b \cdot a \cdot b = a \cdot a \cdot a \cdot b \cdot b \cdot b = a^3 \cdot b^3$. We can apply the same reasoning to the power of 4,5, 6, all the way up to any number n.

The **Quotient Property** is from the product property and negative exponent property which we saw in chapter 1. That is, $\frac{a^m}{a^n} = a^m a^{-n} = a^{m+(-n)} = a^{m-n}$

The **Quotient to a Power Property** follows from the negative exponent property and product to a power property. That is, $(\frac{a}{b})^2 = (ab^{-1})^2 = a^2 b^{-2} = \frac{a^2}{b^2}$

Use the Properties of Exponents to Simplify Expressions with Rational Exponents

The same properties of exponents that we have already used also apply to rational exponents. We will list the Properties of Exponents here to have them for reference as we simplify expressions.

Example 5.1.4

Simplify:

a. $x^{\frac{1}{2}} \cdot x^{\frac{5}{6}}$

b. $\left(z^9\right)^{\frac{2}{3}}$

c. $\frac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}}$

Solution

a. The Product Property tells us that when we multiple the same base, we add the exponents.

$$x^{\frac{1}{2}} \cdot x^{\frac{5}{6}}$$

The bases are the same, so we add the exponents.

$$x^{\frac{1}{2}+\frac{5}{6}}$$

Add the fractions.

$$x^{\frac{8}{6}}$$

Simplify the exponent.

$$x^{\frac{4}{3}}$$

b. The Power Property tells us that when we raise a power to a power, we multiple the exponents.

$$\left(z^9\right)^{\frac{2}{3}}$$

To raise a power to a power, we multiple the exponents.

$$z^{9\cdot\frac{2}{3}}$$

Simplify.

$$z^6$$

c. The Quotient Property tells us that when we divide with the same base, we subtract the exponents.

$$\frac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}}$$

To divide with the same base, we subtract the exponents.

$$\frac{1}{x^{\frac{5}{3}-\frac{1}{3}}}$$

Simplify.

$$\frac{1}{x^{\frac{4}{3}}}$$

Exercise 5.1.5

Simplify:

a. $x^{\frac{1}{6}}\cdot x^{\frac{4}{3}}$

b. $\left(x^6\right)^{\frac{4}{3}}$

c. $\frac{x^{\frac{2}{3}}}{x^{\frac{5}{3}}}$

Answer

a. $x^{\frac{3}{2}}$

b. x^8

c. $\frac{1}{x}$

Exercise 5.1.6

Simplify:

a. $y^{\frac{3}{4}}\cdot y^{\frac{5}{8}}$

b. $\left(m^9\right)^{\frac{2}{9}}$

c. $\frac{d^{\frac{1}{5}}}{d^{\frac{6}{5}}}$

Answer

a. $y^{\frac{11}{8}}$

b. m^2

c. $\frac{1}{d}$

Sometimes we need to use more than one property. In the next example, we will use both the **Product to a Power Property** and then the **Power Property**.

Example 5.1.7

Simplify:

a. $\left(27u^{\frac{1}{2}}\right)^{\frac{2}{3}}$

b. $\left(m^{\frac{2}{3}}n^{\frac{1}{2}}\right)^{\frac{3}{2}}$

Solution:

a.

$$\left(27u^{\frac{1}{2}}\right)^{\frac{2}{3}}$$

First we use the Product to a Power Property.

$$(27)^{\frac{2}{3}}\left(u^{\frac{1}{2}}\right)^{\frac{2}{3}}$$

Rewrite 27 as a power of 3.

$$\left(3^3\right)^{\frac{2}{3}}\left(u^{\frac{1}{2}}\right)^{\frac{2}{3}}$$

To raise a power to a power, we multiple the exponents.

$$\left(3^2\right)\left(u^{\frac{1}{3}}\right)$$

Simplify.

$$9u^{\frac{1}{3}}$$

b.

$$\left(m^{\frac{2}{3}}n^{\frac{1}{2}}\right)^{\frac{3}{2}}$$

First we use the Product to a Power Property.

$$\left(m^{\frac{2}{3}}\right)^{\frac{3}{2}}\left(n^{\frac{1}{2}}\right)^{\frac{3}{2}}$$

To raise a power to a power, we multiply the exponents.

$$mn^{\frac{3}{4}}$$

Exercise 5.1.8

Simplify:

a. $\left(32x^{\frac{1}{3}}\right)^{\frac{3}{5}}$

b. $\left(x^{\frac{3}{4}}y^{\frac{1}{2}}\right)^{\frac{2}{3}}$

Answer

a. $8x^{\frac{1}{5}}$

b. $x^{\frac{1}{2}}y^{\frac{1}{3}}$

Exercise 5.1.9

Simplify:

a. $\left(81n^{\frac{2}{5}}\right)^{\frac{3}{2}}$

b. $\left(a^{\frac{3}{2}}b^{\frac{1}{2}}\right)^{\frac{4}{3}}$

Answer

a. $729n^{\frac{3}{5}}$

b. $a^2b^{\frac{2}{3}}$

We will use both the **Product Property** and the **Quotient Property** in the next example.

Example 5.1.10

Simplify:

a. $\frac{x^{\frac{3}{4}}\cdot x^{-\frac{1}{4}}}{x^{-\frac{6}{4}}}$

b. $\left(\frac{16x^{\frac{4}{3}}y^{-\frac{5}{6}}}{x^{-\frac{2}{3}}y^{\frac{1}{6}}}\right)^{\frac{1}{2}}$

Solution:

a.

$$\frac{x^{\frac{3}{4}}\cdot x^{-\frac{1}{4}}}{x^{-\frac{6}{4}}}$$

Use the Product Property in the numerator, add the exponents.

$$\frac{x^{\frac{2}{4}}}{x^{-\frac{6}{4}}}$$

Use the Quotient Property, subtract the exponents.

$$x^{\frac{8}{4}}$$

Simplify.

$$x^2$$

b.

$$\left(\frac{16x^{\frac{4}{3}}y^{-\frac{5}{6}}}{x^{-\frac{2}{3}}y^{\frac{1}{6}}}\right)^{\frac{1}{2}}$$

Use the Quotient Property, subtract the exponents.

$$\left(\frac{16x^{\frac{6}{3}}}{y^{\frac{6}{6}}}\right)^{\frac{1}{2}}$$

Simplify.

$$\left(\frac{16x^2}{y}\right)^{\frac{1}{2}}$$

Use the Product to a Power Property, multiply the exponents.

$$\frac{4x}{y^{\frac{1}{2}}}$$

Exercise 5.1.11

Simplify:

a. $\frac{m^{\frac{2}{3}} \cdot m^{-\frac{1}{3}}}{m^{-\frac{5}{3}}}$

b. $\left(\frac{25m^{\frac{1}{6}} n^{\frac{11}{6}}}{m^{\frac{2}{3}} n^{-\frac{1}{6}}}\right)^{\frac{1}{2}}$

Answer

a. m^2

b. $\frac{5n}{m^{\frac{1}{4}}}$

Exercise 5.1.12

Simplify:

a. $\frac{u^{\frac{4}{5}} \cdot u^{-\frac{2}{5}}}{u^{-\frac{13}{5}}}$

b. $\left(\frac{27x^{\frac{4}{5}} y^{\frac{1}{6}}}{x^{\frac{1}{5}} y^{-\frac{5}{6}}}\right)^{\frac{1}{3}}$

Answer

a. u^3

b. $3x^{\frac{1}{5}} y^{\frac{1}{3}}$

Access these online resources for additional instruction and practice with simplifying rational exponents.

- Review-Rational Exponents
- Using Laws of Exponents on Radicals: Properties of Rational Exponents

Key Concepts

- **Rational Exponent $a^{\frac{1}{n}}$**
  - If $\sqrt[n]{a}$ is a real number and $n \geq 2$, then $a^{\frac{1}{n}} = \sqrt[n]{a}$.
- **Rational Exponent $a^{\frac{m}{n}}$**
  - For any positive integers m and n, $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$ and $a^{\frac{m}{n}} = \sqrt[n]{a^m}$
- **Properties of Exponents**
  - If a, b are real numbers and m, n are rational numbers, then
    - **Product Property** $a^m \cdot a^n = a^{m+n}$
    - **Power Property** $(a^m)^n = a^{m \cdot n}$
    - **Product to a Power** $(ab)^m = a^m b^m$
    - **Quotient Property** $\frac{a^m}{a^n} = a^{m-n}, a \neq 0$
    - **Zero Exponent Definition** $a^0 = 1, a \neq 0$
    - **Quotient to a Power Property** $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$
    - **Negative Exponent Property** $a^{-n} = \frac{1}{a^n}, a \neq 0$

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5.1E: Exercises

Practice Makes Perfect

Rationalize the denominator

Rationalize the denominator

1. $\frac{1}{\sqrt{7}}$
2. $\frac{1}{\sqrt{8}}$
3. $\frac{4}{\sqrt{11}}$
4. $\frac{2}{\sqrt{8}}$
5. $\frac{1}{1+\sqrt{2}}$
6. $\frac{1}{3-\sqrt{3}}$
7. $\frac{1}{2-\sqrt{8}}$

Answer

1. $\frac{\sqrt{7}}{7}$
2. $\frac{\sqrt{2}}{4}$
3. $\frac{4\sqrt{11}}{11}$
4. $\frac{2\sqrt{2}}{\sqrt{2}}$
5. $-1+\sqrt{2}$
6. $\frac{3+\sqrt{3}}{6}$
7. $\frac{-1-\sqrt{2}}{2}$

Simplify expressions with $a^{\frac{m}{n}}$

In the following exercises, write with a rational exponent.

1. a. $\sqrt{m^5}$ b. $(\sqrt[3]{3y})^7$ c. $\sqrt{\left(\frac{4x}{5y}\right)^3}$
2. a. $\sqrt[5]{u^2}$ b. $(\sqrt[3]{6x})^5$ c. $\sqrt{\left(\frac{18a}{5b}\right)^7}$

Answer

1. a. $m^{\frac{5}{2}}$ b. $(3y)^{\frac{7}{3}}$ c. $\left(\frac{4x}{5y}\right)^{\frac{3}{5}}$
2. a. $u^{\frac{2}{5}}$ b. $(6x)^{\frac{5}{3}}$ c. $\left(\frac{18a}{5b}\right)^{\frac{7}{4}}$

Simplify expressions with $a^{\frac{m}{n}}$

In the following exercises, simplify.

1. a. $64^{\frac{5}{2}}$ b. $81^{\frac{-3}{2}}$ c. $(-27)^{\frac{2}{3}}$
2. a. $32^{\frac{2}{5}}$ b. $27^{-\frac{2}{3}}$ c. $(-25)^{\frac{1}{2}}$
3. a. $9^{\frac{3}{2}}$ b. $-9^{-\frac{3}{2}}$ c. $-9^{\frac{3}{2}}$

Answer

1. a. $32,768$ b. $\frac{1}{729}$ c. 9
2. a. 4 b. $\frac{1}{9}$ c. $5i$
3. a. 27 b. $-\frac{1}{27}$ c. -27

? Use the laws of exponents to simplify expressions with rational exponents

In the following exercises, simplify. Assume all variables are positive.

1. a. $c^{\frac{1}{4}} \cdot c^{\frac{5}{8}}$ b. $\left(p^{12}\right)^{\frac{3}{4}}$ c. $\dfrac{r^{\frac{4}{5}}}{r^{\frac{9}{5}}}$
2. a. $y^{\frac{1}{2}} \cdot y^{\frac{3}{4}}$ b. $\left(x^{12}\right)^{\frac{2}{3}}$ c. $\dfrac{m^{\frac{5}{8}}}{m^{\frac{13}{8}}}$
3. a. $\left(27q^{\frac{3}{2}}\right)^{\frac{4}{3}}$ b. $\left(a^{\frac{1}{3}}b^{\frac{2}{3}}\right)^{\frac{3}{2}}$
4. a. $\left(16u^{\frac{1}{3}}\right)^{\frac{3}{4}}$ b. $\left(4p^{\frac{1}{3}}q^{\frac{1}{2}}\right)^{\frac{3}{2}}$
5. a. $\dfrac{r^{\frac{5}{2}} \cdot r^{-\frac{1}{2}}}{r^{-\frac{3}{2}}}$ b. $\left(\dfrac{36s^{\frac{1}{5}}t^{-\frac{3}{2}}}{s^{-\frac{9}{5}}t^{\frac{1}{2}}}\right)^{\frac{1}{2}}$
6. a. $\dfrac{c^{\frac{5}{3}} \cdot c^{-\frac{1}{3}}}{c^{-\frac{2}{3}}}$ b. $\left(\dfrac{8x^{\frac{5}{3}}y^{-\frac{1}{2}}}{27x^{-\frac{4}{3}}y^{\frac{5}{2}}}\right)^{\frac{1}{3}}$

Answer

1. a. $c^{\frac{7}{8}}$ b. p^9 c. $\frac{1}{r}$
2. a. $y^{\frac{5}{4}}$ b. x^8 c. $\dfrac{1}{m}$
3. a. $81q^2$ b. $a^{\frac{1}{2}}b$
4. a. $8u^{\frac{1}{4}}$ b. $8p^{\frac{1}{2}}q^{\frac{3}{4}}$
5. a. $r^{\frac{7}{2}}$ b. $\dfrac{6s}{t}$
6. a. c^2 b. $\dfrac{2x}{3y}$

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5.2: Solve Radical Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve radical equations
- Solve radical equations with two radicals
- Use radicals in applications

Warm Up

1. Simplify: $(y-3)^2$
2. Solve: $2x-5=0$
3. Solve $n^2-6n+8=0$

Answer

1. y^2-6y+9
2. $\frac{5}{2}$
3. $n=4,2$

Solve Radical Equations

In this section we will solve equations that have a variable in the radicand of a radical expression. An equation of this type is called a **radical equation**.

Definition: RADICAL EQUATION

An equation in which a variable is in the radicand of a radical expression is called a **radical equation**.

As usual, when solving these equations, what we do to one side of an equation we must do to the other side as well. Once we isolate the radical, our strategy will be to raise both sides of the equation to the power of the index. This will eliminate the radical.

Solving radical equations containing an even index by raising both sides to the power of the index may introduce an algebraic solution that would not be a solution to the original radical equation. Again, we call this an extraneous solution as we did when we solved rational equations.

In the next example, we will see how to solve a radical equation. Our strategy is based on raising a radical with index n to the n^{th} power. This will eliminate the radical.

$$\text{For } a \geq 0, (\sqrt[n]{a})^n = a \text{ .}$$

Example 5.2.1

Solve: $\sqrt{5n-4}-9=0$.

Solution:

Table 8.6.1

Step 1: Isolate the radical on one side of the equation.	To isolate the radical, add 9 to both sides. Simplify.	$\sqrt{5n-4}-9=0$ $\sqrt{5n-4}-9+9=0+9$ $\sqrt{5n-4}=9$
Step 2: Raise both sides of the equation to the power of the index.	Since the index of a square root is 2, we square both sides.	$(\sqrt{5n-4})^2=(9)^2$

Step 3: Solve the new equation.	Remember, $(\sqrt{a})^2 = a$.	$5n - 4 = 81$ $5n = 85$ $n = 17$
Step 4: Check the answer in the original equation.		Check the answer. $\sqrt{5n-4} - 9 = 0$ $\sqrt{5(17)-4} - 9 \stackrel{?}{=} 0$ $\sqrt{85-4} - 9 \stackrel{?}{=} 0$ $\sqrt{81} - 9 \stackrel{?}{=} 0$ $9 - 9 = 0$ $0 = 0$ The solution is $n = 17$.

? Exercise 5.2.2

Solve: $\sqrt{3m+2} - 5 = 0$.

Answer

$m = \frac{23}{3}$

? Exercise 5.2.3

Solve: $\sqrt{10z+1} - 2 = 0$.

Answer

$z = \frac{3}{10}$

Solve a Radical Equation With One Radical

1. Isolate the radical on one side of the equation.
2. Raise both sides of the equation to the power of the index.
3. Solve the new equation.
4. Check the answer in the original equation.

When we use a radical sign, it indicates the principal or positive root. If an equation has a radical with an even index equal to a negative number, that equation will have no solution.

✔ Example 5.2.4

Solve: $\sqrt{9k-2} + 1 = 0$.

Solution:

Table 8.6.2

	$\sqrt{9k-2} + 1 = 0$
To isolate the radical, subtract 1 to both sides.	$\sqrt{9k-2} + 1 - 1 = 0 - 1$
Simplify.	$\sqrt{9k-2} = -1$

Because the square root is equal to a negative number, the equation has no solution.

Exercise 5.2.5

Solve: $\sqrt{2r-3}+5=0$.

Answer

no solution

Exercise 5.2.6

Solve: $\sqrt{7s-3}+2=0$.

Answer

no solution

If one side of an equation with a square root is a binomial, we use the Product of Binomial Squares Pattern when we square it.

Definition: BINOMIAL SQUARES

$(a+b)^2=a^2+2ab+b^2$
$(a-b)^2=a^2-2ab+b^2$

Don't forget the middle term!

Example 5.2.7

Solve: $\sqrt{p-1}+1=p$.

Solution:

Table 8.6.3

	$\sqrt{p-1}+1=p$
To isolate the radical, subtract 1 from both sides.	$\sqrt{p-1}+1-1=p-1$
Simplify.	$\sqrt{p-1}=p-1$
Square both sides of the equation.	$\left(\sqrt{p-1}\right)^2=(p-1)^2$
Simplify, using the Product of Binomial Squares Pattern on the right, Then solve the new equation.	$p-1=p^2-2p+1$
It is a quadratic equation, so get zero on one side.	$0=p^2-3p+2$
Factor the right side.	$0=(p-1)(p-2)$
Use the Zero Product Property.	$0=p-1 \quad 0=p-2$
Solve each equation.	$p=1 \quad p=2$
Check the answers.	
$p=1$ $\sqrt{p-1}+1=p$ $\sqrt{1-1}+1\stackrel{?}{=}1$ $\sqrt{0}+1\stackrel{?}{=}1$ $1=1$ ✓ $p=2$ $\sqrt{p-1}+1=p$ $\sqrt{2-1}+1\stackrel{?}{=}2$ $\sqrt{1}+1\stackrel{?}{=}2$ $2=2$ ✓	

The solutions are $p=1, p=2$.

Exercise 5.2.8

Solve: $\sqrt{x-2}+2=x$.

Answer

$x=2, x=3$

Exercise 5.2.9

Solve: $\sqrt{y-5}+5=y$.

Answer

$y=5, y=6$

When the index of the radical is 3, we cube both sides to remove the radical.

$$(\sqrt[3]{a})^3=a$$

Example 5.2.10

Solve: $\sqrt[3]{5x+1}+8=4$.

Solution:

Table 8.6.4

	$\sqrt[3]{5x+1}+8=4$
To isolate the radical, subtract 8 from both sides.	$\sqrt[3]{5x+1}=-4$
Cube both sides of the equation.	$(\sqrt[3]{5x+1})^3=(-4)^3$
Simplify.	$5x+1=-64$
Solve the equation.	$5x=-65$
	$x=-13$
Check the answer.	
$x=-13$ $\sqrt[3]{5x+1}+8=4$ $\sqrt[3]{5(-13)+1}+8\stackrel{?}{=}4$ $\sqrt[3]{-64}+8\stackrel{?}{=}4$ $-4+8\stackrel{?}{=}4$ $4=4\checkmark$	
	The solution is $x=-13$.

Exercise 5.2.11

Solve: $\sqrt[3]{4x-3}+8=5$

Answer

$x=-6$

Exercise 5.2.12

Solve: $\sqrt[3]{6x-10}+1=-3$

Answer

$x=-9$

Sometimes an equation will contain rational exponents instead of a radical. We use the same techniques to solve the equation as when we have a radical. We raise each side of the equation to the power of the denominator of the rational exponent. Since $(a^m)^n = a^{m\cdot n}$, we have for example,

$$\left(x^{\frac{1}{2}}\right)^2 = x, \left(x^{\frac{1}{3}}\right)^3 = x$$

Remember, $x^{\frac{1}{2}} = \sqrt{x}$ and $x^{\frac{1}{3}} = \sqrt[3]{x}$.

Example 5.2.13

Solve: $(3x-2)^{\frac{1}{4}}+3=5$.

Solution:

Table 8.6.5

	$(3x-2)^{\frac{1}{4}}+3=5$
To isolate the term with the rational exponent, subtract 3 from both sides.	$(3x-2)^{\frac{1}{4}}=2$
Raise each side of the equation to the fourth power.	$\left((3x-2)^{\frac{1}{4}}\right)^4=(2)^4$
Simplify.	$3x-2=16$
Solve the equation.	$3x=18$
	$x=6$
Check the answer.	
$x=6$ $(3x-2)^{\frac{1}{4}}+3=5$ $(3\cdot 6-2)^{\frac{1}{4}}+3\stackrel{?}{=}5$ $(16)^{\frac{1}{4}}+3\stackrel{?}{=}5$ $2+3\stackrel{?}{=}5$ $5=5$ ✓	
	The solution is $x=6$.

Exercise 5.2.14

Solve: $(9x+9)^{\frac{1}{4}}-2=1$

Answer

$x=8$

Exercise 5.2.15

Solve: $(4x-8)^{\frac{1}{4}}+5=7$

Answer

$x=6$

Sometimes the solution of a radical equation results in two algebraic solutions, but one of them may be an **extraneous solution**!

Example 5.2.16

Solve: $\sqrt{r+4}-r+2=0$.

Solution:

Table 8.6.6

	$\sqrt{r+4}-r+2=0$
Isolate the radical.	$\sqrt{r+4}=r-2$
Square both sides of the equation.	$(\sqrt{r+4})^2=(r-2)^2$
Simplify and then solve the equation.	$r+4=r^2-4r+4$
If it is a quadratic equation, so get zero on one side.	$0=r^2-5r$
Factor the right side.	$0=r(r-5)$
Use the Zero Product Property.	$0=r \quad 0=r-5$
Solve the equation.	$r=0 \quad r=5$
Check your answer.	
$r=0$, $\sqrt{r+4}-r+2=0$; $\sqrt{0+4}-0+2\stackrel{?}{=}0$; $\sqrt{4}+2\stackrel{?}{=}0$; $4\neq 0$ $r=5$, $\sqrt{r+4}-r+2=0$; $\sqrt{5+4}-5+2\stackrel{?}{=}0$; $\sqrt{9}-3\stackrel{?}{=}0$; $0=0$ ✓	The solution is $r=5$.
	$r=0$ is an extreneous solution.

