TRIGONOMETRY

CK-12 Foundation

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CHAPTER OVERVIEW

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1.1: Pythagorean Theorem

Discover, geometrically prove, and apply the Pythagorean Theorem.

Lengths of Triangle Sides Using the Pythagorean Theorem

You've just signed up to be an architect's assistant in a new office downtown. You're asked to draw a scale model of a sculpture for a business plaza. The sculpture has a large triangular piece where one of the angles between the sides is ninety degrees. This type of triangle is called a ‘‘right triangle’’. The architect you're working for comes into the room and tells you that the sides of the triangle that form the right angle are 9 feet and 12 feet. Can you tell how long the third side is?

Finding the Length of Triangle Sides Using **Pythagorean Theorem**

From Geometry, recall that the Pythagorean Theorem is $a^2 + b^2 = c^2$ where a and b are the legs of a right triangle and c is the hypotenuse. Also, the side opposite the angle is lower case and the angle is upper case. For example, angle A is opposite side a.

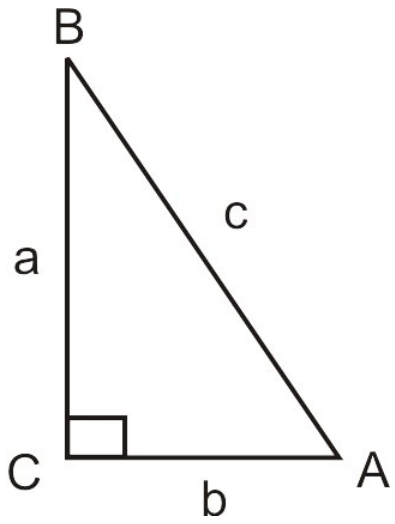
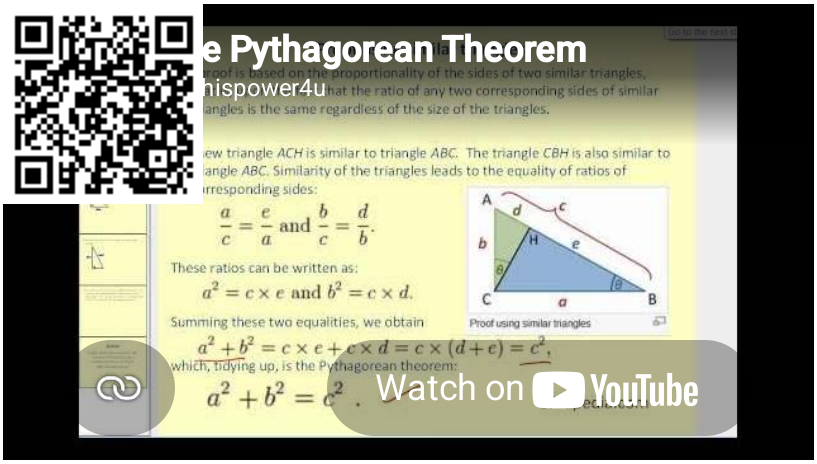


Figure **1.1.1**

The Pythagorean Theorem is used to solve for the sides of a right triangle.



Using the Pythagorean Theorem

1.

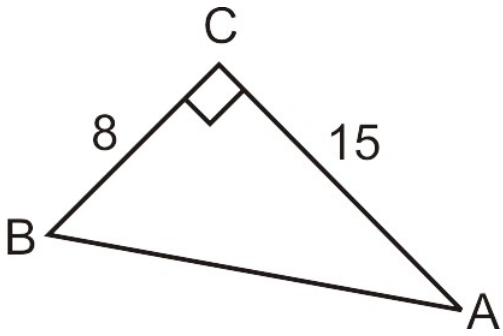


Figure 1.1.2

$a = 8, b = 15$, we need to find the hypotenuse.

$$\begin{aligned} 8^2 + 15^2 &= c^2 \\ 64 + 225 &= c^2 \\ 289 &= c^2 \\ 17 &= c \end{aligned}$$

Notice, we do not include -17 as a solution because a negative number cannot be a side of a triangle.

2.

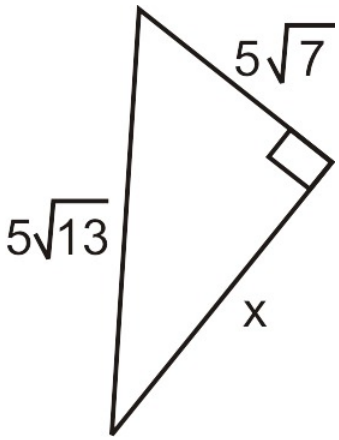


Figure 1.1.3

Use the Pythagorean Theorem to find the missing leg.

$$\begin{aligned} (5\sqrt{7})^2 + x^2 &= (5\sqrt{13})^2 \\ 25 \cdot 7 + x^2 &= 25 \cdot 13 \\ 175 + x^2 &= 325 \\ x^2 &= 150 \\ x &= 5\sqrt{6} \end{aligned}$$

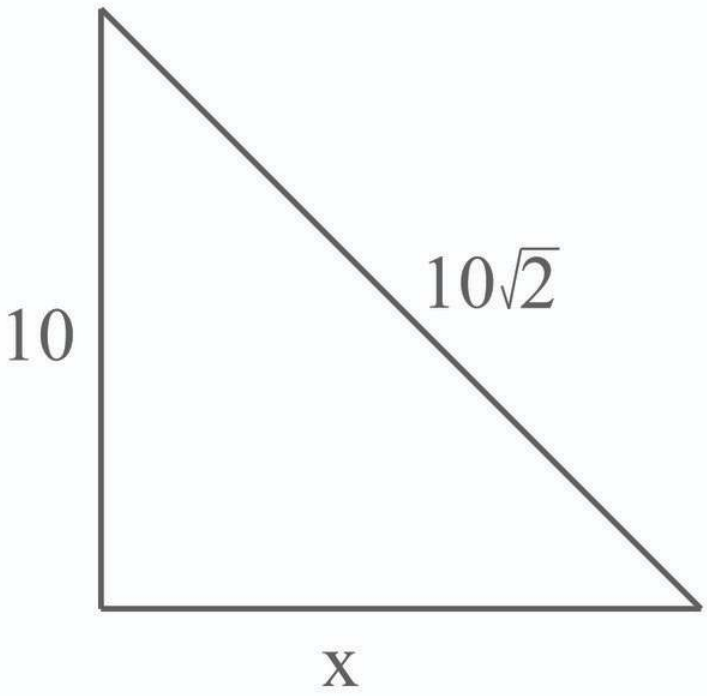


Figure 1.1.4

3. Use the Pythagorean Theorem to find the missing leg in the triangle above.

$$\begin{aligned} 10^2 + x^2 &= (10\sqrt{2})^2 \\ 100 + x^2 &= 100 \cdot 2 \\ 100 + x^2 &= 100 \\ x^2 &= 100 \\ x &= 10 \end{aligned}$$

Example 1.1.1

Earlier, you were given a problem asking you to draw a scale model of a sculpture for a business plaza.

With your knowledge of the Pythagorean Theorem, you can see that the triangle has sides with lengths 9 feet and 12 feet. You work to find the hypotenuse:

Solution

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 9^2 + 12^2 &= c^2 \\ 81 + 144 &= c^2 \\ 225 &= c^2 \\ (\sqrt{225}) &= 15 = c \end{aligned}$$

With the knowledge that the length of the third side of the triangle is 15 feet, you are able to construct your scale model with ease.

Example 1.1.2

Use the Pythagorean Theorem to find the missing side of the following triangle:

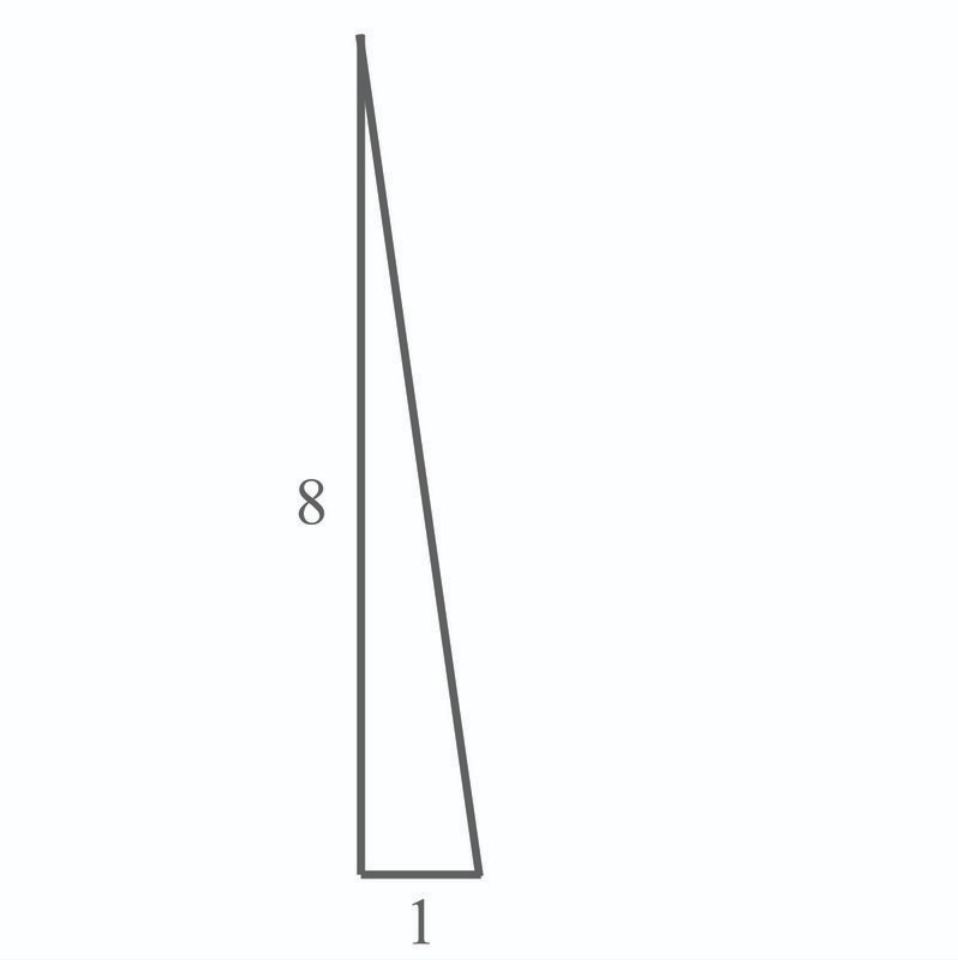


Figure 1.1.5

Solution

$a = 1$, $b = 8$, we need to find the hypotenuse.

$$1^2 + 8^2 = c^2$$
$$1 + 64 = c^2$$
$$65 = c^2$$
$$\sqrt{65} = c$$

Example 1.1.3

Use the Pythagorean Theorem to find the missing side of the following triangle:

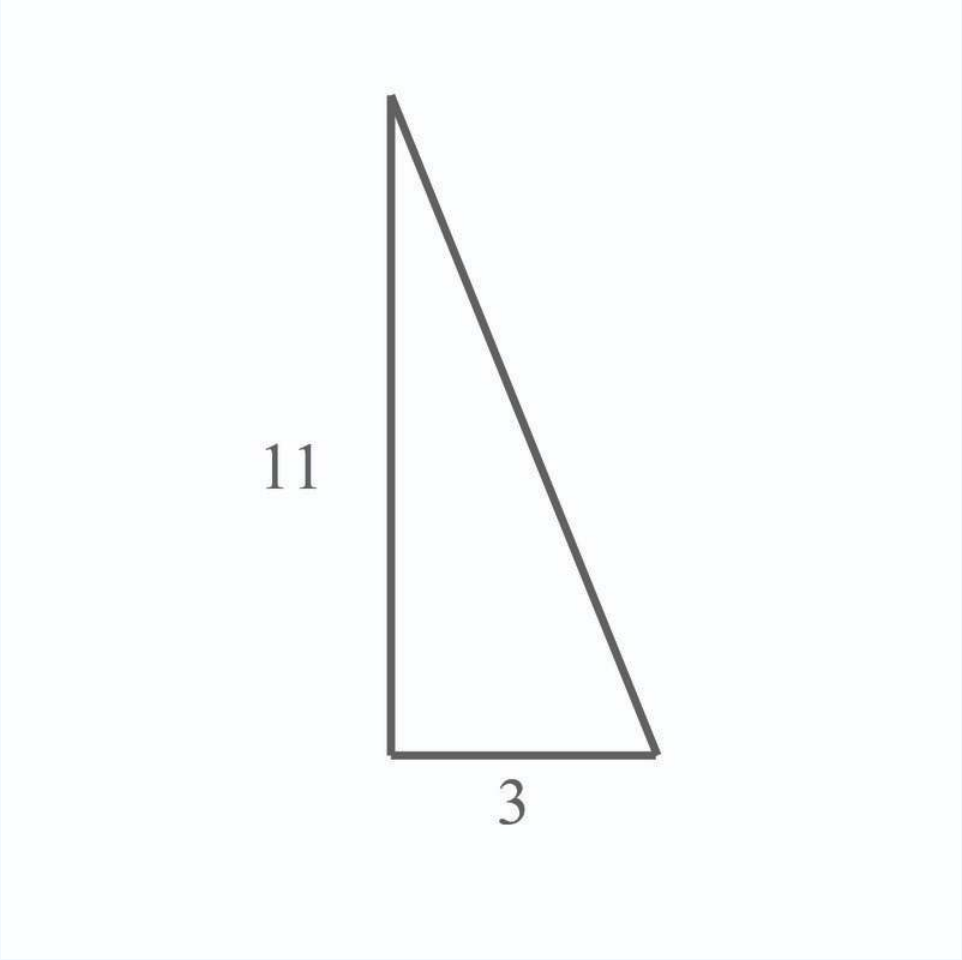


Figure 1.1.6

Solution

$a = 3, b = 11$, we need to find the length of side c, the hypotenuse.

$$3^2 + 11^2 = c^2$$
$$9 + 121 = c^2$$
$$130 = c^2$$
$$\sqrt{130} = c$$

Example 1.1.4

Find the missing side of the right triangle below. Leave the answer in simplest radical form.

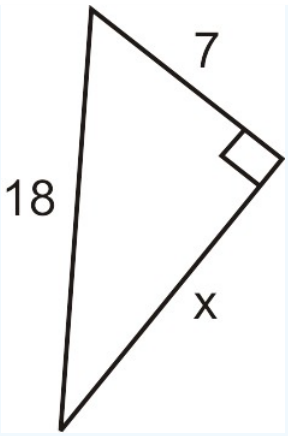


Figure 1.1.7

Solution

$a = 7$, $c = 18$, we need to find the length of side b.

$$7^2 + b^2 = 18^2$$
$$49 + b^2 = 18^2$$
$$324 - 49 = b^2$$
$$275 = b^2$$
$$\sqrt{275} = b$$

Review

Find the missing sides of the right triangles. Leave answers in simplest radical form.

1. If the legs of a right triangle are 3 and 4, then the hypotenuse is _____.
2. If the legs of a right triangle are 6 and 8, then the hypotenuse is _____.
3. If the legs of a right triangle are 5 and 12, then the hypotenuse is _____.
4. If the sides of a square are length 6, then the diagonal is _____.
5. If the sides of a square are 9, then the diagonal is _____.
6. If the sides of a square are x, then the diagonal is _____.
7. If the legs of a right triangle are 3 and 7, then the hypotenuse is _____.
8. If the legs of a right triangle are $2\sqrt{5}$ and 6, then the hypotenuse is _____.
9. If one leg of a right triangle is 4 and the hypotenuse is 8, then the other leg is _____.
10. If one leg of a right triangle is 10 and the hypotenuse is 15, then the other leg is _____.
11. If one leg of a right triangle is $4\sqrt{7}$ and the hypotenuse is $10\sqrt{6}$, then the other leg is _____.
12. If the legs of a right triangle are x and y, then the hypotenuse is _____.

Pythagorean Theorem Proof

Use the picture below to answer the following questions.

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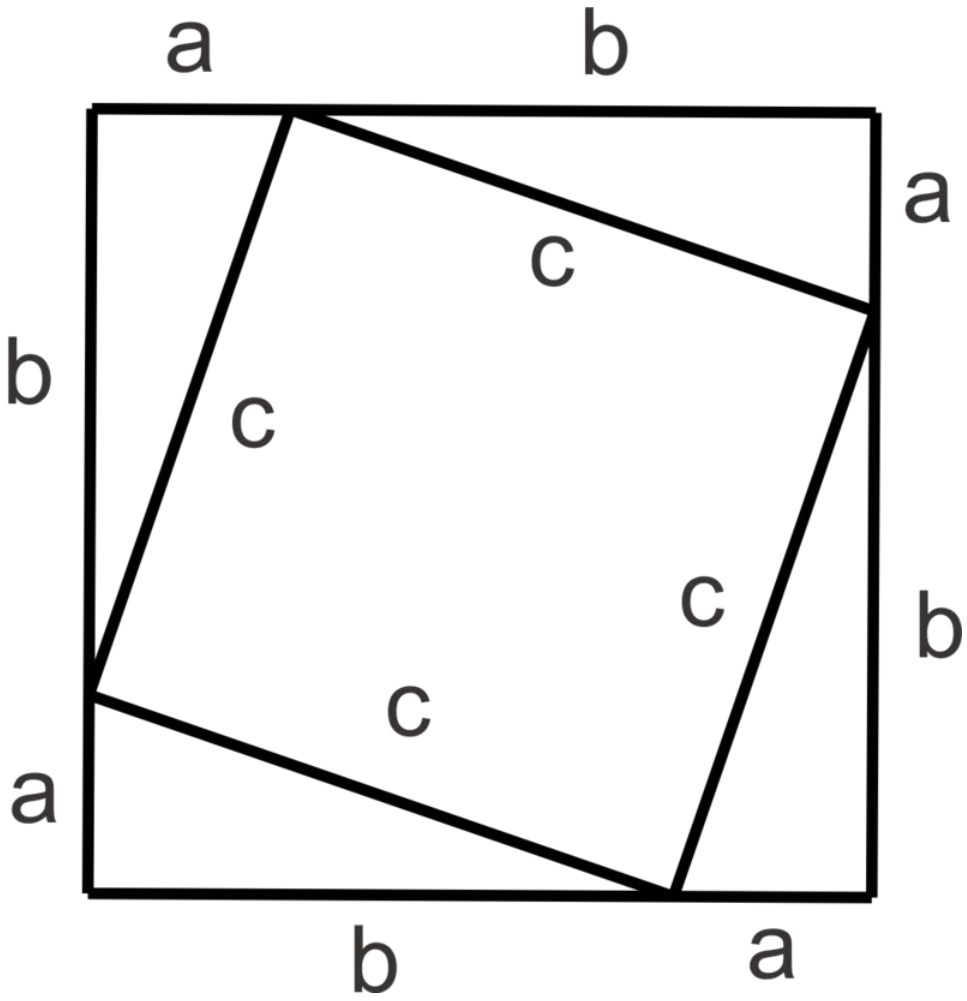


Figure **1.1.8**

13. Find the area of the square in the picture with sides (a+b).
14. Find the sum of the areas of the square with sides c and the right triangles with legs a and b.
15. Explain why the areas found in the previous two problems should be the same value. Then, set the expressions equal to each other and simplify to get the Pythagorean Theorem.

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.1.

Vocabulary

Term	Definition
Pythagorean Theorem	The Pythagorean Theorem is a mathematical relationship between the sides of a right triangle, given by $a^2 + b^2 = c^2$, where a and b are legs of the triangle and c is the hypotenuse of the triangle.

Additional Resources

Video: Proving the Pythagorean Theorem

Practice: Pythagorean Theorem

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1.2: Converse of the Pythagorean Theorem

Identify right triangles using the theorem

Derive and Use the Converse of the Pythagorean Theorem

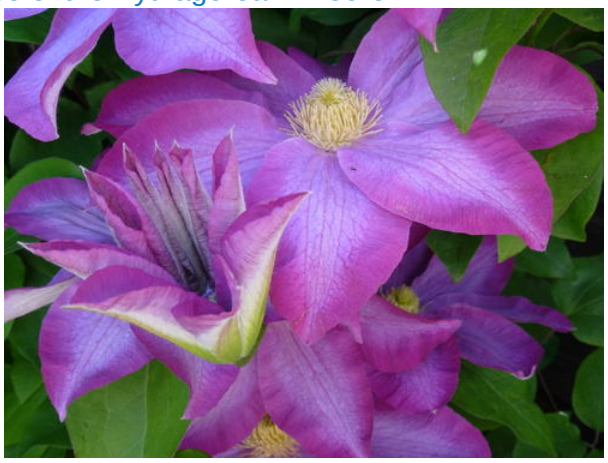


Figure 1.2.1

Brandon has just purchased his first home and is very anxious to plant a flower garden in a triangular section of the backyard. Brandon has inherited a metal fence that his grandfather had made many years ago and he would like to use it in his garden. The fence consists of two pieces that are welded together to form a right angle and another long straight piece. He knows that the pieces of fence will fit in the garden if his triangular garden is a **right triangle**.

"If only I could use the garden measurements of 29 feet, 20 feet and 29 feet to figure out if my garden space forms a right triangle, then I could use Grandpa's fence." he thought to himself.

In this concept, you will learn to derive and use the converse of the **Pythagorean Theorem**.

The Converse of the Pythagorean Theorem

The **Pythagorean Theorem** states that for a right triangle the square of the length of the **hypotenuse** is equal to the sum of the squares of the lengths of the other two sides. This theorem can be modeled by the equation $c^2 = a^2 + b^2$ where 'c' represents the length of the hypotenuse, 'a' represents the length of one leg and 'b' represents the length of the other leg.

The Pythagorean Theorem can be used to calculate the length of an unknown side of a right triangle. The **Converse of the Pythagorean Theorem** states that if the square of the length of one side of a triangle is equal to the sum of the squares of the lengths of the other two sides, then the angle opposite the longest side is a right angle. A triangle that contains a right angle is a right triangle.

Let's look at the converse of the Pythagorean Theorem in action.

Determine whether or not the following triangles (which may not be drawn to scale) are right triangles. If the triangle is a right triangle, make any necessary changes to the triangle and draw it correctly.

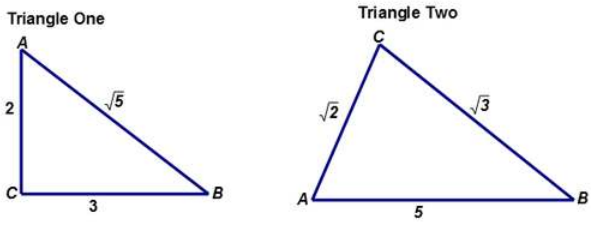


Figure 1.2.2

Let's begin with Triangle One.

First, determine the longest side of the triangle.

The lengths of the sides are 2, 3 and $\sqrt{5}$. The length of side BC is $\sqrt{5} \approx 2.24$ units. The longest side of the triangle is AB which will be the hypotenuse of the triangle if ΔABC is a right triangle.

Next, substitute the values $(2, \sqrt{5}, 3)$ for (a, b, c)respectively into the equation c^2=a^2+b^2.

c^2&=a^2+b^2 \ (3)^2&=(2)^2+(\sqrt{5})^2

Next, square each of the terms indicated in the equation.

$$(3)^2 = (2)^2 + (\sqrt{5})^2$$
$$(3 \times 3) = (2 \times 2) + (\sqrt{5} \times \sqrt{5})$$

Next, evaluate each of the squares.

$$(3 \times 3) = (2 \times 2) + (\sqrt{5} \times \sqrt{5})$$
$$9 = 4 + 5$$

Then, simplify the right side of the equation by adding the numbers.

$$9 = 4 + 5$$
$$9 = 9$$

Both sides of the equation are equal. The square of the length of one side of the triangle is equal to the sum of the squares of the lengths of the other two sides. Triangle One is a right triangle. The length of BC should be 3 and the length of AB should be $\sqrt{5}$ since the right angle is located at C.

Let's look at Triangle Two.

First, determine the longest side of the triangle.

The lengths of the sides are $\sqrt{2}$, $\sqrt{3}$ and 5. The length of side AC is $\sqrt{2}$, approximately 1.41 units in length. The length of side BC is $\sqrt{3}$, approximately 1.73 units in length. The longest side of the triangle is AB which will be the hypotenuse of the triangle if ΔABC is a right triangle.

Next, substitute the values $(\sqrt{2}, \sqrt{3}, 5)$ for (a, b, c) respectively into the equation $c^2 = a^2 + b^2$.

\(

$$c^2 = a^2 + b^2$$
$$(5)2 = (\sqrt{2})^2 + (\sqrt{3})^2 \qquad (1.2.1)$$

、)

Next, square each of the terms indicated in the equation.

$$(5)^2 = (\sqrt{2})^2 + (\sqrt{3})^2$$
$$(5 \times 5) = (\sqrt{2} \times \sqrt{2}) + (\sqrt{3} \times \sqrt{3})$$

Next, evaluate each of the squares.

$$(5 \times 5) = (\sqrt{2} \times \sqrt{2}) + (\sqrt{3} \times \sqrt{3})$$
$$25 = 2 + 3$$

Then, simplify the right side of the equation by adding the numbers.

$$25 = 2 + 3$$
$$25 \neq 5$$

Both sides of the equation are not equal. The square of the length of one side of the triangle is NOT equal to the sum of the squares of the lengths of the other two sides. Triangle Two is not a right triangle.

Example 1.2.1

Earlier, you were given a problem about Brandon's flower garden and his metal fence. He needs to figure out if his garden space is a right triangle. How can Brandon figure this out using the measurements of the space?

Solution

First, determine the values for (a,b,c) of a right triangle using the measurements of 29, 20 and 21 feet.

The longest side will represent 'c' the hypotenuse.

$$a = 20$$
$$b = 21$$
$$c = 29$$

Next, substitute the given values into the Pythagorean Theorem.

$$c^2 = a^2 + b^2$$
$$(29)^2 = (20)^2 + (21)^2$$

Next, square each of the terms indicated in the equation.

$$(29)^2 = (20)^2 + (21)^2$$
$$(29 \times 29) = (20 \times 20) + (21 \times 21)$$

Next, evaluate each of the squares and simplify the equation.

$$(29 \times 29) = (20 \times 20) + (21 \times 21)$$
$$841 = 400 + 441$$
$$841 = 841$$

The square of the length of one side of the triangle is equal to the sum of the squares of the lengths of the other two sides. The garden is a right triangle.

Brandon can use his Grandpa's fence for his flower garden.

Example 1.2.2

Are the set of numbers (5, 8, 10) the lengths of the sides of a right triangle?

Solution

First, determine the values for each side of the right triangle. The length of the longest side will represent 'c,' the hypotenuse. The other two lengths will represent the lengths of the legs 'a' and 'b.'

$$a = 5$$
$$b = 8$$
$$c = 10$$

Next, substitute the given values into the Pythagorean Theorem.

$$c^2 = a^2 + b^2$$
$$(10)^2 = (5)^2 + (8)^2$$

Next, square each of the terms indicated in the equation.

$$(10)^2 = (5)^2 + (8)^2$$
$$(10 \times 10) = (5 \times 5) + (8 \times 8)$$

Next, evaluate each of the squares.

$$(10 \times 10) = (5 \times 5) + (8 \times 8)$$
$$100 = 25 + 64$$

Then, simplify the right side of the equation by adding the numbers.

$$100 = 25 + 64$$
$$100 \neq 89$$

Both sides of the equation are not equal. The square of the length of one side of the triangle is NOT equal to the sum of the squares of the lengths of the other two sides.

The set of numbers (5, 8, 10) are not the lengths of the sides of a right triangle.

Example 1.2.3

Is the following triangle (which may or may not be drawn to scale) a right triangle?

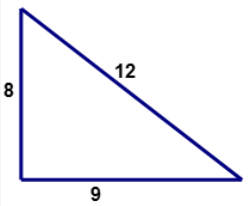


Figure 1.2.3

Solution

First, determine the values for (a,b,c) of a right triangle. The longest side will represent 'c' the hypotenuse.

$$a = 8$$
$$b = 9$$
$$c = 12$$

Next, substitute the given values into the Pythagorean Theorem.

$$c^2 = a^2 + b^2$$
$$(12)^2 = (8)^2 + (9)^2$$

Next, square each of the terms indicated in the equation.

$$(12)^2 = (8)^2 + (9)^2$$
$$(12 \times 12) = (8 \times 8) + (9 \times 9)$$

Next, evaluate each of the squares.

$$(12 \times 12) = (8 \times 8) + (9 \times 9)$$
$$144 = 64 + 81$$

Then, simplify the right side of the equation by adding the numbers.

$$144 = 64 + 81$$
$$144 \neq 145$$

Both sides of the equation are not equal. The square of the length of one side of the triangle is NOT equal to the sum of the squares of the lengths of the other two sides. The triangle is not a right triangle.

Example 1.2.4

Is the following triangle (which may or may not be drawn to scale) a right triangle?

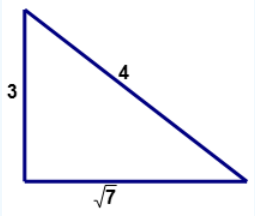


Figure 1.2.4

Solution

First, determine the values for (a,b,c) of a right triangle. The longest side will represent 'c' the hypotenuse.

$$\sqrt{7} \approx 2.65$$
$$a = 3$$
$$b = \sqrt{7}$$
$$c = 4$$

Next, substitute the given values into the Pythagorean Theorem.

$$c^2 = a^2 + b^2$$
$$(4)^2 = (3)^2 + (\sqrt{7})^2$$

Next, square each of the terms indicated in the equation.

$$(4)^2 = (3)^2 + (\sqrt{7})^2$$
$$(4 \times 4) = (3 \times 3) + (\sqrt{7} \times \sqrt{7})$$

Next, evaluate each of the squares.

$$(4 \times 4) = (3 \times 3) + (\sqrt{7} \times \sqrt{7})$$
$$16 = 9 + 7$$

Then, simplify the right side of the equation by adding the numbers.

$$16 = 9 + 7$$
$$16 = 16$$

Both sides of the equation are equal. The square of the length of one side of the triangle is equal to the sum of the squares of the lengths of the other two sides. The triangle is a right triangle.

Review

Think about what you have learned about the Pythagorean Theorem and answer true or false for the following questions.

1. The Pythagorean Theorem will work for an acute triangle with all 60° angles.
2. The Pythagorean Theorem will work for a right triangle.
3. The Pythagorean Theorem will only work if the triangle is a right triangle.
4. The **legs of a right triangle** are considered the two shorter sides of the right triangle.
5. The hypotenuse is the longest side of a right triangle.
6. The converse of the Pythagorean Theorem is used to find the angle measures of an obtuse triangle.
7. A **Pythagorean Triple** is when you multiply all of the angle measures by three.
8. You can use the Pythagorean Theorem to figure out if the side lengths of a triangle make it a right triangle or not.

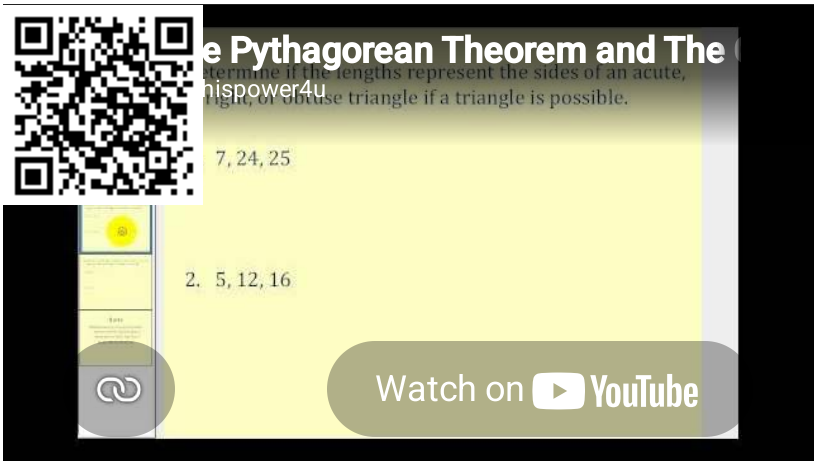
Identify whether or not each of the following values is a Pythagorean Triple. Write yes or no for your answer.

9. (4, 5, 6)
10. (6, 8, 10)
11. (5, 6, 9)
12. (9, 12, 15)
13. (30, 40, 55)
14. (21, 28, 35)
15. (12, 16, 20).

Review (Answers)

To see the Review answers, open this PDF file and look for section 7.7.

Resources



Additional Resources

Video: The Pythagorean Theorem and The Converse of the Pythagorean Theorem

Practice: Converse of the Pythagorean Theorem

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1.3: Pythagorean Triples

Integer triples that make right triangles.

While working as an architect's assistant, you're asked to utilize your knowledge of the Pythagorean Theorem to determine if the lengths of a particular triangular brace support qualify as a **Pythagorean Triple**. You measure the sides of the brace and find them to be 7 inches, 24 inches, and 25 inches. Can you determine if the lengths of the sides of the triangular brace qualify as a Pythagorean Triple?

Pythagorean Triples

Pythagorean Triples are sets of whole numbers for which the Pythagorean Theorem holds true. The most well-known triple is 3, 4, 5. This means that 3 and 4 are the lengths of the legs and 5 is the hypotenuse. *The largest length is always the hypotenuse*. If we were to multiply any triple by a constant, this new triple would still represent sides of a right triangle. Therefore, 6, 8, 10 and 15, 20, 25, among countless others, would represent sides of a right triangle.



Identifying Pythagorean Triples

Determine if the following lengths are Pythagorean Triples.

1. 7, 24, 25

Plug the given numbers into the Pythagorean Theorem.

$$\begin{aligned} 7^2 + 24^2 &\stackrel{?}{=} 25^2 \\ 49 + 576 &= 625 \\ 625 &= 625 \end{aligned}$$

Yes, 7, 24, 25 is a Pythagorean Triple and sides of a right triangle.

2. 9, 40, 41

Plug the given numbers into the Pythagorean Theorem.

$$\begin{aligned} 9^2 + 40^2 &\stackrel{?}{=} 41^2 \\ 81 + 1600 &= 1681 \\ 1681 &= 1681 \end{aligned}$$

Yes, 9, 40, 41 is a Pythagorean Triple and sides of a right triangle.

