MATH 420: DIFFERENTIAL EQUATIONS



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Math 420: Differential Equations

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Glossary

Detailed Licensing

Licensing

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CHAPTER OVERVIEW

1: Introduction

In this chapter, we begin our study of differential equations.

1.1: Basic Concepts

1.1E: Basic Concepts (Exercises)

1.2: Existence and Uniqueness of Solutions

1.2E: Existence and Uniqueness of Solutions (Exercises)

1.3: Direction Fields

1.3E: Direction Fields (Exercises)

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1.1: Basic Concepts

Definition 1.1.1

A *differential equation* is an equation that contains one or more derivatives of an unknown function.

A differential equation is an *ordinary differential equation* if it involves an unknown function of only one variable, or a *partial differential equation* if it involves partial derivatives of a function of more than one variable.

Note: Partial differential equations are studied in another class; in this class we'll consider only ordinary differential equations, and we'll just call them *differential equations.*

Definition 1.1.2

The *order* of a differential equation is the order of the highest derivative that it contains.

Example 1.1.1

$$\begin{aligned} \tfrac{dy}{dx} - y = x \qquad & \text{(first order)}, \\ (x^2 - 5xy)dx + 7xy^2 dy = 0 \qquad & \text{(first order)}, \\ \tfrac{d^2y}{dx^2} + 7x\tfrac{dy}{dx} + 3y = 5x \qquad & \text{(second order)}, \\ x^3 y''' + 7xy^2 = \cos x \qquad & \text{(third order)}, \\ 9x^3 y^{(n)} + 3x^2 y' + 7y = e^x \qquad & \text{(n-th order)}. \end{aligned}$$

The simplest differential equations are first order equations of the form

$$\frac{dy}{dx} = f(x)$$

or equivalently

$$y' = f(x),$$

where f is a known function of x. We already know from calculus how to find functions that satisfy this kind of equation. For example, if

$$y' = x^3,$$

then

$$y = \int x^3\, dx = \frac{x^4}{4} + c,$$

where c is an arbitrary constant. If $n > 1$ we can find functions y that satisfy equations of the form

$$y^{(n)} = f(x) \tag{1.1.1}$$

by repeated integration. Again, this is a calculus problem.

Except for illustrative purposes in this section, there's no need to consider differential equations like Equation 1.1.1 as they can usually be solved with calculus. We'll usually consider differential equations that *can* be written as

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}), \tag{1.1.2}$$

where at least one of the functions y, y', …, $y^{(n-1)}$ actually appears on the right when written in the form 1.1.2. Here are some examples:

Example 1.1.2

$$\begin{aligned} \tfrac{dy}{dx} - x^2 y = 0 \qquad & \text{(first order)}, \\ \tfrac{dy}{dx} + 2xy^2 = -2 \qquad & \text{(first order)}, \\ \tfrac{d^2y}{dx^2} + 2\tfrac{dy}{dx} + y = 2x \qquad & \text{(second order)}, \\ xy''' + y^2 = \sin x \qquad & \text{(third order)}, \\ y^{(n)} + xy' + 3y = x \qquad & \text{(n-th order)}. \end{aligned}$$

Although none of these equations is written as in Equation 1.1.2, all of them *can* be written in this form:

$$\begin{aligned} y' &= x^2 y, \\ y' &= -2 - 2xy^2, \\ y'' &= 2x - 2y' - y, \\ y''' &= \frac{\sin x - y^2}{x}, \\ y^{(n)} &= x - xy' - 3y. \end{aligned}$$

Solutions of Differential Equations

Definition 1.1.3

A *solution* of a differential equation is a function that satisfies the differential equation on **some** open interval; thus, y is a solution of Equation 1.1.2 if y is n times differentiable and

$$y^{(n)}(x) = f(x, y(x), y'(x), \ldots, y^{(n-1)}(x))$$

for all x in some open interval (a, b). In this case, we also say that y *is a solution of Equation* 1.1.2 *on* (a, b).

Caution

A solution of a differential equation must be both continuous and differentiable on an open interval.

Definition 1.1.4

The graph of a **solution** of a differential equation is a *solution curve*. More generally, a curve C is said to be an *integral curve* of a differential equation if every function $y = y(x)$ whose graph is a segment of C is a solution of the differential equation. Thus, any solution curve of a differential equation is an integral curve, but an integral curve need not be a solution curve (see Example 6).

Example 1.1.3

Verify that

$$y = \frac{x^2}{3} + \frac{1}{x} \tag{1.1.3}$$

is a solution of

$$xy' + y = x^2 \tag{1.1.4}$$

on $(0, \infty)$ and on $(-\infty, 0)$.

Solution

Substituting Equation 1.1.3 and

$$y' = \frac{2x}{3} - \frac{1}{x^2}$$

into Equation 1.1.4 yields

$$xy'(x) + y(x) = x\left(\frac{2x}{3} - \frac{1}{x^2}\right) + \left(\frac{x^2}{3} + \frac{1}{x}\right) = x^2$$

for all $x \neq 0$. Therefore y is a solution of Equation 1.1.4 on $(-\infty, 0)$ and $(0, \infty)$. However, y isn't a solution of the differential equation on any open interval that contains $x = 0$, since y is not defined at $x = 0$.

Figure 1.1.2 shows the graph of Equation 1.1.3. The part of the graph of Equation 1.1.3 on $(0, \infty)$ is a solution curve of Equation 1.1.4, as is the part of the graph on $(-\infty, 0)$.

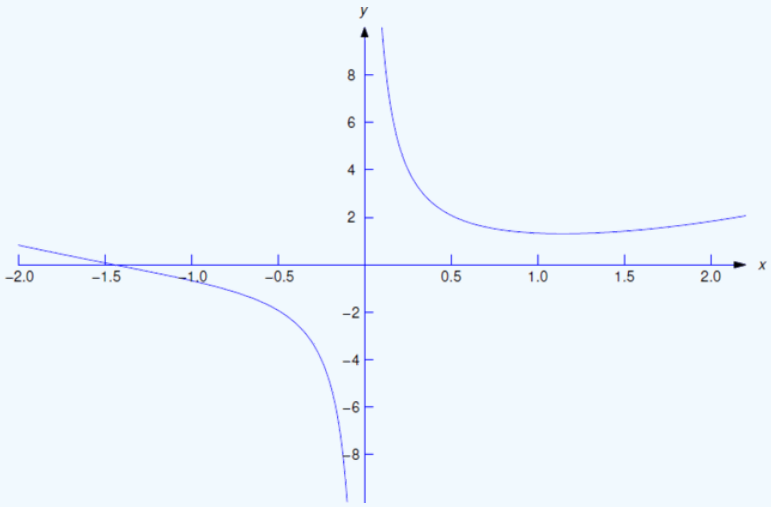


Figure 1.1.2 : $y = \frac{x^2}{3} + \frac{1}{x}$

Example 1.1.4

Show that if c_1 and c_2 are constants then

$$y = (c_1 + c_2 x)e^{-x} + 2x - 4 \tag{1.1.5}$$

is a solution of

$$y'' + 2y' + y = 2x \tag{1.1.6}$$

on $(-\infty, \infty)$.

Solution

Differentiating Equation 1.1.5 twice yields

$$y' = -(c_1 + c_2 x)e^{-x} + c_2 e^{-x} + 2$$

and

$$y'' = (c_1 + c_2 x)e^{-x} - 2c_2 e^{-x},$$

so

$$\begin{aligned} y'' + 2y' + y &= (c_1 + c_2 x)e^{-x} - 2c_2 e^{-x} + 2\left[-(c_1 + c_2 x)e^{-x} + c_2 e^{-x} + 2\right] + (c_1 + c_2 x)e^{-x} + 2x - 4 \\ &= (1 - 2 + 1)(c_1 + c_2 x)e^{-x} + (-2 + 2)c_2 e^{-x} + 4 + 2x - 4 \\ &= 2x \end{aligned}$$

for all values of x. Therefore y is a solution of Equation 1.1.6 on $(-\infty, \infty)$.

Note: Functions that satisfy a differential equation at isolated points are neither interesting nor useful, and since they do not satisfy the equation on an open interval we do not consider them solutions.

Example 1.1.5

For example, $y = x^2$ satisfies

$$xy' + x^2 = 3x$$

if and only if $x = 0$ or $x = 1$, but it is not a solution of this differential equation because it does not satisfy the equation on an open interval.

Example 1.1.6

If a is any positive constant, the circle

$$x^2 + y^2 = a^2 \tag{1.1.7}$$

is an integral curve of

$$y' = -\frac{x}{y}. \tag{1.1.8}$$

Solution

To see this, note that the only functions whose graphs are segments of Equation 1.1.7 are

$$y_1 = \sqrt{a^2 - x^2} \quad \text{and} \quad y_2 = -\sqrt{a^2 - x^2}.$$

We leave it to you to verify that these functions both satisfy Equation 1.1.8 on the open interval $(-a, a)$. However, Equation 1.1.7 is not a solution curve of Equation 1.1.8, since it is not the graph of a function. However, both y_1 and y_2 are each solution curves.

Initial Value Problems

Definition 1.1.5

An nth order differential equation has an infinite *family of solutions* that depend upon the n arbitrary constants $c_1, c_2, \ldots, c_n$. In the absence of additional conditions, there's no reason to prefer one solution of a differential equation over another. However, we'll often be interested in finding a *particular solution* of a differential equation that satisfies one or more specific conditions.

Example 1.1.7

Find a solution of

$$y' = x^3$$

such that $y(1) = 2$.

Solution

At the beginning of this section we saw that the **family of solutions** of the differential equation $y' = x^3$ are

$$y = \frac{x^4}{4} + c.$$

To determine a value of c such that $y(1) = 2$, we set $x = 1$ and $y = 2$ here to obtain

$$2 = y(1) = \frac{1}{4} + c$$

so

$$c = \frac{7}{4}.$$

Therefore the required **particular solution** of the initial value problem is

$$y = \frac{x^4 + 7}{4}.$$

Figure 1.1.2 shows the graph of this solution. Note that imposing the condition $y(1) = 2$ is equivalent to requiring the graph of y to pass through the point $(1, 2)$.

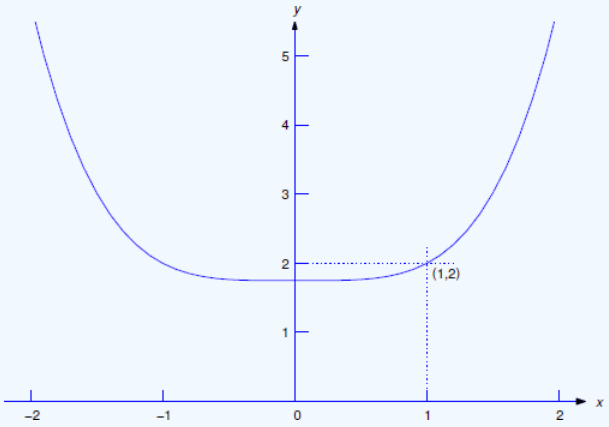


Figure 1.1.2 : $y = \frac{x^2+7}{4}$

Definition 1.1.6

A solution of a differential equation that cannot be obtained from a family of solutions by varying the parameters is called a *singular solution.*

Example 1.1.8

$y = (\frac{x^2}{4} + c)^2$ is a **family of solutions** to the differential equation $y' = xy^{1/2}$ (verify this). But $y = 0$ is also a solution (easily verified) and it cannot be obtained from the family by varying the parameter c. Therefore $y = 0$ is a **singular solution**.

We can rewrite the initial value problem considered in Example 1.1.7 more briefly as

$$y' = x^3, \quad y(1) = 2.$$

The requirement $y(1) = 2$ is an initial condition. Initial value problems can also be posed for higher order differential equations. For example,

$$y'' - 2y' + 3y = e^x, \quad y(0) = 1, \quad y'(0) = 2 \tag{1.1.9}$$

is an initial value problem for a second order differential equation where y and y' are required to have specified values at $x = 0$. In general, an initial value problem for an n-th order differential equation requires y and its first $n - 1$ derivatives to have specified values at some point x_0. These requirements are the initial conditions. We'll denote an initial value problem for a differential equation by writing the initial conditions after the equation, as in Equation 1.1.9. For example, we would write an initial value problem for Equation 1.1.2 as

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}),\ y(x_0) = k_0,\ y'(x_0) = k_1,\ \ldots,\ y^{(n-1)} = k_{n-1}. \tag{1.1.10}$$

Consistent with our earlier definition of a solution of the differential equation in Equation 1.1.10, we say that y is a solution of the initial value problem Equation 1.1.10 if y is n times differentiable and

$$y^{(n)}(x) = f(x, y(x), y'(x), \ldots, y^{(n-1)}(x))$$

for all x in some open interval (a, b) that contains x_0, and y satisfies the initial conditions in Equation 1.1.10.

Definition 1.1.7

The largest open interval that contains x_0 on which y is defined and satisfies the differential equation is the *interval of validity* of y.

Example 1.1.9

In Example 1.1.7 we saw that

$$y = \frac{x^4 + 7}{4} \tag{1.1.11}$$

is a solution of the initial value problem

$$y' = x^3, \quad y(1) = 2.$$

Since the function in Equation 1.1.11 is defined for all x, the interval of validity of this solution is $(-\infty, \infty)$.

Example 1.1.10

In Example 1.1.3 we verified that

$$y = \frac{x^2}{3} + \frac{1}{x} \tag{1.1.12}$$

is a solution of

$$xy' + y = x^2$$

on $(0, \infty)$ and on $(-\infty, 0)$. By evaluating Equation 1.1.12 at $x = \pm 1$, you can see that Equation 1.1.12 is a solution of the initial value problems

$$xy' + y = x^2, \quad y(1) = \frac{4}{3} \tag{1.1.13}$$

and

$$xy' + y = x^2, \quad y(-1) = -\frac{2}{3}. \tag{1.1.14}$$

The interval of validity of Equation 1.1.12 as a solution of Equation 1.1.13 is $(0, \infty)$, since this is the largest interval that contains $x_0 = 1$ on which Equation 1.1.12 is defined. Similarly, the interval of validity of Equation 1.1.12 as a solution of Equation 1.1.14 is $(-\infty, 0)$, since this is the largest interval that contains $x_0 = -1$ on which Equation 1.1.12 is defined.

Free Fall Under Constant Gravity

The term *initial value problem* originated in problems of motion where the independent variable is t (representing elapsed time), and the initial conditions are the position and velocity of an object at the initial (starting) time of an experiment, usually, but not always, 0.

Example 1.1.11

An object falls under the influence of gravity near Earth's surface, where it can be assumed that the magnitude of the acceleration due to gravity is a constant g.

a. Construct a mathematical model for the motion of the object in the form of an initial value problem for a second order differential equation, assuming that the altitude and velocity of the object at time $t = 0$ are known. Assume that gravity is the only force acting on the object.
b. Solve the initial value problem derived above to obtain the altitude as a function of time.

Solution a

Let $y(t)$ be the altitude of the object at time t. Since the acceleration of the object has constant magnitude g and is in the downward (negative) direction, y satisfies the second order equation

$$y'' = -g,$$

where the prime now indicates differentiation with respect to t. If y_0 and v_0 denote the altitude and velocity when $t = 0$, then y is a solution of the initial value problem

$$y'' = -g, \quad y(0) = y_0, \quad y'(0) = v_0. \tag{1.1.15}$$

Solution b

Integrating Equation 1.1.15 twice yields the family of solutions to the differential equation $y'' = -g$,

$$\begin{aligned} y' &= -gt + c_1, \\ y &= -\frac{gt^2}{2} + c_1 t + c_2. \end{aligned}$$

Imposing the initial conditions $y(0) = y_0$ and $y'(0) = v_0$ in these two equations shows that $c_1 = v_0$ and $c_2 = y_0$. Therefore the particular solution of the initial value problem Equation 1.1.15 is

$$y = -\frac{gt^2}{2} + v_0 t + y_0.$$

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1.1E: Basic Concepts (Exercises)

In Exercises **1-5** find the order of the differential equation.

1. $y''-3y'+2y=x^7$
2. $\frac{d^2y}{dx^2}+2x(\frac{dy}{dx})^3+xy=0$
3. $x^3dy-y^7dx=0$
4. $y'-y^7=0$
5. $y''y-(y')^2=2$

In Exercises **6-13** verify that the function is a solution of the differential equation on some interval, for any choice of the arbitrary constants appearing in the function.

6. $y=e^{2x};\quad y'=2y$
7. $y=\frac{x^2}{3}+\frac{c}{x};\quad xy'+y=x^2$
8. $y=\frac{1}{16}x^4;\quad y'=xy^{1/2}$
9. $y=\frac{1}{2}+ce^{-x^2};\quad y'+2xy=x$
10. $y=\tan\left(\frac{x^3}{3}\right);\quad y'=x^2(1+y^2)$
11. $y=(c_1+c_2x)e^x+\sin x+x^2;\quad y''-2y'+y=-2\cos x+x^2-4x+2$
12. $y=c_1e^x+c_2x+\frac{2}{x};\quad (1-x)y''+xy'-y=4(1-x-x^2)x^{-3}$
13. $y=x^{-1/2}(c_1\sin x+c_2\cos x)+4x+8\ ;x^2y''+xy'+\left(x^2-\frac{1}{4}\right)y=4x^3+8x^2+3x-2$

In Exercises **14-19** find the family of solutions of the differential equation.

14. $y'=4x^3-x^2+5$
15. $y'=-x\sin x$
16. $y'=\sin^5x\cos^4x$
17. $y'=x\sec x\tan x$
18. $y'=\frac{x^2}{\sqrt{4-x^2}}$
19. $y'=\frac{2x^2-x+9}{x^3+9x}$

In Exercises **20-22** find the particular solution of the differential equation.

20. $y'=x\sin x^2,\quad y\left(\sqrt{\frac{\pi}{2}}\right)=1$
21. $y'=-xe^x,\quad y(0)=1$
22. $y'=\tan x,\quad y(\pi/4)=3$

In Exercises **23-30** verify that the function is a solution of the initial value problem and give its interval of validity.

23. $y=\tan x;\quad y'=y^2+1,\quad y(\pi)=0$
24. $y=x\cos x;\quad y'=\cos x-y\tan x,\quad y(\pi/4)=\frac{\pi}{4\sqrt{2}}$
25. $y=\frac{1+2\ln x}{x^2}+\frac{1}{2};\quad y'=\frac{x^2-2x^2y+2}{x^3},\quad y(1)=\frac{3}{2}$
26. $y=\tan\left(\frac{x^2}{2}\right);\quad y'=x(1+y^2),\quad y(0)=0$
27. $y=\frac{2}{x-2};\quad y'=\frac{-y(y+1)}{x},\quad y(1)=-2$
28. $y=x^2(1+\ln x);\quad y''=\frac{3xy'-4y}{x^2},\quad y(e)=2e^2,\quad y'(e)=5e$

29. $y=\frac{x^2}{3}+x-1; \quad y''=\frac{x^2-xy'+y+1}{x^2}, \quad y(1)=\frac{1}{3}, \quad y'(1)=\frac{5}{3}$

30. $y=(1+x^2)^{-1/2}; \quad y''=\frac{(x^2-1)y-x(x^2+1)y'}{(x^2+1)^2}, \quad y(0)=1,\ y'(0)=0$

31. Give the interval of validity of $y=\frac{x^2}{1-x}; \quad y''=\frac{2(x+y)(xy'-y)}{x^3}, \quad y(1/2)=1/2, \quad y'(1/2)=3$

32. a. Give the domain of the function $y=x+2\sqrt{x+3}$

b. Verify that $y=x+2\sqrt{x+3}$ is a solution to the differential equation $(y-x)y'=y-x+2$ and give its interval of validity.

33. $y=\sin(\frac{x^2}{2}+c)$ is a family of solutions to the differential equation $y'=x\sqrt{1-y^2}$ on certain intervals, dependent on the value of c.

a. Verify that $y=\sin(\frac{x^2}{2})$ is a particular solution to the initial value problem $y'=x\sqrt{1-y^2}$, $y(0)=0$ and find its interval of validity.

b. By inspection find two singular solutions to $y'=x\sqrt{1-y^2}$ and find the interval of validity of each.

34. Suppose an object is launched from a point 320 feet above the earth with an initial velocity of 128 ft/sec upward, and the only force acting on it thereafter is gravity. Take $g=32$ ft/sec^2.

a. Find the highest altitude attained by the object.
b. Determine how long it takes for the object to fall to the ground.

35. Let a be a nonzero real number.

a. Verify that if c is an arbitrary constant then

$$y=(x-c)^a \tag{A}$$

is a solution of

$$y'=ay^{(a-1)/a} \tag{B}$$

on (c,∞).
b. Why is the interval of validity (c,∞)?

36. Verify that

$$y=\begin{cases} e^x-1, & x\geq 0, \\ 1-e^{-x}, & x<0, \end{cases}$$

is a solution of

$$y'=|y|+1$$

on $(-\infty,\infty)$.

In particular, address the continuity and differentiability of y at $x=0$.

37. a. Verify that if c is any real number then

$$y=c^2+cx+2c+1 \tag{A}$$

is a family of solutions to

$$y'=\frac{-(x+2)+\sqrt{x^2+4x+4y}}{2} \tag{B}$$

on some open interval. Identify the open interval.

b. Verify that

$$y_1=\frac{-x(x+4)}{4}$$

is a singular solution to (B) on some open interval, and identify the open interval. (Note that y_1 can't be obtained by selecting a value of c in (A).)

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1.2: Existence and Uniqueness of Solutions

Although there are methods for solving many differential equations, it is impossible to find useful formulas for the solutions of all of them. Whether we are looking for exact solutions or numerical approximations, it is useful to know conditions that imply the existence and uniqueness of solutions of initial value problems. In this section we state such a condition and illustrate it with examples.

Caution

If you have not had Math 402 (multivariable calculus), then you will need to read Appendices 11.1 and 11.2 before continuing.

Some terminology: an *open rectangle* R is a set of points (x, y) such that

$$a < x < b \quad \text{and} \quad c < y < d$$

(Figure 1.2.1). We'll denote this set by $R : \{a < x < b, c < y < d\}$. "Open" means that the boundary rectangle (indicated by the dashed lines in Figure 1.2.1) is not included in R.

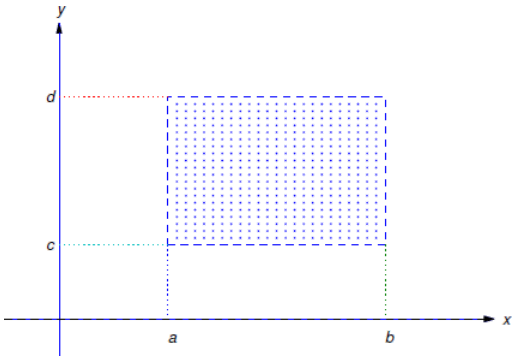


Figure 1.2.1 : An open rectangle

The next theorem gives sufficient conditions for existence and uniqueness of solutions of initial value problems for first order differential equations. We omit the proof, which is beyond the scope of this book.

Theorem 1.2.1 : Existence and Uniqueness

a. If f is continuous on an open rectangle

$$R : \{a < x < b, c < y < d\}$$

that contains (x_0, y_0) then the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{1.2.1}$$

has at least one solution on some open subinterval of (a, b) that contains x_0.

b. If both f and f_y are continuous on R then Equation 1.2.1 has a unique solution on some open subinterval of (a, b) that contains x_0.

It's important to understand exactly what Theorem 1.2.1 says.

- (a) is an *existence theorem*. It guarantees that a solution exists on some open interval that contains x_0, but provides no information on how to find the solution, or to determine the open interval on which it exists. Moreover, (a) provides no information on the number of solutions that Equation 1.2.1 may have. It leaves open the possibility that Equation 1.2.1 may have two or more solutions that differ for values of x arbitrarily close to x_0. We will see in Example 1.2.6 that this can happen.
- (b) is a *uniqueness theorem*. It guarantees that Equation 1.2.1 has a unique solution on some open interval (a, b) that contains x_0. However, if $(a, b) \neq (-\infty, \infty)$, Equation 1.2.1 may have more than one solution on a larger interval that contains (a, b). For example, it may happen that $b < \infty$ and all solutions have the same values on (a, b), but two solutions y_1 and y_2 are defined on some interval (a, b_1) with $b_1 > b$, and have different values for $b < x < b_1$; thus, the graphs of y_1 and y_2 "branch

off" in different directions at $x=b$. (See Example 1.2.7 and Figure 1.2.3). In this case, continuity implies that $y_1(b)=y_2(b)=\overline{y}$ (call their common value y), and y_1 and y_2 are both solutions of the initial value problem

$$y=f(x,y), \quad y(b)=\overline{y} \tag{1.2.2}$$

that differ on every open interval that contains b. Therefore f or f_y must have a discontinuity at some point in each open rectangle that contains (b,y), since if this were not so, 1.2.2 would have a unique solution on some open interval that contains b. We leave it to you to give a similar analysis of the case where $a>-\infty$.

Example 1.2.1

Consider the initial value problem

$$y'=\frac{x^2-y^2}{1+x^2+y^2}, \quad y(x_0)=y_0. \tag{1.2.3}$$

Since

$$f(x,y)=\frac{x^2-y^2}{1+x^2+y^2} \quad \text{and} \quad f_y(x,y)=-\frac{2y(1+2x^2)}{(1+x^2+y^2)^2}$$

are continuous for all (x,y), Theorem 1.2.1 implies that if (x_0,y_0) is arbitrary, then Equation 1.2.3 has a unique solution on some open interval that contains x_0.

Example 1.2.2

Consider the initial value problem

$$y'=\frac{x^2-y^2}{x^2+y^2}, \quad y(x_0)=y_0. \tag{1.2.4}$$

Here

$$f(x,y)=\frac{x^2-y^2}{x^2+y^2} \quad \text{and} \quad f_y(x,y)=-\frac{4x^2y}{(x^2+y^2)^2}$$

are continuous everywhere except at $(0,0)$. If $(x_0,y_0)\neq(0,0)$, there's an open rectangle R that contains (x_0,y_0) that does not contain $(0,0)$. Since f and f_y are continuous on R, Theorem 1.2.1 implies that if $(x_0,y_0)\neq(0,0)$ then Equation 1.2.4 has a unique solution on some open interval that contains x_0.

Example 1.2.3

Consider the initial value problem

$$y'=\frac{x+y}{x-y}, \quad y(x_0)=y_0. \tag{1.2.5}$$

Here

$$f(x,y)=\frac{x+y}{x-y} \quad \text{and} \quad f_y(x,y)=\frac{2x}{(x-y)^2}$$

are continuous everywhere except on the line $y=x$. If $y_0\neq x_0$, there's an open rectangle R that contains (x_0,y_0) that does not intersect the line $y=x$. Since f and f_y are continuous on R, Theorem 1.2.1 implies that if $y_0\neq x_0$, Equation 1.2.5 has a unique solution on some open interval that contains x_0.

Example 1.2.4

We will see later that the solutions of

$$y' = 2xy^2 \tag{1.2.6}$$

are

$$y \equiv 0 \quad \text{and} \quad y = -\frac{1}{x^2 + c},$$

where c is an arbitrary constant. In particular, this implies that no solution of Equation 1.2.6 other than $y \equiv 0$ can equal zero for any value of x. Show that Theorem $1.2.1b$ implies this.

We'll obtain a contradiction by assuming that Equation 1.2.6 has a solution y_1 that equals zero for some value of x, but is not identically zero. If y_1 has this property, there's a point x_0 such that $y_1(x_0) = 0$, but $y_1(x) \neq 0$ for some value of x in every open interval that contains x_0. This means that the initial value problem

$$y' = 2xy^2, \quad y(x_0) = 0 \tag{1.2.7}$$

has two solutions $y \equiv 0$ and $y = y_1$ that differ for some value of x on every open interval that contains x_0. This contradicts Theorem 1.2.1 (b), since in Equation 1.2.6 the functions

$$f(x, y) = 2xy^2 \quad \text{and} \quad f_y(x, y) = 4xy.$$

are both continuous for all (x, y), which implies that Equation 1.2.7 has a unique solution on some open interval that contains x_0.

Note: $y \equiv 0$ is a singular solution since it cannot be obtained from the family $y = -\frac{1}{x^2+c}$, by varying the parameter c.

Example 1.2.5

Consider the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(x_0) = y_0. \tag{1.2.8}$$

a. For what points (x_0, y_0) does Theorem $1.2.1a$ imply that Equation 1.2.8 has a solution?
b. For what points (x_0, y_0) does Theorem $1.2.1b$ imply that Equation 1.2.8 has a unique solution on some open interval that contains x_0?

Solution a

Since

$$f(x, y) = \frac{10}{3}xy^{2/5}$$

is continuous for all (x, y), Theorem 1.2.1 implies that Equation 1.2.8 has a solution for every (x_0, y_0).

Solution b

Here

$$f_y(x, y) = \frac{4}{3}xy^{-3/5}$$

is continuous for all (x, y) with $y \neq 0$. Therefore, if $y_0 \neq 0$ there's an open rectangle on which both f and f_y are continuous, and Theorem 1.2.1 implies that Equation 1.2.8 has a unique solution on some open interval that contains x_0.

If $y = 0$ then $f_y(x, y)$ is undefined, and therefore discontinuous; hence, Theorem 1.2.1 does not apply to Equation 1.2.8 if $y_0 = 0$.

Example 1.2.6

Example 1.2.5 leaves open the possibility that the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(0) = 0 \tag{1.2.9}$$

has more than one solution on every open interval that contains $x_0 = 0$. Show that this is true.

Solution

By inspection, $y \equiv 0$ is a solution of the differential equation

$$y' = \frac{10}{3} x y^{2/5}. \tag{1.2.10}$$

Since $y \equiv 0$ satisfies the initial condition $y(0) = 0$, it is a solution of Equation 1.2.9.

Now suppose y is a solution of Equation 1.2.10 that is not identically zero. We will see in chapter 2 that we can obtain the family of solutions

$$y = (x^2 + c)^{5/3}. \tag{1.2.11}$$

Applying our initial condition gives us a particular solution

$$y = x^{10/3}$$

as a second solution of Equation 1.2.9. Both solutions are defined on $(-\infty, \infty)$, and they differ on every open interval that contains $x_0 = 0$ (Figure 1.2.2).

Note: $y \equiv 0$ is a singular solution since it cannot be obtained from the family by varying c.

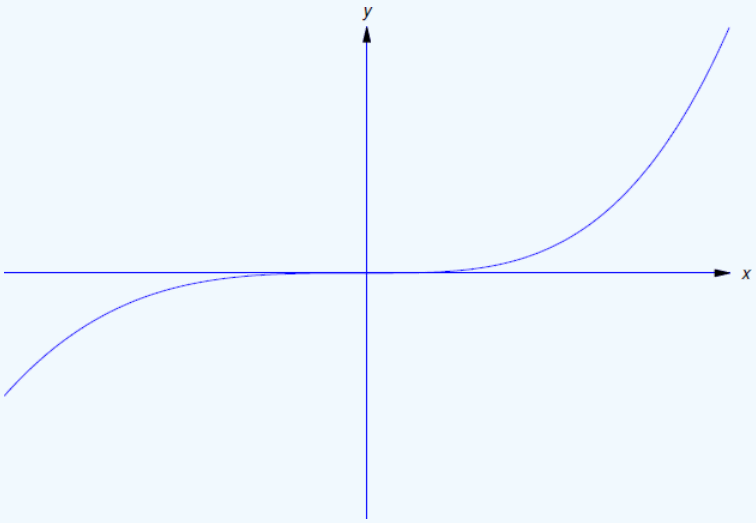


Figure 1.2.2 : Two solutions ($y \equiv 0$ and $y = x^{10/3}$) of Equation 1.2.9 that differ on every interval containing $x_0 = 0$

Example 1.2.7

From Example 1.2.5 , the initial value problem

$$y' = \frac{10}{3} x y^{2/5}, \quad y(0) = -1 \tag{1.2.12}$$

has a unique solution on some open interval that contains $x_0 = 0$. Find a solution and determine the largest open interval (a, b) on which it is unique.

Solution

Let y be any solution of Equation 1.2.12. Because of the initial condition $y(0) = -1$ and the continuity of y, there's an open interval I that contains $x_0 = 0$ on which y has no zeros, and is consequently of the form Equation 1.2.11. Setting $x = 0$ and $y = -1$ in Equation 1.2.11 yields $c = -1$, so

$$y = (x^2 - 1)^{5/3} \tag{1.2.13}$$

for x in I. Therefore every solution of Equation 1.2.12 differs from zero and is given by Equation 1.2.13 on $(-1,1)$; that is, Equation 1.2.13 is the unique solution of Equation 1.2.12 on $(-1,1)$. This is the largest open interval on which Equation 1.2.12 has a unique solution. To see this, note that Equation 1.2.13 is a solution of Equation 1.2.12 on $(-\infty,\infty)$, but the solution below is another solution of Equation 1.2.12 that differs from Equation 1.2.13 on every open interval larger than $(-1,1)$:

$$y=\begin{cases}(x^2-1)^{5/3}, & -1\le x\le 1,\\ 0, & |x|>1.\end{cases}$$

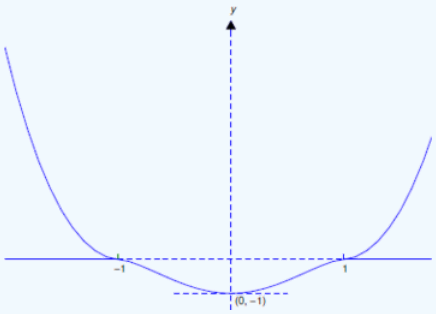


Figure 1.2.3 : Two solutions of Equation 1.2.12 on (−1,1) that coincide on (−1, 1), but on no larger open interval.

Example 1.2.8

From Example 1.2.5 , the initial value problem

$$y'=\frac{10}{3}xy^{2/5},\quad y(0)=1 \tag{1.2.14}$$

has a unique solution on some open interval that contains $x_0=0$. Find the solution and determine the largest open interval on which it is unique.

Solution

Let y be any solution of Equation 1.2.14. Because of the initial condition $y(0)=1$ and the continuity of y, there's an open interval I that contains $x_0=0$ on which y has no zeros, and is consequently of the form Equation 1.2.11. Setting $x=0$ and $y=1$ in Equation 1.2.11 yields $c=1$, so

$$y=(x^2+1)^{5/3} \tag{1.2.15}$$

for x in I. Therefore every solution of Equation 1.2.14 differs from zero and is given by Equation 1.2.15 on $(-\infty,\infty)$; that is, Equation 1.2.15 is the unique solution of Equation 1.2.14 on $(-\infty,\infty)$. Figure 1.2.4) shows the graph of this solution.

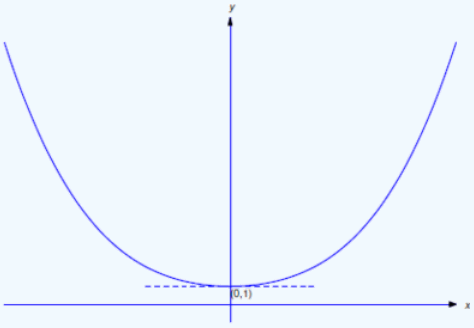


Figure 1.2.4 : The unique solution of Equation 1.2.14.

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1.2E: Existence and Uniqueness of Solutions (Exercises)

In Exercises **1-14**, find all (x_0, y_0) for which Theorem 1.2.1 implies that the initial value problem $y' = f(x, y),\ y(x_0) = y_0$ has (a) a solution and (b) a unique solution on some open interval that contains x_0.

1. $(\sin x)y' = x^2 + y^2$
2. $\frac{dy}{dx} = y^{2/3}$
3. $y' = \tan(xy)$
4. $y' = \frac{x^2+y^2}{\ln(xy)}$
5. $y' = (x^2 + y^2)y^{1/3}$
6. $dy = 2xydx$
7. $(4 - y^2)\frac{dy}{dx} = x^2$
8. $dy = \ln(1 + x^2 + y^2)dx$
9. $y' = \frac{2x+3y}{x-4y}$
10. $dy = (x^2 + y^2)^{1/2}dx$
11. $y' = x(y^2 - 1)^{2/3}$
12. $dy = (x^2 + y^2)^2dx$
13. $\frac{dy}{dx} = (x + y)^{1/2}$
14. $(x - 1)y' = \tan y$

15. Consider the initial value problem

$$y' = 3x(y - 1)^{1/3}, \quad y(x_0) = y_0.$$

a. For what points (x_0, y_0) does Theorem 1.2.1 imply that the IVP has a solution?
b. For what points (x_0, y_0) does Theorem 1.2.1 imply that the IVP has a unique solution on some open interval that contains x_0?

16. Consider the initial value problem

$$y' = 3y^{2/3}, \quad y(0) = 0.$$

a. Does Theorem 1.2.1 guarantee a solution? If so, by inspection, can you find one?

b. Does Theorem 1.2.1 guarantee a unique solution. If not, by inspection, can you find a second solution?

17. Consider the initial value problem

$$\frac{dy}{dx} = y^2$$

a. Verify that the function

$$y = \frac{-1}{x + c}$$

is a family of solutions of the differential equation.

b. Does Theorem 1.2.1 guarantee a solution to the IVP

$$\frac{dy}{dx} = y^2, \quad y(x_0) = y_0$$

for all points (x_0, y_0)? Why or why not?

c. Does Theorem 1.2.1 guarantee a unique solution to the IVP

$$\frac{dy}{dx} = y^2, \quad y(x_0) = y_0$$

for all points (x_0, y_0)? Why or why not?

d. Find a solution from the family that satisfies $y(0) = 1$ and give its interval of validity.

e. Find a solution from the family that satisfies $y(0) = -1$ and give its interval of validity.

f. Find a singular solution to the differential equation that satisfies $y(0) = 0$ and give its interval of validity.

18. Consider the initial value problem

$$\frac{dy}{dx} = 1 + y^2$$

a. Verify that the function

$$y = \tan(x + c)$$

is a family of solutions of the differential equation.

b. Does Theorem 1.2.1 guarantee a solution to the IVP

$$\frac{dy}{dx} = 1 + y^2, \quad y(x_0) = y_0$$

for all points (x_0, y_0)? Why or why not?

c. Does Theorem 1.2.1 guarantee a unique solution to the IVP

$$\frac{dy}{dx} = 1 + y^2, \quad y(x_0) = y_0$$

for all points (x_0, y_0)? Why or why not?

d. Find a solution from the family that satisfies $y(0) = 1$ and give its interval of validity.

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1.3: Direction Fields

It's impossible to find explicit formulas for solutions of some differential equations. Even if there are such formulas, they may be so complicated that they're useless. In this case we may resort to graphical or numerical methods to get some idea of how the solutions of the given equation behave.

In Section 1.2 we talked about the existence of solutions of a first order equation

$$y' = f(x, y). \tag{1.3.1}$$

In this section we'll simply assume that Equation 1.3.1 has solutions and discuss a graphical method for approximating them. In Chapter 3 we discuss numerical methods for obtaining approximate solutions of Equation 1.3.1. Recall that a solution of Equation 1.3.1 is a function $y = y(x)$ such that

$$y'(x) = f(x, y(x))$$

for all values of x in some interval, and an integral curve is either the graph of a solution or is made up of segments that are graphs of solutions. Therefore, not being able to solve Equation 1.3.1 is equivalent to not knowing the equations of integral curves of Equation 1.3.1. However, it is easy to calculate the slopes of these curves. To be specific, the slope of an integral curve of Equation 1.3.1 through a given point (x_0, y_0) is given by the number $f(x_0, y_0)$. This is the basis of *the method of direction fields*.

If f is defined on a set R, we can construct a *direction field* for Equation 1.3.1 in R by drawing a short line segment through each point (x, y) in R with slope $f(x, y)$. Of course, as a practical matter, we can't actually draw line segments through *every* point in R; rather, we must select a finite set of points in R. For example, suppose f is defined on the closed rectangular region

$$R : \{a \le x \le b, c \le y \le d\}.$$

Let

$$a = x_0 < x_1 < \cdots < x_m = b$$

be equally spaced points in $[a, b]$ and

$$c = y_0 < y_1 < \cdots < y_n = d$$

form a *rectangular grid* (Figure 1.3.1). Through each point in the grid we draw a short line segment with slope $f(x_i, y_j)$. The result is an approximation to a direction field for Equation 1.3.1 in R. If the grid points are sufficiently numerous and close together, we can draw approximate integral curves of Equation 1.3.1 by drawing curves through points in the grid tangent to the line segments associated with the points in the grid.

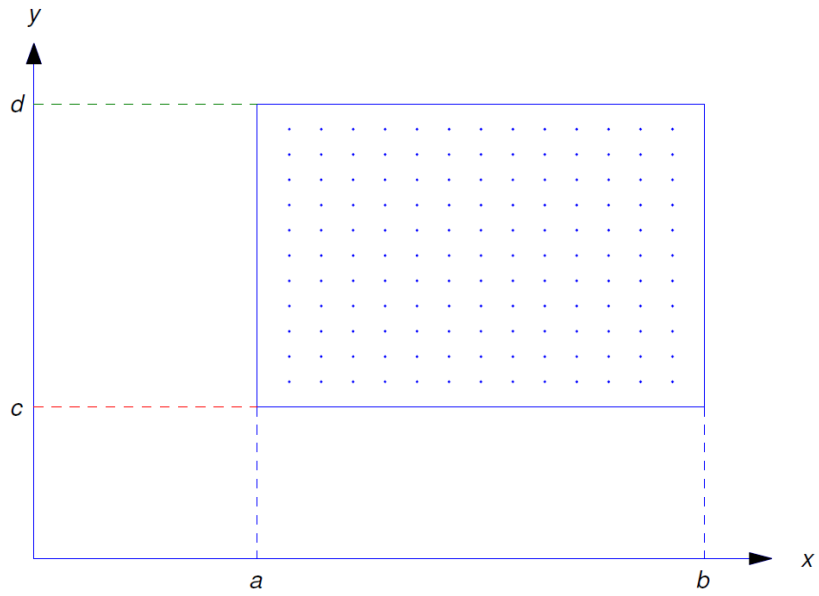


Figure 1.3.1 : A rectangular grid

Unfortunately, approximating a direction field and graphing integral curves in this way is too tedious to be done effectively by hand. However, there is software for doing this. As you'll see, the combination of direction fields and integral curves gives useful insights into the behavior of the solutions of the differential equation even if we can't obtain exact solutions.

We'll study numerical methods for solving a single first order equation Equation 1.3.1 in Chapter 3. These methods can be used to plot solution curves of Equation 1.3.1 in a rectangular region R *if* f *is continuous on* R. Figures 1.3.2 , 1.3.3 , and 1.3.4 show direction fields and solution curves for the differential equations:

- $y' = \frac{x^2-y^2}{1+x^2+y^2}$,
- $y' = 1 + xy^2$, and
- $y' = \frac{x-y}{1+x^2}$.

which are all of the form Equation 1.3.1 with f continuous for all (x, y).

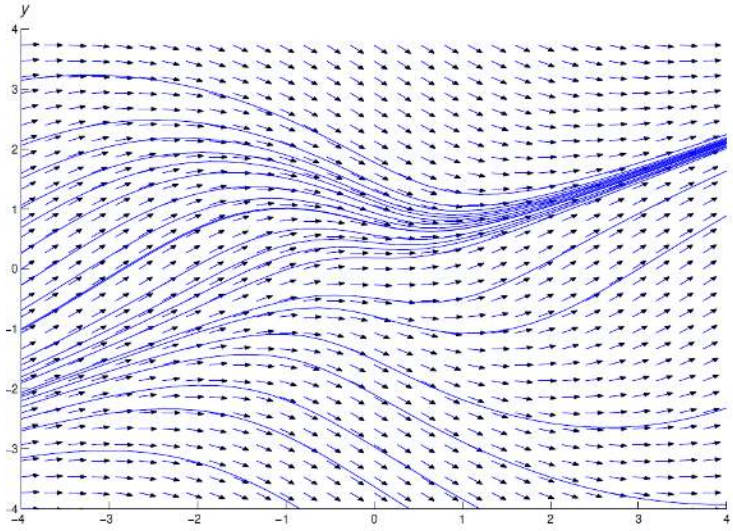


Figure 1.3.2 : A direction and integral curves for $y' = \frac{x^2-y^2}{1+x^2+y^2}$.

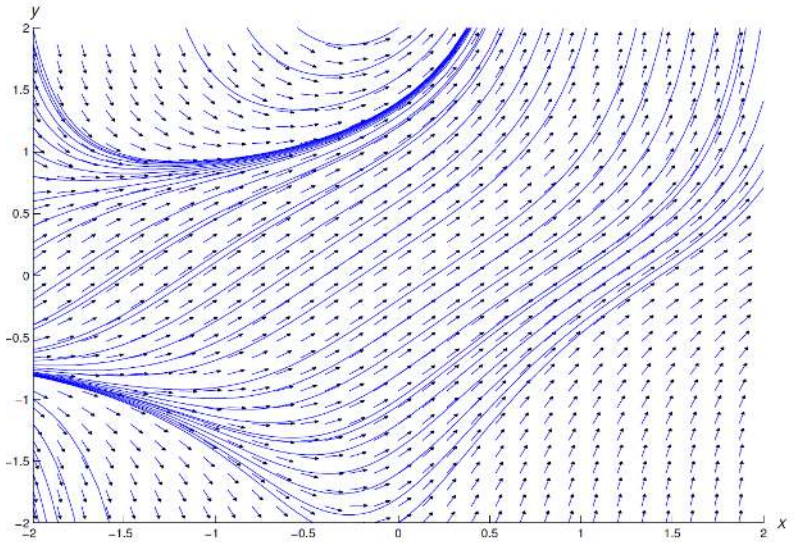


Figure 1.3.3 : A direction and integral curves for $y' = 1 + xy^2$.

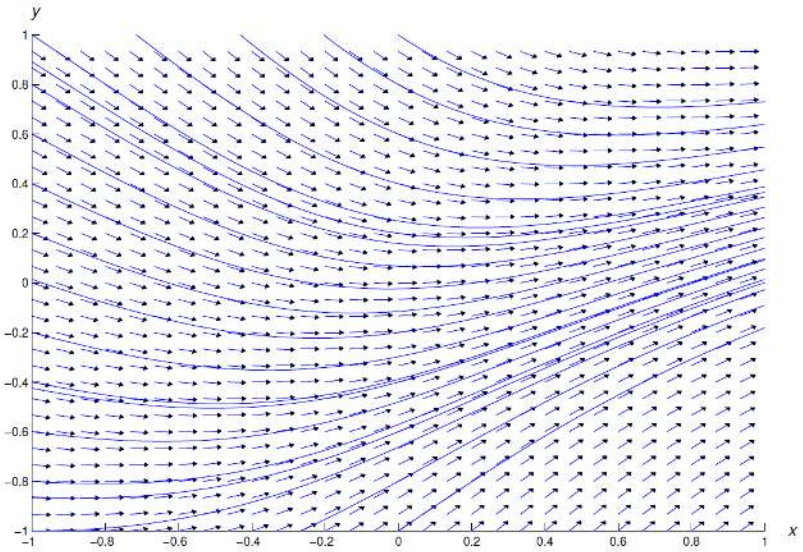


Figure 1.3.4 : A direction and integral curves for $y' = \frac{x-y}{1+x^2}$.

The methods of Chapter 3 will not work for the equation

$$y' = -x/y \tag{1.3.2}$$

if R contains part of the x-axis, since $f(x,y) = -x/y$ is undefined when $y = 0$. Similarly, they will not work for the equation

$$y' = \frac{x^2}{1-x^2-y^2} \tag{1.3.3}$$

if R contains any part of the unit circle $x^2+y^2 = 1$, because the right side of Equation 1.3.3 is undefined if $x^2+y^2 = 1$. However, Equation 1.3.2 and Equation 1.3.3 can written as

$$y' = \frac{A(x,y)}{B(x,y)} \tag{1.3.4}$$

where A and B are continuous on any rectangle R. Because of this, some differential equation software is based on numerically solving pairs of equations of the form

$$\frac{dx}{dt} = B(x,y), \quad \frac{dy}{dt} = A(x,y) \tag{1.3.5}$$

where x and y are regarded as functions of a parameter t. If $x = x(t)$ and $y = y(t)$ satisfy these equations, then

$$y' = \frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{A(x,y)}{B(x,y)},$$

so $y = y(x)$ satisfies Equation 1.3.4.

Equations 1.3.2 and 1.3.3 can be reformulated as in Equation 1.3.4 with

$$\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = x$$

and

$$\frac{dx}{dt} = 1-x^2-y^2, \quad \frac{dy}{dt} = x^2,$$

respectively. Even if f is continuous and otherwise "nice" throughout R, your software may require you to reformulate the equation $y' = f(x,y)$ as

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = f(x,y),$$

which is of the form Equation 1.3.5 with $A(x,y)=f(x,y)$ and $B(x,y)=1$.

Figure 1.3.5 shows a direction field and some integral curves for Equation 1.3.2. As we saw in Example 1.1.1 and will verify again in Section 2.1, the integral curves of Equation 1.3.2 are circles centered at the origin.

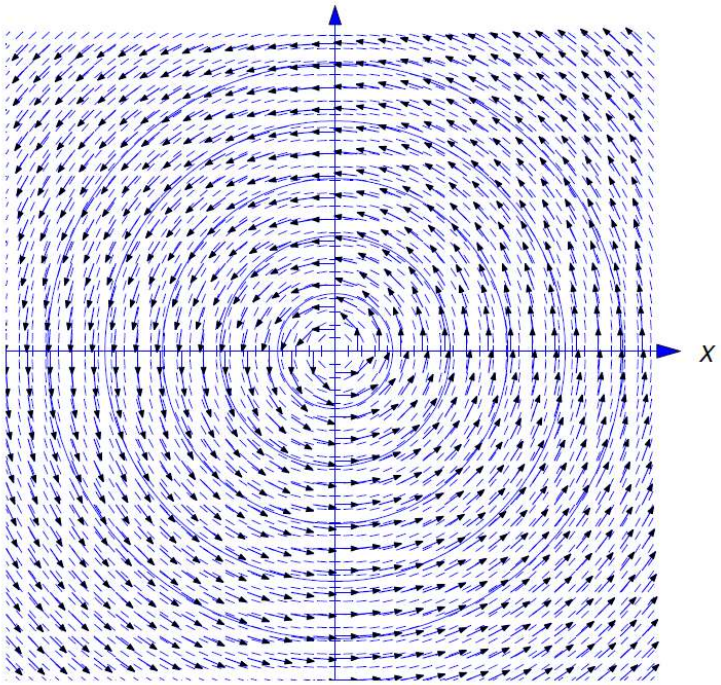


Figure 1.3.5 : A direction field and integral curves for $y'=-\frac{x}{y}$

Figure 1.3.6 shows a direction field and some integral curves for Equation 1.3.3. The integral curves near the top and bottom are solution curves. However, the integral curves near the middle are more complicated. For example, Figure 1.3.7 shows the integral curve through the origin. The vertices of the dashed rectangle are on the circle $x^2+y^2=1$ ($a\approx .846$, $b\approx .533$), where all integral curves of Equation 1.3.3 have infinite slope. There are three solution curves of Equation 1.3.3 on the integral curve in the figure: the segment above the level $y=b$ is the graph of a solution on $(-\infty,a)$, the segment below the level $y=-b$ is the graph of a solution on $(-a,\infty)$, and the segment between these two levels is the graph of a solution on $(-a,a)$.

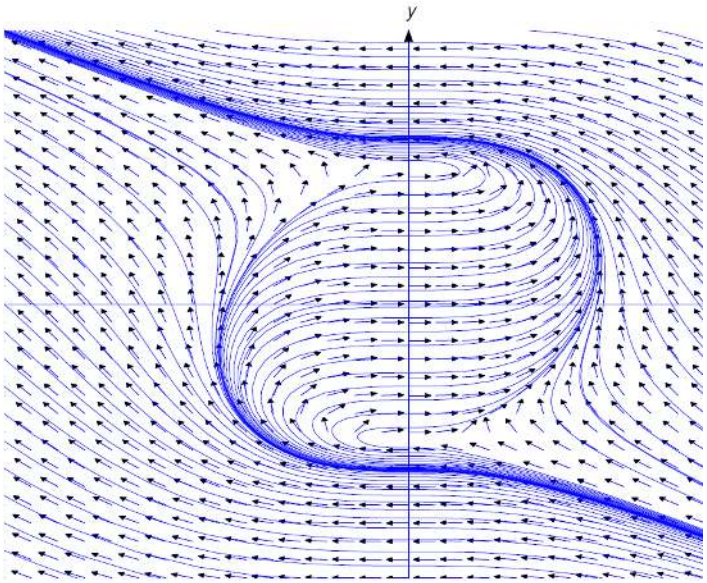


Figure 1.3.6 : A direction field and integral curves for $y'=\dfrac{x^2}{1-x^2-y^2}$.

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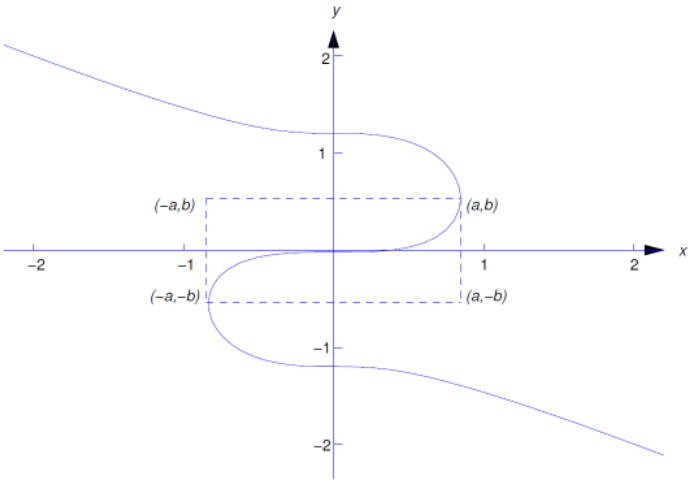


Figure 1.3.7 : A direction field and integral curves for $y' = \dfrac{x^2}{1 - x^2 - y^2}$

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1.3E: Direction Fields (Exercises)

*In Exercises **1–10** a direction field is drawn for the given equation. Sketch a solution for the IVP.*

1. $y' = \frac{x}{y}, \quad y(0) = 1$

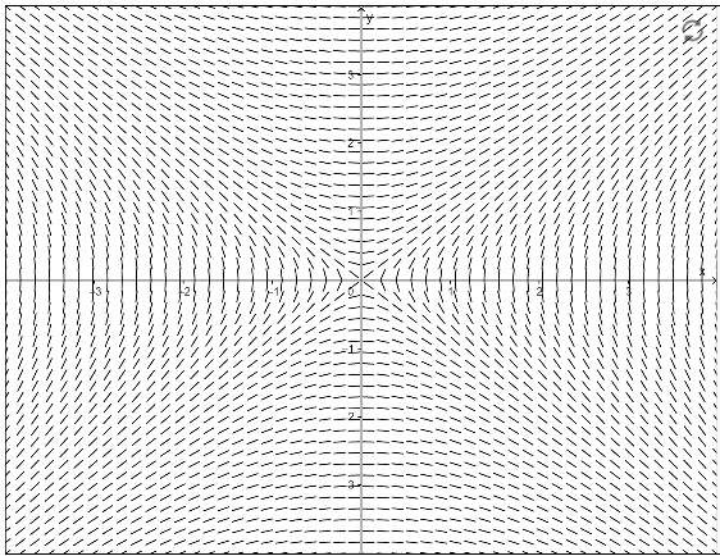


Figure 1.3E.1 : A direction field for $y' = \frac{x}{y}$

2. $y' = \frac{2xy^2}{1+x^2} \quad y(1) = -1$

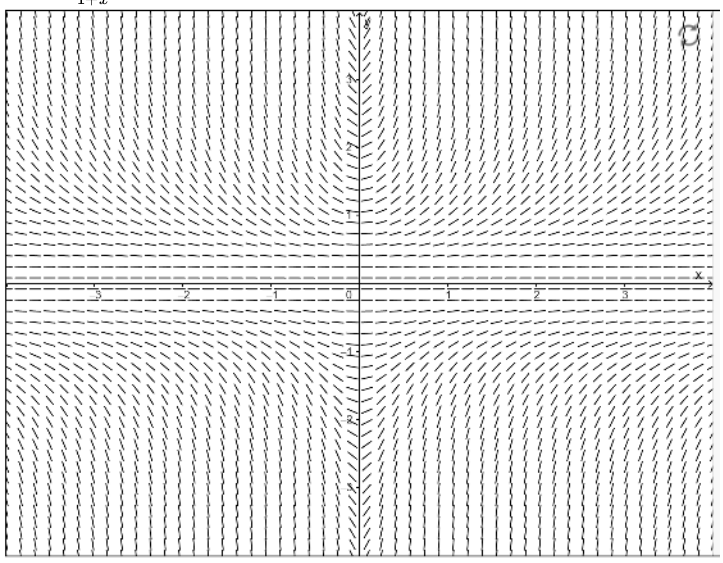


Figure 1.3E.2 : A direction field for $y' = \frac{2xy^2}{1+x^2}$

3. $y' = x^2(1+y^2) \quad y(0) = 0$

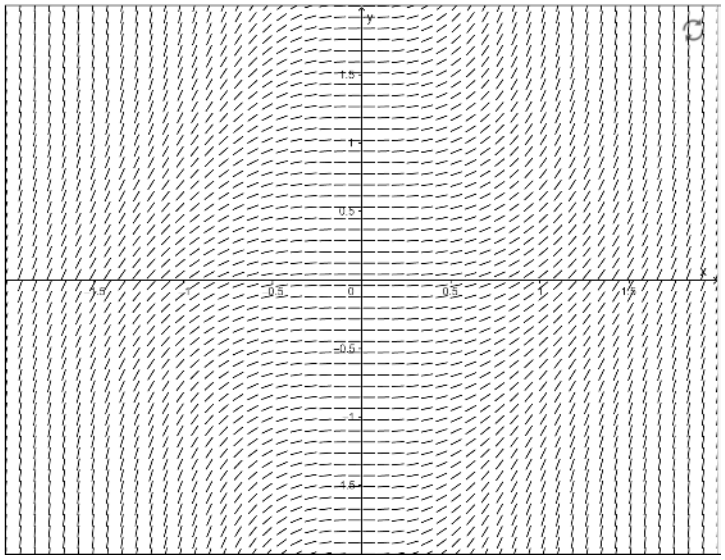


Figure 1.3E.3 : A direction field for $y' = x^2(1+y^2)$

4. $y' = \frac{1}{1+x^2+y^2} \quad y(0) = -1$



Figure 1.3E.4 : A direction field for $y' = \frac{1}{1+x^2+y^2}$

5. $y' = -(2xy^2 + y^3) \quad y(-1) = 1$

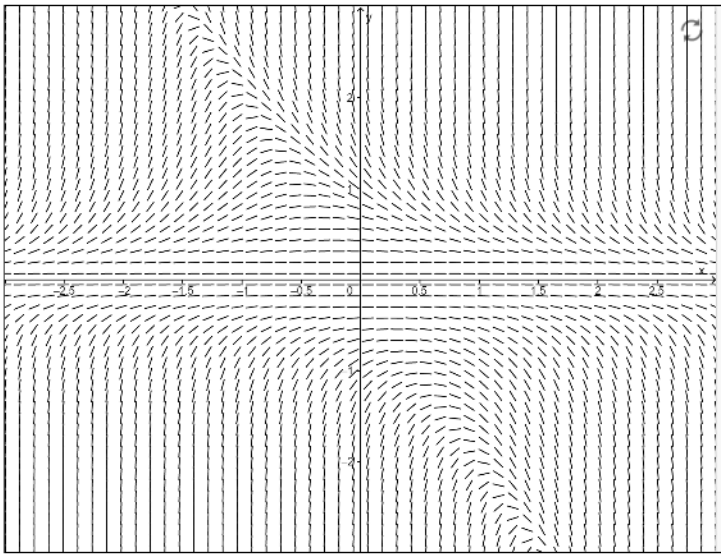


Figure 1.3E.5 : A direction field for $y' = -(2xy^2 + y^3)$

6. $y' = (x^2 + y^2)^{1/2} \quad y(0) = 0$

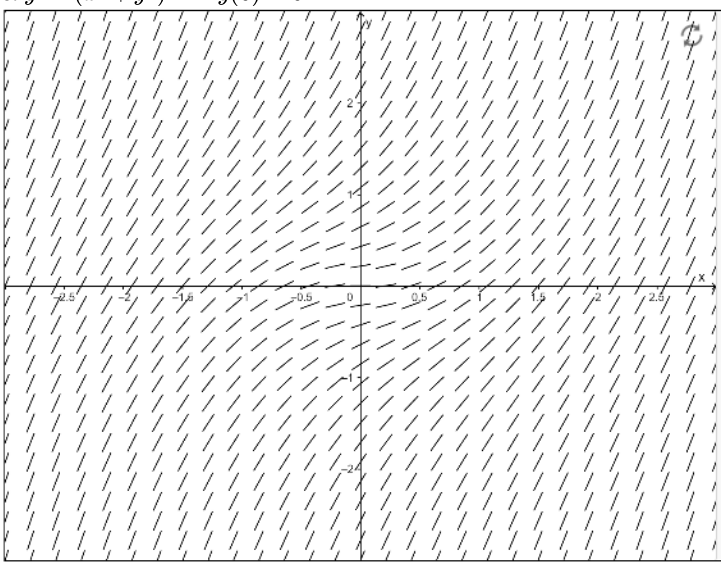


Figure 1.3E.6 : A direction field for $y' = (x^2 + y^2)^{1/2}$

7. $y' = \sin xy \quad y(1) = -1$

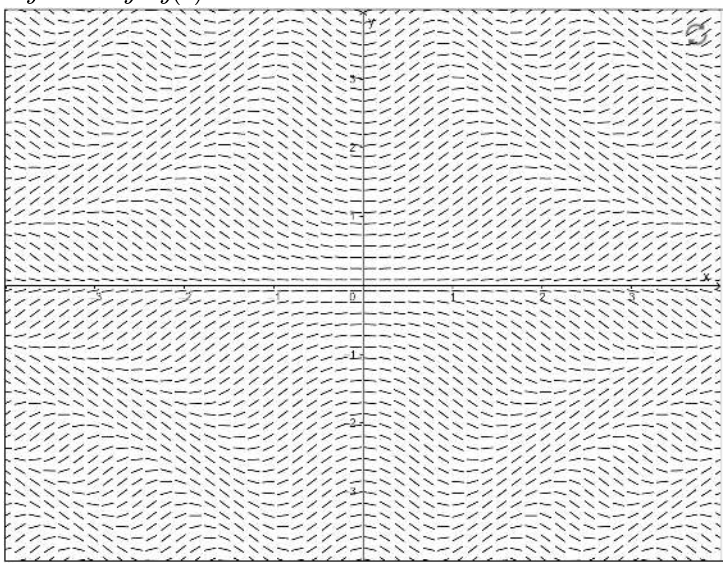


Figure 1.3E.7 : A direction field for $y' = \sin xy$

8. $y' = e^{xy} \quad y(0) = 1$

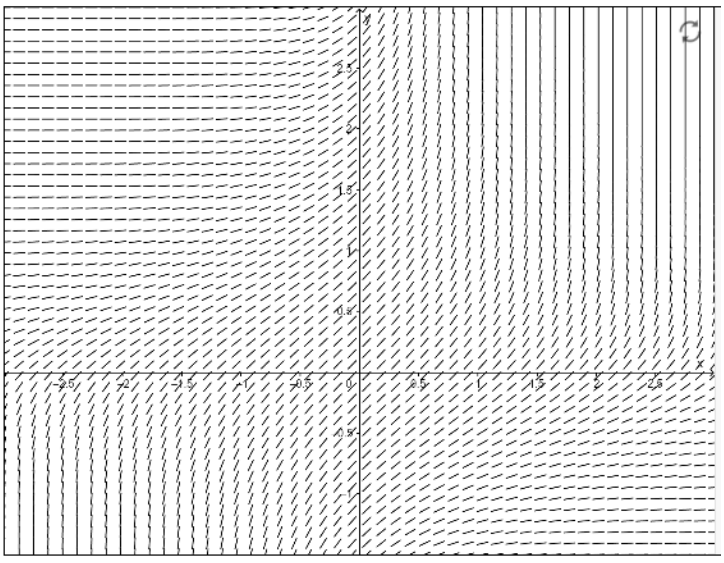


Figure 1.3E.8 : A direction field for $y' = e^{xy}$

9. $y' = x^3y^2 + xy^3 \quad y(0) = 0$

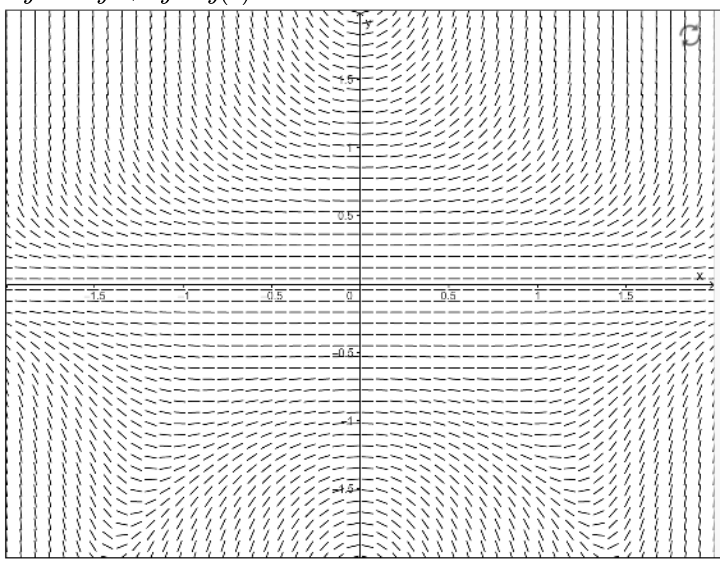


Figure 1.3E.9 : A direction field for $y' = x^3y^2 + xy^3$

10. $y' = \sin(x - 2y) \quad y(0) = 0$

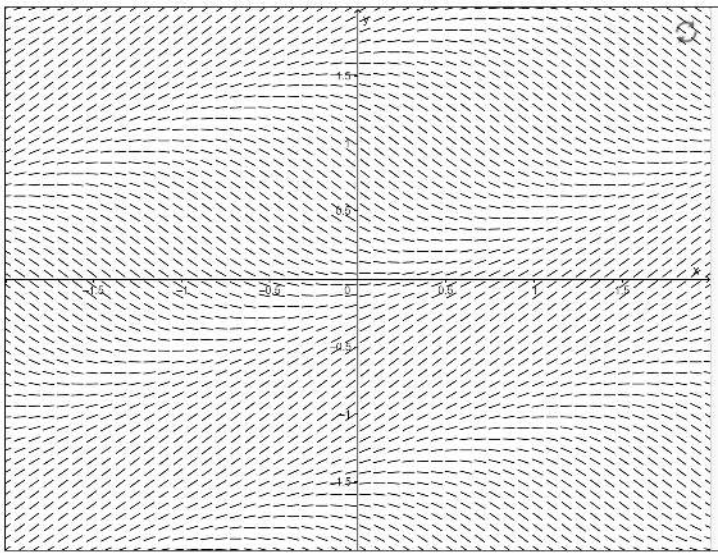


Figure 1.3E.10 : A direction field for $y' = \sin(x - 2y)$

In Exercises 11 - 20 use the slope field generator to construct a solution curve to the given IVP in the indicated rectangular region.

11. $y' = y(y-1), \quad y(1) = 1; \quad \{-1 \le x \le 2, \ -2 \le y \le 2\}$

12. $y' = 2 - 3xy, \quad y(0) = 0; \quad \{-2 \le x \le 2, \ -2 \le y \le 2\}$

13. $y' = xy(y-1), \quad y(0) = 1; \quad \{-2 \le x \le 2, \ -3 \le y \le 4\}$

14. $y' = 3x + y, \quad y(0) = 1; \quad \{-2 \le x \le 2, \ 0 \le y \le 4\}$

15. $y' = y - x^3, \quad y(1) = 1; \quad \{-2 \le x \le 2, \ -2 \le y \le 2\}$

16. $y' = 1 - x^2 - y^2, \quad y(0) = 0; \quad \{-2 \le x \le 2, \ -2 \le y \le 2\}$

17. $y' = x(y^2 - 1), \quad y(1) = 2; \quad \{-3 \le x \le 3, \ -3 \le y \le 3\}$

18. $y' = \frac{xy^2}{y-1}, \quad y(0) = 3; \quad \{-1 \le x \le 1, \ 1 \le y \le 4\}$

19. $y' = \sin y, \quad y(1) = 1; \quad \{-2 \le x \le 2, \ -2 \le y \le 2\}$

20. $y' = \frac{x^2+y^2}{1+x^2+y^2}, \quad y(1) = 0; \quad \{-2 \le x \le 2, \ -2 \le y \le 2\}$

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CHAPTER OVERVIEW

2: First Order Equations

In this chapter, we study first-order differential equations for which there are general methods of solution.

2.1: Separable Equations

2.1E: Separable Equations (Exercises)

2.2: Exact Equations

2.2E: Exact Equations (Exercises)

2.3: Linear First Order Equations

2.3E: Linear First Order Equations (Exercises)

2.4: Solving Differential Equations by Substitutions

2.4E: Solving Differential Equations by Substitution (Exercises)

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2.1: Separable Equations

Definition 2.1.1

A first order differential equation is *separable* if it *can* be written as

$$h(y)dy = g(x)dx, \tag{2.1.1}$$

Rewriting a separable differential equation in the form of Equation 2.1.1 is called *separation of variables.*

Solving this equation is fairly simple as we need only integrate both sides of the equation with respect to the appropriate variable. Integrating both sides of this equation and combining the constants of integration yields our solution.

Example 2.1.1

Solve the equation

$$\frac{dy}{dx} = x(1+y^2).$$

Solution

Separating variables yields

$$\frac{dy}{1+y^2} = xdx.$$

Integrating yields

$$\tan^{-1} y = \frac{x^2}{2} + c$$

Therefore

$$y = \tan\left(\frac{x^2}{2} + c\right).$$

Example 2.1.2

a. Solve the equation

$$y' = -\frac{x}{y}. \tag{2.1.2}$$

b. Solve the initial value problem

$$y' = -\frac{x}{y}, \quad y(1) = 1. \tag{2.1.3}$$

c. Solve the initial value problem

$$y' = -\frac{x}{y}, \quad y(1) = -2. \tag{2.1.4}$$

Solution a

Separating variables in Equation 2.1.2 yields

$$ydy = -xdx.$$

Integrating yields

$$\frac{y^2}{2} = -\frac{x^2}{2} + c, \quad \text{or equivalently} \quad x^2 + y^2 = 2c.$$

The last equation shows that c must be positive if y is to be a solution of Equation 2.1.2 on an open interval. Therefore we let $2c = a^2$ (with $a > 0$) and rewrite the last equation as

$$x^2 + y^2 = a^2.$$

This equation has two differentiable solutions for y in terms of x:

$$y = \sqrt{a^2 - x^2}, \quad -a < x < a, \tag{2.1.5}$$

and

$$y = -\sqrt{a^2 - x^2}, \quad -a < x < a. \tag{2.1.6}$$

The solution curves defined by Equation 2.1.5 are semicircles above the x-axis and those defined by Equation 2.1.6 are semicircles below the x-axis.

Solution b

The solution of Equation 2.1.3 is positive when $x = 1$; hence, it is of the form Equation 2.1.5. Substituting $x = 1$ and $y = 1$ into Equation 2.1.5 to satisfy the initial condition yields $a^2 = 2$; hence, the solution of Equation 2.1.3 is

$$y = \sqrt{2 - x^2}, \quad -\sqrt{2} < x < \sqrt{2}.$$

Solution c

The solution of Equation 2.1.4 is negative when $x = 1$ and is therefore of the form Equation 2.1.6. Substituting $x = 1$ and $y = -2$ into Equation 2.1.6 to satisfy the initial condition yields $a^2 = 5$. Hence, the solution of Equation 2.1.4 is

$$y = -\sqrt{5 - x^2}, \quad -\sqrt{5} < x < \sqrt{5}.$$

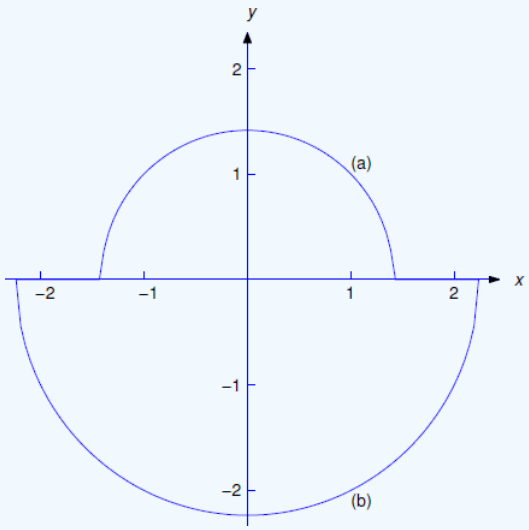


Figure 2.1.1 : **(b)** $y = \sqrt{2 - x^2}, -\sqrt{2} < x < \sqrt{2}$; **(c)** $y = -\sqrt{5 - x^2}, -\sqrt{5} < x < \sqrt{5}$.

Implicit Solutions of Separable Equations

In Examples 2.1.1 and 2.1.2 we were able to solve the equation for the dependent variable y to obtain explicit formulas for solutions of the given separable differential equations. As we'll see in the next example, this isn't always possible. In this situation we must broaden our definition of a solution of a separable equation. The next theorem provides the basis for this modification. We omit the proof, which requires a result from advanced calculus.

Theorem 2.1.1 Implicit Function Theorem

Suppose $g = g(x)$ is continuous on (a, b) and $h = h(y)$ is continuous on (c, d). Let G be an antiderivative of g on (a, b) and let H be an antiderivative of h on (c, d). Let x_0 be an arbitrary point in (a, b), let y_0 be a point in (c, d) such that $h(y_0) \neq 0$, and define

$$c = H(y_0) - G(x_0). \tag{2.1.7}$$

Then there's a function $y = y(x)$ defined on some open interval (a_1, b_1), where $a \le a_1 < x_0 < b_1 \le b$, such that $y(x_0) = y_0$ and

$$H(y) = G(x) + c \tag{2.1.8}$$

for $a_1 < x < b_1$. Therefore y is a solution of the initial value problem

$$h(y)dy = g(x)dx, \quad y(x_0) = y_0. \tag{2.1.9}$$

It's convenient to say that Equation 2.1.8 with c arbitrary is an *implicit solution* of $h(y)dy = g(x)dx$. Curves defined by Equation 2.1.8 are integral curves of $h(y)dy = g(x)dx$. If c satisfies Equation 2.1.7, we'll say that Equation 2.1.8 is an *implicit solution of the initial value problem* Equation 2.1.9. However, keep these points in mind:

- For some choices of c there may not be any differentiable functions y that satisfy Equation 2.1.8.
- The function y in Equation 2.1.8 (not Equation 2.1.8 itself) is a solution of $h(y)dy = g(x)dx$.

Example 2.1.3

a. Find implicit solutions of

$$y' = \frac{2x+1}{5y^4+1}. \tag{2.1.10}$$

b. Find an implicit solution of

$$y' = \frac{2x+1}{5y^4+1}, \quad y(2) = 1. \tag{2.1.11}$$

Solution a

Separating variables yields

$$(5y^4+1)dy = (2x+1)dx.$$

Integrating yields the implicit solution

$$y^5 + y = x^2 + x + c. \tag{2.1.12}$$

of Equation 2.1.10.

Solution b

Imposing the initial condition $y(2) = 1$ in Equation 2.1.12 yields $1+1 = 4+2+c$, so $c = -4$. Therefore

$$y^5 + y = x^2 + x - 4$$

is an implicit solution of the initial value problem Equation 2.1.11. Although more than one differentiable function $y = y(x)$ satisfies Equation 2.1.10 near $x = 1$, it can be shown that there is only one such function that satisfies the initial condition $y(1) = 2$. Figure 2.1.2 shows a direction field and some integral curves for Equation 2.1.10.

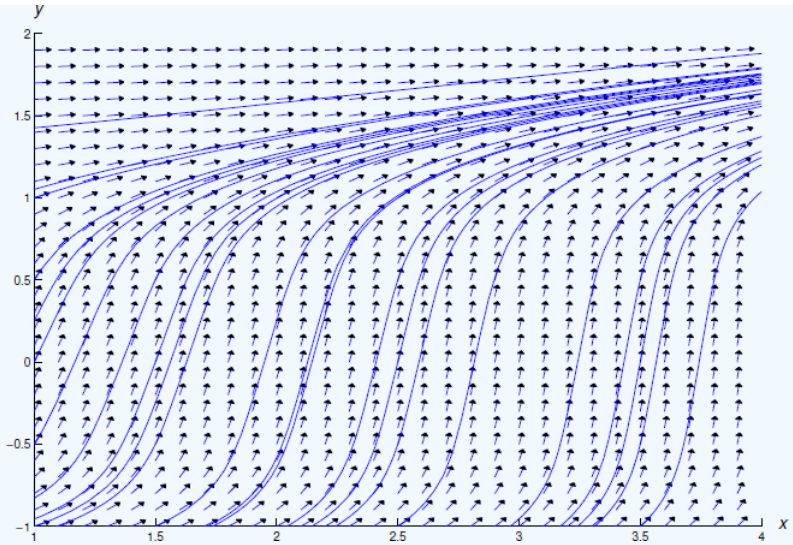


Figure 2.1.2 : A direction field and integral curves for $y' = \frac{2x+1}{5y^4+1}$

Singular Solutions of Separable Equations

An equation of the form

$$y' = g(x)p(y)$$

is separable, since it can be rewritten as

$$\frac{1}{p(y)}dy = g(x)dx.$$

However, the division by $p(y)$ is not legitimate if $p(y) = 0$ for some values of y. The next two examples show how to deal with this problem.

Example 2.1.4

Find all solutions of

$$y' = 2xy^2. \tag{2.1.13}$$

Solution

Here we must divide by $p(y) = y^2$ to separate variables. This isn't legitimate if y is a solution of Equation 2.1.13 that equals zero for some value of x. One such solution can be found by inspection: $y \equiv 0$. Now suppose y is a solution of Equation 2.1.13 that isn't identically zero. Since y is continuous there must be an interval on which y is never zero. Since division by y^2 is legitimate for x in this interval, we can separate variables in Equation 2.1.13 to obtain

$$\frac{dy}{y^2} = 2xdx.$$

Integrating this yields

$$-\frac{1}{y} = x^2 + c,$$

which is equivalent to

$$y = -\frac{1}{x^2 + c}. \tag{2.1.14}$$

We've now shown that if y is a solution of Equation 2.1.13 that is not identically zero, then y must be of the form Equation 2.1.14. By substituting Equation 2.1.14 into Equation 2.1.13, you can verify that Equation 2.1.14 is a solution of Equation

2.1.13. Thus, solutions of Equation 2.1.13 are $y \equiv 0$ and the functions of the form Equation 2.1.14. Note that the solution $y \equiv 0$ isn't of the form Equation 2.1.14 for any value of c. Because of this, $y \equiv 0$ is called a **singular solution** of 2.1.13 because it cannot be obtained from the **family of solutions** 2.1.14.

Figure 2.1.3 shows a direction field and some integral curves for Equation 2.1.13

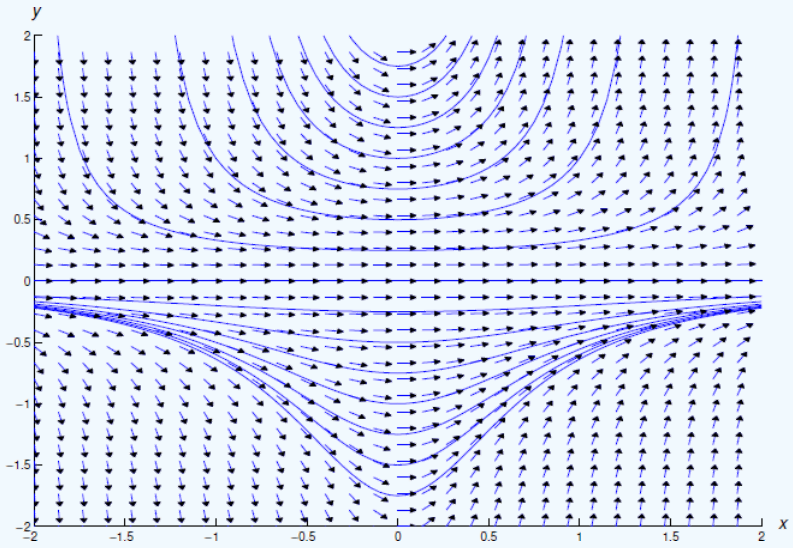


Figure 2.1.3 : A direction field and integral curves for $y' = 2xy^2$

Example 2.1.5

Find all solutions of

$$y' = \frac{1}{2}x(1-y^2). \tag{2.1.15}$$

Here we must divide by $p(y) = 1-y^2$ to separate variables. This isn't legitimate if y is a solution of Equation 2.1.15 that equals ± 1 for some value of x. Two such solutions can be found by inspection: $y \equiv 1$ and $y \equiv -1$. Now suppose y is a solution of Equation 2.1.15 such that $1-y^2$ isn't identically zero. Since $1-y^2$ is continuous there must be an interval on which $1-y^2$ is never zero. Since division by $1-y^2$ is legitimate for x in this interval, we can separate variables in Equation 2.1.15 to obtain

$$\frac{2dy}{y^2-1} = -x\,dx.$$

A partial fraction expansion on the left yields

$$\left[\frac{1}{y-1} - \frac{1}{y+1}\right] y' = -x,$$

and integrating yields

$$\ln\left|\frac{y-1}{y+1}\right| = -\frac{x^2}{2} + k;$$

hence,

$$\left|\frac{y-1}{y+1}\right| = e^k e^{-x^2/2}.$$

Since $y(x) \neq \pm 1$ for x on the interval under discussion, the quantity $(y-1)/(y+1)$ cannot change sign in this interval. Therefore we can rewrite the last equation as

$$\frac{y-1}{y+1} = ce^{-x^2/2},$$

where $c = \pm e^k$, depending upon the sign of $(y-1)/(y+1)$ on the interval. Solving for y yields

$$y = \frac{1 + ce^{-x^2/2}}{1 - ce^{-x^2/2}}. \tag{2.1.16}$$

We've now shown that if y is a solution of Equation 2.1.15 that is not identically equal to ± 1, then y must be as in Equation 2.1.16. By substituting Equation 2.1.16 into Equation 2.1.15 you can verify that Equation 2.1.16 is a solution of Equation 2.1.15. Thus, the solutions of Equation 2.1.15 are $y \equiv 1$, $y \equiv -1$ and the functions of the form Equation 2.1.16. Note that the constant solution $y \equiv 1$ can be obtained from this formula by taking $c = 0$; however, the other constant solution, $y \equiv -1$, cannot be obtained in this way. Thus, $y \equiv 1$ comes from the family of solutions and $y \equiv -1$ is a singular solution.

Figure 2.1.4 shows a direction field and some integrals for Equation 2.1.15.

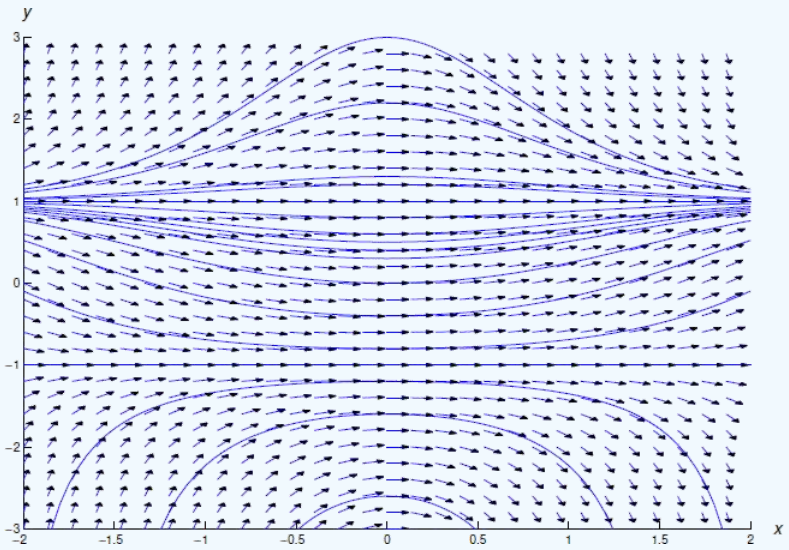


Figure 2.1.4 : A direction field and integral curves for $y' = \frac{x(1-y^2)}{2}$

Example 2.1.6

Solve the initial value problem

$$y' = 2xy^2, \quad y(0) = y_0$$

and determine the interval of validity of the solution.

Solution

First suppose $y_0 \neq 0$. From Example 2.1.4 , we know that y must be of the form

$$y = -\frac{1}{x^2 + c}. \tag{2.1.17}$$

Imposing the initial condition shows that $c = -1/y_0$. Substituting this into Equation 2.1.17 and rearranging terms yields the solution

$$y = \frac{y_0}{1 - y_0 x^2}.$$

This is also the solution if $y_0 = 0$. If $y_0 < 0$, the denominator isn't zero for any value of x, so the the solution is valid on $(-\infty, \infty)$. If $y_0 > 0$, the solution is valid only on $(-1/\sqrt{y_0}, 1/\sqrt{y_0})$ (recall that the interval must always include x_0; in this case 0).

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2.1E: Separable Equations (Exercises)

In Exercises **1-12**, find all solutions.

1. $y' = \frac{3x^2+2x+1}{y-2}$
2. $(\sin x)(\sin y) + (\cos y)\frac{dy}{dx} = 0$
3. $\frac{dy}{dx} = e^{2x+5y}$
4. $xy' + y^2 + y = 0$
5. $\frac{dy}{dx} = y - y^2$
6. $y' \ln y + x^2 y = 0$
7. $(3y^3 + 3y\cos y + 1)dy + \frac{(2x+1)y}{1+x^2}dx = 0$
8. $x^2 y y' = (y^2-1)^{3/2}$
9. $\frac{dy}{dx} = x^2(1+y^2)$
10. $(1+x^2)dy + xydx = 0$
11. $y' = (x-1)(y-1)(y-2)$
12. $(y-1)^2 y' = 2x+3$

In Exercises **13-19** solve the initial value problem.

13. $\frac{dy}{dx} = \frac{x^2+3x+2}{y-2}, \quad y(1) = 4$
14. $y' + x(y^2+y) = 0, \quad y(2) = 1$
15. $x^2\frac{dy}{dx} = y - xy, \quad y(-1) = -1$
16. $(3y^2+4y)dy + (2x+\cos x)dx = 0, \quad y(0) = 1$
17. $y' + \frac{(y+1)(y-1)(y-2)}{x+1} = 0, \quad y(1) = 0$
18. $y' + 2x(y+1) = 0, \quad y(0) = 2$
19. $\frac{dy}{dx} = 2xy(1+y^2), \quad y(0) = 1$

In Exercises **20-27** solve the initial value problem and give the interval of validity of the solution.

20. $(x^2+2)dy + 4x(y^2+2y+1)dx = 0, \quad y(1) = -1$
21. $y' = -2x(y^2-3y+2), \quad y(0) = 3$
22. $y' = \frac{2x}{1+2y}, \quad y(2) = 0$
23. $\frac{dy}{dx} = 2y - y^2, \quad y(0) = 1$
24. $x + yy' = 0, \quad y(3) = -4$
25. $y' + x^2(y+1)(y-2)^2 = 0, \quad y(4) = 2$
26. $(x+1)(x-2)dy + ydx = 0, \quad y(1) = -3$
27. $\frac{dy}{dx} = \frac{-x}{y}, \quad y(4) = -3$
28. Solve $y' = \frac{(1+y^2)}{(1+x^2)}$ explicitly.
29. Solve $\frac{dy}{dx}\sqrt{1-x^2} + \sqrt{1-y^2} = 0$ explicitly.
30. Solve $y' = \frac{\cos x}{\sin y}, \quad y(\pi) = \frac{\pi}{2}$ explicitly.

31. Solve $\frac{dy}{dx} = y^2 - y$ explicitly for the initial condition and give the interval of validity of the solution.
    a. $y(0) = 2$
    b. $y(0) = 0$
32. Solve $\frac{dy}{dx} = (y-1)^2$ for the initial condition
    a. $y(0) = 0$
    b. $y(0) = 1$
33. From Theorem 1.2.1, the initial value problem

$$y' = 3x(y-1)^{1/3}, \quad y(0) = 9$$

has a unique solution on an open interval that contains $x_0 = 0$. Find the solution and determine the largest open interval on which it is unique.

34. From Theorem 1.2.1, the initial value problem

$$y' = 3x(y-1)^{1/3}, \quad y(3) = -7$$

has a unique solution on some open interval that contains $x_0 = 3$. Find the solution and determine the largest open interval on which it is unique

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2.2: Exact Equations

In this section it is convenient to write first order differential equations in the form

$$M(x,y)\,dx + N(x,y)\,dy = 0. \tag{2.2.1}$$

This equation can be interpreted as

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0, \tag{2.2.2}$$

where x is the independent variable and y is the dependent variable, or as

$$M(x,y)\,\frac{dx}{dy} + N(x,y) = 0, \tag{2.2.3}$$

where y is the independent variable and x is the dependent variable. Since the solutions of Equation 2.2.2 and Equation 2.2.3 will often have to be left in implicit form we will say that $F(x,y) = c$ is an implicit solution of Equation 2.2.1 if every differentiable function $y = y(x)$ that satisfies $F(x,y) = c$ is a solution of Equation 2.2.2 and every differentiable function $x = x(y)$ that satisfies $F(x,y) = c$ is a solution of Equation 2.2.3

Here are some examples:

Table 2.2.1 : Examples of Exact Differential Equations in three forms

Equation 2.2.1	Equation 2.2.2	Equation 2.2.3
$3x^2y^2\,dx + 2x^3y\,dy = 0$	$3x^2y^2 + 2x^3y\,\frac{dy}{dx} = 0$	$3x^2y^2\,\frac{dx}{dy} + 2x^3y = 0$
$(x^2+y^2)\,dx + 2xy\,dy = 0$	$(x^2+y^2) + 2xy\,\frac{dy}{dx} = 0$	$(x^2+y^2)\,\frac{dx}{dy} + 2xy = 0$
$3y\sin x\,dx - 2xy\cos x\,dy = 0$	$3y\sin x - 2xy\cos x\,\frac{dy}{dx} = 0$	$3y\sin x\,\frac{dx}{dy} - 2xy\cos x = 0$

We will develop a method for solving Equation 2.2.1 under appropriate assumptions on M and N. This method is an extension of the method of separation of variables. Before stating it we consider an example.

Example 2.2.1

Show that

$$x^4y^3 + x^2y^5 + 2xy = c \tag{2.2.4}$$

is an implicit solution of

$$(4x^3y^3 + 2xy^5 + 2y)\,dx + (3x^4y^2 + 5x^2y^4 + 2x)\,dy = 0. \tag{2.2.5}$$

Solution

Regarding y as a function of x and differentiating Equation 2.2.4 implicitly with respect to x yields

$$(4x^3y^3 + 2xy^5 + 2y) + (3x^4y^2 + 5x^2y^4 + 2x)\,\frac{dy}{dx} = 0.$$

Similarly, regarding x as a function of y and differentiating Equation 2.2.4 implicitly with respect to y yields

$$(4x^3y^3 + 2xy^5 + 2y)\frac{dx}{dy} + (3x^4y^2 + 5x^2y^4 + 2x) = 0.$$

Therefore Equation 2.2.4 is an implicit solution of Equation 2.2.5 in either of these interpretations.

You may think Example 2.2.1 is pointless, since concocting a differential equation that has a given implicit solution is not particularly interesting. However, it illustrates the next important theorem, which we will prove by using implicit differentiation, as in Example 2.2.1 .

Theorem 2.2.1

If $F = F(x, y)$ has continuous partial derivatives F_x and F_y, then

$$F(x, y) = c \tag{2.2.6}$$

(with c as a constant) is an implicit solution of the differential equation

$$F_x(x, y)\, dx + F_y(x, y)\, dy = 0. \tag{2.2.7}$$

Proof:

Regarding y as a function of x and differentiating Equation 2.2.6implicitly with respect to x yields

$$F_x(x, y) + F_y(x, y)\, \frac{dy}{dx} = 0.$$

On the other hand, regarding x as a function of y and differentiating Equation 2.2.6implicitly with respect to y yields

$$F_x(x, y)\, \frac{dx}{dy} + F_y(x, y) = 0.$$

Thus, Equation 2.2.6is an implicit solution of Equation 2.2.7in either of its two possible interpretations.

Definition 2.2.1

We will say that the equation

$$M(x, y)\, dx + N(x, y)\, dy = 0 \tag{2.2.8}$$

is *exact* on an open rectangle R if there's a function $F = F(x, y)$ such that F_x and F_y are continuous, and

$$F_x(x, y) = M(x, y) \quad \text{and} \quad F_y(x, y) = N(x, y) \tag{2.2.9}$$

for all (x, y) in R.

This usage of "exact" is related to its usage in multivariable calculus, where the expression

$$F_x(x, y)\, dx + F_y(x, y)\, dy$$

(obtained by substituting Equation 2.2.9 into the left side of Equation 2.2.8) is the *exact (or total) differential of* F.

Example 2.2.1 shows that it is easy to solve Equation 2.2.8 if it is exact *and* we know a function F that satisfies Equation 2.2.9. The important questions are:

- Question 1. Given an equation Equation 2.2.8, how can we determine whether it is exact?
- Question 2. If Equation 2.2.8 is exact, how do we find a function F satisfying Equation 2.2.9?

To discover the answer to Question 1, assume that there's a function F that satisfies Equation 2.2.9 on some open rectangle R, and in addition that F has continuous mixed partial derivatives F_{xy} and F_{yx}. Then a theorem from multivariable calculus (Math 402) implies that

$$F_{xy} = F_{yx}. \tag{2.2.10}$$

If $F_x = M$ and $F_y = N$, differentiating the first of these equations with respect to y and the second with respect to x yields

$$F_{xy} = M_y \quad \text{and} \quad F_{yx} = N_x. \tag{2.2.11}$$

From Equation 2.2.10 and Equation 2.2.11, we conclude that a necessary condition for exactness is that $M_y = N_x$. This motivates the next theorem, which we state without proof.

Theorem 2.2.2 : The Exactness Condition

Suppose M and N are continuous and have continuous partial derivatives M_y and N_x on an open rectangle R. Then

$$M(x,y)\,dx+N(x,y)\,dy=0$$

is exact on R if and only if

$$M_y(x,y)=N_x(x,y) \tag{2.2.12}$$

for all (x,y) in R..

Note that a separable equation can be written as Equation 2.2.1 as

$$M(x)\,dx+N(y)\,dy=0,$$

so separable equations can be thought of as a special case of an exact equation.

To help you remember the exactness condition, observe that the coefficients of dx and dy are differentiated in Equation 2.2.12 with respect to the "opposite" variables; that is, the coefficient of dx is differentiated with respect to y, while the coefficient of dy is differentiated with respect to x.

Example 2.2.2

Show that the equation

$$3x^2y\,dx+4x^3\,dy=0$$

is not exact on any open rectangle.

Solution

Here

$$M(x,y)=3x^2y \quad \text{and} \quad N(x,y)=4x^3$$

so

$$M_y(x,y)=3x^2 \quad \text{and} \quad N_x(x,y)=12x^2.$$

Therefore $M_y=N_x$ on the line $x=0$, but not on any open rectangle, so there's no function F such that $F_x(x,y)=M(x,y)$ and $F_y(x,y)=N(x,y)$ for all (x,y) on any open rectangle.

The next example illustrates two possible methods for finding a function F that satisfies the condition $F_x=M$ and $F_y=N$ if $M\,dx+N\,dy=0$ is exact.

Example 2.2.3

Solve

$$(4x^3y^3+3x^2)\,dx+(3x^4y^2+6y^2)\,dy=0. \tag{2.2.13}$$

Solution (Method 1)

Here

$$M(x,y)=4x^3y^3+3x^2, \quad N(x,y)=3x^4y^2+6y^2,$$

and

$$M_y(x,y)=N_x(x,y)=12x^3y^2$$

for all (x,y). Therefore Theorem 2.2.2 implies that there's a function F such that

$$F_x(x,y)=M(x,y)=4x^3y^3+3x^2 \tag{2.2.14}$$

and

$$F_y(x,y)=N(x,y)=3x^4y^2+6y^2 \tag{2.2.15}$$

for all (x, y). To find F, we integrate Equation 2.2.14 with respect to x to obtain

$$F(x, y) = x^4y^3 + x^3 + \phi(y), \tag{2.2.16}$$

where $\phi(y)$ is the "constant" of integration. (Here ϕ is "constant" in that it is independent of x, the variable of integration.) If ϕ is any differentiable function of y then F satisfies Equation 2.2.14. To determine ϕ so that F also satisfies Equation 2.2.15, assume that ϕ is differentiable and differentiate F with respect to y. This yields

$$F_y(x, y) = 3x^4y^2 + \phi'(y).$$

Comparing this with Equation 2.2.15 shows that

$$\phi'(y) = 6y^2.$$

We integrate this with respect to y and take the constant of integration to be zero because we are interested only in finding *some* F that satisfies Equation 2.2.14 and Equation 2.2.15. This yields

$$\phi(y) = 2y^3.$$

Substituting this into Equation 2.2.16 yields

$$F(x, y) = x^4y^3 + x^3 + 2y^3. \tag{2.2.17}$$

Now Theorem 2.2.1 implies that

$$x^4y^3 + x^3 + 2y^3 = c$$

is an implicit solution of Equation 2.2.13. Solving this for y yields the explicit solution

$$y = \left(\frac{c - x^3}{2 + x^4}\right)^{1/3}.$$

Solution (Method 2)

Instead of first integrating Equation 2.2.14 with respect to x, we could begin by integrating Equation 2.2.15 with respect to y to obtain

$$F(x, y) = x^4y^3 + 2y^3 + \psi(x), \tag{2.2.18}$$

where ψ is an arbitrary function of x. To determine ψ, we assume that ψ is differentiable and differentiate F with respect to x, which yields

$$F_x(x, y) = 4x^3y^3 + \psi'(x).$$

Comparing this with Equation 2.2.14 shows that

$$\psi'(x) = 3x^2.$$

Integrating this and again taking the constant of integration to be zero yields

$$\psi(x) = x^3.$$

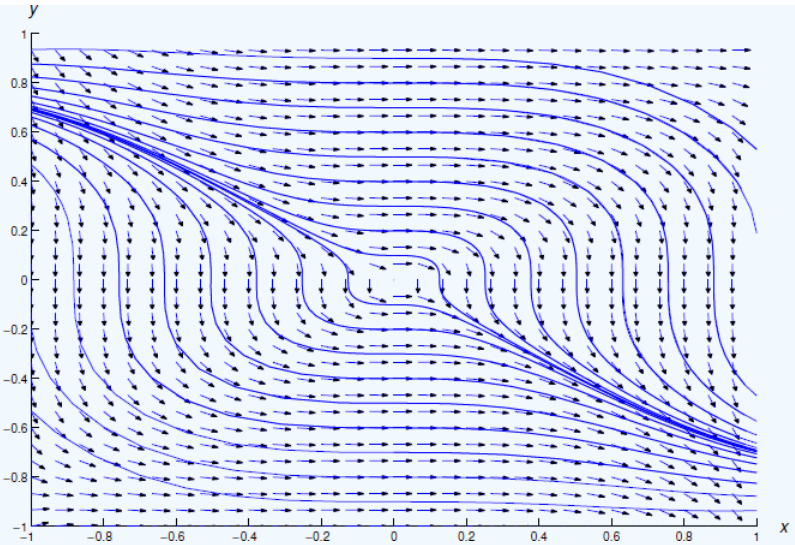


Figure 2.2.1 : A direction field and integral curves for $(4x^3y^3+3x^2)\,dx+(3x^4y^2+6y^2)\,dy=0$

Substituting this into Equation 2.2.18 yields Equation 2.2.17.

Figure 2.2.1 shows a direction field and some integral curves of Equation 2.2.13.

Many equations can be conveniently solved by either of the two methods used in Example 2.2.3 . However, sometimes the integration required in one approach is more difficult than in the other. In such cases we choose the approach that requires the easier integration.

Example 2.2.4

Solve the equation

$$\left(ye^{xy}\tan x+e^{xy}\sec^2 x\right)dx+xe^{xy}\tan x\,dy=0 \tag{2.2.19}$$

Solution

We leave it to you to check that $M_y=N_x$ on any open rectangle where $\tan x$ and $\sec x$ are defined. Here we must find a function F such that

$$F_x(x,y)=ye^{xy}\tan x+e^{xy}\sec^2 x \tag{2.2.20}$$

and

$$F_y(x,y)=xe^{xy}\tan x. \tag{2.2.21}$$

It's difficult to integrate Equation 2.2.20 with respect to x, but easy to integrate Equation 2.2.21 with respect to y. This yields

$$F(x,y)=e^{xy}\tan x+\psi(x). \tag{2.2.22}$$

Differentiating this with respect to x yields

$$F_x(x,y)=ye^{xy}\tan x+e^{xy}\sec^2 x+\psi'(x).$$

Comparing this with Equation 2.2.20 shows that $\psi'(x)=0$. Hence, ψ is a constant, which we can take to be zero in Equation 2.2.22, and

$$e^{xy}\tan x=c,$$

is an implicit solution of Equation 2.2.19.

Attempting to apply our procedure to a differential equation that is not exact will lead to failure. Example 2.2.5 illustrates this.

Example 2.2.5

Verify that the equation

$$3x^2y^2\,dx+6x^3y\,dy=0 \tag{2.2.23}$$

is not exact, and show that the procedure for solving exact equations fails when applied to Equation 2.2.23.

Solution

Here

$$M_y(x,y)=6x^2y \quad \text{and} \quad N_x(x,y)=18x^2y,$$

so Equation 2.2.23 is not exact. Nevertheless, let us try to find a function F such that

$$F_x(x,y)=3x^2y^2 \tag{2.2.24}$$

and

$$F_y(x,y)=6x^3y. \tag{2.2.25}$$

Integrating Equation 2.2.24 with respect to x yields

$$F(x,y)=x^3y^2+\phi(y),$$

and differentiating this with respect to y yields

$$F_y(x,y)=2x^3y+\phi'(y).$$

For this equation to be consistent with Equation 2.2.25,

$$6x^3y=2x^3y+\phi'(y),$$

or

$$\phi'(y)=4x^3y.$$

This is a contradiction, since ϕ' must be a function involving only y. Therefore the procedure fails.

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2.2E: Exact Equations (Exercises)

In Exercises **1-19** determine which equations are exact and, if they are exact, solve them.

1. $6x^2y^2\,dx+4x^3y\,dy=0$
2. $(3y\cos x+4xe^x+2x^2e^x)+(3\sin x+3)\,\frac{dy}{dx}=0$
3. $14x^2y^3\,dx+21x^2y^2\,dy=0$
4. $(2x-2y^2)\,dx=(4xy-12y^2)\,dy$
5. $(x+y)^2\,dx+(x+y)^2\,dy=0$
6. $(4x+7y)\,+(3x+4y)\,\frac{dy}{dx}=0$
7. $(2y\sin x\cos x-y+2y^2e^{xy^2})dx=(x-\sin^2 x-4xye^{xy^2})dy$
8. $(4y\cos x+9xy^2)\,dy=(2y^2\sin x-3y^3+2x)\,dx$
9. $(2x+y)\,dx+(2y+2x)\,dy=0$
10. $(3x^2+2xy+4y^2)\,dx+(x^2+8xy+18y)\,dy=0$
11. $(2x^2+8xy+y^2)\,dx+(2x^2+xy^3/3)\,dy=0$
12. $\left(\frac{1}{x}+2x\right)\,\frac{dx}{dy}+\frac{1}{y}+2y=0$
13. $(y\sin xy+xy^2\cos xy)\,dx+(x\sin xy+xy^2\cos xy)\,dy=0$
14. $\frac{x\,dx}{(x^2+y^2)^{3/2}}+\frac{y\,dy}{(x^2+y^2)^{3/2}}=0$
15. $\left(e^x(x^2y^2+2xy^2)+6x\right)\,dx+(2x^2ye^x+2)\,dy=0$
16. $\left(x^2e^{x^2+y}(2x^2+3)+4x\right)=(12y^2-x^3e^{x^2+y})\,\frac{dy}{dx}$
17. $\left(e^{xy}(x^4y+4x^3)+3y\right)\,dx+(x^5e^{xy}+3x)\,dy=0$
18. $(8y-x^4\sin xy)\,dy=(x^3y\sin xy-3x^2\cos xy-4x)\,dx$
19. $(1+\ln x+\frac{y}{x})dx=(1-\ln x)dy$

In Exercises **20-25** solve the initial value problem.

20. $(4x^3y^2-6x^2y-2x-3)+(2x^4y-2x^3)\,\frac{dy}{dx}=0,\quad y(1)=3$
21. $(-4y\cos x+4\sin x\cos x+\sec^2 x)\,dx=(4\sin x-4y)\,dy,\quad y(\pi/4)=0$
22. $(y^3-1)e^x\,dx+3y^2(e^x+1)\,dy=0,\quad y(0)=0$
23. $\cos x\,dy=(y\sin x+2\cos x-\sin x)\,dx\quad y(0)=1$
24. $(2x-1)(y-1)\,dx+(x+2)(x-3)\,dy=0,\quad y(1)=-1$
25. $\frac{dy}{dx}=\frac{xy^2-\cos x\sin x}{y-yx^2},\quad y(0)=2$

26. Find all functions M such that the equation is exact.
    a. $M(x,y)\,dx+(x^2-y^2)\,dy=0$
    b. $M(x,y)\,dx+2xy\sin x\cos y\,dy=0$
    c. $M(x,y)\,dx+(e^x-e^y\sin x)\,dy=0$

27. Find all functions N such that the equation is exact.
    a. $(x^3y^2+2xy+3y^2)\,dx+N(x,y)\,dy=0$
    b. $(\ln xy+2y\sin x)\,dx+N(x,y)\,dy=0$
    c. $(x\sin x+y\sin y)\,dx+N(x,y)\,dy=0$

28. Prove: If the equations $M_1\,dx+N_1\,dy=0$ and $M_2\,dx+N_2\,dy=0$ are exact on an open rectangle R, so is the equation

$$(M_1 + M_2)\,dx + (N_1 + N_2)\,dy = 0.$$

29. Find conditions on the constants A, B, C, and D such that the equation

$$(Ax + By)\,dx + (Cx + Dy)\,dy = 0$$

is exact.

30. Find conditions on the constants A, B, C, D, E, and F such that the equation

$$(Ax^2 + Bxy + Cy^2)\,dx + (Dx^2 + Exy + Fy^2)\,dy = 0$$

is exact.

31. Rewrite the separable equation

$$h(y)y' = g(x) \tag{A}$$

as an exact equation

$$M(x, y)\,dx + N(x, y)\,dy = 0. \tag{B}$$

Show that applying the method of this section to (B) yields the same solutions that would be obtained by applying the method of separation of variables to (A)

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2.3: Linear First Order Equations

Definition 2.3.1

A first order differential equation is said to be *linear* if it **can** be written in the form

$$y' + p(x)y = f(x). \tag{2.3.1}$$

A first order differential equation that cannot be written like this is *nonlinear*.

Example 2.3.1

The first order equations

$$\begin{aligned} x^2y' + 3y &= x^2 \\ xy' - 8x^2y &= \sin x \\ xy' + (\ln x)y &= 0 \\ y' &= x^2y - 2 \end{aligned}$$

are not in the form in Equation 2.3.1, but they are linear, since they can be rewritten as

$$\begin{aligned} y' + \frac{3}{x^2}y &= 1 \\ y' - 8xy &= \frac{\sin x}{x} \\ y' + \frac{\ln x}{x}y &= 0 \\ y' - x^2y &= -2 \end{aligned}$$

Example 2.3.2

Here are some nonlinear first order equations:

$$\begin{aligned} xy' + 3y^2 &= 2x && \text{(because } y \text{ is squared)} \\ yy' &= 3 && \text{(because of the product } yy') \\ y' + xe^y &= 12 && \text{(because of } e^y) \end{aligned}$$

In the following theorem we discuss how to solve linear differential equations.

Theorem 2.3.1

If p is continuous on (a, b), then multiplying the differential equation

$$y' + p(x)y = 0 \tag{2.3.2}$$

on (a, b) by

$$e^{\int p(x)\,dx}$$

will make the differential equation **exact**.

Note: we call $e^{\int p(x)\,dx}$ an *integrating factor* of the differential equation.

Proof

Consider the first order linear equation

$$y' + p(x)y = 0$$

Rewriting this in the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

we get

$$p(x)y\,dx + dy = 0$$

If we multiply this by $e^{\int p(x)\,dx}$ we get

$$e^{\int p(x)\,dx}p(x)y\,dx + e^{\int p(x)\,dx}\,dy = 0,$$

Note that this is an exact differential equation (verify this) resulting in the following:

$$d[e^{\int p(x)\,dx}y] = 0,$$

Integrating both sides leads to the solution:

$$e^{\int p(x)\,dx}y = c,$$

or

$$y = ce^{-\int p(x)\,dx}$$

Note that in this special case the solution is immediately found from the formula given in the last line of the proof.

Example 2.3.3

a. Find the general solution of

$$xy' + y = 0. \tag{2.3.3}$$

b. Solve the initial value problem

$$xy' + y = 0, \quad y(1) = 3. \tag{2.3.4}$$

Solution a

We rewrite Equation 2.3.3 as

$$y' + \frac{1}{x}y = 0, \tag{2.3.5}$$

where x is restricted to either $(-\infty, 0)$ or $(0, \infty)$. If y is a nontrivial solution of Equation 2.3.5, there must be some open interval I on which y has no zeros. From the last line of the proof, we can readily find the solution:

$$y = ce^{-\int \frac{1}{x}\,dx}$$

which, after integrating and simplifying, leads to the final answer:

$$y = \frac{c}{x} \tag{2.3.6}$$

We have now shown that every solution of Equation 2.3.5 is given by Equation 2.3.6 for some choice of c. (Even though we assumed that y was nontrivial to derive Equation 2.3.6, we can get the trivial solution by setting $c = 0$ in Equation 2.3.6.) Conversely, any function of the form Equation 2.3.6 is a solution of Equation 2.3.5, since differentiating Equation 2.3.6 yields

$$y' = -\frac{c}{x^2},$$

and substituting this and Equation 2.3.6 into Equation 2.3.5 yields

$$y' + \frac{1}{x}y = -\frac{c}{x^2} + \frac{1}{x}\frac{c}{x}$$
$$= -\frac{c}{x^2} + \frac{c}{x^2} = 0.$$

Figure 2.3.2 shows the graphs of some solutions corresponding to various values of c

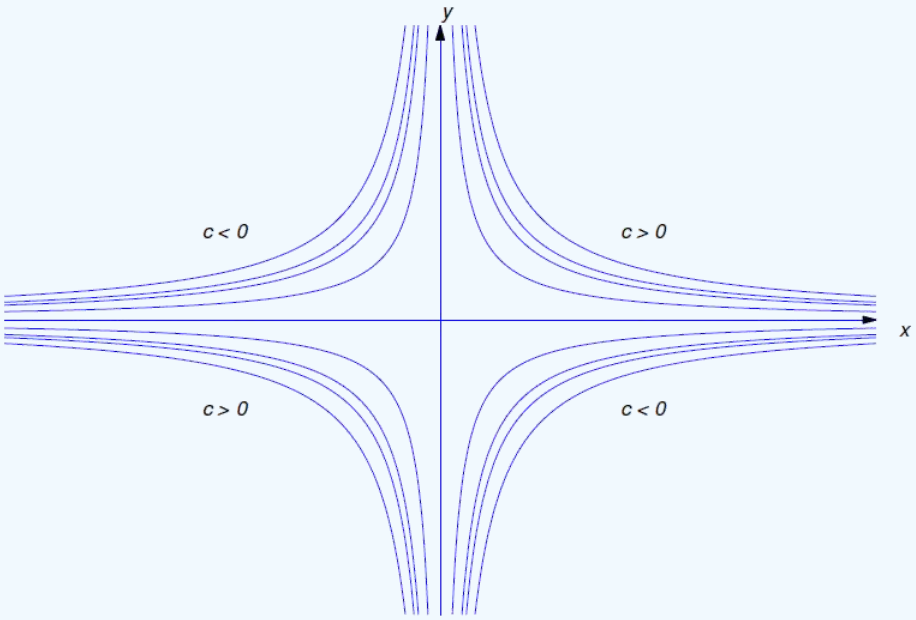


Figure 2.3.2 : Solutions of $xy' + y = 0$ on $(0, \infty)$ and $(-\infty, 0)$

Solution b

Imposing the initial condition $y(1) = 3$ in Equation 2.3.6 yields $c = 3$. Therefore the solution of Equation 2.3.4 is

$$y = \frac{3}{x}.$$

The interval of validity of this solution is $(0, \infty)$.

General Solution of a Linear First Order Equation

Consider the first order linear equation

$$y' + p(x)y = f(x); \tag{2.3.7}$$

Writing this in the form

$$y' = f(x) - p(x)y; \tag{2.3.8}$$

note that, if p and f are continuous on some open interval (a, b) then (by theorem 1.2.1) there exists a unique solution of the form $y = y(x, c)$ that involves x and a parameter c and has the these properties:

- For each fixed value of c, the resulting function of x is a solution of Equation 2.3.7 on (a, b).
- If y is a solution of Equation 2.3.7 on (a, b), then y can be obtained from the formula by choosing c appropriately.

$y = y(x, c)$ is the *general solution* of Equation 2.3.7.

When this has been established, it will follow that an equation of the form

$$P_0(x)y' + P_1(x)y = F(x) \tag{2.3.9}$$

has a general solution on any open interval (a, b) on which P_0, P_1, and F are all continuous and P_0 has no zeros, since in this case we can rewrite Equation 2.3.9 in the form Equation 2.3.7 with $p = P_1/P_0$ and $f = F/P_0$, which are both continuous on (a, b).

To avoid awkward wording in examples and exercises, we will not specify the interval (a, b) when we ask for the general solution of a specific linear first order equation. Let's agree that this always means that we want the general solution on every open interval on which p and f are continuous if the equation is of the form Equation 2.3.7, or on which P_0, P_1, and F are continuous and P_0 has no zeros, if the equation is of the form Equation 2.3.9. We leave it to you to identify these intervals in specific examples and exercises.

For completeness, we point out that if P_0, P_1, and F are all continuous on an open interval (a, b), but P_0 *does* have a zero in (a, b), then Equation 2.3.9 may fail to have a general solution on (a, b) in the sense just defined. Since this isn't a major point that needs to be developed in depth, we will not discuss it further.

The method for solving

$$y' + p(x)y = f(x) \tag{2.3.10}$$

is largely the same in regards to the left hand side of the equation as when $f(x) \equiv 0$, but we do handle it in a more efficient manner, as the next example illustrates.

Example 2.3.4

Find the general solution of

$$y' + 2y = x^3 e^{-2x}. \tag{2.3.11}$$

Solution

We first find the integrating factor:

$$y = e^{\int 2\, dx} = e^{2x}$$

Now multiplying both sides of 2.3.11 by the integrating factor we get

$$e^{2x}y' + 2e^{2x}y = x^3.$$

Now the left hand side of the equation is exact as we saw when $f(x) \equiv 0$, but because this isn't the case here we can take a different and better approach. Note that we can write the above as

$$\frac{d}{dx}[e^{2x}y] = x^3.$$

Note: the left hand side becomes the derivative of the integrating factor times the dependent variable, which, as we saw earlier, will always be the case. Rewriting this we have

$$d[e^{2x}y] = x^3 dx.$$

Integrating both sides gives us

$$e^{2x}y = \frac{x^4}{4} + c.$$

Solving for y we get the general solution of the differential Equation 2.3.11

$$y = \frac{x^4}{4}e^{-2x} + ce^{-2x}.$$

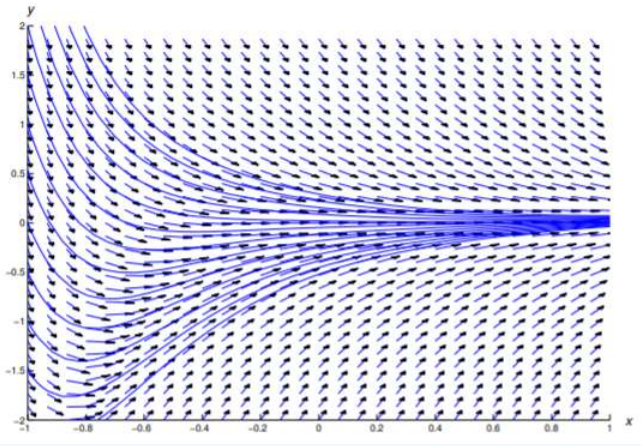


Figure 2.3.3 : A direction field and integral curves for $y' + 2y = x^2 e^{-2x}$

Figure 2.3.3 shows a direction field and some integral curves for Equation 2.3.11.

Example 2.3.5

a. Find the general solution

$$(\sin x)y' + (\cos x)y = x. \tag{2.3.12}$$

b. Solve the initial value problem

$$(\sin x)y' + (\cos x)y = x, \quad y(\pi/2) = 1. \tag{2.3.13}$$

Solution a

We first rewrite Equation 2.3.12 in standard linear form

$$y' + (\cot x)y = x(\csc x)$$

Here $p(x) = \cot x$ and $f(x) = x \csc x$ are both continuous except at the points $x = r\pi$, where r is an integer. Therefore we seek solutions of Equation 2.3.12 on the intervals $(r\pi, (r+1)\pi)$. We first find the integrating factor

$$y = e^{\int \cot x \, dx} = \sin x$$

and multiplying both sides by this we get

$$(\sin x)y' + (\cos x)y = x$$

which we know we can rewrite as

$$\frac{d}{dx}[(\sin x)y] = x$$

and then as

$$d[(\sin x)y] = x dx.$$

Integrating both sides gives us

$$(\sin x)y = \frac{x^2}{2} + c.$$

Solving for y we get the general solution of the differential Equation 2.3.12

$$y = \frac{x^2}{2} \csc x + c \csc x.$$

Solution b

Imposing the initial condition $y(\pi/2) = 1$ yields

$$1 = \frac{\pi^2}{8} + c \quad \text{or} \quad c = 1 - \frac{\pi^2}{8}.$$

Thus,

$$y = \frac{x^2}{2}\csc x + (1 - \pi^2/8)\csc x$$

is a solution of Equation 2.3.13. The interval of validity of this solution is $(0, \pi)$.

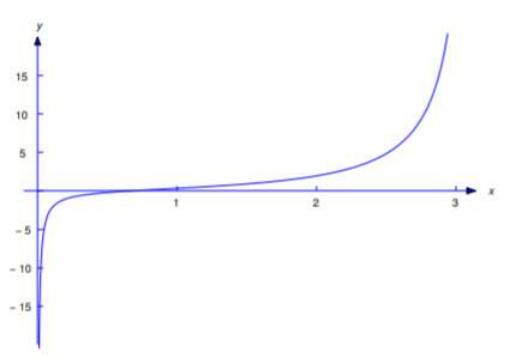


Figure 2.3.4 : Solution of $(\sin x)y' + (\cos x)y = x, \quad y(\pi/2) = 1$

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2.3E: Linear First Order Equations (Exercises)

In Exercises **1-20** find the general solution.

1. $y' + ay = 0$ (a=constant)
2. $\frac{dx}{dt} + x = e^{3t}$
3. $dy = (x^3 - 2xy)dx$
4. $\cos x \frac{dy}{dx} + (\sin x)y = 1$
5. $x^2y' + xy = 1$
6. $\frac{dr}{d\theta} + r\sec\theta = \cos\theta$
7. $y' + \left(\frac{1}{x} - 1\right)y = -\frac{2}{x}$
8. $y' + 2xy = xe^{-x^2}$
9. $\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{e^{-x}}{1+x^2}$
10. $y' + \frac{1}{x}y = \frac{7}{x^2} + 3$
11. $ydx - 4(x + y^6)dy = 0$
12. $\frac{dy}{dx} + \frac{4}{x-1}y = \frac{1}{(x-1)^5} + \frac{\sin x}{(x-1)^4}$
13. $xy' + (1 + 2x^2)y = x^3e^{-x^2}$
14. $xy' + 2y = \frac{2}{x^2} + 1$
15. $\frac{dp}{dt} + (\tan t)p = \cos t$
16. $(1 + x)y' + 2y = \frac{\sin x}{1+x}$
17. $(x - 2)(x - 1)y' - (4x - 3)y = (x - 2)^3$
18. $y^2dx = (e^y - xy^2 - 2xy)dy$
19. $\frac{dy}{dx} + (2\sin x\cos x)y = e^{-\sin^2 x}$
20. $x^2y' + 3xy = e^x$

In Exercises **21-40** solve the initial value problem.

21. $y' + \left(\frac{1+x}{x}\right)y = 0, \quad y(1) = 1$
22. $x\frac{dy}{dx} + \left(1 + \frac{1}{\ln x}\right)y = 0, \quad y(e) = 1$
23. $xdy + (1 + x\cot x)ydx = 0, \quad y\left(\frac{\pi}{2}\right) = 2$
24. $y' - \left(\frac{2x}{1+x^2}\right)y = 0, \quad y(0) = 2$
25. $\frac{dx}{dt} + \frac{k}{t}x = 0, \quad x(1) = 3 \quad (k = \text{constant})$
26. $y' + (\tan kx)y = 0, \quad y(0) = 2 \quad (k = \text{constant})$
27. $y' + 7y = e^{3x}, \quad y(0) = 0$
28. $(1 + x^2)dy = (\frac{2}{1+x^2} - 4xy)dx, \quad y(0) = 1$
29. $xy' + 3y = \frac{2}{x(1+x^2)}, \quad y(-1) = 0$
30. $\frac{dy}{dx} + (\cot x)y = \cos x, \quad y\left(\frac{\pi}{2}\right) = 1$
31. $y' + \frac{1}{x}y = \frac{2}{x^2} + 1, \quad y(-1) = 0$

32. $(x-1)y'+3y=\frac{1}{(x-1)^3}+\frac{\sin x}{(x-1)^2}, \quad y(0)=1$

33. $ydx=(8y^2-2x)dy, \quad y(3)=1$

34. $xy'-2y=-x^2, \quad y(1)=1$

35. $y'+2xy=x, \quad y(0)=3$

36. $(x-1)y'+3y=\frac{1+(x-1)\sec^2 x}{(x-1)^3}, \quad y(0)=-1$

37. $(x+2)y'+4y=\frac{1+2x^2}{x(x+2)^3}, \quad y(-1)=2$

38. $(t^2-1)\frac{dx}{dx}-2tx=t(t^2-1), \quad x(0)=4$

39. $(x^2-5)dy=-2x(x^2-5-y)dx, \quad y(2)=7$

40. $(x+1)\frac{dy}{dx}+y=\ln x, \quad y(1)=10$

41. Experiments indicate that glucose is absorbed by the body at a rate proportional to the amount of glucose present in the bloodstream. Let λ denote the (positive) constant of proportionality. Now suppose glucose is injected into a patient's bloodstream at a constant rate of r units per unit of time. Let $G=G(t)$ be the number of units in the patient's bloodstream at time $t>0$. Then

$$G'=-\lambda G+r,$$

where the first term on the right is due to the absorption of the glucose by the patient's body and the second term is due to the injection. Determine G for $t>0$, given that $G(0)=G_0$. Also, find $\lim_{t\to\infty} G(t)$.

42. Assume that all functions in this exercise are defined on a common interval (a,b).

Prove: If y_1 and y_2 are solutions of

$$y'+p(x)y=f_1(x)$$

and

$$y'+p(x)y=f_2(x)$$

respectively, and c_1 and c_2 are constants, then $y=c_1y_1+c_2y_2$ is a solution of

$$y'+p(x)y=c_1f_1(x)+c_2f_2(x).$$

(This is the *principle of superposition.)*

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2.4: Solving Differential Equations by Substitutions

Bernoulli Equations

Definition 2.4.1

A *Bernoulli equation* is an equation of the form

$$\frac{dy}{dx} + p(x)y = f(x)y^r, \tag{2.4.1}$$

where r can be any real number other than 0 or 1. (Note that Equation 2.4.1 is linear if and only if $r = 0$ or $r = 1$.)

Theorem 2.4.1

The substitution $u = y^{1-r}$, will turn the Bernoulli Equation 2.4.1 into a linear equation.

Proof

Dividing 2.4.1 by y^r yields

$$y^{-r}\frac{dy}{dx} + p(x)y^{1-r} = f(x).$$

If we make the substitution

$$u = y^{1-r}$$

and differentiate with respect to x we get

$$\frac{du}{dx} = (1-r)y^{-r}\frac{dy}{dx}.$$

From this we can see that 2.4.1 becomes

$$\frac{1}{1-r}\frac{du}{dx} + p(x)u = f(x)$$

which is linear and can be solved by the methods of section 2.3.

Example 2.4.1

Solve the Bernoulli equation

$$y' - y = xy^2. \tag{2.4.2}$$

Solution

Dividing by y^2 we get

$$y^{-2}y' - y^{-1} = x.$$

If we now let $u = y^{-1}$, we get $\frac{du}{dx} = -y^{-2}\frac{dy}{dx}$ and substituting into 2.4.2 we get

$$\frac{-du}{dx} - u = x$$

which is a linear equation. Putting it in standard linear form we get

$$\frac{du}{dx} + u = -x$$

and using the method of section 2.3 we get the solution

$$u = -(x-1) + ce^{-x}$$

and substituting back in for u we get

$$y^{-1} = -(x-1) + ce^{-x}$$

and solving for y we get our final solution

$$y = -\frac{1}{x - 1 + ce^{-x}}.$$

Figure 2.4.1 shows the direction field and some integral curves of Equation 2.4.2.

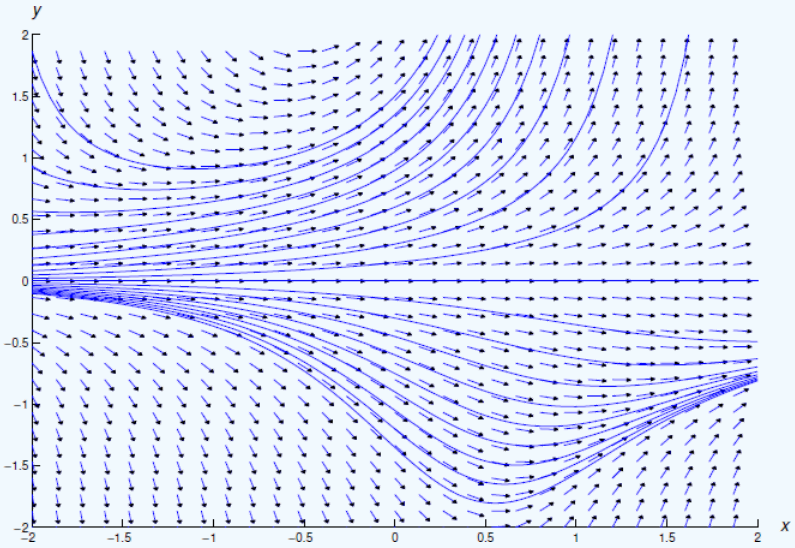


Figure 2.4.1 : A direction field and integral curves for $y' - y = xy^2$

Homogeneous Equations

Definition 2.4.2

We say that a function $f(x, y)$ is *homogeneous of degree n* if

$$f(tx, ty) = t^n f(x, y)$$

Example 2.4.2

Determine if

$$f(x, y) = x^5 + x^2y^3$$

is a homogeneous function and, if so, of what degree.

Solution

Substituting tx for x and ty for y we get

$$f(tx, ty) = (tx)^5 + (tx)^2(ty)^3 = t^5(x^5 + x^2y^3) = t^5 f(x, y).$$

Therefore, $f(x, y)$ is a homogeneous function of degree 5.

Example 2.4.3

Determine if

$$f(x, y) = x^2 + xy^2$$

is a homogeneous function and, if so, of what degree.

Solution

Substituting tx for x and ty for y we get

$$f(tx,ty)=(tx)^2+(tx)(ty)^2=t^2(x^2+txy^2)\neq t^n f(x,y).$$

for any number n. Therefore, $f(x,y)$ is not a homogeneous function.

Note: For this type of differential equation, as in section 2.2, it is convenient to write first order differential equations in the form

$$M(x,y)\,dx+N(x,y)\,dy=0. \tag{2.4.3}$$

Definition 2.4.3

We say that a differential equation of the form $M(x,y)\,dx+N(x,y)\,dy=0$ is *homogeneous* if $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same degree.

Example 2.4.4

Determine if

$$(x^2+xy)dx+y^2dy=0$$

is a homogeneous differential equation.

Solution

Substituting tx for x and ty for y in $M(x,y)$ we get

$$M(tx,ty)=(tx)^2+(tx)(ty)=t^2(x^2+xy)=t^2M(x,y)$$

and $M(x,y)$ is a homogeneous function of degree 2.

Substituting tx for x and ty for y in $N(x,y)$ we get

$$N(tx,ty)=(ty)^2=t^2y^2=t^2N(x,y)$$

and $N(x,y)$ is a homogeneous function of degree 2.

Since both $M(x,y)$ and $N(x,y)$ are homogeneous functions of degree 2, the differential equation is homogeneous.

Theorem 2.4.2

The substitution $y=ux$, where u is a function of x, or $x=vy$, where v is a function of y will make a homogeneous equation separable.

Proof

Note: we will only prove this for the substitution $y=ux$ and leave the substitution $x=vy$ to the reader.

Letting

$$y=ux$$

and differentiating with respect to x we get

$$\frac{dy}{dx}=u+x\frac{du}{dx}$$

which can be rewritten as

$$dy=udx+xdu.$$

Substituting y and dy into 2.4.3 we get

$$M(x,ux)dx+N(x,ux)[udx+xdu]=0.$$

Using the fact that both $M(x,y)$ and $N(x,y)$ are homogeneous of the **same** degree we get

$$x^nM(1,u)dx+x^nN(1,u)[udx+xdu]=0.$$

Expanding and rewriting this we eventually come to

$$x^n[M(1,u)+uN(1,u)]dx=-x^{n+1}N(1,u)du$$

which is clearly a separable differential equation:

$$\frac{1}{x}dx=\frac{-N(1,u)}{M(1,u)+uN(1,u)}du.$$

Example 2.4.5

Solve

$$xdy=(y+xe^{-y/x})dx. \tag{2.4.4}$$

First verify that $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same degree.

Substituting $y=ux$ into Equation 2.4.4 yields

$$x(udx+xdu)=(ux+xe^{-ux/x})dx.$$

Simplifying and separating variables yields

$$e^udu=\frac{1}{x}dx.$$

Integrating yields $e^u=\ln|x|+c$. Therefore $u=\ln(\ln|x|+c)$ and $y=ux=x\ln(\ln|x|+c)$.

Figure 2.4.2 shows a direction field and integral curves for Equation 2.4.4.

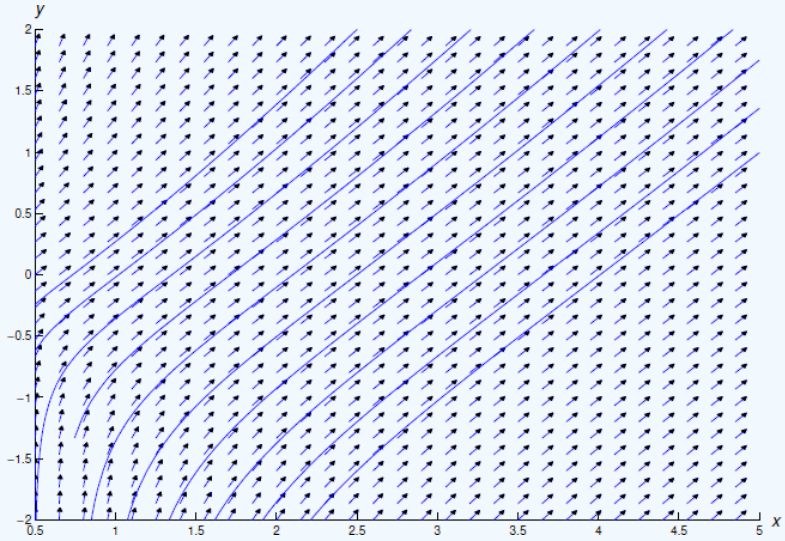


Figure 2.4.2 : A direction field and some integral curves for $xdy=(y+xe^{-y/x})dx$

Equations of the form $\frac{dy}{dx}=f(Ax+By+C)$

Theorem 2.4.3

The substitution

$$u = Ax + By + C$$

will make equations of the form

$$\frac{dy}{dx} = f(Ax + By + C) \tag{2.4.5}$$

separable.

Proof

Consider a differential equation of the form 2.4.5.

Let

$$u = Ax + By + C$$

Taking the derivative with respect to x we get

$$\frac{du}{dx} = A + B\frac{dy}{dx}.$$

Substituting into 2.4.5 we get

$$\frac{1}{B}(\frac{du}{dx} - A) = f(u)$$

which is clearly a separable differential equation:

$$\frac{du}{Bf(u) + A} = dx$$

Example 2.4.6

Solve

$$y' = sec^2(y - x - 3). \tag{2.4.6}$$

Solution

Letting

$$u = y - x - 3$$

and differentiating we get

$$\frac{du}{dx} = \frac{dy}{dx} - 1$$

Substituting into Equation 2.4.6 we get

$$\frac{du}{dx} + 1 = sec^2 u$$

This is now a separable equation which can be solved using the method of section 2.1, resulting in a solution of

$$-\cot u - u = x + c$$

and substituting in for u we get

$$-\cot(y - x - 3) - (y - x - 3) = x + c$$

and simplifying we get

$$\cot(y-x-3) = -y+c$$

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2.4E: Solving Differential Equations by Substitution (Exercises)

Bernoulli Equations

In Exercises **1-8** solve the given equation.

1. $y' + y = y^2$
2. $7x\frac{dy}{dx} - 2y = -\frac{x^2}{y^6}$
3. $x^2y' + 2y = 2e^{1/x}y^{1/2}$
4. $\frac{dy}{dx} = y(xy^3 - 1)$
5. $(1+x^2)y' + 2xy = \frac{1}{(1+x^2)y}$
6. $t^2\frac{dy}{dt} + y^2 = ty$
7. $y' - xy = x^3y^3$
8. $\frac{dy}{dx} - \frac{1+x}{3x}y = y^4$

In Exercises **9-16** solve the initial value problem.

9. $y' - 2y = xy^3, \quad y(0) = 2\sqrt{2}$
10. $y' - xy = xy^{3/2}, \quad y(1) = 4$
11. $xy' + y = x^4y^4, \quad y(1) = 1/2$
12. $y' - 2y = 2y^{1/2}, \quad y(0) = 1$
13. $x^2\frac{dy}{dx} - 2xy = 3y^4, \quad y(1) = \frac{1}{2}$
14. $y' - 4y = \frac{48x}{y^2}, \quad y(0) = 1$
15. $x^2y' + 2xy = y^3, \quad y(1) = 1/\sqrt{2}$
16. $y' - y = xy^{1/2}, \quad y(0) = 4$

Homogeneous Equations

In Exercises **17-24** solve the equation.

17. $y' = \frac{y+x}{x}$
18. $y' = \frac{y^2+2xy}{x^2}$
19. $xy^3y' = y^4 + x^4$
20. $y' = \frac{y}{x} + \sec\frac{y}{x}$
21. $x^2y' = xy + x^2 + y^2$
22. $-ydx + (x + \sqrt{xy})dy = 0$
23. $xyy' = x^2 + 2y^2$
24. $y' = \frac{2y^2 + x^2e^{-(y/x)^2}}{2xy}$

In Exercises **25-32** solve the initial value problem.

25. $y' = \frac{xy+y^2}{x^2}, \quad y(-1) = 2$
26. $xy^2\frac{dy}{dx} = y^3 - x^3, \quad y(1) = 2$
27. $y' = \frac{x^3+y^3}{xy^2}, \quad y(1) = 3$

28. $xyy' + x^2 + y^2 = 0, \quad y(1) = 2$
29. $(x + ye^{\frac{y}{x}})dx - xe^{\frac{y}{x}}dy = 0, \quad y(1) = 0$
30. $y' = \frac{y^2 - 3xy - 5x^2}{x^2}, \quad y(1) = -1$
31. $x^2y' = 2x^2 + y^2 + 4xy, \quad y(1) = 1$
32. $xyy' = 3x^2 + 4y^2, \quad y(1) = \sqrt{3}$

In Exercises **33-39** solve the given equation.

33. $dy = \frac{x+y}{x-y}dx$
34. $(y'x - y)(\ln|y| - \ln|x|) = x$
35. $y' = \frac{y^3 + 2xy^2 + x^2y + x^3}{x(y+x)^2}$
36. $(2x + y)dy = (x + 2y)dx$
37. $\frac{dy}{dx} = \frac{y}{y - 2x}$
38. $y' = \frac{xy^2 + 2y^3}{x^3 + x^2y + xy^2}$
39. $(x^3 + 3xy^2)\frac{dy}{dx} = x^3 + x^2y + 3y^3$

The Substitution u=Ax+By+C

In Exercises **40-45** solve the given equation.

40. $\frac{dy}{dx} = (x + y + 1)^2$
41. $\frac{dy}{dx} = \tan^2(x + y)$
42. $\frac{dy}{dx} = 2 + \sqrt{y - 2x + 3}$
43. $\frac{dy}{dx} = \frac{1 - x - y}{x + y}$
44. $\frac{dy}{dx} = \sin(x + y)$
45. $\frac{dy}{dx} = 1 + e^{y - x + 5}$

In Exercises **46-47** solve the initial value problem.

46. $\frac{dy}{dx} = \cos(x + y), \quad y(0) = \pi/4$
47. $\frac{dy}{dx} = \frac{3x + 2y}{3x + 2y + 2}, \quad y(-1) = -1$

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CHAPTER OVERVIEW

3: Numerical Methods

In this chapter we study numerical methods for approximating solutions of a first order differential equation $y' = f(x, y)$.

3.1: Euler's Method

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3.1: Euler's Method

If an initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{3.1.1}$$

cannot be solved analytically, it is necessary to resort to numerical methods to obtain useful approximations to a solution of Equation 3.1.1. We will consider such methods in this chapter.

We're interested in computing approximate values of the solution of Equation 3.1.1 at equally spaced points x_0, x_1, …, $x_n = b$ in an interval $[x_0, b]$. Thus,

$$x_i = x_0 + ih, \quad i = 0, 1, \ldots, n,$$

where

$$h = \frac{b - x_0}{n}.$$

we will denote the approximate values of the solution at these points by y_0, y_1, …, y_n; thus, y_i is an approximation to $y(x_i)$. we will call

$$e_i = y(x_i) - y_i$$

the *error at the* ith step. Because of the initial condition $y(x_0) = y_0$, we will always have $e_0 = 0$. However, in general $e_i \neq 0$ if $i > 0$.

We encounter two sources of error in applying a numerical method to solve an initial value problem:

- The formulas defining the method are based on some sort of approximation. Errors due to the inaccuracy of the approximation are called *truncation errors*.
- Computers do arithmetic with a fixed number of digits, and therefore make errors in evaluating the formulas defining the numerical methods. Errors due to the computer's inability to do exact arithmetic are called *roundoff errors.*

That said, a careful analysis of error is beyond the scope of this book.

Euler's Method

The simplest numerical method for solving Equation 3.1.1 is *Euler's method*. Euler's method is based on the assumption that the tangent line to the integral curve of Equation 3.1.1 at $(x_i, y(x_i))$ approximates the integral curve over the interval $[x_i, x_{i+1}]$. Since the slope of the integral curve of Equation 3.1.1 at $(x_i, y(x_i))$ is $y'(x_i) = f(x_i, y(x_i))$, the equation of the tangent line to the integral curve at $(x_i, y(x_i))$ is

$$y = y(x_i) + f(x_i, y(x_i))(x - x_i). \tag{3.1.2}$$

Setting $x = x_{i+1} = x_i + h$ in Equation 3.1.2 yields

$$y_{i+1} = y(x_i) + hf(x_i, y(x_i)) \tag{3.1.3}$$

as an approximation to $y(x_{i+1})$. Since $y(x_0) = y_0$ is known, we can use Equation 3.1.3 with $i = 0$ to compute

$$y_1 = y_0 + hf(x_0, y_0).$$

However, setting $i = 1$ in Equation 3.1.3 yields

$$y_2 = y(x_1) + hf(x_1, y(x_1)),$$

which isn't useful, since we *don't know* $y(x_1)$. Therefore we replace $y(x_1)$ by its approximate value y_1 and redefine

$$y_2 = y_1 + hf(x_1, y_1).$$

Having computed y_2, we can compute

$$y_3 = y_2 + hf(x_2, y_2).$$

In general, Euler's method starts with the known value $y(x_0) = y_0$ and computes y_1, y_2, …, y_n successively by with the formula

$$y_{i+1} = y_i + hf(x_i, y_i), \quad 0 \le i \le n-1. \tag{3.1.4}$$

The next example illustrates the computational procedure indicated in Euler's method.

Example 3.1.1

Use Euler's method with $h = 0.1$ to find approximate values for the solution of the initial value problem

$$y' + 2y = x^3 e^{-2x}, \quad y(0) = 1 \tag{3.1.5}$$

at $x = 0.1, 0.2, 0.3$

Solution

We rewrite Equation 3.1.5 as

$$y' = -2y + x^3 e^{-2x}, \quad y(0) = 1,$$

which is of the form Equation 3.1.1, with

$$f(x, y) = -2y + x^3 e^{-2x},\ x_0 = 0,\ \text{and}\ y_0 = 1.$$

Euler's method yields

$$\begin{aligned} y_1 &= y_0 + hf(x_0, y_0) \\ &= 1 + (0.1)f(0, 1) = 1 + (0.1)(-2) = 0.8, \\ y_2 &= y_1 + hf(x_1, y_1) \\ &= 0.8 + (0.1)f(0.1, 0.8) = 0.8 + (0.1)\left(-2(0.8) + (0.1)^3 e^{-0.2}\right) = 0.640081873, \\ y_3 &= y_2 + hf(x_2, y_2) \\ &= 0.640081873 + (0.1)\left(-2(0.640081873) + (0.2)^3 e^{-0.4}\right) = 0.512601754. \end{aligned}$$

We've written the details of these computations to ensure that you understand the procedure. However, in the rest of the examples as well as the exercises in this chapter, we will assume that you can use a calculator to carry out the necessary computations.

Examples Illustrating The Error in Euler's Method

Example 3.1.2

Use Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + 2y = x^3 e^{-2x}, \quad y(0) = 1$$

at $x = 0$, 0.1, 0.2, 0.3, …, 1.0. Compare these approximate values with the values of the exact solution

$$y = \frac{e^{-2x}}{4}(x^4 + 4), \tag{3.1.6}$$

which can be obtained by the method of Section 2.3 (verify this).