Exercise 5.2.17

Solve: $\sqrt{m+9}-m+3=0$

Answer

$m=7$

Exercise 5.2.18

Solve: $\sqrt{n+1}-n+1=0$.

Answer

$n=3$

When there is a coefficient in front of the radical, we isolate the radical by dividing both sides by the coefficient.

Example 5.2.19

Solve: $3\sqrt{3x-5}-8=4$.

Solution:

Table 8.6.7

	$3\sqrt{3x-5}-8=4$
Isolate the radical term.	$3\sqrt{3x-5}=12$
Isolate the radical by dividing both sides by 3.	$\sqrt{3x-5}=4$
Square both sides of the equation.	$(\sqrt{3x-5})^2=(4)^2$
Simplify, then solve the new equation.	$3x-5=16$
	$3x=21$
Solve the equation.	$x=7$
Check the answer.	
$x=7$ $3\sqrt{3x-5}-8=4$ $3\sqrt{3(7)-5}-8\stackrel{?}{=}4$ $3\sqrt{21-5}-8\stackrel{?}{=}4$ $3\sqrt{16}-8\stackrel{?}{=}4$ $3(4)-8\stackrel{?}{=}4$ $4=4$ ✓	
	The solution is $x=7$.

Exercise 5.2.20

Solve: $2\sqrt{4a+4}-16=16$.

Answer

$a=63$

Exercise 5.2.21

Solve: $3\sqrt{2b+3}-25=50$

Answer

$b=311$

Solve Radical Equations with Two Radicals

If the radical equation has two radicals, we start out by isolating one of them. It often works out easiest to isolate the more complicated radical first.

In the next example, when one radical is isolated, the second radical is also isolated.

Example 5.2.22

Solve: $\sqrt[3]{4x-3} = \sqrt[3]{3x+2}$.

Solution:

The radical terms are isolated.

$$\sqrt[3]{4x-3} = \sqrt[3]{3x+2}$$

Since the index is 3, cube both sides of the equation.

$$(\sqrt[3]{4x-3})^3 = (\sqrt[3]{3x+2})^3$$

Simplify, then solve the new equation.

$$\begin{aligned} 4x-3 &= 3x+2 \\ x-3 &= 2 \\ x &= 5 \end{aligned}$$

The solution is $x = 5$.

Check the answer.

We leave it to you to show that 5 checks!

Exercise 5.2.23

Solve: $\sqrt[3]{5x-4} = \sqrt[3]{2x+5}$.

Answer

$x = 3$

Exercise 5.2.24

Solve: $\sqrt[3]{7x+1} = \sqrt[3]{2x-5}$.

Answer

$x = -\frac{6}{5}$

Sometimes after raising both sides of an equation to a power, we still have a variable inside a radical. When that happens, we repeat Step 1 and Step 2 of our procedure. We isolate the radical and raise both sides of the equation to the power of the index again.

Example 5.2.25

Solve: $\sqrt{m}+1 = \sqrt{m+9}$.

Solution:

Table 8.6.8

Step 1: Isolate one of the radical terms on one side of the equation.	The radical on the right is isolated.	$\sqrt{m}+1 = \sqrt{m+9}$
Step 2: Raise both sides of the equation to the power of the index.	We square both sides. Simplify--be very careful as you multiply!	$(\sqrt{m}+1)^2 = (\sqrt{m+9})^2$

Step 3: Are there any more radicals? If yes, repeat Step 1 and Step 2 again. If no, solve the new equation.	There is still a radical in the equation. So we must repeat the previous steps. Isolate the radical term. Here, we can easily isolate the radical by dividing both sides by 2. Square both sides.	$m+2\sqrt{m}+1 = m+9$ $2\sqrt{m} = 8$ $\sqrt{m} = 4$ $(\sqrt{m})^2 = (4)^2$ $m = 16$
Step 4: Check the answer in the original equation.		$\sqrt{m}+1 = \sqrt{m+9}$ $\sqrt{16}+1 \stackrel{?}{=} \sqrt{16+9}$ $4+1 \stackrel{?}{=} 5$ $5 = 5$ The solution is $m = 16$.

Exercise 5.2.26

Solve: $3-\sqrt{x}=\sqrt{x-3}$.

Answer

$x = 4$

Exercise 5.2.27

Solve: $\sqrt{x}+2=\sqrt{x+16}$.

Answer

$x = 9$

We summarize the steps here. We have adjusted our previous steps to include more than one radical in the equation This procedure will now work for any radical equations.

Solve a Radical Equation

1. Isolate one of the radical terms on one side of the equation.
2. Raise both sides of the equation to the power of the index.
3. Are there any more radicals?
   If yes, repeat Step 1 and Step 2 again.
   If no, solve the new equation.
4. Check the answer in the original equation.

Be careful as you square binomials in the next example. Remember the pattern in $(a+b)^2=a^2+2ab+b^2$ or $(a-b)^2=a^2-2ab+b^2$.

Example 5.2.28

Solve: $\sqrt{q-2}+3=\sqrt{4q+1}$.

Solution:

Table 8.6.9

	$\sqrt{q-2}+3=\sqrt{4q+1}$
The radical on the right is isolated. Square both sides.	$(\sqrt{q-2}+3)^2=(\sqrt{4q+1})^2$
Simplify.	$q-2+6\sqrt{q-2}+9=4q+1$

There is still a radical in the equation so we must repeat the previous steps. Isolate the radical.	$6\sqrt{q-2}=3q-6$
Square both sides. It would not help to divide both sides by 6. Remember to square both the 6 and the $\sqrt{q-2}$.	$(6\sqrt{q-2})^2=(3q-6)^2$ $6^2(\sqrt{q-2})^2=(3q)^2-2\cdot 3q\cdot 6+6^2$
Simplify, then solve the new equation.	$36(q-2)=9q^2-36q+36$
Distribute.	$36q-72=9q^2-36q+36$
It is a quadratic equation, so get zero on one side.	$0=9q^2-72q+108$
Factor the right side.	$0=9(q^2-8q+12)$ $0=9(q-6)(q-2)$
Use the Zero Product Property.	$q-6=0 \quad q-2=0$ $q=6 \quad q=2$
The checks are left to you.	The solutions are $q=6$ and $q=2$.

? Exercise 5.2.29

Solve: $\sqrt{x-1}+2=\sqrt{2x+6}$

Answer

$x=5$

? Exercise 5.2.30

Solve: $\sqrt{x}+2=\sqrt{3x+4}$

Answer

$x=0x=4$

Key Concepts

- **Solve a Radical Equation**
  1. Isolate one of the radical terms on one side of the equation.
  2. Raise both sides of the equation to the power of the index.
  3. Are there any more radicals?
     If yes, repeat Step 1 and Step 2 again.
     If no, solve the new equation.
  4. Check the answer in the original equation.

Glossary

radical equation

An equation in which a variable is in the radicand of a radical expression is called a radical equation.

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5.2E: Exercises

Practice Makes Perfect

Solve Radical Equations

In the following exercises, solve.

1. $\sqrt{5x-6}=8$
2. $\sqrt{5x+1}=-3$
3. $\sqrt[3]{2x}=-2$
4. $\sqrt{2m-3}-5=0$
5. $\sqrt{6v-2}-10=0$
6. $\sqrt{4m+2}+2=6$
7. $\sqrt{2u-3}+2=0$
8. $\sqrt{u-3}+3=u$
9. $\sqrt{r-1}=r-1$
10. $\sqrt[3]{6x+4}=4$
11. $\sqrt[3]{4x+5}-2=-5$
12. $(6x+1)^{\frac{1}{2}}-3=4$
13. $(8x+5)^{\frac{1}{3}}+2=-1$
14. $(12x-3)^{\frac{1}{4}}-5=-2$
15. $\sqrt{x+1}-x+1=0$
16. $\sqrt{z+100}-z=-10$
17. $3\sqrt{2x-3}-20=7$
18. $2\sqrt{8r+1}-8=2$

Answer

1. $x=14$
2. no solution
3. $x=-4$
4. $m=14$
5. $v=17$
6. $m=\frac{7}{2}$
7. no solution
8. $u=3, u=4$
9. $r=1, r=2$
10. $x=10$
11. $x=-8$
12. $x=8$
13. $x=-4$
14. $x=7$
15. $x=3$
16. $z=21$
17. $x=42$
18. $r=3$

Solve Radical Equations with Two Radicals

In the following exercises, solve.

19. $\sqrt{3u+7}=\sqrt{5u+1}$
20. $\sqrt{8+2r}=\sqrt{3r+10}$

21. $\sqrt[3]{5x-1}=\sqrt[3]{x+3}$
22. $\sqrt[3]{2x^2+9x-18}=\sqrt[3]{x^2+3x-2}$
23. $\sqrt{a}+2=\sqrt{a+4}$
24. $\sqrt{u}+1=\sqrt{u+4}$
25. $\sqrt{a+5}-\sqrt{a}=1$
26. $\sqrt{2x+1}=1+\sqrt{x}$
27. $\sqrt{2x-1}-\sqrt{x-1}=1$
28. $\sqrt{x+7}-\sqrt{x-5}=2$

Answer

19. $u=3$
20. $r=-2$
21. $x=1$
22. $x=-8, x=2$
23. $a=0$
24. $u=\frac{9}{4}$
25. $a=4$
26. $x=0\ x=4$
27. $x=1\ x=5$
28. $x=9$

Writing Exercises

29. Explain why an equation of the form $\sqrt{x}+1=0$ has no solution.
30. a. Solve the equations $\sqrt{r+4}-r+2=0$.
    b. Explain why one of the "solutions" that was found was not actually a solution to the equation.

Answer

29-30. Answers will vary.

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5.3: Use Radicals in Functions

Learning Objectives

By the end of this section, you will be able to:

- Evaluate a radical function
- Find the domain of a radical function
- Graph radical functions

Warm Up

1. Solve: $1-2x \geq 0$
2. For $f(x)=3x-4$, evaluate $f(2), f(-1), f(0)$
3. For $f(x)=\sqrt{x}$, evaluate $f(4), f(-1), f(0)$

Answer

1. $x \leq -\frac{1}{2}$
2. $f(2)=2, f(-1)=-7, f(0)=-4$
3. $f(4)=2, f(-1)=i, f(0)=0$

Evaluate a Radical Function

In this section we will extend our previous work with functions to include radicals. If a function is defined by a radical expression, we call it a **radical function**.

- The square root function is $f(x)=\sqrt{x}$.
- The cube root function is $f(x)=\sqrt[3]{x}$.

Definition: RADICAL FUNCTION

A **radical function** is a function that is defined by a radical expression.

We will be evaluating higher power roots but first let's review how we evaluated square roots.

Example 5.3.1

For the function $f(x)=\sqrt{2x-1}$, find

a. $f(5)$
b. $f(-2)$

Solution:

a.

$$f(x)=\sqrt{2x-1}$$

To evaluate $f(5)$, substitute 5 for x.

$$f(5)=\sqrt{2\cdot 5-1}$$

Simplify.

$$f(5)=\sqrt{9}$$

Take the square root.

$$f(5)=3$$

b.

$$f(x)=\sqrt{2x-1}$$

To evaluate $f(-2)$, substitute -2 for x.

$$f(-2)=\sqrt{2(-2)-1}$$

Simplify.

$$f(-2)=\sqrt{-5}$$
$$f(-2)=\sqrt{5}i$$

? Exercise 5.3.2

For the function $f(x)=\sqrt{3x-2}$, find

a. $f(6)$
b. $f(0)$

Answer

a. $f(6)=4$
b. $f(0)=\sqrt{2}i$

? Exercise 5.3.3

For the function $g(x)=\sqrt{5x+5}$, find

a. $g(4)$
b. $g(-3)$

Answer

a. $g(4)=5$
b. \(g(-3)=\sqrt{10}i

We follow the same procedure to evaluate cube roots.

✓ Example 5.3.4

For the function $g(x)=\sqrt[3]{x-6}$, find

a. $g(14)$
b. $g(-2)$

Solution:

a.

$$g(x)=\sqrt[3]{x-6}$$

To evaluate $g(14)$, substitute 14 for x.

$$g(14)=\sqrt[3]{14-6}$$

Simplify.

$$g(14)=\sqrt[3]{8}$$

Take the cube root.

$$g(14)=2$$

b.

$$g(x)=\sqrt[3]{x-6}$$

To evaluate $g(-2)$, substitute -2 for x.

$$g(-2) = \sqrt[3]{-2-6}$$

Simplify.

$$g(-2) = \sqrt[3]{-8}$$

Take the cube root.

$$g(-2) = -2$$

? Exercise 5.3.5

For the function $g(x) = \sqrt[3]{3x-4}$, find

a. $g(4)$
b. $g(1)$

Answer

a. $g(4) = 2$
b. $g(1) = -1$

? Exercise 5.3.6

For the function $h(x) = \sqrt[3]{5x-2}$, find

a. $h(2)$
b. $h(-5)$

Answer

a. $h(2) = 2$
b. $h(-5) = -3$

The next example has fourth roots.

✓ Example 5.3.7

For the function $f(x) = \sqrt[4]{5x-4}$, find

a. $f(4)$
b. $f(-12)$

Solution:

a.

$$f(x) = \sqrt[4]{5x-4}$$

To evaluate $f(4)$, substitute 4 for x.

$$f(4) = \sqrt[4]{5 \cdot 4 - 4}$$

Simplify.

$$f(4) = \sqrt[4]{16}$$

Take the fourth root.

$$f(4) = 2$$

b.

$$f(x) = \sqrt[4]{5x-4}$$

To evaluate $f(-12)$, substitute -12 for x.

$$f(-12) = \sqrt[4]{5(-12)-4}$$

Simplify.

$$f(-12) = \sqrt[4]{-64}$$
$$f(-12) = 2\sqrt[4]{4}i$$

? Exercise 5.3.8

For the function $f(x) = \sqrt[4]{3x+4}$, find

a. $f(4)$
b. $f(-1)$

Answer

a. $f(4) = 2$
b. $f(-1) = 1$

? Exercise 5.3.9

For the function $g(x) = \sqrt[4]{5x+1}$, find

a. $g(16)$
b. $g(3)$

Answer

a. $g(16) = 3$
b. $g(3) = 2$

Find the Domain and Range of a Radical Function

To find the **domain** and **range** of radical functions, we use our properties of radicals. For a radical with an even index, we said the radicand had to be greater than or equal to zero as even roots of negative numbers are not real numbers. For an odd index, the radicand can be any real number. We restate the properties here for reference.

Properties of $\sqrt[n]{a}$

When n is an **even** number and:

- $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
- $a < 0$, then $\sqrt[n]{a}$ is an imaginary number

When n is an **odd** number, $\sqrt[n]{a}$ is a real number for all values of a.

So, to find the domain of a radical function with even index, we set the radicand to be greater than or equal to zero. For an odd index radical, the radicand can be any real number.

Domain of a Radical Function

Remember, we are graphing on the real number line with output also on the real number line so there is no room on our graph for imaginary numbers. So if we get an imaginary number then it is not part of our domain for $\sqrt[n]{a}$.

For example, if we have $f(x) = \sqrt{x}$, and we want to plot a point at x=4 then we would have the point $(4, 2)$. If we wanted to plot a point at $x = -1$ then we can't since that is an imaginary number and so we don't include $x = -1$ in our domain.

When the **index** of the radical is **even**, the radicand must be greater than or equal to zero.

When the **index** of the radical is **odd**, the radicand can be any real number.

Example 5.3.10

Find the domain of the function, $f(x)=\sqrt{x}$. Write the domain in interval notation.

Solution

The index of $\sqrt{x}$ is two so we must have whatever is inside the radical to be more than or equal to zero. Only x is inside the radical so $x \geq 0$ is our domain.

In interval notation, this is $[0, \infty)$.

Remember we use brackets [] to include numbers and parenthesis () to exclude numbers or for $\pm\infty$.

Example 5.3.11

Find the domain of the function, $f(x)=\sqrt{3x-4}$. Write the domain in interval notation.

Solution:

Since the function, $f(x)=\sqrt{3x-4}$ has a radical with an index of 2, which is even, we know the radicand must be greater than or equal to 0. We set the radicand to be greater than or equal to 0 and then solve to find the domain.

Solve.

$$\begin{aligned} 3x-4 &\geq 0 \\ 3x &\geq 4 \\ x &\geq \frac{4}{3} \end{aligned}$$

The domain of $f(x)=\sqrt{3x-4}$ is all values $x \geq \frac{4}{3}$ and we write it in interval notation as $\left[\frac{4}{3}, \infty\right)$.

Exercise 5.3.12

Find the domain of the function, $f(x)=\sqrt{6x-5}$. Write the domain in interval notation.

Answer

$\left[\frac{5}{6}, \infty\right)$

Exercise 5.3.13

Find the domain of the function, $f(x)=\sqrt{4-5x}$. Write the domain in interval notation.

Answer

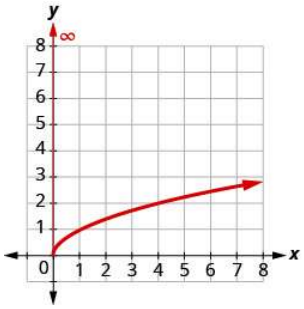
Domain: $\left(-\infty, \frac{4}{5}\right]$

Graph Radical Functions

Before we graph any radical function, we first find the **domain** of the function. For the function, $f(x)=\sqrt{x}$, the index is even, and so the radicand must be greater than or equal to 0.

This tells us the domain is $x \geq 0$ and we write this in interval notation as $[0, \infty)$.

Previously we used point plotting to graph the function, $f(x)=\sqrt{x}$. We chose x-values, substituted them in and then created a chart. Notice we chose points that are perfect squares in order to make taking the square root easier.



x	$f(x)=\sqrt{x}$	$(x, f(x))$
0	0	(0, 0)
1	1	(1, 1)
4	2	(4, 2)
9	3	(9, 3)

Figure 8.7.1

Once we see the graph, we can find the **range** of the function. The y-values of the function are greater than or equal to zero. The range then is $[0, \infty)$.

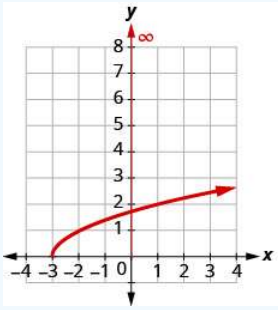
Example 5.3.14

For the function $f(x)=\sqrt{x+3}$,

a. find the domain and range
b. graph the function
c. use the graph to determine the range

Solution:

a. Since the radical has index 2, we know the radicand must be greater than or equal to zero. If $x+3 \geq 0$, then $x \geq -3$. This tells us the domain is all values $x \geq -3$ and written in interval notation as $[-3, \infty)$. While the range is $[0, \infty)$ since $f(-3)=0$ and as x grows to ∞, $f(x)$ grows to ∞.
b. To graph the function, we choose points in the interval $[-3, \infty)$ that will also give us a radicand which will be easy to take the square root.



x	$f(x)=\sqrt{x+3}$	$(x, f(x))$
–3	0	(–3, 0)
–2	1	(–2, 1)
1	2	(1, 2)
6	3	(6, 3)

Figure 8.7.2

c. Looking at the graph, we see the y-values of the function are greater than or equal to zero. The range then is $[0, \infty)$.

Exercise 5.3.15

For the function $f(x)=\sqrt{x+2}$,

a. find the domain and range
b. graph the function
c. use the graph to determine the range

Answer

a. domain: $[-2, \infty)$

range: $[0, \infty$

a.

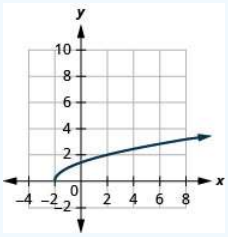


Figure 8.7.3

b. range: $[0, \infty)$

? Exercise 5.3.16

For the function $f(x) = \sqrt{x-2}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Answer

a. domain: $[2, \infty)$
b.

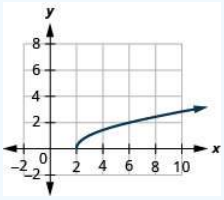


Figure 8.7.4

c. range: $[0, \infty)$

We notice a pattern here. If we add a positive number to x inside the radical, then the graph moves to the left by that number. This is because we shifted the x values by that number. Likewise, if we subtract a positive number from x inside the radical, then the graph moves to the right by that number.

What happens when we add a positive number outside the radical?

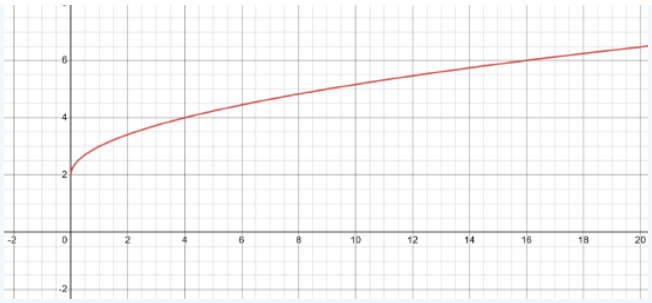
? Exercise 5.3.17

For the function $f(x) = \sqrt{x} + 2$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Answer

a. domain: $[0, \infty$
b.



c. Range: $[2, \infty$

What about if we subtract?

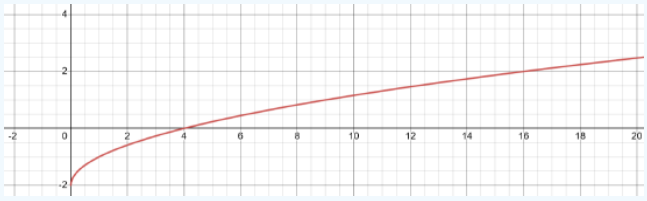
Exercise 5.3.18

For the function $f(x) = \sqrt{x} + 2$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Answer

a. domain: $[0, \infty$
b.



c. Range: $[-2, \infty$

So adding or subtracting a positive number outside the radical makes the function go up or down in accordance to that number.