3. 11, 56, 57

Plug the given numbers into the Pythagorean Theorem.

$$\begin{aligned} 11^2 + 56^2 &\stackrel{?}{=} 57^2 \\ 121 + 3136 &= 3249 \\ 3257 &\neq 3249 \end{aligned}$$

No, 11, 56, 57 do not represent the sides of a right triangle.

Example 1.3.1

Earlier, you were asked to utilize your knowledge of the Pythagorean Theorem to determine if the lengths of a particular triangular brace support qualify as a Pythagorean Triple.

Solution

Since you know that the sides of the brace have lengths of 7, 24, and 25 inches, you can substitute these values in the Pythagorean Theorem. If the Pythagorean Theorem is satisfied, then you know with certainty that these are indeed sides of a triangle with a right angle:

$$\begin{aligned} 7^2 + 24^2 &\stackrel{?}{=} 25^2 \\ 49 + 576 &= 625 \\ 625 &= 625 \end{aligned}$$

The Pythagorean Theorem is satisfied with these values as a lengths of sides of a right triangle. Since each of the sides is a whole number, this is indeed a set of Pythagorean Triples.

Example 1.3.2

Determine if the following lengths are Pythagorean Triples.

5, 10, 13

Solution

Plug the given numbers into the Pythagorean Theorem.

$$\begin{aligned} 5^2 + 10^2 &\stackrel{?}{=} 13^2 \\ 25 + 100 &= 169 \\ 125 &\neq 169 \end{aligned}$$

No, 5, 10, 13 is not a Pythagorean Triple and not the sides of a right triangle.

Example 1.3.3

Determine if the following lengths are Pythagorean Triples.

8, 15, 17

Solution

Plug the given numbers into the Pythagorean Theorem.

$$\begin{aligned} 8^2 + 15^2 &\stackrel{?}{=} 17^2 \\ 64 + 225 &= 289 \\ 289 &= 289 \end{aligned}$$

Yes, 8, 15, 17 is a Pythagorean Triple and sides of a right triangle.

Example 1.3.4

Determine if the following lengths are Pythagorean Triples.

11, 60, 61

Solution

Plug the given numbers into the Pythagorean Theorem.

$$11^2 + 60^2 \stackrel{?}{=} 61^2$$
$$121 + 3600 = 3721$$
$$3721 = 3721$$

Yes, 11, 60, 61 is a Pythagorean Triple and sides of a right triangle.

Review

1. Determine if the following lengths are Pythagorean Triples: 9, 12, 15.
2. Determine if the following lengths are Pythagorean Triples: 10, 24, 36.
3. Determine if the following lengths are Pythagorean Triples: 4, 6, 8.
4. Determine if the following lengths are Pythagorean Triples: 20, 99, 101.
5. Determine if the following lengths are Pythagorean Triples: 21, 99, 101.
6. Determine if the following lengths are Pythagorean Triples: 65, 72, 97.
7. Determine if the following lengths are Pythagorean Triples: 15, 30, 62.
8. Determine if the following lengths are Pythagorean Triples: 9, 39, 40.
9. Determine if the following lengths are Pythagorean Triples: 48, 55, 73.
10. Determine if the following lengths are Pythagorean Triples: 8, 15, 17.
11. Determine if the following lengths are Pythagorean Triples: 13, 84, 85.
12. Determine if the following lengths are Pythagorean Triples: 15, 16, 24.
13. Explain why it might be useful to know some of the basic Pythagorean Triples.
14. Prove that any multiple of 5, 12, 13 will be a Pythagorean Triple.
15. Prove that any multiple of 3, 4, 5 will be a Pythagorean Triple.

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.2.

Vocabulary

Term	Definition
Pythagorean Triple	A Pythagorean Triple is a set of three whole numbers a, b and c that satisfy the Pythagorean Theorem, $a^2 + b^2 = c^2$.

Additional Resources

Video: Pythagorean Triples

Practice: Pythagorean Triples

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1.4: Applications Using the Pythagorean Theorem

Word problems using the Pythagorean Theorem

The Pythagorean Theorem and its **converse** have many applications for finding lengths and distances.

Real-World Application: Diagonal Length

Maria has a rectangular cookie sheet that measures 10 inches×14 inches. Find the length of the diagonal of the cookie sheet.

Draw a sketch:

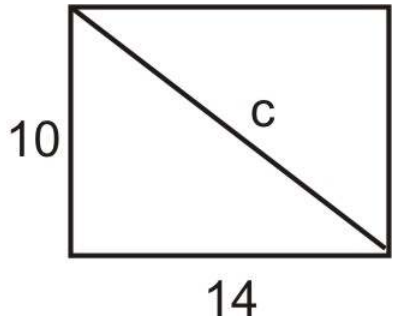


Figure 1.4.1

Define variables: Let $c =$ length of the diagonal.

Write a formula: Use the Pythagorean Theorem: $a^2 + b^2 = c^2$

Solve the equation:

$$10^2 + 14^2 = c^2$$
$$100 + 196 = c^2$$
$$c^2 = 296 \Rightarrow c = \sqrt{296} \Rightarrow c = 2\sqrt{74} \text{ or } c = 17.2 \text{ inches}$$

Check: $10^2 + 14^2 = 100 + 196 = 296$ and $c^2 = 17.2^2 = 296$. The solution checks out.

Real-World Application: Area of Shaded Regions

Find the area of the shaded region in the following diagram:

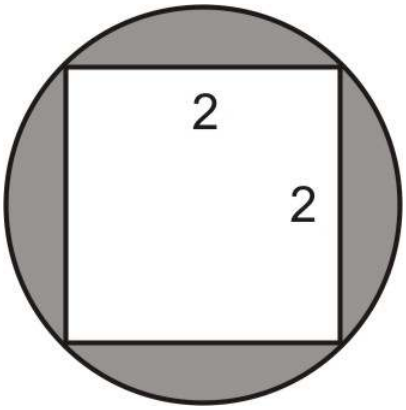


Figure 1.4.2

Draw the diagonal of the square in the figure:

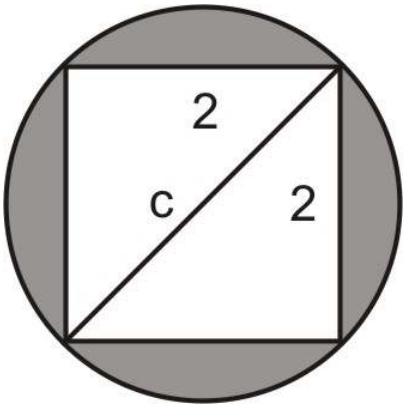


Figure 1.4.3

Notice that the diagonal of the square is also the diameter of the circle.

Define variables: Let $c =$ **diameter of the circle.**

Write the formula: Use the Pythagorean Theorem: $a^2 + b^2 = c^2$.

Solve the equation:

$$2^2 + 2^2 = c^2$$
$$4 + 4 = c^2$$
$$c^2 = 8 \Rightarrow c = \sqrt{8} \Rightarrow c = 2\sqrt{2}$$

The diameter of the circle is $2\sqrt{2}$, therefore the radius $R = \sqrt{2}$.

Area of a circle formula: $A = \pi \cdot R^2 = \pi(\sqrt{2})^2 = 2\pi$.

The area of the shaded region is therefore $2\pi - 4 = 2.28$.

Real-World Application: Measuring the Sides of a Triangle

In a right triangle, one leg is twice as long as the other and the perimeter is 28. What are the measures of the sides of the triangle?

Make a sketch and define variables:

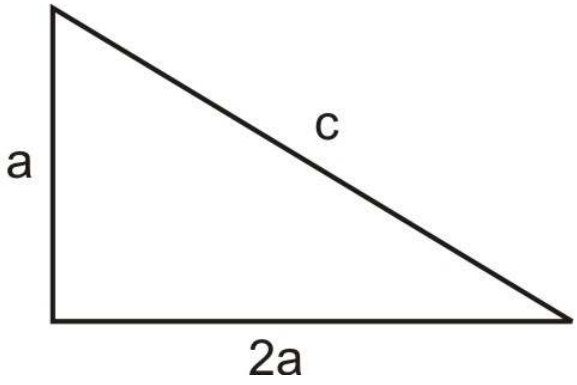


Figure 1.4.4

Let: $a =$ **length of the short leg**

$2a =$ **length of the long leg**

$c =$ **length of the hypotenuse**

Write formulas:

The sides of the triangle are related in two different ways.

The perimeter is 28, so $a + 2a + c = 28 \Rightarrow 3a + c = 28$

The triangle is a right triangle, so the measures of the sides must satisfy the Pythagorean Theorem:

$$a^2 + (2a)^2 = c^2 \Rightarrow a^2 + 4a^2 = c^2 \Rightarrow 5a^2 = c^2$$
$$\text{or} \qquad c = a\sqrt{5} = 2.236a$$

Solve the equation:

Plug the value of c we just obtained into the perimeter equation: $3a + c = 28$

$$3a + 2.236a = 28 \Rightarrow 5.236a = 28 \Rightarrow a = 5.35$$

The short leg is: $a = 5.35$

The long leg is: $2a = 10.70$

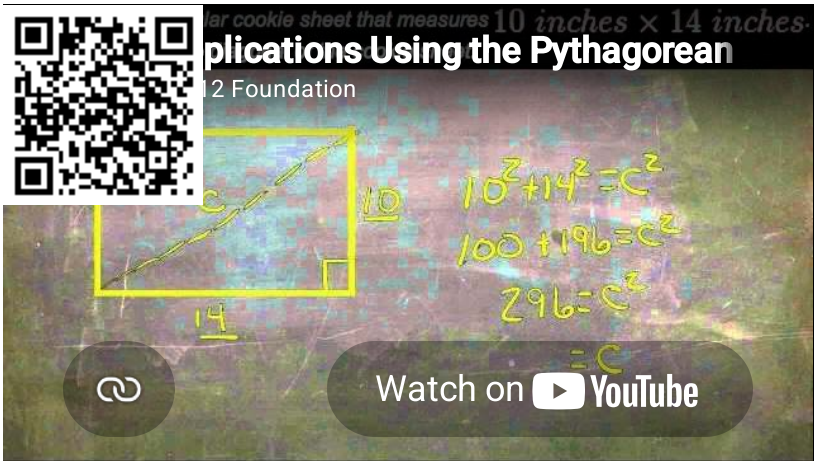
The hypotenuse is: $c = 11.95$

Check: The legs of the triangle should satisfy the Pythagorean Theorem:

$a^2 + b^2 = 5.35^2 + 10.70^2 = 143.1$, $c^2 = 11.95^2 = 142.80$. The results are approximately the same.

The perimeter of the triangle should be 28:

$a + b + c = 5.35 + 10.70 + 11.95 = 28$. **The answer checks out.**



Example 1.4.1

Mike is loading a moving van by walking up a ramp. The ramp is 10 feet long and the bed of the van is 2.5 feet above the ground. How far does the ramp extend past the back of the van?

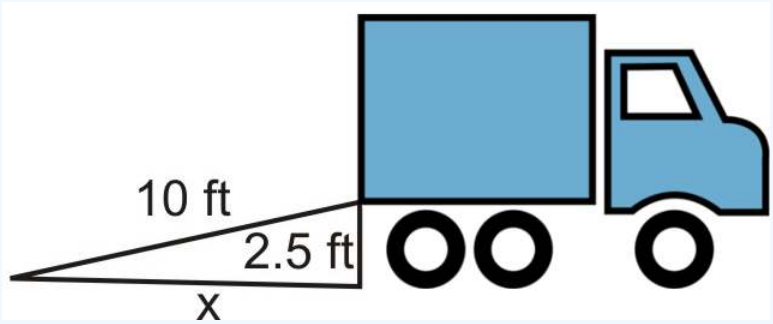


Figure 1.4.5

Solution

Define variables: Let $x =$ **how far the ramp extends past the back of the van** .

Write a formula: Use the Pythagorean Theorem: $x^2 + 2.5^2 = 10^2$

Solve the equation:

$$x^2 + 6.25 = 100$$
$$x^2 = 93.5$$
$$x = \sqrt{93.5} = 9.7 \text{ ft}$$

Check by plugging the result in the Pythagorean Theorem:

$9.7^2 + 2.5^2 = 94.09 + 6.25 = 100.34 \approx 100$. So the ramp is 10 feet long. **The answer checks out.**

Review

1. In order to make a ramp that is 3 ft high and covers 4 ft of ground, how long must the ramp be?
2. A regulation baseball diamond is a square with 90 feet between bases. How far is second base from home plate?
3. Emanuel has a cardboard box that measures 20 cm long $\times 10$ cm wide $\times 8$ cm deep .
    1. What is the length of the diagonal across the bottom of the box?
    2. What is the length of the diagonal from a bottom corner to the opposite top corner?
4. Samuel places a ladder against his house. The base of the ladder is 6 feet from the house and the ladder is 10 feet long.
    1. How high above the ground does the ladder touch the wall of the house?
    2. If the edge of the roof is 10 feet off the ground and sticks out 1.5 feet beyond the wall, how far is it from the edge of the roof to the top of the ladder?
5. Find the area of the triangle below if the area of a triangle is defined as $A = \frac{1}{2}$ base $\times$ height:

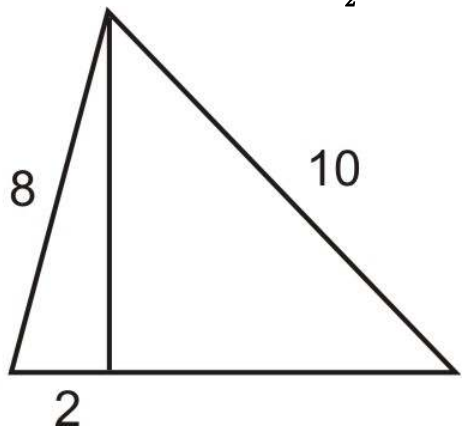


Figure **1.4.6**

6. Instead of walking along the two sides of a rectangular field, Mario decided to cut across the diagonal. He thus saves a distance that is half of the long side of the field.
    a. Find the length of the long side of the field given that the short side is 123 feet.
    b. Find the length of the diagonal.
7. Marcus sails due north and Sandra sails due east from the same starting point. In two hours Marcus' boat is 35 miles from the starting point and Sandra's boat is 28 miles from the starting point.
    a. How far are the boats from each other?
    b. Sandra then sails 21 miles due north while Marcus stays put. How far is Sandra from the original starting point?
    c. How far is Sandra from Marcus now?
8. Determine the area of the circle below. (Hint: the hypotenuse of the triangle is the diameter of the circle.)

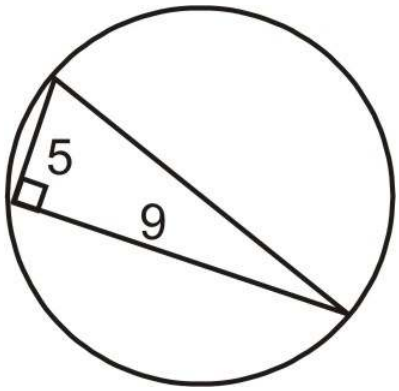


Figure 1.4.7

9. A rectangle's length is 1 in longer than its width and the diagonal has a length of 29 in. What are the lengths of the sides of the rectangle?
10. For an isosceles triangle with sides of the length given, find the length of each hypotenuse:
    a. 1
    b. 2
    c. 3
    d. n

Review (Answers)

To view the Review answers, open this PDF file and look for section 11.10.

Vocabulary

Term	Definition
converse	If a conditional statement is $p \rightarrow q$ (if p, then q), then the converse is $q \rightarrow p$ (if q, then p). Note that the converse of a statement is not true just because the original statement is true.

Additional Resources

Video: Solving Word Problems Involving Square Roots - Overview

Practice: Applications Using the Pythagorean Theorem

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1.5: The Pythagorean Theorem for Area and Perimeter

Find missing sides to calculate the area

Applications of the **Pythagorean Theorem**

Find the Height of an Isosceles Triangle

One way to use The Pythagorean Theorem is to find the height of an isosceles triangle (see Example 1).

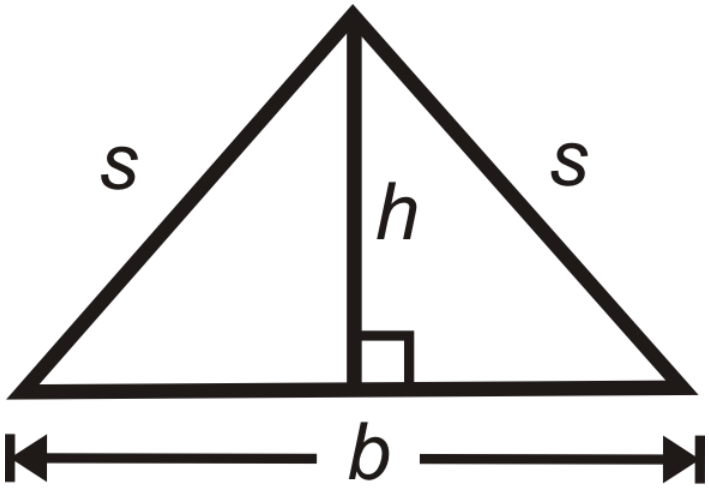


Figure **1.5.1**

Prove the **Distance Formula**

Another application of the Pythagorean Theorem is the Distance Formula. We will prove it here.

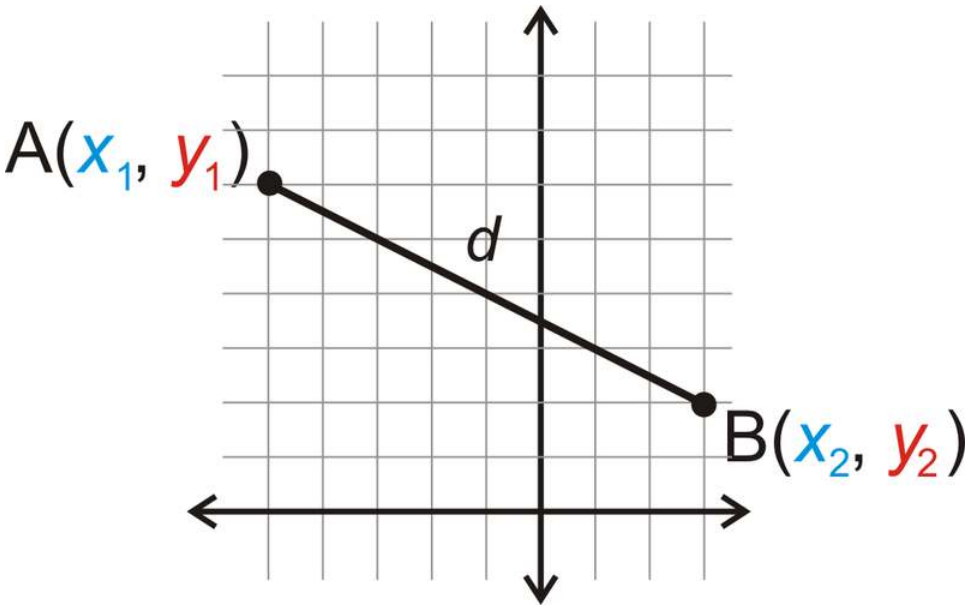


Figure **1.5.2**

Let's start with point $A(x_1, y_1)$ and point $B(x_2, y_2)$. We will call the distance between A and B, d.

Draw the vertical and horizontal lengths to make a right triangle.

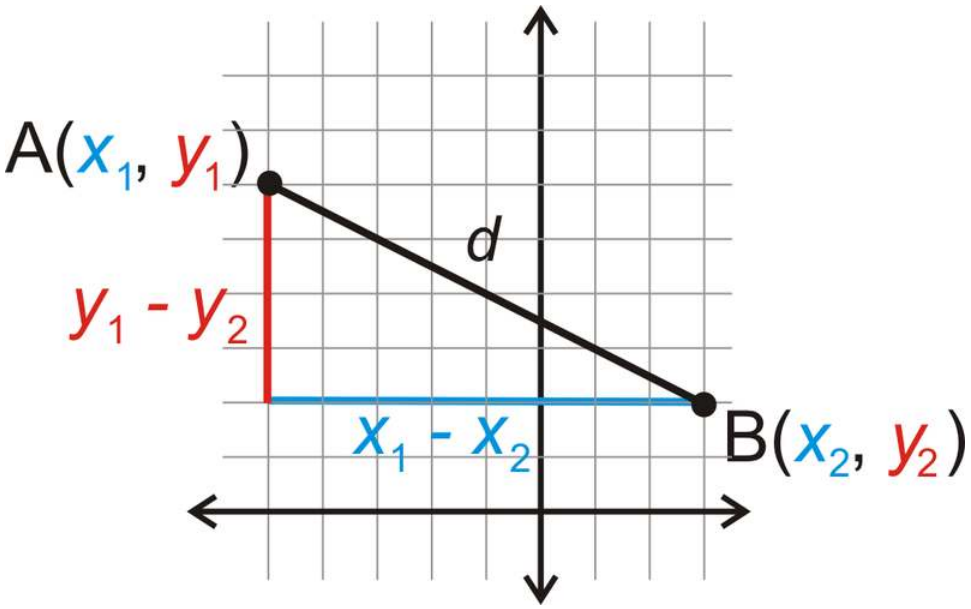


Figure 1.5.3

Now that we have a right triangle, we can use the Pythagorean Theorem to find the hypotenuse, d.

$$d^2 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
$$d = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

Distance Formula: The distance between $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Classify a Triangle as Acute, Right, or Obtuse

We can extend the converse of the Pythagorean Theorem to determine if a triangle is an obtuse or **acute triangle**.

Acute Triangles: If the sum of the squares of the two shorter sides in a right triangle is ***greater*** than the square of the longest side, then the triangle is ***acute.***

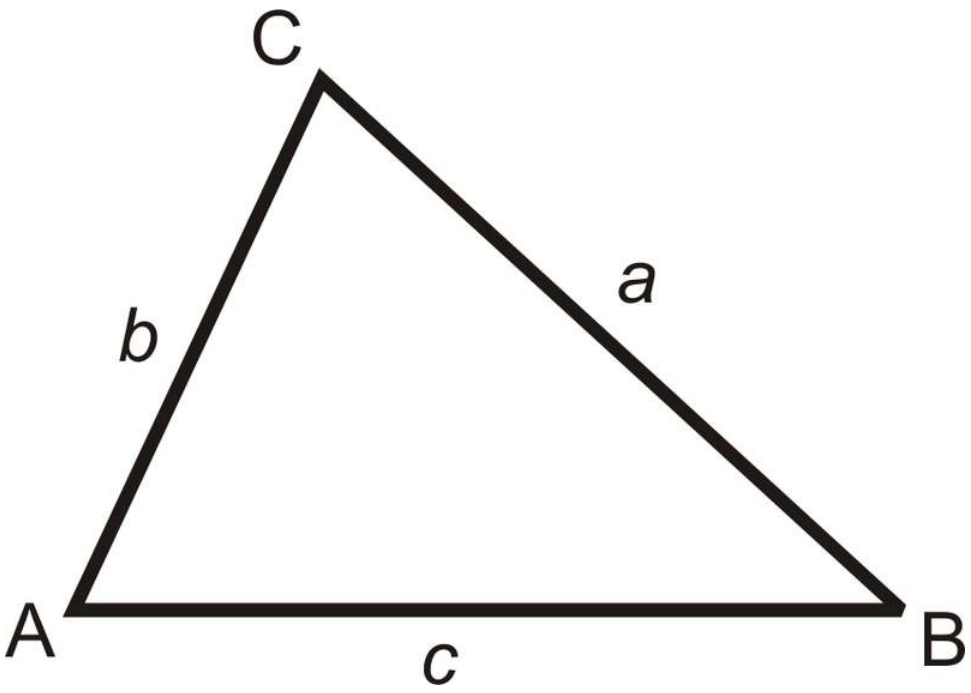


Figure 1.5.4

For $b < c$ and $a < c$, if $a^2 + b^2 > c^2$, then the triangle is acute.

Obtuse Triangles: If the sum of the squares of the two shorter sides in a right triangle is ***less*** than the square of the longest side, then the triangle is ***obtuse.***

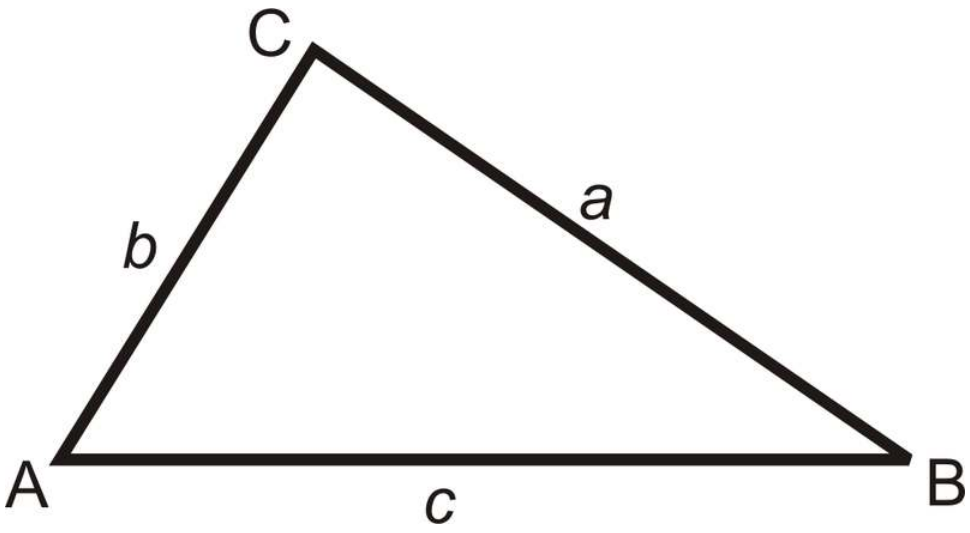
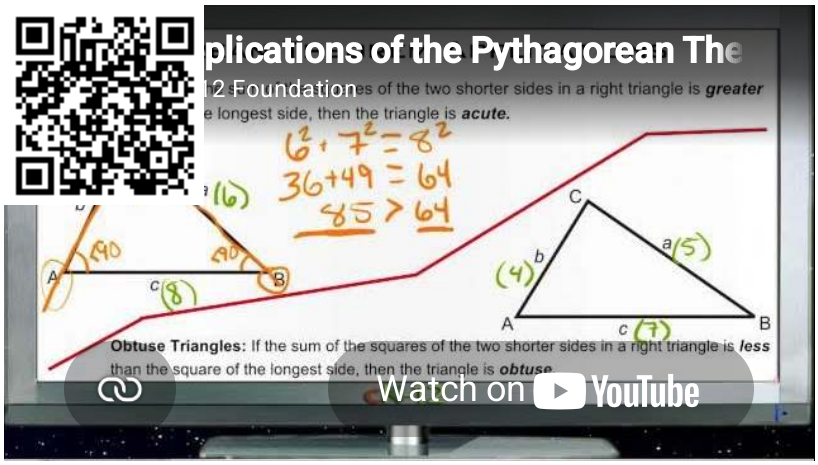


Figure **1.5.5**

For $b < c$ and $a < c$, if $a^2 + b^2 < c^2$, then the triangle is obtuse.

What if you were given an equilateral triangle in which all the sides measured 4 inches? How could you use the Pythagorean Theorem to find the triangle's altitude?



Example 1.5.1

What is the height of the isosceles triangle?

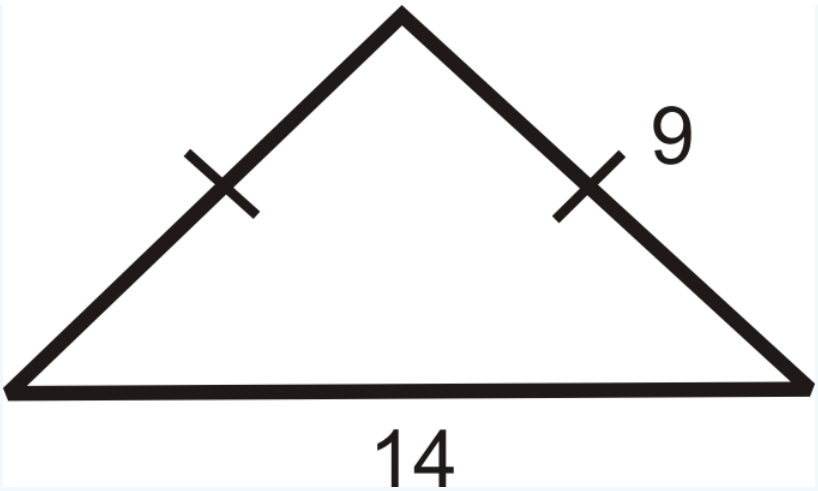


Figure 1.5.6

Solution

Draw the altitude from the **vertex** between the congruent sides, which will bisect the base.

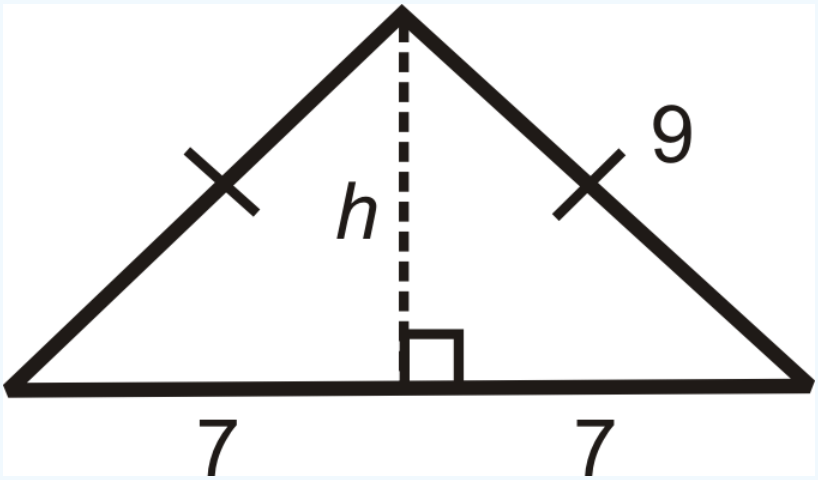


Figure 1.5.7

$$\begin{aligned} 7^2 + h^2 &= 9^2 \\ 49 + h^2 &= 81 \\ h^2 &= 32 \\ h &= \sqrt{32} = \sqrt{16 \cdot 2} = 4\sqrt{2} \end{aligned}$$

Example 1.5.2

Find the distance between $(1, 5)$ and $(5, 2)$.

Solution

Make $A(1, 5)$ and $B(5, 2)$. Plug into the distance formula.

$$\begin{aligned} d &= \sqrt{(1-5)^2 + (5-2)^2} \\ &= \sqrt{(-4)^2 + (3)^2} \\ &= \sqrt{16+9} = \sqrt{25} = 5 \end{aligned}$$

Just like the lengths of the sides of a triangle, distances are always positive.

Example 1.5.3

Graph $A(-4,1)$, $B(3,8)$, and $C(9,6)$. Determine if ΔABC is acute, obtuse, or right.

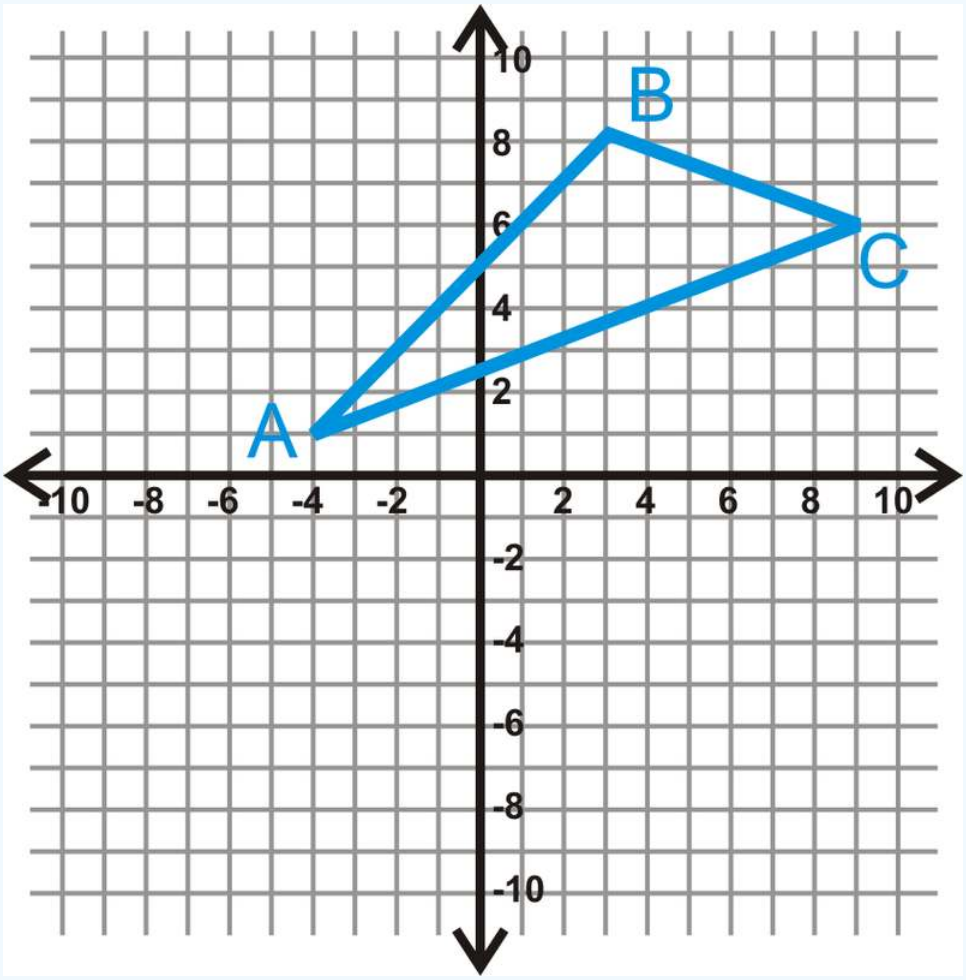


Figure 1.5.8

Solution

Use the distance formula to find the length of each side.

$$AB = \sqrt{(-4-3)^2 + (1-8)^2} = \sqrt{49+49} = \sqrt{98}$$
$$BC = \sqrt{(3-9)^2 + (8-6)^2} = \sqrt{36+4} = \sqrt{40}$$
$$AC = \sqrt{(-4-9)^2 + (1-6)^2} = \sqrt{169+25} = \sqrt{194}$$

Plug these lengths into the Pythagorean Theorem.

$$(\sqrt{98})^2 + (\sqrt{40})^2 ? (\sqrt{194})^2$$
$$98 + 40 ? 194$$
$$138 < 194$$

ΔABC is an **obtuse triangle**.