Table 3.1.1 shows the values of the exact solution Equation 3.1.6 at the specified points, and the approximate values of the solution at these points obtained by Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$. In examining this table, keep in mind that the approximate values in the column corresponding to $h = 0.05$ are actually the results of 20 steps with Euler's method. We haven't listed the estimates of the solution obtained for $x = 0.05$, 0.15, …, since there's nothing to compare them with in the column corresponding to $h = 0.1$. Similarly, the approximate values in the column corresponding to $h = 0.025$ are actually the results of 40 steps with Euler's method.

Table 3.1.1 : Numerical solution of $y' + 2y = x^3 e^{-2x}$, $y(0) = 1$, by Euler's method.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
0.0	1.000000000	1.000000000	1.000000000	1.000000000

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
0.1	0.800000000	0.810005655	0.814518349	0.818751221
0.2	0.640081873	0.656266437	0.663635953	0.670588174
0.3	0.512601754	0.532290981	0.541339495	0.549922980
0.4	0.411563195	0.432887056	0.442774766	0.452204669
0.5	0.332126261	0.353785015	0.363915597	0.373627557
0.6	0.270299502	0.291404256	0.301359885	0.310952904
0.7	0.222745397	0.242707257	0.252202935	0.261398947
0.8	0.186654593	0.205105754	0.213956311	0.222570721
0.9	0.159660776	0.176396883	0.184492463	0.192412038
1.0	0.139778910	0.154715925	0.162003293	0.169169104

You can see from Table 3.1.1 that decreasing the step size improves the accuracy of Euler's method. For example,

$$y_{exact}(1) - y_{approx}(1) \approx \begin{cases} 0.0293 \text{with} h = 0.1, \\ 0.0144 \text{ with } h = 0.05, \\ 0.0071 \text{ with } h = 0.025. \end{cases}$$

Based on this scanty evidence, you might guess that the error in approximating the exact solution at a *fixed value of* x by Euler's method is roughly halved when the step size is halved. You can find more evidence to support this conjecture by examining Table 3.1.2 , which lists the approximate values of $y_{exact} - y_{approx}$ at $x = 0.1, 0.2, \ldots, 1.0$.

Table 3.1.2 : Errors in approximate solutions of $y' + 2y = x^3 e^{-2x}$, $y(0) = 1$, obtained by Euler's method.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$
0.1	0.0187	0.0087	0.0042
0.2	0.0305	0.0143	0.0069
0.3	0.0373	0.0176	0.0085
0.4	0.0406	0.0193	0.0094
0.5	0.0415	0.0198	0.0097
0.6	0.0406	0.0195	0.0095
0.7	0.0386	0.0186	0.0091
0.8	0.0359	0.0174	0.0086
0.9	0.0327	0.0160	0.0079
1.0	0.0293	0.0144	0.0071

Example 3.1.3

Tables 3.1.3 and 3.1.4 show analogous results for the initial value problem

$$y' = -2y^2 + xy + x^2, \; y(0) = 1, \tag{3.1.7}$$

except in this case we cannot solve Equation 3.1.7 exactly. The results in the "Exact" column were obtained by using a more accurate numerical method known as the *Runge-Kutta* method (discussed later) with a small step size. They are exact to eight decimal places.

Table 3.1.3 : Numerical solution of $y' = -2y^2 + xy + x^2,\ y(0) = 1$, by Euler's method.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
0.0	1.000000000	1.000000000	1.000000000	1.000000000
0.1	0.800000000	0.821375000	0.829977007	0.837584494
0.2	0.681000000	0.707795377	0.719226253	0.729641890
0.3	0.605867800	0.633776590	0.646115227	0.657580377
0.4	0.559628676	0.587454526	0.600045701	0.611901791
0.5	0.535376972	0.562906169	0.575556391	0.587575491
0.6	0.529820120	0.557143535	0.569824171	0.581942225
0.7	0.541467455	0.568716935	0.581435423	0.593629526
0.8	0.569732776	0.596951988	0.609684903	0.621907458
0.9	0.614392311	0.641457729	0.654110862	0.666250842
1.0	0.675192037	0.701764495	0.714151626	0.726015790

Since we think it is important in evaluating the accuracy of the numerical methods that we will be studying in this chapter, we often include a column listing values of the exact solution of the initial value problem, even if the directions in the example or exercise don't specifically call for it. If quotation marks are included in the heading, the values were obtained by applying the Runge-Kutta method in a way that's explained in Section 3.3. If quotation marks are not included, the values were obtained from a known formula for the solution. In either case, the values are exact to eight places to the right of the decimal point.

Table 3.1.4 : Errors in approximate solutions of $y' = -2y^2 + xy + x^2,\ y(0) = 1$, obtained by Euler's method.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$
0.1	0.0376	0.0162	0.0076
0.2	0.0486	0.0218	0.0104
0.3	0.0517	0.0238	0.0115
0.4	0.0523	0.0244	0.0119
0.5	0.0522	0.0247	0.0121
0.6	0.0521	0.0248	0.0121
0.7	0.0522	0.0249	0.0122
0.8	0.0522	0.0250	0.0122
0.9	0.0519	0.0248	0.0121
1.0	0.0508	0.0243	0.0119

In the next two sections we will study two other numerical methods for solving initial value problems, the *improved Euler method* and the *Runge- Kutta method*.

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3.1E: Euler's Method (Exercises)

Caution

You will want to save the results of these exercises, since we will revisit them in the next two sections.

In Exercises **1-5** use Euler's method to find approximate values of the solution of the given initial value problem at the points $x_i = x_0 + ih$, where x_0 is the point where the initial condition is imposed and $i = 1, 2, 3$.

1. $y' = 2x^2 + 3y^2 - 2, \quad y(2) = 1; \quad h = 0.05$
2. $y' = y + \sqrt{x^2 + y^2}, \quad y(0) = 1; \quad h = 0.1$
3. $y' + 3y = x^2 - 3xy + y^2, \quad y(0) = 2; \quad h = 0.05$
4. $y' = \frac{1+x}{1-y^2}, \quad y(2) = 3; \quad h = 0.1$
5. $y' + x^2 y = \sin xy, \quad y(1) = \pi; \quad h = 0.2$

6. Use Euler's method with step size $h = 0.1$ to find approximate values of the solution of the initial value problem

$$y' + 3y = 7e^{4x}, \quad y(0) = 2$$

at $x = 0, 0.1, 0.2, 0.3, \ldots, 1.0$. Compare these approximate values with the values of the exact solution $y = e^{4x} + e^{-3x}$, which can be obtained by the method of Section 2.3. Present your results in a table like *Table 3.1.1*.

7. Use Euler's method with step size $h = 0.1$ to find approximate values of the solution of the initial value problem

$$y' + \frac{2}{x}y = \frac{3}{x^3} + 1, \quad y(1) = 1$$

at $x = 1.0, 1.1, 1.2, 1.3, \ldots, 2.0$. Compare these approximate values with the values of the exact solution

$$y = \frac{1}{3x^2}(9 \ln x + x^3 + 2),$$

which can be obtained by the method of Section 2.3. Present your results in a table like *Table 3.1.1*.

8. Use Euler's method with step size $h = 0.05$ to find approximate values of the solution of the initial value problem

$$y' = \frac{y^2 + xy - x^2}{x^2}, \quad y(1) = 2$$

at $x = 1.0, 1.05, 1.10, 1.15, \ldots, 1.5$. Compare these approximate values with the values of the exact solution

$$y = \frac{x(1 + x^2/3)}{1 - x^2/3}$$

obtained by a method in Section 2.4. Present your results in a table like *Table 3.1.1*.

9. It was shown in an example in Section 2.1 that

$$y^5 + y = x^2 + x - 4$$

is an implicit solution of the initial value problem

$$y' = \frac{2x + 1}{5y^4 + 1}, \quad y(2) = 1. \tag{A}$$

Use Euler's method with step size $h = 0.1$ to find approximate values of the solution of (A) at $x = 2.0, 2.1, 2.2, 2.3, \ldots, 3.0$. Present your results in tabular form. To check the error in these approximate values, construct another table of values of the residual

$$R(x, y) = y^5 + y - x^2 - x + 4$$

for each value of (x, y) appearing in the first table.

10. It was previously shown that

$$x^4y^3 + x^2y^5 + 2xy = 4$$

is an implicit solution of the initial value problem

$$y' = -\frac{4x^3y^3 + 2xy^5 + 2y}{3x^4y^2 + 5x^2y^4 + 2x}, \quad y(1) = 1. \tag{A}$$

Use Euler's method with step size $h = 0.1$ to find approximate values of the solution of (A) at $x = 1.0, 1.1, 1.2, 1.3, \ldots, 2.0$. Present your results in tabular form. To check the error in these approximate values, construct another table of values of the residual

$$R(x, y) = x^4y^3 + x^2y^5 + 2xy - 4$$

for each value of (x, y) appearing in the first table.

11. Use Euler's method with step size $h = 0.1$ to find approximate values of the solution of the initial value problem

$$(3y^2 + 4y)y' + 2x + \cos x = 0, \quad y(0) = 1$$

at $x = 0, 0.1, 0.2, 0.3, \ldots, 1.0$.

12. Use Euler's method with step size $h = 0.1$ to find approximate values of the solution of the initial value problem

$$y' + \frac{(y+1)(y-1)(y-2)}{x+1} = 0, \quad y(1) = 0$$

at $x = 1.0, 1.1, 1.2, 1.3, \ldots, 2.0$.

13. Use Euler's method with step size $h = 0.1$ to find approximate values of the solution of the initial value problem

$$y' + 3y = 7e^{-3x}, \quad y(0) = 6$$

at $x = 0, 0.1, 0.2, 0.3, \ldots, 1.0$.

The linear initial value problems in Exercises **14-22** can't be solved exactly in terms of known elementary functions. In each exercise, use Euler's method with the indicated step size to find approximate values of the solution of the given initial value problem at 11 equally spaced points (including the endpoints) in the interval.

14. $y' - 2y = \frac{1}{1+x^2}, \quad y(2) = 2; \quad h = 0.1$ on $[2, 3]$
15. $y' + 2xy = x^2, \quad y(0) = 3; \quad h = 0.2$ on $[0, 2]$
16. $y' + \frac{1}{x}y = \frac{\sin x}{x^2}, \quad y(1) = 2; \quad h = 0.2$ on $[1, 3]$
17. $y' + y = \frac{e^{-x}\tan x}{x}, \quad y(1) = 0; \quad h = 0.05$ on $[1, 1.5]$
18. $y' + \frac{2x}{1+x^2}y = \frac{e^x}{(1+x^2)^2}, \quad y(0) = 1; \quad h = 0.2$ on $[0, 2]$
19. $xy' + (x+1)y = e^{x^2}, \quad y(1) = 2; \quad h = 0.05$ on $[1, 1.5]$
20. $y' + 3y = xy^2(y+1), \quad y(0) = 1; \quad h = 0.1$ on $[0, 1]$
21. $y' - 4y = \frac{x}{y^2(y+1)}, \quad y(0) = 1; \quad h = 0.1$ on $[0, 1]$
22. $y' + 2y = \frac{x^2}{1+y^2}, \quad y(2) = 1; \quad h = 0.1$, on $[2, 3]$

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3.2: The Improved Euler Method

Section 3.1 would seem to imply that we can achieve arbitrarily accurate results with Euler's method by simply choosing the step size sufficiently small. However, this is not a good idea, for two reasons. First, after a certain point decreasing the step size will increase roundoff errors to the point where the accuracy will deteriorate rather than improve. The second and more important reason is that in most applications of numerical methods to an initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0, \tag{3.2.1}$$

the expensive part of the computation is the evaluation of f. Therefore we want methods that give good results for a given number of such evaluations. This is what motivates us to look for numerical methods better than Euler's.

To clarify this point, suppose we want to approximate the value of e by applying Euler's method to the initial value problem

$$y' = y, \quad y(0) = 1$$

(with solution $y = e^x$) on $[0, 1]$, with $h = 1/12$, $1/24$, and $1/48$, respectively. Since each step in Euler's method requires one evaluation of f, the number of evaluations of f in each of these attempts is $n = 12$, 24, and 48, respectively. In each case we accept y_n as an approximation to e. The second column of Table 3.2.1 shows the results. The first column of the table indicates the number of evaluations of f required to obtain the approximation, and the last column contains the value of e rounded to ten significant figures.

In this section we will study the *improved Euler method*, which requires two evaluations of f at each step. We've used this method with $h = 1/6$, $1/12$, and $1/24$. The required number of evaluations of f were 12, 24, and 48, as in the three applications of Euler's method; however, you can see from the third column of Table 3.2.1 that the approximation to e obtained by the improved Euler method with only 12 evaluations of f is better than the approximation obtained by Euler's method with 48 evaluations.

In Section 3.3, we will study the *Runge-Kutta method*, which requires four evaluations of f at each step. We've used this method with $h = 1/3$, $1/6$, and $1/12$. The required number of evaluations of f were again 12, 24, and 48, as in the three applications of Euler's method and the improved Euler method; however, you can see from the fourth column of Table 3.2.1 that the approximation to e obtained by the Runge-Kutta method with only 12 evaluations of f is better than the approximation obtained by the improved Euler method with 48 evaluations.

Table 3.2.1 : Approximations to e obtained by three numerical methods.

n	Euler	Improved Euler	Runge-Kutta	Exact
12	2.613035290	2.707188994	2.718069764	2.718281828
24	2.663731258	2.715327371	2.718266612	2.718281828
48	2.690496599	2.717519565	2.718280809	2.718281828

The Improved Euler Method

The *improved Euler method* for solving the initial value problem Equation 3.2.1 is based on approximating the integral curve of Equation 3.2.1 at $(x_i, y(x_i))$ by the line through $(x_i, y(x_i))$ with slope

$$m_i = \frac{f(x_i, y(x_i)) + f(x_{i+1}, y(x_{i+1}))}{2};$$

that is, m_i is the average of the slopes of the tangents to the integral curve at the endpoints of $[x_i, x_{i+1}]$. The equation of the approximating line is therefore

$$y = y(x_i) + \frac{f(x_i, y(x_i)) + f(x_{i+1}, y(x_{i+1}))}{2}(x - x_i). \tag{3.2.2}$$

Setting $x = x_{i+1} = x_i + h$ in Equation 3.2.2 yields

$$y_{i+1} = y(x_i) + \frac{h}{2}\left(f(x_i, y(x_i)) + f(x_{i+1}, y(x_{i+1}))\right) \tag{3.2.3}$$

as an approximation to $y(x_{i+1})$. As in our derivation of Euler's method, we replace $y(x_i)$ (unknown if $i > 0$) by its approximate value y_i; then Equation 3.2.3 becomes

$$y_{i+1} = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y(x_{i+1}))\right).$$

However, this still will not work, because we do not know $y(x_{i+1})$. We overcome this by replacing $y(x_{i+1})$ by $y_i + hf(x_i, y_i)$, the value that the Euler method would assign to y_{i+1}. Thus, the improved Euler method starts with the known value $y(x_0) = y_0$ and computes $y_1, y_2, \ldots, y_n$ successively with the formula

$$y_{i+1} = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y_i + hf(x_i, y_i))\right). \tag{3.2.4}$$

The computation indicated here can be conveniently organized as follows: given y_i, compute

$$\begin{aligned} k_{1i} &= f(x_i, y_i), \\ k_{2i} &= f\left(x_i + h, y_i + hk_{1i}\right), \\ y_{i+1} &= y_i + \frac{h}{2}(k_{1i} + k_{2i}). \end{aligned}$$

The improved Euler method requires two evaluations of $f(x, y)$ per step, while Euler's method requires only one. However, as we can see the error is significantly reduced.

Example 3.2.1 illustrates the computational procedure indicated in the improved Euler method.

Example 3.2.1

Use the improved Euler method with $h = 0.1$ to find approximate values of the solution of the initial value problem

$$y' + 2y = x^3 e^{-2x}, \quad y(0) = 1 \tag{3.2.5}$$

at $x = 0.1, 0.2, 0.3$

Solution

As in Example 3.1.1, we rewrite Equation 3.2.5 as

$$y' = -2y + x^3 e^{-2x}, \quad y(0) = 1,$$

which is of the form Equation 3.2.1, with

$$f(x, y) = -2y + x^3 e^{-2x}, x_0 = 0, \text{and } y_0 = 1.$$

The improved Euler method yields

$$
\begin{aligned}
k_{10} &= f(x_0, y_0) = f(0, 1) = -2, \\
k_{20} &= f(x_1, y_0 + hk_{10}) = f(0.1, 1 + (0.1)(-2)) \\
&= f(0.1, 0.8) = -2(0.8) + (0.1)^3 e^{-0.2} = -1.599181269, \\
y_1 &= y_0 + \frac{h}{2}(k_{10} + k_{20}), \\
&= 1 + (0.05)(-2 - 1.599181269) = 0.820040937, \\
k_{11} &= f(x_1, y_1) = f(0.1, 0.820040937) = -2(0.820040937) + (0.1)^3 e^{-0.2} = -1.639263142, \\
k_{21} &= f(x_2, y_1 + hk_{11}) = f(0.2, 0.820040937 + 0.1(-1.639263142)), \\
&= f(0.2, 0.656114622) = -2(0.656114622) + (.2)^3 e^{-0.4} = -1.306866684, \\
y_2 &= y_1 + \frac{h}{2}(k_{11} + k_{21}), \\
&= .820040937 + (.05)(-1.639263142 - 1.306866684) = 0.672734445, \\
k_{12} &= f(x_2, y_2) = f(.2, .672734445) = -2(.672734445) + (.2)^3 e^{-.4} = -1.340106330, \\
k_{22} &= f(x_3, y_2 + hk_{12}) = f(.3, .672734445 + .1(-1.340106330)), \\
&= f(.3, .538723812) = -2(.538723812) + (.3)^3 e^{-.6} = -1.062629710, \\
y_3 &= y_2 + \frac{h}{2}(k_{12} + k_{22}) \\
&= .672734445 + (.05)(-1.340106330 - 1.062629710) = 0.552597643.
\end{aligned}
$$

Examples Illustrating The Error in the Improved Euler's Method

Example 3.2.2

Table 3.2.2 shows results of using the improved Euler method with step sizes $h = 0.1$ and $h = 0.05$ to find approximate values of the solution of the initial value problem

$$y' + 2y = x^3 e^{-2x}, \quad y(0) = 1$$

at $x = 0$, 0.1, 0.2, 0.3, …, 1.0. For comparison, it also shows the corresponding approximate values obtained with Euler's method in Example 3.1.2, and the values of the exact solution

$$y = \frac{e^{-2x}}{4}(x^4 + 4).$$

Note: the results obtained by the improved Euler method with $h = 0.1$ are better than those obtained by Euler's method with $h = 0.05$.

	Euler		Improved Euler		Exact
x	$h = 0.1$	$h = 0.05$	$h = 0.1$	$h = 0.05$	Exact
0.0	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
0.1	0.800000000	0.810005655	0.820040937	0.819050572	0.818751221
0.2	0.640081873	0.656266437	0.672734445	0.671086455	0.670588174
0.3	0.512601754	0.532290981	0.552597643	0.550543878	0.549922980
0.4	0.411563195	0.432887056	0.455160637	0.452890616	0.452204669
0.5	0.332126261	0.353785015	0.376681251	0.374335747	0.373627557
0.6	0.270299502	0.291404256	0.313970920	0.311652239	0.310952904
0.7	0.222745397	0.242707257	0.264287611	0.262067624	0.261398947
0.8	0.186654593	0.205105754	0.225267702	0.223194281	0.222570721

	Euler		Improved Euler		Exact
x	$h = 0.1$	$h = 0.05$	$h = 0.1$	$h = 0.05$	Exact
0.9	0.159660776	0.176396883	0.194879501	0.192981757	0.192412038
1.0	0.139778910	0.154715925	0.171388070	0.169680673	0.169169104

Table 3.2.2 : Numerical solution of $y' + 2y = x^3 e^{-2x},\ y(0) = 1$, by Euler's method and the improved Euler method.

Example 3.2.3

Table 3.2.3 shows analogous results for the initial value problem

$$y' = -2y^2 + xy + x^2,\ y(0) = 1.$$

We applied Euler's method to this problem in Example 3.1.3.

Table 3.2.3 : Numerical solution of $y' = -2y^2 + xy + x^2,\ y(0) = 1$, by Euler's method and the improved Euler method.

	Euler		Improved Euler		Exact
x	$h = 0.1$	$h = 0.05$	$h = 0.1$	$h = 0.05$	Exact
0.0	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
0.1	0.800000000	0.821375000	0.840500000	0.838288371	0.837584494
0.2	0.681000000	0.707795377	0.733430846	0.730556677	0.729641890
0.3	0.605867800	0.633776590	0.661600806	0.658552190	0.657580377
0.4	0.559628676	0.587454526	0.615961841	0.612884493	0.611901791
0.5	0.535376972	0.562906169	0.591634742	0.588558952	0.587575491
0.6	0.529820120	0.557143535	0.586006935	0.582927224	0.581942225
0.7	0.541467455	0.568716935	0.597712120	0.594618012	0.593629526
0.8	0.569732776	0.596951988	0.626008824	0.622898279	0.621907458
0.9	0.614392311	0.641457729	0.670351225	0.667237617	0.666250842
1.0	0.675192037	0.701764495	0.730069610	0.726985837	0.726015790

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3.2E: The Improved Euler Method (Exercises)

The following exercises involve initial value problems considered in the exercises in Section 3.1. You'll find it instructive to compare the results that you obtain here with the corresponding results that you obtained in Section 3.1.

Caution

You will want to save the results of these exercises, along with those of section 3.1, since we will revisit them in the next section.

In Exercises **1-5** use the improved Euler method to find approximate values of the solution of the given initial value problem at the points $x_i = x_0 + ih$, where x_0 is the point where the initial condition is imposed and $i = 1, 2, 3$.

1. $y' = 2x^2 + 3y^2 - 2, \quad y(2) = 1; \quad h = 0.05$
2. $y' = y + \sqrt{x^2 + y^2}, \quad y(0) = 1; \quad h = 0.1$
3. $y' + 3y = x^2 - 3xy + y^2, \quad y(0) = 2; \quad h = 0.05$
4. $y' = \frac{1+x}{1-y^2}, \quad y(2) = 3; \quad h = 0.1$
5. $y' + x^2 y = \sin xy, \quad y(1) = \pi; \quad h = 0.2$

6. Use the improved Euler method with step size $h = 0.1$ to find approximate values of the solution of the initial value problem

$$y' + 3y = 7e^{4x}, \quad y(0) = 2$$

at $x = 0, 0.1, 0.2, 0.3, \ldots, 1.0$. Compare these approximate values with the values of the exact solution $y = e^{4x} + e^{-3x}$, which can be obtained by the method of Section 2.3. Present your results in a table like *Table 3.2.2*.

7. Use the improved Euler method with step size $h = 0.1$ to find approximate values of the solution of the initial value problem

$$y' + \frac{2}{x}y = \frac{3}{x^3} + 1, \quad y(1) = 1$$

at $x = 1.0, 1.1, 1.2, 1.3, \ldots, 2.0$. Compare these approximate values with the values of the exact solution

$$y = \frac{1}{3x^2}(9 \ln x + x^3 + 2)$$

which can be obtained by the method of Section 2.3. Present your results in a table like *Table 3.2.2*.

8. Use the improved Euler method with step size $h = 0.05$ to find approximate values of the solution of the initial value problem

$$y' = \frac{y^2 + xy - x^2}{x^2}, \quad y(1) = 2,$$

at $x = 1.0, 1.05, 1.10, 1.15, \ldots, 1.5$. Compare these approximate values with the values of the exact solution

$$y = \frac{x(1 + x^2/3)}{1 - x^2/3}$$

obtained by a method in Section 2.4. Present your results in a table like *Table 3.2.2*.

9. It was shown in an example in Section 2.1 that

$$y^5 + y = x^2 + x - 4$$

is an implicit solution of the initial value problem

$$y' = \frac{2x + 1}{5y^4 + 1}, \quad y(2) = 1. \tag{A}$$

Use the improved Euler method with step size $h = 0.1$ to find approximate values of the solution of (A) at $x = 2.0, 2.1, 2.2, 2.3, \ldots, 3.0$. Present your results in tabular form. To check the error in these approximate values, construct another table of values of the

residual

$$R(x,y)=y^5+y-x^2-x+4$$

for each value of (x,y) appearing in the first table.

10. It was previously shown that

$$x^4y^3+x^2y^5+2xy=4$$

is an implicit solution of the initial value problem

$$y'=-\frac{4x^3y^3+2xy^5+2y}{3x^4y^2+5x^2y^4+2x},\quad y(1)=1. \tag{A}$$

Use the improved Euler method with step size $h=0.1$ to find approximate values of the solution of (A) at $x=1.0$, 1.14, 1.2, 1.3, …, 2.0. Present your results in tabular form. To check the error in these approximate values, construct another table of values of the residual

$$R(x,y)=x^4y^3+x^2y^5+2xy-4$$

for each value of (x,y) appearing in the first table.

11. Use the improved Euler method with step size $h=0.1$ to find approximate values of the solution of the initial value problem

$$(3y^2+4y)y'+2x+\cos x=0,\quad y(0)=1$$

at $x=0$, 0.1, 0.2, 0.3, …, 1.0.

12. Use the improved Euler method with step size $h=0.1$ to find approximate values of the solution of the initial value problem

$$y'+\frac{(y+1)(y-1)(y-2)}{x+1}=0,\quad y(1)=0$$

at $x=1.0$, 1.1, 1.2, 1.3, …, 2.0.

13. Use the improved Euler method with step size $h=0.1$ to find approximate values of the solution of the initial value problem

$$y'+3y=e^{-3x}(1-2x),\quad y(0)=2,$$

at $x=0$, 0.1, 0.2, 0.3, …, 1.0. Compare these approximate values with the values of the exact solution $y=e^{-3x}(2+x-x^2)$, which can be obtained by the method of Section 2.3.

The linear initial value problems in Exercises **14-22** can't be solved exactly in terms of known elementary functions. In each exercise, use the improved Euler's method with the indicated step size to find approximate values of the solution of the given initial value problem at 11 equally spaced points (including the endpoints) in the interval.

14. $y'-2y=\frac{1}{1+x^2},\quad y(2)=2;\quad h=0.1$ on $[2,3]$
15. $y'+2xy=x^2,\quad y(0)=3;\quad h=0.2$ on $[0,2]$
16. $y'+\frac{1}{x}y=\frac{\sin x}{x^2},\quad y(1)=2;\quad h=0.2$ on $[1,3]$
17. $y'+y=\frac{e^{-x}\tan x}{x},\quad y(1)=0;\quad h=0.05$ on $[1,1.5]$
18. $y'+\frac{2x}{1+x^2}y=\frac{e^x}{(1+x^2)^2},\quad y(0)=1;\quad h=0.2$ on $[0,2]$
19. $xy'+(x+1)y=e^{x^2},\quad y(1)=2;\quad h=0.05$ on $[1,1.5]$
20. $y'+3y=xy^2(y+1),\quad y(0)=1;\quad h=0.1$ on $[0,1]$
21. $y'-4y=\frac{x}{y^2(y+1)},\quad y(0)=1;\quad h=0.1$ on $[0,1]$
22. $y'+2y=\frac{x^2}{1+y^2},\quad y(2)=1;\quad h=0.1$, on $[2,3]$

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3.3: The Runge-Kutta Method

Each step in *Euler's method* requires one evaluation of f and each step in the *improved Euler method* requires two evaluations of f. We'll skip methods for which three evaluations of f are used and proceed to the *Runge-Kutta* method, the most widely used method, for which four evaluations of f are used to solve the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0. \tag{3.3.1}$$

The Runge-Kutta method computes approximate values $y_1, y_2, \ldots, y_n$ of the solution of Equation 3.3.1 at $x_0, x_0 + h, \ldots, x_0 + nh$ as follows: Given y_i, compute

$$\begin{aligned} k_{1i} &= f(x_i, y_i), \\ k_{2i} &= f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_{1i}\right), \\ k_{3i} &= f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_{2i}\right), \\ k_{4i} &= f(x_i + h, y_i + hk_{3i}), \end{aligned}$$

and

$$y_{i+1} = y_i + \frac{h}{6}(k_{1i} + 2k_{2i} + 2k_{3i} + k_{4i}).$$

Example 3.3.1 illustrates the computational procedure indicated in the Runge-Kutta method.

Example 3.3.1

Use the Runge-Kutta method with $h = 0.1$ to find approximate values for the solution of the initial value problem

$$y' + 2y = x^3 e^{-2x}, \quad y(0) = 1, \tag{3.3.2}$$

at $x = 0.1, 0.2$.

Solution

Again we rewrite Equation 3.3.2 as

$$y' = -2y + x^3 e^{-2x}, \quad y(0) = 1,$$

which is of the form Equation 3.3.1, with

$$f(x, y) = -2y + x^3 e^{-2x}, \ x_0 = 0, \text{ and } y_0 = 1.$$

The Runge-Kutta method yields

$$
\begin{aligned}
k_{10} &= f(x_0, y_0) = f(0, 1) = -2,\\
k_{20} &= f(x_0 + h/2, y_0 + hk_{10}/2) = f(.05, 1 + (.05)(-2))\\
&= f(.05, .9) = -2(.9) + (.05)^3 e^{-.1} = -1.799886895,\\
k_{30} &= f(x_0 + h/2, y_0 + hk_{20}/2) = f(.05, 1 + (.05)(-1.799886895))\\
&= f(.05, .910005655) = -2(.910005655) + (.05)^3 e^{-.1} = -1.819898206,\\
k_{40} &= f(x_0 + h, y_0 + hk_{30}) = f(.1, 1 + (.1)(-1.819898206))\\
&= f(.1, .818010179) = -2(.818010179) + (.1)^3 e^{-.2} = -1.635201628,\\
y_1 &= y_0 + \frac{h}{6}(k_{10} + 2k_{20} + 2k_{30} + k_{40}),\\
&= 1 + \frac{.1}{6}(-2 + 2(-1.799886895) + 2(-1.819898206) - 1.635201628) = .818753803,\\
k_{11} &= f(x_1, y_1) = f(.1, .818753803) = -2(.818753803)) + (.1)^3 e^{-.2} = -1.636688875,\\
k_{21} &= f(x_1 + h/2, y_1 + hk_{11}/2) = f(.15, .818753803 + (.05)(-1.636688875))\\
&= f(.15, .736919359) = -2(.736919359) + (.15)^3 e^{-.3} = -1.471338457,\\
k_{31} &= f(x_1 + h/2, y_1 + hk_{21}/2) = f(.15, .818753803 + (.05)(-1.471338457))\\
&= f(.15, .745186880) = -2(.745186880) + (.15)^3 e^{-.3} = -1.487873498,\\
k_{41} &= f(x_1 + h, y_1 + hk_{31}) = f(.2, .818753803 + (.1)(-1.487873498))\\
&= f(.2, .669966453) = -2(.669966453) + (.2)^3 e^{-.4} = -1.334570346,\\
y_2 &= y_1 + \frac{h}{6}(k_{11} + 2k_{21} + 2k_{31} + k_{41}),\\
&= .818753803 + \frac{.1}{6}(-1.636688875 + 2(-1.471338457) + 2(-1.487873498) - 1.334570346)\\
&= .670592417.
\end{aligned}
$$

The Runge-Kutta method is sufficiently accurate for most applications.

Examples Illustrating The Error in the Runge-Kutta Method

Example 3.3.2

Table 3.3.1 shows results of using the Runge-Kutta method with step sizes $h = 0.1$ and $h = 0.05$ to find approximate values of the solution of the initial value problem

$$y' + 2y = x^3 e^{-2x}, \quad y(0) = 1$$

at $x = 0,\ 0.1,\ 0.2,\ 0.3,\ \ldots,\ 1.0$. For comparison, it also shows the corresponding approximate values obtained with the improved Euler method in Example 3.2.2, and the values of the exact solution

$$y = \frac{e^{-2x}}{4}(x^4 + 4).$$

The results obtained by the Runge-Kutta method are clearly better than those obtained by the improved Euler method in fact; the results obtained by the Runge-Kutta method with $h = 0.1$ are better than those obtained by the improved Euler method with $h = 0.05$.

Table 3.3.1 : Numerical solution of $y' + 2y = x^3 e^{-2x},\ y(0) = 1$, by the Runge-Kuttta method and the improved Euler method.

	Improved Euler		Runge-Kutta		
x	**h=0.1**	**h=0.05**	**h=0.1**	**h-0.05**	**Exact**
0.0	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
0.1	0.820040937	0.819050572	0.818753803	0.818751370	0.818751221
0.2	0.672734445	0.671086455	0.670592417	0.670588418	0.670588174
0.3	0.552597643	0.550543878	0.549928221	0.549923281	0.549922980

	Improved Euler		Runge-Kutta		
x	h=0.1	h=0.05	h=0.1	h-0.05	Exact
0.4	0.455160637	0.452890616	0.452210430	0.452205001	0.452204669
0.5	0.376681251	0.374335747	0.373633492	0.373627899	0.373627557
0.6	0.313970920	0.311652239	0.310958768	0.310953242	0.310952904
0.7	0.264287611	0.262067624	0.261404568	0.261399270	0.261398947
0.8	0.225267702	0.223194281	0.222575989	0.222571024	0.222570721
0.9	0.194879501	0.192981757	0.192416882	0.192412317	0.192412038
1.0	0.171388070	0.169680673	0.169173489	0.169169356	0.169169104

Example 3.3.3

Table 3.3.2 shows analogous results for the nonlinear initial value problem

$$y' = -2y^2 + xy + x^2,\ y(0) = 1.$$

We applied the improved Euler method to this problem in Example 3.2.3.

Table 3.3.2 : Numerical solution of $y' = -2y^2 + xy + x^2,\ y(0) = 1$, by the Runge-Kuttta method and the improved Euler method.

	Improved Euler		Runge-Kutta		
x	h=0.1	h=0.05	h=0.1	h-0.05	"exact"
0.0	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
0.1	0.840500000	0.838288371	0.837587192	0.837584759	0.837584494
0.2	0.733430846	0.730556677	0.729644487	0.729642155	0.729641890
0.3	0.661600806	0.658552190	0.657582449	0.657580598	0.657580377
0.4	0.615961841	0.612884493	0.611903380	0.611901969	0.611901791
0.5	0.591634742	0.588558952	0.587576716	0.587575635	0.587575491
0.6	0.586006935	0.582927224	0.581943210	0.581942342	0.581942225
0.7	0.597712120	0.594618012	0.593630403	0.593629627	0.593629526
0.8	0.626008824	0.622898279	0.621908378	0.621907553	0.621907458
0.9	0.670351225	0.667237617	0.666251988	0.666250942	0.666250842
1.0	0.730069610	0.726985837	0.726017378	0.726015908	0.726015790

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3.3E: The Runge-Kutta Method (Exercises)

The following exercises involve initial value problems considered in the exercises in Sections 3.1 and 3.2. You'll find it instructive to compare the results that you obtain here with the corresponding results that you obtained in those sections.

In Exercises **1**-**5** use the Runge-Kutta method to find approximate values of the solution of the given initial value problem at the points $x_i = x_0 + ih$, where x_0 is the point where the initial condition is imposed and $i = 1, 2$.

1. $y' = 2x^2 + 3y^2 - 2, \quad y(2) = 1; \quad h = 0.05$
2. $y' = y + \sqrt{x^2 + y^2}\,, \quad y(0) = 1; \quad h = 0.1$
3. $y' + 3y = x^2 - 3xy + y^2, \quad y(0) = 2; \quad h = 0.05$
4. $y' = \frac{1+x}{1-y^2}, \quad y(2) = 3; \quad h = 0.1$
5. $y' + x^2 y = \sin xy, \quad y(1) = \pi; \quad h = 0.2$

6. Use the Runge-Kutta method with step size $h = 0.1$ to find approximate values of the solution of the initial value problem

$$y' + 3y = 7e^{4x}, \quad y(0) = 2,$$

at $x = 0, 0.1, 0.2, 0.3, \ldots, 1.0$. Compare these approximate values with the values of the exact solution $y = e^{4x} + e^{-3x}$, which can be obtained by the method of Section 2.3. Present your results in a table like Table 3.3.1.

7. Use the Runge-Kutta method with step sizes $h = 0.1$ to find approximate values of the solution of the initial value problem

$$y' + \frac{2}{x}y = \frac{3}{x^3} + 1, \quad y(1) = 1$$

at $x = 1.0, 1.1, 1.2, 1.3, \ldots, 2.0$. Compare these approximate values with the values of the exact solution

$$y = \frac{1}{3x^2}(9\ln x + x^3 + 2),$$

which can be obtained by the method of Section 2.3. Present your results in a table like Table 3.3.1.

8. Use the Runge-Kutta method with step sizes $h = 0.05$ to find approximate values of the solution of the initial value problem

$$y' = \frac{y^2 + xy - x^2}{x^2}, \quad y(1) = 2$$

at $x = 1.0, 1.05, 1.10, 1.15 \ldots, 1.5$. Compare these approximate values with the values of the exact solution

$$y = \frac{x(1 + x^2/3)}{1 - x^2/3},$$

obtained by a method in Section 2.4. Present your results in a table like Table 3.3.1.

9. It was shown in an example in Section 2.1 that

$$y^5 + y = x^2 + x - 4$$

is an implicit solution of the initial value problem

$$y' = \frac{2x+1}{5y^4+1}, \quad y(2) = 1. \tag{A}$$

Use the Runge-Kutta method with step size $h = 0.1$ to find approximate values of the solution of (A) at $x = 2.0, 2.1, 2.2, 2.3, \ldots, 3.0$. Present your results in tabular form. To check the error in these approximate values, construct another table of values of the residual

$$R(x, y) = y^5 + y - x^2 - x + 4$$

for each value of (x, y) appearing in the first table.

10. It was previously shown that

$$x^4y^3+x^2y^5+2xy=4$$

is an implicit solution of the initial value problem

$$y'=-\frac{4x^3y^3+2xy^5+2y}{3x^4y^2+5x^2y^4+2x},\quad y(1)=1. \tag{A}$$

Use the Runge-Kutta method with step size $h=0.1$ to find approximate values of the solution of (A) at $x=1.0$, 1.1, 1.2, 1.3, …, 2.0. Present your results in tabular form. To check the error in these approximate values, construct another table of values of the residual

$$R(x,y)=x^4y^3+x^2y^5+2xy-4$$

for each value of (x,y) appearing in the first table.

11. Use the Runge-Kutta method with step size $h=0.1$ to find approximate values of the solution of the initial value problem

$$(3y^2+4y)y'+2x+\cos x=0,\quad y(0)=1$$

at $x=0$, 0.1, 0.2, 0.3, …, 1.0.

12. Use the Runge-Kutta method with step sizes $h=0.1$ to find approximate values of the solution of the initial value problem

$$y'+\frac{(y+1)(y-1)(y-2)}{x+1}=0,\quad y(1)=0$$

at $x=1.0$, 1.1, 1.2, 1.3, …, 2.0.

13. Use the Runge-Kutta method and the Runge-Kutta semilinear method with step size $h=0.1$ to find approximate values of the solution of the initial value problem

$$y'+3y=e^{-3x}(1-4x+3x^2-4x^3),\quad y(0)=-3$$

at $x=0$, 0.1, 0.2, 0.3, …, 1.0. Compare these approximate values with the values of the exact solution $y=-e^{-3x}(3-x+2x^2-x^3+x^4)$, which can be obtained by the method of Section 2.3.

The linear initial value problems in Exercises **14-22** can't be solved exactly in terms of known elementary functions. In each exercise, use the Runge-Kutta method with the indicated step size to find approximate values of the solution of the given initial value problem at 11 equally spaced points (including the endpoints) in the interval.

14. $y'-2y=\frac{1}{1+x^2},\quad y(2)=2;\qquad h=0.1$ on $[2,3]$
15. $y'+2xy=x^2,\quad y(0)=3;\qquad h=0.2$ on $[0,2]$
16. $y'+\frac{1}{x}y=\frac{\sin x}{x^2},\quad y(1)=2;\qquad h=0.2$ on $[1,3]$
17. $y'+y=\frac{e^{-x}\tan x}{x},\quad y(1)=0;\qquad h=0.05$ on $[1,1.5]$
18. $y'+\frac{2x}{1+x^2}y=\frac{e^x}{(1+x^2)^2},\quad y(0)=1;\qquad h=0.2$ on $[0,2]$
19. $xy'+(x+1)y=e^{x^2},\quad y(1)=2;\qquad h=0.05$ on $[1,1.5]$
20. $y'+3y=xy^2(y+1),\quad y(0)=1;\qquad h=0.1$ on $[0,1]$
21. $y'-4y=\frac{x}{y^2(y+1)},\quad y(0)=1;\qquad h=0.1$ on $[0,1]$
22. $y'+2y=\frac{x^2}{1+y^2},\quad y(2)=1;\qquad h=0.1$ on $[2,3]$

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CHAPTER OVERVIEW

4: Applications of First Order Equations

In this chapter, we consider applications of first order differential equations.

4.1: Growth and Decay

4.1E: Growth and Decay (Exercises)

4.2: Cooling and Mixing

4.2E: Cooling and Mixing (Exercises)

4.3: Elementary Mechanics

4.3E: Elementary Mechanics (Exercises)

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4.1: Growth and Decay

This section begins with a discussion of exponential growth and decay, which you have probably already seen in calculus. We consider applications to radioactive decay, carbon dating, and compound interest. We also consider more complicated problems where the rate of change of a quantity is in part proportional to the magnitude of the quantity, but is also influenced by other other factors for example, a radioactive substance is manufactured at a certain rate, but decays at a rate proportional to its mass, or a saver makes regular deposits in a savings account that draws compound interest.

Since the applications in this section deal with functions of time, we'll denote the independent variable by t. If Q is a function of t, Q' will denote the derivative of Q with respect to t; thus,

$$Q' = \frac{dQ}{dt}.$$

Exponential Growth and Decay

One of the most common mathematical models for a physical process is the *exponential model*, where it is assumed that the rate of change of a quantity Q is proportional to Q; thus

$$Q' = aQ, \tag{4.1.1}$$

where a is the constant of proportionality.

This is a simple separable equation and the general solution of Equation 4.1.1 is

$$Q = ce^{at}$$

and the solution of the initial value problem

$$Q' = aQ, \quad Q(0) = Q_0$$

is

$$Q = Q_0 e^{at}. \tag{4.1.2}$$

Since the solutions of $Q' = aQ$ are exponential functions, we say that a quantity Q that satisfies this equation *grows exponentially* if $a > 0$, or *decays exponentially* if $a < 0$ (Figure 4.1.1).

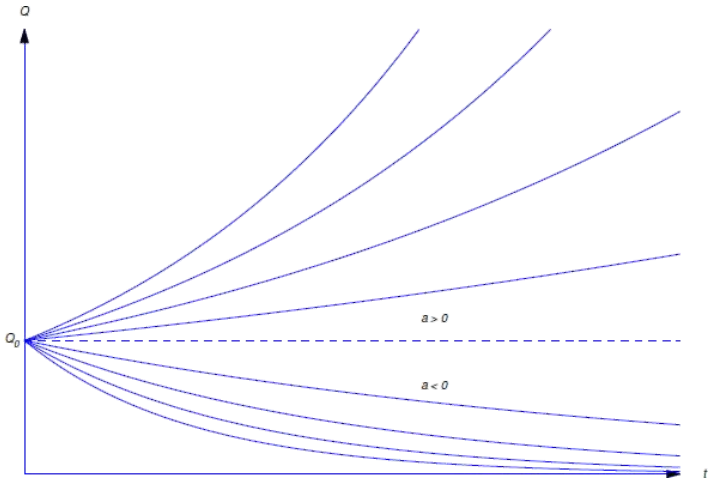


Figure 4.1.1 : Exponential growth and decay

Example 4.1.1

A radioactive substance has a half-life of 1620 years.

a. If its mass is now 4 g (grams), how much will be left 810 years from now?
b. Find the time t_1 when 1.5 g of the substance remain.

Solution a

Using our condition with $t_0 = 0$ and $Q_0 = 4$, we get

$$Q = 4e^{at}.$$

Knowing that $Q(1620) = 2$ we see that $2 = 4e^{at}$, and solving and simplifying we get

$$Q(t) = (\frac{1}{2})^{\frac{t}{1620}}$$

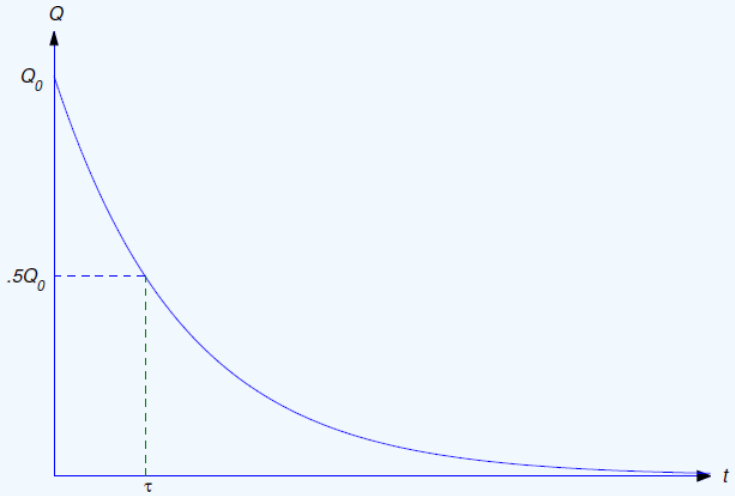


Figure 4.1.2 : Half-life of a radioactive substance

Therefore the mass left after 810 years will be

$$Q(810) \quad = 4(\tfrac{1}{2})^{\frac{810}{1620}} = 2\sqrt{2}\,\text{g}.$$

Solution b

Requiring that $Q(t) = 1.5$ yields

$$\frac{3}{2} = 4(\frac{1}{2})^{\frac{t}{1620}}.$$

Solving for t yields

$$t = 1620\frac{\ln(3/8)}{\ln(1/2)} \approx 2292.4 \text{ years}.$$

Interest Compounded Continuously

Suppose we deposit an amount of money Q_0 in an interest-bearing account and make no further deposits or withdrawals for t years, during which the account bears interest at a constant annual rate r. To calculate the value of the account at the end of t years, we need one more piece of information: how the interest is added to the account, or—as the bankers say—how it is *compounded*. If the interest is compounded annually, the value of the account is multiplied by $1 + r$ at the end of each year. This means that after t years the value of the account is

$$Q(t) = Q_0(1 + r)^t.$$

If interest is compounded semiannually, the value of the account is multiplied by $(1 + r/2)$ every 6 months. Since this occurs twice annually, the value of the account after t years is

$$Q(t) = Q_0\left(1 + \frac{r}{2}\right)^{2t}.$$

In general, if interest is compounded n times per year, the value of the account is multiplied n times per year by $(1 + r/n)$; therefore, the value of the account after t years is

$$Q(t) = Q_0\left(1+\frac{r}{n}\right)^{nt}. \tag{4.1.3}$$

Thus, increasing the frequency of compounding increases the value of the account after a fixed period of time. Table 4.1.1 shows the effect of increasing the number of compoundings over $t = 5$ years on an initial deposit of $Q_0 = 100$ (dollars), at an annual interest rate of 6%.

Table 4.1.1 : The effect of compound interest

n (number of compoundings per year)	$\$100\left(1+\frac{.06}{n}\right)^{5n}$ (value in dollars after 5 years)
1	\$133.82
2	\$134.39
4	\$134.68
8	\$134.83
364	\$134.98

You can see from Table 4.1.1 that the value of the account after 5 years is an increasing function of n. Now suppose the maximum allowable rate of interest on savings accounts is restricted by law, but the time intervals between successive compoundings isn't; then competing banks can attract savers by compounding often. The ultimate step in this direction is to *compound continuously*, by which we mean that $n \to \infty$ in Equation 4.1.3. Since we know from calculus that

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n} = e^r,$$

this yields

$$\begin{aligned} Q(t) &= \lim_{n\to\infty} Q_0\left(1+\tfrac{r}{n}\right)^{nt} = Q_0\left[\lim_{n\to\infty}\left(1+\tfrac{r}{n}\right)^{n}\right]^t \\ &= Q_0e^{rt}. \end{aligned}$$

Observe that $Q = Q_0e^{rt}$ is the solution of the initial value problem

$$Q' = rQ, \quad Q(0) = Q_0;$$

that is, with continuous compounding the value of the account grows exponentially.

Example 4.1.2

If \$150 is deposited in a bank that pays $5\frac{1}{2}\%$ annual interest compounded continuously, the value of the account after t years is

$$Q(t) = 150e^{.055t}$$

dollars. (Note that it is necessary to write the interest rate as a decimal; thus, $r = .055$.) Therefore, after $t = 10$ years the value of the account is

$$Q(10) = 150e^{.55} \approx \$259.99.$$

Example 4.1.3

We wish to accumulate \$10,000 in 10 years by making a single deposit in a savings account bearing $5\frac{1}{2}\%$ annual interest compounded continuously. How much must we deposit in the account?

Solution

The value of the account at time t is

$$Q(t) = Q_0e^{.055t}. \tag{4.1.4}$$

Since we want $Q(10)$ to be \$10,000, the initial deposit Q_0 must satisfy the equation

$$10000 = Q_0 e^{.55}, \tag{4.1.5}$$

obtained by setting $t = 10$ and $Q(10) = 10000$ in Equation 4.1.4. Solving Equation 4.1.5 for Q_0 yields

$$Q_0 = 10000e^{-.55} \approx \$5769.50.$$

Mixed Growth and Decay

Example 4.1.4

A radioactive substance with decay constant k is produced at a constant rate of a units of mass per unit time.

a. Assuming that $Q(0) = Q_0$, find the mass $Q(t)$ of the substance present at time t.
b. Find $\lim_{t\to\infty} Q(t)$.

Solution a:

Here

$$Q' = \text{ rate of increase of } Q + \text{ rate of decrease of } Q.$$

Note: the rate of decrease, as we would expect, will turn out to be negative.

The rate of increase is the constant a. Since Q is radioactive with decay constant k, the rate of decrease is kQ. Therefore

$$Q' = a + kQ.$$

This is a linear first order differential equation. Rewriting it and imposing the initial condition shows that Q is the solution of the initial value problem

$$Q' - kQ = a, \quad Q(0) = Q_0. \tag{4.1.6}$$

Solving we get

$$Q = -\frac{a}{k} + \left(Q_0 + \frac{a}{k}\right) e^{kt}. \tag{4.1.7}$$

Since $k < 0$, $\lim_{t\to\infty} e^{kt} = 0$, so from Equation 4.1.7

$$\lim_{t\to\infty} Q(t) = -\frac{a}{k}.$$

This limit depends only on a and k, and not on Q_0. We say that -a/k is the *steady state* value of Q. From Equation 4.1.7 we also see that Q approaches its steady state value from above if $Q_0 > -a/k$, or from below if $Q_0 < -a/k$. If $Q_0 = -a/k$, then Q remains constant (Figure 4.1.3).

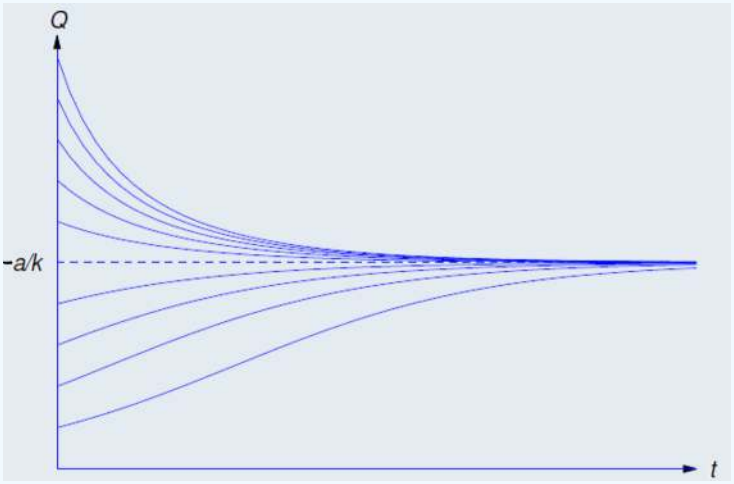


Figure 4.1.3 : $Q(t)$ approaches the steady state value $-a/k$ as $t \to \infty$

Carbon Dating

The fact that Q approaches a steady state value in the situation discussed in Example 4 underlies the method of *carbon dating*, devised by the American chemist and Nobel Prize Winner *W.S. Libby*.

Carbon 12 is stable, but carbon-14, which is produced by cosmic bombardment of nitrogen in the upper atmosphere, is radioactive with a half-life of about 5570 years. Libby assumed that the quantity of carbon-12 in the atmosphere has been constant throughout time, and that the quantity of radioactive carbon-14 achieved its steady state value long ago as a result of its creation and decomposition over millions of years. These assumptions led Libby to conclude that the ratio of carbon-14 to carbon-12 has been nearly constant for a long time. This constant, which we denote by R, has been determined experimentally.

Living cells absorb both carbon-12 and carbon-14 in the proportion in which they are present in the environment. Therefore the ratio of carbon-14 to carbon-12 in a living cell is always R. However, when the cell dies it ceases to absorb carbon, and the ratio of carbon-14 to carbon-12 decreases exponentially as the radioactive carbon-14 decays. This is the basis for the method of carbon dating, as illustrated in the next example.

Example 4.1.5

An archaeologist investigating the site of an ancient village finds a burial ground where the amount of carbon-14 present in individual remains is between 42 and 44% of the amount present in live individuals. Estimate the age of the village and the length of time for which it survived.

Solution

Let $Q = Q(t)$ be the quantity of carbon-14 in an individual set of remains t years after death, and let Q_0 be the quantity that would be present in live individuals. Since carbon-14 decays exponentially with half-life 5570 years, we get

$$Q(t) = Q_0\left(\frac{1}{2}\right)^{\frac{t}{5570}} \tag{4.1.8}$$

where we choose our time scale so that $t = 0$ is the time of death. If we know the present value of Q we can solve this equation for t, the number of years since death occurred.

It is given that $Q = .42Q_0$ in the remains of individuals who died first. Therefore these deaths occurred about

$$t_1 = 5570\frac{\ln .42}{\ln(1/2)} \approx 6971$$

years ago. For the most recent deaths, $Q = .44Q_0$; hence, these deaths occurred about

$$t_2 = 5570\frac{\ln .44}{\ln(1/2)} \approx 6597$$

years ago. Therefore it is reasonable to conclude that the village was founded about 7000 years ago, and lasted for about 400 years.

A Savings Program

Example 4.1.6

A person opens a savings account with an initial deposit of \$1000 and subsequently deposits \$50 per week. Find the value $Q(t)$ of the account at time $t > 0$, assuming that the bank pays 6% interest compounded continuously.

Solution

Observe that Q isn't continuous, since there are 52 discrete deposits per year of \$50 each. To construct a mathematical model for this problem in the form of a differential equation, we make the simplifying assumption that the deposits are made continuously at a rate of \$2600 per year. This is essential, since solutions of differential equations are continuous functions. With this assumption, Q increases continuously at the rate

$$Q' = 2600 + 0.06Q$$

and therefore Q satisfies the differential equation

$$Q' - .06Q = 2600.$$

(Of course, we must recognize that the solution of this equation is an approximation to the true value of Q at any given time. We'll discuss this further below.)

Solving we get

$$Q = -\frac{2600}{.06} + ce^{.06t}. \tag{4.1.9}$$

Setting $t = 0$ and $Q = 1000$ here yields

$$c = 1000 + \frac{2600}{0.06},$$

and substituting this into Equation 4.1.9 yields

$$Q = 1000e^{.06t} + \frac{2600}{.06}(e^{.06t} - 1) \tag{4.1.10}$$

where the first term is the value due to the initial deposit and the second is due to the subsequent weekly deposits.

Mathematical models must be tested for validity by comparing predictions based on them with the actual outcome of experiments. Example 6 is unusual in that we can compute the exact value of the account at any specified time and compare it with the approximate value predicted by Equation 4.1.10. The following table gives a comparison for a ten year period. Each exact answer corresponds to the time of the year-end deposit, and each year is assumed to have exactly 52 weeks.

Table 4.1.3

Year	Approximate Value of Q (Example 4.1.6)	Exact Value of P (*Exercise 4.1.21*)	Error $Q - P$	Percentage Error $(Q - P)/P$
1	\$3741.42	\$3739.87	\$1.55	.0413
2	6652.36	6649.17	3.19	.0479
3	9743.30	9738.37	4.93	.0506
4	13, 025.38	13, 018.60	6.78	.0521
5	16, 510.41	16, 501.66	8.75	.0530
6	20, 210.94	20, 200.11	10.83	.0536
7	24, 140.30	24, 127.25	13.05	.0541
8	28, 312.63	28, 297.23	15.40	.0544
9	32, 742.97	32, 725.07	17.90	.0547
10	37, 447.27	37, 426.72	20.55	.0549

The Logistic Equation

So far we have talked about the exponential growth and decay problem

$$N' = aN$$

When talking about population growth, in particular, there are often real-life restrictions that limit unbounded population growth. The differential equation resulting from this situation is called the *logistic equation*, which we discuss here.

In a logistic problem the coefficient, a, on N is no longer constant. Rather, it is a function of N itself resulting in

$$N' = f(N)N$$

Now, $f(N)$ can take on many different forms, but we will limit ourselves to when it describes a line $f(N) = a - \frac{a}{L}N$ where $f(0) = a$ and $f(L) = 0$ and L is the limiting value of N. Note that $f(0) = a$ as we would expect because in the beginning we would have pure exponential growth. However, as time goes on the growth rate would decrease until we reach the limiting population, leading to $f(L) = 0$.

For the logistic equations in this class we will only consider equations of the form

$$N' = (a - \frac{a}{L}N)N \tag{4.1.11}$$

Example 4.1.7

Sang's Boba Shop decides to advertise by putting flyers around campus at CRC. The time is in weeks and the limiting population that could see the flyers is estimated to be 10,000. If $N(0) = 500$ and $N(1) = 1000$, find

a. The number of people who have seen the flyers at any time.

b. How long it takes for 5000 people to see the flyer.

Solution a:

Using Equation 4.1.11 along with the conditions given we get the following IVP:

$$\frac{dN}{dt} = (a - \frac{a}{10,000}N)N; \quad N(0) = 500, N(1) = 1000$$

Although this is a separable equation, it would require a not so nice partial fractions, so we will use the fact that it is also a Bernoulli equation:

$$\frac{dN}{dt} - aN = \frac{-a}{10,000}N^2$$

Solving this and applying our conditions we get

$$N = \frac{10,000}{1 + 19(\frac{9}{19})^t}$$

Solution b:

Letting N=5000 we get

$$5000 = \frac{10,000}{1 + 19(\frac{9}{19})^t}$$

and solving for t we get

$$t = \frac{\ln(1/19)}{\ln(9/19)} \approx 3.94$$

So, it in about 4 weeks about 5000 people will have seen the flyer.

Nevertheless, Sang's Boba Shop went out of business because Sang is no business person...

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4.1E: Growth and Decay (Exercises)

In Exercises **1-6**, a radioactive substance decays at a rate proportional to the amount present at any time t.

1. The half-life of a radioactive substance is 3200 years. Find the quantity $Q(t)$ of the substance left at time $t>0$ if $Q(0)=20$ g.

2. The half-life of a radioactive substance is 2 days. Find the time required for a given amount of the material to decay to 1/10 of its original mass.

3. A radioactive material loses 25% of its mass in 10 minutes. What is its half-life?

4. A tree contains a known percentage p_0 of a radioactive substance with half-life τ. When the tree dies the substance decays and isn't replaced. If the percentage of the substance in the fossilized remains of such a tree is found to be p_1, how long has the tree been dead?

5. If t_p and t_q are the times required for a radioactive material to decay to $1/p$ and $1/q$ times its original mass (respectively), how are t_p and t_q related?

6. Find the decay constant k for a radioactive substance, given that the mass of the substance is Q_1 at time t_1 and Q_2 at time t_2.

7. A process creates a radioactive substance at the rate of 2 g/hr and the substance decays at a rate proportional to its mass, with constant of proportionality $k=.1(\text{hr})^{-1}$. If $Q(t)$ is the mass of the substance at time t, find $\lim_{t\to\infty} Q(t)$.

8. A bank pays interest continuously at the rate of 6%. How long does it take for a deposit of Q_0 to grow in value to $2Q_0$?

9. At what rate of interest, compounded continuously, will a bank deposit double in value in 8 years?

10. A savings account pays 5% per annum interest compounded continuously. The initial deposit is Q_0 dollars. Assume that there are no subsequent withdrawals or deposits.

    a. How long will it take for the value of the account to triple?
    b. What is Q_0 if the value of the account after 10 years is $100,000 dollars?

11. A candymaker makes 500 pounds of candy per week, while his large family eats the candy at a rate equal to $Q(t)/10$ pounds per week, where $Q(t)$ is the amount of candy present at time t.

    a. Find $Q(t)$ for $t>0$ if the candymaker has 250 pounds of candy at $t=0$.
    b. Find $\lim_{t\to\infty} Q(t)$.

12. Suppose a substance decays at a yearly rate equal to half the square of the mass of the substance present. If we start with 50 g of the substance, how long will it be until only 25 g remain?

13. A super bread dough increases in volume at a rate proportional to the volume V present. If V increases by a factor of 10 in 2 hours and $V(0)=V_0$, find V at any time t. How long will it take for V to increase to $100V_0$?

14. A radioactive substance decays at a rate proportional to the amount present, and half the original quantity Q_0 is left after 1500 years. In how many years would the original amount be reduced to $3Q_0/4$? How much will be left after 2000 years?

15. Phuong can magically create gold continuously at the rate of 1 ounce per hour, but her assistant PINK steals it continuously at the rate of 5% of however much is there per hour. Let $W(t)$ be the number of ounces that Phuong has at time t. Find $W(t)$ and $\lim_{t\to\infty} W(t)$ if $W(0)=1$.

16. A process creates a radioactive substance at the rate of 1 g/hr, and the substance decays at an hourly rate equal to 1/10 of the mass present (expressed in grams). Assuming that there are initially 20 g, find the mass $S(t)$ of the substance present at time t, and find $\lim_{t\to\infty} S(t)$.

17. A tank is empty at $t=0$. Water is added to the tank at the rate of 10 gal/min, but it leaks out at a rate (in gallons per minute) equal to the number of gallons in the tank. What is the smallest capacity the tank can have if this process is to continue forever?

18. A person deposits $25,000 in a bank that pays 5% per year interest, compounded continuously. The person continuously withdraws from the account at the rate of $750 per year. Find $V(t)$, the value of the account at time t after the initial deposit.

19. A person has a fortune that grows at rate proportional to the square root of its worth. Find the worth W of the fortune as a function of t (in years) if it was \$1 million six months ago and is \$4 million today.

20. Let $p = p(t)$ be the quantity of a product present at time t. The product is manufactured continuously at a rate proportional to p, with proportionality constant 1/2, and it is consumed continuously at a rate proportional to p^2, with proportionality constant 1/8. Find $p(t)$ if $p(0) = 100$.

21. A person infected with a virus goes to an isolated island inhabited by 1000 (including himself) people. If the rate of virus spread is proportional to the product of the number of infected people and the number of people not infected, how many people were infected after 6 days if the number of people infected after 4 days is 50.

22. A lake with an estimated limiting population of 6000 fish was stocked with 1000 fish in the year 2000. In 2007 it was estimated that there were 3000 fish. How many fish were in the lake in 2021?

23. In 2010 there were approximately 700 monkeys living on an island. In 2020 there were about 1000 of them. If the estimated limiting population is 7000, how many monkeys should we expect to be living on the island in 2030?

24. In 1994 there were approximately 500 sea otters living near an island in the pacific. In 2005 there were about 700 otters. If it is estimated that the waters around the island can support a maximum of 3000 otters, find how many otters will be living there in 2025.

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4.2: Cooling and Mixing

Newton's Law of Cooling

Newton's law of cooling states that if an object with temperature $T(t)$ at time t is in a medium with temperature $T_m(t)$, the rate of change of T at time t is proportional to $T(t) - T_m(t)$; thus, T satisfies a differential equation of the form

$$T' = k(T - T_m). \tag{4.2.1}$$

For simplicity, in this section we'll assume that the medium is maintained at a constant temperature T_m. This is another example of building a simple mathematical model for a physical phenomenon. Like most mathematical models it has its limitations. For example, it is reasonable to assume that the temperature of a room remains approximately constant if the cooling object is a cup of coffee, but perhaps not if it is a huge cauldron of molten metal.

To solve Equation 4.2.1, we can simply use separation of variables.

Example 4.2.1

A ceramic insulator is baked at 400°C and cooled in a room in which the temperature is 25°C. After 4 minutes the temperature of the insulator is 200°C. What is its temperature after 8 minutes?

Solution

Here $T_m = 25$, so Equation 4.2.1 becomes

$$\frac{dT}{dt} = k(T - 25) \tag{4.2.2}$$

Separating variables and solving we get

$$T = ce^{kt} + 25. \tag{4.2.3}$$

We determine c and k from the stated conditions that $T(0) = 400$ and $T(4) = 200$; solving we get

$$T = 375(7/15)^{(t/4)} + 25,$$

the graph of which is shown in Figure 4.2.1 below.

Therefore the temperature of the insulator after 8 minutes is

$$T(8) = 375(7/15)^{(8/4)} + 25 \approx 107^\circ\text{C}.$$

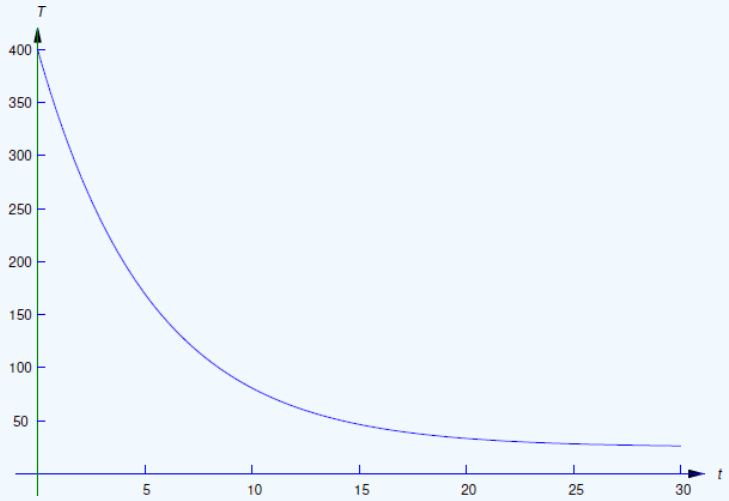


Figure 4.2.1 : $T = 375(7/15)^{(t/4)} + 25$

Example 4.2.2

An object with temperature 72°F is placed outside, where the temperature is -20°F. At 11:05 the temperature of the object is 60°F and at 11:07 its temperature is 50°F. At what time was the object placed outside?

Solution

Let $T(t)$ be the temperature of the object at time t. For convenience, we choose the origin $t_0 = 0$ of the time scale to be 11:05 so that $T(0) = 60$. We must determine the time ϕ when $T(\phi) = 72$. Substituting $T_m = -20$ into Equation 4.2.1 yields

$$\frac{dT}{dt} = k(T + 20) \tag{4.2.4}$$

Separating variables and solving we get

$$T = ce^{kt} - 20. \tag{4.2.5}$$

We determine c and k from the stated conditions that $T(0) = 60$ and $T(2) = 50$; solving we get

$$T = 80(7/8)^{(t/2)} - 20,$$

the graph of which is shown in Figure 4.2.2 below.

and the condition $T(\phi) = 72$ implies that

$$72 = 80(7/8)^{(\phi/2)} - 20;$$

hence,

$$\phi = 2\frac{\ln \frac{92}{80}}{\ln \frac{7}{8}} \approx -2.09 \text{ min}.$$

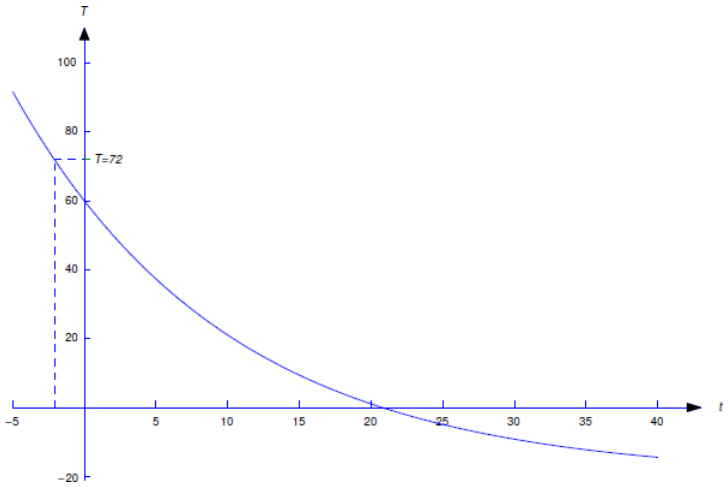


Figure 4.2.2 : $T = 80(7/8)^{(t/2)} - 20$

Therefore the object was placed outside about 2 minutes and 5 seconds before 11:05; that is, at 11:02:55.

Mixing Problems

In the next two examples a saltwater solution with a given concentration (weight of salt per unit volume of solution) is added at a specified rate to a tank that initially contains saltwater with a different concentration. The problem is to determine the quantity of salt in the tank as a function of time. This is an example of a *mixing problem*. To construct a tractable mathematical model for mixing problems we assume in our examples (and most exercises) that the mixture is stirred instantly so that the salt is always uniformly distributed throughout the mixture.

Example 4.2.3

A tank initially contains 40 pounds of salt dissolved in 600 gallons of water. Starting at $t_0 = 0$, water that contains 1/2 pound of salt per gallon is poured into the tank at the rate of 4 gal/min and the mixture is drained from the tank at the same rate

(Figure 4.2.3).

a. Find a differential equation for the quantity $Q(t)$ of salt in the tank at time $t > 0$, and solve the equation to determine $Q(t)$.
b. Find $\lim_{t\to\infty} Q(t)$.

Solution a

To find a differential equation for Q, we must use the given information to derive an expression for Q'. But Q' is the rate of change of the quantity of salt in the tank changes with respect to time; thus, if *rate in* denotes the rate at which salt enters the tank and *rate out* denotes the rate by which it leaves, then

$$Q' = \text{rate in} - \text{rate out}. \tag{4.2.6}$$

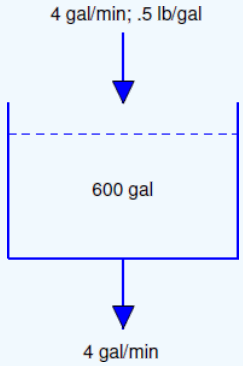


Figure 4.2.3 : A mixing problem

The rate in is

$$\left(\frac{1}{2}\ \text{lb/gal}\right) \times (4\ \text{gal/min}) = 2\ \text{lb/min}.$$

Determining the rate out requires a little more thought. We're removing 4 gallons of the mixture per minute, and there are always 600 gallons in the tank; that is, we are removing $1/150$ of the mixture per minute. Since the salt is evenly distributed in the mixture, we are also removing $1/150$ of the salt per minute. Therefore, if there are $Q(t)$ pounds of salt in the tank at time t, the rate out at any time t is $Q(t)/150$. Alternatively, we can arrive at this conclusion by arguing that

$$\begin{aligned}\text{rate out} &= (\text{concentration}) \times (\text{rate of flow out}) \\ &= (\text{lb/gal}) \times (\text{gal/min}) \\ &= \frac{Q(t)}{600} \times 4 \\ &= \frac{Q(t)}{150}.\end{aligned}$$

We can now write Equation 4.2.6 as

$$Q' = 2 - \frac{Q}{150}.$$

This first order equation can be rewritten as

$$Q' + \frac{Q}{150} = 2,$$

which is a linear equation; solving we get

$$Q = 300 - 260e^{-t/150},$$

the graph of which is shown in Figure 4.2.4 below.

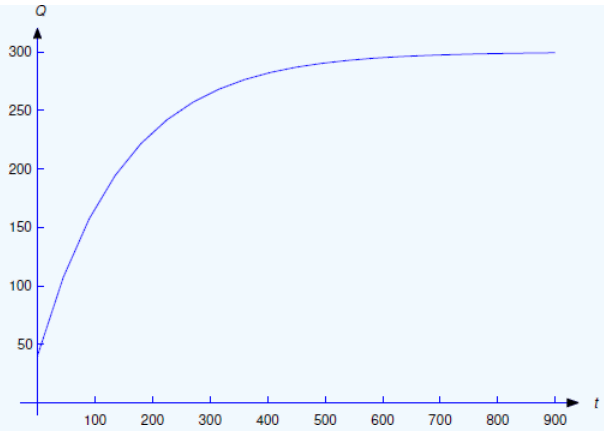


Figure 4.2.4 : $Q = 300 - 260e^{-t/150}$

Solution b

It is easy to see from calculus that $\lim_{t\to\infty} Q(t) = 300$ for any value of $Q(0)$. This is intuitively reasonable, since the incoming solution contains 1/2 pound of salt per gallon and there are always 600 gallons of water in the tank.

Example 4.2.4

A 500-liter tank initially contains 10 g of salt dissolved in 200 liters of water. Starting at $t_0 = 0$, water that contains 1/4 g of salt per liter is poured into the tank at the rate of 4 liters/min and the mixture is drained from the tank at the rate of 2 liters/min (Figure 4.2.5). Find a differential equation for the quantity $Q(t)$ of salt in the tank at time t prior to the time when the tank overflows and find the concentration $K(t)$ (g/liter) of salt in the tank at any such time.

Solution

We first determine the amount $W(t)$ of solution in the tank at any time t prior to overflow. Since $W(0) = 200$ and we are adding 4 liters/min while removing only 2 liters/min, there's a net gain of 2 liters/min in the tank; therefore,

$$W(t) = 2t + 200.$$

Since $W(150) = 500$ liters (capacity of the tank), this formula is valid for $0 \le t \le 150$.

Now let $Q(t)$ be the number of grams of salt in the tank at time t, where $0 \le t \le 150$. As in Example 4.2.3

$$Q' = \text{rate in} - \text{rate out}. \tag{4.2.7}$$

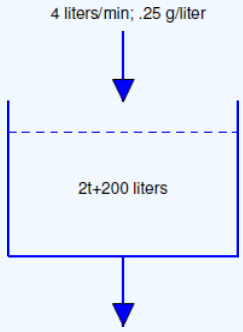


Figure 4.2.5 : Another mixing problem

The rate in is

$$\left(\frac{1}{4}\ \text{g/liter}\right) \times (4\ \text{liters/min}) = 1\ \text{g/min}. \tag{4.2.8}$$

To determine the rate out, we observe that since the mixture is being removed from the tank at the constant rate of 2 liters/min and there are $2t + 200$ liters in the tank at time t, the fraction of the mixture being removed per minute at time t is

$$\frac{2}{2t+200} = \frac{1}{t+100}.$$

We're removing this same fraction of the salt per minute. Therefore, since there are $Q(t)$ grams of salt in the tank at time t,

$$\text{rate out} = \frac{Q(t)}{t+100}. \tag{4.2.9}$$

Alternatively, we can arrive at this conclusion by arguing that

$$\begin{aligned} \text{rate out} &= (\text{concentration}) \times (\text{rate of flow out}) = (\text{g/liter}) \times (\text{liters/min}) \\ &= \tfrac{Q(t)}{2t+200} \times 2 = \tfrac{Q(t)}{t+100}. \end{aligned}$$

Substituting Equation 4.2.8 and Equation 4.2.9 into Equation 4.2.7 yields

$$Q' = 1 - \frac{Q}{t+100}, \quad \text{so} \quad Q' + \frac{1}{t+100}Q = 1, \tag{4.2.10}$$

which is a linear equation; solving we get

$$Q = \frac{t+100}{2} - \frac{4000}{t+100}.$$

Now let $K(t)$ be the concentration of salt at time t. Then

$$K(t) = \frac{1}{4} - \frac{2000}{(t+100)^2}$$

This is shown in Figure 4.2.6 .

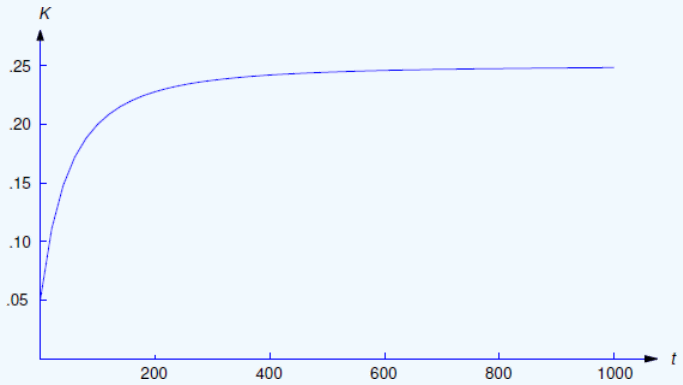


Figure 4.2.6 : $K(t) = \frac{1}{4} - \frac{2000}{(t+100)^2}$

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4.2E: Cooling and Mixing (Exercises)

1. A thermometer is moved from a room where the temperature is 70°F to a freezer where the temperature is $12^\circ F$. After 30 seconds the thermometer reads 40°F. What does it read after 2 minutes?

2. A fluid initially at 100°C is placed outside on a day when the temperature is -10°C, and the temperature of the fluid drops 20°C in one minute. Find the temperature $T(t)$ of the fluid for $t > 0$.

3. At 12:00 pm a thermometer reading 10°F is placed in a room where the temperature is 70°F. It reads 56° when it is placed outside, where the temperature is 5°F, at 12:03. What does it read at 12:05 pm?

4. A thermometer initially reading 212°F is placed in a room where the temperature is 70°F. After 2 minutes the thermometer reads 125°F.

   a. What does the thermometer read after 4 minutes?
   b. When will the thermometer read 72°F?
   c. When will the thermometer read 69°F?

5. An object with initial temperature 150°C is placed outside, where the temperature is 35°C. Its temperatures at 12:15 and 12:20 are 120°C and 90°C, respectively.

   a. At what time was the object placed outside?
   b. When will its temperature be 40°C?

6. An object is placed in a room where the temperature is 20°C. The temperature of the object drops by 5°C in 4 minutes and by 7°C in 8 minutes. What was the temperature of the object when it was initially placed in the room?

7. A cup of boiling water (212° F) is placed outside at 1:00 pm. One minute later the temperature of the water is 152°F. After another minute its temperature is 112°F. What is the outside temperature?

8. A tank initially contains 40 gallons of pure water. A solution with 1 gram of salt per gallon of water is added to the tank at 3 gal/min, and the resulting solution drains out at the same rate. Find the quantity $Q(t)$ of salt in the tank at time $t > 0$.

9. A tank initially contains a solution of 10 pounds of salt in 60 gallons of water. Water with $1/2$ pound of salt per gallon is added to the tank at 6 gal/min, and the resulting solution leaves at the same rate. Find the quantity $Q(t)$ of salt in the tank at time $t > 0$.

10. A tank initially contains 100 liters of a salt solution with a concentration of .1 g/liter. A solution with a salt concentration of .3 g/liter is added to the tank at 5 liters/min, and the resulting mixture is drained out at the same rate. Find the concentration $K(t)$ of salt in the tank as a function of t.

11. A 200 gallon tank initially contains 100 gallons of water with 20 pounds of salt. A salt solution with $1/4$ pound of salt per gallon is added to the tank at 4 gal/min, and the resulting mixture is drained out at 2 gal/min. Find the quantity of salt in the tank as it is about to overflow.

12. Suppose water is added to a tank at 10 gal/min, but leaks out at the rate of $1/5$ gal/min for each gallon in the tank. What is the smallest capacity the tank can have if the process is to continue indefinitely?

13. A chemical reaction in a laboratory with volume V (in ft^3) produces q_1 ft^3/min of a noxious gas as a byproduct. The gas is dangerous at concentrations greater than $\bar{c}$, but harmless at concentrations $\leq \bar{c}$. Intake fans at one end of the laboratory pull in fresh air at the rate of q_2 ft^3/min and exhaust fans at the other end exhaust the mixture of gas and air from the laboratory at the same rate. Assuming that the gas is always uniformly distributed in the room and its initial concentration c_0 is at a safe level, find the smallest value of q_2 required to maintain safe conditions in the laboratory for all time.

14. A 1200-gallon tank initially contains 40 pounds of salt dissolved in 600 gallons of water. Starting at $t_0 = 0$, water that contains $1/2$ pound of salt per gallon is added to the tank at the rate of 6 gal/min and the resulting mixture is drained from the tank at 4 gal/min. Find the quantity $Q(t)$ of salt in the tank at any time $t > 0$ prior to overflow.

15. Tank T_1 initially contain 50 gallons of pure water. Starting at $t_0 = 0$, water that contains 1 pound of salt per gallon is poured into T_1 at the rate of 2 gal/min. The mixture is drained from T_1 at the same rate into a second tank T_2, which initially contains 50 gallons of pure water. Also starting at $t_0 = 0$, a mixture from another source that contains 2 pounds of salt per gallon is poured into T_2 at the rate of 2 gal/min. The mixture is drained from T_2 at the rate of 4 gal/min.

   a. Find a differential equation for the quantity $Q(t)$ of salt in tank T_2 at time $t > 0$.

b. Solve the equation derived in (a) to determine $Q(t)$.
c. Find $\lim_{t\to\infty} Q(t)$.

16. Suppose an object with initial temperature T_0 is placed in a sealed container, which is in turn placed in a medium with temperature T_m. Let the initial temperature of the container be S_0. Assume that the temperature of the object does not affect the temperature of the container, which in turn does not affect the temperature of the medium. (These assumptions are reasonable, for example, if the object is a cup of coffee, the container is a house, and the medium is the atmosphere.)

a. Assuming that the container and the medium have distinct temperature decay constants k and k_m respectively, use Newton's law of cooling to find the temperatures $S(t)$ and $T(t)$ of the container and object at time t.
b. Assuming that the container and the medium have the same temperature decay constant k, use Newton's law of cooling to find the temperatures $S(t)$ and $T(t)$ of the container and object at time t.
c. Find $\lim_{t\to\infty} S(t)$ and $\lim_{t\to\infty} T(t)$.

17. In our previous examples and exercises concerning Newton's law of cooling we assumed that the temperature of the medium remains constant. This model is adequate if the heat lost or gained by the object is insignificant compared to the heat required to cause an appreciable change in the temperature of the medium. If this isn't so, we must use a model that accounts for the heat exchanged between the object and the medium. Let $T = T(t)$ and $T_m = T_m(t)$ be the temperatures of the object and the medium, respectively, and let T_0 and T_{m0} be their initial values. Again, we assume that T and T_m are related by Newton's law of cooling,

$$T' = -k(T - T_m). \tag{A}$$

We also assume that the change in heat of the object as its temperature changes from T_0 to T is $a(T - T_0)$ and that the change in heat of the medium as its temperature changes from T_{m0} to T_m is $a_m(T_m - T_{m0})$, where a and a_m are positive constants depending upon the masses and thermal properties of the object and medium, respectively. If we assume that the total heat of the system consisting of the object and the medium remains constant (that is, energy is conserved), then

$$a(T - T_0) + a_m(T_m - T_{m0}) = 0. \tag{B}$$

a. Equation (A) involves two unknown functions T and T_m. Use (A) and (B) to derive a differential equation involving only T.
b. Find $T(t)$ and $T_m(t)$ for $t > 0$.
c. Find $\lim_{t\to\infty} T(t)$ and $\lim_{t\to\infty} T_m(t)$.

18. Control mechanisms allow fluid to flow into a tank at a rate proportional to the volume V of fluid in the tank, and to flow out at a rate proportional to V^2. Suppose $V(0) = V_0$ and the constants of proportionality are a and b, respectively. Find $V(t)$ for $t > 0$ and find $\lim_{t\to\infty} V(t)$.

19. A small lake containing 100,000 liters of water was contaminated with a pesticide from a train spill and 10 grams of the pesticide ended up in the lake. A giant named Roy, who lives near the lake, likes to keep it well stirred with his very large stick. If pure water is entering the lake at 300 liters per day and the well-mixed lake water leaves the lake at 200 liters per day, find:

a. The amount of the toxic substance in the lake at any time t.

b. How long it will take until the toxic substance is at a "safe" level of 1 gram.

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4.3: Elementary Mechanics

Newton's Second Law of Motion

In this section we consider an object with constant mass m moving along a line under a force F. Let $y = y(t)$ be the displacement of the object from a reference point on the line at time t, and let $v = v(t)$ and $a = a(t)$ be the velocity and acceleration of the object at time t. Thus, $v = y'$ and $a = v' = y''$, where the prime denotes differentiation with respect to t. Newton's second law of motion asserts that the force F and the acceleration a are related by the equation

$$F = ma. \tag{4.3.1}$$

Note: Units

In applications there are three main sets of units in use for length,force, and mass: the cgs, mks, and British systems. All three use the second as the unit of time. Table 4.3.1 shows the other units. Consistent with Equation 4.3.1, the unit of force in each system is defined to be the force required to impart an acceleration of (one unit of length)$/s^2$ to one unit of mass.

Table 4.3.1

Set	Length	Force	Mass
cgs	centimeter (cm)	dyne (d)	gram (g)
mks	meter (m)	newton (N)	kilogram (kg)
British	foot (ft)	pound (lb)	slug (sl)

If we assume that Earth is a perfect sphere with constant mass density, Newton's law of gravitation (discussed later in this section) asserts that the force exerted on an object by Earth's gravitational field is proportional to the mass of the object and inversely proportional to the square of its distance from the center of Earth. However, if the object remains sufficiently close to Earth's surface, we may assume that the gravitational force is constant and equal to its value at the surface. The magnitude of this force is mg, where g is called the *acceleration due to gravity*. (To be completely accurate, g should be called the *magnitude of the acceleration due to gravity at Earth's surface*.) This quantity has been determined experimentally. Approximate values of g are

$$\begin{aligned} g &= 980 \text{ cm/s}^2 && \text{(cgs)} \\ g &= 9.8 \text{ m/s}^2 && \text{(mks)} \\ g &= 32 \text{ ft/s}^2 && \text{(British).} \end{aligned}$$

In general, the force F in Equation 4.3.1 may depend upon t, y, and y'. Since $a = y''$, Equation 4.3.1 can be written in the form

$$my'' = F(t, y, y'), \tag{4.3.2}$$

which is a second order equation. We'll consider this equation with restrictions on F later; however, since Chapter 2 dealt only with first order equations, we consider here only problems in which Equation 4.3.2 can be recast as a first order equation. This is possible if F does not depend on y, so Equation 4.3.2 is of the form

$$my'' = F(t, y').$$

Letting $v = y'$ and $v' = y''$ yields a first order equation for v:

$$mv' = F(t, v). \tag{4.3.3}$$

Solving this equation yields v as a function of t. If we know $y(t_0)$ for some time t_0, we can integrate v to obtain y as a function of t.

Equations of the form Equation 4.3.3 occur in problems involving motion through a resisting medium.

Motion Through a Resisting Medium Under Constant Force

Now we consider an object moving vertically in some medium. We assume that the only forces acting on the object are gravity and resistance from the medium. We also assume that the motion takes place close to Earth's surface and take the upward direction to be positive, so the gravitational force can be assumed to have the constant value $-mg$. We'll see that, under reasonable assumptions on the resisting force, the velocity approaches a limit as $t \to \infty$. We call this limit the *terminal velocity*.

If an object with mass m moves under constant gravitational force through a medium that exerts a resistance with magnitude proportional to the speed of the object (Recall that the speed of an object is $|v|$, the absolute value of its velocity v), then the total force acting on the object is

$$F = -mg + F_1, \tag{4.3.4}$$

where $-mg$ is the force due to gravity and F_1 is the resisting force of the medium, which has magnitude $k|v|$, where k is a positive constant. If the object is moving downward ($v \leq 0$), the resisting force is upward (Figure 4.3.1a), so

$$F_1 = k|v| = k(-v) = -kv.$$

On the other hand, if the object is moving upward ($v \geq 0$), the resisting force is downward (Figure 4.3.1b), so

$$F_1 = -k|v| = -kv.$$

Thus, Equation 4.3.4 can be written as

$$F = -mg - kv, \tag{4.3.5}$$

regardless of the sign of the velocity.

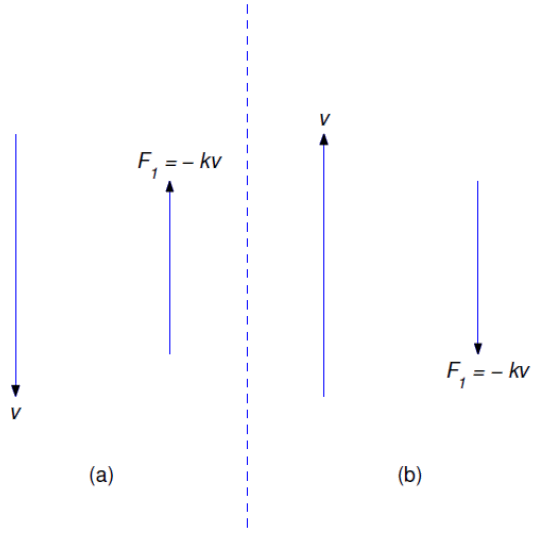


Figure 4.3.1 : Resistive forces

From Newton's second law of motion,

$$F = ma = mv',$$

so Equation 4.3.5 yields

$$mv' = -mg - kv,$$

or

$$v' + \frac{k}{m}v = -g, \tag{4.3.6}$$

which is a linear equation.

Example 4.3.1 Terminal Velocity

A 960-lb object is given an initial upward velocity of 60 ft/s near the surface of Earth. The atmosphere resists the motion with a force of 3 lb for each ft/s of speed. Assuming that the only other force acting on the object is constant gravity, find its velocity v as a function of t, and find its terminal velocity.

Solution

Since $mg = 960$ and $g = 32$, $m = 960/32 = 30$. The atmospheric resistance is $-3v$ lb if v is expressed in feet per second. Therefore

$$30v' = -960 - 3v,$$

which we rewrite as

$$v' + \frac{1}{10}v = -32,$$

which is a linear equation; solving we get

$$v = -320 + ce^{-t/10}. \tag{4.3.7}$$

The initial velocity is 60 ft/s in the upward (positive) direction; hence, $v(0) = 60$. Substituting $t = 0$ and $v = 60$ yields

$$60 = -320 + c,$$

so $c = 380$, and our solution becomes

$$v = -320 + 380e^{-t/10} \text{ ft/s}$$

The terminal velocity is

$$\lim_{t\to\infty} v(t) = -320 \text{ ft/s}.$$

Example 4.3.2

A 10 kg mass is given an initial downward velocity of 4 m/s near Earth's surface. The only forces acting on it are gravity and atmospheric resistance proportional to the square of the speed. Assuming that the resistance is 8 N if the speed is 2 m/s, find the velocity of the object as a function of t, and find the terminal velocity.

Solution

Since the object is falling, the resistance is in the upward (positive) direction. Hence,

$$mv' = -mg + kv^2, \tag{4.3.8}$$

where k is a constant. Since the magnitude of the resistance is 8 N when $v = 2$ m/s,

$$k(2^2) = 8,$$

so $k = 2$ N-s^2/m^2. Since $m = 10$ and $g = 9.8$, Equation 4.3.8 becomes

$$10v' = -98 + 2v^2 = 2(v^2 - 49). \tag{4.3.9}$$

We separate variables to obtain

$$\frac{1}{v^2 - 49}v' = \frac{1}{5}, \tag{4.3.10}$$

which is convenient for the required partial fraction expansion

$$\frac{1}{v^2 - 49} = \frac{1}{(v-7)(v+7)} = \frac{1}{14}\left[\frac{1}{v-7} - \frac{1}{v+7}\right]. \tag{4.3.11}$$

Substituting Equation 4.3.11 into Equation 4.3.10 yields

$$\frac{1}{14}\left[\frac{1}{v-7}-\frac{1}{v+7}\right]v'=\frac{1}{5},$$

so

$$\left[\frac{1}{v-7}-\frac{1}{v+7}\right]dv=\frac{14}{5}dt.$$

Integrating this yields

$$\ln|v-7|-\ln|v+7|=14t/5+k.$$

Therefore

$$\left|\frac{v-7}{v+7}\right|=e^k e^{14t/5}.$$

We can rewrite the last equation as

$$\frac{v-7}{v+7}=ce^{14t/5}, \tag{4.3.12}$$

which is an implicit solution of Equation 4.3.9. Solving this for v yields

$$v=-7\frac{c+e^{-14t/5}}{c-e^{-14t/5}}. \tag{4.3.13}$$

Since $v(0)=-4$, it Equation 4.3.12implies that

$$c=\frac{-4-7}{-4+7}=\frac{-11}{3}.$$

Substituting this into Equation 4.3.13and simplifying yields

$$v=-7\frac{-11+3e^{-14t/5}}{-11-3e^{-14t/5}}.$$

Since $v_0\le 0$, v is defined and negative for all $t>0$. The terminal velocity is

$$\lim_{t\to\infty}v(t)=-7\text{ m/s},$$

which should be noted, is independent of v_0.

Example 4.3.3

A 10-kg mass is launched vertically upward from Earth's surface with an initial velocity of 3 m/s. The only forces acting on the mass are gravity and atmospheric resistance proportional to the square of the speed. Assuming that the atmospheric resistance is 8 N if the speed is 2 m/s, find the time T required for the mass to reach maximum altitude.

Solution

The mass will climb while $v>0$ and reach its maximum altitude when $v=0$. Therefore

$$10v'=-98-2v^2. \tag{4.3.14}$$

Separating variables yields

$$\frac{5}{v^2+49}v'=-1,$$

and integrating this yields

$$\frac{5}{7}\tan^{-1}\frac{v}{7}=-t+c.$$

(Recall that $\tan^{-1} u$ is the number θ such that $-\pi/2 < \theta < \pi/2$ and $\tan\theta = u$.) Since $v(0) = 3$,

$$c = \frac{5}{7}\tan^{-1}\frac{3}{7},$$

so v is defined implicitly by

$$\frac{5}{7}\tan^{-1}\frac{v}{7} = -t + \frac{5}{7}\tan^{-1}\frac{3}{7}, \quad 0 \le t \le T. \tag{4.3.15}$$

Solving this for v yields

$$v = 7\tan\left(-\frac{7t}{5} + \tan^{-1}\frac{3}{7}\right). \tag{4.3.16}$$

Since $v(T) = 0$, we get

$$0 = 7\tan\left(-\frac{7T}{5} + \tan^{-1}\frac{3}{7}\right). \tag{4.3.17}$$

Therefore,

$$-\frac{7T}{5} + \tan^{-1}\frac{3}{7} = 0 \tag{4.3.18}$$

and

$$T = \frac{5}{7}\tan^{-1}\frac{3}{7} \approx 0.289s.$$

Escape Velocity

Suppose a space vehicle is launched vertically from a body and assume that the gravitational forces of all other celestial bodies can be neglected. The motion of the vehicle for $t > 0$ is that of an object with constant mass m under the influence of a body's gravitational force, which we now assume to vary inversely with the square of the distance from body's center; thus, if we take the upward direction to be positive then the gravitational force on the vehicle at an altitude y above the body is

$$F = -\frac{K}{(y+R)^2}, \tag{4.3.19}$$

where R is the body's radius (Figure 4.3.2).

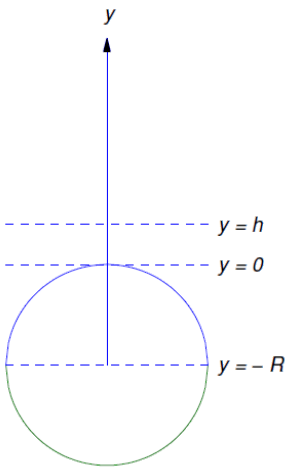


Figure 4.3.2 : Escape velocity

Since $F = -mg$ when $y = 0$, setting $y = 0$ in Equation 4.3.19 yields

$$-mg = -\frac{K}{R^2};$$

therefore $K = mgR^2$ and Equation 4.3.19 can be written more specifically as

$$F = -\frac{mgR^2}{(y+R)^2}. \tag{4.3.20}$$

From Newton's second law of motion,

$$F = m\frac{d^2y}{dt^2},$$

so Equation 4.3.20 implies that

$$\frac{d^2y}{dt^2} = -\frac{gR^2}{(y+R)^2}. \tag{4.3.21}$$

Since Equation 4.3.21 is second order, we cannot solve it by methods discussed so far. However, we are concerned with v rather than y, and v is easier to find. Since $v = y'$ the chain rule implies that

$$\frac{d^2y}{dt^2} = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy}.$$

Substituting this into Equation 4.3.21 yields the first order separable equation

$$v\frac{dv}{dy} = -\frac{gR^2}{(y+R)^2}. \tag{4.3.22}$$

We now consider v to be a function of y with the initial velocity being $v(0) = v_0$. Therefore we can obtain v as a function of y by solving the initial value problem

$$v\frac{dv}{dy} = -\frac{gR^2}{(y+R)^2}, \quad v(0) = v_0.$$

Equation 4.3.22 is a separable equation which yields

$$\frac{v^2}{2} = \frac{gR^2}{y+R} + c. \tag{4.3.23}$$

Since $v(0) = v_0$,

$$c = \frac{v_0^2}{2} - gR,$$

so Equation 4.3.23 becomes

$$\frac{v^2}{2} = \frac{gR^2}{y+R} + \left(\frac{v_0^2}{2} - gR\right).$$

Solving for $v(y)$ we get

$$v(y) = \sqrt{\frac{2gR^2}{y+R} + v_0^2 - 2gR} \tag{4.3.24}$$

If the object is to escape the gravitational field of the body, y needs to not become negative. With this in mind we consider what happens if the object were to keep going indefinitely. We do this by letting y approach ∞ in 4.3.24 which leads to v approaching

$$\sqrt{v_0^2 - 2gR}. \tag{4.3.25}$$

The parenthetical expression in Equation 4.3.25 is nonnegative, so $v_0^2 - 2gR \geq 0$, which gives us $v_0 \geq \sqrt{2gR}$. This proves that there's an escape velocity v_e, which we take to be the minimum value of v_0 which is

$$v_e = \sqrt{2gR}.$$

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4.3E: Elementary Mechanics (Exercises)

Except where directed otherwise, assume that the magnitude of the gravitational force on an object with mass m is constant and equal to mg. In exercises involving vertical motion take the upward direction to be positive.

1. A firefighter who weighs 192 lb slides down an infinitely long fire pole that exerts a frictional resistive force with magnitude proportional to his velocity, with $k = 2.5$ lb-s/ft. Assuming that he starts from rest, find his velocity as a function of time and find his terminal velocity.

2. A firefighter who weighs 192 lb slides down an infinitely long fire pole that exerts a frictional resistive force with magnitude proportional to her velocity, with constant of proportionality k. Find k, given that her terminal velocity is -16 ft/s, and then find her velocity v as a function of t. Assume that she starts from rest.

3. A boat weighs $64,000$ lb. Its propellor produces a constant thrust of $50,000$ lb and the water exerts a resistive force with magnitude proportional to the velocity, with $k = 2000$ lb-s/ft. Assuming that the boat starts from rest, find its velocity as a function of time, and find its terminal velocity.

4. A constant horizontal force of 10 N pushes a 20 kg-mass through a medium that resists its motion with $.5$ N for every m/s of velocity. The initial velocity of the mass is 7 m/s in the direction opposite to the direction of the applied force. Find the velocity of the mass for $t > 0$.

5. A stone weighing $1/2$ lb is thrown upward from an initial height of 5 ft with an initial speed of 32 ft/s. Air resistance is proportional to velocity, with $k = 1/128$ lb-s/ft. Find the maximum height attained by the stone.

6. A 3200-lb car is moving at 64 ft/s down a 30-degree grade when it runs out of fuel. Find its velocity after that if friction exerts a resistive force with magnitude proportional to the square of the velocity, with $k = 1$ lb-s^2/ft^2. Also find its terminal velocity.

7. A 96 lb weight is dropped from rest in a medium that exerts a resistive force with magnitude proportional to the velocity. Find its velocity as a function of time if its terminal velocity is -128 ft/s.

8. An object weighing 256 lb is dropped from rest in a medium that exerts a resistive force with magnitude proportional to the square of the velocity. The magnitude of the resisting force is 1 lb when $|v| = 4$ ft/s. Find v for $t > 0$, and find its terminal velocity.

9. An object with mass m is given an initial velocity $v_0 \leq 0$ in a medium that exerts a resistive force with magnitude proportional to the square of the velocity. Find the velocity of the object for $t > 0$, and find its terminal velocity.

In Exercises **10 and 11**, assume that the force due to gravity is given by Newton's law of gravitation. Take the upward direction to be positive.

10. A space probe is to be launched from Earth. Determine its escape velocity in miles/s. Take Earth's radius to be 3960 miles.

11. A space vehicle is to be launched from the moon, which has a radius of about 1080 miles. The acceleration due to gravity at the surface of the moon is about 5.31 ft/s^2. Find the escape velocity in miles/s.

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CHAPTER OVERVIEW

5: Linear Second Order Equations

In this chapter, we study a particularly important class of second order equations. Because of their many applications in science and engineering, second order differential equation have historically been the most thoroughly studied class of differential equations. Research on the theory of second order differential equations continues to the present day.

5.1: Homogeneous Linear Equations

5.1.1: Homogeneous Linear Equations (Exercises)

5.2: Reduction of Order

5.2.1: Reduction of Order (Exercises)

5.3: Constant Coefficient Homogeneous Equations

5.3.1: Constant Coefficient Homogeneous Equations (Exercises)

5.4: Nonhomogeneous Linear Equations

5.4.1: Nonhomgeneous Linear Equations (Exercises)

5.5: Annihilation

5.5.1: Annihilation (Exercises)

5.6: Variation of Parameters

5.6.1: Variation of Parameters (Exercises)

5.7: Cauchy-Euler Equations

5.7.1: Cauchy-Euler Equations (Exercises)

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5.1: Homogeneous Linear Equations

Caution

If you have not had Math 410 (linear algebra), then you will need to read Appendix 11.3 before starting Chapter 5.

Definition 5.1.1

A second order differential equation is said to be *linear* if it **can** be written as

$$y''+p(x)y'+q(x)y=f(x). \tag{5.1.1}$$

We call the function f on the right a *forcing function*, since in physical applications it is often related to a force acting on some system modeled by the differential equation. We say that Equation 5.1.1 is *homogeneous* if $f\equiv 0$ or *nonhomogeneous* if $f\not\equiv 0$.

Note: This use of homogeneous is different from how we used it in section 2.4, but should not cause confusion.

Since these definitions are like the corresponding definitions in Section 2.3 for the linear first order equation

$$y'+p(x)y=f(x), \tag{5.1.2}$$

it is natural to expect similarities between methods of solving Equation 5.1.1 and Equation 5.1.2. However, solving Equation 5.1.1 is more difficult than solving Equation 5.1.2. In this case it is necessary to solve the homogeneous equation

$$y''+p(x)y'+q(x)y=0 \tag{5.1.3}$$

before we solve the nonhomogeneous Equation 5.1.1, and this section is devoted to studying (not solving) Equation 5.1.3.

The next theorem gives sufficient conditions for existence and uniqueness of solutions of initial value problems for Equation 5.1.3. We omit the proof.

Theorem 5.1.1

Suppose p and q are continuous on an open interval (a,b), let x_0 be any point in (a,b), and let k_0 and k_1 be arbitrary real numbers. Then the initial value problem

$$y''+p(x)y'+q(x)y=0,\ y(x_0)=k_0,\ y'(x_0)=k_1$$

has a unique solution on (a,b).

Since $y\equiv 0$ is obviously a solution of Equation 5.1.3 we call it the *trivial* solution. Any other solution is *nontrivial*. Under the assumptions of Theorem 5.1.1 , the only solution of the initial value problem

$$y''+p(x)y'+q(x)y=0,\ y(x_0)=0,\ y'(x_0)=0$$

on (a,b) is the trivial solution.

The next three examples illustrate concepts that we'll develop later in this section. You shouldn't be concerned with how to *find* the given solutions of the equations in these examples. This will be explained in later sections.

Example 5.1.1

The coefficients of y' and y in

$$y''-y=0 \tag{5.1.4}$$

are the constant functions $p\equiv 0$ and $q\equiv -1$, which are continuous on $(-\infty,\infty)$. Therefore Theorem 5.1.1 implies that every initial value problem for Equation 5.1.4 has a unique solution on $(-\infty,\infty)$.

a. Verify that $y_1=e^x$ and $y_2=e^{-x}$ are solutions of Equation 5.1.4 on $(-\infty,\infty)$.

b. Verify that if c_1 and c_2 are arbitrary constants, $y = c_1e^x + c_2e^{-x}$ is a solution of Equation 5.1.4 on $(-\infty, \infty)$.
c. Solve the initial value problem

$$y'' - y = 0, \quad y(0) = 1, \quad y'(0) = 3. \tag{5.1.5}$$

Solution a

If $y_1 = e^x$ then $y_1' = e^x$ and $y_1'' = e^x$, so

$$y_1'' - y_1 = e^x - e^x = 0.$$

If $y_2 = e^{-x}$, then $y_2' = -e^{-x}$ and $y_2'' = e^{-x}$, so

$$y_2'' - y_2 = e^{-x} - e^{-x} = 0.$$

Solution b

If

$$y = c_1e^x + c_2e^{-x} \tag{5.1.6}$$

then

$$y' = c_1e^x - c_2e^{-x} \tag{5.1.7}$$

and

$$y'' = c_1e^x + c_2e^{-x},$$

so

$$\begin{aligned} y'' - y &= (c_1e^x + c_2e^{-x}) - (c_1e^x + c_2e^{-x}) \\ &= c_1(e^x - e^x) + c_2(e^{-x} - e^{-x}) = 0 \end{aligned}$$

for all x. Therefore $y = c_1e^x + c_2e^{-x}$ is a solution of Equation 5.1.4 on $(-\infty, \infty)$.

Solution c

We can solve Equation 5.1.5 by choosing c_1 and c_2 in Equation 5.1.6 so that $y(0) = 1$ and $y'(0) = 3$. Setting $x = 0$ in Equation 5.1.6 and Equation 5.1.7 shows that this is equivalent to

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1 - c_2 &= 3. \end{aligned}$$

Solving these equations yields $c_1 = 2$ and $c_2 = -1$. Therefore $y = 2e^x - e^{-x}$ is the unique solution of Equation 5.1.5 on $(-\infty, \infty)$.

Theorem 5.1.1 implies that if k_0 and k_1 are arbitrary real numbers then the initial value problem

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1 \tag{5.1.8}$$

has a unique solution on an interval (a, b) that contains x_0, provided that P_0, P_1, and P_2 are continuous and P_0 has no zeros on (a, b). To see this, we rewrite the differential equation in Equation 5.1.8 as

$$y'' + \frac{P_1(x)}{P_0(x)}y' + \frac{P_2(x)}{P_0(x)}y = 0$$

and apply Theorem 5.1.1 with $p = P_1/P_0$ and $q = P_2/P_0$.

Example 5.1.2

The equation

$$x^2y'' + xy' - 4y = 0 \tag{5.1.9}$$

has the form of the differential equation in Equation 5.1.8, with $P_0(x) = x^2$, $P_1(x) = x$, and $P_2(x) = -4$, which are are all continuous on $(-\infty, \infty)$. However, since $P_0(0) = 0$ we must consider solutions of Equation 5.1.9 on $(-\infty, 0)$ and $(0, \infty)$. Since P_0 has no zeros on these intervals, Theorem 5.1.1 implies that the initial value problem

$$x^2y'' + xy' - 4y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

has a unique solution on $(0, \infty)$ if $x_0 > 0$, or on $(-\infty, 0)$ if $x_0 < 0$.

a. Verify that $y_1 = x^2$ is a solution of Equation 5.1.9 on $(-\infty, \infty)$ and $y_2 = 1/x^2$ is a solution of Equation 5.1.9 on $(-\infty, 0)$ and $(0, \infty)$.
b. Verify that if c_1 and c_2 are any constants then $y = c_1x^2 + c_2/x^2$ is a solution of Equation 5.1.9 on $(-\infty, 0)$ and $(0, \infty)$.
c. Solve the initial value problem

$$x^2y'' + xy' - 4y = 0, \quad y(1) = 2, \quad y'(1) = 0. \tag{5.1.10}$$

d. Solve the initial value problem

$$x^2y'' + xy' - 4y = 0, \quad y(-1) = 2, \quad y'(-1) = 0. \tag{5.1.11}$$

Solution a

If $y_1 = x^2$ then $y_1' = 2x$ and $y_1'' = 2$, so

$$x^2y_1'' + xy_1' - 4y_1 = x^2(2) + x(2x) - 4x^2 = 0$$

for x in $(-\infty, \infty)$. If $y_2 = 1/x^2$, then $y_2' = -2/x^3$ and $y_2'' = 6/x^4$, so

$$x^2y_2'' + xy_2' - 4y_2 = x^2\left(\frac{6}{x^4}\right) - x\left(\frac{2}{x^3}\right) - \frac{4}{x^2} = 0$$

for x in $(-\infty, 0)$ or $(0, \infty)$.

Solution b

If

$$y = c_1x^2 + \frac{c_2}{x^2} \tag{5.1.12}$$

then

$$y' = 2c_1x - \frac{2c_2}{x^3} \tag{5.1.13}$$

and

$$y'' = 2c_1 + \frac{6c_2}{x^4},$$

so

$$\begin{aligned} x^2y'' + xy' - 4y &= x^2\left(2c_1 + \frac{6c_2}{x^4}\right) + x\left(2c_1x - \frac{2c_2}{x^3}\right) - 4\left(c_1x^2 + \frac{c_2}{x^2}\right) \\ &= c_1(2x^2 + 2x^2 - 4x^2) + c_2\left(\frac{6}{x^2} - \frac{2}{x^2} - \frac{4}{x^2}\right) \\ &= c_1 \cdot 0 + c_2 \cdot 0 = 0 \end{aligned}$$

for x in $(-\infty, 0)$ or $(0, \infty)$.

Solution c

To solve Equation 5.1.10, we choose c_1 and c_2 in Equation 5.1.12 so that $y(1)=2$ and $y'(1)=0$. Setting $x=1$ in Equation 5.1.12 and Equation 5.1.13 shows that this is equivalent to

$$\begin{aligned} c_1 + \ c_2 &= 2 \\ 2c_1 - 2c_2 &= 0. \end{aligned}$$

Solving these equations yields $c_1=1$ and $c_2=1$. Therefore $y=x^2+1/x^2$ is the unique solution of Equation 5.1.10 on $(0,\infty)$.

Solution d

We can solve Equation 5.1.11 by choosing c_1 and c_2 in Equation 5.1.12 so that $y(-1)=2$ and $y'(-1)=0$. Setting $x=-1$ in Equation 5.1.12 and Equation 5.1.13 shows that this is equivalent to

$$\begin{aligned} c_1 + \ c_2 &= 2 \\ -2c_1 + 2c_2 &= 0. \end{aligned}$$

Solving these equations yields $c_1=1$ and $c_2=1$. Therefore $y=x^2+1/x^2$ is the unique solution of Equation 5.1.11 on $(-\infty,0)$.

Caution

Although the *formulas* for the solutions of Equation 5.1.10 and Equation 5.1.11 are both $y=x^2+1/x^2$, you should not conclude that these two initial value problems have the same solution. Remember that a solution of an initial value problem is defined *on an interval that contains the initial point*; therefore, the solution of Equation 5.1.10 is $y=x^2+1/x^2$ *on the interval* $(0,\infty)$, which contains the initial point $x_0=1$, while the solution of Equation 5.1.11 is $y=x^2+1/x^2$ *on the interval* $(-\infty,0)$, which contains the initial point $x_0=-1$.

Definition 5.1.2

If y_1 and y_2 are defined on an interval (a,b) and c_1 and c_2 are constants, then we call

$$y=c_1y_1+c_2y_2$$

a *linear combination of* y_1 and y_2.

Example 5.1.3

$y=c_1e^x+c_2e^{-x}$ is a linear combination of $y_1=e^x$ and $y_2=e^{-x}$.

$y=c_1x^2+c_2x^{-2}$ is a linear combination of $y_1=x^2$ and $y_2=x^{-2}$.

The next theorem states a fact that we've seen verified in Examples 5.1.1 and 5.1.2 .

Theorem 5.1.2

If y_1 and y_2 are solutions of the homogeneous equation

$$y''+p(x)y'+q(x)y=0 \tag{5.1.14}$$

on (a,b), then any linear combination

$$y=c_1y_1+c_2y_2 \tag{5.1.15}$$

of y_1 and y_2 is also a solution of (5.1.14) on (a,b).

Proof

If

$$y = c_1 y_1 + c_2 y_2$$

then

$$y' = c_1 y_1' + c_2 y_2' \quad \text{and} \quad y'' = c_1 y_1'' + c_2 y_2''.$$

Therefore

$$\begin{aligned} y'' + p(x)y' + q(x)y &= (c_1 y_1'' + c_2 y_2'') + p(x)(c_1 y_1' + c_2 y_2') + q(x)(c_1 y_1 + c_2 y_2) \\ &= c_1 \left(y_1'' + p(x)y_1' + q(x)y_1\right) + c_2 \left(y_2'' + p(x)y_2' + q(x)y_2\right) \\ &= c_1 \cdot 0 + c_2 \cdot 0 = 0, \end{aligned}$$

since y_1 and y_2 are solutions of Equation 5.1.14.

The General Solution of a Homogeneous Linear Second Order Equation

You should note that both of Example 5.1.1 and Example 5.1.2 are **second** order homogeneous differential equations and each had **two** solutions; this is not a coincidence and we will see why this is true later in this chapter. For now though, we need to discuss how these solutions are "independent" from one another.

Definition 5.1.3

We say that two functions y_1 and y_2 defined on an interval (a, b) are *linearly independent on* (a, b) if neither is a constant multiple of the other on (a, b). In particular, this means that neither can be the trivial solution of Equation 5.1.14, since, for example, if $y_1 \equiv 0$ we could write $y_1 = 0y_2$. We'll also say that the set $\{y_1, y_2\}$ *is linearly independent on* (a, b).

Example 5.1.4

Since $e^x / e^{-x} = e^{2x}$ is nonconstant, $y_1 = e^x$ and $y_2 = e^{-x}$ from Example 5.1.1 are linearly independent on $(-\infty, \infty)$.

Since $x^2 / x^{-2} = x^4$ is nonconstant, $y_1 = x^2$ and $y_2 = x^{-2}$ from Example 5.1.2 are linearly independent on $(-\infty, 0)$ and $(0, \infty)$.

Definition 5.1.4

We say that any set of two linearly independent solutions $\{y_1, y_2\}$ of (5.1.14) on (a, b) is a *fundamental set of solutions* of Equation 5.1.14 on (a, b).

Example 5.1.5

$\{y_1 = e^x, y_2 = e^{-x}\}$ is a fundamental set of solutions of the differential equation in Example 5.1.1

$\{y_1 = x^2, y_2 = x^{-2}\}$ is a fundamental set of solutions of the differential equation in Example 5.1.2

Definition 5.1.5

The set of **all** linear combinations of a fundamental set of solutions y_1 and y_2 of Equation 5.1.15 is called the *general solution of* (5.1.14) on (a, b).

Example 5.1.6

$y = c_1 e^x + c_2 e^{-x}$ is the general solution of the differential equation in Example 5.1.1.

$y = c_1 x^2 + c_2 x^{-2}$ is the general solution of the differential equation in Example 5.1.2.

Let's take a moment to discuss why a fundamental set of solutions is so important - we'll also develop an important idea we will use later in the class.

Suppose p and q are continuous on (a,b) and $\{y_1,y_2\}$ is a fundamental set of solutions of the homogeneous equation

$$y''+p(x)y'+q(x)y=0 \tag{5.1.16}$$

on (a,b).

Let x_0 be an arbitrary point in (a,b), and suppose a linear combination of $\{y_1,y_2\}$, $y=c_1y_1+c_2y_2$, is an arbitrary solution of Equation 5.1.16 on (a,b). Then, solving for c_1 and c_2 will give us the **unique** solution of the initial value problem

$$y''+p(x)y'+q(x)y=0, \quad y(x_0)=k_0, \quad y'(x_0)=k_1; \tag{5.1.17}$$

that is, k_0 and k_1 are the numbers obtained by evaluating y and y' at x_0. Moreover, k_0 and k_1 can be any real numbers, since Theorem 5.1.1 implies that Equation 5.1.17 has a solution no matter how k_0 and k_1 are chosen. Therefore $\{y_1,y_2\}$ is a fundamental set of solutions of Equation 5.1.16 on (a,b) if and only if it is possible to write **any** solution of the initial value problem Equation 5.1.17 as $y=c_1y_1+c_2y_2$. This is equivalent to requiring that the system

$$\begin{aligned} c_1y_1(x_0)+c_2y_2(x_0)&=k_0\\ c_1y_1'(x_0)+c_2y_2'(x_0)&=k_1 \end{aligned} \tag{5.1.18}$$

has a solution (c_1,c_2) for every choice of (k_0,k_1). Let's try to solve Equation 5.1.18.

Multiplying the first equation in Equation 5.1.18 by $y_2'(x_0)$ and the second by $y_2(x_0)$ yields

$$\begin{aligned} c_1y_1(x_0)y_2'(x_0)+c_2y_2(x_0)y_2'(x_0)&=y_2'(x_0)k_0\\ c_1y_1'(x_0)y_2(x_0)+c_2y_2'(x_0)y_2(x_0)&=y_2(x_0)k_1, \end{aligned}$$

and subtracting the second equation here from the first yields

$$\left(y_1(x_0)y_2'(x_0)-y_1'(x_0)y_2(x_0)\right)c_1=y_2'(x_0)k_0-y_2(x_0)k_1. \tag{5.1.19}$$

Multiplying the first equation in Equation 5.1.18 by $y_1'(x_0)$ and the second by $y_1(x_0)$ yields

$$\begin{aligned} c_1y_1(x_0)y_1'(x_0)+c_2y_2(x_0)y_1'(x_0)&=y_1'(x_0)k_0\\ c_1y_1'(x_0)y_1(x_0)+c_2y_2'(x_0)y_1(x_0)&=y_1(x_0)k_1, \end{aligned}$$

and subtracting the first equation here from the second yields

$$\left(y_1(x_0)y_2'(x_0)-y_1'(x_0)y_2(x_0)\right)c_2=y_1(x_0)k_1-y_1'(x_0)k_0. \tag{5.1.20}$$

If

$$y_1(x_0)y_2'(x_0)-y_1'(x_0)y_2(x_0)\neq 0 \tag{5.1.21}$$

we can divide Equation 5.1.19 and Equation 5.1.20 through by the quantity on the left to obtain

$$\begin{aligned} c_1&=\frac{y_2'(x_0)k_0-y_2(x_0)k_1}{y_1(x_0)y_2'(x_0)-y_1'(x_0)y_2(x_0)}\\ c_2&=\frac{y_1(x_0)k_1-y_1'(x_0)k_0}{y_1(x_0)y_2'(x_0)-y_1'(x_0)y_2(x_0)}, \end{aligned} \tag{5.1.22}$$

no matter how k_0 and k_1 are chosen. This motivates us to consider conditions on y_1 and y_2 that imply Equation 5.1.21.

This is only possible if $y_1(x_0)y_2'(x_0)-y_1'(x_0)y_2(x_0)\neq 0$ and this is only the case when y_1 and y_2 are linearly independent.

The Wronskian

The expression $y_1y_2'-y_1'y_2$ is very important in this class and we define it formally here.

Definition 5.1.6

The function $W=y_1y_2'-y_1'y_2$ is called the *Wronskian of* $\{y_1,y_2\}$.

The Wronskian is a determinant and is written

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

Theorem 5.1.3

Suppose p and q are continuous on (a,b), let y_1 and y_2 be solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{5.1.23}$$

on (a,b).

Then

a. y_1 and y_2 are a set of linearly independent solutions of 5.1.23 if and only if W$\neq 0$ on (a,b).

b. y_1 and y_2 are a set of linearly dependent solutions 5.1.23 if and only if W$\equiv 0$ on (a,b).

So, y_1 and y_2 are a fundamental set of solutions of 5.1.23 if and only if W$\neq 0$ on (a,b), and $y = c_1y_1 + c_2y_2$ is therefore the general solution of 5.1.23.

Example 5.1.7

Use Theorem 5.1.3 to verify that the solutions from Examples 5.1.1 and 5.1.2 form a fundamental set of solutions to the given differential equations.

a. $y'' - y = 0; \quad y_1 = e^x,\ y_2 = e^{-x}$
b. $x^2y'' + xy' - 4y = 0; \quad y_1 = x^2,\ y_2 = 1/x^2$

Solution a

$$W(x) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = e^x(-e^{-x}) - e^xe^{-x} = -2 \not\equiv 0$$

for all x.

Solution b

$$W(x) = \begin{vmatrix} x^2 & 1/x^2 \\ 2x & -2/x^3 \end{vmatrix} = x^2\left(-\frac{2}{x^3}\right) - 2x\left(\frac{1}{x^2}\right) = -\frac{4}{x} \not\equiv 0. \tag{5.1.24}$$

for all x on $(-\infty, 0)$ or $(0,\infty)$.

Example 5.1.8

Use the Wronskian to show $y_1 = x^3$ and $y_2 = 5x^3$ are linearly dependent.

Solution

$$W(x) = \begin{vmatrix} x^3 & 5x^3 \\ 3x^2 & 15x^2 \end{vmatrix} = x^3\left(15x^2\right) - 3x^2\left(5x^3\right) \equiv 0. \tag{5.1.25}$$

for all x.

The natural question here is: why do we need a Wronskian to determine linear independence if we can just check to see if one solution is a multiple of another solution? While that is true, it won't be so easy to see linear independence if we move to higher order equations, and then it will become indispensable. Additionally, this idea plays a significant role in solving nonhomogeneous equations later in the chapter.

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5.1.1: Homogeneous Linear Equations (Exercises)

1.

a. Verify that $y_1 = e^{2x}$ and $y_2 = e^{5x}$ are solutions of

$$y'' - 7y' + 10y = 0 \qquad \text{(A)}$$

on $(-\infty, \infty)$.

b. Verify that if c_1 and c_2 are arbitrary constants then $y = c_1 e^{2x} + c_2 e^{5x}$ is a solution of (A) on $(-\infty, \infty)$.

c. Solve the initial value problem

$$y'' - 7y' + 10y = 0, \quad y(0) = -1, \quad y'(0) = 1.$$

2.

a. Verify that $y_1 = e^x \cos x$ and $y_2 = e^x \sin x$ are solutions of

$$y'' - 2y' + 2y = 0 \qquad \text{(A)}$$

on $(-\infty, \infty)$.

b. Verify that if c_1 and c_2 are arbitrary constants then $y = c_1 e^x \cos x + c_2 e^x \sin x$ is a solution of (A) on $(-\infty, \infty)$.

c. Solve the initial value problem

$$y'' - 2y' + 2y = 0, \quad y(0) = 3, \quad y'(0) = -2.$$

3.

a. Verify that $y_1 = e^x$ and $y_2 = xe^x$ are solutions of

$$y'' - 2y' + y = 0 \qquad \text{(A)}$$

on $(-\infty, \infty)$.

b. Verify that if c_1 and c_2 are arbitrary constants then $y = e^x(c_1 + c_2 x)$ is a solution of (A) on $(-\infty, \infty)$.

c. Solve the initial value problem

$$y'' - 2y' + y = 0, \quad y(0) = 7, \quad y'(0) = 4.$$

4.

a. Verify that $y_1 = 1/(x-1)$ and $y_2 = 1/(x+1)$ are solutions of

$$(x^2 - 1)y'' + 4xy' + 2y = 0 \qquad \text{(A)}$$

on $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. What is the general solution of (A) on each of these intervals?

b. Solve the initial value problem

$$(x^2 - 1)y'' + 4xy' + 2y = 0, \quad y(0) = -5, \quad y'(0) = 1.$$

What is the interval of validity of the solution?

In Exercises **5-11** compute the Wronskians of the given sets of functions.

5. $\{1, e^x\}$
6. $\{e^x, e^x \sin x\}$
7. $\{x + 1, x^2 + 2\}$
8. $\{x^{1/2}, x^{-1/3}\}$
9. $\{\frac{\sin x}{x}, \frac{\cos x}{x}\}$
10. $\{x \ln|x|, x^2 \ln|x|\}$
11. $\{e^x \cos\sqrt{x}, e^x \sin\sqrt{x}\}$

12. Suppose p and q are continuous on an open interval (a,b) and let x_0 be in (a,b). Use Theorem 5.1.1 to prove that the only solution of the initial value problem

$$y''+p(x)y'+q(x)y=0, \quad y(x_0)=0, \quad y'(x_0)=0$$

on (a,b) is the trivial solution $y\equiv 0$.

13. Suppose P_0, P_1, and P_2 are continuous on (a,b) and let x_0 be in (a,b). Show that if either of the following statements is true then $P_0(x)=0$ for some x in (a,b).

a. The initial value problem

$$P_0(x)y''+P_1(x)y'+P_2(x)y=0, \quad y(x_0)=k_0, \quad y'(x_0)=k_1$$

has more than one solution on (a,b).

b. The initial value problem

$$P_0(x)y''+P_1(x)y'+P_2(x)y=0, \quad y(x_0)=0, \quad y'(x_0)=0$$

has a nontrivial solution on (a,b).

14. Suppose p and q are continuous on (a,b) and y_1 and y_2 are solutions of

$$y''+p(x)y'+q(x)y=0 \tag{A}$$

on (a,b). Let

$$z_1=\alpha y_1+\beta y_2 \quad \text{and} \quad z_2=\gamma y_1+\delta y_2,$$

where α, β, γ, and δ are constants. Prove that if $\{z_1,z_2\}$ is a fundamental set of solutions of (A) on (a,b) then so is $\{y_1,y_2\}$.

15. Suppose p and q are continuous on (a,b) and $\{y_1,y_2\}$ is a fundamental set of solutions of

$$y''+p(x)y'+q(x)y=0 \tag{A}$$

on (a,b). Let

$$z_1=\alpha y_1+\beta y_2 \quad \text{and} \quad z_2=\gamma y_1+\delta y_2,$$

where α,β,γ, and δ are constants. Prove that $\{z_1,z_2\}$ is a fundamental set of solutions of (A) on (a,b) if and only if $\alpha\gamma-\beta\delta\neq 0$.

16. Suppose y_1 is differentiable on an interval (a,b) and $y_2=ky_1$, where k is a constant. Prove that the Wronskian of $\{y_1,y_2\}$ is identically zero on (a,b).

17. Let

$$y_1=x^3 \quad \text{and} \quad y_2=\begin{cases} x^3, & x\geq 0,\\ -x^3, & x<0.\end{cases}$$

Show that the Wronskian of $\{y_1,y_2\}$ is defined and identically zero on $(-\infty,\infty)$.

18. Suppose p and q are continuous on (a,b) and every solution of

$$y''+p(x)y'+q(x)y=0 \tag{A}$$

on (a,b) can be written as a linear combination of the twice differentiable functions $\{y_1,y_2\}$. Use Theorem 5.1.2 to prove that y_1 and y_2 are themselves solutions of (A) on (a,b).

19. Suppose p_1, p_2, q_1, and q_2 are continuous on (a,b) and the equations

$$y''+p_1(x)y'+q_1(x)y=0 \quad \text{and} \quad y''+p_2(x)y'+q_2(x)y=0$$

have the same solutions on (a,b). Prove that $p_1=p_2$ and $q_1=q_2$ on (a,b).

20. Suppose p and q are continuous on (a,b) and $\{y_1,y_2\}$ is a fundamental set of solutions of

$$y''+p(x)y'+q(x)y=0 \tag{A}$$

on (a,b). Prove that if y is a solution of (A) on (a,b), there's exactly one way to choose c_1 and c_2 so that $y=c_1y_1+c_2y_2$ on (a,b).

21. If $y=c_1+c_2x^2$ is a family of solutions of $xy''-y'=0$, show that the initial value problem

$$xy''-y'=0;\quad y(0)=0,\quad y'(0)=1$$

has no solution that can be obtained from this family. Why does this not contradict Theorem 5.1.1?

22. If $y=c_1+c_2x^2$ is a family of solutions of $xy''-y'=0$, show that the initial value problem

$$xy''-y'=0;\quad y(0)=0,\quad y'(0)=0$$

has an infinite number of solutions that can be obtained from this family. Why does this not contradict Theorem 5.1.1?

23. a. Find the largest interval for which the IVP

$$e^xy''+(\cot x)y'+\frac{3}{x-2}y=0;\quad y(1)=2,\quad y'(1)=0$$

is guaranteed a unique solution.

b. Find the largest interval for which the IVP

$$e^xy''+(\cot x)y'+\frac{3}{x-2}y=0;\quad y(3)=2,\quad y'(3)=0$$

is guaranteed a unique solution.

24. a. Find the largest interval for which the IVP

$$(\ln x)y''+(\sec x)y'+\frac{1}{x-5}y=0;\quad y(4)=2,\quad y'(4)=0$$

is guaranteed a unique solution.

b. Find the largest interval for which the IVP

$$(\ln x)y''+(\sec x)y'+\frac{1}{x-5}y=0;\quad y(3/2)=2,\quad y'(3/2)=0$$

is guaranteed a unique solution.

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5.2: Reduction of Order

In this section we give a method for finding the general solution of the homogeneous equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0 \tag{5.2.1}$$

if we happen to know one nontrivial solution y_1 (don't ask how we know this solution - it will make sense later, for now let's just call it magic).

The method is called *reduction of order* because it reduces the task of solving Equation 5.2.1 to solving a first order equation.

A common method used in finding additional solutions is to multiply a known solution by an unknown function, such as

$$y = uy_1 \tag{5.2.2}$$

where u is to be determined so that y satisfies Equation 5.2.1. Substituting Equation 5.2.2 and both

$$\begin{aligned} y' &= u'y_1 + uy_1' \\ y'' &= u''y_1 + 2u'y_1' + uy_1'' \end{aligned}$$

into Equation 5.2.1 yields

$$P_0(x)(u''y_1 + 2u'y_1' + uy_1'') + P_1(x)(u'y_1 + uy_1') + P_2(x)uy_1 = 0.$$

Collecting the coefficients of u, u', and u'' yields

$$(P_0y_1)u'' + (2P_0y_1' + P_1y_1)u' + (P_0y_1'' + P_1y_1' + P_2y_1)u = 0. \tag{5.2.3}$$

However, the coefficient of u is zero, since y_1 satisfies Equation 5.2.1. Therefore Equation 5.2.3 reduces to the form

$$Q_0(x)u'' + Q_1(x)u' = 0.$$

Since there no longer is a "u" term we can reduce the order by making the substitution $z = u'$, which in turn gives $z' = u''$ and leads to

$$Q_0(x)z' + Q_1(x)z = 0. \tag{5.2.4}$$

Since Equation 5.2.4 is a linear first order equation in z, we can solve it for z easily and then integrate the solution to obtain u, and from there obtain y from Equation 5.2.2.

Example 5.2.1

Find the general solution of

$$xy'' - (2x+1)y' + (x+1)y = 0, \tag{5.2.5}$$

given that $y_1 = e^x$ is one solution.

Solution

If we let $y = ue^x$, then $y' = u'e^x + ue^x$ and $y'' = u''e^x + 2u'e^x + ue^x$, so

$$\begin{aligned} xy'' - (2x+1)y' + (x+1)y &= x(u''e^x + 2u'e^x + ue^x) - (2x+1)(u'e^x + ue^x) + (x+1)ue^x \\ &= (xu'' - u')e^x. \end{aligned}$$

Therefore $y = ue^x$ is a solution of Equation 5.2.5 if and only if

$$(xu'' - u')e^x = 0,$$

which is a first order equation in u'. If rewrite it as

$$u'' - \frac{u'}{x} = 0 \tag{5.2.6}$$

and make the substitution $z = u'$, then Equation 5.2.6 becomes

$$z' - \frac{z}{x} = 0.$$

We leave it to you to show (by separation of variables) that

$$z = C_1 x$$

is a solution.

Now using the fact that $u' = z$ and integrating we get

$$u = \frac{C_1}{2}x^2 + C_2$$

and substituting this into $y = ue^x$ we get

$$y = ue^x = \frac{C_1}{2}x^2e^x + C_2e^x. \tag{5.2.7}$$

Note that $y_1 = e^x$ and $y_2 = x^2e^x$ are clearly linearly independent and form a fundamental set of solutions to the homogeneous differential Equation 5.2.5.

Although Equation 5.2.7 is a correct form for the general solution of Equation 5.2.5, it is silly to leave the arbitrary coefficient of x^2e^x as $C_1/2$ where C_1 is an arbitrary constant. Moreover, it is sensible to make the subscripts of the coefficients of $y_1 = e^x$ and $y_2 = x^2e^x$ consistent with the subscripts of the functions themselves. Therefore we rewrite Equation 5.2.7 as

$$y = c_1e^x + c_2x^2e^x$$

by simply renaming the arbitrary constants. This last form is what we'll call the general solution of 5.2.5.

Although we could go through this procedure each time this situation arises, it would be much better if we actually had a formula to generate a second solution, and that's precisely what the next theorem gives us.

Theorem 5.2.1

Consider the differential equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0.$$

Dividing by $P_0(x)$, assuming it is not zero, we get

$$y'' + p(x)y' + q(x)y = 0. \tag{5.2.8}$$

Now suppose we know that y_1 is one solution. Then a second linearly independent solution is given by:

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2}dx \tag{5.2.9}$$

Proof

The proof follows the procedure we used in Example 5.2.1.

Suppose y_1 is a solution to 5.2.8

Let $y = uy_1$, then $y' = u'y_1 + uy_1'$ and $y'' = u''y_1 + 2u'y_1' + uy_1''$, so 5.2.8 becomes

$$\begin{aligned} y'' + p(x)y' + q(x)y &= (u''y_1 + 2u'y_1' + uy_1'') + p(x)(u'y_1 + uy_1') + q(x)uy_1 = 0 \\ &= u''y_1 + 2u'y_1' + u'p(x)y_1 + u(y_1'' + p(x)y_1' + q(x)y_1). \end{aligned}$$

However, we know y_1 is a solution to 5.2.8, so the coefficient of u, $y_1'' + p(x)y_1' + q(x)y_1 = 0$, and we get

$$u''y_1 + 2u'y_1' + u'p(x)y_1 = 0.$$

Since there is no "u" term it is easy to reduce the order by substituing $z = u'$ giving us

$$z'y_1 + 2zy_1' + zp(x)y_1 = 0.$$

Rewriting this as

$$y_1 z' + (2y_1' + p(x)y_1)z = 0$$

and diving by y_1 we get the standard linear form

$$z' + \frac{2y_1' + p(x)y_1}{y_1} z = 0 \tag{5.2.10}$$

So, the integrating factor becomes

$$e^{\int \frac{2y_1' + p(x)y_1}{y_1} dx}$$

$$= e^{\int \frac{2y_1'}{y_1} dx + \int \frac{p(x)y_1}{y_1} dx}$$

$$= e^{\int \frac{2y_1'}{y_1} dx} e^{\int p(x) dx}$$

$$= y_1^2 e^{\int p(x) dx}$$

and multiplying both sides of 5.2.10 by the integrating factor we get

$$d[y_1^2 e^{\int p(x)dx} z] = 0$$

Integrating this we get

$$y_1^2 e^{\int p(x)dx} z = c_1$$

so

$$z = c_1 \frac{1}{y_1^2 e^{\int p(x)dx}}$$

Now substituting u' back in for z we have

$$u' = c_1 \frac{1}{y_1^2 e^{\int p(x)dx}}$$

Integrating this we get

$$u = c_1 \int \frac{dx}{y_1^2 e^{\int p(x)dx}} + c_2$$

Therefore,

$$y = uy_1 = (c_1 \int \frac{dx}{y_1^2 e^{\int p(x)dx}} + c_2)y_1 = c_1 y_1 \int \frac{dx}{y_1^2 e^{\int p(x)dx}} + c_2 y_1.$$

So, we get that the second solution must be

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2} dx$$

which is clearly independent of y_1 as it is not a constant multiple of y_1.

If we wanted to check independence using the Wronskian we would get

$$W(x) = \begin{vmatrix} y_1 & y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2} dx \\ y_1' & y_1' \int \frac{e^{-\int p(x)dx}}{y_1^2} dx + y_1 \frac{e^{-\int p(x)dx}}{y_1^2} \end{vmatrix} = e^{-\int p(x)dx} \not\equiv 0$$

for all x.

Example 5.2.2

a. Find a second linearly independent solution of

$$x^2y'' + xy' - y = 0 \tag{5.2.11}$$

given that $y_1 = x$ is one solution.

b. Find the general solution of 5.2.11

c. Solve the initial value problem

$$x^2y'' + xy' - y = 0, \quad y(1) = 2,\ y'(1) = -3. \tag{5.2.12}$$

Solution a

We first need to divide by the lead coefficient to get the equation in the form 5.2.8:

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0.$$

So, $p(x) = \frac{1}{x}$ and using 5.2.9 we get

$$\begin{aligned} y_2 &= x\int \frac{e^{-\int \frac{1}{x}dx}}{x^2}dx \\ &= x\int \frac{1}{x^3}dx \\ &= \frac{-1}{2x} \end{aligned}$$

Since any constant will be absorbed into an arbitrary constant in the general solution we will take

$$y_2 = \frac{1}{x}$$

Since we used 5.2.9 we know y_2 is independent of y_1.

Solution b

Using the given y_1 and what we just found for y_2 we can form the general solution of 5.2.11

$$y = c_1x + c_2\frac{1}{x}$$

Solution c

Using the general solution from part b and the initial conditions from 5.2.12 we get

$$c_1 + c_2 = 2$$

and

$$c_1 - c_2 = -3$$

Solving this system we get $c_1 = -1/2$ and $c_2 = 5/2$ which gives us our solution to 5.2.12

$$y = \frac{-x}{2} + \frac{5}{2x}$$

which is unique on $(0, \infty)$.

Why is it unique? Why is the interval of validity $(0, \infty)$?

Example 5.2.3

Find the general solution of

$$x^2y'' - 3xy' + 3y = 0, \tag{5.2.13}$$

given that $y_1 = x$ is a solution.

Solution

$$\begin{aligned} y_2 &= x \int \frac{e^{\int \frac{3}{x} dx}}{x^2} dx \\ &= x \int x dx \\ &= \frac{x^3}{2} \end{aligned}$$

So, we will take $y_2 = x^3$ and our general solution is

$$y = c_1 x + c_2 x^3$$

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5.2.1: Reduction of Order (Exercises)

In Exercises **1-30** find the general solution, given that y_1 is a solution.

1. $(2x+1)y''-2y'-(2x+3)y=0; \quad y_1=e^{-x}$
2. $x^2y''+xy'-y=0; \quad y_1=x$
3. $x^2y''-xy'+y=0; \quad y_1=x$
4. $y''-3y'+2y=0; \quad y_1=e^{2x}$
5. $y''-2y'+y=0; \quad y_1=e^x$
6. $4x^2y''+(4x-8x^2)y'+(4x^2-4x-1)y=0; \quad y_1=x^{1/2}e^x$
7. $y''-2y'+2y=0; \quad y_1=e^x\cos x$
8. $y''+4xy'+(4x^2+2)y=0; \quad y_1=e^{-x^2}$
9. $x^2y''+xy'-4y=0; \quad y_1=x^2$
10. $x^2y''+2x(x-1)y'+(x^2-2x+2)y=0; \quad y_1=xe^{-x}$
11. $x^2y''-x(2x-1)y'+(x^2-x-1)y=0; \quad y_1=xe^x$
12. $(1-2x)y''+2y'+(2x-3)y=0; \quad y_1=e^x$
13. $x^2y''-3xy'+4y=0; \quad y_1=x^2$
14. $2xy''+(4x+1)y'+(2x+1)y=0; \quad y_1=e^{-x}$
15. $xy''-(2x+1)y'+(x+1)y=0; \quad y_1=e^x$
16. $4x^2y''-4x(x+1)y'+(2x+3)y=0; \quad y_1=x^{1/2}$
17. $x^2y''-5xy'+8y=0; \quad y_1=x^2$
18. $xy''+(2-2x)y'+(x-2)y=0; \quad y_1=e^x$
19. $x^2y''-4xy'+6y=0; \quad y_1=x^2$
20. $x^2(\ln|x|)^2y''-(2x\ln|x|)y'+(2+\ln|x|)y=0; \quad y_1=\ln|x|$
21. $4xy''+2y'+y=0; \quad y_1=\sin\sqrt{x}$
22. $xy''-(2x+2)y'+(x+2)y=0; \quad y_1=e^x$
23. $x^2y''-(2a-1)xy'+a^2y=0; \quad y_1=x^a$
24. $x^2y''-2xy'+(x^2+2)y=0; \quad y_1=x\sin x$
25. $xy''-(4x+1)y'+(4x+2)y=0; \quad y_1=e^{2x}$
26. $4x^2(\sin x)y''-4x(x\cos x+\sin x)y'+(2x\cos x+3\sin x)y=0; \quad y_1=x^{1/2}$
27. $4x^2y''-4xy'+(3-16x^2)y=0; \quad y_1=x^{1/2}e^{2x}$
28. $(2x+1)xy''-2(2x^2-1)y'-4(x+1)y=0; \quad y_1=1/x$
29. $(x^2-2x)y''+(2-x^2)y'+(2x-2)y=0; \quad y_1=e^x$
30. $xy''-(4x+1)y'+(4x+2)y=0; \quad y_1=e^{2x}$

In Exercises **31-33** solve the initial value problem, given that y_1 is a solution.

31. $x^2y''-3xy'+4y=0, \quad y(-1)=7, \quad y'(-1)=-8; \quad y_1=x^2$
32. $(x+1)^2y''-2(x+1)y'-(x^2+2x-1)y=0, \quad y(0)=1, \quad y'(0)=-1; \quad y_1=(x+1)e^x$
33. $x^2y''+2xy'-2y=0, \quad y(1)=0,\ y'(1)=3; \quad y_1=x$

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5.3: Constant Coefficient Homogeneous Equations

This section deals with homogeneous equations of the special form $ay''+by'+cy=0$, where a, b, and c are constant ($a\neq 0$).

If a, b, and c are real constants and $a\neq 0$, then

$$ay''+by'+cy=0. \tag{5.3.1}$$

is said to be a *constant coefficient equation*.

As we'll see, all solutions of Equation 5.3.1 are defined on $(-\infty,\infty)$. This being the case, we'll omit references to the interval on which solutions are defined, or on which a given set of solutions is a fundamental set, etc., since the interval will always be $(-\infty,\infty)$.

To get an idea of what form the solution might be, let's consider the analogous first order equation

$$ay'+by=0.$$

This is a simple separable equation whose solution is

$$y=e^{\frac{-b}{a}x}$$

Since $\frac{-b}{a}$ is just a constant we will replace it with the constant r and we get a solution of the form

$$y=e^{rx}$$

Mathematicians, being inherently lazy, figured it's worth trying the same solution in 5.3.1 and see if we can make it work.

Thus, if $y=e^{rx}$ then $y'=re^{rx}$ and $y''=r^2e^{rx}$, so

$$ay''+by'+cy=ar^2e^{rx}+bre^{rx}+ce^{rx}=(ar^2+br+c)e^{rx}=0$$

So, solving 5.3.1 is the same thing as solving

$$ar^2+br+c=0. \tag{5.3.2}$$

Note: we can ignore e^{rx} because it is never 0.