We can also do add or subtract a number inside and outside to make it move left/right and up/down.

In our previous work graphing functions, we graphed $f(x) = x^3$ but we did not graph the function $f(x) = \sqrt[3]{x}$. We will do this now in the next example.

Example 5.3.19

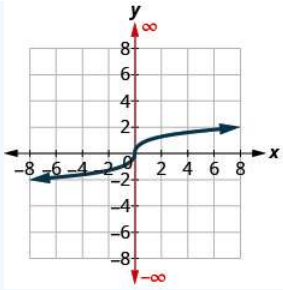
For the function, $f(x) = \sqrt[3]{x}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Solution:

a. Since the radical has index 3, we know the radicand can be any real number. This tells us the domain is all real numbers and written in interval notation as $(-\infty, \infty)$

b. To graph the function, we choose points in the interval $(-\infty, \infty)$ that will also give us a radicand which will be easy to take the cube root.



x	$f(x)=\sqrt[3]{x}$	$(x, f(x))$
–8	–2	(–8, –2)
–1	–1	(–1, –1)
0	0	(0, 0)
1	1	(1, 1)
8	2	(8, 2)

Figure 8.7.5

c. Looking at the graph, we see the y-values of the function are all real numbers. The range then is $(-\infty, \infty)$.

Exercise 5.3.20

For the function $f(x)=-\sqrt[3]{x}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Answer

a. domain: $(-\infty, \infty)$
b.

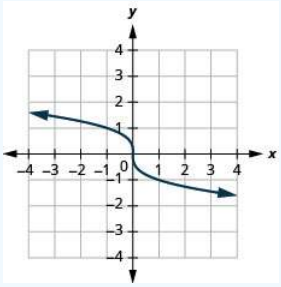


Figure 8.7.6

c. range: $(-\infty, \infty)$

Exercise 5.3.21

For the function $f(x)=\sqrt[3]{x-2}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Answer

a. domain: $(-\infty, \infty)$
b.

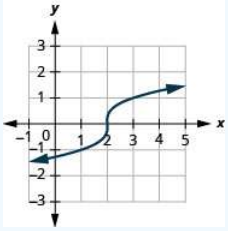


Figure 8.7.7

c. range: $(-\infty, \infty)$

Key Concepts

- **Properties of $\sqrt[n]{a}$**
  - When n is an **even** number and:
    $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
    $a < 0$, then $\sqrt[n]{a}$ is not a real number.
  - When n is an **odd** number, $\sqrt[n]{a}$ is a real number for all values of a.
- **Domain of a Radical Function**
  - When the **index** of the radical is **even**, the radicand must be greater than or equal to zero.
  - When the **index** of the radical is **odd**, the radicand can be any real number.

Glossary

radical function

A radical function is a function that is defined by a radical expression.

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5.3E: Exercises

Practice Makes Perfect

Exercise $5.3E.17$ Evaluate a Radical Function

In the following exercises, evaluate each function.

1. $f(x)=\sqrt{4x-4}$, find
   a. $f(5)$
   b. $f(0)$
2. $g(x)=\sqrt{6x+1}$, find
   a. $g(4)$
   b. $g(8)$
3. $F(x)=\sqrt{3-2x}$, find
   a. $F(1)$
   b. $F(-11)$
4. $G(x)=\sqrt{5x-1}$, find
   a. $G(5)$
   b. $G(2)$
5. $g(x)=\sqrt[3]{2x-4}$, find
   a. $g(6)$
   b. $g(-2)$
6. $h(x)=\sqrt[3]{x^2-4}$, find
   a. $h(-2)$
   b. $h(6)$
7. For the function $f(x)=\sqrt[4]{2x^3}$, find
   a. $f(0)$
   b. $f(2)$
8. For the function $g(x)=\sqrt[4]{4-4x}$, find
   a. $g(1)$
   b. $g(-3)$

Answer

1.
   a. $f(5)=4$
   b. $f(5)=2i$
2.
   a. $g(4)=5$
   b. $g(8)=7$
3.
   a. $F(1)=1$
   b. $F(-11)=5$
4.
   a. $G(5)=2\sqrt{6}$
   b. $G(2)=3$
5.

a. $g(6)=2$
b. $g(-2)=-2$

6.

a. $h(-2)=0$
b. $h(6)=2\sqrt[3]{4}$

7.

a. $f(0)=0$
b. $f(2)=2$

8.

a. $g(1)=0$
b. $g(-3)=2$

Exercise $5.3E.18$ Find the Domain of a Radical Function

In the following exercises, find the domain of the function and write the domain in interval notation.

9. $f(x)=\sqrt{3x-1}$
10. $g(x)=\sqrt{2-3x}$
11. $h(x)=\sqrt{x-2}$
12. $f(x)=\sqrt{-x+3}$
13. $g(x)=\sqrt[3]{8x-1}$
14. $f(x)=\sqrt[3]{4x^2-16}$
15. $F(x)=\sqrt[4]{8x+3}$
16. $G(x)=\sqrt[5]{2x-1}$

Answer

9. $\left[\frac{1}{3},\infty\right)$
10. $\left(-\infty,\frac{2}{3}\right]$
11. $[2,\infty)$
12. $(-\infty,3]$
13. $(-\infty,\infty)$
14. $(-\infty,\infty)$
15. $\left[-\frac{3}{8},\infty\right)$
16. $(-\infty,\infty)$

Exercise $5.3E.19$ graph radical functions

In the following exercises,

a. find the domain of the function
b. graph the function
c. use the graph to determine the range

17. $f(x)=\sqrt{x+1}$
18. $g(x)=\sqrt{x+4}$
19. $f(x)=\sqrt{x}+2$
20. $g(x)=2\sqrt{x}$
21. $f(x)=\sqrt{3-x}$
22. $g(x)=-\sqrt{x}$
23. $f(x)=\sqrt[3]{x+1}$
24. $g(x)=\sqrt[3]{x+2}$

25. $f(x) = \sqrt[3]{x} + 3$
26. $g(x) = \sqrt[3]{x}$
27. $f(x) = 2\sqrt[3]{x}$

Answer

17.

a. domain: $[-1, \infty)$

b.

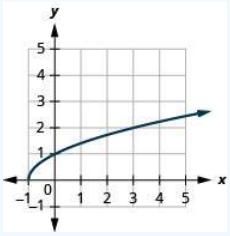


Figure 8.7.8

c. $[0, \infty)$

18.

a. domain: $[-4, \infty)$

b.

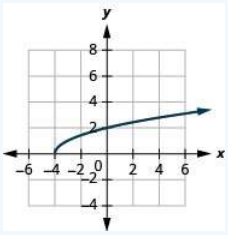


Figure 8.7.9

c. $[0, \infty)$

19.

a. domain: $[0, \infty)$

b.

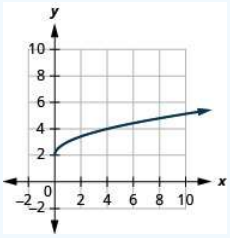


Figure 8.7.10

c. $[2, \infty)$

20.

a. domain: $[0, \infty)$

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b.

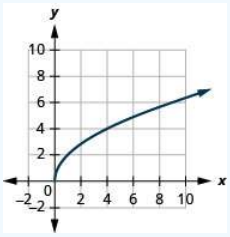


Figure 8.7.11

c. $[0, \infty)$

21.

a. domain: $(-\infty, 3]$

b.

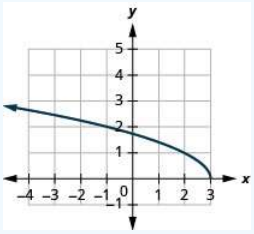


Figure 8.7.12

c. $[0, \infty)$

22.

a. domain: $[0, \infty)$

b.

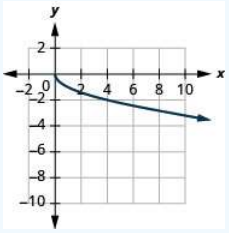


Figure 8.7.13

c. $(-\infty, 0]$

23.

a. domain: $(-\infty, \infty)$

b.

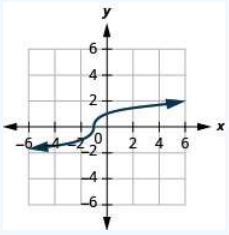


Figure 8.7.14

c. $(-\infty, \infty)$

24.

a. domain: $(-\infty, \infty)$

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b.

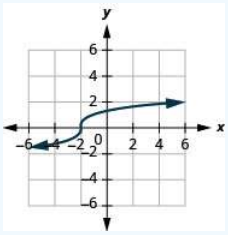


Figure 8.7.15

c. $(-\infty, \infty)$

25.

a. domain: $(-\infty, \infty)$

b.

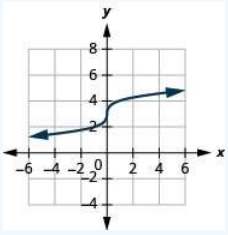


Figure 8.7.16

c. $(-\infty, \infty)$

26.

a. domain: $(-\infty, \infty)$

b.

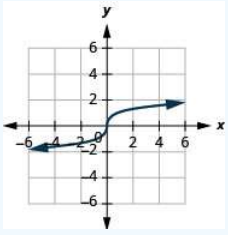


Figure 8.7.17

c. $(-\infty, \infty)$

27.

a. domain: $(-\infty, \infty)$

b.

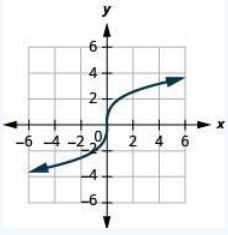


Figure 8.7.18

c. $(-\infty, \infty)$

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Exercise $5.3E.20$ writing exercises

28. Explain how to find the domain of a fourth root function.
29. Explain how to find the domain of a fifth root function.
30. Explain why $y = \sqrt[3]{x}$ is a function.
31. Explain why the process of finding the domain of a radical function with an even index is different from the process when the index is odd.

Answer

Answers may vary

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5.4: Multiply and Divide Rational Expressions

Learning Objectives

- Determine the values for which a rational expression is undefined
- Simplify rational expressions
- Multiply rational expressions
- Divide rational expressions
- Multiply and divide rational functions

Warm-up

a. Simplify: $\frac{90y}{15y^2}$

b. Multiply: $\frac{14}{15} \cdot \frac{6}{35}$

c. Divide: $\frac{12}{10} \div \frac{8}{25}$

Answer

a. $\frac{6}{y}$

b. \drac425

c. \(\dfrac{15}{4}

We previously reviewed the properties of fractions and their operations. We introduced rational numbers, which are just fractions where the numerators and denominators are integers. In this chapter, we will work with fractions whose numerators and denominators are polynomials. We call this kind of expression a **rational expression**.

Definition: RATIONAL EXPRESSION

A rational expression is an expression of the form $\frac{p}{q}$, where p and q are polynomials and $q \neq 0$.

Here are some examples of rational expressions:

$$-\frac{24}{56} \qquad \frac{5x}{12y} \qquad \frac{4x+1}{x^2-9} \qquad \frac{4x^2+3x-1}{2x-8}$$

Notice that the first rational expression listed above, $-\frac{24}{56}$, is just a fraction. Since a constant is a polynomial with degree zero, the ratio of two constants is a rational expression, provided the denominator is not zero.

We will do the same operations with rational expressions that we did with fractions. We will simplify, add, subtract, multiply, divide and use them in applications.

Determine the Values for Which a Rational Expression is Undefined

If the denominator is zero, the rational expression is undefined. The numerator of a rational expression may be 0—but not the denominator.

When we work with a numerical fraction, it is easy to avoid dividing by zero because we can see the number in the denominator. In order to avoid dividing by zero in a rational expression, we must not allow values of the variable that will make the denominator be zero.

So before we begin any operation with a rational expression, we examine it first to find the values that would make the denominator zero. That way, when we solve a rational equation for example, we will know whether the algebraic solutions we find are allowed or not.

DETERMINE THE VALUES FOR WHICH A RATIONAL EXPRESSION IS UNDEFINED.

1. Set the denominator equal to zero.
2. Solve the equation.

EXAMPLE 5.4.1

Determine the value for which each rational expression is undefined:

a. $\frac{8a^2b}{3c}$ b. $\frac{4b-3}{2b+5}$ c. $\frac{x+4}{x^2+5x+6}$.

Solution

The expression will be undefined when the denominator is zero.

a.

$$\frac{8a^2b}{3c}$$

Set the denominator equal to zero and solve for the variable.

$$3c=0$$

$$c=0$$

$\frac{8a^2b}{3c}$ is undefined for $c=0$

b.

$$\frac{4b-3}{2b+5}$$

Set the denominator equal to zero and solve for the variable.

$$2b+5=0$$

$$2b=-5$$

$$b=-\frac{5}{2}$$

$\frac{4b-3}{2b+5}$ is undefined for $b=-\frac{5}{2}$

c.

$$\frac{x+4}{x^2+5x+6}$$

Set the denominator equal to zero and solve for the variable.

$$x^2+5x+6=0$$

$$(x+2)(x+3)=0$$

$$x+2=0 \text{ or } x+3=0$$

$$x=-2 \text{ or } x=-3$$

$\frac{x+4}{x^2+5x+6}$ is undefined for $x=-2$ or $x=-3$

Example 5.4.2

Determine the value for which each rational expression is undefined.

a. $\frac{3y^2}{8x}$ b. $\frac{8n-5}{3n+1}$ c. $\frac{a+10}{a^2+4a+3}$

Answer

a. $x = 0$

b. $n = -\frac{1}{3}$

c. $a = -1, a = -3$

? Example 5.4.3

Determine the value for which each rational expression is undefined.

a. $\frac{4p}{5q}$ b. $\frac{y-1}{3y+2}$ c. $\frac{m-5}{m^2+m-6}$

Answer

a. $q = 0$

b. $y = -\frac{2}{3}$

c. $m = 2, m = -3$

Simplify Rational Expressions

A fraction is considered simplified if there are no common factors, other than 1, in its numerator and denominator. Similarly, a **simplified rational expression** has no common factors, other than 1, in its numerator and denominator.

Definition: SIMPLIFIED RATIONAL EXPRESSION

A rational expression is considered simplified if there are no common factors in its numerator and denominator.

For example,

$\frac{x+2}{x+3}$ is simplified because there are no common factors of $x+2$ and $x+3$.

$\frac{2x}{3x}$ is not simplified because x is a common factor of $2x$ and $3x$.

We use the Equivalent Fractions Property to simplify numerical fractions. We restate it here as we will also use it to simplify rational expressions.

Definition: EQUIVALENT FRACTIONS PROPERTY

If a, b, and c are numbers where $b \neq 0, c \neq 0$,

$$\text{then } \frac{a}{b} = \frac{a \cdot c}{b \cdot c} \text{ and } \frac{a \cdot c}{b \cdot c} = \frac{a}{b}$$

Notice that in the Equivalent Fractions Property, the values that would make the denominators zero are specifically disallowed. We see $b \neq 0, c \neq 0$ clearly stated.

To simplify rational expressions, we first write the numerator and denominator in factored form. Then we remove the common factors using the Equivalent Fractions Property.

Be very careful as you remove common factors. Factors are multiplied to make a product. You can remove a factor from a product. You cannot remove a term from a sum.

$\frac{2 \cdot \cancel{3} \cdot \cancel{7}}{\cancel{3} \cdot 5 \cdot \cancel{7}}$ $\frac{2}{5}$

We removed the common factors 3 and 7. They are factors of the product.

$\frac{3x\cancel{(x-9)}}{5\cancel{(x-9)}}$ where $x \neq 9$ $\frac{3x}{5}$

We removed the common factor $(x-9)$. It is a factor of the product.

$\frac{x+5}{x}$ **NO COMMON FACTORS**

While there is an x in both the numerator and denominator, the x in the numerator is a term of a sum!

Removing the x's from $\frac{x+5}{x}$ would be like cancelling the 2's in the fraction $\frac{2+5}{2}$ so be careful to not do that.

How to Simplify a Rational Expression

Example 5.4.4

Simplify: $\frac{x^2+5x+6}{x^2+8x+12}$

Solution

Step 1. Factor the numerator and denominator completely.	Factor x^2+5x+6 and $x^2+8x+12$	$\frac{x^2+5x+6}{x^2+8x+12}$ $\frac{(x+2)(x+3)}{(x+2)(x+6)}$
Step 2. Simplify by dividing out common factors.	Remove the common factor $x+2$ from the numerator and the denominator.	$\frac{\cancel{(x+2)}(x+3)}{\cancel{(x+2)}(x+6)}$ $\frac{(x+3)}{(x+6)}$ $x \neq -2 \quad x \neq -6$

Example 5.4.5

Simplify: $\frac{x^2-x-2}{x^2-3x+2}$.

Answer

$\frac{x+1}{x-1}, x \neq 2, x \neq 1$

Example 5.4.6

Simplify: $\frac{x^2-3x-10}{x^2+x-2}$.

Answer

$\frac{x-5}{x-1}, x \neq -2, x \neq 1$

We now summarize the steps you should follow to simplify rational expressions.

SIMPLIFY A RATIONAL EXPRESSION

1. Factor the numerator and denominator completely.
2. Simplify by dividing out common factors.

Usually, we leave the simplified rational expression in factored form. This way, it is easy to check that we have removed *all* the common factors.

We'll use the methods we have learned to factor the polynomials in the numerators and denominators in the following examples.

Every time we write a rational expression, we should make a statement disallowing values that would make a denominator zero. However, to let us focus on the work at hand, we will omit writing it in the examples.

Example 5.4.7

Simplify: $\dfrac{3a^2-12ab+12b^2}{6a^2-24b^2}$.

Solution

$$\frac{3a^2-12ab+12b^2}{6a^2-24b^2}$$

Factor the numerator and denominator, first factoring out the GCF.

$$\frac{3(a^2-4ab+4b^2)}{6(a^2-4b^2)}$$

$$\frac{3(a-2b)(a-2b)}{6(a+2b)(a-2b)}$$

Remove the common factors of $a-2b$ and 3.

$$\frac{\cancel{3}(a-2b)\ \cancel{(a-2b)}}{\cancel{3}\cdot 2(a+2b)\ \cancel{(a-2b)}}$$

$$\frac{a-2b}{2(a+2b)}$$

Example 5.4.8

Simplify: $\dfrac{2x^2-12xy+18y^2}{3x^2-27y^2}$.

Answer

$$\frac{2(x-3y)}{3(x+3y)}$$

Example 5.4.9

Simplify: $\dfrac{5x^2-30xy+25y^2}{2x^2-50y^2}$.

Answer

$$\frac{5(x-y)}{2(x+5y)}$$

Now we will see how to simplify a rational expression whose numerator and denominator have opposite factors. We previously introduced opposite notation: the opposite of a is $-a$ and $-a = -1 \cdot a$.

The numerical fraction, say $\frac{7}{-7}$ simplifies to -1. We also recognize that the numerator and denominator are opposites.

The fraction $\frac{a}{-a}$, whose numerator and denominator are opposites also simplifies to -1.

Let's look at the expression $b-a$.	$b-a$
Rewrite.	$-a+b$
Factor out -1.	$-1(a-b)$

This tells us that $b-a$ is the opposite of $a-b$.

In general, we could write the opposite of $a-b$ as $b-a$. So the rational expression $\frac{a-b}{b-a}$ simplifies to -1.

Definition: OPPOSITES IN A RATIONAL EXPRESSION

The opposite of $a-b$ is $b-a$.

$$\frac{a-b}{b-a} = -1 \quad a \neq b$$

An expression and its opposite divide to -1.

We will use this property to simplify rational expressions that contain opposites in their numerators and denominators. Be careful not to treat $a+b$ and $b+a$ as opposites. Recall that in addition, order doesn't matter so $a+b=b+a$. So if $a \neq -b$, then $\frac{a+b}{b+a} = 1$.

Example 5.4.10

Simplify: $\frac{x^2-4x-32}{64-x^2}$

Solution

	$\frac{x^2-4x-32}{64-x^2}$
Factor the numerator and the denominator.	$\frac{(x-8)(x+4)}{(8-x)(8+x)}$
Recognize the factors that are opposites.	$(-1)\frac{(x-8)(x+4)}{(8-x)(8+x)}$
Simplify.	$-\frac{x+4}{x+8}$

Example 5.4.11

Simplify: $\frac{x^2-4x-5}{25-x^2}$

Answer

$-\frac{x+1}{x+5}$

Example 5.4.12

Simplify: $\dfrac{x^2+x-2}{1-x^2}$.

Answer

$-\dfrac{x+2}{x+1}$

Multiply Rational Expressions

To multiply rational expressions, we do just what we did with numerical fractions. We multiply the numerators and multiply the denominators. Then, if there are any common factors, we remove them to simplify the result.

MULTIPLICATION OF RATIONAL EXPRESSIONS

If p, q, r, and s are polynomials where $q \neq 0$, $s \neq 0$, then

$$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$

To multiply rational expressions, multiply the numerators and multiply the denominators.

Remember, throughout this chapter, we will assume that all numerical values that would make the denominator be zero are excluded. We will not write the restrictions for each rational expression, but keep in mind that the denominator can never be zero. So in this next example, $x \neq 0$, $x \neq 3$, and $x \neq 4$.

Example 5.4.13

Simplify: $\dfrac{2x}{x^2-7x+12} \cdot \dfrac{x^2-9}{6x^2}$.