For Examples 4 and 5, determine if the triangles are acute, right or obtuse.

Example 1.5.4

Set the longest side to c.

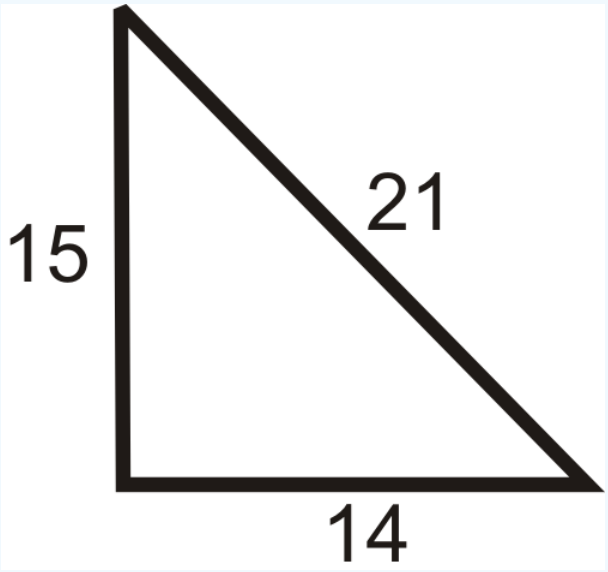


Figure 1.5.9

Solution

$$15^2 + 14^2 ? 21^2$$
$$225 + 196 ? 441$$
$$421 < 441$$

The triangle is obtuse.

Example 1.5.5

Set the longest side to c.

A triangle with side lengths 5, 12, 13.

Solution

$5^2 + 12^2 = 13^2$ so this triangle is right.

Review

Find the height of each isosceles triangle below. Simplify all radicals.

1.

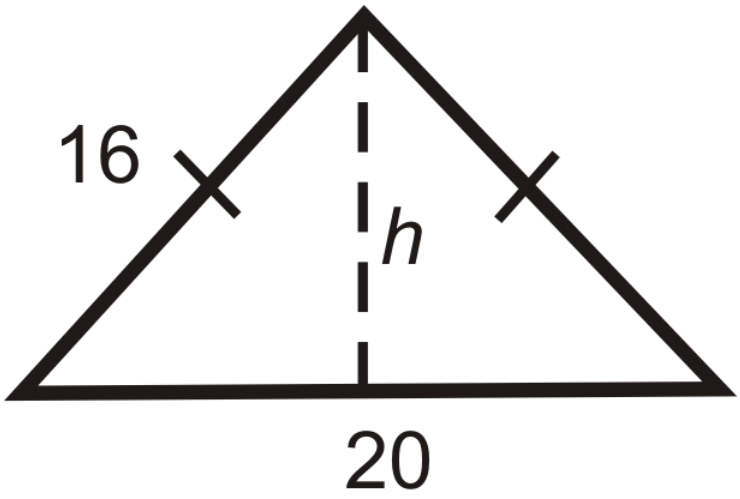


Figure 1.5.10

2.

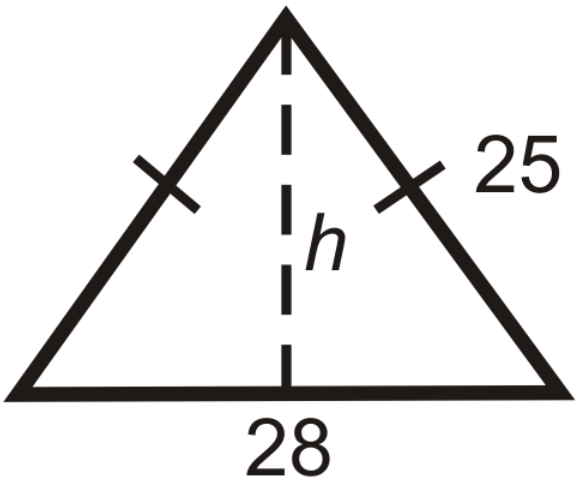


Figure 1.5.11

3.

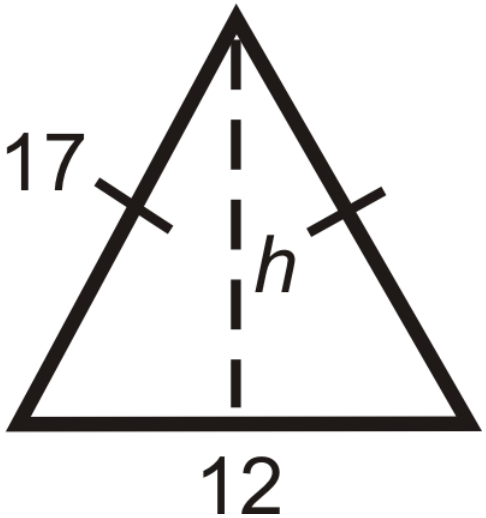


Figure 1.5.12

Find the length between each pair of points.

4. (-1, 6) and (7, 2)

5. (10, -3) and (-12, -6)
6. (1, 3) and (-8, 16)
7. What are the length and width of a 42” HDTV? Round your answer to the nearest tenth.
8. Standard definition TVs have a length and width ratio of 4:3. What are the length and width of a 42” Standard definition TV? Round your answer to the nearest tenth.

Determine whether the following triangles are acute, right or obtuse.

9. 7, 8, 9
10. 14, 48, 50
11. 5, 12, 15
12. 13, 84, 85
13. 20, 20, 24
14. 35, 40, 51
15. 39, 80, 89
16. 20, 21, 38
17. 48, 55, 76

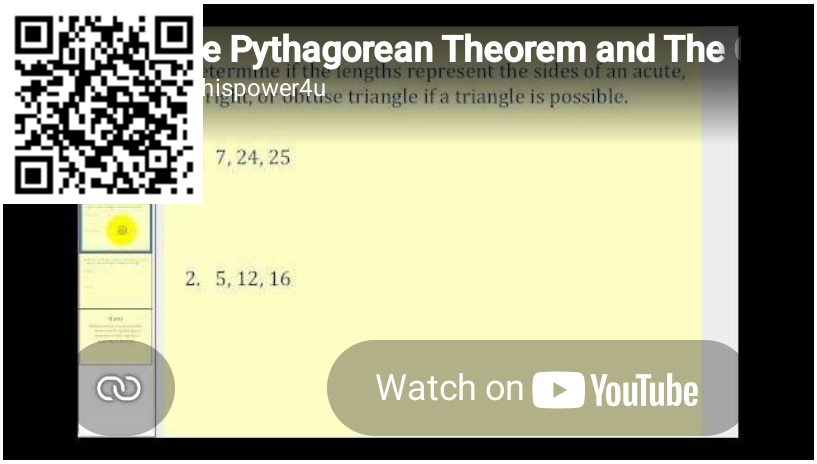
Graph each set of points and determine whether ΔABC is acute, right, or obtuse, using the distance formula.

18. $A(3,-5), B(-5,-8), C(-2,7)$
19. $A(5,3), B(2,-7), C(-1,5)$
20. $A(1,6), B(5,2), C(-2,3)$
21. $A(-6,1), B(-4,-5), C(5,-2)$

Review (Answers)

To see the Review answers, open this PDF file and look for section 8.3.

Resources



Additional Resources

Video: The Pythagorean Theorem and The Converse of the Pythagorean Theorem

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1.6: Distance Formula and the Pythagorean Theorem

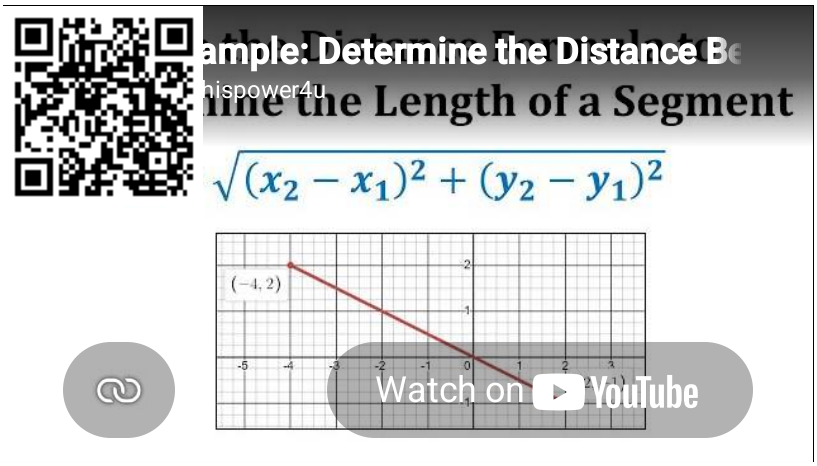
Discover lengths of triangle sides using the Pythagorean Theorem. Identify distance as the hypotenuse of a right triangle. Determine distance between ordered pairs.

Pythagorean Theorem to Determine Distance

While walking to school one day, you decide to use your knowledge of the Pythagorean Theorem to determine how far it is between your home and school. You know that you walk 3 blocks east, and then turn and walk 7 blocks north to get to school. Is it possible to use the Pythagorean Theorem to help you determine the "straight line" distance between your home and school?

Determining the Distance Using the Pythagorean Theorem

You can use the Pythagorean Theorem is to find the distance between two points.



Consider the points (-1, 6) and (5, -3). If we plot these points on a grid and connect them, they make a diagonal line. Draw a vertical line down from (-1, 6) and a horizontal line to the left of (5, -3) to make a right triangle.

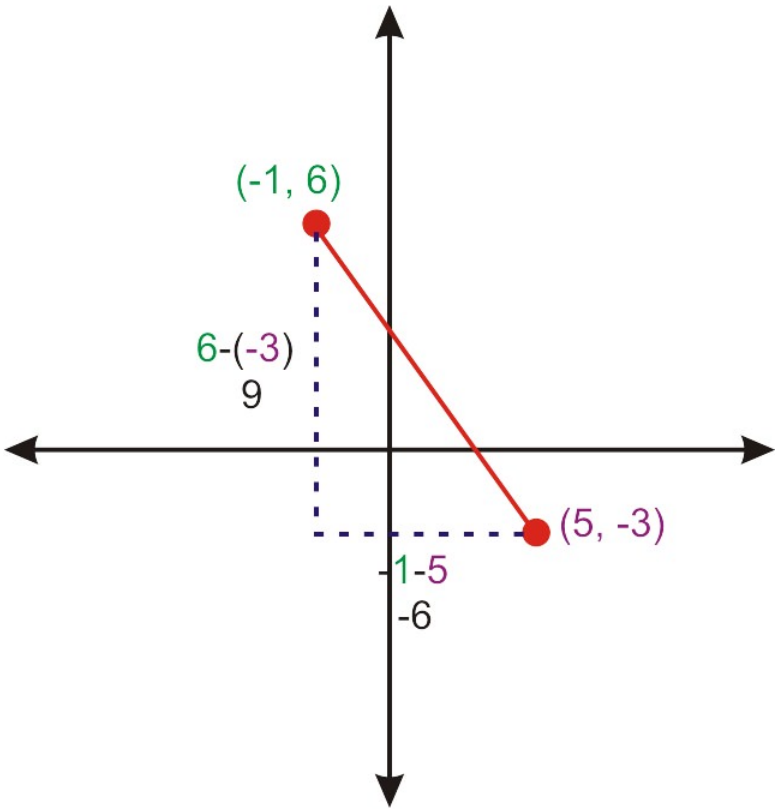


Figure 1.6.1

Now we can find the distance between these two points by using the vertical and horizontal distances that we determined from the graph.

$$\begin{aligned} 9^2 + (-6)^2 &= d^2 \\ 81 + 36 &= d^2 \\ 117 &= d^2 \\ \sqrt{117} &= d \\ 3\sqrt{13} &= d \end{aligned}$$

Notice, that the x−values were subtracted from each other to find the horizontal distance and the y−values were subtracted from each other to find the vertical distance. If this process is generalized for two points (x_1, y_1) and (x_2, y_2), the Distance Formula is derived.

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = d^2$$

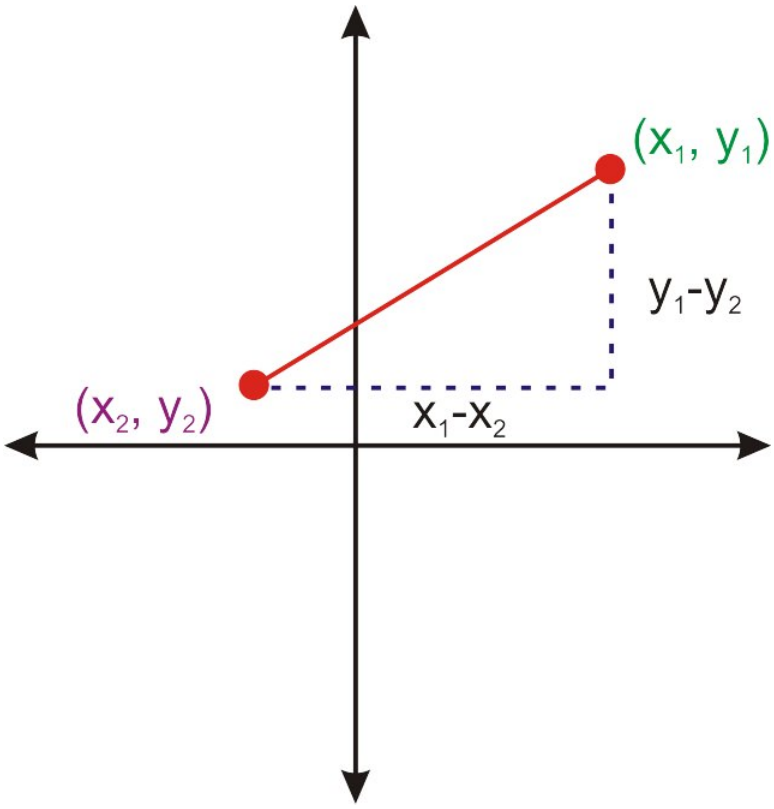


Figure **1.6.2**

This is the Pythagorean Theorem with the vertical and horizontal differences between (x_1, y_1) and (x_2, y_2). Taking the square root of both sides will solve the right hand side for d, the distance.

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = d$$

This is the Distance Formula. The following problems show how to apply the distance formula.

Applying the Distance Formula

1. Find the distance between the two points.

(4, 2) and (-9, 5)

Plug each pair of points into the Distance Formula.

$$\begin{aligned} d &= \sqrt{(4-(-9))^2 + (2-5)^2} \\ &= \sqrt{13^2 + (-3)^2} \\ &= \sqrt{169 + 9} \\ &= \sqrt{178} \end{aligned}$$

2. Find the distance between the two points.

(-10, 3) and (0, -15)

Plug each pair of points into the distance formula.

$$\begin{aligned} d &= \sqrt{(-10-0)^2 + (3-(-15))^2} \\ &= \sqrt{(-10)^2 + (18)^2} \\ &= \sqrt{100 + 324} \\ &= \sqrt{424} \\ &= 2\sqrt{106} \end{aligned}$$

3. Find the distance between the two points.

(3, 1) and (2, -7)

Plug each pair of points into the distance formula.

$$\begin{aligned} d &= \sqrt{(3-2)^2 + (1-(-7))^2} \\ &= \sqrt{(1)^2 + (8)^2} \\ &= \sqrt{1+64} \\ &= \sqrt{65} \end{aligned}$$

Example 1.6.1

Earlier, you were asked to use your knowledge of the Pythagorean Theorem to determine how far it is between your home and school.

Solution

Since you know that the trip to school involves walking 3 blocks east followed by 7 blocks north, you can construct a triangle on a coordinate system out of these lengths, like this:

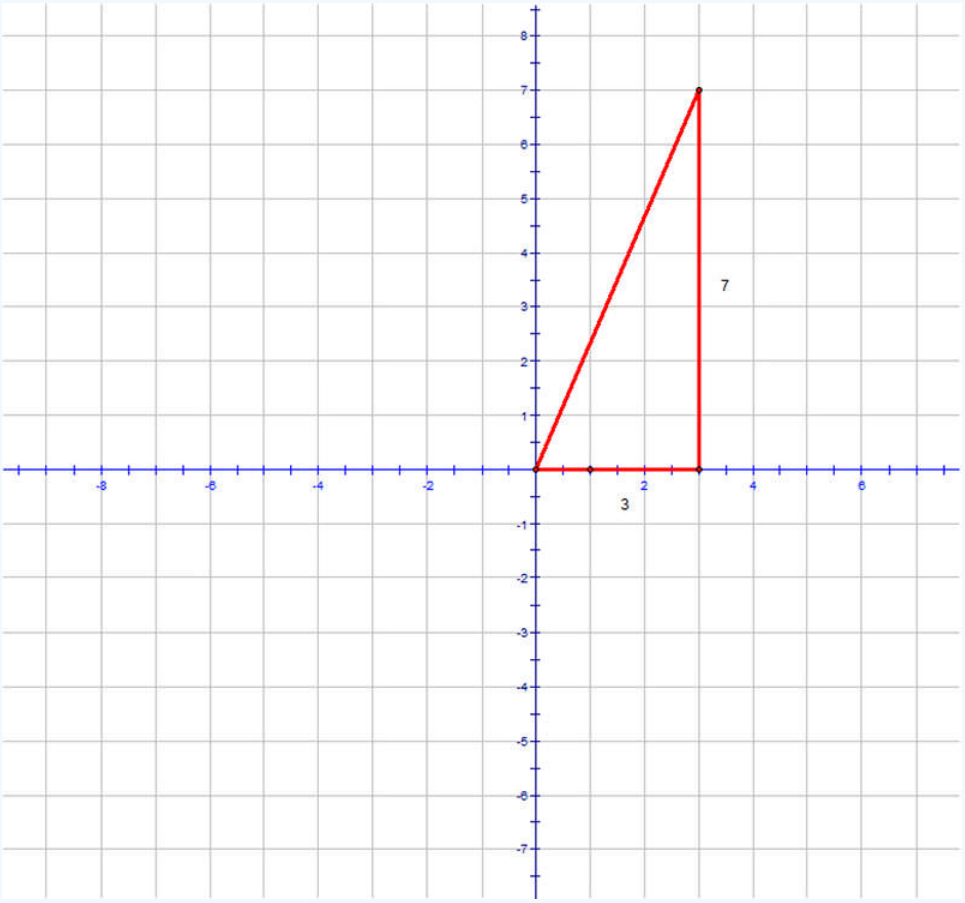


Figure 1.6.3

Since you went three blocks to the east, the school has an "x" coordinate of 3. Likewise, since you went 7 blocks north, the school has a "y" coordinate of 7. To find the straight-line distance to school, you can use the Distance Formula:

$$\begin{aligned} d &= (3-0)^2 + (7-0)^2 \\ &= \sqrt{(3)^2 + (7)^2} \\ &= \sqrt{58} \end{aligned}$$

This is a straight line distance of approximately 7.6 blocks.

Example 1.6.2

Find the distance between the two points.

(3, 1) and (2, -7)

Solution

Plug each pair of points into the distance formula.

$$\begin{aligned} d &= \sqrt{(3-2)^2 + (1-(-7))^2} \\ &= \sqrt{(1)^2 + (8)^2} \\ &= \sqrt{1+64} \\ &= \sqrt{65} \end{aligned}$$

Example 1.6.3

Find the distance between the two points.

(5, -8) and (0, 3)

Solution

Plug each pair of points into the distance formula.

$$\begin{aligned} d &= \sqrt{(5-0)^2 + (-8-(3))^2} \\ &= (5)^2 + (-11)^2 \\ &= \sqrt{25+121} \\ &= \sqrt{146} \end{aligned}$$

Example 1.6.4

Find the distance between the two points.

(2, 6) and (2, 9)

Solution

Plug each pair of points into the distance formula.

$$\begin{aligned} d &= \sqrt{(2-2)^2 + (6-9)^2} \\ &= \sqrt{(0)^2 + (-3)^2} \\ &= \sqrt{9} = 3 \end{aligned}$$

Review

Find the distance between each pair of points. Round each answer to the nearest tenth.

1. (2, 4) and (5, 10)
2. (1, 5) and (8, 9)
3. (-2, 3) and (6, 4)
4. (5, 7) and (5, 10)
5. (8, 12) and (15, 12)
6. (1, -4) and (25, -2)
7. (5, -6) and (3, 7)
8. (12, -9) and (-1, 5)
9. (-3, 14) and (8, 10)
10. (-11, 3) and (-5, 1)
11. (5, 2) and (11, 13)

12. (8, 10) and (9, -6)

Find the perimeter of each triangle. Round each answer to the nearest tenth.

13. $A(3, -5), B(-5, -8), C(-2, 7)$
14. $A(5, 3), B(2, -7), C(-1, 5)$
15. $A(1, 2), B(1, 5), C(4, 5)$

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.4.

Vocabulary

Term	Definition
Distance Formula	The distance between two points (x_1, y_1) and (x_2, y_2) can be defined as $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Additional Resources

Video: The Distance Formula

Practice: Distance Formula and the Pythagorean Theorem

Real World: It All Depends on Distance

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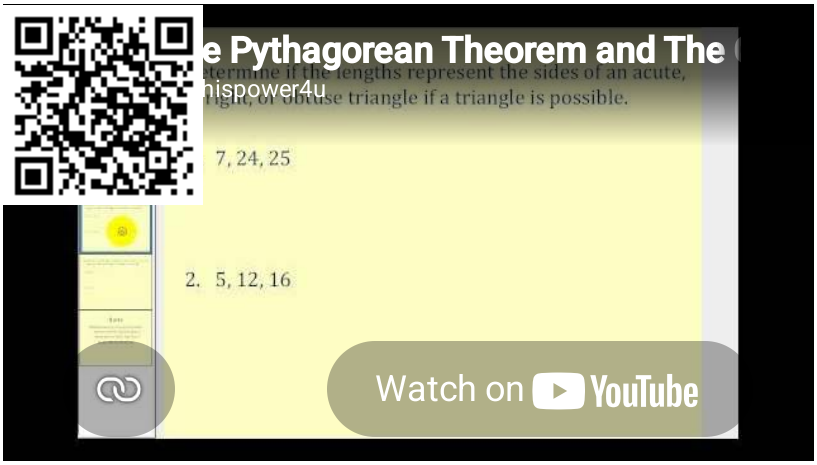
1.7: Pythagorean Theorem to Classify Triangles

Lengths of triangle sides using the Pythagorean Theorem to classify triangles as obtuse, acute or right.

While painting a wall in your home one day, you realize that the wall you are painting seems "tilted", as though it might fall over. You realize that if the wall is standing upright, the angle between the wall and the floor is ninety degrees. After a few careful measurements, you find that the distance from the bottom of the ladder to the wall is 3 feet, the top of the ladder is at a point 10 feet up on the wall, and the ladder is 12 feet long. Can you determine if the wall is still standing upright, or if it is starting to lean?

Classifying Triangles by Using the Pythagorean Theorem

We can use the Pythagorean Theorem to help determine if a triangle is a **right triangle**, if it is acute, or if it is obtuse.



To help you visualize this, think of an equilateral triangle with sides of length 5. We know that this is an **acute triangle**. If you plug in 5 for each number in the Pythagorean Theorem we get $5^2+5^2=5^2$ and $50>25$. Therefore, if $a^2+b^2>c^2$, then lengths a, b, and c make up an acute triangle. Conversely, if $a^2+b^2<c^2$, then lengths a, b, and c make up the sides of an **obtuse triangle**. *It is important to note that the length* "c" *is always the longest.*

Using the Pythagorean Theorem

Determine if the following lengths make an acute, right or obtuse triangle.

1. 5, 6, 7

Plug in each set of lengths into the Pythagorean Theorem.

$$\begin{aligned} 5^2+6^2 \,?\, 7^2 \\ 25+36 \,?\, 49 \\ 61>49 \end{aligned}$$

Because 61>49, this is an acute triangle.

2. 5, 10, 14

Plug in each set of lengths into the Pythagorean Theorem.

$$\begin{aligned} 5^2+10^2 \,?\, 14^2 \\ 25+100 \,?\, 196 \\ 125<196 \end{aligned}$$

Because $125<196$, this is an obtuse triangle.

3. 12, 35, 37

Plug in each set of lengths into the Pythagorean Theorem.

$$\begin{aligned} 12^2+35^2 \,?\, 37^2 \\ 144+1225 \,?\, 1369 \\ 1369=1369 \end{aligned}$$

Because the two sides are equal, this is a right triangle.

NOTE: All of the lengths in the above problem represent the lengths of the sides of a triangle. Recall the Triangle Inequality Theorem from geometry which states: The length of a side in a triangle is less than the sum of the other two sides. For example, 4, 7 and 13 cannot be the sides of a triangle because 4+7 is not greater than 13.

Example 1.7.1

Earlier, you were given a problem asking if the wall is still standing upright, or if it is starting to lean.

Solution

The ladder is making a triangle with the floor as one side, the wall as another, and the ladder itself serves as the hypotenuse. To see if the wall is leaning, you can determine the type of triangle that is made with these lengths (right, acute, or obtuse). If the triangle is a right triangle, then the wall is standing upright. Otherwise, it is leaning.

Plugging the lengths of the sides into the Pythagorean Theorem:

$$\begin{aligned} 3^2 + 10^2 \; ? \; 12^2 \\ 9 + 100 \; ? \; 144 \\ 109 < 144 \end{aligned}$$

Yes, you were right. Because $109 < 144$, this is an obtuse triangle. The wall is leaning with an angle greater than ninety degrees.

Example 1.7.2

Determine if the following lengths make an acute, right or obtuse triangle.

8, 15, 20

Solution

Plug in each set of lengths into the Pythagorean Theorem.

$$\begin{aligned} 8^2 + 15^2 \; ? \; 20^2 \\ 64 + 225 \; ? \; 400 \\ 289 < 400 \end{aligned}$$

Because $289 < 400$, this is an obtuse triangle.

Example 1.7.3

Determine if the following lengths make an acute, right or obtuse triangle.

15, 22, 25

Solution

Plug in each set of lengths into the Pythagorean Theorem.

$$\begin{aligned} 15^2 + 22^2 \; ? \; 25^2 \\ 225 + 484 \; ? \; 625 \\ 709 > 625 \end{aligned}$$

Because $709 > 625$, this is an acute triangle.

Review

Determine if each of the following lengths make a right triangle.

1. 9, 40, 41.
2. 12, 24, 26.
3. 5, 10, 14.
4. 3, $3\sqrt{3}$, 6.

Determine if the following lengths make an acute, right or obtuse triangle.

5. 10, 15, 18.
6. 4, 20, 21.
7. 15, 16, 17.
8. 15, 15, $15\sqrt{2}$.
9. 12, 17, 19.
10. 3, 4, 5.
11. 12, $12\sqrt{3}$, 24.
12. 2, 4, 5.
13. 3, 5, 7.
14. Explain why if $a^2 + b^2 < c^2$ then the triangle is obtuse.
15. Explain why if $a^2 + b^2 > c^2$ then the triangle is acute.

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.3.

Vocabulary

Term	Definition
Acute Triangle	An acute triangle has three angles that each measure less than 90 degrees.
Obtuse Triangle	An obtuse triangle is a triangle with one angle that is greater than 90 degrees.
Right Triangle	A right triangle is a triangle with one 90 degree angle.

Additional Resources

Video: The Pythagorean Theorem and The Converse of the Pythagorean Theorem

Practice: Pythagorean Theorem to Classify Triangles

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1.8: Special Right Triangles and Ratios

Properties of 30-60-90 and 45-45-90 triangles.

The **Pythagorean Theorem** is great for finding the third side of a right triangle when you already know two other sides. There are some triangles like 30-60-90 and 45-45-90 triangles that are so common that it is useful to know the side ratios without doing the Pythagorean Theorem each time. Using these patterns also allows you to totally solve for the missing sides of these special triangles when you only know one side length.

Given a 45-45-90 right triangle with sides 6 inches, 6 inches and x inches, what is the value of x?

Special Right Triangles

There are three types of special right triangles, 30-60-90 triangles, 45-45-90 triangles, and Pythagorean triple triangles.

30-60-90 Triangles

A 30-60-90 right triangle has side ratios x, $x\sqrt{3}$,$2x$.

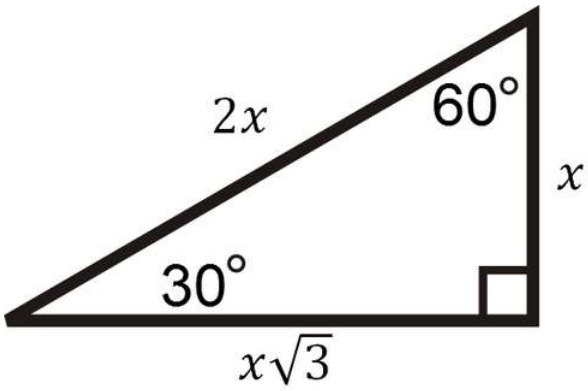


Figure 1.8.1

Confirm with Pythagorean Theorem:

$$\begin{aligned} x^2 + (x\sqrt{3})^2 &= (2x)^2 \\ x^2 + 3x^2 &= 4x^2 \\ 4x^2 &= 4x^2 \end{aligned}$$

45-45-90 Triangles

A 45-45-90 right triangle has side ratios x, x, $x\sqrt{2}$.

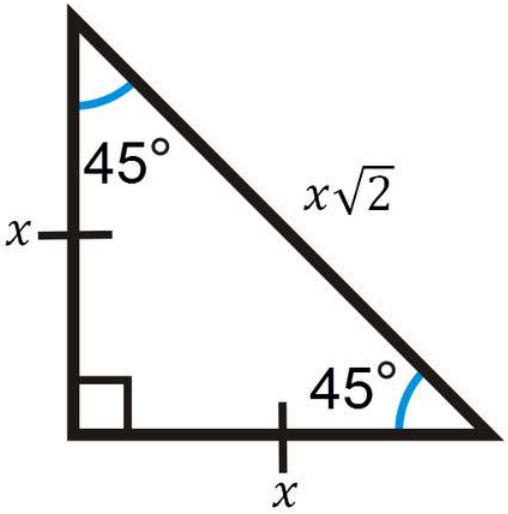


Figure 1.8.2

Confirm with Pythagorean Theorem:

$$x^2 + x^2 = (x\sqrt{2})^2$$
$$2x^2 = 2x^2$$

Note that the order of the side ratios $x, x\sqrt{3}, 2x$ and $x, x, x\sqrt{2}$ is important because each side ratio has a corresponding angle. In all triangles, the smallest sides correspond to smallest angles and largest sides always correspond to the largest angles.

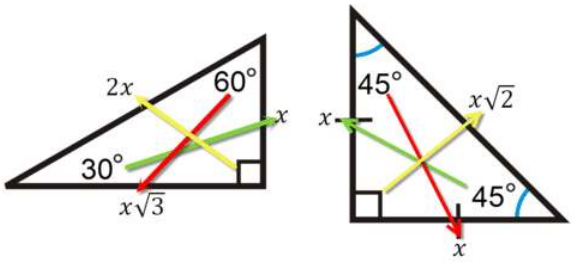


Figure 1.8.3

Pythagorean Triple Triangles

Pythagorean number triples are special right triangles with integer sides. While the angles are not integers, the side ratios are very useful to know because they show up everywhere. Knowing these number triples also saves a lot of time from doing the Pythagorean Theorem repeatedly. Here are some examples of Pythagorean number triples:

- 3, 4, 5
- 5, 12, 13
- 7, 24, 25
- 8, 15, 17
- 9, 40, 41

More Pythagorean number triples can be found by scaling any other **Pythagorean number triple**. For example:

$3, 4, 5 \rightarrow 6, 8, 10$ (scaled by a factor of 2)

Even more Pythagorean number triples can be found by taking any odd integer like 11, squaring it to get 121, halving the result to get 60.5. The original number 11 and the two numbers that are 0.5 above and below (60 and 61) will always be a Pythagorean number triple.

$11^2 + 60^2 = 61^2$

Example 1.8.1

Earlier you were asked about a 45-45-90 right triangle with sides 6 inches, 6 inches and x inches.

Solution

If you can recognize the pattern for 45-45-90 right triangles, a right triangle with legs 6 inches and 6 inches has a hypotenuse that is $6\sqrt{2}$ inches. $x = 6\sqrt{2}$.

Example 1.8.2

A 30-60-90 right triangle has hypotenuse of length 10. What are the lengths of the other two sides?

Solution

The hypotenuse is the side opposite 90. Sometimes it is helpful to draw a picture or make a table.

30	60	90
x	$x\sqrt{3}$	$2x$
		10

From the table you can write very small subsequent equations to solve for the missing sides.

$$10 = 2x$$
$$x = 5$$
$$x\sqrt{3} = 5\sqrt{3}$$

The other sides are 5 and $5\sqrt{3}$.

Example 1.8.3

A 30-60-90 right triangle has a side length of 18 inches corresponding to 60 degrees. What are the lengths of the other two sides?

Solution

Make a table with the side ratios and the information given, then write equations and solve for the missing side lengths.

30	60	90
x	$x\sqrt{3}$	$2x$
	18	

$$18 = x\sqrt{3}$$
$$\frac{18}{\sqrt{3}} = x$$
$$x = 18\sqrt{3} = 18\sqrt{3} \cdot \frac{\sqrt{3}}{\sqrt{3}} = 18\frac{\sqrt{3}}{3} = 6\sqrt{3}$$
$$x = 6\sqrt{3}$$

Note that you need to rationalize denominators.

Now use the calculated x value to solve for $2x$.

$2x = 2(6\sqrt{3})$
$2x = 12\sqrt{3}$

The other sides are $6\sqrt{3}$ and $12\sqrt{3}$.

Example 1.8.4

Using your knowledge of special right triangle ratios, solve for the missing sides of the right triangle.

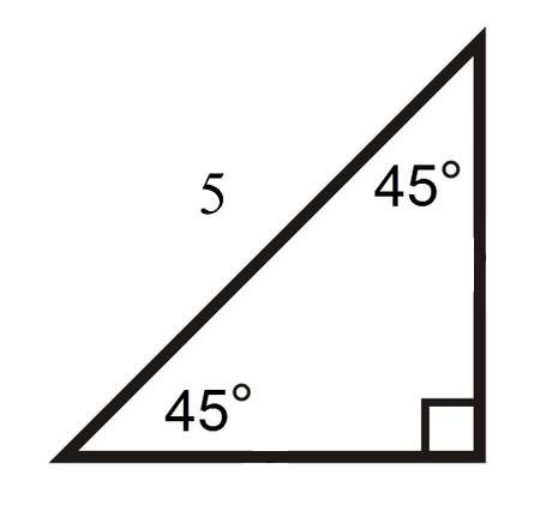


Figure 1.8.4

Solution

The other sides are each $\frac{5\sqrt{2}}{2}$.