5.3.2 is called the *characteristic equation* of Equation 5.3.1.

The roots of the characteristic equation are given by the quadratic formula

$$r=\frac{-b\pm\sqrt{b^2-4ac}}{2a}. \tag{5.3.3}$$

Three Cases

As you'll recall from algebra, there are three possible types of solutions to 5.3.2 and we will exam each case separately

- Case 1: $b^2-4ac>0$, so the characteristic equation has two distinct real roots.
- Case 2: $b^2-4ac=0$, so the characteristic equation has a repeated real root.
- Case 3: $b^2-4ac<0$, so the characteristic equation has complex roots.

Case 1: Distinct Real Roots

Let r_1 and r_2 be distinct real roots of 5.3.2.

As such, we get two solutions, $y_1=e^{r_1x}$ and $y_2=e^{r_2x}$.

The only question is whether or not these are linearly independent of each other. In this case we'll use the Wronskian.

$$W(x)=\begin{vmatrix} e^{r_1x} & e^{r_2x} \\ r_1e^{r_1x} & r_2e^{r_2x}\end{vmatrix}=(r_2-r_1)e^{(r_1+r_2)x}\not\equiv 0$$

for all x since $r_1\neq r_2$ as the roots are distinct.

So, in this case, the general solution is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} \tag{5.3.4}$$

Example 5.3.1

a. Find the general solution of

$$y'' + 6y' + 5y = 0. \tag{5.3.5}$$

b. Solve the initial value problem

$$y'' + 6y' + 5y = 0, \quad y(0) = 3,\ y'(0) = -1. \tag{5.3.6}$$

Solution a

The characteristic equation of Equation 5.3.5 is

$$r^2 + 6r + 5 = (r+1)(r+5) = 0.$$

Since $r = -1$ and $r = -5$ are roots, $y_1 = e^{-x}$ and $y_2 = e^{-5x}$ are solutions of Equation 5.3.5 and the general solution of Equation 5.3.5 is

$$y = c_1 e^{-x} + c_2 e^{-5x}. \tag{5.3.7}$$

Solution b

We must determine c_1 and c_2 in Equation 5.3.7 so that y satisfies the initial conditions in Equation 5.3.6. Differentiating Equation 5.3.7 yields

$$y' = -c_1 e^{-x} - 5c_2 e^{-5x}. \tag{5.3.8}$$

Imposing the initial conditions $y(0) = 3,\ y'(0) = -1$ in Equation 5.3.7 and Equation 5.3.8 yields

$$\begin{aligned} c_1 + c_2 &= 3 \\ -c_1 - 5c_2 &= -1. \end{aligned}$$

The solution of this system is $c_1 = 7/2, c_2 = -1/2$. Therefore the solution of Equation 5.3.6 is

$$y = \frac{7}{2}e^{-x} - \frac{1}{2}e^{-5x}.$$

Figure 5.3.1 is a graph of this solution.

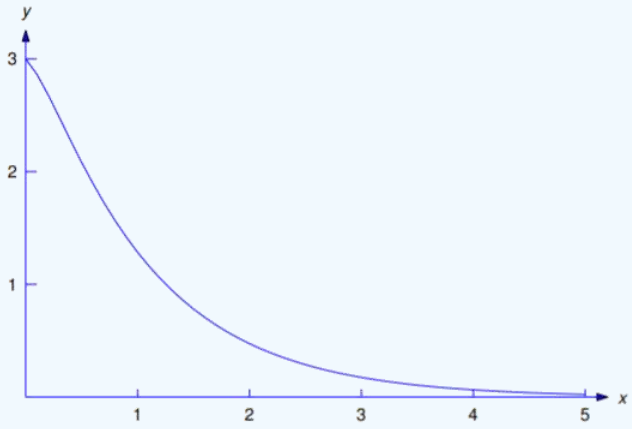


Figure 5.3.1 : $y = \frac{7}{2}e^{-x} - \frac{1}{2}e^{-5x}$

Case 2: A Repeated Real Root

Let r be a repeated real root of 5.3.2. As such, we only get one solution, $y_1 = e^{rx}$. But we know there must be two independent solutions to a second order homogeneous differential equation. So, how can we find a second independent solution if we only know

one? Recall the formula for a second solution that we derived in section 5.2,

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2} dx$$

If we divide 5.3.1 by a we get

$$y'' + \frac{b}{a}y' + \frac{c}{a}y = 0$$

which gives is $p(x) = \frac{b}{a}$, which gives us

$$y_2 = e^{rx} \int \frac{e^{-\int \frac{b}{a}dx}}{e^{2rx}} dx$$

$$= e^{rx} \int \frac{e^{-\frac{b}{a}x}}{e^{2rx}} dx$$

Since we have a repeated root we know $b^2 - 4ac = 0$ in 5.3.3 which gives us $r = \frac{-b}{2a}$ or

$$\frac{-b}{a} = 2r$$

Using this we now get

$$y_2 = e^{rx} \int \frac{e^{2rx}}{e^{2rx}} dx$$

$$= e^{rx} \int dx$$

$$= xe^{rx}$$

Since we already know that the formula gives us a second linearly independent solution, the general solution is

$$y = c_1 e^{rx} + c_2 x e^{rx} \tag{5.3.9}$$

Example 5.3.2

a. Find the general solution of

$$y'' + 6y' + 9y = 0. \tag{5.3.10}$$

b. Solve the initial value problem

$$y'' + 6y' + 9y = 0, \quad y(0) = 3,\ y'(0) = -1. \tag{5.3.11}$$

Solution a

The characteristic equation of Equation 5.3.10 is

$$r^2 + 6r + 9 = (r+3)^2 = 0,$$

so the characteristic equation has the repeated real root $r = -3$. Therefore $y_1 = e^{-3x}$ and $y_2 = xe^{-3x}$ are linearly independent solutions of Equation 5.3.10 and the general solution of Equation 5.3.10 is

$$y = c_1 e^{-3x} + c_2 x e^{-3x}. \tag{5.3.12}$$

Solution b

b. Differentiating Equation 5.3.12 yields

$$y' = -3e^{-3x}(c_1 + c_2 x) + c_2 e^{-3x}. \tag{5.3.13}$$

Imposing the initial conditions $y(0) = 3$, $y'(0) = -1$ in Equation 5.3.12 and Equation 5.3.13 yields $c_1 = 3$ and $-3c_1 + c_2 = -1$, so $c_2 = 8$. Therefore the solution of Equation 5.3.11 is

$$y = 3e^{-3x} + 8xe^{-3x}.$$

Figure 5.3.2 is a graph of this solution.

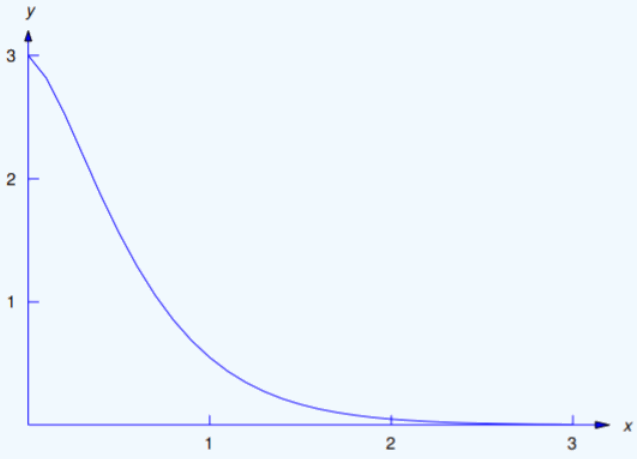


Figure 5.3.2 : $y = 3e^{-3x} + 8xe^{-3x}$

Case 3: Complex Conjugate Roots

Now suppose 5.3.2 has complex conjugate roots $r = \lambda \pm \omega i$. This is similar to case one in that we do have distinct roots and we could say $y_1 = e^{(\lambda+\omega i)x}$ and $y_2 = e^{(\lambda-\omega i)x}$. The problem with this is that it's not very useful in real life applications to have functions of complex numbers. So, we want to see if we can somehow write both solutions as functions of real numbers, and the following theorem gives us exactly what we need.

Theorem 5.3.1 Euler's Identity

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{5.3.14}$$

Proof

See Exercise 23.

We can now rewrite y_1 and y_2 as follows:

$$y_1 = e^{(\lambda+\omega i)x} = e^{\lambda x}e^{i\omega x} = e^{\lambda x}(\cos\omega x + i\sin\omega x) = e^{\lambda x}\cos\omega x + ie^{\lambda x}\sin\omega x)$$

and

$$y_2 = e^{(\lambda-\omega i)x} = e^{\lambda x}e^{-i\omega x} = e^{\lambda x}(\cos(-\omega x) + i\sin(-\omega x)) = e^{\lambda x}\cos\omega x - ie^{\lambda x}\sin\omega x)$$

The latter is using the fact the cos is an even function and sin is an odd function.

Note: the i is no longer **inside** a function.

Recall that if y_1 and y_2 are solutions of a homogeneous equation, then any linear combination is also a solution. So, if we add y_1 and y_2 we get

$$y_3 = 2e^{\lambda x}\cos\omega x$$

and if we subtract y_1 and y_2 we get

$$y_4 = 2ie^{\lambda x}\sin\omega x$$

Now, both 2 and 2i are constants that can be absorbed into the constants c_1 and c_2 of our general solution and since y_3 and y_4 are not constant multiples of each other, they are independent. Thus, our general solution is

$$y = c_1e^{\lambda x}\cos\omega x + c_2e^{\lambda x}\sin\omega x \tag{5.3.15}$$

Example 5.3.3

a. Find the general solution of

$$y'' + 4y' + 13y = 0. \tag{5.3.16}$$

b. Solve the initial value problem

$$y'' + 4y' + 13y = 0, \quad y(0) = 2,\ y'(0) = -3. \tag{5.3.17}$$

Solution a

The characteristic equation of Equation 5.3.16 is

$$r^2 + 4r + 13 = r^2 + 4r + 4 + 9 = (r+2)^2 + 9 = 0.$$

The roots of the characteristic equation are $r = -2 \pm 3i$ and the general solution of 5.3.16 is

$$y = c_1 e^{-2x} \cos 3x + c_2 e^{-2x} \sin 3x$$

Solution b

Imposing the condition $y(0) = 2$ in Equation 5.3.16 shows that $c_1 = 2$. Differentiating Equation 5.3.16 yields

$$y' = -2e^{-2x}(c_1 \cos 3x + c_2 \sin 3x) + 3e^{-2x}(-c_1 \sin 3x + c_2 \cos 3x),$$

and imposing the initial condition $y'(0) = -3$ here yields $-3 = -2c_1 + 3c_2$, so $c_2 = 1/3$. Therefore the solution of Equation 5.3.17 is

$$y = 2e^{-2x} \cos 3x + \frac{1}{3} e^{-2x} \sin 3x.$$

Figure 5.3.3 is a graph of this function.

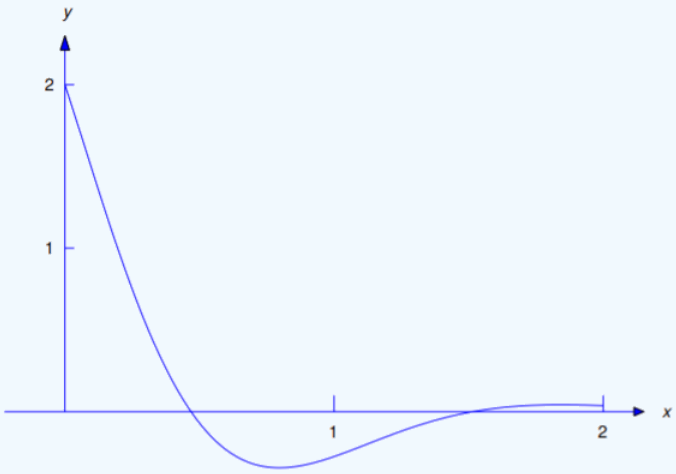


Figure 5.3.3 : $y = 2e^{-2x} \cos 3x + \frac{1}{3} e^{-2x} \sin 3x$

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5.3.1: Constant Coefficient Homogeneous Equations (Exercises)

In Exercises **1-12** find the general solution.

1. $y'' + 5y' - 6y = 0$
2. $y'' - 4y' + 5y = 0$
3. $y'' + 8y' + 7y = 0$
4. $y'' - 4y' + 4y = 0$
5. $y'' + 2y' + 10y = 0$
6. $y'' + 6y' + 10y = 0$
7. $y'' - 8y' + 16y = 0$
8. $y'' + y' = 0$
9. $y'' - 2y' + 3y = 0$
10. $y'' + 6y' + 13y = 0$
11. $4y'' + 4y' + 10y = 0$
12. $10y'' - 3y' - y = 0$

In Exercises **13-21** solve the initial value problem.

13. $y'' + 14y' + 50y = 0, \quad y(0) = 2, \quad y'(0) = -17$
14. $6y'' - y' - y = 0, \quad y(0) = 10, \quad y'(0) = 0$
15. $6y'' + y' - y = 0, \quad y(0) = -1, \quad y'(0) = 3$
16. $4y'' - 4y' - 3y = 0, \quad y(0) = \frac{13}{12}, \quad y'(0) = \frac{23}{24}$
17. $4y'' - 12y' + 9y = 0, \quad y(0) = 3, \quad y'(0) = \frac{5}{2}$
18. $y'' + 7y' + 12y = 0, \quad y(0) = -1, \quad y'(0) = 0$
19. $y'' - 6y' + 9y = 0, \quad y(0) = 0, \quad y'(0) = 2$
20. $36y'' - 12y' + y = 0, \quad y(0) = 3, \quad y'(0) = \frac{5}{2}$
21. $y'' + 4y' + 10y = 0, \quad y(0) = 3, \quad y'(0) = -2$

22. Prove: If the characteristic equation of

$$ay'' + by' + cy = 0 \tag{A}$$

has a repeated negative root or two roots with negative real parts, then every solution of (A) approaches zero as $x \to \infty$.

23. In calculus you learned that e^u, $\cos u$, and $\sin u$ can be represented by the infinite series

$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} = 1 + \frac{u}{1!} + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots + \frac{u^n}{n!} + \cdots \tag{A}$$

$$\cos u = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n}}{(2n)!} = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} + \cdots + (-1)^n \frac{u^{2n}}{(2n)!} + \cdots, \tag{B}$$

and

$$\sin u = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}}{(2n+1)!} = u - \frac{u^3}{3!} + \frac{u^5}{5!} + \cdots + (-1)^n \frac{u^{2n+1}}{(2n+1)!} + \cdots \tag{C}$$

for all real values of u. Even though you have previously considered (A) only for real values of u, we can set $u = i\theta$, where θ is real, to obtain

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}. \tag{D}$$

Given the proper background in the theory of infinite series with complex terms, it can be shown that the series in (D) converges for all real θ.

Recalling that $i^2 = -1$, write enough terms of the sequence $\{i^n\}$ to convince yourself that the sequence is repetitive:

$$1, i, -1, -i, 1, i, -1, -i, 1, i, -1, -i, 1, i, -1, -i, \cdots.$$

Use this to group the terms in (D) as

$$\begin{aligned} e^{i\theta} &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right) \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!}. \end{aligned}$$

By comparing this result with (B) and (C), conclude that

$$e^{i\theta} = \cos\theta + i \sin\theta. \tag{E}$$

This is *Euler's identity*.

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5.4: Nonhomogeneous Linear Equations

We'll now consider the nonhomogeneous linear second order equation

$$y''+p(x)y'+q(x)y=f(x), \tag{5.4.1}$$

where the forcing function f isn't identically zero. The next theorem, an extension of Theorem 5.1.1, gives sufficient conditions for existence and uniqueness of solutions of initial value problems for Equation 5.4.1. We omit the proof, which is beyond the scope of this book.

Theorem 5.4.1 Uniqueness

Suppose p, q, and f are continuous on an open interval (a,b), let x_0 be any point in (a,b), and let k_0 and k_1 be arbitrary real numbers. Then the initial value problem

$$y''+p(x)y'+q(x)y=f(x), \quad y(x_0)=k_0, \quad y'(x_0)=k_1$$

has a unique solution on (a,b).

To find the general solution of Equation 5.4.1 on an interval (a,b) where p, q, and f are continuous, it is necessary to find the general solution of the associated homogeneous equation

$$y''+p(x)y'+q(x)y=0 \tag{5.4.2}$$

on (a,b). We call Equation 5.4.2 the *homogeneous equation* for Equation 5.4.1.

The next theorem shows how to find the general solution of Equation 5.4.1 if we know one solution y_p of Equation 5.4.1 and a fundamental set of solutions of Equation 5.4.2. We call y_p a *particular solution* of Equation 5.4.1; it can be any solution that we can find, one way or another.

Theorem 5.4.2

Suppose p, q, and f are continuous on (a,b). Let y_p be a particular solution of

$$y''+p(x)y'+q(x)y=f(x) \tag{5.4.3}$$

on (a,b), and let $\{y_1,y_2\}$ be a fundamental set of solutions of the homogeneous equation

$$y''+p(x)y'+q(x)y=0 \tag{5.4.4}$$

on (a,b). Then y is a solution of (5.4.3) on (a,b) if and only if

$$y=y_p+c_1y_1+c_2y_2, \tag{5.4.5}$$

where c_1 and c_2 are constants

Proof

We first show that y in Equation 5.4.5 is a solution of Equation 5.4.3 for any choice of the constants c_1 and c_2. Differentiating Equation 5.4.5 twice yields

$$y'=y_p'+c_1y_1'+c_2y_2' \quad \text{and} \quad y''=y_p''+c_1y_1''+c_2y_2'',$$

so

$$\begin{aligned} y''+p(x)y'+q(x)y &= (y_p''+c_1y_1''+c_2y_2'')+p(x)(y_p'+c_1y_1'+c_2y_2')+q(x)(y_p+c_1y_1+c_2y_2) \\ &= (y_p''+p(x)y_p'+q(x)y_p)+c_1(y_1''+p(x)y_1'+q(x)y_1)+c_2(y_2''+p(x)y_2'+q(x)y_2) \\ &= f+c_1\cdot 0+c_2\cdot 0=f, \end{aligned}$$

since y_p satisfies Equation 5.4.3 and y_1 and y_2 satisfy Equation 5.4.4.

Now we'll show that every solution of Equation 5.4.3 has the form Equation 5.4.5 for some choice of the constants c_1 and c_2. Suppose y is a solution of Equation 5.4.3. We'll show that $y-y_p$ is a solution of Equation 5.4.4, and therefore of the form $y-y_p=c_1y_1+c_2y_2$, which implies Equation 5.4.5. To see this, we compute

$$\begin{aligned}(y-y_p)''+p(x)(y-y_p)'+q(x)(y-y_p) &= (y''-y_p'')+p(x)(y'-y_p')+q(x)(y-y_p)\\ &= (y''+p(x)y'+q(x)y)-(y_p''+p(x)y_p'+q(x)y_p)\\ &= f(x)-f(x)=0,\end{aligned}$$

since y and y_p both satisfy Equation 5.4.3.

We say that Equation 5.4.5 is the *general solution of* (5.4.3) on (a,b).

If P_0, P_1, and F are continuous and P_0 has no zeros on (a,b), then Theorem 5.4.2 implies that the general solution of

$$P_0(x)y''+P_1(x)y'+P_2(x)y=F(x) \tag{5.4.6}$$

on (a,b) is $y=y_p+c_1y_1+c_2y_2$, where y_p is a particular solution of Equation 5.4.6 on (a,b) and $\{y_1,y_2\}$ is a fundamental set of solutions of

$$P_0(x)y''+P_1(x)y'+P_2(x)y=0$$

on (a,b). To see this, we rewrite Equation 5.4.6 as

$$y''+\frac{P_1(x)}{P_0(x)}y'+\frac{P_2(x)}{P_0(x)}y=\frac{F(x)}{P_0(x)}$$

and apply Theorem 5.4.2 with $p=P_1/P_0$, $q=P_2/P_0$, and $f=F/P_0$.

To avoid awkward wording in examples and exercises, we will not specify the interval (a,b) when we ask for the general solution of a specific linear second order equation, or for a fundamental set of solutions of a homogeneous linear second order equation. Let's agree that this always means that we want the general solution (or a fundamental set of solutions, as the case may be) on every open interval on which p, q, and f are continuous if the equation is of the form Equation 5.4.3, or on which P_0, P_1, P_2, and F are continuous and P_0 has no zeros, if the equation is of the form Equation 5.4.6. We leave it to you to identify these intervals in specific examples and exercises.

For completeness, we point out that if P_0, P_1, P_2, and F are all continuous on an open interval (a,b), but P_0 *does* have a zero in (a,b), then Equation 5.4.6 may fail to have a general solution on (a,b) in the sense just defined.

The next three examples illustrate concepts that we'll develop later in this chapter. As in section 5.1, you shouldn't be concerned with how to *find* the given particular solutions of the equations in these examples. This will be explained in later sections. In this section we limit ourselves to applications of Theorem 5.4.2 where we are given the particular solution.

Example 5.4.1

a. Find the general solution of

$$y''+y=1. \tag{5.4.7}$$

b. Solve the initial value problem

$$y''+y=1,\quad y(0)=2,\quad y'(0)=7. \tag{5.4.8}$$

Solution

a. We can apply Theorem 5.4.2 with $(a,b)=(-\infty,\infty)$, since the functions $p\equiv0$, $q\equiv1$, and $f\equiv1$ in Equation 5.4.7 are continuous on $(-\infty,\infty)$. By inspection we see that $y_p\equiv1$ is a particular solution of Equation 5.4.7. Since $y_1=\cos x$ and $y_2=\sin x$ form a fundamental set of solutions of the complementary equation $y''+y=0$, the general solution of Equation 5.4.7 is

$$y=c_1\cos x+c_2\sin x+1. \tag{5.4.9}$$

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b. Imposing the initial condition $y(0)=2$ in Equation 5.4.9 yields $2=1+c_1$, so $c_1=1$. Differentiating Equation 5.4.9 yields

$$y'=-c_1\sin x+c_2\cos x.$$

Imposing the initial condition $y'(0)=7$ here yields $c_2=7$, so the solution of Equation 5.4.8 is

$$y=\cos x+7\sin x+1.$$

Figure 5.4.1 is a graph of this function.

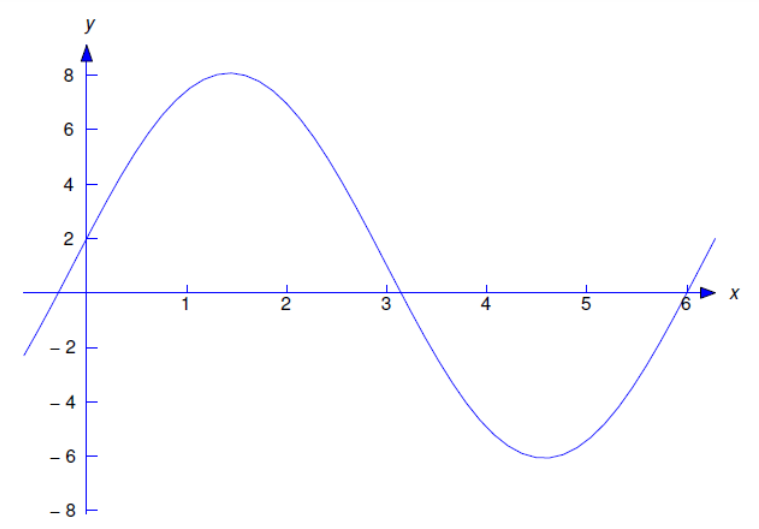


Figure 5.4.1 : $y=\cos x+7\sin x+1$

Example 5.4.2

a. Find the general solution of

$$y''-2y'+y=-3-x+x^2. \tag{5.4.10}$$

b. Solve the initial value problem

$$y''-2y'+y=-3-x+x^2,\quad y(0)=-2,\quad y'(0)=1. \tag{5.4.11}$$

Solution

a. The characteristic equation of the homogeneous equation

$$y''-2y'+y=0$$

is $r^2-2r+1=(r-1)^2$, so $y_1=e^x$ and $y_2=xe^x$ form a fundamental set of solutions of the homogeneous equation.

$y_p=1+3x+x^2$ is a particular solution of Equation 5.4.10 (verify) and Theorem 5.4.2 implies that

$$y=1+3x+x^2+e^x(c_1+c_2x) \tag{5.4.12}$$

is the general solution of Equation 5.4.10.

b. Imposing the initial condition $y(0)=-2$ in Equation 5.4.12 yields $-2=1+c_1$, so $c_1=-3$. Differentiating Equation 5.4.12 yields

$$y'=3+2x+e^x(c_1+c_2x)+c_2e^x,$$

and imposing the initial condition $y'(0)=1$ here yields $1=3+c_1+c_2$, so $c_2=1$. Therefore the solution of Equation 5.4.11 is

$$y=1+3x+x^2-e^x(3-x).$$

Figure 5.4.2 is a graph of this solution.

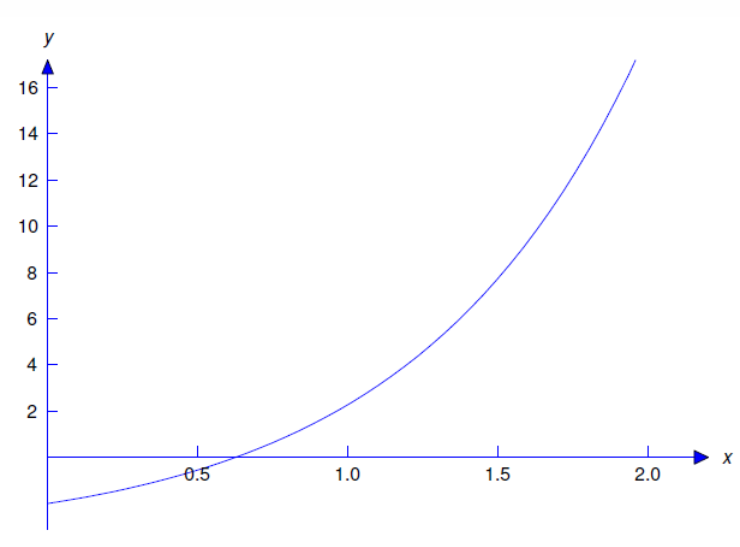


Figure 5.4.2 : $y = 1 + 3x + x^2 - e^x(3 - x)$

Example 5.4.3

Find the general solution of

$$x^2y'' + xy' - 4y = 2x^4 \tag{5.4.13}$$

on $(-\infty, 0)$ and $(0, \infty)$.

Solution

In Example 5.1.2, we verified that $y_1 = x^2$ and $y_2 = 1/x^2$ form a fundamental set of solutions of the homogeneous equation

$$x^2y'' + xy' - 4y = 0$$

on $(-\infty, 0)$ and $(0, \infty)$.

$y_p = x^4/6$ is a particular solution of Equation 5.4.13 (verify) on $(-\infty, \infty)$. Theorem 5.4.2 implies that the general solution of Equation 5.4.13 on $(-\infty, 0)$ and $(0, \infty)$ is

$$y = \frac{x^4}{6} + c_1x^2 + \frac{c_2}{x^2}.$$

The Principle of Superposition

The next theorem enables us to break a nonhomogeneous equation into simpler parts, find a particular solution for each part, and then combine their solutions to obtain a particular solution of the original problem.

Theorem 5.4.3 the principle of superposition

Suppose y_{p_1} is a particular solution of

$$y'' + p(x)y' + q(x)y = f_1(x)$$

on (a, b) and y_{p_2} is a particular solution of

$$y'' + p(x)y' + q(x)y = f_2(x)$$

on (a, b). Then

$$y_p = y_{p_1} + y_{p_2}$$

is a particular solution of

$$y'' + p(x)y' + q(x)y = f_1(x) + f_2(x)$$

on (a, b)

Proof

If $y_p = y_{p_1} + y_{p_2}$ then

$$\begin{aligned} y_p'' + p(x)y_p' + q(x)y_p &= (y_{p_1} + y_{p_2})'' + p(x)(y_{p_1} + y_{p_2})' + q(x)(y_{p_1} + y_{p_2}) \\ &= \left(y_{p_1}'' + p(x)y_{p_1}' + q(x)y_{p_1}\right) + \left(y_{p_2}'' + p(x)y_{p_2}' + q(x)y_{p_2}\right) \\ &= f_1(x) + f_2(x). \end{aligned}$$

It's easy to generalize Theorem 5.4.3 to the equation

$$y'' + p(x)y' + q(x)y = f(x) \tag{5.4.14}$$

where

$$f = f_1 + f_2 + \cdots + f_k;$$

thus, if y_{p_i} is a particular solution of

$$y'' + p(x)y' + q(x)y = f_i(x)$$

on (a, b) for $i = 1, 2, \ldots, k$, then $y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_k}$ is a particular solution of Equation 5.4.14 on (a, b).

Example 5.4.4

The function $y_{p_1} = x^4/15$ is a particular solution of

$$x^2y'' + 4xy' + 2y = 2x^4 \tag{5.4.15}$$

on $(-\infty, \infty)$ and $y_{p_2} = x^2/3$ is a particular solution of

$$x^2y'' + 4xy' + 2y = 4x^2 \tag{5.4.16}$$

on $(-\infty, \infty)$. Use the principle of superposition to find a particular solution of

$$x^2y'' + 4xy' + 2y = 2x^4 + 4x^2 \tag{5.4.17}$$

on $(-\infty, \infty)$.

Solution

The right side $F(x) = 2x^4 + 4x^2$ in Equation 5.4.17 is the sum of the right sides

$$F_1(x) = 2x^4 \quad \text{and} \quad F_2(x) = 4x^2.$$

in Equation 5.4.15 and Equation 5.4.16 Therefore the principle of superposition implies that

$$y_p = y_{p_1} + y_{p_2} = \frac{x^4}{15} + \frac{x^2}{3}$$

is a particular solution of Equation 5.4.17.

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5.4.1: Nonhomgeneous Linear Equations (Exercises)

In Exercises **1-18** verify that the given y_p is particular solution of the nonhomogeneous differential equation given and form the general solution or, where initial values are given, solve the initial value problem.

1. $y'' + 5y' - 6y = 22 + 18x - 18x^2; \quad y_p = -1 + 2x + 3x^2$
2. $y'' - 4y' + 5y = 1 + 5x; \quad y_p = 1 + x$
3. $y'' + 8y' + 7y = -8 - x + 24x^2 + 7x^3; \quad y_p = -x + x^3$
4. $y'' - 4y' + 4y = 2 + 8x - 4x^2; \quad y_p = 1 - x^2$
5. $y'' + 2y' + 10y = 4 + 26x + 6x^2 + 10x^3, \quad y(0) = 2, \quad y'(0) = 91 - x^2; \quad y_p = 2x + x^3$
6. $y'' + 6y' + 10y = 22 + 20x, \quad y(0) = 2,\ y'(0) = -2; \quad y_p = 1 + 2x$
7. $y'' + 5y' - 6y = 6e^{3x}; \quad y_p = \frac{e^{3x}}{3}$
8. $y'' - 4y' + 5y = e^{2x}; \quad y_p = e^{2x}$
9. $y'' + 8y' + 7y = 10e^{-2x}, \quad y(0) = -2,\ y'(0) = 10; \quad y_p = -2e^{-2x}$
10. $y'' - 4y' + 4y = e^x, \quad y(0) = 2, \quad y'(0) = 0; \quad y_p = e^x$
11. $y'' + 2y' + 10y = e^{x/2}; \quad y_p = \frac{4}{45}e^{x/2}$
12. $y'' + 6y' + 10y = e^{-3x}; \quad y_p = e^{-3x}$
13. $y'' - 8y' + 16y = 23\cos x - 7\sin x; \quad y_p = \cos x - \sin x$
14. $y'' + y' = -8\cos 2x + 6\sin 2x; \quad y_p = \cos 2x - 2\sin 2x$
15. $y'' - 2y' + 3y = -6\cos 3x + 6\sin 3x; \quad y_p = \cos 3x$
16. $y'' + 6y' + 13y = 18\cos x + 6\sin x; \quad y_p = \cos x + \sin x$
17. $y'' + 7y' + 12y = -2\cos 2x + 36\sin 2x, \quad y(0) = -3, \quad y'(0) = 3; \quad y_p = -2\cos 2x + \sin 2x$
18. $y'' - 6y' + 9y = 18\cos 3x + 18\sin 3x, \quad y(0) = 2, \quad y'(0) = 2; \quad y_p = \cos 3x - \sin 3x$

In Exercises **19-24** refer to the cited exercises and use the principle of superposition to find a particular solution. Then find the general solution.

19. $y'' + 5y' - 6y = 22 + 18x - 18x^2 + 6e^{3x}$ (See *Exercises 5.4.1* and *5.4.7*.)
20. $y'' - 4y' + 5y = 1 + 5x + e^{2x}$ (See *Exercises 5.4.2* and *5.4.8*)
21. $y'' + 8y' + 7y = -8 - x + 24x^2 + 7x^3 + 10e^{-2x}$ (See *Exercises 5.4.3* and *5.4.9*.)
22. $y'' - 4y' + 4y = 2 + 8x - 4x^2 + e^x$ (See *Exercises 5.4.4* and *5.4.10*)
23. $y'' + 2y' + 10y = 4 + 26x + 6x^2 + 10x^3 + e^{x/2}$ (See *Exercises 5.3.5* and *5.4.11*.)
24. $y'' + 6y' + 10y = 22 + 20x + e^{-3x}$ (See *Exercises 5.4.6* and *5.4.12*.)

25. Prove: If y_{p_1} is a particular solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F_1(x)$$

on (a, b) and y_{p_2} is a particular solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F_2(x)$$

on (a, b), then $y_p = y_{p_1} + y_{p_2}$ is a solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F_1(x) + F_2(x)$$

on (a, b).

26. Suppose p, q, and f are continuous on (a,b). Let y_1, y_2, and y_p be twice differentiable on (a,b), such that $y=c_1y_1+c_2y_2+y_p$ is a solution of

$$y''+p(x)y'+q(x)y=f$$

on (a,b) for every choice of the constants c_1, c_2. Prove that y_1 and y_2 are solutions of the homogeneous equation on (a,b).

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5.5: Annihilation

In this section we consider the constant coefficient equation

$$ay'' + by' + cy = f(x) \tag{5.5.1}$$

From Theorem 5.4.2, the general solution of Equation 5.5.1 is $y = y_p + c_1 y_1 + c_2 y_2$, where y_p is a particular solution of Equation 5.5.1 and $\{y_1, y_2\}$ is a fundamental set of solutions of the homogeneous equation

$$ay'' + by' + cy = 0.$$

In Section 5.3 we showed how to find $\{y_1, y_2\}$. In this section we'll show how to find y_p. The procedure that we'll use is called *the method of annihilation*.

The annihilation method is exactly what the name implies; we will attempt to annihilate $f(x)$ in 5.5.1 (make it 0) and create a new homogeneous equation that we now know how to solve.

Note: again, we are transforming a current equation that we don't know how to solve into one that we do.

So, the next obvious questions are: how, exactly, do we annihilate a function and can we annihilate all functions that we will come across. We'll discuss the first question here and the second in the following section.

Definition 5.5.1

An operator L is an annihilator of $f(x)$ if

$$L(f(x)) = 0.$$

Note

Annihilators in this class involve derivatives and we will use D to stand for derivative; sometimes called a **differential operator** in linear algebra. In other words,

$$D = \frac{d}{dx}, \quad D^2 = \frac{d^2}{dx^2}, \quad etc.$$

Annihilating Polynomials

We will first consider simple polynomials. As noted above, annihilators are derivatives and as you'll recall from calculus, derivatives lower the power on each term of the polynomial, eventually leading to 0 if you differentiate enough.

Example 5.5.1

If $f(x) = x^2$, find an annihilator L that annihilates $f(x)$.

Solution

Annihilators involve derivatives, so let's see what happens if we start repeatedly taking derivatives:

$$D(x^2) = 2x; \quad D(2x) = 2; \quad D(2) = 0$$

So, it takes three derivatives to annihilate $f(x) = x^2$, so in this example $L = D^3$.

Or,

$$D^3(x^2) = 0$$

Theorem 5.5.1

If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, then D^{n+1} annihilates $f(x)$.

Example 5.5.2

Find an annihilator L for $f(x) = 3x^5 - 7x^2 + 2x - 9$

Solution

Since the highest power of x is 5, we will need one more derivative, so $L = D^6$, to annihilate $f(x)$. It is left to the reader to show this.

Caution

Although powers of D higher than $n+1$ will annihilate $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, it is best to choose the smallest annihilator, as we will see later.

Annihilating $e^{\alpha x}$

In the case of functions like $e^{\alpha x}$, derivatives alone do not annihilate it. However, exponential functions are what we call cyclic functions; that is, they come back to themselves eventually, and in the case of the exponential it happens immediately. We can use this to formulate an annihilator for $e^{\alpha x}$.

Example 5.5.3

If $f(x) = e^{5x}$, find an annihilator L that annihilates $f(x)$.

Solution

If we take a derivative we get $D(e^{5x}) = 5e^{5x}$, and you can see that we once again have e^{5x} but multiplied by 5, so to annihilate $f(x)$ we need to address that as well. The following will do just that:

$$(D-5)e^{5x} = D(e^{5x}) - 5(e^{5x}) = 5e^{5x} - 5e^{5x} = 0$$

So, in this example $L = D - 5$.

Note: it may seem odd that we can "distribute" a derivative like a number, but that is because a derivative is what in linear algebra is called a linear transformation. If you have had linear algebra you will recall this; if not...just trust me.

Theorem 5.5.2

If $f(x) = a_n x^n e^{\alpha x} + a_{n-1}x^{n-1}e^{\alpha x} + \ldots + a_1 x e^{\alpha x} + a_0 e^{\alpha x}$, then $(D-\alpha)^{n+1}$ annihilates $f(x)$.

Note: we are adding on to what we did earlier. $D - \alpha$ is for the exponential part and the power $n+1$ is for the polynomial.

Example 5.5.4

Find an annihilator L for $f(x) = x^2 e^{5x} + 2xe^{5x}$

Solution

Since the highest power of x is 2 and α is 5, the annihilator is $L = (D-5)^3$. It is left to the reader to show this.

Annihilating $\sin \beta x$ and $\cos \beta x$

Similar to $e^{\alpha x}$, $\sin \beta x$ and $\cos \beta x$ are cyclic functions; that is, they come back to themselves eventually, and in this case it takes two derivatives to come back to itself. We can use this to formulate an annihilator for $\sin \beta x$ and $\cos \beta x$.

Example 5.5.5

If $f(x) = \sin 3x$, find an annihilator L that annihilates $f(x)$.

Solution

As we did in Example 1, let's see what happens if we start repeatedly taking derivatives:

$$D(\sin 3x) = 3\cos 3x; \quad D(3\cos 3x) = -9\sin 3x$$

Similar to the exponential function, we got back to sine again but multiplied by -9 this time, so to annihilate $f(x)$ we need to address that as well. The following will do just that:

$$(D^2 + 9)\sin 3x = -9\sin 3x + 9\sin 3x = 0.$$

So, in this example $L = D^2 + 9$.

Theorem 5.5.3

If $f(x) = a_n x^n e^{\alpha x}\cos\beta x + \ldots + a_0 e^{\alpha x}\cos\beta x + b_n x^n e^{\alpha x}\sin\beta x + \ldots + b_0 e^{\alpha x}\sin\beta x$ then $((D-\alpha)^2 + \beta^2)^{n+1}$ annihilates $f(x)$.

Note: we are again adding on to what we did earlier. $D^2 + \beta^2$ is for sine and cosine, $D - \alpha$ is for the exponential part, and the power $n+1$ is for the polynomial.

Example 5.5.6

Find an annihilator for $f(x) = x^3 e^{2x}\cos 5x + 3xe^{2x}\cos 5x + 9\sin 5x$

Solution

Since the highest power of x is 3, α is 2, and β is 5, the annihilator is $((D-2)^2 + 25)^4$. It is left to the reader to show this...but...you probably don't want to do that.

Annihilating Linear Combinations of Functions

Our goal again is to take 5.5.1 and annihilate $f(x)$ using the smallest annihilator possible. so, the next question we're going to consider is what happens when we need to annihilate several different types of functions all at once. The next theorem addresses this - the proof is beyond the scope of this class.

Theorem 5.5.4

If L_1 annihilates $f_1(x)$ and L_2 annihilates $f_2(x)$, then $L_1 L_2$ annihilates any linear combination of $f_1(x)$ and $f_2(x)$. In other words,

$$L_1 L_2(c_1 f_1(x) + c_2 f_2(x)) = 0.$$

Note: When dealing with constant coefficient differential equations, as we are in this section, annihilators are commutative. In other words, $L_1 L_2 = L_2 L_1$.

Example 5.5.7

Find an annihilator for $f(x) = 7x^4 + 2x^3 - 5 + 2xe^{2x} + 9e^{2x} - 3x^5 e^{3x}\cos 4x$

Solution

D^5 annihilates $7x^4 + 2x^3 - 5$, $(D-2)^2$ annihilates $2xe^{2x} + 9e^{2x}$, and $((D-3)^2 + 16)^6$ annihilates $-3x^5 e^{3x}\cos 4x$, so $D^5(D-2)^2((D-3)^2 + 16)^6$ annihilates $f(x)$. In other words,

$$D^5(D-2)^2((D-3)^2 + 16)^6(7x^4 + 2x^3 - 5 + 2xe^{2x} + 9e^{2x} - 3x^5 e^{3x}\cos 4x) = 0$$

Again, it would not be fun to show this, but there is no need to do this.

Solving Nonhomogeneous Differential Equations Using Annihilation

Note

We can write differential equations using the differential operator $D = \frac{d}{dx}$ as well. For example:

$$y'' - y' - 6y = D^2y - Dy - 6y = (D-3)(D+2)y$$

Not that we know how to annihilate several types of common functions, let's see how we can use this idea to solve nonhomogeneous differential equations of the form 5.5.1. We will first solve the homogeneous version of 5.5.1 before we solve the nonhomogeneous version. The reason will quickly become apparent in Example 5.5.9.

Example 5.5.8

Find the general solution of

$$y'' - 7y' + 12y = 4e^{2x}. \tag{5.5.2}$$

Solution

We first solve

$$y'' - 7y' + 12y = 0 \tag{5.5.3}$$

by finding the characteristic equation $r^2 - 7r + 12 = (r-3)(r-4) = 0$, which gives us a fundamental set of solutions $\{e^{3x}, e^{4x}\}$.

Therefore,

$$y_h = c_1e^{3x} + c_2e^{4x} \tag{5.5.4}$$

We now go back to Equation 5.5.2 and write it using differential operator notation:

$$D^2y - 7Dy + 12y = 4e^{2x}$$

or

$$(D-4)(D-3)y = 4e^{2x}$$

Note: The factorization of the differential operators is exactly the same as the characteristic equation - upon inspection the reason why should be obvious.

Now, our goal here is to take this nonhomogeneous equation and make it into a new homogeneous equation. We do this by annihilating $4e^{2x}$ by applying $D-2$ to both sides we get

$$(D-2)(D-4)(D-3)y = (D-2)4e^{2x}$$

giving us

$$(D-2)(D-4)(D-3)y = 0$$

This new homogeneous equation has the characteristic equation $(r-2)(r-3)(r-4) = 0$ giving us roots $r = 2, 3, 4$.

So, we would get solutions $\{e^{2x}, e^{3x}, e^{4x}\}$. However, we know that $\{e^{3x}, e^{4x}\}$ are homogeneous solutions and will give us 0 when we use them in 5.5.2. So, we can disregard those two solutions and focus on e^{2x}. So, we are going to look at solutions of the form

$$y_p = Ae^{2x}$$

Note: we are using A instead of c_3 to remind us that we must actually find A. In other words, we need to find the value of A so that when we substitute it for y in our nonhomogeneous equation we get $4e^{2x}$.

So, if $y_p = Ae^{2x}$, then $y_p' = 2Ae^{2x}$, and $y_p'' = 4Ae^{2x}$. Substituting these into 5.5.2 gives us

$$4Ae^{2x} - 7(2Ae^{2x}) + 12(Ae^{2x}) = 4e^{2x}$$

which gives us

$$2Ae^{2x} = 4e^{2x}$$

which implies $2A = 4$, or $A = 2$. Therefore, we get

$$y_p = 2e^{2x}$$

and the general solution of 5.5.2 is

$$y = c_1 e^{3x} + c_2 e^{4x} + 2e^{2x}$$

Example 5.5.9

Find the general solution of

$$y'' - 7y' + 12y = 5e^{4x}. \tag{5.5.5}$$

Solution

We first solve $y'' - 7y' + 12y = 0$, which is the same homogeneous equation that we had in Example 5.5.8; therefore, we have $y_h = c_1 e^{3x} + c_2 e^{4x}$.

Now writing 5.5.5 using differential operator notation we get

$$(D-3)(D-4)y = 5e^{4x}$$

We know $D-4$ annihilates $5e^{4x}$, so applying it to both sides we get

$$(D-3)(D-4)^2 y = 0$$

and the roots $r = 3, 4, 4$. Now we have used 3 and 4 in the homogenous solution, so the new root is the second 4. Now recall from repeated roots that

$$y_p = Axe^{4x}.$$

Note: This is why it's important that we solve the homogeneous equation before the nonhomogeneous.

So, if $y_p = Axe^{4x}$, then $y_p' = Ae^{4x} + 4Axe^{4x}$, and $y_p'' = 8Ae^{4x} + 16Axe^{4x}$. Substituting these into 5.5.5 gives us

$$8Ae^{2x} + 16Axe^{4x} - 7(Ae^{4x} + 4Axe^{4x}) + 12(Axe^{4x}) = 5e^{4x}$$

which gives us

$$Ae^{4x} = 5e^{4x}$$

which implies $A = 5$. Therefore, we get

$$y_p = 5xe^{4x}$$

and the general solution of 5.5.5 is

$$y = c_1 e^{3x} + c_2 e^{4x} + 5xe^{4x}$$

Example 5.5.10

Find the general solution of

$$y'' - 8y' + 16y = 2e^{4x}. \tag{5.5.6}$$

Solution

Since the characteristic equation of the homogeneous equation

$$y'' - 8y' + 16y = 0 \tag{5.5.7}$$

is $r^2 - 8r + 16 = (r-4)^2 = 0$, both $y_1 = e^{4x}$ and $y_2 = xe^{4x}$ are solutions of Equation 5.5.7.

Turning to 5.5.6 and annihilating $2e^{4x}$ we get

$$(D-4)^3 y = 0$$

giving us a new characteristic equation $(r-4)^3 = 0$.

Since the new root is the third time 4 is a root,

$$y_p = Ax^2 e^{4x}$$

So, if $y_p = Ax^2 e^{4x}$, then $y_p' = 2Axe^{4x} + 4Ax^2 e^{4x}$, and $y_p'' = 2Ae^{4x} + 16Axe^{4x} + 16Ax^2 e^{4x}$. Substituting these into 5.5.6 gives us

$$2Ae^{4x} + 16Axe^{4x} + 16Ax^2 e^{4x} - 8(2Axe^{4x} + 4Ax^2 e^{4x}) + 16(Ax^2 e^{4x}) = 2e^{4x}$$

which gives us

$$2Ae^{4x} = 5e^{4x}$$

which implies $A = 5/2$. Therefore, we get

$$y_p = \frac{5}{2} x^2 e^{4x}$$

and the general solution of 5.5.5 is

$$y = c_1 e^{4x} + c_2 xe^{4x} + \frac{5}{2} x^2 e^{4x}$$

Example 5.5.11

Solve the initial value problem

$$y'' + y = 4x + 10\sin x; \quad y(\pi) = 0, y'(\pi) = 2. \tag{5.5.8}$$

Solution

We'll leave the homogeneous part to the reader and just start there:

$$y_h = c_1 \cos x + c_2 \sin x$$

Turning to 5.5.8 and annihilating $4x + 10\sin x$ we get

$$D^2 (D^2 + 1)^2 y = 0$$

giving us a new characteristic equation $r^2 (r^2 + 1)^2 = 0$.

So, we now have roots $r = 0, 0, \pm i, \pm i$. Since the new roots are $0, 0, \pm i$ and since it's the second time $\pm i$ are roots, we have

$$y_p = A + Bx + Cx\cos x + Dx\sin x.$$

Following the same process as above we end up with $A = 0, B = 4, C = -5, D = 0$. Therefore, we get

$$y_p = 4x - 5x\cos x$$

and the general solution of 5.5.8 is

$$y = c_1 \cos x + c_2 \sin x + 4x - 5x\cos x$$

Turning to our initial conditions we have $y(\pi) = -c_1 + 9\pi = 0$, so $c_1 = 9\pi$.

Substituting c_1 and taking the derivative we get $y' = -9\pi \sin x + c_2 \cos x + 4 - 5\cos x + 5x\sin x$ and using

$y'(\pi) = -c_2 + 9 = 2$, so $c_2 = 7$ and we get the solution to our initial value problem:

$$y = 9\pi \cos x + 7 \sin x + 4x - 5x \cos x$$

Using the Principle of Superposition

The next example shows how to combine the method of annihilation and Theorem 5.4.3, the principle of superposition.

Example 5.5.12

Find a particular solution of

$$y'' - 7y' + 12y = 4e^{2x} + 5e^{4x}. \tag{5.5.9}$$

Solution

In Example 5.5.8 we found that $y_{p_1} = 2e^{2x}$ is a particular solution of

$$y'' - 7y' + 12y = 4e^{2x},$$

and in Example 5.5.9 we found that $y_{p_2} = 5xe^{4x}$ is a particular solution of

$$y'' - 7y' + 12y = 5e^{4x}.$$

Therefore the principle of superposition implies that $y_p = 2e^{2x} + 5xe^{4x}$ is a particular solution of Equation 5.5.9.

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5.5.1: Annihilation (Exercises)

In Exercises **1-51** find the general solution.

1. $y'' - 9y = 54$
2. $y'' + y' = 3$
3. $y'' + 4y' + 4y = 2x + 6$
4. $y'' - y' - 12y = e^{4x}$
5. $y'' - 2y' - 3y = 4e^{x} - 9$
6. $y'' + 25y = 6\sin x$
7. $y'' + 6y' + 9y = -xe^{4x}$
8. $y'' - y = x^{2}e^{x} + 5$
9. $y'' - 2y' + 5y = e^{x}\sin x$
10. $y'' + 25y = 20\sin 5x$
11. $y'' + y' + y = x\sin x$
12. $y'' - 3y' + 2y = e^{3x}(1 + x)$
13. $y'' - 6y' + 5y = e^{-3x}(35 - 8x)$
14. $y'' - 2y' - 3y = e^{x}(-8 + 3x)$
15. $y'' + 2y' + y = e^{2x}(-7 - 15x + 9x^{2})$
16. $y'' + 4y = e^{-x}(7 - 4x + 5x^{2})$
17. $y'' - y' - 2y = e^{x}(9 + 2x - 4x^{2})$
18. $y'' - 4y' - 5y = -6xe^{-x}$
19. $y'' - 3y' + 2y = e^{x}(3 - 4x)$
20. $y'' + y' - 12y = e^{3x}(-6 + 7x)$
21. $2y'' - 3y' - 2y = e^{2x}(-6 + 10x)$
22. $y'' + 2y' + y = e^{-x}(2 + 3x)$
23. $y'' - 2y' + y = e^{x}(1 - 6x)$
24. $y'' - 4y' + 4y = e^{2x}(1 - 3x + 6x^{2})$
25. $9y'' + 6y' + y = e^{-x/3}(2 - 4x + 4x^{2})$
26. $y'' - 3y' + 2y = e^{3x}(1 + x)$
27. $y'' - 6y' + 8y = e^{x}(11 - 6x)$
28. $y'' + 6y' + 9y = e^{2x}(3 - 5x)$
29. $y'' + 2y' - 3y = -16xe^{x}$
30. $y'' - 2y' + y = e^{x}(2 - 12x)$
31. $y'' + 3y' + 2y = 7\cos x - \sin x$
32. $y'' + 3y' + y = (2 - 6x)\cos x - 9\sin x$
33. $y'' + 2y' + y = e^{x}(6\cos x + 17\sin x)$
34. $y'' + 3y' - 2y = -e^{2x}(5\cos 2x + 9\sin 2x)$
35. $y'' - y' + y = e^{x}(2 + x)\sin x$
36. $y'' + 3y' - 2y = e^{-2x}\left[(4 + 20x)\cos 3x + (26 - 32x)\sin 3x\right]$

37. $y''+4y=-12\cos 2x-4\sin 2x$

38. $y''+y=(-4+8x)\cos x+(8-4x)\sin x$

39. $4y''+y=-4\cos x/2-8x\sin x/2$

40. $y''+2y'+2y=e^{-x}(8\cos x-6\sin x)$

41. $y''-2y'+5y=e^{x}\left[(6+8x)\cos 2x+(6-8x)\sin 2x\right]$

42. $y''+2y'+y=8x^2\cos x-4x\sin x$

43. $y''+3y'+2y=(12+20x+10x^2)\cos x+8x\sin x$

44. $y''+3y'+2y=(1-x-4x^2)\cos 2x-(1+7x+2x^2)\sin 2x$

45. $y''-5y'+6y=-e^{x}\left[(4+6x-x^2)\cos x-(2-4x+3x^2)\sin x\right]$

46. $y''-2y'+y=-e^{x}\left[(3+4x-x^2)\cos x+(3-4x-x^2)\sin x\right]$

47. $y''-2y'+2y=e^{x}\left[(2-2x-6x^2)\cos x+(2-10x+6x^2)\sin x\right]$

48. $y''+2y'+y=e^{-x}\left[(5-2x)\cos x-(3+3x)\sin x\right]$

49. $y''+9y=-6\cos 3x-12\sin 3x$

50. $y''+3y'+2y=(1-x-4x^2)\cos 2x-(1+7x+2x^2)\sin 2x$

51. $y''+4y'+3y=e^{-x}\left[(2+x+x^2)\cos x+(5+4x+2x^2)\sin x\right]$

In Exercises **52-66** solve the initial value problem.

52. $y''-64y=16,\quad y(0)=1,\quad y'(0)=0$

53. $y''-5y'=x-2,\quad y(0)=0,\quad y'(0)=2$

54. $y''+y=8\cos 2x-4\sin x,\quad y(\frac{\pi}{2})=-1,\quad y'(\frac{\pi}{2})=0$

55. $y''-4y'+8y=x^3,\quad y(0)=2,\quad y'(0)=4$

56. $y''-4y'-5y=9e^{2x}(1+x),\quad y(0)=0,\quad y'(0)=-10$

57. $y''+3y'-4y=e^{2x}(7+6x),\quad y(0)=2,\quad y'(0)=8$

58. $y''+4y'+3y=-e^{-x}(2+8x),\quad y(0)=1,\quad y'(0)=2$

59. $y''-3y'-10y=7e^{-2x},\quad y(0)=1,\quad y'(0)=-17$

60. $y''-7y'+6y=-e^{x}(17\cos x-7\sin x),\quad y(0)=4,\ y'(0)=2$

61. $y''-2y'+2y=-e^{x}(6\cos x+4\sin x),\quad y(0)=1,\ y'(0)=4$

62. $y''+6y'+10y=-40e^{x}\sin x,\quad y(0)=2,\quad y'(0)=-3$

63. $y''-6y'+10y=-e^{3x}(6\cos x+4\sin x),\quad y(0)=2,\quad y'(0)=7$

64. $y''-3y'+2y=e^{3x}\left[21\cos x-(11+10x)\sin x\right],\ y(0)=0,\quad y'(0)=6$

65. $y''+4y=-e^{-2x}\left[(4-7x)\cos x+(2-4x)\sin x\right],\ y(0)=3,\quad y'(0)=1$

66. $y''+4y'+4y=2\cos 2x+3\sin 2x+e^{-x},\quad y(0)=-1,\ y'(0)=2$

In Exercises **67-72** solve a. and b. and use the principle of superposition to solve c.

67. a. $y''+y'+y=xe^{x}$

    b. $y''+y'+y=e^{-x}(1+2x)$

    c. $y''+y'+y=xe^{x}+e^{-x}(1+2x)$

68. a. $y''-7y'+12y=-e^{x}(17-42x)$

    b. $y''-7y'+12y=-e^{3x}$

    c. $y''-7y'+12y=-e^{x}(17-42x)-e^{3x}$

69. a. $y''-8y'+16y=6xe^{4x}$

b. $y''-8y'+16y=2+16x+16x^2$

c. $y''-8y'+16y=6xe^{4x}+2+16x+16x^2$

70. a. $y''-3y'+2y=-e^{2x}(3+4x)$

b. $y''-3y'+2y=-e^{x}$

c. $y''-3y'+2y=-e^{2x}(3+4x)-e^{x}$

71. a. $y''-2y'+2y=e^{x}(1+x)$

b. $y''-2y'+2y=e^{-x}(2-8x+5x^2)$

c. $y''-2y'+2y=e^{x}(1+x)+e^{-x}(2-8x+5x^2)$

72. a. $y''+y=e^{-x}(2-4x+2x^2)$

b. $y''+y=e^{3x}(8-12x-10x^2)$

c. $y''+y=e^{-x}(2-4x+2x^2)+e^{3x}(8-12x-10x^2)$

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5.6: Variation of Parameters

Caution

If you have not had Math 410 (linear algebra), then you may want to review Appendix 11.3 again before starting this section.

In this section we'll discuss the method of *variation of parameters* to find a particular solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x) \tag{5.6.1}$$

if we know a fundamental set $\{y_1, y_2\}$ of solutions of the homogeneous equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0. \tag{5.6.2}$$

Having found a particular solution y_p by this method, we can write the general solution of Equation 5.6.1 as

$$y = y_p + c_1y_1 + c_2y_2.$$

Since we previously only needed solutions of Equation 5.6.2 and annihilation to find the general solution of Equation 5.6.1, it is natural to ask why we are interested in another method to achieve the same goal. Here's the answer:

- We cannot annihilate all types of functions $F(x)$ that may occur in Equation 5.6.1.
- Annihilation is more complicated when we do not have constant coefficients. Note the coefficients in Equation 5.6.1 are not necessarily constants.
- Variation of parameters generalizes naturally to a method for finding particular solutions of linear systems of equations in Chapter 10, while annihilation doesn't.
- Although a detailed discussion of this is beyond the scope of this book, variation of parameters is a powerful theoretical tool used by researchers in differential equations.

We'll now derive the method. As usual, we consider solutions of Equation 5.6.1 and Equation 5.6.2 on an interval (a, b) where P_0, P_1, P_2, and F are continuous and P_0 has no zeros. Suppose that $\{y_1, y_2\}$ is a fundamental set of solutions of the homogeneous equation Equation 5.6.2. We use a similar idea to what we did in section 5.2. We look for a particular solution of Equation 5.6.1 in the form

$$y_p = u_1y_1 + u_2y_2 \tag{5.6.3}$$

where u_1 and u_2 are functions to be determined so that y_p satisfies Equation 5.6.1. You may not think this is a good idea, since there are now two unknown functions to be determined, rather than one like we had in section 5.2. However, since u_1 and u_2 have to satisfy only one condition (that y_p is a solution of Equation 5.6.1), we can impose a second condition that produces a convenient simplification, as follows.

Differentiating Equation 5.6.3 yields

$$y_p' = u_1y_1' + u_2y_2' + u_1'y_1 + u_2'y_2. \tag{5.6.4}$$

As our second condition on u_1 and u_2 we require that

$$u_1'y_1 + u_2'y_2 = 0. \tag{5.6.5}$$

Then Equation 5.6.4 becomes

$$y_p' = u_1y_1' + u_2y_2'; \tag{5.6.6}$$

that is, Equation 5.6.5 permits us to differentiate y_p (once!) as if u_1 and u_2 are constants. Differentiating Equation 5.6.6 yields

$$y_p'' = u_1y_1'' + u_2y_2'' + u_1'y_1' + u_2'y_2'. \tag{5.6.7}$$

(There are no terms involving u_1'' and u_2'' here, as there would be if we hadn't required Equation 5.6.5.) Substituting Equation 5.6.3, Equation 5.6.6, and Equation 5.6.7 into Equation 5.6.1 and collecting the coefficients of u_1 and u_2 yields

$$u_1(P_0y_1'' + P_1y_1' + P_2y_1) + u_2(P_0y_2'' + P_1y_2' + P_2y_2) + P_0(u_1'y_1' + u_2'y_2') = F.$$

As in the derivation of the method of reduction of order, the coefficients of u_1 and u_2 here are both zero because y_1 and y_2 satisfy the homogeneous equation. Hence, we can rewrite the last equation as

$$P_0(u_1'y_1' + u_2'y_2') = F. \tag{5.6.8}$$

Therefore y_p in Equation 5.6.3 satisfies Equation 5.6.1 if

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= \tfrac{F}{P_0}, \end{aligned}$$

where the first equation is the same as Equation 5.6.5 and the second is from Equation 5.6.8.

From Cramer's Rule (covered in linear algebra) you can always solve Equation 5.6.9 for u_1' and u_2'. To make things a little cleaner we will replace $\frac{F}{P_0}$ with simply $f(x)$ as we did when using annihilation, and rearrange the factors slightly, giving us:

$$\begin{aligned} y_1u_1' + y_2u_2' &= 0 \\ y_1'u_1' + y_2'u_2' &= f(x) \end{aligned} \tag{5.6.9}$$

Now, letting

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix},$$

Cramer's rule then says that

$$u_1' = \frac{W_1}{W}, \quad u_2' = \frac{W_2}{W}$$

.

Since $\{y_1, y_2\}$ is a fundamental set of solutions of Equation 5.6.2 on (a, b), Theorem 5.1.3 implies that the Wronskian $W = y_1y_2' - y_1'y_2$ has no zeros on (a, b).

We can now obtain u_1 and u_2 by integrating u_1' and u_2'. The constants of integration can be taken to be zero, since any choice of u_1 and u_2 in Equation 5.6.3 will suffice.

Example 5.6.1

Find a particular solution y_p of

$$x^2y'' - 2xy' + 2y = x^{9/2}, \tag{5.6.10}$$

given that $y_1 = x$ and $y_2 = x^2$ are solutions of the complementary equation

$$x^2y'' - 2xy' + 2y = 0.$$

Then find the general solution of Equation 5.6.10.

Solution

Note that we have variable coefficients **and** a function that cannot be annihilated.

We set

$$y_p = u_1x + u_2x^2,$$

where

$$\begin{aligned} xu_1' + \ x^2u_2' &= 0 \\ u_1' + 2x\ u_2' &= \frac{x^{9/2}}{x^2} = x^{5/2}. \end{aligned}$$

Now, letting

$$W=\begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}=x^2, \quad W_1=\begin{vmatrix} 0 & x^2 \\ x^{5/2} & 2x \end{vmatrix}=-x^{9/2}, \quad W_2=\begin{vmatrix} x & 0 \\ 1 & x^{5/2} \end{vmatrix}=x^{7/2},$$

So, $u_1'=\frac{W_1}{W}=-x^{5/2}$ and $u_2'=\frac{W_2}{W}=x^{3/2}$. Integrating and taking the constants of integration to be zero yields

$$u_1=-\frac{2}{7}x^{7/2} \quad \text{and} \quad u_2=\frac{2}{5}x^{5/2}.$$

Therefore

$$y_p=u_1x+u_2x^2=-\frac{2}{7}x^{7/2}x+\frac{2}{5}x^{5/2}x^2=\frac{4}{35}x^{9/2},$$

and the general solution of Equation 5.6.10 is

$$y=\frac{4}{35}x^{9/2}+c_1x+c_2x^2.$$

Example 5.6.2

Find a particular solution y_p of

$$(x-1)y''-xy'+y=(x-1)^2, \tag{5.6.11}$$

given that $y_1=x$ and $y_2=e^x$ are solutions of the homogeneous equation

$$(x-1)y''-xy'+y=0.$$

Then find the general solution of Equation 5.6.11.

Solution

We set

$$y_p=u_1x+u_2e^x,$$

where

$$\begin{aligned} xu_1'+e^xu_2' &= 0 \\ u_1'+e^xu_2' &= \frac{(x-1)^2}{x-1}=x-1. \end{aligned}$$

Now, letting

$$W=\begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix}=xe^x-e^x, \quad W_1=\begin{vmatrix} 0 & e^x \\ x-1 & e^x \end{vmatrix}=-e^x(x-1), \quad W_2=\begin{vmatrix} x & 0 \\ 1 & x-1 \end{vmatrix}=x(x-1),$$

So, $u_1'=\frac{W_1}{W}=-1$ and $u_2'=\frac{W_2}{W}=xe^{-x}$. Integrating and taking the constants of integration to be zero yields

$$u_1=-x \quad \text{and} \quad u_2=-(x+1)e^{-x}.$$

Therefore

$$y_p=u_1x+u_2e^x=(-x)x+(-(x+1)e^{-x})e^x=-x^2-x-1,$$

so the general solution of Equation 5.6.11 is

$$y=y_p+c_1x+c_2e^x=-x^2-x-1+c_1x+c_2e^x=-x^2-1+(c_1-1)x+c_2e^x. \tag{5.6.12}$$

However, since c_1 is an arbitrary constant, so is c_1-1; therefore, we improve the appearance of this result by renaming the constant and writing the general solution as

$$y=-x^2-1+c_1x+c_2e^x. \tag{5.6.13}$$

Example 5.6.3

Find a particular solution of

$$y''+3y'+2y=\frac{1}{1+e^x}. \tag{5.6.14}$$

Then find the general solution.

Solution

The characteristic polynomial of the homogeneous equation

$$y''+3y'+2y=0 \tag{5.6.15}$$

is $r^2+3r+2=(r+1)(r+2)=0$, so $y_1=e^{-x}$ and $y_2=e^{-2x}$ form a fundamental set of solutions of Equation 5.6.15. We look for a particular solution of Equation 5.6.14 in the form

$$y_p=u_1e^{-x}+u_2e^{-2x},$$

where

$$\begin{aligned} e^{-x}u_1'+\ e^{-2x}u_2' &=0\\ -e^{-x}u_1'-2e^{-2x}u_2' &=\frac{1}{1+e^x}. \end{aligned}$$

Now, letting

$$W=\begin{vmatrix} e^{-x} & e^{-2x}\\ -e^{-x} & -2e^{-2x}\end{vmatrix}=-e^{-3x},\quad W_1=\begin{vmatrix} 0 & e^{-2x}\\ \frac{1}{1+e^x} & -2e^{-2x}\end{vmatrix}=\frac{-e^{-2x}}{1+e^x},\quad W_2=\begin{vmatrix} e^{-x} & 0\\ -e^{-x} & \frac{1}{1+e^x}\end{vmatrix}=\frac{e^{-x}}{1+e^x},$$

So, $u_1'=\frac{W_1}{W}=\frac{e^x}{e^x+1}$ and $u_2'=\frac{W_2}{W}=\frac{-e^{2x}}{e^x+1}$.

Integrating by means of the substitution $v=e^x$ and taking the constants of integration to be zero yields

$$u_1=\int\frac{e^x}{1+e^x}\,dx=\int\frac{dv}{1+v}=\ln(1+v)=\ln(1+e^x)$$

and

$$\begin{aligned} u_2\ &=-\int\frac{e^{2x}}{1+e^x}\,dx=-\int\frac{v}{1+v}\,dv=\int\left[\frac{1}{1+v}-1\right]\,dv\\ &=\ln(1+v)-v=\ln(1+e^x)-e^x. \end{aligned}$$

Therefore

$$\begin{aligned} y_p &=u_1e^{-x}+u_2e^{-2x}\\ &=[\ln(1+e^x)]e^{-x}+[\ln(1+e^x)-e^x]\,e^{-2x}, \end{aligned}$$

so

$$y_p=\left(e^{-x}+e^{-2x}\right)\ln(1+e^x)-e^{-x}.$$

Since the last term on the right satisfies the homogeneous equation, we drop it and redefine

$$y_p=\left(e^{-x}+e^{-2x}\right)\ln(1+e^x).$$

The general solution of Equation 5.6.14 is

$$y=y_p+c_1e^{-x}+c_2e^{-2x}=\left(e^{-x}+e^{-2x}\right)\ln(1+e^x)+c_1e^{-x}+c_2e^{-2x}.$$

Example 5.6.4

Solve the initial value problem

$$(x^2-1)y''+4xy'+2y=\frac{2}{x+1},\quad y(0)=-1,\quad y'(0)=-5, \tag{5.6.16}$$

given that

$$y_1=\frac{1}{x-1}\quad\text{and}\quad y_2=\frac{1}{x+1}$$

are solutions of the homogeneous equation

$$(x^2-1)y''+4xy'+2y=0.$$

Solution

We first use variation of parameters to find a particular solution of

$$(x^2-1)y''+4xy'+2y=\frac{2}{x+1}$$

on $(-1,1)$ in the form

$$y_p=\frac{u_1}{x-1}+\frac{u_2}{x+1},$$

where

$$\frac{u_1'}{x-1}+\frac{u_2'}{x+1}=0 \tag{5.6.17}$$

$$-\frac{u_1'}{(x-1)^2}-\frac{u_2'}{(x+1)^2}=\frac{2}{(x+1)(x^2-1)}=\frac{2}{(x+1)^2(x-1)}$$

Now, letting

$$W=\begin{vmatrix}\frac{1}{x-1} & \frac{1}{x+1}\\ \frac{-1}{(x-1)^2} & \frac{-1}{(x+1)^2}\end{vmatrix}=\frac{2}{(x+1)^2(x-1)^2},\quad W_1=\begin{vmatrix}0 & \frac{1}{x+1}\\ \frac{2}{(x+1)^2(x-1)} & \frac{-1}{(x+1)^2}\end{vmatrix}=\frac{-2}{(x+1)^3(x-1)},$$

$$W_2=\begin{vmatrix}\frac{1}{x-1} & 0\\ \frac{-1}{(x-1)^2} & \frac{2}{(x+1)^2(x-1)}\end{vmatrix}=\frac{2}{(x+1)^2(x-1)^2}.$$

So, $u_1'=\frac{W_1}{W}=-\frac{x-1}{x+1}$ and $u_2'=\frac{W_2}{W}=1$.

Integrating and taking the constants of integration to be zero yields

$$\begin{aligned}u_1&=-\int\frac{x-1}{x+1}\,dx=-\int\frac{x+1-2}{x+1}\,dx\\&=\int\left[\frac{2}{x+1}-1\right]dx=2\ln(x+1)-x\end{aligned}$$

and

$$u_2=\int dx=x.$$

Therefore

$$\begin{aligned}y_p&=\frac{u_1}{x-1}+\frac{u_2}{x+1}=[2\ln(x+1)-x]\,\frac{1}{x-1}+x\,\frac{1}{x+1}\\&=\frac{2\ln(x+1)}{x-1}+x\left[\frac{1}{x+1}-\frac{1}{x-1}\right]=\frac{2\ln(x+1)}{x-1}-\frac{2x}{(x+1)(x-1)}.\end{aligned}$$

However, since

$$\frac{2x}{(x+1)(x-1)} = \left[\frac{1}{x+1} + \frac{1}{x-1}\right]$$

is a solution of the homogeneous equation, we redefine

$$y_p = \frac{2\ln(x+1)}{x-1}.$$

Therefore the general solution of Equation 5.6.18 is

$$y = \frac{2\ln(x+1)}{x-1} + \frac{c_1}{x-1} + \frac{c_2}{x+1}. \tag{5.6.18}$$

Differentiating this yields

$$y' = \frac{2}{x^2-1} - \frac{2\ln(x+1)}{(x-1)^2} - \frac{c_1}{(x-1)^2} - \frac{c_2}{(x+1)^2}.$$

Setting $x = 0$ in the last two equations and imposing the initial conditions $y(0) = -1$ and $y'(0) = -5$ yields the system

$$\begin{aligned} -c_1 + c_2 &= -1 \\ -2 - c_1 - c_2 &= -5. \end{aligned}$$

The solution of this system is $c_1 = 2,\ c_2 = 1$. Substituting these into Equation 5.6.18 yields

$$\begin{aligned} y &= \frac{2\ln(x+1)}{x-1} + \frac{2}{x-1} + \frac{1}{x+1} \\ &= \frac{2\ln(x+1)}{x-1} + \frac{3x+1}{x^2-1} \end{aligned}$$

as the solution of Equation 5.6.16. Figure 5.6.1 is a graph of the solution.

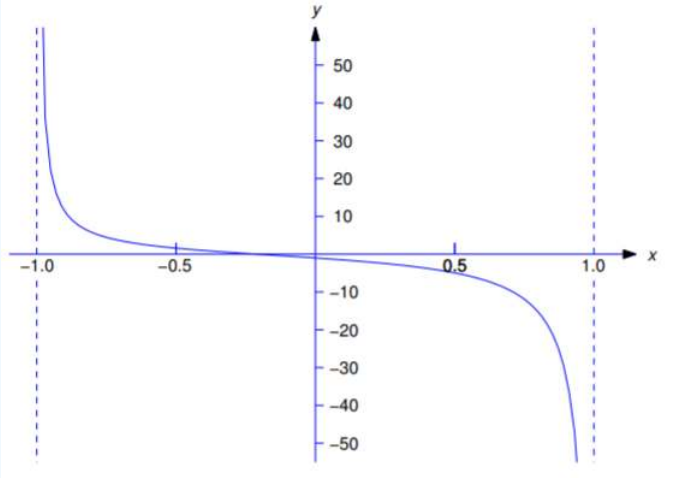


Figure 5.6.1 : $y = \frac{2\ln(x+1)}{x-1} + \frac{3x+1}{x^2-1}$

We've now considered two methods for solving nonhomogeneous linear equations: annihilation and variation of parameters. It's natural to ask which method is best for a given problem. The method of annihilation should be used for constant coefficient equations with forcing functions that are linear combinations of polynomials multiplied by functions of the form $e^{\alpha x}$, $e^{\lambda x}\cos\omega x$, or $e^{\lambda x}\sin\omega x$. Although variation of parameters can be used to solve such problems, it will be more difficult except in the most trivial cases, because of the integrations involved.

If the equation isn't a constant coefficient equation or the forcing function isn't of the form just specified, the method of annihilation does not apply and the choice is necessarily variation of parameters.

Using the Principle of Superposition

The next example shows how to combine the method of annihilation and variation of parameters, along with Theorem 5.4.3, the principle of superposition.

Example 5.6.5

Find the general solution of

$$y''-2y'+y=4x^2-3+\frac{e^x}{x} \tag{5.6.19}$$

on $(0,\infty)$.

Solution

Solving the homogeneous equation $y''-2y'+y=0$, we get the characteristic equation $(r-1)^2=0$ which gives us

$$y_h=c_1e^x+c_2xe^x$$

Now, when we turn to 5.6.19 it might appear that we must use variation of parameters to solve it because we cannot annihilate the entire right hand side of 5.6.19. However, we can annihilate the first two terms and use variation for the last term, and then use the principle of superposition to find y_p.

Let's first consider

$$y''-2y'+y=4x^2-3. \tag{5.6.20}$$

Writing it as

$$(D-1)^2y=4x^2-3$$

and using D^3 to annihilate the right hand side we get

$$D^3(D-1)^2y=0$$

which gives us a characteristic equation of $r^3(r-1)^2=0$ and roots $0,0,0,1,1$. We have used $1,1$ so that gives us

$$y_{p_1}=A+Bx+Cx^2$$

which gives us $y'_{p_1}=B+2Cx$ and $y''_{p_1}=2C$. Substituting into 5.6.20 gives us

$$2C-2(B+2Cx)+(A+Bx+Cx^2)=4x^2-3$$

which gives us $A=21, B=16, C=4$ and therefore

$$y_{p_1}=21+16x+4x^2$$

is the solution to 5.6.20.

Let's now consider

$$y''-2y'+y=\frac{e^x}{x}. \tag{5.6.21}$$

Here we are forced to use variation of parameters with

$$W=\begin{vmatrix} e^x & xe^x \\ e^x & xe^x+e^x \end{vmatrix}=e^{2x},\quad W_1=\begin{vmatrix} 0 & xe^x \\ \frac{e^x}{x} & xe^x+e^x \end{vmatrix}=-e^{2x},\quad W_2=\begin{vmatrix} e^x & 0 \\ e^x & \frac{e^x}{x} \end{vmatrix}=\frac{e^{2x}}{x},$$

So, $u_1'=\frac{W_1}{W}=-1$ and $u_2'=\frac{W_2}{W}=\frac{1}{x}$. Integrating and taking the constants of integration to be zero yields

$$u_1=-x \quad \text{and} \quad u_2=\ln x.$$

Therefore,

$$y_{p_2}=u_1x+u_2x^2=-xe^x+xe^x\ln x,$$

is the solution to 5.6.21.

Putting y_{p_1} and y_{p_2} together we get

$$y_p = y_{p_1} + y_{p_2} = 21 + 16x + 4x^2 - xe^x + xe^x \ln x.$$

The term $-xe^x$ can be absorbed into y_h, so the general solution of Equation 5.6.19 is

$$y = c_1 e^x + c_2 xe^x + 21 + 16x + 4x^2 + xe^x \ln x.$$

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5.6.1: Variation of Parameters (Exercises)

In Exercises **1-6** find the general solution.

1. $y''+9y=\tan 3x$
2. $y''+4y=\sin 2x\sec^2 2x$
3. $y''-3y'+2y=\frac{4}{1+e^{-x}}$
4. $y''-2y'+2y=3e^x\sec x$
5. $y''-2y'+y=14x^{3/2}e^x$
6. $y''-y=\frac{4e^{-x}}{1-e^{-2x}}$

In Exercises **7-29** find the general solution given the solutions y_1, y_2 of the homogeneous equation.

7. $x^2y''+xy'-y=2x^2+2;\quad y_1=x,\quad y_2=\frac{1}{x}$
8. $xy''+(2-2x)y'+(x-2)y=e^{2x};\quad y_1=e^x,\quad y_2=\frac{e^x}{x}$
9. $4x^2y''+(4x-8x^2)y'+(4x^2-4x-1)y=4x^{1/2}e^x,\quad x>0\ ;y_1=x^{1/2}e^x,\ y_2=x^{-1/2}e^x$
10. $y''+4xy'+(4x^2+2)y=4e^{-x(x+2)};\quad y_1=e^{-x^2},\quad y_2=xe^{-x^2}$
11. $x^2y''-4xy'+6y=x^{5/2},\ x>0;\quad y_1=x^2,\ y_2=x^3$
12. $x^2y''-3xy'+3y=2x^4\sin x;\quad y_1=x,\ y_2=x^3$
13. $(2x+1)y''-2y'-(2x+3)y=(2x+1)^2e^{-x};\quad y_1=e^{-x},\quad y_2=xe^x$
14. $4xy''+2y'+y=\sin\sqrt{x};\quad y_1=\cos\sqrt{x},\quad y_2=\sin\sqrt{x}$
15. $xy''-(2x+2)y'+(x+2)y=6x^3e^x;\quad y_1=e^x,\quad y_2=x^3e^x$
16. $x^2y''-(2a-1)xy'+a^2y=x^{a+1};\quad y_1=x^a,\quad y_2=x^a\ln x$
17. $x^2y''-2xy'+(x^2+2)y=x^3\cos x;\quad y_1=x\cos x,\quad y_2=x\sin x$
18. $xy''-y'-4x^3y=8x^5;\quad y_1=e^{x^2},\ y_2=e^{-x^2}$
19. $(\sin x)y''+(2\sin x-\cos x)y'+(\sin x-\cos x)y=e^{-x};\quad y_1=e^{-x},\quad y_2=e^{-x}\cos x$
20. $4x^2y''-4xy'+(3-16x^2)y=8x^{5/2};\quad y_1=\sqrt{x}e^{2x},\ y_2=\sqrt{x}e^{-2x}$
21. $4x^2y''-4xy'+(4x^2+3)y=x^{7/2};\quad y_1=\sqrt{x}\sin x,\ y_2=\sqrt{x}\cos x$
22. $x^2y''-2xy'-(x^2-2)y=3x^4;\quad y_1=xe^x,\ y_2=xe^{-x}$
23. $x^2y''-2x(x+1)y'+(x^2+2x+2)y=x^3e^x;\quad y_1=xe^x,\quad y_2=x^2e^x$
24. $x^2y''-xy'-3y=x^{3/2};\quad y_1=1/x,\quad y_2=x^3$
25. $x^2y''-x(x+4)y'+2(x+3)y=x^4e^x;\quad y_1=x^2,\quad y_2=x^2e^x$
26. $x^2y''-2x(x+2)y'+(x^2+4x+6)y=2xe^x;\quad y_1=x^2e^x,\quad y_2=x^3e^x$
27. $x^2y''-4xy'+(x^2+6)y=x^4;\quad y_1=x^2\cos x,\quad y_2=x^2\sin x$
28. $(x-1)y''-xy'+y=2(x-1)^2e^x;\quad y_1=x,\quad y_2=e^x$
29. $4x^2y''-4x(x+1)y'+(2x+3)y=x^{5/2}e^x;\quad y_1=\sqrt{x},\quad y_2=\sqrt{x}e^x$

In Exercises **30-35** solve the initial value problem, given y_1, y_2 are solutions of the homogeneous equation.

30. $(3x-1)y''-(3x+2)y'-(6x-8)y=(3x-1)^2e^{2x},\quad y(0)=1,\ y'(0)=2\ ;y_1=e^{2x},\ y_2=xe^{-x}$
31. $(x-1)^2y''-2(x-1)y'+2y=(x-1)^2,\quad y(0)=3,\quad y'(0)=-6\ ;y_1=x-1\ , y_2=x^2-1$
32. $(x-1)^2y''-(x^2-1)y'+(x+1)y=(x-1)^3e^x,\quad y(0)=4,\quad y'(0)=-6\ ;y_1=(x-1)e^x,\quad y_2=x-1$
33. $(x^2-1)y''+4xy'+2y=2x,\quad y(0)=0,\ y'(0)=-2;\quad y_1=\frac{1}{x-1},\ y_2=\frac{1}{x+1}$

34. $x^2y''+2xy'-2y=-2x^2, \quad y(1)=1,\ y'(1)=-1; \quad y_1=x,\ y_2=\frac{1}{x^2}$

35. $(x+1)(2x+3)y''+2(x+2)y'-2y=(2x+3)^2, \quad y(0)=0, \quad y'(0)=0 \quad ; y_1=x+2, \quad y_2=\frac{1}{x+1}$

36. Solve $3y''-6y'+30y=15\sin x+e^x\tan 3x$

37. Solve $y''+y=\sin x+\sec x$

38. Solve $y''+3y'+2y=6x^2+\frac{1}{1+e^x}+\sin e^x$

39. Solve $y''-2y'+y=x^3+4x+\frac{e^x}{1+x^2}$

40. Solve $y''-2y'+y=e^x+e^x\tan^{-1}x$

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5.7: Cauchy-Euler Equations

A special class of linear differential equations that is of interest are Cauchy-Euler equations, defined as follows.

Definition 5.7.1 Cauchy-Euler Equations

A second order *Cauchy-Euler equation* is an equation that **can** be written in the form

$$ax^2y'' + bxy' + cy = 0, \tag{5.7.1}$$

where a, b, and c are real constants and $a \neq 0$.

Theorem 5.1.1 implies that Equation 5.7.1 has solutions defined on $(0, \infty)$ and $(-\infty, 0)$, since Equation 5.7.1 can be rewritten as

$$ay'' + \frac{b}{x}y' + \frac{c}{x^2}y = 0.$$

For convenience we'll restrict our attention to the interval $(0, \infty)$. (*Exercise 7.4.40* deals with solutions of Equation 5.7.1 on $(-\infty, 0)$.)

The key to finding solutions on $(0, \infty)$ is that if $x > 0$ then we can guess that $y = x^r$ is defined as a real-valued function on $(0, \infty)$ for all values of r, and it may lead to a solution. Substituting $y = x^r$ into Equation 5.7.1 produces

$$\begin{aligned} ax^2(x^r)'' + bx(x^r)' + cx^r &= ax^2r(r-1)x^{r-2} + bxrx^{r-1} + cx^r \\ &= [ar(r-1) + br + c]x^r. \end{aligned} \tag{5.7.2}$$

The equation

$$ar(r-1) + br + c = ar^2 + (b-a)r + c = 0 \tag{5.7.3}$$

is called the *characteristic equation* of Equation 5.7.1. From Equation 5.7.2 we can see that $y = x^r$ is a solution of Equation 5.7.1 on $(0, \infty)$ if and only if $ar(r-1) + br + c = 0$.

The roots of the characteristic equation are given by the quadratic formula

$$r = \frac{-(b-a) \pm \sqrt{(b-a)^2 - 4ac}}{2a}. \tag{5.7.4}$$

Three Cases

As you'll recall from algebra, there are three possible types of solutions to 5.7.3 and we will exam each case separately

- Case 1: $(b-a)^2 - 4ac > 0$, so the characteristic equation has two distinct real roots.
- Case 2: $(b-a)^2 - 4ac = 0$, so the characteristic equation has a repeated real root.
- Case 3: $(b-a)^2 - 4ac < 0$, so the characteristic equation has complex roots.

Case 1: Distinct Real Roots

Let r_1 and r_2 be distinct real roots of 5.7.3.

Therefore, $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$ form a fundamental set of solutions of Equation 5.7.1 on $(0, \infty)$, since $y_2/y_1 = x^{r_2 - r_1}$ is nonconstant. In this case

$$y = c_1x^{r_1} + c_2x^{r_2}$$

is the general solution of Equation 5.7.1 on $(0, \infty)$.

Note: we could also use the Wronskian to show independence:

$$W(x) = \begin{vmatrix} x^{r_1} & x^{r_2} \\ r_1x^{r_1-1} & r_2x^{r_2-1} \end{vmatrix} = (r_2 - r_1)x^{(r_1+r_2-1)} \not\equiv 0$$

for all x on $(0, \infty)$ since $r_1 \neq r_2$.

Example 5.7.1

Find the general solution of

$$x^2y'' - xy' - 8y = 0 \tag{5.7.5}$$

on $(0, \infty)$.

Solution

The characteristic equation of 5.7.5 is

$$r(r-1) - r - 8 = (r-4)(r+2) = 0.$$

Therefore $y_1 = x^4$ and $y_2 = x^{-2}$ are solutions of Equation 5.7.5 on $(0, \infty)$, and its general solution on $(0, \infty)$ is

$$y = c_1x^4 + \frac{c_2}{x^2}.$$

Example 5.7.2

Find the general solution of

$$6x^2y'' + 5xy' - y = 0 \tag{5.7.6}$$

on $(0, \infty)$.

Solution

The characteristic equation of 5.7.6 is

$$6r(r-1) + 5r - 1 = (2r-1)(3r+1) = 0.$$

Therefore the general solution of Equation 5.7.6 on $(0, \infty)$ is

$$y = c_1x^{1/2} + c_2x^{-1/3}.$$

Case 2: A Repeated Real Root

Let r be a repeated real root of 5.7.3. As such, we only get one solution, $y_1 = x^r$. But we know there must be two independent solutions to a second order homogeneous differential equation. So, how can we find a second independent solution if we only know one? Recall the formula for a second solution that we derived in section 5.2,

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2} dx$$

If we divide 5.7.2 by a we get

$$y'' + \frac{b}{ax}y' + \frac{c}{ax^2}y = 0$$

which gives is $p(x) = \frac{b}{ax}$, which gives us

$$y_2 = x^r \int \frac{e^{-\int \frac{b}{ax}dx}}{x^{2r}} dx$$

$$= x^r \int \frac{x^{-\frac{b}{a}}}{x^{2r}} dx$$

Since we have a repeated root we know $(b-a)^2 - 4ac = 0$ in 5.7.4 which gives us $r = \frac{-(b-a)}{2a}$ or

$$\frac{-b}{a} = 2r - 1$$

Using this we now get

$$y_2 = x^r \int \frac{x^{2r-1}}{x^{2r}} dx$$
$$= x^r \int \frac{1}{x} dx$$
$$= x^r \ln x$$

Since we already know that the formula gives us a second linearly independent solution, the general solution is

$$y = c_1 x^r + c_2 x^r \ln x \tag{5.7.7}$$

on $(0, \infty)$.

Example 5.7.3

Find the general solution of

$$x^2 y'' - 5xy' + 9y = 0 \tag{5.7.8}$$

on $(0, \infty)$.

Solution

The characteristic equation of 5.7.8 is

$$r(r-1) - 5r + 9 = (r-3)^2 = 0.$$

Therefore the general solution of Equation 5.7.8 on $(0, \infty)$ is

$$y = x^3(c_1 + c_2 \ln x).$$

Example 5.7.4

a. Find the general solution of

$$x^2 y'' + 3xy' + y = 0 \tag{5.7.9}$$

on $(0, \infty)$.

b. Solve the initial value problem

$$x^2 y'' + 3xy' + y = 0. \quad y(1) = 0, y'(1) = 1 \tag{5.7.10}$$

on $(0, \infty)$.

Solution a

The characteristic equation of 5.7.9 is

$$r(r-1) + 3r + 1 = (r+1)^2 = 0.$$

Therefore the general solution of Equation 5.7.9 on $(0, \infty)$ is

$$y = x^{-1}(c_1 + c_2 \ln x).$$

Solution b

At $x = 1$ we know the values of y and y'. Using the general solution of 5.7.9, we first have that

$$0 = y(1) = c_1$$

Thus, we have so far that $y = c_2 x^{-1} \ln x$.

Now, using the second condition and

$$y'(x) = c_2(1 - \ln x)x^{-2}$$

we have

$$1 = y'(1) = c_2$$

Therefore, the solution of 5.7.10 on $(0, \infty)$ is

$$y = x^{-1} \ln x$$

.

Case 3: Complex Conjugate Roots

Now suppose 5.7.3 has complex conjugate roots $r = \lambda \pm \omega i$. This is similar to case one in that we do have distinct roots and we could say $y_1 = x^{(\lambda+\omega i)}$ and $y_2 = x^{(\lambda-\omega i)}$. The problem with this is that it's not very useful in real life applications to have functions of complex numbers. So, we want to see if we can somehow write both solutions as functions of real numbers, and as we did earlier in the chapter we will use Euler's Identity.

We will now rewrite y_1 and y_2 as follows:

$$y_1 = x^{(\lambda+\omega i)} = e^{\ln x^{\lambda+\omega i}} = e^{(\lambda+\omega i)\ln x} = e^{\lambda lnx} e^{i\omega \ln x} = e^{\lambda lnx}(\cos(\omega \ln x) + i \sin(\omega \ln x)) = e^{\lambda lnx} \cos(\omega \ln x) + ie^{\lambda lnx} \sin(\omega \ln x)$$

$$y_2 = x^{(\lambda-\omega i)} = e^{\ln x^{\lambda-\omega i}} = e^{(\lambda-\omega i)\ln x} = e^{\lambda lnx} e^{-i\omega \ln x} = e^{\lambda lnx}(\cos(-\omega \ln x) + i \sin(-\omega \ln x)) = e^{\lambda lnx} \cos(\omega \ln x) - ie^{\lambda lnx} \sin(\omega \ln x)$$

The latter is using the fact the cos is an even function and sin is an odd function.

Note: the i is no longer **inside** a function.

Recall that if y_1 and y_2 are solutions of a homogeneous equation, then any linear combination is also a solution. So, if we add y_1 and y_2 we get

$$y_3 = 2e^{\lambda lnx} \cos(\omega \ln x) = 2e^{lnx^{\lambda}} \cos(\omega \ln x) = 2x^{\lambda} \cos(\omega \ln x)$$

and if we subtract y_1 and y_2 we get

$$y_4 = 2ie^{\lambda lnx} \sin(\omega \ln x) = 2ie^{lnx^{\lambda}} \sin(\omega \ln x) = 2ix^{\lambda} \sin(\omega \ln x)$$

Now, both 2 and 2i are constants that can be absorbed into the constants c_1 and c_2 of our general solution and since y_3 and y_4 are not constant multiples of each other, they are independent. Thus, our general solution is

$$y = c_1 x^{\lambda} \cos(\omega \ln x) + c_2 x^{\lambda} \sin(\omega \ln x) \tag{5.7.11}$$

Example 5.7.5

Find the general solution of

$$x^2 y'' + 3xy' + 2y = 0 \tag{5.7.12}$$

on $(0, \infty)$.

Solution

The characteristic equation of 5.7.12 is

$$r(r-1) + 3r + 2 = (r+1)^2 + 1 = 0.$$

The roots of the characteristic equation are $r = -1 \pm i$ and the general solution of Equation 5.7.12 on $(0, \infty)$ is

$$y = \frac{1}{x} \left[c_1 \cos(\ln x) + c_2 \sin(\ln x) \right].$$

Example 5.7.6

Find the general solution of

$$x^2y'' - xy' + 5y = 0 \tag{5.7.13}$$

on $(0, \infty)$.

Solution

The characteristic equation of 5.7.13 is

$$r(r-1) - r + 5 = (r-1)^2 + 4 = 0.$$

The roots of the characteristic equation are $r = 1 \pm 2i$ and the general solution of Equation 5.7.13 on $(0, \infty)$ is

$$y = x\left[c_1 \cos(2\ln x) + c_2 \sin(2\ln x)\right].$$

Nonhomogeneous Cauchy-Euler Equations

We can solve nonhomogeneous Cauchy-Euler equations using the Method of Variation of Parameters. We will demonstrate this with a couple examples.

Example 5.7.7

Find the general solution of

$$x^2y'' - xy' - 3y = 2x^2 \tag{5.7.14}$$

on $(0, \infty)$.

Solution

First we find the solution of the homogeneous equation

$$x^2y'' - xy' - 3y = 0. \tag{5.7.15}$$

The characteristic equation of 5.7.15 is $r^2 - 2r - 3 = 0$. So, the roots are $r = -1, 3$ and the solution of 5.7.15 is

$$y_h = c_1x^{-1} + c_2x^3.$$

We set

$$y_p = u_1x^{-1} + u_2x^3,$$

where

$$\begin{aligned} x^{-1}u_1' + x^3u_2' &= 0 \\ -x^{-2}u_1' + 3x^2u_2' &= \frac{2x^2}{x^2} = 2. \end{aligned}$$

Now, letting

$$W = \begin{vmatrix} x^{-1} & x^3 \\ -x^{-2} & 3x^2 \end{vmatrix} = 4x, \quad W_1 = \begin{vmatrix} 0 & x^3 \\ 2 & 3x^2 \end{vmatrix} = -2x^3, \quad W_2 = \begin{vmatrix} x^{-1} & 0 \\ -x^{-2} & 2 \end{vmatrix} = 2x^{-1},$$

So, $u_1' = \frac{W_1}{W} = -\frac{x^2}{2}$ and $u_2' = \frac{W_2}{W} = \frac{x^{-2}}{2}$. Integrating and taking the constants of integration to be zero yields

$$u_1 = -\frac{x^3}{6} \quad \text{and} \quad u_2 = -\frac{x^{-1}}{2}.$$

Therefore

$$y_p = u_1x^{-1} + u_2x^3 = -\frac{x^3}{6}x^{-1} - \frac{x^{-1}}{2}x^3 = -\frac{2}{3}x^2,$$

so the general solution of Equation 5.7.14 is

$$y = y_p + c_1 x^{-1} + c_2 x^3 = -\frac{2}{3}x^2 + c_1 x^{-1} + c_2 x^3$$

on $(0, \infty)$.

Example 5.7.8

Find the general solution of

$$x^2 y'' - xy' - 3y = 2x^3 \tag{5.7.16}$$

on $(0, \infty)$.

Solution

5.7.16 has the same homogenous solution as 5.7.14, so

$$y_h = c_1 x^{-1} + c_2 x^3.$$

We set

$$y_p = u_1 x^{-1} + u_2 x^3,$$

where

$$\begin{aligned} x^{-1} u_1' + x^3 u_2' &= 0 \\ -x^{-2} u_1' + 3x^2 u_2' &= \frac{2x^2}{x^2} = 2x. \end{aligned}$$

Now, letting

$$W = \begin{vmatrix} x^{-1} & x^3 \\ -x^{-2} & 3x^2 \end{vmatrix} = 4x, \quad W_1 = \begin{vmatrix} 0 & x^3 \\ 2x & 3x^2 \end{vmatrix} = -2x^4, \quad W_2 = \begin{vmatrix} x^{-1} & 0 \\ -x^{-2} & 2x \end{vmatrix} = 2,$$

So, $u_1' = \frac{W_1}{W} = -\frac{x^3}{2}$ and $u_2' = \frac{W_2}{W} = \frac{2}{4x}$. Integrating and taking the constants of integration to be zero yields

$$u_1 = -\frac{x^4}{8} \quad \text{and} \quad u_2 = \frac{1}{2}\ln x.$$

Therefore

$$y_p = u_1 x^{-1} + u_2 x^3 = -\frac{x^4}{8}x^{-1} + (\frac{1}{2}\ln x)x^3 = -\frac{x^3}{8} + \frac{x^3}{2}\ln x,$$

However, since c_2 is an arbitrary constant, so is $c_2 - \frac{1}{8}$; therefore, we improve the appearance of this result by renaming the constant and writing the general solution of Equation 5.7.16 as

$$y = y_p + c_1 x^{-1} + c_2 x^3 = \frac{x^3}{2}\ln x + c_1 x^{-1} + c_2 x^3$$

on $(0, \infty)$.

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5.7.1: Cauchy-Euler Equations (Exercises)

In Exercises **1-40** find the general solution of the given equation on $(0, \infty)$.

1. $x^2y'' + 7xy' + 8y = 0$
2. $x^2y'' - 7xy' + 7y = 0$
3. $x^2y'' - xy' + y = 0$
4. $x^2y'' + 5xy' + 4y = 0$
5. $x^2y'' + xy' + y = 0$
6. $x^2y'' - 3xy' + 13y = 0$
7. $x^2y'' + 3xy' - 3y = 0$
8. $12x^2y'' - 5xy'' + 6y = 0$
9. $4x^2y'' + 8xy' + y = 0$
10. $3x^2y'' - xy' + y = 0$
11. $2x^2y'' - 3xy' + 2y = 0$
12. $x^2y'' + 3xy' + 5y = 0$
13. $9x^2y'' + 15xy' + y = 0$
14. $x^2y'' - xy' + 10y = 0$
15. $x^2y'' - 6y = 0$
16. $2x^2y'' + 3xy' - y = 0$
17. $x^2y'' - 3xy' + 4y = 0$
18. $2x^2y'' + 10xy' + 9y = 0$
19. $x^2y'' - 2y = 0$
20. $xy'' + y' = 0$
21. $x^2y'' + xy' + 4y = 0$
22. $x^2y'' - 3xy' - 2y = 0$
23. $25x^2y'' + 25xy' + y = 0$
24. $x^2y'' + 5xy' + 4y = 0$
25. $3x^2y'' + 6xy' + y = 0$
26. $x^2y'' + 3xy' - 3y = 0$
27. $2x^2y'' + 5xy' + y = 0$
28. $4x^2y'' + y = 0$
29. $xy'' - 4y' = x^4$
30. $x^2y'' - xy' + y = 2x$
31. $x^2y'' - xy' - 3y = 2x^2$
32. $x^2y'' - xy' - 3y = 2x^3$
33. $x^2y'' - 4xy' + 6y = 2x^4 + x^2$
34. $x^2y'' - 3xy' + 3y = 2x^4e^x$
35. $x^2y'' - xy' + y = \ln x$
36. $2x^2y'' + 5xy' + y = x^2 + x$

37. $x^2y''+5xy'+4y=2x^3$

38. $x^2y''+10xy'+8y=x^2$

39. $x^2y''-4xy'+6y=\ln x^2$

40. $x^2y''-3xy'+13y=4+3x$

In Exercises **41-49** find the solution of the given initial value problem on $(0,\infty)$.

41. $x^2y''+3xy'=0;\quad y(1)=0,\quad y'(1)=4$

42. $x^2y''-5xy'+8y=0;\quad y(2)=32,\quad y'(2)=0$

43. $x^2y''+xy'+y=0;\quad y(1)=1,\quad y'(1)=2$

44. $x^2y''-3xy'+4y=0;\quad y(1)=5,\quad y'(1)=3$

45. $x^2y''+3xy'+y=0;\quad y(1)=0,\quad y'(1)=1$

46. $xy''+y'=x;\quad y(1)=1,\quad y'(1)=-1/2$

47. $x^2y''-5xy'+8y=8x^6;\quad y(\frac{1}{2})=0,\quad y'(\frac{1}{2})=0$

48. $x^2y''-3xy'+3y=2x^4e^x;\quad y(1)=0,\quad y'(1)=0$

49. $x^2y''-xy'+y=\ln x;\quad y(1)=2,\quad y'(1)=2$

50. Prove that if $y=y(x)$ satisfies the Cauchy-Euler equation

$$ax^2\frac{d^2y}{dx^2}+bx\frac{dy}{dx}+cy=0 \tag{A}$$

on $(0,\infty)$ then $y=y(t)$, where $t=-x$, satisfies A on $(-\infty,0)$.

In Exercises **51-52** find the solution of the given initial value problem on $(-\infty,0)$.

51. $4x^2y''+y=0;\quad y(-1)=2,\quad y'(-1)=4$

52. $x^2y''-4xy'+6y=0;\quad y(-2)=8,\quad y'(-2)=0$

53. A nontrivial solution of

$$P_0(x)y''+P_1(x)y'+P_2(x)y=0$$

is said to be *oscillatory* on an interval (a,b) if it has infinitely many zeros on (a,b). Otherwise y is said to be *nonoscillatory* on (a,b). Prove that the equation

$$x^2y''+ky=0\quad (k=\text{ constant})$$

has oscillatory solutions on $(0,\infty)$ if and only if $k>1/4$.

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CHAPTER OVERVIEW

6: Applications of Linear Second Order Equations

In this Chapter, we study applications of linear second order equations.

6.1: Spring-Mass Problems (Without Damping)

6.1E: Spring-Mass Problems (Without Damping) (Exercises)

6.2: Spring-Mass Problems (with Damping)

6.2E: Spring-Mass Problems (With Damping) (Exercises)

6.3: The RLC Circuit

6.3E: The RLC Circuit (Exercises)

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6.1: Spring-Mass Problems (Without Damping)

We consider the motion of an object of mass m, suspended from a spring of negligible mass. We say that the spring–mass system is in *equilibrium* when the object is at rest and the forces acting on it sum to zero. The position of the object in this case is the *equilibrium position*. We define y to be the displacement of the object from its equilibrium position (Figure 6.1.1), measured positive upward.

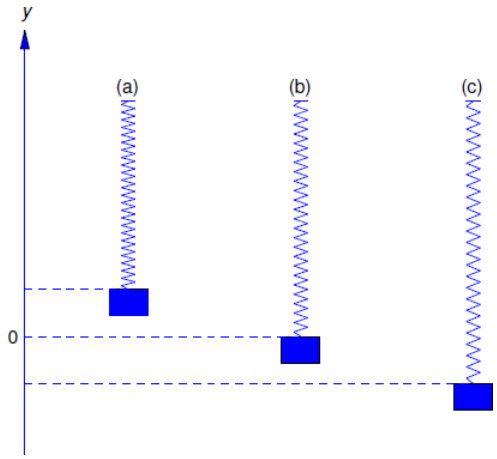


Figure 6.1.1 : (a) $y > 0$ (b) $y = 0$, (c) $y < 0$

Our model accounts for the following kinds of forces acting on the object:

- The force $-mg$, due to gravity.
- A force F_s exerted by the spring resisting change in its length. The *natural length* of the spring is its length with no mass attached. We assume that the spring obeys *Hooke's law*: If the length of the spring is changed by an amount ΔL from its natural length, then the spring exerts a force $F_s = k\Delta L$, where k is a positive number called the *spring constant*. If the spring is stretched then $\Delta L > 0$ and $F_s > 0$, so the spring force is upward, while if the spring is compressed then $\Delta L < 0$ and $F_s < 0$, so the spring force is downward.
- A *damping force* $F_d = -cy'$ that resists the motion with a force proportional to the velocity of the object. It may be due to air resistance or friction in the spring. However, a convenient way to visualize a damping force is to assume that the object is rigidly attached to a piston with negligible mass immersed in a cylinder (called a *dashpot*) filled with a viscous liquid (Figure 6.1.2). As the piston moves, the liquid exerts a damping force. We say that the motion is *undamped* if $c = 0$, or *damped* if $c > 0$.
- An external force F, other than the force due to gravity, that may vary with t, but is independent of displacement and velocity. We say that the motion is *free* if $F = 0$, or *forced* if $F \neq 0$.



Figure 6.1.2 : A spring – mass system with damping

From Newton's second law of motion,

$$my'' = -mg + F_d + F_s + F = -mg - cy' + F_s + F. \tag{6.1.1}$$

We must now relate F_s to y. In the absence of external forces the object stretches the spring by an amount Δl to assume its equilibrium position (Figure 6.1.3). Since the sum of the forces acting on the object is then zero, Hooke's Law implies that $mg = k\Delta l$. If the object is displaced y units from its equilibrium position, the total change in the length of the spring is $\Delta L = \Delta l - y$, so Hooke's law implies that

$$F_s = k\Delta L = k\Delta l - ky.$$

Substituting this into Equation 6.1.1 yields

$$my'' = -mg - cy' + k\Delta l - ky + F.$$

Since $mg = k\Delta l$ this can be written as

$$my'' + cy' + ky = F. \tag{6.1.2}$$

We call this *the equation of motion*.

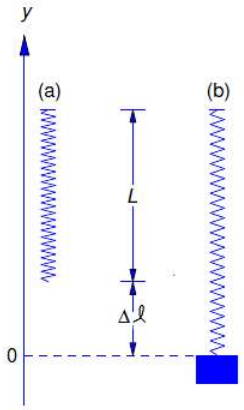


Figure 6.1.3 :**(a)** Natural length of spring **(b)** Spring stretched by mass

Simple Harmonic Motion

Throughout the rest of this section we'll consider spring–mass systems without damping; that is, $c = 0$. We'll consider systems with damping in the next section. We first consider the case where the motion is also free; that is, $F = 0$. We begin with an example.

Example 6.1.1

An object stretches a spring 6 inches in equilibrium.

a. Set up the equation of motion and find its general solution.
b. Find the displacement of the object for $t > 0$ if it is initially displaced 18 inches above equilibrium and given a downward velocity of 3 ft/s.

Solution a

Setting $c = 0$ and $F = 0$ in Equation 6.1.2 yields the equation of motion

$$my'' + ky = 0,$$

which we rewrite as

$$y'' + \frac{k}{m}y = 0. \tag{6.1.3}$$

Although we would need the weight of the object to obtain k from the equation $mg = k\Delta l$ we can obtain k/m from Δl alone; thus, $k/m = g/\Delta l$. Consistent with the units used in the problem statement, we take $g = 32$ ft/s^2. Although Δl is stated in inches, we must convert it to feet to be consistent with this choice of g; that is, $\Delta l = 1/2$ ft. Therefore

$$\frac{k}{m} = \frac{32}{1/2} = 64$$

and Equation 6.1.3 becomes

$$y'' + 64y = 0. \tag{6.1.4}$$

The characteristic equation of Equation 6.1.4 is

$$r^2 + 64 = 0,$$

which has the zeros $r = \pm 8i$. Therefore the general solution of Equation 6.1.4 is

$$y = c_1 \cos 8t + c_2 \sin 8t. \tag{6.1.5}$$

Solution b

The initial upward displacement of 18 inches is positive and must be expressed in feet. The initial downward velocity is negative; thus,

$$y(0) = \frac{3}{2} \quad \text{and} \quad y'(0) = -3.$$

Differentiating Equation 6.1.5 yields

$$y' = -8c_1 \sin 8t + 8c_2 \cos 8t. \tag{6.1.6}$$

Setting $t = 0$ in Equation 6.1.5 and Equation 6.1.6 and imposing the initial conditions shows that $c_1 = 3/2$ and $c_2 = -3/8$. Therefore

$$y = \frac{3}{2} \cos 8t - \frac{3}{8} \sin 8t,$$

where y is in feet (Figure 6.1.4).

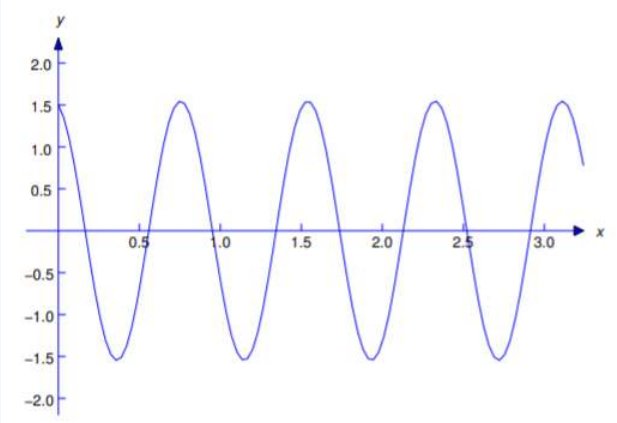


Figure 6.1.4 : $y = \frac{3}{2}\cos 8t - \frac{3}{8}\sin 8t$

Amplitude-Phase Form of Solution

There are times when the solution in the last example is not adequate for answering certain questions easily. Here, we will talk about another form which is simpler and easier to use for many applications.

Consider the general equation

$$my'' + ky = 0$$

where m and k are arbitrary positive numbers.

The characteristic equation is

$$mr^2 + k = 0,$$

which has the zeros $r = \pm i\sqrt{k/m}$.

Letting $\omega_0 = \sqrt{k/m}$ we get $r = \pm\omega_0 i$.

The general solution of this equation is

$$y = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t. \tag{6.1.7}$$

We can rewrite this in a more useful form by defining

$$A = \sqrt{c_1^2 + c_2^2}, \tag{6.1.8}$$

and

$$c_1 = A\sin\phi \quad \text{and} \quad c_2 = A\cos\phi. \tag{6.1.9}$$

Substituting from Equation 6.1.9 into Equation 6.1.7 and applying the identity

$$\sin\omega_0 t\cos\phi + \cos\omega_0 t\sin\phi = \sin(\omega_0 t + \phi)$$

yields

$$y = A\sin(\omega_0 t + \phi). \tag{6.1.10}$$

From Equation 6.1.8 and Equation 6.1.9 we see that the A and ϕ can be interpreted as polar coordinates of the point with rectangular coordinates (c_2, c_1) (Figure 6.1.5). Given c_1 and c_2, we can compute A from Equation 6.1.8. From Equation 6.1.9, we see that ϕ is related to c_1 and c_2 by

$$\tan\phi = \frac{c_1}{c_2}$$

The motion described by Equation 6.1.7 or Equation 6.1.10 is *simple harmonic motion*. We see from either of these equations that the motion is periodic, with period

$$T = 2\pi/\omega_0.$$

This is the time required for the object to complete one full cycle of oscillation (for example, to move from its highest position to its lowest position and back to its highest position). Since the highest and lowest positions of the object are $y = A$ and $y = -A$, we say that A is the *amplitude* of the oscillation. The angle ϕ in Equation 6.1.10 is the *phase angle*. It's measured in radians. Equation 6.1.10 is the *amplitude–phase form* of the displacement. If t is in seconds then ω_0 is in radians per second (rad/s); it is the *frequency* of the motion. It is also called the *natural frequency* of the spring–mass system without damping.

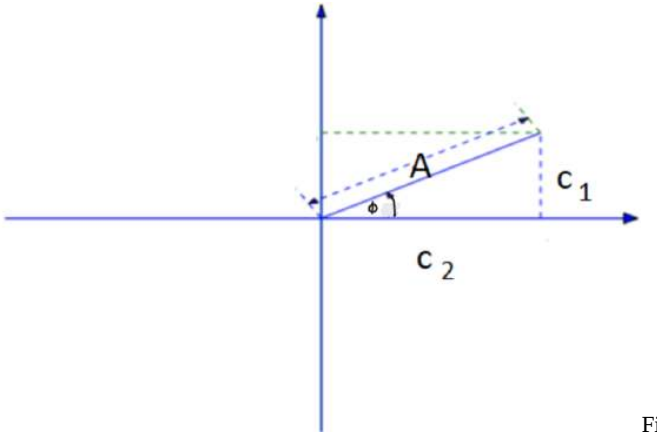


Figure 6.1.5 : $A = \sqrt{c_1^2 + c_2^2}$; $c_1 = A\sin\phi$; $c_2 = A\cos\phi$

Example 6.1.2

We found the displacement of the object in Example 6.1.1 to be

$$y = \frac{3}{2}\cos 8t - \frac{3}{8}\sin 8t.$$

Find the frequency, period, amplitude, and phase angle of the motion.

Solution

The frequency is $\omega_0 = 8$ rad/s, and the period is $T = 2\pi/\omega_0 = \pi/4$ s. Since $c_1 = 3/2$ and $c_2 = -3/8$, the amplitude is

$$A = \sqrt{c_1^2 + c_2^2} = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{3}{8}\right)^2} = \frac{3}{8}\sqrt{17}.$$

The phase angle is determined by

$$\tan\phi = \frac{\frac{3}{2}}{-\frac{3}{8}} = -4 \tag{6.1.11}$$

So,

$$\phi = \pi - \tan^{-1} 4$$

and the amplitude-phase form is

$$y = \frac{3}{8}\sqrt{17}\sin(8t + \pi - \tan^{-1} 4)$$

Example 6.1.3

The natural length of a spring is 1 m. An object is attached to it and the length of the spring increases to 102 cm when the object is in equilibrium. Then the object is initially displaced downward 1 cm and given an upward velocity of 14 cm/s. Find the displacement for $t > 0$. Also, find the natural frequency, period, amplitude, and phase angle of the resulting motion. Express the answers in cgs units.

Solution

In cgs units $g = 980$ cm/s^2. Since $\Delta l = 2$ cm, $\omega_0^2 = g/\Delta l = 490$. Therefore

$$y'' + 490y = 0, \quad y(0) = -1, \quad y'(0) = 14.$$

The general solution of the differential equation is

$$y = c_1 \cos 7\sqrt{10}t + c_2 \sin 7\sqrt{10}t,$$

so

$$y' = 7\sqrt{10}\,(-c_1 \sin 7\sqrt{10}t + c_2 \cos 7\sqrt{10}t)\,.$$

Substituting the initial conditions into the last two equations yields $c_1 = -1$ and $c_2 = 2/\sqrt{10}$. Hence,

$$y = -\cos 7\sqrt{10}t + \frac{2}{\sqrt{10}}\sin 7\sqrt{10}t.$$

The frequency is $7\sqrt{10}$ rad/s, and the period is $T = 2\pi/(7\sqrt{10})$ s. The amplitude is

$$A = \sqrt{c_1^2 + c_2^2} = \sqrt{(-1)^2 + \left(\frac{2}{\sqrt{10}}\right)^2} = \sqrt{\frac{7}{5}}\text{cm}$$

The phase angle is determined by

$$\tan\phi = \frac{-1}{\frac{2}{\sqrt{10}}} = -\frac{\sqrt{10}}{2}$$

Therefore ϕ is in the fourth quadrant and

$$\phi = -\tan^{-1}\frac{\sqrt{10}}{2}$$

and the amplitude-phase form is

$$y = \sqrt{\frac{7}{5}}\sin(10t - \tan^{-1}\frac{\sqrt{10}}{2})$$

Undamped Forced Oscillation

In many mechanical problems a device is subjected to periodic external forces. For example, soldiers marching in cadence on a bridge cause periodic disturbances in the bridge, and the engines of a propeller driven aircraft cause periodic disturbances in its wings. In the absence of sufficient damping forces, such disturbances – even if small in magnitude – can cause structural breakdown if they are at certain critical frequencies. To illustrate, this we'll consider the motion of an object in a spring–mass system without damping, subject to an external force

$$F(t) = F_0\cos\omega t$$

where F_0 is a constant. In this case the equation of motion Equation 6.1.2 is

$$my'' + ky = F_0\cos\omega t,$$

which we rewrite as

$$y'' + \omega_0^2 y = \frac{F_0}{m}\cos\omega t \tag{6.1.12}$$

with $\omega_0 = \sqrt{k/m}$. We'll see from the next two examples that the solutions of Equation 6.1.12 with $\omega \neq \omega_0$ behave very differently from the solutions with $\omega = \omega_0$.

Example 6.1.4

Solve the initial value problem

$$y'' + \omega_0^2 y = \frac{F_0}{m}\cos\omega t, \quad y(0) = 0, \quad y'(0) = 0, \tag{6.1.13}$$

given that $\omega \neq \omega_0$.

Solution

We've solved the homogeneous equation $y'' + \omega_0^2 y = 0$ earlier (see above) and came up with the homogeneous solution

$$y_h = c_1\cos\omega_0 t + c_2\sin\omega_0 t.$$

We obtain a particular solution of Equation 6.1.12 by the method of annihilation. Since $\omega \neq \omega_0$, $\cos\omega t$ isn't a solution of the homogeneous equation. So,

$$y_p = A\cos\omega t + B\sin\omega t.$$

Since

$$y_p'' = -\omega^2(A\cos\omega t + B\sin\omega t),$$

$$y_p'' + \omega_0^2 y_p = \frac{F_0}{m}\cos\omega t$$

if and only if

$$(\omega_0^2 - \omega^2)(A\cos\omega t + B\sin\omega t) = \frac{F_0}{m}\cos\omega t.$$

This holds if and only if

$$A = \frac{F_0}{m(\omega_0^2 - \omega^2)} \quad \text{and} \quad B = 0,$$

so

$$y_p = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t.$$

The general solution of Equation 6.1.12is

$$y = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t + c_1 \cos \omega_0 t + c_2 \sin \omega_0 t, \tag{6.1.14}$$

so

$$y' = \frac{-\omega F_0}{m(\omega_0^2 - \omega^2)} \sin \omega t + \omega_0(-c_1 \sin \omega_0 t + c_2 \cos \omega_0 t).$$

The initial conditions $y(0) = 0$ and $y'(0) = 0$ in Equation 6.1.13imply that

$$c_1 = -\frac{F_0}{m(\omega_0^2 - \omega^2)} \quad \text{and} \quad c_2 = 0.$$

Substituting these into Equation 6.1.14yields

$$y = \frac{F_0}{m(\omega_0^2 - \omega^2)} (\cos \omega t - \cos \omega_0 t). \tag{6.1.15}$$

It is revealing to write this in a different form. We start with the trigonometric identities

$$\begin{aligned} \cos(\alpha - \beta) &= \cos \alpha \cos \beta + \sin \alpha \sin \beta \\ \cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta. \end{aligned}$$

Subtracting the second identity from the first yields

$$\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2 \sin \alpha \sin \beta \tag{6.1.16}$$

Now let

$$\alpha - \beta = \omega t \quad \text{and} \quad \alpha + \beta = \omega_0 t, \tag{6.1.17}$$

so that

$$\alpha = \frac{(\omega_0 + \omega)t}{2} \quad \text{and} \quad \beta = \frac{(\omega_0 - \omega)t}{2}. \tag{6.1.18}$$

Substituting Equation 6.1.17and Equation 6.1.18into Equation 6.1.16yields

$$\cos \omega t - \cos \omega_0 t = 2 \sin \frac{(\omega_0 - \omega)t}{2} \sin \frac{(\omega_0 + \omega)t}{2},$$

and substituting this into Equation 6.1.15yields

$$y = R(t) \sin \frac{(\omega_0 + \omega)t}{2}, \tag{6.1.19}$$

where

$$R(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{(\omega_0 - \omega)t}{2}. \tag{6.1.20}$$

From Equation 6.1.19we can regard y as a sinusoidal variation with frequency $(\omega_0 + \omega)/2$ and variable amplitude $|R(t)|$. In Figure 6.1.6 the dashed curve above the t axis is $y = |R(t)|$, the dashed curve below the t axis is $y = -|R(t)|$, and the displacement y appears as an oscillation bounded by them. The oscillation of y for t on an interval between successive zeros of $R(t)$ is called a *beat*.

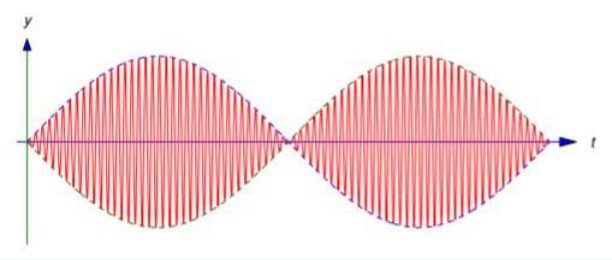


Figure 6.1.6 : Undamped oscillation with beats

You can see from Equation 6.1.19 and Equation 6.1.20 that

$$|y(t)| \le \frac{2|F_0|}{m|\omega_0^2 - \omega^2|};$$

moreover, if $\omega + \omega_0$ is sufficiently large compared with $\omega - \omega_0$, then $|y|$ assumes values close to (perhaps equal to) this upper bound during each beat. However, the oscillation remains bounded for all t. (This assumes that the spring can withstand deflections of this size and continues to obey Hooke's law.) The next example shows that this isn't so if $\omega = \omega_0$.

Example 6.1.5

Find the general solution of

$$y'' + \omega_0^2 y = \frac{F_0}{m}\cos\omega_0 t. \tag{6.1.21}$$

Solution

This equation has the same homogeneous solution as Example 4, so we will jump directly to the particular solution, which we again solve using annihilation.

Since $\cos\omega_0 t$ is a solution of the homogeneous equation, the form for y_p is

$$y_p = t(A\cos\omega_0 t + B\sin\omega_0 t). \tag{6.1.22}$$

Then

$$y_p' = A\cos\omega_0 t + B\sin\omega_0 t + \omega_0 t(-A\sin\omega_0 t + B\cos\omega_0 t)$$

and

$$y_p'' = 2\omega_0(-A\sin\omega_0 t + B\cos\omega_0 t) - \omega_0^2 t(A\cos\omega_0 t + B\sin\omega_0 t). \tag{6.1.23}$$

From Equation 6.1.22 and Equation 6.1.23, we see that y_p satisfies Equation 6.1.21 if

$$-2A\omega_0\sin\omega_0 t + 2B\omega_0\cos\omega_0 t = \frac{F_0}{m}\cos\omega_0 t;$$

that is, if

$$A = 0 \quad \text{and} \quad B = \frac{F_0}{2m\omega_0}.$$

Therefore

$$y_p = \frac{F_0 t}{2m\omega_0}\sin\omega_0 t$$

is a particular solution of Equation 6.1.21. The general solution of Equation 6.1.21 is

$$y = \frac{F_0 t}{2m\omega_0}\sin\omega_0 t + c_1\cos\omega_0 t + c_2\sin\omega_0 t.$$

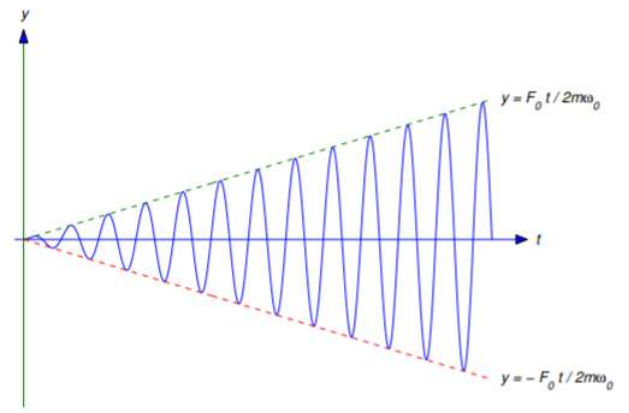


Figure 6.1.7 : Undamped displacement due to resonance.

The graph of y_p is shown in Figure 6.1.7 , where it can be seen that y_p oscillates between the dashed lines

$$y = \frac{F_0 t}{2m\omega_0} \quad \text{and} \quad y = -\frac{F_0 t}{2m\omega_0}$$

with increasing amplitude that approaches ∞ as $t \to \infty$. Of course, this means that the spring must eventually fail to obey Hooke's law or break.

This phenomenon of unbounded displacements of a spring–mass system in response to a periodic forcing function at its natural frequency is called *resonance*. More complicated mechanical structures can also exhibit resonance–like phenomena. For example, rhythmic oscillations of a suspension bridge by wind forces or of an airplane wing by periodic vibrations of reciprocating engines can cause damage or even failure if the frequencies of the disturbances are close to critical frequencies determined by the parameters of the mechanical system in question.

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6.1E: Spring-Mass Problems (Without Damping) (Exercises)

In the following exercises assume that there's no damping.

1. An object stretches a spring 4 inches in equilibrium. Find its displacement for $t > 0$ if it is initially displaced 36 inches above equilibrium and given a downward velocity of 2 ft/s.

2. An object stretches a spring 1.2 inches in equilibrium. Find its displacement for $t > 0$ if it is initially displaced 3 inches below equilibrium and given a downward velocity of 2 ft/s.

3. A spring with natural length $.5$ m has length 50.5 cm with a mass of 2 gm suspended from it. The mass is initially displaced 1.5 cm below equilibrium and released with zero velocity. Find its displacement for $t > 0$.

4. An object stretches a spring 6 inches in equilibrium. Find its displacement for $t > 0$ if it is initially displaced 3 inches above equilibrium and given a downward velocity of 6 inches/s. Find the frequency, period, amplitude and phase angle of the motion.

5. An object stretches a spring 5 cm in equilibrium. It is initially displaced 10 cm above equilibrium and given an upward velocity of $.25$ m/s. Find its displacement for $t > 0$. Find the frequency, period, amplitude, and phase angle of the motion.

6. A 10 kg mass stretches a spring 70 cm in equilibrium. Suppose a 2 kg mass is attached to the spring, initially displaced 25 cm below equilibrium, and given an upward velocity of 2 m/s. Find its displacement for $t > 0$. Find the frequency, period, amplitude, and phase angle of the motion.

7. A weight stretches a spring 1.5 inches in equilibrium. The weight is initially displaced 8 inches above equilibrium and given a downward velocity of 4 ft/s. Find its displacement for $t > 0$.

8. A weight stretches a spring 6 inches in equilibrium. The weight is initially displaced 6 inches above equilibrium and given a downward velocity of 3 ft/s. Find its displacement for $t > 0$.

9. A spring–mass system has natural frequency $7\sqrt{10}$ rad/s. The natural length of the spring is $.7$ m. What is the length of the spring when the mass is in equilibrium?

10. A 64 lb weight is attached to a spring with constant $k = 8$ lb/ft and subjected to an external force $F(t) = 2\sin t$. The weight is initially displaced 6 inches above equilibrium and given an upward velocity of 2 ft/s. Find its displacement for $t > 0$.

11. A unit mass hangs in equilibrium from a spring with constant $k = 1/16$. Starting at $t = 0$, a force $F(t) = 3\sin t$ is applied to the mass. Find its displacement for $t > 0$.

12. A 4 lb weight stretches a spring 1 ft in equilibrium. An external force $F(t) = .25\sin 8t$ lb is applied to the weight, which is initially displaced 4 inches above equilibrium and given a downward velocity of 1 ft/s. Find its displacement for $t > 0$.

13. A 2 lb weight stretches a spring 6 inches in equilibrium. An external force $F(t) = \sin 8t$ lb is applied to the weight, which is released from rest 2 inches below equilibrium. Find its displacement for $t > 0$.

14. A 10 gm mass suspended on a spring moves in simple harmonic motion with period 4 s. Find the period of the simple harmonic motion of a 20 gm mass suspended from the same spring.

15. A 6 lb weight stretches a spring 6 inches in equilibrium. Suppose an external force $F(t) = \frac{3}{16}\sin \omega t + \frac{3}{8}\cos \omega t$ lb is applied to the weight. For what value of ω will the displacement be unbounded? Find the displacement if ω has this value. Assume that the motion starts from equilibrium with zero initial velocity.

16. A 6 lb weight stretches a spring 4 inches in equilibrium. Suppose an external force $F(t) = 4\sin \omega t - 6\cos \omega t$ lb is applied to the weight. For what value of ω will the displacement be unbounded? Find the displacement if ω has this value. Assume that the motion starts from equilibrium with zero initial velocity.

17. A mass of one kg is attached to a spring with constant $k = 4$ N/m. An external force $F(t) = -\cos \omega t - 2\sin \omega t$ n is applied to the mass. Find the displacement y for $t > 0$ if ω equals the natural frequency of the spring–mass system. Assume that the mass is initially displaced 3 m above equilibrium and given an upward velocity of 450 cm/s.

18. An object is in simple harmonic motion with frequency ω_0, with $y(0) = y_0$ and $y'(0) = v_0$. Find its displacement for $t > 0$. Also, find the amplitude of the oscillation and the phase angle.

19. Two objects suspended from identical springs are set into motion. The period of one object is twice the period of the other. How are the weights of the two objects related?

20. Two objects suspended from identical springs are set into motion. The weight of one object is twice the weight of the other. How are the periods of the resulting motions related?

21. Two identical objects suspended from different springs are set into motion. The period of one motion is 3 times the period of the other. How are the two spring constants related?

22. A 24 pound weight is attached to the end of a spring, stretching it 4 inches. If the object is released from rest 3 inches above equilibrium, find the displacement for $t > 0$.

23. A 20 pound weight is attached to the end of a spring, stretching it 6 inches. If the object is released from rest 6 inches below equilibrium, find

a. the displacement for $t > 0$.

b. the times the object passes through equilibrium.

c. the time the object passes through equilibrium heading downward for the second time.

24. An 8 pound weight is attached to the end of a spring, stretching it 8 feet. If the object is released with a downward velocity of $3/2$ ft/s 6 inches below equilibrium, find

a. the displacement for $t > 0$.

b. write the equation of motion in the Amplitude-Phase Form.

c. the times the object passes through equilibrium.

d. the time the object passes through equilibrium heading upward for the third time.

25. A 64 pound weight is attached to the end of a spring, stretching it 0.32 foot. If the object is released 8 inches above equilibrium with a downward velocity of 5 ft/s, find

a. the displacement for $t > 0$.

b. write the equation of motion in the Amplitude-Phase Form.

c. the amplitude and period.

d. how many complete vibrations will the weight complete after 3π seconds.

e. the times the object passes through equilibrium.

f. the time the object passes through equilibrium heading downward for the second time.

g. the speed when the object passes through equilibrium.

h. the times the object is 5 inches below equilibrium.

26. A mass of 2 kilograms is attached to a spring whose constant is 32 N/m. At time $t = 0$, with the object in equilibrium, a force of $68e^{-2t}\cos 4t$ is impressed on the object. Find the displacement for $t > 0$.

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6.2: Spring-Mass Problems (with Damping)

Free Vibrations with Damping

In this section we consider the motion of an object in a spring–mass system with damping. We start with unforced motion, so the equation of motion is

$$my'' + cy' + ky = 0. \tag{6.2.1}$$

Now suppose the object is displaced from equilibrium and given an initial velocity. Intuition suggests that if the damping force is sufficiently weak the resulting motion will be oscillatory, as in the undamped case considered in the previous section, while if it is sufficiently strong the object may just move slowly toward the equilibrium position without ever reaching it. We'll now confirm these intuitive ideas mathematically. The characteristic equation of Equation 6.2.1 is

$$mr^2 + cr + k = 0.$$

The roots of this equation are

$$r_1 = \frac{-c - \sqrt{c^2 - 4mk}}{2m} \quad \text{and} \quad r_2 = \frac{-c + \sqrt{c^2 - 4mk}}{2m}. \tag{6.2.2}$$

We saw in Section 5.3 that the form of the solution of Equation 6.2.1 depends upon whether $c^2 - 4mk$ is positive, negative, or zero. We'll now consider these three cases.

Underdamped Motion

We say the motion is *underdamped* if $c < \sqrt{4mk}$. In this case r_1 and r_2 in Equation 6.2.2 are complex conjugates, which we write as

$$r_1 = -\frac{c}{2m} - \omega_1\, i \quad \text{and} \quad r_2 = -\frac{c}{2m} + \omega_1\, i,$$

where

$$\omega_1 = \frac{\sqrt{4mk - c^2}}{2m}.$$

The general solution of Equation 6.2.1 in this case is

$$y = e^{-ct/2m}(c_1 \cos \omega_1 t + c_2 \sin \omega_1 t).$$

By the method used in Section 6.1 to derive the amplitude–phase form of the displacement of an object in simple harmonic motion, we can rewrite this equation as

$$y = Ae^{-ct/2m} \sin(\omega_1 t + \phi), \tag{6.2.3}$$

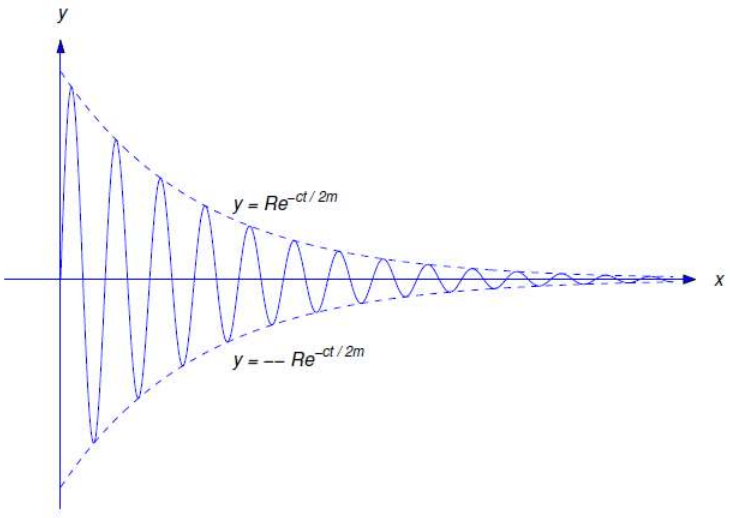


Figure 6.2.1 : Underdamped motion

where

$$A = \sqrt{c_1^2 + c_2^2}, \quad \tan\phi = \frac{c_1}{c_2}.$$

The factor $Ae^{-ct/2m}$ in Equation 6.2.3 is called the *time–varying amplitude* of the motion, the quantity ω_1 is called the *frequency*, and $T = 2\pi/\omega_1$ (which is the period of the sine function in Equation 6.2.3) is called the *quasi–period*. A typical graph of Equation 6.2.3 is shown in Figure 6.2.1 . As illustrated in that figure, the graph of y oscillates between the dashed exponential curves $y = \pm Ae^{-ct/2m}$.

Critically Damped Motion

We say the motion is *critically damped* if $c = \sqrt{4mk}$. In this case $r_1 = r_2 = -c/2m$ and the general solution of Equation 6.2.1 is

$$y = e^{-ct/2m}(c_1 + c_2 t).$$

Again $\lim_{t\to\infty} y(t) = 0$ and the motion is nonoscillatory, since y can't equal zero for more than one value of t unless $c_1 = c_2 = 0$. (*Exercise 6.2.22*).

Overdamped Motion

We say the motion is *overdamped* if $c > \sqrt{4mk}$. In this case the zeros r_1 and r_2 of the characteristic polynomial are real, with $r_1 < r_2 < 0$, and the general solution of Equation 6.2.1 is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Again $\lim_{t\to\infty} y(t) = 0$ as in the underdamped case, but the motion isn't oscillatory, since y can't equal zero for more than one value of t unless $c_1 = c_2 = 0$. (*Exercise 6.2.23.*)

Example 6.2.1

Suppose a 64 lb weight stretches a spring 6 inches in equilibrium and a dashpot provides a damping force of c lb for each ft/sec of velocity.

a. Write the equation of motion of the object and determine the value of c for which the motion is critically damped.
b. Find the displacement y for $t > 0$ if the motion is critically damped and the initial conditions are $y(0) = 1$ and $y'(0) = 20$.
c. Find the displacement y for $t > 0$ if the motion is critically damped and the initial conditions are $y(0) = 1$ and $y'(0) = -20$.

Solution a

Here $m = 2$ slugs and $k = 64/.5 = 128$ lb/ft. Therefore the equation of motion Equation 6.2.1 is

$$2y'' + cy' + 128y = 0. \tag{6.2.4}$$

The characteristic equation is

$$2r^2 + cr + 128 = 0,$$

which has roots

$$r = \frac{-c \pm \sqrt{c^2 - 8 \cdot 128}}{4}.$$

Therefore the damping is critical if

$$c = \sqrt{8 \cdot 128} = 32 \text{ lb-sec/ft}.$$

Solution b

Setting $c = 32$ in Equation 6.2.4 and cancelling the common factor 2 yields

$$y'' + 16y + 64y = 0.$$

The characteristic equation is

$$r^2 + 16r + 64y = (r+8)^2 = 0.$$

Hence, the general solution is

$$y = e^{-8t}(c_1 + c_2 t). \tag{6.2.5}$$

Differentiating this yields

$$y' = -8e^{-8t} + c_2 e^{-8t}. \tag{6.2.6}$$

6.2.2 Imposing the initial conditions $y(0) = 1$ and $y'(0) = 20$ in the last two equations shows that $1 = c_1$ and $20 = -8 + c_2$. Hence, the solution of the initial value problem is

$$y = e^{-8t}(1 + 28t).$$

Therefore the object approaches equilibrium from above as $t \to \infty$. There's no oscillation.

Solution c

Imposing the initial conditions $y(0) = 1$ and $y'(0) = -20$ in Equation 6.2.5 and Equation 6.2.6 yields $1 = c_1$ and $-20 = -8 + c_2$. Hence, the solution of this initial value problem is

$$y = e^{-8t}(1 - 12t).$$

Therefore the object moves downward through equilibrium just once, and then approaches equilibrium from below as $t \to \infty$. Again, there's no oscillation. The solutions of these two initial value problems are graphed in Figure 6.2.2 .

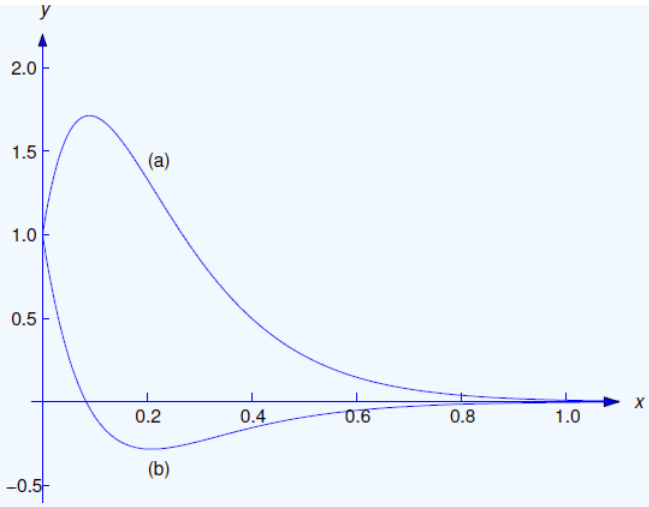


Figure 6.2.2 : (a) $y = e^{-8t}(1 + 28t)$ and (b) $y = e^{-8t}(1 - 12t)$

Example 6.2.2

Find the displacement of the object in Example 6.2.1 if the damping constant is $c = 4$ lb–sec/ft and the initial conditions are $y(0) = 1.5$ ft and $y'(0) = -3$ ft/sec.

Solution

With $c = 4$, the equation of motion Equation 6.2.4 becomes

$$y'' + 2y' + 64y = 0 \tag{6.2.7}$$

after cancelling the common factor 2. The characteristic equation

$$r^2 + 2r + 64 = 0$$

has complex conjugate roots

$$r = \frac{-2 \pm \sqrt{4 - 4 \cdot 64}}{2} = -1 \pm 3\sqrt{7}i.$$

Therefore the motion is underdamped and the general solution of Equation 6.2.7 is

$$y = e^{-t}(c_1 \cos 3\sqrt{7}t + c_2 \sin 3\sqrt{7}t).$$

Differentiating this yields

$$y' = -y + 3\sqrt{7}e^{-t}(-c_1 \sin 3\sqrt{7}t + c_2 \cos 3\sqrt{7}t).$$

Imposing the initial conditions $y(0) = 1.5$ and $y'(0) = -3$ in the last two equations yields $1.5 = c_1$ and $-3 = -1.5 + 3\sqrt{7}c_2$. Hence, the solution of the initial value problem is

$$y = e^{-t}\left(\frac{3}{2}\cos 3\sqrt{7}t - \frac{1}{2\sqrt{7}}\sin 3\sqrt{7}t\right). \tag{6.2.8}$$

The amplitude of the function in parentheses is

$$A = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{1}{2\sqrt{7}}\right)^2} = \sqrt{\frac{9}{4} + \frac{1}{4 \cdot 7}} = \sqrt{\frac{64}{4 \cdot 7}} = \frac{4}{\sqrt{7}}$$

and

$$\tan\phi = -3\sqrt{7}.$$

Therefore $\phi = \pi - \tan^{-1} 3\sqrt{7}$ radians.

So, the general solution is $y = \frac{4}{\sqrt{7}} \sin(\frac{3}{\sqrt{7}}t + \pi - \tan^{-1} 3\sqrt{7})$.

Example 6.2.3

Let the damping constant in Example 6.2.1 be $c = 40$ lb–sec/ft. Find the displacement y for $t > 0$ if $y(0) = 1$ and $y'(0) = 1$.

Solution

With $c = 40$, the equation of motion Equation 6.2.4 reduces to

$$y'' + 20y' + 64y = 0 \tag{6.2.9}$$

after cancelling the common factor 2. The characteristic equation

$$r^2 + 20r + 64 = (r + 16)(r + 4) = 0$$

has the roots $r_1 = -4$ and $r_2 = -16$. Therefore the general solution of Equation 6.2.9 is

$$y = c_1 e^{-4t} + c_2 e^{-16t}. \tag{6.2.10}$$

Differentiating this yields

$$y' = -4e^{-4t} - 16c_2 e^{-16t}.$$

The last two equations and the initial conditions $y(0) = 1$ and $y'(0) = 1$ imply that

$$\begin{aligned} c_1 \quad + \quad c_2 \quad &= 1 \\ -4c_1 \quad - \quad 16c_2 \quad &= 1. \end{aligned}$$

The solution of this system is $c_1 = 17/12$, $c_2 = -5/12$. Substituting these into Equation 6.2.10 yields

$$y = \frac{17}{12} e^{-4t} - \frac{5}{12} e^{-16t}$$

as the solution of the given initial value problem (Figure 6.2.3).

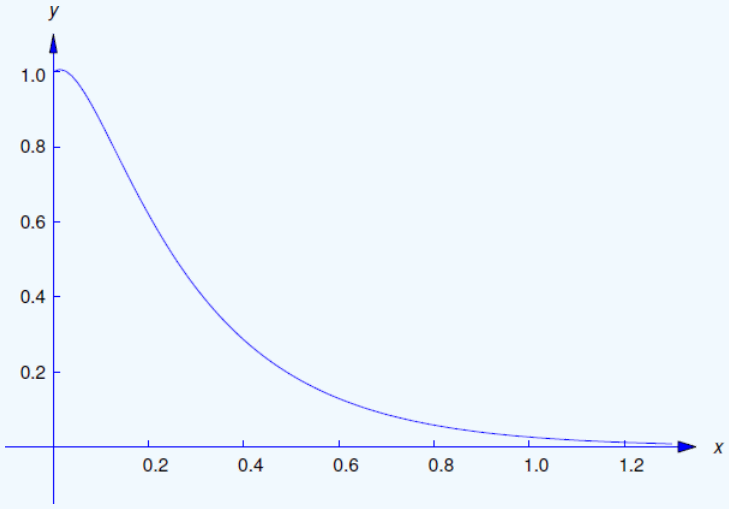


Figure 6.2.3 : $y = \frac{17}{12} e^{-4t} - \frac{5}{12} e^{-16t}$

Forced Vibrations with Damping

Now we consider the motion of an object in a spring-mass system with damping, under the influence of a periodic forcing function $F(t) = F_0 \cos \omega t$, so that the equation of motion is

$$my'' + cy' + ky = F_0 \cos \omega t. \tag{6.2.11}$$

In Section 6.1 we considered this equation with $c = 0$ and found that the resulting displacement y assumed arbitrarily large values in the case of resonance (that is, when $\omega = \omega_0 = \sqrt{k/m}$). Here we'll see that in the presence of damping the displacement

remains bounded for all t, and the initial conditions have little effect on the motion as $t \to \infty$. In fact, we'll see that for large t the displacement is closely approximated by a function of the form

$$y = A\sin(\omega t + \phi), \tag{6.2.12}$$

where the amplitude A depends upon m, c, k, F_0, and ω. We're interested in the following question:

QUESTION

Assuming that m, c, k, and F_0 are held constant, what value of ω produces the largest amplitude A in Equation 6.2.12, and what is this largest amplitude?