Solution

Step 1. Factor each numerator and denominator completely.	Factor x^2-9 and $x^2-7x+12$.	$\dfrac{2x}{x^2-7x+12} \cdot \dfrac{x^2-9}{6x^2}$ $\dfrac{2x}{(x-3)(x-4)} \cdot \dfrac{(x-3)(x+3)}{6x^2}$
Step 2. Multiply the numerators and denominators.	Multiply the numerators and denominators. It is helpful to write the monomials first.	$\dfrac{2x(x-3)(x+3)}{6x^2(x-3)(x-4)}$
Step 3. Simplify by dividing out common factors.	Divide out the common factors. Leave the denominator in factored form.	$\dfrac{2 \cdot x \cdot (x-3)(x+3)}{2 \cdot 3 \cdot x \cdot x \cdot (x-3)(x-4)}$ $\dfrac{(x+3)}{3x(x-4)}$

Example 5.4.14

Simplify: $\dfrac{5x}{x^2+5x+6} \cdot \dfrac{x^2-4}{10x}$.

Answer

$\dfrac{x-2}{2(x+3)}$

Example 5.4.15

Simplify: $\frac{9x^2}{x^2+11x+30} \cdot \frac{x^2-36}{3x^2}$.

Answer

$\frac{3(x-6)}{x+5}$

MULTIPLY RATIONAL EXPRESSIONS.

1. Factor each numerator and denominator completely.
2. Multiply the numerators and denominators.
3. Simplify by dividing out common factors.

Example 5.4.16

Multiply: $\frac{3a^2-8a-3}{a^2-25} \cdot \frac{a^2+10a+25}{3a^2-14a-5}$.

Solution

$$\frac{3a^2-8a-3}{a^2-25} \cdot \frac{a^2+10a+25}{3a^2-14a-5}$$

Factor the numerators and denominators and then multiply.

$$\frac{(3a+1)(a-3)(a+5)(a+5)}{(a-5)(a+5)(3a+1)(a-5)}$$

Simplify by dividing out common factors.

$$\frac{\cancel{(3a+1)}(a-3)\ \cancel{(a+5)}(a+5)}{(a-5)\ \cancel{(a+5)}\ \cancel{(3a+1)}(a-5)}$$

Simplify.

$$\frac{(a-3)(a+5)}{(a-5)(a-5)}$$

Rewrite $(a-5)(a-5)$ using an exponent.

$$\frac{(a-3)(a+5)}{(a-5)^2}$$

Example 5.4.17

Simplify: $\frac{2x^2+5x-12}{x^2-16} \cdot \frac{x^2-8x+16}{2x^2-13x+15}$.

Answer

$\frac{x-4}{x-5}$

Example 5.4.18

Simplify: $\frac{4b^2+7b-2}{1-b^2} \cdot \frac{b^2-2b+1}{4b^2+15b-4}$.

Answer

$$-\frac{(b+2)(b-1)}{(1+b)(b+4)}$$

Divide Rational Expressions

Just like we did for numerical fractions, to divide rational expressions, we multiply the first fraction by the reciprocal of the second.

DIVISION OF RATIONAL EXPRESSIONS

If p, q, r, and s are polynomials where $q \neq 0$, $r \neq 0$, $s \neq 0$, then

$$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r}$$

To divide rational expressions, multiply the first fraction by the reciprocal of the second.

Once we rewrite the division as multiplication of the first expression by the reciprocal of the second, we then factor everything and look for common factors.

Example 5.4.19

Divide: $\frac{p^3+q^3}{2p^2+2pq+2q^2} \div \frac{p^2-q^2}{6}$.

Solution

Step 1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.	"Flip" the second fraction and change the division sign to multiplication.	$\frac{p^3+q^3}{2p^2+2pq+2q^2} \div \frac{p^2-q^2}{6}$ $\frac{p^3+q^3}{2p^2+2pq+2q^2} \cdot \frac{6}{p^2-q^2}$
Step 2. Factor the numerators and denominators completely.	Factor the numerators and denominators.	$\frac{(p+q)(p^2-pq+q^2)}{2(p^2+pq+q^2)} \cdot \frac{2 \cdot 3}{(p-q)(p+q)}$
Step 3. Multiply the numerators and denominators.	Multiply the numerators and multiply the denominators.	$\frac{(p+q)(p^2-pq+q^2)2 \cdot 3}{2(p^2+pq+q^2)(p-q)(p+q)}$
Step 4. Simplify by dividing out common factors.	Divide out the common factors.	$\frac{\cancel{(p+q)}(p^2-pq+q^2)\cancel{2} \cdot 3}{\cancel{2}(p^2+pq+q^2)(p-q)\cancel{(p+q)}}$ $\frac{3(p^2-pq+q^2)}{(p-q)(p^2+pq+q^2)}$

Example 5.4.20

Simplify: $\frac{x^3-8}{3x^2-6x+12} \div \frac{x^2-4}{6}$.

Answer

$$\frac{2(x^2+2x+4)}{(x+2)(x^2-2x+4)}$$

Example 5.4.21

Simplify: $\dfrac{2z^2}{z^2-1} \div \dfrac{z^3-z^2+z}{z^3+1}$.

Answer

$$\frac{2z}{z-1}$$

DIVISION OF RATIONAL EXPRESSIONS

1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.
2. Factor the numerators and denominators completely.
3. Multiply the numerators and denominators together.
4. Simplify by dividing out common factors.

Recall from Use the Language of Algebra that a compound fraction (or complex fraction) is a fraction that contains a fraction in the numerator, the denominator or both. Also, remember a fraction bar means division. A complex fraction is another way of writing division of two fractions.

Example 5.4.22

Divide: $\dfrac{\dfrac{6x^2-7x+2}{4x-8}}{\dfrac{2x^2-7x+3}{x^2-5x+6}}$.

Solution

$$\frac{\dfrac{6x^2-7x+2}{4x-8}}{\dfrac{2x^2-7x+3}{x^2-5x+6}}$$

Rewrite with a division sign. $\dfrac{6x^2-7x+2}{4x-8} \div \dfrac{2x^2-7x+3}{x^2-5x+6}$

Rewrite as product of first times reciprocal of second. $\dfrac{6x^2-7x+2}{4x-8} \cdot \dfrac{x^2-5x+6}{2x^2-7x+3}$

Factor the numerators and the denominators, and then multiply. $\dfrac{(2x-1)(3x-2)(x-2)(x-3)}{4(x-2)(2x-1)(x-3)}$

Simplify by dividing out common factors. $\dfrac{\cancel{(2x-1)}(3x-2)\ \cancel{(x-2)}\ \cancel{(x-3)}}{4\ \cancel{(x-2)}\ \cancel{(2x-1)}\ \cancel{(x-3)}}$

Simplify. $\dfrac{3x-2}{4}$

Example 5.4.23

Simplify: $\dfrac{\dfrac{3x^2+7x+2}{4x+24}}{\dfrac{3x^2-14x-5}{x^2+x-30}}$.

Answer

$\dfrac{x+2}{4}$

? Example 5.4.24

Simplify: $\dfrac{\dfrac{y^2-36}{2y^2+11y-6}}{\dfrac{2y^2-2y-60}{8y-4}}$.

Answer

$\dfrac{2}{y+5}$

Key Concepts

- **Determine the values for which a rational expression is undefined.**
  1. Set the denominator equal to zero.
  2. Solve the equation.
- **Equivalent Fractions Property**
  If a, b, and c are numbers where $b \neq 0$, $c \neq 0$, then

  $$\frac{a}{b} = \frac{a \cdot c}{b \cdot c} \text{ and } \frac{a \cdot c}{b \cdot c} = \frac{a}{b}.$$
- **How to simplify a rational expression.**
  1. Factor the numerator and denominator completely.
  2. Simplify by dividing out common factors.
- **Opposites in a Rational Expression**
  The opposite of $a-b$ is $b-a$.

  $$\frac{a-b}{b-a} = -1 \qquad a \neq b$$

  An expression and its opposite divide to -1.
- **Multiplication of Rational Expressions**
  If p, q, r, and s are polynomials where $q \neq 0$, $s \neq 0$, then

  $$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$
- **How to multiply rational expressions.**
  1. Factor each numerator and denominator completely.
  2. Multiply the numerators and denominators.
  3. Simplify by dividing out common factors.
- **Division of Rational Expressions**
  If p, q, r, and s are polynomials where $q \neq 0$, $r \neq 0$, $s \neq 0$, then

$$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r}$$

- **How to divide rational expressions.**
  1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.
  2. Factor the numerators and denominators completely.
  3. Multiply the numerators and denominators together.
  4. Simplify by dividing out common factors.
- **How to determine the domain of a rational function.**
  1. Set the denominator equal to zero.
  2. Solve the equation.
  3. The domain is all real numbers excluding the values found in Step 2.

Glossary

rational expression

A rational expression is an expression of the form $\frac{p}{q}$, where p and q are polynomials and $q \neq 0$.

simplified rational expression

A simplified rational expression has no common factors, other than 1, in its numerator and denominator.

rational function

A rational function is a function of the form $R(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions and $q(x)$ is not zero.

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5.4E: Exercises

Practice Makes Perfect

Undefined Values

In the following exercises, determine the values for which the rational expression is undefined.

1. $\frac{2x^2}{z}$
2. $\frac{4p-1}{6p-5}$
3. $\frac{n-3}{n^2+2n-8}$
4. $\frac{4x^2y}{3y}$
5. $\frac{3x-2}{2x+1}$
6. $\frac{u-1}{u^2-3u-28}$

Answer

1. $z=0$
2. $p=\frac{5}{6}$
3. $n=-4, n=2$
4. $y=0$
5. $x=-\frac{1}{2}$
6. $u=-4, u=7$

Simplify Rational Expressions

In the following exercises, simplify each rational expression.

7. $\frac{x+1+y}{y+x+1}$
8. $\frac{8m^3n}{12mn^2}$
9. $\frac{8n-96}{3n-36}$
10. $\frac{x^2+4x-5}{x^2-2x+1}$
11. $\frac{a^2-4}{a^2+6a-16}$
12. $\frac{p^3+3p^2+4p+12}{p^2+p-6}$
13. $\frac{8b^2-32b}{2b^2-6b-80}$
14. $\frac{3m^2+30mn+75n^2}{4m^2-100n^2}$
15. $\frac{a-5}{5-a}$
16. $\frac{20-5y}{y^2-16}$

17. $\dfrac{z^2-9z+20}{16-z^2}$

Answer

7. 1
8. $\dfrac{2m^2}{3n}$
9. $\dfrac{8}{3}$
10. $\dfrac{x+5}{x-1}$
11. $\dfrac{a+2}{a+8}$
12. $\dfrac{p^2+4}{p-2}$
13. $\dfrac{4b(b-4)}{(b+5)(b-8)}$
14. $\dfrac{3(m+5n)}{4(m-5n)}$
15. -1
16. $\dfrac{-5}{y+4}$
17. $-\dfrac{z-5}{4+z}$

Multiply Rational Expressions

In the following exercises, multiply the rational expressions.

? Multiply Rational Expressions

In the following exercises, multiply the rational expressions.

18. $\dfrac{12}{16}\cdot\dfrac{4}{10}$
19. $\dfrac{5x^2y^4}{12xy^3}\cdot\dfrac{6x^2}{20y^2}$
20. $\dfrac{5p^2}{p^2-5p-36}\cdot\dfrac{p^2-16}{10p}$
21. $\dfrac{2y^2-10y}{y^2+10y+25}\cdot\dfrac{y+5}{6y}$
22. $\dfrac{28-4b}{3b-3}\cdot\dfrac{b^2+8b-9}{b^2-49}$
23. $\dfrac{c^2-10c+25}{c^2-25}\cdot\dfrac{c^2+10c+25}{3c^2-14c-5}$
24. $\dfrac{2m^2-3m-2}{2m^2+7m+3}\cdot\dfrac{3m^2-14m+15}{3m^2+17m-20}$

Answer

18. $\dfrac{3}{10}$
19. $\dfrac{x^3}{8y}$
20. $\dfrac{p(p-4)}{2(p-9)}$

21. $\dfrac{y-5}{3(y+5)}$
22. $\dfrac{-4(b+9)}{3(b+7)}$
23. $\dfrac{c+5}{3c+1}$
24. $\dfrac{(m-3)(m-2)(3m-5)}{(m-1)(m+3)(3m+20)}$

? Divide Rational Expressions

In the following exercises, divide the rational expressions.

25. $\dfrac{v-5}{11-v} \div \dfrac{v^2-25}{v-11}$
26. $\dfrac{3s^2}{s^2-16} \div \dfrac{s^3+4s^2+16s}{s^3-64}$
27. $\dfrac{p^3+q^3}{3p^2+3pq+3q^2} \div \dfrac{p^2-q^2}{12}$
28. $\dfrac{x^2+3x-10}{4x} \div (2x^2+20x+50)$
29. $\dfrac{\dfrac{2a^2-a-21}{5a+20}}{\dfrac{a^2+7a+12}{a^2+8a+16}}$
30. $\dfrac{\dfrac{12c^2-12}{2c^2-3c+1}}{\dfrac{4c+4}{6c^2-13c+5}}$
31. $\dfrac{10m^2+80m}{3m-9} \cdot \dfrac{m^2+4m-21}{m^2-9m+20} \div \dfrac{5m^2+10m}{2m-10}$
32. $\dfrac{12p^2+3p}{p+3} \div \dfrac{p^2+2p-63}{p^2-p-12} \cdot \dfrac{p-7}{9p^3-9p^2}$

Answer

25. $-\dfrac{1}{v+5}$
26. $\dfrac{3s}{s+4}$
27. $\dfrac{4(p^2-pq+q^2)}{(p-q)(p^2+pq+q^2)}$
28. $\dfrac{x-2}{8x(x+5)}$
29. $\dfrac{2a-7}{5}$
30. $3(3c-5)$
31. $\dfrac{4(m+8)(m+7)}{3(m-4)(m+2)}$
32. $\dfrac{(4p+1)(p-4)}{3p(p+9)(p-1)}$

Writing Exercises

33. Explain how you find the values of x for which the rational expression $\frac{x^2-x-20}{x^2-4}$ is undefined.

34.

a. Multiply $\frac{7}{4} \cdot \frac{9}{10}$ and explain all your steps.

b. Multiply $\frac{n}{n-3} \cdot \frac{9}{n+3}$ and explain all your steps.

c. Evaluate your answer to part b. when $n = 7$. Did you get the same answer you got in part a.? Why or why not?

Answer

Answers will vary.

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5.5: Add and Subtract Rational Expressions

Summary

By the end of this section, you will be able to:

- Add and subtract rational expressions with a common denominator
- Add and subtract rational expressions whose denominators are opposites
- Find the least common denominator of rational expressions
- Add and subtract rational expressions with unlike denominators
- Add and subtract rational functions

Warm up

1. Add: $\frac{7}{10}+\frac{8}{15}$
2. Subtract: $\frac{3x}{4}-\frac{8}{9}$
3. Subtract: $6(2x+1)-4(x-5)$

Answer

1. $\frac{37}{30}$
2. $\drac27x-3236$
3. $8x+26$

Add and Subtract Rational Expressions with a Common Denominator

What is the first step you take when you add numerical fractions? You check if they have a common denominator. If they do, you add the numerators and place the sum over the common denominator. If they do not have a common denominator, you find one before you add.

It is the same with rational expressions. To add rational expressions, they must have a common denominator. When the denominators are the same, you add the numerators and place the sum over the common denominator.

RATIONAL EXPRESSION ADDITION AND SUBTRACTION

If p, q, and r are polynomials where $r\neq 0$, then

$$\frac{p}{r}+\frac{q}{r}=\frac{p+q}{r} \quad \text{and} \quad \frac{p}{r}-\frac{q}{r}=\frac{p-q}{r}$$

To add or subtract rational expressions with a common denominator, add or subtract the numerators and place the result over the common denominator.

We always simplify rational expressions. Be sure to factor, if possible, after you subtract the numerators so you can identify any common factors.

Remember, too, we do not allow values that would make the denominator zero. What value of x should be excluded in the next example?

Example 5.5.1

Add: $\frac{11x+28}{x+4}+\frac{x^2}{x+4}$.

Solution

Since the denominator is $x+4$, we must exclude the value $x=-4$.

	$\frac{11x+28}{x+4}+\frac{x^2}{x+4},\ x\neq -4$
The fractions have a common denominator, so add the numerators and place the sum over the common denominator.	$\frac{11x+28+x^2}{x+4}$
Write the degrees in descending order.	$\frac{x^2+11x+28}{x+4}$
Factor the numerator.	$\frac{(x+4)(x+7)}{x+4}$
Simplify by removing common factors.	$\frac{\cancel{(x+4)}(x+7)}{\cancel{x+4}}$
Simplify.	$x+7$

The expression simplifies to $x+7$ but the original expression had a denominator of $x+4$ so $x\neq -4$.

? Example 5.5.2

Simplify: $\frac{9x+14}{x+7}+\frac{x^2}{x+7}$.

Answer

$x+2$

? Example 5.5.3

Simplify: $\frac{x^2+8x}{x+5}+\frac{15}{x+5}$.

Answer

$x+3$

To subtract rational expressions, they must also have a common denominator. When the denominators are the same, you subtract the numerators and place the difference over the common denominator. Be careful of the signs when you subtract a binomial or trinomial.

✓ Example 5.5.4

Subtract: $\frac{5x^2-7x+3}{x^2-3x+18}-\frac{4x^2+x-9}{x^2-3x+18}$.

Solution

	$\frac{5x^2-7x+3}{x^2-3x+18}-\frac{4x^2+x-9}{x^2-3x+18}$
Subtract the numerators and place the difference over the common denominator.	$\frac{5x^2-7x+3-(4x^2+x-9)}{x^2-3x+18}$
Distribute the sign in the numerator.	$\frac{5x^2-7x+3-4x^2-x+9}{x^2-3x-18}$
Combine like terms.	$\frac{x^2-8x+12}{x^2-3x-18}$
Factor the numerator and the denominator.	$\frac{(x-2)(x-6)}{(x+3)(x-6)}$
Simplify by removing common factors.	$\frac{(x-2)\ \cancel{(x-6)}}{(x+3)\ \cancel{(x-6)}}$
	$(x-2)(x+3)$

? Example 5.5.5

Subtract: $\frac{4x^2-11x+8}{x^2-3x+2}-\frac{3x^2+x-3}{x^2-3x+2}$.

Answer

$\frac{x-11}{x-2}$

? Example 5.5.6

Subtract: $\frac{6x^2-x+20}{x^2-81}-\frac{5x^2+11x-7}{x^2-81}$.

Answer

$\frac{x-3}{x+9}$

Add and Subtract Rational Expressions Whose Denominators are Opposites

When the denominators of two rational expressions are opposites, it is easy to get a common denominator. We just have to multiply one of the fractions by $\frac{-1}{-1}$.

Let's see how this works.

	$\frac{7}{d}+\frac{5}{-d}$
Multiply the second fraction by $\frac{-1}{-1}$.	$\frac{7}{d}+\frac{(-1)5}{(-1)(-d)}$
The denominators are the same.	$\frac{7}{d}+\frac{-5}{d}$

Simplify.	$\frac{2}{d}$

Be careful with the signs as you work with the opposites when the fractions are being subtracted.

Example 5.5.7

Subtract: $\frac{m^2-6m}{m^2-1}-\frac{3m+2}{1-m^2}$.

Solution

	$\frac{m^2-6m}{m^2-1}-\frac{3m+2}{1-m^2}$
The denominators are opposites, so multiply the second fraction by $\frac{-1}{-1}$.	$\frac{m^2-6m}{m^2-1}-\frac{-1(3m+2)}{-1(1-m^2)}$
Simplify the second fraction.	$\frac{m^2-6m}{m^2-1}-\frac{-3m-2}{m^2-1}$
The denominators are the same. Subtract the numerators.	$\frac{m^2-6m-(-3m-2)}{m^2-1}$
Distribute.	$\frac{m^2-6m+3m+2}{m^2-1}$
Combine like terms.	$\frac{m^2-3m+2}{m^2-1}$
Factor the numerator and denominator.	$\frac{(m-1)(m-2)}{(m-1)(m+1)}$
Simplify by removing common factors.	$\frac{(m-1)(m-2)}{(m-1)(m+1)}$
Simplify.	$\frac{m-2}{m+1}$

Example 5.5.8

Subtract: $\frac{y^2-5y}{y^2-4}-\frac{6y-6}{4-y^2}$.

Answer

$\frac{y+3}{y+2}$

Example 5.5.9

Subtract: $\frac{2n^2+8n-1}{n^2-1}-\frac{n^2-7n-1}{1-n^2}$.

Answer

$\frac{3n-2}{n-1}$

Find the Least Common Denominator of Rational Expressions

When we add or subtract rational expressions with unlike denominators, we will need to get common denominators. If we review the procedure we used with numerical fractions, we will know what to do with rational expressions.

Let's look at this example: $\frac{7}{12}+\frac{5}{18}$. Since the denominators are not the same, the first step was to find the least common denominator (LCD).

To find the LCD of the fractions, we factored 12 and 18 into primes, lining up any common primes in columns. Then we "brought down" one prime from each column. Finally, we multiplied the factors to find the LCD.

When we add numerical fractions, once we found the LCD, we rewrote each fraction as an equivalent fraction with the LCD by multiplying the numerator and denominator by the same number. We are now ready to add.

$$\frac{7}{12}+\frac{5}{18}$$
$$\frac{7\cdot 3}{12\cdot 3}+\frac{5\cdot 2}{18\cdot 2}$$
$$\frac{21}{36}+\frac{10}{36}$$

$$\begin{aligned}12&=2\cdot 2\cdot 3\\18&=2\cdot\ \ 3\cdot 3\\\text{LCD}&=2\cdot 2\cdot 3\cdot 3\\\text{LCD}&=36\end{aligned}$$

We do the same thing for rational expressions. However, we leave the LCD in factored form.

FIND THE LEAST COMMON DENOMINATOR OF RATIONAL EXPRESSIONS.

1. Factor each denominator completely.
2. List the factors of each denominator. Match factors vertically when possible.
3. Bring down the columns by including all factors, but do not include common factors twice.
4. Write the LCD as the product of the factors.

Remember, we always exclude values that would make the denominator zero. What values of xx should we exclude in this next example?

Example 5.5.10

a. Find the LCD for the expressions $\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$ and b. rewrite them as equivalent rational expressions with the lowest common denominator.

Solution

a.