45	45	90
x	x	$x\sqrt{2}$
		5

$$x\sqrt{2} = 5$$
$$x = \frac{5}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$$

The other sides are each $5\frac{\sqrt{2}}{2}$.

Example 1.8.1

Using your knowledge of special right triangle ratios, solve for the missing sides of the right triangle.

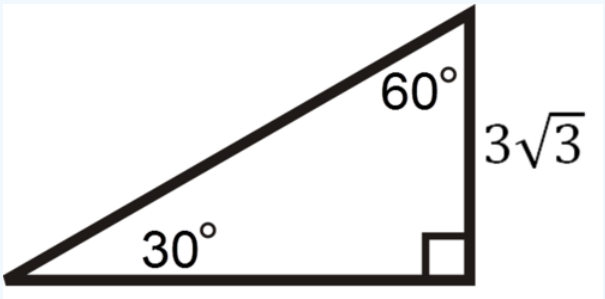


Figure 1.8.5

Solution

The other sides are 9 and $6\sqrt{3}$.

30	60	90
x	$x\sqrt{3}$	$2x$
$3\sqrt{3}$		

$$x = 3\sqrt{3}$$
$$2x = 6\sqrt{3}$$
$$x\sqrt{3} = 3\sqrt{3} \cdot \sqrt{3} = 9$$

The other sides are 9 and $6\sqrt{3}$.

Review

For 1-4, find the missing sides of the **45-45-90 triangle** based on the information given in each row.

Problem Number	Side Opposite $45°$	Side Opposite $45°$	Side Opposite $90°$
1.	3		
2.		7.2	
3.			16
4.	$5\sqrt{2}$		

For 5-8, find the missing sides of the **30-60-90 triangle** based on the information given in each row.

Problem Number	Side Opposite $30°$	Side Opposite $60°$	Side Opposite $90°$
5.	$3\sqrt{2}$		
6.		4	
7.			15
8.			$12\sqrt{3}$

Use the picture below for 9-11.

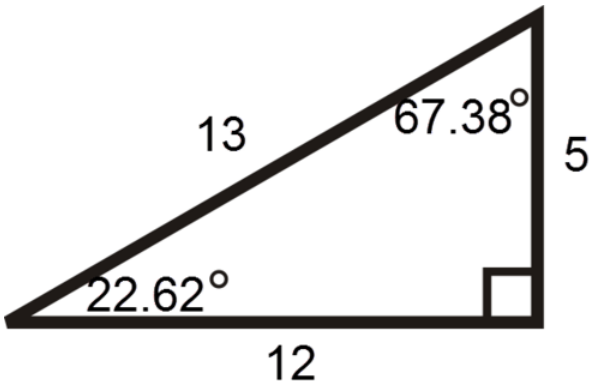


Figure $1.8.6$

9. Which angle corresponds to the side that is 12 units?
10. Which side corresponds to the right angle?
11. Which angle corresponds to the side that is 5 units?

12. A right triangle has an angle of $\frac{\pi}{6}$ radians and a hypotenuse of 20 inches. What are the lengths of the other two sides of the triangle?
13. A triangle has two angles that measure $\frac{\pi}{4}$ radians. The longest side is 3 inches long. What are the lengths of the other two sides?

For 14-19, verify the Pythagorean Number Triple using the Pythagorean Theorem.

14. 3, 4, 5
15. 5, 12, 13
16. 7, 24, 25
17. 8, 15, 17
18. 9, 40, 41
19. 6, 8, 10
20. Find another Pythagorean Number Triple using the method explained for finding "11, 60, 61".

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.3.

Vocabulary

Term	Definition
30-60-90 Triangle	A 30-60-90 triangle is a special right triangle with angles of $30°$, $60°$, and $90°$.
45-45-90 Triangle	A 45-45-90 triangle is a special right triangle with angles of $45°$, $45°$, and $90°$.
Pythagorean number triple	A Pythagorean number triple is a set of three whole numbers a,b and c that satisfy the Pythagorean Theorem, $a^2 + b^2 = c^2$.
Pythagorean Theorem	The Pythagorean Theorem is a mathematical relationship between the sides of a right triangle, given by $a^2 + b^2 = c^2$, where a and b are legs of the triangle and c is the hypotenuse of the triangle.

Additional Resources

Video: Solving Special Right Triangles

Practice: Special Right Triangles and Ratios

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1.9: 30-60-90 Right Triangles

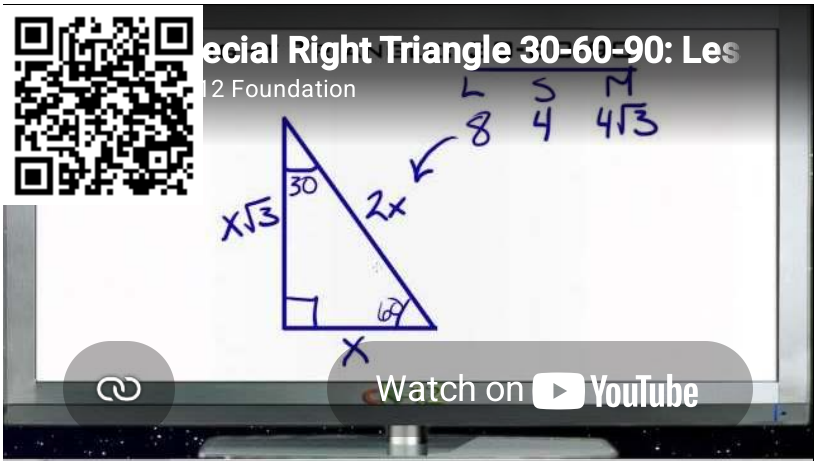
Hypotenuse equals twice the smallest leg, while the larger leg is *sqrt3* times the smallest.

One of the two special right triangles is called a **30-60-90 triangle**, after its three angles.

30-60-90 Theorem: If a triangle has angle measures **30°**, **60°** and **90°**, then the sides are in the ratio $x : x\sqrt{3} : 2x$.

The shorter leg is always x, the longer leg is always $x\sqrt{3}$, and the **hypotenuse** is always $2x$. If you ever forget these theorems, you can still use the **Pythagorean Theorem**.

What if you were given a 30-60-90 right triangle and the length of one of its side? How could you figure out the lengths of its other sides?



Example 1.9.1

Find the value of x and y.

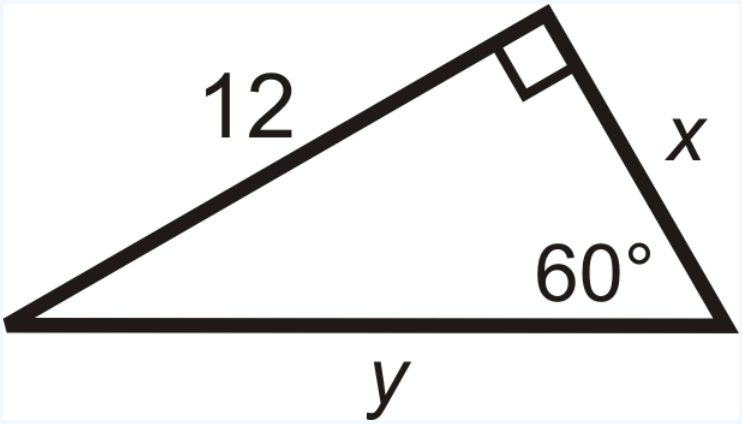


Figure 1.9.1

Solution

We are given the longer leg.

$$x\sqrt{3} = 12$$

$$x = 12\sqrt{3} \cdot \frac{\sqrt{3}}{\sqrt{3}} = 12\frac{\sqrt{3}}{3} = 4\sqrt{3}$$

The hypotenuse is $y = 2(4\sqrt{3}) = 8\sqrt{3}$

Example 1.9.2

Find the value of x and y.

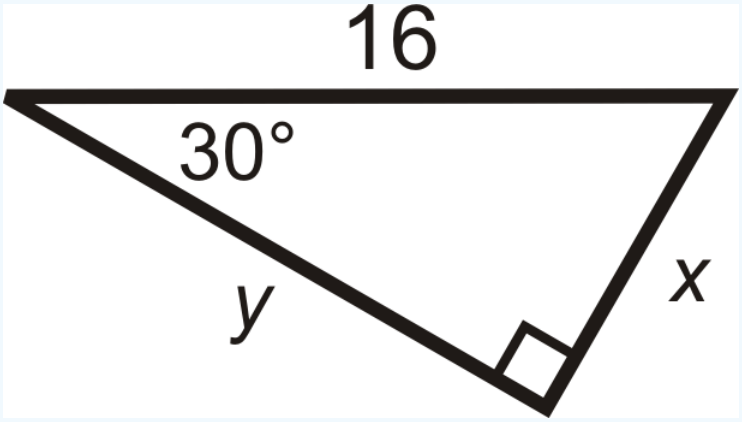


Figure 1.9.2

Solution

We are given the hypotenuse.

$$2x = 16$$
$$x = 8$$
$$\text{The longer leg is } y = 8 \cdot \sqrt{3} \quad = 8\sqrt{3}$$

Example 1.9.3

Find the length of the missing sides.

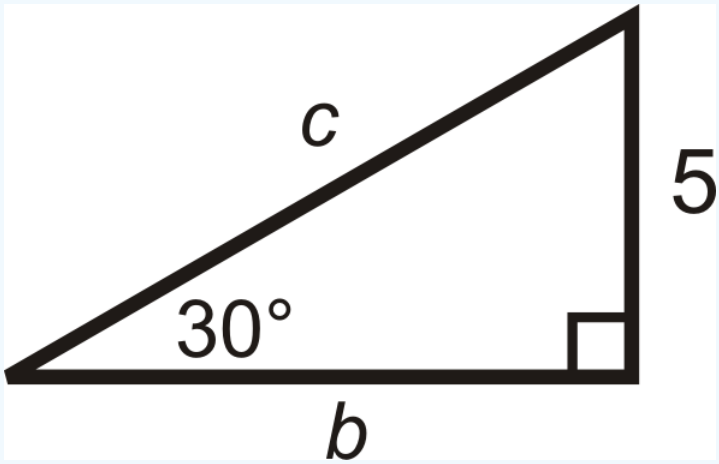


Figure 1.9.3

Solution

We are given the shorter leg. If $x = 5$, then the longer leg, $b = 5\sqrt{3}$, and the hypotenuse, $c = 2(5) = 10$.

Example 1.9.4

Find the length of the missing sides.

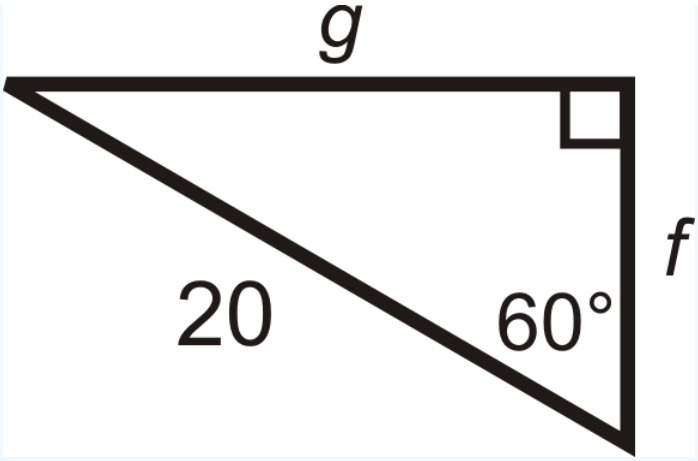


Figure 1.9.4

Solution

We are given the hypotenuse. $2x = 20$, so the shorter leg, $f = \frac{20}{2} = 10$, and the longer leg, $g = 10\sqrt{3}$.

Example 1.9.5

A rectangle has sides 4 and $4\sqrt{3}$. What is the length of the diagonal?

Solution

If you are not given a picture, draw one.

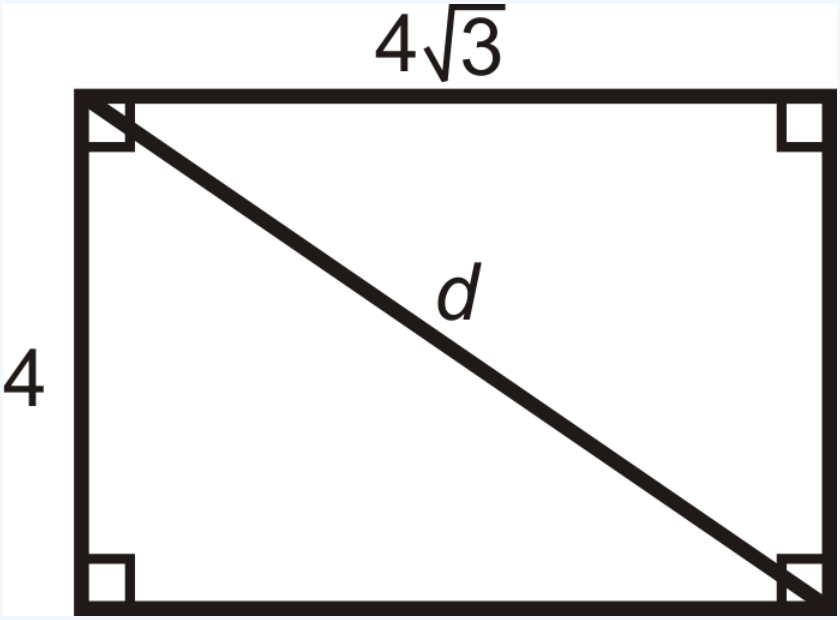


Figure 1.9.5

The two lengths are x, $x\sqrt{3}$, so the diagonal would be $2x$, or $2(4) = 8$.

If you did not recognize this is a 30-60-90 triangle, you can use the Pythagorean Theorem too.

$$4^2 + (4\sqrt{3})^2 = d^2$$
$$16 + 48 = d^2$$
$$d = \sqrt{64} = 8$$

Review

1. In a 30-60-90 triangle, if the shorter leg is 5, then the longer leg is _____ and the hypotenuse is _____.
2. In a 30-60-90 triangle, if the shorter leg is x, then the longer leg is _____ and the hypotenuse is _____.
3. A rectangle has sides of length 6 and $6\sqrt{3}$. What is the length of the diagonal?
4. Two (opposite) sides of a rectangle are 10 and the diagonal is 20. What is the length of the other two sides?

For questions 5-12, find the lengths of the missing sides. Simplify all radicals.

5.

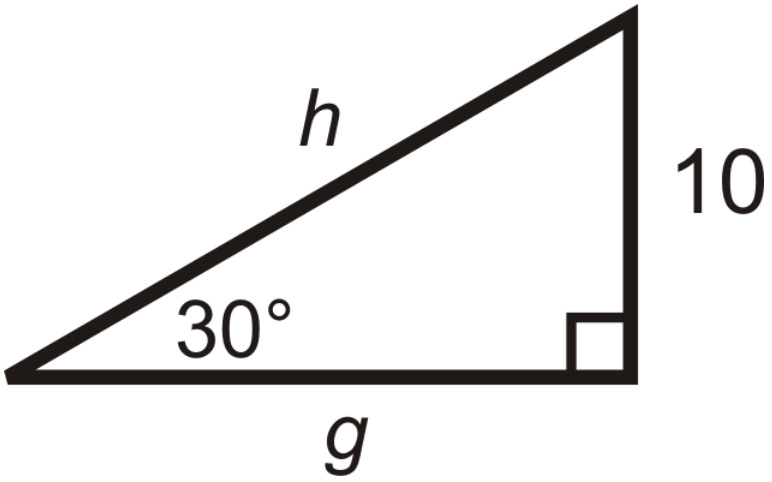


Figure 1.9.6

6.

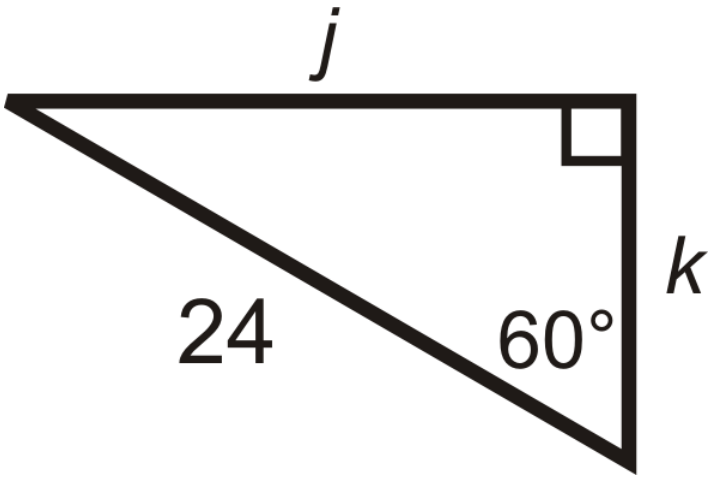


Figure 1.9.7

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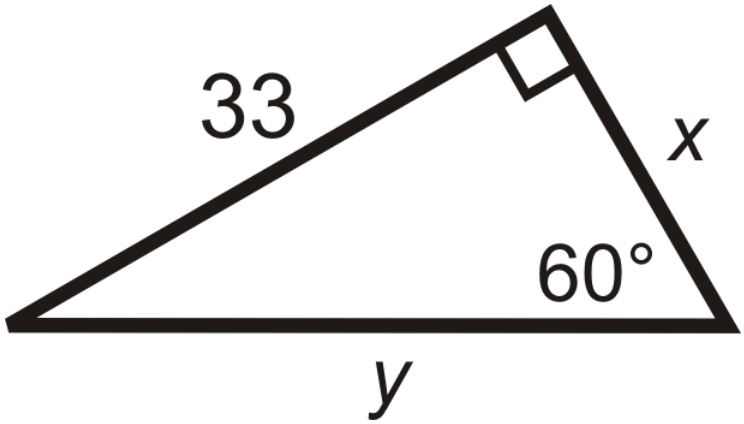


Figure 1.9.8

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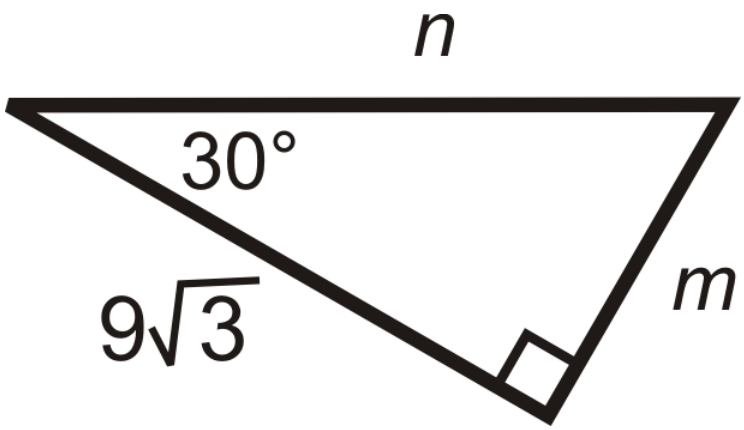


Figure 1.9.9

9.

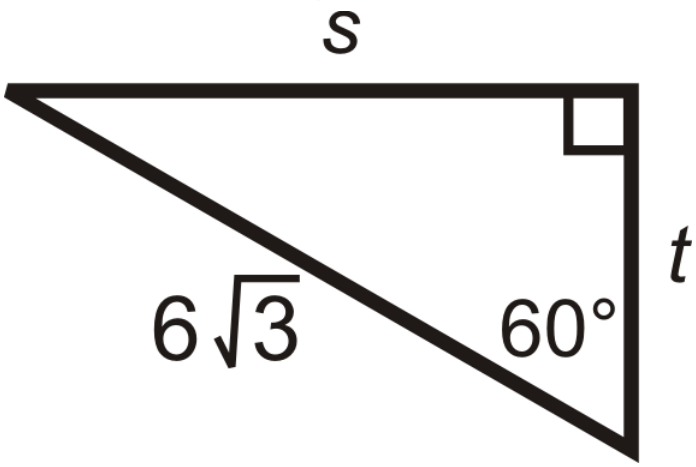


Figure 1.9.10

10.

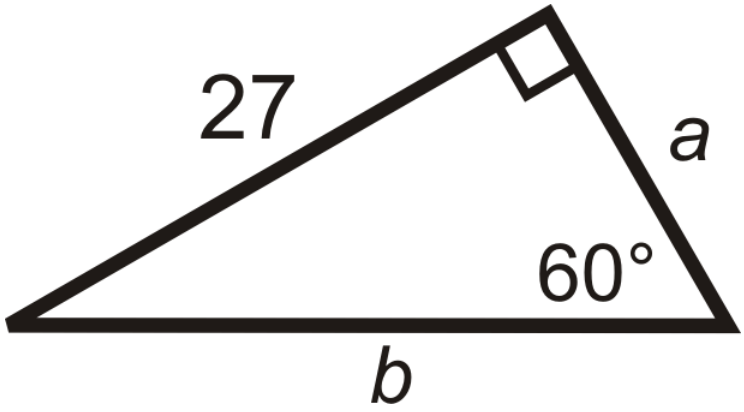


Figure 1.9.11

11.

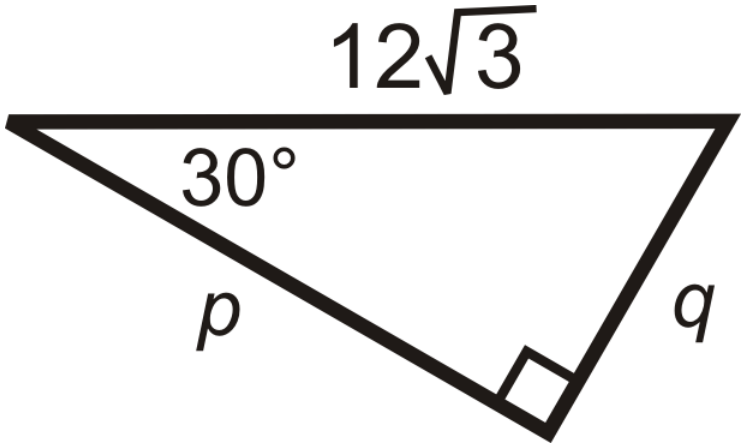


Figure 1.9.12

12.

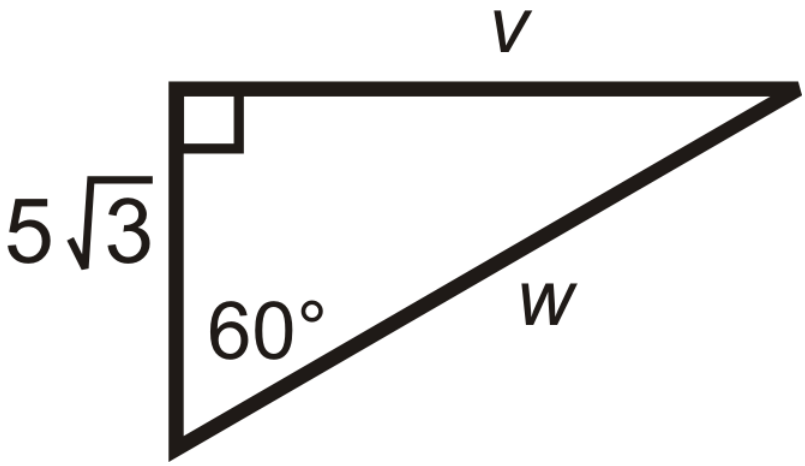
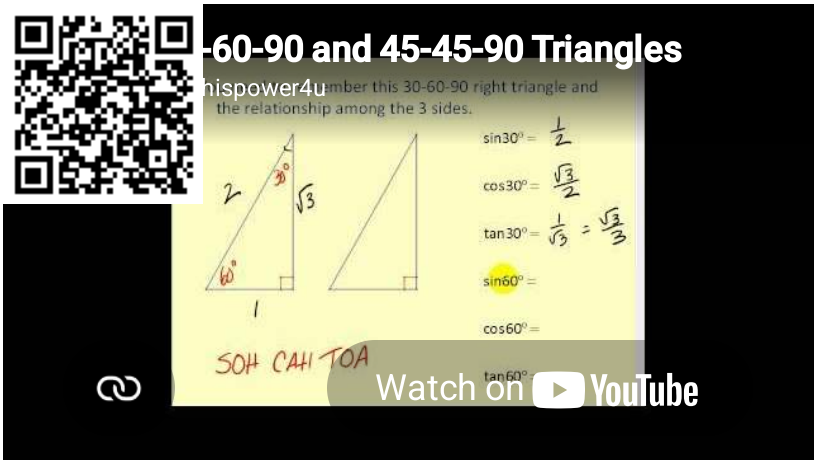


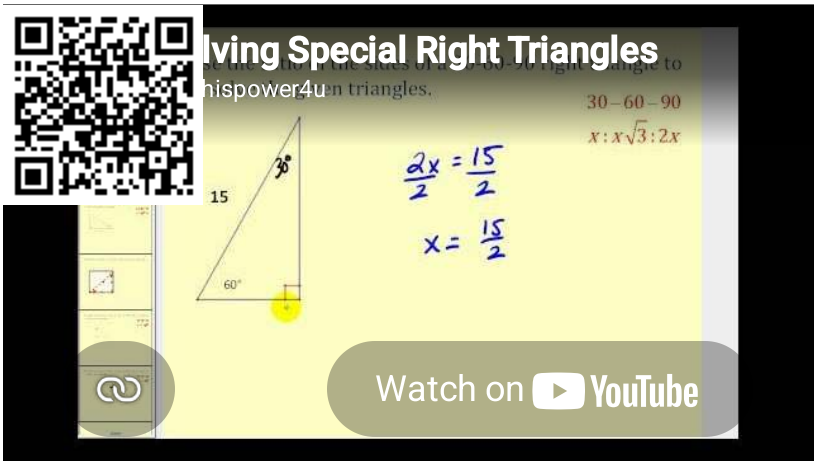
Figure 1.9.13

Review (Answers)

To see the Review answers, open this PDF file and look for section 8.6.

Resources





Vocabulary

Term	Definition
30-60-90 Theorem	If a triangle has angle measures of 30, 60, and 90 degrees, then the sides are in the ratio $x : x\sqrt{3} : 2x$
30-60-90 Triangle	A 30-60-90 triangle is a special right triangle with angles of $30°$, $60°$, and $90°$.
Hypotenuse	The hypotenuse of a right triangle is the longest side of the right triangle. It is across from the right angle.
Legs of a Right Triangle	The legs of a right triangle are the two shorter sides of the right triangle. Legs are adjacent to the right angle.
Pythagorean Theorem	The Pythagorean Theorem is a mathematical relationship between the sides of a right triangle, given by $a^2 + b^2 = c^2$, where a and b are legs of the triangle and c is the hypotenuse of the triangle.
Radical	The $\sqrt{}$, or square root, sign.

Additional Resources

Video: Solving Special Right Triangles

Activities: 30-60-90 Right Triangles Discussion Questions

Study Aids: Special Right Triangles Study Guide

Practice: 30-60-90 Right Triangles

Real World: Fighting the War on Drugs Using Geometry and Special Triangles

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1.10: 45-45-90 Right Triangles

Leg times $\sqrt{2}$ equals hypotenuse.

45-45-90 Right Triangles

A right triangle with congruent legs and acute angles is an **Isosceles Right Triangle**. This triangle is also called a **45-45-90 triangle** (named after the angle measures).

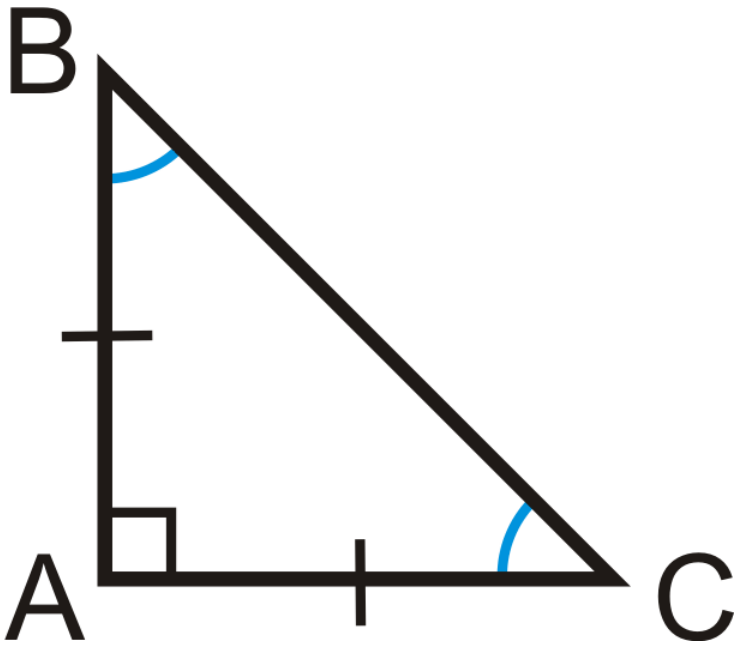
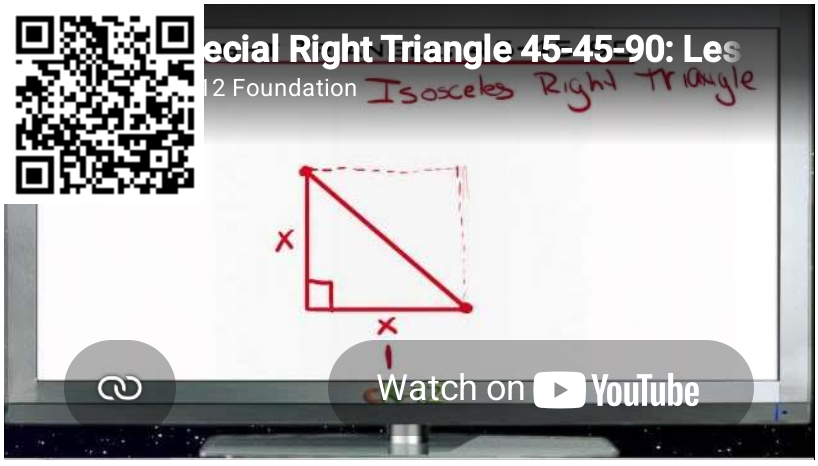


Figure **1.10.1**

ΔABC is a right triangle with $m\angle A = 90°$, $\overline{AB} \cong \overline{AC}$ and $m\angle B = m\angle C = 45°$.

45-45-90 Theorem: If a right triangle is isosceles, then its sides are in the ratio $x : x : x\sqrt{2}$. For any isosceles right triangle, the legs are x and the **hypotenuse** is always $x\sqrt{2}$.

What if you were given an isosceles right triangle and the length of one of its sides? How could you figure out the lengths of its other sides?



Example 1.10.1

Find the length of x.

Solution

Use the $x : x : x\sqrt{2}$ ratio.

Here, we are given the hypotenuse. Solve for x in the ratio.

$$x\sqrt{2} = 16$$

$$x = 16\sqrt{2} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{16\sqrt{2}}{2} = 8\sqrt{2}$$

Example 1.10.2

Find the length of x, where x is the hypotenuse of a 45-45-90 triangle with leg lengths of $5\sqrt{3}$.

Solution

Use the $x : x : x\sqrt{2}$ ratio.

$x = 5\sqrt{3} \cdot \sqrt{2} = 5\sqrt{6}$

Example 1.10.3

Find the length of the missing side.

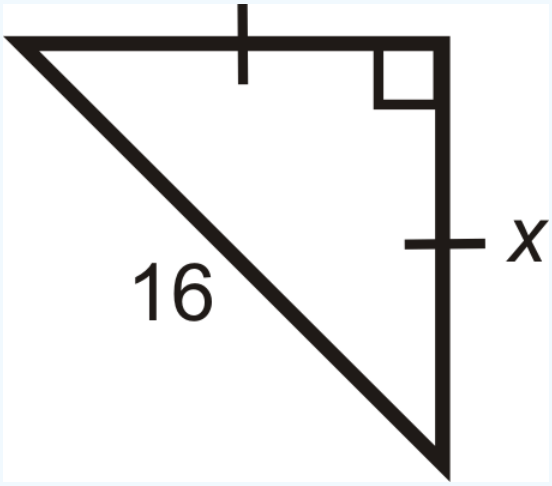


Figure 1.10.2

Solution

Use the $x : x : x\sqrt{2}$ ratio. $TV = 6$ because it is equal to ST. So, $SV = 6 \cdot \sqrt{2} = 6\sqrt{2}$.

Example 1.10.4

Find the length of the missing side.

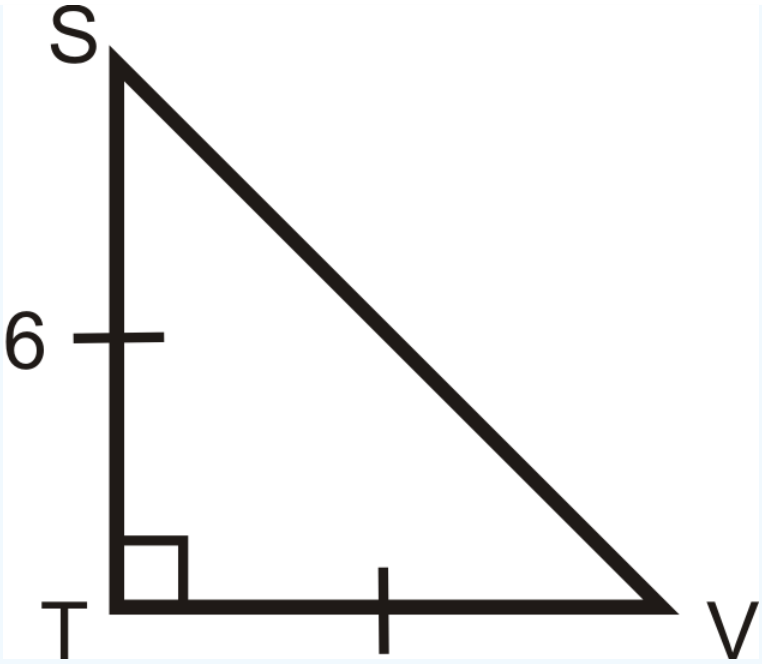


Figure 1.10.3

Solution

Use the $x : x : x\sqrt{2}$ ratio. $AB = 9\sqrt{2}$ because it is equal to AC. So, $BC = 9\sqrt{2} \cdot \sqrt{2} = 9 \cdot 2 = 18$.