To answer this question, we must solve Equation 6.2.11 and determine A in terms of F_0, ω_0, ω, and c. We can obtain a particular solution of Equation 6.2.11 by the method of annihilation. Since $\cos\omega t$ does not satisfy the homogeneous equation

$$my'' + cy' + ky = 0,$$

we can obtain a particular solution of Equation 6.2.11 in the form

$$y_p = A\cos\omega t + B\sin\omega t. \tag{6.2.13}$$

Differentiating this yields

$$y_p' = \omega(-A\sin\omega t + B\cos\omega t)$$

and

$$y_p'' = -\omega^2(A\cos\omega t + B\sin\omega t).$$

From the last three equations,

$$my_p'' + cy_p' + ky_p = (-m\omega^2 A + c\omega B + kA)\cos\omega t + (-m\omega^2 B - c\omega A + kB)\sin\omega t,$$

so y_p satisfies Equation 6.2.11 if

$$\begin{aligned} (k - m\omega^2)A + \quad c\omega B \quad &= F_0 \\ -c\omega A \quad + (k - m\omega^2)B \quad &= 0. \end{aligned}$$

Solving for A and B and substituting the results into Equation 6.2.13 yields

$$y_p = \frac{F_0}{(k - m\omega^2)^2 + c^2\omega^2}\left[(k - m\omega^2)\cos\omega t + c\omega\sin\omega t\right],$$

which can be written in amplitude–phase form as

$$y_p = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}}\sin(\omega t + \phi), \tag{6.2.14}$$

where

$$\tan\phi = \frac{k - m\omega^2}{c\omega}. \tag{6.2.15}$$

To compare this with the undamped forced vibration that we considered in Section 6.1 it is useful to write

$$k - m\omega^2 = m\left(\frac{k}{m} - \omega^2\right) = m(\omega_0^2 - \omega^2), \tag{6.2.16}$$

where $\omega_0 = \sqrt{k/m}$ is the natural angular frequency of the undamped simple harmonic motion of an object with mass m on a spring with constant k. Substituting Equation 6.2.16 into Equation 6.2.14 yields

$$y_p = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2}}\sin(\omega t + \phi). \tag{6.2.17}$$

The solution of an initial value problem

$$my'' + cy' + ky = F_0 \cos \omega t, \quad y(0) = y_0, \quad y'(0) = v_0,$$

is of the form $y = y_h + y_p$, where y_h has one of the three forms

$$\begin{aligned} y_h &= e^{-ct/2m}(c_1 \cos \omega_1 t + c_2 \sin \omega_1 t), \\ y_h &= e^{-ct/2m}(c_1 + c_2 t), \\ y_h &= c_1 e^{r_1 t} + c_2 e^{r_2 t} \ (r_1, r_2 < 0). \end{aligned}$$

In all three cases $\lim_{t \to \infty} y_c(t) = 0$ for any choice of c_1 and c_2. For this reason we say that y_h is the *transient component* of the solution y. The behavior of y for large t is determined by y_p, which we call the *steady state component* of y. Thus, for large t the motion is like simple harmonic motion at the frequency of the external force.

The amplitude of y_p in Equation 6.2.17 is

$$A = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2}}, \tag{6.2.18}$$

which is finite for all ω; that is, the presence of damping precludes the phenomenon of resonance that we encountered in studying undamped vibrations under a periodic forcing function. We'll now find the value $\omega_{\max}$ of ω for which A is maximized. This is the value of ω for which the function

$$\rho(\omega) = m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2$$

in the denominator of Equation 6.2.18 attains its minimum value. By rewriting this as

$$\rho(\omega) = m^2(\omega_0^4 + \omega^4) + (c^2 - 2m^2\omega_0^2)\omega^2, \tag{6.2.19}$$

you can see that ρ is a strictly increasing function of ω^2 if

$$c \ge \sqrt{2m^2\omega_0^2} = \sqrt{2mk}.$$

(Recall that $\omega_0^2 = k/m$). Therefore $\omega_{\max} = 0$ if this inequality holds. From Equation 6.2.15, you can see that $\phi = \pi/2$ if $\omega = 0$. In this case, Equation 6.2.14 reduces to

$$y_p = \frac{F_0}{\sqrt{m^2\omega_0^4}} = \frac{F_0}{k},$$

which is consistent with Hooke's law: if the mass is subjected to a constant force F_0, its displacement should approach a constant y_p such that $ky_p = F_0$. Now suppose $c < \sqrt{2mk}$. Then, from Equation 6.2.19,

$$\rho'(\omega) = 2\omega(2m^2\omega^2 + c^2 - 2m^2\omega_0^2),$$

and $\omega_{\max}$ is the value of ω for which the expression in parentheses equals zero; that is,

$$\omega_{\max} = \sqrt{\omega_0^2 - \frac{c^2}{2m^2}} = \sqrt{\frac{k}{m}\left(1 - \frac{c^2}{2km}\right)}.$$

(To see that $\rho(\omega_{\max})$ is the minimum value of $\rho(\omega)$, note that $\rho'(\omega) < 0$ if $\omega < \omega_{\max}$ and $\rho'(\omega) > 0$ if $\omega > \omega_{\max}$.) Substituting $\omega = \omega_{\max}$ in Equation 6.2.18 and simplifying shows that the maximum amplitude $A_{\max}$ is

$$A_{\max} = \frac{2mF_0}{c\sqrt{4mk - c^2}} \quad \text{if} \quad c < \sqrt{2mk}.$$

We summarize our results as follows.

Theorem 6.2.1

Suppose we consider the amplitude A of the steady state component of the solution of

$$my'' + cy' + ky = F_0 \cos \omega t$$

as a function of ω.

a. If $c \geq \sqrt{2mk}$, the maximum amplitude is $A_{\max} = F_0/k$ and it is attained when $\omega = \omega_{\max} = 0$.
b. If $c < \sqrt{2mk}$, the maximum amplitude is

$$A_{\max} = \frac{2mF_0}{c\sqrt{4mk - c^2}}, \tag{6.2.20}$$

and it is attained when

$$\omega = \omega_{\max} = \sqrt{\frac{k}{m}\left(1 - \frac{c^2}{2km}\right)}. \tag{6.2.21}$$

Note that $A_{\max}$ and $\omega_{\max}$ are continuous functions of c, for $c \geq 0$, since Equation 6.2.20 and Equation 6.2.21 reduce to $A_{\max} = F_0/k$ and $\omega_{\max} = 0$ if $c = \sqrt{2km}$.

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6.2E: Spring-Mass Problems (With Damping) (Exercises)

1. A 64 lb object stretches a spring 4 ft in equilibrium. It is attached to a dashpot with damping constant $c = 8$ lb-sec/ft. The object is initially displaced 18 inches above equilibrium and given a downward velocity of 4 ft/sec. Find its displacement and time–varying amplitude for $t > 0$.

2. A 16 lb weight is attached to a spring with natural length 5 ft. With the weight attached, the spring measures 8.2 ft. The weight is initially displaced 3 ft below equilibrium and given an upward velocity of 2 ft/sec. Find its displacement for $t > 0$ if the medium resists the motion with a force of one lb for each ft/sec of velocity. Also, find its time–varying amplitude.

3. An 8 lb weight stretches a spring 1.5 inches. It is attached to a dashpot with damping constant $c = 8$ lb-sec/ft. The weight is initially displaced 3 inches above equilibrium and given an upward velocity of 6 ft/sec. Find its displacement for $t > 0$.

4. A 96 lb weight stretches a spring 3.2 ft in equilibrium. It is attached to a dashpot with damping constant c=18 lb-sec/ft. The weight is initially displaced 15 inches below equilibrium and given a downward velocity of 12 ft/sec. Find its displacement for $t > 0$.

5. A 16 lb weight stretches a spring 6 inches in equilibrium. It is attached to a damping mechanism with constant c. Find all values of c such that the free vibration of the weight has infinitely many oscillations.

6. An 8 lb weight stretches a spring .32 ft. The weight is initially displaced 6 inches above equilibrium and given an upward velocity of 4 ft/sec. Find its displacement for $t > 0$ if the medium exerts a damping force of 1.5 lb for each ft/sec of velocity.

7. A 32 lb weight stretches a spring 2 ft in equilibrium. It is attached to a dashpot with constant $c = 8$ lb-sec/ft. The weight is initially displaced 8 inches below equilibrium and released from rest. Find its displacement for $t > 0$.

8. A mass of 20 gm stretches a spring 5 cm. The spring is attached to a dashpot with damping constant 400 dyne sec/cm. Determine the displacement for $t > 0$ if the mass is initially displaced 9 cm above equilibrium and released from rest.

9. A 64 lb weight is suspended from a spring with constant $k = 25$ lb/ft. It is initially displaced 18 inches above equilibrium and released from rest. Find its displacement for $t > 0$ if the medium resists the motion with 6 lb of force for each ft/sec of velocity.

10. A 32 lb weight stretches a spring 1 ft in equilibrium. The weight is initially displaced 6 inches above equilibrium and given a downward velocity of 3 ft/sec. Find its displacement for $t > 0$ if the medium resists the motion with a force equal to 3 times the speed in ft/sec.

11. An 8 lb weight stretches a spring 2 inches. It is attached to a dashpot with damping constant $c = 4$ lb-sec/ft. The weight is initially displaced 3 inches above equilibrium and given a downward velocity of 4 ft/sec. Find its displacement for $t > 0$.

12. A 2 lb weight stretches a spring .32 ft. The weight is initially displaced 4 inches below equilibrium and given an upward velocity of 5 ft/sec. The medium provides damping with constant $c = 1/8$ lb-sec/ft. Find the displacement for $t > 0$.

13. An 8 lb weight stretches a spring 8 inches in equilibrium. It is attached to a dashpot with damping constant $c = .5$ lb-sec/ft and subjected to an external force $F(t) = 4\cos 2t$ lb. Determine the steady state component of the displacement for $t > 0$.

14. A 32 lb weight stretches a spring 1 ft in equilibrium. It is attached to a dashpot with constant $c = 12$ lb-sec/ft. The weight is initially displaced 8 inches above equilibrium and released from rest. Find its displacement for $t > 0$.

15. A mass of one kg stretches a spring 49 cm in equilibrium. A dashpot attached to the spring supplies a damping force of 4 N for each m/sec of speed. The mass is initially displaced 10 cm above equilibrium and given a downward velocity of 1 m/sec. Find its displacement for $t > 0$.

16. A mass of 100 grams stretches a spring 98 cm in equilibrium. A dashpot attached to the spring supplies a damping force of 600 dynes for each cm/sec of speed. The mass is initially displaced 10 cm above equilibrium and given a downward velocity of 1 m/sec. Find its displacement for $t > 0$.

17. A 192 lb weight is suspended from a spring with constant $k = 6$ lb/ft and subjected to an external force $F(t) = 8\cos 3t$ lb. Find the steady state component of the displacement for $t > 0$ if the medium resists the motion with a force equal to 8 times the speed in ft/sec.

18. A 2 gm mass is attached to a spring with constant 20 dyne/cm. Find the steady state component of the displacement if the mass is subjected to an external force $F(t) = 3\cos 4t - 5\sin 4t$ dynes and a dashpot supplies 4 dynes of damping for each cm/sec of velocity.

19. A 96 lb weight is attached to a spring with constant 12 lb/ft. Find the steady state component of the displacement if the mass is subjected to an external force $F(t) = 18\cos t - 9\sin t$ lb and a dashpot supplies 24 lb of damping for each ft/sec of velocity.

20. A mass of one kg stretches a spring 49 cm in equilibrium. It is attached to a dashpot that supplies a damping force of 4 N for each m/sec of speed. Find the steady state component of its displacement if it is subjected to an external force $F(t) = 8\sin 2t - 6\cos 2t$ N.

21. A mass m is suspended from a spring with constant k and subjected to an external force $F(t) = \alpha\cos\omega_0 t + \beta\sin\omega_0 t$, where ω_0 is the natural frequency of the spring–mass system without damping. Find the steady state component of the displacement if a dashpot with constant c supplies damping.

22. Show that if c_1 and c_2 are not both zero then

$$y = e^{r_1 t}(c_1 + c_2 t) \tag{6.2E.1}$$

can't equal zero for more than one value of t.

23. Show that if c_1 and c_2 are not both zero then

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} \tag{6.2E.2}$$

can't equal zero for more than one value of t.

24. A 4 pound object is attached to a spring whose constant is 2 lb/ft and a dashpot supplies a damping equal to the velocity. If the object is released 1 foot above equilibrium with a downward velocity of 8 ft/s, find

a. the time the object passes through equilibrium.

b. the time when the object is furthest from the equilibrium, and its position at that time.

25. A 1 kilogram object is attached to a spring whose constant is 16 N/m and a dashpot supplies a damping equal to 10 times the velocity. If the object is released from rest 1 meter below equilibrium, find the equation of motion.

26. A 1 kilogram object is attached to a spring whose constant is 16 N/m and a dashpot supplies a damping equal to 10 times the velocity. If the object is released 1 meter below equilibrium with an upward velocity of 12 m/s, find the equation of motion.

27. A 3.2 pound object is attached to a spring whose constant is $1/2$ lb/ft and a dashpot supplies a damping equal to 0.4 times the velocity. If the object is released from rest 1 foot above equilibrium, find

a. the equation of motion.

b. write the equation of motion in the Amplitude-Phase Form.

c. the time the object passes through equilibrium.

d. the first time the object passes through equilibrium heading upward.

28. A 16 pound object stretches a spring $8/3$ feet and a dashpot supplies a damping equal to $1/2$ the velocity. Find the equation of motion if an external force of $10\cos 3t$ drives the object which is released from rest 2 feet below equilibrium.

29. A object with mass 1 slug stretches a spring 2 feet and a dashpot supplies a damping equal to 8 times the velocity. Find the equation of motion if an external force of $8\sin 4t$ drives the object which is released from rest at equilibrium.

30. A object with mass 2 kilograms is attached to a spring whose constant is 32 N/m. Find the equation of motion if an external force of $68e^{-2t}\cos 4t$ drives the object which is released from rest at equilibrium.

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6.3: The RLC Circuit

In this section we consider the RLC circuit, shown schematically in Figure 6.3.1 . As we'll see, the RLC circuit is an electrical analog of a spring-mass system with damping.

Nothing happens while the switch is open (dashed line). When the switch is closed (solid line) we say that the *circuit is closed*. Differences in electrical potential in a closed circuit cause current to flow in the circuit. The battery or generator in Figure 6.3.1 creates a difference in electrical potential $E = E(t)$ between its two terminals, which we've marked arbitrarily as positive and negative. (We could just as well interchange the markings.) We'll say that $E(t) > 0$ if the potential at the positive terminal is greater than the potential at the negative terminal, $E(t) < 0$ if the potential at the positive terminal is less than the potential at the negative terminal, and $E(t) = 0$ if the potential is the same at the two terminals. We call E the *impressed voltage*.

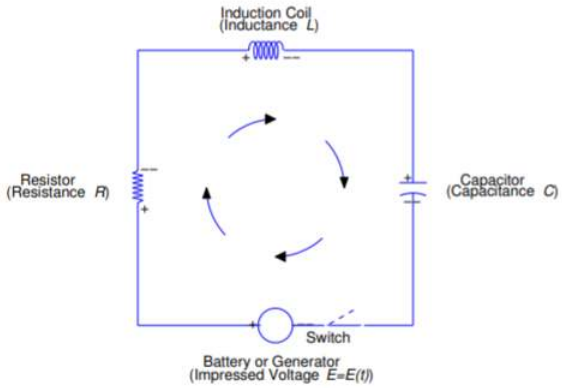


Figure 6.3.1 : An RLC circuit

At any time t, the same current flows in all points of the circuit. We denote current by $I = I(t)$. We say that $I(t) > 0$ if the direction of flow is around the circuit from the positive terminal of the battery or generator back to the negative terminal, as indicated by the arrows in Figure 6.3.1 $I(t) < 0$ if the flow is in the opposite direction, and $I(t) = 0$ if no current flows at time t.

Differences in potential occur at the resistor, induction coil, and capacitor in Figure 6.3.1 . Note that the two sides of each of these components are also identified as positive and negative. The *voltage drop across* each component is defined to be the potential on the positive side of the component minus the potential on the negative side. This terminology is somewhat misleading, since "drop" suggests a decrease even though changes in potential are signed quantities and therefore may be increases. Nevertheless, we'll go along with tradition and call them voltage drops. The voltage drop across the resistor in Figure 6.3.1 is given by

$$V_R = IR, \tag{6.3.1}$$

where I is current and R is a positive constant, the *resistance* of the resistor. The voltage drop across the induction coil is given by

$$V_I = L\frac{dI}{dt} = LI', \tag{6.3.2}$$

where L is a positive constant, the *inductance* of the coil.

A capacitor stores electrical charge $Q = Q(t)$, which is related to the current in the circuit by the equation

$$Q(t) = Q_0 + \int_0^t I(\tau)\, d\tau, \tag{6.3.3}$$

where Q_0 is the charge on the capacitor at $t = 0$. The voltage drop across a capacitor is given by

$$V_C = \frac{Q}{C}, \tag{6.3.4}$$

where C is a positive constant, the *capacitance* of the capacitor.

Table 6.3.1 names the units for the quantities that we've discussed. The units are defined so that

$$\begin{aligned} 1\text{volt} &= 1\text{ampere} \cdot 1\text{ohm} \\ &= 1\text{henry} \cdot 1\ \text{ampere/second} \\ &= 1\text{coulomb/farad} \end{aligned}$$

and

$$1\text{ampere} = 1\text{coulomb/second}.$$

Table 6.3.1 : Electrical Units

Symbol	Name	Unit
E	Impressed Voltage	volt
I	Current	ampere
Q	Charge	coulomb
R	Resistance	ohm
L	Inductance	henry
C	Capacitance	farad

According to *Kirchoff's law*, the sum of the voltage drops in a closed RLC circuit equals the impressed voltage. Therefore, from Equation 6.3.1, Equation 6.3.2, and Equation 6.3.4,

$$LI' + RI + \frac{1}{C}Q = E(t). \tag{6.3.5}$$

This equation contains two unknowns, the current I in the circuit and the charge Q on the capacitor. However, Equation 6.3.3 implies that $Q' = I$, so Equation 6.3.5 can be converted into the second order equation

$$LQ'' + RQ' + \frac{1}{C}Q = E(t) \tag{6.3.6}$$

in Q. To find the current flowing in an RLC circuit, we solve Equation 6.3.6 for Q and then differentiate the solution to obtain I.

In Sections 6.1 and 6.2 we encountered the equation

$$my'' + cy' + ky = F(t) \tag{6.3.7}$$

in connection with spring-mass systems. Except for notation this equation is the same as Equation 6.3.6. The correspondence between electrical and mechanical quantities connected with Equation 6.3.6 and Equation 6.3.7 is shown in Table 6.3.2 .

Table 6.3.2 *: Electrical and Mechanical Units*

Electrical	Mechanical
charge Q	displacement y
current I	velocity y'
impressed voltage $E(t)$	external force $F(t)$
inductance L	mass m
resistance R	damping c
1/capacitance $1/C$	spring constant k

The equivalence between Equation 6.3.6 and Equation 6.3.7 is an example of how mathematics unifies fundamental similarities in diverse physical phenomena. Since we've already studied the properties of solutions of Equation 6.3.7 in Sections 6.1 and 6.2, we can obtain results concerning solutions of Equation 6.3.6 by simply changing notation, according to Table 6.3.1 .

Free Oscillations

We say that an RLC circuit is in *free oscillation* if $E(t) = 0$ for $t > 0$, so that Equation 6.3.6 becomes

$$LQ'' + RQ' + \frac{1}{C}Q = 0. \tag{6.3.8}$$

The characteristic equation of Equation 6.3.8 is

$$Lr^2 + Rr + \frac{1}{C} = 0,$$

with roots

$$r_1 = \frac{-R - \sqrt{R^2 - 4L/C}}{2L} \quad \text{and} \quad r_2 = \frac{-R + \sqrt{R^2 - 4L/C}}{2L}. \tag{6.3.9}$$

There are three cases to consider, all analogous to the cases considered in Section 6.2 for free vibrations of a damped spring-mass system.

Case 1: Underdamped Motion

The oscillation is *underdamped* if $R < \sqrt{4L/C}$. In this case, r_1 and r_2 in Equation 6.3.9 are complex conjugates, which we write as

$$r_1 = -\frac{R}{2L} + i\omega_1 \quad \text{and} \quad r_2 = -\frac{R}{2L} - i\omega_1,$$

where

$$\omega_1 = \frac{\sqrt{4L/C - R^2}}{2L}.$$

The general solution of Equation 6.3.8 is

$$Q = e^{-Rt/2L}(c_1 \cos \omega_1 t + c_2 \sin \omega_1 t),$$

which we can write as

$$Q = Ae^{-Rt/2L} \sin(\omega_1 t + \phi), \tag{6.3.10}$$

where

$$A = \sqrt{c_1^2 + c_2^2}, \quad \tan \phi = \frac{c_1}{c_2}.$$

In the idealized case where $R = 0$, the solution Equation 6.3.10 reduces to

$$Q = A \sin\left(\frac{t}{\sqrt{LC}} + \phi\right),$$

which is analogous to the simple harmonic motion of an undamped spring-mass system in free vibration.

Actual RLC circuits are usually underdamped, so the case we've just considered is the most important. However, for completeness we'll consider the other two possibilities.

Case 2: Critically Damped

The oscillation is *critically damped* if $R = \sqrt{4L/C}$. In this case, $r_1 = r_2 = -R/2L$ and the general solution of Equation 6.3.8 is

$$Q = e^{-Rt/2L}(c_1 + c_2 t). \tag{6.3.11}$$

Case 3: Overdamped Motion

The oscillation is *overdamped* if $R > \sqrt{4L/C}$. In this case, the zeros r_1 and r_2 of the characteristic polynomial are real, with $r_1 < r_2 < 0$ (see 6.3.9), and the general solution of 6.3.8 is

$$Q = c_1 e^{r_1 t} + c_2 e^{r_2 t}. \tag{6.3.12}$$

If $R \neq 0$, the exponentials in Equation 6.3.10, Equation 6.3.11, and Equation 6.3.12 are negative, so the solution of any homogeneous initial value problem

$$LQ'' + RQ' + \frac{1}{C}Q = 0, \quad Q(0) = Q_0, \quad Q'(0) = I_0,$$

approaches zero exponentially as $t \to \infty$. Thus, all such solutions are *transient*, in the sense defined Section 6.2 in the discussion of forced vibrations of a spring-mass system with damping.

Example 6.3.1

At $t = 0$ a current of 2 amperes flows in an RLC circuit with resistance $R = 40$ ohms, inductance $L = .2$ henrys, and capacitance $C = 10^{-5}$ farads. Find the current flowing in the circuit at $t > 0$ if the initial charge on the capacitor is 1 coulomb. Assume that $E(t) = 0$ for $t > 0$.

Solution

The equation for the charge Q is

$$\frac{1}{5}Q'' + 40Q' + 10000Q = 0,$$

or

$$Q'' + 200Q' + 50000Q = 0. \tag{6.3.13}$$

Therefore we must solve the initial value problem

$$Q'' + 200Q' + 50000Q = 0, \quad Q(0) = 1, \quad Q'(0) = 2. \tag{6.3.14}$$

The desired current is the derivative of the solution of this initial value problem.

The characteristic equation of Equation 6.3.13 is

$$r^2 + 200r + 50000 = 0,$$

which has complex zeros $r = -100 \pm 200i$. Therefore the general solution of Equation 6.3.13 is

$$Q = e^{-100t}(c_1 \cos 200t + c_2 \sin 200t). \tag{6.3.15}$$

Differentiating this and collecting like terms yields

$$Q' = -e^{-100t}\left[(100c_1 - 200c_2)\cos 200t + (100c_2 + 200c_1)\sin 200t\right]. \tag{6.3.16}$$

To find the solution of the initial value problem Equation 6.3.14, we set $t = 0$ in Equation 6.3.15 and Equation 6.3.16 to obtain

$$c_1 = Q(0) = 1 \quad \text{and} \quad -100c_1 + 200c_2 = Q'(0) = 2;$$

therefore, $c_1 = 1$ and $c_2 = 51/100$, so

$$Q = e^{-100t}\left(\cos 200t + \frac{51}{100}\sin 200t\right)$$

is the solution of Equation 6.3.14. Differentiating this yields

$$I = e^{-100t}(2\cos 200t - 251\sin 200t).$$

Forced Oscillations With Damping

An initial value problem for Equation 6.3.6 has the form

$$LQ'' + RQ' + \frac{1}{C}Q = E(t), \quad Q(0) = Q_0, \quad Q'(0) = I_0, \tag{6.3.17}$$

where Q_0 is the initial charge on the capacitor and I_0 is the initial current in the circuit. We've already seen that if $E \equiv 0$ then all solutions of Equation 6.3.17 are transient. If $E \not\equiv 0$, we know that the solution of Equation 6.3.17 has the form $Q = Q_h + Q_p$,

where Q_h satisfies the homogeneous equation, and approaches zero exponentially as $t \to \infty$ for any initial conditions, while Q_p depends only on E and is independent of the initial conditions. As in the case of forced oscillations of a spring-mass system with damping, we call Q_p the *steady state* charge on the capacitor of the RLC circuit. Since $I = Q' = Q_h' + Q_p'$ and Q_h' also tends to zero exponentially as $t \to \infty$, we say that $I_h = Q_h'$ is the *transient* current and $I_p = Q_p'$ is the *steady state* current. In most applications we are interested only in the steady state charge and current.

Example 6.3.2

Find the amplitude-phase form of the steady state current in the RLC circuit in Figure 6.3.1 if the impressed voltage, provided by an alternating current generator, is $E(t) = E_0 \cos \omega t$.

Solution

We'll first find the steady state charge on the capacitor as a particular solution of

$$LQ'' + RQ' + \frac{1}{C}Q = E_0 \cos \omega t.$$

To do, this we'll simply reinterpret a result obtained in Section 6.2, where we found that the steady state solution of

$$my'' + cy' + ky = F_0 \cos \omega t$$

is

$$y_p = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \sin(\omega t + \phi),$$

where

$$\tan \phi = \frac{k - m\omega^2}{c\omega}.$$

(see Equations 6.3.14 and Equation 6.3.15) By making the appropriate changes in the symbols (according to Table 6.3.2) yields the steady state charge

$$Q_p = \frac{E_0}{\sqrt{(1/C - L\omega^2)^2 + R^2\omega^2}} \sin(\omega t + \phi),$$

where

$$\tan \phi = \frac{R\omega}{1/C - L\omega^2}.$$

Therefore the steady state current in the circuit is

$$I_p = Q_p' = \frac{\omega E_0}{\sqrt{(1/C - L\omega^2)^2 + R^2\omega^2}} \cos(\omega t + \phi).$$

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6.3E: The RLC Circuit (Exercises)

In Exercises **1-5** find the current in the RLC circuit, assuming that $E(t) = 0$ for $t > 0$.

1. $R = 3$ ohms; $L = .1$ henrys; $C = .01$ farads; $Q_0 = 0$ coulombs; $I_0 = 2$ amperes.
2. $R = 2$ ohms; $L = .05$ henrys; $C = .01$ farads'; $Q_0 = 2$ coulombs; $I_0 = -2$ amperes.
3. $R = 2$ ohms; $L = .1$ henrys; $C = .01$ farads; $Q_0 = 2$ coulombs; $I_0 = 0$ amperes.
4. $R = 6$ ohms; $L = .1$ henrys; $C = .004$ farads'; $Q_0 = 3$ coulombs; $I_0 = -10$ amperes.
5. $R = 4$ ohms; $L = .05$ henrys; $C = .008$ farads; $Q_0 = -1$ coulombs; $I_0 = 2$ amperes.

In Exercises **6-10** find the steady state current in the circuit described by the equation.

6. $\frac{1}{10}Q'' + 3Q' + 100Q = 5\cos 10t - 5\sin 10t$
7. $\frac{1}{20}Q'' + 2Q' + 100Q = 10\cos 25t - 5\sin 25t$
8. $\frac{1}{10}Q'' + 2Q' + 100Q = 3\cos 50t - 6\sin 50t$
9. $\frac{1}{10}Q'' + 6Q' + 250Q = 10\cos 100t + 30\sin 100t$
10. $\frac{1}{20}Q'' + 4Q' + 125Q = 15\cos 30t - 30\sin 30t$

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CHAPTER OVERVIEW

7: Linear Higher Order Differential Equations

This chapter extend the results obtained in Chapter 5 for linear second order equations to linear higher order equations.

7.1: Introduction to Linear Higher Order Equations

7.1.1: Introduction to Linear Higher Order Equations (Exercises)

7.2: Higher Order Homogeneous Equations

7.2.1: Higher Order Homogeneous Equations (Exercises)

7.3: Annihilation for Higher Order Equations

7.3.1: Annihilation for Higher Order Equations (Exercises)

7.4: Variation of Parameters for Higher Order Equations

7.4.1: Variation of Parameters for Higher Order Equations (Exercises)

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7.1: Introduction to Linear Higher Order Equations

Caution

If you have not had Math 410 (linear algebra), then you will need to read Appendix 11.4 before starting Chapter 7.

Definition 7.1.1

An nth order differential equation is said to be *linear* if it can be written in the form

$$y^{(n)}+p_1(x)y^{(n-1)}+\cdots+p_n(x)y=f(x). \tag{7.1.1}$$

We considered equations of this form with $n=1$ in Section 2.3 and with $n=2$ in Chapter 5. In this chapter n is an arbitrary positive integer. In this section we sketch the general theory of linear nth order equations. Since this theory has already been discussed for $n=2$ in Sections 5.1 and 5.4, we'll omit proofs.

For convenience, we consider linear differential equations written as

$$P_0(x)y^{(n)}+P_1(x)y^{(n-1)}+\cdots+P_n(x)y=F(x), \tag{7.1.2}$$

which can be rewritten as Equation 7.1.1 on any interval on which P_0 has no zeros, with $p_1=P_1/P_0$, ..., $p_n=P_n/P_0$, and $f=F/P_0$.

The next theorem is analogous to Theorem 5.4.1.

Theorem 7.1.1

Suppose Equation 7.1.1 has continuous coefficients on (a,b), let x_0 be a point in (a,b), and let k_0, k_1, ..., k_{n-1} be arbitrary real numbers. Then the initial value problem

$$y^{(n)}+p_1(x)y^{(n-1)}+\cdots+p_n(x)y=f(x),\quad y(x_0)=k_0,\quad y'(x_0)=k_1,\ldots,\quad y^{(n-1)}(x_0)=k_{n-1}$$

has a unique solution on (a,b).

Homogeneous Equations

Equation 7.1.2 is said to be *homogeneous* if $f(x)\equiv 0$ and *nonhomogeneous* otherwise. Since $y\equiv 0$ is obviously a solution, we call it the *trivial* solution. Any other solution is *nontrivial*.

If y_1, y_2, ..., y_n are defined on (a,b) and c_1, c_2, ..., c_n are constants, then

$$y=c_1y_1+c_2y_2+\cdots+c_ny_n \tag{7.1.3}$$

is a *linear combination* of $\{y_1,y_2\ldots,y_n\}$. It's easy to show that if y_1, y_2, ..., y_n are solutions of the homogeneous equation on (a,b), then so is any linear combination of $\{y_1,y_2,\ldots,y_n\}$. (See the proof of Theorem 5.1.2.) We say that $\{y_1,y_2,\ldots,y_n\}$ is a *fundamental set of solutions of the homogenous equation* on (a,b) if every solution on (a,b) can be written as a linear combination of n linearly independent solutions $\{y_1,y_2,\ldots,y_n\}$, as in Equation 7.1.3. In this case we say that Equation 7.1.3 is the *general solution of the homogeneous equation* on (a,b).

The problem we have here is determining whether the set of solutions is independent or not. With only two, it is simple because we only need to see if one is a constant multiple of the other. However, once we get beyond two it's more complicated and typically requires knowledge of linear algebra, and this is where the idea of the Wronskian becomes more prominent.

Recall from section 5.1, we defined the *Wronskian of* $\{y_1,y_2\}$ as the function $W=y_1y_2'-y_1'y_2$.

$$W=\begin{vmatrix} y_1 & y_2\\ y_1' & y_2'\end{vmatrix}=y_1y_2'-y_1'y_2$$

which you can now see is just a determinant. Note that the first row is made up of y_1 and y_2 and the second is made up of their derivatives. This idea extends to the following definition.

Definition 7.1.2

The *Wronskian of* $\{y_1, y_2 \ldots, y_n\}$ is the determinant

$$W = \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix}.$$

Note that the first row is made up of y_1, y_2,...,y_n, the second row is made up of their derivatives, the third row is made up of the second derivatives, etc. You will always end up with an $n \times n$ matrix.

The next theorem is analogous to Theorem 5.1.3.

Theorem 7.1.2

Suppose $y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0$ has continuous coefficients on (a, b) and let y_1, y_2, ..., y_n be n solutions on (a, b). Then

a. $\{y_1, y_2, \ldots, y_n\}$ is linearly independent on (a, b) if and only if W$\not\equiv 0$ on (a, b).
b. $\{y_1, y_2, \ldots, y_n\}$ is linearly dependent on (a, b) if and only if W$\equiv 0$ on (a, b).

So, $\{y_1, y_2, \ldots, y_n\}$ is a fundamental set of solutions of if and only if W$\not\equiv 0$ on (a, b), and $y = c_1y_1 + c_2y_2 + \ldots + y_n$ is therefore the general solution of $y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0$.

The next two examples illustrate concepts that we'll develop later in this section. You shouldn't be concerned with how to *find* the given solutions of the equations in these examples. This will be explained in later sections.

Example 7.1.1

The equation

$$x^3y''' - x^2y'' - 2xy' + 6y = 0 \tag{7.1.4}$$

has the solutions $y_1 = x^2$, $y_2 = x^3$, and $y_3 = 1/x$ on $(-\infty, 0)$ and $(0, \infty)$. Show that $\{y_1, y_2, y_3\}$ is linearly independent on $(-\infty, 0)$ and $(0, \infty)$. Then find the general solution of Equation 7.1.4 on $(-\infty, 0)$ and $(0, \infty)$.

Solution

$$W = \begin{vmatrix} x^2 & x^3 & \frac{1}{x} \\ 2x & 3x^2 & \frac{-1}{x^2} \\ 2 & 6x & \frac{2}{x^3} \end{vmatrix} = 12x$$

Therefore $W(x) \neq 0$ on $(-\infty, 0)$ and $(0, \infty)$ and $\{x^2, x^3, \frac{1}{x}\}$ is linearly independent on $(-\infty, 0)$ and $(0, \infty)$ thereby forming a fundamental set of solutions to 7.1.4. Therefore, the general solution to 7.1.4 is

$$y = c_1x^2 + c_2x^3 + c_3\frac{1}{x}.$$

Example 7.1.2

The equation

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0 \tag{7.1.5}$$

has the solutions $y_1 = e^x$, $y_2 = e^{-x}$, $y_3 = e^{2x}$, and $y_4 = e^{-3x}$ on $(-\infty, \infty)$. (Verify.) Show that $\{y_1, y_2, y_3, y_4\}$ is linearly independent on $(-\infty, \infty)$. Then find the general solution of Equation 7.1.5.

Solution

$$W = \begin{vmatrix} e^x & e^{-x} & e^{2x} & e^{-3x} \\ e^x & -e^{-x} & 2e^{2x} & -3e^{-3x} \\ e^x & e^{-x} & 4e^{2x} & 9e^{-3x} \\ e^x & -e^{-x} & 8e^{2x} & -27e^{-3x} \end{vmatrix} = 240e^{-x}$$

Therefore $W(x) \neq 0$ on $(-\infty, \infty)$ and $\{e^x, e^{-x}, e^{2x}, e^{3x}\}$ is linearly independent on $(-\infty, \infty)$ thereby forming a fundamental set of solutions to 7.1.5. Therefore, the general solution to 7.1.5 is

$$y = c_1 e^x + c_e^{-x} + c_3 e^{2x} + c_4 e^{3x}.$$

General Solution of a Nonhomogeneous Equation

The next theorem is analogous to Theorem 5.4.2. It shows how to find the general solution of $y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = f(x)$ if we know a particular solution of $y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = f(x)$ and a fundamental set of solutions of the homogeneous *equation* $y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0$.

Theorem 7.1.3

Suppose $p_1(x)$,...,$p_n(x)$, and $f(x)$ are continuous on (a, b). Let y_p be a particular solution of

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = f(x) \tag{7.1.6}$$

on (a, b), and let $\{y_1, y_2, \ldots, y_n\}$ be a fundamental set of solutions of the homogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0 \tag{7.1.7}$$

on (a, b). Then y is a solution of (7.1.6) on (a, b) if and only if

$$y = y_p + c_1 y_1 + c_2 y_2 + \ldots c_n y_n \tag{7.1.8}$$

where c_1,c_2,....,c_n are constant.

Proof

We first show that y in Equation 7.1.8 is a solution of Equation 7.1.6 for any choice of the constants c_1 and c_2. Differentiating Equation 7.1.8 twice yields

$$y' = y_p' + c_1 y_1' \cdots + c_n y_n' \quad \text{and} \quad y'' = y_p'' + c_1 y_1'' \cdots + c_n y_n'',$$

so

$$\begin{aligned} y'' + p(x)y' + q(x)y &= (y_p'' + c_1 y_1'' \cdots + c_n y_n'') + p(x)(y_p' + c_1 y_1' \cdots + c_n y_n') + q(x)(y_p + c_1 y_1 \cdots + c_n y_n) \\ &= (y_p'' + p(x)y_p' + q(x)y_p) + c_1(y_1'' + p(x)y_1' + q(x)y_1) \cdots + c_n(y_n'' + p(x)y_n' + q(x)y_n) \\ &= f + c_1 \cdot 0 \cdots + c_n \cdot 0 = f, \end{aligned}$$

since y_p satisfies Equation 7.1.6 and y_1 and y_2 satisfy Equation 7.1.7.

Now we'll show that every solution of Equation 7.1.6 has the form Equation 7.1.8 for some choice of the constants c_1 and c_2. Suppose y is a solution of Equation 7.1.6. We'll show that $y - y_p$ is a solution of Equation 7.1.7, and therefore of the form $y - y_p = c_1 y_1 \cdots + c_n y_n$, which implies Equation 7.1.8. To see this, we compute

$$\begin{aligned} (y - y_p)'' + p(x)(y - y_p)' + q(x)(y - y_p) &= (y'' - y_p'') + p(x)(y' - y_p') + q(x)(y - y_p) \\ &= (y'' + p(x)y' + q(x)y) - (y_p'' + p(x)y_p' + q(x)y_p) \\ &= f(x) - f(x) = 0, \end{aligned}$$

since y and y_p both satisfy Equation 7.1.6.

We say that Equation 7.1.8 is the *general solution of* (7.1.6) on (a, b).

The next theorem is analogous to Theorem 5.4.3.

Theorem 7.1.4 The Principle of Superposition

Suppose for each $i = 1, 2, \ldots, r$, the function y_{p_i} is a particular solution of $y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = f_i(x)$ on (a, b). Then

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_r}$$

is a particular solution of

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = f_1(x) + f_2(x) + \cdots + f_r(x)$$

on (a, b).

We'll apply Theorems 7.1.6 and 7.1.7 throughout the rest of this chapter.

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7.1.1: Introduction to Linear Higher Order Equations (Exercises)

In Exercises **1-6** verify that the given function is the solution of the initial value problem.

1. Verify that $y = -6x - 8x^2 - 3x^3 + \frac{1}{x}$ satisfies the IVP:

$$x^3y''' - 3x^2y'' + 6xy' - 6y = \frac{-24}{x}; \quad y(-1) = 0, \quad y'(-1) = 0, \quad y''(-1) = 0$$

2. Verify that $y = x + \dfrac{1}{2x}$ satisfies the IVP:

$$y''' - \frac{1}{x}y'' - y' + \frac{1}{x}y = \frac{x^2 - 4}{x^4}, \quad y(1) = \frac{3}{2}, \quad y'(1) = \frac{1}{2}, \quad y''(1) = 1$$

3. Verify that $y = -x^2 - 2 + 2e^{(x-1)} - e^{-(x-1)} + 4x$ satisfies the IVP:

$$xy''' - y'' - xy' + y = x^2, \quad y(1) = 2, \quad y'(1) = 5, \quad y''(1) = -1$$

4. Verify that $y = 2x^2 \ln x - x^{1/2} + 2x^{-1/2} + 4x^2$ satisfies the IVP:

$$4x^3y''' + 4x^2y'' - 5xy' + 2y = 30x^2, \quad y(1) = 5, \quad y'(1) = \frac{17}{2}, \quad y''(1) = \frac{63}{4}$$

5. Verify that $y = x^4 \ln x + x - 2x^2 + 3x^3 - 4x^4$ satisfies the IVP

$$x^4y^{(4)} - 4x^3y''' + 12x^2y'' - 24xy' + 24y = 6x^4, \quad y(1) = -2, \quad y'(1) = -9, \quad y''(1) = -27, \quad y'''(1) = -52$$

6. Verify that $y = 9 - 12x + 6x^2 - 8x^3$ satisfies the IVP:

$$xy^{(4)} - y''' - 4xy'' + 4y' = 96x^2, \quad y(1) = -5, \quad y'(1) = -24, \quad y''(1) = -36; \quad y'''(1) = -48$$

7. Using the solution from Example 7.1.1, solve the initial value problem

$$x^3y''' - x^2y'' - 2xy' + 6y = 0, \quad y(-1) = -4, \quad y'(-1) = -14, \quad y''(-1) = -20.$$

8. Using the solution from Example 7.1.2, solve the initial value problem

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0, \quad y(0) = 5, \quad y'(0) = -6, \quad y''(0) = 10, \quad y'''(0) - 36.$$

9.

a. Verify that the function

$$y = c_1x^3 + c_2x^2 + \frac{c_3}{x}$$

satisfies

$$x^3y''' - x^2y'' - 2xy' + 6y = 0 \tag{A}$$

if c_1, c_2, and c_3 are constants.

b. Use (a) to find solutions y_1, y_2, and y_3 of (A) such that

$$y_1(1) = 1, \quad y_1'(1) = 0, \quad y_1''(1) = 0$$
$$y_2(1) = 0, \quad y_2'(1) = 1, \quad y_2''(1) = 0$$
$$y_3(1) = 0, \quad y_3'(1) = 0, \quad y_3''(1) = 1.$$

In Exercises **10-16** verify that the given functions are solutions of the given equation, and show that they form a fundamental set of solutions of the equation on any interval on which the equation has continuous coefficients.

10. $y''' + y'' - y' - y = 0; \quad \{e^x, e^{-x}, xe^{-x}\}$
11. $y''' - 3y'' + 7y' - 5y = 0; \quad \{e^x, e^x\cos 2x, e^x \sin 2x\}$
12. $xy''' - y'' - xy' + y = 0; \quad \{e^x, e^{-x}, x\}$
13. $x^2y''' + 2xy'' - (x^2 + 2)y = 0; \quad \{e^x/x, e^{-x}/x, 1\}$

14. $(x^2-2x+2)y'''-x^2y''+2xy'-2y=0; \quad \{x,\ x^2,\ e^x\}$

15. $(2x-1)y^{(4)}-4xy'''+(5-2x)y''+4xy'-4y=0; \quad \{x,\ e^x,\ e^{-x},e^{2x}\}$

16. $xy^{(4)}-y'''-4xy'+4y'=0; \quad \{1,x^2,\ e^{2x},\ e^{-2x}\}$

In Exercise **17-27** compute the Wronskian of the given set of functions and what your result says about the linear independence of the set.

17. $\{1,\ e^x,\ e^{-x}\}$

18. $\{e^x,\ e^x\sin x,\ e^x\cos x\}$

19. $\{2,\ x+1,\ x^2+2\}$

20. $\{x,\ x\ln x,\ 1/x\}$

21. $\{1,\ x,\ \frac{x^2}{2!},\ \frac{x^3}{3!},\cdots,\ \frac{x^n}{n!}\}$

22. $\{e^x,\ e^{-x},\ x\}$

23. $\{e^x/x,\ e^{-x}/x,\ 1\}$

24. $\{x,\ x^2,\ e^x\}$

25. $\{x,\ x^3,\ 1/x,\ 1/x^2\}$

26. $\{e^x,\ e^{-x},\ x,\ e^{2x}\}$

27. $\{e^{2x},\ e^{-2x},\ 1,\ x^2\}$

28. Suppose $y^{(n)}+p_1(x)y^{(n-1)}+\cdots+p_n(x)y=f(x)$ has continuous coefficients on (a,b) and x_0 is in (a,b). Use Theorem 7.1.1 to prove that $y\equiv 0$ is the only solution of the initial value problem

$$Ly=0, \quad y(x_0)=0, \quad y'(x_0)=0,\dots,y^{(n-1)}(x_0)=0,$$

on (a,b).

29. Prove: If

$$y=c_1y_1+c_2y_2+\cdots+c_ky_k+y_p$$

is a solution of a linear equation $y^{(n)}+p_1(x)y^{(n-1)}+\cdots+p_n(x)y=f(x)$ for every choice of the constants c_1, c_2,..., c_k, then y_i is a solution to $y^{(n)}+p_1(x)y^{(n-1)}+\cdots+p_n(x)y=0$ for $1\le i\le k$.

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7.2: Higher Order Homogeneous Equations

Higher Order Constant Coefficient Homogeneous Equations

If a_0, a_1, …, a_n are constants and $a_0 \neq 0$, then

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = F(x)$$

is said to be a *constant coefficient equation*. In this section we consider the homogeneous constant coefficient equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0. \tag{7.2.1}$$

The theorems in Section 7.1 all apply with $(a, b) = (-\infty, \infty)$.

As in Section 5.3 we use $y = e^{rx}$ as our solution and from substituting repeated derivatives into 7.2.1 we get the characteristic equation

$$a_0 r^n + a_1 r^{n-1} + \cdots + a_n = 0 \tag{7.2.2}$$

We saw in Section 5.3 that when $n = 2$ the solutions of Equation 7.2.1 are determined by the solutions of the characteristic equation. This is also true when $n > 2$.

Note

The discussion we had in 5.3 regarding distinct, repeating, and complex roots is valid here as well. Additionally, distinct roots always lead to independent solutions, repeated roots multiply the repeated solution by x each time a root is repeated, thereby leading to independent solutions, and repeated complex roots are handled the same way as repeated real roots. We will illustrate each idea in the following examples.

Example 7.2.1

a. Find the general solution of

$$y''' - 6y'' + 11y' - 6y = 0. \tag{7.2.3}$$

b. Solve the initial value problem

$$y''' - 6y'' + 11y' - 6y = 0, \quad y(0) = 4, \quad y'(0) = 5, \quad y''(0) = 9. \tag{7.2.4}$$

Solution a

The characteristic equation of Equation 7.2.3 is

$$r^3 - 6r^2 + 11r - 6 = (r-1)(r-2)(r-3) = 0.$$

Therefore $\{e^x, e^{2x}, e^{3x}\}$ is a fundamental set of solutions of Equation 7.2.3 and the general solution of Equation 7.2.3 is

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}. \tag{7.2.5}$$

Solution b

We must determine c_1, c_2 and c_3 in Equation 7.2.5 so that y satisfies the initial conditions in Equation 7.2.4. Differentiating Equation 7.2.5 twice yields

$$\begin{aligned} y' &= c_1 e^x + 2c_2 e^{2x} + 3c_3 e^{3x} \\ y'' &= c_1 e^x + 4c_2 e^{2x} + 9c_3 e^{3x}. \end{aligned} \tag{7.2.6}$$

Setting $x = 0$ in Equation 7.2.5 and Equation 7.2.6 and imposing the initial conditions yields

$$\begin{aligned} c_1 + c_2 + c_3 &= 4 \\ c_1 + 2c_2 + 3c_3 &= 5 \\ c_1 + 4c_2 + 9c_3 &= 9. \end{aligned}$$

The solution of this system is $c_1 = 4$, $c_2 = -1$, $c_3 = 1$. Therefore the solution of Equation 7.2.4 is

$$y = 4e^x - e^{2x} + e^{3x}$$

(Figure 7.2.1).

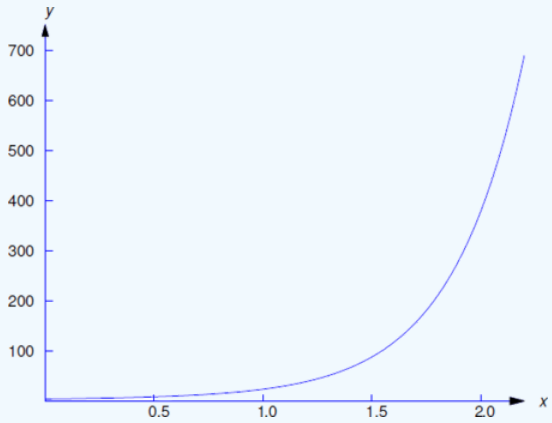


Figure 7.2.1 : $y = 4e^x - e^{2x} + e^{3x}$

Example 7.2.2

Find the general solution of

$$y''' + 3y'' + 3y' + y = 0. \tag{7.2.7}$$

Solution

The characteristic equation of Equation 7.2.7 is

$$r^3 + 3r^2 + 3r + 1 = (r+1)^3 = 0.$$

Therefore $\{e^{-x}, xe^{-x}, x^2e^{-x}\}$ is a fundamental set of solutions of Equation 7.2.7 and the general solution of Equation 7.2.7 is

$$y = e^{-x}(c_1 + c_2x + c_3x^2).$$

Example 7.2.3

Find the general solution of

$$y''' - y'' + y' - y = 0. \tag{7.2.8}$$

Solution

The characteristic equation of Equation 7.2.8 is

$$r^3 - r^2 + r - 1 = (r-1)(r^2+1) = 0.$$

Therefore $\{e^x, \cos x, \sin x\}$ is a fundamental set of solutions of Equation 7.2.8 and the general solution of Equation 7.2.8 is

$$y = c_1e^x + c_2\cos x + c_3\sin x \tag{7.2.9}$$

.

Note

To help with the notation when we use annihilation we will often use differential operator notation, as the following examples illustrate.

Example 7.2.4

Find the general solution of

$$y^{(4)} - 16y = 0. \tag{7.2.10}$$

Solution

The characteristic equation of Equation 7.2.10 is

$$r^4 - 16 = (r^2 - 4)(r^2 + 4) = (r-2)(r+2)(r^2+4) = 0.$$

Using differential operator notation, Equation 7.2.10 can be written as

$$(D^4 - 16)y = 0$$

or

$$(D^2 - 4)(D^2 + 4)y = 0$$

or

$$(D-2)(D+2)(D^2+4)y = 0.$$

Note that the differential equation written in differential operator notation mirrors the characteristic equation, and we can see the solutions from either.

Hence, $\{e^{2x}, e^{-2x}, \cos 2x, \sin 2x\}$ is a fundamental set of solutions of Equation 7.2.10 and

$$y_1 = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 2x + c_4 \sin 2x$$

is the general solution of Equation 7.2.10.

Example 7.2.5

Find the general solution of

$$(D^2 + 4D + 13)^3 y = 0. \tag{7.2.11}$$

Solution

The characteristic equation of Equation 7.2.11 is

$$(r^2 + 4r + 13)^3 = \left((r+2)^2 + 9\right)^3 = 0.$$

Therefore Equation 7.2.11 can be be written as

$$[(D+2)^2 + 9]^3 y = 0.$$

The roots are $r = -2 \pm 3i, -2 \pm 3i, -2 \pm 3i$

The general solution of Equation 7.2.11 is

$$y = (a_1 + a_2 x + a_3 x^2)e^{-2x} \cos 3x + (b_1 + b_2 x + b_3 x^2)e^{-2x} \sin 3x.$$

Example 7.2.6

Find the general solution of

$$y^{(4)} + 4y''' + 6y'' + 4y' = 0. \tag{7.2.12}$$

Solution

The characteristic equation of Equation 7.2.12 is

$$r^4 + 4r^3 + 6r^2 + 4r = r(r^3 + 4r^2 + 6r + 4) = r(r+2)(r^2 + 2r + 2) = r(r+2)[(r+1)^2 + 1] = 0.$$

Therefore Equation 7.2.12 can be written as

$$[(D+1)^2+1](D+2)Dy=0.$$

Therefore the general solution of Equation 7.2.12 is

$$y=e^{-x}(c_1\cos x+c_2\sin x)+c_3e^{-2x}+c_4.$$

Example 7.2.7

Find the general solution of

$$[(D+1)^2+1]^2(D-1)^3(D+1)D^2y=0. \tag{7.2.13}$$

Solution

At this point you should be able to identify the roots immediately as

$$r=-1\pm i,-1\pm i,1,1,1,-1,0,0.$$

The general solution of Equation 7.2.13 is therefore

$$y=c_1e^{-x}\cos x+c_2e^{-x}\sin x+c_3xe^{-x}\cos x+c_4xe^{-x}\sin x+c_5e^{x}+c_6xe^{x}+c_7x^2e^{x}+c_8e^{-x}+c_9+c_{10}x.$$

Higher Order Cauchy-Euler Equations

If a_0, a_1, …, a_n are constants and $a_0\neq 0$, then

$$a_0x^ny^{(n)}+a_1x^{n-1}y^{(n-1)}+\cdots+a_{n-1}xy'+a_ny=F(x)$$

is said to be a Cauchy-Euler equation. In this section we consider the homogeneous Cauchy-Euler Equation

$$a_0x^ny^{(n)}+a_1x^{n-1}y^{(n-1)}+\cdots+a_{n-1}xy'+a_ny=0. \tag{7.2.14}$$

The theorems in Section 7.1 all apply with (a,b)=$(-\infty,0)$ and $(0,\infty)$.

As in Section 5.7 we use $y=x^r$ as our solution and from substituting repeated derivatives into 7.2.14 we get the characteristic equation

$$a_0r(r-1)\cdots(r-(n-1))+a_1r(r-1)\cdots(r-(n-2))+\cdots+a_{n-1}r+a_n=0 \tag{7.2.15}$$

We saw in Section 5.7 that when $n=2$ the solutions of Equation 7.2.14 are determined by the solutions of the characteristic equation. This is also true when $n>2$.

Note

The discussion we had in 5.7 regarding distinct, repeating, and complex roots is valid here as well. Additionally, distinct roots always lead to independent solutions, repeated roots multiply the repeated solution by $\ln x$ each time a root is repeated, thereby leading to independent solutions, and repeated complex roots are handled the same way as repeated real roots. We will illustrate each idea in the following examples.

Example 7.2.8

Find the general solution of

$$x^3y'''-3x^2y''+6xy'-6y=0 \tag{7.2.16}$$

on $(0,\infty)$.

Solution

The characteristic equation of Equation 7.2.16 is

$$r(r-1)(r-2)-3r(r-1)+6r-6=(r-1)(r-2)(r-3)=0.$$

Therefore $\{x,x^2,x^3\}$ is a fundamental set of solutions of Equation 7.2.16 and the general solution of Equation 7.2.16 is

$$y = c_1 x + c_2 x^2, + c_3 x^3.$$

Example 7.2.9

Find the general solution of

$$x^3 y''' + x y' - y = 0 \tag{7.2.17}$$

on $(0, \infty)$.

Solution

The characteristic equation of Equation 7.2.17 is

$$r(r-1)(r-2) + r - 1 = (r-1)^3 = 0.$$

Therefore $\{x, x \ln x, x \ln^2 x\}$ is a fundamental set of solutions of Equation 7.2.17 and the general solution of Equation 7.2.17 is

$$y = c_1 x + c_2 x \ln x, + c_3 x \ln^2 x.$$

Example 7.2.10

Find the general solution of

$$x^3 y''' + 5x^2 y'' + 7x y' + 8y = 0 \tag{7.2.18}$$

on $(0, \infty)$.

Solution

The characteristic equation of Equation 7.2.18 is

$$r(r-1)(r-2) + 5r(r-1) + 7r + 8 = (r+2)(r^2+4) = 0.$$

Therefore $\{x^{-2}, \cos(2 \ln x), \sin(2 \ln x)\}$ is a fundamental set of solutions of Equation 7.2.18 and the general solution of Equation 7.2.18 is

$$y = c_1 x^{-2} + c_2 \cos(2 \ln x) + c_3 \sin(2 \ln x).$$

Example 7.2.11

Find the general solution of a tenth order Cauchy-Euler Equation that has the following roots for its characteristic equation:

$$r = -3 \pm 5i, -3 \pm 5i, -3 \pm 5i, 3, 0, 0, 0.$$

Solution

$$y = c_1 x^{-3} \cos(5 \ln x) + c_2 x^{-3} \sin(5 \ln x) + c_3 x^{-3} \cos(5 \ln x) \ln x + c_4 x^{-3} \sin(5 \ln x) \ln x + c_5 x^{-3} \cos(5 \ln x) \ln^2 x$$
$$+ c_6 x^{-3} \sin(5 \ln x) \ln^2 x + c_7 x^3 + c_8 + c_9 \ln x + c_{10} \ln^2 x$$

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7.2.1: Higher Order Homogeneous Equations (Exercises)

In Exercises **1-14** find the general solution.

1. $y''' - 3y'' + 3y' - y = 0$
2. $y^{(4)} + 8y'' - 9y = 0$
3. $y''' - y'' + 16y' - 16y = 0$
4. $2y''' + 3y'' - 2y' - 3y = 0$
5. $y''' + 5y'' + 9y' + 5y = 0$
6. $4y''' - 8y'' + 5y' - y = 0$
7. $27y''' + 27y'' + 9y' + y = 0$
8. $y^{(4)} + y'' = 0$
9. $y^{(4)} - 16y = 0$
10. $y^{(4)} + 12y'' + 36y = 0$
11. $16y^{(4)} - 72y'' + 81y = 0$
12. $6y^{(4)} + 5y''' + 7y'' + 5y' + y = 0$
13. $4y^{(4)} + 12y''' + 3y'' - 13y' - 6y = 0$
14. $y^{(4)} - 4y''' + 7y'' - 6y' + 2y = 0$

In Exercises **15-27** solve the initial value problem.

15. $y''' - 2y'' + 4y' - 8y = 0, \quad y(0) = 2, \quad y'(0) = -2, \; y''(0) = 0$
16. $y''' + 3y'' - y' - 3y = 0, \quad y(0) = 0, \quad y'(0) = 14, \quad y''(0) = -40$
17. $y''' - y'' - y' + y = 0, \quad y(0) = -2, \quad y'(0) = 9, \quad y''(0) = 4$
18. $y''' - 2y' - 4y = 0, \quad y(0) = 6, \quad y'(0) = 3, \quad y''(0) = 22$
19. $3y''' - y'' - 7y' + 5y = 0, \quad y(0) = \frac{14}{5}, \quad y'(0) = 0, \quad y''(0) = 10$
20. $y''' - 6y'' + 12y' - 8y = 0, \quad y(0) = 1, \quad y'(0) = -1, \quad y''(0) = -4$
21. $2y''' - 11y'' + 12y' + 9y = 0, \quad y(0) = 6, \quad y'(0) = 3, \quad y''(0) = 13$
22. $8y''' - 4y'' - 2y' + y = 0, \quad y(0) = 4, \quad y'(0) = -3, \quad y''(0) = -1$
23. $y^{(4)} - 16y = 0, \quad y(0) = 2, \; y'(0) = 2, \; y''(0) = -2, \; y'''(0) = 0$
24. $y^{(4)} - 6y''' + 7y'' + 6y' - 8y = 0, \quad y(0) = -2, \quad y'(0) = -8, \quad y''(0) = -14, \quad y'''(0) = -62$
25. $4y^{(4)} - 13y'' + 9y = 0, \quad y(0) = 1, \quad y'(0) = 3, \quad y''(0) = 1, \quad y'''(0) = 3$
26. $y^{(4)} + 2y''' - 2y'' - 8y' - 8y = 0, \quad y(0) = 5, \quad y'(0) = -2, \quad y''(0) = 6, \quad y'''(0) = 8$
27. $4y^{(4)} + 8y''' + 19y'' + 32y' + 12y = 0, \quad y(0) = 3, \quad y'(0) = -3, \quad y''(0) = -\frac{7}{2}, \quad y'''(0) = \frac{31}{4}$

In Exercises **28-33** find a fundamental set of solutions of the given equation, and verify that it is a fundamental set by evaluating its Wronskian. Form the general solution.

28. $(D-1)^2(D-2)y = 0$
29. $(D^2+4)(D-3)y = 0$
30. $(D^2+2D+2)(D-1)y = 0$
31. $D^3(D-1)y = 0$
32. $(D^2-1)(D^2+1)y = 0$

33. $(D^2-2D+2)(D^2+1)y=0$

In Exercises **34-43** find a fundamental set of solutions and form the general solution.

34. $(D^2+6D+13)(D-2)^2D^3y=0$
35. $(D-1)^2(2D-1)^3(D^2+1)y=0$
36. $(D^2+9)^3D^2y=0$
37. $(D-2)^3(D+1)^2Dy=0$
38. $(D^2+1)(D^2+9)^2(D-2)y=0$
39. $(D^4-16)^2y=0$
40. $(4D^2+4D+9)^3y=0$
41. $D^3(D-2)^2(D^2+4)^2y=0$
42. $(4D^2+1)^2(9D^2+4)^3y=0$

In Exercises **43-48** find the general solution on $(0,\infty)$

43. $x^3y'''+5x^2y''+7xy'+8y=0$
44. $x^3y'''-6y=0$
45. $x^3y'''+xy'-y=0$
46. $xy^{(4)}+6y'''=0$
47. $x^4y^{(4)}+6x^3y'''+9x^2y''+3xy'+y=0$
48. $x^3y'''-3x^2y''+6xy'-6y=0$

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7.3: Annihilation for Higher Order Equations

Here we consider the constant coefficient nonhomogeneous equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = F(x). \tag{7.3.1}$$

The theory and methods are the same as in Section 5.5 and will be omitted here, and we will proceed directly to a few examples.

Example 7.3.1

Find the general solution of

$$y''' + y'' = e^x \cos x \tag{7.3.2}$$

Solution

We first solve

$$y''' + y'' = 0$$

Its characteristic equation is

$$r^3 + r^2 = r^2(r+1) = 0$$

which gives us our homogeneous solution

$$y_h = c_1 + c_2 x + c_3 e^{-x}.$$

Now turning to the nonhomogeneous equation and writing it using differential operator notation we get

$$D^2(D+1)y = e^x \cos x.$$

We now annihilate the right hand side using $((D-1)^2+1)$ to turn it into a new homogeneous equation:

$$((D-1)^2+1)D^2(D+1)y = 0$$

So, our particular solution takes the form

$$y_p = Ae^x \cos x + Be^x \sin x$$

and $y_p' = Ae^x \cos x - Ae^x \sin x + Be^x \sin x + Be^x \cos x$, $y_p'' = -2Ae^x \sin x + 2Be^x \cos x$, and $y_p''' = -2Ae^x \sin x - 2Ae^x \cos x + 2Be^x \cos x - 2Be^x \sin x$

Substituting into 7.3.2 gives us

$$(-2Ae^x \sin x - 2Ae^x \cos x + 2Be^x \cos x - 2Be^x \sin x) + (-2Ae^x \sin x + 2Be^x \cos x) = e^x \cos x$$

This leads to

$$(-2A+4B)e^x \cos x + (-4A-2B)e^x \sin x = e^x \cos x.$$

So, $-2A+4B=1$ and $-4A-2B=0$ giving us

$$A = -\frac{1}{10}, B = \frac{1}{5}$$

and

$$y_p = -\frac{1}{10}e^x \cos x + \frac{1}{5}e^x \sin x$$

and the general solution is

$$y = c_1 + c_2 x + c_3 e^{-x} - \frac{1}{10}e^x \cos x + \frac{1}{5}e^x \sin x.$$

Example 7.3.2

Find the general solution of

$$y''' - 3y'' + 3y' - y = e^x - x + 16. \tag{7.3.3}$$

Solution

We first solve

$$y''' - 3y'' + 3y' - y = 0$$

Its characteristic equation is

$$r^3 - 3r^2 + 3r - 1 = (r-1)^3 = 0$$

which gives us our homogeneous solution

$$y_h = c_1 e^x + c_2 x e^x + c_3 x^2 e^x.$$

Now turning to the nonhomogeneous equation and writing it using differential operator notation we get

$$(D-1)^3 y = e^x - x + 16.$$

We now annihilate the right hand side using $(D-1)D^2$ to turn it into a new homogeneous equation:

$$(D-1)^4 D^2 y = 0$$

Note that while the root 0 is occurring for the first and second time, the root 1 is occurring for the fourth time. So, our particular solution takes the form

$$y_p = Ax^3 e^x + Bx + C$$

and $y_p' = Ax^3 e^x + 3Ax^2 e^x + B$, $y_p'' = Ax^3 e^x + 6Ax^2 e^x + 6Axe^x$, and $y_p''' = Ax^3 e^x + 9Ax^2 e^x + 18Axe^x + 6Ae^x$

Substituting into 7.3.3 gives us

$$(Ax^3 e^x + 9Ax^2 e^x + 18Axe^x + 6Ae^x) - 3(Ax^3 e^x + 6Ax^2 e^x + 6Axe^x) + 3(Ax^3 e^x + 3Ax^2 e^x + B)$$
$$-(Ax^3 e^x + Bx + C) = e^x - x + 16$$

This leads to

$$6Ae^x - Bx + (3B - C) = e^x - x + 16$$

So,

$$A = \frac{1}{6}, B = 1, C = -13$$

and

$$y_p = \frac{1}{6}e^x + x - 13$$

and the general solution is

$$y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x + \frac{1}{6}e^x + x - 13$$

Example 7.3.3

Find the general solution of

$$y''' - 4y'' + 4y' = 5x^2 - 6x + 4x^2 e^{2x} + 3e^{5x}. \tag{7.3.4}$$

Solution

We first solve

$$y''' - 4y'' + 4y' = 0$$

Its characteristic equation is

$$r^3 - 4r^2 + 4r = r(r-2)^2 = 0$$

which gives us our homogeneous solution

$$y_h = c_1 + c_2 e^{2x} + c_3 x e^{2x}.$$

Now turning to the nonhomogeneous equation and writing it using differential operator notation we get

$$D(D-2)^2 y = 5x^2 - 6x + 4x^2 e^{2x} + 3e^{5x}.$$

We now annihilate the right hand side using $D^3(D-2)^3(D-5)$ to turn it into a new homogeneous equation:

$$D^4(D-2)^5(D-5)y = 0$$

Note that while the root 5 is occurring for the first time, the root 0 is occurring for the second, third, and fourth time and the root 2 is occurring for the third, fourth, and fifth time. So, our particular solution takes the form

$$y_p = Ax + Bx^2 + Cx^3 + De^{5x} + Ex^2 e^{2x} + Fx^3 e^{2x} + Gx^4 e^{2x}$$

and $y_p' = A + 2Bx + 3Cx^2 + 5De^{5x} + 2Ex^2 e^{2x} + 2Exe^{2x} + 2Fx^3 e^{2x} + 3Fx^2 e^{2x} + 2Gx^4 e^{2x} + 4Gx^3 e^{2x}$,
$y_p'' = 2B + 6Cx + 25De^{5x} + 4Ex^2 e^{2x} + 8Exe^{2x} + 2Ee^{2x} + 4Fx^3 e^{2x} + 12Fx^2 e^{2x} + 6Fxe^{2x} + 4Gx^4 e^{2x}$, and
$+ 16Gx^3 e^{2x} + 12Gx^2 e^{2x}$
$y_p''' = 6C + 125De^{5x} + 8Ex^2 e^{2x} + 24Exe^{2x} + 12Ee^{2x} + 8Fx^3 e^{2x} + 36Fx^2 e^{2x} + 36Fxe^{2x} + 6Fe^{2x} + 8Gx^4 e^{2x}$
$+ 48Gx^3 e^{2x} + 72Gx^2 e^{2x} + 24Gxe^{2x}$

Substituting into 7.3.4 gives us

$$\begin{aligned}(6C + 125De^{5x} + 8Ex^2 e^{2x} + 24Exe^{2x} + 12Ee^{2x} + 8Fx^3 e^{2x} + 36Fx^2 e^{2x} + 36Fxe^{2x} + 6Fe^{2x} + 8Gx^4 e^{2x} \\ + 48Gx^3 e^{2x} + 72Gx^2 e^{2x} + 24Gxe^{2x}) - 4(2B + 6Cx + 25De^{5x} + 4Ex^2 e^{2x} + 8Exe^{2x} + 2Ee^{2x} + 4Fx^3 e^{2x} \\ + 12Fx^2 e^{2x} + 6Fxe^{2x} + 4Gx^4 e^{2x} + 16Gx^3 e^{2x} + 12Gx^2 e^{2x}) + 4(A + 2Bx + 3Cx^2 + 5De^{5x} + 2Ex^2 e^{2x} + 2Exe^{2x} \\ + 2Fx^3 e^{2x} + 3Fx^2 e^{2x} + 2Gx^4 e^{2x} + 4Gx^3 e^{2x}) = 5x^2 - 6x + 4x^2 e^{2x} + 3e^{5x}.\end{aligned}$$

This leads to

$$\begin{aligned}(4A - 8B + 6C) + (8B - 24C)x + 12Cx^2 + 45De^{5x} + 24Gx^2 e^{2x} + (12F + 24G)xe^{2x} + (4E + 6F)e^{2x} = 5x^2 \\ - 6x + 4x^2 e^{2x} + 3e^{5x}\end{aligned}$$

So,

$$A = \frac{3}{8}, B = \frac{1}{2}, C = \frac{5}{12}, D = \frac{1}{15}, E = \frac{1}{2}, F = -\frac{1}{3}, G = \frac{1}{6}$$

and

$$y_p = \frac{3}{8}x + \frac{1}{2}x^2 + \frac{5}{12}x^3 + \frac{1}{15}e^{5x} + \frac{1}{2}x^2 e^{2x} - \frac{1}{3}x^3 e^{2x} + \frac{1}{6}x^4 e^{2x}$$

and the general solution is

$$y = c_1 + c_2 e^{2x} + c_3 x e^{2x} + \frac{3}{8}x + \frac{1}{2}x^2 + \frac{5}{12}x^3 + \frac{1}{15}e^{5x} + \frac{1}{2}x^2 e^{2x} - \frac{1}{3}x^3 e^{2x} + \frac{1}{6}x^4 e^{2x}.$$

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7.3.1: Annihilation for Higher Order Equations (Exercises)

In Exercises **1-76** find the general solution.

1. $y''' + y'' = 8x^2$
2. $y''' + 8y'' = -6x^2 + 9x + 2$
3. $y''' - y'' + y' - y = xe^x - e^{-x} + 7$
4. $y''' - 3y'' + 3y' - y = e^x - x + 16$
5. $2y''' - 3y'' - 3y' + 2y = (e^x + e^{-x})^2$
6. $y^{(4)} - 2y''' + y'' = e^x + 1$
7. $y^{(4)} - 4y'' = 5x^2 - e^{2x}$
8. $y''' - 6y'' = 3 - \cos x$
9. $y''' - 2y'' - 4y' + 8y = 6xe^{2x}$
10. $y''' - 3y'' + 3y' - y = x - 4e^x$
11. $y''' - y'' - 4y' + 4y = 5 - e^x + e^{2x}$
12. $y^{(4)} + 2y'' + y = (x-1)^2$
13. $y^{(4)} - y'' = 4x + 2xe^{-x}$
14. $y''' - 6y'' + 11y' - 6y = -e^{-x}(4 + 76x - 24x^2)$ x.
15. $y''' - 2y'' - 5y' + 6y = e^{-3x}(32 - 23x + 6x^2)$
16. $4y''' + 8y'' - y' - 2y = -e^x(4 + 45x + 9x^2)$
17. $y''' + 3y'' - y' - 3y = e^{-2x}(2 - 17x + 3x^2)$
18. $y''' + 3y'' - y' - 3y = e^x(-1 + 2x + 24x^2 + 16x^3)$
19. $y''' + y'' - 2y = e^x(14 + 34x + 15x^2)$
20. $4y''' + 8y'' - y' - 2y = -e^{-2x}(1 - 15x)$
21. $y''' - y'' - y' + y = e^x(7 + 6x)$
22. $2y''' - 7y'' + 4y' + 4y = e^{2x}(17 + 30x)$
23. $y''' - 5y'' + 3y' + 9y = 2e^{3x}(11 - 24x^2)$
24. $y''' - 7y'' + 8y' + 16y = 2e^{4x}(13 + 15x)$
25. $8y''' - 12y'' + 6y' - y = e^{x/2}(1 + 4x)$
26. $y^{(4)} + 3y''' - 3y'' - 7y' + 6y = -e^{-x}(12 + 8x - 8x^2)$
27. $y^{(4)} + 3y''' + y'' - 3y' - 2y = -3e^{2x}(11 + 12x)$
28. $y^{(4)} + 8y''' + 24y'' + 32y' = -16e^{-2x}(1 + x + x^2 - x^3)$
29. $4y^{(4)} - 11y'' - 9y' - 2y = -e^x(1 - 6x)$
30. $2y^{(4)} + 5y''' - 5y' - 2y = 18e^x(5 + 2x)$
31. $y^{(4)} - 5y'' + 4y = e^x(3 + x - 3x^2)$
32. $y^{(4)} - 2y''' - 3y'' + 4y' + 4y = e^{2x}(13 + 33x + 18x^2)$
33. $y^{(4)} - 3y''' + 4y' = e^{2x}(15 + 26x + 12x^2)$
34. $y^{(4)} - 2y''' + 2y' - y = e^x(1 + x)$
35. $2y^{(4)} - 5y''' + 3y'' + y' - y = e^x(11 + 12x)$

36. $y^{(4)}+3y'''+3y''+y'=e^{-x}(5-24x+10x^2)$
37. $y^{(4)}-7y'''+18y''-20y'+8y=e^{2x}(3-8x-5x^2)$
38. $y'''-y''-4y'+4y=e^{-x}\left[(16+10x)\cos x+(30-10x)\sin x\right]$
39. $y'''+y''-4y'-4y=e^{-x}\left[(1-22x)\cos 2x-(1+6x)\sin 2x\right]$
40. $y'''-y''+2y'-2y=e^{2x}[(27+5x-x^2)\cos x+(2+13x+9x^2)\sin x]$
41. $y'''-2y''+y'-2y=-e^{x}[(9-5x+4x^2)\cos 2x-(6-5x-3x^2)\sin 2x]$
42. $y'''+3y''+4y'+12y=8\cos 2x-16\sin 2x$
43. $y'''-y''+2y=e^{x}[(20+4x)\cos x-(12+12x)\sin x]$
44. $y'''-7y''+20y'-24y=-e^{2x}[(13-8x)\cos 2x-(8-4x)\sin 2x]$
45. $y'''-6y''+18y'=-e^{3x}[(2-3x)\cos 3x-(3+3x)\sin 3x]$
46. $y^{(4)}+2y'''-2y''-8y'-8y=e^{x}(8\cos x+16\sin x)$
47. $y^{(4)}-3y'''+2y''+2y'-4y=e^{x}(2\cos 2x-\sin 2x)$
48. $y^{(4)}-8y'''+24y''-32y'+15y=e^{2x}(15x\cos 2x+32\sin 2x)$
49. $y^{(4)}+6y'''+13y''+12y'+4y=e^{-x}[(4-x)\cos x-(5+x)\sin x]$
50. $y^{(4)}+3y'''+2y''-2y'-4y=-e^{-x}(\cos x-\sin x)$
51. $y^{(4)}-5y'''+13y''-19y'+10y=e^{x}(\cos 2x+\sin 2x)$
52. $y^{(4)}+8y'''+32y''+64y'+39y=e^{-2x}[(4-15x)\cos 3x-(4+15x)\sin 3x]$
53. $y^{(4)}-5y'''+13y''-19y'+10y=e^{x}[(7+8x)\cos 2x+(8-4x)\sin 2x]$
54. $y^{(4)}+4y'''+8y''+8y'+4y=-2e^{-x}(\cos x-2\sin x)$
55. $y^{(4)}-8y'''+32y''-64y'+64y=e^{2x}(\cos 2x-\sin 2x)$
56. $y^{(4)}-8y'''+26y''-40y'+25y=e^{2x}[3\cos x-(1+3x)\sin x]$
57. $y'''-4y''+5y'-2y=e^{2x}-4e^{x}-2\cos x+4\sin x$
58. $y'''-y''+y'-y=5e^{2x}+2e^{x}-4\cos x+4\sin x$
59. $y'''-y'=-2(1+x)+4e^{x}-6e^{-x}+96e^{3x}$
60. $y'''-4y''+9y'-10y=10e^{2x}+20e^{x}\sin 2x-10$
61. $y'''+3y''+3y'+y=12e^{-x}+9\cos 2x-13\sin 2x$
62. $y'''+y''-y'-y=4e^{-x}(1-6x)-2x\cos x+2(1+x)\sin x$
63. $y^{(4)}-5y''+4y=-12e^{x}+6e^{-x}+10\cos x$
64. $y^{(4)}-4y'''+11y''-14y'+10y=-e^{x}(\sin x+2\cos 2x)$
65. $y^{(4)}+2y'''-3y''-4y'+4y=2e^{x}(1+x)+e^{-2x}$
66. $y^{(4)}+5y'''+9y''+7y'+2y=e^{-x}(30+24x)-e^{-2x}$
67. $y^{(4)}-4y'''+7y''-6y'+2y=e^{x}(12x-2\cos x+2\sin x)$
68. $y'''-y''-y'+y=e^{2x}(10+3x)$
69. $y'''+y''-2y=-e^{3x}(9+67x+17x^2)$
70. $y'''-6y''+11y'-6y=e^{2x}(5-4x-3x^2)$
71. $y'''+2y''+y'=-2e^{-x}(7-18x+6x^2)$
72. $y'''-3y''+3y'-y=e^{x}(1+x)$
73. $y^{(4)}-2y''+y=-e^{-x}(4-9x+3x^2)$

74. $y''' + 2y'' - y' - 2y = e^{-2x}\left[(23 - 2x)\cos x + (8 - 9x)\sin x\right]$

75. $y^{(4)} - 3y''' + 4y'' - 2y' = e^{x}\left[(28 + 6x)\cos 2x + (11 - 12x)\sin 2x\right]$

76. $y^{(4)} - 4y''' + 14y'' - 20y' + 25y = e^{x}\left[(2 + 6x)\cos 2x + 3\sin 2x\right]$

In Exercises **77-86** solve the initial value problem.

77. $y''' - 2y'' + y' = xe^{x} + 5, \quad y(0) = 2, \quad y'(0) = 2, \quad y''(0) = -1$

78. $y^{(4)} - y''' = x + e^{x}, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$

79. $y''' - 2y'' + y' = 2 - 24e^{x} + 40e^{5x}, \quad y(0) = \frac{1}{2}, \quad y'(0) = \frac{5}{2}, \quad y''(0) = -\frac{9}{2}$

80. $y''' + 8y = 2x - 5 + 8e^{-2x}, \quad y(0) = -5, \quad y'(0) = 3, \quad y''(0) = -4$

81. $y''' - 2y'' - 5y' + 6y = 2e^{x}(1 - 6x), \quad y(0) = 2, \quad y'(0) = 7, \quad y''(0) = 9$

82. $y''' - y'' - y' + y = -e^{-x}(4 - 8x), \quad y(0) = 2, \quad y'(0) = 0, \quad y''(0) = 0$

83. $4y''' - 3y' - y = e^{-x/2}(2 - 3x), \quad y(0) = -1, \quad y'(0) = 15, \quad y''(0) = -17$

84. $y^{(4)} + 2y''' + 2y'' + 2y' + y = e^{-x}(20 - 12x), \; y(0) = 3, \; y'(0) = -4, \; y''(0) = 7, \; y'''(0) = -22$

85. $y''' + 2y'' + y' + 2y = 30\cos x - 10\sin x, \quad y(0) = 3, \quad y'(0) = -4, \quad y''(0) = 16$

86. $y^{(4)} - 3y''' + 5y'' - 2y' = -2e^{x}(\cos x - \sin x), \; y(0) = 2, \; y'(0) = 0, \; y''(0) = -1, \; y'''(0) = -5$

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7.4: Variation of Parameters for Higher Order Equations

Derivation of the Method

Caution

If you have not had Math 410 (linear algebra), then you may want to review Appendix 11.4 again before starting this section.

Although the method of variation of parameters for higher order equations is similar to what we did in Section 5.6 for second order equations, there are differences that we'll need to discuss here. For convenience, we consider linear differential equations written as

$$P_0(x)y^{(n)}+P_1(x)y^{(n-1)}+\cdots+P_n(x)y=F(x), \tag{7.4.1}$$

which can be rewritten as

$$y^{(n)}+p_1(x)y^{(n-1)}+\cdots+p_n(x)y=f(x). \tag{7.4.2}$$

on any interval on which P_0 has no zeros, with $p_1=P_1/P_0$, ..., $p_n=P_n/P_0$ and $f=F/P_0$. For simplicity, throughout this chapter we'll abbreviate the left side of Equation 7.4.2 by Ly; that is,

$$Ly=y^{(n)}+p_1(x)y^{(n-1)}+\cdots+p_n(x)y.$$

When we speak of solutions of this equation and its homogeneous equation $Ly=0$, we mean solutions on (a,b). We'll show how to use the method of variation of parameters to find a particular solution of $Ly=f(x)$, provided that we know a fundamental set of solutions $\{y_1,y_2,\dots,y_n\}$ of $Ly=0$.

We seek a particular solution of $Ly=f(x)$ in the form

$$y_p=u_1y_1+u_2y_2+\cdots+u_ny_n \tag{7.4.3}$$

where $\{y_1,y_2,\dots,y_n\}$ is a known fundamental set of solutions of the homogeneous equation

$$y^{(n)}+p_1(x)y^{(n-1)}+\cdots+p_n(x)y=0$$

and u_1, u_2, ..., u_n are functions to be determined. We begin, as we did in Section 5.6, by imposing conditions that make our calculations easier. These conditions are similar to what we did in Section 5.6.

We impose the following $n-1$ conditions on $u_1,u_2,\dots,u_n$:

$$\begin{aligned} u_1'y_1+u_2'y_2+&\quad\cdots\quad+u_n'y_n=0\\ u_1'y_1'+u_2'y_2'+&\quad\cdots\quad+u_n'y_n'=0\\ &\quad\vdots\\ u_1'y_1^{(n-2)}+u_2'y_2^{(n-2)}+&\quad\cdots\quad+u_n'y_n^{(n-2)}=0. \end{aligned} \tag{7.4.4}$$

These conditions lead to simple formulas for the first $n-1$ derivatives of y_p:

$$y_p^{(r)}=u_1y_1^{(r)}+u_2y_2^{(r)}\cdots+u_ny_n^{(r)},\ 0\le r\le n-1. \tag{7.4.5}$$

These formulas are easy to remember, since they look as though we obtained them by differentiating Equation 7.4.3 $n-1$ times while treating u_1, u_2, ..., u_n as constants. To see that Equation 7.4.4 implies Equation 7.4.5, we first differentiate Equation 7.4.3 to obtain

$$y_p'=u_1y_1'+u_2y_2'+\cdots+u_ny_n'+u_1'y_1+u_2'y_2+\cdots+u_n'y_n,$$

which reduces to

$$y_p'=u_1y_1'+u_2y_2'+\cdots+u_ny_n'$$

because of the first equation in Equation 7.4.4. Differentiating this yields

$$y_p''=u_1y_1''+u_2y_2''+\cdots+u_ny_n''+u_1'y_1'+u_2'y_2'+\cdots+u_n'y_n',$$

which reduces to

$$y_p''=u_1y_1''+u_2y_2''+\cdots+u_ny_n''$$

because of the second equation in Equation 7.4.4. Continuing in this way yields Equation 7.4.5.

The last equation in Equation 7.4.5 is

$$y_p^{(n-1)} = u_1 y_1^{(n-1)} + u_2 y_2^{(n-1)} + \cdots + u_n y_n^{(n-1)}.$$

Differentiating this yields

$$y_p^{(n)} = u_1 y_1^{(n)} + u_2 y_2^{(n)} + \cdots + u_n y_n^{(n)} + u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)}.$$

Substituting this and Equation 7.4.5 into Equation 7.4.2 yields

$$u_1 L y_1 + u_2 L y_2 + \cdots + u_n L y_n + \left(u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)}\right) = f(x).$$

Since $Ly_i = 0\ (1 \le i \le n)$, this reduces to

$$u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)} = f(x).$$

Combining this equation with Equation 7.4.4 shows that

$$y_p = u_1 y_1 + u_2 y_2 + \cdots + u_n y_n$$

is a solution of Equation 7.4.2 if

$$\begin{aligned}
u_1' y_1 + u_2' y_2 + \quad & \cdots \quad + u_n' y_n = 0 \\
u_1' y_1' + u_2' y_2' + \quad & \cdots \quad + u_n' y_n' = 0 \\
& \vdots \\
u_1' y_1^{(n-2)} + u_2' y_2^{(n-2)} + \quad & \cdots \quad + u_n' y_n^{(n-2)} = 0 \\
u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \quad & \cdots \quad + u_n' y_n^{(n-1)} = f(x),
\end{aligned}$$

where $W = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(x) \end{bmatrix}$.

The determinant of W is the Wronskian of the fundamental set of solutions $\{y_1, y_2, \ldots, y_n\}$, which we know has no zeros on (a, b) since the solutions are linearly independent. Solving by Cramer's rule yields

$$u_j' = \frac{W_j}{W}, \quad 1 \le j \le n, \tag{7.4.6}$$

where W_j is the determinant obtained by replacing the j-th column of W with $\mathbf{b}$.

Having obtained u_1', u_2', $\ldots$, u_n', we can integrate to obtain u_1, u_2, $\ldots$, u_n. As in Section 5.6, we take the constants of integration to be zero, and we drop any linear combination of $\{y_1, y_2, \ldots, y_n\}$ that may appear in y_p.

Third Order Equations

If $n = 3$, then

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}.$$

Therefore

$$W_1=\begin{vmatrix}0&y_2&y_3\\0&y_2'&y_3'\\f(x)&y_2''&y_3''\end{vmatrix},\quad W_2=\begin{vmatrix}y_1&0&y_3\\y_1'&0&y_3'\\y_1''&f(x)&y_3''\end{vmatrix},\quad W_3=\begin{vmatrix}y_1&y_2&0\\y_1'&y_2'&0\\y_1''&y_2''&f(x)\end{vmatrix},$$

and Equation 7.4.6 becomes

$$u_1'=\frac{W_1}{W},\quad u_2'=\frac{W_2}{W},\quad u_3'=\frac{W_3}{W}. \tag{7.4.7}$$

Example 7.4.1

Find the general solution of

$$xy'''-y''-xy'+y=8x^2e^x, \tag{7.4.8}$$

given that $y_1=x$, $y_2=e^x$, and $y_3=e^{-x}$ form a fundamental set of solutions of the homogeneous equation. Then find the general solution of Equation 7.4.8.

Solution

We seek a particular solution of Equation 7.4.8 of the form

$$y_p=u_1x+u_2e^x+u_3e^{-x}.$$

The Wronskian of $\{y_1,y_2,y_3\}$ is

$$W(x)=\begin{vmatrix}x&e^x&e^{-x}\\1&e^x&-e^{-x}\\0&e^x&e^{-x}\end{vmatrix}=2x,$$

so

$$W_1=\begin{vmatrix}0&e^x&e^{-x}\\0&e^x&-e^{-x}\\8xe^x&e^x&e^{-x}\end{vmatrix}=-16xe^x,\quad W_2=\begin{vmatrix}x&0&e^{-x}\\1&0&-e^{-x}\\0&8xe^x&e^{-x}\end{vmatrix}=8x(x+1),\quad W_3=\begin{vmatrix}x&e^x&0\\1&e^x&0\\0&e^x&8xe^x\end{vmatrix}=8xe^{2x}(x-1).$$

Therefore, from Equation 7.4.7

$$u_1'=\frac{W_1}{W}=-8e^x,\quad u_2'=\frac{W_2}{W}=4(x+1),\quad u_3'=\frac{W_3}{W}=4e^{2x}(x-1).$$

Integrating and taking the constants of integration to be zero yields

$$u_1=-8e^x,\quad u_2=2(x+1)^2,u_3=e^{2x}(2x-3).$$

Hence,

$$\begin{aligned}y_p&=u_1y_1+u_2y_2+u_3y_3\\&=(-8e^x)x+e^x(2(x+1)^2)+e^{-x}\left(e^{2x}(2x-3)\right)\\&=e^x(2x^2-2x-1).\end{aligned}$$

Since $-e^x$ is a solution of the complementary equation, we redefine

$$y_p=2xe^x(x-1).$$

Therefore the general solution of Equation 7.4.8 is

$$y=2xe^x(x-1)+c_1x+c_2e^x+c_3e^{-x}.$$

Fourth Order Equations

If $n=4$, then

$$W=\begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix},$$

Therefore

$$W_1=\begin{vmatrix} 0 & y_2 & y_3 & y_4 \\ 0 & y_2' & y_3' & y_4' \\ 0 & y_2'' & y_3'' & y_4'' \\ f(x) & y_2''' & y_3''' & y_4''' \end{vmatrix}, \quad W_2=\begin{vmatrix} y_1 & 0 & y_3 & y_4 \\ y_1' & 0 & y_3' & y_4' \\ y_1'' & 0 & y_3'' & y_4'' \\ y_1''' & f(x) & y_3''' & y_4''' \end{vmatrix},$$

$$W_3=\begin{vmatrix} y_1 & y_2 & 0 & y_4 \\ y_1' & y_2' & 0 & y_4' \\ y_1'' & y_2'' & 0 & y_4'' \\ y_1''' & y_2''' & f(x) & y_4''' \end{vmatrix}, \quad W_4=\begin{vmatrix} y_1 & y_2 & y_3 & 0 \\ y_1' & y_2' & y_3' & 0 \\ y_1'' & y_2'' & y_3'' & 0 \\ y_1''' & y_2''' & y_3''' & f(x) \end{vmatrix},$$

and Equation 7.4.6 becomes

$$u_1'=\frac{W_1}{W}, \quad u_2'=\frac{W_2}{W}, \quad u_3'=\frac{W_3}{W}, \quad u_4'=\frac{W_4}{W}. \tag{7.4.9}$$

Example 7.4.2

Find the general solution of

$$x^4y^{(4)}+6x^3y'''+2x^2y''-4xy'+4y=12x^2, \tag{7.4.10}$$

given that $y_1=x$, $y_2=x^2$, $y_3=1/x$ and $y_4=1/x^2$ form a fundamental set of solutions of the homogeneous equation. Then find the general solution of Equation 7.4.10 on $(-\infty,0)$ and $(0,\infty)$.

Solution

We seek a particular solution of Equation 7.4.10 of the form

$$y_p=u_1x+u_2x^2+\frac{u_3}{x}+\frac{u_4}{x^2}.$$

The Wronskian of $\{y_1,y_2,y_3,y_4\}$ is

$$W(x)=\begin{vmatrix} x & x^2 & 1/x & -1/x^2 \\ 1 & 2x & -1/x^2 & -2/x^3 \\ 0 & 2 & 2/x^3 & 6/x^4 \\ 0 & 0 & -6/x^4 & -24/x^5 \end{vmatrix}=-72/x^6,$$

so

$$W_1=\begin{vmatrix} 0 & x^2 & 1/x & 1/x^2 \\ 0 & 2x & -1/x^2 & -2/x^3 \\ 0 & 2 & 2/x^3 & 6/x^4 \\ 12/x^2 & 0 & -6/x^4 & -24/x^5, \end{vmatrix}=144/x^6, \quad W_2=\begin{vmatrix} x & 0 & 1/x & 1/x^2 \\ 1 & 0 & -1/x^2 & -2/x^3 \\ 0 & 0 & 2/x^3 & 6/x^4 \\ 0 & 12/x^2 & -6/x^4 & -24/x^5 \end{vmatrix}=-72/x^7,$$

$$W_3 = \begin{vmatrix} x & x^2 & 0 & -1/x^2 \\ 1 & 2x & 0 & -2/x^3 \\ 0 & 2 & 0 & 6/x^4 \\ 0 & 0 & 12/x^2 & -24/x^5 \end{vmatrix} = -144/x^4, \quad W_4 = \begin{vmatrix} x & x^2 & 1/x & 0 \\ 1 & 2x & -1/x^2 & 0 \\ 0 & 2 & 2/x^3 & 0 \\ 0 & 0 & -6/x^4 & 12/x^2 \end{vmatrix} = 72/x^3,$$

Therefore, from Equation 7.4.9,

$$u_1' = \frac{W_1}{W} = -2, \quad u_2' = \frac{W_2}{W} = 1/x, \quad u_3' = \frac{W_3}{W} = 2x^2, \quad u_4' = \frac{W_4}{W} = -x^3.$$

Integrating these and taking the constants of integration to be zero yields

$$u_1 = -2x, \quad u_2 = \ln|x|, \quad u_3 = \frac{2x^3}{3}, u_4 = -\frac{x^4}{4}.$$

Hence,

$$\begin{aligned} y_p &= u_1y_1 + u_2y_2 + u_3y_3 + u_4y_4 \\ &= (-2x)x + (\ln|x|)x^2 + \frac{2x^3}{3}\frac{1}{x} + \left(-\frac{x^4}{4}\right)\frac{1}{x^2} \\ &= x^2\ln|x| - \frac{19x^2}{12}. \end{aligned}$$

Since $-19x^2/12$is a solution of the homogeneous equation, we redefine

$$y_p = x^2\ln|x|.$$

Therefore

$$y = x^2\ln|x| + c_1x + c_2x^2 + \frac{c_3}{x} + \frac{c_4}{x^2}$$

is the general solution of Equation 7.4.10on $(-\infty, 0)$ and $(0, \infty)$.

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7.4.1: Variation of Parameters for Higher Order Equations (Exercises)

In Exercises **1-7** find the general solution.

1. $x^3y'''-3x^2y''+6xy'-6y=2x$
2. $4x^3y'''+4x^2y''-5xy'+2y=30x^2$
3. $x^3y'''+x^2y''-2xy'+2y=x^2$
4. $xy^{(4)}+4y'''=6\ln|x|$
5. $16x^4y^{(4)}+96x^3y'''+72x^2y''-24xy'+9y=96x^{5/2}$
6. $x^4y^{(4)}-4x^3y'''+12x^2y''-24xy'+24y=x^4$
7. $x^4y^{(4)}+6x^3y'''+2x^2y''-4xy'+4y=12x^2$

In Exercises **8-14** solve the initial value problem.

8. $x^3y'''-2x^2y''+3xy'-3y=4x,\quad y(1)=4,\quad y'(1)=4,\quad y''(1)=2$
9. $x^3y'''-5x^2y''+14xy'-18y=x^3,\quad y(1)=0,\quad y'(1)=1,\quad y''(1)=7$
10. $x^3y'''-6x^2y''+16xy'-16y=9x^4,\quad y(1)=2,\quad y'(1)=1,\quad y''(1)=5$
11. $x^3y'''+x^2y''-2xy'+2y=x(x+1),\quad y(-1)=-6,\quad y'(-1)=\frac{43}{6},\quad y''(-1)=-\frac{5}{2}$
12. $x^4y^{(4)}+3x^3y'''-x^2y''+2xy'-2y=9x^2,\quad y(1)=-7,\quad y'(1)=-11,\quad y''(1)=-5,\quad y'''(1)=6$
13. $4x^4y^{(4)}+24x^3y'''+23x^2y''-xy'+y=6x,\quad y(1)=2,\quad y'(1)=0,\quad y''(1)=4,\quad y'''(1)=-\frac{37}{4}$
14. $x^4y^4+5x^3y'''-3x^2y''-6xy'+6y=40x^3,\quad y(-1)=-1,\ y'(-1)=-7\ ,y''(-1)=-1,\quad y'''(-1)=-31\)$

In exercises **15-28** find the general solution, given the fundamental set of solutions of the homogeneous equation.

15. $x^3y'''-x^2(x+3)y''+2x(x+3)y'-2(x+3)y=-4x^4\ ;\{x,\ x^2,\ xe^x\}$
16. $y'''+6xy''+(6+12x^2)y'+(12x+8x^3)y=x^{1/2}e^{-x^2}\ ;\{e^{-x^2},\ xe^{-x^2},\ x^2e^{-x^2}\}$
17. $x^2y'''+2xy''-(x^2+2)y'=2x^2\ ;\{1,\ e^x/x,\ e^{-x}/x\}$
18. $x^3y'''-3x^2(x+1)y''+3x(x^2+2x+2)y'-(x^3+3x^2+6x+6)y=x^4e^{-3x}\ ;\{xe^x,\ x^2e^x,\ x^3e^x\}$
19. $x(x^2-2)y'''+(x^2-6)y''+x(2-x^2)y'+(6-x^2)y=2(x^2-2)^2\ ;\{e^x,\ e^{-x},\ 1/x\}$
20. $xy'''-(x-3)y''-(x+2)y'+(x-1)y=-4e^{-x}\ ;\{e^x,\ e^x/x,\ e^{-x}/x\}$
21. $x(x^2-1)y'''+(5x^2+1)y''+2xy'-2y=12x^2\ ;\{x,\ 1/(x-1),\ 1/(x+1)\}$
22. $x(1-x)y'''+(x^2-3x+3)y''+xy'-y=2(x-1)^2\ ;\{x,\ 1/x,e^x/x\}$
23. $xy'''-y''-xy'+y=x^2\ ;\{x,\ e^x,\ e^{-x}\}$
24. $x(x^2-6)y^{(4)}+2(x^2-12)y'''+x(6-x^2)y''+2(12-x^2)y'=2(x^2-6)^2\ ;\{1,\ 1/x,\ e^x,\ e^{-x}\}$
25. $x^4y^{(4)}-4x^3y'''+2x^2(6-x^2)y''+4x(x^2-6)y'+(x^4-4x^2+24)y=4x^5e^x\ ;\{xe^x,\ x^2e^x,\ xe^{-x},\ x^2e^{-x}\}$
26. $xy^{(4)}+4y'''-2xy''-4y'+xy=4e^x\ ;\{e^x,\ e^{-x},\ e^x/x,\ e^{-x}/x\}$
27. $xy^{(4)}+(4-6x)y'''+(13x-18)y''+(26-12x)y'+(4x-12)y=3e^x\ ;\{e^x,\ e^{2x},\ e^x/x,\ e^{2x}/x\}$
28. $x^4y^{(4)}-4x^3y'''+x^2(12-x^2)y''+2x(x^2-12)y'+2(12-x^2)y=2x^5\ ;\{x,\ x^2,\ xe^x,\ xe^{-x}\}$

In Exercises **29-33** solve the initial value problem, given the fundamental set of solutions of the homogeneous equation.

29. $(5-6x)y'''+(12x-4)y''+(6x-23)y'+(22-12x)y=-(6x-5)^2e^x\quad \{y(0)=-4,\quad y'(0)=-\frac{3}{2},$
$y''(0)=-19;\{e^x,\ e^{2x},\ xe^{-x}\}$
30. $(x^2-2x+2)y'''-x^2y''+2xy'-2y=(x^2-2x+2)^2,\quad y(0)=0,\quad y'(0)=5\quad ,y''(0)=0;\{x,\ x^2,\ e^x\}$

31. $(3x-1)y'''-(12x-1)y''+9(x+1)y'-9y=2e^x(3x-1)^2, \quad y(0)=\frac{3}{4}, \quad y'(0)=\frac{5}{4},$
$y''(0)=\frac{1}{4}; \{x+1,\ e^x,\ e^{3x}\}$
32. $(x^2-2)y'''-2xy''+(2-x^2)y'+2xy=2(x^2-2)^2, \quad y(0)=1, \quad y'(0)=-5 \quad ,y''(0)=5; \{x^2,\ e^x,\ e^{-x}\}$
33. $(2x-1)y^{(4)}-4xy'''+(5-2x)y''+4xy'-4y=6(2x-1)^2, \quad y(0)=\frac{55}{4}, \quad y'(0)=0, \quad y''(0)=13,$
$y'''(0)=1; \{x,\ e^x,\ e^{-x},\ e^{2x}\}$

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CHAPTER OVERVIEW

8: Series Solutions of Linear Second Order Equations

In this Chapter, we study a class of second order differential equations that occur in many applications, but cannot be solved in closed form in terms of elementary functions.

8.1: Prelude to Series Solutions of Linear Second Order Equations

8.2: Review of Power Series

8.2.1: Review of Power Series (Exercises)

8.3: Series Solutions About an Ordinary Point

8.3.1: Series Solutions About an Ordinary Point (Exercises)

8.4: Series Solutions About a Regular Singular Point

8.4.1: Series Solutions About a Regular Singular Point (Exercises)

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8.1: Prelude to Series Solutions of Linear Second Order Equations

In this Chapter, we study a class of second order differential equations that occur in many applications, but cannot be solved in closed form in terms of elementary functions. Here are some examples:

1. **Bessel's equation** which occurs in problems displaying cylindrical symmetry, such as diffraction of light through a circular aperture, propagation of electromagnetic radiation through a coaxial cable, and vibrations of a circular drum head.

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

2. **Airy's equation** which occurs in astronomy and quantum physics.

$$y'' - xy = 0,$$

3. **Legendre's equation** which occurs in problems displaying spherical symmetry, particularly in electromagnetism.

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0,$$

These equations and others considered in this chapter can be written in the form

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x) \tag{8.1.1}$$

where P_0, P_1, and P_2 are functions with no common factor. For most equations that occur in applications, these are polynomials of degree two or less, although the methods that we'll develop can be extended to the case where the coefficient functions are polynomials of arbitrary degree, or even power series that converge in some circle around the origin in the complex plane.

Since Equation 8.1.1 does not in general have closed form solutions, we seek series representations for solutions. We'll see that if $P_0(0) \neq 0$ then solutions of 8.1.1 can be written as power series

$$y = \sum_{n=0}^{\infty} a_n x^n$$

that converge in an open interval centered at $x = 0$.

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8.2: Review of Power Series

Many applications give rise to differential equations with solutions that can't be expressed in terms of elementary functions such as polynomials, rational functions, exponential and logarithmic functions, and trigonometric and inverse trigonometric functions. The solutions of some of the most important of these equations can be expressed in terms of power series. In this section we review relevant properties of power series. We'll omit proofs, which can be found in any standard calculus text.

Definition 8.2.1

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n, \tag{8.2.1}$$

where x_0 and a_0, a_1, ..., a_n, ...are constants, is called a *power series in* $x-x_0$. We say that the power series Equation 8.2.1 *converges* for a given x if the limit

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n(x-x_0)^n$$

exists; otherwise, we say that the power series *diverges* for the given x.

A power series in $x-x_0$ must converge if $x=x_0$, since the positive powers of $x-x_0$ are all zero in this case. This may be the only value of x for which the power series converges. However, the next theorem shows that if the power series converges for some $x\neq x_0$ then the set of all values of x for which it converges forms an interval.

Theorem 8.2.1

For any power series

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n,$$

exactly one of these statements is true:

I. The power series converges only for $x=x_0$.
II. The power series converges for all values of x.
III. There's a positive number R such that the power series converges if $|x-x_0|<R$ and diverges if $|x-x_0|>R$.

In case (iii) we say that R is the *radius of convergence* of the power series. For convenience, we include the other two cases in this definition by defining $R=0$ in case (i) and $R=\infty$ in case (ii). We define the *open interval of convergence* of $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ to be

$$(x_0-R,x_0+R) \quad \text{if} \quad 0<R<\infty, \quad \text{or} \quad (-\infty,\infty) \quad \text{if} \quad R=\infty.$$

If R is finite, no general statement can be made concerning convergence at the endpoints $x=x_0\pm R$ of the open interval of convergence; the series may converge at one or both points, or diverge at both.

As you'll recall from calculus, the next theorem provides a useful method for determining the radius of convergence of a power series.

Theorem 8.2.2 Ratio Test

Consider the series $\sum_{n=0}^{\infty} c_n$ and suppose:

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = L,$$

I. If $L<1$, then $\sum_{n=0}^{\infty} c_n$ converges.

II. If $L > 1$ or the limit approaches ∞, then $\sum_{n=0}^{\infty} c_n$ diverges.

III. If $L = 1$, then the ratio test gives no information and we have to consider another test.

Example 8.2.1

Find the radius of convergence of the series:

a. $$\sum_{n=0}^{\infty} n! x^n$$

b. $$\sum_{n=10}^{\infty} (-1)^n \frac{x^n}{n!}$$

c. $$\sum_{n=0}^{\infty} 2^n n^2 (x-1)^n.$$

Solution a

Here $c_n = n! x^n$, so

$$\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)! x^{n+1}}{n! x^n} \right| = \lim_{n\to\infty} (n+1)|x|.$$

This limit will approach ∞ for all x except $x = 0$, where the limit would be 0, so the series only converges at the point $x = 0$.

Hence, $R = 0$.

Solution b

Here $c_n = \frac{(-1)^n x^n}{n!}$, so

$$\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = \lim_{n\to\infty} \left| \frac{(n)! x^{n+1}}{(n+1)! x^n} \right| = |x| \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$

for all x. Hence, $R = \infty$.

Solution c

Here $c_n = 2^n n^2 (x-1)^n$, so

$$\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = \lim_{n\to\infty} \left| \frac{2^{n+1}(n+1)^2 (x-1)^{n+1}}{2^n n^2 (x-1)^n} \right| = 2|x-1| \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = 2|x-1|$$

which converges when $2|x-1| < 1$ which is true when $|x-1| < \frac{1}{2}$.

Hence, $R = 1/2$.

Taylor Series

Definition 8.2.2

If a function f has derivatives of all orders at a point $x = x_0$, then the *Taylor series of f about x_0* is defined by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n.$$

In the special case where $x_0 = 0$, this series is also called the *Maclaurin series of f*:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Taylor series for most of the common elementary functions converge to the functions on their open intervals of convergence. For example, you should be familiar with the following Maclaurin series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty \tag{8.2.2}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty \tag{8.2.3}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad -\infty < x < \infty \tag{8.2.4}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad -1 < x < 1 \tag{8.2.5}$$

Differentiation of Power Series

A power series with a positive radius of convergence defines a function

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

on its open interval of convergence. We say that the series *represents* f on the open interval of convergence. A function f represented by a power series may be a familiar elementary function as in Equations 8.2.2 - 8.2.5; however, it often happens that f isn't a familiar function, so the series actually *defines* f.

The next theorem (again, from calculus) shows that a function represented by a power series has derivatives of all orders on the open interval of convergence of the power series, and provides power series representations of the derivatives.

Theorem 8.2.3

A power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

with positive radius of convergence R has derivatives of all orders in its open interval of convergence, and successive derivatives can be obtained by repeatedly differentiating term by term; that is,

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}, \tag{8.2.6}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}, \tag{8.2.7}$$

$$\vdots$$

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n (x - x_0)^{n-k}. \tag{8.2.8}$$

Moreover, all of these series have the same radius of convergence R.

Example 8.2.2

Let $f(x) = \sin x$. From Equation 8.2.3,

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

From Equation 8.2.6,

$$f'(x) = \sum_{n=0}^{\infty} (-1)^n \frac{d}{dx}\left[\frac{x^{2n+1}}{(2n+1)!}\right] = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!},$$

which is the series Equation 8.2.4 for $\cos x$.

Uniqueness of Power Series

The next theorem shows that if f is *defined* by a power series in $x - x_0$ with a positive radius of convergence, then the power series is the Taylor series of f about x_0.

Theorem 8.2.4

If the power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

has a positive radius of convergence, then

$$a_n = \frac{f^{(n)}(x_0)}{n!}; \tag{8.2.9}$$

that is, $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is the Taylor series of f about x_0.

This result can be obtained by setting $x = x_0$ in Equation 8.2.8, which yields

$$f^{(k)}(x_0) = k(k-1)\cdots 1 \cdot a_k = k! a_k.$$

This implies that

$$a_k = \frac{f^{(k)}(x_0)}{k!}.$$

Except for notation, this is the same as Equation 8.2.9.

Shifting the Summation Index in a Power Series

For any integer k, the power series

$$\sum_{n=n_0}^{\infty} b_n (x - x_0)^{n-k}$$

can be rewritten as

$$\sum_{n=n_0-k}^{\infty} b_{n+k} (x - x_0)^n$$

that is, replacing n by $n + k$ in the general term and subtracting k from the lower limit of summation leaves the series unchanged.

Example 8.2.3

Rewrite the following power series from Equation 8.2.7 and Equation 8.2.8 so that the general term in each is a constant multiple of $(x - x_0)^n$:

$$(a) \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2} \quad (b) \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n (x - x_0)^{n-k}.$$

Solution a

Replacing n by $n+2$ in the general term and subtracting 2 from the lower limit of summation yields

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-x_0)^n.$$

Solution b

Replacing n by $n+k$ in the general term and subtracting k from the lower limit of summation yields

$$\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(x-x_0)^{n-k} = \sum_{n=0}^{\infty}(n+k)(n+k-1)\cdots(n+1)a_{n+k}(x-x_0)^n.$$

Example 8.2.4

Given that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

write the function xf'' as a power series in which the general term is a constant multiple of x^n.

Solution

From Theorem 8.2.4 with $x_0 = 0$,

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Therefore

$$xf''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1}.$$

Replacing n by $n+1$ in the general term and subtracting 1 from the lower limit of summation yields

$$xf''(x) = \sum_{n=1}^{\infty}(n+1)na_{n+1}x^n.$$

Linear Combinations of Power Series

If a power series is multiplied by a constant, then the constant can be placed inside the summation; that is,

$$c\sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{n=0}^{\infty} ca_n(x-x_0)^n.$$

Two power series

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} b_n(x-x_0)^n$$

with positive radii of convergence can be added term by term at points common to their open intervals of convergence; thus, if the first series converges for $|x-x_0| < R_1$ and the second converges for $|x-x_0| < R_2$, then

$$f(x)+g(x) = \sum_{n=0}^{\infty}(a_n+b_n)(x-x_0)^n$$

for $|x-x_0| < R$, where R is the smaller of R_1 and R_2. More generally, linear combinations of power series can be formed term by term; for example,

$$c_1 f(x) + c_2 g(x) = \sum_{n=0}^{\infty} (c_1 a_n + c_2 b_n)(x - x_0)^n.$$

Example 8.2.5

Find the Maclaurin series for $\cosh x$ as a linear combination of the Maclaurin series for e^x and e^{-x}.

Solution

By definition,

$$\cosh x = \frac{1}{2} e^x + \frac{1}{2} e^{-x}.$$

Since

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{and} \quad e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!},$$

it follows that

$$\cosh x = \sum_{n=0}^{\infty} \frac{1}{2}[1 + (-1)^n] \frac{x^n}{n!}. \tag{8.2.10}$$

Since

$$\frac{1}{2}[1 + (-1)^n] = \begin{cases} 1 & \text{if } n = 2m, \text{ an even integer,} \\ 0 & \text{if } n = 2m+1, \text{ an odd integer,} \end{cases}$$

we can rewrite Equation 8.2.10 more simply as

$$\cosh x = \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!}.$$

This result is valid on $(-\infty, \infty)$, since this is the open interval of convergence of the Maclaurin series for both e^x and e^{-x}.

Example 8.2.6

Suppose

$$y = \sum_{n=0}^{\infty} a_n x^n$$

on an open interval I that contains the origin.

a. Express

$$(2 - x)y'' + 2y$$

as a power series in x on I.

b. Use the result of (a) to find necessary and sufficient conditions on the coefficients $\{a_n\}$ for y to be a solution of the homogeneous equation

$$(2 - x)y'' + 2y = 0 \tag{8.2.11}$$

on I.

Solution a

From Equation 8.2.7 with $x_0 = 0$,

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Therefore

$$(2-x)y''+2y=2y''-xy''+2y=\sum_{n=2}^{\infty}2n(n-1)a_nx^{n-2}-\sum_{n=2}^{\infty}n(n-1)a_nx^{n-1}+\sum_{n=0}^{\infty}2a_nx^n.$$

Since series are just polynomials, adding them is just like adding polynomials; that is, we add like terms. To add series we want to "run out" terms until each series starts at the same power of x.

So, we get

$$\sum_{n=2}^{\infty}2n(n-1)a_nx^{n-2}-\sum_{n=2}^{\infty}n(n-1)a_nx^{n-1}+\sum_{n=0}^{\infty}2a_nx^n$$

$$=4a_2+\sum_{n=3}^{\infty}2n(n-1)a_nx^{n-2}-\sum_{n=2}^{\infty}n(n-1)a_nx^{n-1}+2a_0+\sum_{n=1}^{\infty}2a_nx^n$$

To combine the three series we need to shift indices so all series start at the same value of n; it doesn't matter where we chose to start the series. In this case let's choose a start of $n=2$. So, we will need to reindex the first and third series; thus,

$$\sum_{n=3}^{\infty}2n(n-1)a_nx^{n-2}=\sum_{n=2}^{\infty}2(n+1)na_{n+1}x^{n-1}$$

and

$$\sum_{n=1}^{\infty}2a_nx^n=\sum_{n=2}^{\infty}2a_{n-1}x^{n-1}$$

Now all three series will start with $n=2$; thus,

$$(2-x)y''+2y=(4a_2+2a_0)+\sum_{n=2}^{\infty}[2(n+1)na_{n+1}-n(n-1)a_n+2a_{n-1}]x^{n-1}. \tag{8.2.12}$$

Solution b

From Equation 8.2.12 we see that y satisfies Equation 8.2.11 on I if

$$4a_2+2a_0=0$$

and

$$2(n+1)na_{n+1}-n(n-1)a_n+2a_{n-1}=0,\quad n=2,3,4,\dots.$$

Example 8.2.7

Suppose

$$y=\sum_{n=0}^{\infty}a_n(x-1)^n$$

on an open interval I that contains $x_0=1$. Express the function

$$(1+x)y''+2(x-1)^2y'+3y \tag{8.2.13}$$

as a power series in $x-1$ on I.

Solution

Since we want a power series in $x-1$, we rewrite the coefficient of y'' in Equation 8.2.13 as $1+x=2+(x-1)$, so Equation 8.2.13 becomes

$$2y''+(x-1)y''+2(x-1)^2y'+3y.$$

From Equation 8.2.6 and Equation 8.2.7 with $x_0 = 1$,

$$y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}.$$

Therefore

$$\begin{aligned} 2y'' &= \sum_{n=2}^{\infty} 2n(n-1) a_n (x-1)^{n-2}, \\ (x-1)y'' &= \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-1}, \\ 2(x-1)^2 y' &= \sum_{n=1}^{\infty} 2n a_n (x-1)^{n+1}, \\ 3y &= \sum_{n=0}^{\infty} 3 a_n (x-1)^n. \end{aligned}$$

Running out terms so all series start at the same power of $x-1$ gives us

$$\begin{aligned} 2y'' &= \sum_{n=2}^{\infty} 2n(n-1) a_n (x-1)^{n-2} = 4a_2 + 12a_3(x-1) + \sum_{n=4}^{\infty} 2n(n-1) a_n (x-1)^{n-2}, \\ (x-1)y'' &= \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-1} = 2a_2(x-1) + \sum_{n=3}^{\infty} n(n-1) a_n (x-1)^{n-1}, \\ 2(x-1)^2 y' &= \sum_{n=1}^{\infty} 2n a_n (x-1)^{n+1}, \\ 3y &= \sum_{n=0}^{\infty} 3 a_n (x-1)^n = 3a_0 + 3a_1(x-1) + \sum_{n=2}^{\infty} 3 a_n (x-1)^n. \end{aligned}$$

We now need to shift indices so all four series start at the same value of n; let's choose to start at $n = 1$, so we need to reindex the first, second, and fourth series as the third one already starts at $n = 1$. This yields

$$\begin{aligned} 2y'' &= \sum_{n=2}^{\infty} 2n(n-1) a_n (x-1)^{n-2} = 4a_2 + 12a_3(x-1) + \sum_{n=1}^{\infty} 2(n+3)(n+2) a_{n+3} (x-1)^{n+1}, \\ (x-1)y'' &= \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-1} = 2a_2(x-1) + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^{n+1}, \\ 2(x-1)^2 y' &= \sum_{n=1}^{\infty} 2n a_n (x-1)^{n+1}, \\ 3y &= \sum_{n=0}^{\infty} 3 a_n (x-1)^n = 3a_0 + 3a_1(x-1) + \sum_{n=1}^{\infty} 3 a_{n+1} (x-1)^{n+1}. \end{aligned}$$

We can now add these together, giving us

$$\begin{aligned} 4a_2 + 12a_3(x-1) + \sum_{n=1}^{\infty} 2(n+3)(n+2) a_{n+3} (x-1)^{n+1} + 2a_2(x-1) + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^{n+1} + \\ \sum_{n=1}^{\infty} 2n a_n (x-1)^{n+1} + 3a_0 + 3a_1(x-1) + \sum_{n=1}^{\infty} 3 a_{n+1} (x-1)^{n+1} \end{aligned}$$

and combining like terms gives us

$$\begin{aligned} (1+x)y'' + 2(x-1)^2 y' + 3y = 2y'' + (x-1)y'' + 2(x-1)^2 y' + 3y = (4a_2 + 3a_0) + (12a_3 + 2a_2 + 3a_1)(x-1) \\ + \sum_{n=1}^{\infty} [2(n+3)(n+2) a_{n+3} + (n+2)(n+1) a_{n+2} + 2n a_n + 3a_{n+1}] (x-1)^{n+1} \end{aligned}$$

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8.2.1: Review of Power Series (Exercises)

In Exercises **1-18** use Theorem 8.2.2 (the ratio test) to find the radius of convergence R. If $R>0$, find the open interval of convergence.

1. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n}(x-1)^n$
2. $\sum_{n=0}^{\infty} 2^n n(x-2)^n$
3. $\sum_{n=0}^{\infty} \frac{n!}{9^n}x^n$
4. $\sum_{n=0}^{\infty} \frac{n(n+1)}{16^n}(x-2)^n$
5. $\sum_{n=0}^{\infty} (-1)^n \frac{7^n}{n!}x^n$
6. $\sum_{n=0}^{\infty} \frac{3^n}{4^{n+1}(n+1)^2}(x+7)^n$
7. $\sum_{m=0}^{\infty} (-1)^m (3m+1)(x-1)^{2m+1}$
8. $\sum_{m=0}^{\infty} (-1)^m \frac{m(2m+1)}{2^m}(x+2)^{2m}$
9. $\sum_{m=0}^{\infty} \frac{m!}{(2m)!}(x-1)^{2m}$
10. $\sum_{m=0}^{\infty} (-1)^m \frac{m!}{9^m}(x+8)^{2m}$
11. $\sum_{m=0}^{\infty} (-1)^m \frac{(2m-1)}{3^m}x^{2m+1}$
12. $\sum_{m=0}^{\infty} (x-1)^{2m}$
13. $\sum_{m=0}^{\infty} \frac{(-1)^m}{(27)^m}(x-3)^{3m+2}$
14. $\sum_{m=0}^{\infty} \frac{x^{7m+6}}{m}$
15. $\sum_{m=0}^{\infty} \frac{9^m(m+1)}{(m+2)}(x-3)^{4m+2}$
16. $\sum_{m=0}^{\infty} (-1)^m \frac{2^m}{m!}x^{4m+3}$
17. $\sum_{m=0}^{\infty} \frac{m!}{(26)^m}(x+1)^{4m+3}$
18. $\sum_{m=0}^{\infty} \frac{(-1)^m}{8^m m(m+1)}(x-1)^{3m+1}$

In Exercises **19-23** express as a power series in x.

19. $(2+x)y''+xy'+3y$

20. $(1+3x^2)y''+3x^2y'-2y$

21. $(1+2x^2)y''+(2-3x)y'+4y$

22. $(1+x^2)y''+(2-x)y'+3y$

23. $(1+3x^2)y''-2xy'+4y$

24. Suppose $y(x)=\sum_{n=0}^{\infty}a_n(x+1)^n$ on an open interval that contains $x_0=-1$. Find a power series in $x+1$ for

$$xy''+(4+2x)y'+(2+x)y.$$

25. Suppose $y(x)=\sum_{n=0}^{\infty}a_n(x-2)^n$ on an open interval that contains $x_0=2$. Find a power series in $x-2$ for

$$x^2y''+2xy'-3xy.$$

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8.3: Series Solutions About an Ordinary Point

Many physical applications give rise to second order linear differential equations of the form

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x). \tag{8.3.1}$$

Often the solutions of these equations can't be expressed in terms of familiar elementary functions. Therefore we'll consider the problem of representing solutions of Equation 8.3.1 with series.

We'll divide 8.3.1 by $P_0(x)$ to get the form we'll use throughout the remainder of this chapter:

$$y'' + p(x)y' + q(x)y = f(x). \tag{8.3.2}$$

Definition 8.3.1

A function $g(x)$ is *analytic* at x_0 if it can be expressed as a **convergent** power series at x_0 (with a positive or infinite radius of convergence) that is infinitely differentiable:

$$g(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Homogeneous Equations

Definition 8.3.2

Let

$$y'' + p(x)y' + q(x)y = 0.$$

We say that x_0 is an *ordinary point* of Equation 8.3.2 if $p(x)$ and $q(x)$ are both analytic at x_0. Otherwise, x_0 is a *singular point*.

Note: Since polynomials are continuous everywhere, if 8.3.1 has polynomial coefficients, then $p(x) = \frac{P_1(x)}{P_0(x)}$, $q(x) = \frac{P_2(x)}{P_0(x)}$, and $f(x) = \frac{F(x)}{P_0(x)}$ from 8.3.2 are analytic at any point x_0 that isn't a zero of $P_0(x)$. Therefore, if the polynomial $P_0(x_0) \neq 0$, then x_0 is an ordinary point of Equation 8.3.2 when $P_0(x)$, $P_1(x)$, and $P_2(x)$ are all polynomials.

For Legendre's equation,

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0,$$

$x_0 = 1$ and $x_0 = -1$ are singular points and all other points are ordinary points. For Bessel's equation,

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0,$$

$x_0 = 0$ is a singular point and all other points are ordinary points. If P_0 is a nonzero constant as in Airy's equation,

$$y'' - xy = 0,$$

then every point is an ordinary point.

Since polynomials are continuous everywhere, P_1/P_0 and P_2/P_0 are continuous at any point x_0 that isn't a zero of P_0. Therefore, if x_0 is an ordinary point of Equation 8.3.1 and a_0 and a_1 are arbitrary real numbers, then the initial value problem

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \quad y(x_0) = a_0, \quad y'(x_0) = a_1 \tag{8.3.3}$$

has a unique solution on the largest open interval that contains x_0 and does not contain any zeros of P_0. To see this, we rewrite the differential equation in Equation 8.3.3 as

$$y''+\frac{P_1(x)}{P_0(x)}y'+\frac{P_2(x)}{P_0(x)}y=0$$

and apply Theorem 5.1.1 with $p=P_1/P_0$ and $q=P_2/P_0$.

In the remainder of this section we consider the problem of representing solutions of Equation 8.3.1 by power series that converge for values of x near an ordinary point x_0.

We state the next theorem without proof.

Theorem 8.3.1

Let

$$y''+p(x)y'+q(x)y=0 \tag{8.3.4}$$

Let x_0 be an ordinary point of 8.3.4 and ρ be the distance from x_0 to the nearest singular point of 8.3.4. Then every solution of 8.3.4

can be represented by a power series

$$y=\sum_{n=0}^{\infty}a_n(x-x_0)^n \tag{8.3.5}$$

that converges at least on the open interval $(x_0-\rho,x_0+\rho)$.

Furthermore, 8.3.5 will generate the two linearly independent solutions of 8.3.4.

We call Equation 8.3.5 a *power series solution in* $x-x_0$ of Equation 8.3.4. We'll now develop a method for finding power series solutions of Equation 8.3.4.

Theorem 8.3.1 implies that every solution of

$$y''+p(x)y'+q(x)y=0 \tag{8.3.6}$$

on $(x_0-\rho,x_0+\rho)$ can be written as

$$y=\sum_{n=0}^{\infty}a_n(x-x_0)^n=a_0+a_1(x-x_0)+a_2(x-x_0)^2+\cdots.$$

Setting $x=x_0$ in this series and in the series

$$y'=\sum_{n=1}^{\infty}na_n(x-x_0)^{n-1}=a_1+2a_2(x-x_0)+3a_3(x-x_0)^2+\cdots$$

shows that $y(x_0)=a_0$ and $y'(x_0)=a_1$. Since every initial value problem Equation 8.3.3 has a unique solution, this means that a_0 and a_1 can be chosen arbitrarily, and a_2, a_3, …are uniquely determined by them.

To find $a_2,a_3,a_4,\cdots$, we write P_0, P_1, and P_2 in powers of $x-x_0$, substitute

$$\begin{aligned}y&=\sum_{n=0}^{\infty}a_n(x-x_0)^n,\\ y'&=\sum_{n=1}^{\infty}na_n(x-x_0)^{n-1},\\ y''&=\sum_{n=2}^{\infty}n(n-1)a_n(x-x_0)^{n-2}\end{aligned}$$

into Equation 8.3.6, and collect the coefficients of like powers of $x-x_0$. This yields

$$\sum_{n=0}^{\infty}b_n(x-x_0)^n=0, \tag{8.3.7}$$

where $\{b_0, b_1, \ldots, b_n, \ldots\}$ are expressed in terms of $\{a_0, a_1, \ldots, a_n, \ldots\}$ and the coefficients of P_0, P_1, and P_2, written in powers of $x - x_0$. Equation 8.3.7 is satisfied if and only if $b_n = 0$ for $n \geq 0$, and all power series solutions in $x - x_0$ of 8.3.6 can be obtained by choosing a_0 and a_1 arbitrarily and computing a_2, a_3, …, successively so that $b_n = 0$ for $n \geq 0$. For simplicity, we call the power series obtained this way *the power series in* $x - x_0$ for the general solution of 8.3.6.

Example 8.3.1

Let x_0 be an arbitrary real number. Find the power series in $x - x_0$ for the general solution of

$$y'' + y = 0. \tag{8.3.8}$$

Solution

Here

$$y'' + y = 0.$$

Since x_0 is an ordinary point (for all x_0) of 8.3.8, $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$, will generate two linearly independent solutions to 8.3.8.

If

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

then

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2},$$

so

$$\sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2} + \sum_{n=0}^{\infty} a_n (x - x_0)^n = 0.$$

To collect coefficients of like powers of $x - x_0$, we shift the summation index in the first sum. This yields

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x - x_0)^n + \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} + a_n](x - x_0)^n = 0,$$

Therefore

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}, \quad n \geq 0, \tag{8.3.9}$$

where a_0 and a_1 are arbitrary.

Computing the coefficients of $x - x_0$ from Equation 8.3.9 yields

$$\begin{aligned}
a_2 &= -\frac{a_0}{2 \cdot 1}, \\
a_3 &= -\frac{a_1}{3 \cdot 2}, \\
a_4 &= -\frac{a_2}{4 \cdot 3} = -\frac{1}{4 \cdot 3}\left(-\frac{a_0}{2 \cdot 1}\right) = \frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1}, \\
a_5 &= -\frac{a_3}{5 \cdot 4} = -\frac{1}{5 \cdot 4}\left(-\frac{a_1}{3 \cdot 2}\right) = \frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2}, \cdots
\end{aligned}$$

Thus, the general solution of Equation 8.3.8 can be written as

$$\begin{aligned} y &= a_0 + a_1(x-x_0) - \frac{a_0}{2\cdot 1}(x-x_0)^2 - \frac{a_1}{3\cdot 2}(x-x_0)^3 + \frac{a_0}{4\cdot 3\cdot 2\cdot 1}(x-x_0)^4 + \frac{a_1}{5\cdot 4\cdot 3\cdot 2}(x-x_0)^5 + \cdots \\ &= a_0\left(1 - \frac{(x-x_0)^2}{2\cdot 1} + \frac{(x-x_0)^4}{4\cdot 3\cdot 2\cdot 1} + \cdots\right) + a_1\left((x-x_0) - \frac{(x-x_0)^3}{3\cdot 2} + \frac{(x-x_0)^5}{5\cdot 4\cdot 3\cdot 2} + \cdots\right). \end{aligned}$$

Writing these series in closed form we get

$$y = a_0 \sum_{m=0}^{\infty} (-1)^m \frac{(x-x_0)^{2m}}{(2m)!} + a_1 \sum_{m=0}^{\infty} (-1)^m \frac{(x-x_0)^{2m+1}}{(2m+1)!}. \tag{8.3.10}$$

If we recall from calculus that

$$\sum_{m=0}^{\infty} (-1)^m \frac{(x-x_0)^{2m}}{(2m)!} = \cos(x-x_0) \quad \text{and} \quad \sum_{m=0}^{\infty} (-1)^m \frac{(x-x_0)^{2m+1}}{(2m+1)!} = \sin(x-x_0),$$

then Equation 8.3.10 becomes

$$y = a_0 \cos(x-x_0) + a_1 \sin(x-x_0),$$

Equations like Equation 8.3.9, which define a given coefficient in the sequence $\{a_n\}$ in terms of one or more coefficients with lesser indices are called *recurrence relations*. When we use a recurrence relation to compute terms of a sequence we are computing *recursively*.

In the example above we were able to obtain a closed formula for coefficients in the power series solution. In many cases this is very difficult, or impossible, and we must settle for computing a finite number of terms in the series, as the following example illustrates.

Example 8.3.2

Find the power series in x for the general solution of

$$(1+2x^2)y'' + 6xy' + 2y = 0. \tag{8.3.11}$$

Solution

Here

$$(1+2x^2)y'' + 6xy' + 2y = y'' + 2x^2y'' + 6xy' + 2y = 0.$$

Since 0 is an ordinary point of 8.3.11, $y = \sum_{n=0}^{\infty} a_n x^n$, will generate two linearly independent solutions to 8.3.11.

If

$$y = \sum_{n=0}^{\infty} a_n x^n$$

then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2},$$

so, we get

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 2x^2 \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 6x \sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} +$$
$$\sum_{n=2}^{\infty} 2n(n-1)a_n x^n + \sum_{n=1}^{\infty} 6n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0.$$

Running out terms we get

$$2a_2+6a_3x+\sum_{n=4}^{\infty}n(n-1)a_nx^{n-2}+\sum_{n=2}^{\infty}2n(n-1)a_nx^n+6a_1x+\sum_{n=2}^{\infty}6na_nx^n+2a_0+2a_1x+\sum_{n=2}^{\infty}2a_nx^n=0.$$

Reindexing we get

$$2a_2+6a_3x+\sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=2}^{\infty}2n(n-1)a_nx^n+6a_1x+\sum_{n=2}^{\infty}6na_nx^n+2a_0+2a_1x+\sum_{n=2}^{\infty}2a_nx^n=0.$$

We now collect like terms to obtain

$$(2a_2+2a_0)+(6a_3+8a_1)x+\sum_{n=2}^{\infty}[(n+2)(n+1)a_{n+2}+(2n^2+4n+2)a_n]x^n=0$$

Equating coefficients we get

$$\begin{aligned}a_2&=-a_0,\\ a_3&=-\frac{4}{3}a_1,\\ a_{n+2}&=-\frac{2(n+1)^2}{(n+2)(n+1)}a_n=-\frac{2(n+1)}{n+2}a_n,\quad n=2,3,4,\cdots\end{aligned}$$

Now substituting $n=2,3,4,\cdots$ into the last equation yields

$$\begin{aligned}a_4&=-\frac{3}{2}a_2=\frac{3}{2}a_0,\\ a_5&=-\frac{8}{5}a_3=\frac{32}{15}a_1,\\ a_6&=-\frac{5}{3}a_4=-\frac{5}{2}a_0,\\ a_7&=-\frac{12}{7}a_5=-\frac{128}{35}a_1,\cdots\end{aligned}$$

Thus,

$$\begin{aligned}y&=a_0+a_1x-a_0x^2-\frac{4}{3}a_1x^3+\frac{3}{2}a_0x^4+\frac{32}{15}a_1x^5-\frac{5}{2}a_0x^6-\frac{128}{35}a_1x^7+\cdots\\ &=a_0(1-x^2+\frac{3}{2}x^4-\frac{5}{2}x^6+\cdots)+a_1(x-\frac{4}{3}x^3+\frac{32}{15}x^5-\frac{128}{35}x^7+\cdots)\end{aligned}$$

is the power series in x for the general solution of Equation 8.3.11. Since $P_0(x)=1+2x^2$ has no real zeros, Theorem 5.1.1 implies that every solution of Equation 8.3.11 is defined on $(-\infty,\infty)$.

Example 8.3.3

Find the power series in $x-1$ for the general solution of

$$(2+4x-2x^2)y''-12(x-1)y'-12y=0. \tag{8.3.12}$$

Solution

We must first write the coefficient $P_0(x)=2+4x-x^2$ in powers of $x-1$. To do this, we write $x=(x-1)+1$ in $P_0(x)$ and then expand the terms, collecting powers of $x-1$; thus,

$$\begin{aligned}2+4x-2x^2&=2+4[(x-1)+1]-2[(x-1)+1]^2\\ &=4-2(x-1)^2.\end{aligned}$$

Therefore we can rewrite Equation 8.3.12 as

$$\left(4-2(x-1)^2\right)y''-12(x-1)y'-12y=0,$$

Since $x=1$ is an ordinary point, we will generate two linearly independent solutions to 8.3.12 from

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n(x-1)^n, \\ y' &= \sum_{n=1}^{\infty} n a_n(x-1)^{n-1}, \\ y'' &= \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} \end{aligned}$$

So, we get

$$\begin{aligned} y \quad &= 4\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - 2(x-1)^2\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - 12(x-1)\sum_{n=1}^{\infty} n a_n(x-1)^{n-1} - 12 \\ &\qquad \sum_{n=0}^{\infty} a_n(x-1)^n \\ &= \sum_{n=2}^{\infty} 4n(n-1)a_n(x-1)^{n-2} - \sum_{n=2}^{\infty} 2n(n-1)a_n(x-1)^n - \sum_{n=1}^{\infty} 12n a_n(x-1)^n - \sum_{n=0}^{\infty} 12a_n(x-1)^n \end{aligned}$$

Running out terms we get

$8a_2+24a_3(x-1)+\sum_{n=4}^{\infty} 4n(n-1)a_n(x-1)^{n-2}-\sum_{n=2}^{\infty} 2n(n-1)a_n(x-1)^n-12a_1(x-1)-$
$\sum_{n=2}^{\infty} 12na_n(x-1)^n-12a_0-12a_1(x-1)-\sum_{n=2}^{\infty} 12a_n(x-1)^n=0.$

Reindexing we get

$8a_2+24a_3(x-1)+\sum_{n=2}^{\infty} 4(n+2)(n+1)a_{n+2}(x-1)^n-\sum_{n=2}^{\infty} 2n(n-1)a_n(x-1)^n-12a_1(x-1)-$
$\sum_{n=2}^{\infty} 12na_n(x-1)^n-12a_0-12a_1(x-1)-\sum_{n=2}^{\infty} 12a_n(x-1)^n=0.$

We now collect like terms to obtain

$$(8a_2-12a_0)+(24a_3-24a_1)(x-1)+\sum_{n=2}^{\infty}[4(n+2)(n+1)a_{n+2}-(2n^2+10n+12)a_n](x-1)^n=0$$

Equating coefficients we get

$$\begin{aligned} a_2 &= \frac{3}{2}a_0, \\ a_3 &= a_1, \\ a_{n+2} &= \frac{2(n+2)(n+3)}{4(n+2)(n+1)}a_n = \frac{n+3}{2(n+1)}a_n, \quad n=2,3,4,\cdots \end{aligned}$$

Now substituting $n=2,3,4,\cdots$ into the last equation yields

$$\begin{aligned} a_4 &= \frac{5}{6}a_2 = \frac{5}{4}a_0, \\ a_5 &= \frac{3}{4}a_3 = \frac{3}{4}a_1, \\ a_6 &= \frac{7}{10}a_4 = \frac{7}{8}a_0, \\ a_7 &= \frac{2}{3}a_5 = \frac{1}{2}a_1, \cdots \end{aligned}$$

Thus,

$$\begin{aligned} y &= a_0 + a_1(x-1) + \frac{3}{2}a_0(x-1)^2 + a_1(x-1)^3 + \frac{5}{4}a_0(x-1)^4 + \frac{3}{4}a_1(x-1)^5 + \frac{7}{8}a_0(x-1)^6 + \frac{1}{2}a_1(x-1)^7 + \\ &\quad \cdots \\ &= a_0\left(1 + \frac{3}{2}(x-1)^2 + \frac{5}{4}(x-1)^4 + \frac{7}{8}(x-1)^6 + \cdots\right) + a_1\left((x-1) + (x-1)^3 + \frac{3}{4}(x-1)^5 + \frac{1}{2}(x-1)^7 + \cdots\right) \end{aligned}$$

which implies that the power series in $x-1$ for the general solution of Equation 8.3.12 is

$$y = a_0 \sum_{m=0}^{\infty} \frac{2m+1}{2^m}(x-1)^{2m} + a_1 \sum_{m=0}^{\infty} \frac{m+1}{2^m}(x-1)^{2m+1}.$$

Since $P_0(x) = 2 + 4x - 2x^2$ has real zeros at $1 \pm \sqrt{2}$, Theorem 5.1.1 implies that every solution of Equation 8.3.12 is defined on $(-\infty, 1-\sqrt{2})$, $(1-\sqrt{2}, 1+\sqrt{2})$, or $(1+\sqrt{2}, \infty)$.

The next example illustrates an initial value problem.

Example 8.3.4

Find the power series in x for the general solution of

$$(1+2x^2)y'' + 6xy' + 2y = 0, \quad y(0) = 2, \quad y'(0) = -3. \tag{8.3.13}$$

Solution

From Example 6 we know the solution to the differential equation is

$$y = a_0\left(1 - x^2 + \frac{3}{2}x^4 - \frac{5}{2}x^6 + \cdots\right) + a_1\left(x - \frac{4}{3}x^3 + \frac{32}{15}x^5 - \frac{128}{35}x^7 + \cdots\right)$$

The condition $y(0) = 2$ yields $a_0 = 2$.

Differentiating our solution gives us

$$y' = a_0\left(-2x + 6x^3 - 15x^5 + \cdots\right) + a_1\left(1 - 4x^2 + \frac{32}{3}x^4 - \frac{128}{5}x^6 + \cdots\right)$$

The condition $y'(0) = -3$ yields $a_1 = -3$.

So the solution to 8.3.13 is

$$y = 2\left(1 - x^2 + \frac{3}{2}x^4 - \frac{5}{2}x^6 + \cdots\right) - 3\left(x - \frac{4}{3}x^3 + \frac{32}{15}x^5 - \frac{128}{35}x^7 + \cdots\right)$$

Nonhomogeneous Equations

Definition: 8.3.3

Let

$$y'' + p(x)y' + q(x)y = f(x).$$

We say that x_0 is an *ordinary point* of Equation 8.3.2 if $p(x)$, $q(x)$, and $f(x)$ are all analytic at x_0. Otherwise, x_0 is a *singular point*.

Theorem 8.3.2

Let

$$y'' + p(x)y' + q(x)y = f(x) \tag{8.3.14}$$

Let x_0 be an ordinary point of 8.3.14 and ρ be the distance from x_0 to the nearest singular point of 8.3.14. Then every solution of 8.3.14

can be represented by a power series

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{8.3.15}$$

that converges at least on the open interval $(x_0 - \rho, x_0 + \rho)$.

Furthermore, 8.3.15 will generate the two linearly independent solutions of the homogeneous part of 8.3.14 AND a particular solution to the nonhomogeneous part of 8.3.14.

Example 8.3.5

Find the power series in x for the general solution of

$$y'' + xy' + y = \frac{1}{1-x} \tag{8.3.16}$$

Solution

The series for $\frac{1}{1-x}$ is given by

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

which converges on $(-1, 1)$.

Since $x = 0$ is an ordinary point, Theorem 2 says we we will generate both the homogeneous and the nonhomogeneous solutions of 8.3.16 from

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n (x-1)^n, \\ y' &= \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}, \\ y'' &= \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} \end{aligned}$$

so, we get

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} x^n.$$

Running out terms we get

$$2a_2 + \sum_{n=3}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + a_0 + \sum_{n=1}^{\infty} a_n x^n = 1 + \sum_{n=1}^{\infty} x^n.$$

Reindexing we get

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + a_0 + \sum_{n=1}^{\infty} a_n x^n = 1 + \sum_{n=1}^{\infty} x^n.$$

We now collect like terms to obtain

$$(2a_2 + a_0) + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} + (n+1) a_n] x^n = 1 + \sum_{n=1}^{\infty} x^n.$$

Equating coefficients we get

$$\begin{aligned} a_2 &= -\frac{1}{2}a_0 + \frac{1}{2}, \\ a_{n+2} &= \frac{-a_n}{n+2} + \frac{1}{(n+2)(n+1)}, \quad n = 1, 2, 3, \cdots \end{aligned}$$

Now substituting $n = 1, 2, 3, \cdots$ into the last equation yields

$$\begin{aligned} a_3 &= -\frac{1}{3}a_1 + \frac{1}{6}, \\ a_4 &= -\frac{1}{4}a_2 + \frac{1}{12} = \frac{1}{8}a_0 - \frac{1}{24}, \\ a_5 &= -\frac{1}{5}a_3 + \frac{1}{20} = \frac{1}{15}a_1 + \frac{1}{60}, \cdots \end{aligned}$$

Thus,

$$\begin{aligned} y &= a_0 + a_1 x - (\frac{1}{2}a_0 - \frac{1}{2})x^2 - (\frac{1}{3}a_1 - \frac{1}{6})x^3 + (\frac{1}{8}a_0 - \frac{1}{24})x^4 + (\frac{1}{15}a_1 + \frac{1}{60})x^5 + \cdots \\ &= a_0(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + \cdots) + a_1(x - \frac{1}{3}x^3 + \frac{1}{15}x^5 + \cdots) + (\frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 + \frac{1}{60}x^5 + \cdots) \end{aligned}$$

is the power series in x for the general solution of Equation 8.3.16 defined on $(-1, 1)$.

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8.3.1: Series Solutions About an Ordinary Point (Exercises)

In Exercises **1-11** find the series in x for the general solution.