Find the LCD for $\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$.	
Factor each denominator completely, lining up common factors. Bring down the columns.	$x^2-2x-3=(x+1)(x-3)$ $x^2+4x+3=(x+1)(x+3)$ $\text{LCD}\quad=(x+1)(x-3)(x+3)$
Write the LCD as the product of the factors.	The LCD is $(x+1)(x-3)(x+3)$.

b.

	$\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$
Factor each denominator.	$\frac{8}{(x+1)(x-3)}, \frac{3x}{(x+1)(x+3)}$
Multiply each denominator by the 'missing' LCD factor and multiply each numerator by the same factor.	$\frac{8(x+3)}{(x+1)(x-3)(x+3)}, \frac{3x(x-3)}{(x+1)(x+3)(x-3)}$
Simplify the numerators.	$\frac{8x+24}{(x+1)(x-3)(x+3)}, \frac{3x^2-9x}{(x+1)(x+3)(x-3)}$

? Example 5.5.11

a. Find the LCD for the expressions $\frac{2}{x^2-x-12}$, $\frac{1}{x^2-16}$ b. rewrite them as equivalent rational expressions with the lowest common denominator.

Answer

a. $(x-4)(x+3)(x+4)$

b. $\frac{2x+8}{(x-4)(x+3)(x+4)}$, $\frac{x+3}{(x-4)(x+3)(x+4)}$

? Example 5.5.12

a. Find the LCD for the expressions $\frac{3x}{x^2-3x+10}$, $\frac{5}{x^2+3x+2}$ b. rewrite them as equivalent rational expressions with the lowest common denominator.

Answer

a. $(x+2)(x-5)(x+1)$

b. $\frac{3x^2+3x}{(x+2)(x-5)(x+1)}$, $\frac{5x-25}{(x+2)(x-5)(x+1)}$

Add and Subtract Rational Expressions with Unlike Denominators

Now we have all the steps we need to add or subtract rational expressions with unlike denominators.

✓ Example 5.5.13

Add: $\frac{3}{x-3}+\frac{2}{x-2}$.

Solution

Step 1. Determine if the expressions have a common denominator. • Yes–go to step 2. • No–Rewrite each rational expression with the LCD. ◦ Find the LCD. ◦ Rewrite each rational expression as an equivalent rational expression with the LCD.	No. Find the LCD of $(x-3)$ and $(x-2)$. Change into equivalent rational expressions with the LCD, $(x-3)$ and $(x-2)$. Keep the denominators factored!	$x-3 : (x-3)$ ⬭ $x-2$ ⬭ : $(x-2)$ LCD : $(x-3)(x-2)$ $\frac{3}{x-3}+\frac{2}{x-2}$ $\frac{3(x-2)}{(x-3)(x-2)}+\frac{2(x-3)}{(x-2)(x-3)}$ $\frac{3x-6}{(x-3)(x-2)}+\frac{2x-6}{(x-2)(x-3)}$
Step 2. Add or subtract the rational expressions.	Add the numerators and place the sum over the common denominator.	$\frac{3x-6+2x-6}{(x-3)(x-2)}$
Step 3. Simplify, if possible.	Because $5x-12$ cannot be factored, the answer is simplified.	$\frac{5x-12}{(x-3)(x-2)}$

Example 5.5.14

Add: $\frac{2}{x-2}+\frac{5}{x+3}$.

Answer

$$\frac{7x-4}{(x-2)(x+3)}$$

Example 5.5.15

Add: $\frac{4}{m+3}+\frac{3}{m+4}$.

Answer

$$\frac{7m+25}{(m+3)(m+4)}$$

The steps used to add rational expressions are summarized here.

ADD OR SUBTRACT RATIONAL EXPRESSIONS.

1. Determine if the expressions have a common denominator.
   - **Yes** – go to step 2.
   - **No** – Rewrite each rational expression with the LCD.
     - Find the LCD.
     - Rewrite each rational expression as an equivalent rational expression with the LCD.
2. Add or subtract the rational expressions.
3. Simplify, if possible.

Avoid the temptation to simplify too soon. In the example above, we must leave the first rational expression as $\frac{3x-6}{(x-3)(x-2)}$ to be able to add it to $\frac{2x-6}{(x-2)(x-3)}$. Simplify *only* after you have combined the numerators.

Example 5.5.16

Add: $\frac{8}{x^2-2x-3}+\frac{3x}{x^2+4x+3}$.

Solution

	$\frac{8}{x^2-2x-3}+\frac{3x}{x^2+4x+3}$
Do the expressions have a common denominator?	No.
Find the LCD.	$x^2-2x-3=(x+1)(x-3)$ $x^2+4x+3=(x+1)\quad(x+3)$ $LCD=(x+1)(x-3)(x+3)$
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{8(x+3)}{(x+1)(x-3)(x+3)}+\frac{3x(x-3)}{(x+1)(x+3)(x-3)}$
Simplify the numerators.	$\frac{8x+24}{(x+1)(x-3)(x+3)}+\frac{3x^2-9x}{(x+1)(x+3)(x-3)}$

Add the rational expressions.	$\frac{8x+24+3x^2-9x}{(x+1)(x-3)(x+3)}$
Simplify the numerator.	$\frac{3x^2-x+24}{(x+1)(x-3)(x+3)}$
	The numerator is prime, so there are no common factors.

Example 5.5.17

Add: $\frac{1}{m^2-m-2}+\frac{5m}{m^2+3m+2}$.

Answer

$$\frac{5m^2-9m+2}{(m+1)(m-2)(m+2)}$$

Example 5.5.18

Add: $\frac{2n}{n^2-3n-10}+\frac{6}{n^2+5n+6}$.

Answer

$$\frac{2n^2+12n-30}{(n+2)(n-5)(n+3)}$$

The process we use to subtract rational expressions with different denominators is the same as for addition. We just have to be very careful of the signs when subtracting the numerators.

Example 5.5.19

Subtract: $\frac{8y}{y^2-16}-\frac{4}{y-4}$.

Solution

	$\frac{8y}{y^2-16}-\frac{4}{y-4}$
Do the expressions have a common denominator?	No.
Find the LCD.	$y^2-16=(y-4)(y+4)$ $y-4=y-4$ $LCD=(y-4)(y+4)$
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{8y}{(y-4)(y+4)}-\frac{4(y+4)}{(y-4)(y+4)}$
Simplify the numerators.	$\frac{8y}{(y-4)(y+4)}-\frac{4y+16}{(y-4)(y+4)}$
Subtract the rational expressions.	$\frac{8y-4y-16}{(y-4)(y+4)}$
Simplify the numerator.	$\frac{4y-16}{(y-4)(y+4)}$
Factor the numerator to look for common factors.	$\frac{4(y-4)}{(y-4)(y+4)}$

Remove common factors	$\frac{4(y-4)}{(y-4)(y+4)}$
Simplify.	$\frac{4}{(y+4)}$

Example 5.5.20

Subtract: $\frac{2x}{x^2-4}-\frac{1}{x+2}$.

Answer

$\frac{1}{x-2}$

Example 5.5.21

Subtract: $\frac{3}{z+3}-\frac{6z}{z^2-9}$.

Answer

$\frac{-3}{z-3}$

Key Concepts

- **Rational Expression Addition and Subtraction**
  If p, q, and r are polynomials where $r \neq 0$, then
  $$\frac{p}{r}+\frac{q}{r}=\frac{p+q}{r} \quad \text{and} \quad \frac{p}{r}-\frac{q}{r}=\frac{p-q}{r}$$
- **How to find the least common denominator of rational expressions.**
  1. Factor each expression completely.
  2. List the factors of each expression. Match factors vertically when possible.
  3. Bring down the columns.
  4. Write the LCD as the product of the factors.
- **How to add or subtract rational expressions.**
  1. Determine if the expressions have a common denominator.
     - Yes – go to step 2.
     - No – Rewrite each rational expression with the LCD.
       - Find the LCD.
       - Rewrite each rational expression as an equivalent rational expression with the LCD.
  2. Add or subtract the rational expressions.
  3. Simplify, if possible.

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5.5E: Exercises

Practice Makes Perfect

Add and Subtract with Common Denominator

In the following exercises, add or subtract.

1. $\frac{2}{15}+\frac{7}{15}$
2. $\frac{3c}{4c-5}+\frac{5}{4c-5}$
3. $\frac{2r^2}{2r-1}+\frac{15r-8}{2r-1}$
4. $\frac{2w^2}{w^2-16}+\frac{8w}{w^2-16}$
5. $\frac{9a^2}{3a-7}-\frac{49}{3a-7}$
6. $\frac{3m^2}{6m-30}-\frac{21m-30}{6m-30}$
7. $\frac{6p^2+3p+4}{p^2+4p-5}-\frac{5p^2+p+7}{p^2+4p-5}$
8. $\frac{5r^2+7r-33}{r^2-49}-\frac{4r^2+5r+30}{r^2-49}$

Answer

1. $\frac{3}{5}$
2. $\frac{3c+5}{4c-5}$
3. $r+8$
4. $\frac{2w}{w-4}$
5. $3a+7$
6. $\frac{m-2}{2}$
7. $\frac{p+3}{p+5}$
8. $\frac{r+9}{r+7}$

Add and Subtract with Opposite Denominators

In the following exercises, add or subtract.

9. $\frac{10v}{2v-1}+\frac{2v+4}{1-2v}$
10. $\frac{10x^2+16x-7}{8x-3}+\frac{2x^2+3x-1}{3-8x}$
11. $\frac{z^2+6z}{z^2-25}-\frac{3z+20}{25-z^2}$
12. $\frac{2b^2+30b-13}{b^2-49}-\frac{2b^2-5b-8}{49-b^2}$

Answer

9. 4
10. $x+2$

11. $\frac{z+4}{z-5}$
12. $\frac{4b-3}{b-7}$

Add and Subtract with Unlike Denominators.

In the following exercises, perform the indicated operations.

13. $\frac{7}{10x^2y}+\frac{4}{15xy^2}$
14. $\frac{3}{r+4}+\frac{2}{r-5}$
15. $\frac{5}{3w-2}+\frac{2}{w+1}$
16. $\frac{2y}{y+3}+\frac{3}{y-1}$
17. $\frac{5b}{a^2b-2a^2}+\frac{2b}{b^2-4}$
18. $\frac{-3m}{3m-3}+\frac{5m}{m^2+3m-4}$
19. $\frac{3r}{r^2+7r+6}+\frac{9}{r^2+4r+3}$
20. $\frac{t}{t-6}-\frac{t-2}{t+6}$
21. $\frac{5a}{a+3}-\frac{a+2}{a+6}$
22. $\frac{6}{m+6}-\frac{12m}{m^2-36}$
23. $\frac{-9p-17}{p^2-4p-21}-\frac{p+1}{7-p}$
24. $\frac{-2r-16}{r^2+6r-16}-\frac{5}{2-r}$
25. $\frac{2x+7}{10x-1}+3$
26. $\frac{3}{x^2-3x-4}-\frac{2}{x^2-5x+4}$
27. $\frac{5}{x^2+8x-9}-\frac{4}{x^2+10x+9}$
28. $\frac{5a}{a-2}+\frac{9}{a}-\frac{2a+18}{a^2-2a}$
29. $\frac{c}{c+2}+\frac{5}{c-2}-\frac{10c}{c^2-4}$
30. $\frac{3d}{d+2}+\frac{4}{d}-\frac{d+8}{d^2+2d}$

Answer

13. $\frac{21y+8x}{30x^2y^2}$
14. $\frac{5r-7}{(r+4)(r-5)}$
15. $\frac{11w+1}{(3w-2)(w+1)}$
16. $\frac{2y^2+y+9}{(y+3)(y-1)}$
17. $\frac{b(5b+10+2a^2)}{a^2(b-2)(b+2)}$

18. $-\dfrac{m}{m+4}$
19. $\dfrac{3(r^2+6r+18)}{(r+1)(r+6)(r+3)}$
20. $\dfrac{2(7t-6)}{(t-6)(t+6)}$
21. $\dfrac{4a^2+25a-6}{(a+3)(a+6)}$
22. $\dfrac{-6}{m-6}$
23. $\dfrac{p+2}{p+3}$
24. $\dfrac{3}{r-2}$
25. $\dfrac{4(8x+1)}{10x-1}$
26. $\dfrac{x-5}{(x-4)(x+1)(x-1)}$
27. $\dfrac{1}{(x-1)(x+1)}$
28. $\dfrac{5a^2+7a-36}{a(a-2)}$
29. $\dfrac{c-5}{c+2}$
30. $\dfrac{3(d+1)}{d+2}$

? Writing Exercises

31. Donald thinks that $\dfrac{3}{x}+\dfrac{4}{x}$ is $\dfrac{7}{2x}$. Is Donald correct? Explain.
32. Explain how you find the Least Common Denominator of x^2+5x+4 and x^2-16.
33. Simplify the expression $\dfrac{4}{n^2+6n+9}-\dfrac{1}{n^2-9}$ and explain all your steps.

Answer

Answers will vary

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5.6: Solve Rational Equations

Learning Objectives

- Solve rational equations
- Use rational functions
- Solve a rational equation for a specific variable

Warm up

a. Solve: $\frac{1}{6}x+\frac{1}{2}=\frac{1}{3}$
b. Solve: $n^2-5n-36=0$
c. Solve $5x+2x=10$

Answer

a. -1
b. $n=9, n=-4$
c. $x=\frac{10}{7}$

After defining the terms ‘expression’ and ‘equation’ earlier, we have used them throughout this book. We have simplified many kinds of expressions and solved many kinds of equations. We have simplified many rational expressions so far in this chapter. Now we will solve a **rational equation**.

Definition: RATIONAL EQUATION

A **rational equation** is an equation that contains a rational expression.

You must make sure to know the difference between rational expressions and rational equations. The equation contains an equal sign.

Rational Expression	Rational Equation
$\frac{1}{8}x+\frac{1}{2}$	$\frac{1}{8}x+\frac{1}{2}=\frac{1}{4}$
$\frac{y+6}{y^2-36}$	$\frac{y+6}{y^2-36}=y+1$
$\frac{1}{n-3}+\frac{1}{n+4}$	$\frac{1}{n-3}+\frac{1}{n+4}=\frac{15}{n^2+n-12}$

Solve Rational Equations

We have already solved linear equations that contained fractions. We found the LCD of all the fractions in the equation and then multiplied both sides of the equation by the LCD to “clear” the fractions.

We will use the same strategy to solve rational equations. We will multiply both sides of the equation by the LCD. Then, we will have an equation that does not contain rational expressions and thus is much easier for us to solve. But because the original equation may have a variable in a denominator, we must be careful that we don’t end up with a solution that would make a denominator equal to zero.

So before we begin solving a rational equation, we examine it first to find the values that would make any denominators zero. That way, when we solve a rational equation we will know if there are any algebraic solutions we must discard.

An algebraic solution to a rational equation that would cause any of the rational expressions to be undefined is called an **extraneous solution to a rational equation**.

Definition: EXTRANEOUS Solution TO A RATIONAL EQUATION

An **extraneous solution to a rational equation** is an algebraic solution that would cause any of the expressions in the original equation to be undefined.

We note any possible extraneous solutions, c, by writing $x \neq c$ next to the equation.

Example 5.6.1

Solve:

$$\frac{1}{x}+\frac{1}{3}=\frac{5}{6}$$

Solution

Step 1. Note any value of the variable that would make any denominator zero.

If $x=0$, then $\frac{1}{x}$ is undefined. So we'll write $x \neq 0$ next to the equation.

$$\frac{1}{x}+\frac{1}{3}=\frac{5}{6}, x \neq 0$$

Step 2. Find the least common denominator of all denominators in the equation.

Find the LCD of $\frac{1}{x}$, $\frac{1}{3}$, and $\frac{5}{6}$

The LCD is $6x$.

Step 3. Clear the fractions by multiplying both sides of the equation by the LCD.

Multiply both sides of the equation by the LCD, $6x$.

$$6x \cdot \left(\frac{1}{x}+\frac{1}{3}\right)=6x \cdot \left(\frac{5}{6}\right)$$

Use the Distributive Property.

$$6x \cdot \frac{1}{x}+6x \cdot \frac{1}{3}=6x \cdot \left(\frac{5}{6}\right)$$

Simplify - and notice, no more fractions!

$$6+2x=5x$$

Step 4. Solve the resulting equation.

Simplify.

$$6=3x$$
$$2=x$$

Step 5. Check.

If any values found in Step 1 are algebraic solutions, discard them. Check any remaining solutions in the original equation.

We did not get 0 as an algebraic solution.

$$\frac{1}{x}+\frac{1}{3}=\frac{5}{6}$$

We substitute $x=2$ into the original equation.

$$\frac{1}{2}+\frac{1}{3} \stackrel{?}{=} \frac{5}{6}$$
$$\frac{3}{6}+\frac{2}{6} \stackrel{?}{=} \frac{5}{6}$$
$$\frac{5}{6}=\frac{5}{6}\checkmark$$

The solution is $x=2$

Exercise 5.6.2

Solve:

$$\frac{1}{y}+\frac{2}{3}=\frac{1}{5}$$

Answer

$$y=-\frac{15}{7}$$

Exercise 5.6.3

Solve:

$$\frac{2}{3}+\frac{1}{5}=\frac{1}{x}$$

Answer

$$x=\frac{15}{3}$$

The steps of this method are shown.

how to Solve equations with rational expressions.

- Step 1. Note any value of the variable that would make any denominator zero.
- Step 2. Find the least common denominator of all denominators in the equation.
- Step 3. Clear the fractions by multiplying both sides of the equation by the LCD.
- Step 4. Solve the resulting equation.
- Step 5. Check:
  - If any values found in Step 1 are algebraic solutions, discard them.
  - Check any remaining solutions in the original equation.

We always start by noting the values that would cause any denominators to be zero.

Example 5.6.4

Solve:

$$1-\frac{5}{y}=-\frac{6}{y^2}$$

Solution

Note any value of the variable that would make any denominator zero.

$$1-\frac{5}{y}=-\frac{6}{y^2}, y\neq 0$$

Find the least common denominator of all denominators in the equation. The LCD is y^2.

Clear the fractions by multiplying both sides of the equation by the LCD.

$$y^2\left(1-\frac{5}{y}\right)=y^2\left(-\frac{6}{y^2}\right)$$

Distribute.

$$y^2\cdot 1-y^2\left(\frac{5}{y}\right)=y^2\left(-\frac{6}{y^2}\right)$$

Multiply.

$$y^2-5y=-6$$

Solve the resulting equation. First write the quadratic equation in standard form.

$$y^2-5y+6=0$$

Factor.

$$(y-2)(y-3)=0$$

Use the Zero Product Property.

$$y-2=0 \text{ or } y-3=0$$

Solve.

$$y=2 \text{ or } y=3$$

Check. We did not get 0 as an algebraic solution.

Check $y=2$ and $y=3$ in the original equation.

$$1-\frac{5}{y}=-\frac{6}{y^2} \qquad 1-\frac{5}{y}=-\frac{6}{y^2}$$

$$1-\frac{5}{2}\stackrel{?}{=}-\frac{6}{2^2} \qquad 1-\frac{5}{3}\stackrel{?}{=}-\frac{6}{3^2}$$

$$1-\frac{5}{2}\stackrel{?}{=}-\frac{6}{4} \qquad 1-\frac{5}{3}\stackrel{?}{=}-\frac{6}{9}$$

$$\frac{2}{2}-\frac{5}{2}\stackrel{?}{=}-\frac{6}{4} \qquad \frac{3}{3}-\frac{5}{3}\stackrel{?}{=}-\frac{6}{9}$$

$$-\frac{3}{2}\stackrel{?}{=}-\frac{6}{4} \qquad -\frac{2}{3}\stackrel{?}{=}-\frac{6}{9}$$

$$-\frac{3}{2}=-\frac{3}{2}\surd \qquad -\frac{2}{3}=-\frac{2}{3}\surd$$

The solution is $y=2, y=3$

? Exercise 5.6.5

Solve:

$$1-\frac{2}{x}=\frac{15}{x^2}$$

Answer

$x=-3, x=5$

Exercise 5.6.6

Solve:

$$1-\frac{4}{y}=\frac{12}{y^2}$$

Answer

$y=-2, y=6$

In the next example, the last denominators is a difference of squares. Remember to factor it first to find the LCD.

Example 5.6.7

Solve:

$$\frac{2}{x+2}+\frac{4}{x-2}=\frac{x-1}{x^2-4}$$

Solution

Note any value of the variable that would make any denominator zero.

$$\frac{2}{x+2}+\frac{4}{x-2}=\frac{x-1}{(x+2)(x-2)}, x\neq -2, x\neq 2$$

Find the least common denominator of all denominators in the equation. The LCD is $(x+2)(x-2)$.

Clear the fractions by multiplying both sides of the equation by the LCD.

$$(x+2)(x-2)\left(\frac{2}{x+2}+\frac{4}{x-2}\right)=(x+2)(x-2)\left(\frac{x-1}{x^2-4}\right)$$

Distribute.

$$(x+2)(x-2)\frac{2}{x+2}+(x+2)(x-2)\frac{4}{x-2}=(x+2)(x-2)\left(\frac{x-1}{x^2-4}\right)$$

Remove common factors.

$$\cancel{(x+2)}(x-2)\frac{2}{\cancel{x+2}}+(x+2)\ \cancel{(x-2)}\frac{4}{\cancel{x-2}}=\ \cancel{(x+2)(x-2)}\left(\frac{x-1}{\cancel{x^2-4}}\right)$$

Simplify.

$$2(x-2)+4(x+2)=x-1$$

Distribute.

$$2x-4+4x+8=x-1$$

Solve.

$$\begin{aligned}6x+4&=x-1\\5x&=-5\\x&=-1\end{aligned}$$

Check: We did not get 2 or −2 as algebraic solutions.

Check $x=-1$ in the original equation.

$$\frac{2}{x+2}+\frac{4}{x-2}=\frac{x-1}{x^2-4}$$
$$\frac{2}{(-1)+2}+\frac{4}{(-1)-2}\stackrel{?}{=}\frac{(-1)-1}{(-1)^2-4}$$
$$\frac{2}{1}+\frac{4}{-3}\stackrel{?}{=}\frac{-2}{-3}$$
$$\frac{6}{3}-\frac{4}{3}\stackrel{?}{=}\frac{2}{3}$$
$$\frac{2}{3}=\frac{2}{3}\surd$$

The solution is $x=-1$.