Example 1.10.5

A square has a diagonal with length 10, what are the lengths of the sides?

Solution

Draw a picture.

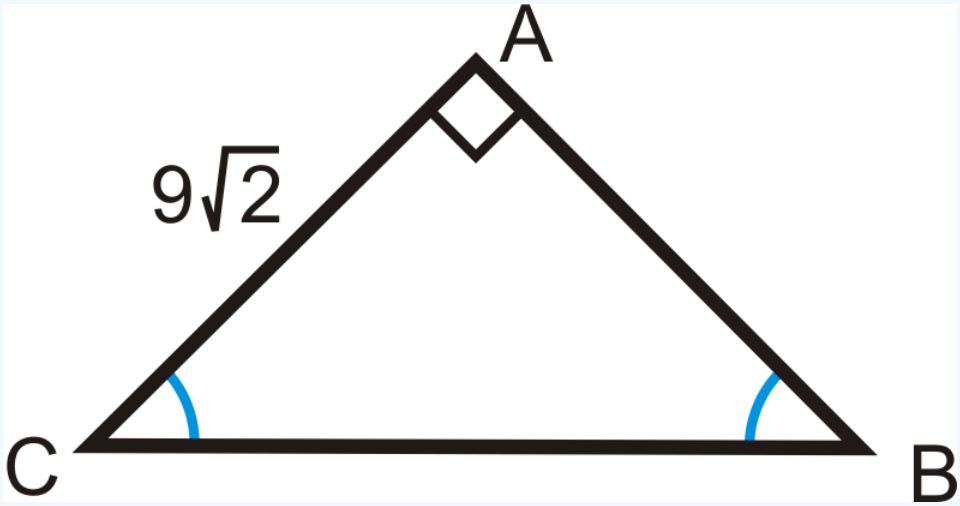


Figure 1.10.4

We know half of a square is a 45-45-90 triangle, so $10 = s\sqrt{2}$.

$$s\sqrt{2} = 10$$

$$s = 10\sqrt{2} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2}$$

Review

1. In an isosceles right triangle, if a leg is 4, then the hypotenuse is _____.
2. In an isosceles right triangle, if a leg is x, then the hypotenuse is _____.
3. A square has sides of length 15. What is the length of the diagonal?
4. A square's diagonal is 22. What is the length of each side?

For questions 5-11, find the lengths of the missing sides. Simplify all radicals.

5.

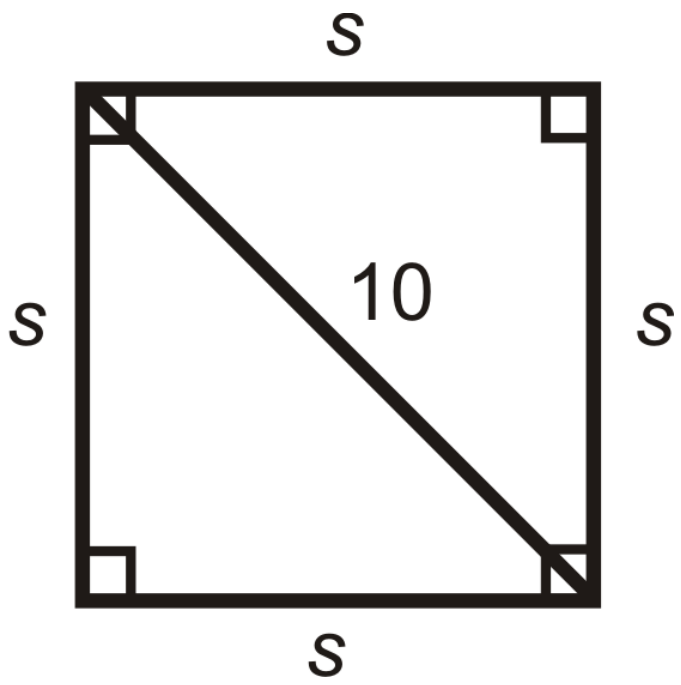


Figure **1.10.5**

6.

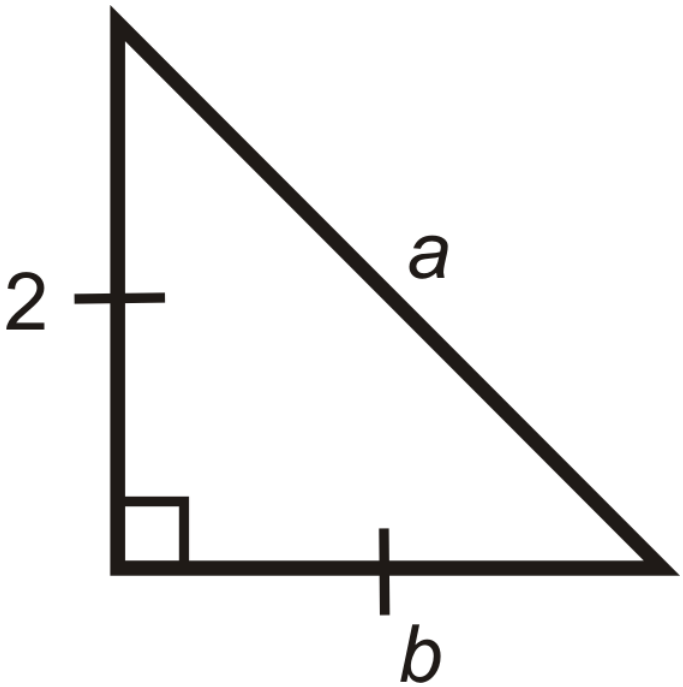


Figure **1.10.6**

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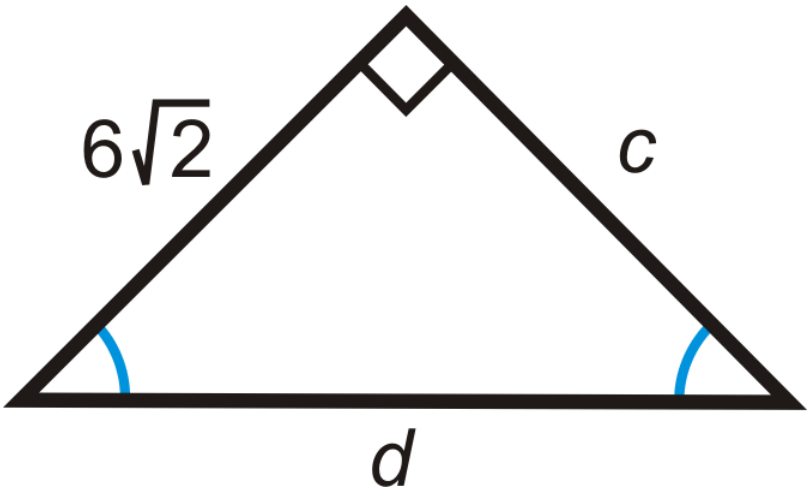


Figure **1.10.7**

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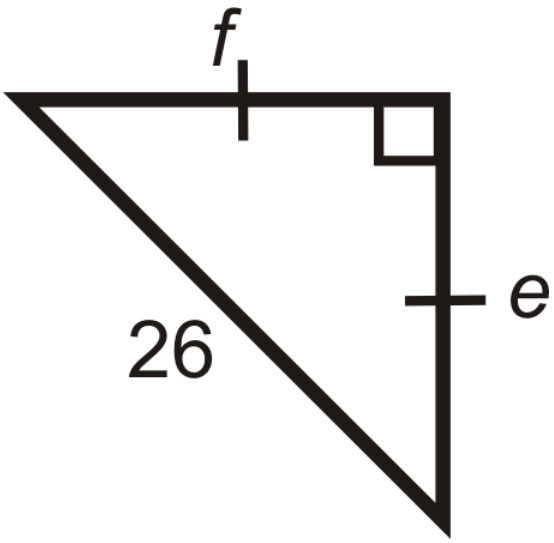


Figure **1.10.8**

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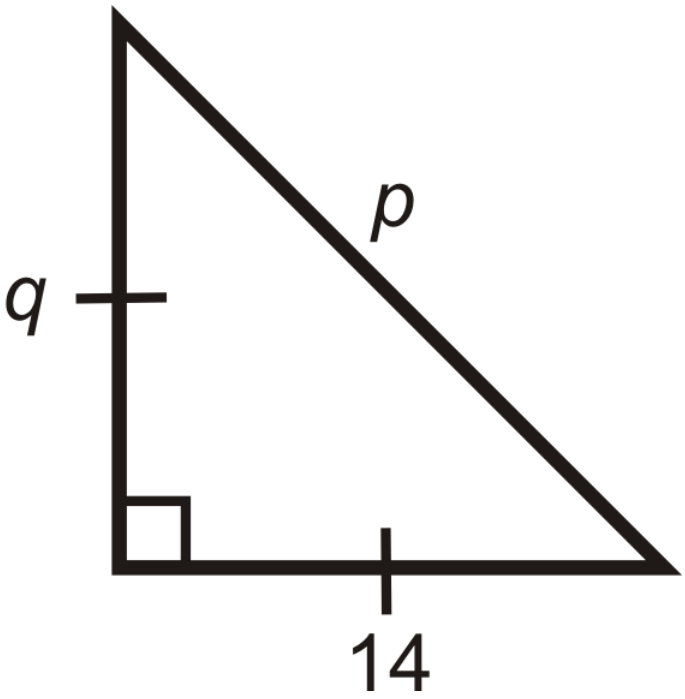


Figure **1.10.9**

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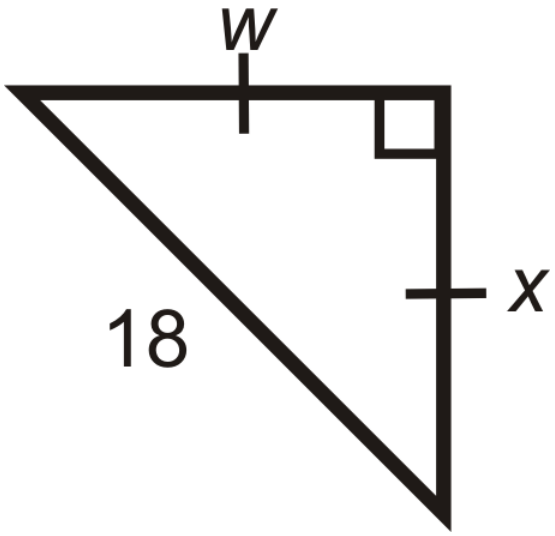


Figure **1.10.10**

11.

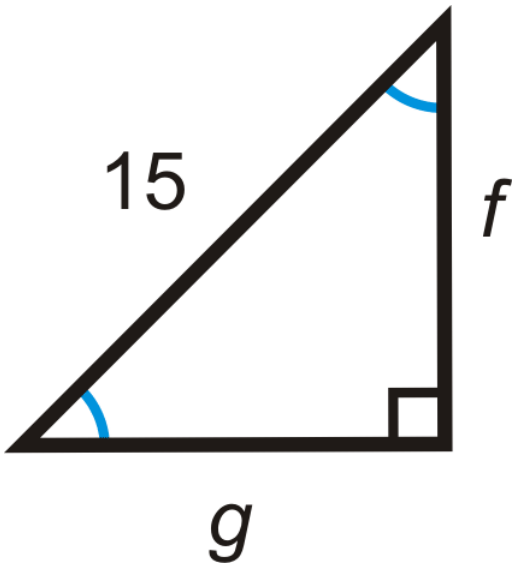
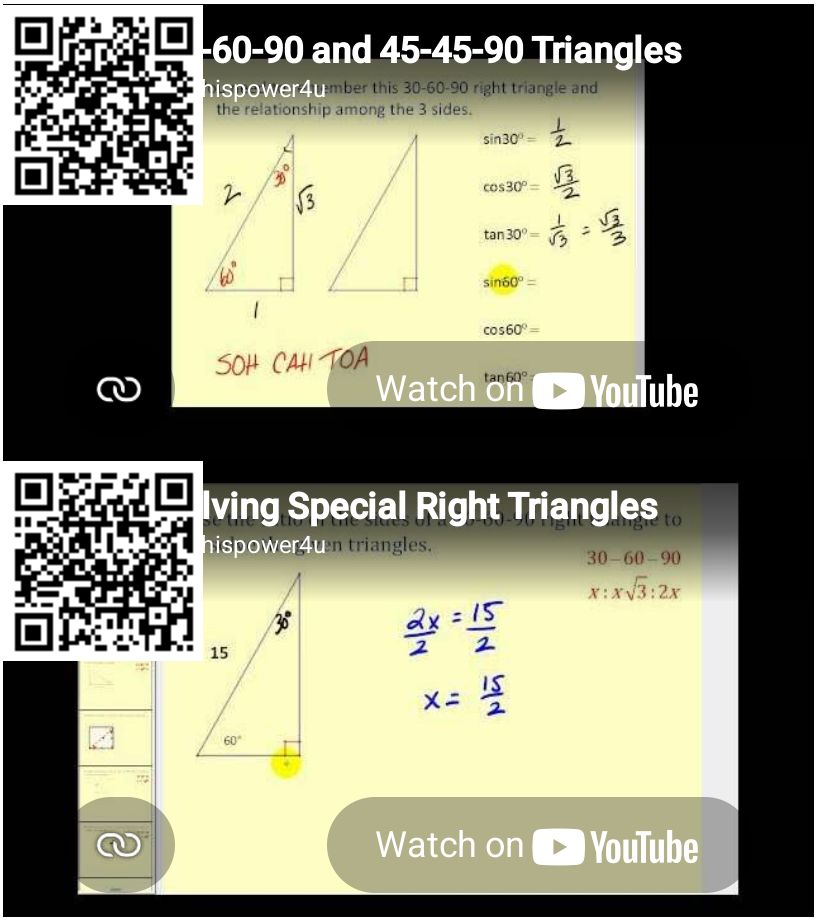


Figure **1.10.11**

Review (Answers)

To see the Review answers, open this PDF file and look for section 8.5.

Resources



Vocabulary

Term	Definition
45-45-90 Theorem	For any isosceles right triangle, if the legs are x units long, the hypotenuse is always $x\sqrt{2}$.
45-45-90 Triangle	A 45-45-90 triangle is a special right triangle with angles of $45°$, $45°$, and $90°$.
Hypotenuse	The hypotenuse of a right triangle is the longest side of the right triangle. It is across from the right angle.
Legs of a Right Triangle	The legs of a right triangle are the two shorter sides of the right triangle. Legs are adjacent to the right angle.
Radical	The √, or square root, sign.

Additional Resources

Video: Solving Special Right Triangles

Activities: 45-45-90 Right Triangles Discussion Questions

Study Aids: Special Right Triangles Study Guide

Practice: 45-45-90 Right Triangles

Real World: Fighting the War on Drugs Using Geometry and Special Triangles

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2.1.4: COS

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2.1.7: SEC CSC COT

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2.1: Trig Functions

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2.1.1: Right Triangle Trigonometry

Sine, cosine, tangent, and other ratios of sides of a right triangle.

Sine, Cosine, and Tangent

Trigonometry is the study of the relationships between the sides and angles of right triangles. The legs are called ***adjacent*** or ***opposite*** depending on which **acute angle** is being used.

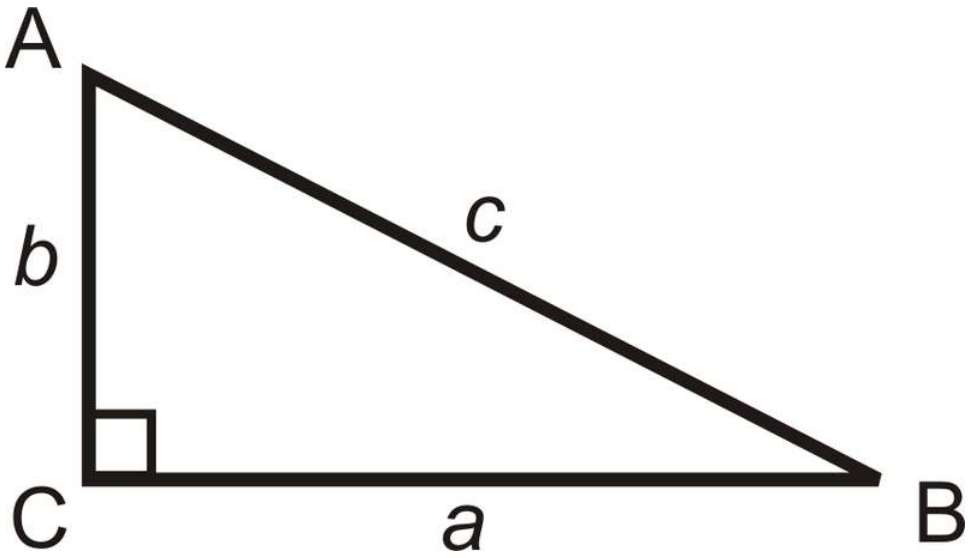


Figure **2.1.1.1**

a is adjacent to $\angle B$ $\quad$ a is opposite $\angle A$
b is adjacent to $\angle A$ $\quad$ b is opposite $\angle B$
c is the hypotenuse

The three basic **trigonometric ratios** are called sine, cosine and tangent. For right triangle $\triangle ABC$, we have:

$$\text{sine Ratio: } \frac{\text{opposite leg}}{\text{hypotenuse}} \qquad \sin A = \frac{a}{c} \text{ or } \sin B = \frac{b}{c}$$

$$\text{cosine Ratio: } \frac{\text{adjacent leg}}{\text{hypotenuse}} \qquad \cos A = \frac{b}{c} \text{ or } \cos B = \frac{a}{c}$$

$$\text{Tangent Ratio: } \frac{\text{opposite leg}}{\text{adjacent leg}} \qquad \tan A = \frac{a}{b} \text{ or } \tan B = \frac{b}{a}$$

An easy way to remember ratios is to use SOH-CAH-TOA.

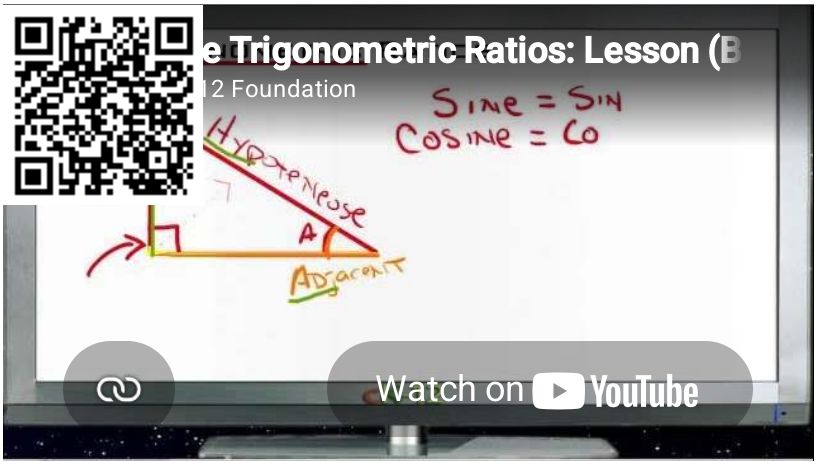
$$\text{Sine} = \frac{\text{Opposite}}{\text{Hypotenuse}} \qquad \text{Cosine} = \frac{\text{Adjacent}}{\text{Hypotenuse}} \qquad \text{Tangent} = \frac{\text{Opposite}}{\text{Adjacent}}$$

Figure **2.1.1.2**

A few important points:

- Always **reduce ratios** (fractions) when you can.
- Use the **Pythagorean Theorem** to find the missing side (if there is one).
- If there is a **radical** in the denominator, **rationalize the denominator.**

What if you were given a right triangle and told that its sides measure 3, 4, and 5 inches? How could you find the sine, cosine, and tangent of one of the triangle's non-right angles?



Example 2.1.1.1

Find the sine, cosine and tangent ratios of $\angle A$.

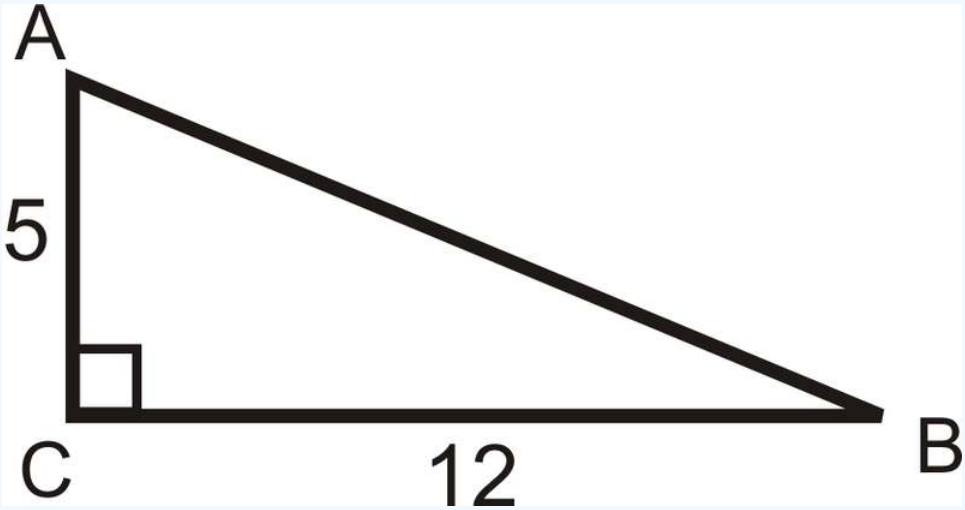


Figure 2.1.1.3

Solution

First, we need to use the Pythagorean Theorem to find the length of the **hypotenuse**.

$$5^2 + 12^2 = c^2$$
$$13 = c$$

$$\sin A = \frac{\textit{leg opposite } \angle A}{\text{hypotenuse}} = \frac{12}{13} \qquad \cos A = \frac{\text{leg adjacent to } \angle A}{\text{hypotenuse}} = \frac{5}{13},$$
$$\tan A = \frac{\text{leg opposite } \angle A}{\text{leg adjacent to } \angle A} = \frac{12}{5}$$

Example 2.1.1.2

Find the sine, cosine, and tangent of $\angle B$.

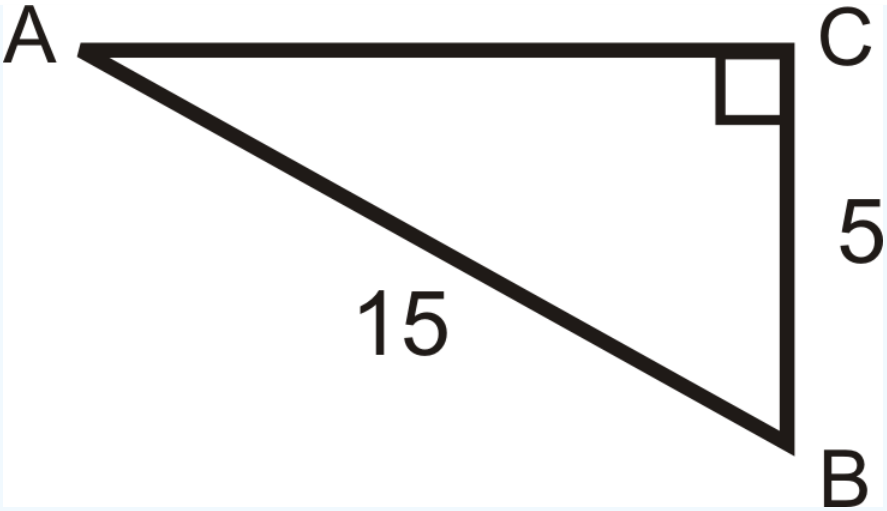


Figure 2.1.1.4

Find the length of the missing side.

Solution

$$AC^2 + 5^2 = 15^2$$
$$AC^2 = 200$$
$$AC = 10\sqrt{2}$$

$$\sin B = \frac{10\sqrt{2}}{15} = \frac{2\sqrt{2}}{3} \quad \cos B = \frac{5}{15} = \frac{1}{3} \quad \tan B = \frac{10\sqrt{2}}{5} = 2\sqrt{2}$$

Example 2.1.1.3

Find the sine, cosine and tangent of $30°$.

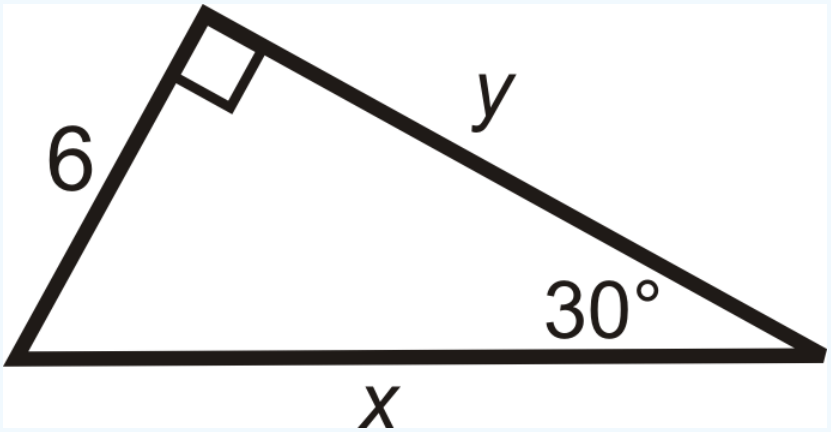


Figure 2.1.1.5

Solution

This is a 30-60-90 triangle. The short leg is 6, $y = 6\sqrt{3}$ and $x = 12$.

$$\sin 30° = \frac{6}{12} = \frac{1}{2} \quad \cos 30° = \frac{6\sqrt{3}}{12} = \frac{\sqrt{3}}{2} \quad \tan 30° = \frac{6}{6\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Example 2.1.1.4

Answer the questions about the following image. Reduce all fractions.

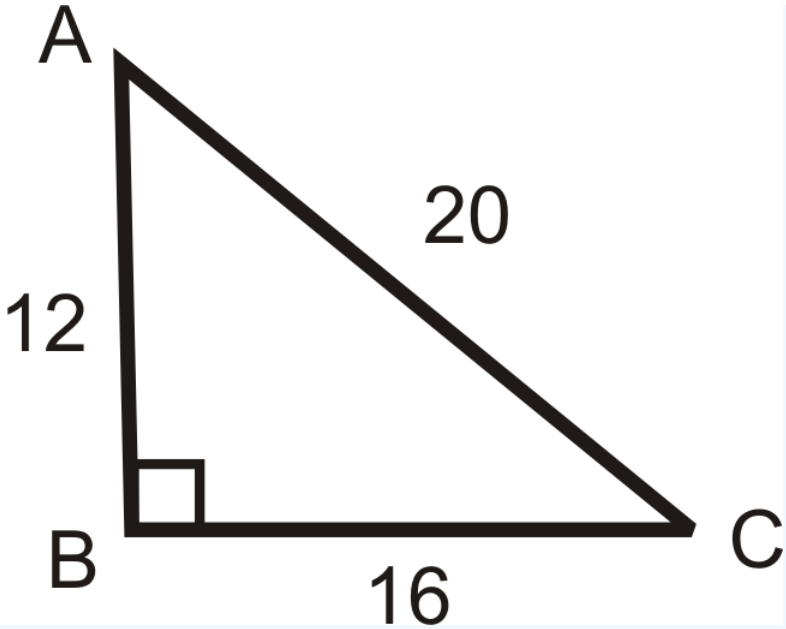


Figure **2.1.1.6**

What is $\sin A$, $\cos A$, and $\tan A$?

Solution

$$\sin A = \frac{16}{20} = \frac{4}{5}$$
$$\cos A = \frac{12}{20} = \frac{3}{5}$$
$$\tan A = \frac{16}{12} = \frac{4}{3}$$

Review

Use the diagram to fill in the blanks below.

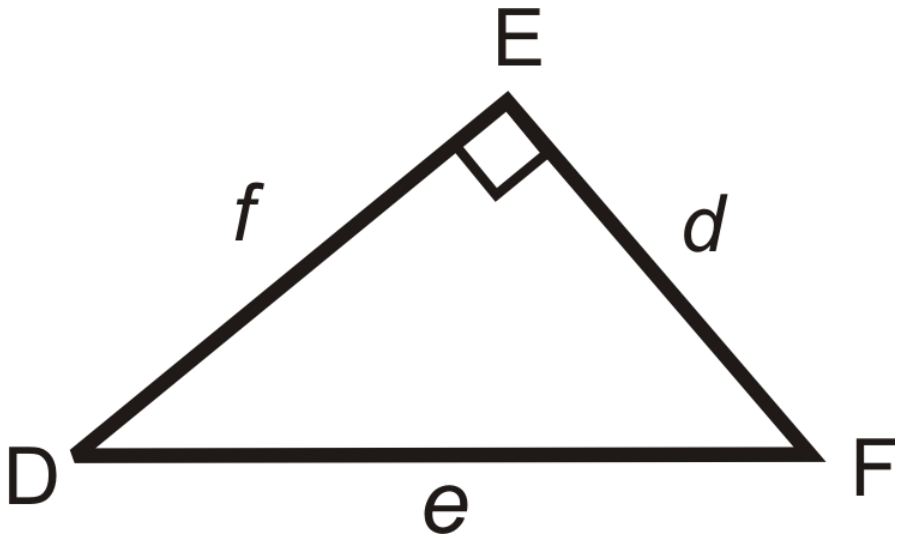


Figure **2.1.1.7**

1. $\tan D = \frac{?}{?}$
2. $\sin F = \frac{?}{?}$
3. $\tan F = \frac{?}{?}$

4. $\cos F = \frac{?}{?}$
5. $\sin D = \frac{?}{?}$
6. $\cos D = \frac{?}{?}$

From questions 1-6, we can conclude the following. Fill in the blanks.

7. $\cos$ ____ $= \sin F$ and $\sin$ ____ $= \cos F$.
8. $\tan D$ and $\tan F$ are _____ of each other.

Find the sine, cosine and tangent of $\angle A$. Reduce all fractions and radicals.

9.

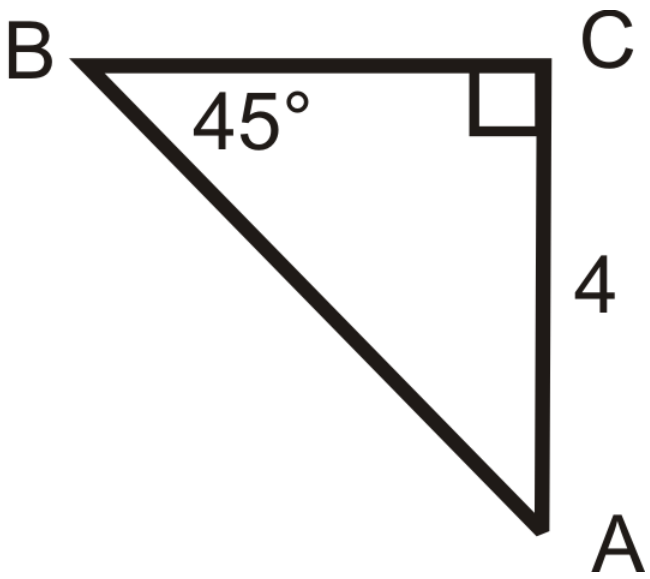


Figure 2.1.1.8

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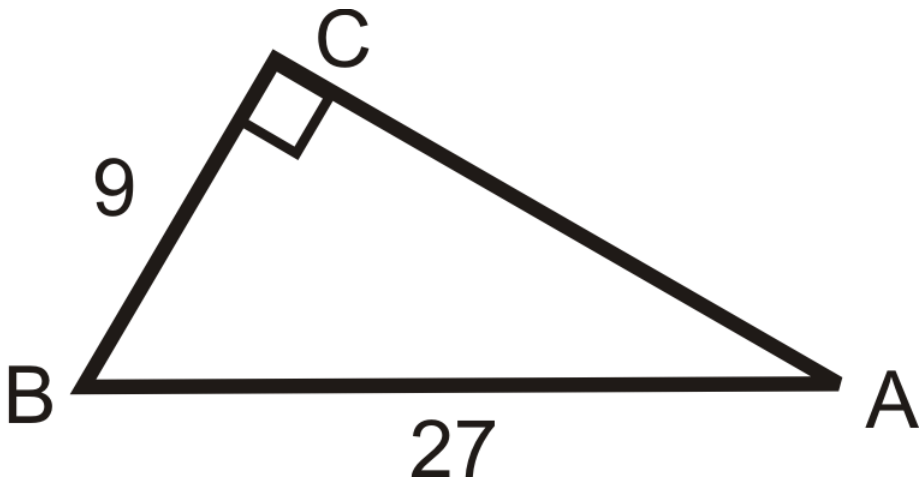


Figure 2.1.1.9

11.