1. $(1+x^2)y''+6xy'+6y=0$
2. $(1+x^2)y''+2xy'-2y=0$
3. $(1+x^2)y''-8xy'+20y=0$
4. $(1-x^2)y''-8xy'-12y=0$
5. $(1+x^2)y''-10xy'+28y=0$
6. $y''-xy=0$
7. $y''-2xy'+y=0$
8. $y''+x^2y'+xy=0$
9. $(x-1)y''+y'=0$
10. $y''-(x+1)y'-y=0$
11. $(x^2+2)y''+3xy'-y=0$

In Exercises **12-23** find the power series solution $y=\sum_{n=0}^{\infty}a_nx^n$ of the initial value problem.

12. $(1+x^2)y''+xy'+y=0, \quad y(0)=2, \quad y'(0)=-1$
13. $(1+2x^2)y''-9xy'-6y=0, \quad y(0)=1, \quad y'(0)=-1$
14. $(1+8x^2)y''+2y=0, \quad y(0)=2, \quad y'(0)=-1$
15. $(1+3x)y''+xy'+2y=0, \quad y(0)=2, \quad y'(0)=-3$
16. $(1+x+2x^2)y''+(2+8x)y'+4y=0, \quad y(0)=-1, \quad y'(0)=2$
17. $(1-2x^2)y''+(2-6x)y'-2y=0, \quad y(0)=1, \quad y'(0)=0$
18. $(1+x+3x^2)y''+(2+15x)y'+12y=0, \quad y(0)=0, \quad y'(0)=1$
19. $(2+x)y''+(1+x)y'+3y=0, \quad y(0)=4, \quad y'(0)=3$
20. $(3+3x+x^2)y''+(6+4x)y'+2y=0, \quad y(0)=7, \quad y'(0)=3$
21. $(4+x)y''+(2+x)y'+2y=0, \quad y(0)=2, \quad y'(0)=5$
22. $(x-1)y''-xy'+y=0, \quad y(0)=-2, \quad y'(0)=6$
23. $y''-2xy'+8y=0, \quad y(0)=3, \quad y'(0)=0$

In Exercises **24 and 25** find the power series in $x-x_0$ for the general solution.

24. $y''-y=0; \quad x_0=3$
25. $(11-8x+2x^2)y''-16(x-2)y'+36y=0; \quad x_0=2$

In Exercises **26-63** find the power series solution $y=\sum_{n=0}^{\infty}a_n(x-x_0)^n$ for the solution of the initial value problem. Take x_0 to be the point where the initial conditions are imposed.

26. $(x^2-4)y''-xy'-3y=0, \quad y(0)=-1, \quad y'(0)=2$
27. $y''+(x-3)y'+3y=0, \quad y(3)=-2, \quad y'(3)=3$
28. $(5-6x+3x^2)y''+(x-1)y'+12y=0, \quad y(1)=-1, \quad y'(1)=1$
29. $(4x^2-24x+37)y''+y=0, \quad y(3)=4, \quad y'(3)=-6$
30. $(x^2-8x+14)y''-8(x-4)y'+20y=0, \quad y(4)=3, \quad y'(4)=-4$
31. $(2x^2+4x+5)y''-20(x+1)y'+60y=0, \quad y(-1)=3, \quad y'(-1)=-3$
32. $(2-3x+2x^2)y''-(4-6x)y'+2y=0, \quad y(1)=1, \quad y'(1)=-1$

33. $(3x+2x^2)y''+10(1+x)y'+8y=0, \quad y(-1)=1, \quad y'(-1)=-1$

34. $(1-x+x^2)y''-(1-4x)y'+2y=0, \quad y(1)=2, \quad y'(1)=-1$

35. $(2+x)y''+(2+x)y'+y=0, \quad y(-1)=-2, \quad y'(-1)=3$

36. $x^2y''-(6-7x)y'+8y=0, \quad y(1)=1, \quad y'(1)=-2$

37. $(2+4x)y''-4y'-(6+4x)y=0, \quad y(0)=2, \quad y'(0)=-7$

38. $(1+2x)y''-(1-2x)y'-(3-2x)y=0, \quad y(1)=1, \quad y'(1)=-2$

39. $(5+2x)y''-y'+(5+x)y=0, \quad y(-2)=2, \quad y'(-2)=-1$

40. $(4+x)y''-(4+2x)y'+(6+x)y=0, \quad y(-3)=2, \quad y'(-3)=-2$

41. $(2+3x)y''-xy'+2xy=0, \quad y(0)=-1, \quad y'(0)=2$

42. $(3+2x)y''+3y'-xy=0, \quad y(-1)=2, \quad y'(-1)=-3$

43. $(3+2x)y''-3y'-(2+x)y=0, \quad y(-2)=-2, \quad y'(-2)=3$

44. $(10-2x)y''+(1+x)y=0, \quad y(2)=2, \quad y'(2)=-4$

45. $(7+x)y''+(8+2x)y'+(5+x)y=0, \quad y(-4)=1, \quad y'(-4)=2$

46. $(6+4x)y''+(1+2x)y=0, \quad y(-1)=-1, \quad y'(-1)=2$

47. $y''+2xy'+(3+2x^2)y=0, \quad y(0)=1, \quad y'(0)=-2$

48. $y''-3xy'+(5+2x^2)y=0, \quad y(0)=1, \quad y'(0)=-2$

49. $y''+5xy'-(3-x^2)y=0, \quad y(0)=6, \quad y'(0)=-2$

50. $y''-2xy'-(2+3x^2)y=0, \quad y(0)=2, \quad y'(0)=-5$

51. $y''-3xy'+(2+4x^2)y=0, \quad y(0)=3, \quad y'(0)=6$

52. $2y''+5xy'+(4+2x^2)y=0, \quad y(0)=3, \quad y'(0)=-2$

53. $3y''+2xy'+(4-x^2)y=0, \quad y(0)=-2, \quad y'(0)=3$

54. $(1+x)y''+x^2y'+(1+2x)y=0, \quad y(0)-2, \quad y'(0)=3$

55. $y''+(1+2x+x^2)y'+2y=0, \quad y(0)=2, \quad y'(0)=3$

56. $(1+x^2)y''+(2+x^2)y'+xy=0, \quad y(0)=-3, \quad y'(0)=5$

57. $(1+x)y''+(1-3x+2x^2)y'-(x-4)y=0, \quad y(1)=-2, \quad y'(1)=3$

58. $y''+(13+12x+3x^2)y'+(5+2x)y=0, \quad y(-2)=2, \quad y'(-2)=-3$

59. $(1+2x+3x^2)y''+(2-x^2)y'+(1+x)y=0, \quad y(0)=1, \quad y'(0)=-2$

60. $(3+4x+x^2)y''-(5+4x-x^2)y'-(2+x)y=0, \quad y(-2)=2, \quad y'(-2)=-1$

61. $(1+2x+x^2)y''+(1-x)y=0, \quad y(0)=2, \quad y'(0)=-1$

62. $(x-2x^2)y''+(1+3x-x^2)y'+(2+x)y=0, \quad y(1)=1, \quad y'(1)=0$

63. $(16-11x+2x^2)y''+(10-6x+x^2)y'-(2-x)y=0, \quad y(3)=1, \quad y'(3)=-2$

64. a. Find a power series in x for the general solution of

$$(1+x^2)y''+4xy'+2y=0. \tag{A}$$

b. Use (a) and the formula

$$\frac{1}{1-r}=1+r+r^2+\cdots+r^n+\cdots \quad (-1<r<1)$$

for the sum of a geometric series to find a closed form expression for the general solution of (A) on $(-1,1)$.

c. Show that the expression obtained in (b) is actually the general solution of of (A) on $(-\infty, \infty)$.

65. Find power series in x for the solutions y_1 and y_2 of

$$y'' + 4xy' + (2 + 4x^2)y = 0$$

such that $y_1(0) = 1$, $y_1'(0) = 0$, $y_2(0) = 0$, $y_2'(0) = 1$, and identify y_1 and y_2 in terms of familiar elementary functions.

In Exercises **66 and 67** find the power series in x for the general solution.

66. $y'' - xy = 1$

67. $y'' + xy' - y = e^x$; where $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

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8.4: Series Solutions About a Regular Singular Point

In this section we'll continue to study equations of the form

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0 \tag{8.4.1}$$

but the emphasis will be different from that of Sections 8.3, where we obtained solutions of Equation 8.4.1 near an ordinary point x_0 in the form of power series in $x - x_0$. If x_0 is a singular point of Equation 8.4.1, then the solutions can't, in general, be represented by power series in $x - x_0$. Nevertheless, it is often necessary in physical applications to study the behavior of solutions of Equation 8.4.1 near a singular point. Although this can be difficult in the absence of some sort of assumption on the nature of the singular point, equations that satisfy the requirements of the next definition can be solved by series methods discussed here. Fortunately, many equations arising in applications satisfy these requirements.

Definition 8.4.1 Regular Singular Points

Let

$$y'' + p(x)y' + q(x)y = 0. \tag{8.4.2}$$

If x_0 is a singular point of 8.4.2, then we say that x_0 is a *regular singular point* of Equation 8.4.2 if $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are both analytic at x_0. Otherwise, x_0 is an *irregular singular point*.

Example 8.4.1 Bessel's equation

Bessel's equation,

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \tag{8.4.3}$$

which can be written in the form 8.4.2 as

$$y'' + \frac{1}{x}y' + \frac{(x^2 - \nu^2)}{x^2}y = 0,$$

has the singular point $x_0 = 0$. Since $xp(x) = 1$ and $x^2q(x) = (x^2 - \nu^2)$ are analytic at 0, it follows that $x_0 = 0$ is a regular singular point of Equation 8.4.3.

Example 8.4.2 Legendre's equation

Rewriting Legendre's equation,

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0, \tag{8.4.4}$$

in the form 8.4.2 yields

$$y'' - \frac{2x}{1 - x^2}y' + \frac{\alpha(\alpha + 1)}{1 - x^2}y = 0.$$

It's clear that 8.4.4 has the singular points $x_0 = \pm 1$. Since $(x - 1)p(x) = \frac{2x}{x+1}$ and $(x - 1)^2 q(x) = -\frac{\alpha(\alpha+1)(x-1)}{(x+1)}$, $x_0 = 1$ is a regular singular point of Equation 8.4.4. We leave it to you to show that $x_0 = -1$ is also a regular singular point of Equation 8.4.4.

Example 8.4.3

The equation

$$x^3y'' + xy' + y = 0$$

has an irregular singular point at $x_0 = 0$. (Verify.)

For convenience we restrict our attention to the case where $x_0 = 0$ is a regular singular point of Equation 8.4.2. This isn't really a restriction, since if $x_0 \neq 0$ is a regular singular point of Equation 8.4.2 then introducing the new independent variable $t = x - x_0$ and the new unknown $Y(t) = y(t + x_0)$ leads to a differential equation that has a regular singular point at $t_0 = 0$.

The Method of Frobenius

The method of Frobenius deals with three distinct cases satisfying the assumptions introduced above. In all three cases 8.4.2 has at least one solution of the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r},$$

where r need not be an integer. The problem is that the three possibilities each require a different approach to obtain a second linearly independent solution that we need to form a fundamental pair of solutions of 8.4.2.

The method we will use to find solutions for regular singular points of 8.4.2 is called *the method of Frobenius*, and we'll call them *Frobenius solutions*.

It can be shown that the power series $x^r \sum_{n=0}^{\infty} a_n x^n$ yields a Frobenius solution of Equation 8.4.2 that converges on some open interval $(-\rho, \rho)$, where $0 < \rho \leq \infty$. The method of Frobenius will give us what is called an *indicial equation*, and it's the nature of these roots that leads us to the three cases mentioned above.

Note

In this class we only deal with real roots of the indicial equation, of which there are three cases. If 0 is a regular singular point of 8.4.2, then we have the following:

Case 1: If the roots of the indicial equation are real numbers that differ by a **noninteger**, then $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ will give both of the linearly independent solutions to 8.4.2.

Case 2: If the roots of the indicial equation are real numbers that differ by a **nonzero integer**, then $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ will give **at least one** of the two linearly independent solutions to 8.4.2.

Case 3: If the roots of the indicial equation are real numbers that are the **same**, then $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ will give only one of the two linearly independent solutions to 8.4.2.

Example 8.4.4

Find a fundamental set of Frobenius solutions and give the general solution of

$$3xy'' + y' - y = 0 \tag{8.4.5}$$

about $x_0 = 0$ on $(0, \infty)$.

Solution

Note that $x_0 = 0$ is a regular singular point of 8.4.5. We now let

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}.$$

Differentiating this series twice gives us

$$y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}.$$

Substituting these into 8.4.5 yields

$$3x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}+\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}-\sum_{n=0}^{\infty}a_nx^{n+r}=\sum_{n=0}^{\infty}3(n+r)(n+r-1)a_nx^{n+r-1}+$$
$$\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}-\sum_{n=0}^{\infty}a_nx^{n+r}=0$$

Running out terms we get

$$3r(r-1)a_0x^{r-1}+\sum_{n=1}^{\infty}3(n+r)(n+r-1)a_nx^{n+r-1}+ra_0x^{r-1}+\sum_{n=1}^{\infty}(n+r)a_nx^{n+r-1}-\sum_{n=0}^{\infty}a_nx^{n+r}=0.$$

Reindexing we get

$$3r(r-1)a_0x^{r-1}+\sum_{n=1}^{\infty}3(n+r)(n+r-1)a_nx^{n+r-1}+ra_0x^{r-1}+\sum_{n=1}^{\infty}(n+r)a_nx^{n+r-1}-\sum_{n=1}^{\infty}a_{n-1}x^{n+r-1}=0.$$

We now collect like terms to obtain

$$(3r^2-3r+r)a_0x^{r-1}+\sum_{n=1}^{\infty}[3(n+r)(n+r-1)a_n+(n+r)a_n-a_{n-1}]x^{n+r-1}=(3r^2-2r)a_0x^{r-1}+$$
$$\sum_{n=1}^{\infty}[(n+r)(3n+3r-2)a_n-a_{n-1}]x^{n+r-1}=0.$$

Equating coefficients gives us the following

$$(3r^2-2r)a_0=0 \tag{8.4.6}$$

and

$$(n+r)(3n+3r-2)a_n-a_{n-1}=0$$

Equation 8.4.6 is our indicial equation; let's take a closer look as to what it says.

$$(3r^2-2r)a_0=r(3r-2)a_0=0.$$

So, we have $a_0=0$, $r=0$, or $r=2/3$. $a_0=0$ will only give us the trivial solution which is always linearly dependent, so we focus on $r=0$ and $r=2/3$. Note that these differ by a **noninteger** and, according to Frobenius, will lead to two linearly independent solutions of Equation 8.4.5.

For the case $r=0$:

a_0 is a free variable (meaning it can be anything we choose) and $a_n=\frac{a_{n-1}}{n(3n-2)}, \quad n=1,2,3,\cdots$

Now substituting $n=1,2,3,\cdots$ into the last equation yields

$$\begin{aligned}a_1&=a_0,\\ a_2&=\frac{1}{8}a_1=\frac{1}{8}a_0\cdots\end{aligned}$$

This leads us to one solution of 8.4.5:

$$\begin{aligned}y_1&=x^0(a_0+a_0x+\frac{1}{8}a_0x^2+\cdots)\\&=a_0(1+x+\frac{1}{8}x^2+\cdots)\end{aligned}$$

Notice that $r=0$ gives us the form of an ordinary point power series,

$$y=x^r\sum_{n=0}^{\infty}a_nx^n=x^0\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}a_nx^n,$$

which only yielded one solution instead of the two we would be guaranteed if $x_0 = 0$ was an ordinary point. As you may guess, $r = 2/3$ will yield the second linearly independent solution of 8.4.5.

For the case $r = 2/3$:

a_0 is once again a free variable and $a_n = \frac{a_{n-1}}{n(3n+2)}, \quad n = 1, 2, 3, \cdots$

Now substituting $n = 1, 2, 3, \cdots$ into the last equation yields

$$\begin{aligned} a_1 &= \frac{1}{5}a_0, \\ a_2 &= \frac{1}{16}a_1 = \frac{1}{80}a_0 \cdots \end{aligned}$$

This leads us to the second linearly independent solution of 8.4.5:

$$\begin{aligned} y_2 &= x^{2/3}(a_0 + \frac{1}{5}a_0 x + \frac{1}{80}a_0 x^2 + \cdots) \\ &= a_0 x^{2/3}(1 + \frac{1}{5}x + \frac{1}{80}x^2 + \cdots) \end{aligned}$$

Note that since we used two separate r values, the a_0 in each is actually different. So, we need to change one or both of the a_0 to get our general solution of 8.4.5:

$$y = c_1(1 + x + \frac{1}{8}x^2 + \cdots) + c_2 x^{2/3}(1 + \frac{1}{5}x + \frac{1}{80}x^2 + \cdots)$$

Example 8.4.5

Find a fundamental set of Frobenius solutions and give the general solution of

$$xy'' + y = 0 \tag{8.4.7}$$

about $x_0 = 0$ on $(0, \infty)$.

Solution

Note that $x_0 = 0$ is a regular singular point of 8.4.7. We now let

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}.$$

Differentiating this series twice gives us

$$y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}.$$

Substituting these into 8.4.7 yields

$$x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + \sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

Running out terms we get

$$r(r-1)a_0 x^{r-1} + \sum_{n=1}^{\infty} (n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

Reindexing we get

$$r(r-1)a_0 x^{r-1} + \sum_{n=1}^{\infty} (n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0.$$

We now collect like terms to obtain

$$r(r-1)a_0x^{r-1}+\sum_{n=1}^{\infty}[(n+r)(n+r-1)a_n+a_{n-1}]x^{n+r-1}=0.$$

Equating coefficients gives us the following

$$r(r-1)a_0=0 \tag{8.4.8}$$

and

$$(n+r)(n+r-1)a_n+a_{n-1}=0$$

Equation 8.4.8 is our indicial equation; let's take a closer look as to what it says.

$$r(r-1)a_0=0.$$

As in Example 6, $a_0=0$ will only give us the trivial solution which is always linearly dependent, so we focus on $r=0$ and $r=1$. Note that these differ by an **integer** and, according to Frobenius, we have...no idea what is going to happen...so, we proceed...with hope, but no guarantee, that the method of Frobenius will give us a second linearly independent solution.

For the case $r=0$:

a_0 is a free variable and $n(n-1)a_n+a_{n-1}=0, \quad n=1,2,3,\cdots$

At this point we would normally solve for a_n, but that creates an issue when $n=1$. So, we need to deal with $n=1$ before we solve for a_n:

$n=1$ gives us $0a_1+a_0=0$, which means $a_0=0$ and now a_1 is free.

So, now solving for a_n we get

$a_n=-\frac{a_{n-1}}{n(n-1)}, \quad n=2,3,4,\cdots$

Now substituting $n=2,3,4,\cdots$ into the last equation yields

$$\begin{aligned} a_2 &= -\frac{1}{2}a_1, \\ a_3 &= -\frac{1}{6}a_2=\frac{1}{12}a_1, \\ a_4 &= -\frac{1}{12}a_3=-\frac{1}{144}a_1\cdots \end{aligned}$$

This leads us to one solution of 8.4.7:

$$\begin{aligned} y_1 &= x^0(a_1x-\frac{1}{2}a_1x^2+\frac{1}{12}a_1x^3-\frac{1}{144}a_1x^4+\cdots) \\ &= a_1(x-\frac{1}{2}x^2+\frac{1}{12}x^3-\frac{1}{144}x^4+\cdots) \end{aligned}$$

As in Example 6, note that $r=0$ gives us the form of an ordinary point power series

$$y=x^r\sum_{n=0}^{\infty}a_nx^n=x^0\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}a_nx^n,$$

which only yielded one solution instead of the two we would be guaranteed if $x_0=0$ was an ordinary point. Now, we hope $r=1$ will yield the second linearly independent solution of 8.4.7.

For the case $r=1$:

a_0 is once again a free variable and $a_n=\frac{-a_{n-1}}{n(n+1)}, \quad n=1,2,3,\cdots$

Now substituting $n=1,2,3,\cdots$ into the last equation yields

$$a_1 = -\frac{1}{2}a_0,$$
$$a_2 = -\frac{1}{6}a_1 = \frac{1}{12}a_0,$$
$$a_3 = -\frac{1}{12}a_2 = -\frac{1}{144}a_0, \cdots$$

This leads us to what we **hope** is the second linearly independent solution of 8.4.7:

$$\begin{aligned} y_2 &= x(a_0 - \frac{1}{2}a_0 x + \frac{1}{12}a_0 x^2 - \frac{1}{144}a_0 x^3 + \cdots) \\ &= a_0(x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 \cdots) \\ &= y_1 \end{aligned}$$

However, as you can see, this is the same solution we obtained from $r = 0$, so the method of Frobenius did not give us the second linearly independent solution we need. So, that leaves us with a problem since we know there must be two linearly independent solutions to 8.4.7, but at this point we only have one. If you recall, in section 5.2 we found that if we knew one solution, we could find a second linearly independent solution from

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2} dx$$

While this may seem daunting, and it is cumbersome, it is fairly straightforward. To make it easier to follow we will do this in parts.

We first note that $p(x) = 0$ and $y_1 = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots$.

So,

$$e^{-\int p(x)dx} = e^{-\int 0dx} = c.$$

To make our calculations easier we will let $c = 1$ (the constant would be absorbed later anyway). So, $e^{-\int p(x)dx} = e^{-\int 0dx} = 1$.

Now we calculate

$$y_1^2 = (x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots)(x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots) = x^2 - x^3 + \frac{5}{12}x^4 - \frac{7}{72}x^5 + \cdots$$

.

So,

$$\frac{e^{-\int p(x)dx}}{y_1^2} = \frac{1}{x^2 - x^3 + \frac{5}{12}x^4 - \frac{7}{72}x^5 + \cdots} = \frac{1}{x^2} + \frac{1}{x} + \frac{7}{12} + \frac{19}{72}x + \cdots$$

This leads to

$$\int \frac{e^{-\int p(x)dx}}{y_1^2} dx = \int (\frac{1}{x^2} + \frac{1}{x} + \frac{7}{12} + \frac{19}{72}x + \cdots)dx = -\frac{1}{x} + \ln x + \frac{7}{12}x + \frac{19}{144}x^2 + \cdots$$

This finally leads us to

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2} dx = y_1(-\frac{1}{x} + \ln x + \frac{7}{12}x + \frac{19}{144}x^2 + \cdots) = y_1 \ln x + y_1(-\frac{1}{x} + \frac{7}{12}x + \frac{19}{144}x^2 + \cdots)$$

and our general solution to 8.4.7 is

$$y = c_1 y_1 + c_2[y_1 \ln x + y_1(-\frac{1}{x} + \frac{7}{12}x + \frac{19}{144}x^2 + \cdots)]$$

We will not discuss the third case, when the roots of the indicial equation are the same, since this clearly leads to only one linearly independent solution and would therefore require the exact same method of Example 7.

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8.4.1: Series Solutions About a Regular Singular Point (Exercises)

In Exercises **1-60** find the series in x for the general solution.

1. $2xy''-y'+2y=0$
2. $4xy''+\frac{1}{2}y'+y=0$
3. $3xy''+(2-x)y'-y=0$
4. $2xy''-(3+2x)y'+y=0$
5. $9x^2y''+9x^2y'+2y=0$
6. $xy''+2y'-xy=0$
7. $xy''-xy'+y=0$
8. $xy''+(1-x)y'-y=0$
9. $2xy''+5y'+xy=0$
10. $2x^2y''-xy'+(x^2+1)y=0$
11. $x^2y''-(x-\frac{2}{9})y=0$
12. $x^2y''+xy'+(x^2-\frac{4}{9})y=0$
13. $2x^2y''+3xy'+(2x-1)y=0$
14. $x^2y''+xy'+(x^2-\frac{1}{4})y=0$
15. $y''+\frac{3}{x}y'-2y=0$
16. $xy''+y'+y=0$
17. $2x^2(1+x+x^2)y''+x(3+3x+5x^2)y'-y=0$
18. $3x^2y''+2x(1+x-2x^2)y'+(2x-8x^2)y=0$
19. $x^2(3+3x+x^2)y''+x(5+8x+7x^2)y'-(1-2x-9x^2)y=0$
20. $4x^2y''+x(7+2x+4x^2)y'-(1-4x-7x^2)y=0$
21. $12x^2(1+x)y''+x(11+35x+3x^2)y'-(1-10x-5x^2)y=0$
22. $x^2(5+x+10x^2)y''+x(4+3x+48x^2)y'+(x+36x^2)y=0$
23. $8x^2y''-2x(3-4x-x^2)y'+(3+6x+x^2)y=0$
24. $18x^2(1+x)y''+3x(5+11x+x^2)y'-(1-2x-5x^2)y=0$
25. $x(3+x+x^2)y''+(4+x-x^2)y'+xy=0$
26. $10x^2(1+x+2x^2)y''+x(13+13x+66x^2)y'-(1+4x+10x^2)y=0$
27. $x^2(8+x)y''+x(2+3x)y'+(1+x)y=0$
28. $x^2(3+4x)y''+x(11+4x)y'-(3+4x)y=0$
29. $2x^2(2+3x)y''+x(4+11x)y'-(1-x)y=0$
30. $x^2(2+x)y''+5x(1-x)y'-(2-8x)y$
31. $x^2(6+x)y''+x(11+4x)y'+(1+2x)y=0$
32. $6x^2y''+x(1+6x^2)y'+(1+9x^2)y=0$
33. $x^2(8+x^2)y''+7x(2+x^2)y'-(2-9x^2)y=0$
34. $9x^2(1+x^2)y''+3x(3+13x^2)y'-(1-25x^2)y=0$
35. $4x^2(1+x^2)y''+4x(1+6x^2)y'-(1-25x^2)y=0$

36. $8x^2(1+2x^2)y''+2x(5+34x^2)y'-(1-30x^2)y=0$
37. $x^2y''-x(1-x)y'+(1-x^2)y=0$
38. $x^2(1+x+2x^2)y'+x(3+6x+7x^2)y'+(1+6x-3x^2)y=0$
39. $x^2(1+2x+x^2)y''+x(1+3x+4x^2)y'-x(1-2x)y=0$
40. $4x^2(1+x+x^2)y''+12x^2(1+x)y'+(1+3x+3x^2)y=0$
41. $x^2(1+x+x^2)y''-x(1-4x-2x^2)y'+y=0$
42. $9x^2y''+3x(5+3x-2x^2)y'+(1+12x-14x^2)y=0$
43. $x^2y''+x(1+x+x^2)y'+x(2-x)y=0$
44. $x^2(1+2x)y''+x(5+14x+3x^2)y'+(4+18x+12x^2)y=0$
45. $4x^2y''+2x(4+x+x^2)y'+(1+5x+3x^2)y=0$
46. $16x^2y''+4x(6+x+2x^2)y'+(1+5x+18x^2)y=0$
47. $9x^2(1+x)y''+3x(5+11x-x^2)y'+(1+16x-7x^2)y=0$
48. $x^2(1+2x)y''+x(5+9x)y'+(4+3x)y=0$
49. $x^2(1-2x)y''-x(5+4x)y'+(9+4x)y=0$
50. $x^2(1+4x)y''-x(1-4x)y'+(1+x)y=0$
51. $x^2(1+x)y''+x(1+2x)y'+xy=0$
52. $x^2(1-x)y''+x(7+x)y'+(9-x)y=0$
53. $x^2(1+x^2)y''+x(3+8x^2)y'+(1+12x^2)y$
54. $x^2y''-x(1-x^2)y'+(1+x^2)y=0$
55. $x^2(1-2x^2)y''+x(5-9x^2)y'+(4-3x^2)y=0$
56. $x^2(2+x^2)y''+x(14-x^2)y'+2(9+x^2)y=0$
57. $x^2(1+x^2)y''+x(3+7x^2)y'+(1+8x^2)y=0$
58. $x^2(1-2x)y''+3xy'+(1+4x)y=0$
59. $x(1+x)y''+(1-x)y'+y=0$
60. $x^2(1+x^2)y''-x(1+9x^2)y'+(1+25x^2)y=0$

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CHAPTER OVERVIEW

9: Laplace Transforms

In this chapter we study the method of Laplace transforms, which illustrates one of the basic problem solving techniques in mathematics: transform a difficult problem into an easier one, solve the latter, and then use its solution to obtain a solution of the original problem. The method discussed here transforms an initial value problem for a constant coefficient equation into an algebraic equation whose solution can then be used to solve the initial value problem. In some cases this method is merely an alternative procedure for solving problems that can be solved equally well by methods that we considered previously; however, in other cases the method of Laplace transforms is more efficient than the methods previously discussed. This is especially true in physical problems dealing with discontinuous forcing functions.

9.1: Introduction to the Laplace Transform

9.1.1: Introduction to the Laplace Transform (Exercises)

9.2: The Inverse Laplace Transform

9.2.1: The Inverse Laplace Transform (Exercises)

9.3: Solutions of Initial Value Problems

9.3.1: Solution of Initial Value Problems (Exercises)

9.4: The First Shifting Theorem

9.4.1: The First Shifting Theorem (Exercises)

9.5: The Second Shifting Theorem and Piecewise Continuous Forcing Functions

9.5.1: The Second Shifting Theorem and Piecewise Continuous Forcing Functions (Exercises)

9.6: Convolution and Periodic Functions

9.6.1: Convolution and Periodic Functions (Exercises)

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9.1: Introduction to the Laplace Transform

Definition of the Laplace Transform

To define the Laplace transform, we first recall the definition of an improper integral. If g is integrable over the interval $[a, T]$ for every $T > a$, then the *improper integral of g over* $[a, \infty)$ is defined as

$$\int_a^\infty g(t)\,dt = \lim_{T\to\infty} \int_a^T g(t)\,dt. \tag{9.1.1}$$

We say that the improper integral *converges* if the limit in Equation 9.1.1 exists; otherwise, we say that the improper integral *diverges* or *does not exist*. We can now turn to the definition of the Laplace transform of a function f.

Definition 9.1.1 Laplace Transform

Let f be defined for $t \geq 0$ and let s be a real number. Then the *Laplace transform* of f is the function F defined by

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt, \tag{9.1.2}$$

for those values of s for which the improper integral converges.

It is important to keep in mind that the variable of integration in Equation 9.1.2 is t, while s is a parameter independent of t. We use t as the independent variable for f because in applications the Laplace transform is usually applied to functions of time.

The Laplace transform can be viewed as an operator $\mathcal{L}$ that transforms the function $f = f(t)$ into the function $F = F(s)$. Thus, Equation 9.1.2 can be expressed as

$$F(s) = \mathcal{L}(f(t)).$$

The functions f and F form a *transform pair*, which we'll sometimes denote by

$$f(t) \leftrightarrow F(s).$$

Computation of Some Simple Laplace Transforms

Example 9.1.1

Find the Laplace transform of $f(t) = 1$.

Solution

From Equation 9.1.2 with $f(t) = 1$ and $s \neq 0$,

$$F(s) = \int_0^\infty e^{-st}(1)dt = \lim_{T\to\infty} \int_0^T e^{-st}\,dt = \lim_{T\to\infty} \left(-\frac{1}{s}e^{-st}\right)\Big|_0^T = \lim_{T\to\infty} \frac{1-e^{-sT}}{s} = \begin{cases} \frac{1}{s}, & s > 0, \\ \infty, & s < 0. \end{cases}$$

If $s = 0$ the integrand reduces to the constant 1, and

$$\int_0^\infty 1dt = \lim_{T\to\infty} \int_0^T 1\,dt = \lim_{T\to\infty} T = \infty.$$

Therefore $F(0)$ is undefined, and

$$F(s) = \int_0^\infty e^{-st}\,dt = \frac{1}{s}, \quad s > 0.$$

This result can be written in operator notation as

$$\mathcal{L}(1) = \frac{1}{s}, \quad s > 0,$$

or as the transform pair

$$1 \leftrightarrow \frac{1}{s}, \quad s > 0.$$

Example 9.1.2

Find the Laplace transform of $f(t) = t$.

Solution

From Equation 9.1.2 with $f(t) = t$ and $s \neq 0$,

$$\begin{aligned} F(s) = \int_0^\infty e^{-st}(t)dt &= \lim_{T\to\infty} \int_0^T te^{-st}dt \\ &= \lim_{T\to\infty} \left(-\frac{te^{-st}}{s}\Big|_0^T + \frac{1}{s}\int_0^T e^{-st}dt \right) \\ &= \lim_{T\to\infty} \left(\frac{-Te^{-sT}}{s} + 0 - \frac{e^{-st}}{s^2}\Big|_0^T \right) \\ &= \lim_{T\to\infty} \left(\frac{-Te^{-sT}}{s} - \frac{e^{-sT}}{s^2} + \frac{1}{s^2} \right) \\ &= \begin{cases} \frac{1}{s^2}, & s > 0, \\ \infty, & s < 0. \end{cases} \end{aligned}$$

Note: $\lim_{T\to\infty} \frac{-Te^{-sT}}{s}$ was evaluated using L'Hôpital's rule.

If $s = 0$, the integral becomes

$$\int_0^\infty t\,dt = \lim_{T\to\infty} \int_0^T tdt = \lim_{T\to\infty} \left(\frac{t^2}{2} \right)\Big|_0^T = \lim_{T\to\infty} \frac{T^2}{2} = \infty.$$

Therefore $F(0)$ is undefined and

$$F(s) = \frac{1}{s^2}, \quad s > 0.$$

This result can also be written as

$$\mathcal{L}(t) = \frac{1}{s^2}, \quad s > 0,$$

or as the transform pair

$$t \leftrightarrow \frac{1}{s^2}, \quad s > 0.$$

Example 9.1.3

Find the Laplace transform of $f(t) = e^{at}$, where a is a constant.

Solution

From Equation 9.1.2 with $f(t) = e^{at}$,

$$\begin{aligned} F(s) = \int_0^\infty e^{-st}e^{at}dt = \lim_{T\to\infty} \int_0^T e^{-(s-a)t}dt &= \lim_{T\to\infty} \left(-\frac{e^{-(s-a)t}}{s-a} \right)\Big|_0^T = \lim_{T\to\infty} \left(-\frac{e^{-(s-a)T}}{s-a} + \frac{1}{s-a} \right) \\ &= \begin{cases} \frac{1}{s-a}, & s > a, \\ \infty, & s < a. \end{cases} \end{aligned}$$

If $s = a$, the integral becomes

$$\int_0^\infty 1dt = \lim_{T\to\infty}\int_0^T 1\,dt = \lim_{T\to\infty} T = \infty.$$

Therefore $F(a)$ is undefined and

$$F(s) = \frac{1}{s-a}, \quad s > a.$$

This can also be written as

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}, \quad s > a.$$

or as the transform pair

$$e^{at} \leftrightarrow \frac{1}{s-a}, \quad s > a.$$

Tables of Laplace Transforms

Extensive tables of Laplace transforms have been compiled and are commonly used in applications. The brief table of Laplace transforms that you need to know are listed here and will be adequate for our purposes.

Table 9.1.1

$f(t)$	$F(s)$	
1	$\frac{1}{s}$	$(s>0)$
t^n	$\frac{n!}{s^{n+1}}$	$(s>0)$
e^{at}	$\frac{1}{s-a}$	$(s>a)$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	$(s>0)$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$(s>0)$
$\cosh bt$	$\frac{s}{s^2-b^2}$	$(s>\lvert b\rvert)$
$\sinh bt$	$\frac{b}{s^2-b^2}$	$(s>\lvert b\rvert)$

Example 9.1.4

Use the table of Laplace transforms to find $\mathcal{L}(\cosh 7t)$.

Solution

$$\mathcal{L}(\cosh 7t) = \frac{s}{s^2-49}.$$

Linearity of the Laplace Transform

The next theorem presents an important property of the Laplace transform. If you have had linear algebra you should recognize that the Laplace transform is a linear transformation.

Theorem 9.1.1 Linearity Property

Suppose $\mathcal{L}(f_i)$ is defined for $s > s_i$, $1 \le i \le n)$. Let s_0 be the largest of the numbers s_1, s_2, …,s_n, and let c_1, c_2,…, c_n be constants. Then

$$\mathcal{L}(c_1 f_1 + c_2 f_2 + \cdots + c_n f_n) = c_1\mathcal{L}(f_1) + c_2\mathcal{L}(f_2) + \cdots + c_n\mathcal{L}(f_n) \text{ for } s > s_0.$$

Proof

If $s > s_0$ then

$$\begin{aligned}\mathcal{L}(c_1 f_1 + c_2 f_2 + \cdots + c_n f_n) &= \int_0^\infty e^{-st}\left(c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_n(t)\right)\, dt \\ &= c_1\int_0^\infty e^{-st} f_1(t)\, dt + c_2\int_0^\infty e^{-st} f_2(t)\, dt + \cdots + c_n\int_0^\infty e^{-st} f_n(t)\, dt \\ &= c_1\mathcal{L}(f_1) + c_2\mathcal{L}(f_2) + \cdots + c_n\mathcal{L}(f_n).\end{aligned}$$

Example 9.1.5

Use Theorem 9.1.1 and the known Laplace transform

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$

to prove that for $b \neq 0$,

$$\mathcal{L}(\cosh bt) = \frac{s}{s^2 - b^2}, \quad s > |b|.$$

Solution

By definition,

$$\cosh bt = \frac{e^{bt} + e^{-bt}}{2}.$$

Therefore

$$\begin{aligned}\mathcal{L}(\cosh bt) &= \mathcal{L}\left(\tfrac{1}{2}e^{bt} + \tfrac{1}{2}e^{-bt}\right) \\ &= \tfrac{1}{2}\mathcal{L}(e^{bt}) + \tfrac{1}{2}\mathcal{L}(e^{-bt}) \qquad \text{(linearity property)} \qquad (9.1.3) \\ &= \tfrac{1}{2}\,\tfrac{1}{s-b} + \tfrac{1}{2}\,\tfrac{1}{s+b},\end{aligned}$$

where the first transform on the right is defined for $s > b$ and the second for $s > -b$; hence, both are defined for $s > |b|$. Simplifying the last expression in Equation 9.1.3 yields

$$\mathcal{L}(\cosh bt) = \frac{s}{s^2 - b^2}, \quad s > |b|.$$

Existence of Laplace Transforms

Not every function has a Laplace transform. For example, it can be shown (*Exercise 19*) that

$$\int_0^\infty e^{-st} e^{t^2}\, dt = \infty$$

for every real number s. Hence, the function $f(t) = e^{t^2}$ does not have a Laplace transform.

Our next objective is to establish conditions that ensure the existence of the Laplace transform of a function. We first review some relevant definitions from calculus.

Recall that a limit

$$\lim_{t \to t_0} f(t)$$

exists if and only if the one-sided limits

$$\lim_{t\to t_0-} f(t) \quad \text{and} \quad \lim_{t\to t_0+} f(t)$$

both exist and are equal; in this case,

$$\lim_{t\to t_0} f(t) = \lim_{t\to t_0-} f(t) = \lim_{t\to t_0+} f(t).$$

Recall also that f is continuous at a point t_0 in an open interval (a, b) if and only if

$$\lim_{t\to t_0} f(t) = f(t_0),$$

which is equivalent to

$$\lim_{t\to t_0+} f(t) = \lim_{t\to t_0-} f(t) = f(t_0). \tag{9.1.4}$$

For simplicity, we define

$$f(t_0+) = \lim_{t\to t_0+} f(t) \quad \text{and} \quad f(t_0-) = \lim_{t\to t_0-} f(t),$$

so Equation 9.1.4 can be expressed as

$$f(t_0+) = f(t_0-) = f(t_0).$$

If $f(t_0+)$ and $f(t_0-)$ have finite but distinct values, we say that f has a *jump discontinuity* at t_0, and

$$f(t_0+) - f(t_0-)$$

is called the *jump* in f at t_0 (Figure 9.1.1).

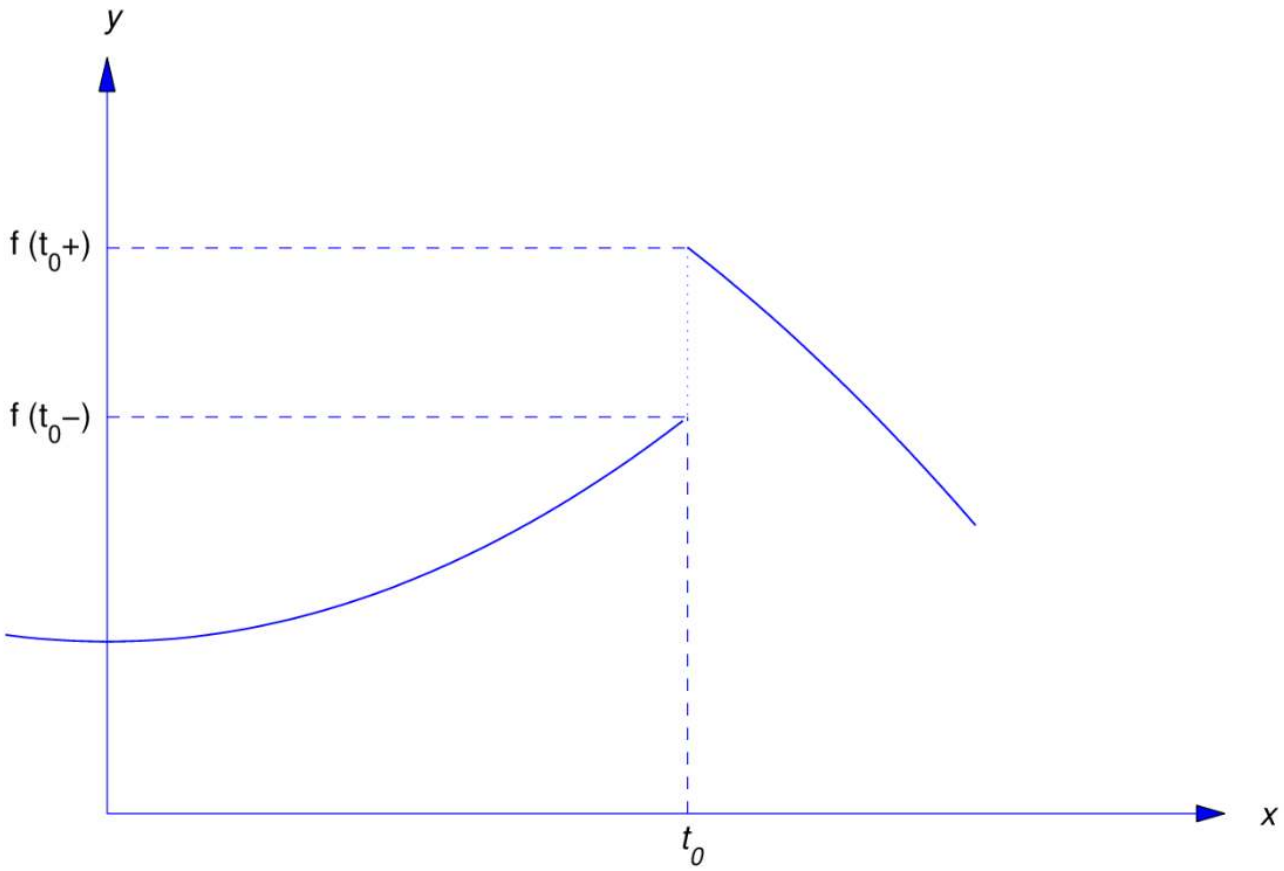


Figure 9.1.1 : A jump discontinuity.

If $f(t_0+)$ and $f(t_0-)$ are finite and equal, but either f isn't defined at t_0 or it is defined but

$$f(t_0) \neq f(t_0+) = f(t_0-),$$

we say that f has a *removable discontinuity* at t_0 (Figure 9.1.2). This terminology is appropriate since a function f with a removable discontinuity at t_0 can be made continuous at t_0 by **redefining**

$$f(t_0) = f(t_0+) = f(t_0-).$$

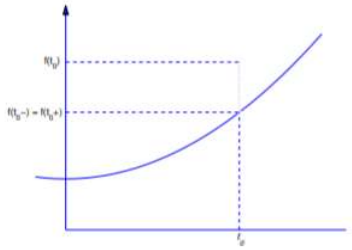


Figure 9.1.2

Note

We know from calculus that a definite integral isn't affected by changing the values of its integrand at isolated points. Therefore, redefining a function f to make it continuous at removable discontinuities does not change $\mathcal{L}(f)$.

Definition 9.1.2 Piecewise Continuous

- A function f is said to be *piecewise continuous* on a finite closed interval $[0,T]$ if $f(0+)$ and $f(T-)$ are finite and f is continuous on the open interval $(0,T)$ except possibly at finitely many points, where f may have jump discontinuities or removable discontinuities.
- A function f is said to be *piecewise continuous* on the infinite interval $[0,\infty)$ if it is piecewise continuous on $[0,T]$ for every $T>0$.

Figure 9.1.3 shows the graph of a typical piecewise continuous function.

It is shown in calculus that if a function is piecewise continuous on a finite closed interval then it is integrable on that interval. But if f is piecewise continuous on $[0,\infty)$, then so is $e^{-st}f(t)$, and therefore

$$\int_0^T e^{-st} f(t)\,dt$$

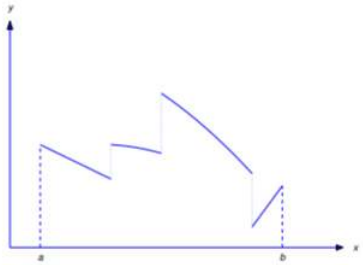


Figure 9.1.3

exists for every $T>0$. However, piecewise continuity alone does not guarantee that the improper integral

$$\int_0^\infty e^{-st} f(t)\,dt = \lim_{T\to\infty} \int_0^T e^{-st} f(t)\,dt \tag{9.1.5}$$

converges for s in some interval (s_0,∞). For example, we noted earlier that Equation 9.1.5 diverges for all s if $f(t)=e^{t^2}$. Stated informally, this occurs because e^{t^2} increases too rapidly as $t\to\infty$. The next definition provides a constraint on the growth of a function that guarantees convergence of its Laplace transform for s in some interval (s_0,∞).

Definition 9.1.3 Exponential order

A function f is said to be *of exponential order* s_0 if there are constants M and t_0 such that

$$|f(t)| \le Me^{s_0 t}, \quad t \ge t_0. \tag{9.1.6}$$

In situations where the specific value of s_0 is irrelevant we say simply that f is *of exponential order*.

The next theorem gives useful sufficient conditions for a function f to have a Laplace transform.

Theorem 9.1.2

If f is piecewise continuous on $[0, \infty)$ and of exponential order s_0, then $\mathcal{L}(f)$ is defined for $s > s_0$.

Note

We emphasize that the conditions of Theorem 9.1.6 are sufficient, but *not necessary*, for f to have a Laplace transform. For example, *Exercise 9.1.23* shows that f may have a Laplace transform even though f isn't of exponential order

Example 9.1.6

If f is bounded on some interval $[t_0, \infty)$, say

$$|f(t)| \le M, \quad t \ge t_0,$$

then Equation 9.1.6 holds with $s_0 = 0$, so f is of exponential order zero. Thus, for example, $\sin \omega t$ and $\cos \omega t$ are of exponential order zero, and Theorem 9.1.2 implies that $\mathcal{L}(\sin \omega t)$ and $\mathcal{L}(\cos \omega t)$ exist for $s > 0$.

Example 9.1.7

It can be shown that if $\lim_{t\to\infty} e^{-s_0 t} f(t)$ exists and is finite then f is of exponential order s_0. If α is any real number and $s_0 > 0$ then $f(t) = t^\alpha$ is of exponential order s_0, since

$$\lim_{t\to\infty} e^{-s_0 t} t^\alpha = 0,$$

by L'Hôpital's rule. If $\alpha \ge 0$, f is also continuous on $[0, \infty)$. Therefore Theorem 9.1.2 imply that $\mathcal{L}(t^\alpha)$ exists for $s \ge s_0$. However, since s_0 is an arbitrary positive number, this really implies that $\mathcal{L}(t^\alpha)$ exists for all $s > 0$. This is consistent with the results of Example 9.1.2 .

Example 9.1.8

Find the Laplace transform of the piecewise continuous function

$$f(t) = \begin{cases} 1, & 0 \le t < 1, \\ -3e^{-t}, & t \ge 1. \end{cases}$$

Solution

Since f is defined by different formulas on $[0, 1)$ and $[1, \infty)$, we write

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt = \int_0^1 e^{-st}(1)\, dt + \int_1^\infty e^{-st}(-3e^{-t})\, dt.$$

Since

$$\int_0^1 e^{-st}\, dt = \begin{cases} \frac{1-e^{-s}}{s} & s \ne 0 \\ 1 & s = 0 \end{cases}$$

and

$$\int_1^\infty e^{-st}(-3e^{-t})\,dt = -3\lim_{T\to\infty}\int_1^T e^{-(s+1)t}\,dt = -3\lim_{T\to\infty}\left(-\frac{1}{s+1}e^{-(s+1)t}\right)\Big|_1^T = -3$$

$$\lim_{T\to\infty}\left(-\frac{1}{s+1}e^{-(s+1)T} + \frac{1}{s+1}e^{-(s+1)}\right) = -\frac{3e^{-(s+1)}}{s+1}, \quad s > -1,$$

it follows that

$$F(s) = \begin{cases} \frac{1-e^{-s}}{s} - 3\frac{e^{-(s+1)}}{s+1} & s > -1, s \neq 0 \\ 1 - \frac{3}{e} & s = 0 \end{cases}$$

This is consistent with Theorem 9.1.2 , since

$$|f(t)| \le 3e^{-t}, \quad t \ge 1,$$

and therefore f is of exponential order $s_0 = -1$.

Example 9.1.9

We stated earlier that

$$\int_0^\infty e^{-st}e^{t^2}\,dt = \infty$$

for all s, so Theorem 9.1.2 implies that $f(t) = e^{t^2}$ is not of exponential order, since

$$\lim_{t\to\infty}\frac{e^{t^2}}{Me^{s_0t}} = \lim_{t\to\infty}\frac{1}{M}e^{t^2-s_0t} = \infty,$$

so

$$e^{t^2} > Me^{s_0t}$$

for sufficiently large values of t, for any choice of M and s_0.

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9.1.1: Introduction to the Laplace Transform (Exercises)

In Exercises **1-5** find the Laplace transform from the definition: $F(s) = \int_0^\infty e^{-st} f(t)\, dt$.

1. t
2. te^{-t}
3. $\sinh bt$
4. $e^{2t} - 3e^t$
5. t^2

In Exercises **6-10** use the table of Laplace transforms to find the Laplace transforms of the following functions.

6. $\sin^2 t$
7. $\cos^2 2t$
8. $\sin t \cos t$
9. $\cos 2t - \cos 3t$
10. $\sin 2t + \cos 4t$

In Exercises **11-14** graph the following piecewise continuous functions and evaluate $f(t+)$, $f(t-)$, and $f(t)$ at each point of discontinuity.

11. $f(t) = \begin{cases} -t, & 0 \le t < 2, \\ t-4, & 2 \le t < 3, \\ 1, & t \ge 3. \end{cases}$
12. $f(t) = \begin{cases} t^2+2, & 0 \le t < 1, \\ 4, & t = 1, \\ t, & t > 1. \end{cases}$
13. $f(t) = \begin{cases} \sin t, & 0 \le t < \pi/2, \\ 2\sin t, & \pi/2 \le t < \pi, \\ \cos t, & t \ge \pi. \end{cases}$
14. $f(t) = \begin{cases} t, & 0 \le t < 1, \\ 2, & t = 1, \\ 2-t, & 1 \le t < 2, \\ 3, & t = 2, \\ 6, & t > 2. \end{cases}$

In Exercises **15-18** find the Laplace transform:

15. $f(t) = \begin{cases} e^{-t}, & 0 \le t < 1, \\ e^{-2t}, & t \ge 1. \end{cases}$
16. $f(t) = \begin{cases} 1, & 0 \le t < 4, \\ t, & t \ge 4. \end{cases}$
17. $f(t) = \begin{cases} t, & 0 \le t < 1, \\ 1, & t \ge 1. \end{cases}$
18. $f(t) = \begin{cases} te^t, & 0 \le t < 1, \\ e^t, & t \ge 1. \end{cases}$

19. Prove that

$$\int_0^\infty e^{-st} e^{t^2}\, dt = \infty$$

for every real number s.

20. Recall the next theorem from calculus.

Theorem 9.1.1.1

Let g be integrable on $[0,T]$ for every $T>0$. Suppose there's a function w defined on some interval $[u,\infty)$ (with $u\ge 0$) such that $|g(t)|\le w(t)$ for $t\ge u$ and $\int_u^\infty w(t)\,dt$ converges. Then $\int_0^\infty g(t)\,dt$ converges.

Use Theorem 9.1.1.1 to show that if f is piecewise continuous on $[0,\infty)$ and of exponential order s_0, then f has a Laplace transform $F(s)$ defined for $s>s_0$.

21. Prove: If f is piecewise continuous and of exponential order then $\lim_{s\to\infty} F(s) = 0$.

22. Prove: If f is continuous on $[0,\infty)$ and of exponential order $s_0>0$, then

$$\mathcal{L}\left(\int_0^t f(u)\,du\right)=\frac{1}{s}\mathcal{L}(f),\quad s>s_0.$$

HINT: *Use integration by parts to evaluate the transform on the left.*

23. Show that the function

$$f(t)=te^{t^2}\cos(e^{t^2})$$

has a Laplace transform defined for $s>0$, even though f isn't of exponential order.

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9.2: The Inverse Laplace Transform

Definition of the Inverse Laplace Transform

In Section 9.1 we defined the Laplace transform of f by

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt.$$

We'll also say that f is an *inverse Laplace Transform* of F, and write

$$f = \mathcal{L}^{-1}(F).$$

To solve differential equations with the Laplace transform, we must be able to obtain f from its transform F. There's a formula for doing this, but we can't use it because it requires the theory of functions of a complex variable. Fortunately, we can use the table of Laplace transforms to find inverse transforms that we'll need.

Example 9.2.1

Use the table of Laplace transforms to find

a. $$\mathcal{L}^{-1}\left(\frac{1}{s^2-1}\right)$$

b. $$\mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right).$$

Solution a

Setting $b = 1$ in the transform pair

$$\sinh bt \leftrightarrow \frac{b}{s^2 - b^2}$$

shows that

$$\mathcal{L}^{-1}\left(\frac{1}{s^2-1}\right) = \sinh t.$$

Solution b

Setting $\omega = 3$ in the transform pair

$$\cos \omega t \leftrightarrow \frac{s}{s^2 + \omega^2}$$

shows that

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right) = \cos 3t.$$

The next theorem enables us to find inverse transforms of linear combinations of transforms in the table. We omit the proof.

Theorem 9.2.1 : Linearity Property

If F_1, F_2, …, F_n are Laplace transforms and c_1, c_2, …, c_n are constants, then

$$\mathcal{L}^{-1}(c_1F_1 + c_2F_2 + \cdots + c_nF_n) = c_1\mathcal{L}^{-1}(F_1) + c_2\mathcal{L}^{-1}(F_2) + \cdots + c_n\mathcal{L}^{-1}F_n.$$

Example 9.2.2

Find

$$\mathcal{L}^{-1}\left(\frac{8}{s+5}+\frac{7}{s^2+3}\right).$$

Solution

From the table of Laplace transforms in Section 9.1,

$$e^{at}\leftrightarrow\frac{1}{s-a}\quad\text{and}\quad\sin\omega t\leftrightarrow\frac{\omega}{s^2+\omega^2}.$$

Theorem 9.2.1 with $a=-5$ and $\omega=\sqrt{3}$ yields

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{8}{s+5}+\frac{7}{s^2+3}\right)&=8\mathcal{L}^{-1}\left(\frac{1}{s+5}\right)+7\mathcal{L}^{-1}\left(\frac{1}{s^2+3}\right)\\&=8\mathcal{L}^{-1}\left(\frac{1}{s+5}\right)+\frac{7}{\sqrt{3}}\mathcal{L}^{-1}\left(\frac{\sqrt{3}}{s^2+3}\right)\\&=8e^{-5t}+\frac{7}{\sqrt{3}}\sin\sqrt{3}t.\end{aligned}$$

Note

We'll often write inverse Laplace transforms of specific functions without explicitly stating how they are obtained. In such cases you should refer to the table of Laplace transforms in Section 9.1.

Inverse Laplace Transforms of Rational Functions

Using the Laplace transform to solve differential equations often requires finding the inverse transform of a rational function

$$F(s)=\frac{P(s)}{Q(s)},$$

where P and Q are polynomials in s with no common factors. Since it can be shown that $\lim_{s\to\infty}F(s)=0$ if F is a Laplace transform, we need only consider the case where $\text{degree}(P)<\text{degree}(Q)$. To obtain $\mathcal{L}^{-1}(F)$, we find the partial fraction expansion of F, obtain inverse transforms of the individual terms in the expansion from the table of Laplace transforms, and use the linearity property of the inverse transform. The next two examples illustrate this.

Example 9.2.3

Find the inverse Laplace transform of

$$F(s)=\frac{3s+2}{s^2-3s+2}. \tag{9.2.1}$$

Solution

Factoring the denominator in Equation 9.2.1 yields

$$F(s)=\frac{3s+2}{(s-1)(s-2)}. \tag{9.2.2}$$

The form for the partial fraction expansion is

$$\frac{3s+2}{(s-1)(s-2)}=\frac{A}{s-1}+\frac{B}{s-2}. \tag{9.2.3}$$

Multiplying this by $(s-1)(s-2)$ yields

$$3s+2=A(s-2)+B(s-1).$$

Taking a limit as $s\to1$ yields $A=-5$ and taking a limit as $s\to2$ yields $B=8$.

$$F(s) = -\frac{5}{s-1} + \frac{8}{s-2}$$

and

$$\mathcal{L}^{-1}(F) = -5\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) + 8\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = -5e^{t} + 8e^{2t}.$$

Example 9.2.4

Find the inverse Laplace transform of

$$F(s) = \frac{6 + (s+1)(s^2 - 5s + 11)}{s(s-1)(s-2)(s+1)}. \tag{9.2.4}$$

Solution

The partial fraction expansion of Equation 9.2.4 is of the form

$$F(s) = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2} + \frac{D}{s+1}. \tag{9.2.5}$$

Multiplying this by $s(s-1)(s-2)(s+1)$ yields

$$6 + (s+1)(s^2 - 5s + 11) = A(s-1)(s-2)(s+1) + Bs(s-2)(s+1) + Cs(s-1)(s+1) + Ds(s-1)(s-2).$$

Taking a limit as $s \to 0$ yields $A = \frac{17}{2}$, the limit as $s \to 1$ yields $B = -10$, the limit as $s \to 2$ yields $C = \frac{7}{2}$, and the limit as $s \to -1$ yields $D = -1$.

Therefore

$$F(s) = \frac{17}{2}\,\frac{1}{s} - \frac{10}{s-1} + \frac{7}{2}\,\frac{1}{s-2} - \frac{1}{s+1}$$

and

$$\begin{aligned}\mathcal{L}^{-1}(F) &= \frac{17}{2}\mathcal{L}^{-1}\left(\frac{1}{s}\right) - 10\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) + \frac{7}{2}\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) \\ &= \frac{17}{2} - 10e^{t} + \frac{7}{2}e^{2t} - e^{-t}.\end{aligned}$$

Example 9.2.5

Find the inverse Laplace transform of

$$F(s) = \frac{8 + 3s}{(s^2+1)(s^2+4)}. \tag{9.2.6}$$

Solution

The form for the partial fraction expansion is

$$F(s) = \frac{A + Bs}{s^2+1} + \frac{C + Ds}{s^2+4}.$$

which leads to

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{1}{3}\,\frac{1}{s^2+1} - \frac{1}{3}\,\frac{1}{s^2+4}.$$

Multiplying by $8 + 3s$ gives us

$$\frac{8+3s}{(s^2+1)(s^2+4)} = \frac{8+3s}{3}\,\frac{1}{s^2+1} - \frac{8+3s}{3}\,\frac{1}{s^2+4}.$$

Therefore

$$\mathcal{L}^{-1}(F) = \frac{8}{3}\sin t + \cos t - \frac{4}{3}\sin 2t - \cos 2t.$$

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9.2.1: The Inverse Laplace Transform (Exercises)

In Exercises **1-55** use the table of Laplace transforms to find the inverse Laplace transform.

1. $\frac{1}{s^3}$
2. $\frac{1}{s^4}$
3. $\frac{1}{s^2}-\frac{48}{s^5}$
4. $\left(\frac{2}{s}-\frac{1}{s^3}\right)^2$
5. $\frac{(s+1)^3}{s^4}$
6. $\frac{(s+2)^2}{s^3}$
7. $\frac{1}{s^2}-\frac{1}{s}+\frac{1}{s-2}$
8. $\frac{4}{s}+\frac{6}{s^5}-\frac{1}{s+8}$
9. $\frac{1}{4s+1}$
10. $\frac{1}{5s-2}$
11. $\frac{5}{s^2+49}$
12. $\frac{10s}{s^2+16}$
13. $\frac{4s}{4s^2+1}$
14. $\frac{1}{4s^2+1}$
15. $\frac{2s-6}{s^2+9}$
16. $\frac{s+1}{s^2+2}$
17. $\frac{1}{s^2+3s}$
18. $\frac{s+1}{s^2-4s}$
19. $\frac{s}{s^2+2s-3}$
20. $\frac{1}{s^2+s-20}$
21. $\frac{0.9s}{(s-0.1)(s+0.2)}$
22. $\frac{s-3}{(s-\sqrt{3})(s+\sqrt{3})}$
23. $\frac{s}{(s-2)(s-3)(s-6)}$
24. $\frac{s^2+1}{s(s-1)(s+1)(s-2)}$
25. $\frac{1}{s^3+5s}$
26. $\frac{s}{(s^2+4)(s+2)}$
27. $\frac{2s-4}{(s^2+s)(s^2+1)}$
28. $\frac{1}{(s^4-9)}$
29. $\frac{s}{(s^2+4)(s+2)}$
30. $\frac{6s+3}{(s^2+1)(s^2+4)}$

31. $\frac{2}{s^2+9}$
32. $\frac{2s+1}{s^2+9}$
33. $\frac{s+1}{s^2-9}$
34. $\frac{1}{s}-\frac{s}{s^2+1}$
35. $\frac{3s+4}{s^2-1}$
36. $\frac{3}{s-1}+\frac{4s+1}{s^2+9}$
37. $\frac{3-(s+1)(s-2)}{(s+1)(s+2)(s-2)}$
38. $\frac{7+(s+4)(18-3s)}{(s-3)(s-1)(s+4)}$
39. $\frac{2+(s-2)(3-2s)}{(s-2)(s+2)(s-3)}$
40. $\frac{3-(s-1)(s+1)}{(s+4)(s-2)(s-1)}$
41. $\frac{3+(s-2)(10-2s-s^2)}{(s-2)(s+2)(s-1)(s+3)}$
42. $\frac{3+(s-3)(2s^2+s-21)}{(s-3)(s-1)(s+4)(s-2)}$
43. $\frac{2+3s}{(s^2+1)(s+2)(s+1)}$

47. $\frac{3s+2}{(s^2+4)(s^2+9)}$
48. $\frac{-4s+1}{(s^2+1)(s^2+16)}$
49. $\frac{5s+3}{(s^2+1)(s^2+4)}$
50. $\frac{-s+1}{(4s^2+1)(s^2+1)}$
51. $\frac{17s-34}{(s^2+16)(16s^2+1)}$
52. $\frac{2s-1}{(4s^2+1)(9s^2+1)}$
53. $\frac{1}{s(s^2+1)}$
54. $\frac{2s+1}{(s^2+1)(s-1)(s-3)}$
55. $\frac{s-6}{(s^2-1)(s^2+4)}$

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9.3: Solutions of Initial Value Problems

Laplace Transforms of Derivatives

In the rest of this chapter we'll use the Laplace transform to solve initial value problems for constant coefficient second order equations. To do this, we must know how the Laplace transform of f' is related to the Laplace transform of f. The next theorem answers this question.

Theorem 9.3.1

Suppose f is continuous on $[0,\infty)$ and of exponential order s_0, and f' is piecewise continuous on $[0,\infty)$. Then f and f' have Laplace transforms for $s > s_0$, and

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0). \tag{9.3.1}$$

Proof

We know from Theorem 9.1.2 that $\mathcal{L}(f)$ is defined for $s > s_0$. We first consider the case where f' is continuous on $[0,\infty)$. Integration by parts yields

$$\begin{aligned}\int_0^T e^{-st} f'(t)\,dt &= e^{-st} f(t)\Big|_0^T + s\int_0^T e^{-st} f(t)\,dt \\ &= e^{-sT} f(T) - f(0) + s\int_0^T e^{-st} f(t)\,dt \end{aligned} \tag{9.3.2}$$

for any $T > 0$. Since f is of exponential order s_0, $\lim\limits_{T\to\infty} e^{-sT} f(T) = 0$ and the integral on the right side of Equation 9.3.2 converges as $T \to \infty$ if $s > s_0$. Therefore

$$\begin{aligned}\int_0^\infty e^{-st} f'(t)\,dt &= -f(0) + s\int_0^\infty e^{-st} f(t)\,dt \\ &= -f(0) + s\mathcal{L}(f),\end{aligned}$$

which proves Equation 9.3.1.

Suppose $T > 0$ and f' is only piecewise continuous on $[0,T]$, with discontinuities at $t_1 < t_2 < \cdots < t_{n-1}$. For convenience, let $t_0 = 0$ and $t_n = T$. Integrating by parts yields

$$\begin{aligned}\int_{t_{i-1}}^{t_i} e^{-st} f'(t)\,dt &= e^{-st} f(t)\Big|_{t_{i-1}}^{t_i} + s\int_{t_{i-1}}^{t_i} e^{-st} f(t)\,dt \\ &= e^{-st_i} f(t_i) - e^{-st_{i-1}} f(t_{i-1}) + s\int_{t_{i-1}}^{t_i} e^{-st} f(t)\,dt.\end{aligned}$$

Summing both sides of this equation from $i = 1$ to n and noting that

$$\begin{aligned}&\left(e^{-st_1} f(t_1) - e^{-st_0} f(t_0)\right) + \left(e^{-st_2} f(t_2) - e^{-st_1} f(t_1)\right) + \cdots + \left(e^{-st_N} f(t_N) - e^{-st_{N-1}} f(t_{N-1})\right) \\ &= e^{-st_N} f(t_N) - e^{-st_0} f(t_0) = e^{-sT} f(T) - f(0)\end{aligned}$$

yields Equation 9.3.2, so Equation 9.3.1 follows as before.

Example 9.3.1

In section 9.1 we saw that

$$\mathcal{L}(\cos\omega t) = \frac{s}{s^2+\omega^2}.$$

Solution

Applying Equation 9.3.1 with $f(t) = \cos\omega t$ shows that

$$\mathcal{L}(\sin(\omega t)) = \mathcal{L}\left(\frac{-1}{\omega}\frac{d}{dt}\cos(\omega t)\right) = \frac{-1}{\omega}\mathcal{L}\left(\frac{d}{dt}\cos(\omega t)\right) = \frac{-1}{\omega}\left(s\mathcal{L}(\cos(\omega t) - \cos(0)\right) = \frac{-1}{\omega}\left(s\frac{s}{s^2+\omega^2} - 1\right)$$
$$= \frac{-1}{\omega}\frac{-\omega^2}{s^2+\omega^2} = \frac{\omega}{s^2+\omega^2}.$$

Therefore

$$\mathcal{L}(\sin\omega t) = \frac{\omega}{s^2+\omega^2},$$

which agrees with the corresponding result obtained in section 9.1.

Using separation of variables from Chapter 2 we can show that the solution of the initial value problem

$$y' = ay, \quad y(0) = y_0, \tag{9.3.3}$$

is $y = y_0 e^{at}$. We'll now obtain this result by using the Laplace transform.

Let $Y(s) = \mathcal{L}(y)$ be the Laplace transform of the unknown solution of Equation 9.3.3. Taking Laplace transforms of both sides of Equation 9.3.3 yields

$$\mathcal{L}(y') = \mathcal{L}(ay),$$

which, by Theorem 9.3.1 , can be rewritten as

$$s\mathcal{L}(y) - y(0) = a\mathcal{L}(y),$$

or

$$sY(s) - y_0 = aY(s).$$

Solving for $Y(s)$ yields

$$Y(s) = \frac{y_0}{s-a},$$

so

$$y = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{y_0}{s-a}\right) = y_0\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = y_0 e^{at},$$

which agrees with the known result.

We need the next theorem to solve second order differential equations using the Laplace transform.

Theorem 9.3.2

Suppose f and f' are continuous on $[0,\infty)$ and of exponential order s_0, and that f'' is piecewise continuous on $[0,\infty)$. Then f, f', and f'' have Laplace transforms for $s > s_0$,

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0), \tag{9.3.4}$$

and

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0). \tag{9.3.5}$$

Proof

Theorem 9.3.1 implies that $\mathcal{L}(f')$ exists and satisfies Equation 9.3.4 for $s > s_0$. To prove that $\mathcal{L}(f'')$ exists and satisfies Equation 9.3.5 for $s > s_0$, we first apply Theorem 9.3.1 to $g = f'$. Since g satisfies the hypotheses of Theorem 9.3.1 , we conclude that $\mathcal{L}(g')$ is defined and satisfies

$$\mathcal{L}(g') = s\mathcal{L}(g) - g(0)$$

for $s > s_0$. However, since $g' = f''$, this can be rewritten as

$$\mathcal{L}(f'') = s\mathcal{L}(f') - f'(0) = s(s\mathcal{L}(f) - f(0)) - f'(0) = s^2\mathcal{L}(f) - sf(0) - f'(0),$$

which is Equation 9.3.5

Solving Second Order Equations with the Laplace Transform

We'll now use the Laplace transform to solve initial value problems for second order equations.

Example 9.3.2

Use the Laplace transform to solve the initial value problem

$$y'' - 6y' + 5y = 3e^{2t}, \quad y(0) = 2, \quad y'(0) = 3. \tag{9.3.6}$$

Solution

Taking Laplace transforms of both sides of the differential equation in Equation 9.3.6 yields

$$\mathcal{L}(y'' - 6y' + 5y) = \mathcal{L}\left(3e^{2t}\right) = \frac{3}{s-2},$$

which we rewrite as

$$\mathcal{L}(y'') - 6\mathcal{L}(y') + 5\mathcal{L}(y) = \frac{3}{s-2}. \tag{9.3.7}$$

Now denote $\mathcal{L}(y) = Y(s)$. Theorem 9.3.2 and the initial conditions in Equation 9.3.6 imply that

$$\mathcal{L}(y') = sY(s) - y(0) = sY(s) - 2$$

and

$$\mathcal{L}(y'') = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2s - 3.$$

Substituting from the last two equations into Equation 9.3.7 yields

$$\left(s^2Y(s) - 2s - 3\right) - 6\left(sY(s) - 2\right) + 5Y(s) = \frac{3}{s-2}.$$

Therefore

$$(s^2 - 6s + 5)Y(s) = \frac{3}{s-2} + (2s+3) + 6(-2), \tag{9.3.8}$$

so

$$(s-5)(s-1)Y(s) = \frac{3 + (s-2)(2s-9)}{s-2},$$

and

$$Y(s) = \frac{3 + (s-2)(2s-9)}{(s-2)(s-5)(s-1)}.$$

Partial fraction expansion gives us

$$Y(s) = -\frac{1}{s-2} + \frac{1}{2}\frac{1}{s-5} + \frac{5}{2}\frac{1}{s-1},$$

and taking the inverse transform of this yields

$$y = -e^{2t} + \frac{1}{2}e^{5t} + \frac{5}{2}e^{t}$$

as the solution of Equation 9.3.6.

From here on out we will solve these equations in a more streamlined manner, as the next example illustrates.

Example 9.3.3

Use the Laplace transform to solve the initial value problem

$$2y'' + 3y' + y = 8e^{-2t}, \quad y(0) = -4,\ y'(0) = 2. \tag{9.3.9}$$

Solution

Taking Laplace transforms of both sides of the differential equation in Equation 9.3.9 yields

$$2s^2Y(s) - 2sy(0) - 2y'(0) + 3sY(s) - 3y(0) + Y(s) = \frac{8}{s+2}.$$

Moving all $Y(s)$ terms to the left and all other terms to the right yields

$$(2s^2 + 3s + 1)Y(s) = \frac{8}{s+2} + 2(2 - 4s) + 3(-4).$$

which becomes

$$(2s+1)(s+1)Y(s) = \frac{8}{s+2} + 2(2 - 4s) + 3(-4).$$

Solving for $Y(s)$ yields

$$Y(s) = \frac{4\,(1 - (s+2)(s+1))}{(s+1/2)(s+1)(s+2)}.$$

The partial fraction expansion gives us

$$Y(s) = \frac{4}{3}\frac{1}{s+1/2} - \frac{8}{s+1} + \frac{8}{3}\frac{1}{s+2},$$

so the solution of Equation 9.3.9 is

$$y = \mathcal{L}^{-1}(Y(s)) = \frac{4}{3}e^{-t/2} - 8e^{-t} + \frac{8}{3}e^{-2t}$$

(Figure 9.3.1).

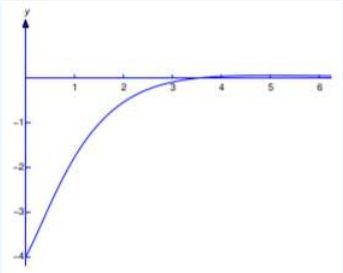


Figure 9.3.1 : $y = \frac{4}{3}e^{-t/2} - 8e^{-t} + \frac{8}{3}e^{-2t}$

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9.3.1: Solution of Initial Value Problems (Exercises)

In Exercises **1-35** use the Laplace transform to solve the initial value problem.

1. $y' - y = 1, \quad y(0) = 0$
2. $2y' + y = 0, \quad y(0) = -3$
3. $y' + 6y = e^{4t}, \quad y(0) = 2$
4. $y' - y = 2\cos 5t, \quad y(0) = 0$
5. $y'' + 5y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 0$
6. $y'' - 4y' = 6e^{3t} - 3e^{-t}, \quad y(0) = 1, \quad y'(0) = -1$
7. $y'' + y = \sqrt{2}\sin\sqrt{2}t, \quad y(0) = 10, \quad y'(0) = 0$
8. $y'' + 9y = e^{t}, \quad y(0) = 0, \quad y'(0) = 0$
9. $2y''' + 3y'' - 3y' - 2y = e^{-t}, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1$
10. $y''' + 2y'' - y' - 2y = \sin 3t, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1$
11. $y'' + 3y' + 2y = e^{t}, \quad y(0) = 1, \quad y'(0) = -6$
12. $y'' - y' - 6y = 2, \quad y(0) = 1, \quad y'(0) = 0$
13. $y'' + y' - 2y = 2e^{3t}, \quad y(0) = -1, \quad y'(0) = 4$
14. $y'' - 4y = 2e^{3t}, \quad y(0) = 1, \quad y'(0) = -1$
15. $y'' + y' - 2y = e^{3t}, \quad y(0) = 1, \quad y'(0) = -1$
16. $y'' + 3y' + 2y = 6e^{t}, \quad y(0) = 1, \quad y'(0) = -1$
17. $y'' + y = \sin 2t, \quad y(0) = 0, \quad y'(0) = 1$
18. $y'' - 3y' + 2y = 2e^{3t}, \quad y(0) = 1, \quad y'(0) = -1$
19. $y'' - 3y' + 2y = e^{4t}, \quad y(0) = 1, \quad y'(0) = -2$
20. $y'' - 3y' + 2y = e^{3t}, \quad y(0) = -1, \quad y'(0) = -4$
21. $y'' + 3y' + 2y = 2e^{t}, \quad y(0) = 0, \quad y'(0) = -1$
22. $y'' + y' - 2y = -4, \quad y(0) = 2, \quad y'(0) = 3$
23. $y'' + 4y = 4, \quad y(0) = 0, \quad y'(0) = 1$
24. $y'' - y' - 6y = 2, \quad y(0) = 1, \quad y'(0) = 0$
25. $y'' + 3y' + 2y = e^{t}, \quad y(0) = 0, \quad y'(0) = 1$
26. $y'' - y = 1, \quad y(0) = 1, \quad y'(0) = 0$
27. $y'' + 4y = 3\sin t, \quad y(0) = 1, \quad y'(0) = -1$
28. $y'' + y' = 2e^{3t}, \quad y(0) = -1, \quad y'(0) = 4$
29. $y'' + y = 1, \quad y(0) = 2, \quad y'(0) = 0$
30. $y'' + y = t, \quad y(0) = 0, \quad y'(0) = 2$
31. $y'' + y = t - 3\sin 2t, \quad y(0) = 1, \quad y'(0) = -3$
32. $y'' + 5y' + 6y = 2e^{-t}, \quad y(0) = 1, \quad y'(0) = 3$
33. $y'' - 2y' - 3y = 10\cos t, \quad y(0) = 2, \quad y'(0) = 7$
34. $2y'' - 3y' - 2y = 4e^{t}, \quad y(0) = 1, \; y'(0) = -2$
35. $6y'' - y' - y = 3e^{2t}, \quad y(0) = 0, \; y'(0) = 0$

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9.4: The First Shifting Theorem

In this section we look at theorems that will allow us to take transforms of more varied functions, which will allow us to solve more varied initial value problems as well.

Theorem 9.4.1 First Shifting Theorem

If

$$\mathcal{L}(f(t)) = F(s) \tag{9.4.1}$$

then

$$\mathcal{L}(e^{at} f(t)) = F(s-a). \tag{9.4.2}$$

Proof

$$\mathcal{L}(e^{at} f(t)) = \int_0^\infty e^{-st} e^{at} f(t) dt = \int_0^\infty e^{-(s-a)t} f(t) dt = F(s-a)$$

Example 9.4.1

Find $\mathcal{L}(t^3 e^{4t})$.

Solution

We know $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$.

Setting $n = 3$ in the above and $a = 4$ in the First Shifting Theorem yields

$$\mathcal{L}(t^3 e^{4t}) = \frac{3!}{(s-4)^4} = \frac{6}{(s-4)^4}.$$

Example 9.4.2

Find $\mathcal{L}(e^{7t} \cosh 2t)$.

Solution

We know $\mathcal{L} \cosh bt = \frac{s}{s^2 - b^2}$.

Setting $b = 2$ in the above and $a = 7$ in the First Shifting Theorem yields

$$\mathcal{L}(e^{7t} \cosh 2t) = \frac{s-7}{(s-7)^2 - 4}.$$

Example 9.4.3

Find

$$\mathcal{L}^{-1}\left(\frac{3s+8}{s^2+2s+5}\right).$$

Solution

Completing the square in the denominator yields

$$\frac{3s+8}{s^2+2s+5} = \frac{3s+8}{(s+1)^2+4}.$$

So,

$$\mathcal{L}^{-1}\left(\frac{3s+8}{(s+1)^2+4}\right) = \mathcal{L}^{-1}\left(\frac{3s+3}{(s+1)^2+4}\right)+\mathcal{L}^{-1}\left(\frac{5}{(s+1)^2+4}\right)$$
$$= 3\mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+4}\right)+\frac{5}{2}\mathcal{L}^{-1}\left(\frac{2}{(s+1)^2+4}\right)$$
$$= e^{-t}(3\cos 2t+\frac{5}{2}\sin 2t).$$

Example 9.4.4

Find

$$\mathcal{L}^{-1}\left(\frac{8-(s+2)(4s+10)}{(s+1)(s+2)^2}\right). \tag{9.4.3}$$

Solution

The form for the partial fraction expansion is

$$\left(\frac{8-(s+2)(4s+10)}{(s+1)(s+2)^2}\right) = \frac{A}{s+1}+\frac{B}{s+2}+\frac{C}{(s+2)^2}.$$

Partial fraction decomposition yields

$$\left(\frac{8-(s+2)(4s+10)}{(s+1)(s+2)^2}\right) = \frac{2}{s+1}-\frac{6}{s+2}-\frac{8}{(s+2)^2}$$

giving us

$$\mathcal{L}^{-1}\left(\frac{8-(s+2)(4s+10)}{(s+1)(s+2)^2}\right) = 2\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)-6\mathcal{L}^{-1}\left(\frac{1}{s+2}\right)-8\mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right)$$
$$= 2e^{-t}-6e^{-2t}-8te^{-2t}.$$

Example 9.4.5

Find

$$\mathcal{L}^{-1}\left(\frac{s^2-5s+7}{(s+2)^3}\right).$$

Solution

The form for the partial fraction expansion is

$$\frac{s^2-5s+7}{(s+2)^3} = \frac{A}{s+2}+\frac{B}{(s+2)^2}+\frac{C}{(s+2)^3}.$$

The easiest way to obtain A, B, and C is to expand the numerator in powers of $s+2$. This yields

$$s^2-5s+7 = [(s+2)-2]^2-5[(s+2)-2]+7 = (s+2)^2-9(s+2)+21.$$

Therefore

$$\frac{s^2-5s+7}{(s+2)^3} = \frac{(s+2)^2-9(s+2)+21}{(s+2)^3}$$
$$= \frac{1}{s+2}-\frac{9}{(s+2)^2}+\frac{21}{(s+2)^3}$$

and

$$\mathcal{L}^{-1}\left(\frac{s^2-5s+7}{(s+2)^3}\right) = \mathcal{L}^{-1}\left(\frac{1}{s+2}\right) - 9\mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right) + \frac{21}{2}\mathcal{L}^{-1}\left(\frac{2}{(s+2)^3}\right)$$
$$= e^{-2t}\left(1-9t+\frac{21}{2}t^2\right).$$

Example 9.4.6

Find

$$L^{-1}\left(\frac{1-s(5+3s)}{s\left[(s+1)^2+1\right]}\right).$$

Solution

One form for the partial fraction expansion of $\frac{1-s(5+3s)}{s\left[(s+1)^2+1\right]}$ is

$$\frac{1-s(5+3s)}{s\left[(s+1)^2+1\right]} = \frac{A}{s} + \frac{Bs+C}{(s+1)^2+1}. \tag{9.4.4}$$

However, we see from the table of Laplace transforms that the inverse transform of the second fraction on the right of Equation 9.4.4 will be a linear combination of the inverse transforms

$$e^{-t}\cos t \quad \text{and} \quad e^{-t}\sin t$$

of

$$\frac{s+1}{(s+1)^2+1} \quad \text{and} \quad \frac{1}{(s+1)^2+1}$$

respectively. Therefore, instead of Equation 9.4.4 we write

$$F(s) = \frac{A}{s} + \frac{B(s+1)+C}{(s+1)^2+1}. \tag{9.4.5}$$

Partial fraction decomposition yields

$$\frac{1-s(5+3s)}{s\left[(s+1)^2+1\right]} = \frac{1}{2s} - \frac{7}{2}\,\frac{s+1}{(s+1)^2+1} - \frac{5}{2}\,\frac{1}{(s+1)^2+1}.$$

Therefore

$$\mathcal{L}^{-1}\left(\frac{1-s(5+3s)}{s\left[(s+1)^2+1\right]}\right) = \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s}\right) - \frac{7}{2}\mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+1}\right) - \frac{5}{2}\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2+1}\right)$$
$$= \frac{1}{2} - \frac{7}{2}e^{-t}\cos t - \frac{5}{2}e^{-t}\sin t.$$

Example 9.4.7

Solve the initial value problem

$$y''+2y'+2y = 1, \quad y(0) = -3,\ y'(0) = 1. \tag{9.4.6}$$

Solution

Taking Laplace transforms of both sides of the differential equation in Equation 9.4.6 yields

$$s^2Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) + 2Y(s) = \frac{1}{s}.$$

This becomes

$$\left[(s+1)^2+1\right]Y(s)=\frac{1}{s}+1\cdot(1-3s)+2(-3).$$

Solving for $Y(s)$ yields

$$Y(s)=\frac{1-s(5+3s)}{s\left[(s+1)^2+1\right]}.$$

In Example 9.4.6 we found the inverse transform of this function to be

$$y=\frac{1}{2}-\frac{7}{2}e^{-t}\cos t-\frac{5}{2}e^{-t}\sin t$$

(Figure 9.4.2), which is therefore the solution of Equation 9.4.6.

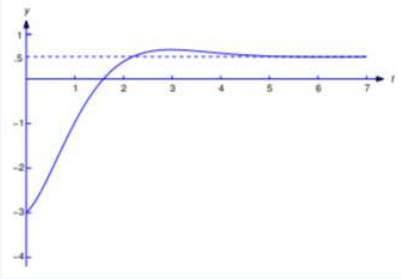


Figure 9.4.2 : $y=\frac{1}{2}-\frac{7}{2}e^{-t}\cos t-\frac{5}{2}e^{-t}\sin t$

Example 9.4.8

Solve the initial value problem

$$y''-6y'+9y=t^2e^{3t},\quad y(0)=2,\ y'(0)=6. \tag{9.4.7}$$

Solution

Taking Laplace transforms of both sides of the differential equation in Equation 9.4.7 yields

$$s^2Y(s)-sy(0)-y'(0)-6sY(s)+6y(0)+9Y(s)=\frac{2}{(s-3)^3}.$$

This becomes

$$(s-3)^2Y(s)=\frac{2}{(s-3)^3}+2s-6.$$

Solving for $Y(s)$ yields

$$Y(s)=\frac{2}{(s-3)^5}+\frac{2s}{(s-3)^2}-\frac{6}{(s-3)^2}=\frac{1}{12}\frac{4!}{(s-3)^5}+\frac{2(s-3)}{(s-3)^2}-\frac{6}{(s-3)^2}=\frac{1}{12}\frac{4!}{(s-3)^5}+\frac{2}{s-3}-\frac{6}{(s-3)^2}.$$

Taking the inverse Laplace transform gives us

$$y=\frac{1}{12}t^4e^{3t}+2e^{3t}-3te^{3t}$$

which is therefore the solution of Equation 9.4.7.

Example 9.4.9

Solve the initial value problem

$$y''+4y'+6y=1+e^{-t},\quad y(0)=0,\ y'(0)=0. \tag{9.4.8}$$

Solution

Taking Laplace transforms of both sides of the differential equation in Equation 9.4.8 yields

$$s^2Y(s)-sy(0)-y'(0)+4sY(s)-4y(0)+6Y(s)=\frac{1}{s}+\frac{1}{s+1}.$$

This becomes

$$[(s+2)^2+2]Y(s)=\frac{1}{s}+\frac{1}{s+1}.$$

Solving for $Y(s)$ yields

$$Y(s)=\frac{1}{s[(s+2)^2+2]}+\frac{1}{(s+1)[(s+2)^2+2]}=\frac{1/6}{s}+\frac{1/3}{s+1}-\frac{1}{2}\frac{s+2}{(s+2)^2+2}-\frac{2}{3}\frac{1}{(s+2)^2+2}.$$

Taking the inverse Laplace transform gives us

$$y=\frac{1}{6}+\frac{1}{3}e^{-t}-\frac{1}{2}e^{-2t}\cos\sqrt{2}t-\frac{\sqrt{2}}{3}e^{-2t}\sin\sqrt{2}t$$

which is therefore the solution of Equation 9.4.8.

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9.4.1: The First Shifting Theorem (Exercises)

In Exercises **1-5** find the Laplace transform of the function.

1. te^{10t}
2. t^3e^{-2t}
3. $e^t \sin 3t$
4. $e^{6t}(1+3t-t^2)$
5. $(1+e^t-3e^{-2t})\cosh 4t$

In Exercises **6-34** find the inverse Laplace transform of the function.

6. $\frac{3}{(s-7)^4}$
7. $\frac{2s-4}{s^2-4s+13}$
8. $\frac{1}{s^2+4s+20}$
9. $\frac{1}{(s-2)^2-4}$
10. $\frac{2}{(s-3)^2-9}$
11. $\frac{2s+3}{(s-7)^4}$
12. $\frac{s^2-1}{(s-2)^6}$
13. $\frac{s+5}{s^2+6s+18}$
14. $\frac{s}{s^2+2s+1}$
15. $\frac{s^3+2s^2-s-3}{(s+1)^4}$
16. $\frac{2s+3}{(s-1)^2+4}$
17. $\frac{3}{(s+2)^2}-\frac{2s+6}{s^2+4}$
18. $\frac{3s^2+2s+1}{(s^2+1)(s^2+2s+2)}$
19. $\frac{3s+2}{(s-2)(s^2+2s+5)}$
20. $\frac{17s-15}{(s^2-2s+5)(s^2+2s+10)}$
21. $\frac{8s+56}{(s^2-6s+13)(s^2+2s+5)}$
22. $\frac{s+9}{(s^2+4s+5)(s^2-4s+13)}$
23. $\frac{3s-2}{(s^2-4s+5)(s^2-6s+13)}$
24. $\frac{3s-1}{(s^2-2s+2)(s^2+2s+5)}$
25. $\frac{20s+40}{(4s^2-4s+5)(4s^2+4s+5)}$
26. $\frac{1}{(s-1)(s^2-2s+17)}$
27. $\frac{3s+2}{(s-2)(s^2+2s+10)}$
28. $\frac{34-17s}{(2s-1)(s^2-2s+5)}$
29. $\frac{s+2}{(s-3)(s^2+2s+5)}$

30. $\frac{2s-2}{(s-2)(s^2+2s+10)}$
31. $\frac{s+2}{(s^2+2s+2)(s^2-1)}$
32. $\frac{2s-1}{(s^2-2s+2)(s+1)(s-2)}$
33. $\frac{2s-3}{s(s-2)(s^2-2s+5)}$
34. $\frac{5s-15}{(s^2-4s+13)(s-2)(s-1)}$

In Exercises **35-49** solve the initial value problem using Laplace transforms.

35. $y'+4y=e^{-4t}, \quad y(0)=2,$
36. $y''+2y'+y=0, \quad y(0)=1,\ y'(0)=1$
37. $y''-6y'+9y=t, \quad y(0)=0,\ y'(0)=1$
38. $y''-6y'+13y=0, \quad y(0)=0,\ y'(0)=-3$
39. $y''-y'=e^t\cos t, \quad y(0)=0,\ y'(0)=0$
40. $y''+2y'+y=6\sin t-4\cos t, \quad y(0)=-1,\ y'(0)=1$
41. $y''-5y'+6y=10e^t\cos t, \quad y(0)=2, \quad y'(0)=1$
42. $y''+2y'+2y=2t, \quad y(0)=2, \quad y'(0)=-7$
43. $y''-2y'+2y=5\sin t+10\cos t, \quad y(0)=1,\ y'(0)=2$
44. $y''+4y'+13y=10e^{-t}-36e^t, \quad y(0)=0,\ y'(0)=-16$
45. $y''+4y'+5y=e^{-t}(\cos t+3\sin t), \quad y(0)=0, \quad y'(0)=4$
46. $2y''+2y'+y=2t, \quad y(0)=1,\ y'(0)=-1$
47. $4y''-4y'+5y=4\sin t-4\cos t, \quad y(0)=0,\ y'(0)=11/17$
48. $4y''+4y'+y=3\sin t+\cos t, \quad y(0)=2,\ y'(0)=-1$
49. $9y''+6y'+y=3e^{3t}, \quad y(0)=0,\ y'(0)=-3$

50. A 4 pound weight stretches a spring 2 feet. The weight is released from rest 18 inches above equilibrium, and the motion takes place in a medium offering a damping force equal to $7/8$ times the instantaneous velocity. Use Laplace transforms to find the equation of motion.

51. Given that $f(t)\leftrightarrow F(s)$, find the inverse Laplace transform of $F(as-b)$, where $a>0$.

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9.5: The Second Shifting Theorem and Piecewise Continuous Forcing Functions

Laplace Transforms of Piecewise Continuous Functions

We'll need to consider initial value problems

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1,$$

where a, b, and c are constants and f is piecewise continuous. Here we'll develop procedures to find Laplace transforms of piecewise continuous functions, and to find the piecewise continuous inverses of Laplace transforms, which will allow us to solve these initial value problems..

Definition 9.5.1 Unit Step Function

For $a > 0$, the *unit step function* is given by

$$\mathcal{U}(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a. \end{cases} \tag{9.5.1}$$

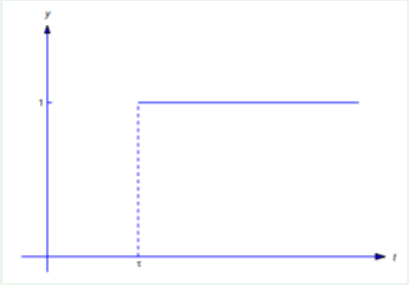


Figure 9.5.2 : $y = \mathcal{U}(t-a)$

We'll now develop a systematic way to find the Laplace transform of a piecewise continuous function.

The step function enables us to represent piecewise continuous functions conveniently. For example, consider the function

$$f(t) = \begin{cases} f_0(t), & 0 \leq t < t_1, \\ f_1(t), & t \geq t_1, \end{cases} \tag{9.5.2}$$

where we assume that f_0 and f_1 are defined on $[0, \infty)$, even though they equal f only on the indicated intervals. This assumption enables us to rewrite Equation 9.5.2 as

$$f(t) = f_0(t) + \mathcal{U}(t-t_1)\left(f_1(t) - f_0(t)\right). \tag{9.5.3}$$

To verify this, note that if $t < t_1$ then $\mathcal{U}(t-t_1) = 0$ and Equation 9.5.3 becomes

$$f(t) = f_0(t) + (0)\left(f_1(t) - f_0(t)\right) = f_0(t).$$

If $t \geq t_1$ then $\mathcal{U}(t-t_1) = 1$ and Equation 9.5.3 becomes

$$f(t) = f_0(t) + (1)\left(f_1(t) - f_0(t)\right) = f_1(t).$$

We need the next theorem to show how Equation 9.5.3 can be used to find $\mathcal{L}(f)$.

Theorem 9.5.1

Let g be defined on $[0, \infty)$. Suppose $a \geq 0$ and $\mathcal{L}\left(g(t+a)\right)$ exists for $s > s_0$. Then $\mathcal{L}\left(\mathcal{U}(t-a)g(t)\right)$ exists for $s > s_0$, and

$$\mathcal{L}(\mathcal{U}(t-a)g(t)) = e^{-sa}\mathcal{L}\left(g(t+a)\right).$$

Proof

By definition,

$$\mathcal{L}\left(\mathcal{U}(t-a)g(t)\right)=\int_0^\infty e^{-st}\mathcal{U}(t-a)g(t)\,dt.$$

From this and the definition of $\mathcal{U}(t-a)$,

$$\mathcal{L}\left(\mathcal{U}(t-a)g(t)\right)=\int_0^a e^{-st}\,(0)\,dt+\int_a^\infty e^{-st}\,g(t)\,dt.$$

The first integral on the right equals zero. Introducing the new variable of integration $x=t-a$ in the second integral yields

$$\mathcal{L}\left(\mathcal{U}(t-a)g(t)\right)=\int_0^\infty e^{-s(x+a)}\,g(x+a)\,dx=e^{-sa}\int_0^\infty e^{-sx}\,g(x+a)\,dx.$$

Changing the name of the variable of integration in the last integral from x to t yields

$$\mathcal{L}\left(\mathcal{U}(t-a)g(t)\right)=e^{-sa}\int_0^\infty e^{-st}\,g(t+a)\,dt=e^{-sa}\,\mathcal{L}(g(t+a)).$$

Example 9.5.1

Find

$$\mathcal{L}\left(\mathcal{U}(t-1)(t^2+1)\right).$$

Solution

Here $a=1$ and $g(t)=t^2+1$, so

$$g(t+1)=(t+1)^2+1=t^2+2t+2.$$

Since

$$\mathcal{L}\left(g(t+1)\right)=\frac{2}{s^3}+\frac{2}{s^2}+\frac{2}{s},$$

Theorem 9.5.1 implies that

$$\mathcal{L}\left(\mathcal{U}(t-1)(t^2+1)\right)=e^{-s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{2}{s}\right).$$

Example 9.5.2

Find the Laplace transform of the function

$$f(t)=\begin{cases}2t+1, & 0\le t<2,\\ 3t, & t\ge 2,\end{cases}$$

Solution

We first write f in the form Equation 9.5.3 as

$$f(t)=2t+1+\mathcal{U}(t-2)(t-1).$$

Therefore

$$\begin{aligned}\mathcal{L}(f) &=\mathcal{L}(2t+1)+\mathcal{L}\left(\mathcal{U}(t-2)(t-1)\right)\\ &=\mathcal{L}(2t+1)+e^{-2s}\mathcal{L}(t+1)\quad\text{(from Theorem 9.5.1)}\\ &=\frac{2}{s^2}+\frac{1}{s}+e^{-2s}\left(\frac{1}{s^2}+\frac{1}{s}\right)\end{aligned}.$$

Formula Equation 9.5.3 can be extended to more general piecewise continuous functions. For example, we can write

$$f(t)=\begin{cases} f_0(t), & 0\le t<t_1,\\ f_1(t), & t_1\le t<t_2,\\ f_2(t), & t\ge t_2,\end{cases}$$

as

$$f(t)=f_0(t)+\mathcal{U}(t-t_1)\,(f_1(t)-f_0(t))+\mathcal{U}(t-t_2)\,(f_2(t)-f_1(t))$$

if f_0, f_1, and f_2 are all defined on $[0,\infty)$.

This idea can be extended for any number of pieces of a piecewise continuous function.

Example 9.5.3

Find the Laplace transform of

$$f(t)=\begin{cases} 1, & 0\le t<2,\\ -2t+1, & 2\le t<3,\\ 3t, & 3\le t<5,\\ t-1, & t\ge 5\end{cases} \tag{9.5.4}$$

(Figure 9.5.3).

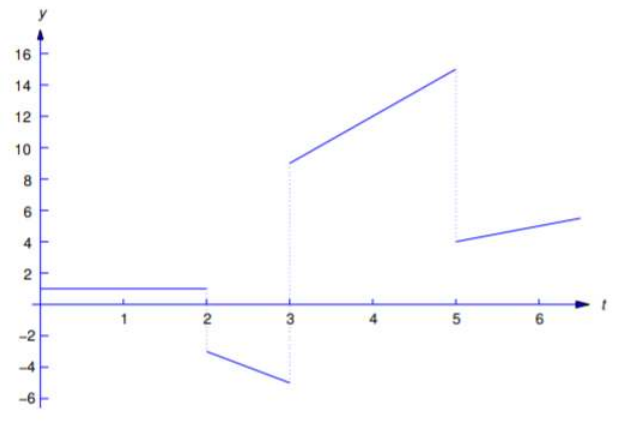


Figure 9.5.3 : The piecewise continuous function Equation 9.5.4

Solution

In terms of step functions,

$$f(t) \quad =1+\mathcal{U}(t-2)(-2t+1-1)+\mathcal{U}(t-3)(3t+2t-1)+\mathcal{U}(t-5)(t-1-3t),$$

or

$$f(t)=1-2\mathcal{U}(t-2)t+\mathcal{U}(t-3)(5t-1)-\mathcal{U}(t-5)(2t+1).$$

Now Theorem 9.5.1 implies that

$$\begin{aligned}\mathcal{L}(f) &=\mathcal{L}(1)-2e^{-2s}\mathcal{L}(t+2)+e^{-3s}\mathcal{L}\,(5(t+3)-1)-e^{-5s}\mathcal{L}\,(2(t+5)+1)\\ &=\mathcal{L}(1)-2e^{-2s}\mathcal{L}(t+2)+e^{-3s}\mathcal{L}(5t+14)-e^{-5s}\mathcal{L}(2t+11)\\ &=\frac{1}{s}-2e^{-2s}\left(\frac{1}{s^2}+\frac{2}{s}\right)+e^{-3s}\left(\frac{5}{s^2}+\frac{14}{s}\right)-e^{-5s}\left(\frac{2}{s^2}+\frac{11}{s}\right).\end{aligned}$$

The trigonometric identities

$$\sin(A+B)=\sin A\cos B+\cos A\sin B \tag{9.5.5}$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B \tag{9.5.6}$$

are useful in problems that involve shifting the arguments of trigonometric functions. We'll use these identities in the next example.

Example 9.5.4

Find the Laplace transform of

$$f(t) = \begin{cases} \sin t, & 0 \le t < \frac{\pi}{2} \\ \cos t - 3\sin t, & \frac{\pi}{2} \le t < \pi \\ 3\cos t, & t \ge \pi \end{cases} \tag{9.5.7}$$

(Figure 9.5.4).

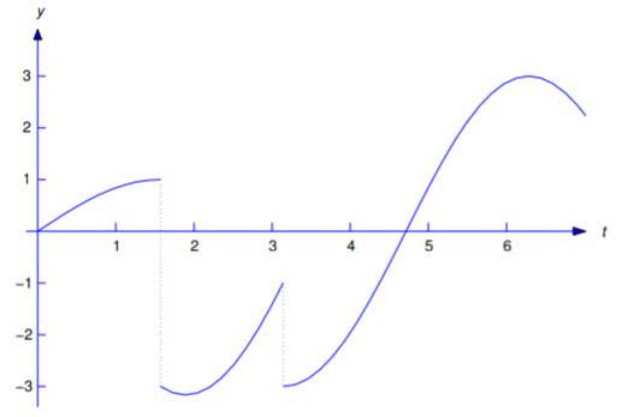


Figure 9.5.4 : The piecewise continuous function Equation 9.5.7.

Solution

In terms of step functions,

$$f(t) = \sin t + \mathcal{U}(t-\pi/2)(\cos t - 4\sin t) + \mathcal{U}(t-\pi)(2\cos t + 3\sin t).$$

Now Theorem 9.5.1 implies that

$$\begin{aligned} \mathcal{L}(f) \quad = \quad & \mathcal{L}(\sin t) + e^{-\frac{\pi}{2}s}\mathcal{L}\left(\cos\left(t+\tfrac{\pi}{2}\right) - 4\sin\left(t+\tfrac{\pi}{2}\right)\right) \\ & + e^{-\pi s}\mathcal{L}\left(2\cos(t+\pi) + 3\sin(t+\pi)\right). \end{aligned} \tag{9.5.8}$$

Since

$$\cos\left(t+\frac{\pi}{2}\right) - 4\sin\left(t+\frac{\pi}{2}\right) = -\sin t - 4\cos t$$

and

$$2\cos(t+\pi) + 3\sin(t+\pi) = -2\cos t - 3\sin t,$$

we see from Equation 9.5.8 that

$$\begin{aligned} \mathcal{L}(f) \ &= \mathcal{L}(\sin t) - e^{-\pi s/2}\mathcal{L}(\sin t + 4\cos t) - e^{-\pi s}\mathcal{L}(2\cos t + 3\sin t) \\ &= \frac{1}{s^2+1} - e^{-\frac{\pi}{2}s}\left(\frac{1+4s}{s^2+1}\right) - e^{-\pi s}\left(\frac{3+2s}{s^2+1}\right). \end{aligned}$$

The Second Shifting Theorem

Replacing $g(t)$ by $g(t-a)$ in Theorem 9.5.2 yields the next theorem.

Theorem 9.5.2 The Second Shifting Theorem

If $a \geq 0$ and $\mathcal{L}(g)$ exists for $s > s_0$ then $\mathcal{L}\left(\mathcal{U}(t-a)g(t-a)\right)$ exists for $s > s_0$ and

$$\mathcal{L}(\mathcal{U}(t-a)g(t-a)) = e^{-sa}\mathcal{L}(g(t)),$$

or, equivalently,

$$\text{if } g(t) \leftrightarrow G(s), \text{ then } \mathcal{U}(t-a)g(t-a) \leftrightarrow e^{-sa}G(s). \tag{9.5.9}$$

Note

Recall that the First Shifting Theorem states that multiplying a function by e^{at} corresponds to shifting the argument of its transform by a units. The Second Shifting Theorem states that multiplying a Laplace transform by the exponential e^{-as} corresponds to shifting the argument of the inverse transform by a units.

Example 9.5.5

Use Equation 9.5.9 to find

$$\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2}\right).$$

Solution

To apply Equation 9.5.9 we let $a = 2$ and $G(s) = 1/s^2$. Then $g(t) = t$ and Equation 9.5.9 implies that

$$\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2}\right) = \mathcal{U}(t-2)(t-2).$$

Example 9.5.6

Find the inverse Laplace transform h of

$$H(s) = \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{2}{s}\right) + e^{-4s}\left(\frac{4}{s^3} + \frac{1}{s}\right),$$

and find distinct formulas for h on appropriate intervals.

Solution

Let

$$G_0(s) = \frac{1}{s^2}, \quad G_1(s) = \frac{1}{s^2} + \frac{2}{s}, \quad G_2(s) = \frac{4}{s^3} + \frac{1}{s}.$$

Then

$$g_0(t) = t, \ g_1(t) = t+2, \ g_2(t) = 2t^2 + 1.$$

Hence, Equation 9.5.9 and the linearity of $\mathcal{L}^{-1}$ imply that

$$\begin{aligned} h(t) &= \mathcal{L}^{-1}\left(G_0(s)\right) - \mathcal{L}^{-1}\left(e^{-s}G_1(s)\right) + \mathcal{L}^{-1}\left(e^{-4s}G_2(s)\right) \\ &= t - \mathcal{U}(t-1)\left[(t-1)+2\right] + \mathcal{U}(t-4)\left[2(t-4)^2+1\right] \\ &= t - \mathcal{U}(t-1)(t+1) + \mathcal{U}(t-4)(2t^2 - 16t + 33), \end{aligned}$$

which can also be written as

$$h(t) = \begin{cases} t, & 0 \leq t < 1, \\ -1, & 1 \leq t < 4, \\ 2t^2 - 16t + 32, & t \geq 4. \end{cases}$$

Example 9.5.7

Find the inverse transform of

$$H(s)=\frac{2s}{s^2+4}-e^{-\frac{\pi}{2}s}\frac{3s+1}{s^2+9}+e^{-\pi s}\frac{s+1}{s^2+6s+10}.$$

Solution

Let

$$G_0(s)=\frac{2s}{s^2+4},\quad G_1(s)=-\frac{(3s+1)}{s^2+9},$$

and

$$G_2(s)=\frac{s+1}{s^2+6s+10}=\frac{(s+3)-2}{(s+3)^2+1}.$$

Then

$$g_0(t)=2\cos 2t,\quad g_1(t)=-3\cos 3t-\frac{1}{3}\sin 3t,$$

and

$$g_2(t)=e^{-3t}(\cos t-2\sin t).$$

Therefore Equation 9.5.9 and the linearity of $\mathcal{L}^{-1}$ imply that

$$h(t)\quad=2\cos 2t-\mathcal{U}(t-\pi/2)\left[3\cos 3(t-\pi/2)+\frac{1}{3}\sin 3\left(t-\frac{\pi}{2}\right)\right]+\mathcal{U}(t-\pi)e^{-3(t-\pi)}\left[\cos(t-\pi)-2\sin(t-\pi)\right].$$

Using the trigonometric identities Equation 9.5.5 and Equation 9.5.6, we can rewrite this as

$$h(t)\quad=2\cos 2t+\mathcal{U}(t-\pi/2)\left(3\sin 3t-\frac{1}{3}\cos 3t\right)+\mathcal{U}(t-\pi)e^{-3(t-\pi)}(\cos t-2\sin t)\tag{9.5.10}$$

(Figure 9.5.5).

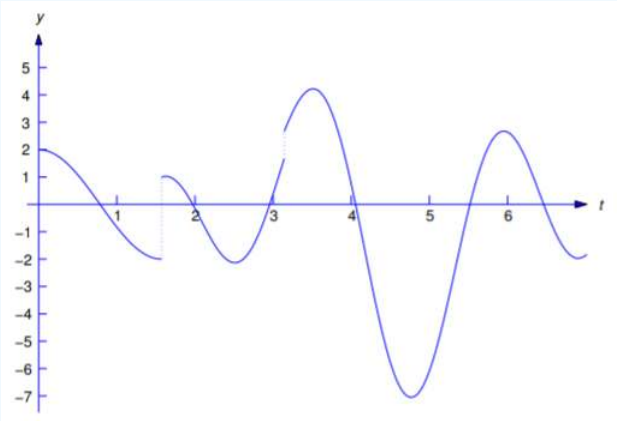


Figure 9.5.5 : The piecewise continuous function Equation 9.5.10.

We'll now consider initial value problems of the form

$$ay''+by'+cy=f(t),\quad y(0)=k_0,\quad y'(0)=k_1,\tag{9.5.11}$$

where a, b, and c are constants ($a\neq 0$) and f is piecewise continuous on $[0,\infty)$. Problems of this kind occur in situations where the input to a physical system undergoes instantaneous changes, as when a switch is turned on or off or the forces acting on the system change abruptly.

It can be shown that the differential equation in Equation 9.5.11 has no solutions on an open interval that contains a jump discontinuity of f. Therefore we must define what we mean by a solution of Equation 9.5.11 on $[0,\infty)$ in the case where f has jump discontinuities. The next theorem motivates our definition. We omit the proof.

Theorem 9.5.3

Suppose a, b, and c are constants $(a \neq 0)$, and f is piecewise continuous on $[0,\infty)$. with jump discontinuities at $t_1, \ldots, t_n$, where

$$0 < t_1 < \cdots < t_n.$$

Let k_0 and k_1 be arbitrary real numbers. Then there is a unique function y defined on $[0,\infty)$ with these properties:

a. $y(0) = k_0$ and $y'(0) = k_1$.
b. y and y' are continuous on $[0,\infty)$.
c. y'' is defined on every open subinterval of $[0,\infty)$ that does not contain any of the points $t_1, \ldots, t_n$, and

$$ay'' + by' + cy = f(t)$$

on every such subinterval.
d. y'' has limits from the right and left at $t_1, \ldots, t_n$.

We define the function y of Theorem 9.5.3 to be the solution of the initial value problem Equation 9.5.11.

We begin by considering initial value problems of the form

$$ay'' + by' + cy = \begin{cases} f_0(t), & 0 \le t < t_1, \\ f_1(t), & t \ge t_1, \end{cases} \qquad y(0) = k_0, \quad y'(0) = k_1, \tag{9.5.12}$$

where the forcing function has a single jump discontinuity at t_1.

It can be shown that y' exists and is continuous at t_1. The next example illustrates this and shows how to solve this kind of intitial value problem **without** Laplace transforms.

Example 9.5.8

Solve the initial value problem

$$y'' + y = f(t), \quad y(0) = 2,\ y'(0) = -1, \tag{9.5.13}$$

where

$$f(t) = \begin{cases} 1, & 0 \le t < \frac{\pi}{2}, \\ -1, & t \ge \frac{\pi}{2}. \end{cases}$$

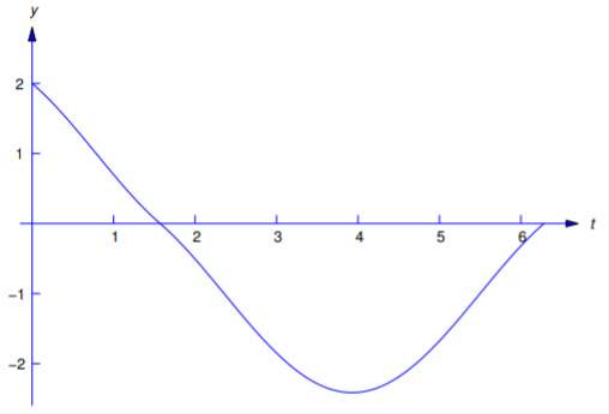


Figure 9.5.1 : Graph of Equation 9.5.14

Solution

The initial value problem when $0 \le t < \frac{\pi}{2}$ is

$$y'' + y = 1, \quad y(0) = 2, \quad y'(0) = -1.$$

We leave it to you to verify that its solution is

$$y_0 = 1 + \cos t - \sin t.$$

To solve the initial value problem when $t \ge \frac{\pi}{2}$ we need to find new initial conditions, and we use $t = \pi/2$ to get $y_0(\pi/2) = 0$ and $y_0'(\pi/2) = -1$, so the second initial value problem is

$$y'' + y = -1, \quad y\left(\frac{\pi}{2}\right) = 0, \; y'\left(\frac{\pi}{2}\right) = -1.$$

We leave it to you to verify that the solution of this problem is

$$y_1 = -1 + \cos t + \sin t.$$

Hence, the solution of Equation 9.5.13 is

$$y = \begin{cases} 1 + \cos t - \sin t, & 0 \le t < \frac{\pi}{2}, \\ -1 + \cos t + \sin t, & t \ge \frac{\pi}{2} \end{cases} \tag{9.5.14}$$

This can quickly become unwieldy if we have many pieces, and it is this kind of initial value problem where Laplace transforms are particularly effective.

If f_0 and f_1 are defined on $[0, \infty)$, we can rewrite Equation 9.5.12 as

$$ay'' + by' + cy = f_0(t) + \mathcal{U}(t - t_1)\left(f_1(t) - f_0(t)\right), \quad y(0) = k_0, \quad y'(0) = k_1,$$

and apply the method of Laplace transforms. We'll now solve the problem considered in Example 9.5.8 by this method.

Example 9.5.9

Use the Laplace transform to solve the initial value problem

$$y'' + y = f(t), \quad y(0) = 2, \; y'(0) = -1, \tag{9.5.15}$$

where

$$f(t) = \begin{cases} 1, & 0 \le t < \frac{\pi}{2}, \\ -1, & \frac{t \ge \pi}{2.} \end{cases}$$

Solution

Here

$$f(t) = 1 - 2\mathcal{U}\left(t - \frac{\pi}{2}\right),$$

so Theorem 9.5.1 (with $g(t) = 1$) implies that

$$\mathcal{L}(f) = \frac{1 - 2e^{-\pi s/2}}{s}.$$

Therefore, transforming Equation 9.5.15 yields

$$(s^2 + 1)Y(s) = \frac{1 - 2e^{-\pi s/2}}{s} - 1 + 2s,$$

so

$$Y(s) = (1 - 2e^{-\pi s/2})G(s) + \frac{2s - 1}{s^2 + 1}, \tag{9.5.16}$$

with

$$G(s)=\frac{1}{s(s^2+1)}.$$

The form for the partial fraction expansion of G is

$$\frac{1}{s(s^2+1)}=\frac{A}{s}+\frac{Bs+C}{s^2+1}. \tag{9.5.17}$$

Multiplying through by $s(s^2+1)$ yields

$$A(s^2+1)+(Bs+C)s=1,$$

or

$$(A+B)s^2+Cs+A=1.$$

Equating coefficients of like powers of s on the two sides of this equation shows that $A=1$, $B=-A=-1$ and $C=0$. Hence, from Equation 9.5.17,

$$G(s)=\frac{1}{s}-\frac{s}{s^2+1}.$$

Therefore

$$g(t)=1-\cos t.$$

From this, Equation 9.5.16, and Theorem 8.4.2,

$$y=1-\cos t-2\mathcal{U}\left(t-\frac{\pi}{2}\right)\left(1-\cos\left(t-\frac{\pi}{2}\right)\right)+2\cos t-\sin t.$$

Simplifying this (recalling that $\cos(t-\pi/2)=\sin t$) yields

$$y=1+\cos t-\sin t-2\mathcal{U}\left(t-\frac{\pi}{2}\right)(1-\sin t),$$

or

$$y=\begin{cases}1+\cos t-\sin t, & 0\le t<\frac{\pi}{2}\\ -1+\cos t+\sin t, & t\ge\frac{\pi}{2}\end{cases}$$

which is the result obtained in Example 9.5.8 .

Note

It isn't obvious that using the Laplace transform to solve Equation 9.5.12 as we did in Example 9.5.9 yields a function y with the properties stated in Theorem 9.5.3 ; that is, such that y and y' are continuous on $[0,\infty)$ and y'' has limits from the right and left at t_1. However, this is true if f_0 and f_1 are continuous and of exponential order on $[0,\infty)$.

Example 9.5.10

Solve the initial value problem

$$y''-y=f(t),\quad y(0)=-1,\ y'(0)=2, \tag{9.5.18}$$

where

$$f(t)=\begin{cases}t, & 0\le t<1,\\ 1, & t\ge 1.\end{cases}$$

Solution

Here

$$f(t) = t - \mathcal{U}(t-1)(t-1),$$

so

$$\begin{aligned}\mathcal{L}(f) &= \mathcal{L}(t) - \mathcal{L}\left(\mathcal{U}(t-1)(t-1)\right)\\ &= \mathcal{L}(t) - e^{-s}\mathcal{L}(t)\\ &= \frac{1}{s^2} - \frac{e^{-s}}{s^2}.\end{aligned}$$

Since transforming Equation 9.5.18 yields

$$(s^2-1)Y(s) = \mathcal{L}(f) + 2 - s,$$

we see that

$$Y(s) = (1-e^{-s})H(s) + \frac{2-s}{s^2-1}, \tag{9.5.19}$$

where

$$H(s) = \frac{1}{s^2(s^2-1)} = \frac{1}{s^2-1} - \frac{1}{s^2};$$

therefore

$$h(t) = \sinh t - t. \tag{9.5.20}$$

Since

$$\mathcal{L}^{-1}\left(\frac{2-s}{s^2-1}\right) = 2\sinh t - \cosh t,$$

we conclude from Equation 9.5.19, Equation 9.5.20, and Theorem 9.5.2 that

$$y = \sinh t - t - \mathcal{U}(t-1)\left(\sinh(t-1) - t + 1\right) + 2\sinh t - \cosh t,$$

or

$$y = 3\sinh t - \cosh t - t - \mathcal{U}(t-1)\left(\sinh(t-1) - t + 1\right) \tag{9.5.21}$$

We leave it to you to verify that y and y' are continuous and y'' has limits from the right and left at $t_1 = 1$.

Example 9.5.11

Solve the initial value problem

$$y'' + y = f(t), \quad y(0) = 0,\ y'(0) = 0, \tag{9.5.22}$$

where

$$f(t) = \begin{cases} 0, & 0 \le t < \frac{\pi}{4} \\ \cos 2t & \frac{\pi}{4} \le t < \pi \\ 0, & t \ge \pi \end{cases}$$

Solution

Here

$$f(t) = \mathcal{U}(t-\pi/4)\cos 2t - \mathcal{U}(t-\pi)\cos 2t,$$

so

$$\begin{aligned}\mathcal{L}(f) &= \mathcal{L}\left(\mathcal{U}(t-\pi/4)\cos 2t\right) - \mathcal{L}\left(\mathcal{U}(t-\pi)\cos 2t\right)\\ &= e^{-\pi s/4}\mathcal{L}\left(\cos 2(t+\pi/4)\right) - e^{-\pi s}\mathcal{L}\left(\cos 2(t+\pi)\right)\\ &= -e^{-\pi s/4}\mathcal{L}(\sin 2t) - e^{-\pi s}\mathcal{L}(\cos 2t)\\ &= -\frac{2e^{-\pi s/4}}{s^2+4} - \frac{se^{-\pi s}}{s^2+4}.\end{aligned}$$

Since transforming Equation 9.5.22 yields

$$(s^2+1)Y(s) = \mathcal{L}(f),$$

we see that

$$Y(s) = e^{-\pi s/4}H_1(s) + e^{-\pi s}H_2(s), \tag{9.5.23}$$

where

$$H_1(s) = -\frac{2}{(s^2+1)(s^2+4)} \quad \text{and} \quad H_2(s) = -\frac{s}{(s^2+1)(s^2+4)}. \tag{9.5.24}$$

To simplify the required partial fraction expansions, we first write

$$\frac{1}{(x+1)(x+4)} = \frac{1}{3}\left[\frac{1}{x+1} - \frac{1}{x+4}\right].$$

Setting $x = s^2$ and substituting the result in Equation 9.5.24 yields

$$H_1(s) = -\frac{2}{3}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right] \quad \text{and} \quad H_2(s) = -\frac{1}{3}\left[\frac{s}{s^2+1} - \frac{s}{s^2+4}\right].$$

The inverse transforms are

$$h_1(t) = -\frac{2}{3}\sin t + \frac{1}{3}\sin 2t \quad \text{and} \quad h_2(t) = -\frac{1}{3}\cos t + \frac{1}{3}\cos 2t.$$

From Equation 9.5.23 and Theorem 9.5.2 ,

$$y = \mathcal{U}\left(t-\frac{\pi}{4}\right)h_1\left(t-\frac{\pi}{4}\right) + \mathcal{U}(t-\pi)h_2(t-\pi). \tag{9.5.25}$$

Since

$$\begin{aligned}h_1\left(t-\frac{\pi}{4}\right) &= -\frac{2}{3}\sin\left(t-\frac{\pi}{4}\right) + \frac{1}{3}\sin 2\left(t-\frac{\pi}{4}\right)\\ &= -\frac{\sqrt{2}}{3}(\sin t - \cos t) - \frac{1}{3}\cos 2t\end{aligned}$$

and

$$\begin{aligned}h_2(t-\pi) &= -\frac{1}{3}\cos(t-\pi) + \frac{1}{3}\cos 2(t-\pi)\\ &= \frac{1}{3}\cos t + \frac{1}{3}\cos 2t,\end{aligned}$$

Equation 9.5.25 can be rewritten as

$$y = -\frac{1}{3}\mathcal{U}\left(t-\frac{\pi}{4}\right)\left(\sqrt{2}(\sin t - \cos t) + \cos 2t\right) + \frac{1}{3}\mathcal{U}(t-\pi)(\cos t + \cos 2t)$$

or

$$\begin{cases} 0, & 0 \le t < \frac{\pi}{4} \\ -\frac{\sqrt{2}}{3}(\sin t - \cos t) - \frac{1}{3}\cos 2t, & \frac{\pi}{4} \le t < \pi, \\ -\frac{\sqrt{2}}{3}\sin t + \frac{1+\sqrt{2}}{3}\cos t, & t \ge \pi \end{cases} \tag{9.5.26}$$

We leave it to you to verify that y and y' are continuous and y'' has limits from the right and left at $t_1 = \pi/4$ and $t_2 = \pi$ (Figure 9.5.2).

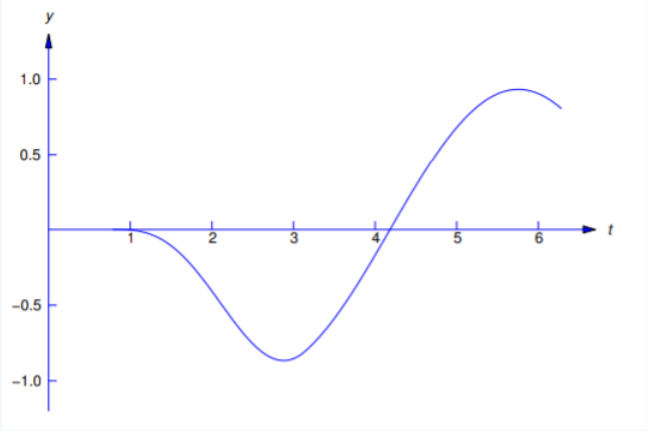


Figure 9.5.2 : Graph of Equation 9.5.26

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9.5.1: The Second Shifting Theorem and Piecewise Continuous Forcing Functions (Exercises)

In Exercises **1-18** express the given function f in terms of unit step functions, and then find $\mathcal{L}(f)$.

1. $f(t)=\begin{cases}1, & 0\le t<4,\\ t, & t\ge 4.\end{cases}$
2. $f(t)=\begin{cases}t, & 0\le t<1,\\ 1, & t\ge 1.\end{cases}$
3. $f(t)=\begin{cases}2t-1, & 0\le t<2,\\ t, & t\ge 2.\end{cases}$
4. $f(t)=\begin{cases}1, & 0\le t<1,\\ t+2, & t\ge 1.\end{cases}$
5. $f(t)=\begin{cases}t-1, & 0\le t<2,\\ 4, & t\ge 2.\end{cases}$
6. $f(t)=\begin{cases}t^2, & 0\le t<1,\\ 0, & t\ge 1.\end{cases}$
7. $f(t)=\begin{cases}0, & 0\le t<2,\\ t^2+3t, & t\ge 2.\end{cases}$
8. $f(t)=\begin{cases}t^2+2, & 0\le t<1,\\ t, & t\ge 1.\end{cases}$
9. $f(t)=\begin{cases}te^t, & 0\le t<1,\\ e^t, & t\ge 1.\end{cases}$
10. $f(t)=\begin{cases}e^{-t}, & 0\le t<1,\\ e^{-2t}, & t\ge 1.\end{cases}$
11. $f(t)=\begin{cases}-t, & 0\le t<2,\\ t-4, & 2\le t<3,\\ 1, & t\ge 3.\end{cases}$
12. $f(t)=\begin{cases}0, & 0\le t<1,\\ t, & 1\le t<2,\\ 0, & t\ge 2.\end{cases}$
13. $f(t)=\begin{cases}t, & 0\le t<1,\\ t^2, & 1\le t<2,\\ 0, & t\ge 2.\end{cases}$
14. $f(t)=\begin{cases}t, & 0\le t<1,\\ 2-t, & 1\le t<2,\\ 6, & t>2.\end{cases}$
15. $f(t)=\begin{cases}\sin t, & 0\le t<\frac{\pi}{2}\\ 2\sin t, & \frac{\pi}{2}\le t<\pi\\ \cos t, & t\ge\pi\end{cases}$

16. $f(t)=\begin{cases} 2, & 0\le t<1,\\ -2t+2, & 1\le t<3,\\ 3t, & t\ge 3.\end{cases}$

17. $f(t)=\begin{cases} 3, & 0\le t<2,\\ 3t+2, & 2\le t<4,\\ 4t, & t\ge 4.\end{cases}$

18. $f(t)=\begin{cases} (t+1)^2, & 0\le t<1,\\ (t+2)^2, & t\ge 1.\end{cases}$

In Exercises **19-28** find the inverse Laplace transforms and write the answers as piecewise functions.

19. $H(s)=\frac{e^{-2s}}{s-2}$

20. $H(s)=\frac{e^{-s}}{s(s+1)}$

21. $H(s)=\frac{e^{-s}}{s^3}+\frac{e^{-2s}}{s^2}$

22. $H(s)=\left(\frac{2}{s}+\frac{1}{s^2}\right)+e^{-s}\left(\frac{3}{s}-\frac{1}{s^2}\right)+e^{-3s}\left(\frac{1}{s}+\frac{1}{s^2}\right)$

23. $H(s)=\left(\frac{5}{s}-\frac{1}{s^2}\right)+e^{-3s}\left(\frac{6}{s}+\frac{7}{s^2}\right)+\frac{3e^{-6s}}{s^3}$

24. $H(s)=\frac{e^{-\pi s}(1-2s)}{s^2+4s+5}$

25. $H(s)=\left(\frac{1}{s}-\frac{s}{s^2+1}\right)+e^{-\frac{\pi}{2}s}\left(\frac{3s-1}{s^2+1}\right)$

26. $H(s)=e^{-2s}\left[\frac{3(s-3)}{(s+1)(s-2)}-\frac{s+1}{(s-1)(s-2)}\right]$

27. $H(s)=\frac{1}{s}+\frac{1}{s^2}+e^{-s}\left(\frac{3}{s}+\frac{2}{s^2}\right)+e^{-3s}\left(\frac{4}{s}+\frac{3}{s^2}\right)$

28. $H(s)=\frac{1}{s}-\frac{2}{s^3}+e^{-2s}\left(\frac{3}{s}-\frac{1}{s^3}\right)+\frac{e^{-4s}}{s^2}$

In Exercises **29-48** use the Laplace transform to solve the initial value problem.

29. $y''+y=\begin{cases} 3, & 0\le t<\pi,\\ 0, & t\ge \pi,\end{cases}\qquad y(0)=0,\quad y'(0)=0$

30. $y''+y=\begin{cases} 3, & 0\le t<4,\\ 2t-5, & t\ge 4,\end{cases}\qquad y(0)=1,\quad y'(0)=0$

31. $y''-2y'=\begin{cases} 4, & 0\le t<1,\\ 6, & t\ge 1,\end{cases}\qquad y(0)=-6,\quad y'(0)=1$

32. $y''-y=\begin{cases} e^{2t}, & 0\le t<2,\\ 1, & t\ge 2,\end{cases}\qquad y(0)=3,\quad y'(0)=-1$

33. $y''-3y'+2y=\begin{cases} 0, & 0\le t<1,\\ 1, & 1\le t<2,\\ -1, & t\ge 2,\end{cases}\qquad y(0)=-3,\quad y'(0)=1$

34. $y''+4y=\begin{cases} |\sin t|, & 0\le t<2\pi,\\ 0, & t\ge 2\pi,\end{cases}\qquad y(0)=-3,\quad y'(0)=1$

35. $y''-5y'+4y=\begin{cases}1, & 0\le t<1\\ -1, & 1\le t<2,\\ 0, & t\ge 2,\end{cases}\qquad y(0)=3,\quad y'(0)=-5$

36. $y''+9y=\begin{cases}\cos t, & 0\le t<\frac{3\pi}{2},\\ \sin t, & t\ge\frac{3\pi}{2},\end{cases}\qquad y(0)=0,\ y'(0)=0$

37. $y''+4y=\begin{cases}t, & 0\le t<\frac{\pi}{2},\\ \pi, & t\ge\frac{\pi}{2},\end{cases}\qquad y(0)=0,\ y'(0)=0$

38. $y''+y=\begin{cases}t, & 0\le t<\pi,\\ -t, & t\ge\pi,\end{cases}\qquad y(0)=0,\ y'(0)=0$

39. $y''-3y'+2y=\begin{cases}0, & 0\le t<2,\\ 2t-4, & t\ge 2,\end{cases},\quad y(0)=0,\quad y'(0)=0$

40. $y''+y=\begin{cases}t, & 0\le t<2\pi,\\ -2t, & t\ge 2\pi,\end{cases}\qquad y(0)=1,\quad y'(0)=2$

41. $y''+3y'+2y=\begin{cases}1, & 0\le t<2,\\ -1, & t\ge 2,\end{cases}\qquad y(0)=0,\ y'(0)=0$

42. $y''-4y'+3y=\begin{cases}-1, & 0\le t<1,\\ 1, & t\ge 1,\end{cases}\qquad y(0)=0,\ y'(0)=0$

43. $y''+2y'+y=\begin{cases}e^t, & 0\le t<1,\\ e^t-1, & t\ge 1,\end{cases}\qquad y(0)=3,\ y'(0)=-1$

44. $y''+2y'+y=\begin{cases}4e^t, & 0\le t<1,\\ 0, & t\ge 1,\end{cases}\qquad y(0)=0,\ y'(0)=0$

45. $y''+3y'+2y=\begin{cases}e^{-t}, & 0\le t<1,\\ 0, & t\ge 1,\end{cases}\qquad y(0)=1,\ y'(0)=-1$

46. $y''-4y'+4y=\begin{cases}e^{2t}, & 0\le t<2,\\ -e^{2t}, & t\ge 2,\end{cases}\qquad y(0)=0,\ y'(0)=-1$

47. $y''=\begin{cases}t^2, & 0\le t<1,\\ -t, & 1\le t<2,\\ t+1, & t\ge 2,\end{cases}\quad y(0)=1,\ y'(0)=0$

48. $y''+2y'+2y=\begin{cases}1, & 0\le t<2\pi,\\ t, & 2\pi\le t<3\pi,\\ -1, & t\ge 3\pi,\end{cases}\quad y(0)=2,\quad y'(0)=-1$

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9.6: Convolution and Periodic Functions

Derivative of a Transform

We have previously seen that if want to find $\mathcal{L}(e^{at}f(t))$ we take $\mathcal{L}(f(t))$ to get $F(s)$ and then shift s to get $\mathcal{L}(e^{at}f(t)) = F(s-a)$. So, as long as we know the transform of $f(t)$ we can easily find the transform of $e^{at}f(t)$.

We now consider what to do if we want to find $\mathcal{L}(t^n f(t))$. This is similar in that if we know the transform of $f(t)$ we can find the transform of $t^n f(t)$, though how we go about it is different.

Theorem 9.6.1 Derivative of a Transform

If

$$\mathcal{L}(f(t)) = F(s) \tag{9.6.1}$$

then

$$\mathcal{L}(t^n f(t)) = (-1)^n \frac{d^n}{ds^n} F(s). \tag{9.6.2}$$

Proof (by induction)

If n=1, then

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty e^{-st}f(t)dt = \int_0^\infty \frac{\partial}{\partial s}e^{-st}f(t)dt = \int_0^\infty -te^{-st}f(t)dt = \int_0^\infty -e^{-st}(tf(t))dt.$$

This gives us

$$\mathcal{L}(tf(t)) = -\frac{d}{ds}\mathcal{F}(s)$$

which proves the theorem for $n = 1$.

If $\frac{d^n}{ds^n}F(s) = (-1)^n\mathcal{L}(t^n f(t))$, then

$$\frac{d^{n+1}}{ds^{n+1}}F(s) = \frac{d}{ds}\left(\frac{d^n}{ds^n}F(s)\right) = \frac{d}{ds}\int_0^\infty e^{-st}(-1)^n t^n f(t)dt = \int_0^\infty \frac{\partial}{\partial s}e^{-st}(-1)^n t^n f(t)dt = \int_0^\infty -te^{-st}(-1)^n t^n f(t)dt =$$
$$\int_0^\infty (-1)^{n+1}e^{-st}(t^{n+1}f(t))dt.$$

This gives us

$$\mathcal{L}(t^{n+1}f(t)) = (-1)^{n+1}\frac{d^{n+1}}{ds^{n+1}}\mathcal{F}(s).$$

Therefore, $\mathcal{L}(t^n f(t)) = (-1)^n \frac{d^n}{ds^n}F(s)$ for all $n \geq 1$.

Example 9.6.1

Find $\mathcal{L}(t\sin t)$

Solution

$$\mathcal{L}(t\sin t) = -\frac{d}{ds}\mathcal{L}(\sin t) = -\frac{d}{ds}\frac{1}{s^2+1} = \frac{2s}{(s^2+1)^2}.$$

Example 9.6.2

Solve the initial value problem:

$$y'' + 16y = \cos 4t, \quad y(0) = 2, \quad y'(0) = 3. \tag{9.6.3}$$

Solution

Taking Laplace transforms we get

$$s^2Y(s) - sy(0) - y'(0) + 16Y(s) = \frac{s}{s^2+16}.$$

This becomes

$$(s^2+16)Y(s) = \frac{s}{s^2+16} + 2s + 3.$$

Solving for $Y(s)$ yields

$$Y(s) = \frac{s}{(s^2+16)^2} + \frac{2s}{s^2+16} + \frac{3}{s^2+16}.$$

At this point we would take the inverse Laplace transform, but we have an issue with the the inverse of $\frac{s}{(s^2+16)^2}$ since partial fraction decomposition will bring us right back to where we started. This is where Theorem 1 comes in:

$$\mathcal{L}^{-1}\left(\frac{s}{(s^2+16)^2}\right) = \mathcal{L}^{-1}\left(-\frac{1}{2}\frac{d}{ds}\frac{1}{s^2+16}\right) = \frac{1}{8}t\sin 4t.$$

Taking the inverse Laplace transform of the remaining two terms gives us

$$y = \frac{1}{8}t\sin 4t + 2\cos 4t + \frac{3}{4}\sin 4t$$

which is therefore the solution of Equation 9.6.3.

Convolution

We now consider the problem of finding the inverse Laplace transform of a product $H(s) = F(s)G(s)$, where $F(s)$ and $G(s)$ are the Laplace transforms of known functions $f(t)$ and $g(t)$. To motivate our interest in this problem, consider the initial value problem

$$ay'' + by' + cy = f(t), \quad y(0) = 0, \quad y'(0) = 0.$$

Taking Laplace transforms yields

$$(as^2 + bs + c)Y(s) = F(s),$$

so

$$Y(s) = F(s)G(s), \tag{9.6.4}$$

where

$$G(s) = \frac{1}{as^2+bs+c}.$$

Until now we haven't needed to consider the factorization indicated in Equation 9.6.4, but there will be times when we cannot take the inverse Laplace transform with techniques we have learned to this point.

To motivate the formula for $\mathcal{L}^{-1}(F(s)G(s))$, consider the initial value problem

$$y' - ay = f(t), \quad y(0) = 0, \tag{9.6.5}$$

where $f(t)$ is unknown. We first solve without using the Laplace transform. Equation 9.6.5 is a first order linear equation with integrating factor e^{-at}.

Using the methods of Section 2.3 to solve we get

$$y(t) = e^{at}\int_0^t e^{-au}f(u)\,du = \int_0^t e^{a(t-u)}f(u)\,du. \tag{9.6.6}$$

Now we'll use the Laplace transform to solve Equation 9.6.5 and compare the result to Equation 9.6.6. Taking Laplace transforms in Equation 9.6.5 yields

$$(s-a)Y(s) = F(s),$$

so

$$Y(s) = F(s)\frac{1}{s-a},$$

which implies that

$$y(t) = \mathcal{L}^{-1}\left(F(s)\frac{1}{s-a}\right). \tag{9.6.7}$$

If we now let $g(t) = e^{at}$, so that

$$G(s) = \frac{1}{s-a},$$

then Equation 9.6.6 and Equation 9.6.7 can be written as

$$y(t) = \int_0^t f(u)g(t-u)\,du$$

and

$$y(t) = \mathcal{L}^{-1}(F(s)G(s)),$$

respectively. Therefore

$$\mathcal{L}^{-1}(F(s)G(s)) = \int_0^t f(u)g(t-u)\,du \tag{9.6.8}$$

in this case.

This motivates the next definition.

Definition 9.6.1 Convolution

The **convolution** $f*g$ of two functions f and g is defined by

$$(f*g)(t) = \int_0^t f(u)g(t-u)\,du.$$

It can be shown (*Exercise 8.6.76*) that $f*g = g*f$; that is,

$$\int_0^t f(t-u)g(u)\,du = \int_0^t f(u)g(t-u)\,du.$$

Equation 9.6.5 shows that $\mathcal{L}^{-1}(FG) = f*g$ in the special case where $g(t) = e^{at}$. This next theorem states that this is true in general. The proof is beyond the scope of this book.

Theorem 9.6.2 The Convolution Theorem

If $\mathcal{L}(f) = F$ and $\mathcal{L}(g) = G$, then

$$\mathcal{L}(f*g) = FG.$$

Example 9.6.3

Let

$$f(t) = e^{at} \quad \text{and} \quad g(t) = e^{bt} \qquad (a \neq b).$$

Verify that $\mathcal{L}(f*g) = \mathcal{L}(f)\mathcal{L}(g)$, as implied by the convolution theorem.

Solution

We first compute

$$(f*g) = \int_0^t e^u e^{b(t-u)}\,du = e^{bt}\int_0^t e^{(a-b)u}du = e^{bt}\frac{e^{a-b}u}{a-b}\Big|_0^t = \frac{e^{bt}[e^{(a-b)t}-1]}{a-b} = \frac{e^{at}-e^{bt}}{a-b}$$

Since

$$e^{at} \leftrightarrow \frac{1}{s-a} \quad \text{and} \quad e^{bt} \leftrightarrow \frac{1}{s-b},$$

it follows that

$$\begin{aligned} \mathcal{L}(f*g) &= \frac{1}{a-b}\left[\frac{1}{s-a} - \frac{1}{s-b}\right] \\ &= \frac{1}{(s-a)(s-b)} \\ &= \mathcal{L}(e^{at})\mathcal{L}(e^{bt}) = \mathcal{L}(f)\mathcal{L}(g). \end{aligned}$$

Example 9.6.4

Solve the initial value problem:

$$y'' + 16y = \sin 4t, \quad y(0) = 2, \quad y'(0) = 3. \tag{9.6.9}$$

Solution

Taking Laplace transforms we get

$$s^2Y(s) - sy(0) - y'(0) + 16Y(s) = \frac{4}{s^2+16}.$$

This becomes

$$(s^2+16)Y(s) = \frac{4}{s^2+16} + 2s + 3.$$

Solving for $Y(s)$ yields

$$Y(s) = \frac{4}{(s^2+16)^2} + \frac{2s}{s^2+16} + \frac{3}{s^2+16}.$$

At this point we would take the inverse Laplace transform, but we have an issue with the the inverse of $\frac{4}{(s^2+16)^2}$ since partial fraction decomposition will bring us right back to where we started. This is where Theorem 2 comes in:

$$\mathcal{L}^{-1}\left(\frac{4}{(s^2+16)^2}\right) = \mathcal{L}^{-1}\left(\frac{1}{2s}\frac{-d}{ds}\frac{4}{s^2+16}\right) = \frac{1}{2} * t\sin 4t.$$

We know convolution is commutative, which leads to

$$\frac{1}{2} * t\sin 4t = t\sin 4t * \frac{1}{2} = \int_0^t (u\sin 4u)\frac{1}{2}\,du = \frac{-1}{8}u\cos 4u + \frac{1}{32}\sin 4u\Big|_0^t = \frac{-1}{8}t\cos 4t + \frac{1}{32}\sin 4t$$

Taking the inverse Laplace transform of the remaining two terms and adding them gives us

$$y = \frac{-1}{8}t\cos 4t + \frac{1}{32}\sin 4t + 2\cos 4t + \frac{3}{4}\sin 4t = \frac{-1}{8}t\cos 4t + 2\cos 4t + \frac{25}{32}\sin 4t$$

which is therefore the solution of Equation 9.6.9.

Volterra Integral Equations

An equation of the form

$$y(t) = f(t) + \int_0^t k(t-u)y(u)\,du \tag{9.6.10}$$

is a **Volterra integral equation**. Here f and k are given functions and y is unknown. Since the integral on the right is a convolution integral, the convolution theorem provides a convenient formula for solving Equation 9.6.10. Taking Laplace transforms in Equation 9.6.10yields

$$Y(s) = F(s) + K(s)Y(s),$$

and solving this for $Y(s)$ yields

$$Y(s) = \frac{F(s)}{1 - K(s)}.$$

We then obtain the solution of Equation 9.6.10 as $y = \mathcal{L}^{-1}(Y)$.

Example 9.6.5

Solve the integral equation

$$y(t) = 1 + 2\int_0^t e^{-2(t-u)} y(u)\, du. \tag{9.6.11}$$

Solution

We immediately note that the integral in Equation 9.6.11 is a convolution and write it as follows,

$$y(t) = 1 + 2e^{-2t} * y(t). \tag{9.6.12}$$

Taking Laplace transforms in Equation 9.6.12 yields

$$Y(s) = \frac{1}{s} + \frac{2}{s+2}Y(s),$$

and solving this for $Y(s)$ yields

$$Y(s) = \frac{1}{s} + \frac{2}{s^2}.$$

Hence,

$$y(t) = 1 + 2t.$$

is the solution to Equation 9.6.11.

Transform of a Periodic Function

Theorem 9.6.3

Suppose f is continuous on $[0, T]$ and $f(t+T) = f(t)$ for all $t \geq 0$. (We say in this case that f is *periodic with period* T.)