Exercise 5.6.8

Solve:

$$\frac{2}{x+1}+\frac{1}{x-1}=\frac{1}{x^2-1}$$

Answer

$x=\frac{2}{3}$

Exercise 5.6.9

Solve:

$$\frac{5}{y+3}+\frac{2}{y-3}=\frac{5}{y^2-9}$$

Answer

$y=2$

In the next example, the first denominator is a trinomial. Remember to factor it first to find the LCD.

Example 5.6.10

Solve:

$$\frac{m+11}{m^2-5m+4}=\frac{5}{m-4}-\frac{3}{m-1}$$

Solution

Note any value of the variable that would make any denominator zero. Use the factored form of the quadratic denominator.

$$\frac{m+11}{(m-4)(m-1)}=\frac{5}{m-4}-\frac{3}{m-1}, m\neq 4, m\neq 1$$

Find the least common denominator of all denominators in the equation. The LCD is $(m-4)(m-1)$

Clear the fractions by multiplying both sides of the equation by the LCD.

$$(m-4)(m-1)\left(\frac{m+11}{(m-4)(m-1)}\right)=(m-4)(m-1)\left(\frac{5}{m-4}-\frac{3}{m-1}\right)$$

Distribute.

$$(m-4)(m-1)\left(\frac{m+11}{(m-4)(m-1)}\right)=(m-4)(m-1)\frac{5}{m-4}-(m-4)(m-1)\frac{3}{m-1}$$

Remove common factors.

$$\cancel{(m-4)(m-1)}\left(\frac{m+11}{\cancel{(m-4)(m-1)}}\right)=\cancel{(m-4)}(m-1)\frac{5}{\cancel{m-4}}-(m-4)\ \cancel{(m-1)}\frac{3}{\cancel{m-1}}$$

Simplify.

$$m+11=5(m-1)-3(m-4)$$

Solve the resulting equation.

$$\begin{aligned} m+11 &= 5m-5-3m+12 \\ 4 &= m \end{aligned}$$

Check. The only algebraic solution was 4, but we said that 4 would make a denominator equal to zero. The algebraic solution is an extraneous solution.

There is no solution to this equation.

? Exercise 5.6.11

Solve:

$$\frac{x+13}{x^2-7x+10}=\frac{6}{x-5}-\frac{4}{x-2}$$

Answer

There is no solution.

? Exercise 5.6.13

Solve:

$$\frac{y-6}{y^2+3y-4}=\frac{2}{y+4}+\frac{7}{y-1}$$

Answer

There is no solution.

The equation we solved in the previous example had only one algebraic solution, but it was an extraneous solution. That left us with no solution to the equation. In the next example we get two algebraic solutions. Here one or both could be extraneous solutions.

✓ Example 5.6.14

Solve:

$$\frac{y}{y+6}=\frac{72}{y^2-36}+4$$

Solution

Factor all the denominators, so we can note any value of the variable that would make any denominator zero.

$$\frac{y}{y+6}=\frac{72}{(y-6)(y+6)}+4, y\neq 6, y\neq -6$$

Find the least common denominator. The LCD is $(y-6)(y+6)$

Clear the fractions.

$$(y-6)(y+6)\left(\frac{y}{y+6}\right)=(y-6)(y+6)\left(\frac{72}{(y-6)(y+6)}+4\right)$$

Simplify.

$$(y-6)\cdot y=72+(y-6)(y+6)\cdot 4$$

Simplify.

$$y(y-6)=72+4\left(y^2-36\right)$$

Solve the resulting equation.

$$\begin{aligned} y^2-6y &= 72+4y^2-144 \\ 0 &= 3y^2+6y-72 \\ 0 &= 3\left(y^2+2y-24\right) \\ 0 &= 3(y+6)(y-4) \\ y &= -6, y=4 \end{aligned}$$

Check.

$y=-6$ is an extraneous solution. Check $y=4$ in the original equation.

$$\begin{aligned} \frac{y}{y+6} &= \frac{72}{y^2-36}+4 \\ \frac{4}{4+6} &\stackrel{?}{=} \frac{72}{4^2-36}+4 \\ \frac{4}{10} &\stackrel{?}{=} \frac{72}{-20}+4 \\ \frac{4}{10} &\stackrel{?}{=} -\frac{36}{10}+\frac{40}{10} \\ \frac{4}{10} &= \frac{4}{10}\surd \end{aligned}$$

The solution is $y=4$.

Exercise 5.6.15

Solve:

$$\frac{x}{x+4}=\frac{32}{x^2-16}+5$$

Answer

$x=3$

Exercise 5.6.16

Solve:

$$\frac{y}{y+8}=\frac{128}{y^2-64}+9$$

Answer

$y=7$

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5.6E: Exercises

Solve Rational Equations

In the following exercises, solve each rational equation.

? Solve Rational Equations

Solve the following

1. $\frac{1}{a}+\frac{2}{5}=\frac{1}{2}$
2. $\frac{4}{5}+\frac{1}{4}=\frac{2}{v}$
3. $1-\frac{2}{m}=\frac{8}{m^2}$
4. $1+\frac{9}{p}=\frac{-20}{p^2}$
5. $\frac{5}{3v-2}=\frac{7}{4v}$
6. $\frac{3}{x+4}+\frac{7}{x-4}=\frac{8}{x^2-16}$
7. $\frac{8}{z-10}-\frac{7}{z+10}=\frac{5}{z^2-100}$
8. $\frac{-10}{q-2}-\frac{7}{q+4}=1$
9. $\frac{v-10}{v^2-5v+4}=\frac{3}{v-1}-\frac{6}{v-4}$
10. $\frac{x-10}{x^2+8x+12}=\frac{3}{x+2}+\frac{4}{x+6}$
11. $\frac{b+3}{3b}+\frac{b}{24}=\frac{1}{b}$
12. $\frac{d}{d+3}=\frac{18}{d^2-9}+4$
13. $\frac{n}{n+2}-3=\frac{8}{n^2-4}$
14. $\frac{q}{3q-9}-\frac{3}{4q+12}=\frac{7q^2+6q+63}{24q^2-216}$
15. $\frac{s}{2s+6}-\frac{2}{5s+5}=\frac{5s^2-3s-7}{10s^2+40s+30}$
16. $\frac{2}{x^2+2x-8}-\frac{1}{x^2+9x+20}=\frac{4}{x^2+3x-10}$
17. $\frac{3}{x^2-5x-6}+\frac{3}{x^2-7x+6}=\frac{6}{x^2-1}$

Answer

1. $a=10$
2. $v=\frac{40}{21}$
3. $m=-2,\ m=4$
4. $p=-5,\ p=-4$
5. $v=14$
6. $x=-\frac{4}{5}$
7. $z=-145$
8. $q=-18,\ q=-1$
9. no solution
10. no solution

11. $b = -8$
12. $d = 2$
13. $n = 1$
14. no solution
15. $s = \frac{5}{4}$
16. $x = -\frac{4}{3}$
17. no solution

Writing Exercises

18. Your class mate is having trouble in this section. Write down the steps you would use to explain how to solve a rational equation.
19. Alek thinks the equation $\frac{y}{y+6} = \frac{72}{y^2-36} + 4$ has two solutions, $y = -6$ and $y = 4$. Explain why Alek is wrong.

Answer

Answers will vary.

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5.7: Solve Systems of Linear Equations with Two Variables

Learning Objectives

By the end of this section, you will be able to:

- Determine whether an ordered pair is a solution of a system of equations
- Solve a system of linear equations by graphing
- Solve a system of equations by substitution
- Solve a system of equations by elimination
- Choose the most convenient method to solve a system of linear equations

Warm-up

1. For the equation $y = \frac{2}{3}x - 4$, is $(6, 0)$ a solution? Is $(-3, -2)$ a solution?
2. Find the slope and *y*-intercept of the line $3x - y = 12$.
3. Find the *x*- and *y*-intercepts of the line $2x - 3y = 12$.

Add exercises text here.

Answer

1. $(6, 0)$ is a solution but $(-3, -2$ is not a solution.
2. The slope is 3 and the y-int is (-12,0)
3. The x-int is (6,0) and the y-int is (0,-4)

Determine Whether an Ordered Pair is a Solution of a System of Equations

Previously, we learned how to solve linear equations with one variable. Now we will work with two or more linear equations grouped together, which is known as a **system of linear equations**.

Definition: SYSTEM OF LINEAR EQUATIONS

When two or more linear equations are grouped together, they form a **system of linear equations**.

In this section, we will focus our work on systems of two linear equations in two unknowns. We will solve larger systems of equations later in this chapter.

An example of a system of two linear equations is shown below. We use a brace to show the two equations are grouped together to form a system of equations.

$$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$$

A linear equation in two variables, such as $2x + y = 7$, has an infinite number of solutions. Its graph is a line. Remember, every point on the line is a solution to the equation and every solution to the equation is a point on the line.

To solve a system of two linear equations, we want to find the values of the variables that are solutions to *both* equations. In other words, we are looking for the ordered pairs (x, y) that make both equations true. These are called the **solutions of a system of equations**.

Definition: Solution TO A SYSTEM OF EQUATIONS

The **solutions of a system of equations** are the values of the variables that make *all* the equations true. A solution of a system of two linear equations is represented by an ordered pair (x, y).

To determine if an ordered pair is a solution to a system of two equations, we substitute the values of the variables into each equation. If the ordered pair makes both equations true, it is a solution to the system.

Example 5.7.1

Determine whether the ordered pair is a solution to the system $\begin{cases} x-y=-1 \\ 2x-y=-5 \end{cases}$.

ⓐ $(-2,-1)$ ⓑ $(-4,-3)$

Answer

ⓐ

$$\begin{cases} x-y=-1 \\ 2x-y=-5 \end{cases}$$

We substitute $x=-2$ and $y=-1$ into both equations.

$x-y=-1$	$2x-y=-5$
$-2-(-1)\stackrel{?}{=}-1$	$2\cdot-2-(-1)\stackrel{?}{=}-5$
$-1=-1$ ✓	$5\neq-5$
(–2, –1) does not make both equations true.	(–2, –1) is not a solution.

ⓑ

We substitute $x=-4$ and $y=-3$ into both equations.

$x-y=-1$	$2x-y=-5$
$-4-(-3)\stackrel{?}{=}-1$	$2\cdot(-4)-(-3)\stackrel{?}{=}-5$
$-1=-1$ ✓	$-5=-5$ ✓
(–4, –3) does make both equations true.	(–4, –3) is a solution.

Example 5.7.2

Determine whether the ordered pair is a solution to the system $\begin{cases} x+y=0 \\ x+2y=-5 \end{cases}$.

ⓐ $(1,-3)$ ⓑ $(0,0)$

Answer

ⓐ yes ⓑ no

Example 5.7.3

Determine whether the ordered pair is a solution to the system $\begin{cases} -3y=-8 \\ -3x-y=4 \end{cases}$.

ⓐ $(2,-2)$ ⓑ $(-2,2)$

Answer

ⓐ no ⓑ yes

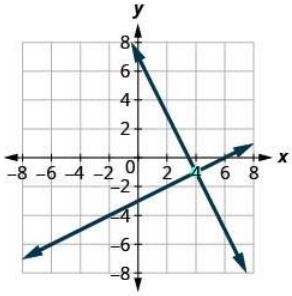
Solve a System of Linear Equations by Graphing

In this section, we will use three methods to solve a system of linear equations. The first method we'll use is graphing.

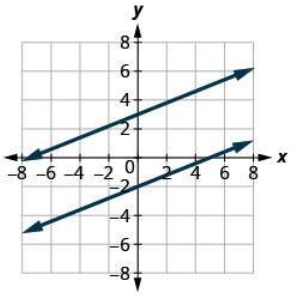
The graph of a linear equation is a line. Each point on the line is a solution to the equation. For a system of two equations, we will graph two lines. Then we can see all the points that are solutions to each equation. And, by finding what the lines have in common, we'll find the solution to the system.

Most linear equations in one variable have one solution, but we saw that some equations, called contradictions, have no solutions and for other equations, called identities, all numbers are solutions.

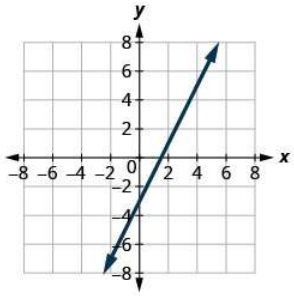
Similarly, when we solve a system of two linear equations represented by a graph of two lines in the same plane, there are three possible cases, as shown.



The lines intersect.
Intersecting lines have one point in common.
There is one solution to this system.



The lines are parallel.
Parallel lines have no points in common.
There is no solution to this system.



Both equations give the same line.
Because we have just one line, there are infinitely many solutions.

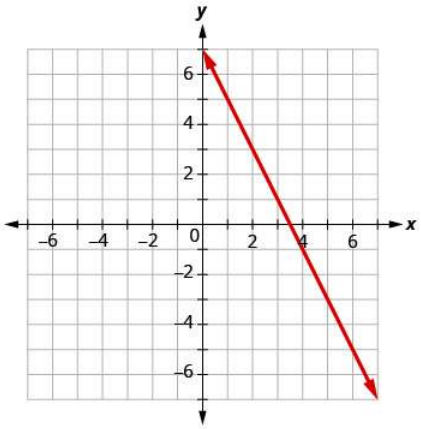
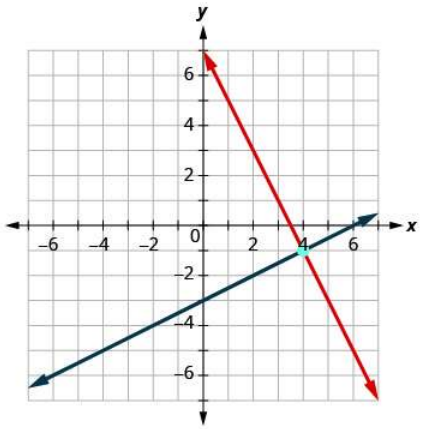
Figure 5.7.1

Each time we demonstrate a new method, we will use it on the same system of linear equations. At the end of the section you'll decide which method was the most convenient way to solve this system.

? Example 5.7.4

Solve the system by graphing $\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$.

Answer

Step 1. Graph the first equation.	To graph the first line, write the equation in slope–intercept form. $2x + y = 7$ $y = -2x + 7$ $m = -2 \quad b = 7$	$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$ 
Step 2. Graph the second equation on the same rectangular coordinate system.	To graph the second line, use intercepts. $x - 2y = 6$ $(0, -3) \quad (6, 0)$	
Step 3. Determine whether the lines intersect, are parallel, or are the same line.	Look at the graph of the lines.	The lines intersect.
Step 4. Identify the solution to the system. • If the lines intersect, identify the point of intersection. Check to make sure it is a solution to both equations. This is the solution to the system. • If the lines are parallel, the system has no solution. • If the lines are the same, the system has an infinite number of solutions.	Since the lines intersect, find the point of intersection. Check the point in both equations.	The lines intersect at $(4, -1)$. $2x + y = 7$ $2(4) + (-1) \stackrel{?}{=} 7$ $8 - 1 \stackrel{?}{=} 7$ $7 = 7$ ✓ $x - 2y = 6$ $4 - 2(-1) \stackrel{?}{=} 6$ $6 = 6$ ✓ The solution is $(4, -1)$.

Example 5.7.5

Solve the system by graphing: $\begin{cases} x-3y=-3 \\ x+y=5 \end{cases}$.

Answer

$(3, 2)$

Example 5.7.6

Solve the system by graphing: $\begin{cases} -x+y=1 \\ 3x+2y=12 \end{cases}$

Answer

$(2, 3)$

The steps to use to solve a system of linear equations by graphing are shown here.

SOLVE A SYSTEM OF LINEAR EQUATIONS BY GRAPHING.

1. Graph the first equation.
2. Graph the second equation on the same rectangular coordinate system.
3. Determine whether the lines intersect, are parallel, or are the same line.
4. Identify the solution to the system.
   - If the lines intersect, identify the point of intersection. This is the solution to the system.
   - If the lines are parallel, the system has no solution.
   - If the lines are the same, the system has an infinite number of solutions.
5. Check the solution in both equations.

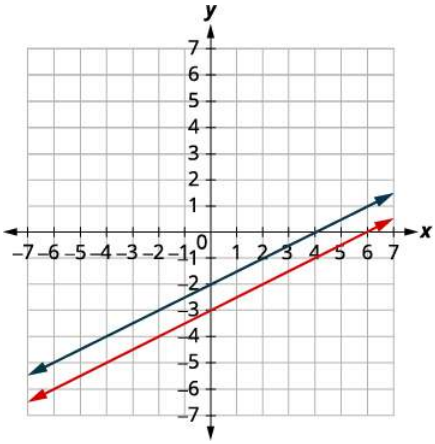
In all the systems of linear equations so far, the lines intersected and the solution was one point. In the next two examples, we'll look at a system of equations that has no solution and at a system of equations that has an infinite number of solutions.

Example 5.7.7

Solve the system by graphing: $\begin{cases} y=\frac{1}{2}x-3 \\ x-2y=4 \end{cases}$.

Answer

	$\begin{cases} y=\frac{1}{2}x-3 \\ x-2y=4 \end{cases}$
To graph the first equation, we will use its slope and *y*-intercept.	$y=\frac{1}{2}x-3$ $m=\frac{1}{2}$ $b=-3$
To graph the second equation, we will use the intercepts.	$x-2y=4$
	$\begin{array}{\|c\|c\|} \hline x & y \\ \hline 0 & -2 \\ \hline 4 & 0 \\ \hline \end{array}$

Graph the lines.	
Determine the points of intersection.	The lines are parallel. Since no point is on both lines, there is no ordered pair that makes both equations true. There is no solution to this system.

Example 5.7.8

Solve the system by graphing: $\begin{cases} y = -\frac{1}{4}x + 2 \\ x + 4y = 4 \end{cases}$.

Answer

no solution

Example 5.7.9

Solve the system by graphing: $\begin{cases} y = 3x - 1 \\ 6x - 2y = 6 \end{cases}$.

Answer

no solution

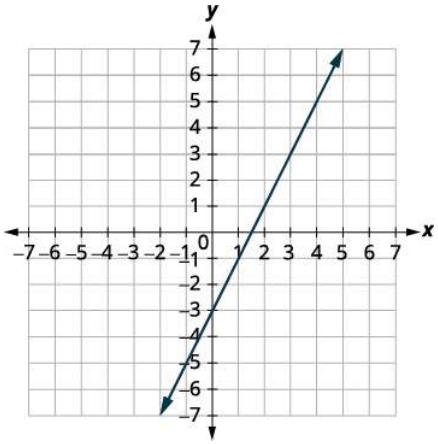
Sometimes the equations in a system represent the same line. Since every point on the line makes both equations true, there are infinitely many ordered pairs that make both equations true. There are infinitely many solutions to the system.

Example 5.7.10

Solve the system by graphing: $\begin{cases} y = 2x - 3 \\ -6x + 3y = 9 \end{cases}$.

Answer

	$\begin{cases} y = 2x - 3 \\ -6x + 3y = -9 \end{cases}$
Find the slope and y-intercept of the first equation.	$y = 2x - 3$ $m = 2$ $b = -3$
Find the intercepts of the second equation.	$-6x + 3y = -9$

	$\begin{array}{\|c\|c\|}\hline x & y \\ \hline 0 & -3 \\ \hline \frac{3}{2} & 0 \\ \hline \end{array}$
Graph the lines.	
	The lines are the same! Since every point on the line makes both equations true, there are infinitely many ordered pairs that make both equations true. There are infinitely many solutions to this system.

If you write the second equation in slope-intercept form, you may recognize that the equations have the same slope and same *y*-intercept.

Example 5.7.11

Solve the system by graphing: $\begin{cases} y = -3x - 6 \\ 6x + 2y = -12 \end{cases}$.

Answer

infinitely many solutions

Example 5.7.12

Solve the system by graphing: $\begin{cases} y = \frac{1}{2}x - 4 \\ 2x - 4y = 16 \end{cases}$.

Answer

infinitely many solutions

When we graphed the second line in the last example, we drew it right over the first line. We say the two lines are **coincident**. Coincident lines have the same slope and same *y*-intercept.

Definition: COINCIDENT LINES

Coincident lines have the same slope and same *y*-intercept.

If a system of equations represents two intersecting lines, it has one solution.

If the equations represent coincident lines, and the system has infinitely many solutions.

A system of equations that has at least one solution is called a *consistent* system.

A system with parallel lines has no solution because they never intersect. We call a system of equations like this *inconsistent.* It has no solution.

Definition: CONSISTENT AND INCONSISTEN SYSTEMS

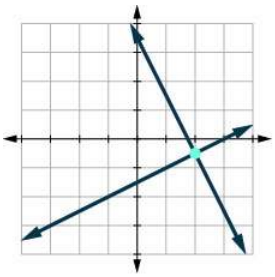
A **consistent system of equations** is a system of equations with at least one solution.

An **inconsistent system of equations** is a system of equations with no solution.

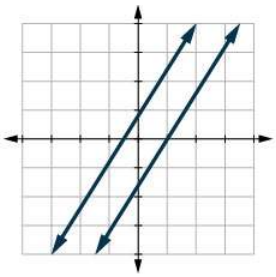
We also categorize the equations in a system of equations by calling the equations *independent* or *dependent*. If two equations are independent, they each have their own set of solutions. Intersecting lines and parallel lines are independent.

If two equations are dependent, all the solutions of one equation are also solutions of the other equation. When we graph two dependent equations, we get coincident lines.

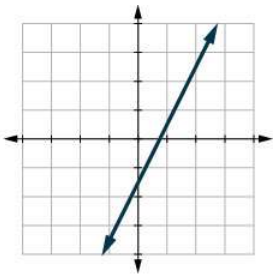
Let's sum this up by looking at the graphs of the three types of systems. See below and Table.