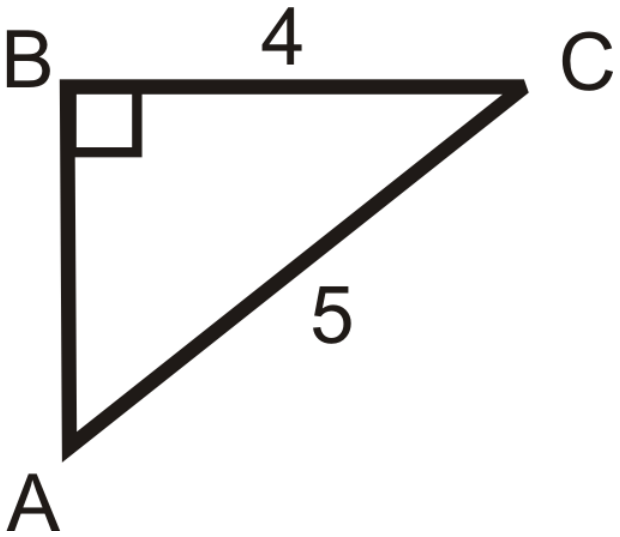


Figure **2.1.1.10**

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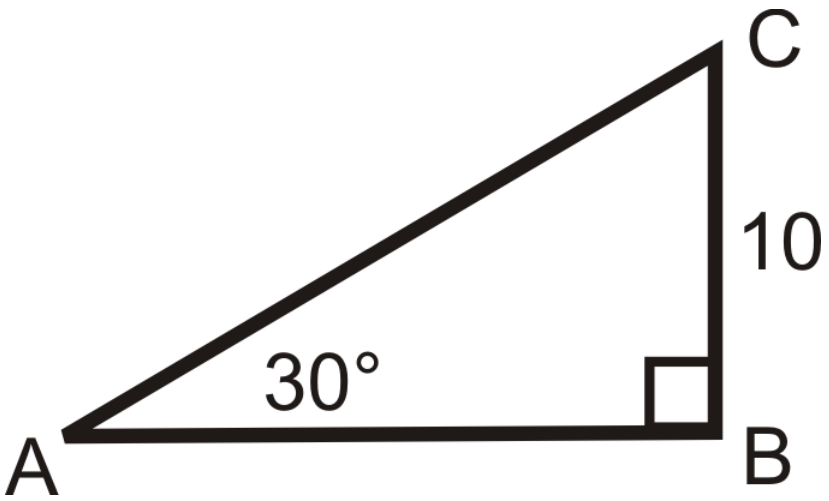


Figure **2.1.1.11**

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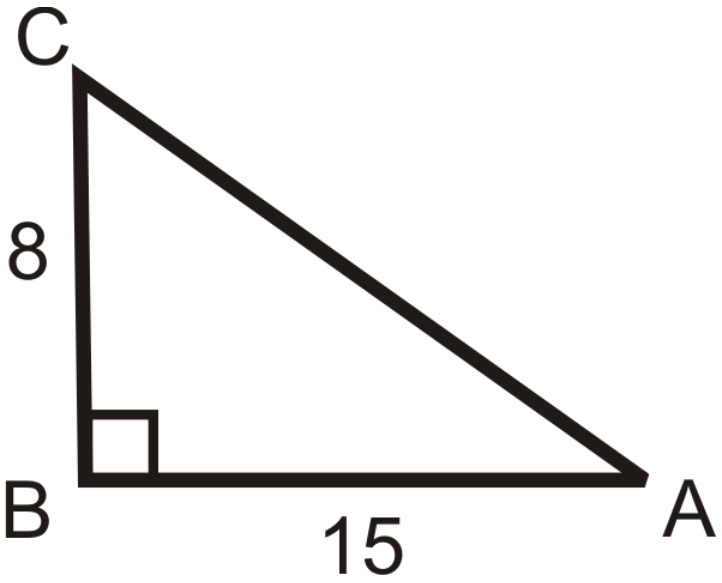
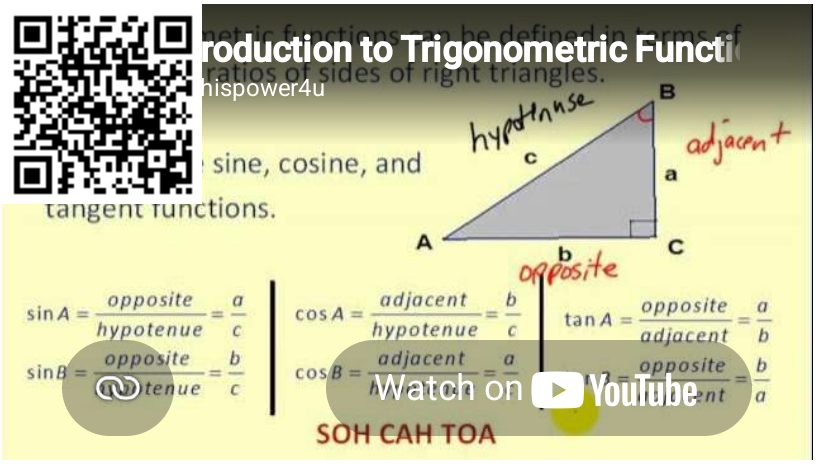


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Review (Answers)

To see the Review answers, open this PDF file and look for section 8.7.

Resources



Vocabulary

Term	Definition
Acute Angle	An acute angle is an angle with a measure of less than 90 degrees.
Adjacent Angles	Two angles are adjacent if they share a side and vertex. The word 'adjacent' means 'beside' or 'next-to'.
Hypotenuse	The hypotenuse of a right triangle is the longest side of the right triangle. It is across from the right angle.
Legs of a Right Triangle	The legs of a right triangle are the two shorter sides of the right triangle. Legs are adjacent to the right angle.
opposite	The opposite of a number x is $-x$. A number and its opposite always sum to zero.
Pythagorean Theorem	The Pythagorean Theorem is a mathematical relationship between the sides of a right triangle, given by $a^2+b^2=c^2$, where a and b are legs of the triangle and c is the hypotenuse of the triangle.
Radical	The Missing argument for \sqrt, or square root, sign.
sine	The sine of an angle in a right triangle is a value found by dividing the length of the side opposite the given angle by the length of the hypotenuse.
Trigonometric Ratios	Ratios that help us to unders\tan d the relationships between sides and angles of right triangles.

Additional Resources

Video: Introduction to Trigonometric Functions U\sin g Triangles

Activities: Sine, cosine, Tangent Discussion Questions

Study Aids: Trigonometric Ratios Study Guide

Practice: Right Triangle Trigonometry

Real World: Sine cosine Tangent

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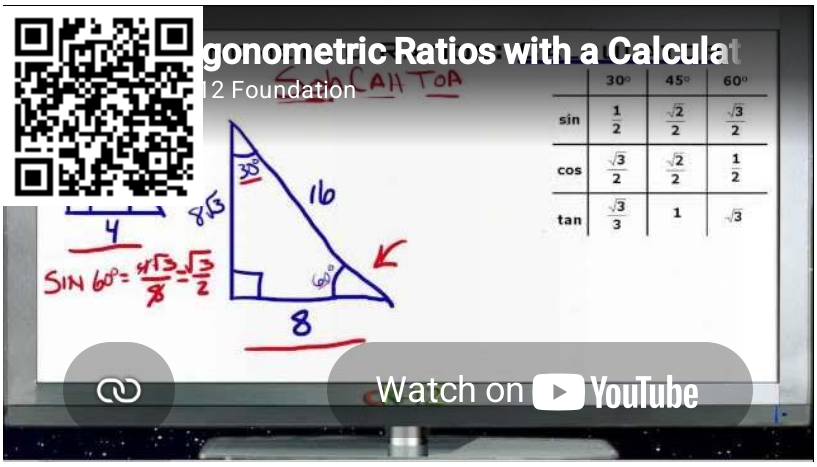
2.1.2: Calculator Trig Functions

Trigonometric ratios based on sides of right triangles in relation to an angle.

Trigonometric Ratios with a Calculator

There is a fixed sine, cosine, and tangent value for every angle, from $0°$ to $90°$. Your scientific (or graphing) calculator knows all the trigonometric values for any angle. Your calculator, should have [SIN], [COS], and [TAN] buttons. You can use your calculator and the trigonometric ratios is to find the missing sides of a right triangle by setting up a trig equation.

What if you were given a 20-70-90 triangle? How could you find the sine, cosine, and tangent of the $20°$ and $70°$ angles?



Example 2.1.2.1

Find the length of the missing sides and round your answers to the nearest tenth:

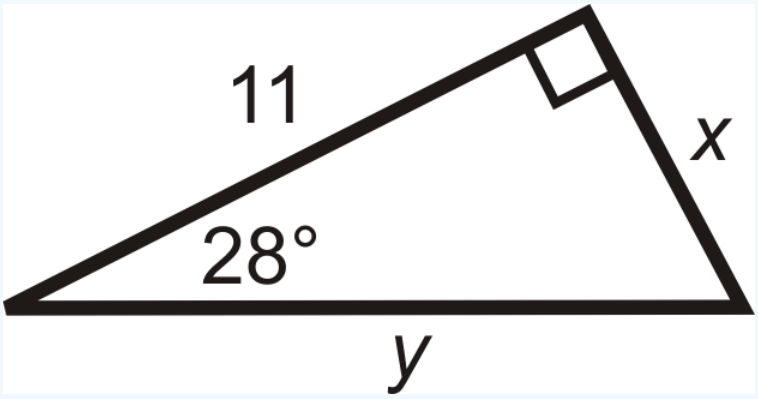


Figure 2.1.2.1

Solution

Use tangent for x and cosine for y.

$$\begin{aligned} \tan 28° &= \frac{x}{11} & \cos 28° &= \frac{11}{y} \\ 11 \cdot \tan 28° &= x & \frac{11}{\cos 28°} &= y \\ x &\approx 5.8 & y &\approx 12.5 \end{aligned}$$

Example 2.1.2.2

Find the length of the missing sides and round your answers to the nearest tenth:

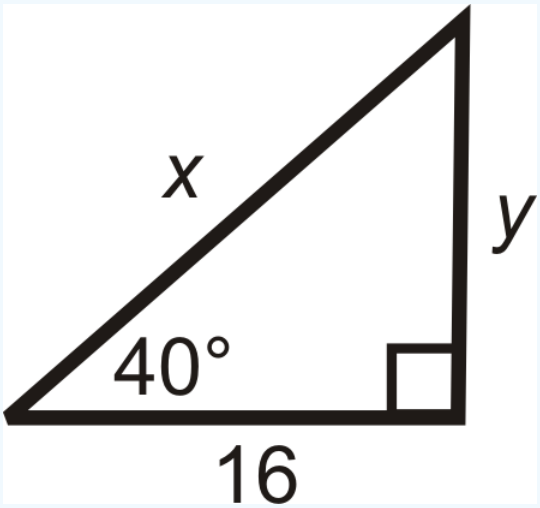


Figure 2.1.2.2

Solution

Use tangent for y and cosine for x.

$$\tan 40^\circ = \frac{y}{16} \qquad \cos 40^\circ = \frac{16}{x}$$
$$16 \cdot \tan 40^\circ = y \qquad \frac{16}{\cos 40^\circ} = x$$
$$y \approx 13.4 \qquad x \approx 20.9$$

Example 2.1.2.3

Find the trigonometric values, using your calculator:

$sin 78^\circ$, $\cos 60^\circ$, $\tan 15^\circ$

Solution

Round to 4 decimal places.

Depending on your calculator, you enter the degree and then press the trig button or the other way around. Also, make sure the mode of your calculator is in *DEGREES*.

$$\sin 78^\circ = 0.97815$$
$$\cos 60^\circ = 0.5$$
$$\tan 15^\circ = 0.26795$$

Example 2.1.2.4

Find the value of each variable. Round your answer to the nearest tenth.

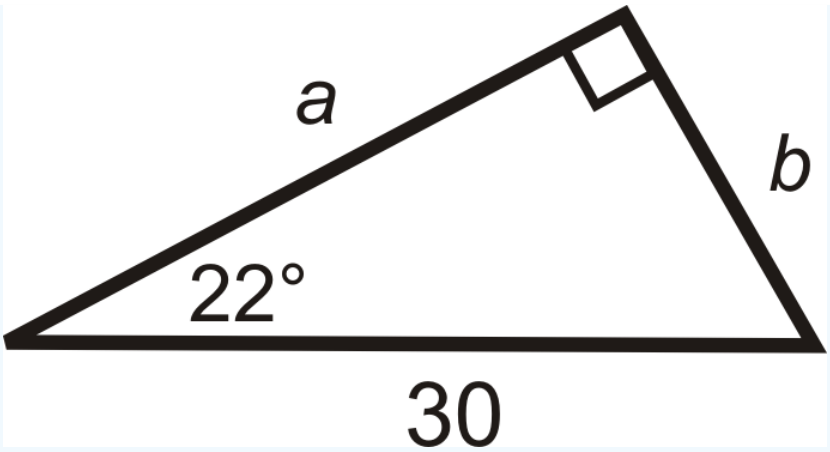


Figure 2.1.2.3

Solution

We are given the **hypotenuse**. Use *sine* to find b, and *cosine* to find a. Use your calculator to evaluate the sine and cosine of the angles.

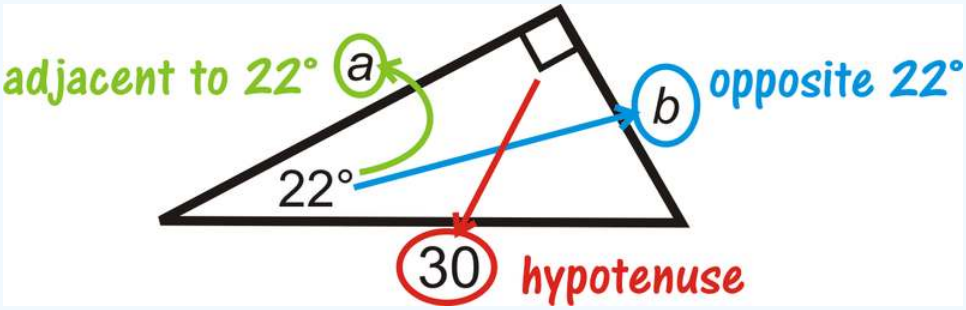


Figure 2.1.2.4

$$\begin{aligned} \sin 22^\circ &= \frac{b}{30} & \cos 22^\circ &= \frac{a}{30} \\ 30 \cdot \sin 22^\circ &= b & 30 \cdot \cos 22^\circ &= a \\ b &\approx 11.2 & a &\approx 27.8 \end{aligned}$$

Example 2.1.2.5

Find the value of each variable. Round your answer to the nearest tenth.

Solution

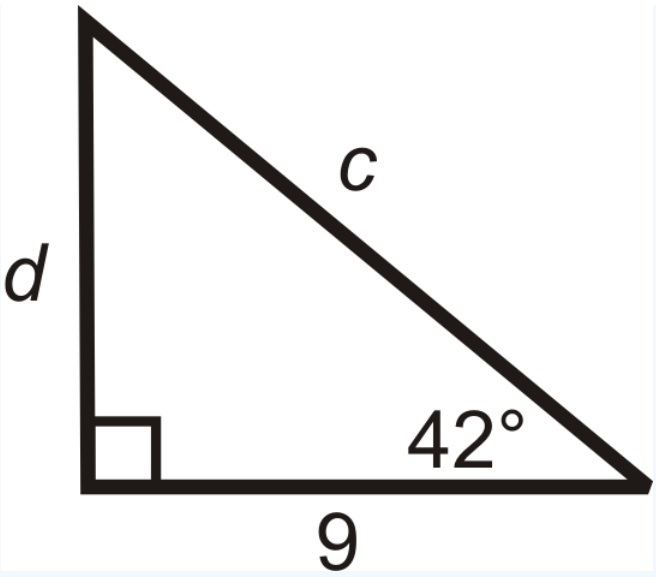


Figure **2.1.2.5**

We are given the adjacent leg to $42°$. To find c, use *cosine* and use *tangent* to find d.

$$\begin{aligned} \cos 42° &= \frac{adjacent}{\text{hypotenuse}} = \frac{9}{c} & \tan 42° &= \frac{\text{opposite}}{\text{adjacent}} = \frac{d}{9} \\ c \cdot \cos 42° &= 9 & 9 \cdot \tan 42° &= d \\ c &= \frac{9}{\cos 42°} \approx 12.1 & d &\approx 27.0 \end{aligned}$$

Any time you use trigonometric ratios, use only the information that you are given. This will result in the most accurate answers.

Review

Use your calculator to find the value of each trig function below. Round to four decimal places.

1. $\sin 24°$
2. $\cos 45°$
3. $\tan 88°$
4. $\sin 43°$
5. $\tan 12°$
6. $\cos 79°$
7. $\sin 82°$

Find the length of the missing sides. Round your answers to the nearest tenth.

8.

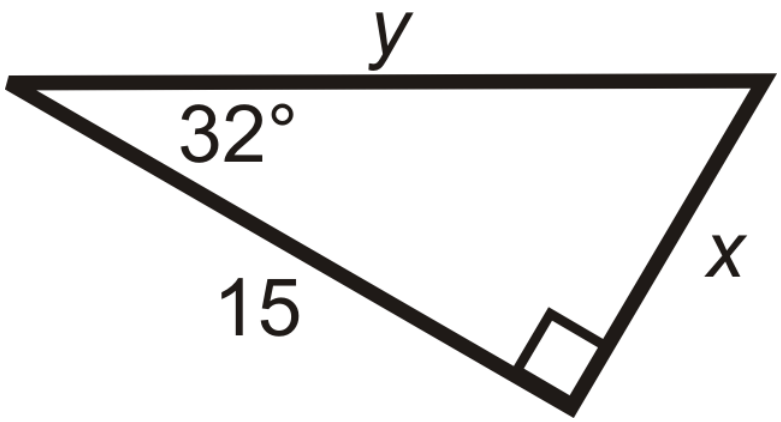


Figure **2.1.2.6**

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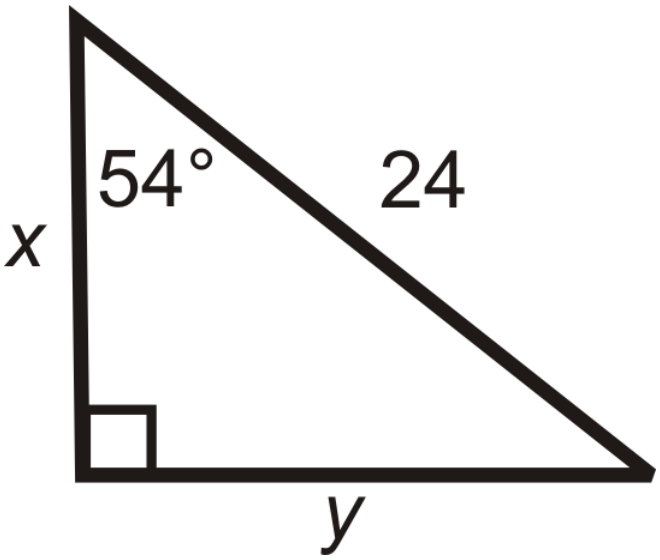


Figure **2.1.2.7**

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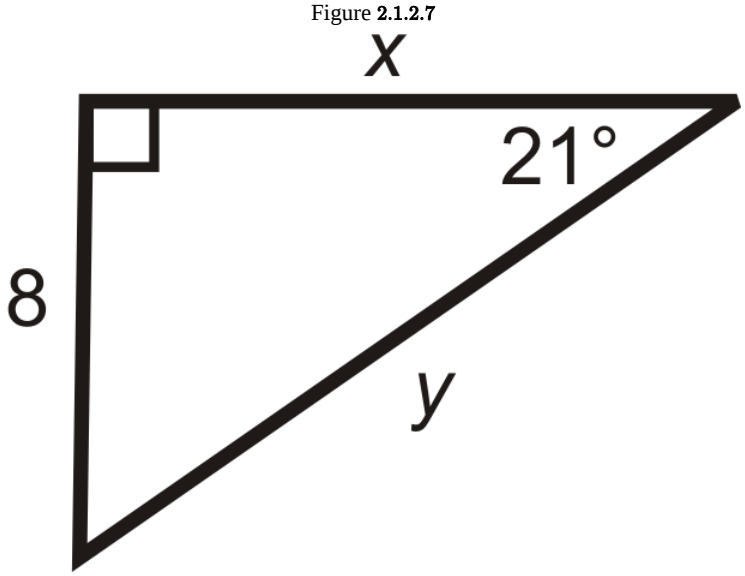
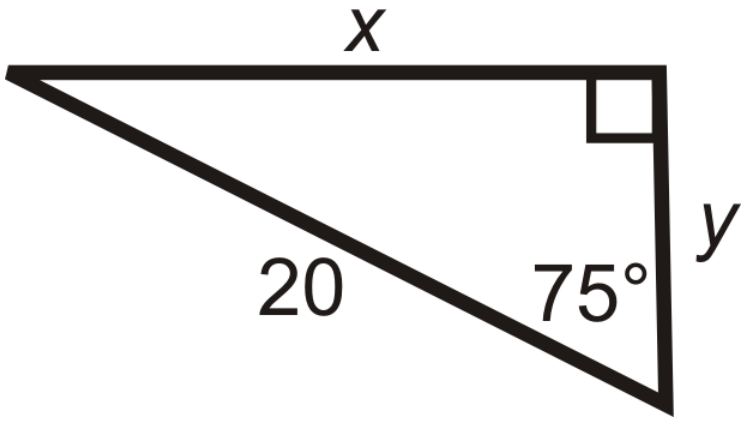


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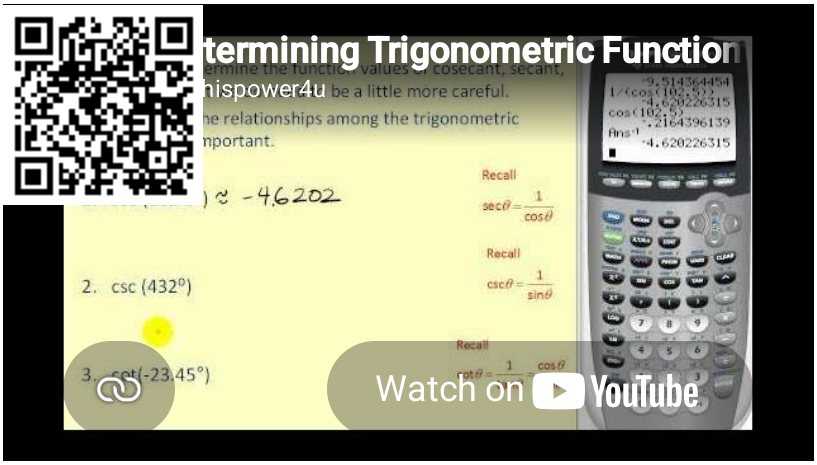
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Review (Answers)

To see the Review answers, open this PDF file and look for section 8.8.

Resources



Vocabulary

Term	Definition
trigonometry	The study of the relationships between the sides and angles of right triangles.
Hypotenuse	The hypotenuse of a right triangle is the longest side of the right triangle. It is across from the right angle.
Legs of a Right Triangle	The legs of a right triangle are the two shorter sides of the right triangle. Legs are adjacent to the right angle.
Trigonometric Ratios	Ratios that help us to understand the relationships between sides and angles of right triangles.

Additional Resources

Video: Introduction to Trigonometric Functions Using Triangles

Activities: Trigonometric Ratios with a Calculator Discussion Questions

Practice: Calculator Trig Functions

Real World: Sine Cosine Tangent

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2.1.3: SIN

Trigonometric ratios based on sides of right triangles in relation to an angle.

Understanding Sines

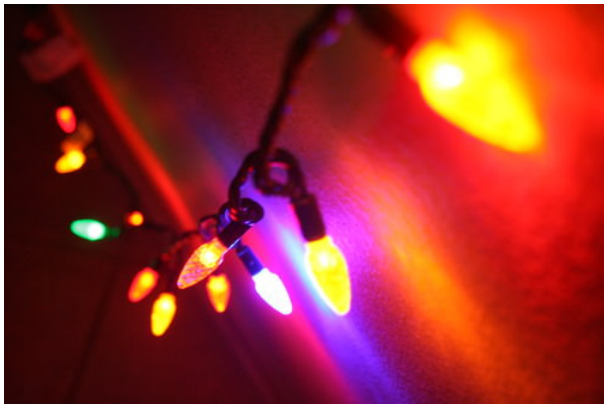


Figure **2.1.3.1**

Jacob has to string lights around the eaves of his flat-roofed shed. Since he is afraid of heights he wants to plan this endeavor very carefully. He knows that for safety reasons, a ladder should make an angle of 75° or less with the ground. If Jacob has a ladder 6.5 meters long to place against the side of the shed that is 2.5 meters high, then what angle will the ladder make with the ground? Jacob needs to be certain that the angle meets safety requirements before he puts one foot on the ladder. How can he figure out the measure of the angle between the ladder and the ground?

In this concept, you will learn to understand the trigonometric ratio of **sine**.

Sines

Trigonometry is a branch of mathematics used to determine the lengths of sides and the measure of angles with great accuracy. You will only consider the results obtained by using right angled triangles.

The side of a right triangle located opposite the 90^{\circ} angle is the longest side of the triangle and is called the hypotenuse. The two shorter sides of the triangle, often called legs, have specific names in reference to the location of one of the acute angles of the right triangle.

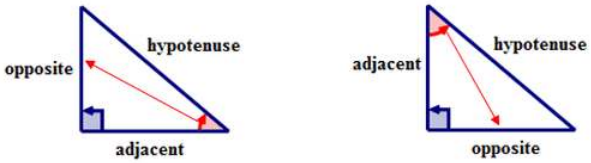


Figure **2.1.3.2**

In both triangles the hypotenuse is opposite the right angle. In the first triangle the angle marked in red is called the **reference angle** and it is used to name the legs of the triangle. The measure of the **reference angle** can be given or its measure may have to be calculated. Whatever the case, one of the acute angle will be referred to in the question or problem. The side across from the reference angle is called the **opposite side**. The third side is called the **adjacent side** and it is the side of the triangle, along with the hypotenuse, that forms the reference angle.

In the second triangle the angle marked in red is called the **reference angle** and it is used to name the legs of this triangle. The side across from the reference angle is called the **opposite side**. The third side is called the **adjacent side** and it is the side of the triangle, along with the hypotenuse, that forms the reference angle.

The location of the hypotenuse never changes but the location of the opposite and adjacent sides depends upon which acute angle is the reference angle.

In each given diagram, label each side of the right triangle as the:

Hypotenuse (H); Side *Opposite* reference angle (O); Side *Adjacent* to reference angle (A)

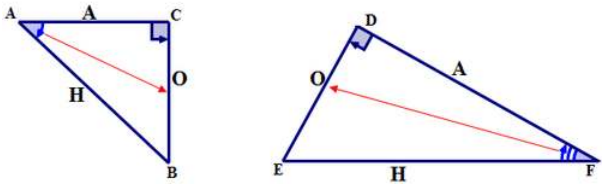


Figure **2.1.3.3**

A ratio of the lengths of the sides of a right triangle is called a **trigonometric ratio**. The ratio of the length of the side opposite the reference angle to the length of the hypotenuse is known as the s**ine ratio**. The sine ratio is related to the reference angle of the right triangle, not to the right angle.

The sine ratio for the first triangle can be written in words as:

sine of $\angle A = \frac{\text{side opposite} \angle A}{\text{hypotenuse}}$ Or in a shortened form as sine of $\angle A = \frac{opposite}{hypotenuse}$

The sine ratio for the first triangle can be written in symbols as:

$\sin A = \frac{BC}{AB}$

The sine ratio for the second triangle can be written in words as:

sine of $\angle F = \frac{\text{side opposite} \angle F}{\text{hypotenuse}}$ Or in a shortened form as sine of $\angle F = \frac{opposite}{hypotenuse}$

The sine ratio for the first triangle can be written in symbols as:

$\sin F = \frac{DE}{EF}$

Let's determine the value for the sine ratio of each of the acute angles using the following right triangle. Express the sine ratio in words and symbols. Then, using the values on the corresponding sides, replace the symbols with the numbers and express the ratio first as a fraction and then as a decimal rounded to the nearest ten thousandth (four places after the decimal).

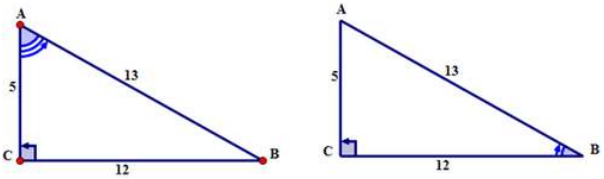


Figure **2.1.3.4**

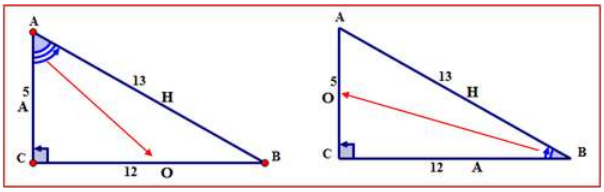


Figure **2.1.3.5**

For the first triangle:

First, with the respect to the reference angle A, name the sides of the triangle.

The hypotenuse is across from the right angle. The opposite side is across from $\angle A$. The side alongside $\angle A$ is the adjacent. The sides are labeled with the letters **H, O, A** respectively.

Next, write the sine ratio for $\angle A$ in all of the required forms.

Words:

sine of $\angle A = \frac{opposite}{hypotenuse}$

Symbols:

$$\sin A = \frac{BC}{AB}$$

Fraction:

$$\sin A = \frac{12}{13}$$

Decimal:

$$\sin A = 0.9231$$

The second triangle is the first triangle with \angle B as the reference angle. Notice that the locations of the opposite and adjacent sides have changed from where they were when \angle A was the reference angle.

First, with the respect to the reference angle B, name the sides of the triangle.

The hypotenuse is across from the right angle. The opposite side is across from $\angle B$. The side alongside $\angle B$ is the adjacent. The sides are labeled with the letters **H, O, A** respectively.

Next, write the sine ratio for $\angle B$ in all of the required forms.

Words:

$$\text{sine of } \angle B = \frac{opposite}{hypotenuse}$$

Symbols:

$$\sin B = \frac{AC}{AB}$$

Fraction:

$$\sin B = \frac{5}{13}$$

Decimal:

$$\sin B = 0.3846$$

If sinB=0.3846, then the measure of \angle B can be found using the TI-84 calculator.

Remember you measure angles in units called degrees. Press mode and look at the screen.



Figure **2.1.3.6**

In the fourth row of the screen, DEGREE is not highlighted.

Follow the Key Press History below to put the calculator in DEGREE mode.

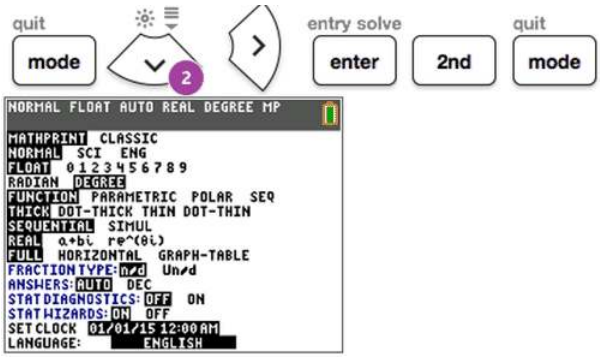


Figure \(\
PageIndex
{7
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In the fourth row of the screen, DEGREE is now highlighted.

Next, follow the Key Press History below to find the measure of $\angle B$.

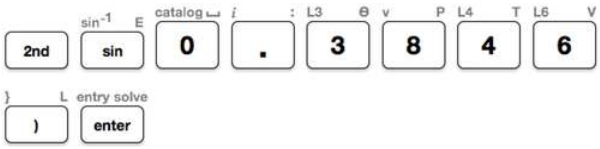


Figure **2.1.3.8**

Then, look at the screen of the calculator:

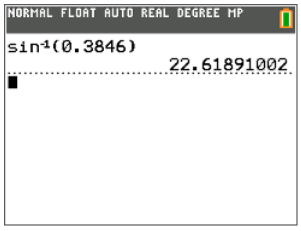


Figure **2.1.3.9**

Round the measure of the angle to the nearest tenth.

$\angle B = 22.6°$

The answer is $22.6°$

Example 2.1.3.1

Earlier, you were given a problem about Jacob and the angle of his ladder. Jacob needs to make sure that the ladder makes an angle of less than $75°$ with the ground? Jacob will have to use the sine ratio to figure this out.

Solution

First, draw and label a right triangle with the information given.

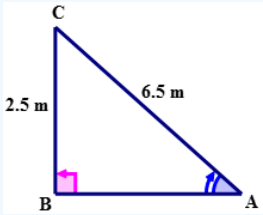


Figure **2.1.3.10**

Next, name the sides of the triangle with reference to $\angle A$.

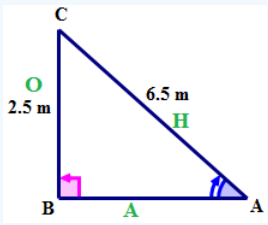


Figure 2.1.3.11

Next, express the ratio $\sin A$ using words.

$$\sin A = \frac{opposite}{hypotenuse}$$

Next, express the ratio using symbols.

$$\sin A = \frac{BC}{AC}$$

Next, express the ratio as a fraction using the corresponding values of the sides.

$$\sin A = \frac{2.5}{6.5}$$

Next, express the ratio as a decimal.

$$\sin A = 0.3846$$

Next, use the sin−1 function on the TI-calculator to find the measure of \angle A.

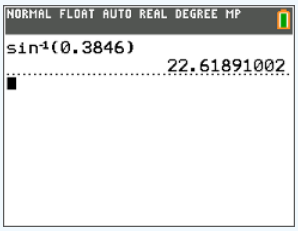


Figure 2.1.3.12

Then, write the measure of $\angle A$ to the nearest tenth.

$$\angle A = 22.6^\circ$$

The answer is 22.6°.

It is safe for Jacob to go up the ladder.

Example 2.1.3.2

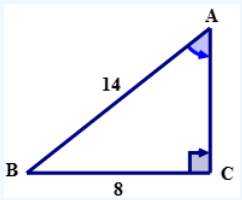


Figure 2.1.3.13

Use the sine ratio to calculate the measure of $\angle A$ to the nearest tenth.

Solution

First, using $\angle A$ name the sides of the triangle.

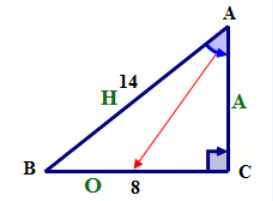


Figure **2.1.3.14**

Next, write the sine ratio in words.

$$\sin A = \frac{opposite}{hypotenuse}$$

Next, write the ratio using symbols.

$$\sin A = \frac{BC}{AB}$$

Next, express the ratio as a fraction using the values of the corresponding sides.

$$\sin A = \frac{8}{14}$$

Next, express the ratio as a decimal rounded to four places after the decimal.

$$\sin A = 0.5714$$

Then, use the inverse sine function ($\sin^{-1}$) on the TI calculator to find the measure of $\angle A$.

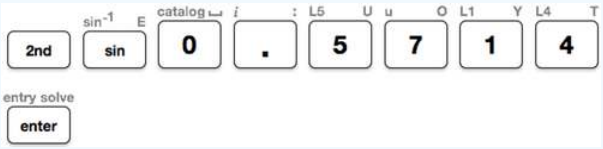


Figure **2.1.3.15**

Then, look at the screen and round the measure of the angle to the nearest tenth.

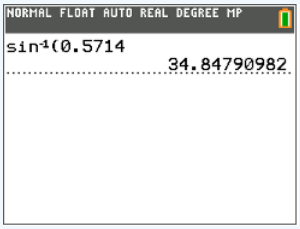


Figure **2.1.3.16**

Notice, it is not necessary to close the bracket after the decimal has been entered into the calculator.

$\angle A = 34.9°$

The answer is **34.9°**

✔ Example 2.1.3.3

For the following right triangle, express sinB in words, symbols, as a fraction and as a decimal.

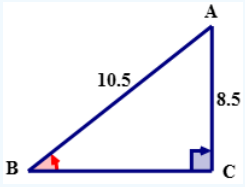


Figure **2.1.3.17**

Solution

First, name the sides of the triangle using the reference angle B.