The Laplace transform of f is defined for $s > 0$ and

$$\mathcal{L}(f) = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\, dt, \quad s > 0.$$

Proof

$$\begin{aligned} \mathcal{L}(f) &= \int_0^\infty e^{-st} f(t)\, dt \\ &= \int_0^T e^{-st} f(t)\, dt + \int_T^\infty e^{-st} f(t)\, dt \end{aligned}$$

Letting $t = w + T$ in the second integral above we get

$$\int_T^\infty e^{-st} f(t)\, dt = \int_0^\infty e^{-s(w+T)} f(w+T)\, dw = e^{-sT} \int_0^\infty e^{-sw} f(w)\, dw = e^{-sT}\mathcal{L}(f)$$

So we now have

$$\mathcal{L}(f) = \int_0^T e^{-st} f(t)\, dt + e^{-sT}\mathcal{L}(f),$$

and solving for $\mathcal{L}(f)$ gives us our result:

$$\mathcal{L}(f) = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt, \quad s > 0.$$

Example 9.6.6

Find $\mathcal{L}|\sin t|$

Solution

$|\sin t|$ is periodic with period $T = \pi$. The previous theorem gives us

$$\begin{aligned}
\mathcal{L}|\sin t| &= \frac{1}{1-e^{-s\pi}} \int_0^\pi e^{-st} |\sin t|\, dt \\
&= \frac{1}{1-e^{-s\pi}} \int_0^\pi e^{-st} \sin t\, dt \\
&= \frac{1}{1-e^{-s\pi}} \left(\int_0^\pi e^{-st} \sin t\, dt + \int_\pi^\infty e^{-st}(0)\, dt \right) \\
&= \frac{1}{1-e^{-s\pi}} \mathcal{L}\left(\sin t - \sin t \mathcal{U}(t-\pi)\right) \\
&= \frac{1}{1-e^{-s\pi}} \mathcal{L}\left(\sin t + \sin(t-\pi)\mathcal{U}(t-\pi)\right) \\
&= \frac{1}{1-e^{-s\pi}} \left(\frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1} \right) \\
&= \frac{1}{1-e^{-s\pi}} \frac{1+e^{-\pi s}}{s^2+1}
\end{aligned}$$

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9.6.1: Convolution and Periodic Functions (Exercises)

In Exercises **1-27** find the Laplace transform of the given function.

1. $f(t) = t\cos 2t$
2. $f(t) = t^2 \sinh t$
3. $f(t) = te^{2t}\sin 6t$
4. $f(t) = t\sinh 3t$
5. $f(t) = t^2\cos t$
6. $f(t) = te^{-3t}\cos 3t$
7. $\int_0^t e^u\,du$
8. $\int_0^t e^{-u}\cos u\,du$
9. $\int_0^t ue^{t-u}\,du$
10. $t\int_0^t \sin u\,du$
11. $\int_0^t \cos u\,du$
12. $\int_0^t u\sin u\,du$
13. $\int_0^t \sin u\cos(t-u)\,du$
14. $t\int_0^t ue^{-u}\,du$
15. $\int_0^t \sin au\cos b(t-u)\,du$
16. $\int_0^t e^u\sin a(t-u)\,du$
17. $\int_0^t \sinh au\cosh a(t-u)\,du$
18. $e^t\int_0^t \sin\omega u\cos\omega(t-u)\,du$
19. $e^t\int_0^t u^2(t-u)e^u\,du$
20. $e^{-t}\int_0^t e^{-u}u\cos\omega(t-u)\,du$
21. $e^t\int_0^t e^{2u}\sinh(t-u)\,du$
22. $\int_0^t ue^{2u}\sin 2(t-u)\,du$
23. $\int_0^t (t-u)^3e^u\,du$
24. $\int_0^t u^6e^{-(t-u)}\sin 3(t-u)\,du$
25. $\int_0^t u^2(t-u)^3\,du$
26. $\int_0^t (t-u)^7e^{-u}\sin 2u\,du$
27. $\int_0^t (t-u)^4\sin 2u\,du$

In Exercises **28-31** find the inverse Laplace transform of the given function.

28. $\frac{4s}{(s^2+1)^2}$
29. $\frac{s^2-1}{(s^2+1)^2}$
30. $\frac{12s-24}{(s^2-4s+85)^2}$
31. $\frac{s^2-4s+3}{(s^2-4s+5)^2}$

In Exercises **32-45** express the inverse transform as an integral.

32. $\frac{1}{s^2(s^2+4)}$
33. $\frac{s}{(s+2)(s^2+9)}$
34. $\frac{s}{(s^2+4)(s^2+9)}$
35. $\frac{s}{(s^2+1)^2}$
36. $\frac{1}{s(s-a)}$
37. $\frac{1}{(s+1)(s^2+2s+2)}$
38. $\frac{1}{(s+1)^2(s^2+4s+5)}$
39. $\frac{1}{(s-1)^3(s+2)^2}$
40. $\frac{s-1}{s^2(s^2-2s+2)}$
41. $\frac{s(s+3)}{(s^2+4)(s^2+6s+10)}$
42. $\frac{1}{(s-3)^5 s^6}$
43. $\frac{1}{(s-1)^3(s^2+4)}$
44. $\frac{1}{s^2(s-2)^3}$
45. $\frac{1}{s^7(s-2)^6}$

In Exercises **46-53** solve the initial value problem using Laplace transforms.

46. $y''+y=4\sin t+6\cos t, \quad y(0)=-6,\ y'(0)=2$
47. $y''+4y=8\sin 2t+9\cos t, \quad y(0)=1, \quad y'(0)=0$
48. $y'+y=t\sin t, \quad y(0)=0$
49. $y''+9y=\cos 3t, \quad y(0)=2, \quad y'(0)=5$
50. $y''+16y=\begin{cases}\cos 4t, & 0\le t<\pi,\\ 0, & t\ge\pi.\end{cases}, \quad y(0)=0, \quad y'(0)=1$
51. $y'-y=te^t\sin t, \quad y(0)=0$
52. $y''+y=\sin t, \quad y(0)=1, \quad y'(0)=-1$
53. $y''+y=\begin{cases}1, & 0\le t<\pi/2,\\ \sin t, & t\ge\pi/2.\end{cases}, \quad y(0)=1, \quad y'(0)=0$

In Exercises **54-69** solve the integral equation.

54. $y(t)+\int_0^t (t-u)y(u)\,du=t$
55. $y(t)=2t-4\int_0^t \sin u\, y(t-u)\,du$
56. $y(t)=te^t+\int_0^t uy(t-u)\,du$
57. $y(t)+2\int_0^t y(u)\cos(t-u)\,du=4e^{-t}+\sin t$
58. $y(t)+\int_0^t y(u)\,du=1$
59. $y(t)=\cos t+\int_0^t e^{-u}y(t-u)\,du$
60. $y(t)=1+t-\frac{8}{3}\int_0^t (u-t)^3 y(u)\,du$
61. $t-2y(t)=\int_0^t (e^u-e^{-u})y(t-u)\,du$

62. $y'(t) = 1 - \sin t - \int_0^t y(u)\,du, \quad y(0) = 0$

63. $y'(t) + 6y(t) + 9\int_0^t y(u)\,du = 1, \quad y(0) = 0$

64. $y(t) = t - \int_0^t (t-u)y(u)\,du$

65. $y(t) = \sin t - 2\int_0^t \cos(t-u)y(u)\,du$

66. $y(t) = 1 + 2\int_0^t y(u)\cos(t-u)\,du$

67. $y(t) = t + \int_0^t y(u)e^{-(t-u)}\,du$

68. $y'(t) = t + \int_0^t y(u)\cos(t-u)\,du,\ y(0) = 4$

69. $y(t) = \cos t - \sin t + \int_0^t y(u)\sin(t-u)\,du$

In Exercises **70-75** find the Laplace transforms of the given periodic functions.

70. $f(t) = \begin{cases} 1, & 0 \le t < a, \\ -1, & a \le t < 2a, \end{cases} \qquad f(t+2a) = f(t), \quad t \ge 0$

71. $f(t) = \begin{cases} 1, & 0 \le t < a, \\ 0, & a \le t < 2a, \end{cases} \qquad f(t+2a) = f(t), \quad t \ge 0$

72. $f(t) = \frac{a}{b}t, \quad 0 \le t < b. \qquad f(t+b) = f(t), \quad t \ge 0$

73. $f(t) = \begin{cases} t, & 0 \le t < 1, \\ 2-t, & 1 \le t < 2, \end{cases} \qquad f(t+2) = f(t), \quad t \ge 0$

74. $f(t) = |\sin t|$

75. $f(t) = \begin{cases} \sin t, & 0 \le t < \pi, \\ 0, & \pi \le t < 2\pi, \end{cases} \qquad f(t+2\pi) = f(t), \quad t \ge 0$

76. Prove that $f * g = g * f$.

77. Prove that if $f(t) \leftrightarrow F(s)$ then

$$\int_0^t f(u)\,du \leftrightarrow \frac{F(s)}{s}.$$

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CHAPTER OVERVIEW

10: Linear Systems of Differential Equations

In this chapter we consider systems of differential equations involving more than one unknown function. Such systems arise in many physical applications.

10.1: Linear Systems of Differential Equations

10.1: Linear Systems of Differential Equations (Exercises)

10.2: Basic Theory of Homogeneous Linear Systems

10.2.1: Basic Theory of Homogeneous Linear Systems (Exercises)

10.3: Constant Coefficient Homogeneous Systems I

10.3.1: Constant Coefficient Homogeneous Systems I (Exercises)

10.4: Constant Coefficient Homogeneous Systems II

10.4.1: Constant Coefficient Homogeneous Systems II (Exercises)

10.5: Constant Coefficient Homogeneous Systems III

10.5.1: Constant Coefficient Homogeneous Systems III (Exercises)

10.6: Variation of Parameters for Nonhomogeneous Linear Systems

10.6.1: Variation of Parameters for Nonhomogeneous Linear Systems (Exercises)

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10.1: Linear Systems of Differential Equations

A first order system of differential equations that can be written in the form

$$\begin{aligned} y_1' &= a_{11}(t)y_1 + a_{12}(t)y_2 + \cdots + a_{1n}(t)y_n + f_1(t) \\ y_2' &= a_{21}(t)y_1 + a_{22}(t)y_2 + \cdots + a_{2n}(t)y_n + f_2(t) \\ &\vdots \\ y_n' &= a_{n1}(t)y_1 + a_{n2}(t)y_2 + \cdots + a_{nn}(t)y_n + f_n(t) \end{aligned} \tag{10.1.1}$$

is called a *linear system*.

The linear system Equation 10.1.1 can be written in matrix form as

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ \vdots \\ y_n'(t) \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

or more briefly as

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t), \tag{10.1.2}$$

where

$$\mathbf{y} = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix}, \quad A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

We call A the *coefficient matrix* of Equation 10.1.2 and $\mathbf{f}$ the *forcing function*. We'll say that A and $\mathbf{f}$ are *continuous* if their entries are continuous. If $\mathbf{f} = \mathbf{0}$, then Equation 10.1.2 is *homogeneous*; otherwise, Equation 10.1.2 is *nonhomogeneous*.

An initial value problem for Equation 10.1.2 consists of finding a solution of Equation 10.1.2 that equals a given constant vector

$$\mathbf{k} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$$

at some initial point t_0. We write this initial value problem as

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t), \quad \mathbf{y}(t_0) = \mathbf{k}.$$

The next theorem gives sufficient conditions for the existence and uniqueness of solutions of initial value problems for Equation 10.1.2. We omit the proof.

Theorem 10.1.1 Existence and Uniqueness

Suppose the coefficient matrix A and the forcing function $\mathbf{f}$ are continuous on (a, b), let t_0 be in (a, b), and let $\mathbf{k}$ be an arbitrary constant n-vector. Then the initial value problem

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t), \quad \mathbf{y}(t_0) = \mathbf{k}$$

has a unique solution on (a, b).

Example 10.1.1

a. Write the system

$$\begin{aligned} y_1' &= y_1 + 2y_2 + 2e^{4t} \\ y_2' &= 2y_1 + y_2 + e^{4t} \end{aligned} \tag{10.1.3}$$

in matrix form and conclude from Theorem 10.1.1 that every initial value problem for Equation 10.1.3 has a unique solution on $(-\infty, \infty)$.

b. Verify that

$$\mathbf{y} = \frac{1}{5}\begin{bmatrix} 8 \\ 7 \end{bmatrix} e^{4t} + c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} \tag{10.1.4}$$

is a solution of Equation 10.1.3 for all values of the constants c_1 and c_2.

c. Find the solution of the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{y}(0) = \frac{1}{5}\begin{bmatrix} 3 \\ 22 \end{bmatrix}. \tag{10.1.5}$$

Solution a

The system Equation 10.1.3 can be written in matrix form as

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t}.$$

An initial value problem for Equation 10.1.3 can be written as

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t}, \quad y(t_0) = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}.$$

Since the coefficient matrix and the forcing function are both continuous on $(-\infty, \infty)$, Theorem 10.1.1 implies that this problem has a unique solution on $(-\infty, \infty)$.

Solution b

If $\mathbf{y}$ is given by Equation 10.1.4, then

$$\begin{aligned} A\mathbf{y} + \mathbf{f} &= \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 8 \\ 7 \end{bmatrix} e^{4t} + c_1 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t} \\ &= \frac{1}{5}\begin{bmatrix} 22 \\ 23 \end{bmatrix} e^{4t} + c_1 \begin{bmatrix} 3 \\ 3 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t} \\ &= \frac{1}{5}\begin{bmatrix} 32 \\ 28 \end{bmatrix} e^{4t} + 3c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} - c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} \\ &= \mathbf{y}'. \end{aligned}$$

Solution c

We must choose c_1 and c_2 in Equation 10.1.4 so that

$$\frac{1}{5}\begin{bmatrix} 8 \\ 7 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 \\ 22 \end{bmatrix},$$

which is equivalent to

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

Solving this system yields $c_1 = 1$, $c_2 = -2$, so

$$\mathbf{y} = \frac{1}{5}\begin{bmatrix} 8 \\ 7 \end{bmatrix} e^{4t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} - 2\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

is the solution of Equation 10.1.5.

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10.1: Linear Systems of Differential Equations (Exercises)

In Exercises **1-4** rewrite the system in matrix form and verify that the given vector function satisfies the system for any choice of the constants c_1 and c_2.

1. $\begin{aligned} y_1' &= 2y_1 + 4y_2 \\ y_2' &= 4y_1 + 2y_2; \end{aligned}$ $\quad \mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}$

2. $\begin{aligned} y_1' &= -2y_1 - 2y_2 \\ y_2' &= -5y_1 + y_2; \end{aligned}$ $\quad \mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} -2 \\ 5 \end{bmatrix} e^{3t}$

3. $\begin{aligned} y_1' &= -4y_1 - 10y_2 \\ y_2' &= 3y_1 + 7y_2; \end{aligned}$ $\quad \mathbf{y} = c_1 \begin{bmatrix} -5 \\ 3 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{t}$

4. $\begin{aligned} y_1' &= 2y_1 + y_2 \\ y_2' &= y_1 + 2y_2; \end{aligned}$ $\quad \mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t}$

In Exercises **5-8** rewrite the system in matrix form and verify that the given vector function satisfies the system for any choice of the constants c_1, c_2, and c_3.

5. $\begin{aligned} y_1' &= -y_1 + 2y_2 + 3y_3 \\ y_2' &= y_2 + 6y_3 \\ y_3' &= -2y_3; \end{aligned}$ $\quad \mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{-2t}$

6. $\begin{aligned} y_1' &= 2y_2 + 2y_3 \\ y_2' &= 2y_1 + 2y_3 \\ y_3' &= 2y_1 + 2y_2; \end{aligned}$ $\quad \mathbf{y} = c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{-2t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{4t}$

7. $\begin{aligned} y_1' &= -y_1 + 2y_2 + 2y_3 \\ y_2' &= 2y_1 - y_2 + 2y_3 \\ y_3' &= 2y_1 + 2y_2 - y_3; \end{aligned}$ $\quad \mathbf{y} = c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{-3t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{3t}$

8. $\begin{aligned} y_1' &= 3y_1 - y_2 - y_3 \\ y_2' &= -2y_1 + 3y_2 + 2y_3 \\ y_3' &= 4y_1 - y_2 - 2y_3; \end{aligned}$ $\quad \mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} e^{-t}$

In Exercises **9-10** rewrite the initial value problem in matrix form and verify that the given vector function is a solution.

9. $\begin{aligned} y_1' &= y_1 + y_2 \\ y_2' &= -2y_1 + 4y_2, \end{aligned}$ $\quad \begin{aligned} y_1(0) &= 1 \\ y_2(0) &= 0; \end{aligned}$ $\quad \mathbf{y} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$

10. $\begin{aligned} y_1' &= 5y_1 + 3y_2 \\ y_2' &= -y_1 + y_2, \end{aligned} \quad \begin{aligned} y_1(0) &= 12 \\ y_2(0) &= -6; \end{aligned} \quad \mathbf{y} = 3\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + 3\begin{bmatrix} 3 \\ -1 \end{bmatrix} e^{4t}$

In Exercises **11-12** rewrite the initial value problem in matrix form and verify that the given vector function is a solution.

11. $\begin{aligned} y_1' &= 6y_1 + 4y_2 + 4y_3 \\ y_2' &= -7y_1 - 2y_2 - y_3 \\ y_3' &= 7y_1 + 4y_2 + 3y_3, \end{aligned} \quad \begin{aligned} y_1(0) &= 3 \\ y_2(0) &= -6 \\ y_3(0) &= 4; \end{aligned} \quad \mathbf{y} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{6t} + 2\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{-t}$

12. $\begin{aligned} y_1' &= 8y_1 + 7y_2 + 7y_3 \\ y_2' &= -5y_1 - 6y_2 - 9y_3 \\ y_3' &= 5y_1 + 7y_2 + 10y_3, \end{aligned} \quad \begin{aligned} y_1(0) &= 2 \\ y_2(0) &= -4 \\ y_3(0) &= 3; \end{aligned} \quad \mathbf{y} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{8t} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{3t} + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{t}$

In Exercises **13-14** rewrite the system in matrix form and verify that the given vector function satisfies the system for any choice of the constants c_1 and c_2.

13. $\begin{aligned} y_1' &= -3y_1 + 2y_2 + 3 - 2t \\ y_2' &= -5y_1 + 3y_2 + 6 - 3t; \end{aligned} \quad \mathbf{y} = c_1\begin{bmatrix} 2\cos t \\ 3\cos t - \sin t \end{bmatrix} + c_2\begin{bmatrix} 2\sin t \\ 3\sin t + \cos t \end{bmatrix} + \begin{bmatrix} 1 \\ t \end{bmatrix}$

14. $\begin{aligned} y_1' &= 3y_1 + y_2 - 5e^t \\ y_2' &= -y_1 + y_2 + e^t; \end{aligned} \quad \mathbf{y} = c_1\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2t} + c_2\begin{bmatrix} 1+t \\ -t \end{bmatrix} e^{2t} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{t}$

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10.2: Basic Theory of Homogeneous Linear Systems

In this section we consider homogeneous linear systems $\mathbf{y}' = A(t)\mathbf{y}$, where $A = A(t)$ is a continuous $n \times n$ matrix function on an interval (a, b). The theory of linear homogeneous systems has much in common with the theory of linear homogeneous scalar equations, which we've previously considered in earlier sections. The theory requires knowledge of linear algebra and will therefore be omitted; however, one should note the similarity to the theory of linear homogeneous scalar equations.

Whenever we refer to solutions of $\mathbf{y}' = A(t)\mathbf{y}$ we'll mean solutions on (a, b). Since $\mathbf{y} \equiv \mathbf{0}$ is obviously a solution of $\mathbf{y}' = A(t)\mathbf{y}$, we call it the *trivial* solution. Any other solution is *nontrivial*.

Definition 10.2.1

If $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$ are vector functions defined on an interval (a, b) and $c_1, c_2, \ldots, c_n$ are constants, then

$$\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n \tag{10.2.1}$$

is a *linear combination of* $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$.

It's easy show that if $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$ are solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b), then so is any linear combination of $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$ (*Exercise 10.2.1*).

Definition 10.2.2

We say that a set $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ of n-vector functions is *linearly independent* on (a, b) if the only constants $c_1, c_2, \ldots, c_n$ such that

$$c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t) + \cdots + c_n\mathbf{y}_n(t) = 0, \quad a < t < b, \tag{10.2.2}$$

are $c_1 = c_2 = \cdots = c_n = 0$. If Equation 10.2.2 holds for some set of constants $c_1, c_2, \ldots, c_n$ that are not all zero, then $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ is *linearly dependent* on (a, b)

The next definition will help to characterize fundamental sets of solutions of $\mathbf{y}' = A(t)\mathbf{y}$.

Definition 10.2.3

We say that $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ is a *fundamental set of solutions* of an $n \times n$ system $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) if $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ is linear independent and every solution of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) can be written as a linear combination of $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$, as in Equation 10.2.1. In this case we say that Equation 10.2.1 is the *general solution of* $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b).

Theorem 10.2.1

Suppose the $n \times n$ matrix $A(t)$ is continuous on (a, b). Then a set $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ of n solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) is a fundamental set if and only if the determinant

$$\begin{vmatrix} \mathbf{y}_1 & \mathbf{y}_2 & \mathbf{y}_3 & \cdots & \mathbf{y}_n \end{vmatrix} \neq 0$$

on (a, b).

Example 10.2.1

Show that if

$$\mathbf{y}_1 = \begin{bmatrix} e^t \\ 0 \\ e^{-t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} 0 \\ e^{3t} \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{y}_3 = \begin{bmatrix} e^{2t} \\ e^{3t} \\ 0 \end{bmatrix}$$

are solutions to some 3×3 system $\mathbf{y}' = A(t)\mathbf{y}$, then they form a fundamental set of solutions on every interval (a, b).

Solution

Expanding the determinant of this system in cofactors of the entries of the first row yields

$$\begin{vmatrix} e^t & 0 & e^{2t} \\ 0 & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{vmatrix} = e^t \begin{vmatrix} e^{3t} & e^{3t} \\ 1 & 0 \end{vmatrix} - 0 \begin{vmatrix} 0 & e^{3t} \\ e^{-t} & 0 \end{vmatrix} + e^{2t} \begin{vmatrix} 0 & e^{3t} \\ e^{-t} & 1 \end{vmatrix}$$
$$= e^t(-e^{3t}) + e^{2t}(-e^{2t}) = -2e^{4t}.$$

Since this determinant is never zero, $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ must form a fundamental set of solutions.

We can use the method in Example 10.2.1 to test n solutions $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ of any $n \times n$ system $\mathbf{y}' = A(t)\mathbf{y}$ for a fundammental set on an interval (a, b) on which A is continuous. To explain this (and for other purposes later), it is useful to write a linear combination of $\mathbf{y}_1$, $\mathbf{y}_2$, ..., $\mathbf{y}_n$ in a different way. We first write the vector functions in terms of their components as

$$\mathbf{y}_1 = \begin{bmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{n1} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} y_{12} \\ y_{22} \\ \vdots \\ y_{n2} \end{bmatrix}, \ldots, \quad \mathbf{y}_n = \begin{bmatrix} y_{1n} \\ y_{2n} \\ \vdots \\ y_{nn} \end{bmatrix}.$$

If

$$\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n$$

then

$$\mathbf{y} = c_1 \begin{bmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{n1} \end{bmatrix} + c_2 \begin{bmatrix} y_{12} \\ y_{22} \\ \vdots \\ y_{n2} \end{bmatrix} + \cdots + c_n \begin{bmatrix} y_{1n} \\ y_{2n} \\ \vdots \\ y_{nn} \end{bmatrix}$$
$$= \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

This shows that

$$c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n = Y\mathbf{c}, \tag{10.2.3}$$

where

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

and

$$Y = [\mathbf{y}_1 \ \mathbf{y}_2 \ \cdots \ \mathbf{y}_n] = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix}; \tag{10.2.4}$$

that is, the columns of Y are the vector functions $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$.

For reference below, note that

$$\begin{aligned} Y' &= [\mathbf{y}_1' \ \mathbf{y}_2' \ \cdots \ \mathbf{y}_n'] \\ &= [A\mathbf{y}_1 \ A\mathbf{y}_2 \ \cdots \ A\mathbf{y}_n] \\ &= A[\mathbf{y}_1 \ \mathbf{y}_2 \ \cdots \ \mathbf{y}_n] = AY; \end{aligned}$$

that is, Y satisfies the matrix differential equation

$$Y' = AY.$$

The determinant of Y,

$$W = \begin{vmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{vmatrix} \tag{10.2.5}$$

is called the *Wronskian* of $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$. It can be shown that this definition is analogous to definitions of the Wronskian of scalar functions given earlier.

The next theorem is analogous to earlier theorems as well.

Theorem 10.2.2

Suppose the $n \times n$ matrix $A = A(t)$ is continuous on (a, b) and let $\mathbf{y}_1$, $\mathbf{y}_2$, ...,$\mathbf{y}_n$ be solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b). Then the following statements are equivalent; that is, they are either all true or all false:

a. The general solution of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) is $\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n$, where c_1, c_2, ..., c_n are arbitrary constants.
b. $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ is a fundamental set of solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b).
c. $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ is linearly independent on (a, b).
d. The Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ is nonzero at all points in (a, b).

We say that Y in Equation 10.2.4 is a *fundamental matrix* for $\mathbf{y}' = A(t)\mathbf{y}$ if any (and therefore all) of the statements (**a**)-(**d**) of Theorem 10.2.2 are true for the columns of Y. In this case, Equation 10.2.3 implies that the general solution of $\mathbf{y}' = A(t)\mathbf{y}$ can be written as $\mathbf{y} = Y\mathbf{c}$, where $\mathbf{c}$ is an arbitrary constant n-vector.

Example 10.2.2

The vector functions

$$\mathbf{y}_1 = \begin{bmatrix} -e^{2t} \\ 2e^{2t} \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix}$$

are solutions of the constant coefficient system

$$\mathbf{y}' = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix} \mathbf{y} \tag{10.2.6}$$

on $(-\infty, \infty)$. (Verify.)

a. Compute the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2\}$ directly from the definition Equation 10.2.5
b. Find the general solution of Equation 10.2.6.
c. Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 4 \\ -5 \end{bmatrix}. \tag{10.2.7}$$

Solution a

From Equation 10.2.5

$$W(t) = \begin{vmatrix} -e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t} \end{vmatrix} = e^{t}. \tag{10.2.8}$$

Solution b

Since $W(t) \neq 0$, Theorem 10.2.2 implies that $\{\mathbf{y}_1, \mathbf{y}_2\}$ is a fundamental set of solutions of Equation 10.2.6 and

$$Y = \begin{bmatrix} -e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t} \end{bmatrix}$$

is a fundamental matrix for Equation 10.2.6. Therefore the general solution of Equation 10.2.6 is

$$\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 = c_1 \begin{bmatrix} -e^{2t} \\ 2e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix} = \begin{bmatrix} -e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \tag{10.2.9}$$

Solution c

Setting $t = 0$ in Equation 10.2.9 and imposing the initial condition in Equation 10.2.7 yields

$$c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}.$$

Thus,

$$\begin{aligned} -c_1 - c_2 &= 4 \\ 2c_1 + c_2 &= -5. \end{aligned}$$

The solution of this system is $c_1 = -1$, $c_2 = -3$. Substituting these values into Equation 10.2.9 yields

$$\mathbf{y} = -\begin{bmatrix} -e^{2t} \\ 2e^{2t} \end{bmatrix} - 3\begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix} = \begin{bmatrix} e^{2t} + 3e^{-t} \\ -2e^{2t} - 3e^{-t} \end{bmatrix}$$

as the solution of Equation 10.2.7.

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10.2.1: Basic Theory of Homogeneous Linear Systems (Exercises)

1. Prove: If $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$ are solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b), then any linear combination of $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$ is also a solution of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b).

2. Let

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}, \quad \mathbf{y}_1 = \begin{bmatrix} e^{6t} \\ e^{6t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}.$$

a. Verify that $\{\mathbf{y}_1, \mathbf{y}_2\}$ is a fundamental set of solutions for $\mathbf{y}' = A\mathbf{y}$.
b. Solve the initial value problem

$$\mathbf{y}' = A\mathbf{y}, \quad \mathbf{y}(0) = \mathbf{k}. \tag{A}$$

3. Repeat *Exercise 10.2.2* with

$$A = \begin{bmatrix} -2 & -2 \\ -5 & 1 \end{bmatrix}, \quad \mathbf{y}_1 = \begin{bmatrix} e^{-4t} \\ e^{-4t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} -2e^{3t} \\ 5e^{3t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 10 \\ -4 \end{bmatrix}.$$

4. Repeat *Exercise 10.2.2* with

$$A = \begin{bmatrix} -4 & -10 \\ 3 & 7 \end{bmatrix}, \quad \mathbf{y}_1 = \begin{bmatrix} -5e^{2t} \\ 3e^{2t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} 2e^{t} \\ -e^{t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} -19 \\ 11 \end{bmatrix}.$$

5. Repeat *Exercise 10.2.2* with

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{y}_1 = \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}.$$

6. Let

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix},$$

$$\mathbf{y}_1 = \begin{bmatrix} e^{2t} \\ 0 \\ e^{2t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} e^{3t} \\ -e^{3t} \\ e^{3t} \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} e^{-t} \\ -3e^{-t} \\ 7e^{-t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 2 \\ -7 \\ 20 \end{bmatrix}.$$

a. Verify that $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is a fundamental set of solutions for $\mathbf{y}' = A\mathbf{y}$.
b. Solve the initial value problem

$$\mathbf{y}' = A\mathbf{y}, \quad \mathbf{y}(0) = \mathbf{k}. \tag{A}$$

7. Repeat *Exercise 10.2.6* with

$$A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix},$$

$$\mathbf{y}_1 = \begin{bmatrix} -e^{-2t} \\ 0 \\ e^{-2t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} -e^{-2t} \\ e^{-2t} \\ 0 \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} e^{4t} \\ e^{4t} \\ e^{4t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 \\ -9 \\ 12 \end{bmatrix}.$$

8. Repeat *Exercise 10.2.6* with

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix},$$

$$\mathbf{y}_1 = \begin{bmatrix} e^t \\ e^t \\ 0 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} e^{-t} \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} e^{-2t} \\ -2e^{-2t} \\ e^{-2t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 5 \\ 5 \\ -1 \end{bmatrix}.$$

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10.3: Constant Coefficient Homogeneous Systems I

Caution

If you have not had Math 410 (linear algebra), then you may want to review Appendix 11.5 and 11.6 before starting this section.

We'll now begin our study of the homogeneous system

$$\mathbf{y}' = A\mathbf{y}, \tag{10.3.1}$$

where A is an $n \times n$ constant matrix. Since A is continuous on $(-\infty, \infty)$, Theorem 10.1.1 implies that all solutions of Equation 10.3.1 are defined on $(-\infty, \infty)$. Therefore, when we speak of solutions of $\mathbf{y}' = A\mathbf{y}$, we'll mean solutions on $(-\infty, \infty)$.

In this section we assume that all the eigenvalues of A are real and that A has a set of n linearly independent eigenvectors. In the next two sections we consider the cases where some of the eigenvalues of A are complex, or where A does not have n linearly independent eigenvectors.

In Example 10.2.2 we showed that the vector functions

$$\mathbf{y}_1 = \begin{bmatrix} -e^{2t} \\ 2e^{2t} \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix}$$

form a fundamental set of solutions of the system

$$\mathbf{y}' = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix} \mathbf{y}, \tag{10.3.2}$$

but we did not show how we obtained $\mathbf{y}_1$ and $\mathbf{y}_2$ in the first place. To see how these solutions can be obtained we write Equation 10.3.2 as

$$\begin{aligned} y_1' &= -4y_1 - 3y_2 \\ y_2' &= 6y_1 + 5y_2 \end{aligned} \tag{10.3.3}$$

and look for solutions of the form

$$y_1 = x_1 e^{\lambda t} \quad \text{and} \quad y_2 = x_2 e^{\lambda t}, \tag{10.3.4}$$

where x_1, x_2, and λ are constants to be determined. Differentiating Equation 10.3.4 yields

$$y_1' = \lambda x_1 e^{\lambda t} \quad \text{and} \quad y_2' = \lambda x_2 e^{\lambda t}.$$

Substituting this and Equation 10.3.4 into Equation 10.3.3 and canceling the common factor $e^{\lambda t}$ yields

$$\begin{aligned} -4x_1 - 3x_2 &= \lambda x_1 \\ 6x_1 + 5x_2 &= \lambda x_2. \end{aligned}$$

For a given λ, this is a homogeneous algebraic system, since it can be rewritten as

$$\begin{aligned} (-4-\lambda)x_1 - 3x_2 &= 0 \\ 6x_1 + (5-\lambda)x_2 &= 0. \end{aligned} \tag{10.3.5}$$

The trivial solution $x_1 = x_2 = 0$ of this system isn't useful, since it corresponds to the trivial solution $y_1 \equiv y_2 \equiv 0$ of Equation 10.3.3, which can't be part of a fundamental set of solutions of Equation 10.3.2. Therefore we consider only those values of λ for which Equation 10.3.5 has nontrivial solutions. These are the values of λ for which the determinant of Equation 10.3.5 is zero; that is,

$$\begin{aligned} \begin{vmatrix} -4-\lambda & -3 \\ 6 & 5-\lambda \end{vmatrix} &= (-4-\lambda)(5-\lambda) + 18 \\ &= \lambda^2 - \lambda - 2 \\ &= (\lambda - 2)(\lambda + 1) = 0, \end{aligned}$$

which has the solutions $\lambda_1 = 2$ and $\lambda_2 = -1$.

Taking $\lambda = 2$ in Equation 10.3.5 yields

$$\begin{aligned} -6x_1 - 3x_2 &= 0 \\ 6x_1 + 3x_2 &= 0, \end{aligned}$$

which implies that $x_1 = -x_2/2$, where x_2 can be chosen arbitrarily. Choosing $x_2 = 2$ yields the solution $y_1 = -e^{2t}$, $y_2 = 2e^{2t}$ of Equation 10.3.3. We can write this solution in vector form as

$$\mathbf{y}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{2t}. \tag{10.3.6}$$

Taking $\lambda = -1$ in Equation 10.3.5 yields the system

$$\begin{aligned} -3x_1 - 3x_2 &= 0 \\ 6x_1 + 6x_2 &= 0, \end{aligned}$$

so $x_1 = -x_2$. Taking $x_2 = 1$ here yields the solution $y_1 = -e^{-t}$, $y_2 = e^{-t}$ of Equation 10.3.3. We can write this solution in vector form as

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}. \tag{10.3.7}$$

In Equation 10.3.6 and Equation 10.3.7 the constant coefficients in the arguments of the exponential functions are the eigenvalues of the coefficient matrix in Equation 10.3.2, and the vector coefficients of the exponential functions are associated eigenvectors. This illustrates the next theorem.

Theorem 10.3.1

Suppose the $n \times n$ constant matrix A has n real eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ (which need not be distinct) with associated linearly independent eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$. Then the functions

$$\mathbf{y}_1 = \mathbf{x}_1 e^{\lambda_1 t},\ \mathbf{y}_2 = \mathbf{x}_2 e^{\lambda_2 t},\ \dots,\ \mathbf{y}_n = \mathbf{x}_n e^{\lambda_n t}$$

form a fundamental set of solutions of $\mathbf{y}' = A\mathbf{y}$; that is, the general solution of this system is

$$\mathbf{y} = c_1\mathbf{x}_1 e^{\lambda_1 t} + c_2\mathbf{x}_2 e^{\lambda_2 t} + \cdots + c_n\mathbf{x}_n e^{\lambda_n t}.$$

Proof (requires knowledge of linear algebra)

Differentiating $\mathbf{y}_i = \mathbf{x}_i e^{\lambda_i t}$ and recalling that $A\mathbf{x}_i = \lambda_i \mathbf{x}_i$ yields

$$\mathbf{y}_i' = \lambda_i \mathbf{x}_i e^{\lambda_i t} = A\mathbf{x}_i e^{\lambda_i t} = A\mathbf{y}_i.$$

This shows that $\mathbf{y}_i$ is a solution of $\mathbf{y}' = A\mathbf{y}$.

The Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is

$$\begin{vmatrix} x_{11}e^{\lambda_1 t} & x_{12}e^{\lambda_2 t} & \cdots & x_{1n}e^{\lambda_n t} \\ x_{21}e^{\lambda_1 t} & x_{22}e^{\lambda_2 t} & \cdots & x_{2n}e^{\lambda_n t} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1}e^{\lambda_1 t} & x_{n2}e^{\lambda_2 t} & \cdots & x_{nn}e^{\lambda x_n t} \end{vmatrix} = e^{\lambda_1 t}e^{\lambda_2 t}\cdots e^{\lambda_n t} \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix}.$$

Since the columns of the determinant on the right are $\mathbf{x}_1$, $\mathbf{x}_2$, ..., $\mathbf{x}_n$, which are assumed to be linearly independent, the determinant is nonzero. Therefore Theorem 10.2.1 implies that $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is a fundamental set of solutions of $\mathbf{y}' = A\mathbf{y}$.

Example 10.3.1

a. Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \mathbf{y}. \tag{10.3.8}$$

b. Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 5 \\ -1 \end{bmatrix}. \tag{10.3.9}$$

Solution a

The characteristic polynomial of the coefficient matrix A in Equation 10.3.8 is

$$\begin{aligned} \begin{vmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{vmatrix} &= (\lambda-2)^2 - 16 \\ &= (\lambda-2-4)(\lambda-2+4) \\ &= (\lambda-6)(\lambda+2). \end{aligned}$$

Hence, $\lambda_1 = 6$ and $\lambda_2 = -2$ are eigenvalues of A. To obtain the eigenvectors, we must solve the system

$$\begin{bmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{10.3.10}$$

with $\lambda = 6$ and $\lambda = -2$. Setting $\lambda = 6$ in Equation 10.3.10 yields

$$\begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which implies that $x_1 = x_2$. Taking $x_2 = 1$ yields the eigenvector

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

so

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t}$$

is a solution of Equation 10.3.8. Setting $\lambda = -2$ in Equation 10.3.10 yields

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which implies that $x_1 = -x_2$. Taking $x_2 = 1$ yields the eigenvector

$$\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

so

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$$

is a solution of Equation 10.3.8. From Theorem 10.2.2, the general solution of Equation 10.3.8 is

$$\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}. \tag{10.3.11}$$

Solution b

To satisfy the initial condition in Equation 10.3.9, we must choose c_1 and c_2 in Equation 10.3.11 so that

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}.$$

This is equivalent to the system

$$\begin{aligned} c_1 - c_2 &= \ \ 5 \\ c_1 + c_2 &= -1, \end{aligned}$$

so $c_1 = 2, c_2 = -3$. Therefore the solution of Equation 10.3.9is

$$\mathbf{y} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t} - 3\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t},$$

or, in terms of components,

$$y_1 = 2e^{6t} + 3e^{-2t}, \quad y_2 = 2e^{6t} - 3e^{-2t}.$$

Example 10.3.2

a. Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix} \mathbf{y}. \tag{10.3.12}$$

b. Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 2 \\ -1 \\ 8 \end{bmatrix}. \tag{10.3.13}$$

Solution a

The characteristic polynomial of the coefficient matrix A in Equation 10.3.12is

$$\begin{vmatrix} 3-\lambda & -1 & -1 \\ -2 & 3-\lambda & 2 \\ 4 & -1 & -2-\lambda \end{vmatrix} = -(\lambda-2)(\lambda-3)(\lambda+1).$$

Hence, the eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = 3$, and $\lambda_3 = -1$. To find the eigenvectors, we must solve the system

$$\begin{bmatrix} 3-\lambda & -1 & -1 \\ -2 & 3-\lambda & 2 \\ 4 & -1 & -2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{10.3.14}$$

with $\lambda = 2, 3, -1$. With $\lambda = 2$, the augmented matrix of Equation 10.3.14is

$$\left[\begin{array}{ccc:c} 1 & -1 & -1 & 0 \\ -2 & 1 & 2 & 0 \\ 4 & -1 & -4 & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccc:c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, $x_1 = x_3$ and $x_2 = 0$. Taking $x_3 = 1$ yields

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t}$$

as a solution of Equation 10.3.12 With $\lambda = 3$, the augmented matrix of Equation 10.3.14is

$$\left[\begin{array}{rrr|c} 0 & -1 & -1 & 0 \\ -2 & 0 & 2 & 0 \\ 4 & -1 & -5 & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, $x_1 = x_3$ and $x_2 = -x_3$. Taking $x_3 = 1$ yields

$$\mathbf{y}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{3t}$$

as a solution of Equation 10.3.12 With $\lambda = -1$, the augmented matrix of Equation 10.3.14is

$$\left[\begin{array}{rrr|c} 4 & -1 & -1 & 0 \\ -2 & 4 & 2 & 0 \\ 4 & -1 & -1 & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{7} & 0 \\ 0 & 1 & \frac{3}{7} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, $x_1 = x_3/7$ and $x_2 = -3x_3/7$. Taking $x_3 = 7$ yields

$$\mathbf{y}_3 = \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} e^{-t}$$

as a solution of Equation 10.3.12 By Theorem 10.2.2, the general solution of Equation 10.3.12is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} e^{-t},$$

which can also be written as

$$\mathbf{y} = \begin{bmatrix} e^{2t} & e^{3t} & e^{-t} \\ 0 & -e^{3t} & -3e^{-t} \\ e^{2t} & e^{3t} & 7e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}. \tag{10.3.15}$$

Solution b

To satisfy the initial condition in Equation 10.3.13we must choose c_1, c_2, c_3 in Equation 10.3.15so that

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \\ 1 & 1 & 7 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 8 \end{bmatrix}.$$

Solving this system yields $c_1 = 3$, $c_2 = -2$, $c_3 = 1$. Hence, the solution of Equation 10.3.13is

$$\mathbf{y} = \begin{bmatrix} e^{2t} & e^{3t} & e^{-t} \\ 0 & -e^{3t} & -3e^{-t} \\ e^{2t} & e^{3t} & 7e^{-t} \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$
$$= 3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} - 2\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{3t} + \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} e^{-t}.$$

Example 10.3.3

Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix} \mathbf{y}. \tag{10.3.16}$$

Solution

The characteristic polynomial of the coefficient matrix A in Equation 10.3.16is

$$\begin{vmatrix} -3-\lambda & 2 & 2 \\ 2 & -3-\lambda & 2 \\ 2 & 2 & -3-\lambda \end{vmatrix} = -(\lambda-1)(\lambda+5)^2.$$

Hence, $\lambda_1 = 1$ is an eigenvalue of multiplicity 1, while $\lambda_2 = -5$ is an eigenvalue of multiplicity 2. Eigenvectors associated with $\lambda_1 = 1$ are solutions of the system with augmented matrix

$$\left[\begin{array}{ccc:c} -4 & 2 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ 2 & 2 & -4 & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccc:c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, $x_1 = x_2 = x_3$, and we choose $x_3 = 1$ to obtain the solution

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t \tag{10.3.17}$$

of Equation 10.3.16 Eigenvectors associated with $\lambda_2 = -5$ are solutions of the system with augmented matrix

$$\left[\begin{array}{ccc:c} 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right].$$

Hence, the components of these eigenvectors need only satisfy the single condition

$$x_1 + x_2 + x_3 = 0.$$

Since there's only one equation here, we can choose x_2 and x_3 arbitrarily. We obtain one eigenvector by choosing $x_2 = 0$ and $x_3 = 1$, and another by choosing $x_2 = 1$ and $x_3 = 0$. In both cases $x_1 = -1$. Therefore

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

are linearly independent eigenvectors associated with $\lambda_2 = -5$, and the corresponding solutions of Equation 10.3.16are

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-5t} \quad \text{and} \quad \mathbf{y}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-5t}.$$

Because of this and Equation 10.3.17, Theorem 10.2.2 implies that the general solution of Equation 10.3.16is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-5t} + c_3 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-5t}.$$

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10.3.1: Constant Coefficient Homogeneous Systems I (Exercises)

In Exercises **1-29** find the general solution.

1. $\begin{aligned} y_1' &= y_1 + 2y_2 \\ y_2' &= 4y_1 + 3y_2 \end{aligned}$

2. $\begin{aligned} y_1' &= 2y_1 + 2y_2 \\ y_2' &= y_1 + 3y_2 \end{aligned}$

3. $\begin{aligned} y_1' &= -4y_1 + 2y_2 \\ y_2' &= -\frac{5}{2}y_1 + 2y_2 \end{aligned}$

4. $\begin{aligned} y_1' &= -\frac{5}{2}y_1 + 2y_2 \\ y_2' &= \frac{3}{4}y_1 - 2y_2 \end{aligned}$

5. $\mathbf{y}' = \begin{bmatrix} 10 & -5 \\ 8 & -12 \end{bmatrix} \mathbf{y}$

6. $\mathbf{y}' = \begin{bmatrix} -6 & 2 \\ -3 & 1 \end{bmatrix} \mathbf{y}$

7. $\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y}$

8. $\mathbf{y}' = \frac{1}{4}\begin{bmatrix} -5 & 3 \\ 3 & -5 \end{bmatrix} \mathbf{y}$

9. $\mathbf{y}' = \frac{1}{5}\begin{bmatrix} -4 & 3 \\ -2 & -11 \end{bmatrix} \mathbf{y}$

10. $\mathbf{y}' = \begin{bmatrix} -1 & -4 \\ -1 & -1 \end{bmatrix} \mathbf{y}$

11. $\mathbf{y}' = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} \mathbf{y}$

12. $\mathbf{y}' = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix} \mathbf{y}$

13. $\mathbf{y}' = \begin{bmatrix} -6 & -3 \\ 1 & -2 \end{bmatrix} \mathbf{y}$

$$\begin{aligned} y_1' &= y_1 + y_2 - y_3 \\ y_2' &= 2y_2 \\ y_3' &= y_2 - y_3 \end{aligned}$$

14.

$$\begin{aligned} y_1' &= 2y_1 - 7y_2 \\ y_2' &= 5y_1 + 10y_2 + 4y_3 \\ y_3' &= 5y_2 + 2y_3 \end{aligned}$$

15.

$$\begin{aligned} y_1' &= 3y_1 - y_2 - y_3 \\ y_2' &= y_1 + y_2 - y_3 \\ y_3' &= y_1 - y_2 + y_3 \end{aligned}$$

16.

$$\begin{aligned} y_1' &= 3y_1 + 2y_2 + 4y_3 \\ y_2' &= 2y_1 + 2y_3 \\ y_3' &= 4y_1 + 2y_2 + 3y_3 \end{aligned}$$

17.

18. $\mathbf{y}' = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{bmatrix} \mathbf{y}$

19. $\mathbf{y}' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \mathbf{y}$

20. $\mathbf{y}' = \begin{bmatrix} -1 & 1 & 0 \\ \frac{3}{4} & -\frac{3}{2} & 3 \\ \frac{1}{8} & \frac{1}{4} & -\frac{1}{2} \end{bmatrix} \mathbf{y}$

21. $\mathbf{y}' = \begin{bmatrix} -1 & 4 & 2 \\ 4 & -1 & -2 \\ 0 & 0 & 6 \end{bmatrix} \mathbf{y}$

22. $\mathbf{y}' = \begin{bmatrix} 1 & -1 & -2 \\ 1 & -2 & -3 \\ -4 & 1 & -1 \end{bmatrix} \mathbf{y}$

23. $\mathbf{y}' = \begin{bmatrix} -6 & -4 & -8 \\ -4 & 0 & -4 \\ -8 & -4 & -6 \end{bmatrix} \mathbf{y}$

24. $\mathbf{y}' = \begin{bmatrix} 3 & 5 & 8 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix} \mathbf{y}$

25. $\mathbf{y}' = \begin{bmatrix} 1 & -1 & 2 \\ 12 & -4 & 10 \\ -6 & 1 & -7 \end{bmatrix} \mathbf{y}$

26. $\mathbf{y}' = \begin{bmatrix} 4 & -1 & -4 \\ 4 & -3 & -2 \\ 1 & -1 & -1 \end{bmatrix} \mathbf{y}$

27. $\mathbf{y}' = \begin{bmatrix} -2 & 2 & -6 \\ 2 & 6 & 2 \\ -2 & -2 & 2 \end{bmatrix} \mathbf{y}$

28. $\mathbf{y}' = \begin{bmatrix} 3 & 2 & -2 \\ -2 & 7 & -2 \\ -10 & 10 & -5 \end{bmatrix} \mathbf{y}$

29. $\mathbf{y}' = \begin{bmatrix} 3 & 1 & -1 \\ 3 & 5 & 1 \\ -6 & 2 & 4 \end{bmatrix} \mathbf{y}$

In Exercises **30-44** solve the initial value problem.

30. $\mathbf{y}' = \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

31. $\mathbf{y}' = \begin{bmatrix} -7 & 4 \\ -6 & 7 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$

32. $\mathbf{y}' = \frac{1}{6} \begin{bmatrix} 7 & 2 \\ -2 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

33. $\mathbf{y}' = \begin{bmatrix} 21 & -12 \\ 24 & -15 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

34. $\mathbf{y}' = \begin{bmatrix} -7 & 4 \\ -6 & 7 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$

35. $\mathbf{y}' = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$

36. $\mathbf{y}' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$

37. $\mathbf{y}' = \frac{1}{6}\begin{bmatrix} 1 & 2 & 0 \\ 4 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix}$

38. $\mathbf{y}' = \frac{1}{3}\begin{bmatrix} 2 & -2 & 3 \\ -4 & 4 & 3 \\ 2 & 1 & 0 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$

39. $\mathbf{y}' = \begin{bmatrix} 6 & -3 & -8 \\ 2 & 1 & -2 \\ 3 & -3 & -5 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$

40. $\mathbf{y}' = \frac{1}{3}\begin{bmatrix} 2 & 4 & -7 \\ 1 & 5 & -5 \\ -4 & 4 & -1 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$

41. $\mathbf{y}' = \begin{bmatrix} 3 & 0 & 1 \\ 11 & -2 & 7 \\ 1 & 0 & 3 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 2 \\ 7 \\ 6 \end{bmatrix}$

42. $\mathbf{y}' = \begin{bmatrix} -2 & -5 & -1 \\ -4 & -1 & 1 \\ 4 & 5 & 3 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 8 \\ -10 \\ -4 \end{bmatrix}$

43. $\mathbf{y}' = \begin{bmatrix} 3 & -1 & 0 \\ 4 & -2 & 0 \\ 4 & -4 & 2 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 7 \\ 10 \\ 2 \end{bmatrix}$

44. $\mathbf{y}' = \begin{bmatrix} -2 & 2 & 6 \\ 2 & 6 & 2 \\ -2 & -2 & 2 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 6 \\ -10 \\ 7 \end{bmatrix}$

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10.4: Constant Coefficient Homogeneous Systems II

We saw in Section 10.3 that if an $n \times n$ constant matrix A has n real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (which need not be distinct) with associated linearly independent eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, then the general solution of $\mathbf{y}' = A\mathbf{y}$ is

$$\mathbf{y} = c_1\mathbf{x}_1 e^{\lambda_1 t} + c_2\mathbf{x}_2 e^{\lambda_2 t} + \cdots + c_n\mathbf{x}_n e^{\lambda_n t}.$$

In this section we consider the case where A has n real eigenvalues, but does not have n linearly independent eigenvectors. It is shown in linear algebra that this occurs if and only if A has at least one eigenvalue of multiplicity $r > 1$ such that the associated eigenspace has dimension less than r. In this case A is said to be *defective*. Since it is beyond the scope of this book to give a complete analysis of systems with defective coefficient matrices, we will restrict our attention to some commonly occurring special cases.

Example 10.4.1

Show that the system

$$\mathbf{y}' = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix} \mathbf{y} \tag{10.4.1}$$

does not have a fundamental set of solutions of the form $\{\mathbf{x}_1 e^{\lambda_1 t}, \mathbf{x}_2 e^{\lambda_2 t}\}$, where λ_1 and λ_2 are eigenvalues of the coefficient matrix A of Equation 10.4.1 and $\mathbf{x}_1$ and $\mathbf{x}_2$ are associated linearly independent eigenvectors.

Solution

The characteristic polynomial of A is

$$\begin{aligned} \begin{vmatrix} 11-\lambda & \\ 4 & -9-\lambda \end{vmatrix} &= (\lambda - 11)(\lambda + 9) + 100 \\ &= \lambda^2 - 2\lambda + 1 \\ &= (\lambda - 1)^2. \end{aligned}$$

Hence, $\lambda = 1$ is the only eigenvalue of A. The augmented matrix of the system $(A - I)\mathbf{x} = \mathbf{0}$ is

$$\left[\begin{array}{cc:c} 10 & -25 & 0 \\ 4 & -10 & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{cc:c} 1 & -\frac{5}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Hence, $x_1 = 5x_2/2$ where x_2 is arbitrary. Therefore all eigenvectors of A are scalar multiples of $\mathbf{x_1} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, so A does not have a set of two linearly independent eigenvectors.

From Example 10.4.1 , we know that all scalar multiples of $\mathbf{y_1} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \mathbf{e^t}$ are solutions of Equation 10.4.1; however, to find the general solution we must find a second solution $\mathbf{y_2}$ such that $\{\mathbf{y_1}, \mathbf{y_2}\}$ is linearly independent. Based on your recollection of the procedure for solving a constant coefficient scalar equation

$$ay'' + by' + cy = 0$$

in the case where the characteristic polynomial has a repeated root, you might expect to obtain a second solution of Equation 10.4.1 by multiplying the first solution by t. However, this yields $\mathbf{y}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix} te^t$, which does not quite work, since

$$\mathbf{y}_2' = \begin{bmatrix} 5 \\ 2 \end{bmatrix} (te^t + e^t), \quad \text{while} \quad \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix} \mathbf{y}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix} te^t.$$

However, the idea does incorporate this idea as the next theorem shows.

Theorem 10.4.1

Suppose the $n \times n$ matrix A has an eigenvalue λ_1 of multiplicity ≥ 2 and the associated eigenspace has dimension 1; that is, all λ_1-eigenvectors of A are scalar multiples of a single eigenvector $\mathbf{x}$. Then there are infinitely many vectors $\mathbf{u}$ such that

$$(A - \lambda_1 I)\mathbf{u} = \mathbf{x}. \tag{10.4.2}$$

Moreover, if $\mathbf{u}$ is any such vector then

$$\mathbf{y}_1 = \mathbf{x}e^{\lambda_1 t} \quad \text{and} \quad \mathbf{y}_2 = \mathbf{u}e^{\lambda_1 t} + \mathbf{x}te^{\lambda_1 t} \tag{10.4.3}$$

are linearly independent solutions of $\mathbf{y}' = A\mathbf{y}$.

A complete proof of this theorem is beyond the scope of this book. The difficulty is in proving that there's a vector $\mathbf{u}$ satisfying Equation 10.4.2, since $\det(A - \lambda_1 I) = 0$. We'll take this without proof and verify the other assertions of the theorem. We already know that $\mathbf{y}_1$ in Equation 10.4.3 is a solution of $\mathbf{y}' = A\mathbf{y}$. To see that $\mathbf{y}_2$ is also a solution, we compute

$$\begin{aligned} \mathbf{y}_2' - A\mathbf{y}_2 &= \lambda_1 \mathbf{u}e^{\lambda_1 t} + \mathbf{x}e^{\lambda_1 t} + \lambda_1 \mathbf{x}te^{\lambda_1 t} - A\mathbf{u}e^{\lambda_1 t} - A\mathbf{x}te^{\lambda_1 t} \\ &= (\lambda_1 \mathbf{u} + \mathbf{x} - A\mathbf{u})e^{\lambda_1 t} + (\lambda_1 \mathbf{x} - A\mathbf{x})te^{\lambda_1 t}. \end{aligned}$$

Since $A\mathbf{x} = \lambda_1 \mathbf{x}$, this can be written as

$$\mathbf{y}_2' - A\mathbf{y}_2 = -\left((A - \lambda_1 I)\mathbf{u} - \mathbf{x}\right) e^{\lambda_1 t},$$

and now Equation 10.4.2 implies that $\mathbf{y}_2' = A\mathbf{y}_2$. To see that $\mathbf{y}_1$ and $\mathbf{y}_2$ are linearly independent, suppose c_1 and c_2 are constants such that

$$c_1\mathbf{y}_1 + c_2\mathbf{y}_2 = c_1\mathbf{x}e^{\lambda_1 t} + c_2(\mathbf{u}e^{\lambda_1 t} + \mathbf{x}te^{\lambda_1 t}) = \mathbf{0}. \tag{10.4.4}$$

We must show that $c_1 = c_2 = 0$. Multiplying Equation 10.4.4 by $e^{-\lambda_1 t}$ shows that

$$c_1\mathbf{x} + c_2(\mathbf{u} + \mathbf{x}t) = \mathbf{0}. \tag{10.4.5}$$

By differentiating this with respect to t, we see that $c_2\mathbf{x} = \mathbf{0}$, which implies $c_2 = 0$, because $\mathbf{x} \neq \mathbf{0}$. Substituting $c_2 = 0$ into Equation 10.4.5 yields $c_1\mathbf{x} = \mathbf{0}$, which implies that $c_1 = 0$, again because $\mathbf{x} \neq \mathbf{0}$

Example 10.4.2

Use Theorem 10.4.1 to find the general solution of the system

$$\mathbf{y}' = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix} \mathbf{y} \tag{10.4.6}$$

considered in Example 10.4.1 .

Solution

In Example 10.4.1 we saw that $\lambda_1 = 1$ is an eigenvalue of multiplicity 2 of the coefficient matrix A in Equation 10.4.6, and that all of the eigenvectors of A are multiples of

$$\mathbf{x} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

Therefore

$$\mathbf{y}_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^t$$

is a solution of Equation 10.4.6. From Theorem 10.4.1 , a second solution is given by $\mathbf{y}_2 = \mathbf{u}e^t + \mathbf{x}te^t$, where $(A-I)\mathbf{u} = \mathbf{x}$. The augmented matrix of this system is

$$\left[\begin{array}{cc|c} 10 & -25 & 5 \\ 4 & -10 & 2 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{cc|c} 1 & -\frac{5}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{array}\right],$$

Therefore the components of $\mathbf{u}$ must satisfy

$$u_1 - \frac{5}{2}u_2 = \frac{1}{2},$$

where u_2 is arbitrary. We choose $u_2 = 0$, so that $u_1 = 1/2$ and

$$\mathbf{u} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}.$$

Thus,

$$\mathbf{y}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{e^t}{2} + \begin{bmatrix} 5 \\ 2 \end{bmatrix} te^t.$$

Since $\mathbf{y}_1$ and $\mathbf{y}_2$ are linearly independent by Theorem 10.4.1 , they form a fundamental set of solutions of Equation 10.4.6. Therefore the general solution of Equation 10.4.6 is

$$\mathbf{y} = c_1 \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{e^t}{2} + \begin{bmatrix} 5 \\ 2 \end{bmatrix} te^t \right).$$

Example 10.4.3

Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 3 & 4 & -10 \\ 2 & 1 & -2 \\ 2 & 2 & -5 \end{bmatrix} \mathbf{y}. \tag{10.4.7}$$

Solution

The characteristic polynomial of the coefficient matrix A in Equation 10.4.7 is

$$\begin{vmatrix} 3-\lambda & 4 & -10 \\ 2 & 1-\lambda & -2 \\ 2 & 2 & -5-\lambda \end{vmatrix} = -(\lambda-1)(\lambda+1)^2.$$

Hence, the eigenvalues are $\lambda_1 = 1$ with multiplicity 1 and $\lambda_2 = -1$ with multiplicity 2. Eigenvectors associated with $\lambda_1 = 1$ must satisfy $(A-I)\mathbf{x} = \mathbf{0}$. The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} 2 & 4 & -10 & 0 \\ 2 & 0 & -2 & 0 \\ 2 & 2 & -6 & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, $x_1 = x_3$ and $x_2 = 2x_3$, where x_3 is arbitrary. Choosing $x_3 = 1$ yields the eigenvector

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Therefore

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^t$$

is a solution of Equation 10.4.7. Eigenvectors associated with $\lambda_2 = -1$ satisfy $(A + I)\mathbf{x} = \mathbf{0}$. The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} 4 & 4 & -10 & 0 \\ 2 & 2 & -2 & 0 \\ 2 & 2 & -4 & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, $x_3 = 0$ and $x_1 = -x_2$, where x_2 is arbitrary. Choosing $x_2 = 1$ yields the eigenvector

$$\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},$$

so

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-t}$$

is a solution of Equation 10.4.7. Since all the eigenvectors of A associated with $\lambda_2 = -1$ are multiples of $\mathbf{x}_2$, we must now use Theorem 10.4.1 to find a third solution of Equation 10.4.7 in the form

$$\mathbf{y}_3 = \mathbf{u}e^{-t} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} te^{-t}, \tag{10.4.8}$$

where $\mathbf{u}$ is a solution of $(A + I)\mathbf{u} = \mathbf{x}_2$. The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} 4 & 4 & -10 & -1 \\ 2 & 2 & -2 & 1 \\ 2 & 2 & -4 & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, $u_3 = 1/2$ and $u_1 = 1 - u_2$, where u_2 is arbitrary. Choosing $u_2 = 0$ yields

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ \frac{1}{2} \end{bmatrix},$$

and substituting this into Equation 10.4.8 yields the solution

$$\mathbf{y}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \frac{e^{-t}}{2} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} te^{-t}$$

of Equation 10.4.7. Since the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ at $t = 0$ is

$$\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{2},$$

$\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is a fundamental set of solutions of Equation 10.4.7. Therefore the general solution of Equation 10.4.7 is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-t} + c_3 \left(\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \frac{e^{-t}}{2} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} te^{-t} \right).$$

Theorem 10.4.2

Suppose the $n \times n$ matrix A has an eigenvalue λ_1 of multiplicity ≥ 3 and the associated eigenspace is one–dimensional; that is, all eigenvectors associated with λ_1 are scalar multiples of the eigenvector $\mathbf{x}$. Then there are infinitely many vectors $\mathbf{u}$ such that

$$(A - \lambda_1 I)\mathbf{u} = \mathbf{x}, \tag{10.4.9}$$

and, if $\mathbf{u}$ is any such vector, there are infinitely many vectors $\mathbf{v}$ such that

$$(A - \lambda_1 I)\mathbf{v} = \mathbf{u}. \tag{10.4.10}$$

If $\mathbf{u}$ satisfies Equation 10.4.9 and $\mathbf{v}$ satisfies Equation 10.4.10, then

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{x}e^{\lambda_1 t}, \\ \mathbf{y}_2 &= \mathbf{u}e^{\lambda_1 t} + \mathbf{x}te^{\lambda_1 t}, \text{ and} \\ \mathbf{y}_3 &= \mathbf{v}e^{\lambda_1 t} + \mathbf{u}te^{\lambda_1 t} + \mathbf{x}\frac{t^2 e^{\lambda_1 t}}{2} \end{aligned}$$

are linearly independent solutions of $\mathbf{y}' = A\mathbf{y}$.

Example 10.4.4

Use Theorem 10.4.2 to find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & -1 \\ 0 & 2 & 2 \end{bmatrix} \mathbf{y}. \tag{10.4.11}$$

Solution

The characteristic polynomial of the coefficient matrix A in Equation 10.4.11is

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 0 & 2 & 2-\lambda \end{vmatrix} = -(\lambda-2)^3.$$

Hence, $\lambda_1 = 2$ is an eigenvalue of multiplicity 3. The associated eigenvectors satisfy $(A-2I)\mathbf{x} = \mathbf{0}$. The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, $x_1 = x_3$ and $x_2 = 0$, so the eigenvectors are all scalar multiples of

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t}$$

is a solution of Equation 10.4.11. We now find a second solution of Equation 10.4.11in the form

$$\mathbf{y}_2 = \mathbf{u}e^{2t} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} te^{2t},$$

where $\mathbf{u}$ satisfies $(A-2I)\mathbf{u} = \mathbf{x}_1$. The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Letting $u_3 = 0$ yields $u_1 = -1/2$ and $u_2 = 1/2$; hence,

$$\mathbf{u} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

and

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \frac{e^{2t}}{2} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} te^{2t}$$

is a solution of Equation 10.4.11. We now find a third solution of Equation 10.4.11in the form

$$\mathbf{y}_3 = \mathbf{v}e^{2t} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \frac{te^{2t}}{2} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \frac{t^2e^{2t}}{2}$$

where $\mathbf{v}$ satisfies $(A-2I)\mathbf{v} = \mathbf{u}$. The augmented matrix of this system is

$$\left[\begin{array}{ccc:c} -1 & 1 & 1 & -\frac{1}{2} \\ 1 & 1 & -1 & \frac{1}{2} \\ 0 & 2 & 0 & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccc:c} 1 & 0 & -1 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Letting $v_3 = 0$ yields $v_1 = 1/2$ and $v_2 = 0$; hence,

$$\mathbf{v} = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore

$$\mathbf{y}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{e^{2t}}{2} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \frac{te^{2t}}{2} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \frac{t^2e^{2t}}{2}$$

is a solution of Equation 10.4.11. Since $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ are linearly independent by Theorem 10.4.2 , they form a fundamental set of solutions of Equation 10.4.11. Therefore the general solution of Equation 10.4.11is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_2 \left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \frac{e^{2t}}{2} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} te^{2t} \right) + c_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{e^{2t}}{2} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \frac{te^{2t}}{2} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \frac{t^2e^{2t}}{2} \right)$$

Theorem 10.4.3

Suppose the $n \times n$ matrix A has an eigenvalue λ_1 of multiplicity ≥ 3 and the associated eigenspace is two–dimensional; that is, all eigenvectors of A associated with λ_1 are linear combinations of two linearly independent eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. Then there are constants α and β (not both zero) such that if

$$\mathbf{x}_3 = \alpha\mathbf{x}_1 + \beta\mathbf{x}_2, \tag{10.4.12}$$

then there are infinitely many vectors $\mathbf{u}$ such that

$$(A - \lambda_1 I)\mathbf{u} = \mathbf{x}_3. \tag{10.4.13}$$

If $\mathbf{u}$ satisfies Equation 10.4.13, then

$$\begin{aligned}\mathbf{y}_1 &= \mathbf{x}_1 e^{\lambda_1 t},\\ \mathbf{y}_2 &= \mathbf{x}_2 e^{\lambda_1 t}, \text{and}\\ \mathbf{y}_3 &= \mathbf{u} e^{\lambda_1 t} + \mathbf{x}_3 t e^{\lambda_1 t}\end{aligned}$$

are linearly independent solutions of $\mathbf{y}' = A\mathbf{y}$.

We omit the proof of this theorem.

Example 10.4.5

Use Theorem 10.4.3 to find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix} \mathbf{y}. \tag{10.4.14}$$

Solution

The characteristic polynomial of the coefficient matrix A in Equation 10.4.14is

$$\begin{vmatrix} -\lambda & 0 & 1 \\ -1 & 1-\lambda & 1 \\ -1 & 0 & 2-\lambda \end{vmatrix} = -(\lambda-1)^3.$$

Hence, $\lambda_1 = 1$ is an eigenvalue of multiplicity 3. The associated eigenvectors satisfy $(A-I)\mathbf{x} = \mathbf{0}$. The augmented matrix of this system is

$$\left[\begin{array}{ccc:c} -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccc:c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, $x_1 = x_3$ and x_2 is arbitrary, so the eigenvectors are of the form

$$\mathbf{x}_1 = \begin{bmatrix} x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{10.4.15}$$

form a basis for the eigenspace, and

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^t \quad \text{and} \quad \mathbf{y}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^t$$

are linearly independent solutions of Equation 10.4.14 To find a third linearly independent solution of Equation 10.4.14, we must find constants α and β (not both zero) such that the system

$$(A-I)\mathbf{u}=\alpha\mathbf{x}_1+\beta\mathbf{x}_2 \tag{10.4.16}$$

has a solution $\mathbf{u}$. The augmented matrix of this system is

$$\left[\begin{array}{cccc|c} -1 & 0 & 1 & \vdots & \alpha \\ -1 & 0 & 1 & \vdots & \beta \\ -1 & 0 & 1 & \vdots & \alpha \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & \vdots & -\alpha \\ 0 & 0 & 0 & \vdots & \beta-\alpha \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]. \tag{10.4.17}$$

Therefore Equation 10.4.16has a solution if and only if $\beta=\alpha$, where α is arbitrary. If $\alpha=\beta=1$ then Equation 10.4.12and Equation 10.4.15yield

$$\mathbf{x}_3=\mathbf{x}_1+\mathbf{x}_2=\begin{bmatrix}1\\0\\1\end{bmatrix}+\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix},$$

and the augmented matrix Equation 10.4.17becomes

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & \vdots & -1 \\ 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right].$$

This implies that $u_1=-1+u_3$, while u_2 and u_3 are arbitrary. Choosing $u_2=u_3=0$ yields

$$\mathbf{u}=\begin{bmatrix}-1\\0\\0\end{bmatrix}.$$

Therefore Equation ??? implies that

$$\mathbf{y}_3=\mathbf{u}e^t+\mathbf{x}_3te^t=\begin{bmatrix}-1\\0\\0\end{bmatrix}e^t+\begin{bmatrix}1\\1\\1\end{bmatrix}te^t$$

is a solution of Equation 10.4.14 Since $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ are linearly independent by Theorem 10.4.3 , they form a fundamental set of solutions for Equation 10.4.14 Therefore the general solution of Equation 10.4.14is

$$\mathbf{y}=c_1\begin{bmatrix}1\\0\\1\end{bmatrix}e^t+c_2\begin{bmatrix}0\\1\\0\end{bmatrix}e^t+c_3\left(\begin{bmatrix}-1\\0\\0\end{bmatrix}e^t+\begin{bmatrix}1\\1\\1\end{bmatrix}te^t\right). \tag{10.4.18}$$

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10.4.1: Constant Coefficient Homogeneous Systems II (Exercises)

In Exercises **1-18** find the general solution.

1. $\begin{aligned} y_1' &= 3y_1 - y_2 \\ y_2' &= 9y_1 - 3y_2 \end{aligned}$

2. $\begin{aligned} y_1' &= -6y_1 + 5y_2 \\ y_2' &= -5y_1 + 4y_2 \end{aligned}$

3. $\mathbf{y}' = \begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix} \mathbf{y}$

4. $\mathbf{y}' = \begin{bmatrix} 12 & -9 \\ 4 & 0 \end{bmatrix} \mathbf{y}$

5. $\mathbf{y}' = \begin{bmatrix} 3 & 4 \\ -1 & 7 \end{bmatrix} \mathbf{y}$

6. $\mathbf{y}' = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \mathbf{y}$

7. $\mathbf{y}' = \begin{bmatrix} -7 & 4 \\ -1 & -11 \end{bmatrix} \mathbf{y}$

8. $\mathbf{y}' = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{y}$

9. $\mathbf{y}' = \begin{bmatrix} 4 & 12 \\ -3 & -8 \end{bmatrix} \mathbf{y}$

10. $\mathbf{y}' = \begin{bmatrix} -10 & 9 \\ -4 & 2 \end{bmatrix} \mathbf{y}$

11. $\mathbf{y}' = \begin{bmatrix} -13 & 16 \\ -9 & 11 \end{bmatrix} \mathbf{y}$

12. $\mathbf{y}' = \begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix} \mathbf{y}$

13. $\mathbf{y}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix} \mathbf{y}$

14. $\mathbf{y}' = \begin{bmatrix} 0 & 2 & 1 \\ -4 & 6 & 1 \\ 0 & 4 & 2 \end{bmatrix} \mathbf{y}$

15. $\mathbf{y}' = \frac{1}{3} \begin{bmatrix} 1 & 1 & -3 \\ -4 & -4 & 3 \\ -2 & 1 & 0 \end{bmatrix} \mathbf{y}$

16. $\mathbf{y}' = \begin{bmatrix} -1 & 1 & -1 \\ -2 & 0 & 2 \\ -1 & 3 & -1 \end{bmatrix} \mathbf{y}$

17. $\mathbf{y}' = \begin{bmatrix} 4 & -2 & -2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \mathbf{y}$

18. $\mathbf{y}' = \begin{bmatrix} 6 & -5 & 3 \\ 2 & -1 & 3 \\ 2 & 1 & 1 \end{bmatrix} \mathbf{y}$

In Exercises **19-30** solve the initial value problem.

19. $\mathbf{y}' = \begin{bmatrix} 2 & 4 \\ -1 & 6 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$

20. $\mathbf{y}' = \begin{bmatrix} -11 & 8 \\ -2 & -3 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

21. $\mathbf{y}' = \begin{bmatrix} 15 & -9 \\ 16 & -9 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$

22. $\mathbf{y}' = \begin{bmatrix} -3 & -4 \\ 1 & -7 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

23. $\mathbf{y}' = \begin{bmatrix} -7 & 24 \\ -6 & 17 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

24. $\mathbf{y}' = \begin{bmatrix} -7 & 3 \\ -3 & -1 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

25. $\mathbf{y}' = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 6 \\ 5 \\ -7 \end{bmatrix}$

26. $\mathbf{y}' = \begin{bmatrix} -2 & 2 & 1 \\ -2 & 2 & 1 \\ -3 & 3 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} -6 \\ -2 \\ 0 \end{bmatrix}$

27. $\mathbf{y}' = \begin{bmatrix} -7 & -4 & 4 \\ -1 & 0 & 1 \\ -9 & -5 & 6 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} -6 \\ 9 \\ -1 \end{bmatrix}$

28. $\mathbf{y}' = \begin{bmatrix} -1 & -4 & -1 \\ 3 & 6 & 1 \\ -3 & -2 & 3 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$

29. $\mathbf{y}' = \begin{bmatrix} 4 & -8 & -4 \\ -3 & -1 & -3 \\ 1 & -1 & 9 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} -4 \\ 1 \\ -3 \end{bmatrix}$

30. $\mathbf{y}' = \begin{bmatrix} -5 & -1 & 11 \\ -7 & 1 & 13 \\ -4 & 0 & 8 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$

The coefficient matrices in Exercises **31-41** have eigenvalues of multiplicity 3. Find the general solution.

31. $\mathbf{y}' = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{y}$

32. $\mathbf{y}' = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix} \mathbf{y}$

33. $\mathbf{y}' = \begin{bmatrix} 5 & -1 & 1 \\ -1 & 9 & -3 \\ -2 & 2 & 4 \end{bmatrix} \mathbf{y}$

34. $\mathbf{y}' = \begin{bmatrix} 1 & 10 & -12 \\ 2 & 2 & 3 \\ 2 & -1 & 6 \end{bmatrix} \mathbf{y}$

35. $\mathbf{y}' = \begin{bmatrix} -6 & -4 & -4 \\ 2 & -1 & 1 \\ 2 & 3 & 1 \end{bmatrix} \mathbf{y}$

36. $\mathbf{y}' = \begin{bmatrix} 0 & 2 & -2 \\ -1 & 5 & -3 \\ 1 & 1 & 1 \end{bmatrix} \mathbf{y}$

37. $\mathbf{y}' = \begin{bmatrix} -2 & -12 & 10 \\ 2 & -24 & 11 \\ 2 & -24 & 8 \end{bmatrix} \mathbf{y}$

38. $\mathbf{y}' = \begin{bmatrix} -1 & -12 & 8 \\ 1 & -9 & 4 \\ 1 & -6 & 1 \end{bmatrix} \mathbf{y}$

39. $\mathbf{y}' = \begin{bmatrix} -4 & 0 & -1 \\ -1 & -3 & -1 \\ 1 & 0 & -2 \end{bmatrix} \mathbf{y}$

40. $\mathbf{y}' = \begin{bmatrix} -3 & -3 & 4 \\ 4 & 5 & -8 \\ 2 & 3 & -5 \end{bmatrix} \mathbf{y}$

41. $\mathbf{y}' = \begin{bmatrix} -3 & -1 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & -2 \end{bmatrix} \mathbf{y}$

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10.5: Constant Coefficient Homogeneous Systems III

We now consider the system $\mathbf{y}' = A\mathbf{y}$, where A has a complex eigenvalue $\lambda = \alpha + i\beta$ with $\beta \neq 0$. We continue to assume that A has real entries, so the characteristic polynomial of A has real coefficients. This implies that $\overline{\lambda} = \alpha - i\beta$ is also an eigenvalue of A.

An eigenvector $\mathbf{x}$ of A associated with $\lambda = \alpha + i\beta$ will have complex entries, so we'll write

$$\mathbf{x} = \mathbf{u} + i\mathbf{v}$$

where $\mathbf{u}$ and $\mathbf{v}$ have real entries; that is, $\mathbf{u}$ and $\mathbf{v}$ are the real and imaginary parts of $\mathbf{x}$. Since $A\mathbf{x} = \lambda\mathbf{x}$,

$$A(\mathbf{u} + i\mathbf{v}) = (\alpha + i\beta)(\mathbf{u} + i\mathbf{v}). \tag{10.5.1}$$

Taking complex conjugates here and recalling that A has real entries yields

$$A(\mathbf{u} - i\mathbf{v}) = (\alpha - i\beta)(\mathbf{u} - i\mathbf{v}),$$

which shows that $\mathbf{x} = \mathbf{u} - i\mathbf{v}$ is an eigenvector associated with $\overline{\lambda} = \alpha - i\beta$. The complex conjugate eigenvalues λ and $\overline{\lambda}$ can be separately associated with linearly independent solutions $\mathbf{y}' = A\mathbf{y}$; however, we will not pursue this approach, since solutions obtained in this way turn out to be complex–valued. Instead, we'll obtain solutions of $\mathbf{y}' = A\mathbf{y}$ in the form

$$\mathbf{y} = f_1\mathbf{u} + f_2\mathbf{v} \tag{10.5.2}$$

where f_1 and f_2 are real–valued scalar functions. The next theorem shows how to do this.

Theorem 10.5.1

Let A be an $n \times n$ matrix with real entries. Let $\lambda = \alpha + i\beta$ $(\beta \neq 0)$ be a complex eigenvalue of A and let $\mathbf{x} = \mathbf{u} + i\mathbf{v}$ be an associated eigenvector, where $\mathbf{u}$ and $\mathbf{v}$ have real components. Then $\mathbf{u}$ and $\mathbf{v}$ are both nonzero and

$$\mathbf{y}_1 = e^{\alpha t}(\mathbf{u}\cos\beta t - \mathbf{v}\sin\beta t) \quad \text{and} \quad \mathbf{y}_2 = e^{\alpha t}(\mathbf{u}\sin\beta t + \mathbf{v}\cos\beta t),$$

which are the real and imaginary parts of

$$e^{\alpha t}(\cos\beta t + i\sin\beta t)(\mathbf{u} + i\mathbf{v}), \tag{10.5.3}$$

are linearly independent solutions of $\mathbf{y}' = A\mathbf{y}$.

Proof

A function of the form Equation 10.5.2is a solution of $\mathbf{y}' = A\mathbf{y}$ if and only if

$$f_1'\mathbf{u} + f_2'\mathbf{v} = f_1 A\mathbf{u} + f_2 A\mathbf{v}. \tag{10.5.4}$$

Carrying out the multiplication indicated on the right side of Equation 10.5.1and collecting the real and imaginary parts of the result yields

$$A(\mathbf{u} + i\mathbf{v}) = (\alpha\mathbf{u} - \beta\mathbf{v}) + i(\alpha\mathbf{v} + \beta\mathbf{u}).$$

Equating real and imaginary parts on the two sides of this equation yields

$$\begin{aligned} A\mathbf{u} &= \alpha\mathbf{u} - \beta\mathbf{v} \\ A\mathbf{v} &= \alpha\mathbf{v} + \beta\mathbf{u}. \end{aligned}$$

We leave it to you to show from this that $\mathbf{u}$ and $\mathbf{v}$ are both nonzero. Substituting from these equations into Equation 10.5.4 yields

$$\begin{aligned} f_1'\mathbf{u} + f_2'\mathbf{v} &= f_1(\alpha\mathbf{u} - \beta\mathbf{v}) + f_2(\alpha\mathbf{v} + \beta\mathbf{u}) \\ &= (\alpha f_1 + \beta f_2)\mathbf{u} + (-\beta f_1 + \alpha f_2)\mathbf{v}. \end{aligned}$$

This is true if

$$\begin{aligned} f_1' &= \alpha f_1 + \beta f_2 \\ f_2' &= -\beta f_1 + \alpha f_2, \end{aligned} \quad \text{or equivalently} \quad \begin{aligned} f_1' - \alpha f_1 &= \beta f_2 \\ f_2' - \alpha f_2 &= -\beta f_1. \end{aligned}$$

If we let $f_1 = g_1 e^{\alpha t}$ and $f_2 = g_2 e^{\alpha t}$, where g_1 and g_2 are to be determined, then the last two equations become

$$\begin{aligned} g_1' &= \beta g_2 \\ g_2' &= -\beta g_1, \end{aligned}$$

which implies that

$$g_1'' = \beta g_2' = -\beta^2 g_1,$$

so

$$g_1'' + \beta^2 g_1 = 0.$$

The general solution of this equation is

$$g_1 = c_1 \cos \beta t + c_2 \sin \beta t.$$

Moreover, since $g_2 = g_1'/\beta$,

$$g_2 = -c_1 \sin \beta t + c_2 \cos \beta t.$$

Multiplying g_1 and g_2 by $e^{\alpha t}$ shows that

$$\begin{aligned} f_1 &= e^{\alpha t}(\ \ c_1 \cos \beta t + c_2 \sin \beta t), \\ f_2 &= e^{\alpha t}(-c_1 \sin \beta t + c_2 \cos \beta t). \end{aligned}$$

Substituting these into Equation 10.5.2shows that

$$\begin{aligned} \mathbf{y} &= e^{\alpha t}\left[(c_1 \cos \beta t + c_2 \sin \beta t)\mathbf{u} + (-c_1 \sin \beta t + c_2 \cos \beta t)\mathbf{v}\right] \\ &= c_1 e^{\alpha t}(\mathbf{u} \cos \beta t - \mathbf{v} \sin \beta t) + c_2 e^{\alpha t}(\mathbf{u} \sin \beta t + \mathbf{v} \cos \beta t) \end{aligned} \tag{10.5.5}$$

is a solution of $\mathbf{y}' = A\mathbf{y}$ for any choice of the constants c_1 and c_2. In particular, by first taking $c_1 = 1$ and $c_2 = 0$ and then taking $c_1 = 0$ and $c_2 = 1$, we see that $\mathbf{y}_1$ and $\mathbf{y}_2$ are solutions of $\mathbf{y}' = A\mathbf{y}$.

Example 10.5.1

Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 4 & -5 \\ 5 & -2 \end{bmatrix} \mathbf{y}. \tag{10.5.6}$$

Solution

The characteristic polynomial of the coefficient matrix A in Equation 10.5.6is

$$\begin{vmatrix} 4-\lambda & -5 \\ 5 & -2-\lambda \end{vmatrix} = (\lambda - 1)^2 + 16.$$

Hence, $\lambda = 1 + 4i$ is an eigenvalue of A. The associated eigenvectors satisfy $(A - (1+4i)\,I)\,\mathbf{x} = \mathbf{0}$. The augmented matrix of this system is

$$\left[\begin{array}{cc:c} 3-4i & -5 & 0 \\ 5 & -3-4i & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{cc:c} 1 & -\frac{3+4i}{5} & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Therefore $x_1 = (3+4i)x_2/5$. Taking $x_2 = 5$ yields $x_1 = 3+4i$, so

$$\mathbf{x} = \begin{bmatrix} 3+4i \\ 5 \end{bmatrix}$$

is an eigenvector. The real and imaginary parts of

$$e^t(\cos 4t + i \sin 4t) \begin{bmatrix} 3+4i \\ 5 \end{bmatrix}$$

are

$$\mathbf{y}_1 = e^t \begin{bmatrix} 3\cos 4t - 4\sin 4t \\ 5\cos 4t \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = e^t \begin{bmatrix} 3\sin 4t + 4\cos 4t \\ 5\sin 4t \end{bmatrix},$$

which are linearly independent solutions of Equation 10.5.6. The general solution of Equation 10.5.6 is

$$\mathbf{y} = c_1 e^t \begin{bmatrix} 3\cos 4t - 4\sin 4t \\ 5\cos 4t \end{bmatrix} + c_2 e^t \begin{bmatrix} 3\sin 4t + 4\cos 4t \\ 5\sin 4t \end{bmatrix}.$$

Example 10.5.2

Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} -14 & 39 \\ -6 & 16 \end{bmatrix} \mathbf{y}. \tag{10.5.7}$$

Solution

The characteristic polynomial of the coefficient matrix A in Equation 10.5.7 is

$$\begin{vmatrix} -14-\lambda & 39 \\ -6 & 16-\lambda \end{vmatrix} = (\lambda-1)^2 + 9.$$

Hence, $\lambda = 1+3i$ is an eigenvalue of A. The associated eigenvectors satisfy $(A-(1+3i)I)\,\mathbf{x} = \mathbf{0}$. The augmented augmented matrix of this system is

$$\left[\begin{array}{cc:c} -15-3i & 39 & 0 \\ -6 & 15-3i & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{cc:c} 1 & \frac{-5+i}{2} & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Therefore $x_1 = (5-i)/2$. Taking $x_2 = 2$ yields $x_1 = 5-i$, so

$$\mathbf{x} = \begin{bmatrix} 5-i \\ 2 \end{bmatrix}$$

is an eigenvector. The real and imaginary parts of

$$e^t(\cos 3t + i \sin 3t) \begin{bmatrix} 5-i \\ 2 \end{bmatrix}$$

are

$$\mathbf{y}_1 = e^t \begin{bmatrix} \sin 3t + 5\cos 3t \\ 2\cos 3t \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = e^t \begin{bmatrix} -\cos 3t + 5\sin 3t \\ 2\sin 3t \end{bmatrix},$$

which are linearly independent solutions of Equation 10.5.7. The general solution of Equation 10.5.7 is

$$\mathbf{y} = c_1 e^t \begin{bmatrix} \sin 3t + 5\cos 3t \\ 2\cos 3t \end{bmatrix} + c_2 e^t \begin{bmatrix} -\cos 3t + 5\sin 3t \\ 2\sin 3t \end{bmatrix}.$$

Example 10.5.3

Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} -5 & 5 & 4 \\ -8 & 7 & 6 \\ 1 & 0 & 0 \end{bmatrix} \mathbf{y}. \tag{10.5.8}$$

Solution

The characteristic polynomial of the coefficient matrix A in Equation 10.5.8 is

$$\begin{vmatrix} -5-\lambda & 5 & 4 \\ -8 & 7-\lambda & 6 \\ 1 & 0 & -\lambda \end{vmatrix} = -(\lambda-2)(\lambda^2+1).$$

Hence, the eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = i$, and $\lambda_3 = -i$. The augmented matrix of $(A-2I)\mathbf{x} = \mathbf{0}$ is

$$\left[\begin{array}{ccc:c} -7 & 5 & 4 & 0 \\ -8 & 5 & 6 & 0 \\ 1 & 0 & -2 & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccc:c} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Therefore $x_1 = x_2 = 2x_3$. Taking $x_3 = 1$ yields

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix},$$

so

$$\mathbf{y}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} e^{2t}$$

is a solution of Equation 10.5.8.

The augmented matrix of $(A-iI)\mathbf{x} = \mathbf{0}$ is

$$\left[\begin{array}{ccc:c} -5-i & 5 & 4 & 0 \\ -8 & 7-i & 6 & 0 \\ 1 & 0 & -i & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccc:c} 1 & 0 & -i & 0 \\ 0 & 1 & 1-i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Therefore $x_1 = ix_3$ and $x_2 = -(1-i)x_3$. Taking $x_3 = 1$ yields the eigenvector

$$\mathbf{x}_2 = \begin{bmatrix} i \\ -1+i \\ 1 \end{bmatrix}.$$

The real and imaginary parts of

$$(\cos t + i \sin t) \begin{bmatrix} i \\ -1+i \\ 1 \end{bmatrix}$$

are

$$\mathbf{y}_2 = \begin{bmatrix} -\sin t \\ -\cos t - \sin t \\ \cos t \end{bmatrix} \quad \text{and} \quad \mathbf{y}_3 = \begin{bmatrix} \cos t \\ \cos t - \sin t \\ \sin t \end{bmatrix},$$

which are solutions of Equation 10.5.8. Since the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ at $t = 0$ is

$$\begin{vmatrix} 2 & 0 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 1,$$

$\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is a fundamental set of solutions of Equation 10.5.8. The general solution of Equation 10.5.8 is

$$\mathbf{y} = c_1 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} -\sin t \\ -\cos t - \sin t \\ \cos t \end{bmatrix} + c_3 \begin{bmatrix} \cos t \\ \cos t - \sin t \\ \sin t \end{bmatrix}.$$

Example 10.5.4

Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 3 & 2 \\ 1 & -1 & 2 \end{bmatrix} \mathbf{y}. \tag{10.5.9}$$

Solution

The characteristic polynomial of the coefficient matrix A in Equation 10.5.9 is

$$\begin{vmatrix} 1-\lambda & -1 & -2 \\ 1 & 3-\lambda & 2 \\ 1 & -1 & 2-\lambda \end{vmatrix} = -(\lambda - 2)\left((\lambda-2)^2 + 4\right).$$

Hence, the eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = 2 + 2i$, and $\lambda_3 = 2 - 2i$. The augmented matrix of $(A - 2I)\mathbf{x} = \mathbf{0}$ is

$$\left[\begin{array}{ccc:c} -1 & -1 & -2 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccc:c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Therefore $x_1 = x_2 = -x_3$. Taking $x_3 = 1$ yields

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix},$$

so

$$\mathbf{y}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{2t}$$

is a solution of Equation 10.5.9. The augmented matrix of $(A-(2+2i)I)\,\mathbf{x} = \mathbf{0}$ is

$$\left[\begin{array}{ccc:c} -1-2i & -1 & -2 & 0 \\ 1 & 1-2i & 2 & 0 \\ 1 & -1 & -2i & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccc:c} 1 & 0 & -i & 0 \\ 0 & 1 & i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Therefore $x_1 = ix_3$ and $x_2 = -ix_3$. Taking $x_3 = 1$ yields the eigenvector

$$\mathbf{x}_2 = \begin{bmatrix} i \\ -i \\ 1 \end{bmatrix}$$

The real and imaginary parts of

$$e^{2t}(\cos 2t + i \sin 2t) \begin{bmatrix} i \\ -i \\ 1 \end{bmatrix}$$

are

$$\mathbf{y}_2 = e^{2t} \begin{bmatrix} -\sin 2t \\ \sin 2t \\ \cos 2t \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = e^{2t} \begin{bmatrix} \cos 2t \\ -\cos 2t \\ \sin 2t \end{bmatrix},$$

which are solutions of Equation 10.5.9. Since the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ at $t = 0$ is

$$\begin{vmatrix} -1 & 0 & 1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{vmatrix} = -2,$$

$\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is a fundamental set of solutions of Equation 10.5.9. The general solution of Equation 10.5.9 is

$$\mathbf{y} = c_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{2t} + c_2 e^{2t} \begin{bmatrix} -\sin 2t \\ \sin 2t \\ \cos 2t \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} \cos 2t \\ -\cos 2t \\ \sin 2t \end{bmatrix}.$$

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10.5.1: Constant Coefficient Homogeneous Systems III (Exercises)

In Exercises **1-28** find the general solution.

1. $\begin{aligned} y_1' &= 6y_1 - y_2 \\ y_2' &= 5y_1 + 2y_2 \end{aligned}$

2. $\begin{aligned} y_1' &= y_1 + y_2 \\ y_2' &= -2y_1 - y_2 \end{aligned}$

3. $\begin{aligned} y_1' &= 5y_1 + y_2 \\ y_2' &= -2y_1 + 3y_2 \end{aligned}$

4. $\begin{aligned} y_1' &= 4y_1 + 5y_2 \\ y_2' &= -2y_1 + 6y_2 \end{aligned}$

5. $\mathbf{y}' = \begin{bmatrix} 4 & -5 \\ 5 & -4 \end{bmatrix} \mathbf{y}$

6. $\mathbf{y}' = \begin{bmatrix} 1 & -8 \\ 1 & -3 \end{bmatrix} \mathbf{y}$

7. $\mathbf{y}' = \begin{bmatrix} -1 & 2 \\ -5 & 5 \end{bmatrix} \mathbf{y}$

8. $\mathbf{y}' = \begin{bmatrix} -11 & 4 \\ -26 & 9 \end{bmatrix} \mathbf{y}$

9. $\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix} \mathbf{y}$

10. $\mathbf{y}' = \begin{bmatrix} 5 & -6 \\ 3 & -1 \end{bmatrix} \mathbf{y}$

11. $\mathbf{y}' = \begin{bmatrix} 5 & -4 \\ 10 & 1 \end{bmatrix} \mathbf{y}$

12. $\mathbf{y}' = \frac{1}{3} \begin{bmatrix} 7 & -5 \\ 2 & 5 \end{bmatrix} \mathbf{y}$

13. $\mathbf{y}' = \begin{bmatrix} 3 & 2 \\ -5 & 1 \end{bmatrix} \mathbf{y}$

14. $\mathbf{y}' = \begin{bmatrix} 34 & 52 \\ -20 & -30 \end{bmatrix} \mathbf{y}$

15.
$$\begin{aligned} y_1' &= y_3 \\ y_2' &= -y_3 \\ y_3' &= y_2 \end{aligned}$$

16.
$$\begin{aligned} y_1' &= 2y_1 + y_2 + 2y_3 \\ y_2' &= 3y_1 + 6y_3 \\ y_3' &= -4y_1 - 3y_3 \end{aligned}$$

17. $\mathbf{y}' = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \mathbf{y}$

18. $\mathbf{y}' = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 6 & 0 \\ -4 & 0 & 4 \end{bmatrix} \mathbf{y}$

19. $\mathbf{y}' = \begin{bmatrix} 2 & 5 & 1 \\ -5 & -6 & 4 \\ 0 & 0 & 2 \end{bmatrix} \mathbf{y}$

20. $\mathbf{y}' = \begin{bmatrix} 2 & 4 & 4 \\ -1 & -2 & 0 \\ -1 & 0 & 2 \end{bmatrix} \mathbf{y}$

21. $\mathbf{y}' = \begin{bmatrix} 3 & -3 & 1 \\ 0 & 2 & 2 \\ 5 & 1 & 1 \end{bmatrix} \mathbf{y}$

22. $\mathbf{y}' = \begin{bmatrix} -3 & 3 & 1 \\ 1 & -5 & -3 \\ -3 & 7 & 3 \end{bmatrix} \mathbf{y}$

23. $\mathbf{y}' = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \mathbf{y}$

24. $\mathbf{y}' = \begin{bmatrix} -3 & 1 & -3 \\ 4 & -1 & 2 \\ 4 & -2 & 3 \end{bmatrix} \mathbf{y}$

25. $\mathbf{y}' = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ -1 & -2 & -1 \end{bmatrix} \mathbf{y}$

26. $\mathbf{y}' = \begin{bmatrix} 3 & -4 & -2 \\ -5 & 7 & -8 \\ -10 & 13 & -8 \end{bmatrix} \mathbf{y}$

27. $\mathbf{y}' = \begin{bmatrix} 6 & 0 & -3 \\ -3 & 3 & 3 \\ 1 & -2 & 6 \end{bmatrix} \mathbf{y}$

28. $\mathbf{y}' = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 2 & -1 \\ 1 & 0 & 0 \end{bmatrix} \mathbf{y}$

In Exercises **29**-**38** solve the initial value problem.

29. $\mathbf{y}' = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$

30. $\mathbf{y}' = \begin{bmatrix} 4 & -6 \\ 3 & -2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

31. $\mathbf{y}' = \begin{bmatrix} 7 & 15 \\ -3 & 1 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

32. $\mathbf{y}' = \begin{bmatrix} 7 & -15 \\ 3 & -5 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 17 \\ 7 \end{bmatrix}$

33. $\mathbf{y}' = \frac{1}{6} \begin{bmatrix} 4 & -2 \\ 5 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

34. $\mathbf{y}' = \begin{bmatrix} 1 & -12 & -14 \\ 1 & 2 & -3 \\ 1 & 1 & -2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 4 \\ 6 \\ -7 \end{bmatrix}$

35. $\mathbf{y}' = \begin{bmatrix} 5 & 2 & -1 \\ -3 & 2 & 2 \\ 1 & 3 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix}$

36. $\mathbf{y}' = \begin{bmatrix} 4 & 4 & 0 \\ 8 & 10 & -20 \\ 2 & 3 & -2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 8 \\ 6 \\ 5 \end{bmatrix}$

37. $\mathbf{y}' = \begin{bmatrix} 1 & 15 & -15 \\ -6 & 18 & -22 \\ -3 & 11 & -15 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 15 \\ 17 \\ 10 \end{bmatrix}$

38. $\mathbf{y}' = \begin{bmatrix} 4 & -4 & 4 \\ -10 & 3 & 15 \\ 2 & -3 & 1 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 16 \\ 14 \\ 6 \end{bmatrix}$

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10.6: Variation of Parameters for Nonhomogeneous Linear Systems

We now consider the nonhomogeneous linear system

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t),$$

where A is an $n \times n$ matrix function and $\mathbf{f}$ is an n-vector forcing function. Associated with this system is the *homogeneous system* $\mathbf{y}' = A(t)\mathbf{y}$.

The next theorem is analogous to earlier theorems. It shows how to find the general solution of $\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t)$ if we know a particular solution of $\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t)$ and a fundamental set of solutions of the homogeneous system.

Theorem 10.6.1

Suppose the $n \times n$ matrix function A and the n-vector function $\mathbf{f}$ are continuous on (a, b). Let $\mathbf{y}_p$ be a particular solution of $\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t)$ on (a, b), and let $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ be a fundamental set of solutions of the homogeneous equation $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b). Then $\mathbf{y}$ is a solution of $\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t)$ on (a, b) if and only if

$$\mathbf{y} = \mathbf{y}_p + c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n,$$

where c_1, c_2, ..., c_n are constants.

Finding a Particular Solution of a Nonhomogeneous System

We now discuss an extension of the method of variation of parameters to linear nonhomogeneous systems. This method will produce a particular solution of a nonhomogenous system $\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t)$ provided that we know a fundamental matrix for the homogeneous system. To derive the method, suppose Y is a fundamental matrix for the homogeneous system; that is,

$$Y = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix},$$

where

$$\mathbf{y}_1 = \begin{bmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{n1} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} y_{12} \\ y_{22} \\ \vdots \\ y_{n2} \end{bmatrix}, \quad \cdots, \quad \mathbf{y}_n = \begin{bmatrix} y_{1n} \\ y_{2n} \\ \vdots \\ y_{nn} \end{bmatrix}$$

is a fundamental set of solutions of the homogeneous system. In Section 10.2 we saw that $Y' = A(t)Y$. We seek a particular solution of

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t) \tag{10.6.1}$$

of the form

$$\mathbf{y}_p = Y\mathbf{u}, \tag{10.6.2}$$

where $\mathbf{u}$ is to be determined. Differentiating Equation 10.6.2 yields

$$\begin{aligned} \mathbf{y}_p' &= Y'\mathbf{u} + Y\mathbf{u}' \\ &= AY\mathbf{u} + Y\mathbf{u}' \text{ (since } Y' = AY) \\ &= A\mathbf{y}_p + Y\mathbf{u}' \text{ (since } Y\mathbf{u} = \mathbf{y}_p). \end{aligned}$$

Comparing this with Equation 10.6.1 shows that $\mathbf{y}_p = Y\mathbf{u}$ is a solution of Equation 10.6.1 if and only if

$$Y\mathbf{u}' = \mathbf{f}.$$

Thus, we can find a particular solution $\mathbf{y}_p$ by solving this equation for $\mathbf{u}'$, integrating to obtain $\mathbf{u}$, and computing $Y\mathbf{u}$. We can take all constants of integration to be zero, since any particular solution will suffice.

The proof is analogous to the method of variation of parameters discussed earlier for scalar linear equations and will be omitted here.

Example 10.6.1

a. Find a particular solution of the system

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 2e^{4t} \\ e^{4t} \end{bmatrix}, \tag{10.6.3}$$

which we considered in Example 10.1.1.

b. Find the general solution of Equation 10.6.3.

Solution a

The homogeneous system is

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y}. \tag{10.6.4}$$

The characteristic polynomial of the coefficient matrix is

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (\lambda+1)(\lambda-3).$$

Using the method of Section 10.3, we find that

$$\mathbf{y}_1 = \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$$

are linearly independent solutions of Equation 10.6.4. Therefore

$$Y = \begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{bmatrix}$$

is a fundamental matrix for Equation 10.6.4. We seek a particular solution $\mathbf{y}_p = Y\mathbf{u}$ of Equation 10.6.3, where $Y\mathbf{u}' = \mathbf{f}$; that is,

$$\begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 2e^{4t} \\ e^{4t} \end{bmatrix}.$$

The determinant of Y is the Wronskian

$$\begin{vmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{vmatrix} = -2e^{2t}.$$

By Cramer's rule,

$$\begin{aligned} u_1' &= -\frac{1}{2e^{2t}} \begin{vmatrix} 2e^4 & e^{-t} \\ e^{4t} & -e^{-t} \end{vmatrix} = \frac{3e^{3t}}{2e^{2t}} = \frac{3}{2}e^t, \\ u_2' &= -\frac{1}{2e^{2t}} \begin{vmatrix} e^{3t} & 2e^{4t} \\ e^{3t} & e^{4t} \end{vmatrix} = \frac{e^{7t}}{2e^{2t}} = \frac{1}{2}e^{5t}. \end{aligned}$$

Therefore

$$\mathbf{u}' = \frac{1}{2}\begin{bmatrix} 3e^t \\ e^{5t} \end{bmatrix}.$$

Integrating and taking the constants of integration to be zero yields

$$\mathbf{u} = \frac{1}{10}\begin{bmatrix} 15e^t \\ e^{5t} \end{bmatrix},$$

so

$$\mathbf{y}_p = Y\mathbf{u} = \frac{1}{10}\begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{bmatrix}\begin{bmatrix} 15e^t \\ e^{5t} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 8e^{4t} \\ 7e^{4t} \end{bmatrix}$$

is a particular solution of Equation 10.6.3.

Solution b

From Theorem 10.6.1 , the general solution of Equation 10.6.3 is

$$\mathbf{y} = \mathbf{y}_p + c_1\mathbf{y}_1 + c_2\mathbf{y}_2 = \frac{1}{5}\begin{bmatrix} 8e^{4t} \\ 7e^{4t} \end{bmatrix} + c_1\begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} + c_2\begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}, \tag{10.6.5}$$

which can also be written as

$$\mathbf{y} = \frac{1}{5}\begin{bmatrix} 8e^{4t} \\ 7e^{4t} \end{bmatrix} + \begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{bmatrix}\mathbf{c},$$

where $\mathbf{c}$ is an arbitrary constant vector.

Writing Equation 10.6.5 in terms of coordinates yields

$$\begin{aligned} y_1 &= \frac{8}{5}e^{4t} + c_1e^{3t} + c_2e^{-t} \\ y_2 &= \frac{7}{5}e^{4t} + c_1e^{3t} - c_2e^{-t}, \end{aligned}$$

so our result is consistent with Example 10.1.1.

Example 10.6.2

a. Find a particular solution of

$$\mathbf{y}' = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}\mathbf{y} + \begin{bmatrix} e^t \\ 0 \\ e^{-t} \end{bmatrix}. \tag{10.6.6}$$

b. Find the general solution of Equation 10.6.6.

Solution a

The homogeneous system for Equation 10.6.6 is

$$\mathbf{y}' = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}\mathbf{y}. \tag{10.6.7}$$

The characteristic polynomial of the coefficient matrix is

$$\begin{vmatrix} 2-\lambda & -1 & -1 \\ 1 & -\lambda & -1 \\ 1 & -1 & -\lambda \end{vmatrix} = -\lambda(\lambda-1)^2.$$

Using the method of Section 10.3, we find that

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} e^t \\ e^t \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{y}_3 = \begin{bmatrix} e^t \\ 0 \\ e^t \end{bmatrix}$$

are linearly independent solutions of Equation 10.6.7. Therefore

$$Y = \begin{bmatrix} 1 & e^t & e^t \\ 1 & e^t & 0 \\ 1 & 0 & e^t \end{bmatrix}$$

is a fundamental matrix for Equation 10.6.7. We seek a particular solution $\mathbf{y}_p = Y\mathbf{u}$ of Equation 10.6.6, where $Y\mathbf{u}' = \mathbf{f}$; that is,

$$\begin{bmatrix} 1 & e^t & e^t \\ 1 & e^t & 0 \\ 1 & 0 & e^t \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} = \begin{bmatrix} e^t \\ 0 \\ e^{-t} \end{bmatrix}.$$

The determinant of Y is the Wronskian

$$\begin{vmatrix} 1 & e^t & e^t \\ 1 & e^t & 0 \\ 1 & 0 & e^t \end{vmatrix} = -e^{2t}.$$

Thus, by Cramer's rule,

$$\begin{aligned} u_1' &= -\tfrac{1}{e^{2t}} \begin{vmatrix} e^t & e^t & e^t \\ 0 & e^t & 0 \\ e^{-t} & 0 & e^t \end{vmatrix} = -\tfrac{e^{3t}-e^t}{e^{2t}} = e^{-t} - e^t \\ u_2' &= -\tfrac{1}{e^{2t}} \begin{vmatrix} 1 & e^t & e^t \\ 1 & 0 & 0 \\ 1 & e^{-t} & e^t \end{vmatrix} = -\tfrac{1-e^{2t}}{e^{2t}} = 1 - e^{-2t} \\ u_3' &= -\tfrac{1}{e^{2t}} \begin{vmatrix} 1 & e^t & e^t \\ 1 & e^t & 0 \\ 1 & 0 & e^{-t} \end{vmatrix} = \tfrac{e^{2t}}{e^{2t}} = 1 \end{aligned}$$

Therefore

$$\mathbf{u}' = \begin{bmatrix} e^{-t} - e^t \\ 1 - e^{-2t} \\ 1 \end{bmatrix}.$$

Integrating and taking the constants of integration to be zero yields

$$\mathbf{u} = \begin{bmatrix} -e^t - e^{-t} \\ \frac{e^{-2t}}{2} + t \\ t \end{bmatrix}$$

so

$$\mathbf{y}_p = Y\mathbf{u} = \begin{bmatrix} 1 & e^t & e^t \\ 1 & e^t & 0 \\ 1 & 0 & e^t \end{bmatrix} \begin{bmatrix} -e^t - e^{-t} \\ \frac{e^{-2t}}{2} + t \\ t \end{bmatrix} = \begin{bmatrix} e^t(2t-1) - \frac{e^{-t}}{2} \\ e^t(t-1) - \frac{e^{-t}}{2} \\ e^t(t-1) - e^{-t} \end{bmatrix}$$

is a particular solution of Equation 10.6.6.

Solution b

From Theorem 10.6.1 the general solution of Equation 10.6.6 is

$$\mathbf{y} = \mathbf{y}_p + c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + c_3\mathbf{y}_3 = \begin{bmatrix} e^t(2t-1) - \frac{e^{-t}}{2} \\ e^t(t-1) - \frac{e^{-t}}{2} \\ e^t(t-1) - e^{-t} \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} e^t \\ e^t \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} e^t \\ 0 \\ e^t \end{bmatrix}$$

which can be written as

$$\mathbf{y} = \mathbf{y}_p + Y\mathbf{c} = \begin{bmatrix} e^t(2t-1) - \frac{e^{-t}}{2} \\ e^t(t-1) - \frac{e^{-t}}{2} \\ e^t(t-1) - e^{-t} \end{bmatrix} + \begin{bmatrix} 1 & e^t & e^t \\ 1 & e^t & 0 \\ 1 & 0 & e^t \end{bmatrix} \mathbf{c}$$

where $\mathbf{c}$ is an arbitrary constant vector.

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10.6.1: Variation of Parameters for Nonhomogeneous Linear Systems (Exercises)

In Exercises **1-20** find the general solution.

1. $\begin{aligned} y_1' &= 3y_1 - 3y_2 + 4 \\ y_2' &= 2y_1 - 2y_2 - 1 \end{aligned}$

2. $\begin{aligned} y_1' &= 2y_1 - y_2 \\ y_2' &= 3y_1 - 2y_2 + 4t \end{aligned}$

3. $\mathbf{y}' = \begin{bmatrix} 3 & -5 \\ \frac{3}{4} & -1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t/2}$

4. $\mathbf{y}' = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix} \mathbf{y} + \begin{bmatrix} \sin 2t \\ 2\cos 2t \end{bmatrix} e^{2t}$

5. $\mathbf{y}' = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t}$

6. $\mathbf{y}' = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 2 \\ e^{-3t} \end{bmatrix}$

7. $\mathbf{y}' = \begin{bmatrix} 1 & 8 \\ 1 & -1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 12 \\ 12 \end{bmatrix} t$

8. $\mathbf{y}' = \begin{bmatrix} 1 & 8 \\ 1 & -1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} e^{-t} \\ te^{t} \end{bmatrix}$

9. $\mathbf{y}' = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}$

10. $\mathbf{y}' = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

11. $\mathbf{y}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{y} + \begin{bmatrix} \sec t \\ 0 \end{bmatrix}$

12. $\mathbf{y}' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} e^{t}$

13. $\mathbf{y}' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} e^{t}$

14. $\mathbf{y}' = \begin{bmatrix} 2 & -2 \\ 8 & -6 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} \frac{e^{-2t}}{t}$

15. $\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 0 \\ \sec t \tan t \end{bmatrix}$

16. $\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 1 \\ \cot t \end{bmatrix}$

17. $\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ -\frac{1}{2} & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} \csc t \\ \sec t \end{bmatrix} e^t$

18. $\mathbf{y}' = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} \tan t \\ 1 \end{bmatrix}$

19. $\mathbf{y}' = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \mathbf{y} + \begin{bmatrix} e^t \\ e^{2t} \\ te^{3t} \end{bmatrix}$

20. $\mathbf{y}' = \begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 0 \\ t \\ 2e^t \end{bmatrix}$

In Exercises **21-22** solve the initial value problem.

21. $\mathbf{y}' = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 4e^{2t} \\ 4e^{4t} \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

22. $\mathbf{y}' = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 1/t \\ 1/t \end{bmatrix}, \quad \mathbf{y}(1) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

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CHAPTER OVERVIEW

11: Appendices

In this chapter we review topics that you need to know for this class, and introduce topics that you may not have previously covered that you will need to be successful in Math 420.

11.1: Functions of Several Variables

11.1: Functions of Several Variables (Exercises)

11.2: Partial Derivatives

11.2.1: Partial Derivatives (Exercises)

11.3: Determinants and Cramer's Rule for 2 X 2 Matrices

11.3.1: Determinants and Cramer's Rule for 2 X 2 Matrices (Exercises)

11.4: Determinants and Cramer's Rule for n x n Matrices

11.4.1: Determinants and Cramer's Rule for n x n Matrices (Exercises)

11.5: Elementary Row Operations and Gaussian Elimination

11.5.1: Elementary Row Operations and Gaussian Elimination (Exercises)

11.6: Eigenvalues and Eigenvectors

11.6.1: Eigenvalues and Eigenvectors (Exercises)

11.7: A Brief Table of Integrals

11.8: A Brief Table of Trigonometric Identities and Logarithmic Laws

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11.1: Functions of Several Variables

In Math 400 (Calculus I), and innumerable courses before it, we studied functions of one variable. A function of this type might represent the temperature over a given time interval, the position of a car as a function of time, or the altitude of a jet plane as it travels from New York to San Francisco. In each of these examples, the function has one independent variable.

Suppose, however, that we have a quantity that depends on more than one variable. For example, temperature can depend on location and the time of day, or a company's profit model might depend on the number of units sold and the amount of money spent on advertising.

When dealing with a function of more than one independent variable, several questions naturally arise. For example, how do we calculate limits of functions of more than one variable? The definition of derivative we used before involved a limit. Does the new definition of derivative involve limits as well? Do the rules of differentiation apply in this context? How to we integrate such functions? All these questions that pertain, in particular to Math 420, are answered in this chapter.

Our first step is to explain what a function of more than one variable is, starting with functions of two independent variables. This step includes identifying the domain and range of such functions.

Functions of Two Variables

The definition of a function of two variables is very similar to the definition for a function of one variable. The main difference is that, instead of mapping values of one variable to values of another variable, we map ordered pairs of variables to another variable.

Definition: Function of Two Variables

A *function of two variables* $z = f(x, y)$ maps each ordered pair (x, y) in a subset D of the real plane R^2 to a unique real number z. The set D is called the domain of the function. The range of f is the set of all real numbers z that has at least one ordered pair $(x, y) \in D$ such that $f(x, y) = z$ as shown in Figure 11.1.1.

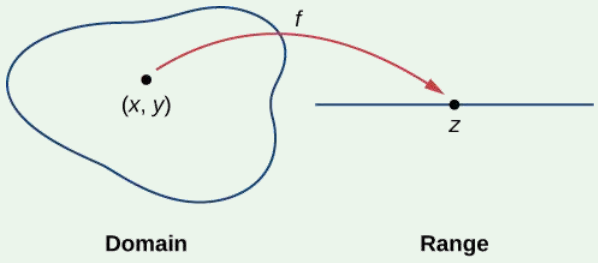


Figure 11.1.1: The domain of a function of two variables consists of ordered pairs (x, y).

Determining the domain of a function of two variables involves taking into account any domain restrictions that may exist. Let's take a look.

Example 11.1.1: Domain and Range for Functions of Two Variables

Find the domain and range of each of the following functions:

a. $f(x, y) = 3x + 5y + 2$
b. $g(x, y) = \sqrt{9 - x^2 - y^2}$

Solution a.

This is an example of a function in two variables. There are no values or combinations of x and y that cause $f(x, y)$ to be undefined, so the domain of f is R^2. To determine the range, first pick a value for z. We need to find a solution to the equation $f(x, y) = z$, or $3x - 5y + 2 = z$. One such solution can be obtained by first setting $y = 0$, which yields the equation $3x + 2 = z$. The solution to this equation is $x = \dfrac{z-2}{3}$, which gives the ordered pair $\left(\dfrac{z-2}{3}, 0\right)$ as a solution to the equation $f(x, y) = z$ for any value of z. Therefore, the range of the function is all real numbers, or R.

Note: we could do something similar by setting $x = 0$, or x or y to any real number.

Solution b.

For the function $g(x,y)$ to have a real value, the quantity under the square root must be nonnegative:

$$9-x^2-y^2 \geq 0.$$

This inequality can be written in the form

$$x^2+y^2 \leq 9.$$

Therefore, the domain of $g(x,y)$ is $\{(x,y) \in R^2 \mid x^2+y^2 \leq 9\}$. The graph of this set of points can be described as a disk of radius 3 centered at the origin. The domain includes the boundary circle as shown in the following graph.

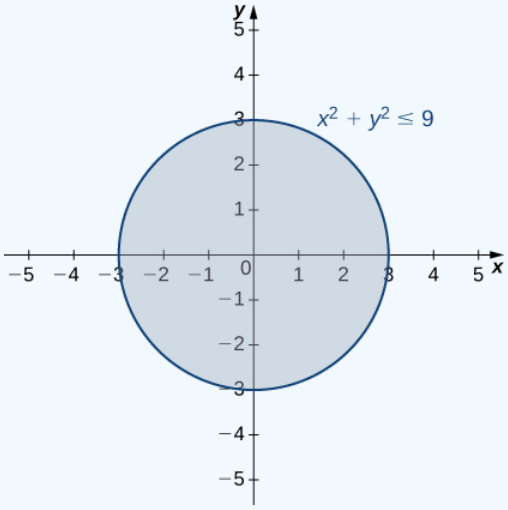


Figure 11.1.2: The domain of the function $g(x,y)=\sqrt{9-x^2-y^2}$ is a closed disk of radius 3.

To determine the range of $g(x,y)=\sqrt{9-x^2-y^2}$ we start with a point (x_0,y_0) on the boundary of the domain, which is defined by the relation $x^2+y^2=9$. It follows that $x_0^2+y_0^2=9$ and

$$\begin{aligned} g(x_0,y_0) &= \sqrt{9-x_0^2-y_0^2} \\ &= \sqrt{9-(x_0^2+y_0^2)} \\ &= \sqrt{9-9} \\ &= 0. \end{aligned}$$

If $x_0^2+y_0^2=0$ (in other words, $x_0=y_0=0$) , then

$$\begin{aligned} g(x_0,y_0) &= \sqrt{9-x_0^2-y_0^2} \\ &= \sqrt{9-(x_0^2+y_0^2)} \\ &= \sqrt{9-0}=3. \end{aligned}$$

This is the maximum value of the function. Given any value c between 0 and 3, we can find an entire set of points inside the domain of g such that $g(x,y)=c$:

$$\begin{aligned} \sqrt{9-x^2-y^2} &= c \\ 9-x^2-y^2 &= c^2 \\ x^2+y^2 &= 9-c^2. \end{aligned}$$

Since $9-c^2>0$, this describes a circle of radius $\sqrt{9-c^2}$ centered at the origin. Any point on this circle satisfies the equation $g(x,y)=c$. Therefore, the range of this function can be written in interval notation as $[0,3]$.

? Exercise 11.1.1

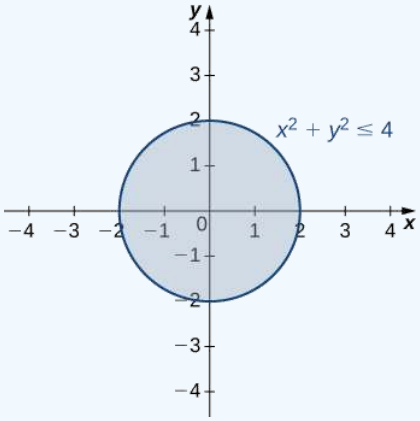
Find the domain and range of the function $f(x,y)=\sqrt{36-9x^2-9y^2}$.

Hint

Determine the set of ordered pairs that do not make the radicand negative.

Solution

The domain is $\{(x,y)|x^2+y^2\leq 4\}$ the shaded circle defined by the inequality $x^2+y^2\leq 4$, which has a circle of radius 2 as its boundary. The range is $[0,6]$.



Note: The remainder of this section is not necessary for Math 420. I am leaving it for those who may wish to have a better understanding of the concepts discussed here.

Graphing Functions of Two Variables

Suppose we wish to graph the function $z=(x,y)$. This function has two independent variables (x and y) and one dependent variable (z). When graphing a function $y=f(x)$ of one variable, we use the Cartesian plane. We are able to graph any ordered pair (x,y) in the plane, and every point in the plane has an ordered pair (x,y) associated with it. With a function of two variables, each ordered pair (x,y) in the domain of the function is mapped to a real number z. Therefore, the graph of the function f consists of ordered triples (x,y,z). The graph of a function $z=(x,y)$ of two variables is called a surface.

To understand more completely the concept of plotting a set of ordered triples to obtain a surface in three-dimensional space, imagine the (x,y) coordinate system laying flat. Then, every point in the domain of the function f has a unique *z*-value associated with it. If *z* is positive, then the graphed point is located above the *xy*-plane, if *z* is negative, then the graphed point is located below the *xy*-plane. The set of all the graphed points becomes the two-dimensional surface that is the graph of the function f.

✓ Example 11.1.2: Graphing Functions of Two Variables

Create a graph of each of the following functions:

a. $g(x,y)=\sqrt{9-x^2-y^2}$
b. $f(x,y)=x^2+y^2$

Solution

a. In Example 11.1.2 we determined that the domain of $g(x,y)=\sqrt{9-x^2-y^2}$ is $\{(x,y)\in R^2\mid x^2+y^2\leq 9\}$ and the range is $\{z\in R^2\mid 0\leq z\leq 3\}$. When $x^2+y^2=9$ we have $g(x,y)=0$. Therefore any point on the circle of radius 3 centered at the origin in the xy-plane maps to $z=0$ in R^3. If $x^2+y^2=8$, then $g(x,y)=1$, so any point on the circle of radius $2\sqrt{2}$ centered at the origin in the xy-plane maps to $z=1$ in R^3. As x^2+y^2 gets closer to zero, the value of z approaches 3. When $x^2+y^2=0$, then $g(x,y)=3$. This is the origin in the xy-plane If x^2+y^2 is equal to any other value between 0 and 9, then $g(x,y)$ equals some other constant between 0 and 3. The surface described by this function is a hemisphere centered at the origin with radius 3 as shown in the following graph.

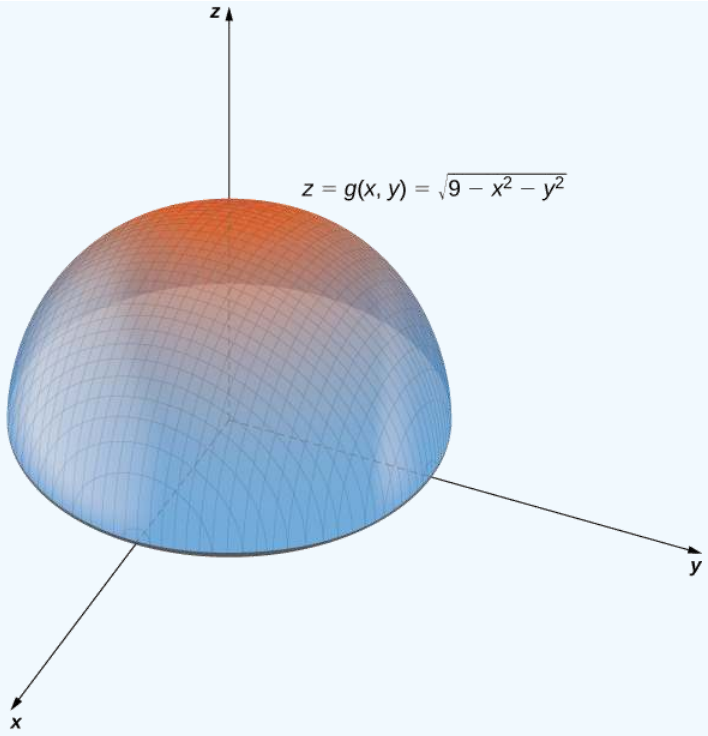


Figure 11.1.3: Graph of the hemisphere represented by the given function of two variables.

b. This function also contains the expression $x^2 + y^2$. Setting this expression equal to various values starting at zero, we obtain circles of increasing radius. The minimum value of $f(x, y) = x^2 + y^2$ is zero (attained when $x = y = 0$. When $x = 0$, the function becomes $z = y^2$, and when $y = 0$, then the function becomes $z = x^2$. These are cross-sections of the graph, and are parabolas. The name of the graph of $f(x, y) = x^2 + y^2$ is a **paraboloid**. The graph of f appears in the following graph.

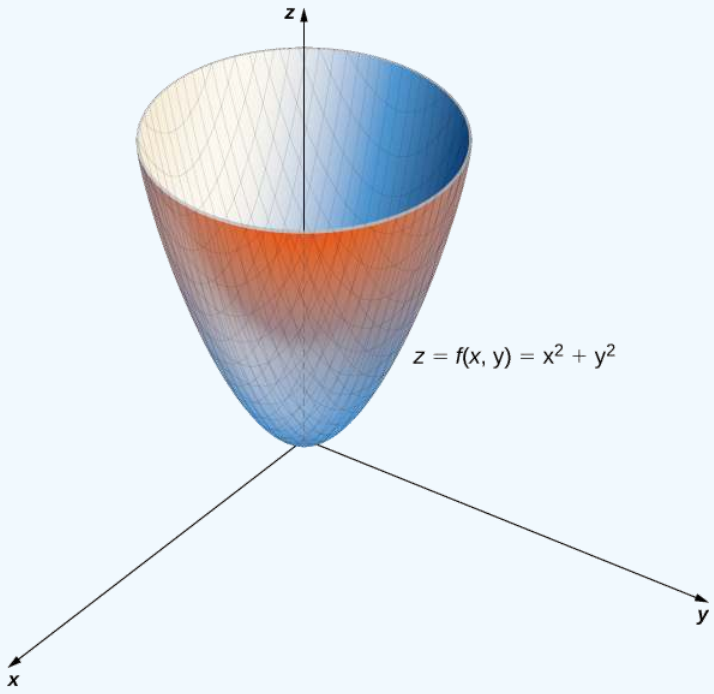


Figure 11.1.4: A paraboloid is the graph of the given function of two variables.

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11.1: Functions of Several Variables (Exercises)

For the following exercises, evaluate each function at the indicated values.

1) $W(x,y)=4x^2+y^2$. Find $W(2,-1), W(-3,6)$

Answer

$W(2,-1)=17, \quad W(-3,6)=72$

2) $W(x,y)=4x^2+y^2$. Find $W(2+h,3+h)$.

3) The volume of a right circular cylinder is calculated by a function of two variables, $V(x,y)=\pi x^2 y$, where x is the radius of the right circular cylinder and y represents the height of the cylinder. Evaluate $V(2,5)$ and explain what this means.

Answer

$V(2,5)=20\pi$ units^3 This is the volume when the radius is 2 and the height is 5.

4) An oxygen tank is constructed of a right cylinder of height y and radius x with two hemispheres of radius x mounted on the top and bottom of the cylinder. Express the volume of the cylinder as a function of two variables, x and y, find $V(10,2)$, and explain what this means.

5) A thin plate made of iron is located in the xy-plane The temperature T in degrees Celsius at a point $P(x,y)$ is inversely proportional to the square of its distance from the origin. Express T as a function of x and y.

Answer

$$T(x,y)=\frac{k}{x^2+y^2}$$

6) Refer to the preceding problem. Using the temperature function found there, determine the proportionality constant if the temperature at point $P(1,2)$ is $50^\circ C$. Use this constant to determine the temperature at point $Q(3,4)$.

For exercises 7 - 18, find the domain and range of the given function.

7) $V(x,y)=4x^2+y^2$

Answer

Domain: $\{(x,y)\,|\,x\in\mathbb{R}, \mathrm{y}\in\mathbb{R}\}$ That is, all points in the xy-plane
Range: $[0,\infty)$

8) $f(x,y)=\sqrt{x^2+y^2-4}$

Answer

Domain: $\{(x,y)\,|\,x^2+y^2\geq 4\}$
Range: $[0,\infty)$

9) $f(x,y)=4\ln(y^2-x)$

Answer

Domain: $\{(x,y)\,|\,x<y^2\}$
Range: $(-\infty,\infty)$

10) $g(x,y)=\sqrt{16-4x^2-y^2}$

Answer

Domain: $\{(x,y)\,|\,\frac{x^2}{4}+\frac{y^2}{16}\leq 1\}$
Range: $[0,4]$

11) $z = \arccos(y - x)$

Answer

Domain: $\{(x, y) \mid x - 1 \leq y \leq x + 1\}$ That is, all points between the graphs of $y = x - 1$ and $y = x + 1$.
Range: $[0, \pi]$

12) $f(x, y) = \dfrac{y + 2}{x^2}$

Answer

Domain: $\{(x, y) \mid x \neq 0\}$
Range: $(-\infty, \infty)$

13) $g(x, y) = \sqrt{16 - 4x^2 - y^2}$

Answer

$\{z \mid 0 \leq z \leq 4\}$ or in interval notation: $[0, 4]$

14) $V(x, y) = 4x^2 + y^2$

15) $z = y^2 - x^2$

Answer

The set $\mathbb{R}$

16) $z = \sqrt{100 - 4x^2 - 25y^2}$

Answer

Domain: $\{(x, y) \mid \dfrac{x^2}{25} + \dfrac{y^2}{4} \leq 1\}$
Range: $[0, 10]$

17) $z = \ln(x - y^2)$

18) $f(x, y) = \cos\sqrt{x^2 + y^2}$

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11.2: Partial Derivatives

Derivatives of a Function of Two Variables

When studying derivatives of functions of one variable, we found that one interpretation of the derivative is an instantaneous rate of change of y as a function of x. Leibniz notation for the derivative is dy/dx, which implies that y is the dependent variable and x is the independent variable. For a function $z = f(x,y)$ of two variables, x and y are the independent variables and z is the dependent variable. This raises two questions right away: How do we adapt Leibniz notation for functions of two variables? Also, what is an interpretation of the derivative? The answer lies in partial derivatives.

Definition: Partial Derivatives

Let $f(x,y)$ be a function of two variables. Then the *partial derivative* of f with respect to x, written as $\partial f/\partial x$ or f_x, is defined as

$$\frac{\partial f}{\partial x} = f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \tag{11.2.1}$$

The partial derivative of f with respect to y, written as $\partial f/\partial y$, or f_y, is defined as

$$\frac{\partial f}{\partial y} = f_y(x,y) = \lim_{k\to 0} \frac{f(x,y+k)-f(x,y)}{k}. \tag{11.2.2}$$

This definition shows two differences already. First, the notation changes, in the sense that we still use a version of Leibniz notation, but the d in the original notation is replaced with the symbol ∂. (This rounded "d" is usually called "partial," so $\partial f/\partial x$ is spoken as the "partial of f with respect to x.") This is the first hint that we are dealing with partial derivatives. Second, we now have two different derivatives we can take, since there are two different independent variables. Depending on which variable we choose, we can come up with different partial derivatives altogether, and often do.

Example 11.2.1: Calculating Partial Derivatives from the Definition

Use the definition of the partial derivative as a limit to calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the function

$$f(x,y) = x^2 - 3xy + 2y^2 - 4x + 5y - 12.$$

Solution

First, calculate $f(x+h,y)$.

$$\begin{aligned} f(x+h,y) &= (x+h)^2 - 3(x+h)y + 2y^2 - 4(x+h) + 5y - 12 \\ &= x^2 + 2xh + h^2 - 3xy - 3hy + 2y^2 - 4x - 4h + 5y - 12. \end{aligned}$$

Next, substitute this into Equation 11.2.1 and simplify:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \\ &= \lim_{h\to 0} \frac{(x^2+2xh+h^2-3xy-3hy+2y^2-4x-4h+5y-12)-(x^2-3xy+2y^2-4x+5y-12)}{h} \\ &= \lim_{h\to 0} \frac{x^2+2xh+h^2-3xy-3hy+2y^2-4x-4h+5y-12-x^2+3xy-2y^2+4x-5y+12}{h} \\ &= \lim_{h\to 0} \frac{2xh+h^2-3hy-4h}{h} \\ &= \lim_{h\to 0} \frac{h(2x+h-3y-4)}{h} \\ &= \lim_{h\to 0} (2x+h-3y-4) \\ &= 2x-3y-4. \end{aligned}$$

To calculate $\dfrac{\partial f}{\partial y}$, first calculate $f(x, y+h)$:

$$\begin{aligned} f(x+h, y) &= x^2 - 3x(y+h) + 2(y+h)^2 - 4x + 5(y+h) - 12 \\ &= x^2 - 3xy - 3xh + 2y^2 + 4yh + 2h^2 - 4x + 5y + 5h - 12. \end{aligned}$$

Next, substitute this into Equation 11.2.2 and simplify:

$$\begin{aligned} \frac{\partial f}{\partial y} &= \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h} \\ &= \lim_{h \to 0} \frac{(x^2 - 3xy - 3xh + 2y^2 + 4yh + 2h^2 - 4x + 5y + 5h - 12) - (x^2 - 3xy + 2y^2 - 4x + 5y - 12)}{h} \\ &= \lim_{h \to 0} \frac{x^2 - 3xy - 3xh + 2y^2 + 4yh + 2h^2 - 4x + 5y + 5h - 12 - x^2 + 3xy - 2y^2 + 4x - 5y + 12}{h} \\ &= \lim_{h \to 0} \frac{-3xh + 4yh + 2h^2 + 5h}{h} \\ &= \lim_{h \to 0} \frac{h(-3x + 4y + 2h + 5)}{h} \\ &= \lim_{h \to 0} (-3x + 4y + 2h + 5) \\ &= -3x + 4y + 5 \end{aligned}$$

Exercise 11.2.1

Use the definition of the partial derivative as a limit to calculate $\partial f / \partial x$ and $\partial f / \partial y$ for the function

$$f(x, y) = 4x^2 + 2xy - y^2 + 3x - 2y + 5.$$

Hint

Use Equations 11.2.1 and 11.2.2 from the definition of partial derivatives.

Answer

$$\frac{\partial f}{\partial x} = 8x + 2y + 3$$

$$\frac{\partial f}{\partial y} = 2x - 2y - 2$$

Note: All differentiation rules you learned in Math 400 (product rule, quotient rule, chain rule, etc.) still apply.

Example 11.2.2: Calculating Partial Derivatives

Calculate $\partial f / \partial x$ and $\partial f / \partial y$ for the following functions by holding the opposite variable constant then differentiating:

a. $f(x, y) = x^2 - 3xy + 2y^2 - 4x + 5y - 12$
b. $g(x, y) = \sin(x^2 y - 2x + 4)$

Solution:

a. To calculate $\partial f / \partial x$, treat the variable y as a constant. Then differentiate $f(x, y)$ with respect to x using the sum, difference, and power rules:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left[x^2 - 3xy + 2y^2 - 4x + 5y - 12\right] \\ &= \frac{\partial}{\partial x}[x^2] - \frac{\partial}{\partial x}[3xy] + \frac{\partial}{\partial x}[2y^2] - \frac{\partial}{\partial x}[4x] + \frac{\partial}{\partial x}[5y] - \frac{\partial}{\partial x}[12] \\ &= 2x - 3y + 0 - 4 + 0 - 0 \\ &= 2x - 3y - 4. \end{aligned}$$

The derivatives of the third, fifth, and sixth terms are all zero because they do not contain the variable x, so they are treated as constant terms. The derivative of the second term is equal to the coefficient of x, which is $-3y$. Calculating $\partial f/\partial y$:

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left[x^2 - 3xy + 2y^2 - 4x + 5y - 12\right] \\ &= \frac{\partial}{\partial y}[x^2] - \frac{\partial}{\partial y}[3xy] + \frac{\partial}{\partial y}[2y^2] - \frac{\partial}{\partial y}[4x] + \frac{\partial}{\partial y}[5y] - \frac{\partial}{\partial y}[12] \\ &= -3x + 4y - 0 + 5 - 0 \\ &= -3x + 4y + 5.\end{aligned}$$

These are the same answers obtained in Example 11.2.1.

b. To calculate $\partial g/\partial x$, treat the variable y as a constant. Then differentiate $g(x, y)$ with respect to x using the chain rule and power rule:

$$\begin{aligned}\frac{\partial g}{\partial x} &= \frac{\partial}{\partial x}\left[\sin(x^2y - 2x + 4)\right] \\ &= \cos(x^2y - 2x + 4)\frac{\partial}{\partial x}[x^2y - 2x + 4] \\ &= (2xy - 2)\cos(x^2y - 2x + 4).\end{aligned}$$

To calculate $\partial g/\partial y$, treat the variable x as a constant. Then differentiate $g(x, y)$ with respect to y using the chain rule and power rule:

$$\begin{aligned}\frac{\partial g}{\partial y} &= \frac{\partial}{\partial y}\left[\sin(x^2y - 2x + 4)\right] \\ &= \cos(x^2y - 2x + 4)\frac{\partial}{\partial y}[x^2y - 2x + 4] \\ &= x^2\cos(x^2y - 2x + 4).\end{aligned}$$

? Exercise 11.2.2

Calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the function

$$f(x, y) = \tan(x^3 - 3x^2y^2 + 2y^4)$$

by holding the opposite variable constant, then differentiating.

Hint

Use Equations 11.2.1and 11.2.1from the definition of partial derivatives.

Answer

$$\frac{\partial f}{\partial x} = (3x^2 - 6xy^2)\sec^2(x^3 - 3x^2y^2 + 2y^4)$$
$$\frac{\partial f}{\partial y} = (-6x^2y + 8y^3)\sec^2(x^3 - 3x^2y^2 + 2y^4)$$

Note: The remainder of this section is not necessary for Math 420. I am leaving it for those who may wish to have a better understanding of the concepts discussed here.

How can we interpret these partial derivatives? Recall that the graph of a function of two variables is a surface in R^3. If we remove the limit from the definition of the partial derivative with respect to x, the difference quotient remains:

$$\frac{f(x + h, y) - f(x, y)}{h}. \tag{11.2.3}$$

This resembles the difference quotient for the derivative of a function of one variable, except for the presence of the y variable. Figure $11.2.1$ illustrates a surface described by an arbitrary function $z = f(x, y)$.

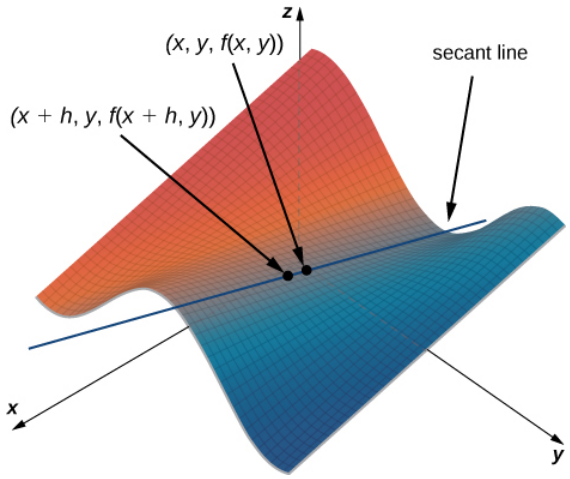


Figure 11.2.1: Secant line passing through the points $(x, y, f(x, y))$ and $(x + h, y, f(x + h, y))$.

In Figure $11.2.1$, the value of h is positive. If we graph $f(x, y)$ and $f(x + h, y)$ for an arbitrary point (x, y), then the slope of the secant line passing through these two points is given by

$$\frac{f(x + h, y) - f(x, y)}{h}. \tag{11.2.4}$$

This line is parallel to the x-axis. Therefore, the slope of the secant line represents an average rate of change of the function f as we travel parallel to the x-axis. As h approaches zero, the slope of the secant line approaches the slope of the tangent line.

If we choose to change y instead of x by the same incremental value h, then the secant line is parallel to the y-axis and so is the tangent line. Therefore, $\partial f/\partial x$ represents the slope of the tangent line passing through the point $(x, y, f(x, y))$parallel to the x-axis and $\partial f/\partial y$ represents the slope of the tangent line passing through the point $(x, y, f(x, y))$parallel to the y-axis. If we wish to find the slope of a tangent line passing through the same point in any other direction, then we need what are called directional derivatives, which is another topic from Math 402 (multivariable calculus).

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11.2.1: Partial Derivatives (Exercises)

In the following exercise, calculate the partial derivative using the limit definitions only.

1) $\dfrac{\partial z}{\partial y}$ for $z = x^2 - 3xy + y^2$

Answer

$$\frac{\partial z}{\partial y} = -3x + 2y$$

In exercises 2 - 13, calculate the requested partial derivatives.

2) $\dfrac{\partial z}{\partial x}$ for $z = \sin(3x)\cos(3y)$

3) $\dfrac{\partial z}{\partial y}$ for $z = \sin(3x)\cos(3y)$

Answer

$$\frac{\partial z}{\partial y} = -3\sin(3x)\sin(3y)$$

4) $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for $z = x^8 e^3 y$

5) $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for $z = \ln(x^6 + y^4)$

Answer

$$\frac{\partial z}{\partial x} = \frac{6x^5}{x^6+y^4}; \quad \frac{\partial z}{\partial y} = \frac{4y^3}{x^6+y^4}$$

6) Find $f_y(x, y)$ for $f(x, y) = e^{xy}\cos(x)\sin(y)$.

7) Let $z = e^{xy}$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Answer

$$\frac{\partial z}{\partial x} = ye^{xy}; \quad \frac{\partial z}{\partial y} = xe^{xy}$$

8) Let $z = \ln(\frac{x}{y})$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

9) Let $z = \tan(2x - y)$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Answer

$$\frac{\partial z}{\partial x} = 2\sec^2(2x - y), \quad \frac{\partial z}{\partial y} = -\sec^2(2x - y)$$

10) Let $z = \sinh(2x + 3y)$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

11) Let $f(x, y) = \arctan(\frac{y}{x})$. Evaluate $f_x(2, -2)$ and $f_y(2, -2)$.

Answer

$$f_x(2, -2) = \tfrac{1}{4} = f_y(2, -2)$$

12) Let $f(x,y)=\dfrac{xy}{x-y}$. Find $f_x(2,-2)$ and $f_y(2,-2)$.

13) Find $\dfrac{\partial z}{\partial x}$ at $(0,1)$ for $z=e^{-x}cos(y)$.

14) Express the volume of a right circular cylinder as a function of two variables:

a. its radius r and its height h.

b. Show that the rate of change of the volume of the cylinder with respect to its radius is the product of its circumference multiplied by its height.

c. Show that the rate of change of the volume of the cylinder with respect to its height is equal to the area of the circular base.

Answer

$a.\ V(r,h)=\pi r^2 h$

$b.\ \dfrac{\partial V}{\partial r}=2\pi rh$

$c.\ \dfrac{\partial V}{\partial h}=\pi r^2$

15) The function $P(T,V)=\dfrac{nRT}{V}$ gives the pressure at a point in a gas as a function of temperature T and volume V. The letters n and R are constants. Find $\dfrac{\partial P}{\partial V}$ and $\dfrac{\partial P}{\partial T}$, and explain what these quantities represent.

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11.3: Determinants and Cramer's Rule for 2 X 2 Matrices

2×2 Determinants

Let A be an $n \times n$ matrix. That is, let A be a square matrix. The **determinant** of A, denoted by $\det(A)$ is a very important number which we will explore throughout this section.

If A is a 2×2 matrix, the determinant is given by the following formula.

Definition 11.3.1 Determinant of a Two By Two Matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$\det(A) = ad - bc$$

The determinant is also often denoted by enclosing the matrix with two vertical lines. Thus

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The following is an example of finding the determinant of a 2×2 matrix.

Example 11.3.1

Find $\det(A)$ for the matrix $A = \begin{bmatrix} 2 & 4 \\ -1 & 6 \end{bmatrix}$.

Solution

From Definition 11.3.1,

$$\det(A) = (2)(6) - (4)(-1) = 12 + 4 = 16$$

Cramer's Rule for a 2×2 Matrix

Cramer's Rule gives a way to solve systems of equations using determinants. Although it may seem as though it is a more difficult way to solve systems of equations when the coefficients are constants, it is a much more efficient way to solve systems with functions as coefficients, which is predominantly what we see in differential equations. We omit the proof as it requires a significant amount of linear algebra.

Theorem 11.3.1 Cramer's Rule for a 2×2 Matrix

Consider the following system of equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

Let A_1 be the matrix where the first column of A has been replaced with $\mathbf{b}$ and A_2 the matrix where the second column of A has been replaced with $\mathbf{b}$.

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

and

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

Example 11.3.2

Use Cramer's Rule to solve

$$\begin{aligned} 3x_1 + 5x_2 &= 19 \\ 7x_1 - 4x_2 &= 13 \end{aligned}$$

Solution

We have Let $A = \begin{bmatrix} 3 & 5 \\ 7 & -4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 19 \\ 13 \end{bmatrix}$.

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{\begin{vmatrix} 19 & 5 \\ 13 & -4 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 7 & -4 \end{vmatrix}} = \frac{-141}{-47} = 3$$

and

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{\begin{vmatrix} 3 & 19 \\ 7 & 13 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 7 & -4 \end{vmatrix}} = \frac{-94}{-47} = 2$$

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11.3.1: Determinants and Cramer's Rule for 2 X 2 Matrices (Exercises)

In Exercises **1**-**3** find the determinant of the given matrix.

1. $\begin{bmatrix} 1 & -3 \\ 3 & 2 \end{bmatrix}$
2. $\begin{bmatrix} 4 & 3 \\ 5 & -2 \end{bmatrix}$
3. $\begin{bmatrix} 4 & 3 \\ 6 & 2 \end{bmatrix}$

In Exercises **4**-**6** use Cramer's Rule to find the solution to the given systems.

4. $\begin{aligned} x+2y &= 1 \\ 2x-y &= 2 \end{aligned}$
5. $\begin{aligned} 5x-8y &= -3 \\ 7x-y &= 6 \end{aligned}$
6. $\begin{aligned} 3x+4y &= 23 \\ 4x-3y &= 14 \end{aligned}$

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11.4: Determinants and Cramer's Rule for n x n Matrices

$n \times n$ Determinants

The 2×2 determinant we looked at in Section 11.3 can be used to find the determinant of larger matrices. We will now explore how to find the determinant of a 3×3 matrix, using several tools, including the 2×2 determinant.