Intersecting



Parallel



Coincident

Lines	Intersecting	Parallel	Coincident
Number of solutions	1 point	No solution	Infinitely many
Consistent/inconsistent	Consistent	Inconsistent	Consistent
Dependent/ independent	Independent	Independent	Dependent

Solving systems of linear equations by graphing is a good way to visualize the types of solutions that may result. However, there are many cases where solving a system by graphing is inconvenient or imprecise. If the graphs extend beyond the small grid with x and y both between -10 and 10, graphing the lines may be cumbersome. And if the solutions to the system are not integers, it can be hard to read their values precisely from a graph.

Solve a System of Equations by Substitution

We will now solve systems of linear equations by the substitution method.

We will use the same system we used first for graphing.

$$\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$$

We will first solve one of the equations for either x or y. We can choose either equation and solve for either variable—but we'll try to make a choice that will keep the work easy.

Then we substitute that expression into the other equation. The result is an equation with just one variable—and we know how to solve those!

After we find the value of one variable, we will substitute that value into one of the original equations and solve for the other variable. Finally, we check our solution and make sure it makes both equations true.

Example 5.7.13

Solve the system by substitution: $\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$

Answer

Step 1. Solve one of the equations for either variable.	We'll solve the first equation for y.	$2x+y=7$ $x-2y=6$ $2x+y=7$ $y=7-2x$
Step 2. Substitute the expression from Step 1 into the other equation.	We replace y in the second equation with the expression $7-2x$.	$x-2y=6$ $x-2(7-2x)=6$
Step 3. Solve the resulting equation.	Now we have an equation with just 1 variable. We know how to solve this!	$x-2(7-2x)=6$ $x-14+4x=6$ $5x=20$ $x=4$
Step 4. Substitute the solution from Step 3 into one of the original equations to find the other variable.	We'll use the first equation and replace x with 4.	$2x+y=7$ $2(4)+y=7$ $8+y=7$ $y=-1$
Step 5. Write the solution as an ordered pair.	The ordered pair is (x, y).	$(4, -1)$
Step 6. Check that the ordered pair is a solution to **both** original equations.	Substitute $x=4, y=-1$ into both equations and make sure they are both true.	$2x+y=7$ $2(4)+(-1)\stackrel{?}{=}7$ $7=7$ ✓ $x-2y=6$ $4-2(-1)\stackrel{?}{=}6$ $6=6$ ✓ Both equations are true. $(4, -1)$ is the solution to the system.

Example 5.7.14

Solve the system by substitution: $\begin{cases} -2x+y=-11 \\ x+3y=9 \end{cases}$

Answer

$(6, 1)$

Example 5.7.15

Solve the system by substitution: $\begin{cases} 2x+y=-1 \\ 4x+3y=3 \end{cases}$

Answer

$(-3, 5)$

SOLVE A SYSTEM OF EQUATIONS BY SUBSTITUTION.

1. Solve one of the equations for either variable.
2. Substitute the expression from Step 1 into the other equation.
3. Solve the resulting equation.
4. Substitute the solution in Step 3 into either of the original equations to find the other variable.
5. Write the solution as an ordered pair.
6. Check that the ordered pair is a solution to **both** original equations.

Be very careful with the signs in the next example.

Example 5.7.16

Solve the system by substitution: $\begin{cases} 4x+2y=4 \\ 6x-y=8 \end{cases}$

Answer

We need to solve one equation for one variable. We will solve the first equation for y.

	$4x+2y=4$ $6x-\ y=8$
Solve the first equation for y. Substitute $-2x+2$ for y in the second equation.	$4x+2y=4$ $2y=-4x+4$ $y=-2x+2$ $6x-y=8$
Replace the y with $-2x+2$.	$6x-(-2x+2)=8$
Solve the equation for x.	$6x+2x-2=8$ $8x-2=8$ $8x=10$ $x=\frac{5}{4}$
Substitute $x=54$ into $4x+2y=4$ to find y.	$4x+2y=4$ $4\left(\frac{5}{4}\right)+2y=4$ $5+2y=4$ $2y=-1$ $y=-\frac{1}{2}$
	The ordered pair is $(54, -12)$.
Check the ordered pair in both equations.	$4x+2y=4$ $4\left(\frac{5}{4}\right)+2\left(-\frac{1}{2}\right)\stackrel{?}{=}4$ $5-1\stackrel{?}{=}4$ $4=4$ ✓ $6x-y=8$ $6\left(\frac{5}{4}\right)-\left(-\frac{1}{2}\right)\stackrel{?}{=}8$ $\frac{15}{4}-\left(-\frac{1}{2}\right)\stackrel{?}{=}8$ $\frac{16}{2}\stackrel{?}{=}8$ $8=8$ ✓
	The solution is $(54, -12)$.

Example 5.7.17

Solve the system by substitution: $\begin{cases} x-4y=-4 \\ -3x+4y=0 \end{cases}$

Answer

$(2, 32)$

Example 5.7.18

Solve the system by substitution: $\begin{cases} 4x-y=0 \\ 2x-3y=5 \end{cases}$

Answer

$(-12, -2)$

Solve a System of Equations by Elimination

We have solved systems of linear equations by graphing and by substitution. Graphing works well when the variable coefficients are small and the solution has integer values. Substitution works well when we can easily solve one equation for one of the variables and not have too many fractions in the resulting expression.

The third method of solving systems of linear equations is called the Elimination Method. When we solved a system by substitution, we started with two equations and two variables and reduced it to one equation with one variable. This is what we'll do with the elimination method, too, but we'll have a different way to get there.

The Elimination Method is based on the Addition Property of Equality. The Addition Property of Equality says that when you add the same quantity to both sides of an equation, you still have equality. We will extend the Addition Property of Equality to say that when you add equal quantities to both sides of an equation, the results are equal.

For any expressions *a*, *b*, *c*, and *d*.

$$\begin{aligned} &\text{if} && a=b \\ &\text{and} && c=d \\ &\text{then} && a+c=b+d. \end{aligned}$$

To solve a system of equations by elimination, we start with both equations in standard form. Then we decide which variable will be easiest to eliminate. How do we decide? We want to have the coefficients of one variable be opposites, so that we can add the equations together and eliminate that variable.

Notice how that works when we add these two equations together:

$$\begin{array}{l} \begin{cases} 3x+y=5 \\ \underline{2x-y=0} \end{cases} \\ 5x=5 \end{array}$$

The *y*'s add to zero and we have one equation with one variable.

Let's try another one:

$$\begin{cases} x+4y=2 \\ 2x+5y=-2 \end{cases}$$

This time we don't see a variable that can be immediately eliminated if we add the equations.

But if we multiply the first equation by -2, we will make the coefficients of *x* opposites. We must multiply every term on both sides of the equation by -2.

$$\begin{aligned} -2(x+4y) &= -2(2) \\ 2x+5y &= -2 \end{aligned}$$

Then rewrite the system of equations.

$$\begin{aligned} -2x-8y &= -4 \\ 2x+5y &= -2 \end{aligned}$$

Now we see that the coefficients of the x terms are opposites, so x will be eliminated when we add these two equations.

$$\begin{array}{r} -2x-8y=-4 \\ 2x+5y=-2 \\ \hline -3y=-6 \end{array}$$

Once we get an equation with just one variable, we solve it. Then we substitute that value into one of the original equations to solve for the remaining variable. And, as always, we check our answer to make sure it is a solution to both of the original equations.

Now we'll see how to use elimination to solve the same system of equations we solved by graphing and by substitution.

Exercise 5.7.19

Solve the system by elimination: $\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$

Answer

Step 1. Write both equations in standard form. • If any coefficients are fractions, clear them.	Both equations are in standard form, $Ax+By=C$. There are no fractions.	$\begin{aligned} 2x+y &= 7 \\ x-2y &= 6 \end{aligned}$
Step 2. Make the coefficients of one variable opposites. • Decide which variable you will eliminate. • Multiply one or both equations so that the coefficients of that variable are opposites.	We can eliminate the y's by multiplying the first equation by 2. Multiply both sides of $2x+y=7$ by 2.	$\begin{aligned} 2x+y &= 7 \\ x-2y &= 6 \end{aligned}$ $\begin{aligned} 2(2x+y) &= 2(7) \\ x-2y &= 6 \end{aligned}$
Step 3. Add the equations resulting from Step 2 to eliminate one variable.	We add the x's, y's, and constants.	$\begin{array}{r} 4x+2y=14 \\ x-2y=6 \\ \hline 5x \quad\quad =20 \end{array}$
Step 4. Solve for the remaining variable.	Solve for x.	$x=4$
Step 5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.	Substitute $x=4$ into the second equation, $x-2y=6$. Then solve for y.	$\begin{aligned} x-2y &= 6 \\ 4-2y &= 6 \\ -2y &= 2 \\ y &= -1 \end{aligned}$
Step 6. Write the solution as an ordered pair.	Write it as (x, y).	$(4, -1)$
Step 7. Check that the ordered pair is a solution to **both** original equations.	Substitute $x=4$, $y=-1$ into $2x+y=7$ and $x-2y=6$. Do they make both equations true? Yes!	$2x+y=7$ $\quad$ $x-2y=6$ $2(4)+(-1) \stackrel{?}{=} 7$ $\quad$ $4-2(-1) \stackrel{?}{=} 6$ $7=7$ ✓ $\quad$ $6=6$ ✓ The solution is $(4, -1)$.

Exercise 5.7.20

Solve the system by elimination: $\begin{cases} 3x+y=5 \\ 2x-3y=7 \end{cases}$

Answer

$(2,-1)$

Exercise 5.7.21

Solve the system by elimination: $\begin{cases} 4x+y=-5 \\ -2x-2y=-2 \end{cases}$

Answer

$(-2,3)$

The steps are listed here for easy reference.

SOLVE A SYSTEM OF EQUATIONS BY ELIMINATION.

1. Write both equations in standard form. If any coefficients are fractions, clear them.
2. Make the coefficients of one variable opposites.
   - Decide which variable you will eliminate.
   - Multiply one or both equations so that the coefficients of that variable are opposites.
3. Add the equations resulting from Step 2 to eliminate one variable.
4. Solve for the remaining variable.
5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.
6. Write the solution as an ordered pair.
7. Check that the ordered pair is a solution to **both** original equations.

Now we'll do an example where we need to multiply both equations by constants in order to make the coefficients of one variable opposites.

Exercise 5.7.22

Solve the system by elimination: $\begin{cases} 4x-3y=9 \\ 7x+2y=-6 \end{cases}$

Answer

In this example, we cannot multiply just one equation by any constant to get opposite coefficients. So we will strategically multiply both equations by different constants to get the opposites.

	$4x-3y=9$ $7x+2y=-6$
Both equations are in standard form. To get opposite coefficients of y, we will multiply the first equation by 2 and the second equation by 3.	$2(4x-3y)=2(9)$ $3(7x+2y)=3(-6)$
Simplify.	$8x-6y=18$ $21x+6y=-18$

Add the two equations to eliminate y.	$\begin{array}{rcl} 8x-6y &=& 18 \\ 21x+6y &=& -18 \\ \hline 29x &=& 0 \end{array}$
Solve for x.	$x=0$ $7x+2y=-6$
Substitute $x=0$ into one of the original equations.	$7\cdot 0+2y=-6$
Solve for y.	$\begin{array}{l} 2y=-6 \\ y=-3 \end{array}$
Write the solution as an ordered pair.	The ordered pair is $(0,-3)$.
Check that the ordered pair is a solution to **both** original equations. $4x-3y=9$: $4(0)-3(-3)\stackrel{?}{=}9$, $9=9$ ✓ $7x+2y=-6$: $7(0)+2(-3)\stackrel{?}{=}-6$, $-6=-6$ ✓	
	The solution is $(0,-3)$.

Exercise 5.7.23

Solve the system by elimination: $\begin{cases} 3x-4y=-9 \\ 5x+3y=14 \end{cases}$

Answer

$(1,3)$

Exercise 5.7.24

Solve each system by elimination: $\begin{cases} 7x+8y=4 \\ 3x-5y=27 \end{cases}$

Answer

$(4,-3)$

When the system of equations contains fractions, we will first clear the fractions by multiplying each equation by the LCD of all the fractions in the equation.

Exercise 5.7.25

Solve the system by elimination: $\begin{cases} x+\frac{1}{2}y=6 \\ \frac{3}{2}x+\frac{2}{3}y=\frac{17}{2} \end{cases}$

Answer

In this example, both equations have fractions. Our first step will be to multiply each equation by the LCD of all the fractions in the equation to clear the fractions.

	$\begin{array}{l} x+\frac{1}{2}y=6 \\ \frac{3}{2}x+\frac{2}{3}y=\frac{17}{2} \end{array}$

To clear the fractions, multiply each equation by its LCD.	$2\left(x+\frac{1}{2}y\right)=2(6)$ $6\left(\frac{3}{2}x+\frac{2}{3}y\right)=6\left(\frac{17}{2}\right)$
Simplify.	$2x+y=12$ $9x+4y=51$
Now we are ready to eliminate one of the variables. Notice that both equations are in standard form.	
We can eliminate y by multiplying the top equation by -4.	$-4(2x+y)=-4(12)$ $9x+4y=51$
Simplify and add. Substitute $x=3$ into one of the original equations.	$-8x-4y=-48$ $9x+4y=51$ $x=3$ $x+\frac{1}{2}y=6$
Solve for y.	$3+\frac{1}{2}y=6$ $\frac{1}{2}y=3$ $y=6$
Write the solution as an ordered pair.	The ordered pair is $(3,6)$.
Check that the ordered pair is a solution to both original equations. $x+\frac{1}{2}y=6$ $3+\frac{1}{2}(6)\stackrel{?}{=}6$ $3+3\stackrel{?}{=}6$ $6=6$ ✓ $\frac{3}{2}x+\frac{2}{3}y=\frac{17}{2}$ $\frac{3}{2}(3)+\frac{2}{3}(6)\stackrel{?}{=}\frac{17}{2}$ $\frac{9}{2}+4\stackrel{?}{=}\frac{17}{2}$ $\frac{9}{2}+\frac{8}{2}\stackrel{?}{=}\frac{17}{2}$ $\frac{17}{2}=\frac{17}{2}$ ✓	
	The solution is $(3,6)$.

? Exercise 5.7.26

Solve each system by elimination: $\begin{cases}\frac{1}{3}x-\frac{1}{2}y=1\\ \frac{3}{4}x-y=\frac{5}{2}\end{cases}$

Answer

$(6,2)$

? Exercise 5.7.27

Solve each system by elimination: $\begin{cases}x+\frac{3}{5}y=-\frac{1}{5}\\ -\frac{1}{2}x-\frac{2}{3}y=\frac{5}{6}\end{cases}$

Answer

$(1,-2)$

When we solved the system by graphing, we saw that not all systems of linear equations have a single ordered pair as a solution. When the two equations were really the same line, there were infinitely many solutions. We called that a consistent system. When the two equations described parallel lines, there was no solution. We called that an inconsistent system.

The same is true using substitution or elimination. If the equation at the end of substitution or elimination is a true statement, we have a consistent but dependent system and the system of equations has infinitely many solutions. If the equation at the end of substitution or elimination is a false statement, we have an inconsistent system and the system of equations has no solution.

? Exercise 5.7.28

Solve the system by elimination: $\begin{cases} 3x+4y=12 \\ y=3-\frac{3}{4}x \end{cases}$

Answer

$$\begin{cases} 3x+4y=12 \\ y=3-\frac{3}{4}x \end{cases}$$

Write the second equation in standard form. $\begin{cases} 3x+4y=12 \\ \frac{3}{4}x+y=3 \end{cases}$

Clear the fractions by multiplying the second equation by 4. $\begin{cases} 3x+4y=12 \\ 4(\frac{3}{4}x+y)=4(3) \end{cases}$

Simplify. $\begin{cases} 3x+4y=12 \\ 3x+4y=12 \end{cases}$

To eliminate a variable, we multiply the second equation by -1. Simplify and add.

$$\begin{cases} 3x+4y=12 \\ \underline{-3x-4y=-12} \end{cases}$$
$$0=0$$

This is a true statement. The equations are consistent but dependent. Their graphs would be the same line. The system has infinitely many solutions.

After we cleared the fractions in the second equation, did you notice that the two equations were the same? That means we have coincident lines.

? Exercise 5.7.29

Solve the system by elimination: $\begin{cases} 5x-3y=15 \\ 5y=-5+\frac{5}{3}x \end{cases}$

Answer

infinitely many solutions

? Exercise 5.7.30

Solve the system by elimination: $\begin{cases} x+2y=6 \\ y=-\frac{1}{2}x+3 \end{cases}$

Answer

infinitely many solutions

Choose the Most Convenient Method to Solve a System of Linear Equations

When you solve a system of linear equations in in an application, you will not be told which method to use. You will need to make that decision yourself. So you'll want to choose the method that is easiest to do and minimizes your chance of making mistakes.

Choose the Most Convenient Method to Solve a System of Linear Equations

Graphing	Substitution	Elimination
Use when you need a picture of the situation.	Use when one equation is already solved or can be easily solved for one variable.	Use when the equations are in standard form.

Example 5.7.31

For each system of linear equations, decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

ⓐ $\begin{cases} 3x+8y=40 \\ 7x-4y=-32 \end{cases}$ ⓑ $\begin{cases} 5x+6y=12 \\ y=\frac{2}{3}x-1 \end{cases}$

Answer

ⓐ

$$\begin{cases} 3x+8y=40 \\ 7x-4y=-32 \end{cases}$$

Since both equations are in standard form, using elimination will be most convenient.

ⓑ

$$\begin{cases} 5x+6y=12 \\ y=\frac{2}{3}x-1 \end{cases}$$

Since one equation is already solved for y, using substitution will be most convenient.

Example 5.7.32

For each system of linear equations decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

ⓐ $\begin{cases} 4x-5y=-32 \\ 3x+2y=-1 \end{cases}$ ⓑ $\begin{cases} x=2y-1 \\ 3x-5y=-7 \end{cases}$

Answer

ⓐ Since both equations are in standard form, using elimination will be most convenient. ⓑ Since one equation is already solved for x, using substitution will be most convenient.

Example 5.7.33

For each system of linear equations decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

ⓐ $\begin{cases} y=2x-1 \\ 3x-4y=-6 \end{cases}$ ⓑ $\begin{cases} 6x-2y=12 \\ 3x+7y=-13 \end{cases}$

Answer

ⓐ Since one equation is already solved for y, using substitution will be most convenient. ⓑ Since both equations are in standard form, using elimination will be most convenient.

Key Concepts

- **How to solve a system of linear equations by graphing.**
  1. Graph the first equation.
  2. Graph the second equation on the same rectangular coordinate system.
  3. Determine whether the lines intersect, are parallel, or are the same line.
  4. Identify the solution to the system.
     If the lines intersect, identify the point of intersection. This is the solution to the system.
     If the lines are parallel, the system has no solution.
     If the lines are the same, the system has an infinite number of solutions.
  5. Check the solution in both equations.
- **How to solve a system of equations by substitution.**
  1. Solve one of the equations for either variable.
  2. Substitute the expression from Step 1 into the other equation.
  3. Solve the resulting equation.
  4. Substitute the solution in Step 3 into either of the original equations to find the other variable.
  5. Write the solution as an ordered pair.
  6. Check that the ordered pair is a solution to **both** original equations.
- **How to solve a system of equations by elimination.**
  1. Write both equations in standard form. If any coefficients are fractions, clear them.
  2. Make the coefficients of one variable opposites.
     Decide which variable you will eliminate.
     Multiply one or both equations so that the coefficients of that variable are opposites.
  3. Add the equations resulting from Step 2 to eliminate one variable.
  4. Solve for the remaining variable.
  5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.
  6. Write the solution as an ordered pair.
  7. Check that the ordered pair is a solution to **both** original equations.

Choose the Most Convenient Method to Solve a System of Linear Equations

Graphing	Substitution	Elimination
Use when you need a picture of the situation.	Use when one equation is already solved or can be easily solved for one variable.	Use when the equations are in standard form.

Glossary

coincident lines

Coincident lines have the same slope and same y-intercept.

consistent and inconsistent systems

Consistent system of equations is a system of equations with at least one solution; inconsistent system of equations is a system of equations with no solution.

solutions of a system of equations

Solutions of a system of equations are the values of the variables that make *all* the equations true; solution is represented by an ordered pair (x,y).(x,y).

system of linear equations

When two or more linear equations are grouped together, they form a system of linear equations.

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5.7E: Exercises

Practice Makes Perfect

Determine if Whether an Ordered Pair is a Solution

In the following exercises, determine if the following points are solutions to the given system of equations.

1. $\begin{cases} 2x-6y=0 \\ 3x-4y=5 \end{cases}$

   ⓐ $(3,1)$ ⓑ $(-3,4)$

2. $\begin{cases} x+y=2 \\ y=x \end{cases}$

   ⓐ $(8,7)$ ⓑ $(1,1)$

Answer

1. ⓐ yes ⓑ no
2. ⓐ no ⓑ yes

Solve by Substitution

In the following exercises, solve the systems of equations by substitution.

3. $\begin{cases} 2x+y=-2 \\ 3x-y=7 \end{cases}$
4. $\begin{cases} x-3y=-9 \\ 2x+5y=4 \end{cases}$
5. $\begin{cases} -2x+2y=6 \\ y=-3x+1 \end{cases}$
6. $\begin{cases} 3x+4y=1 \\ y=-\frac{2}{5}x+2 \end{cases}$
7. $\begin{cases} 4x+y=10 \\ x-2y=-20 \end{cases}$
8. $\begin{cases} y=x-6 \\ y=-\frac{3}{2}x+4 \end{cases}$
9. $\begin{cases} 2x-16y=8 \\ -x-8y=-4 \end{cases}$
10. $\begin{cases} y=-\frac{2}{3}x+5 \\ 2x+3y=11 \end{cases}$
11. $\begin{cases} x=-3y+4 \\ 2x+6y=8 \end{cases}$

Answer

3. $(1,-4)$
4. $(-3,2)$
5. $(-1/2,5/2)$
6. $(-5,4)$
7. $(0,10)$
8. $(4,-2)$
9. $(4,0)$
10. no solution

11. infinite solutions with solution set: $\{(x,y)|y=-\frac{1}{3}x+4\}$

Solve by Elimination

In the following exercises, solve the systems of equations by elimination.