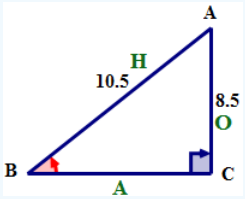


Figure 2.1.3.18

Next, express the ratio sinB using words.

$$\sin B = \frac{opposite}{hypotenuse}$$

Next, express the ratio using symbols.

$$\sin B = \frac{AC}{AB}$$

Next, express the ratio as a fraction using the corresponding values of the sides.

$$\sin B = \frac{8.5}{10.5}$$

Then, express the ratio as a decimal to the ten thousandth.

$$\sin B = 0.8095$$

The answer is 0.8095.

Example 2.1.3.4

If $\sin A = 0.3872$, what is the measure of $\angle A$ to the nearest tenth.

Solution

First, use the TI-calculator and follow the key press history.

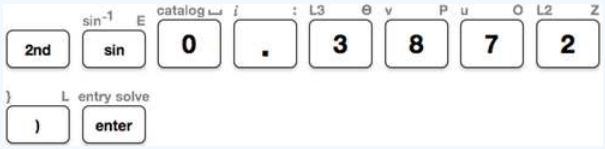


Figure 2.1.3.19

Next, look at the screen of the calculator.

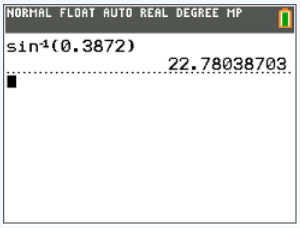


Figure 2.1.3.20

Then, write the measure of $\angle A$ to the nearest tenth.

$$\angle A = 22.8°$$

The answer is 22.8°.

Example 2.1.3.5

Determine the measure of \angle B in the following triangle.

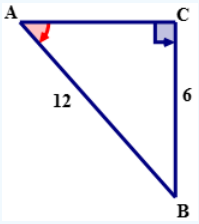


Figure 2.1.3.21

Solution

First, name the sides of the triangle.

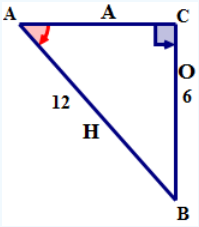


Figure 2.1.3.22

Next, express the ratio sin A using words.

$$sinA = \frac{opposite}{hypotenuse}$$

Next, express the ratio using symbols.

$$\sin A = \frac{BC}{AB}$$

Next, express the ratio as a fraction using the corresponding values of the sides.

$$\sin A = \frac{6}{12}$$

Next, express the ratio as a decimal.

$$\sin A = 0.5$$

Next, use the sin−1 function on the TI-calculator to find the measure of $\angle A$.

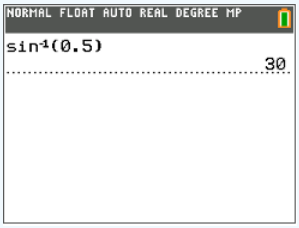


Figure 2.1.3.23

Then, write the measure of $\angle A$.

$$\angle A = 30°$$

The answer is $30°$.

Review

Solve each problem.

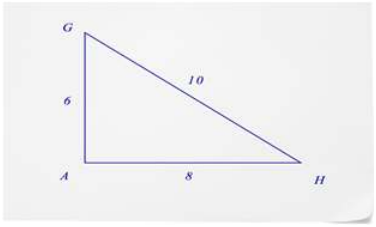


Figure **2.1.3.24**

1. What is the sine of $\angle G$?
2. What is the sine of $\angle H$?
3. Can you find the sine of $\angle A$?

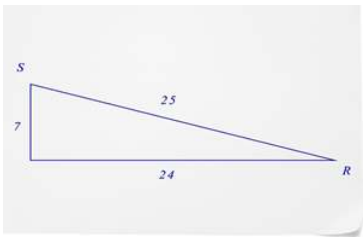


Figure **2.1.3.25**

4. What is the sine of $\angle R$?
5. What is the sine of $\angle S$?

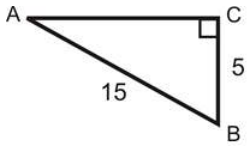


Figure **2.1.3.26**

6. What is the sine of $\angle A$?
7. What is the sine of $\angle B$?
8. What is the length of the missing side rounded to the nearest hundredth?

Answer each question true or false.

9. You can use the Pythagorean Theorem to find the length of a missing side in a right triangle.
10. A right triangle must have a 90 degree angle.
11. The sine ratio is the hypotenuse over the opposite side.
12. If you know only side length, then you can figure out all the side lengths.
13. Sine ratio has to do with side lengths.
14. The hypotenuse is always opposite the right angle.
15. You must be given all three side lengths to figure out the sine ratio.

Review (Answers)

To see the Review answers, open this PDF file and look for section 7.13.

Vocabulary

Term	Definition
sine	The sine of an angle in a right triangle is a value found by dividing the length of the side opposite the given angle by the length of the hypotenuse.
Trigonometric Ratios	Ratios that help us to understand the relationships between sides and angles of right triangles.

Additional Resources

Video: Introduction to Trigonometric Functions Using Triangles

Activities: Trigonometric Ratios with a Calculator Discussion Questions

Practice: Calculator Trig Functions

Real World: Sine Cosine Tangent

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2.1.4: COS

Determine cosine given side lengths

Understanding Cosines



Figure 2.1.4.1

Roger has placed a ladder that is 12 feet in length against the side of the house such that the foot of the ladder is 5 feet from the base of the house. As Roger prepares to climb the ladder to clean the rain gutters of the house, his neighbor yells to him, “You had better adjust that ladder. The angle between the ladder and the ground has to be less than $75°$.”

Roger stepped back and looked at the position of the ladder. “He could be right,” thought Roger. “I’ve got to figure out the size of that angle before I climb up that ladder.”

How can Roger calculate the measure of the angle?

In this concept, you will learn to understand the trigonometric ratio **Cosine**.

Cosine

Trigonometry is a branch of mathematics used to determine the lengths of sides and the measure of angles with great accuracy. A ratio of the lengths of the sides of a right triangle is called a **trigonometric ratio**. The ratio of the length of the side alongside the reference angle to the length of the hypotenuse is known as the **Cosine ratio**. An acute angle of a right triangle is formed by the hypotenuse and one of the legs of the triangle. This leg is called the adjacent side of the reference angle. The Cosine ratio is the ratio of the adjacent side to the hypotenuse.

In each given diagram, label each side of the right triangle as the:

Hypotenuse (H); Side *Opposite* reference angle (O); Side *Adjacent* to reference angle (A).

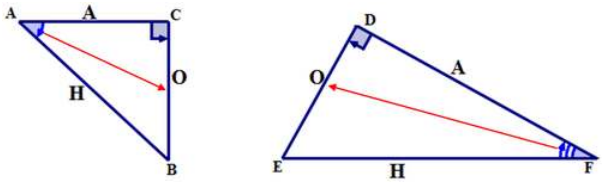


Figure 2.1.4.2

The Cosine ratio for the first triangle can be written in words as:

cosine of $\angle A = \frac{\text{side adjacent } \angle A}{\text{hypotenuse}}$ Or in a shortened form as cosine of $\angle A = \frac{\text{adjacent}}{\text{hypotenuse}}$

The cosine ratio for the first triangle can be written in symbols as:

$\cos A = \frac{AC}{AB}$

The Cosine ratio for the second triangle can be written in words as:

cosine of $\angle F = \frac{\text{side adjacent } \angle F}{\text{hypotenuse}}$ Or in a shortened form as cosine of $\angle F = \frac{\text{adjacent}}{\text{hypotenuse}}$

The Cosine ratio for the second triangle can be written in symbols as:

$\cos F = \frac{DF}{EF}$

Let's determine the value for the cosine ratio of each of the acute angles using the following right triangle. Express the cosine ratio in words and symbols. Then, using the values on the corresponding sides, replace the symbols with the numbers and express the ratio first as a fraction and then as a decimal rounded to the nearest ten thousandth (four places after the decimal).

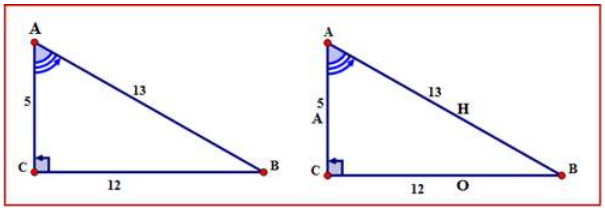


Figure **2.1.4.3**

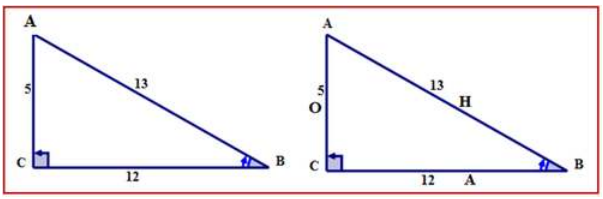


Figure **2.1.4.4**

ΔABC with $\angle A$ as the reference angle.

First, name the sides of the triangle.

The hypotenuse is across from the right angle. The opposite side is across from \angle A. The side alongside \angle A is the adjacent. The sides are labeled with the letters **H, O, A** respectively.

Next, write the cosine ratio for $\angle A$ in all of the required forms.

Words:

cosine of $\angle A = \frac{\text{adjacent}}{\text{hypotenuse}}$

Symbols:

$\cos A = \frac{AC}{AB}$

Fraction:

$\cos A = \frac{5}{13}$

Decimal:

$\cos A = 0.3846$

ΔABC with $\angle B$ as the reference angle.

Notice that the locations of the opposite and adjacent sides have changed from where they were when $\angle A$ was the reference angle.

First, name the sides of the triangle.

The hypotenuse is across from the right angle. The opposite side is across from $\angle B$. The side alongside $\angle B$ is the adjacent. The sides are labeled with the letters **H, O, A** respectively.

Next, write the cosine ratio for $\angle B$ in all of the required forms.

Words:

cosine of $\angle B = \dfrac{\text{adjacent}}{\text{hypotenuse}}$

Symbols:

$\cos B = \dfrac{BC}{AB}$

Fraction:

$\cos B = \dfrac{12}{13}$

Decimal:

$\cos B = 0.9231$

If $\cos A = 0.3846$, then the measure of $\angle A$ can be found using the inverse cosine function on the TI- calculator.

First, follow the Key Press History below to calculate the measure of $\angle A$.

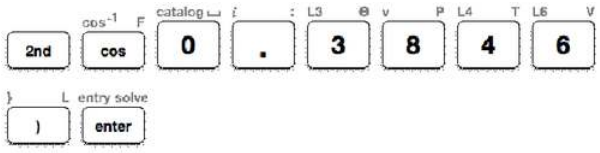


Figure 2.1.4.5

Next, look at the calculator screen to see the measure of $\angle A$.

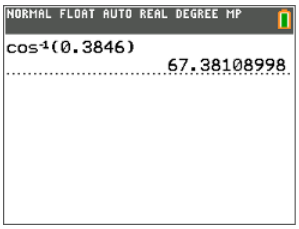


Figure 2.1.4.6

Then, write the measure of $\angle A$ to the nearest tenth.

$\angle A = 67.4°$

The answer is $67.4°$

Example 2.1.4.1

Earlier, you were given a problem about Roger and the ladder. He needs to figure out the angle the ladder makes with the ground.

How can Roger figure out the measure of the angle?

Solution

He can use the cosine ratio.

First, draw and label a right triangle to model the problem.

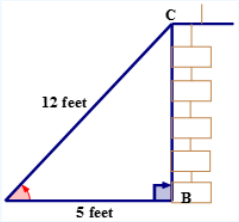


Figure 2.1.4.7

Next, name the sides of the triangle.

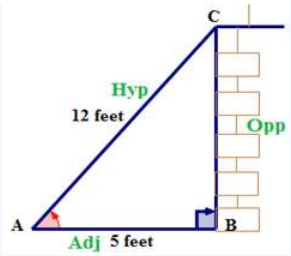


Figure 2.1.4.8

Next, write the cosine ratio in words.

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}}$$

Next, write the ratio using symbols.

$$\cos A = \frac{AB}{AC}$$

Next, express the ratio as a fraction using the values of the corresponding sides.

$$\cos A = \frac{5}{12}$$

Next, express the ratio as a decimal rounded to four places after the decimal.

$$\cos A = 0.4167$$

Next, use the inverse cosine function (cos−1) on the TI calculator to find the measure of $\angle A$.

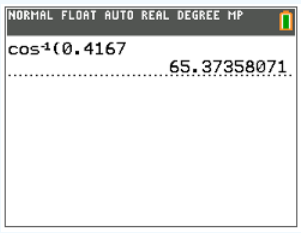


Figure 2.1.4.9

Then, write the measure of $\angle A$ to the nearest tenth.

$$\angle A = 65.4^\circ$$

The answer is 65.4°

The measure of $\angle A$ is 65.4°. It is less than 75° so Roger can repair the rain gutters.

Example 2.1.4.2

Use the cosine ratio to calculate the measure of $\angle A$ to the nearest tenth.

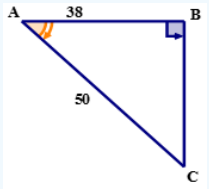


Figure 2.1.4.10

Solution

First, using the reference angle name the sides of the triangle.

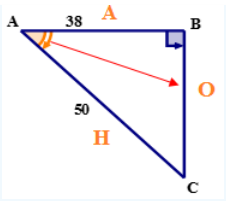


Figure 2.1.4.11

Next, write the cosine ratio in words.

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}}$$

Next, write the ratio using symbols.

$$\cos A = \frac{AB}{AC}$$

Next, express the ratio as a fraction using the values of the corresponding sides.

$$\cos A = \frac{38}{50}$$

Next, express the ratio as a decimal rounded to four places after the decimal.

$$\cos A = 0.76$$

The decimal terminated. You could write the decimal as 0.7600 but this is not necessary.

Next, use the inverse cosine function ($\cos^{-1}$) on the TI calculator to find the measure of $\angle A$.

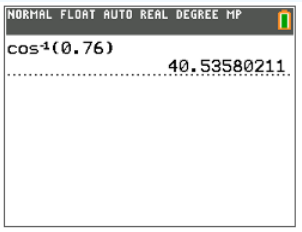


Figure 2.1.4.12

Then, write the value of $\angle A$ to the nearest tenth.

$$\angle A = 40.5°$$

The answer is $40.5°$

Example 2.1.4.3

If $\cos B = 0.7984$, what is the measure of $\angle B$ to the nearest tenth.

Solution

First, use the TI-calculator to determine the measure of $\angle B$ by using the inverse cosine function (\(\cos^{-1}) on the calculator. This function is above the cos button on the calculator. To access this function press

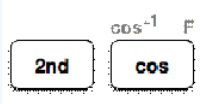


Figure 2.1.4.13

Next, enter the decimal 0.7984 in the parenthesis where the cursor is flashing.



Figure 2.1.4.14

Next, press enter and the calculator screen will display the measure of the angle.

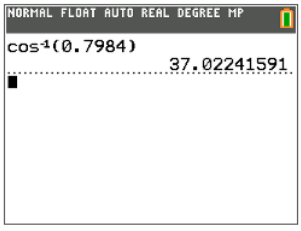


Figure 2.1.4.15

Then, write the measure of $\angle B$ to the nearest tenth.

$\angle B = 32.0°$.

The answer is $32.0°$

Example 2.1.4.4

For the following solution that shows using the cosine ratio to calculate the measure of \angle B, draw and label the right triangle ABC that was used to do the solution.

$$\begin{aligned} \cos B &= \frac{\text{adjacent}}{\text{hypotenuse}} \\ \cos B &= \frac{BC}{AB} \\ \cos B &= \frac{36}{76.7} \\ \cos B &= 0.4694 \\ \cos^{-1}(\cos B) &= \cos^{-1}(0.4694) \\ \angle B &= 62.00454372 \\ \angle B &= 62.0° \end{aligned}$$

Solution

First, write down what you know from the solution.

ΔABC is a right triangle.

$\overline{BC}$ is the adjacent side of the triangle.

$\overline{BC}$ has a length of 36.

$\overline{AB}$ is the hypotenuse of the triangle.

$\overline{AB}$ has a length of 76.7.

$\angle B$ is the reference angle.

Next, use the information you have written down to draw and label the triangle.

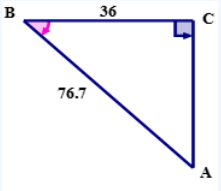


Figure 2.1.4.16

The above triangle represents the problem.

Example 2.1.4.5

For the given right triangle, use the cosine ratio to calculate the measure of $\angle A$ to the nearest tenth.

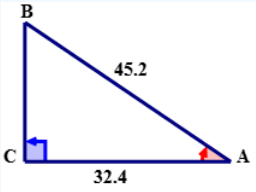


Figure 2.1.4.17

Solution

First, using the reference angle name the sides of the triangle.

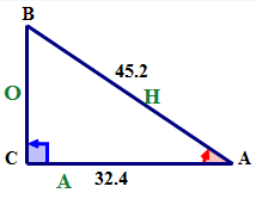


Figure 2.1.4.18

Next, write the cosine ratio in words.

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}}$$

Next, write the ratio using symbols.

$$\cos A = \frac{AC}{AB}$$

Next, express the ratio as a fraction using the values of the corresponding sides.

$$\cos A = \frac{32.4}{45.2}$$

Next, express the ratio as a decimal rounded to four places after the decimal.

$$\cos A = 0.7168$$

Next, use the inverse cosine function ($\cos^{-1}$) on the TI calculator to find the measure of $\angle A$.

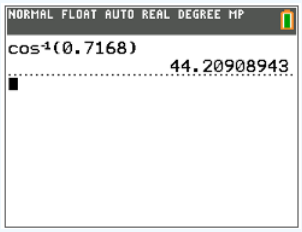


Figure 2.1.4.19

Then, write the measure of $\angle A$ to the nearest tenth.

$\angle A = 44.2°$.

The answer is $44.2°$

Review

Solve each problem.

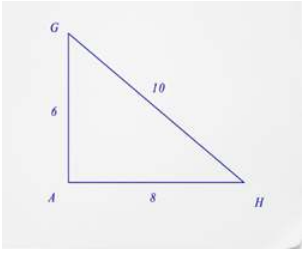


Figure **2.1.4.20**

1. What is the cosine of $\angle G$?
2. What is the cosine of $\angle H$?
3. How do you identify a cosine?

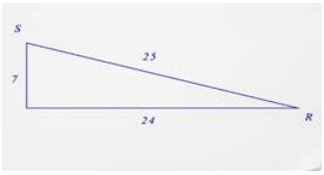


Figure **2.1.4.21**

4. What is the cosine of $\angle R$?
5. What is the cosine of $\angle S$?

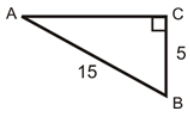


Figure **2.1.4.1**

6. What is the cosine of $\angle A$?
7. What is the cosine of $\angle B$?
8. What is the length of the missing side rounded to the nearest hundredth?

Answer each of the following questions.

9. Is the cosine related to an angle?
10. Do you need to know side lengths of a triangle to write a cosine?
11. Which side lengths do you need?
12. If the cosine was $\frac{5}{20}$ what would the numerical value of the cosine be?
13. If the cosine was $\frac{5}{25}$ what would the numerical value of the cosine be?
14. If the cosine was $\frac{3}{33}$ what would the numerical value of the cosine be?
15. If the cosine was $\frac{12}{14}$ what would the numerical value of the cosine be?

Review (Answers)

To see the Review answers, open this PDF file and look for section 7.14.

Vocabulary

Term	Definition
cosine	The cosine of an angle in a right triangle is a value found by dividing the length of the side adjacent the given angle by the length of the hypotenuse.
Trigonometric Ratios	Ratios that help us to understand the relationships between sides and angles of right triangles.

Additional Resources

Video: Basic Trigonometry

Practice: COS

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2.1.5: Sine and Cosine of Complementary Angles

Sine of an angle equals the cosine of its complement.

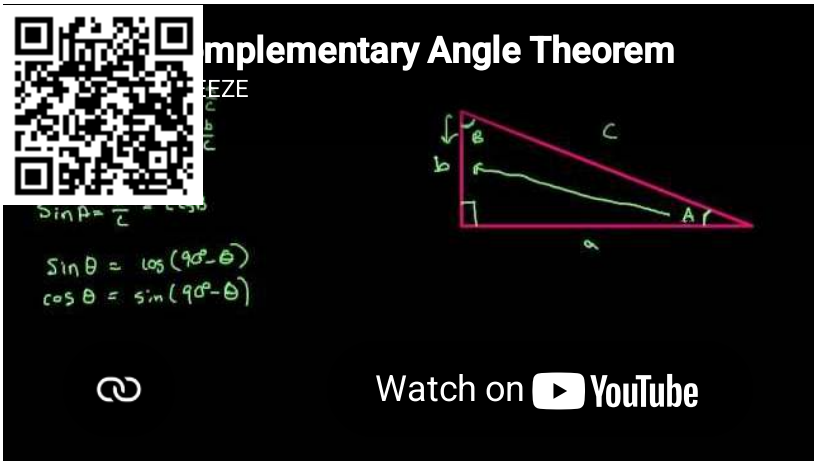
ΔABC is a right triangle with $m\angle C = 90°$ and $\sin A = k$. What is $\cos B$?

Sine and **Cosine** of **Complementary Angles**

Recall that the **sine** and **cosine** of angles are ratios of pairs of sides in right triangles.

- The **sine** of an angle in a right triangle is the ratio of the side *opposite* the angle to the *hypotenuse*.
- The **cosine** of an angle in a right triangle is the ratio of the side *adjacent* to the angle to the *hypotenuse*.

In the problems below, you will explore how the sine and cosine of the angles in a right triangle are related.



Let's take a look at a few example problems.

1. Consider the right triangle below. Find the sine and cosine of angles A and B in terms of a,b, and c. What do you notice?

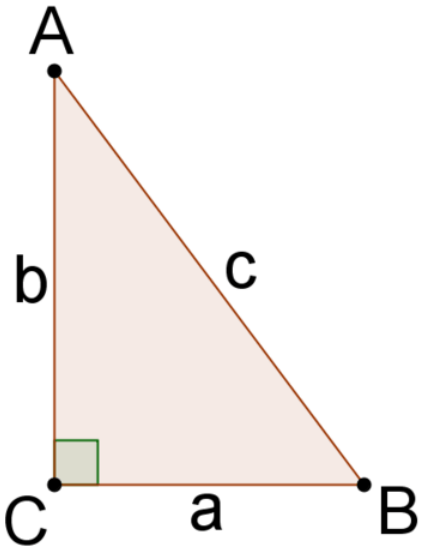


Figure **2.1.5.1**

$\sin A = \frac{a}{c}$, $\sin B = \frac{b}{c}$, $\cos A = \frac{b}{c}$, $\cos B = \frac{a}{c}$ Note that $\sin A = \cos B$ and $\sin B = \cos A$.

2. Consider the triangle from the previous problem. How is $\angle A$ related to $\angle B$?

The sum of the measures of the three angles in a triangle is $180°$. This means that $m\angle A + m\angle B + m\angle C = 180°$. $\angle C$ is a right angle so $m\angle C = 90°$. Therefore, $m\angle A + m\angle B = 90°$. Angles A and B are complementary angles because their sum is $90°$.

In #1 you saw that $\sin A = \cos B$ and $\sin B = \cos A$. This means that the sine and cosine of *complementary angles* are equal.

3. Find $80°$ and $\cos 10°$. Explain the result.

$\sin 80° \approx 0.985$ and $\cos 10° \approx 0.985$. $\sin 80° = \cos 10°$ because $80°$ and $10°$ are complementary angle measures. $\sin 80°$ and $\cos 10°$ are the ratios of the same sides of a right triangle, as shown below.

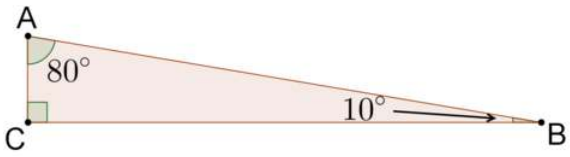


Figure **2.1.5.2**

Example 2.1.5.1

Earlier, you were asked what is $\cos B$.

ΔABC is a right triangle with $m\angle C = 90°$ and $\sin A = k$. What is $\cos B$?

Solution

$\angle A$ and $\angle B$ are complementary because they are the two non-right angles of a right triangle. This means that $\sin A = \cos B$ and $\sin B = \cos A$. If $\sin A = k$, then $\cos B = k$ as well.

Example 2.1.5.2

If $\sin 30° = \frac{1}{2}$, $\cos \stackrel{?}{=} \frac{1}{2}$.

Solution

The sine and cosine of **complementary** angles are equal. $90° - 30° = 60°$ is complementary to $30°$. Therefore, $\cos 60° = \frac{1}{2}$.

Example 2.1.5.3

Consider the right triangle below. Find $\tan A$ and $\tan B$.

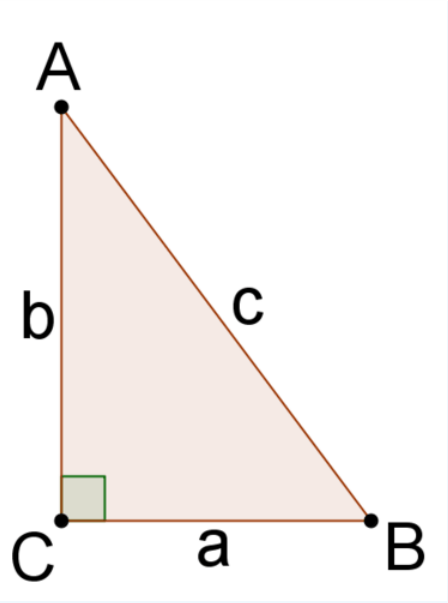


Figure 2.1.5.3

Solution

$\tan A = \frac{a}{b}$ and $\tan B = \frac{b}{a}$.

Example 2.1.5.4

In general, what is the relationship between the tangents of complementary angles?

Solution

In general, the tangents of complementary angles are reciprocals.

Review

1. How are the two non-right angles in a right triangle related? Explain.
2. How are the sine and cosine of complementary angles related? Explain.
3. How are the tangents of complementary angles related? Explain.

Let A and B be the two non-right angles in a right triangle.

4. If $\tan A = \frac{1}{2}$, what is $\tan B$?
5. If $\sin A = \frac{7}{10}$, what is $\cos B$?
6. If $\cos A = \frac{1}{4}$ what is $\sin B$?
7. If $\sin A = \frac{3}{5}$, $\cos \stackrel{?}{=} \frac{3}{5}$?
8. Simplify $\frac{\sin A + \cos B}{2}$.
9. If $\tan A = \frac{2}{3}$ what is $\tan B$?
10. If $\tan B = \frac{1}{5}$, what is $\tan A$? Which angle is bigger, $\angle A$ or $\angle B$?

Solve for θ.

11. $\cos 30^\circ = \sin \theta$
12. $\sin 75^\circ = \cos \theta$

13. $\cos 52^\circ = \sin \theta$

14. $\sin 18^\circ = \cos \theta$

15. $\cos 49^\circ = \sin \theta$

Review (Answers)

To see the Review answers, open this PDF file and look for section 7.3.

Vocabulary

Term	Definition
Complementary angles	Complementary angles are a pair of angles with a sum of 90°.
cosine	The cosine of an angle in a right triangle is a value found by dividing the length of the side adjacent the given angle by the length of the hypotenuse.
sine	The sine of an angle in a right triangle is a value found by dividing the length of the side opposite the given angle by the length of the hypotenuse.

Additional Resources

Practice: Sine and Cosine of Complementary Angles

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2.1.6: TAN

Explore the tangent ratio and use it to find missing sides of right triangles.

As the measure of an angle increases between $0°$ and $90°$, how does the **tangent** ratio of the angle change?

Tangent Ratio

Recall that one way to show that two triangles are **similar** is to show that they have two pairs of **congruent** angles. This means that two right triangles will be similar if they have one pair of congruent non-right angles.

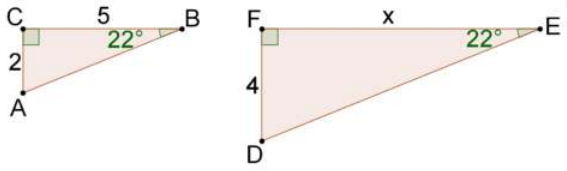


Figure 2.1.6.1

The two right triangles above are similar because they have two pairs of congruent angles. This means that their corresponding sides are proportional. $\overline{DF}$ and $\overline{AC}$ are corresponding sides because they are both opposite the $22°$ angle. $\frac{DF}{AC} = \frac{4}{2} = 2$, so the scale factor between the two triangles is 2. This means that $x = 10$, because $\frac{FE}{CB} = \frac{10}{5} = 2$.

The ratio between the two legs of any $22°$ right triangle will always be the same, because all $22°$ right triangles are similar. The ratio of the **length of the leg opposite** the $22°$ angle to the **length of the leg adjacent to** the $22°$ angle will be $\frac{2}{5} = 0.4$. You can use this fact to find a missing side of another $22°$ right triangle.

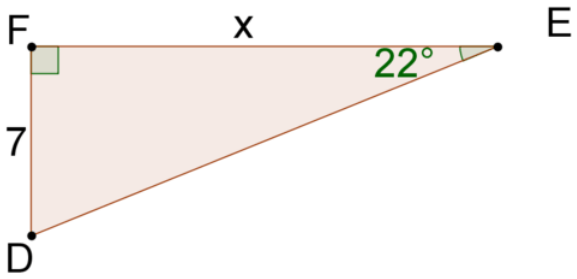
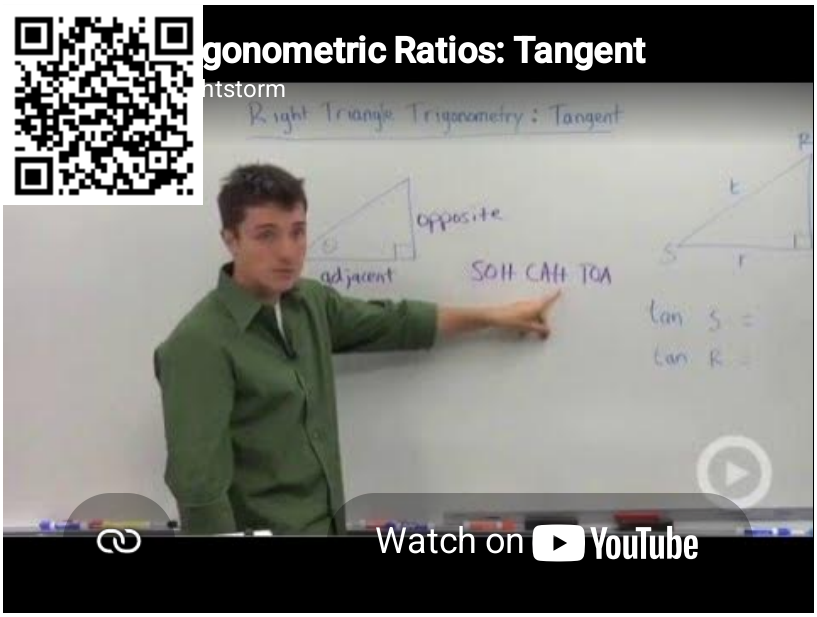


Figure 2.1.6.2

Because this is a $22°$ right triangle, you know that $\frac{\text{opposite leg}}{\text{adjacent leg}} = \frac{2}{5} = 0.4$.

$$\begin{aligned} \frac{\text{opposite leg}}{\text{adjacent leg}} &= 0.4 \\ \frac{7}{x} &= 0.4 \\ 0.4x &= 7 \\ x &= 17.5 \end{aligned}$$

The ratio between the opposite leg and the adjacent leg for a given angle in a right triangle is called the **tangent ratio**. Your scientific or graphing calculator has **tangent** programmed into it, so that you can determine the \dfrac{\text{opposite leg}}{\text{adjacent leg}}\) ratio for any angle within a right triangle. The abbreviation for **tangent** is **tan**.



Calculating Tangent Functions

Use your calculator to find the tangent of $75°$. What does this value represent?

Make sure your calculator is in degree mode. Then, type "$\tan(75)$".

$tan(75°) \approx 3.732$

This means that the ratio of the ***length of the opposite leg to the length of the adjacent leg*** for a $75°$ angle within a right triangle will be approximately 3.732.

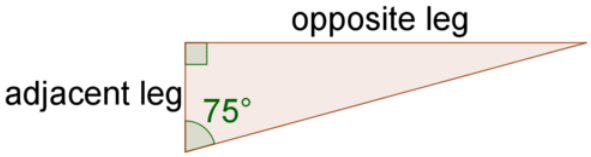


Figure **2.1.6.3**

Solving for Unknown Values

1. Solve for x.

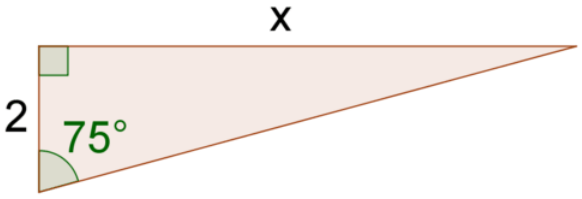


Figure **2.1.6.4**

From the previous problem, you know that the ratio $\frac{\text{opposite leg}}{\text{adjacent leg}} \approx 3.732$. You can use this to solve for x.

$$\frac{\text{opposite leg}}{\text{adjacent leg}} \approx 3.732$$
$$\frac{x}{2} \approx 3.732$$
$$x \approx 7.464$$

2. Solve for x and y.

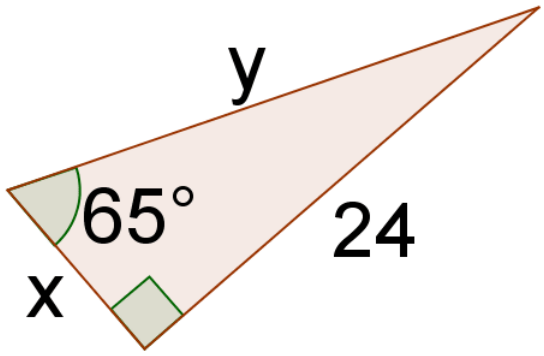


Figure 2.1.6.5

You can use the $65°$ angle to find the correct ratio between 24 and x.

$$\begin{aligned}\tan(65°) &= \frac{\text{opposite leg}}{\text{adjacent leg}}\\ 2.145 &\approx \frac{24}{x}\\ x &\approx \frac{24}{2.145}\\ x &\approx 11.189\end{aligned}$$

Note that this answer is only approximate because you rounded the value of $\tan 65°$. An exact answer will include "tan". The exact answer is:

$$x = \frac{24}{\tan 65°}$$

To solve for y, you can use the Pythagorean Theorem because this is a right triangle.

$$\begin{aligned}11.189^2 + 24^2 &= y^2\\ 701.194 &= y^2\\ 26.48 &= y\end{aligned}$$

Example 2.1.6.1

Earlier, you were asked how does the tangent ratio of the angle change.