We begin with the following definition.

Definition 11.4.1 The ij^{th} Minor of a Matrix

Let A be a 3×3 matrix. The ij^{th} **minor** of A, denoted as $minor(A)_{ij}$, is the determinant of the 2×2 matrix which results from deleting the i^{th} row and the j^{th} column of A.

In general, if A is an $n \times n$ matrix, then the ij^{th} minor of A is the determinant of the $n-1 \times n-1$ matrix which results from deleting the i^{th} row and the j^{th} column of A.

Hence, there is a minor associated with each entry of A. Consider the following example which demonstrates this definition.

Example 11.4.1

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Find $minor(A)_{12}$ and $minor(A)_{23}$.

Solution

First we will find $minor(A)_{12}$. By Definition 11.4.1, this is the determinant of the 2×2 matrix which results when you delete the first row and the second column. This minor is given by

$$minor(A)_{12} = \det \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} = -2$$

Similarly, $minor(A)_{23}$ is the determinant of the 2×2 matrix which results when you delete the second row and the third column. This minor is therefore

$$minor(A)_{23} = \det \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = -4$$

Finding the other minors of A is left as an exercise.

The ij^{th} minor of a matrix A is used in another important definition, given next.

Definition 11.4.2 The ij^{th} Cofactor of a Matrix

Suppose A is an $n \times n$ matrix. The ij^{th} **cofactor**, denoted by C_{ij} is defined to be

$$C_{ij} = (-1)^{i+j} minor(A)_{ij}$$

It is also convenient to refer to the cofactor of an entry of a matrix as follows. If a_{ij} is the ij^{th} entry of the matrix, then its cofactor is just C_{ij}.

Example 11.4.2

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}.$$

Find C_{12} and C_{23}.

Solution

We will use Definition 11.4.2 to compute these cofactors.

First, we will compute C_{12}:

$$C_{12} = (-1)^{1+2} \det \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} = (-1)(-2) = 2.$$

Similarly, we can find C_{23}:

$$C_{23} = (-1)^{2+3} \det \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = (-1)(-4) = 4.$$

You may wish to find the remaining cofactors for the above matrix. Remember that there is a cofactor for every entry in the matrix.

We have now established the tools we need to find the determinant of a 3×3 matrix.

Definition 11.4.3 The Determinant of a Three By Three Matrix

Let A be a 3×3 matrix. Then, $\det(A)$ is calculated by picking a row (or column) and taking the product of each entry in that row (column) with its cofactor and adding these products together.

This process when applied to the i^{th} row (column) is known as **expanding along the i^{th} row (column)** and is given by

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3}$$

When calculating the determinant, you can choose to expand any row or any column. Regardless of your choice, you will always get the same number which is the determinant of the matrix A. This method of evaluating a determinant by expanding along a row or a column is called **Cofactor Expansion**.

Consider the following example.

Example 11.4.3

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Find $\det(A)$ using the method of Cofactor Expansion.

Solution

First, we will calculate $\det(A)$ by expanding along the first column. Using Definition 11.4.3, we take the 1 in the first column and multiply it by its cofactor,

$$1(-1)^{1+1} \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = (1)(1)(-1) = -1$$

Similarly, we take the 4 in the first column and multiply it by its cofactor, as well as with the 3 in the first column. Finally, we add these numbers together, as given in the following equation.

$$\det(A) = 1\overbrace{(-1)^{1+1}\begin{vmatrix}3 & 2\\ 2 & 1\end{vmatrix}}^{C_{11}} + 4\overbrace{(-1)^{2+1}\begin{vmatrix}2 & 3\\ 2 & 1\end{vmatrix}}^{C_{21}} + 3\overbrace{(-1)^{3+1}\begin{vmatrix}2 & 3\\ 3 & 2\end{vmatrix}}^{C_{31}}$$

Calculating each of these, we obtain

$$\det(A) = 1(1)(-1) + 4(-1)(-4) + 3(1)(-5) = -1 + 16 + -15 = 0$$

Hence, $\det(A) = 0$.

As mentioned in Definition 11.4.3, we can choose to expand along any row or column. Let's try now by expanding along the second row. Here, we take the 4 in the second row and multiply it to its cofactor, then add this to the 3 in the second row multiplied by its cofactor, and the 2 in the second row multiplied by its cofactor. The calculation is as follows.

$$\det(A) = 4\overbrace{(-1)^{2+1}\begin{vmatrix}2 & 3\\ 2 & 1\end{vmatrix}}^{C_{21}} + 3\overbrace{(-1)^{2+2}\begin{vmatrix}1 & 3\\ 3 & 1\end{vmatrix}}^{C_{22}} + 2\overbrace{(-1)^{2+3}\begin{vmatrix}1 & 2\\ 3 & 2\end{vmatrix}}^{C_{23}}$$

Calculating each of these products, we obtain

$$\det(A) = 4(-1)(-4) + 3(1)(-8) + 2(-1)(-4) = 0$$

You can see that for both methods, we obtained $\det(A) = 0$.

As mentioned above, we will always come up with the same value for $\det(A)$ regardless of the row or column we choose to expand along. You should try to compute the above determinant by expanding along other rows and columns. This is a good way to check your work, because you should come up with the same number each time.

We present this idea formally in the following theorem, which we state without proof.

Theorem 11.4.1 The Determinant is Well Defined

Expanding the $n \times n$ matrix along any row or column always gives the same answer, which is the determinant.

There will be times when we need to find the determinant of a matrix with function entries instead of constants; we give an example here.

Example 11.4.4

Suppose

$$A(t) = \begin{bmatrix} e^t & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{bmatrix}.$$

Find the determinant of $A(t)$.

Solution

It is clear we should expand about the first row or column and either way we get:

$\det(A(t)) = e^t(\cos^2 t + \sin^2 t) = e^t$.

We have now looked at the determinant of 2×2 and 3×3 matrices. It turns out that the method used to calculate the determinant of a 3×3 matrix can be used to calculate the determinant of any sized matrix.

For example, the ij^{th} minor of a 4×4 matrix is the determinant of the 3×3 matrix you obtain when you delete the i^{th} row and the j^{th} column. Just as with the 3×3 determinant, we can compute the determinant of a 4×4 matrix by Cofactor Expansion, along any row or column

Consider the following example.

Example 11.4.5

Find $\det(A)$ where

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 4 & 2 & 3 \\ 1 & 3 & 4 & 5 \\ 3 & 4 & 3 & 2 \end{bmatrix}$$

Solution

As in the case of a 3×3 matrix, you can expand this along any row or column. Lets pick the third column. Then, using Cofactor Expansion,

$$\det(A) = 3(-1)^{1+3}\begin{vmatrix} 5 & 4 & 3 \\ 1 & 3 & 5 \\ 3 & 4 & 2 \end{vmatrix} + 2(-1)^{2+3}\begin{vmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 3 & 4 & 2 \end{vmatrix} +$$

$$4(-1)^{3+3}\begin{vmatrix} 1 & 2 & 4 \\ 5 & 4 & 3 \\ 3 & 4 & 2 \end{vmatrix} + 3(-1)^{4+3}\begin{vmatrix} 1 & 2 & 4 \\ 5 & 4 & 3 \\ 1 & 3 & 5 \end{vmatrix}$$

Now, you can calculate each 3×3 determinant using Cofactor Expansion, as we did above. You should complete these as an exercise and verify that $\det(A) = -12$.

The following provides a formal definition for the determinant of an $n \times n$ matrix. You may wish to take a moment and consider the above definitions for 2×2 and 3×3 determinants in context of this definition.

Definition 11.4.4 The Determinant of an $n \times n$ Matrix

Let A be an $n \times n$ matrix where $n \geq 2$ and suppose the determinant of an $(n-1) \times (n-1)$ has been defined. Then

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{i=1}^{n} a_{ij} C_{ij}$$

The first formula consists of expanding the determinant along the i^{th} row and the second expands the determinant along the j^{th} column.

The Determinant of a Triangular Matrix

There is a certain type of matrix for which finding the determinant is a very simple procedure. Consider the following definition.

Definition 11.4.5 Triangular Matrices

An $n \times n$ matrix A is upper triangular if $a_{ij} = 0$ whenever $i > j$. Thus the entries of such a matrix below the main diagonal equal 0, as shown. Here, $*$ refers to **any** number.

$$\begin{bmatrix} * & * & \cdots & * \\ 0 & * & \cdots & \vdots \\ \vdots & \vdots & \ddots & * \\ 0 & \cdots & 0 & * \end{bmatrix}$$

A lower triangular matrix is defined similarly as a matrix for which all entries above the main diagonal are equal to zero.

The following theorem provides a useful way to calculate the determinant of a triangular matrix.

Theorem 11.4.2 Determinant of a Triangular Matrix

Let A be an upper or lower triangular matrix. Then $\det(A)$ is obtained by taking the product of the entries on the main diagonal.

The verification of this Theorem can be done by computing the determinant using Cofactor Expansion along the first row or column.

Consider the following example.

Example 11.4.6

Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 77 \\ 0 & 2 & 6 & 7 \\ 0 & 0 & 3 & 33.7 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Find $\det(A)$.

Solution

From Theorem 11.4.2, it suffices to take the product of the elements on the main diagonal. Thus $\det(A) = (1)(2)(3)(-1) = -6$.

Without using Theorem 11.4.2, you could use Cofactor Expansion. We will expand along the first column. This gives

$$\begin{aligned} \det(A) = & 1(-1)^{1+1}\begin{vmatrix} 2 & 6 & 7 \\ 0 & 3 & 33.7 \\ 0 & 0 & -1 \end{vmatrix} + 0(-1)^{2+1}\begin{vmatrix} 2 & 3 & 77 \\ 0 & 3 & 33.7 \\ 0 & 0 & -1 \end{vmatrix} + \\ & 0(-1)^{3+1}\begin{vmatrix} 2 & 3 & 77 \\ 2 & 6 & 7 \\ 0 & 0 & -1 \end{vmatrix} + 0(-1)^{4+1}\begin{vmatrix} 2 & 3 & 77 \\ 2 & 6 & 7 \\ 0 & 3 & 33.7 \end{vmatrix} \\ = & 1(-1)^{1+1}\begin{vmatrix} 2 & 6 & 7 \\ 0 & 3 & 33.7 \\ 0 & 0 & -1 \end{vmatrix} \\ = & \begin{vmatrix} 2 & 6 & 7 \\ 0 & 3 & 33.7 \\ 0 & 0 & -1 \end{vmatrix} \end{aligned}$$

Now find the determinant of this 3×3 matrix, by expanding along the first column to obtain

$$\begin{aligned} \det(A) &= 2(-1)^{1+1}\begin{vmatrix} 3 & 33.7 \\ 0 & -1 \end{vmatrix} + 0(-1)^{2+1}\begin{vmatrix} 6 & 7 \\ 0 & -1 \end{vmatrix} + 0(-1)^{3+1}\begin{vmatrix} 6 & 7 \\ 3 & 33.7 \end{vmatrix} \\ &= (1)(2)\begin{vmatrix} 3 & 33.7 \\ 0 & -1 \end{vmatrix} \end{aligned}$$

Next find the determinant of this 2×2 matrix, which is just $(3)(-1)$. Putting all these steps together, we have

$$\det(A) = (1)(2)(3)(-1) = -6$$

which is just the product of the entries down the main diagonal of the original matrix.

You can see that while both methods result in the same answer, Theorem 11.4.2 provides a much quicker result.

Cramer's Rule for an $n \times n$ Matrix

Cramer's Rule for $n \times n$ matrices is similar to what we did in Section 11.3 for 2×2 matrices; we state the generalized theorem here.

Theorem 11.4.3: Cramer's Rule for an $n \times n$ Matrix

Consider the following system of equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \vdots & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$.

Let A_i be the matrix where the i^{th} column of A has been replaced with $\mathbf{b}$.

Then

$$x_i = \frac{\det(A_i)}{\det(A)}$$

Example 11.4.7

Use Cramer's Rule to solve:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 3x_1 + 2x_2 + x_3 &= 2 \\ 2x_1 - 3x_2 + 2x_3 &= 3 \end{aligned}$$

We have $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & -3 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Solution

In order to find x_1, we calculate

$$x_1 = \frac{\det(A_1)}{\det(A)}$$

where A_1 is the matrix obtained from replacing the first column of A with $\mathbf{b}$.

Hence, A_1 is given by

$$A_1 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 3 & -3 & 2 \end{bmatrix}$$

Therefore,

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{\begin{vmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 3 & -3 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & -3 & 2 \end{vmatrix}} = \frac{1}{2}$$

Similarly, to find x_2 we construct A_2 by replacing the second column of A with $\mathbf{b}$. Hence, A_2 is given by

$$A_2 = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

Therefore,

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 3 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & -3 & 2 \end{vmatrix}} = -\frac{1}{7}$$

Similarly, A_3 is constructed by replacing the third column of A with $\mathbf{b}$. Then, A_3 is given by

$$A_3 = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 2 \\ 2 & -3 & 3 \end{bmatrix}$$

Therefore, x_3 is calculated as follows.

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{\begin{vmatrix} 1 & 2 & 1 \\ 3 & 2 & 2 \\ 2 & -3 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & -3 & 2 \end{vmatrix}} = \frac{11}{14}$$

Again, there will be times when we need to solve systems with functions as coefficients instead of constants; we give an example here.

Example 11.4.8

Use Cramer's Rule to solve for x_3:

$$\begin{aligned} x_1 &= 1 \\ (e^t \cos t)x_2 + (e^t \sin t)x_3 &= t \\ (-e^t \sin t)x_2 + (e^t \cos t)x_3 &= t^2 \end{aligned}$$

We have $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t \cos t & e^t \sin t \\ 0 & -e^t \sin t & e^t \cos t \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix}$.

Solution

We are asked to find the value of x_3 in the solution. We will solve using Cramer's rule. Thus

$$x_3 = \frac{\begin{vmatrix} 1 & 0 & 1 \\ 0 & e^t \cos t & t \\ 0 & -e^t \sin t & t^2 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 \\ 0 & e^t \cos t & e^t \sin t \\ 0 & -e^t \sin t & e^t \cos t \end{vmatrix}} = \frac{t^2 e^t \cos t + t e^t \sin t}{e^{2t} \cos^2 t + e^{2t} \sin^2 t} = t^2 e^{-t} \cos t + t e^{-t} \sin t$$

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11.4.1: Determinants and Cramer's Rule for n x n Matrices (Exercises)

1. Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ -2 & 5 & 1 \end{bmatrix}$. Find the following.

   a. $minor(A)_{11}$
   b. $minor(A)_{21}$
   c. $minor(A)_{32}$
   d. C_{11}
   e. C_{21}
   f. C_{32}

2. Find the determinants of the following matrices.

   a. $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \\ 0 & 9 & 8 \end{bmatrix}$
   b. $\begin{bmatrix} 4 & 3 & 2 \\ 1 & 7 & 8 \\ 3 & -9 & 3 \end{bmatrix}$
   c. $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & 3 & 2 & 3 \\ 4 & 1 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{bmatrix}$

Answer

   a. The answer is 31.
   b. The answer is 375.
   c. The answer is -2.

3. Find the following determinant by expanding along the first row and second column.

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix}$$

Answer

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix} = 6$$

4. Find the following determinant by expanding along the first column and third row.

$$\begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{vmatrix}$$

Answer

$$\begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 2$$

5. Find the following determinant by expanding along the second row and first column.

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix}$$

Answer

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix} = 6$$

6. Compute the determinant by cofactor expansion. Pick the easiest row or column to use.

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 1 & 3 & 1 \end{vmatrix}$$

Answer

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 1 & 3 & 1 \end{vmatrix} = -4$$

7. Find the determinant of the following matrices.

a. $A = \begin{bmatrix} 1 & -34 \\ 0 & 2 \end{bmatrix}$

b. $A = \begin{bmatrix} 4 & 3 & 14 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

c. $A = \begin{bmatrix} 2 & 3 & 15 & 0 \\ 0 & 4 & 1 & 7 \\ 0 & 0 & -3 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

8. Use Cramer's rule to find the solution to

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 - x_2 - x_3 &= 2 \\ x_1 + x_3 &= 1 \end{aligned}$$

Answer

Solution is: $[x_1 = 1, x_2 = 0, x_3 = 0]$. For example,

$$y = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & -1 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & -1 \\ 1 & 0 & 1 \end{vmatrix}} = 0$$

9. Use Cramer's rule to find the solution to

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 14 \\ 2x_1 - x_2 + x_3 &= 0 \\ 6x_1 + 3x_2 + 4x_3 &= 1 \end{aligned}$$

10. Use Cramer's rule to find the solution to

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 4 \\ 10x_1 - 2x_2 + 2x_3 &= -1 \\ 6x_1 - 2x_2 + 4x_3 &= 8 \end{aligned}$$

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11.5: Elementary Row Operations and Gaussian Elimination

In this section and the next we will briefly go over the computational aspects from linear algebra necessary for chapter 10. If you have not had linear algebra, then you are unlikely to understand why everything is true, but you need only be able to perform the calculations to be able to solve the problems that occur in chapter 10. For a thorough understanding of these topics it is best to take a class in linear algebra - highly recommended for anyone studying mathematics, physics, engineering...or anything.

Consider the following system of equations:

$$\begin{aligned} r+b+g &= 30 \\ r &= 2g \\ b &= r+g \end{aligned} \tag{11.5.1}$$

Let's rewrite these equations so that all variables are on the left of the equal sign and all constants are on the right. Also, for a bit more consistency, let's list the variables in alphabetical order in each equation. Therefore we can write the equations as

$$\begin{array}{rcrcrcr} b & + & g & + & r & = & 30 \\ & - & 2g & + & r & = & 0 \\ -b & + & g & + & r & = & 0 \end{array}. \tag{11.5.2}$$

There isn't just one "right" way of finding the solution to this system of equations, but for simplicity we will only consider one very simple method.

Caution

The methods discussed below are **not** what we would do in linear algebra, though they suffice for our needs in this class.

First, lets add the first and last equations together, and write the result as a new third equation. This gives us:

$$\begin{array}{rcrcrcr} b & + & g & + & r & = & 30 \\ & - & 2g & + & r & = & 0 \\ & & 2g & + & 2r & = & 30 \end{array}.$$

A nice feature of this is that the only equation with a b in it is the first equation.

Now let's multiply the second equation by $-\frac{1}{2}$. This gives

$$\begin{array}{rcrcrcr} b & + & g & + & r & = & 30 \\ & & g & - & 1/2r & = & 0 \\ & & 2g & + & 2r & = & 30 \end{array}.$$

Let's now do two steps in a row; our goal is to get rid of the g's in the first and third equations. In order to remove the g in the first equation, let's multiply the second equation by -1 and add that to the first equation, replacing the first equation with that sum. To remove the g in the third equation, let's multiply the second equation by -2 and add that to the third equation, replacing the third equation. Our new system of equations now becomes

$$\begin{array}{rcrcrcr} b & + & & & 3/2r & = & 30 \\ & & g & - & 1/2r & = & 0 \\ & & & & 3r & = & 30 \end{array}.$$

Clearly we can multiply the third equation by $\frac{1}{3}$ and find that $r = 10$; let's make this our new third equation, giving

$$\begin{array}{rcrcrcr} b & + & & & 3/2r & = & 30 \\ & & g & - & 1/2r & = & 0 \\ & & & & r & = & 10 \end{array}.$$

Now let's get rid of the r's in the first and second equation. To remove the r in the first equation, let's multiply the third equation by $-\frac{3}{2}$ and add the result to the first equation, replacing the first equation with that sum. To remove the r in the second equation,

we can multiply the third equation by $\frac{1}{2}$ and add that to the second equation, replacing the second equation with that sum. This gives us:

$$\begin{aligned} b &= 15 \\ g &= 5 \\ r &= 10 \end{aligned}.$$

Clearly we have discovered our solution.

Let's consider the idea that all that really matters are the coefficients and the constants. There is nothing special about the letters b, g and r; we could have used x, y and z or x_1, x_2 and x_3. And even then, since we wrote our equations so carefully, we really didn't need to write the variable names at all as long as we put things "in the right place."

Let's look again at our system of equations in (11.5.2) and write the coefficients and the constants in a rectangular array. This time we won't ignore the zeros, but rather write them out.

$$\begin{aligned} b &+ g &+ r &= 30 \\ &- 2g &+ r &= 0 \\ -b &+ g &+ r &= 0 \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 1 & 1 & 30 \\ 0 & -2 & 1 & 0 \\ -1 & 1 & 1 & 0 \end{bmatrix}$$

Notice how even the equal signs are gone; we don't need them, for we know that the last *column* contains the constants.

We have just created a *matrix*. The definition of matrix is remarkable only in how unremarkable it seems.

Definition: Matrix

A *matrix* is a rectangular array of numbers.
The horizontal lines of numbers form *rows* and the vertical lines of numbers form *columns*. A matrix with m rows and n columns is said to be an $m \times n$ matrix ("an m by n matrix").
The entries of an $m \times n$ matrix are indexed as follows:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}.$$

That is, a_{32} means "the number in the third row and second column."

In the future, we'll want to create matrices with just the coefficients of a system of linear equations and leave out the constants. Therefore, when we include the constants, we often refer to the resulting matrix as an *augmented matrix*.

We can use augmented matrices to find solutions to linear equations by using essentially the same steps we used above. Every time we used the word "equation" above, substitute the word "row," as we show below. The comments explain how we get from the current set of equations (or matrix) to the one on the next line.

We can use a shorthand to describe matrix operations; let R_1, R_2 represent "row 1" and "row 2," respectively. We can write "add row 1 to row 3, and replace row 3 with that sum" as "$R_1 + R_3 \to R_3$." The expression "$R_1 \leftrightarrow R_2$" means "interchange row 1 and row 2."

$\begin{aligned} b &+ g &+ r &= 30 \\ &- 2g &+ r &= 0 \\ -b &+ g &+ r &= 0 \end{aligned}$	$\begin{bmatrix} 1 & 1 & 1 & 30 \\ 0 & -2 & 1 & 0 \\ -1 & 1 & 1 & 0 \end{bmatrix}$
Replace equation 3 with the sum of equations 1 and 3	Replace row 3 with the sum of rows 1 and 3. $(R_1 + R_3 \to R_3)$

$\begin{aligned} b &+ g &+ r &= 30 \\ -\,2g &+ r &= 0 \\ 2g &+ 2r &= 30 \end{aligned}$	$\begin{bmatrix} 1 & 1 & 1 & 30 \\ 0 & -2 & 1 & 0 \\ 0 & 2 & 2 & 30 \end{bmatrix}$
Multiply equation 2 by $-\frac{1}{2}$	Multiply row 2 by $-\frac{1}{2}$ $\left(-\frac{1}{2}R_2 \to R_2\right)$
$\begin{aligned} b &+ g &+ r &= 30 \\ g &+ -1/2r &= 0 \\ 2g &+ 2r &= 30 \end{aligned}$	$\begin{bmatrix} 0 & 1 & 1 & 30 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 2 & 2 & 30 \end{bmatrix}$
Replace equation 1 with the sum of (-1) times equation 2 plus equation 1; Replace equation 3 with the sum of (-2) times equation 2 plus equation 3	Replace row 1 with the sum of (-1) times row 2 plus row 1 $\left(-R_2 + R_1 \to R_1\right)$; Replace row 3 with the sum of (-2) times row 2 plus row 3 $\left(-2R_2 + R_3 \to R_3\right)$
$\begin{aligned} b &+ 3/2r &= 30 \\ g &- 1/2r &= 0 \\ 3r &= 30 \end{aligned}$	$\begin{bmatrix} 1 & 0 & \frac{3}{2} & 30 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 3 & 30 \end{bmatrix}$
Multiply equation 3 by $\frac{1}{3}$	Multiply row 3 by $\frac{1}{3}$ $\left(\frac{1}{3}R_3 \to R_3\right)$
$\begin{aligned} b &+ 3/2r &= 30 \\ g &- 1/2r &= 0 \\ r &= 10 \end{aligned}$	$\begin{bmatrix} 1 & 0 & \frac{3}{2} & 30 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 10 \end{bmatrix}$
Replace equation 2 with the sum of $\frac{1}{2}$ times equation 3 plus equation 2; Replace equation 1 with the sum of $-\frac{3}{2}$ times equation 3 plus equation 1	Replace row 2 with the sum of $\frac{1}{2}$ times row 3 plus row 2 $\left(\frac{1}{2}R_3 + R_2 \to R_2\right)$; Replace row 1 with the sum of $-\frac{3}{2}$ times row 3 plus row 1 $\left(-\frac{3}{2}R_3 + R_1 \to R_1\right)$
$\begin{aligned} b &= 15 \\ g &= 5 \\ r &= 10 \end{aligned}$	$\begin{bmatrix} 1 & 0 & 0 & 15 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 10 \end{bmatrix}$

The final matrix contains the same solution information as we have on the left in the form of equations. Recall that the first column of our matrices held the coefficients of the b variable; the second and third columns held the coefficients of the g and r variables, respectively. Therefore, the first row of the matrix can be interpreted as "$b + 0g + 0r = 15$," or more concisely, "$b = 15$."

Let's practice this idea again.

Example 11.5.1

Find a solution to the following system of linear equations by simultaneously manipulating the equations and the corresponding augmented matrices.

$$\begin{aligned} x_1 &+ x_2 &+ x_3 &= 0 \\ 2x_1 &+ 2x_2 &+ x_3 &= 0 \\ -1x_1 &+ x_2 &- 2x_3 &= 2 \end{aligned}$$

Solution

We'll first convert this system of equations into a matrix, then we'll proceed by manipulating the system of equations (and hence the matrix) to find a solution. Again, there is not just one "right" way of proceeding; we'll choose a method that is pretty efficient, but other methods certainly exist (and may be "better"). The method we use here, though, is the method that we will be using in chapter 10.

The given system and its corresponding augmented matrix are seen below.

Original system of equations	Corresponding matrix
$\begin{aligned} x_1 &+ x_2 + x_3 = 0 \\ 2x_1 &+ 2x_2 + x_3 = 0 \\ -1x_1 &+ x_2 - 2x_3 = 2 \end{aligned}$	$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 2 & 1 & 0 \\ -1 & 1 & -2 & 2 \end{bmatrix}$

We'll proceed by trying to get the x_1 out of the second and third equation.

Replace equation 2 with the sum of (-2) times equation 1 plus equation 2; Replace equation 3 with the sum of equation 1 and equation 3	Replace row 2 with the sum of (-2) times row 1 plus row 2 $(-2R_1 + R_2 \to R_2)$; Replace row 3 with the sum of row 1 and row 3 $(R_1 + R_3 \to R_3)$
$\begin{aligned} x_1 &+ x_2 + x_3 = 0 \\ &\quad -x_3 = 0 \\ 2x_2 &- x_3 = 2 \end{aligned}$	$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 2 & -1 & 2 \end{bmatrix}$

Notice that the second equation no longer contains x_2. We'll exchange the order of the equations so that we can follow the convention of solving for the second variable in the second equation.

Interchange equations 2 and 3	Interchange rows 2 and 3 $R_2 \leftrightarrow R_3$
$\begin{aligned} x_1 &+ x_2 + x_3 = 0 \\ 2x_2 &- x_3 = 2 \\ &\quad -x_3 = 0 \end{aligned}$	$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & -1 & 0 \end{bmatrix}$
Multiply equation 2 by $\frac{1}{2}$	Multiply row 2 by $\frac{1}{2}$ $\left(\frac{1}{2}R_2 \to R_2\right)$
$\begin{aligned} x_1 &+ x_2 + x_3 = 0 \\ x_2 &- \tfrac{1}{2}x_3 = 1 \\ &\quad -x_3 = 0 \end{aligned}$	$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$
Multiply equation 3 by -1	Multiply row 3 by -1 $(-1R_3 \to R_3)$
$\begin{aligned} x_1 &+ x_2 + x_3 = 0 \\ x_2 &- \tfrac{1}{2}x_3 = 1 \\ &\quad x_3 = 0 \end{aligned}$	$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Notice that the last equation (and also the last row of the matrix) show that $x_3 = 0$. Knowing this would allow us to simply eliminate the x_3 from the first two equations. However, we will formally do this by manipulating the equations (and rows) as we have previously.

Replace equation 1 with the sum of (-1) times equation 3 plus equation 1; Replace equation 2 with the sum of $\frac{1}{2}$ times equation 3 plus equation 2	Replace row 1 with the sum of (-1) times row 3 plus row 1 $(-R_3 + R_1 \to R_1)$; Replace row 2 with the sum of $\frac{1}{2}$ times row 3 plus row 2 $\left(\frac{1}{2}R_3 + R_2 \to R_2\right)$
$\begin{aligned} x_1 &+ x_2 = 0 \\ x_2 &= 1 \\ x_3 &= 0 \end{aligned}$	$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Notice how the second equation shows that $x_2 = 1$. All that remains to do is to solve for x_1.

Replace equation 1 with the sum of (-1) times equation 2 plus equation 1	Replace row 1 with the sum of (-1) times row 2 plus row 1 $(-R_2 + R_1 \to R_1)$
$\begin{aligned} x_1 &= -1 \\ x_2 &= 1 \\ x_3 &= 0 \end{aligned}$	$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Obviously the equations on the left tell us that $x_1 = -1$, $x_2 = 1$ and $x_3 = 0$, and notice how the matrix on the right tells us the same information.

We didn't justify our ability to manipulate our equations in the above three ways; it seems rather obvious that we should be able to do that. In that sense, these operations are "elementary." These operations are *elementary* in another sense; they are *fundamental* – they form the basis for much of what we will do. Since these operations are so important, we list them again here in the context of matrices.

Key Idea 11.5.1: Elementary Row Operations

1. Add a scalar multiple of one row to another row, and replace the latter row with that sum
2. Multiply one row by a nonzero scalar
3. Swap the position of two rows

Given any system of linear equations, we can find a solution (if one exists) by using these three row operations. Elementary row operations give us a new linear system, but the solution to the new system is the same as the old. We can use these operations as much as we want and not change the solution. This brings to mind two good questions:

1. Since we can use these operations as much as we want, how do we know when to stop? (Where are we supposed to "go" with these operations?)
2. Is there an efficient way of using these operations? (How do we get "there" the fastest?)

We'll answer the first question first. Most of the time we will want to take our original matrix and, using the elementary row operations, put it into something called *reduced row echelon form*. This is our "destination," for this form allows us to readily identify whether or not a solution exists, and in the case that it does, what that solution is.

When we manipulated matrices above to find solutions, we were unwittingly putting the matrix into reduced row echelon form. However, not all solutions come in such a simple manner as we'll see shortly.

Definition: Reduced Row Echelon Form

A matrix is in *reduced row echelon form* if its entries satisfy the following conditions.

1. The first nonzero entry in each row is a 1 (called a *leading 1*).
2. Each leading 1 comes in a column to the right of the leading 1s in rows above it.
3. All rows of all 0s come at the bottom of the matrix.
4. If a column contains a leading 1, then all other entries in that column are 0.

A matrix that satisfies the first three conditions is said to be in *row echelon form*.

Example 11.5.2

Which of the following matrices is in reduced row echelon form?

a. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

c. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

f. $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

g. $\begin{bmatrix} 0 & 1 & 2 & 3 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

h. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Solution

The matrices in a), b), c), d) and g) are all in reduced row echelon form. Check to see that each satisfies the necessary conditions.

The matrix in e) is not in reduced row echelon form since the row of all zeros is not at the bottom. The matrix in f) is not in reduced row echelon form since the first nonzero entries in rows 2 and 3 are not 1. Finally, the matrix in h) is not in reduced row echelon form since the first entry in column 2 is not zero; the second 1 in column 2 is a leading one, hence all other entries in that column should be 0.

We've just answered the first question: most of the time we are "going to" reduced row echelon form. This brings us to the second question, "How do we get there quickly?" We address the second question now.

There is no one "right" way of using these operations to transform a matrix into reduced row echelon form. However, there is a general technique that works very well in that it is very efficient (so we don't waste time on unnecessary steps). This technique is called *Gaussian elimination*. It is named in honor of the great mathematician Karl Friedrich Gauss.

While this technique isn't very difficult to use, it is one of those things that is easier understood by watching it being used than explained as a series of steps. With this in mind, we will go through one more example highlighting important steps and then we'll explain the procedure in detail.

Example 11.5.3

Put the augmented matrix of the following system of linear equations into reduced row echelon form.

$$\begin{aligned} -3x_1 - 3x_2 + 9x_3 &= 12 \\ 2x_1 + 2x_2 - 4x_3 &= -2 \\ -2x_2 - 4x_3 &= -8 \end{aligned}$$

Solution

We start by converting the linear system into an augmented matrix.

$$\begin{bmatrix} \boxed{-3} & -3 & 9 & 12 \\ 2 & 2 & -4 & -2 \\ 0 & -2 & -4 & -8 \end{bmatrix}$$

Our next step is to change the entry in the box to a 1. To do this, let's multiply row 1 by $-\frac{1}{3}$.

$-\frac{1}{3}R_1 \to R_1$	$\begin{bmatrix} 1 & 1 & -3 & -4 \\ 2 & 2 & -4 & -2 \\ 0 & -2 & -4 & -8 \end{bmatrix}$

We have now created a *leading 1*; that is, the first entry in the first row is a 1. Our next step is to put zeros under this 1. To do this, we'll use the elementary row operation given below.

$-2R_1 + R_2 \to R_2$	$\begin{bmatrix} 1 & 1 & -3 & -4 \\ 0 & \boxed{0} & 2 & 6 \\ 0 & -2 & -4 & -8 \end{bmatrix}$

Once this is accomplished, we shift our focus from the leading one down one row, and to the right one column, to the position that is boxed. We again want to put a 1 in this position. We can use any elementary row operations, but we need to restrict ourselves to using only the second row and any rows below it. Probably the simplest thing we can do is interchange rows 2 and 3, and then scale the new second row so that there is a 1 in the desired position.

$R_2 \leftrightarrow R_3$	$\begin{bmatrix} 1 & 1 & -3 & -4 \\ 0 & \boxed{-2} & -4 & -8 \\ 0 & 0 & 2 & 6 \end{bmatrix}$
$-\frac{1}{2}R_2 \to R_2$	$\begin{bmatrix} 1 & 1 & -3 & -4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \boxed{2} & 6 \end{bmatrix}$

We have now created another leading 1, this time in the second row. Our next desire is to put zeros underneath it, but this has already been accomplished by our previous steps. Therefore we again shift our attention to the right one column and down one row, to the next position put in the box. We want that to be a 1. A simple scaling will accomplish this.

$\frac{1}{2}R_3 \to R_3$	$\begin{bmatrix} 1 & 1 & -3 & -4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

This ends what we will refer to as the *forward steps*. Our next task is to use the elementary row operations and go back and put zeros above our leading 1s. This is referred to as the *backward steps*. These steps are given below.

$3R_3 + R_1 \to R_1$ $-2R_3 + R_2 \to R_2$	$\begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$
$-R_2 + R_1 \to R_1$	$\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

It is now easy to read off the solution as $x_1 = 7$, $x_2 = -2$ and $x_3 = 3$.

In all of our practice so far, we've only encountered systems of linear equations with exactly one solution. Is this always going to be the case? Could we ever have systems with more than one solution? If so, how many solutions could there be? Could we have systems without a solution? These are some of the questions we'll address now.

Example 11.5.4

Find the solution to the linear system

$$\begin{aligned} x_1 + x_2 &= 1 \\ 2x_1 + 2x_2 &= 2 \end{aligned}.$$

Solution

Create the corresponding augmented matrix, and then put the matrix into reduced row echelon form.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \xrightarrow[\text{rref}]{} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Now convert the reduced matrix back into equations. In this case, we only have one equation,

$$x_1 + x_2 = 1$$

or, equivalently,

$$\begin{aligned} x_1 &= 1 - x_2 \\ x_2 &\text{ is free.} \end{aligned} \tag{11.5.3}$$

We have just introduced a new term, the word *free*. It is used to stress that idea that x_2 can take on *any* value; we are "free" to choose any value for x_2. Once this value is chosen, the value of x_1 is determined. We have infinite choices for the value of x_2, so therefore we have infinite solutions.

For example, if we set $x_2 = 0$, then $x_1 = 1$; if we set $x_2 = 5$, then $x_1 = -4$.

Let's try another example, one that uses more variables.

Example 11.5.5

Find the solution to the linear system

$$\begin{array}{rcrcrcl} & & x_2 & - & x_3 & = & 3 \\ x_1 & & & + & 2x_3 & = & 2 \\ & & -3x_2 & + & 3x_3 & = & -9 \end{array}.$$

Solution

To find the solution, put the corresponding matrix into reduced row echelon form.

$$\begin{bmatrix} 0 & 1 & -1 & 3 \\ 1 & 0 & 2 & 2 \\ 0 & -3 & 3 & -9 \end{bmatrix} \xrightarrow[\text{rref}]{} \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now convert this reduced matrix back into equations. We have

$$\begin{aligned} x_1 + 2x_3 &= 2 \\ x_2 - x_3 &= 3 \end{aligned} \tag{11.5.4}$$

or, equivalently,

$$\begin{aligned} x_1 &= 2 - 2x_3 \\ x_2 &= 3 + x_3 \\ x_3 &\text{ is free.} \end{aligned} \tag{11.5.5}$$

These two equations tell us that the values of x_1 and x_2 depend on what x_3 is. As we saw before, there is no restriction on what x_3 must be; it is "free" to take on the value of any real number. Once x_3 is chosen, we have a solution. Since we have infinite choices for the value of x_3, we have infinite solutions.

As examples, $x_1 = 2$, $x_2 = 3$, $x_3 = 0$ is one solution; $x_1 = -2$, $x_2 = 5$, $x_3 = 2$ is another solution. Try plugging these values back into the original equations to verify that these indeed are solutions.

We have now seen examples of systems with exactly one solution and others with infinite solutions. How will we recognize that a system does not have a solution? Let's find out through an example.

Example 11.5.3

Find the solution to the linear system

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 1 \\ x_1 & + & 2x_2 & + & x_3 & = & 2\,. \\ 2x_1 & + & 3x_2 & + & 2x_3 & = & 0 \end{array}$$

Solution

We start by putting the corresponding matrix into reduced row echelon form.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 3 & 2 & 0 \end{bmatrix} \xrightarrow[\text{rref}]{} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now let us take the reduced matrix and write out the corresponding equations. The first two rows give us the equations

$$\begin{aligned} x_1 + x_3 &= 0 \\ x_2 &= 0. \end{aligned} \tag{11.5.6}$$

So far, so good. However the last row gives us the equation

$$0x_1 + 0x_2 + 0x_3 = 1$$

or, more concisely, $0 = 1$. Obviously, this is not true; we have reached a contradiction. Therefore, no solution exists.

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11.5.1: Elementary Row Operations and Gaussian Elimination (Exercises)

In Exercises 11.5.1.1- 11.5.1.4 state whether or not the given matrices are in reduced row echelon form. If it is not, state why.

Exercise 11.5.1.1

a. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

Answer

a. yes
b. no
c. no
d. yes

Exercise 11.5.1.2

a. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Answer

a. yes
b. yes
c. no
d. yes

Exercise 11.5.1.3

a. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

Answer

a. no
b. yes
c. yes
d. yes

Exercise 11.5.1.4

a. $\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

Answer

1. no
2. yes
3. yes
4. yes

In Exercises 11.5.1.5- 11.5.1.22 use Gaussian Elimination to put the given matrix into reduced row echelon form.

Exercise 11.5.1.5

$\begin{bmatrix} 1 & 2 \\ -3 & -5 \end{bmatrix}$

Answer

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Exercise 11.5.1.6

$\begin{bmatrix} 2 & -2 \\ 3 & -2 \end{bmatrix}$

Answer

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Exercise 11.5.1.7

$\begin{bmatrix} 4 & 12 \\ -2 & -6 \end{bmatrix}$

Answer

$$\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$$

Exercise 11.5.1.8

$$\begin{bmatrix} -5 & 7 \\ 10 & 14 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 1 & -7/5 \\ 0 & 0 \end{bmatrix}$$

Exercise 11.5.1.9

$$\begin{bmatrix} -1 & 1 & 4 \\ -2 & 1 & 1 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 7 \end{bmatrix}$$

Exercise 11.5.1.10

$$\begin{bmatrix} 7 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 5 \end{bmatrix}$$

Exercise 11.5.1.11

$$\begin{bmatrix} 3 & -3 & 6 \\ -1 & 1 & -2 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Exercise 11.5.1.12

$$\begin{bmatrix} 4 & 5 & -6 \\ -12 & -15 & 18 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 1 & \frac{5}{4} & -\frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Exercise 11.5.1.13

$$\begin{bmatrix} -2 & -4 & -8 \\ -2 & -3 & -5 \\ 2 & 3 & 6 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise 11.5.1.14

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise 11.5.1.15

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \\ -1 & -3 & 0 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise 11.5.1.16

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 6 & 9 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise 11.5.1.17

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & -1 & -1 & 1 \\ -1 & 1 & 1 & 0 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercise 11.5.1.18

$$\begin{bmatrix} 2 & -1 & 1 & 5 \\ 3 & 1 & 6 & -1 \\ 3 & 0 & 5 & 0 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

Exercise 11.5.1.19

$$\begin{bmatrix} 1 & 1 & -1 & 7 \\ 2 & 1 & 0 & 10 \\ 3 & 2 & -1 & 17 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercise 11.5.1.20

$$\begin{bmatrix} 4 & 1 & 8 & 15 \\ 1 & 1 & 2 & 7 \\ 3 & 1 & 5 & 11 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercise 11.5.1.21

$$\begin{bmatrix} 2 & 2 & 1 & 3 & 1 & 4 \\ 1 & 1 & 1 & 3 & 1 & 4 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 4 \end{bmatrix}$$

Exercise 11.5.1.22

$$\begin{bmatrix} 1 & -1 & 3 & 1 & -2 & 9 \\ 2 & -2 & 6 & 1 & -2 & 13 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 1 & -1 & 3 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & -2 & 5 \end{bmatrix}$$

In Exercises 11.5.1.23- 11.5.1.36, find the solution to the given linear system. If the system has infinite solution, give 2 solutions.

? Exercise 11.5.1.23

$$\begin{aligned} 2x_1 + 4x_2 &= 2 \\ x_1 + 2x_2 &= 1 \end{aligned}$$

Answer

$x_1 = 1 - 2x_2$; x_2 is free. Possible solutions: $x_1 = 1$, $x_2 = 0$ and $x_1 = -1$, $x_2 = 1$.

? Exercise 11.5.1.24

$$\begin{aligned} -x_1 + 5x_2 &= 3 \\ 2x_1 - 10x_2 &= -6 \end{aligned}$$

Answer

$x_1 = -3 + 5x_2$; x_2 is free. Possible solutions: $x_1 = 3$, $x_2 = 0$ and $x_1 = -8$, $x_2 = -1$

? Exercise 11.5.1.25

$$\begin{aligned} x_1 + x_2 &= 3 \\ 2x_1 + x_2 &= 4 \end{aligned}$$

Answer

$x_1 = 1$; $x_2 = 2$

? Exercise 11.5.1.26

$$\begin{aligned} -3x_1 + 7x_2 &= -7 \\ 2x_1 - 8x_2 &= 8 \end{aligned}$$

Answer

$x_1 = 0$; $x_2 = -1$

? Exercise 11.5.1.27

$$\begin{aligned} 2x_1 + 3x_2 &= 1 \\ -2x_1 - 3x_2 &= 1 \end{aligned}$$

Answer

No solution; the system is inconsistent.

? Exercise 11.5.1.28

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ -x_1 - 2x_2 &= 5 \end{aligned}$$

Answer

No solution; the system is inconsistent.

Exercise 11.5.1.29

$$\begin{aligned} -2x_1 &+ 4x_2 + 4x_3 &= 6 \\ x_1 &- 3x_2 + 2x_3 &= 1 \end{aligned}$$

Answer

$x_1 = -11 + 10x_3$; $x_2 = -4 + 4x_3$; x_3 is free. Possible solutions: $x_1 = -11$, $x_2 = -4$, $x_3 = 0$ and $x_1 = -1$, $x_2 = 0$ and $x_3 = 1$.

Exercise 11.5.1.30

$$\begin{aligned} -x_1 &+ 2x_2 + 2x_3 &= 2 \\ 2x_1 &+ 5x_2 + x_3 &= 2 \end{aligned}$$

Answer

$x_1 = -\frac{2}{3} + \frac{8}{9}x_3$; $x_2 = \frac{2}{3} - \frac{5}{9}x_3$; x_3 is free. Possible solutions: $x_1 = -\frac{2}{3}$, $x_2 = \frac{2}{3}$, $x_3 = 0$ and $x_1 = \frac{4}{9}$, $x_2 = -\frac{1}{9}$, $x_3 = 1$

Exercise 11.5.1.31

$$\begin{aligned} -x_1 - x_2 + x_3 + x_4 &= 0 \\ -2x_1 - 2x_2 + x_3 &= -1 \end{aligned}$$

Answer

$x_1 = 1 - x_2 - x_4$; x_2 is free; $x_3 = 1 - 2x_4$; x_4 is free. Possible solutions: $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, $x_4 = 0$ and $x_1 = -2$, $x_2 = 1$, $x_3 = -3$, $x_4 = 2$

Exercise 11.5.1.32

$$\begin{aligned} x_1 + x_2 + 6x_3 + 9x_4 &= 0 \\ -x_1 - x_3 - 2x_4 &= -3 \end{aligned}$$

Answer

$x_1 = 3 - x_3 - 2x_4$; $x_2 = -3 - 5x_3 - 7x_4$; x_3 is free; x_4 is free. Possible solutions: $x_1 = 3$, $x_2 = -3$, $x_3 = 0$, $x_4 = 0$ and $x_1 = 0$, $x_2 = -5$, $x_3 = -1$, $x_4 = 1$

Exercise 11.5.1.33

$$\begin{aligned} 2x_1 &+ x_2 + 2x_3 &= 0 \\ x_1 &+ x_2 + 3x_3 &= 1 \\ 3x_1 &+ 2x_2 + 5x_3 &= 3 \end{aligned}$$

Answer

No solution; the system is inconsistent.

Exercise 11.5.1.34

$$\begin{aligned} x_1 &+ 3x_2 + 3x_3 = 1 \\ 2x_1 &- x_2 + 2x_3 = -1 \\ 4x_1 &+ 5x_2 + 8x_3 = 2 \end{aligned}$$

Answer

No solution; the system is inconsistent.

Exercise 11.5.1.35

$$\begin{aligned} x_1 &+ 2x_2 + 2x_3 = 1 \\ 2x_1 &+ x_2 + 3x_3 = 1 \\ 3x_1 &+ 3x_2 + 5x_3 = 2 \end{aligned}$$

Answer

$x_1 = \frac{1}{3} - \frac{4}{3}x_3$; $x_2 = \frac{1}{3} - \frac{1}{3}x_3$; x_3 is free. Possible solutions: $x_1 = \frac{1}{3}$, $x_2 = \frac{1}{3}$, $x_3 = 0$ and $x_1 = -1$, $x_2 = 0$, $x_3 = 1$

Exercise 11.5.1.36

$$\begin{aligned} 2x_1 &+ 4x_2 + 6x_3 = 2 \\ 1x_1 &+ 2x_2 + 3x_3 = 1 \\ -3x_1 &- 6x_2 - 9x_3 = -3 \end{aligned}$$

Answer

$x_1 = 1 - 2x_2 - 3x_3$; x_2 is free; x_3 is free. Possible solutions: $x_1 = 1$, $x_2 = 0$, $x_3 = 0$ and $x_1 = 8$, $x_2 = 1$, $x_3 = -3$

In Exercises 11.5.1.37- 11.5.1.40 state for which values of k the given system will have exactly 1 solution, infinite solutions, or no solution.

Exercise 11.5.1.37

$$\begin{aligned} x_1 &+ 2x_2 = 1 \\ 2x_1 &+ 4x_2 = k \end{aligned}$$

Answer

Never exactly 1 solution; infinite solutions if $k = 2$; no solution if $k \neq 2$.

Exercise 11.5.1.38

$$\begin{aligned} x_1 &+ 2x_2 = 1 \\ x_1 &+ kx_2 = 1 \end{aligned}$$

Answer

Exactly 1 solution if $k \neq 2$; infinite solutions if $k = 2$; never no solution.

Exercise 11.5.1.39

$$\begin{aligned} x_1 &+ 2x_2 = 1 \\ x_1 &+ kx_2 = 2 \end{aligned}$$

Answer

Exactly 1 solution if $k \neq 2$; no solution if $k = 2$; never infinite solutions.

Exercise 11.5.1.40

$$\begin{aligned} x_1 &+ 2x_2 = 1 \\ x_1 &+ 3x_2 = k \end{aligned}$$

Answer

Exactly 1 solution for all k.

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11.6: Eigenvalues and Eigenvectors

Note

In this section, as well as in chapter 10, we will write systems of equations as $A\vec{x} = \vec{b}$ where the matrix A is made up of the coefficients of the variables, $\vec{x}$ represents the variables x_1, x_2, x_3, etc., and $\vec{b}$ is made up of the constants on the right side of the equation. For example, the system

$$\begin{array}{rcrcrcl} 3x_1 & + & 4x_2 & + & 5x_3 & = & 7 \\ -x_1 & + & x_2 & - & 3x_3 & = & 1 \\ 2x_1 & - & 2x_2 & + & 3x_3 & = & 5 \end{array}$$

will be written as $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 3 & 4 & 5 \\ -1 & 1 & -3 \\ 2 & -2 & 3 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \text{and} \quad \vec{b} = \begin{bmatrix} 7 \\ 1 \\ 5 \end{bmatrix}. \tag{11.6.1}$$

$\vec{x}$ and $\vec{b}$ are called $n \times 1$ column vectors.

Definition: Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix, $\vec{x}$ a nonzero $n \times 1$ column vector and λ a scalar. If

$$A\vec{x} = \lambda\vec{x},$$

then $\vec{x}$ is an *eigenvector* of A and λ is an *eigenvalue* of A.

The word “eigen” is German for “proper” or “characteristic.” Therefore, an *eigenvector* of A is a “characteristic vector of A.” This vector tells us something about A.

Note that our definition requires that A be a square matrix. Also note that $\vec{x}$ must be nonzero. Why? What if $\vec{x} = \vec{0}$? Then *no matter what* λ is, $A\vec{x} = \lambda\vec{x}$. This would then imply that *every number* is an eigenvalue; if every number is an eigenvalue, then we wouldn’t need a definition for it. Therefore we specify that $\vec{x} \neq \vec{0}$.

Our last comment before trying to find eigenvalues and eigenvectors for given matrices deals with “why we care.” Did we stumble upon a mathematical curiosity, or does this somehow help us build better bridges, heal the sick, send astronauts into orbit, design optical equipment, and understand quantum mechanics? The answer, of course, is “Yes." This is a wonderful topic in and of itself: we need no external application to appreciate its worth. At the same time, it has many, many applications to “the real world.” A simple Internet search on “applications of eigenvalues” will confirm this.

Before we can talk about how to find eigenvalues and eigenvectors we need the following definition:

Definition: Identity Matrix

The $n \times n$ matrix with 1’s on the diagonal and zeros elsewhere is the $n \times n$ *identity matrix*, denoted I_n. When the context makes the dimension of the identity clear, the subscript is generally omitted.

We show a few identity matrices below.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Given a square matrix A, we want to find a nonzero vector $\vec{x}$ and a scalar λ such that $A\vec{x} = \lambda\vec{x}$. We will solve this using the skills we developed in Section 11.5.

$$\begin{aligned} A\vec{x} &= \lambda\vec{x} && \text{original equation} \\ A\vec{x} - \lambda\vec{x} &= \vec{0} && \text{subtract } \lambda\vec{x} \text{ from both sides} \\ (A-\lambda I)\vec{x} &= \vec{0} && \text{factor out } \vec{x} \end{aligned} \tag{11.6.2}$$

Think about this last factorization. We are likely tempted to say

$$A\vec{x} - \lambda\vec{x} = (A-\lambda)\vec{x},$$

but this really doesn't make sense. After all, what does "a matrix minus a number" mean? We need the identity matrix in order for this to be logical.

Let us now think about the equation $(A-\lambda I)\vec{x} = \vec{0}$. While it looks complicated, it really is just matrix equation of the type we solved in the previous section.

This type of equation always has a solution, namely, $\vec{x} = \vec{0}$. However, we want $\vec{x}$ to be an eigenvector and, by the definition, eigenvectors cannot be $\vec{0}$.

This means that we want solutions to $(A-\lambda I)\vec{x} = \vec{0}$ other than $\vec{x} = \vec{0}$.

A theorem in linear algebra tells us that for this to happen, we need $\det(A-\lambda I) = 0$.

Let's start our practice of this theory by finding λ such that $\det(A-\lambda I) = 0$; that is, let's find the eigenvalues of a matrix.

Example 11.6.1

Find the eigenvalues of A, that is, find λ such that $\det(A-\lambda I) = 0$, where

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}.$$

Solution

First, we write out what $A-\lambda I$ is:

$$\begin{aligned} A-\lambda I &= \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \\ &= \begin{bmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{bmatrix} \end{aligned} \tag{11.6.3}$$

Therefore,

$$\begin{aligned} \det(A-\lambda I) &= \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} \\ &= (1-\lambda)(3-\lambda) - 8 \\ &= \lambda^2 - 4\lambda - 5 \end{aligned} \tag{11.6.4}$$

Since we want $\det(A-\lambda I) = 0$, we want $\lambda^2 - 4\lambda - 5 = 0$. This is a simple quadratic equation that is easy to factor:

$$\begin{aligned} \lambda^2 - 4\lambda - 5 &= 0 \\ (\lambda-5)(\lambda+1) &= 0 \\ \lambda &= -1,\ 5 \end{aligned} \tag{11.6.5}$$

According to our above work, $\det(A-\lambda I)$ when $\lambda = -1,\ 5$. Thus, the eigenvalues of A are -1 and 5.

Example 11.6.2

Find $\vec{x}$ such that $A\vec{x} = 5\vec{x}$, where

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}.$$

Solution

Recall that our algebra from before showed that if

$$A\vec{x} = \lambda\vec{x} \quad \text{then} \quad (A - \lambda I)\vec{x} = \vec{0}.$$

Therefore, we need to solve the equation $(A - \lambda I)\vec{x} = \vec{0}$ for $\vec{x}$ when $\lambda = 5$.

$$\begin{aligned} A - 5I &= \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \end{aligned} \tag{11.6.6}$$

To solve $(A - 5I)\vec{x} = \vec{0}$, we form the augmented matrix and put it into reduced row echelon form:

$$\begin{bmatrix} -4 & 4 & 0 \\ 2 & -2 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus

$$\begin{aligned} x_1 &= x_2 \\ x_2 &\text{ is free} \end{aligned} \tag{11.6.7}$$

and

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We have infinite solutions to the equation $A\vec{x} = 5\vec{x}$; any nonzero scalar multiple of the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a solution. We can do a few examples to confirm this:

$$\begin{aligned} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} &= \begin{bmatrix} 10 \\ 10 \end{bmatrix} = 5\begin{bmatrix} 2 \\ 2 \end{bmatrix}; \\ \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 7 \\ 7 \end{bmatrix} &= \begin{bmatrix} 35 \\ 35 \end{bmatrix} = 5\begin{bmatrix} 7 \\ 7 \end{bmatrix}; \\ \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -3 \\ -3 \end{bmatrix} &= \begin{bmatrix} -15 \\ -15 \end{bmatrix} = 5\begin{bmatrix} -3 \\ -3 \end{bmatrix}. \end{aligned} \tag{11.6.8}$$

Our method of finding the eigenvalues of a matrix A boils down to determining which values of λ give the matrix $(A - \lambda I)$ a determinant of 0. In computing $\det(A - \lambda I)$, we get a polynomial in λ whose roots are the eigenvalues of A. This polynomial is important and so it gets its own name.

Definition: Characteristic Polynomial

Let A be an $n \times n$ matrix. The *characteristic polynomial* of A is the n^{th} degree polynomial $p(\lambda) = \det(A - \lambda I)$.

Our definition just states *what* the characteristic polynomial is. We know from our work so far *why* we care: the roots of the characteristic polynomial of an $n \times n$ matrix A are the eigenvalues of A.

In Examples 11.6.1 and 11.6.2, we found eigenvalues and eigenvectors, respectively, of a given matrix. That is, given a matrix A, we found values λ and vectors $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$. The steps that follow outline the general procedure for finding eigenvalues and eigenvectors; we'll follow this up with some examples.

Key Idea 11.6.1: Finding Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix.

1. To find the eigenvalues of A, compute $p(\lambda)$, the characteristic polynomial of A, set it equal to 0, then solve for λ.
2. To find the eigenvectors of A, *for each eigenvalue* solve the system $(A - \lambda I)\vec{x} = \vec{0}$.

Example 11.6.3

Find the eigenvalues of A, and for each eigenvalue, find an eigenvector where

$$A = \begin{bmatrix} -3 & 15 \\ 3 & 9 \end{bmatrix}.$$

Solution

To find the eigenvalues, we must compute $\det(A - \lambda I)$ and set it equal to 0.

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -3-\lambda & 15 \\ 3 & 9-\lambda \end{vmatrix} \\ &= (-3-\lambda)(9-\lambda) - 45 \\ &= \lambda^2 - 6\lambda - 27 - 45 \\ &= \lambda^2 - 6\lambda - 72 \\ &= (\lambda - 12)(\lambda + 6) \end{aligned} \tag{11.6.9}$$

Therefore, $\det(A - \lambda I) = 0$ when $\lambda = -6$ and 12; these are our eigenvalues. (We should note that $p(\lambda) = \lambda^2 - 6\lambda - 72$ is our characteristic polynomial.) It sometimes helps to give them "names," so we'll say $\lambda_1 = -6$ and $\lambda_2 = 12$. Now we find eigenvectors.

For $\lambda_1 = -6$:

We need to solve the equation $(A - (-6)I)\vec{x} = \vec{0}$. To do this, we form the appropriate augmented matrix and put it into reduced row echelon form.

$$\begin{bmatrix} 3 & 15 & 0 \\ 3 & 15 & 0 \end{bmatrix} \overset{\text{rref}}{\rightarrow} \begin{bmatrix} 1 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Our solution is

$$\begin{aligned} x_1 &= -5x_2 \\ x_2 &\text{ is free;} \end{aligned} \tag{11.6.10}$$

in vector form, we have

$$\vec{x} = x_2 \begin{bmatrix} -5 \\ 1 \end{bmatrix}.$$

We may pick any nonzero value for x_2 to get an eigenvector; a simple option is $x_2 = 1$. Thus we have the eigenvector

$$\vec{x_1} = \begin{bmatrix} -5 \\ 1 \end{bmatrix}.$$

(We used the notation $\vec{x_1}$ to associate this eigenvector with the eigenvalue λ_1.)

We now repeat this process to find an eigenvector for $\lambda_2 = 12$:

In solving $(A - 12I)\vec{x} = \vec{0}$, we find

$$\begin{bmatrix} -15 & 15 & 0 \\ 3 & -3 & 0 \end{bmatrix} \overset{\text{rref}}{\rightarrow} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In vector form, we have

$$\vec{x} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Again, we may pick any nonzero value for x_2, and so we choose $x_2 = 1$. Thus an eigenvector for λ_2 is

$$\vec{x_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

To summarize, we have:

$$\text{eigenvalue } \lambda_1 = -6 \text{ with eigenvector } \vec{x_1} = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$$

and

$$\text{eigenvalue } \lambda_2 = 12 \text{ with eigenvector } \vec{x_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We should take a moment and check our work: is it true that $A\vec{x_1} = \lambda_1 \vec{x_1}$?

$$\begin{aligned} A\vec{x_1} &= \begin{bmatrix} -3 & 15 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} -5 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 30 \\ -6 \end{bmatrix} \\ &= (-6) \begin{bmatrix} -5 \\ 1 \end{bmatrix} \\ &= \lambda_1 \vec{x_1}. \end{aligned} \tag{11.6.11}$$

Yes; it appears we have truly found an eigenvalue/eigenvector pair for the matrix A.

Let's do another example.

Example 11.6.4

Let $A = \begin{bmatrix} -3 & 0 \\ 5 & 1 \end{bmatrix}$. Find the eigenvalues of A and an eigenvector for each eigenvalue.

Solution

We first compute the characteristic polynomial, set it equal to 0, then solve for λ.

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -3-\lambda & 0 \\ 5 & 1-\lambda \end{vmatrix} \\ &= (-3-\lambda)(1-\lambda) \end{aligned} \tag{11.6.12}$$

From this, we see that $\det(A - \lambda I) = 0$ when $\lambda = -3, 1$. We'll set $\lambda_1 = -3$ and $\lambda_2 = 1$.

Finding an eigenvector for λ_1:

We solve $(A - (-3)I)\vec{x} = \vec{0}$ for $\vec{x}$ by row reducing the appropriate matrix:

$$\begin{bmatrix} 0 & 0 & 0 \\ 5 & 4 & 0 \end{bmatrix} \overset{\text{rref}}{\rightarrow} \begin{bmatrix} 1 & 5/4 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Our solution, in vector form, is

$$\vec{x} = x_2 \begin{bmatrix} -5/4 \\ 1 \end{bmatrix}.$$

Again, we can pick any nonzero value for x_2; a nice choice would eliminate the fraction. Therefore we pick $x_2 = 4$, and find

$$\vec{x_1} = \begin{bmatrix} -5 \\ 4 \end{bmatrix}.$$

Finding an eigenvector for λ_2:

We solve $(A - (1)I)\vec{x} = \vec{0}$ for $\vec{x}$ by row reducing the appropriate matrix:

$$\begin{bmatrix} -4 & 0 & 0 \\ 5 & 0 & 0 \end{bmatrix} \overset{\text{rref}}{\rightarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Our first row tells us that $x_1 = 0$, and we see that no rows/equations involve x_2. We conclude that x_2 is free. Therefore, our solution, in vector form, is

$$\vec{x} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We pick $x_2 = 1$, and find

$$\vec{x_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

To summarize, we have:

$$\text{eigenvalue } \lambda_1 = -3 \text{ with eigenvector } \vec{x_1} = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$$

and

$$\text{eigenvalue } \lambda_2 = 1 \text{ with eigenvector } \vec{x_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

So far, our examples have involved 2×2 matrices. Let's do an example with a 3×3 matrix.

Example 11.6.5

Find the eigenvalues of A, and for each eigenvalue, give one eigenvector, where

$$A = \begin{bmatrix} -7 & -2 & 10 \\ -3 & 2 & 3 \\ -6 & -2 & 9 \end{bmatrix}.$$

Solution

We first compute the characteristic polynomial, set it equal to 0, then solve for λ. A warning: this process is rather long. We'll use cofactor expansion along the first row; don't get bogged down with the arithmetic that comes from each step; just try to get the basic idea of what was done from step to step.

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -7-\lambda & -2 & 10 \\ -3 & 2-\lambda & 3 \\ -6 & -2 & 9-\lambda \end{vmatrix} \\ &= (-7-\lambda)\begin{vmatrix} 2-\lambda & 3 \\ -2 & 9-\lambda \end{vmatrix} - (-2)\begin{vmatrix} -3 & 3 \\ -6 & 9-\lambda \end{vmatrix} + 10\begin{vmatrix} -3 & 2-\lambda \\ -6 & -2 \end{vmatrix} \qquad (11.6.13) \\ &= (-7-\lambda)(\lambda^2 - 11\lambda + 24) + 2(3\lambda - 9) + 10(-6\lambda + 18) \\ &= -\lambda^3 + 4\lambda^2 - \lambda - 6 \\ &= -(\lambda+1)(\lambda-2)(\lambda-3) \end{aligned}$$

In the last step we factored the characteristic polynomial $-\lambda^3 + 4\lambda^2 - \lambda - 6$.

Our eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 2$ and $\lambda_3 = 3$. We now find corresponding eigenvectors.

For $\lambda_1 = -1$:

We need to solve the equation $(A - (-1)I)\vec{x} = \vec{0}$. To do this, we form the appropriate augmented matrix and put it into reduced row echelon form.

$$\begin{bmatrix} -6 & -2 & 10 & 0 \\ -3 & 3 & 3 & 0 \\ -6 & -2 & 10 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1.5 & 0 \\ 0 & 1 & -.5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Our solution, in vector form, is

$$\vec{x} = x_3 \begin{bmatrix} 3/2 \\ 1/2 \\ 1 \end{bmatrix}.$$

We can pick any nonzero value for x_3; a nice choice would get rid of the fractions. So we'll set $x_3 = 2$ and choose $\vec{x_1} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ as our eigenvector.

For $\lambda_2 = 2$:

We need to solve the equation $(A - 2I)\vec{x} = \vec{0}$. To do this, we form the appropriate augmented matrix and put it into reduced row echelon form.

$$\begin{bmatrix} -9 & -2 & 10 & 0 \\ -3 & 0 & 3 & 0 \\ -6 & -2 & 7 & 0 \end{bmatrix} \overset{\text{rref}}{\rightarrow} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -.5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Our solution, in vector form, is

$$\vec{x} = x_3 \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}.$$

We can pick any nonzero value for x_3; again, a nice choice would get rid of the fractions. So we'll set $x_3 = 2$ and choose $\vec{x_2} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ as our eigenvector.

For $\lambda_3 = 3$:

We need to solve the equation $(A - 3I)\vec{x} = \vec{0}$. To do this, we form the appropriate augmented matrix and put it into reduced row echelon form.

$$\begin{bmatrix} -10 & -2 & 10 & 0 \\ -3 & -1 & 3 & 0 \\ -6 & -2 & 6 & 0 \end{bmatrix} \overset{\text{rref}}{\rightarrow} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Our solution, in vector form, is (note that $x_2 = 0$):

$$\vec{x} = x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

We can pick any nonzero value for x_3; an easy choice is $x_3 = 1$, so $\vec{x_3} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ as our eigenvector.

To summarize, we have the following eigenvalue/eigenvector pairs:

$$\text{eigenvalue } \lambda_1 = -1 \text{ with eigenvector } \vec{x_1} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$\text{eigenvale } \lambda_2 = 2 \text{ with eigenvector } \vec{x_2} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$\text{eigenvalue } \lambda_3 = 3 \text{ with eigenvector } \vec{x_3} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Let's practice once more.

Example 11.6.6

Find the eigenvalues of A, and for each eigenvalue, give one eigenvector, where

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 6 \\ 0 & 3 & 4 \end{bmatrix}.$$

Solution

We first compute the characteristic polynomial, set it equal to 0, then solve for λ. We'll leave the details to you and just give the result.

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 2-\lambda & -1 & 1 \\ 0 & 1-\lambda & 6 \\ 0 & 3 & 4-\lambda \end{vmatrix} \\ &= (2-\lambda)(\lambda - 7)(\lambda + 2) \end{aligned} \tag{11.6.14}$$

Our eigenvalues are $\lambda_1 = -2$, $\lambda_2 = 2$ and $\lambda_3 = 7$. We now find corresponding eigenvectors.

For $\lambda_1 = -2$:

We need to solve the equation $(A - (-2)I)\vec{x} = \vec{0}$. To do this, we form the appropriate augmented matrix and put it into reduced row echelon form.

$$\begin{bmatrix} 4 & -1 & 1 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 3 & 6 & 0 \end{bmatrix} \xrightarrow[\text{rref}]{} \begin{bmatrix} 1 & 0 & 3/4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Our solution, in vector form, is

$$\vec{x} = x_3 \begin{bmatrix} -3/4 \\ -2 \\ 1 \end{bmatrix}.$$

We can pick any nonzero value for x_3; a nice choice would get rid of the fractions. So we'll set $x_3 = 4$ and choose $\vec{x_1} = \begin{bmatrix} -3 \\ -8 \\ 4 \end{bmatrix}$ as our eigenvector.

For $\lambda_2 = 2$:

We need to solve the equation $(A - 2I)\vec{x} = \vec{0}$. To do this, we form the appropriate augmented matrix and put it into reduced row echelon form.

$$\begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & -1 & 6 & 0 \\ 0 & 3 & 2 & 0 \end{bmatrix} \xrightarrow[\text{rref}]{} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first two rows tell us that $x_2 = 0$ and $x_3 = 0$, respectively. Notice that no row/equation uses x_1; we conclude that it is free. Therefore, our solution in vector form is

$$\vec{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

We can pick any nonzero value for x_1; an easy choice is $x_1 = 1$ and choose $\overrightarrow{x_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ as our eigenvector.

For $\lambda_3 = 7$:

We need to solve the equation $(A - 7I)\vec{x} = \vec{0}$. To do this, we form the appropriate augmented matrix and put it into reduced row echelon form.

$$\begin{bmatrix} -5 & -1 & 1 & 0 \\ 0 & -6 & 6 & 0 \\ 0 & 3 & -3 & 0 \end{bmatrix} \overrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Our solution, in vector form, is (note that $x_1 = 0$):

$$\vec{x} = x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

We can pick any nonzero value for x_3; an easy choice is $x_3 = 1$, so $\overrightarrow{x_3} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ as our eigenvector.

To summarize, we have the following eigenvalue/eigenvector pairs:

$$\text{eigenvalue } \lambda_1 = -2 \text{ with eigenvector } \overrightarrow{x_1} = \begin{bmatrix} -3 \\ -8 \\ 4 \end{bmatrix}$$

$$\text{eigenvalue } \lambda_2 = 2 \text{ with eigenvector } \overrightarrow{x_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{eigenvalue } \lambda_3 = 7 \text{ with eigenvector } \overrightarrow{x_3} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

We have only talked about eigenvalues that are distinct. However, they can repeat, or even be complex, but the ideas are similar to the above and we will discuss them as they come up in Chapter 10.

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11.6.1: Eigenvalues and Eigenvectors (Exercises)

In Exercises 11.6.1.1 - 11.6.1.6, a matrix A and one of its eigenvectors are given. Find the eigenvalue of A for the given eigenvector.

Exercise 11.6.1.1

$$A = \begin{bmatrix} 9 & 8 \\ -6 & -5 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

Answer

$\lambda = 3$

Exercise 11.6.1.2

$$A = \begin{bmatrix} 19 & -6 \\ 48 & -15 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Answer

$\lambda = 1$

Exercise 11.6.1.3

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Answer

$\lambda = 0$

Exercise 11.6.1.4

$$A = \begin{bmatrix} -11 & -19 & 14 \\ -6 & -8 & 6 \\ -12 & -22 & 15 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

Answer

$\lambda = -5$

Exercise 11.6.1.5

$$A = \begin{bmatrix} -7 & 1 & 3 \\ 10 & 2 & -3 \\ -20 & -14 & 1 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

Answer

$\lambda = 3$

Exercise 11.6.1.6

$$A = \begin{bmatrix} -12 & -10 & 0 \\ 15 & 13 & 0 \\ 15 & 18 & -5 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Answer

$\lambda = -2$

In Exercises 11.6.1.7 – 11.6.1.11, a matrix A and one of its eigenvalues are given. Find an eigenvector of A for the given eigenvalue.

Exercise 11.6.1.7

$$A = \begin{bmatrix} 16 & 6 \\ -18 & -5 \end{bmatrix} \quad \lambda = 4$$

Answer

$$\vec{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Exercise 11.6.1.8

$$A = \begin{bmatrix} -2 & 6 \\ -9 & 13 \end{bmatrix} \quad \lambda = 7$$

Answer

$$\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Exercise 11.6.1.9

$$A = \begin{bmatrix} -16 & -28 & -19 \\ 42 & 69 & 46 \\ -42 & -72 & -49 \end{bmatrix} \quad \lambda = 5$$

Answer

$$\vec{x} = \begin{bmatrix} 3 \\ -7 \\ 7 \end{bmatrix}$$

Exercise 11.6.1.10

$$A = \begin{bmatrix} 7 & -5 & -10 \\ 6 & 2 & -6 \\ 2 & -5 & -5 \end{bmatrix} \quad \lambda = -3$$

Answer

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Exercise 11.6.1.11

$$A = \begin{bmatrix} 4 & 5 & -3 \\ -7 & -8 & 3 \\ 1 & -5 & 8 \end{bmatrix} \quad \lambda = 2$$

Answer

$$\vec{x} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

In Exercises 11.6.1.12– 11.6.1.28 find the eigenvalues of the given matrix. For each eigenvalue, give an eigenvector.

Exercise 11.6.1.12

$$\begin{bmatrix} -1 & -4 \\ -3 & -2 \end{bmatrix}$$

Answer

$$\lambda_1 = -5 \text{ with } \vec{x_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix};$$

$$\lambda_2 = 2 \text{ with } \vec{x_2} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

Exercise 11.6.1.13

$$\begin{bmatrix} -4 & 72 \\ -1 & 13 \end{bmatrix}$$

Answer

$$\lambda_1 = 4 \text{ with } \vec{x_1} = \begin{bmatrix} 9 \\ 1 \end{bmatrix};$$

$$\lambda_2 = 5 \text{ with } \vec{x_2} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$$

Exercise 11.6.1.14

$$\begin{bmatrix} 2 & -12 \\ 2 & -8 \end{bmatrix}$$

Answer

$$\lambda_1 = -4 \text{ with } \vec{x_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix};$$

$$\lambda_2 = -2 \text{ with } \vec{x_2} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Exercise 11.6.1.15

$$\begin{bmatrix} 3 & 12 \\ 1 & -1 \end{bmatrix}$$

Answer

$\lambda_1 = -3$ with $\vec{x_1} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$;

$\lambda_2 = 5$ with $\vec{x_2} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$

? Exercise 11.6.1.16

$\begin{bmatrix} 5 & 9 \\ -1 & -5 \end{bmatrix}$

Answer

$\lambda_1 = -4$ with $\vec{x_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$;

$\lambda_2 = 4$ with $\vec{x_2} = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$

? Exercise 11.6.1.17

$\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$

Answer

$\lambda_1 = 2$ with $\vec{x_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$;

$\lambda_2 = 4$ with $\vec{x_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

? Exercise 11.6.1.18

$\begin{bmatrix} 0 & 1 \\ 25 & 0 \end{bmatrix}$

Answer

$\lambda_1 = -5$ with $\vec{x_1} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$;

$\lambda_2 = 5$ with $\vec{x_2} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

? Exercise 11.6.1.19

$\begin{bmatrix} -3 & 1 \\ 0 & -1 \end{bmatrix}$

Answer

$\lambda_1 = -1$ with $\vec{x_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$;

$\lambda_2 = -3$ with $\vec{x_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Exercise 11.6.1.20

$$\begin{bmatrix} 1 & -2 & -3 \\ 0 & 3 & 0 \\ 0 & -1 & -1 \end{bmatrix}$$

Answer

$$\lambda_1 = -1 \text{ with } \vec{x_1} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix};$$

$$\lambda_2 = 1 \text{ with } \vec{x_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_3 = 3 \text{ with } \vec{x_3} = \begin{bmatrix} 5 \\ -8 \\ 2 \end{bmatrix}$$

Exercise 11.6.1.21

$$\begin{bmatrix} 5 & -2 & 3 \\ 0 & 4 & 0 \\ 0 & -1 & 3 \end{bmatrix}$$

Answer

$$\lambda_1 = 3 \text{ with } \vec{x_1} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix};$$

$$\lambda_2 = 4 \text{ with } \vec{x_2} = \begin{bmatrix} -5 \\ -1 \\ 1 \end{bmatrix}$$

$$\lambda_3 = 5 \text{ with } \vec{x_3} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Exercise 11.6.1.22

$$\begin{bmatrix} 1 & 0 & 12 \\ 2 & -5 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Answer

$$\lambda_1 = -5 \text{ with } \vec{x_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix};$$

$$\lambda_2 = -2 \text{ with } \vec{x_2} = \begin{bmatrix} -12 \\ -8 \\ 3 \end{bmatrix}$$

$\lambda_3 = 5$ with $\vec{x_3} = \begin{bmatrix} 15 \\ 3 \\ 5 \end{bmatrix}$

Exercise 11.6.1.23

$\begin{bmatrix} 1 & 0 & -18 \\ -4 & 3 & -1 \\ 1 & 0 & -8 \end{bmatrix}$

Answer

$\lambda_1 = -5$ with $\vec{x_1} = \begin{bmatrix} 24 \\ 13 \\ 8 \end{bmatrix}$;

$\lambda_2 = -2$ with $\vec{x_2} = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}$

$\lambda_3 = 3$ with $\vec{x_3} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Exercise 11.6.1.24

$\begin{bmatrix} -1 & 18 & 0 \\ 1 & 2 & 0 \\ 5 & -3 & -1 \end{bmatrix}$

Answer

$\lambda_1 = -4$ with $\vec{x_1} = \begin{bmatrix} -6 \\ 1 \\ 11 \end{bmatrix}$;

$\lambda_2 = -1$ with $\vec{x_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\lambda_3 = 5$ with $\vec{x_3} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

Exercise 11.6.1.25

$\begin{bmatrix} 5 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 5 & -2 \end{bmatrix}$

Answer

$\lambda_1 = -2$ with $\vec{x_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$;

$\lambda_2 = 1$ with $\vec{x_2} = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}$

$\lambda_3 = 5$ with $\vec{x_3} = \begin{bmatrix} 28 \\ 7 \\ 1 \end{bmatrix}$

? Exercise 11.6.1.26

$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 7 \end{bmatrix}$

Answer

$\lambda_1 = 2$ with $\vec{x_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$;

$\lambda_2 = 3$ with $\vec{x_2} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$\lambda_3 = 7$ with $\vec{x_3} = \begin{bmatrix} -1 \\ 15 \\ 10 \end{bmatrix}$

? Exercise 11.6.1.27

$\begin{bmatrix} 3 & 5 & -5 \\ -2 & 3 & 2 \\ -2 & 5 & 0 \end{bmatrix}$

Answer

$\lambda_1 = -2$ with $\vec{x_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$;

$\lambda_2 = 3$ with $\vec{x_2} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$;

$\lambda_3 = 5$ with $\vec{x_3} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

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11.7: A Brief Table of Integrals

These are integrals you should know in this class. Any other integral you should be able to derive from techniques of integration you learned in Math 400 and 401 (single variable calculus).

1. $\int u^n du = \frac{u^{n+1}}{n+1} + c, \quad n \neq -1$
2. $\int \frac{1}{u} du = \ln|u| + c$
3. $\int e^u du = e^u + c$
4. $\int a^u du = \frac{a^u}{\ln a} + c$
5. $\int \cos u \, du = \sin u + c$
6. $\int \sin u \, du = -\cos u + c$
7. $\int \tan u \, du = -\ln|\cos u| + c$
8. $\int \cot u \, du = \ln|\sin u| + c$
9. $\int \sec^2 u \, du = \tan u + c$
10. $\int \csc^2 u \, du = -\cot u + c$
11. $\int \sec u \, du = \ln|\sec u + \tan u| + c$
12. $\int \csc u \, du = \ln|\csc u - \cot u| + c$
13. $\int \frac{1}{a^2+u^2} du = \frac{1}{a} \tan^{-1} \frac{u}{a} + c$
14. $\int \frac{1}{\sqrt{a^2-u^2}} du = \sin^{-1} \frac{u}{a} + c \text{ with } a > 0$
15. $\int \frac{1}{u\sqrt{u^2-a^2}} du = \frac{1}{a} \sec^{-1} \frac{u}{a} + c$
16. $\int \cosh u \, du = \sinh u + c$
17. $\int \sinh u \, du = \cosh u + c$

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11.8: A Brief Table of Trigonometric Identities and Logarithmic Laws

These are trigonometric identities and logarithmic laws you should know in this class.

Trigonometric Identities

1. $\csc x = \frac{1}{\sin x}$
2. $\sec x = \frac{1}{\cos x}$
3. $\cot x = \frac{1}{\tan x}$
4. $\tan x = \frac{\sin x}{\cos x}$
5. $\cot x = \frac{\cos x}{\sin x}$
6. $\sin(-x) = -\sin x$
7. $\cos(-x) = \cos x$
8. $\tan(-x) = -\tan x$
9. $\sin^2 x + \cos^2 x = 1$
10. $1 + \tan^2 x = \sec^2 x$
11. $1 + \cot^2 x = \csc^2 x$
12. $\sin 2x = 2 \sin x \cos x$
13. $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
14. $\cos^2 x = \frac{1+\cos 2x}{2}$
15. $\sin^2 x = \frac{1-\cos 2x}{2}$

Logarithmic Laws

1. $\log_a(xy) = \log_a x + \log_a y$
2. $\log_a \frac{x}{y} = \log_a x - \log_a y$
3. $\log_a x^r = r \log_a x$

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1: Section 1.1 Answers

1. 2

2. 2

3. 1

4. 1

5. 2

14. $x^4 - \frac{1}{3}x^3 + 5x + c$

15. $y = x\cos x - \sin x + c$

16. $y = -\frac{1}{9}\cos^9 x + \frac{2}{7}\cos^7 x - \frac{1}{5}\cos^5 x + c$

17. $x\sec x - \ln|\sec x + \tan x| + c$

18. $2\sin^{-1}\frac{x}{2} - \frac{x\sqrt{4-x^2}}{2} + c$

19. $\ln|x| + \frac{1}{2}\ln(x^2+9) - \frac{1}{3}\tan^{-1}\frac{x}{3} + c$

20. $y = 1 - \frac{1}{2}\cos x^2$

21. $y = -(x-1)e^x$

22. $y = 3 - \ln(\sqrt{2}\cos x)$

23. $(\frac{\pi}{2}, \frac{3\pi}{2})$

24. $(-\frac{\pi}{2}, \frac{\pi}{2})$

25. $(0, \infty)$

26. $(-\sqrt{\pi}, \sqrt{\pi})$

27. $(0, 2)$

28. $(0, \infty)$

29. $(0, \infty)$

30. $(-\infty, \infty)$

31. $(0, 1)$

32. a. $[-3, \infty)$

b. $(-3, \infty)$

33. a. $[-\sqrt{\pi}, \sqrt{\pi}]$

b. $y = -1$ on $(-\infty, \infty)$ and $y = 1$ on $(-\infty, \infty)$

34. a. 576 ft

b. 10 s

37. a. $(-2c - 2, \infty)$

b. $(-\infty, \infty)$

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2: Section 1.2 Answers

1. a. $x_0 \neq k\pi \quad (k = \text{integer})$; all y_0

b. $x_0 \neq k\pi \quad (k = \text{integer})$; all y_0

2. a. all (x_0, y_0)

b. all x_0; all $y_0 \neq 0$

3. a. $x_0 y_0 \neq (2k+1)\frac{\pi}{2} \quad (k = \text{integer})$

b. $x_0 y_0 \neq (2k+1)\frac{\pi}{2} \quad (k = \text{integer})$

4. a. $x_0 y_0 > 0$ and $x_0 y_0 \neq 1$

b. $x_0 y_0 > 0$ and $x_0 y_0 \neq 1$

5. a. all (x_0, y_0)

b. (x_0, y_0) with $y_0 \neq 0$

6. a. all (x_0, y_0)

b. all (x_0, y_0)

7. a. all x_0 with $y_0 \neq \pm 2$

b. all x_0 with $y_0 \neq \pm 2$

8. a. all (x_0, y_0)

b. all (x_0, y_0)

9. a. (x_0, y_0) such that $x_0 \neq 4y_0$

b. (x_0, y_0) such that $x_0 \neq 4y_0$

10. a. all (x_0, y_0)

b. $(x_0, y_0) \neq (0, 0)$

11. a. all (x_0, y_0)

b. all (x_0, y_0) with $y_0 \neq \pm 1$

12. a. all (x_0, y_0)

b. all (x_0, y_0)

13. a. all (x_0, y_0) such that $y_0 > -x_0$

b. all (x_0, y_0) such that $y_0 > -x_0$

14. a. all (x_0, y_0) with $x_0 \neq 1, \quad y_0 \neq (2k+1)\frac{\pi}{2} (k = \text{integer})$

b. all (x_0, y_0) with $x_0 \neq 1, \quad y_0 \neq (2k+1)\frac{\pi}{2} (k = \text{integer})$

15. a. all (x_0, y_0)

b. all (x_0, y_0) with $y_0 \neq 1$

16. a. yes, $y = 0$

b. no, $y = x^3$

17. d. $y = -\frac{1}{x-1}$ on $(-\infty, 1)$

e. $y = -\frac{1}{x+1}$ on $(-1, \infty)$

f. $y = 0$ on $(-\infty, \infty)$

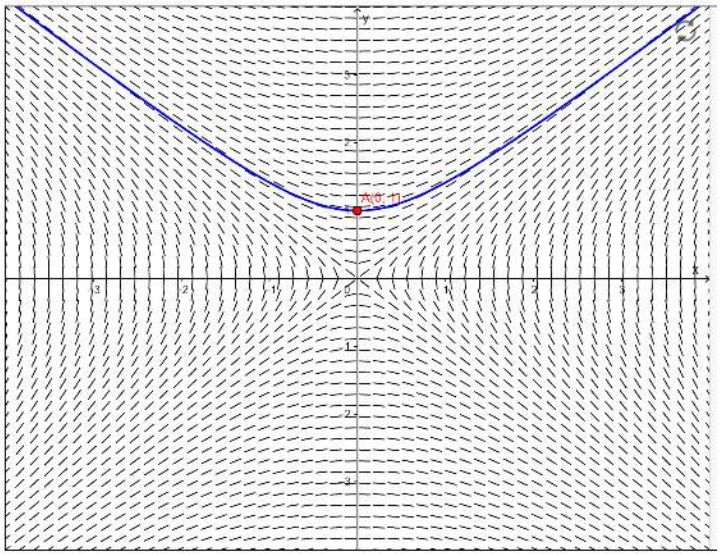
18. d. $y = \tan(x + \frac{\pi}{4})$ on $(-\frac{3\pi}{4}, \frac{\pi}{4})$

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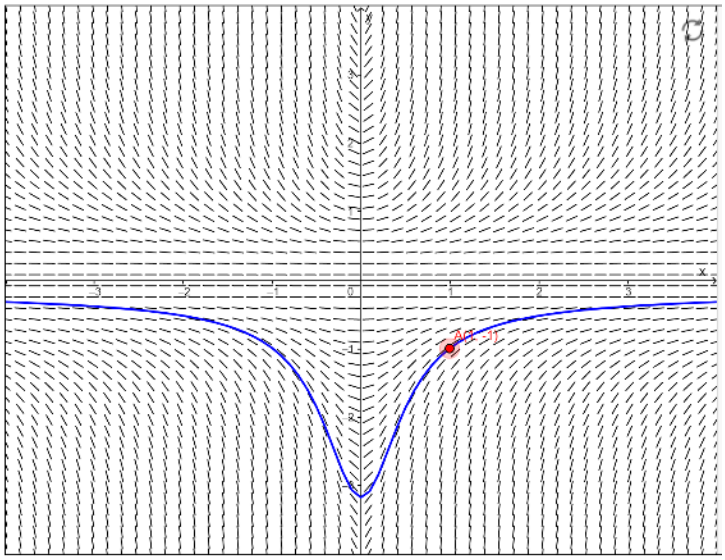
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3: Section 1.3 Answers

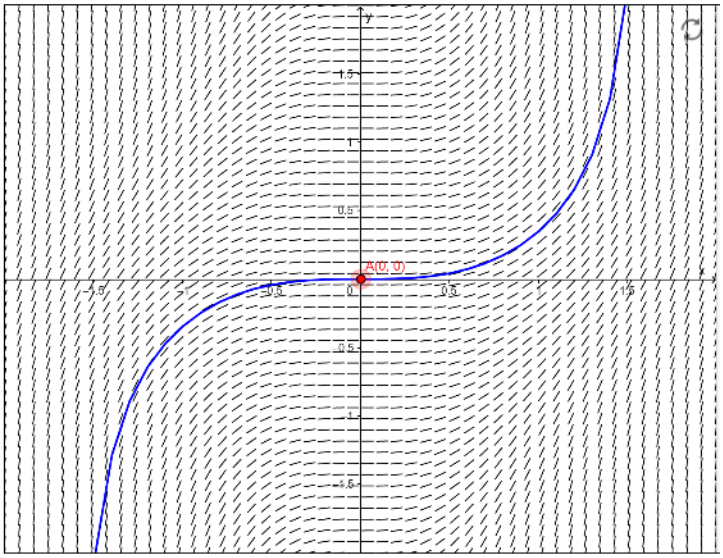
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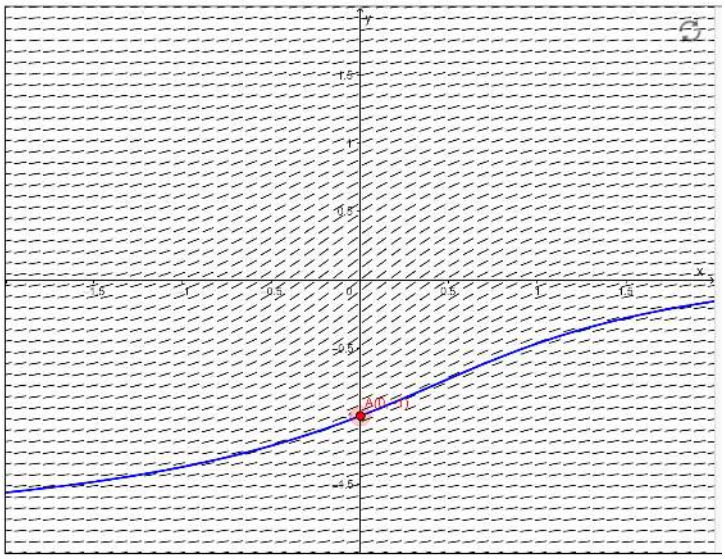
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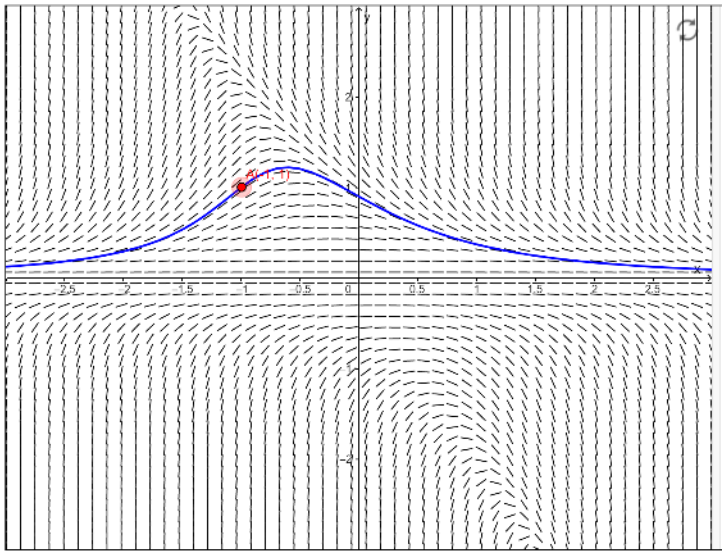
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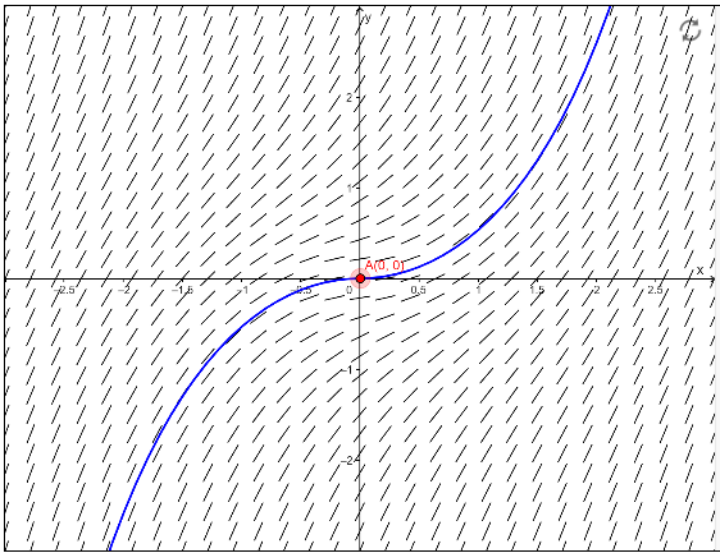
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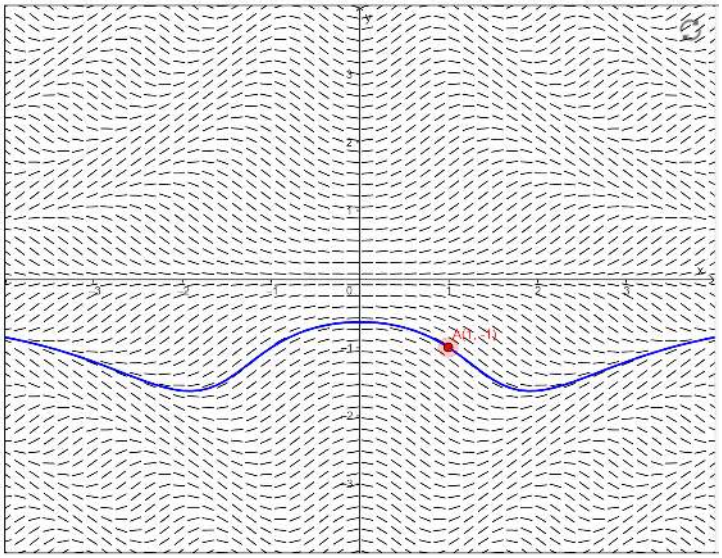
5.


y
x
A(-1, 1)
2
1
-1
-2
2.5
2
1.5
1
0.5
0
0.5
1
1.5
2
2.5

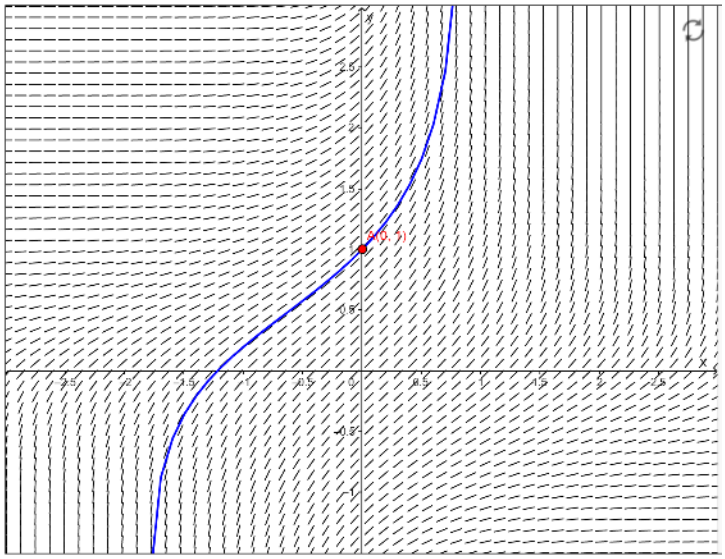
6.


y
x
A(0, 0)
2
1
-1
-2
2.5
2
1.5
1
0.5
0
0.5
1
1.5
2
2.5

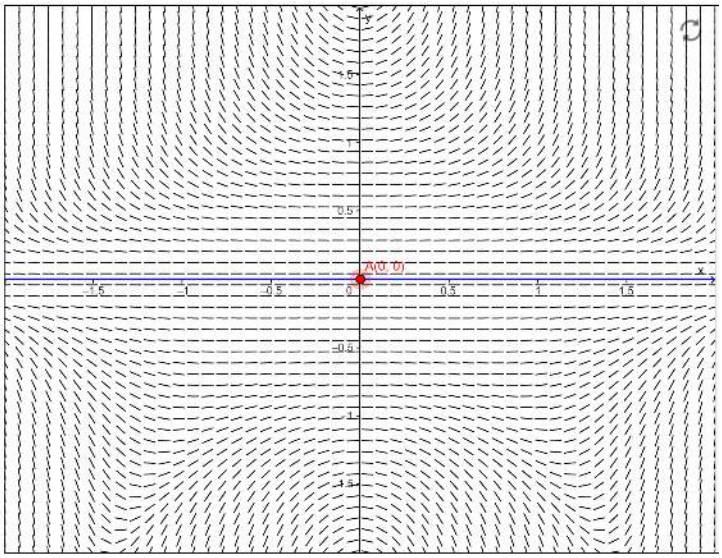
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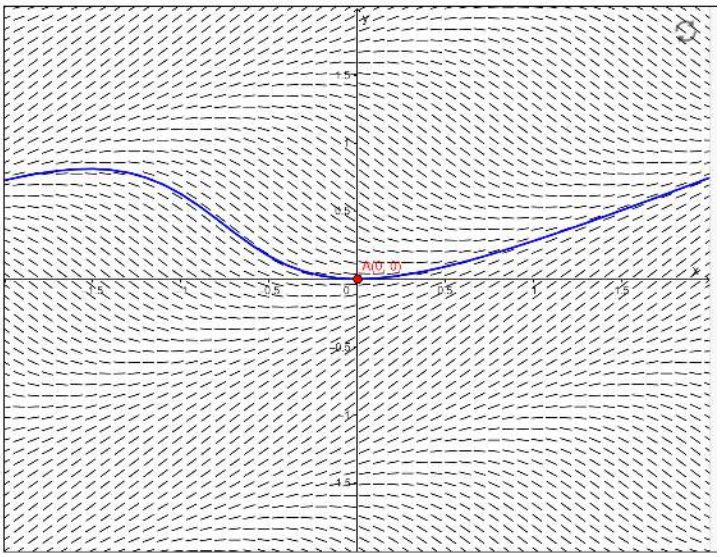
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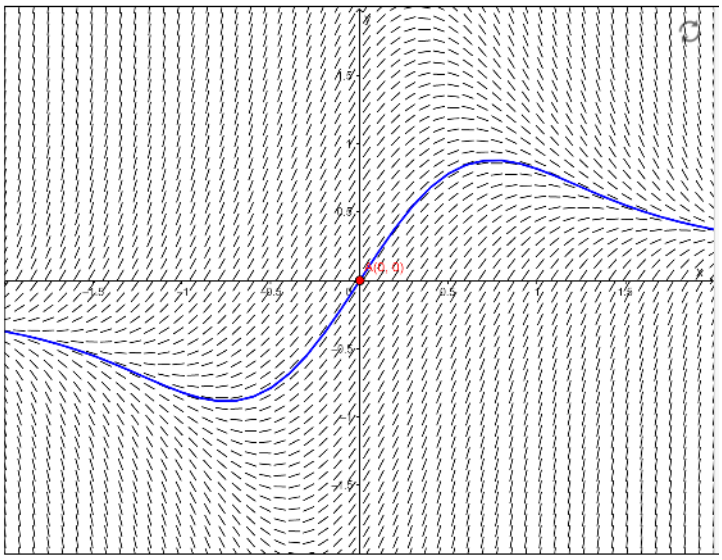


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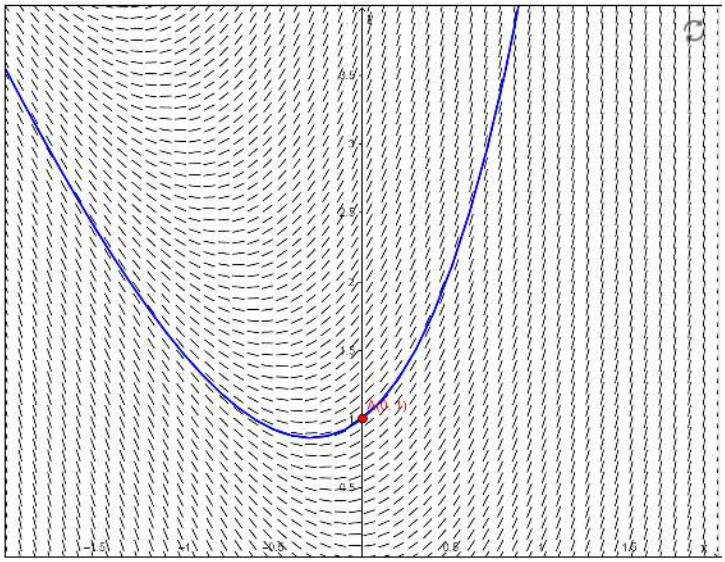
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12.

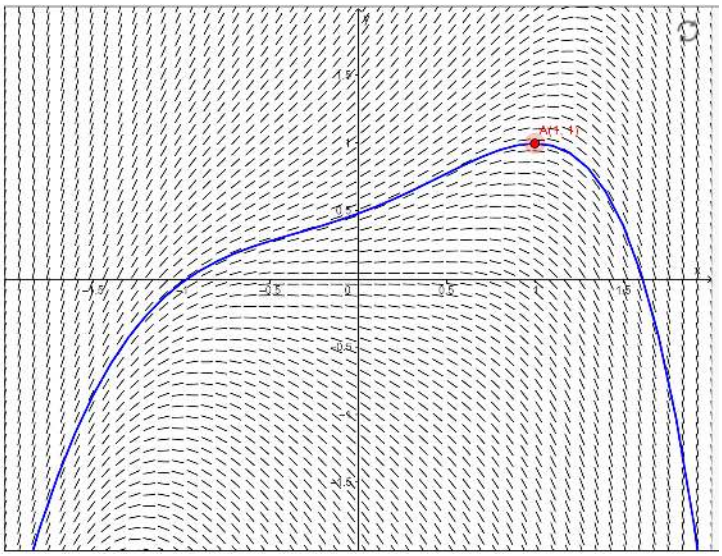


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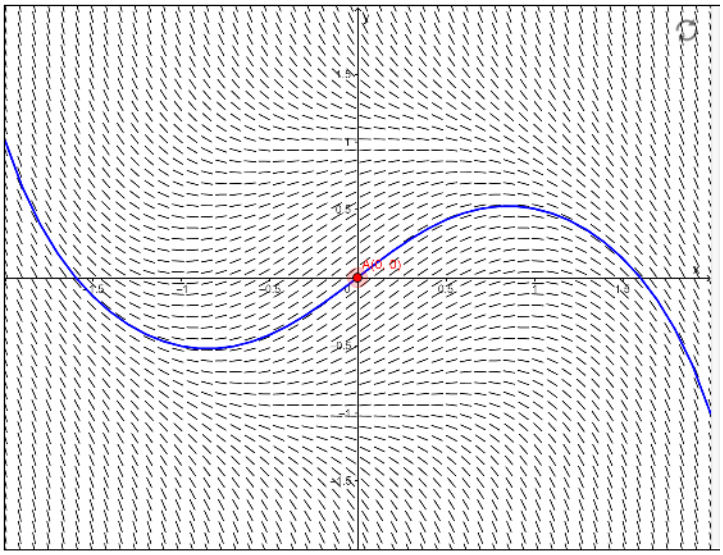
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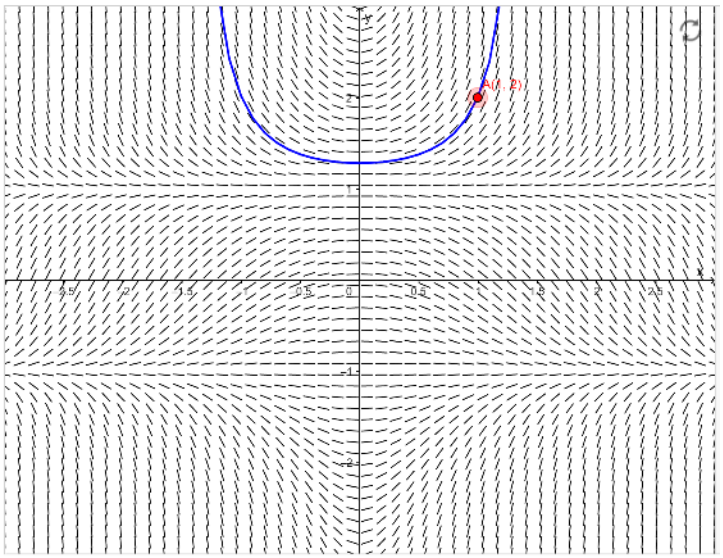
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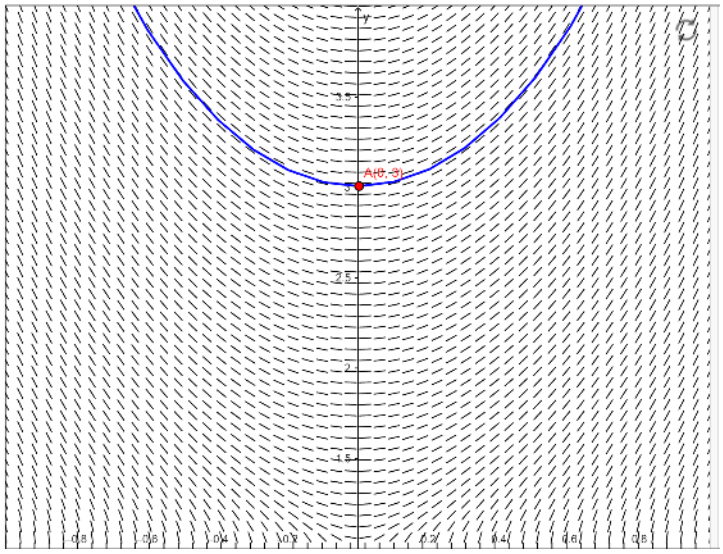
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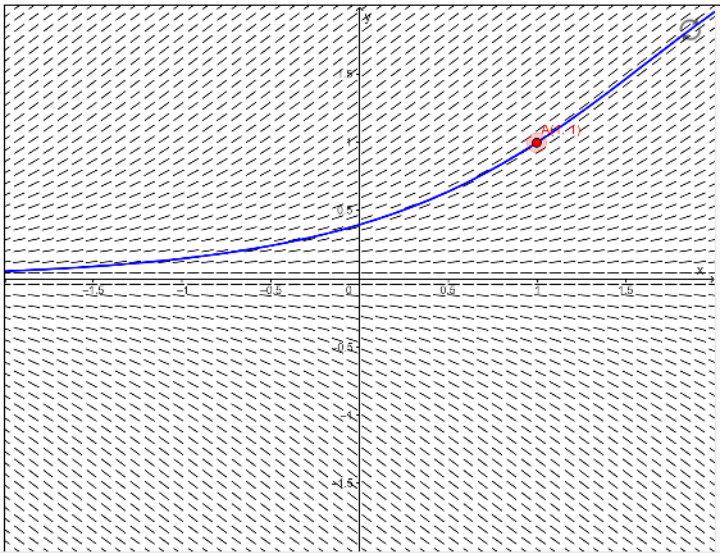
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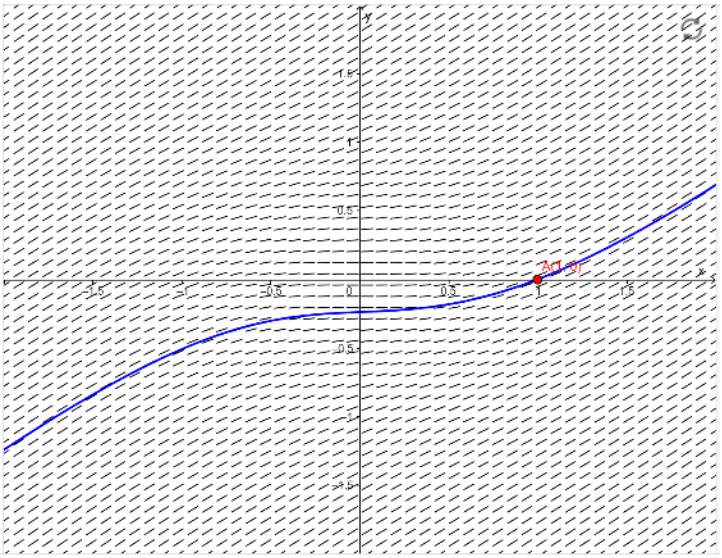
18.



19.



20.



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4: Section 2.1 Answers

1. $y = 2 \pm \sqrt{2(x^3 + x^2 + x + c)}$
2. $\ln(|\sin y|) = \cos x + c; \quad y \equiv k\pi, \quad k = \text{integer}$
3. $-\frac{1}{5}e^{-5y} = \frac{1}{2}e^{2x} + c$
4. $y = \frac{c}{x-c}; \quad y \equiv -1$
5. $\frac{y}{y-1} = ce^x; \quad y \equiv 1$
6. $\frac{(\ln y)^2}{2} = -\frac{x^3}{3} + c$
7. $y^3 + 3\sin y + \ln|y| + \ln(1+x^2) + \tan^{-1} x = c; \quad y \equiv 0$
8. $(y^2-1)^{-1/2} = \frac{1}{x} + c; \quad y \equiv \pm 1$
9. $y = \tan\left(\frac{x^3}{3} + c\right)$
10. $y = \frac{c}{\sqrt{1+x^2}}$
11. $\frac{y-2}{y-1} = ce^{(x^2-2x)/2}; \quad y \equiv 1$
12. $y = 1 + (3x^2 + 9x + c)^{1/3}$
13. $y = 2 + \sqrt{\frac{2}{3}x^3 + 3x^2 + 4x - \frac{11}{3}}$
14. $y = \frac{e^{-(x^2-4)/2}}{2 - e^{-(x^2-4)/2}}$
15. $xy = e^{-(1+1/x)}$
16. $y^3 + 2y^2 + x^2 + \sin x = 3$
17. $(y+1)(y-1)^{-3}(y-2)^2 = -256(x+1)^{-6}$
18. $y = -1 + 3e^{-x^2}$
19. $\frac{y^2}{y^2+1} = \frac{1}{2}e^{2x^2}$
20. $y \equiv -1; \quad (-\infty, \infty)$
21. $\frac{y-2}{y-1} = \frac{1}{2}e^{-x^2}; \quad (-\infty, \infty)$
22. $y = \frac{-1+\sqrt{4x^2-15}}{2}; \quad (\frac{\sqrt{15}}{2}, \infty)$
23. $y = \frac{2}{1+e^{-2x}}; \quad (-\infty, \infty)$
24. $y = -\sqrt{25 - x^2}; \quad (-5, 5)$
25. $y \equiv 2; \quad (-\infty, \infty)$
26. $y = 3\left(\frac{x+1}{2x-4}\right)^{1/3}; \quad (-\infty, 2)$
27. $y = -\sqrt{25 - x^2}; \quad (-5, 5)$
28. $y = \frac{x+c}{1-cx}$
29. $y = -x\cos c + \sqrt{1-x^2}\,\sin c; \quad y \equiv 1; \; y \equiv -1$
30. $y = -x + 3\pi/2$
31. a. $y = \frac{1}{1 - \frac{1}{2}e^x}; \quad (-\infty, \ln 2)$

    b. $y \equiv 0; \quad (-\infty, \infty)$

32. a. $y = \frac{x}{x+1}; \quad (-1, \infty)$
b. $y \equiv 1; \quad (-\infty, \infty)$

33. $y = 1 + (x^2+4)^{3/2}; \quad (-\infty, \infty)$

34. $y = \begin{cases} 1, & 0 < x \le \sqrt{5} \\ 1-(x^2-5)^{3/2}, & \sqrt{5} < x < \infty \end{cases}$

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5: Section 2.2 Answers

1. $2x^3y^2 = c$

2. $3y\sin x + 2x^2e^x + 3y = c$

3. Not exact

4. $x^2 - 2xy^2 + 4y^3 = c$

5. $x + y = c$

6. Not exact

7. $y\sin^2 x - xy + 2e^{xy^2} = c$

8. $2y^2\cos x + 3xy^3 - x^2 = c$

9. Not exact

10. $x^3 + x^2y + 4xy^2 + 9y^2 = c$

11. Not exact

12. $\ln|xy| + x^2 + y^2 = c$

13. Not exact

14. $x^2 + y^2 = c$

15. $x^2y^2e^x + 2y + 3x^2 = c$

16. $x^3e^{x^2+y} - 4y^3 + 2x^2 = c$

17. $x^4e^{xy} + 3xy = c$

18. $x^3\cos xy + 4y^2 + 2x^2 = c$

19. $x\ln x + y\ln x - y = c$

20. $y = \frac{x+\sqrt{2x^2+3x-1}}{x^2}$

21. $y = \sin x - \sqrt{1 - \frac{\tan x}{2}}$

22. $y = \left(\frac{e^x-1}{e^x+1}\right)^{1/3}$

23. $y = 1 + 2\tan x$

24. $y = \frac{x^2-x+6}{(x+2)(x-3)}$

25. $y^2(1-x^2) - \cos^2 x = 3$

26. a. $M(x,y) = 2xy + f(x)$
b. $M(x,y) = 2(\sin x + x\cos x)(y\sin y + \cos y) + f(x)$
c. $M(x,y) = ye^x - e^y\cos x + f(x)$

27. a. $N(x,y) = \frac{x^4y}{2} + x^2 + 6xy + g(y)$
b. $N(x,y) = \frac{x}{y} - 2\cos x + g(y)$
c. $N(x,y) = x(\sin y + y\cos y) + g(y)$

29. $B = C$

30. $B = 2D, \quad E = 2C$

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6: Section 2.3 Answers

1. $y = e^{-ax}$
2. $x = \frac{1}{4}e^{3t} + ce^{-t}$
3. $y = \frac{1}{2}x^2 - \frac{1}{2} + ce^{-x^2}$
4. $y = \sin x + c\cos x$
5. $y = x^{-1}\ln x + cx^{-1}$
6. $(\sec\theta + \tan\theta)r = \theta - \cos\theta + c$
7. $y = \frac{2}{x} + \frac{c}{x}e^{x}$
8. $y = e^{-x^2}\left(\frac{x^2}{2} + c\right)$
9. $y = -\frac{e^{-x}+c}{1+x^2}$
10. $\frac{7\ln|x|}{x} + \frac{3}{2}x + \frac{c}{x}$
11. $x = 2y^6 + cy^4$
12. $y = (x-1)^{-4}(\ln|x-1| - \cos x + c)$
13. $y = e^{-x^2}\left(\frac{x^3}{4} + \frac{c}{x}\right)$
14. $y = \frac{2\ln|x|}{x^2} + \frac{1}{2} + \frac{c}{x^2}$
15. $p = (t+c)\cos t$
16. $y = \frac{c-\cos x}{(1+x)^2}$
17. $y = -\frac{1}{2}\frac{(x-2)^3}{(x-1)} + c\frac{(x-2)^5}{(x-1)}$
18. $x = \frac{1}{2y^2}e^{y} + \frac{c}{y^2}e^{-y}$
19. $y = (x+c)e^{-\sin^2 x}$
20. $y = \frac{e^x}{x^2} - \frac{e^x}{x^3} + \frac{c}{x^2}$
21. $y = \frac{e^{-(x-1)}}{x}$
22. $y = \frac{e}{x\ln x}$
23. $y = \frac{\pi}{x\sin x}$
24. $y = 2(1+x^2)$
25. $x = 3t^{-k}$
26. $y = 2(\cos kx)^{1/k}$
27. $y = \frac{e^{3x}-e^{-7x}}{10}$
28. $y = \frac{2x+1}{(1+x^2)^2}$
29. $y = \frac{1}{x^3}\ln\left(\frac{1+x^2}{2}\right)$
30. $y = \frac{1}{2}(\sin x + \csc x)$
31. $y = \frac{2\ln|x|}{x} + \frac{x}{2} - \frac{1}{2x}$
32. $y = (x-1)^{-3}[\ln(1-x) - \cos x]$

33. $x = 2y^2 + \frac{1}{y^2}$
34. $y = x^2(1 - \ln x)$
35. $y = \frac{1}{2} + \frac{5}{2}e^{-x^2}$
36. $y = \frac{\ln|x-1| + \tan x + 1}{(x-1)^3}$
37. $y = \frac{\ln|x| + x^2 + 1}{(x+2)^4}$
38. $x = (t^2 - 1)\left(\frac{1}{2}\ln|t^2 - 1| - 4\right)$
39. $y = -(x^2 - 5)(7 + \ln|x^2 - 5|)$
40. $(x+1)y = x\ln x - x + 21$
41. $G = \frac{r}{\lambda} + \left(G_0 - \frac{r}{\lambda}\right)e^{-\lambda t}$ $\lim_{t\to\infty} G(t) = \frac{r}{\lambda}$

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7: Section 2.4 Answers

1. $y = \frac{1}{1-ce^x}$
2. $y = x^{2/7}(c - \ln|x|)^{1/7}$
3. $y = e^{2/x}(c - 1/x)^2$
4. $y^{-3} = x + \frac{1}{3} + ce^{3x}$
5. $y = \pm\frac{\sqrt{2x+c}}{1+x^2}$
6. $e^{t/y} = ct$
7. $y = \pm(1 - x^2 + ce^{-x^2})^{-1/2}$
8. $y = \left[\frac{x}{3(1-x)+ce^{-x}}\right]^{1/3}$
9. $y = \frac{2\sqrt{2}}{\sqrt{1-4x}}$
10. $y = \left[-1 + \frac{3}{2}e^{-(x^2-1)/4}\right]^{-2}$
11. $y = \frac{1}{x(11-3x)^{1/3}}$
12. $y = (2e^x - 1)^2$
13. $y^{-3} = -\frac{9}{5}x^{-1} + \frac{49}{5}x^{-6}$
14. $y = (2e^{12x} - 1 - 12x)^{1/3}$
15. $y = \left[\frac{5x}{2(1+4x^5)}\right]^{1/2}$
16. $y = (4e^{x/2} - x - 2)^2$
17. $y = x(\ln|x| + c)$
18. $\ln\left|\frac{y}{x}\right| - \ln\left|\frac{y}{x} + 1\right| = \ln|x| + c; \quad y = -x$
19. $y = \pm x(4\ln|x| + c)^{1/4}$
20. $y = x\sin^{-1}(\ln|x| + c)$
21. $y = x\tan(\ln|x| + c)$
22. $4x = y(\ln|y| - c)^2$
23. $y = \pm x\sqrt{cx^2 - 1}$
24. $e^{(y/x)^2} = \ln|x| + c$
25. $y = -\frac{2x}{2\ln|x|+1}$
26. $y^3 + 3x^3\ln|x| = 8x^3$
27. $y = x(3\ln x + 27)^{1/3}$
28. $y = \frac{1}{x}\left(\frac{9-x^4}{2}\right)^{1/2}$
29. $\ln|x| = e^{y/x} - 1$
30. $y = -x$
31. $y = -\frac{x(4x-3)}{(2x-3)}$
32. $y = x\sqrt{4x^6 - 1}$

33. $\tan^{-1}\frac{y}{x}-\frac{1}{2}\ln(x^2+y^2)=c$
34. $(x+y)\ln|x|+y(1-\ln|y|)+cx=0$
35. $(y+x)^3=3x^3(\ln|x|+c)$
36. $(y+x)=c(y-x)^3;\quad y=x;\quad y=-x$
37. $y^2(y-3x)=c;\quad y\equiv 0;\quad y=3x$
38. $(x-y)^3(x+y)=cy^2x^4;\quad y=0;\quad y=x;\quad y=-x$
39. $\frac{y}{x}+\frac{y^3}{x^3}=\ln|x|+c$
40. $y=-x-1+\tan(x+c)$
41. $2y-2x+\sin 2(x+y)=c$
42. $4y-8x+12=(x+c)^2$
43. $(x+y)^2=2x+c$
44. $\tan(x+y)-\sec(x+y)=x+c$
45. $-e^{-y+x-5}=x+c$
46. $-\cot(x+y)+\csc(x+y)=x+\sqrt{2}-1$
47. $4\ln|15x+10y+6|=10x-10y+4\ln 19$

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8: Section 3.1 Answers

1. $y_1 = 1.450000000,\ y_2 = 2.085625000,\ y_3 = 3.079099746$

2. $y_1 = 1.200000000,\ y_2 = 1.440415946,\ y_3 = 1.729880994$

3. $y_1 = 1.900000000,\ y_2 = 1.781375000,\ y_3 = 1.646612970$

4. $y_1 = 2.962500000,\ y_2 = 2.922635828,\ y_3 = 2.880205639$

5. $y_1 = 2.513274123,\ y_2 = 1.814517822,\ y_3 = 1.216364496$

6.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
1.0	48.298147362	51.492825643	53.076673685	54.647937102

7.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
2.0	1.390242009	1.370996758	1.361921132	1.353193719

8.

x	$h = 0.05$	$h = 0.025$	$h = 0.0125$	Exact
1.50	7.886170437	8.852463793	9.548039907	10.500000000

9.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.1$	$h = 0.05$	$h = 0.025$
3.0	1.469458241	1.462514486	1.459217010	**0.3210**	**0.1537**	**0.0753**
	Approximate Solutions			Residuals		

10.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.1$	$h = 0.05$	$h = 0.025$
2.0	0.473456737	0.483227470	0.487986391	-0.3129	-0.1563	-0.0781
	Approximate Solutions			Residuals		

11.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	0.691066797	0.676269516	0.668327471	0.659957689

12.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
2.0	-0.772381768	-0.761510960	-0.756179726	-0.750912371

13.

Euler's Method

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
1.0	0.538871178	0.593002325	0.620131525	0.647231889

14.

Euler's Method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
3.0	12.804226135	13.912944662	14.559623055	15.282004826

15.

Euler's method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
2.0	0.867565004	0.885719263	0.895024772	0.904276722

16.

Euler's method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
3.0	0.922094379	0.945604800	0.956752868	0.967523153

17.

Euler's method				
x	$h = 0.0500$	$h = 0.0250$	$h = 0.0125$	"Exact"
1.50	0.319892131	0.330797109	0.337020123	0.343780513

18.

Euler's method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
2.0	0.754572560	0.743869878	0.738303914	0.732638628

19.

Euler's method				
x	$h = 0.0500$	$h = 0.0250$	$h = 0.0125$	"Exact"
1.50	2.175959970	2.210259554	2.227207500	2.244023982

20.

Euler's method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	0.032105117	0.043997045	0.050159310	0.056415515

21.

Euler's method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	28.987816656	38.426957516	45.367269688	54.729594761

22.

Euler's method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
3.0	1.361427907	1.361320824	1.361332589	1.361383810

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9: Section 3.2 Answers

1. $y_1 = 1.542812500,\ y_2 = 2.421622101,\ y_3 = 4.208020541$

2. $y_1 = 1.220207973,\ y_2 = 1.489578775.\ y_3 = 1.819337186$

3. $y_1 = 1.890687500,\ y_2 = 1.763784003,\ y_3 = 1.622698378$

4. $y_1 = 2.961317914,\ y_2 = 2.920132727,\ y_3 = 2.876213748$

5. $y_1 = 2.478055238,\ y_2 = 1.844042564,\ y_3 = 1.313882333$

6.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
1.0	56.134480009	55.003390448	54.734674836	54.647937102

7.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
2.0	1.353501839	1.353288493	1.353219485	1.353193719

8.

x	$h = 0.5$	$h = 0.025$	$h = 0.0125$	Exact
1.50	10.141969585	10.396770409	10.472502111	10.500000000

9.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.1$	$h = 0.05$	$h = 0.025$
3.0	1.455674816	1.455935127	1.456001289	-0.00818	-0.00207	-0.000518
	Approximate Solutions			Residuals		

10.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.1$	$h = 0.05$	$h = 0.025$
2.0	0.492862999	0.492709931	0.492674855	0.00335	0.000777	0.000187
	Approximate Solutions			Residuals		

11.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	0.660268159	0.660028505	0.659974464	0.659957689

12.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
2.0	-0.749751364	-0.750637632	-0.750845571	-0.750912371

13.

Improved Euler method

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
1.0	0.105660401	0.100924399	0.099893685	0.099574137

14.

Improved Euler method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
3.0	15.107600968	15.234856000	15.269755072	15.282004826

15.

Improved Euler method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
2.0	0.924335375	0.907866081	0.905058201	0.904276722

16.

Improved Euler method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
3.0	0.967473721	0.967510790	0.967520062	0.967523153

17.

Improved Euler method				
x	$h = 0.0500$	$h = 0.0250$	$h = 0.0125$	"Exact"
1.50	0.349176060	0.345171664	0.344131282	0.343780513

18.

Improved Euler method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
2.0	0.732679223	0.732721613	0.732667905	0.732638628

19.

Improved Euler method				
x	$h = 0.0500$	$h = 0.0250$	$h = 0.0125$	"Exact"
1.50	2.247880315	2.244975181	2.244260143	2.244023982

20.

Improved Euler method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	0.059071894	0.056999028	0.056553023	0.056415515

21.

Improved Euler method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	50.534556346	53.483947013	54.391544440	54.729594761

22.

Improved Euler method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
3.0	1.361395309	1.361379259	1.361382239	1.361383810

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10: Section 3.3 Answers

1. $y_1 = 1.550598190,\ y_2 = 2.469649729$

2. $y_1 = 1.221551366,\ y_2 = 1.492920208$

3. $y_1 = 1.890339767,\ y_2 = 1.763094323$

4. $y_1 = 2.961316248,\ y_2 = 2.920128958$

5. $y_1 = 2.475605264,\ y_2 = 1.825992433$

6.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
1.0	54.654509699	54.648344019	54.647962328	54.647937102

7.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
2.0	1.353191745	1.353193606	1.353193712	1.353193719

8.

x	$h = 0.05$	$h = 0.025$	$h = 0.0125$	Exact
1.50	10.498658198	10.499906266	10.499993820	10.500000000

9.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.1$	$h = 0.05$	$h = 0.025$
3.0	1.456023907	1.456023403	1.456023379	0.0000124	0.000000611	0.0000000333
	Approximate Solutions			Residuals		

10.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.1$	$h = 0.05$	$h = 0.025$
2.0	0.492663789	0.492663738	0.492663736	0.000000902	0.0000000508	0.00000000302
	Approximate Solutions			Residuals		

11.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	0.659957046	0.659957646	0.659957686	0.659957689

12.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
2.0	-0.750911103	-0.750912294	-0.750912367	-0.750912371

13.

Runge-Kutta method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
0.0	-3.000000000	-3.000000000	-3.000000000	-3.000000000
0.1	-2.162598011	-2.162526572	-2.162522707	-2.162522468
0.2	-1.577172164	-1.577070939	-1.577065457	-1.577065117
0.3	-1.163350794	-1.163242678	-1.163236817	-1.163236453
0.4	-0.868030294	-0.867927182	-0.867921588	-0.867921241
0.5	-0.655542739	-0.655450183	-0.655445157	-0.655444845
0.6	-0.501535352	-0.501455325	-0.501450977	-0.501450707
0.7	-0.389127673	-0.389060213	-0.389056546	-0.389056318
0.8	-0.306468018	-0.306412184	-0.306409148	-0.306408959
0.9	-0.245153433	-0.245107859	-0.245105379	-0.245105226
1.0	-0.199187198	-0.199150401	-0.199148398	-0.199148273

14.

Runge-Kutta method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
3.0	15.281660036	15.281981407	15.282003300	15.282004826

15.

Runge-Kutta method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
2.0	0.904678156	0.904295772	0.904277759	0.904276722

16.

Runge-Kutta method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
3.0	0.967523147	0.967523152	0.967523153	0.967523153

17.

Runge-Kutta method				
x	$h = 0.0500$	$h = 0.0250$	$h = 0.0125$	"Exact"
1.50	0.343839158	0.343784814	0.343780796	0.343780513

18.

Runge-Kutta method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
2.0	0.732633229	0.732638318	0.732638609	0.732638628

19.

Runge-Kutta method				
x	$h = 0.0500$	$h = 0.0250$	$h = 0.0125$	"Exact"
1.50	2.244025683	2.244024088	2.244023989	2.244023982

20.

Runge-Kutta method

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	0.056426886	0.056416137	0.056415552	0.056415515

21.

Runge-Kutta method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	54.695901186	54.727111858	54.729426250	54.729594761

22.