12. $\begin{cases} 6x-5y=-1 \\ 2x+y=13 \end{cases}$
13. $\begin{cases} 5x-3y=-1 \\ 2x-y=2 \end{cases}$
14. $\begin{cases} 4x-3y=3 \\ 2x+5y=-31 \end{cases}$
15. $\begin{cases} 11x+9y=-5 \\ 7x+5y=-1 \end{cases}$
16. $\begin{cases} 2x+9y=-4 \\ 3x+13y=-7 \end{cases}$
17. $\begin{cases} x+\frac{1}{2}y=\frac{3}{2} \\ \frac{1}{5}x-\frac{1}{5}y=3 \end{cases}$
18. $\begin{cases} \frac{1}{3}x-y=-3 \\ \frac{2}{3}x+\frac{5}{2}y=3 \end{cases}$
19. $\begin{cases} x-4y=-1 \\ -3x+12y=3 \end{cases}$

Answer

12. $(4,5)$
13. $(7,12)$
14. $(-3,-5)$
15. $(2,-3)$
16. $(-11,2)$
17. $(6,-9)$
18. $(-3,2)$
19. infinitely many solutions with solution set: $\{(x,y)|x-4y=-1\}$

Writing Exercises

Writing Exercises

20. In a system of linear equations, the two equations have the same intercepts. Describe the possible solutions to the system.
21. Solve the system of equations by substitution and explain all your steps in words: $\begin{cases} 3x+y=1 \\ 2x=y-8 \end{cases}$

Answer

Answers will vary

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5.8: Applications

Use Systems in Applications

Systems can be very useful for solving word problems involving several variables. Here is an example of a problem you will see in the near future.

Example 5.8.1

1 pound of cereal A contains 6g of potassium and 2g of calcium. 1 pound of cereal B contains 7g of potassium and 4g of calcium. How many pounds of each cereal is needed to have a mixture of 51g of potassium and 22g of calcium?

Solution

Write this word problem as a system of equations	We have "a" pounds of cereal A and that contains $6 \cdot a$ grams of potassium. We have "b" pounds of cereal B and that contains $7 \cdot b$ grams of potassium. Combining these, we should get 51 g of potassium. So that's $6a + 7b = 51$. Likewise, We have "a" pounds of cereal A and that contains $2 \cdot a$ grams of calcium. We have "b" pounds of cereal B and that contains $4 \cdot b$ grams of calcium. Combining these, we should get 22 g of potassium. So that's $2a + 4b = 22$. $6a + 7b = 51$ $2a + 4b = 22$
Solve the system of equations using either substitution or elimination. We will be using elimination for this.	$6a + 7b = 51$ $2a + 4b = 22 \Rightarrow 3(2a + 4b) = 3(22)$ $6a + 7b = 51$ $6a + 12b = 66$ subtract the bottom from the top $5b = 15 \Rightarrow b = 3$ $2a + 4(3) = 22 \Rightarrow a = 5$
State the answer	We need 5 pounds of cereal A and 3 pounds of cereal B.

Let's do some more examples.

Exercise 5.8.2

1 pound of cereal A contains 10g of potassium and 1g of calcium. 1 pound of cereal B contains 4g of potassium and 5g of calcium. How many pounds of each cereal is needed to have a mixture of 44g of potassium and 32g of calcium?

Answer

We need 2 pounds of cereal A and 6 pounds of cereal B.

Exercise 5.8.3

A pound of cereal A contains 14g of potassium and 3g of calcium. 1 pound of cereal B contains 7g of potassium and 4g of calcium. How many pounds of each cereal is needed to have a mixture of 70g of potassium and 25g of calcium?

Answer

We need 3 pounds of cereal A and 4 pounds of cereal B.

Exercise 5.8.4

A pound of cereal A contains 8g of potassium and 3g of calcium. 1 pound of cereal B contains 3g of potassium and 2g of calcium. How many pounds of each cereal is needed to have a mixture of 77g of potassium and 22g of calcium?

Answer

We need 4 pounds of cereal A and 5 pounds of cereal B.

Rates in Applications

Suppose we have two people painting a house. Person A takes 2 hours to paint a house and Person B takes 3 hours to paint a house, how long will it take for them to paint a house working together? The solution to this isn't immediately obvious so lets try a simpler problem.

Say Person A and B both take 2 hours to paint a house. Intuitively, we want to say they take 1 hour to paint a house if they work together but why? Well both of them work at a rate of half a house per hour so if we give Person A and Person B "x" amount of hours, add their progress together, we should get 1 house. So $x\frac{1}{2}+x\frac{1}{2}=1$. We can add fractions, so we see that the answer is x=1 hour.

Let's take another intuitive example. Say both are painting a shed and both take an hour working separately, what about together? Intuitively we want to say that they will be done in half an hour. Thinking about it in rates, we get $x(1)+x(1)=1 \Rightarrow 2x=1 \Rightarrow x=\frac{1}{2}$.

Okay, we are ready to come back to the original problem.

Example 5.8.1

Suppose we have two people painting a house. Person A takes 2 hours to paint a house and Person B takes 3 hours to paint a house, how long will it take for them to paint a house working together?

Solution

Person A works at a rate of $\frac{1}{2}$ house per hour. Person B works at a rate of $\frac{1}{3}$.

$$x\frac{1}{2}+x\frac{1}{3}=1$$

$$x\frac{3}{6}+x\frac{2}{6}=1$$

$$x\frac{5}{6}=1$$

$$x=\frac{6}{5}$$

So it will take $\frac{6}{5}$ hours.

Let's try some more examples.

Exercise 5.8.1

Suppose we have two people painting a house. Person A takes 5 hours to paint a house and Person B takes 3 hours to paint a house, how long will it take for them to paint a house working together?

Answer

$\frac{15}{8}$ hours

Exercise 5.8.1

Suppose we have two people painting a house. Person A takes 5 hours to paint a house and Person B takes 4 hours to paint a house, how long will it take for them to paint a house working together?

Answer

$\frac{20}{9}$ hours

Exercise 5.8.1

Suppose we have two people painting a house. Person A takes 6 hours to paint a house and Person B takes 3 hours to paint a house, how long will it take for them to paint a house working together?

Answer

2 hours

Use Radicals in Applications

As you progress through your college courses, you'll encounter formulas that include radicals in many disciplines. We will modify our Problem Solving Strategy for Geometry Applications slightly to give us a plan for solving applications with formulas from any discipline.

Use a Problem Solving Strategy for Applications With Formulas

1. **Read** the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.
2. **Identify** what we are looking for.
3. **Name** what we are looking for by choosing a variable to represent it.
4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
5. **Solve the equation** using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

One application of radicals has to do with the effect of **gravity** on falling objects. The formula allows us to determine how long it will take a fallen object to hit the gound.

Definition 5.8.2

Falling Objects

On Earth, if an object is dropped from a height of h feet, the time in seconds it will take to reach the ground is found by using the formula

$$t = \frac{\sqrt{h}}{4}$$

For example, if an object is dropped from a height of 64 feet, we can find the time it takes to reach the ground by substituting $h = 64$ into the formula.

Table 8.6.10

	$t = \frac{\sqrt{h}}{4}$
	$t = \frac{\sqrt{64}}{4}$
Take the square root of 64.	$t = \frac{8}{4}$
Simplify the fraction.	$t = 2$

It would take 2 seconds for an object dropped from a height of 64 feet to reach the ground.

Example 5.8.11

Marissa dropped her sunglasses from a bridge 400 feet above a river. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the sunglasses to reach the river.

Solution:

Table 8.6.11

Step 1: **Read** the problem.	
Step 2: **Identify** what we are looking.	The time it takes for the sunglasses to reach the river.
Step 3: **Name** what we are looking.	Let (t=\) time.
Step 4: **Translate** into an equation by writing the appropriate formula. Substitute in the given information.	$t = \frac{\sqrt{h}}{4}$, and $h = 400$ $t = \frac{\sqrt{400}}{4}$
Step 5: **Solve the equation**.	$t = \frac{20}{4}$
	$t = 5$
Step 6: **Check** the answer in the problem and make sure it makes sense.	$5 \stackrel{?}{=} \frac{\sqrt{400}}{4}$ $5 \stackrel{?}{=} \frac{20}{4}$ $5 = 5$ ✓
Does 5 seconds seem like a reasonable length of time?	Yes.
Step 7: **Answer** the equation.	It will take 5 seconds for the sunglasses to reach the river.

Exercise 5.8.21

A helicopter dropped a rescue package from a height of $1,296$ feet. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the package to reach the ground.

Answer

9 seconds

Exercise 5.8.22

A window washer dropped a squeegee from a platform 196 feet above the sidewalk. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the squeegee to reach the sidewalk.

Answer

3.5 seconds

Police officers investigating car accidents measure the length of the skid marks on the pavement. Then they use square roots to determine the **speed**, in miles per hour, a car was going before applying the brakes.

Definition 5.8.3

Skid Marks and Speed of a Car

If the length of the skid marks is d feet, then the speed, s, of the car before the brakes were applied can be found by using the formula

$$s = \sqrt{24d}$$

Example 5.8.12

After a car accident, the skid marks for one car measured 190 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Solution:

Table 8.6.12

Step 1: **Read** the problem.	
Step 2: **Identify** what we are looking for.	The speed of a car.
Step 3: **Name** what we are looking for.	Let $s =$ the speed.
Step 4: **Translate** into an equation by writing the appropriate formula. Substitute in the given information.	$s = \sqrt{24d}$, and $d = 190$ $s = \sqrt{24(190)}$
Step 5: **Solve the equation**.	$s = \sqrt{4{,}560}$
	$s = 67.52777\ldots$
Round to 1 decimal place.	$s \approx 67.5$
	$67.5 \stackrel{?}{\approx} \sqrt{24(190)}$ $67.5 \stackrel{?}{\approx} \sqrt{4560}$ $67.5 \approx 67.5277\ldots$ ✓
	The speed of the car before the brakes were applied was 67.5 miles per hour.

Exercise 5.8.23

An accident investigator measured the skid marks of the car. The length of the skid marks was 76 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Answer

42.7 feet

Exercise 5.8.24

The skid marks of a vehicle involved in an accident were 122 feet long. Use the formula $s = \sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

Answer

54.1 feet

Solve Maximum and Minimum Applications

Knowing that the **vertex** of a parabola is the lowest or highest point of the parabola gives us an easy way to determine the minimum or maximum value of a quadratic function. The *y*-coordinate of the vertex is the **minimum** value of a parabola that opens upward. It is the **maximum** value of a parabola that opens downward. See *Figure 9.6.124*.

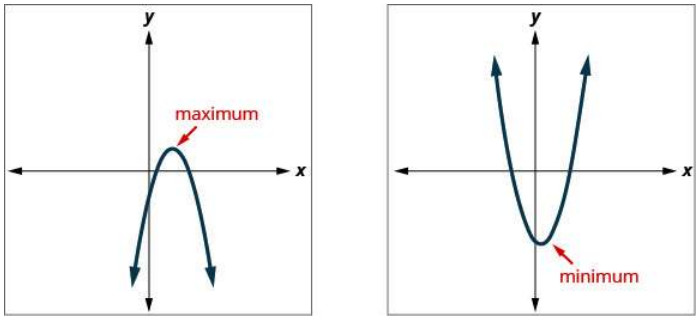


Figure 9.6.124

Minimum or Maximum Values of a Quadratic Function

The ***y*-coordinate of the vertex** of the graph of a quadratic function is the

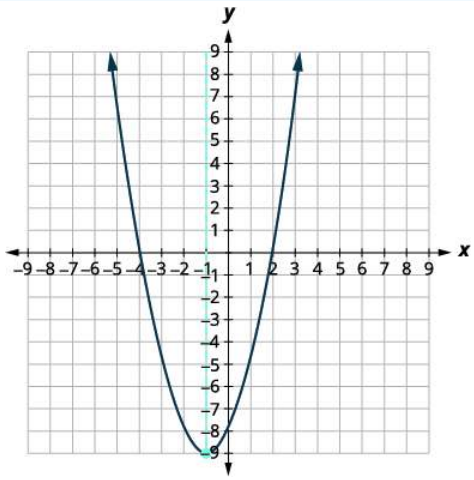
- *minimum* value of the quadratic equation if the parabola opens *upward*.
- *maximum* value of the quadratic equation if the parabola opens *downward*.

Example 5.8.10

Find the minimum or maximum value of the quadratic function $f(x) = x^2 + 2x - 8$.

Solution:

	$f(x) = x^2 + 2x - 8$
Since a is positive, the parabola opens upward. The quadratic equation has a minimum.	
Find the equation of the axis of symmetry.	$x = -\frac{b}{2a}$
	$x = -\frac{2}{2\times 1}$
	$x = -1$
	The equation of the axis of symmetry is $x = -1$.
The vertex is on the line $x = -1$.	$f(x) = x^2 + 2x - 8$
Find $f(-1)$.	$f(-1) = (-1)^2 + 2(-1) - 8$
	$f(-1) = 1 - 2 - 8$
	$f(-1) = -9$
	The vertex is $(-1, -9)$.
Since the parabola has a minimum, the y-coordinate of the vertex is the minimum y-value of the quadratic equation. The minimum value of the quadratic is -9 and it occurs when $x = -1$.	



Show the graph to verify the result.

? Exercise 5.8.19

Find the maximum or minimum value of the quadratic function $f(x) = x^2 - 8x + 12$.

Answer

The minimum value of the quadratic function is -4 and it occurs when $x = 4$.

? Exercise 5.8.20

Find the maximum or minimum value of the quadratic function $f(x) = -4x^2 + 16x - 11$.

Answer

The maximum value of the quadratic function is 5 and it occurs when $x = 2$.

We have used the formula

$$h(t) = -16t^2 + v_0 t + h_0$$

to calculate the height in feet, h, of an object shot upwards into the air with initial velocity, v_0, after t seconds .

This formula is a quadratic function, so its graph is a parabola. By solving for the coordinates of the vertex (t, h), we can find how long it will take the object to reach its maximum height. Then we can calculate the maximum height.

✓ Example 5.8.11

The quadratic equation $h(t) = -16t^2 + 176t + 4$ models the height of a volleyball hit straight upwards with velocity 176 feet per second from a height of 4 feet.

a. How many seconds will it take the volleyball to reach its maximum height?
b. Find the maximum height of the volleyball.

Solution:

$$h(t) = -16t^2 + 176t + 4$$

Since a is negative, the parabola opens downward. The quadratic function has a maximum.

a. Find the equation of the axis of symmetry.

$$t = -\frac{b}{2a}$$
$$t = -\frac{176}{2(-16)}$$
$$t = 5.5$$

The equation of the axis of symmetry is $t = 5.5$.

The vertex is on the line $t = 5.5$.

The maximum occurs when $t = 5.5$ seconds.

b. Find $h(5.5)$.

$$h(t) = -16t^2 + 176t + 4$$
$$h(t) = -16(5.5)^2 + 176(5.5) + 4$$

Use a calculator to simplify.

$$h(t) = 488$$

The vertex is $(5.5, 488)$.

Since the parabola has a maximum, the h-coordinate of the vertex is the maximum value of the quadratic function.

The maximum value of the quadratic is 488 feet and it occurs when $t = 5.5$ seconds.

After 5.5 seconds, the volleyball will reach its maximum height of 488 feet.

Exercise 5.8.21

Solve, rounding answers to the nearest tenth.

The quadratic function $h(t) = -16t^2 + 128t + 32$ is used to find the height of a stone thrown upward from a height of 32 feet at a rate of 128 ft/sec. How long will it take for the stone to reach its maximum height? What is the maximum height?

Answer

It will take 4 seconds for the stone to reach its maximum height of 288 feet.

Exercise 5.8.22

A path of a toy rocket thrown upward from the ground at a rate of 208 ft/sec is modeled by the quadratic function of $h(t) = -16t^2 + 208t$. When will the rocket reach its maximum height? What will be the maximum height?

Answer

It will take 6.5 seconds for the rocket to reach its maximum height of 676 feet.

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5.8E: Exercises

Practice Makes Perfect

Systems

Solve using systems.

1. 1 pound of cereal A contains 7g of potassium and 2g of calcium. 1 pound of cereal B contains 5g of potassium and 3g of calcium. How many pounds of each cereal is needed to have a mixture of 17g of potassium and 8g of calcium?
2. 1 pound of cereal A contains 5g of potassium and 2g of calcium. 1 pound of cereal B contains 6g of potassium and 1g of calcium. How many pounds of each cereal is needed to have a mixture of 21g of potassium and 7g of calcium?
3. 1 pound of cereal A contains 2g of potassium and 2g of calcium. 1 pound of cereal B contains 3g of potassium and 1g of calcium. How many pounds of each cereal is needed to have a mixture of 10g of potassium and 6g of calcium?
4. 1 pound of cereal A contains 6g of potassium and 1g of calcium. 1 pound of cereal B contains 3g of potassium and 3g of calcium. How many pounds of each cereal is needed to have a mixture of 57g of potassium and 22g of calcium?
5. 1 pound of cereal A contains 5g of potassium and 2g of calcium. 1 pound of cereal B contains 3g of potassium and 3g of calcium. How many pounds of each cereal is needed to have a mixture of 8g of potassium and 5g of calcium?

Answer

1. We need 1 pounds of cereal A and 2 pounds of cereal B.
2. We need 3 pounds of cereal A and 1 pounds of cereal B.
3. We need 2 pounds of cereal A and 2 pounds of cereal B.
4. We need 7 pounds of cereal A and 5 pounds of cereal B.
5. We need 1 pounds of cereal A and 1 pounds of cereal B.

Rates

Solve

6. Suppose we have two people painting a house. Person A takes 2 hours to paint a house and Person B takes 1 hour to paint a house, how long will it take for them to paint a house working together?
7. Suppose we have two people painting a house. Person A takes 3 hours to paint a house and Person B takes 5 hours to paint a house, how long will it take for them to paint a house working together?
8. Suppose we have two people painting a house. Person A takes 4 hours to paint a house and Person B takes 5 hours to paint a house, how long will it take for them to paint a house working together?
9. Suppose we have two people painting a house. Person A takes 3 hours to paint a house and Person B takes 6 hours to paint a house, how long will it take for them to paint a house working together?
10. Suppose we have two people painting a house. Person A takes 1 hour to paint a house and Person B takes 5 hours to paint a house, how long will it take for them to paint a house working together?

Answer

6. It will take $\frac{2}{3}$ hours
7. It will take $\frac{15}{8}$ hours
8. It will take $\frac{20}{9}$ hours
9. It will take 2 hours
10. It will take $\frac{5}{6}$ hours

Radicals

1. **Landscaping** Reed wants to have a square garden plot in his backyard. He has enough compost to cover an area of 75 square feet. Use the formula $s = \sqrt{A}$ to find the length of each side of his garden. Round your answer to the nearest tenth of a foot.

2. **Landscaping** Vince wants to make a square patio in his yard. He has enough concrete to pave an area of 130 square feet. Use the formula $s = \sqrt{A}$ to find the length of each side of his patio. Round your answer to the nearest tenth of a foot.
3. **Gravity** A hang glider dropped his cell phone from a height of 350 feet. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the cell phone to reach the ground.
4. **Gravity** A construction worker dropped a hammer while building the Grand Canyon skywalk, 4000 feet above the Colorado River. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the hammer to reach the river.
5. **Accident investigation** The skid marks for a car involved in an accident measured 216 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.
6. **Accident investigation** An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 175 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

Answer

1. 8.7 feet
2. 11.4feet
3. 4.7 seconds
4. 15.8seconds
5. 72 feet/second
6. 64.8feet/second

? Maximum and minimum

1. An arrow is shot vertically upward from a platform 45 feet high at a rate of 168 ft/sec. Use the quadratic function $h(t) = -16t^2 + 168t + 45$ find how long it will take the arrow to reach its maximum height, and then find the maximum height.
2. A stone is thrown vertically upward from a platform that is 20 feet height at a rate of 160 ft/sec. Use the quadratic function $h(t) = -16t^2 + 160t + 20$ to find how long it will take the stone to reach its maximum height, and then find the maximum height.
3. A ball is thrown vertically upward from the ground with an initial velocity of 109 ft/sec. Use the quadratic function $h(t) = -16t^2 + 109t + 0$ to find how long it will take for the ball to reach its maximum height, and then find the maximum height.
4. A ball is thrown vertically upward from the ground with an initial velocity of 122 ft/sec. Use the quadratic function $h(t) = -16t^2 + 122t + 0$ to find how long it will take for the ball to reach its maximum height, and then find the maximum height.
5. A computer store owner estimates that by charging x dollars each for a certain computer, he can sell $40 - x$ computers each week. The quadratic function $R(x) = -x^2 + 40x$ is used to find the revenue, R, received when the selling price of a computer is x, Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue.
6. A retailer who sells backpacks estimates that by selling them for x dollars each, he will be able to sell $100 - x$ backpacks a month. The quadratic function $R(x) = -x^2 + 100x$ is used to find the R, received when the selling price of a backpack is x. Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue.

Answer

1. In 5.3 sec the arrow will reach maximum height of 486 ft.
2. In 5 sec the stone will reach maximum height of 420 ft.
3. In 3.4 seconds the ball will reach its maximum height of 185.6feet.
4. In 3.8 seconds the ball will reach its maximum height of 232.6feet.
5. 20 computers will give the maximum of $400 in receipts.
6. 50 backpacks will give the maximum of $2500 in receipts.

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CHAPTER OVERVIEW

Instructor Resources

Course outline 1

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Course outline 1

Practice problems for Test 1

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Sample exams for Test 1

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- test 1.pdf

Practice problems for Test 2

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Sample exams for Test 2

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Practice problems for Test 3

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Sample exams for Test 3

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- 3.1: Use a General Strategy to Solve Linear Equations

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