Runge-Kutta method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
3.0	1.361384082	1.361383812	1.361383809	1.361383810

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11: Section 4.1 Answers

1. $Q = 20e^{-(t \ln 2)/3200}$

2. $\frac{2 \ln 10}{\ln 2}$ days

3. $\tau = 10 \frac{\ln 2}{\ln 4/3}$ minutes

4. $\tau \frac{\ln(p_0/p_1)}{\ln 2}$

5. $\frac{t_p}{t_q} = \frac{\ln p}{\ln q}$

6. $k = \frac{1}{t_2 - t_1} \ln \frac{Q_1}{Q_2}$

7. 20 g

8. $\frac{50 \ln 2}{3}$ yrs

9. $\frac{25}{2} \ln 2$

10. a. $= 20 \ln 3$ yr
b. $Q_0 = 100000e^{-5}$

11. a. $Q(t) = 5000 - 4750e^{-t/10}$
b. 5000lbs

12. $\frac{1}{25}$ yrs

13. $V = V_0 e^{t \ln 10/2}$ 4 hours

14. $\frac{1500 \ln \frac{4}{3}}{\ln 2}$ yrs; $2^{-4/3} Q_0$

15. $W(t) = 20 - 19e^{-t/20}$; $\lim_{t \to \infty} W(t) = 20$ ounces

16. $S(t) = 10(1 + e^{-t/10})$; $\lim_{t \to \infty} S(t) = 10$ g

17. 10 gallons

18. $V(t) = 15000 + 10000e^{t/20}$

19. $W(t) = 4 \times 10^6 (t+1)^2$ dollars t years from now

20. $p = \frac{100}{25 - 24e^{-t/2}}$

21. $p(t) = \frac{1000}{1 + 999e^{-.9906t}}$ and $p(6) = 276$ people

22. 5769 fish

23. 1400 monkeys

24. 1185 otters

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12: Section 4.2 Answers

1. $\approx 15.15^\circ\mathrm{F}$

2. $T = -10 + 110e^{-t\ln\frac{11}{9}}$

3. $\approx 24.33^\circ\mathrm{F}$

4. a. $\approx 91.30^\circ\mathrm{F}$
b. 8.99 minutes after being placed outs
c. never

5. a. $12 : 11 : 32$
b. $12 : 47 : 33$

6. $(85/3)^\circ\mathrm{C}$

7. $32^\circ\mathrm{F}$

8. $Q(t) = 40(1 - e^{-3t/40})$

9. $Q(t) = 30 - 20e^{-t/10}$

10. $K(t) = .3 - .2e^{-t/20}$

11. $Q(50) = 47.5$ (pounds)

12. 50 gallons

13. $\min q_2 = q_1\sqrt{c}$

14. $Q = t + 300 - \frac{234\times 10^5}{(t+300)^2}, \quad 0 \le t \le 300$

15. a. $Q' + \frac{2}{25}Q = 6 - 2e^{-t/25}$
b. $Q = 75 - 50e^{-t/25} - 25e^{-2t/25}$
c. 75

16. a. $T = T_m + (T_0 - T_m)e^{-kt} + \frac{k(S_0 - T_m)}{(k - k_m)}(e^{-kmt} - e^{-kt})$
b. $T = T_m + k(S_0 - T_m)te^{-kt} + (T_0 - T_m)e^{-kt}$
c. $\lim_{t\to\infty} T(t) = \lim_{t\to\infty} S(t) = T_m$

17. a. $T' = -k(1 + \frac{a}{a_m})T + k(T_{m0} + \frac{a}{a_m}T_0)$
b. $T = \frac{aT_0 + a_mT_{m0}}{a + a_m} + \frac{a_m(T_0 - T_{m0})}{a + a_m}e^{-k(1 + a/a_m)t}, \quad T_m = \frac{aT_0 + a_mT_{m0}}{a + a_m} + \frac{a(T_{m0} - T_0)}{a + a_m}e^{-k(a + a/a_m)t}$
c. $\lim_{t\to\infty} T(t) = \lim_{t\to\infty} T_m(t) = \frac{aT_0 + a_mT_{m0}}{a + a_m}$

18. $V = \frac{a}{b}\frac{V_0}{V_0 - (V_0 - a/b)e^{-at}}; \quad \lim_{t\to\infty} V(t) = a/b$

19. $P(t) = \frac{10{,}000{,}000}{(1000 + t)^2}; \quad t = 2162$ days, or almost 6 years

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13: Section 4.3 Answers

1. $v = -\frac{384}{5}(1 - e^{-5t/12}); \quad -\frac{384}{5}\text{ft/s}$
2. $k = 12; \quad v = -16(1 - e^{-2t})$
3. $v = 25(1 - e^{-t}); \quad 25\text{ft/s}$
4. $v = 20 - 27e^{-t/40}$
5. ≈ 17.10 ft
6. $v = -\frac{40(13+3e^{-4t/5})}{13-3e^{-4t/5}}; \quad -40\text{ft/s}$
7. $v = -128(1 - e^{-t/4})$
8. $v = -\frac{64(1-e^{-t})}{1+e^{-t}}; \quad -64\text{ft/s}$
9. $v = \alpha\frac{v_0(1+e^{-\beta t})-\alpha(1-e^{-\beta t})}{\alpha(1+e^{-\beta t})-v_0(1-e^{-\beta t})}; \quad -\alpha$ where $\alpha = \sqrt{\frac{mg}{k}}$ and $\beta = 2\sqrt{\frac{kg}{m}}$
10. $\approx 6.93\text{miles/s}$
11. $\approx 1.47\text{miles/s}$

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14: Section 5.1 Answers

1. c. $y = -2e^{2x} + e^{5x}$

2. c. $y = e^{3x}(3\cos x - 5\sin x)$

3. c. $y = e^{x}(7 - 3x)$

4. a. $y = \frac{c_1}{x-1} + \frac{c_2}{x+1}$
b. $y = \frac{2}{x-1} - \frac{3}{x+1};\ (-1, 1)$

5. e^{x}

6. $e^{2x}\cos x$

7. $x^2 + 2x - 2$

8. $-\frac{5}{6}x^{-5/6}$

9. $-\frac{1}{x^2}$

10. $(x\ln|x|)^2$

11. $\frac{e^{2x}}{2\sqrt{x}}$

23. a. $(0, 2)$

b. $(2, \pi)$

24. a. $(\pi/2, 3\pi/2)$

b. $(1, \pi/2)$

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15: Section 5.2 Answers

1. $y = c_1 e^{-x} + c_2 x e^{x}$
2. $y = c_1 x + \frac{c_2}{x}$
3. $y = c_1 x + c_2 x \ln|x|$
4. $y = c_1 e^{2x} + c_2 e^{x}$
5. $y = c_1 e^{x} + c_2 x e^{x}$
6. $y = c_1 x^{1/2} e^{x} + c_2 x^{-1/2} e^{x}$
7. $y = c_1 e^{x} \cos x + c_2 e^{x} \sin x$
8. $y = c_1 e^{-x^2} + c_2 x e^{-x^2}$
9. $y = c_1 x^2 + \frac{c_2}{x^2}$
10. $y = c_1 x e^{-x} + c_2 x^2 e^{-x}$
11. $y = c_1 x e^{x} + \frac{e^{x}}{x}$
12. $y = c_1 e^{x} + c_2 x e^{-x}$
13. $y = c_1 x^2 + c_2 x^2 \ln|x|$
14. $y = c_1 e^{-x} + c_2 x^{1/2} e^{-x}$
15. $y = c_1 e^{x} + c_2 x^2 e^{x}$
16. $y = c_1 x^{1/2} + c_2 x^{1/2} e^{x}$
17. $y = c_1 x^2 + c_2 x^4$
18. $y = c_1 e^{x} + c_2 \frac{e^{x}}{x}$
19. $y = c_1 x^2 + c_2 x^3$
20. $y = c_1 \ln|x| + c_2 x \ln|x|$
21. $y = c_1 \sin\sqrt{x} + c_2 \cos\sqrt{x}$
22. $y = c_1 e^{x} + c_2 x^3 e^{x}$
23. $y = c_1 x^{a} + c_2 x^{a} \ln|x|$
24. $y = c_1 x \sin x + c_2 x \cos x$
25. $y = c_1 e^{2x} + c_2 x^2 e^{2x}$
26. $y = c_1 x^{1/2} + c_2 x^{1/2} \cos x$
27. $y = c_1 x^{1/2} e^{2x} + c_2 x^{1/2} e^{-2x}$
28. $y = c_1 \frac{1}{x} + c_2 e^{2x}$
29. $y = c_1 e^{x} + c_2 x^2$
30. $y = c_1 e^{2x} + c_2 x^2 e^{2x}$
31. $y = 7x^2 - 6x^2 \ln|x|$
32. $y = \frac{-1}{2}(x+1)e^{x} + \frac{3}{2}(x+1)e^{-x}$
33. $y = -x^{-2} + x$

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16: Section 5.3 Answers

1. $y = c_1 e^{-6x} + c_2 e^{x}$
2. $y = e^{2x}(c_1 \cos x + c_2 \sin x)$
3. $y = c_1 e^{-7x} + c_2 e^{-x}$
4. $y = e^{2x}(c_1 + c_2 x)$
5. $y = e^{-x}(c_1 \cos 3x + c_2 \sin 3x)$
6. $y = e^{-3x}(c_1 \cos x + c_2 \sin x)$
7. $y = e^{4x}(c_1 + c_2 x)$
8. $y = c_1 + c_2 e^{-x}$
9. $y = e^{x}(c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x)$
10. $y = e^{-3x}(c_1 \cos 2x + c_2 \sin 2x)$
11. $y = e^{-x/2}\left(c_1 \cos \frac{3x}{2} + c_2 \sin \frac{3x}{2}\right)$
12. $y = c_1 e^{-x/5} + c_2 e^{x/2}$
13. $y = e^{-7x}(2 \cos x - 3 \sin x)$
14. $y = 4e^{x/2} + 6e^{-x/3}$
15. $y = 3e^{x/3} - 4e^{-x/2}$
16. $y = \frac{e^{-x/2}}{3} + \frac{3e^{3x/2}}{4}$
17. $y = e^{3x/2}(3 - 2x)$
18. $y = 3e^{-4x} - 4e^{-3x}$
19. $y = 2xe^{3x}$
20. $y = e^{x/6}(3 + 2x)$
21. $y = e^{-2x}\left(3 \cos \sqrt{6}x + \frac{2\sqrt{6}}{3} \sin \sqrt{6}x\right)$

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17: Section 5.4 Answers

1. $y = -1 + 2x + 3x^2 + c_1 e^{-6x} + c_2 e^{x}$
2. $y = 1 + x + e^{2x}(c_1 \cos x + c_2 \sin x)$
3. $y = -x + x^3 + c_1 e^{-7x} + c_2 e^{-x}$
4. $y = 1 - x^2 + e^{2x}(c_1 + c_2 x)$
5. $y = 2x + x^3 + e^{-x}(2 \cos 3x + 3 \sin 3x)$
6. $y = 1 + 2x + e^{-3x}(\cos x - \sin x)$
7. $y = c_1 e^{-6x} + c_2 e^{x} + \frac{e^{3x}}{3}$
8. $y = c_1 e^{2x} \cos x + c_2 e^{2x} \sin x + e^{2x}$
9. $y = -2e^{-2x} - e^{-7x} + e^{-x}$
10. $y = e^{x} + e^{2x}(1 - 3x)$
11. $y = \frac{4}{45} e^{x/2} + e^{-x}(c_1 \cos 3x + c_2 \sin 3x)$
12. $y = e^{-3x}(1 + c_1 \cos x + c_2 \sin x)$
13. $y = \cos x - \sin x + e^{4x}(c_1 + c_2 x)$
14. $y = \cos 2x - 2 \sin 2x + c_1 + c_2 e^{-x}$
15. $y = \cos 3x + e^{x}(c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x)$
16. $y = \cos x + \sin x + e^{-3x}(c_1 \cos 2x + c_2 \sin 2x)$
17. $y = -2 \cos 2x + \sin 2x + c_1 e^{-4x} + c_2 e^{-3x};\ y = -2 \cos 2x + \sin 2x + 2e^{-4x} - 3e^{-3x}$
18. $y = \cos 3x - \sin 3x + e^{3x}(c_1 + c_2 x)\ y = \cos 3x - \sin 3x + e^{3x}(1 + 2x)$
19. $y_p = -1 + 2x + 3x^2 + \frac{e^{3x}}{3};\ y = -1 + 2x + 3x^2 + \frac{e^{3x}}{3} + c_1 e^{-6x} + c_2 e^{x}$
20. $y_p = 1 + x + e^{2x};\ y = 1 + x + e^{2x}(1 + c_1 \cos x + c_2 \sin x)$
21. $y_p = -x + x^3 - 2e^{-2x};\ y = -x + x^3 - 2e^{-2x} + c_1 e^{-7x} + c_2 e^{-x}$
22. $y_p = 1 - x^2 + e^{x};\ y = 1 - x^2 + e^{x} + e^{2x}(c_1 + c_2 x)$
23. $y_p = 2x + x^3 + \frac{4}{45} e^{x/2};\ y = 2x + x^3 + \frac{4}{45} e^{x/2} + e^{-x}(c_1 \cos 3x + c_2 \sin 3x)$
24. $y_p = 1 + 2x + e^{-3x};\ y = 1 + 2x + e^{-3x}(1 + c_1 \cos x + c_2 \sin x)$

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18: Section 5.5 Answers

1. $y = c_1 e^{-3x} + c_2 e^{3x} - 6$
2. $y = c_1 + c_2 e^{-x} + 3x$
3. $y = c_1 e^{-2x} + c_2 x e^{-2x} + \frac{1}{2}x + 1$
4. $y = c_1 e^{-3x} + c_2 e^{4x} + \frac{1}{7} x e^{4x}$
5. $y = c_1 e^{-x} + c_2 e^{3x} - e^{x} + 3$
6. $y = c_1 \cos 5x + c_2 \sin 5x + \frac{1}{4} \sin x$
7. $y = c_1 e^{-3x} + c_2 x e^{-3x} - \frac{1}{49} x e^{4x} + \frac{2}{343} e^{4x}$
8. $y = c_1 e^{-x} + c_2 e^{x} + \frac{1}{6} x^3 e^{x} - \frac{1}{4} x^2 e^{x} + \frac{1}{4} x e^{x} - 5$
9. $y = c_1 e^{x} \cos 2x + c_2 e^{x} \sin 2x + \frac{1}{3} e^{x} \sin x$
10. $y = c_1 \cos 5x + c_2 \sin 5x - 2x \cos 5x$
11. $y = c_1 e^{-x/2} \cos \frac{\sqrt{3}}{2} x + c_2 e^{-x/2} \sin \frac{\sqrt{3}}{2} x + \sin x + 2 \cos x - x \cos x$
12. $y = c_1 e^{x} + c_2 e^{2x} + e^{3x} \left(-\frac{1}{4} + \frac{x}{2} \right)$
13. $y = c_1 e^{x} + c_2 e^{5x} + e^{-3x} \left(1 - \frac{x}{4} \right)$
14. $y = c_1 e^{3x} + c_2 e^{-x} + e^{x} \left(2 - \frac{3x}{4} \right)$
15. $y = c_1 e^{-x} + c_2 x e^{-x} + e^{2x} (1 - 3x + x^2)$
16. $y = c_1 \cos 2x + c_2 \sin 2x + e^{-x} (1 + x^2)$
17. $y = c_1 e^{2x} + c_2 e^{-x} + e^{x} (-2 + x + 2x^2)$
18. $y = c_1 e^{5x} + c_2 e^{-x} + x e^{-x} \left(\frac{1}{6} + \frac{x}{2} \right)$
19. $y = c_1 e^{x} + c_2 e^{2x} + x e^{x} (1 + 2x)$
20. $y = c_1 e^{-4x} + c_2 e^{3x} + x e^{3x} \left(-1 + \frac{x}{2} \right)$
21. $y = c_1 e^{-x/2} + c_2 e^{2x} + x e^{2x} (-2 + x)$
22. $y = c_1 e^{-x} + c_2 x e^{-x} + x^2 e^{-x} \left(1 + \frac{x}{2} \right)$
23. $y = c_1 e^{x} + c_2 x e^{x} + x^2 e^{x} \left(\frac{1}{2} - x \right)$
24. $y = c_1 e^{2x} + c_2 x e^{2x} + \frac{x^2 e^{2x}}{2} (1 - x + x^2)$
25. $y = c_1 e^{-x/3} + c_2 x e^{-x/3} + \frac{x^2 e^{-x/3}}{27} (3 - 2x + x^2)$
26. $y = \frac{e^{3x}}{4} (-1 + 2x) + c_1 e^{x} + c_2 e^{2x}$
27. $y = e^{x} (1 - 2x) + c_1 e^{2x} + c_2 e^{4x}$
28. $y = \frac{e^{2x}}{5} (1 - x) + e^{-3x} (c_1 + c_2 x)$
29. $y = x e^{x} (1 - 2x) + c_1 e^{x} + c_2 e^{-3x}$
30. $y = e^{x} \left[x^2 (1 - 2x) + c_1 + c_2 x \right]$
31. $y = c_1 e^{-x} + c_2 e^{-2x} + \cos x + 2 \sin x$
32. $y = c_1 e^{(-3+\sqrt{5})x/2} + c_2 e^{(-3-\sqrt{5})x/2} + \cos x + (2 - 2x) \sin x$
33. $y = c_1 e^{-x} + c_2 x e^{-x} + e^{x} (-2 \cos x + 3 \sin x)$
34. $y = c_1 e^{(-3+\sqrt{17})x/2} + c_2 e^{(-3-\sqrt{17})x/2} + \frac{e^{2x}}{2} (\cos 2x - \sin 2x)$
35. $y = c_1 e^{x/2} \cos \frac{\sqrt{3}}{2} x + c_2 e^{x/2} \sin \frac{\sqrt{3}}{2} x - e^{x} (x \cos x - \sin x)$

36. $y = c_1 e^{(-3+\sqrt{17})x/2} + c_2 e^{(-3-\sqrt{17})x/2} + e^{-2x}(1-2x)(\cos 3x - \sin 3x)$

37. $y = c_1 \cos 2x + c_2 \sin 2x + x(\cos 2x - 3\sin 2x)$

38. $y = c_1 e^x + c_2 x e^x + -x[(2-x)\cos x + (3-2x)\sin x]$

39. $y = c_1 \cos \frac{x}{2} + c_2 \sin \frac{x}{2} + x\left[x\cos\left(\frac{x}{2}\right) - 3\sin\left(\frac{x}{2}\right)\right]$

40. $y = c_1 e^{-x}\cos x + c_2 e^{-x}\sin x + x e^{-x}(3\cos x + 4\sin x)$

41. $y = c_1 e^x \cos 2x + c_2 e^x \sin 2x + x e^x[(-1+x)\cos 2x + (1+x)\sin 2x]$

42. $y = c_1 e^{-x} + c_2 x e^{-x} - (14 - 10x)\cos x - (2 + 8x - 4x^2)\sin x$

43. $y = c_1 e^{-x} + c_2 e^{-2x} + (1 + 2x + x^2)\cos x + (1 + 3x^2)\sin x$

44. $y = c_1 e^{-x} + c_2 e^{-2x} + \frac{x^2}{2}(\cos 2x - \sin 2x)$

45. $y = c_1 e^{2x} + c_2 e^{3x} e^x (x^2 \cos x + 2\sin x)$

46. $y = c_1 e^x + c_2 x e^x + e^x(1 - x^2)(\cos x + \sin x)$

47. $y = c_1 e^x \cos x + c_2 e^x \sin x + e^x(x^2 - x^3)(\cos x + \sin x)$

48. $y = c_1 e^{-x} + c_2 x e^{-x} + e^{-x}[(1+2x)\cos x - (1-3x)\sin x]$

49. $y = c_1 \cos 3x + c_2 \sin 3x + x(2\cos 3x - \sin 3x)$

50. $y = c_1 e^{-x} + c_2 e^{-2x} - x^3 \cos x + (x + 2x^2)\sin x$

51. $y = c_1 e^{-x} + c_2 e^{-3x} - e^{-x}[(x + x^2)\cos x - (1 + 2x)\sin x]$

52. $y = \frac{5}{8}e^{-8x} + \frac{5}{8}e^{8x} - \frac{1}{4}$

53. $y = -\frac{41}{125} + \frac{41}{125}e^{5x} - \frac{1}{10}x^2 + \frac{9}{25}x$

54. $y = -\pi\cos x - \frac{11}{3}\sin x - \frac{8}{3}\cos 2x + 2x\cos x$

55. $y = 2e^{2x}\cos 2x - \frac{3}{64}e^{2x}\sin 2x + \frac{1}{8}x^3 + \frac{3}{16}x^2 + \frac{3}{32}x$

56. $y = -e^{2x}(1+x) + 2e^{-x} - e^{5x}$

57. $y = xe^{2x} + 3e^x - e^{-4x}$

58. $y = e^{-x}(2 + x - 2x^2) - e^{-3x}$

59. $y = e^{-2x}(3 - x) - 2e^{5x}$

60. $y = e^x(2\cos x + 3\sin x) + 3e^x - e^{6x}$

61. $y = e^x[(1+2x)\cos x + (1-3x)\sin x]$

62. $y = e^x(\cos x - 2\sin x) + e^{-3x}(\cos x + \sin x)$

63. $y = e^{3x}[(2+2x)\cos x - (1+3x)\sin x]$

64. $y = e^{3x}[(2+3x)\cos x + (4-x)\sin x] + 3e^x - 5e^{2x}$

65. $y = xe^{-2x}\cos x + 3\cos 2x$

66. $y = -\frac{3}{8}\cos 2x + \frac{1}{4}\sin 2x + e^{-x} - \frac{13}{8}e^{-2x} - \frac{3}{4}xe^{-2x}$

67. a. $y = c_1 e^{-x/2}\cos\frac{\sqrt{3}}{2}x + c_2 e^{-x/2}\cos\frac{\sqrt{3}}{2}x - \frac{e^x}{3}(1-x)$
b. $y = c_1 e^{-x/2}\cos\frac{\sqrt{3}}{2}x + c_2 e^{-x/2}\cos\frac{\sqrt{3}}{2}x + e^{-x}(3+2x)$
c. $y = c_1 e^{-x/2}\cos\frac{\sqrt{3}}{2}x + c_2 e^{-x/2}\cos\frac{\sqrt{3}}{2}x - \frac{e^x}{3}(1-x) + e^{-x}(3+2x)$

68. a. $y = c_1 e^{4x} + c_2 e^{3x} + e^x(3+7x)$
b. $y = c_1 e^{4x} + c_2 e^{3x} + xe^{3x}$
c. $y = c_1 e^{4x} + c_2 e^{3x} + e^x(3+7x) + xe^{3x}$

69. a. $y = c_1 e^{4x} + c_2 x e^{4x} + x^3 e^{4x}$
b. $y = c_1 e^{4x} + c_2 x e^{4x} + 1 + 2x + x^2$
c. $y = c_1 e^{4x} + c_2 x e^{4x} + x^3 e^{4x} + 1 + 2x + x^2$

70. a. $y = c_1 e^{x} + c_2 e^{2x} + x e^{2x}(1 - 2x)$
b. $y = c_1 e^{x} + c_2 e^{2x} + x e^{x}$
c. $y = c_1 e^{x} + c_2 e^{2x} + x e^{2x}(1 - 2x) + x e^{x}$

71. a. $y = c_1 e^{x} \cos x + c_2 e^{x} \sin x + e^{x}(1 + x)$
b. $y = c_1 e^{x} \cos x + c_2 e^{x} \sin x + x^2 e^{-x}$
c. $y = c_1 e^{x} \cos x + c_2 e^{x} \sin x + e^{x}(1 + x) + x^2 e^{-x}$

72. a. $y = c_1 \cos x + c_2 \sin x + x^2 e^{-x}$
b. $y = c_1 \cos x + c_2 \sin x + e^{3x}(1 - x^2)$
c. $y = c_1 \cos x + c_2 \sin x + x^2 e^{-x} + e^{3x}(1 - x^2)$

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19: Section 5.6 Answers

1. $y = c_1 \cos 3x + c_2 \sin 3x + \frac{-\cos 3x \ln|\sec 3x + \tan 3x|}{9}$
2. $y = c_1 \cos 2x + c_2 \sin 2x - \frac{\sin 2x \ln|\cos 2x|}{4} + \frac{x \cos 2x}{2}$
3. $y = c_1 e^x + c_2 e^{2x} + 4e^x(1+e^x)\ln(1+e^{-x})$
4. $y = c_1 e^x \cos x + c_2 e^x \sin x + 3e^x + \cos x \ln|\cos x| + x \sin x$
5. $y = c_1 e^x + c_2 x e^x + \frac{8}{5} x^{7/2} e^x$
6. $y = c_1 e^x + c_2 e^{-x} + e^x \ln(1 - e^{-2x}) - e^{-x} \ln(e^{2x} - 1)$
7. $y = c_1 x + c_2 \frac{1}{x} + \frac{2(x^2-3)}{3}$
8. $y = c_1 e^x + c_2 \frac{e^x}{x} + \frac{e^{2x}}{x}$
9. $y = c_1 x^{1/2} e^x + c_2 x^{-1/2} e^x + x^{1/2} e^x \ln x$
10. $y = c_1 e^{-x^2} + c_2 x e^{-x^2} + e^{-x(x+2)}$
11. $y = c_1 x^2 + c_2 x^3 - 4x^{5/2}$
12. $y = c_1 x + c_2 x^3 - 2x^2 \sin x - 2x \cos x$
13. $y = c_1 e^{-x} + c_2 x e^x - \frac{x e^{-x}(x+1)}{2}$
14. $y = c_1 \cos \sqrt{x} + c_2 \sin \sqrt{x} - \frac{\sqrt{x} \cos \sqrt{x}}{2}$
15. $y = c_1 e^x + c_2 x^3 e^x + \frac{3x^4 e^x}{2}$
16. $y = c_1 x^a + c_2 x^a \ln x + x^{a+1}$
17. $y = c_1 x \cos x + c_2 x \sin x + \frac{x^2 \sin x}{2}$
18. $y = c_1 e^{x^2} + c_2 e^{-x^2} - 2x^2$
19. $y = c_1 e^{-x} + c_2 e^{-x} \cos x - e^{-x} \sin x$
20. $y = c_1 \sqrt{x} e^{2x} + c_2 \sqrt{x} e^{-2x} - \frac{\sqrt{x}}{2}$
21. $y = c_1 \sqrt{x} \sin x + c_2 \sqrt{x} \cos x + \frac{x^{3/2}}{4}$
22. $y = c_1 x e^x + c_2 x e^{-x} - 3x^2$
23. $y = c_1 x e^x + c_2 x^2 e^x + \frac{x^3 e^x}{2}$
24. $y = c_1 \frac{1}{x} + + c_2 x^3 - \frac{4x^{3/2}}{15}$
25. $y = c_1 x^2 + c_2 x^2 e^x + x^3 e^x$
26. $y = c_1 x^2 e^x + c_2 x^3 e^x + x e^x$
27. $y = c_1 x^2 \cos x + c_2 x^2 \sin x + x^2$
28. $y = c_1 x + c_2 e^x + x e^x (x-2)$
29. $y = c_1 \sqrt{x} + c_2 \sqrt{x} e^x + \sqrt{x} e^x (x-1)/4$
30. $y = \frac{e^{2x}(3x^2 - 2x + 6)}{6} + \frac{x e^{-x}}{3}$
31. $y = (x-1)^2 \ln(1-x) + 2x^2 - 5x + 3$
32. $y = (x^2 - 1)e^x - 5(x-1)$
33. $y = \frac{x(x^2+6)}{3(x^2-1)}$
34. $y = -\frac{x^2}{2} + x + \frac{1}{2x^2}$

35. $y = \frac{x^2(4x+9)}{6(x+1)}$

36. $y = c_1 e^x \cos 3x + c_2 e^x \sin 3x + \frac{2}{17}\cos x + \frac{9}{17}\sin x - \frac{1}{27}e^x \cos 3x \ln|\sec 3x + \tan 3x|$

37. $y = c_1 \cos x + c_2 \sin x - \frac{1}{2}x\cos x + x \sin x + \cos x \ln|\cos x|$

38. $y = c_1 e^{-x} + c_2 e^{-2x} + 3x^2 - 9x + \frac{21}{2} - e^{-2x}\sin e^x + (e^{-x} + e^{-2x})\ln(1+e^x)$

39. $y = c_1 e^x + c_2 x e^x + x^3 + 6x^2 + 22x + 32 - \frac{1}{2}e^x \ln(1+x^2) + x e^x \tan^{-1} x$

40. $y = c_1 e^x + c_2 x e^x + \frac{1}{2}x^2 e^x + \frac{1}{2}x^2 e^x \tan^{-1} x - \frac{1}{2}e^x \tan^{-1} x - \frac{1}{2}x e^x \ln(1+x^2)$

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20: Section 5.7 Answers

1. $y = x_1 x^{-4} + c_2 x^{-2}$
2. $y = c_1 x + c_2 x^7$
3. $y = x(c_1 + c_2 \ln x)$
4. $y = x^{-2}(c_1 + c_2 \ln x)$
5. $y = c_1 \cos(\ln x) + c_2 \sin(\ln x)$
6. $y = x^2[c_1 \cos(3 \ln x) + c_2 \sin(3 \ln x)]$
7. $y = c_1 x + \frac{c_2}{x^3}$
8. $y = c_1 x^{2/3} + c_2 x^{3/4}$
9. $y = x^{-1/2}(c_1 + c_2 \ln x)$
10. $y = c_1 x + c_2 x^{1/3}$
11. $y = c_1 x^2 + c_2 x^{1/2}$
12. $y = \frac{1}{x}[c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x]$
13. $y = x^{-1/3}(c_1 + c_2 \ln x)$
14. $y = x[c_1 \cos(3 \ln x) + c_2 \sin(3 \ln x)]$
15. $y = c_1 x^3 + \frac{x_2}{x^2}$
16. $y = \frac{c_1}{x} + c_2 x^{1/2}$
17. $y = x^2(c_1 + c_2 \ln x)$
18. $y = \frac{1}{x^2}\left[c_1 \cos\left(\frac{1}{\sqrt{2}} \ln x\right) + c_2 \sin\left(\frac{1}{\sqrt{2}} \ln x\right)\right]$
19. $y = c_1 x^{-1} + c_2 x^2$
20. $y = c_1 + c_2 \ln x$
21. $y = c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x)$
22. $y = c_1 x^{2-\sqrt{6}} + c_2 x^{2+\sqrt{6}}$
23. $y = c_1 \cos(\frac{1}{5} \ln x) + c_2 \sin(\frac{1}{5} \ln x)$
24. $y = c_1 x^{-2} + c_2 x^{-2} \ln x$
25. $y = c_1 x^{-1/2} \cos(\frac{\sqrt{3}}{6} \ln x) + c_2 x^{-1/2} \sin(\frac{\sqrt{3}}{6} \ln x)$
26. $y = c_1 x^{-3} + c_2 x^1$
27. $y = c_1 x^{(-3+\sqrt{5})/2} + c_2 x^{(-3-\sqrt{5})/2}$
28. $y = c_1 \cos(\frac{1}{2} \ln x) + c_2 \sin(\frac{1}{2} \ln x)$
29. $y = c_1 + c_2 x^5 + \frac{1}{5} x^5 \ln x$
30. $y = c_1 x + c_2 x \ln x + x \ln^2 x$
31. $y = c_1 x^{-1} + c_2 x^3 - \frac{2}{3} x^2$
32. $y = c_1 x^{-1} + c_2 x^3 + \frac{1}{2} x^3 \ln x$
33. $y = c_1 x^2 + c_2 x^3 + x^4 - x^2 \ln x$
34. $y = c_1 x + c_2 x^3 + 2x^2 e^x - 2x e^x$
35. $y = c_1 x + c_2 x \ln x + 2 + \ln x$

36. $y = c_1 x^{-1/2} + c_2 x^{-1} + \frac{1}{15} x^2 + \frac{1}{6} x$
37. $y = c_1 x^{-2} + c_2 x^{-2} \ln x + \frac{2}{25} x^3$
38. $y = c_1 x^{-1} + c_2 x^{-8} + \frac{1}{30} x^2$
39. $y = c_1 x^2 + c_2 x^3 + \frac{1}{3} \ln x + \frac{5}{18}$
40. $y = c_1 x^2 \cos(3 \ln x) + c_2 x^2 \sin(3 \ln x) + \frac{3}{10} x + \frac{4}{13}$
41. $y = 2 - 2x^{-2}$
42. $y = 16x^2 - 2x^4$
43. $y = \cos(\ln x) + 2 \sin(\ln x)$
44. $y = 5x^2 - 7x^2 \ln x$
45. $y = x^{-1} \ln x$
46. $y = \frac{3}{4} - \ln x + \frac{1}{4} x^2$
47. $y = \frac{1}{16} x^2 - \frac{1}{2} x^4 + x^6$
48. $y = ex - ex^3 + 2x^2 e^x - 2xe^x$
49. $y = x \ln x + 2 + \ln x$
51. $y = 2\sqrt{-x} - 5\sqrt{-x} \ln(-x)$
52. $y = 6x^2 + 2x^3$

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21: Section 6.1 Answers

1. $y = 3\cos 4\sqrt{6}t - \frac{1}{2\sqrt{6}}\sin 4\sqrt{6}t$ ft

2. $y = -\frac{1}{4}\cos 8\sqrt{5}t - \frac{1}{4\sqrt{5}}\sin 8\sqrt{5}t$ ft

3. $y = -1.5\cos 14\sqrt{10}t$ cm

4. $y = \frac{1}{4}\cos 8t - \frac{1}{16}\sin 8t$ ft; $R = \frac{\sqrt{17}}{16}$ ft; $\omega_0 = 8$ rad/s; $T = \pi/4$ s; $\phi = \pi - \tan^{-1} 4$ rad

5. $y = 10\cos 14t + \frac{25}{14}\sin 14t$ cm; $R = \frac{5}{14}\sqrt{809}$ cm; $\omega_0 = 14$ rad/s; $T = \pi/7$ s; $\phi \approx 1.39$ rad $\approx 79.88^\circ$

6. $y = -\frac{1}{4}\cos\sqrt{70}t + \frac{2}{\sqrt{70}}\sin\sqrt{70}t$ m; $R = \frac{1}{4}\sqrt{\frac{67}{35}}$ m; $\omega_0 = \sqrt{70}$ rad/s; $T = 2\pi/\sqrt{70}$ s; $\phi \approx 5.48$ rad $\approx 313.72^\circ$

7. $y = \frac{2}{3}\cos 16t - \frac{1}{4}\sin 16t$ ft

8. $y = \frac{1}{2}\cos 8t - \frac{3}{8}\sin 8t$ ft

9. .72 m

10. $y = \frac{1}{3}\sin t + \frac{1}{2}\cos 2t + \frac{5}{6}\sin 2t$ ft

11. $y = \frac{16}{5}\left(4\sin\frac{t}{4} - \sin t\right)$

12. $y = -\frac{1}{16}\sin 8t + \frac{1}{3}\cos 4\sqrt{2}t - \frac{1}{8\sqrt{2}}\sin 4\sqrt{2}t$

13. $y = -t\cos 8t - \frac{1}{6}\cos 8t + \frac{1}{8}\sin 8t$ ft

14. $T = 4\sqrt{2}$ s

15. $\omega = 8$ rad/s $y = -\frac{t}{16}(-\cos 8t + 2\sin 8t) + \frac{1}{128}\sin 8t$ ft

16. $\omega = 4\sqrt{6}$ rad/s; $y = -\frac{t}{\sqrt{6}}\left[\frac{8}{3}\cos 4\sqrt{6}t + 4\sin 4\sqrt{6}t\right] + \frac{1}{9}\sin 4\sqrt{6}t$ ft

17. $y = \frac{t}{2}\cos 2t - \frac{t}{4}\sin 2t + 3\cos 2t + 2\sin 2t$ m

18. $y = y_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t$; $R = \frac{1}{\omega_0}\sqrt{(\omega_0 y_0)^2 + (v_0)^2}$; $\cos\phi = \frac{y_0\omega_0}{\sqrt{(\omega_0 y_0)^2 + (v_0)^2}}$; $\sin\phi = \frac{v_0}{\sqrt{(\omega_0 y_0)^2 + (v_0)^2}}$

19. The object with the longer period weighs four times as much as the other.

20. $T_2 = \sqrt{2}T_1$, where T_1 is the period of the smaller object

21. $k_1 = 9k_2$, where k_1 is the spring constant of the system with the shorter period.

22. $y = \frac{1}{4}\cos 4\sqrt{6}t$

23. a. $y = -\frac{1}{2}\cos 8t$
b. $t = \frac{(2n+1)\pi}{16}$ for $n = 0, 1, 2, \ldots$
c. $t = \frac{7\pi}{16}$

24. a. $y = -\frac{1}{2}\cos 2t - \frac{3}{4}\sin 2t$
b. $y = \frac{\sqrt{13}}{4}\sin(2t + \pi + \tan^{-1}\frac{2}{3})$
c. $t = \frac{(n-1)\pi - \tan^{-1}\frac{2}{3}}{2}$ for $n = 2, 3, 4, \ldots$
d. $t = \frac{5\pi - \tan^{-1}\frac{2}{3}}{2}$

25. a. $y = \frac{2}{3}\cos 10t - \frac{1}{2}\sin 10t$
b. $y = \frac{5}{6}\sin(10t + \pi - \tan^{-1}\frac{4}{3})$
c. The amplitude is $\frac{5}{6}$ and the period is $\frac{\pi}{5}$ s
d. 15
e. $t = \frac{(n-1)\pi + \tan^{-1}\frac{4}{3}}{10}$ s for $n = 1, 2, 3, \ldots$
f. $t = \frac{2\pi + \tan^{-1}\frac{4}{3}}{10}$ s

g. $\frac{25}{3}$ ft/s

h. $t = \frac{1}{10}(\frac{\pi}{6} + 2n\pi + \tan^{-1}\frac{4}{3})$ and $t = \frac{1}{10}(\frac{5\pi}{6} + 2n\pi + \tan^{-1}\frac{4}{3})$ for $n = 0, 1, 2, \ldots$

26. $y = -\frac{1}{2}\cos 4t + \frac{9}{4}\sin 4t + \frac{1}{2}e^{-2t}\cos 4t - 2e^{-2t}\sin 4t$

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22: Section 6.2 Answers

1. $y = \frac{e^{-2t}}{2}(3\cos 2t - \sin 2t)$ ft; $\sqrt{\frac{5}{2}}e^{-2t}$ ft
2. $y = -e^{-t}\left(3\cos 3t + \frac{1}{3}\sin 3t\right)$ ft $\frac{\sqrt{82}}{3}e^{-t}$ ft
3. $y = e^{-16t}\left(\frac{1}{4} + 10t\right)$ ft
4. $y = -\frac{e^{-3t}}{4}(5\cos t + 63\sin t)$ ft
5. $0 \le c < 8$ lb-sec/ft
6. $y = \frac{1}{2}e^{-3t}\left(\cos\sqrt{91}t + \frac{11}{\sqrt{91}}\sin\sqrt{91}t\right)$ ft
7. $y = -\frac{e^{-4t}}{3}(2 + 8t)$ ft
8. $y = e^{-10t}\left(9\cos 4\sqrt{6}t + \frac{45}{2\sqrt{6}}\sin 4\sqrt{6}t\right)$ cm
9. $y = e^{-3t/2}\left(\frac{3}{2}\cos\frac{\sqrt{41}}{2}t + \frac{9}{2\sqrt{41}}\sin\frac{\sqrt{41}}{2}t\right)$ ft
10. $y = e^{-\frac{3}{2}t}\left(\frac{1}{2}\cos\frac{\sqrt{119}}{2}t - \frac{9}{2\sqrt{119}}\sin\frac{\sqrt{119}}{2}t\right)$ ft
11. $y = e^{-8t}\left(\frac{1}{4}\cos 8\sqrt{2}t - \frac{1}{4\sqrt{2}}\sin 8\sqrt{2}t\right)$ ft
12. $y = e^{-t}\left(-\frac{1}{3}\cos 3\sqrt{11}t + \frac{14}{9\sqrt{11}}\sin 3\sqrt{11}t\right)$ ft
13. $y_p = \frac{22}{61}\cos 2t + \frac{2}{61}\sin 2t$ ft
14. $y = -\frac{2}{3}(e^{-8t} - 2e^{-4t})$
15. $y = e^{-2t}\left(\frac{1}{10}\cos 4t - \frac{1}{5}\sin 4t\right)$ m
16. $y = e^{-3t}(10\cos t - 70\sin t)$ cm
17. $y_p = -\frac{2}{15}\cos 3t + \frac{1}{15}\sin 3t$ ft
18. $y_p = \frac{11}{100}\cos 4t + \frac{27}{100}\sin 4t$ cm
19. $y_p = \frac{42}{73}\cos t + \frac{39}{73}\sin t$ ft
20. $y = -\frac{1}{2}\cos 2t + \frac{1}{4}\sin 2t$ m
21. $y_p = \frac{1}{c\omega_0}(-\beta\cos\omega_0 t + \alpha\sin\omega_0 t)$
24. a. $t = 1/4$ s
    b. $t = 1/2$ s and $y = -e^{-2}$ ft
25. $y = -\frac{4}{3}e^{-2t} + \frac{1}{3}e^{-8t}$
26. $y = \frac{2}{3}e^{-2t} - \frac{5}{3}e^{-8t}$
27. a. $y = e^{-2t}\cos t + 2e^{-2t}\sin t$
    b. $y = \sqrt{5}e^{-2t}\sin(t + \tan^{-1}\frac{1}{2})$
    c. $t = n\pi - \tan^{-1}\frac{1}{2}$
    d. $t = 2\pi - \tan^{-1}\frac{1}{2}$ s
28. $y = -\frac{16}{3}e^{-t/2}\cos\frac{\sqrt{47}}{2}t - \frac{76}{3\sqrt{47}}e^{-t/2}\sin\frac{\sqrt{47}}{2}t + \frac{10}{3}\cos 3t + \frac{10}{3}\sin 3t$
29. $y = \frac{1}{4}e^{-4t} + te^{-4t} - \frac{1}{4}\cos 4t$
30. $y = \frac{1}{2}\cos 4t - \frac{9}{4}\sin 4t - \frac{1}{2}e^{-2t}\cos 4t + 2e^{-2t}\sin 4t$

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23: Section 6.3 Answers

1. $I = e^{-15t}\left(2\cos 5\sqrt{31}t - \frac{6}{\sqrt{31}}\sin 5\sqrt{31}t\right)$
2. $I = e^{-20t}(2\cos 40t - 101\sin 40t)$
3. $I = -\frac{200}{3}e^{-10t}\sin 30t$
4. $I = -10e^{-30t}(\cos 40t + 18\sin 40t)$
5. $I = -e^{-40t}(2\cos 30t - 86\sin 30t)$
6. $I_p = -\frac{1}{3}(\cos 10t + 2\sin 10t)$
7. $I_p = \frac{20}{37}(\cos 25t - 6\sin 25t)$
8. $I_p = \frac{3}{13}(8\cos 50t - \sin 50t)$
9. $I_p = \frac{20}{123}(17\sin 100t - 11\cos 100t)$
10. $I_p = -\frac{45}{52}(\cos 30t + 8\sin 30t)$

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24: Section 7.1 Answers

7. $y = \frac{22}{3}x^2 + 3x^3 + \frac{25}{3x}$

8. $y = 2e^x + 3e^{-x} - e^{2x} + e^{-3x}$

9. b. $y_1 = -\frac{1}{2}x^3 + x^2 + \frac{1}{2x}, \quad y_2 = \frac{1}{3}x^2 - \frac{1}{3x}, \quad y_3 = \frac{1}{4}x^3 - \frac{1}{3}x^2 + \frac{1}{12x}$

17. 2

18. $-e^{3x}$

19. 4

20. $4/x^2$

21. 1

22. $2x$

23. $2/x^2$

24. $e^x(x^2 - 2x + 2)$

25. $-240/x^5$

26. $6e^{2x}(2x - 1)$

27. $-128x$

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25: Section 7.2 Answers

1. $y = e^x(c_1 + c_2x + c_3x^2)$
2. $y = c_1e^x + c_2e^{-x} + c_3\cos 3x + c_4\sin 3x$
3. $y = c_1e^x + c_2\cos 4x + c_3\sin 4x$
4. $y = c_1e^x + c_2e^{-x} + c_3e^{-3x/2}$
5. $y = c_1e^{-x} + e^{-2x}(c_1\cos x + c_2\sin x)$
6. $y = c_1e^x + e^{x/2}(c_2 + c_3x)$
7. $y = e^{-x/3}(c_1 + c_2x + c_3x^2)$
8. $y = c_1 + c_2x + c_3\cos x + c_4\sin x$
9. $y = c_1e^{2x} + c_2e^{-2x} + c_3\cos 2x + c_4\sin 2x$
10. $y = (c_1 + c_2x)\cos\sqrt{6}x + (c_3 + c_4x)\sin\sqrt{6}x$
11. $y = e^{3x/2}(c_1 + c_2x) + e^{-3x/2}(c_3 + c_4x)$
12. $y = c_1e^{-x/2} + c_2e^{-x/3} + c_3\cos x + c_4\sin x$
13. $y = c_1e^x + c_2e^{-2x} + c_3e^{-x/2} + c_4e^{-3x/2}$
14. $y = e^x(c_1 + c_2x + c_3\cos x + c_4\sin x)$
15. $y = \cos 2x - 2\sin 2x + e^{2x}$
16. $y = 2e^x + 3e^{-x} - 5e^{-3x}$
17. $y = 2e^x + 3xe^x - 4e^{-x}$
18. $y = 2e^{-x}\cos x - 3e^{-x}\sin x + 4e^{2x}$
19. $y = \frac{9}{5}e^{-5x/3} + e^x(1 + 2x)$
20. $y = e^{2x}(1 - 3x + 2x^2)$
21. $y = e^{3x}(2 - x) + 4e^{-x/2}$
22. $y = e^{x/2}(1 - 2x) + 3e^{-x/2}$
23. $y = \frac{1}{8}(5e^{2x} + e^{-2x} + 10\cos 2x + 4\sin 2x)$
24. $y = -4e^x + e^{2x} - e^{4x} + 2e^{-x}$
25. $y = 2e^x - e^{-x}$
26. $y = e^{2x} + e^{-2x} + e^{-x}(3\cos x + \sin x)$
27. $y = 2e^{-x/2} + \cos 2x - \sin 2x$
28. $\{e^x, xe^x, e^{2x}\}$ 1
29. $\{\cos 2x, \sin 2x, e^{3x}\}$: 26
30. $\{e^{-x}\cos x, e^{-x}\sin x, e^x\}$: 5
31. $\{1, x, x^2, e^x\}$ $2e^x$
32. $\{e^x, e^{-x}, \cos x, \sin x\}$ 8
33. $\{\cos x, \sin x, e^x\cos x, e^x\sin x\}$: 5
34. $\{e^{-3x}\cos 2x, e^{-3x}\sin 2x, e^{2x}, xe^{2x}, 1, x, x^2\}$
35. $\{e^x, xe^x, e^{x/2}, xe^{x/2}, x^2e^{x/2}, \cos x, \sin x\}$
36. $\{\cos 3x, x\cos 3x, x^2\cos 3x, \sin 3x, x\sin 3x, x^2\sin 3x, 1, x\}$

37. $\{e^{2x}, xe^{2x}, x^2e^{2x}, e^{-x}, xe^{-x}, 1\}$
38. $\{\cos x, \sin x, \cos 3x, x\cos 3x, \sin 3x, x\sin 3x, e^{2x}\}$
39. $\{e^{2x}, xe^{2x}, e^{-2x}, xe^{-2x}, \cos 2x, x\cos 2x, \sin 2x, x\sin 2x\}$
40. $\{e^{-x/2}\cos\sqrt{2}x, xe^{-x/2}\cos\sqrt{2}x, x^2e^{-x/2}\cos\sqrt{2}x, e^{-x/2}\sin\sqrt{2}x, xe^{-x/2}\sin\sqrt{2}x, x^2e^{-x/2}\sin\sqrt{2}x\}$
41. $\{1, x, x^2, e^{2x}, xe^{2x}, \cos 2x, x\cos 2x, \sin 2x, x\sin 2x\}$
42. $\{\cos(x/2), x\cos(x/2), \sin(x/2), x\sin(x/2), \cos 2x/3x\cos(2x/3), x^2\cos(2x/3), \sin(2x/3), x\sin(2x/3), x^2\sin(2x/3)\}$
43. $y = c_1x^{-2} + c_2\cos(2\ln x) + c_3\sin(2\ln x)$
44. $y = c_1x^3 + c_2\cos(\sqrt{2}\ln x) + c_3\sin(\sqrt{2}\ln x)$
45. $y = c_1x + c_2x\ln x + c_3x\ln^2 x$
46. $y = c_1 + c_2x + c_3x^2 + c_4x^{-3}$
47. $y = c_1\cos(\ln x) + c_2\sin(\ln x) + c_3\ln x\cos(\ln x) + c_4\ln x\sin(\ln x)$
48. $y = c_1x + c_2x^2 + c_3x^3$

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26: Section 7.3 Answers

1. $y = c_1 + c_2 x + c_3 e^{-x} + \frac{2}{3}x^4 - \frac{8}{3}x^3 + 8x^2$
2. $y = c_1 + c_2 x + c_3 e^{-8x} + \frac{11}{256}x^2 + \frac{7}{32}x^3 - \frac{1}{16}x^4$
3. $y = c_1 e^x + c_2 \cos x + c_3 \sin x - 7 + \frac{1}{4}e^{-x} - \frac{1}{2}xe^x + \frac{1}{4}x^2 e^x$
4. $y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x - 13 + x + \frac{1}{6}x^3 e^x$
5. $y = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{x/2} + 1 + \frac{1}{9}xe^{2x} - \frac{1}{20}e^{-2x}$
6. $y = c_1 + c_2 x + c_3 e^x + c_4 x e^x + \frac{1}{2}x^2 + \frac{1}{2}x^2 e^x$
7. $y = c_1 + c_2 x + c_3 e^{2x} + c_4 e^{-2x} - \frac{5}{16}x^2 - \frac{5}{48}x^4 - \frac{1}{16}xe^{2x}$
8. $y = c_1 + c_2 x + c_3 e^{6x} - \frac{1}{4}x^2 - \frac{6}{37}\cos x + \frac{1}{37}\sin x$
9. $y = c_1 e^{2x} + c_2 x e^{2x} + c_3 e^{-2x} + \frac{1}{4}x^3 e^{2x} - \frac{3}{16}x^2 e^{2x}$
10. $y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x - x - 3 - \frac{2}{3}x^3 e^x$
11. $y = c_1 e^x + c_2 e^{2x} + c_3 e^{-2x} + \frac{5}{4} + \frac{1}{3}xe^x + \frac{1}{4}xe^{2x}$
12. $y = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x + x^2 - 2x - 3$
13. $y = c_1 + c_2 x + c_3 e^x + c_4 e^{-x} - \frac{2}{3}x^3 - \frac{1}{2}x^2 e^{-x} - \frac{5}{2}xe^{-x}$
14. $y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} + e^{-x}(2 + x - x^2)$
15. $y = c_1 e^x + c_2 e^{3x} + c_3 e^{-2x} - \frac{e^{-3x}}{4}(3 - x + x^2)$
16. $y = c_1 e^{-2x} + c_2 e^{-x/2} + c_3 e^{x/2} + e^x(1 + x - x^2)$
17. $y = c_1 e^x + c_2 e^{-x} + c_3 e^{-3x} + e^{-2x}(1 - 5x + x^2)$
18. $y = c_1 e^x + c_2 e^{-x} + c_3 e^{-3x} - \frac{xe^x}{2}(1 - x + x^2 - x^3)$
19. $y = c_1 e^x + c_2 e^{-x}\cos x + c_3 e^{-x}\cos x + x^2 e^x(1 + x)$
20. $y = c_1 e^{-2x} + c_2 e^{-x/2} + c_3 e^{x/2} + \frac{xe^{-2x}}{2}(2 + x)$
21. $y = c_1 e^x + c_2 x e^x + c_3 e^{-x} + \frac{x^2 e^x}{2}(2 + x)$
22. $y = c_1 e^{2x} + c_2 x e^{2x} + c_3 e^{-x/2} + \frac{x^2 e^{2x}}{2}(1 + 2x)$
23. $y = c_1 e^{3x} + c_2 x e^{3x} + c_3 e^{-x} + x^2 e^{3x}(2 + x - x^2)$
24. $y = c_1 e^{4x} + c_2 x e^{4x} + c_3 e^{-x} + x^2 e^{4x}(2 + x)$
25. $y = c_1 e^{x/2} + c_2 x e^{x/2} + c_3 x^2 e^{x/2} + \frac{x^3 e^{x/2}}{48}(1 + x)$
26. $y = c_1 e^x + c_2 x e^x + c_3 e^{-2x} + c_4 e^{-3x} + e^{-x}(1 - 2x + x^2)$
27. $y = c_1 e^{-x} + c_2 x e^{-x} + c_3 e^x + c_4 e^{-2x} + e^{2x}(1 - x)$
28. $y = c_1 + c_2 e^{-4x} + c_3 e^{-2x}\cos 2x + c_4 e^{-2x}\sin 2x + e^{-2x}(1 + x + x^2 - x^3)$
29. $y = c_1 e^{-x/2} + c_2 x e^{-x/2} + c_3 e^{-x} + c_4 e^{2x} + \frac{e^x}{3}(1 - x)$
30. $y = c_1 e^{-x/2} + c_2 e^{-x} + c_3 e^{-2x} + c_4 e^x + x e^x(2 + x)$
31. $y = c_1 e^{-x} + c_2 e^x + c_3 e^{-2x} + c_4 e^{2x} + \frac{xe^x}{6}(1 + x^2)$
32. $y = c_1 e^{-x} + c_2 x e^{-x} + c_3 e^{2x} + c_4 x e^{2x} + \frac{x^2 e^{2x}}{6}(1 + x + x^2)$
33. $y = c_1 + c_2 e^{-x} + c_3 e^{2x} + c_4 x e^{2x} + \frac{x^2 e^{2x}}{6}(3 + x + x^2)$
34. $y = c_1 e^{-x} + c_2 e^x + c_3 x e^x + c_4 x^2 e^x + \frac{x^3 e^x}{48}(2 + x)$

35. $y = c_1 e^{-x/2} + c_2 e^{x} + c_3 x e^{x} + c_4 x^2 e^{x} + \frac{x^3 e^{x}}{6}(1+x)$

36. $y = c_1 + c_2 e^{-x} + c_3 x e^{-x} + c_4 x^2 e^{-x} - \frac{x^3 e^{-x}}{6}(1-x+x^2)$

37. $y = c_1 e^{x} + c_2 e^{2x} + c_3 x e^{2x} + c_4 x^2 e^{2x} + \frac{x^3 e^{2x}}{12}(2+x-x^2)$

38. $y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{-2x} + e^{-x}\left[(1+x)\cos x + (2-x)\sin x\right]$

39. $y = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{-2x} + e^{-x}\left[(1-x)\cos 2x + (1+x)\sin 2x\right]$

40. $y = c_1 e^{x} + c_2 \cos(\sqrt{2}x) + c_3 \sin(\sqrt{2}x) + e^{2x}\left[(1+x-x^2)\cos x + (1+2x)\sin x\right]$

41. $y = c_1 e^{2x} + c_2 \cos x + c_3 \sin x + \frac{e^{x}}{2}\left[(1+x)\cos 2x + (1-x+x^2)\sin 2x\right]$

42. $y = c_1 e^{-3x} + c_2 \cos 2x + c_3 \sin 2x + \frac{x}{13}(8\cos 2x + 14\sin 2x)$

43. $y = c_1 e^{-x} + c_2 e^{x}\cos x + c_3 e^{x}\sin x + x e^{x}\left[(1+x)\cos x + (3+x)\sin x\right]$

44. $y = c_1 e^{3x} + c_2 e^{2x}\cos 2x + c_3 e^{2x}\sin 2x + \frac{x e^{2x}}{2}\left[(3-x)\cos 2x + \sin 2x\right]$

45. $y = c_1 + c_2 e^{3x}\cos 3x + c_3 e^{3x}\sin 3x - \frac{x e^{3x}}{12}(x\cos 3x + \sin 3x)$

46. $y = c_1 e^{-2x} + c_2 e^{2x} + c_3 e^{-x}\cos x + c_4 e^{-x}\sin x - \frac{e^{x}}{10}(\cos x + 7\sin x)$

47. $y = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{x}\cos x + c_4 e^{x}\sin x + \frac{e^{x}}{12}(\cos 2x - \sin 2x)$

48. $y = c_1 e^{x} + c_2 e^{3x} + c_3 e^{2x}\cos x + c_4 e^{2x}\sin x + x e^{2x}\cos 2x$

49. $y = c_1 e^{-x} + c_2 x e^{-x} + c_3 e^{-2x} + c_4 x e^{-2x} - \frac{e^{-x}}{2}\left[(1+x)\cos x + (2-x)\sin x\right]$

50. $y = c_1 e^{-2x} + c_2 e^{x} + c_3 e^{-x}\cos x + c_4 e^{-x}\sin x + \frac{x e^{-x}}{10}(\cos x + 2\sin x)$

51. $y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{x}\cos 2x + c_4 e^{x}\sin 2x + \frac{x e^{x}}{4-}(3\cos 2x - \sin 2x)$

52. $y = c_1 e^{-x} + c_2 e^{-3x} + c_3 e^{-2x}\cos 3x + c_4 e^{-2x}\sin 3x + \frac{x e^{-2x}}{8}\left[(1-x)\cos 3x + (1+x)\sin 3x\right]$

53. $y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{x}\cos 2x + c_4 e^{x}\sin 2x - \frac{x e^{x}}{4}(1+x)\sin 2x$

54. $y = c_1 e^{-x}\cos x + c_2 e^{-x}\sin x + c_3 x e^{-x}\cos x + c_4 x e^{-x}\sin x + \frac{x^2 e^{-x}}{4}(\cos x - 2\sin x)$

55. $y = c_1 e^{2x}\cos 2x + c_2 e^{2x}\sin 2x + c_3 x e^{2x}\cos 2x + c_4 x e^{2x}\sin 2x - \frac{x^2 e^{2x}}{32}(\cos 2x - \sin 2x)$

56. $y = c_1 e^{2x}\cos x + c_2 e^{2x}\sin x + c_3 x e^{2x}\cos x + c_4 x e^{2x}\sin x + \frac{x^2 e^{2x}}{8}(1+x)\sin x$

57. $y = c_1 e^{x} + c_2 x e^{x} + c_3 e^{2x} + 2x^2 e^{x} + x e^{2x} - \cos x$

58. $y = c_1 e^{x} + c_2 \cos x + c_3 \sin x + e^{2x} + x e^{x} + 2x\cos x$

59. $y = c_1 + c_2 e^{x} + c_3 e^{-x} + 2x + x^2 + 2x e^{x} - 3x e^{-x} + 4e^{3x}$

60. $y = c_1 e^{2x} + c_2 e^{x}\cos x + c_3 e^{x}\sin x + x e^{x}(\cos 2x - 2\sin 2x) + 2x e^{2x} + 1$

61. $y = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x} + x^2 e^{-2x}(1+2x) - \cos 2x + \sin 2x$

62. $y = c_1 e^{-x} + c_2 x e^{-x} + c_3 e^{x} + 2x^2(1+x)e^{-x} + x\cos x - 2\sin x$

63. $y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{-2x} + c_4 e^{2x} + 2x e^{x} + x e^{2x} + \cos x$

64. $y = c_1 e^{x}\cos x + c_2 e^{x}\sin x + c_3 e^{x}\cos 2x + c_4 e^{x}\sin 2x + \frac{x e^{x}}{6}(\cos x + \sin 2x)$

65. $y = c_1 e^{x} + c_2 x e^{x} + c_3 e^{-2x} + c_4 x e^{-2x} + \frac{x^2}{54}\left[(2+2x)e^{x} + 3e^{-2x}\right]$

66. $y = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x} + c_4 e^{2x} + x^3(1+x)e^{-x} + x e^{-2x}$

67. $y = c_1 e^{x} + c_2 x e^{x} + c_3 e^{x}\cos x + c_4 e^{x}\sin x + x e^{x}(2x^2 + \cos x + \sin x)$

68. $y = e^{2x}(1+x) + c_1 e^{-x} + e^{x}(c_2 + c_3 x)$

69. $y = e^{3x}\left(1 - x - \frac{x^2}{2}\right) + c_1 e^{x} + e^{-x}(c_2\cos x + c_3\sin x)$

70. $y = x e^{2x}(1+x)^2 + c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$

71. $y = x^2 e^{-x}(1-x)^2 + c_1 + e^{-x}(c_2 + c_3 x)$
72. $y = \frac{x^3 e^x}{24}(4+x) + e^x(c_1 + c_2 x + c_3 x^2)$
73. $y = \frac{x^2 e^{-x}}{16}(1 + 2x - x^2) + e^x(c_1 + c_2 x) + e^{-x}(c_3 + c_4 x)$
74. $y = e^{-2x}\left[\left(1 + \frac{x}{2}\right)\cos x + \left(\frac{3}{2} - 2x\right)\sin x\right] + c_1 e^x + c_2 e^{-x} + c_3 e^{-2x}$
75. $y = -x e^x \sin 2x + c_1 + c_2 e^x + e^x(c_3 \cos x + c_4 \sin x)$
76. $y = -\frac{x^2 e^x}{16}(1+x)\cos 2x + e^x\left[(c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x\right]$
77. $y = 8 - 6e^x + 3xe^x + 5x - \frac{1}{2}x^2 e^x + \frac{1}{6}x^3 e^x$
78. $y = 2 + x - 2e^x - \frac{1}{6}x^3 - \frac{1}{24}x^4 + xe^x$
79. $y = 11 - 11e^x + 9xe^x + 2x - 12x^2 e^x + \frac{1}{2}e^{5x}$
80. $y = -\frac{23}{12}e^{-2x} - \frac{59}{24}e^x \cos(\sqrt{3}x) + \frac{17}{72}\sqrt{3}e^x \sin(\sqrt{3}x) + \frac{1}{4}x - \frac{5}{8} + \frac{2}{3}xe^{-2x}$
81. $y = (x^2 + 2)e^x - e^{-2x} + e^{3x}$
82. $y = e^{-x}(1 + x + x^2) + (1-x)e^x$
83. $y = \left(\frac{x^2}{12} + 16\right)xe^{-x/2} - e^x$
84. $y = (2-x)(x^2+1)e^{-x} + \cos x - \sin x$
85. $y = (2-x)\cos x - (1-7x)\sin x + e^{-2x}$
86. $y = 2 + e^x\left[(1+x)\cos x - \sin x - 1\right]$

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27: Section 7.4 Answers

1. $y = c_1 x + c_2 x^2 + c_3 x^3 + x \ln |x|$
2. $y = c_1 \sqrt{x} + c_2 \frac{1}{\sqrt{x}} + c_3 x^2 + 2x^2 \ln |x|$
3. $y = c_1 x + c_2 x^2 + c_3 \frac{1}{x} + \frac{x^2 \ln |x|}{3}$
4. $y = c_1 + c_2 x + c_3 x^2 + c_4 \frac{1}{x} + \frac{1}{4} x^3 \ln |x| - \frac{25}{48} x^3$
5. $y = c_1 \sqrt{x} + c_2 \frac{1}{\sqrt{x}} + c_3 x^{3/2} + c_4 x^{-3/2} + \frac{x^{5/2}}{4}$
6. $y = c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \frac{x^4 \ln |x|}{6}$
7. $y = c_1 x + c_2 x^2 + c_3 \frac{1}{x} + c_4 \frac{1}{x^2} + x^2 \ln |x|$
8. $y = -x(\ln x)^2 + 3x + x^3 - 2x \ln x$
9. $y = \frac{x^3}{2} (\ln |x|)^2 + x^2 - x^3 + 2x^3 \ln |x|$
10. $y = \frac{3}{2} x^4 (\ln x)^2 + 3x - x^4 + 2x^4 \ln x$
11. $y = \left(\frac{x^2}{3} - \frac{x}{2} \right) \ln |x| + 4x - 2x^2$
12. $y = 3x^2 \ln x - 7x^2$
13. $y = x \ln x + x - \sqrt{x} + \frac{1}{x} + \frac{1}{\sqrt{x}}$
14. $y = x^3 \ln |x| + x - 2x^3 + \frac{1}{x} - \frac{1}{x^2}$
15. $y = c_1 x + c_2 x^2 + c_3 x e^x + 2x^3$
16. $y = c_1 e^{-x^2} + c_2 x e^{-x^2} + c_3 x^2 e^{-x^2} + \frac{8}{105} x^{7/2} e^{-x^2}$
17. $y = c_1 + c_2 \frac{e^x}{x} + c_3 \frac{e^{-x}}{x} - \frac{2(x^2+2)}{x}$
18. $y = c_1 x e^x + c_2 x^2 e^x + c_3 x^3 e^x + - \frac{x e^{-3x}}{64}$
19. $y = c_1 e^x + c_2 e^{-x} + c_3 \frac{1}{x} - \frac{2x^2}{3}$
20. $y = c_1 e^x + c_2 \frac{e^x}{x} + c_3 \frac{e^{-x}}{x} - \frac{e^{-x}(x+1)}{x}$
21. $y = c_1 x + c_2 \frac{1}{x-1} + c_3 \frac{1}{x+1} + x^2 + 1$
22. $y = c_1 x + c_2 \frac{1}{x} + c_3 \frac{e^x}{x} + \frac{2x^2+6}{3}$
23. $y = c_1 x + c_2 e^x + c_3 e^{-x} - x^2 - 2$
24. $y = c_1 + c_2 \frac{1}{x} + c_3 e^x + c_4 e^{-x} + \frac{x(12-x^2)}{6}$
25. $y = c_1 x e^x + c_2 x^2 e^x + c_3 x e^{-x} + c_4 x^2 e^{-x} + \frac{x^3 e^x}{2}$
26. $y = c_1 e^x + c_2 e^{-x} + c_3 \frac{e^x}{x} + c_4 \frac{e^{-x}}{x} + \frac{x e^x}{2}$
27. $y = c_1 e^x + c_2 e^{2x} + c_3 \frac{e^x}{x} + c_4 \frac{e^{2x}}{x} + \frac{3x e^x}{2}$
28. $y = c_1 x + c_2 x^2 + c_3 x e^x + c_4 x e^{-x} - x^3$
29. $y = -\frac{1}{2}(3x+1) x e^x - 3e^x - e^{2x} + 4x e^{-x}$
30. $y = -\frac{x^4+12}{6} + 3x - x^2 + 2e^x$
31. $y = -\frac{x e^x (1+3x)}{2} + \frac{x+1}{2} - \frac{e^x}{4} + \frac{e^{3x}}{2}$
32. $y = -8x + 2x^2 - 2x^3 + 2e^x - e^{-x}$

33. $y = \frac{3(4x^2+9)}{2} + \frac{x}{2} - \frac{e^x}{2} + \frac{e^{-x}}{2} + \frac{e^{2x}}{4}$

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28: Section 8.2 Answers

1. $R = 2;\ I = (-1, 3)$
2. $R = 1/2;\ I = (3/2, 5/2)$
3. $R = 0$
4. $R = 16;\ I = (-14, 18)$
5. $R = \infty;\ I = (-\infty, \infty)$
6. $R = 4/3;\ I = (-25/3, -17/3)$
7. $R = 1;\ I = (0, 2)$
8. $R = \sqrt{2};\ I = (-2 - \sqrt{2}, -2 + \sqrt{2})$
9. $R = \infty;\ I = (-\infty, \infty)$
10. $R = 0$
11. $R = \sqrt{3};\ I = (-\sqrt{3}, \sqrt{3})$
12. $R = 1;\ I = (0, 2)$
13. $R = 3;\ I = (0, 6)$
14. $R = 1;\ I = (-1, 1)$
15. $R = 1/\sqrt{3};\ I = (3 - 1/\sqrt{3}, 3 + 1/\sqrt{3})$
16. $R = \infty;\ I = (-\infty, \infty)$
17. $R = 0$
18. $R = 2;\ I = (-1, 3)$
19. $(4a_2 + 3a_0) + \sum_{n=1}^{\infty}[2(n+2)(n+1)a_{n+2} + (n+1)na_{n+1} + (n+3)a_n]x^n$
20. $(2a_2 - 2a_0) + (6a_3 - 2a_1)x + \sum_{n=2}^{\infty}[(n+2)(n+1)a_{n+2} + [3n(n-1) - 2]a_n + 3(n-1)a_{n-1}]x^n$
21. $(2a_2 - a_1 + 4a_0) + (6a_3 + 4a_2 + 4a_1)x + \sum_{n=2}^{\infty}[(n+2)(n+1)a_{n+2} + 2(n+1)a_{n+1} + (2n^2 - 5n + 4)a_n]x^n$
22. $(2a_2 + 2a_1 + 3a_0) + (6a_3 + 4a_2 + 2a_1)x + \sum_{n=2}^{\infty}[(n+2)(n+1)a_{n+2} + 2(n+1)a_{n+1} + (n^2 - 2n + 3)a_n]x^n$
23. $(2a_2 + 4a_0) + (6a_3 + 2a_1)x + \sum_{n=2}^{\infty}[(n+2)(n+1)a_{n+2} + (3n^2 - 5n + 4)a_n]x^n$
24. $(-2a_2 + 2a_1 + a_0) + \sum_{n=1}^{\infty}[-(n+2)(n+1)a_{n+2} + (n+1)(n+2)a_{n+1} + (2n+1)a_n + a_{n-1}](x+1)^n$
25. $(8a_2 + 4a_1 - 6a_0) + (24a_3 + 16a_2 - 4a_1 - 3a_0)(x-2) +$
$\sum_{n=2}^{\infty}[4(n+2)(n+1)a_{n+2} + 4(n+1)^2a_{n+1} + (n^2 + n - 6)a_n - 3a_{n-1}](x-2)^n$

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29: Section 8.3 Answers

1. $y = a_0 \sum_{m=0}^{\infty} (-1)^m (2m+1) x^{2m} + a_1 \sum_{m=0}^{\infty} (-1)^m (m+1) x^{2m+1}$
2. $y = a_0 \sum_{m=0}^{\infty} (-1)^{m+1} \frac{x^{2m}}{2m-1} + a_1 x$
3. $y = a_0 (1 - 10x^2 + 5x^4) + a_1 \left(x - 2x^3 + \frac{1}{5} x^5\right)$
4. $y = a_0 \sum_{m=0}^{\infty} (m+1)(2m+1) x^{2m} + \frac{a_1}{3} \sum_{m=0}^{\infty} (m+1)(2m+3) x^{2m+1}$
5. $y = a_0 \left(1 - 14x^2 + \frac{35}{3} x^4\right) + a_1 \left(x - 3x^3 + \frac{3}{5} x^5 + \frac{1}{35} x^7\right)$
6. $y = a_0 (1 + \frac{1}{6} x^3 + \frac{1}{180} x^6 + \cdots) + a_1 (x + \frac{1}{12} x^4 + \frac{1}{504} x^7 + \cdots)$
7. $y = a_0 (1 - \frac{1}{2} x^2 - \frac{1}{8} x^4 + \cdots) + a_1 (x + \frac{1}{6} x^3 + \frac{1}{24} x^5 + \cdots)$
8. $y = a_0 (1 - \frac{1}{6} x^3 + \frac{1}{45} x^6 + \cdots) + a_1 (x - \frac{1}{6} x^4 + \frac{5}{252} x^7 + \cdots)$
9. $y = a_0 + a_1 \sum_{m=1}^{\infty} \frac{x^m}{m}$
10. $y = a_0 (1 + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \cdots) + a_1 (x + \frac{1}{2} x^2 + \frac{1}{2} x^3 + \cdots)$
11. $y = a_0 (1 + \frac{1}{4} x^2 - \frac{7}{96} x^4 + \cdots) + a_1 (x - \frac{1}{6} x^3 + \frac{7}{120} x^5 + \cdots)$
12. $y = 2 - x - x^2 + \frac{1}{3} x^3 + \frac{5}{12} x^4 - \frac{1}{6} x^5 - \frac{17}{72} x^6 + \frac{13}{126} x^7 + \ldots$
13. $y = 1 - x + 3x^2 - \frac{5}{2} x^3 + 5x^4 - \frac{21}{8} x^5 + 3x^6 - \frac{11}{16} x^7 + \ldots$
14. $y = 2 - x - 2x^2 + \frac{1}{3} x^3 + 3x^4 - \frac{5}{6} x^5 - \frac{49}{5} x^6 + \frac{45}{14} x^7 + \ldots$
15. $y = 2 - 3x - 2x^2 + \frac{7}{2} x^3 - \frac{55}{12} x^4 + \frac{59}{8} x^5 - \frac{83}{6} x^6 + \frac{9547}{336} x^7 + \ldots$
16. $y = -1 + 2x - 4x^3 + 4x^4 + 4x^5 - 12x^6 + 4x^7 + \ldots$
17. $y = 1 + x^2 - \frac{2}{3} x^3 + \frac{11}{6} x^4 - \frac{9}{5} x^5 + \frac{329}{90} x^6 - \frac{1301}{315} x^7 + \ldots$
18. $y = x - x^2 - \frac{7}{2} x^3 + \frac{15}{2} x^4 + \frac{45}{8} x^5 - \frac{261}{8} x^6 + \frac{207}{16} x^7 + \ldots$
19. $y = 4 + 3x - \frac{15}{4} x^2 + \frac{1}{4} x^3 + \frac{11}{16} x^4 - \frac{5}{16} x^5 + \frac{1}{20} x^6 + \frac{1}{120} x^7 + \ldots$
20. $y = 7 + 3x - \frac{16}{3} x^2 + \frac{13}{3} x^3 - \frac{23}{9} x^4 + \frac{10}{9} x^5 - \frac{7}{27} x^6 - \frac{1}{9} x^7 + \ldots$
21. $y = 2 + 5x - \frac{7}{4} x^2 - \frac{3}{16} x^3 + \frac{37}{192} x^4 - \frac{7}{192} x^5 - \frac{1}{1920} x^6 + \frac{19}{11520} x^7 + \ldots$
22. $y = 8x - 2e^x$
23. $3 - 12x^2 + 4x^4$
24. $y = a_0 \sum_{m=0}^{\infty} \frac{(x-3)^{2m}}{(2m)!} + a_1 \sum_{m=0}^{\infty} \frac{(x-3)^{2m+1}}{(2m+1)!}$
25. $y = a_0 \left(1 - 6(x-2)^2 + \frac{4}{3}(x-2)^4 + \frac{8}{135}(x-2)^6\right) + a_1 \left((x-2) - \frac{10}{9}(x-2)^3\right)$
26. $y = -1 + 2x + \frac{3}{8} x^2 - \frac{1}{3} x^3 - \frac{3}{128} x^4 - \frac{1}{1024} x^6 + \ldots$
27. $y = -2 + 3(x-3) + 3(x-3)^2 - 2(x-3)^3 - \frac{5}{4}(x-3)^4 + \frac{3}{5}(x-3)^5 + \frac{7}{24}(x-3)^6 - \frac{4}{35}(x-3)^7 + \ldots$
28. $y = -1 + (x-1) + 3(x-1)^2 - \frac{5}{2}(x-1)^3 - \frac{27}{4}(x-1)^4 + \frac{21}{4}(x-1)^5 + \frac{27}{2}(x-1)^6 - \frac{81}{8}(x-1)^7 + \ldots$
29. $y = 4 - 6(x-3) - 2(x-3)^2 + (x-3)^3 + \frac{3}{2}(x-3)^4 - \frac{5}{4}(x-3)^5 - \frac{49}{20}(x-3)^6 + \frac{135}{56}(x-3)^7 + \ldots$
30. $y = 3 - 4(x-4) + 15(x-4)^2 - 4(x-4)^3 + \frac{15}{4}(x-4)^4 - \frac{1}{5}(x-4)^5 + \ldots$
31. $y = 3 - 3(x+1) - 30(x+1)^2 + \frac{20}{3}(x+1)^3 + 20(x+1)^4 - \frac{4}{3}(x+1)^5 - \frac{8}{9}(x+1)^6 + \ldots$
32. $y = 1 - (x-1) + \frac{4}{3}(x-1)^3 - \frac{4}{3}(x-1)^4 - \frac{4}{5}(x-1)^5 + \frac{136}{45}(x-1)^6 - \frac{104}{63}(x-1)^7 + \ldots$
33. $y = 1 - (x+1) + 4(x+1)^2 - \frac{13}{3}(x+1)^3 + \frac{77}{6}(x+1)^4 - \frac{278}{15}(x+1)^5 + \frac{1942}{45}(x+1)^6 - \frac{23332}{315}(x+1)^7 + \ldots$
34. $y = 2 - (x-1) - \frac{1}{2}(x-1)^2 + \frac{5}{3}(x-1)^3 - \frac{19}{12}(x-1)^4 + \frac{7}{30}(x-1)^5 + \frac{59}{45}(x-1)^6 - \frac{1091}{630}(x-1)^7 + \ldots$

35. $y=-2+3(x+1)-\frac{1}{2}(x+1)^2-\frac{2}{3}(x+1)^3+\frac{5}{8}(x+1)^4-\frac{11}{30}(x+1)^5+\frac{29}{144}(x+1)^6-\frac{101}{840}(x+1)^7+\ldots$
36. $y=1-2(x-1)-3(x-1)^2+8(x-1)^3-4(x-1)^4-\frac{42}{5}(x-1)^5+19(x-1)^6-\frac{604}{35}(x-1)^7+\ldots$
37. $y=2-7x-4x^2-\frac{17}{6}x^3-\frac{3}{4}x^4-\frac{9}{40}x^5+\ldots$
38. $y=1-2(x-1)+\frac{1}{2}(x-1)^2-\frac{1}{6}(x-1)^3+\frac{5}{36}(x-1)^4-\frac{73}{1080}(x-1)^5+\ldots$
39. $y=2-(x+2)-\frac{7}{2}(x+2)^2+\frac{4}{3}(x+2)^3-\frac{1}{24}(x+2)^4+\frac{1}{60}(x+2)^5+\ldots$
40. $y=2-2(x+3)-(x+3)^2+(x+3)^3-\frac{11}{12}(x+3)^4+\frac{67}{60}(x+3)^5+\ldots$
41. $y=-1+2x+\frac{1}{3}x^3-\frac{5}{12}x^4+\frac{2}{5}x^5+\ldots$
42. $y=2-3(x+1)+\frac{7}{2}(x+1)^2-5(x+1)^3+\frac{197}{24}(x+1)^4-\frac{287}{20}(x+1)^5+\ldots$
43. $y=-2+3(x+2)-\frac{9}{2}(x+2)^2+\frac{11}{6}(x+2)^3+\frac{5}{24}(x+2)^4+\frac{7}{20}(x+2)^5+\ldots$
44. $y=2-4(x-2)-\frac{1}{2}(x-2)^2+\frac{2}{9}(x-2)^3+\frac{49}{432}(x-2)^4+\frac{23}{1080}(x-2)^5+\ldots$
45. $y=1+2(x+4)-\frac{1}{6}(x+4)^2-\frac{10}{27}(x+4)^3+\frac{19}{648}(x+4)^4+\frac{13}{324}(x+4)^5+\ldots$
46. $y=-1+2(x+1)-\frac{1}{4}(x+1)^2+\frac{1}{2}(x+1)^3-\frac{65}{96}(x+1)^4+\frac{67}{80}(x+1)^5+\ldots$
47. $y=1-2x-\frac{3}{2}x^2+\frac{5}{3}x^3+\frac{17}{24}x^4-\frac{11}{20}x^5+\ldots$
48. $y=1-2x-\frac{5}{2}x^2+\frac{2}{3}x^3-\frac{3}{8}x^4+\frac{1}{3}x^5+\ldots$
49. $y=6-2x+9x^2+\frac{2}{3}x^3-\frac{23}{4}x^4-\frac{3}{10}x^5+\ldots$
50. $y=2-5x+2x^2-\frac{10}{3}x^3+\frac{3}{2}x^4-\frac{25}{12}x^5+\ldots$
51. $y=3+6x-3x^2+x^3-2x^4-\frac{17}{20}x^5+\ldots$
52. $y=3-2x-3x^2+\frac{3}{2}x^3+\frac{3}{2}x^4-\frac{49}{80}x^5+\ldots$
53. $y=-2+3x+\frac{4}{3}x^2-x^3-\frac{19}{54}x^4+\frac{13}{60}x^5+\ldots$
54. $y=-2+3x+x^2-\frac{1}{6}x^3-\frac{3}{4}x^4+\frac{31}{120}x^5+\ldots$
55. $y=2+3x-\frac{7}{2}x^2-\frac{5}{6}x^3+\frac{41}{24}x^4+\frac{41}{120}x^5+\ldots$
56. $y=-3+5x-5x^2+\frac{23}{6}x^3-\frac{23}{12}x^4+\frac{11}{30}x^5+\ldots$
57. $y=-2+3(x-1)+\frac{3}{2}(x-1)^2-\frac{17}{12}(x-1)^3-\frac{1}{12}(x-1)^4+\frac{1}{8}(x-1)^5+\ldots$
58. $y=2-3(x+2)+\frac{1}{2}(x+2)^2-\frac{1}{3}(x+2)^3+\frac{31}{24}(x+2)^4-\frac{53}{120}(x+2)^5+\ldots$
59. $y=1-2x+\frac{3}{2}x^2-\frac{11}{6}x^3+\frac{15}{8}x^4-\frac{71}{60}x^5+\ldots$
60. $y=2-(x+2)-\frac{7}{2}(x+2)^2-\frac{43}{6}(x+2)^3-\frac{203}{24}(x+2)^4-\frac{167}{30}(x+2)^5+\ldots$
61. $y=2-x-x^2+\frac{7}{6}x^3-x^4+\frac{89}{120}x^5+\ldots$
62. $y=1+\frac{3}{2}(x-1)^2+\frac{1}{6}(x-1)^3-\frac{1}{8}(x-1)^5+\ldots$
63. $y=1-2(x-3)+\frac{1}{2}(x-3)^2-\frac{1}{6}(x-3)^3+\frac{1}{4}(x-3)^4-\frac{1}{6}(x-3)^5+\ldots$
64. a. $y=a_0\sum_{m=0}^{\infty}(-1)^m x^{2m}+a_1\sum_{m=0}^{\infty}(-1)^m x^{2m+1}$
    b. $y=\frac{a_0+a_1x}{1+x^2}$
65. $y_1=\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{m!}=e^{-x^2}$, $y_2=\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{m!}=xe^{-x^2}$
66. $y=a_0(1+\frac{1}{6}x^3+\frac{1}{180}x^6+\cdots)+a_1(x+\frac{1}{12}x^4+\frac{1}{504}x^7+\cdots)+(\frac{1}{2}x^2+\frac{1}{40}x^5+\frac{1}{2240}x^8+\cdots)$
67. $y=a_0(1+\frac{1}{2}x^2-\frac{1}{24}x^4+\cdots)+a_1x+(\frac{1}{2}x^2+\frac{1}{6}x^3-\frac{1}{120}x^5+\cdots)$

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30: Section 8.4 Answers

1. $y = c_1(1 + 2x - 2x^2 + \frac{4}{9}x^3 + \cdots) + c_2x^{3/2}(1 - \frac{2}{5}x + \frac{2}{35}x^2 - \frac{4}{945}x^3 + \cdots)$
2. $y = c_1(1 - 2x + \frac{2}{9}x^2 - \frac{4}{459}x^3 + \cdots) + c_2x^{7/8}(1 - \frac{2}{15}x + \frac{2}{345}x^2 - \frac{4}{32085}x^3 + \cdots)$
3. $y = c_1(1 + \frac{1}{2}x + \frac{1}{10}x^2 + \frac{1}{80}x^3 + \cdots) + c_2x^{1/3}(1 + \frac{1}{3}x + \frac{1}{18}x^2 + \frac{1}{162}x^3 + \cdots)$
4. $y = c_1(1 + \frac{1}{3}x - \frac{1}{6}x^2 - \frac{1}{6}x^3 + \cdots) + c_2x^{5/2}(1 + \frac{4}{7}x + \frac{4}{21}x^2 + \frac{32}{693}x^3 + \cdots)$
5. $y = c_1x^{1/3}(1 - \frac{1}{2}x + \frac{1}{5}x^2 - \frac{7}{120}x^3 + \cdots) + c_2x^{2/3}(1 - \frac{1}{2}x + \frac{5}{28}x^2 - \frac{1}{21}x^3 + \cdots)$
6. $y = c_1\frac{\sinh x}{x} + c_2\frac{\cosh x}{x}$
7. $y = c_1x + c_2(x\ln x - 1 + \frac{1}{2}x^2 + \frac{1}{12}x^3 + \frac{1}{72}x^4 + \cdots)$
8. $y = c_1e^x + c_2[e^x\ln x + e^x(-x + \frac{1}{4}x^2 - \frac{1}{18}x^3 + \frac{1}{96}x^4 + \cdots)]$
9. $y = c_1(1 - \frac{1}{14}x^2 + \frac{1}{616}x^4 + \cdots) + c_2x^{-3/2}(1 - \frac{1}{2}x^2 + \frac{1}{40}x^4 + \cdots)$
10. $y = c_1x(1 - \frac{1}{10}x^2 + \frac{1}{360}x^4 + \cdots) + c_2x^{1/2}(1 - \frac{1}{6}x^2 + \frac{1}{168}x^4 + \cdots)$
11. $y = c_1x^{1/3}(1 + \frac{3}{2}x + \frac{9}{20}x^2 + \cdots) + c_2x^{2/3}(1 + \frac{3}{4}x + \frac{9}{56}x^2 + \cdots)$
12. $y = c_1x^{-2/3}(1 - \frac{3}{4}x^2 + \frac{9}{128}x^4 + \cdots) + c_2x^{2/3}(1 - \frac{3}{20}x^2 + \frac{9}{1280}x^4 + \cdots)$
13. $y = c_1x^{-1}(1 + 2x - 2x^2 + \cdots) + c_2x^{1/2}(1 - \frac{2}{5}x + \frac{2}{35}x^2 + \cdots)$
14. $y - c_1x^{-1/2}\cos x + c_2x^{-1/2}\sin x$
15. $y = c_1(1 + \frac{1}{4}x^2 + \frac{1}{48}x^4 + \cdots) + c_2[-\frac{1}{2}y_1\ln x + y_1(-\frac{1}{2x^2} + \frac{7}{96}x^2 - \frac{19}{2304}x^4 + \cdots)]$
16. $y = c_1(1 - x + \frac{1}{4}x^2 + \cdots) + c_2[y_1\ln x + y_1(2x + \frac{5}{4}x^2 + \frac{23}{27}x^3 + \cdots)]$
17. $y = c_1x^{1/2}\left(1 - \frac{1}{5}x - \frac{2}{35}x^2 + \frac{31}{315}x^3 + \ldots\right) + c_2x^{-1}\left(1 + x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \ldots\right)$
18. $y = c_1x^{1/3}\left(1 - \frac{2}{3}x + \frac{8}{9}x^2 - \frac{40}{81}x^3 + \ldots\right) + c_2(1 - x + \frac{6}{5}x^2 - \frac{4}{5}x^3 + \ldots)$
19. $y = c_1x^{1/3}\left(1 - \frac{4}{7}x - \frac{7}{45}x^2 + \frac{970}{2457}x^3 + \ldots\right) + c_2x^{-1}\left(1 - x^2 + \frac{2}{3}x^3 + \ldots\right)$
20. $y = c_1x^{1/4}\left(1 - \frac{1}{2}x - \frac{19}{104}x^2 + \frac{1571}{10608}x^3 + \ldots\right) + c_2x^{-1}\left(1 + 2x - \frac{11}{6}x^2 - \frac{1}{7}x^3 + \ldots\right)$
21. $y = c_1x^{1/3}\left(1 - x + \frac{28}{31}x^2 - \frac{1111}{1333}x^3 + \ldots\right) + c_2x^{-1/4}\left(1 - x + \frac{7}{8}x^2 - \frac{19}{24}x^3 + \ldots\right)$
22. $y = c_1x^{1/5}\left(1 - \frac{6}{25}x - \frac{1217}{625}x^2 + \frac{41972}{46875}x^3 + \ldots\right) + c_2(x - \frac{1}{4}x^2 - \frac{35}{18}x^3 + \frac{11}{12}x^4 + \ldots)$
23. $y = c_1x^{3/2}\left(1 - x + \frac{11}{26}x^2 - \frac{109}{1326}x^3 + \ldots\right) + c_2x^{1/4}\left(1 + 4x - \frac{131}{24}x^2 + \frac{39}{14}x^3 + \ldots\right)$
24. $y = c_1x^{1/3}\left(1 - \frac{1}{3}x + \frac{2}{15}x^2 - \frac{5}{63}x^3 + \ldots\right) + c_2x^{-1/6}\left(1 - \frac{1}{12}x^2 + \frac{1}{18}x^3 + \ldots\right)$
25. $y = c_1(1 - \frac{1}{14}x^2 + \frac{1}{105}x^3 + \ldots) + c_2x^{-1/3}\left(1 - \frac{1}{18}x - \frac{71}{405}x^2 + \frac{719}{34992}x^3 + \ldots\right)$
26. $y = c_1x^{1/5}\left(1 + \frac{3}{17}x - \frac{7}{153}x^2 - \frac{547}{5661}x^3 + \ldots\right) + c_2x^{-1/2}\left(1 + x + \frac{14}{13}x^2 - \frac{556}{897}x^3 + \ldots\right)$
27. $y = c_1x^{1/2}\left(1 - \frac{9}{40}x + \frac{5}{128}x^2 - \frac{245}{39936}x^3 + \ldots\right) + c_2x^{1/4}\left(1 - \frac{25}{96}x + \frac{675}{14336}x^2 - \frac{38025}{5046272}x^3 + \ldots\right)$
28. $y = c_1x^{1/3}\left(1 + \frac{32}{117}x - \frac{28}{1053}x^2 + \frac{4480}{540189}x^3 + \ldots\right) + c_2x^{-3}\left(1 + \frac{32}{7}x + \frac{48}{7}x^2\right)$
29. $y = c_1x^{1/2}\left(1 - \frac{5}{8}x + \frac{55}{96}x^2 - \frac{935}{1536}x^3 + \ldots\right) + c_2x^{-1/2}\left(1 + \frac{1}{4}x - \frac{5}{32}x^2 - \frac{55}{384}x^3 + \ldots\right)$
30. $y = c_1x^{1/2}\left(1 - \frac{3}{4}x + \frac{5}{96}x^2 + \frac{5}{4224}x^3 + \ldots\right) + c_2x^{-2}(1 + 8x + 60x^2 - 160x^3 + \ldots)$
31. $y = c_1x^{-1/3}\left(1 - \frac{10}{63}x + \frac{200}{7371}x^2 - \frac{17600}{3781323}x^3 + \ldots\right) + c_2x^{-1/2}\left(1 - \frac{3}{20}x + \frac{9}{352}x^2 - \frac{105}{23936}x^3 + \ldots\right)$
32. $y = c_1x^{1/2}\left(1 - \frac{6}{13}x^2 + \frac{36}{325}x^4 - \frac{216}{12025}x^6 + \ldots\right) + c_2x^{1/3}\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \ldots\right)$
33. $y = c_1x^{1/4}\left(1 - \frac{13}{64}x^2 + \frac{273}{8192}x^4 - \frac{2639}{524288}x^6 + \ldots\right) + c_2x^{-1}\left(1 - \frac{1}{3}x^2 + \frac{2}{33}x^4 - \frac{2}{209}x^6 + \ldots\right)$
34. $y = c_1x^{1/3}\left(1 - \frac{3}{4}x^2 + \frac{9}{14}x^4 - \frac{81}{140}x^6 + \ldots\right) + c_2x^{-1/3}\left(1 - \frac{2}{3}x^2 + \frac{5}{9}x^4 - \frac{40}{81}x^6 + \ldots\right)$

35. $y = c_1 x^{1/2}\left(1 - \frac{3}{2}x^2 + \frac{15}{8}x^4 - \frac{35}{16}x^6 + \ldots\right) + c_2 x^{-1/2}\left(1 - 2x^2 + \frac{8}{3}x^4 - \frac{16}{5}x^6 + \ldots\right)$
36. $y = c_1 x^{1/4}\left(1 - x^2 + \frac{3}{2}x^4 - \frac{5}{2}x^6 + \ldots\right) + c_2 x^{-1/2}\left(1 - \frac{2}{5}x^2 + \frac{36}{65}x^4 - \frac{408}{455}x^6 + \ldots\right)$
37. $y = c_1 x\left(1 - x + \frac{3}{4}x^2 - \frac{13}{36}x^3 + \ldots\right) + c_2[y_1 \ln x + x^2\left(1 - x + \frac{65}{108}x^2 + \ldots\right)]$
38. $y = c_1 x^{-1}\left(1 - 2x + \frac{9}{2}x^2 - \frac{20}{3}x^3 + \ldots\right) + c_2(y_1 \ln x + 1 - \frac{15}{4}x + \frac{133}{18}x^2 + \ldots)$
39. $y = c_1(1 + x - x^2 + \frac{1}{3}x^3 + \ldots) + c_2[y_1 \ln x - x\left(3 - \frac{1}{2}x - \frac{31}{18}x^2 + \ldots\right)]$
40. $y = c_1 x^{1/2}\left(1 - 2x + \frac{5}{2}x^2 - 2x^3 + \ldots\right) + c_2[y_1 \ln x + x^{3/2}\left(1 - \frac{9}{4}x + \frac{17}{6}x^2 + \ldots\right)]$
41. $y = c_1 x\left(1 - 4x + \frac{19}{2}x^2 - \frac{49}{3}x^3 + \ldots\right) + c_2[y_1 \ln x + x^2\left(3 - \frac{43}{4}x + \frac{208}{9}x^2 + \ldots\right)]$
42. $y = c_1 x^{-1/3}\left(1 - x + \frac{5}{6}x^2 - \frac{1}{2}x^3 + \ldots\right) + c_2[y_1 \ln x + x^{2/3}\left(1 - \frac{11}{12}x + \frac{25}{36}x^2 + \ldots\right)]$
43. $y = c_1(1 - 2x + \frac{7}{4}x^2 - \frac{7}{9}x^3 + \ldots) + c_2[y_1 \ln x + x\left(3 - \frac{15}{4}x + \frac{239}{108}x^2 + \ldots\right)]$
44. $y = c_1 x^{-2}\left(1 - 2x + \frac{5}{2}x^2 - 3x^3 + \ldots\right) + c_2(y_1 \ln x + \frac{3}{4} - \frac{13}{6}x + \ldots)$
45. $y = c_1 x^{-1/2}\left(1 - x + \frac{1}{4}x^2 + \frac{1}{18}x^3 + \ldots\right) + c_2[y_1 \ln x + x^{1/2}\left(\frac{3}{2} - \frac{13}{16}x + \frac{1}{54}x^2 + \ldots\right)]$
46. $y = c_1 x^{-1/4}\left(1 - \frac{1}{4}x - \frac{7}{32}x^2 + \frac{23}{384}x^3 + \ldots\right) + c_2[y_1 \ln x + x^{3/4}\left(\frac{1}{4} + \frac{5}{64}x - \frac{157}{2304}x^2 + \ldots\right)]$
47. $y = c_1 x^{-1/3}\left(1 - x + \frac{7}{6}x^2 - \frac{23}{18}x^3 + \ldots\right) + c_2[y_1 \ln x - x^{5/3}\left(\frac{1}{12} - \frac{13}{108}x \ldots\right)]$
48. $y = c_1 x^{-2}\left(1 + 3x + \frac{3}{2}^2 - \frac{1}{2}x^3 + \ldots\right) + c_2[y_1 \ln x - 5x^{-1}\left(1 + \frac{5}{4}x - \frac{1}{4}x^2 + \ldots\right)]$
49. $y = c_1 x^3(1 + 20x + 180x^2 + 1120x^3 + \ldots) + c_2[y_1 \ln x - x^4\left(26 + 324x + 69683x^2 + \ldots\right)]$
50. $y = c_1 x\left(1 - 5x + \frac{85}{4}x^2 - \frac{3145}{36}x^3 + \ldots\right) + c_2[y_1 \ln x + x^2\left(2 - \frac{39}{4}x + \frac{4499}{108}x^2 + \ldots\right)]$
51. $y = c_1(1 - x + \frac{3}{4}x^2 - \frac{7}{12}x^3 + \ldots) + c_2[y_1 \ln x + x\left(1 - \frac{3}{4}x + \frac{5}{9}x^2 + \ldots\right)]$
52. $y = c_1 x^{-3}(1 + 16x + 36x^2 + 16x^3 + \ldots) + c_2[y_1 \ln x - x^{-2}\left(40 + 150x + \frac{280}{3}x^2 + \ldots\right)]$
53. $y = c_1 x^{-1}\left(1 - \frac{3}{2}x^2 + \frac{15}{8}x^4 - \frac{35}{16}x^6 + \ldots\right) + c_2[y_1 \ln x + x\left(\frac{1}{4} - \frac{13}{32}x^2 + \frac{101}{192}x^4 + \ldots\right)]$
54. $y = c_1 x\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \ldots\right) + c_2[y_1 \ln x + x^3\left(\frac{1}{4} - \frac{3}{32}x^2 + \frac{11}{576}x^4 + \ldots\right)]$
55. $y = c_1 x^{-2}\left(1 - \frac{3}{4}x^2 - \frac{9}{64}x^4 - \frac{25}{256}x^6 + \ldots\right) + c_2(y_1 \ln x + \frac{1}{2} - \frac{21}{128}x^2 - \frac{215}{1536}x^4 + \ldots)$
56. $y = c_1 x^{-3}\left(1 - \frac{17}{8}x^2 + \frac{85}{256}x^4 - \frac{85}{18432}x^6 + \ldots\right) + c_2[y_1 \ln x + x^{-1}\left(\frac{25}{8} - \frac{471}{512}x^2 + \frac{1583}{110592}x^4 + \ldots\right)]$
57. $y = c_1 x^{-1}\left(1 - \frac{3}{4}x^2 + \frac{45}{64}x^4 - \frac{175}{256}x^6 + \ldots\right) + c_2[y_1 \ln x - x\left(\frac{1}{4} - \frac{33}{128}x^2 + \frac{395}{1536}x^4 + \ldots\right)]$
58. $y = c_1 \frac{1}{x} + c_2(y_1 \ln x - 6 + 6x - \frac{8}{3}x^2)$
59. $y = c_1(1 - x) + c_2(y_1 \ln x + 4x)$
60. $y = c_1(x - 4x^3 + x^5) + c_2(y_1 \ln x + 6x^3 - 3x^5)$

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31: Section 9.1 Answers

1. $\frac{1}{s^2}$
2. $\frac{1}{(s+1)^2}$
3. $\frac{b}{s^2-b^2}$
4. $\frac{-2s+5}{(s-1)(s-2)}$
5. $\frac{2}{s^3}$
6. $\frac{2}{s(s^2+4)}$
7. $\frac{s^2+8}{s(s^2+16)}$
8. $\frac{1}{(s^2+4)}$
9. $\frac{5s}{(s^2+4)(s^2+9)}$
10. $\frac{s^3+2s^2+4s+32}{(s^2+4)(s^2+16)}$
11. $f(3-)=-1,\ f(3)=f(3+)=1$
12. $f(1-)=3,\ f(1)=4,\ f(1+)=1$
13. $f\left(\frac{\pi}{2}-\right)=1,\ f\left(\frac{\pi}{2}\right)=f\left(\frac{\pi}{2}+\right)=2,\ f(\pi-)=0,\ f(\pi)=f(\pi+)=-1$
14. $f(1-)=1,\ f(1)=2,\ f(1+)=1,\ f(2-)=0,\ f(2)=3,\ f(2+)=6$
15. $\frac{1-e^{-(s+1)}}{s+1}+\frac{e^{-(s+2)}}{s+2}$
16. $\frac{1}{s}+e^{-4s}\left(\frac{1}{s^2}+\frac{3}{s}\right)$
17. $\frac{1-e^{-s}}{s^2}$
18. $\frac{1-e^{-(s-1)}}{(s-1)^2}$

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32: Section 9.2 Answers

1. $\frac{1}{2}t^2$
2. $\frac{1}{6}t^3$
3. $t-2t^4$
4. $4t-\frac{2}{3}t^3+\frac{1}{120}t^5$
5. $1+3t+\frac{3}{2}t^2+\frac{1}{6}t^3$
6. $1+4t+2t^2$
7. $t-1+e^{2t}$
8. $4+\frac{1}{4}t^4-e^{-8t}$
9. $\frac{1}{4}e^{-t/4}$
10. $\frac{1}{5}e^{2t/5}$
11. $\frac{5}{7}\sin 7t$
12. $10\cos 4t$
13. $\cos\frac{t}{2}$
14. $\frac{1}{2}\sin\frac{t}{2}$
15. $2\cos 3t-2\sin 3t$
16. $\cos\sqrt{2}t+\frac{\sqrt{2}}{2}\sin\sqrt{2}t$
17. $\frac{1}{3}-\frac{1}{3}e^{-3t}$
18. $-\frac{1}{4}+\frac{5}{4}e^{4t}$
19. $\frac{1}{4}e^{t}+\frac{3}{4}e^{-3t}$
20. $\frac{1}{9}e^{4t}-\frac{1}{9}e^{-5t}$
21. $0.3e^{0.1t}+0.6e^{-0.2t}$
22. $\cosh\sqrt{3}t-\sqrt{3}\sinh\sqrt{3}t$
23. $\frac{1}{2}e^{2t}-e^{3t}+\frac{1}{2}e^{6t}$
24. $\frac{1}{2}-e^{t}-\frac{1}{3}e^{-t}+\frac{5}{6}e^{2t}$
25. $\frac{1}{5}-\frac{1}{5}\cos\sqrt{5}t$
26. $\frac{1}{4}\cos 2t+\frac{1}{4}\sin 2t-\frac{1}{4}e^{-2t}$
27. $-4+3e^{-t}+\cos t+3\sin t$
28. $\frac{1}{6\sqrt{3}}\sinh\sqrt{3}t-\frac{1}{6\sqrt{3}}\sin\sqrt{3}t$
29. $\frac{1}{4}\cos 2t+\frac{1}{4}\sin 2t-\frac{1}{4}e^{-2t}$
30. $2\cos t+\sin t-2\cos 2t-\frac{1}{2}\sin 2t$
31. $\frac{2}{3}\sin 3t$
32. $2\cos 3t+\frac{1}{3}\sin 3t$
33. $\cosh 3t+\frac{1}{3}\sinh 3t$
34. $1-\cos t$
35. $3\cosh t+4\sinh t$

36. $3e^{t}+4\cos 3t+\frac{1}{3}\sin 3t$
37. $\frac{1}{4}e^{2t}-\frac{1}{4}e^{-2t}-e^{-t}$
38. $\frac{1}{5}e^{-4t}-\frac{41}{5}e^{t}+5e^{3t}$
39. $-\frac{1}{2}e^{2t}-\frac{13}{10}e^{-2t}-\frac{1}{5}e^{3t}$
40. $-\frac{2}{5}e^{-4t}-\frac{3}{5}e^{t}$
41. $\frac{3}{20}e^{2t}-\frac{37}{12}e^{-2t}+\frac{1}{3}e^{t}+\frac{8}{5}e^{-3t}$
42. $\frac{39}{10}e^{t}+\frac{3}{14}e^{3t}+\frac{23}{105}e^{-4t}-\frac{7}{3}e^{2t}$
43. $\frac{4}{5}e^{-2t}-\frac{1}{2}e^{-t}-\frac{3}{10}\cos t+\frac{11}{10}\sin t$

47. $\frac{3}{5}\cos 2t+\frac{1}{5}\sin 2t-\frac{3}{5}\cos 3t-\frac{2}{15}\sin 3t$
48. $-\frac{4}{15}\cos t+\frac{1}{15}\sin t+\frac{4}{15}\cos 4t-\frac{1}{60}\sin 4t$
49. $\frac{5}{3}\cos t+\sin t-\frac{5}{3}\cos 2t-\frac{1}{2}\sin 2t$
50. $-\frac{1}{3}\cos\frac{t}{2}+\frac{2}{3}\sin\frac{t}{2}+\frac{1}{3}\cos t-\frac{1}{3}\sin t$
51. $\frac{1}{15}\cos\frac{t}{4}-\frac{8}{15}\sin\frac{t}{4}-\frac{1}{15}\cos 4t+\frac{1}{30}\sin 4t$
52. $\frac{2}{5}\cos\frac{t}{3}-\frac{3}{5}\sin\frac{t}{3}-\frac{2}{5}\cos\frac{t}{2}+\frac{2}{5}\sin\frac{t}{2}$
53. $1-\cos t$
54. $-\frac{3}{10}\sin t+\frac{2}{5}\cos t-\frac{3}{4}e^{t}+\frac{7}{20}e^{3t}$
55. $-\frac{1}{2}e^{t}+\frac{7}{10}e^{-t}-\frac{1}{5}\cos 2t+\frac{3}{5}\sin 2t$

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33: Section 9.3 Answers

1. $y = -1 + e^{t}$
2. $y = -3e^{-t/2}$
3. $y = \frac{1}{10}e^{4t} + \frac{19}{10}e^{-6t}$
4. $y = \frac{1}{13}e^{t} - \frac{1}{13}\cos 5t + \frac{5}{13}\sin 5t$
5. $y = \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t}$
6. $y = \frac{5}{2} - 2e^{3t} - \frac{3}{5}e^{-t} + \frac{11}{10}e^{4t}$
7. $y = 10\cos t + 2\sin t - \sqrt{2}\sin\sqrt{2}t$
8. $y = \frac{1}{10}e^{t} - \frac{1}{30}\sin 3t - \frac{1}{10}\cos 3t$
9. $y = \frac{1}{2}e^{-t} + \frac{5}{18}e^{t} - \frac{8}{9}e^{-t/2} + \frac{1}{9}e^{-2t}$
10. $y = \frac{13}{60}e^{t} - \frac{13}{20}e^{-t} + \frac{16}{39}e^{-2t} + \frac{3}{130}\cos 3t - \frac{1}{65}\sin 3t$
11. $y = \frac{1}{6}e^{t} - \frac{9}{2}e^{-t} + \frac{16}{3}e^{-2t}$
12. $y = -\frac{1}{3} + \frac{8}{15}e^{3t} + \frac{4}{5}e^{-2t}$
13. $y = -\frac{23}{15}e^{-2t} + \frac{1}{3}e^{t} + \frac{1}{5}e^{3t}$
14. $y = -\frac{1}{4}e^{2t} + \frac{17}{20}e^{-2t} + \frac{2}{5}e^{3t}$
15. $y = \frac{11}{15}e^{-2t} + \frac{1}{6}e^{t} + \frac{1}{10}e^{3t}$
16. $y = e^{t} + 2e^{-2t} - 2e^{-t}$
17. $y = \frac{5}{3}\sin t - \frac{1}{3}\sin 2t$
18. $y = 4e^{t} - 4e^{2t} + e^{3t}$
19. $y = -\frac{7}{2}e^{2t} + \frac{13}{3}e^{t} + \frac{1}{6}e^{4t}$
20. $y = \frac{5}{2}e^{t} - 4e^{2t} + \frac{1}{2}e^{3t}$
21. $y = \frac{1}{3}e^{t} - 2e^{-t} + \frac{5}{3}e^{-2t}$
22. $y = 2 - e^{-2t} + e^{t}$
23. $y = 1 - \cos 2t + \frac{1}{2}\sin 2t$
24. $y = -\frac{1}{3} + \frac{8}{15}e^{3t} + \frac{4}{5}e^{-2t}$
25. $y = \frac{1}{6}e^{t} - \frac{2}{3}e^{-2t} + \frac{1}{2}e^{-t}$
26. $y = -1 + e^{t} + e^{-t}$
27. $y = \cos 2t - \sin 2t + \sin t$
28. $y = \frac{7}{3} - \frac{7}{2}e^{-t} + \frac{1}{6}e^{3t}$
29. $y = 1 + \cos t$
30. $y = t + \sin t$
31. $y = t - 6\sin t + \cos t + \sin 2t$
32. $y = e^{-t} + 4e^{-2t} - 4e^{-3t}$
33. $y = -\sin t - 2\cos t + 3e^{3t} + e^{-t}$
34. $y = \frac{1}{5}e^{2t} - \frac{4}{3}e^{t} + \frac{32}{15}e^{-t/2}$
35. $y = \frac{1}{7}e^{2t} - \frac{2}{5}e^{t/2} + \frac{9}{35}e^{-t/3}$

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34: Section 9.4 Answers

1. $\frac{1}{(s-10)^2}$
2. $\frac{6}{(s+2)^4}$
3. $\frac{3}{(s-1)^2+9}$
4. $\frac{1}{s-6}+\frac{3}{(s-6)^2}-\frac{2}{(s-6)^3}$
5. $\frac{s}{s^2-16}+\frac{s-1}{(s-1)^2-16}-3\frac{s+2}{(s+2)^2-16}$
6. $\frac{t^3e^{7t}}{2}$
7. $2e^{2t}\cos 3t$
8. $\frac{e^{-2t}}{4}\sin 4t$
9. $\frac{e^{2t}}{2}\sinh 2t$
10. $\frac{2e^{3t}}{3}\sinh 3t$
11. $t^2e^{7t}+\frac{17}{6}t^3e^{7t}$
12. $e^{2t}\left(\frac{1}{6}t^3+\frac{1}{6}t^4+\frac{1}{40}t^5\right)$
13. $e^{-3t}\left(\cos 3t+\frac{2}{3}\sin 3t\right)$
14. $(1-t)e^{-t}$
15. $\left(1-t-t^2-\frac{1}{6}t^3\right)e^{-t}$
16. $e^t\left(2\cos 2t+\frac{5}{2}\sin 2t\right)$
17. $3te^{-2t}-2\cos 2t-3\sin 2t$
18. $\frac{2}{5}\sin t+\frac{6}{5}\cos t+\frac{7}{5}e^{-t}\sin t-\frac{6}{5}e^{-t}\cos t$
19. $\frac{8}{13}e^{2t}-\frac{8}{13}e^{-t}\cos 2t+\frac{15}{26}e^{-t}\sin 2t$
20. $e^t(\cos 2t+\sin 2t)-e^{-t}\left(\cos 3t+\frac{4}{3}\sin 3t\right)$
21. $e^{3t}\left(-\cos 2t+\frac{3}{2}\sin 2t\right)+e^{-t}\left(\cos 2t+\frac{1}{2}\sin 2t\right)$
22. $e^{-2t}\left(\frac{1}{8}\cos t+\frac{1}{4}\sin t\right)-e^{2t}\left(\frac{1}{8}\cos 3t-\frac{1}{12}\sin 3t\right)$
23. $e^{2t}\left(\cos t+\frac{1}{2}\sin t\right)-e^{3t}\left(\cos 2t-\frac{1}{4}\sin 2t\right)$

24 $e^t\left(\frac{1}{5}\cos t+\frac{2}{5}\sin t\right)-e^{-t}\left(\frac{1}{5}\cos 2t+\frac{2}{5}\sin 2t\right)$

25. $e^{t/2}\left(-\cos t+\frac{9}{8}\sin t\right)+e^{-t/2}\left(\cos t-\frac{1}{8}\sin t\right)$
26. $\frac{e^t}{16}(1-\cos 4t)$
27. $\frac{4}{9}e^{2t}+\frac{5}{9}e^{-t}\sin 3t-\frac{4}{9}e^{-t}\cos 3t$
28. $3e^{t/2}-\frac{7}{2}e^t\sin 2t-3e^t\cos 2t$
29. $\frac{1}{4}e^{3t}-\frac{1}{4}e^{-t}\cos 2t$
30. $\frac{1}{9}e^{2t}-\frac{1}{9}e^{-t}\cos 3t+\frac{5}{9}e^{-t}\sin 3t$
31. $-\frac{3}{5}e^{-t}\sin t+\frac{1}{5}e^{-t}\cos t-\frac{1}{2}e^{-t}+\frac{3}{10}e^t$
32. $-\frac{1}{10}e^t\sin t-\frac{7}{10}e^t\cos t+\frac{1}{5}e^{-t}+\frac{1}{2}e^{2t}$
33. $\frac{3}{10}+\frac{1}{10}e^{2t}+\frac{1}{10}e^t\sin 2t-\frac{2}{5}e^t\cos 2t$

34. $-\frac{4}{9}e^{2t}\cos 3t+\frac{1}{3}e^{2t}\sin 3t-\frac{5}{9}e^{2t}+e^{t}$
35. $y=te^{-4t}+2e^{-4t}$
36. $y=e^{-t}+2te^{-t}$
37. $y=\frac{1}{9}t+\frac{2}{27}-\frac{2}{27}e^{3t}+\frac{10}{9}te^{3t}$
38. $y=-\frac{3}{2}e^{3t}\sin 2t$
39. $\frac{1}{2}-\frac{1}{2}e^{t}\cos t+\frac{1}{2}e^{t}\sin t$
40. $y=-3\cos t-2\sin t+e^{-t}(2+5t)$
41. $y=e^{t}(\cos t-3\sin t)+e^{3t}$
42. $y=-1+t+e^{-t}(3\cos t-5\sin t)$
43. $y=4\cos t-3\sin t-e^{t}(3\cos t-8\sin t)$
44. $y=e^{-t}-2e^{t}+e^{-2t}(\cos 3t-11/3\sin 3t)$
45. $y=e^{-t}(\sin t-\cos t)+e^{-2t}(\cos t+4\sin t)$
46. $y=e^{-t/2}(5\cos(t/2)-\sin(t/2))+2t-4$
47. $y=\frac{1}{17}(12\cos t+20\sin t-3e^{t/2}(4\cos t+\sin t))$
48. $y=\frac{e^{-t/2}}{10}(5t+26)-\frac{1}{5}(3\cos t+\sin t)$
49. $y=\frac{1}{100}(3e^{3t}-e^{t/3}(3+310t))$
50. $y=\frac{3}{2}e^{-7t/2}\cos\frac{\sqrt{15}}{2}t+\frac{7\sqrt{15}}{10}e^{-7t/2}\sin\frac{\sqrt{15}}{2}t$
51. $\frac{1}{a}e^{\frac{b}{a}t}f\left(\frac{t}{a}\right)$

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35: Section 9.5 Answers

1. $1+\mathcal{U}(t-4)(t-1)$; $\quad \frac{1}{s}+e^{-4s}\left(\frac{1}{s^2}+\frac{3}{s}\right)$

2. $t+\mathcal{U}(t-1)(1-t)$; $\quad \frac{1-e^{-s}}{s^2}$

3. $2t-1-\mathcal{U}(t-2)(t-1)$; $\quad \left(\frac{2}{s^2}-\frac{1}{s}\right)-e^{-2s}\left(\frac{1}{s^2}+\frac{1}{s}\right)$

4. $1+\mathcal{U}(t-1)(t+1)$; $\quad \frac{1}{s}+e^{-s}\left(\frac{1}{s^2}+\frac{2}{s}\right)$

5. $t-1+\mathcal{U}(t-2)(5-t)$; $\quad \frac{1}{s^2}-\frac{1}{s}-e^{-2s}\left(\frac{1}{s^2}-\frac{3}{s}\right)$

6. $t^2(1-\mathcal{U}(t-1))$; $\quad \frac{2}{s^3}-e^{-s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right)$

7. $\mathcal{U}(t-2)(t^2+3t)$; $\quad e^{-2s}\left(\frac{2}{s^3}+\frac{7}{s^2}+\frac{10}{s}\right)$

8. $t^2+2+\mathcal{U}(t-1)(t-t^2-2)$; $\quad \frac{2}{s^3}+\frac{2}{s}-e^{-s}\left(\frac{2}{s^3}+\frac{1}{s^2}+\frac{2}{s}\right)$

9. $te^t+\mathcal{U}(t-1)(e^t-te^t)$; $\quad \frac{1-e^{-(s-1)}}{(s-1)^2}$

10. $e^{-t}+\mathcal{U}(t-1)(e^{-2t}-e^{-t})$; $\quad \frac{1-e^{-(s+1)}}{s+1}+\frac{e^{-(s+2)}}{s+2}$

11. $-t+2\mathcal{U}(t-2)(t-2)-\mathcal{U}(t-3)(t-5)$; $\quad -\frac{1}{s^2}+\frac{2e^{-2s}}{s^2}+e^{-3s}\left(\frac{2}{s}-\frac{1}{s^2}\right)$

12. $[\mathcal{U}(t-1)-\mathcal{U}(t-2)]\,t$; $\quad e^{-s}\left(\frac{1}{s^2}+\frac{1}{s}\right)-e^{-2s}\left(\frac{1}{s^2}+\frac{2}{s}\right)$

13. $t+\mathcal{U}(t-1)(t^2-t)-\mathcal{U}(t-2)t^2$; $\quad \frac{1}{s^2}+e^{-s}\left(\frac{2}{s^3}+\frac{1}{s^2}\right)-e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s}\right)$

14. $t+\mathcal{U}(t-1)(2-2t)+\mathcal{U}(t-2)(4+t)$; $\quad \frac{1}{s^2}-2\frac{e^{-s}}{s^2}+e^{-2s}\left(\frac{1}{s^2}+\frac{6}{s}\right)$

15. $\sin t+\mathcal{U}(t-\pi/2)\sin t+\mathcal{U}(t-\pi)(\cos t-2\sin t)$; $\quad \frac{1+e^{-\frac{\pi}{2}s}s-e^{-\pi s}(s-2)}{s^2+1}$

16. $2-2\mathcal{U}(t-1)t+\mathcal{U}(t-3)(5t-2)$; $\quad \frac{2}{s}-e^{-s}\left(\frac{2}{s^2}+\frac{2}{s}\right)+e^{-3s}\left(\frac{5}{s^2}+\frac{13}{s}\right)$

17. $3+\mathcal{U}(t-2)(3t-1)+\mathcal{U}(t-4)(t-2)$; $\quad \frac{3}{s}+e^{-2s}\left(\frac{3}{s^2}+\frac{5}{s}\right)+e^{-4s}\left(\frac{1}{s^2}+\frac{2}{s}\right)$

18. $(t+1)^2+\mathcal{U}(t-1)(2t+3)$; $\quad \frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}+e^{-s}\left(\frac{2}{s^2}+\frac{5}{s}\right)$

19. $\mathcal{U}(t-2)e^{2(t-2)}=\begin{cases}0, & 0\le t<2,\\ e^{2(t-2)}, & t\ge 2\end{cases}$

20. $\mathcal{U}(t-1)\left(1-e^{-(t-1)}\right)=\begin{cases}0, & 0\le t<1,\\ 1-e^{-(t-1)}, & t\ge 1\end{cases}$

21. $\mathcal{U}(t-1)\frac{(t-1)^2}{2}+\mathcal{U}(t-2)(t-2)=\begin{cases}0, & 0\le t<1,\\ \frac{(t-1)^2}{2}, & 1\le t<2,\\ \frac{t^2-3}{2}, & t\ge 2\end{cases}$

22. $2+t+\mathcal{U}(t-1)(4-t)+\mathcal{U}(t-3)(t-2)=\begin{cases}2+t, & 0\le t<1,\\ 6, & 1\le t<3,\\ t+4, & t\ge 3\end{cases}$

23. $5-t+\mathcal{U}(t-3)(7t-15)+\frac{3}{2}\mathcal{U}(t-6)(t-6)^2=\begin{cases}5-t, & 0\le t<3,\\ 6t-10, & 3\le t<6,\\ 44-12t+\frac{3}{2}t^2, & t\ge 6\end{cases}$

24. $\mathcal{U}(t-\pi)e^{-2(t-\pi)}(2\cos t-5\sin t)=\begin{cases}0, & 0\le t<\pi,\\ e^{-2(t-\pi)}(2\cos t-5\sin t) & t\ge\pi\end{cases}$

25. $1-\cos t+\mathcal{U}(t-\pi/2)(3\sin t+\cos t)=\begin{cases}1-\cos t, & 0\le t<\frac{\pi}{2},\\ 1+3\sin t, & t\ge\frac{\pi}{2}\end{cases}$

26. $\mathcal{U}(t-2)(4e^{-(t-2)}-4e^{2(t-2)}+2e^{(t-2)})=\begin{cases}0, & 0\le t<2,\\ 4e^{-(t-2)}-4e^{2(t-2)}+2e^{(t-2)}, & t\ge 2\end{cases}$

27. $1+t+\mathcal{U}(t-1)(2t+1)+\mathcal{U}(t-3)(3t-5)=\begin{cases}t+1, & 0\le t<1,\\ 3t+2, & 1\le t<3,\\ 6t-3, & t\ge 3\end{cases}$

28. $1-t^2+\mathcal{U}(t-2)\left(-\frac{t^2}{2}+2t+1\right)+\mathcal{U}(t-4)(t-4)=\begin{cases}1-t^2, & 0\le t<2,\\ -\frac{3t^2}{2}+2t+2, & 2\le t<4,\\ -\frac{3t^2}{2}+3t-2, & t\ge 4\end{cases}$

29. $y=3(1-\cos t)-3\mathcal{U}(t-\pi)(1+\cos t)$

30. $y=3-2\cos t+2\mathcal{U}(t-4)(t-4-\sin(t-4))$

31. $y=-\frac{15}{2}+\frac{3}{2}e^{2t}-2t+\frac{\mathcal{U}(t-1)}{2}(e^{2(t-1)}-2t+1)$

32. $y=\frac{1}{2}e^{t}+\frac{13}{6}e^{-t}+\frac{1}{3}e^{2t}+\mathcal{U}(t-2)\left(-1+\frac{1}{2}e^{t-2}+\frac{1}{2}e^{-(t-2)}+\frac{1}{2}e^{t+2}-\frac{1}{6}e^{-(t-6)}-\frac{1}{3}e^{2t}\right)$

33. $y=-7e^{t}+4e^{2t}+\mathcal{U}(t-1)\left(\frac{1}{2}-e^{t-1}+\frac{1}{2}e^{2(t-1)}\right)-2\mathcal{U}(t-2)\left(\frac{1}{2}-e^{t-2}+\frac{1}{2}e^{2(t-2)}\right)$

34. $y=\frac{1}{3}\sin 2t-3\cos 2t+\frac{1}{3}\sin t-2\mathcal{U}(t-\pi)\left(\frac{1}{3}\sin t+\frac{1}{6}\sin 2t\right)+\mathcal{U}(t-2\pi)\left(\frac{1}{3}\sin t-\frac{1}{6}\sin 2t\right)$

35. $y=\frac{1}{4}-\frac{31}{12}e^{4t}+\frac{16}{3}e^{t}+\mathcal{U}(t-1)\left(\frac{2}{3}e^{t-1}-\frac{1}{6}e^{4(t-1)}-\frac{1}{2}\right)+\mathcal{U}(t-2)\left(\frac{1}{4}+\frac{1}{12}e^{4(t-2)}-\frac{1}{3}e^{t-2}\right)$

36. $y=\frac{1}{8}(\cos t-\cos 3t)+\frac{1}{8}\mathcal{U}\left(t-\frac{3\pi}{2}\right)\left(\sin t-\cos t+\sin 3t-\frac{1}{3}\cos 3t\right)$

37. $y=\frac{t}{4}-\frac{1}{8}\sin 2t+\frac{1}{8}\mathcal{U}\left(t-\frac{\pi}{2}\right)(\pi\cos 2t-\sin 2t+2\pi-2t)$

38. $y=t-\sin t-2\mathcal{U}(t-\pi)(t+\sin t+\pi\cos t)$

39. $y=\mathcal{U}(t-2)\left(t-\frac{1}{2}+\frac{e^{2(t-2)}}{2}-2e^{t-2}\right)$

40. $y=t+\sin t+\cos t-\mathcal{U}(t-2\pi)(3t-3\sin t-6\pi\cos t)$

41. $y=\frac{1}{2}+\frac{1}{2}e^{-2t}-e^{-t}+\mathcal{U}(t-2)\left(2e^{-(t-2)}-e^{-2(t-2)}-1\right)$

42. $y=-\frac{1}{3}-\frac{1}{6}e^{3t}+\frac{1}{2}e^{t}+\mathcal{U}(t-1)\left(\frac{2}{3}+\frac{1}{3}e^{3(t-1)}-e^{t-1}\right)$

43. $y=\frac{1}{4}(e^{t}+e^{-t}(11+6t))+\mathcal{U}(t-1)(te^{-(t-1)}-1)$

44. $y=e^{t}-e^{-t}-2te^{-t}-\mathcal{U}(t-1)\left(e^{t}-e^{-(t-2)}-2(t-1)e^{-(t-2)}\right)$

45. $y=te^{-t}+e^{-2t}+\mathcal{U}(t-1)\left(e^{-t}(2-t)-e^{-(2t-1)}\right)$

46. $y=\frac{t^2e^{2t}}{2}-te^{2t}-\mathcal{U}(t-2)(t-2)^2e^{2t}$

47. $y=\frac{t^4}{12}+1-\frac{1}{12}\mathcal{U}(t-1)(t^4+2t^3-10t+7)+\frac{1}{6}\mathcal{U}(t-2)(2t^3+3t^2-36t+44)$

48. $y=\frac{1}{2}e^{-t}(3\cos t+\sin t)+\frac{1}{2}-\mathcal{U}(t-2\pi)\left(e^{-(t-2\pi)}\left((\pi-1)\cos t+\frac{2\pi-1}{2}\sin t\right)+1-\frac{t}{2}\right)$
$-\frac{1}{2}\mathcal{U}(t-3\pi)(e^{-(t-3\pi)}(3\pi\cos t+(3\pi+1)\sin t)+t)$

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36: Section 9.6 Answers

1. $\frac{s^2-4}{(s^2+4)^2}$
2. $\frac{6s^2+2}{(s^2-1)^3}$
3. $\frac{12s-24}{((s-2)^2+36)^2}$
4. $\frac{6s}{(s^2-9)^2}$
5. $\frac{2s(s^2-3)}{(s^2+1)^3}$
6. $\frac{(s+3)^2-9}{((s+3)^2+9)^2}$
7. $\frac{1}{s(s-1)}$
8. $\frac{s+1}{s[(s+1)^2+1]}$
9. $\frac{1}{s^2(s-1)}$
10. $\frac{3s^2+1}{s^2(s^2+1)^2}$
11. $\frac{1}{s^2+1}$
12. $\frac{2}{(s^2+1)^2}$
13. $\frac{s}{(s^2+1)^2}$
14. $\frac{3s+1}{s^2(s+1)^3}$
15. $\frac{as}{(s^2+a^2)(s^2+b^2)}$
16. $\frac{a}{(s-1)(s^2+a^2)}$
17. $\frac{as}{(s^2-a^2)^2}$
18. $\frac{(s-1)\omega}{((s-1)^2+\omega^2)^2}$
19. $\frac{2}{(s-2)^3(s-1)^2}$
20. $\frac{s+1}{(s+2)^2\left[(s+1)^2+\omega^2\right]}$
21. $\frac{1}{(s-3)((s-1)^2-1)}$
22. $\frac{2}{(s-2)^2(s^2+4)}$
23. $\frac{6}{s^4(s-1)}$
24. $\frac{3\cdot 6!}{s^7\left[(s+1)^2+9\right]}$
25. $\frac{12}{s^7}$
26. $\frac{2\cdot 7!}{s^8\left[(s+1)^2+4\right]}$
27. $\frac{48}{s^5(s^2+4)}$
28. $2t\sin t$

29. $t\cos t$
30. $\frac{2te^{2t}}{3}\sin 9t$
31. $e^{2t}t\cos t$
32. $\frac{1}{2}\int_0^t u\sin 2(t-u)du$
33. $\int_0^t e^{-2u}\cos 3(t-u)du$
34. $\frac{1}{2}\int_0^t \sin 2u\cos 3(t-u)du$ or $\frac{1}{3}\int_0^t \sin 3u\cos 2(t-u)du$
35. $\int_0^t \cos u\sin(t-u)du$
36. $\int_0^t e^{au}du$
37. $e^{-t}\int_0^t \sin(t-u)du$
38. $e^{-2t}\int_0^t ue^u\sin(t-u)du$
39. $\frac{e^{-2t}}{2}\int_0^t u^2(t-u)e^{3u}du$
40. $\int_0^t (t-u)e^u\cos udu$
41. $\int_0^t e^{-3u}\cos u\cos 2(t-u)du$
42. $\frac{1}{4!5!}\int_0^t u^4(t-u)^5e^{3u}du$
43. $\frac{1}{4}\int_0^t u^2e^u\sin 2(t-u)du$
44. $\frac{1}{2}\int_0^t u(t-u)^2e^{2(t-u)}du$
45. $\frac{1}{5!6!}\int_0^t (t-u)^5e^{2(t-u)}u^6du$
46. $y=(3t+4)\sin t-(2t+6)\cos t$
47. $y=-(2t+2)\cos 2t+\sin 2t+3\cos t$
48. $y=-\frac{1}{2}e^{-t}+\frac{1}{2}\cos t-\frac{1}{2}t\cos t+\frac{1}{2}t\sin t$
49. $y=2\cos 3t+\frac{5}{3}\sin 3t+\frac{1}{6}t\sin 3t$
50. $y=\frac{1}{4}\sin 4t+\frac{1}{8}t\sin 4t-\frac{1}{8}(t-\pi)\sin 4(t-\pi)\mathcal{U}(t-\pi)$
51. $y=e^t\sin t-te^t\cos t$
52. $y=\cos t-\frac{1}{2}\sin t-\frac{1}{2}t\cos t$
53. $y=1-(1-\cos(t-\pi/2))\mathcal{U}(t-\pi/2)+\frac{1}{2}(t-\pi/2)\sin(t-\pi/2)\mathcal{U}(t-\pi/2)$
54. $y=\sin t$
55. $y=\frac{2}{5}t+\frac{8}{5\sqrt{5}}\sin\sqrt{5}t$
56. $y=\frac{1}{8}e^t+\frac{3}{4}te^t+\frac{1}{4}t^2e^t-\frac{1}{8}e^{-t}$
57. $y=4e^{-t}-7te^{-t}+4t^2e^{-t}$
58. $y=e^{-t}$
59. $y=\cos t+\sin t$
60. $y=\frac{1}{2}\cos 2t+\frac{1}{4}\sin 2t+\frac{1}{2}\cosh 2t+\frac{1}{4}\sinh 2t$
61. $y=\frac{1}{2}t-\frac{1}{12}t^3$
62. $y=\sin t-\frac{1}{2}t\sin t$
63. $y=te^{-3t}$
64. $y=\sin t$

65. $y = te^{-t}$

66. $y = 1 + 2te^{t}$

67. $y = t + \frac{t^2}{2}$

68. $y = 4 + \frac{5}{2}t^2 + \frac{1}{24}t^4$

69. $y = 1 - t$

70. $\frac{1-e^{-as}}{s(1+e^{-as})}$

71. $\frac{1}{s(1+e^{-as})}$

72. $\frac{a}{s}\left(\frac{1}{bs} - \frac{1}{e^{bs}-1}\right)$

73. $\frac{1-e^{-s}}{s^2(1-e^{-2s})}$

74. $\frac{1}{s^2+1}\coth\frac{\pi s}{2}$

75. $\frac{1}{s^2+1}\frac{1}{1-e^{-\pi s}}$

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37: Section 10.1 Answers

1. $\mathbf{y}' = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \mathbf{y}$

2. $\mathbf{y}' = \begin{bmatrix} -2 & -2 \\ -5 & 1 \end{bmatrix} \mathbf{y}$

3. $\mathbf{y}' = \begin{bmatrix} -4 & -10 \\ 3 & 7 \end{bmatrix} \mathbf{y}$

4. $\mathbf{y}' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{y}$

5. $\mathbf{y}' = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix} \mathbf{y}$

6. $\mathbf{y}' = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix} \mathbf{y}$

7. $\mathbf{y}' = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \mathbf{y}$

8. $\mathbf{y}' = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix} \mathbf{y}$

9. $\mathbf{y}' = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

10. $\mathbf{y}' = \begin{bmatrix} 5 & 3 \\ -1 & 1 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 9 \\ -5 \end{bmatrix}$

11. $\mathbf{y}' = \begin{bmatrix} 6 & 4 & 4 \\ -7 & -2 & -1 \\ 7 & 4 & 3 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 3 \\ -6 \\ 4 \end{bmatrix}$

12. $\mathbf{y}' = \begin{bmatrix} 8 & 7 & 7 \\ -5 & -6 & -9 \\ 5 & 7 & 10 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$

13. $\mathbf{y}' = \begin{bmatrix} -3 & 2 \\ -5 & 3 \end{bmatrix} + \begin{bmatrix} 3-2t \\ 6-3t \end{bmatrix}$

14. $\mathbf{y}' = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} -5e^t \\ e^t \end{bmatrix}$

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38: Section 10.2 Answers

2. b. $\mathbf{y} = \begin{bmatrix} 3e^{6t} - 6e^{-2t} \\ 3e^{6t} + 6e^{-2t} \end{bmatrix}$

3. b. $\mathbf{y} = \begin{bmatrix} 6e^{-4t} + 4e^{3t} \\ 6e^{-4t} - 10e^{3t} \end{bmatrix}$

4. b. $\mathbf{y} = \begin{bmatrix} -15e^{2t} - 4e^{t} \\ 9e^{2t} + 2e^{t} \end{bmatrix}$

5. b. $\mathbf{y} = \begin{bmatrix} 5e^{3t} - 3e^{t} \\ 5e^{3t} + 3e^{t} \end{bmatrix}$

6. b. $\mathbf{y} = \begin{bmatrix} e^{2t} - 2e^{3t} + 3e^{-t} \\ 2e^{3t} - 9e^{-t} \\ e^{2t} - 2e^{3t} + 21e^{-t} \end{bmatrix}$

7. b. $\mathbf{y} = \frac{1}{3}\begin{bmatrix} -e^{-2t} + e^{4t} \\ -10e^{-2t} + e^{4t} \\ 11e^{-2t} + e^{4t} \end{bmatrix}$

8. b. $\mathbf{y} = \begin{bmatrix} 3e^{t} + 3e^{-t} - e^{-2t} \\ 3e^{t} + 2e^{-2t} \\ -e^{-2t} \end{bmatrix}$

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1. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$
2. $\mathbf{y} = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}$
3. $\mathbf{y} = c_1 \begin{bmatrix} 2 \\ 5 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-3t}$
4. $\mathbf{y} = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-7t/2} + c_2 \begin{bmatrix} 4 \\ 3 \end{bmatrix} e^{-t}$
5. $\mathbf{y} = c_1 \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^{8t} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-10t}$
6. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-5t}$
7. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$
8. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t/2} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$
9. $\mathbf{y} = c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-2t}$
10. $\mathbf{y} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{t}$
11. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} -4 \\ 1 \end{bmatrix} e^{3t}$
12. $\mathbf{y} = c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}$
13. $\mathbf{y} = c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}$
14. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} e^{-t}$
15. $\mathbf{y} = c_1 \begin{bmatrix} 4 \\ 0 \\ -5 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} -7 \\ 3 \\ 5 \end{bmatrix} e^{5t} + c_3 \begin{bmatrix} -7 \\ 5 \\ 5 \end{bmatrix} e^{7t}$
16. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t}$
17. $\mathbf{y} = c_1 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} e^{8t} + c_2 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} e^{-t}$
18. $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} e^{-2t}$

19. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{t} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t}$

20. $\mathbf{y} = c_1 \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -12 \\ 6 \\ 5 \end{bmatrix} e^{-t/2} + c_3 \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} e^{-3t/2}$

21. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{-5t} + c_3 \begin{bmatrix} 2 \\ -2 \\ 11 \end{bmatrix} e^{6t}$

22. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{2t}$

23. $\mathbf{y} = c_1 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} e^{-16t} + c_2 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{2t}$

24. $\mathbf{y} = c_1 \begin{bmatrix} -2 \\ -4 \\ 3 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-2t} + c_3 \begin{bmatrix} -7 \\ -5 \\ 4 \end{bmatrix} e^{2t}$

25. $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} e^{-3t} + c_3 \begin{bmatrix} -2 \\ -6 \\ 3 \end{bmatrix} e^{-5t}$

26. $\mathbf{y} = c_1 \begin{bmatrix} 11 \\ 7 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{-2t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{-t}$

27. $\mathbf{y} = c_1 \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{6t} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{4t}$

28. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{5t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{5t}$

29. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} e^{6t} + c_3 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} e^{6t}$

30. $\mathbf{y} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} e^{-t/2} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} e^{t/2}$

31. $\mathbf{y} = -\begin{bmatrix} 2 \\ 6 \end{bmatrix} e^{5t} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} e^{-5t}$

32. $\mathbf{y} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} e^{t/2} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{t}$

33. $\mathbf{y} = \begin{bmatrix} 7 \\ 7 \end{bmatrix} e^{9t} - \begin{bmatrix} 2 \\ 4 \end{bmatrix} e^{-3t}$

34. $\mathbf{y} = \begin{bmatrix} 3 \\ 9 \end{bmatrix} e^{5t} - \begin{bmatrix} 4 \\ 2 \end{bmatrix} e^{-5t}$

35. $\mathbf{y} = \begin{bmatrix} -5 \\ 3 \\ -2 \end{bmatrix} e^{2t} + \begin{bmatrix} 5 \\ 0 \\ \frac{5}{2} \end{bmatrix} e^{3t} + \begin{bmatrix} 1 \\ 0 \\ -\frac{1}{2} \end{bmatrix} e^{-t}$

36. $\mathbf{y} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} e^{-t} + \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} e^{t} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} e^{t}$

37. $\mathbf{y} = \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix} e^{t/2} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{t/2} + \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} e^{-t/2}$

38. $\mathbf{y} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} e^{t} + \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} e^{-t}$

39. $\mathbf{y} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} e^{t} - \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} e^{-2t} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{3t}$

40. $\mathbf{y} = -\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{t} + \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t}$

41. $\mathbf{y} = \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} e^{2t} - \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} e^{-2t} + \begin{bmatrix} 4 \\ 12 \\ 4 \end{bmatrix} e^{4t}$

42. $\mathbf{y} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{-6t} + \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} e^{2t} + \begin{bmatrix} 7 \\ -7 \\ -7 \end{bmatrix} e^{4t}$

43. $\mathbf{y} = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix} e^{-t} + \begin{bmatrix} 6 \\ 6 \\ -2 \end{bmatrix} e^{2t}$

44. $\mathbf{y} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ -9 \\ 6 \end{bmatrix} e^{4t} + \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} e^{2t}$

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40: Section 10.4 Answers

1. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \left(\begin{bmatrix} 1 \\ 3 \end{bmatrix} t + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$
2. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} + \begin{bmatrix} 0 \\ \frac{1}{5} \end{bmatrix} e^{-t} \right)$
3. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{2t} + \begin{bmatrix} -\frac{1}{3} \\ 0 \end{bmatrix} e^{2t} \right)$
4. $\mathbf{y} = c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{6t} + c_2 \left(\begin{bmatrix} 3 \\ 2 \end{bmatrix} te^{6t} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} e^{6t} \right)$
5. $\mathbf{y} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t} + c_2 \left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{5t} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} te^{5t} \right)$
6. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} \right)$
7. $\mathbf{y} = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-9t} + c_2 \left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{-9t} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} te^{-9t} \right)$
8. $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2t} + c_2 \left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{2t} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} te^{2t} \right)$
9. $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} \frac{e^{-2t}}{3} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} te^{-2t} \right)$
10. $\mathbf{y} = c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{-4t} + c_2 \left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} \frac{e^{-4t}}{2} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} te^{-4t} \right)$
11. $\mathbf{y} = c_1 \begin{bmatrix} 4 \\ 3 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} \frac{e^{-t}}{3} + \begin{bmatrix} 4 \\ 3 \end{bmatrix} te^{-t} \right)$
12. $\mathbf{y} = c_1 \begin{bmatrix} -4 \\ -5 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} e^{5t} + c_3 \left(\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} te^{5t} + \begin{bmatrix} \frac{5}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} e^{5t} \right)$
13. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{2t} + c_3 \left(\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} e^{2t} \right)$
14. $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} e^{4t} + c_3 \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \frac{e^{4t}}{2} + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} te^{4t} \right)$
15. $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{-t} + c_3 \left(\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} te^{-t} \right)$
16. $\mathbf{y} = c_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{-2t} + c_3 \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \frac{e^{-2t}}{2} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} te^{-2t} \right)$
17. $\mathbf{y} = c_1 \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{4t} + c_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{e^{4t}}{2} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} te^{4t} \right)$

18. $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{4t} + c_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{e^{4t}}{2} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} te^{4t} \right)$

19. $\mathbf{y} = \begin{bmatrix} -1 \\ 6 \end{bmatrix} e^{4t} + \begin{bmatrix} 26 \\ 13 \end{bmatrix} te^{4t}$

20. $\mathbf{y} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} e^{-7t} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} te^{-7t}$

21. $\mathbf{y} = \begin{bmatrix} 5 \\ 8 \end{bmatrix} e^{3t} - \begin{bmatrix} 12 \\ 16 \end{bmatrix} te^{3t}$

22. $\mathbf{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{-5t} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} te^{-5t}$

23. $\mathbf{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{5t} - \begin{bmatrix} 12 \\ 6 \end{bmatrix} te^{5t}$

24. $\mathbf{y} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} e^{-4t} + \begin{bmatrix} 6 \\ 6 \end{bmatrix} te^{-4t}$

25. $\mathbf{y} = \begin{bmatrix} 4 \\ 8 \\ -6 \end{bmatrix} e^{t} + \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} e^{-2t} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} te^{-2t}$

26. $\mathbf{y} = \begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix} e^{2t} - \begin{bmatrix} 9 \\ 5 \\ 6 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} t$

27. $\mathbf{y} = -\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} e^{-3t} + \begin{bmatrix} -4 \\ 9 \\ 1 \end{bmatrix} e^{t} - \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} te^{t}$

28. $\mathbf{y} = \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix} e^{4t} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} te^{2t}$

29. $\mathbf{y} = -\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{-4t} + \begin{bmatrix} -3 \\ 2 \\ -3 \end{bmatrix} e^{8t} + \begin{bmatrix} 8 \\ 0 \\ -8 \end{bmatrix} te^{8t}$

30. $\mathbf{y} = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} e^{4t} - \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} + \begin{bmatrix} 8 \\ 4 \\ 4 \end{bmatrix} t$

31. $\mathbf{y} = c_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{t} + c_2 \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} te^{t} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{t} \right) + c_3 \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \frac{t^2}{2} e^{t} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} te^{t} + \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} e^{t} \right)$

32. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{4t} + c_2 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} te^{4t} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{4t} \right) + c_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{t^2}{2} e^{4t} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} te^{4t} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{4t} \right)$

33. $\mathbf{y} = c_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{6t} + c_2 \left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \frac{e^{6t}}{4} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} te^{6t} \right) + c_3 \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \frac{e^{6t}}{8} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \frac{te^{6t}}{4} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \frac{t^2 e^{6t}}{2} \right)$

34. $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{e^{3t}}{2} + \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} te^{3t} \right) + c_3 \left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \frac{e^{3t}}{36} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{te^{3t}}{2} + \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \frac{t^2 e^{3t}}{2} \right)$

35. $\mathbf{y} = c_1 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-2t} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} te^{-2t} \right) + c_3 \left(\begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} \frac{e^{-2t}}{4} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} te^{-2t} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \frac{t^2 e^{-2t}}{2} \right)$

36. $\mathbf{y} = c_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \frac{e^{2t}}{2} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} te^{2t} \right) + c_3 \left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \frac{e^{2t}}{8} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \frac{te^{2t}}{2} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \frac{t^2 e^{2t}}{2} \right)$

37. $\mathbf{y} = c_1 \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} e^{-6t} + c_2 \left(- \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix} \frac{e^{-6t}}{6} + \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} te^{-6t} \right) + c_3 \left(- \begin{bmatrix} 12 \\ 1 \\ 0 \end{bmatrix} \frac{e^{-6t}}{36} - \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix} \frac{te^{-6t}}{6} + \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \frac{t^2 e^{-6t}}{2} \right)$

38. $\mathbf{y} = c_1 \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix} e^{-3t} + c_3 \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{-3t} + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} te^{-3t} \right)$

39. $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-3t} + c_3 \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{-3t} + \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} te^{-3t} \right)$

40. $\mathbf{y} = c_1 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix} e^{-t} + c_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{e^{-t}}{2} + \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} te^{-t} \right)$

41. $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-2t} + c_3 \left(\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} e^{-2t} + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} te^{-2t} \right)$

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1. $\mathbf{y} = c_1 e^{4t} \begin{bmatrix} 2\cos t - \sin t \\ 5\cos t \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 2\sin t + \cos t \\ 5\sin t \end{bmatrix}$
2. $\mathbf{y} = c_1 \begin{bmatrix} \sin t - \cos t \\ 2\cos t \end{bmatrix} + c_2 \begin{bmatrix} -\cos t - \sin t \\ 2\sin t \end{bmatrix}$
3. $\mathbf{y} = c_1 e^{4t} \begin{bmatrix} \sin t - \cos t \\ 2\cos t \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} -\sin t - \cos t \\ 2\sin t \end{bmatrix}$
4. $\mathbf{y} = c_1 e^{5t} \begin{bmatrix} \cos 3t + 3\sin 3t \\ 2\cos 3t \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} \sin 3t - 3\cos 3t \\ 2\cos 3t \end{bmatrix}$
5. $\mathbf{y} = c_1 \begin{bmatrix} 4\cos 3t - 3\sin 3t \\ 5\cos 3t \end{bmatrix} + c_2 \begin{bmatrix} 4\sin 3t + 3\cos 3t \\ 5\sin 3t \end{bmatrix}$
6. $\mathbf{y} = c_1 e^{-t} \begin{bmatrix} 2\cos 2t - 2\sin 2t \\ \cos 2t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 2\cos 2t + 2\sin 2t \\ \sin 2t \end{bmatrix}$
7. $\mathbf{y} = c_1 e^{2t} \begin{bmatrix} 3\cos t + \sin t \\ 5\cos t \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 3\sin t - \cos t \\ 5\sin t \end{bmatrix}$
8. $\mathbf{y} = c_1 e^{-t} \begin{bmatrix} 5\cos 2t + \sin 2t \\ 13\cos 2t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 5\sin 2t - \cos 2t \\ 13\sin 2t \end{bmatrix}$
9. $\mathbf{y} = c_1 e^{3t} \begin{bmatrix} \cos 2t + \sin 2t \\ 2\cos 2t \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} \sin 2t - \cos 2t \\ 2\sin 2t \end{bmatrix}$
10. $\mathbf{y} = c_1 e^{2t} \begin{bmatrix} \cos 3t - \sin 3t \\ \cos 3t \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} \sin 3t + \cos 3t \\ \sin 3t \end{bmatrix}$
11. $\mathbf{y} = c_1 e^{3t} \begin{bmatrix} \cos 6t - 3\sin 6t \\ 5\cos 6t \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} \sin 6t + 3\cos 6t \\ 5\sin 6t \end{bmatrix}$
12. $\mathbf{y} = c_1 e^{2t} \begin{bmatrix} \cos t - 3\sin t \\ 2\cos t \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} \sin t + 3\cos t \\ 2\sin t \end{bmatrix}$
13. $\mathbf{y} = c_1 e^{2t} \begin{bmatrix} 3\sin 3t - \cos 3t \\ 5\cos 3t \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} -3\cos 3t - \sin 3t \\ 5\sin 3t \end{bmatrix}$
14. $\mathbf{y} = c_1 e^{2t} \begin{bmatrix} \sin 4t - 8\cos 4t \\ 5\cos 4t \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} -\cos 4t - 8\sin 4t \\ 5\sin 4t \end{bmatrix}$
15. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ -\sin t \\ \cos t \end{bmatrix} + c_3 \begin{bmatrix} -\cos t \\ \cos t \\ \sin t \end{bmatrix}$
16. $\mathbf{y} = c_1 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} e^{-3t} + c_2 e^{t} \begin{bmatrix} -2\cos 2t + \sin 2t \\ 3\sin 2t \\ 2\cos 2t \end{bmatrix} + c_3 e^{t} \begin{bmatrix} -\cos 2t - 2\sin 2t \\ -3\cos 2t \\ 2\sin 2t \end{bmatrix}$
17. $\mathbf{y} = c_1 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} e^{t} + c_2 e^{t} \begin{bmatrix} \cos t \\ -\sin t \\ -\sin t \end{bmatrix} + c_3 e^{t} \begin{bmatrix} \sin t \\ \cos t \\ \cos t \end{bmatrix}$
18. $\mathbf{y} = c_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{6t} + c_2 e^{4t} \begin{bmatrix} \sin 2t \\ 0 \\ 2\cos 2t \end{bmatrix} + c_3 e^{4t} \begin{bmatrix} -\cos 2t \\ 0 \\ 2\sin 2t \end{bmatrix}$
19. $\mathbf{y} = c_1 \begin{bmatrix} 28 \\ -5 \\ 25 \end{bmatrix} e^{2t} + c_2 e^{-2t} \begin{bmatrix} 4\cos 3t - 3\sin 3t \\ -5\cos 3t \\ 0 \end{bmatrix} + c_3 e^{-2t} \begin{bmatrix} 4\sin 3t + 3\cos 3t \\ -5\sin 3t \\ 0 \end{bmatrix}$

20. $\mathbf{y} = c_1 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} -2\cos 2t + 2\sin 2t \\ \cos 2t \\ \cos 2t \end{bmatrix} + c_3 \begin{bmatrix} -2\cos 2t - 2\sin 2t \\ \sin 2t \\ \sin 2t \end{bmatrix}$

21. $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} e^{-2t} + c_2 e^{4t} \begin{bmatrix} \cos 2t - \sin 2t \\ \cos 2t + \sin 2t \\ 2\cos 2t \end{bmatrix} + c_3 e^{4t} \begin{bmatrix} \sin 2t + \cos 2t \\ \sin 2t - \cos 2t \\ 2\sin 2t \end{bmatrix}$

22. $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{-t} + c_2 e^{-2t} \begin{bmatrix} \cos 2t - \sin 2t \\ -\cos 2t - \sin 2t \\ 2\cos 2t \end{bmatrix} + c_3 e^{-2t} \begin{bmatrix} \sin 2t + \cos 2t \\ -\sin 2t + \cos 2t \\ 2\sin 2t \end{bmatrix}$

23. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t} + c_2 e^{t} \begin{bmatrix} -\sin t \\ \sin t \\ \cos t \end{bmatrix} + c_3 e^{t} \begin{bmatrix} \cos t \\ -\cos t \\ \sin t \end{bmatrix}$

24. $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} e^{t} + c_2 e^{-t} \begin{bmatrix} -\sin 2t - \cos 2t \\ 2\cos 2t \\ 2\cos 2t \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} \cos 2t - \sin 2t \\ 2\sin 2t \\ 2\sin 2t \end{bmatrix}$

25. $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 e^{t} \begin{bmatrix} \sin t \\ -\cos t \\ \cos t \end{bmatrix} + c_3 e^{t} \begin{bmatrix} -\cos t \\ -\sin t \\ \sin t \end{bmatrix}$

26. $\mathbf{y} = c_1 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} e^{-2t} + c_2 e^{2t} \begin{bmatrix} -\cos 3t - \sin 3t \\ -\sin 3t \\ \cos 3t \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} -\sin 3t + \cos 3t \\ \cos 3t \\ \sin 3t \end{bmatrix}$

27. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{3t} + c_2 e^{6t} \begin{bmatrix} -\sin 3t \\ \sin 3t \\ \cos 3t \end{bmatrix} + c_3 e^{6t} \begin{bmatrix} \cos 3t \\ -\cos 3t \\ \sin 3t \end{bmatrix}$

28. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{t} + c_2 e^{t} \begin{bmatrix} 2\cos t - 2\sin t \\ \cos t - \sin t \\ 2\cos t \end{bmatrix} + c_3 e^{t} \begin{bmatrix} 2\sin t + 2\cos t \\ \cos t + \sin t \\ 2\sin t \end{bmatrix}$

29. $\mathbf{y} = e^{5t} \begin{bmatrix} -2\cos 2t + 5\sin 2t \\ -12\cos 2t + \sin 2t \end{bmatrix}$

30. $\mathbf{y} = e^{t} \begin{bmatrix} 5\cos 3t + \sin 3t \\ 2\cos 3t + 3\sin 3t \end{bmatrix}$

31. $\mathbf{y} = e^{4t} \begin{bmatrix} 5\cos 6t + 5\sin 6t \\ \cos 6t - 3\sin 6t \end{bmatrix}$

32. $\mathbf{y} = e^{t} \begin{bmatrix} 17\cos 3t - \sin 3t \\ 7\cos 3t + 3\sin 3t \end{bmatrix}$

33. $\mathbf{y} = e^{t/2} \begin{bmatrix} \cos(t/2) + \sin(t/2) \\ -\cos(t/2) + 2\sin(t/2) \end{bmatrix}$

34. $\mathbf{y} = \begin{bmatrix} -25 \\ 7 \\ -6 \end{bmatrix} e^{t} + \begin{bmatrix} 29\cos 5t + 11\sin t \\ -\cos 5t + 6\sin 5t \\ -\cos 5t + 6\sin 5t \end{bmatrix}$

35. $\mathbf{y} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} e^{t} + e^{4t} \begin{bmatrix} 3\cos t + \sin t \\ \cos t - 3\sin t \\ 4\cos t - 2\sin t \end{bmatrix}$

36. $\mathbf{y} = \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix} e^{8t} + e^{2t} \begin{bmatrix} 4\cos 2t + 8\sin 2t \\ -6\sin 2t + 2\cos 2t \\ 3\cos 2t + \sin 3t \end{bmatrix}$

37. $\mathbf{y} = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} e^{-4t} + e^{4t} \begin{bmatrix} 15\cos 6t + 10\sin 6t \\ 14\cos 6t - 8\sin 6t \\ 7\cos 6t - 4\sin 6t \end{bmatrix}$

38. $\mathbf{y} = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix} e^{8t} + \begin{bmatrix} 10\cos 4t - 4\sin 4t \\ 17\cos 4t - \sin 4t \\ 3\cos 4t - 7\sin 4t \end{bmatrix}$

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1. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^t - \begin{bmatrix} 11 \\ 11 \end{bmatrix} t - \begin{bmatrix} 15 \\ 10 \end{bmatrix}$

2. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t} + \begin{bmatrix} 4 \\ 8 \end{bmatrix} t + \begin{bmatrix} 0 \\ -4 \end{bmatrix}$

3. $\mathbf{y} = c_1 \begin{bmatrix} 10 \\ 3 \end{bmatrix} e^{3t/2} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{t/2} + \begin{bmatrix} -13/2 \\ -13/4 \end{bmatrix} te^{t/2} + \begin{bmatrix} -15/2 \\ -9/4 \end{bmatrix} e^{t/2}$

4. $\mathbf{y} = c_1 \begin{bmatrix} -\sin 2t \\ 2\cos 2t \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} \cos 2t \\ 2\sin 2t \end{bmatrix} e^{2t} + \begin{bmatrix} -\frac{1}{8}\sin 2t \cos 4t - \frac{1}{8}\cos 2t \cos 4t \\ \frac{1}{4}\cos 2t \sin 4t - \frac{1}{4}\sin 2t \cos 4t \end{bmatrix} e^{2t}$

5. $\mathbf{y} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} te^t + \begin{bmatrix} 3 \\ 3 \end{bmatrix} e^t$

6. $\mathbf{y} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} 1/10 \\ -3/20 \end{bmatrix} e^{-3t} - \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

7. $\mathbf{y} = c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-3t} + \begin{bmatrix} -12 \\ 0 \end{bmatrix} t - \begin{bmatrix} 4/3 \\ 4/3 \end{bmatrix}$

8. $\mathbf{y} = c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-3t} + \begin{bmatrix} -te^t - \frac{1}{4}e^t \\ -\frac{1}{8}e^{-t} - \frac{1}{8}e^t \end{bmatrix}$

9. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} e^t \right) + \begin{bmatrix} 1/2 \\ -2 \end{bmatrix} e^{-t}$

10. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} e^t \right) + \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

11. $\mathbf{y} = c_1 \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix} + \begin{bmatrix} t\cos t + \sin t \ln|\sec t| \\ t\sin t - \cos t \ln|\sec t| \end{bmatrix}$

12. $\mathbf{y} = c_1 \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} e^t + c_2 \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} e^t + \begin{bmatrix} -3 \\ 3 \end{bmatrix} e^t$

13. $\mathbf{y} = c_1 \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} e^t + c_2 \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} e^t + \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} te^t$

14. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-2t} + c_2 \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} te^{-2t} + \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} e^{-2t} \right) + \begin{bmatrix} 2t + \ln t - 2t \ln t \\ 4t + 3\ln t - 4t \ln t \end{bmatrix} e^{-2t}$

15. $\mathbf{y} = c_1 \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} t + \begin{bmatrix} -\sin t \\ \sin t \tan t \end{bmatrix} + \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} \ln|\sec t|$

16. $\mathbf{y} = c_1 \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + \begin{bmatrix} \sin t \ln|\csc t - \cot t| \\ \cos t \ln|\csc t - \cot t| \end{bmatrix}$

17. $\mathbf{y} = c_1 \begin{bmatrix} 2\sin t \\ \cos t \end{bmatrix} e^t + c_2 \begin{bmatrix} 2\cos t \\ -\sin t \end{bmatrix} e^t + \begin{bmatrix} 3\sin t \\ \frac{3}{2}\cos t \end{bmatrix} te^t + \begin{bmatrix} \cos t \\ -\frac{1}{2}\sin t \end{bmatrix} e^t \ln|\sin t| + \begin{bmatrix} -2\cos t \\ \sin t \end{bmatrix} e^t \ln|\sec t|$

18. $\mathbf{y} = c_1 \begin{bmatrix} \cos t - \sin t \\ \cos t \end{bmatrix} + c_2 \begin{bmatrix} \cos t + \sin t \\ \sin t \end{bmatrix} + \begin{bmatrix} 3\sin t \cos t - \cos^2 t - 2\sin^2 t + (\sin t - \cos t)\ln|\sec t + \tan t| \\ \sin^2 t - \cos^2 t - \cos t \ln|\sec t + \tan t| \end{bmatrix}$

19. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{3t} + \begin{bmatrix} -\frac{1}{4}e^{2t} + \frac{1}{2}te^{2t} \\ -e^t + \frac{1}{4}e^{2t} + \frac{1}{2}te^{2t} \\ \frac{1}{2}t^2 e^{3t} \end{bmatrix}$

20. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} -1/2 \\ -1 \\ -1/2 \end{bmatrix} t + \begin{bmatrix} -3/4 \\ -1 \\ -3/4 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} te^t$

21. $\mathbf{y} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} te^{2t} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} te^{4t} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{4t}$

22. $\mathbf{y} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} t - \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \ln t$

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