Solution

As the measure of an angle increases between $0°$ and $90°$, how does the tangent ratio of the angle change?



Figure 2.1.6.6

As an angle increases, the length of its opposite leg increases. Therefore, $\frac{\text{opposite leg}}{\text{adjacent leg}}$ increases and thus the value of the tangent ratio increases.

Example 2.1.6.2

Tangent tells you the ratio of the two legs of a right triangle with a given angle. Why does the tangent ratio not work in the same way for non-right triangles?

Solution

Two right triangles with a $32°$ angle will be similar. Two non-right triangles with a $32°$ angle will not necessarily be similar. The tangent ratio works for right triangles because all right triangles with a given angle are similar. The tangent ratio doesn't work

in the same way for non-right triangles because not all non-right triangles with a given angle are similar. **You can only use the tangent ratio for right triangles.**

Example 2.1.6.3

Use your calculator to find the tangent of $45°$. What does this value represent? Why does this value make sense?

Solution

$\tan(45°) = 1$. This means that the ratio of the length of the opposite leg to the length of the adjacent leg is equal to 1 for right triangles with a $45°$ angle.

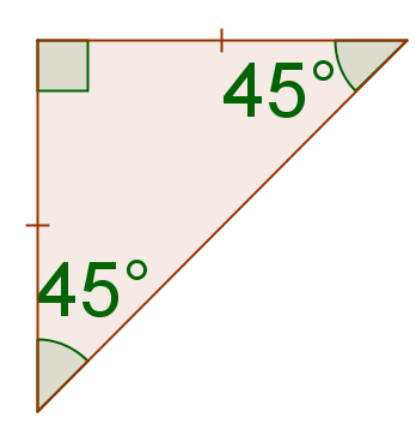


Figure 2.1.6.7

This should make sense because right triangles with a $45°$ angle are isosceles. The legs of an isosceles triangle are congruent, so the ratio between them will be 1.

Example 2.1.6.4

Solve for x.

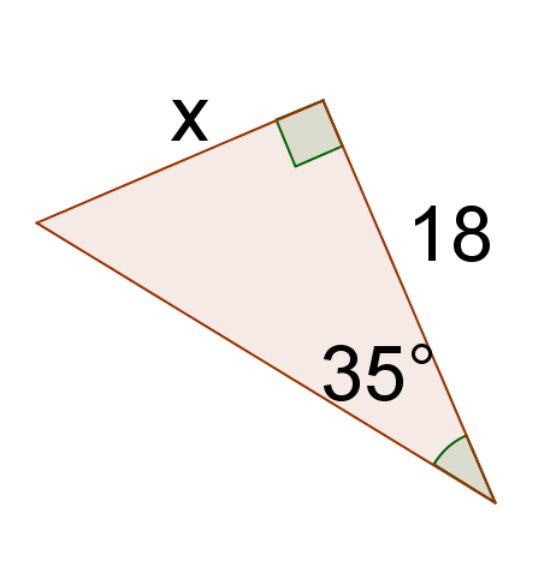


Figure 2.1.6.8

Solution

Use the tangent ratio of a $35°$ angle.

$$\tan(35^\circ) = \frac{\text{opposite leg}}{\text{adjacent leg}}$$
$$\tan(35^\circ) = \frac{x}{18}$$
$$x = 18\tan(35^\circ)$$
$$x \approx 12.604$$

Review

1. Why are all right triangles with a 40° angle similar? What does this have to do with the tangent ratio?

2. Find the tangent of 40°.

3. Solve for x.

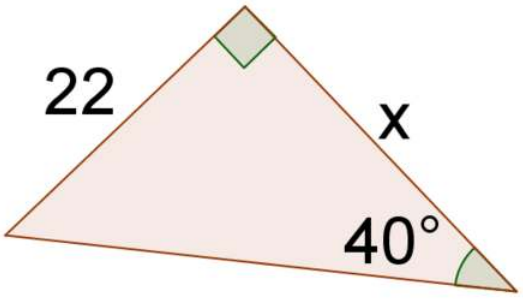


Figure 2.1.6.9

4. Find the tangent of 80°.

5. Solve for x.

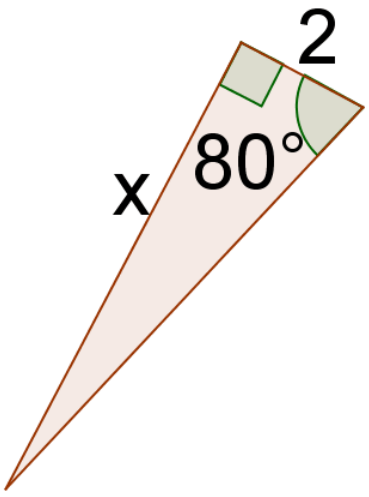


Figure 2.1.6.10

6. Find the tangent of 10°.

7. Solve for x.

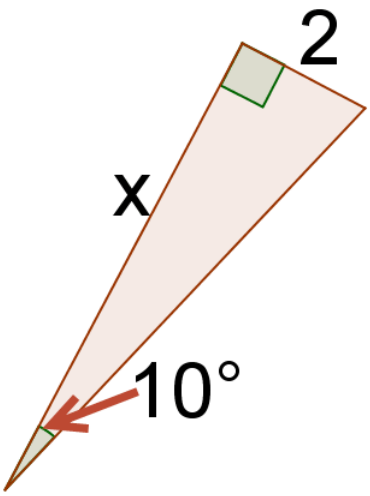


Figure 2.1.6.11

8. Your answer to #5 should be the same as your answer to #7. Why?

9. Find the tangent of $27°$.

10. Solve for x.

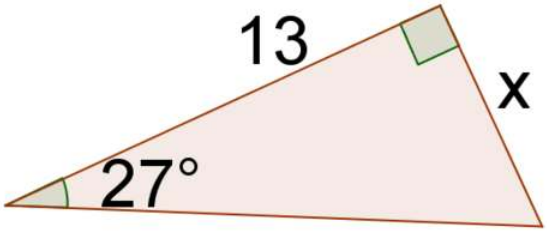


Figure 2.1.6.12

11. Find the tangent of $42°$.

12. Solve for x.

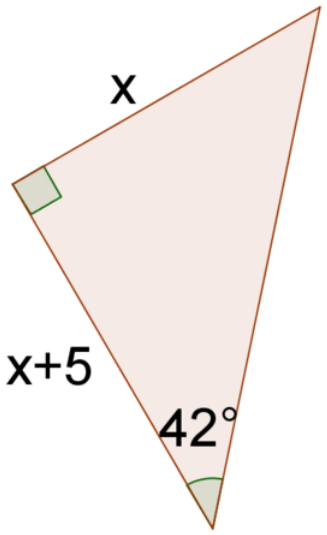


Figure 2.1.6.13

13. A right triangle has a $42°$ angle. The base of the triangle, adjacent to the $42°$ angle, is 5 inches. Find the area of the triangle.

14. Recall that the ratios between the sides of a 30-60-90 triangle are $1 : \sqrt{3} : 2$. Find the tangent of $30°$. Explain how this matches the ratios for a 30-60-90 triangle.

15. Explain why it makes sense that the value of the tangent ratio increases as the angle goes from $0°$ to $90°$.

Review (Answers)

To see the Review answers, open this PDF file and look for section 7.1.

Vocabulary

Term	Definition
AA Similarity Postulate	If two angles in one triangle are congruent to two angles in another triangle, then the two triangles are similar.
Congruent	Congruent figures are identical in size, shape and measure.
Similar	Two figures are similar if they have the same shape, but not necessarily the same size.
Tangent	The tangent of an angle in a right triangle is a value found by dividing the length of the side opposite the given angle by the length of the side adjacent to the given angle.

Additional Resources

Video: Basic Trigonometry

Practice: TAN

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2.1.7: SEC CSC COT

Secant, cosecant, cotangent values of common angles

While working to paint your grandfather's staircase, you are looking at the triangular shape made by the wall that support the stairs. The staircase looks like this:



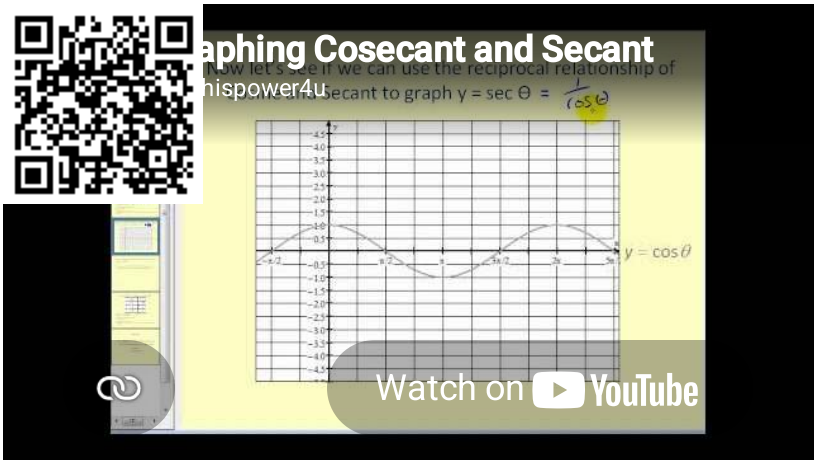
Figure **2.1.7.1**

You are thinking about all of the possible relationships between sides. You already know that there are three common relationships, called sine, cosine, and tangent.

How many others can you find?

secant, **cosecant** and **cotangent** Functions

We can define three more functions also based on a right triangle. They are the reciprocals of sine, cosine and tangent.



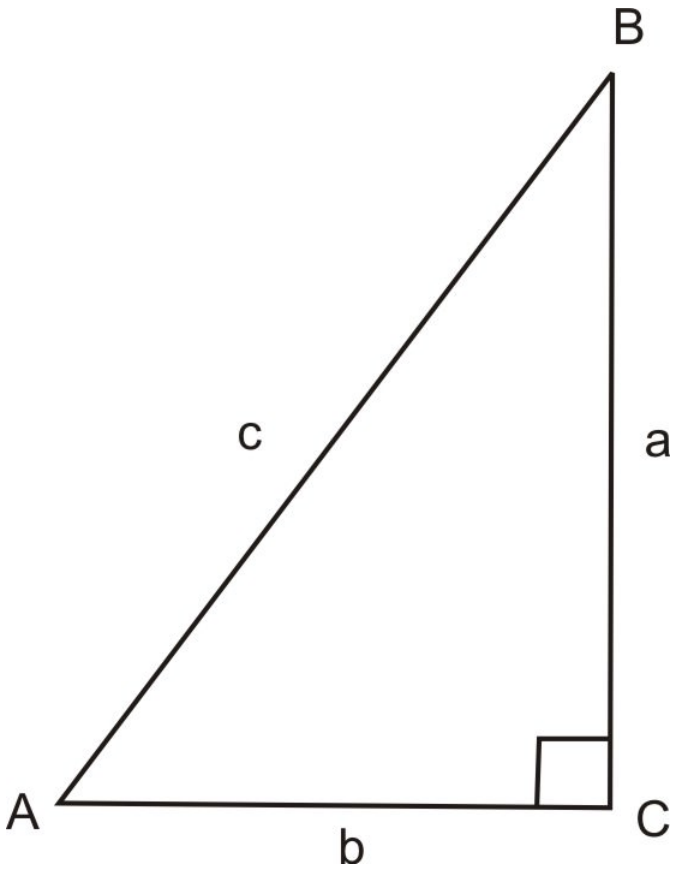


Figure **2.1.7.2**

If $\sin A = \frac{a}{c}$, then the definition of cosecant, or csc, is $\csc A = \frac{c}{a}$.

If $\cos A = \frac{b}{c}$, then the definition of secant, or sec, is $\sec A = \frac{c}{b}$.

If $\tan A = \frac{a}{b}$, then the definition of cotangent, or cot, is $\cot A = \frac{b}{a}$.

Use the definition of secant, cosecant and cotangent to solve the following problems.

1. Find the secant, cosecant, and cotangent of angle B.

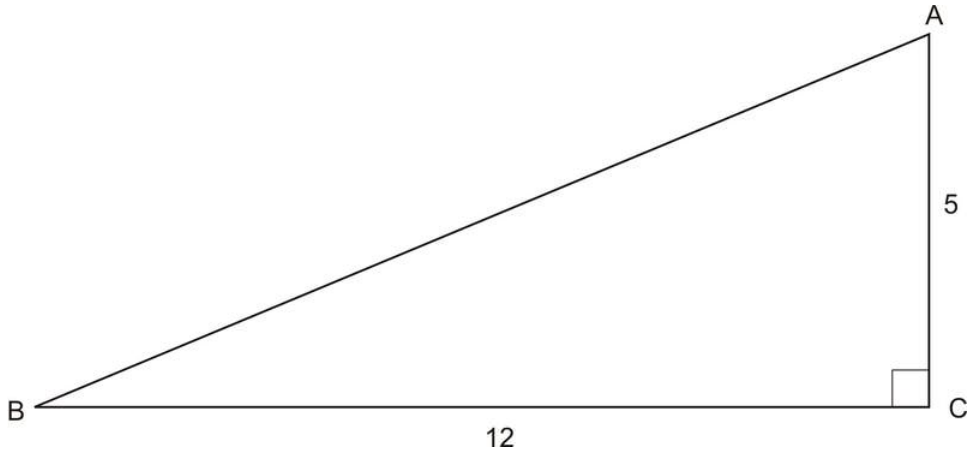


Figure **2.1.7.3**

First, we must find the length of the hypotenuse. We can do this using the Pythagorean Theorem:

$$5^2 + 12^2 = H^2$$
$$25 + 144 = H^2$$
$$169 = H^2$$
$$H = 13$$

Now we can find the secant, cosecant, and cotangent of angle B:

$$\sec B = \frac{\text{hypotenuse}}{\text{adjacent side}} = \frac{13}{12}$$
$$\csc B = \frac{\text{hypotenuse}}{\text{opposite side}} = \frac{13}{5}$$
$$\cot B = \frac{\text{adjacent side}}{\text{opposite side}} = \frac{12}{5}$$

2. Find the secant, cosecant, and cotangent of angle A

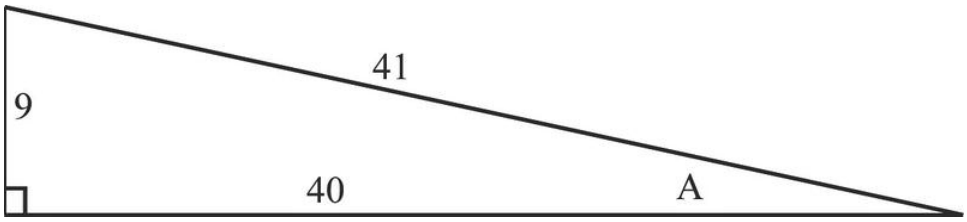


Figure 2.1.7.4

$$\sec A = \frac{\text{hypotenuse}}{\text{adjacent side}} = \frac{41}{40}$$
$$\csc A = \frac{\text{hypotenuse}}{\text{opposite side}} = \frac{41}{9}$$
$$\cot A = \frac{\text{adjacent side}}{\text{opposite side}} = \frac{40}{9}$$

3. Find the sine, cosine, and tangent of angle A, and then use this to construct the secant, cosecant, and cotangent of the angle

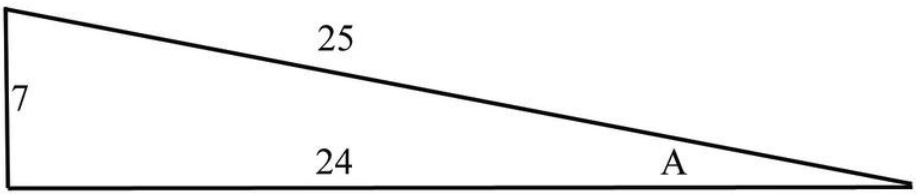


Figure 2.1.7.5

$$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{7}{25}$$
$$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{24}{25}$$
$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{7}{24}$$

Since we know that cosecant is the reciprocal of sine, secant is the reciprocal of sine, and cotangent is the reciprocal of tangent, we can construct these functions as follows:

$$\sec A = \frac{1}{\cos A} = \frac{25}{24}$$
$$\csc A = \frac{1}{\sin A} = \frac{25}{7}$$
$$\cot A = \frac{1}{\tan A} = \frac{24}{7}$$

Example 2.1.7.1

Earlier, you were given a problem about your grandfather's staircase.

Looking at a triangle-like the shape of the wall supporting your grandfather's staircase:



Figure 2.1.7.6

Solution

We can see that there are several ways to make relationships between the sides. In this case, we are only interested in ratios between the sides, which means one side will be divided by another. We've already seen some functions, such as:

1) The side opposite the angle divided by the hypotenuse (the sine function)

2) The side adjacent the angle divided by the hypotenuse (the cosine function)

3) The side opposite the angle divided by adjacent side (the tangent function)

In this section we introduced the reciprocal of the above trig functions. These are found by taking ratios between the same sides shown above, except reversing the numerator and denominator:

4) The hypotenuse divided by the side opposite the angle (the cosecant function)

5) The hypotenuse divided by the side adjacent to the angle (the secant function)

6) The adjacent side divided by the opposite side (the cotangent function)

Use the figure below to help solve the following examples.

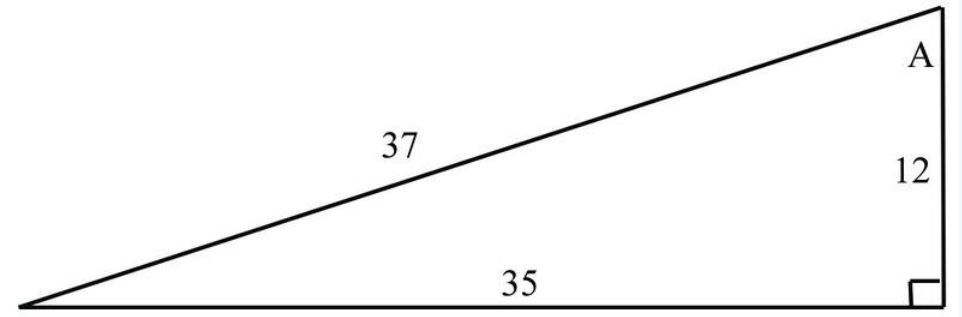


Figure 2.1.7.7

Example 2.1.7.2

Find the secant $\angle A$

Solution

The secant function is defined to be $\frac{1}{\cos}$. Since $\cos = \frac{\text{adjacent}}{\text{hypotenuse}}$, $\sec = \frac{\text{hypotenuse}}{\text{adjacent}}$.

$$\sec = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{37}{12} \approx 3.08$$

Example 2.1.7.3

Find the cosecant of $\angle A$

Solution

The cosecant function is defined to be $\frac{1}{\sin}$. Since $\sin = \frac{\text{opposite}}{\text{hypotenuse}}$, $\csc = \frac{\text{hypotenuse}}{\text{opposite}}$.

$$\csc = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{37}{35} \approx 1.06$$

Example 2.1.7.4

Find the cotangent of $\angle A$

Solution

The cotangent function is defined to be $\frac{1}{\tan}$. Since $\tan = \frac{\text{opposite}}{\text{adjacent}}$, $\cot = \frac{\text{adjacent}}{\text{opposite}}$.

$$\cot = \frac{\text{adjacent}}{\text{opposite}} = \frac{12}{35} \approx .34$$

Review

Use the diagram below for questions 1-3.

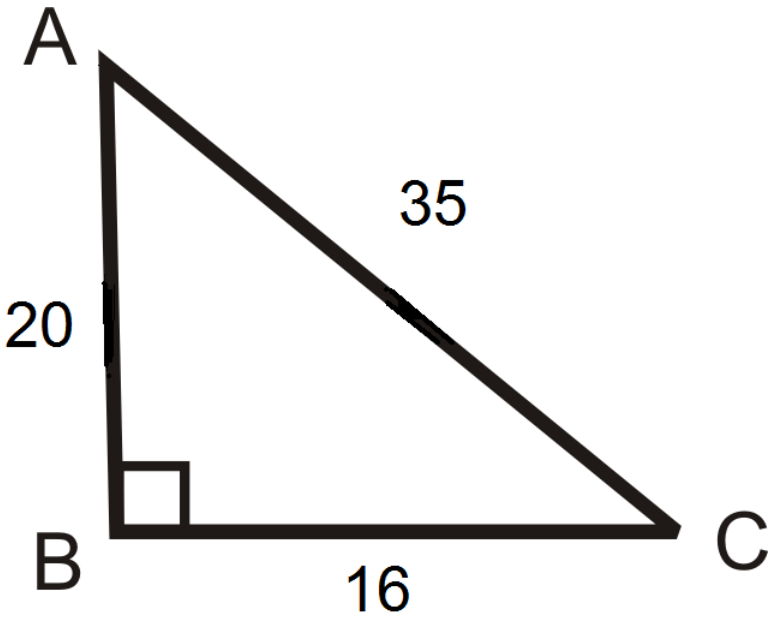


Figure **2.1.7.8**

1. Find $\csc A$ and $\csc C$.
2. Find $\sec A$ and $\sec C$.
3. Find $\cot A$ and $\cot C$.

Use the diagram to fill in the blanks below.

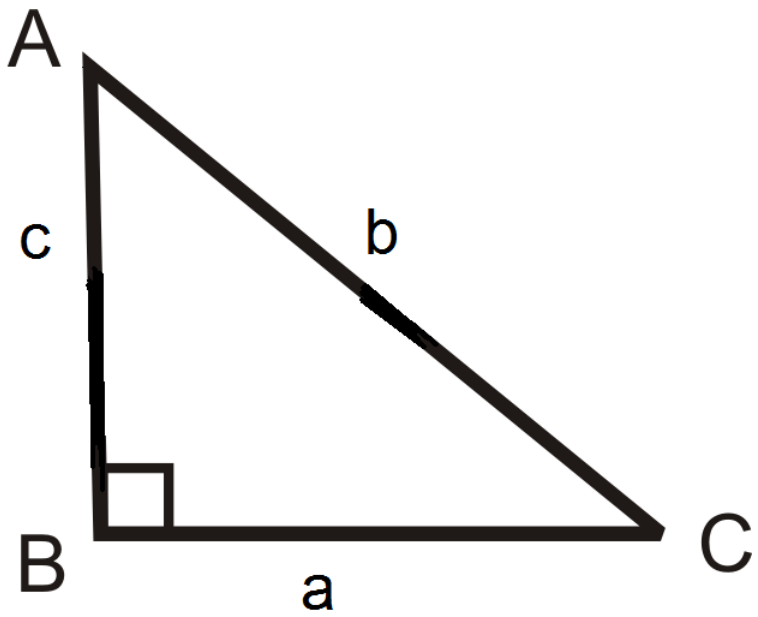


Figure **2.1.7.9**

4. $\cot A = \frac{?}{?}$
5. $\csc C = \frac{?}{?}$
6. $\cot C = \frac{?}{?}$
7. $\sec C = \frac{?}{?}$
8. $\csc A = \frac{?}{?}$
9. $\sec A = \frac{?}{?}$

From questions 4-9, we can conclude the following. Fill in the blanks.

10. '_' allowed only in math mode and '_' allowed only in math mode.
11. $\cot A$ and $\cot C$ are _____ of each other.
12. Explain why the csc of an angle will always be greater than 1.
13. Use your knowledge of 45-45-90 triangles to find the cosecant, secant, and cotangent of a 45 degree angle.
14. Use your knowledge of 30-60-90 triangles to find the cosecant, secant, and cotangent of a 30 degree angle.
15. Use your knowledge of 30-60-90 triangles to find the cosecant, secant, and cotangent of a 60 degree angle.
16. As the degree of an angle increases, will the cotangent of the angle increase or decrease? Explain.

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.9.

Vocabulary

Term	Definition
cosecant	The cosecant of an angle in a right triangle is a relationship found by dividing the length of the hypotenuse by the length of the side opposite to the given angle. This is the reciprocal of the sine function.
cotangent	The cotangent of an angle in a right triangle is a relationship found by dividing the length of the side adjacent to the given angle by the length of the side opposite to the given angle. This is the reciprocal of the tangent function.
secant	The secant of an angle in a right triangle is the value found by dividing length of the hypotenuse by the length of the side adjacent the given angle. The secant ratio is the reciprocal of the cosine ratio.

Additional Resources

Video: Graphing cosecant and secant

Practice: SEC CSC COT

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2.2: Solving Triangles

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2.2.1: The Pythagorean Theorem and Trigonometry

Using trigonometric functions or the pythagorean theorem based on given information.

A 40-foot-tall tree casts a shadow of 80 feet. What is the angle of elevation from the end of the shadow to the top of the tree with respect to the ground?

Application Problems

When solving word problems, it is important to understand the terminology used to describe angles. In trigonometric problems, the terms angle of elevation and angle of depression are commonly used. Both of these angles are always measured from a horizontal line as shown in the diagrams below.

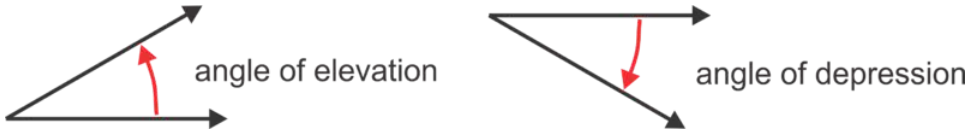


Figure **2.2.1.1**

Let's solve the following problems.

1. An airplane approaching an airport spots the runway at an angle of depression of $25°$. If the airplane is 15,000 ft above the ground, how far (ground distance) is the plane from the runway? Give your answer to the nearest 100 ft.

Make a diagram to illustrate the situation described and then use a trigonometric ratio to solve. Keep in mind that an angle of depression is down from a horizontal line of sight-in this case a horizontal line from the pilot of the plane parallel to the ground.

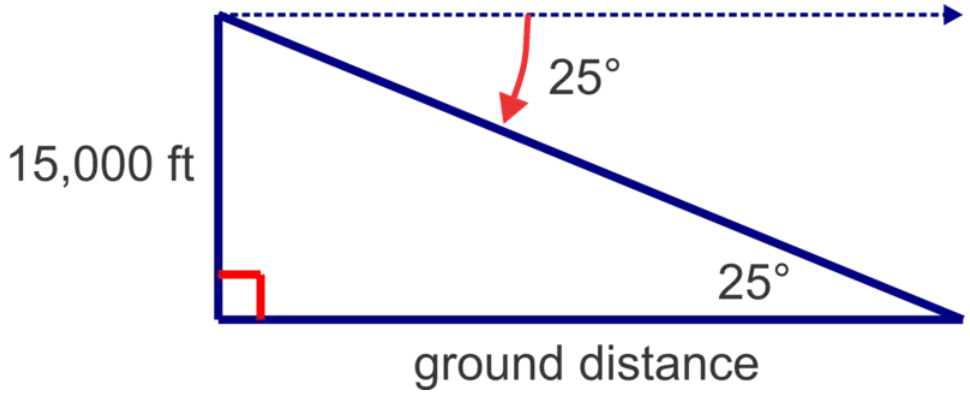


Figure **2.2.1.2**

Note that the angle of depression and the alternate interior angle will be congruent, so the angle in the triangle is also $25°$.

From the picture, we can see that we should use the tangent ratio to find the ground distance.

$$\tan 25° = \frac{15000}{d}$$
$$d = \frac{15000}{\tan 25°} \approx 32,200 \text{ ft}$$

2. Rachel spots a bird in a tree at an angle of elevation of $30°$. If Rachel is 20 ft from the base of the tree, how high up in the tree is the bird? Give your answer to the nearest tenth of a foot.

Make a diagram to illustrate the situation. Keep in mind that there will be a right triangle and that the right angle is formed by the ground and the trunk of the tree.

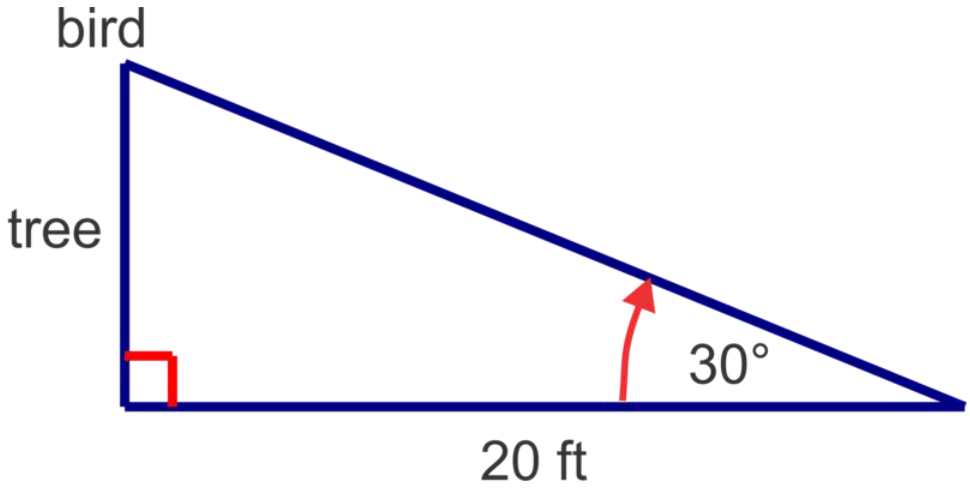


Figure 2.2.1.3

Here we can use the tangent ratio to solve for the height of the bird

$$tan30^\circ = \frac{h}{20}$$
$$h = 20 \tan 30^\circ \approx 11.5 \text{ ft}$$

3. A 12 ft ladder is leaning against a house and reaches 10 ft up the side of the house. To the nearest degree, what angle does the ladder make with the ground?

In this problem, we will need to find an angle. By making a sketch of the triangle we can see which inverse trigonometric ratio to use.

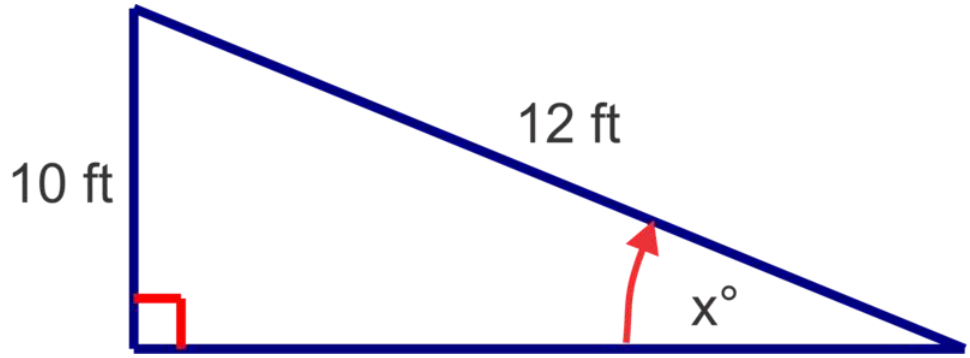
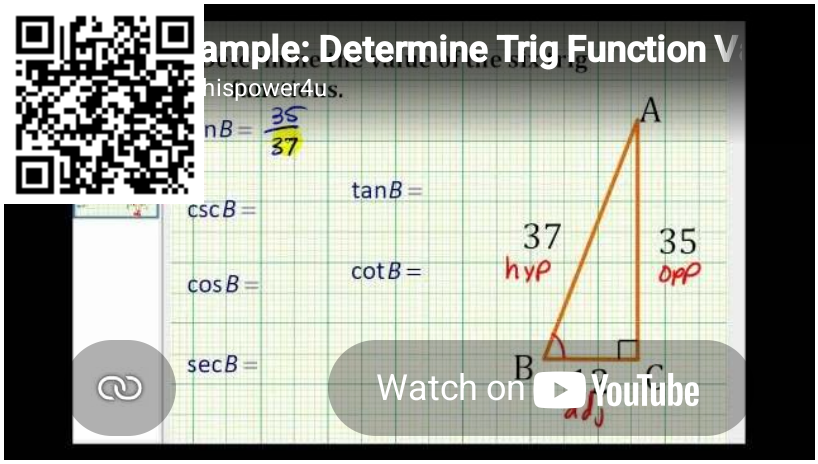


Figure 2.2.1.4

$$sinx^\circ = \frac{10}{12}$$
$$sin^{-1}(\frac{10}{12}) \approx 56^\circ$$



Example 2.2.1.1

Earlier, you were asked to find the angle of elevation from the end of the shadow to the top of the tree with respect to the ground.

Solution

If you draw this situation, you will see that we are dealing with a right triangle. The side opposite the angle of elevation is 40. The side adjacent to the angle is 80. Therefore, we can use the tangent to find the angle of elevation.

$$\tan x^\circ = \frac{40}{80} = \frac{1}{2}$$
$$tan^{-1}(\frac{1}{2}) = \approx 26.57^\circ$$

Example 2.2.1.2

A ramp makes a 20° angle with the ground. If door the ramp leads to is 2 ft above the ground, how long is the ramp? Give your answer to the nearest tenth of a foot.

Solution

$$\sin 20^\circ = \frac{2}{x}$$
$$x = \frac{2}{sin20^\circ} \approx 5.8 \text{ ft}$$

Example 2.2.1.3

Charlie lets out 90 ft of kite string. If the angle of elevation of the string is 70^{\circ}, approximately how high is the kite? Give your answer to the nearest foot.

Solution

$$\sin 70^\circ = \frac{x}{90}$$
$$x = 90 \sin 70^\circ \approx 85 \text{ ft}$$

Example 2.2.1.4

A ship's sonar spots a wreckage at an angle of depression of 32^{\circ}. If the depth of the ocean is about 250 ft, how far is the wreckage (measured along the surface of the water) from the ship, to the nearest foot.

Solution

$$\tan 32^\circ = \frac{250}{x}$$
$$x = \frac{250}{\tan 32^\circ} \approx 400 \text{ ft}$$

Review

Use the Pythagorean Theorem and/or trigonometry to solve the following word problems.

1. A square has sides of length 8 inches. To the nearest tenth of an inch, what is the length of its diagonal?
2. Layne spots a sailboat from her fifth floor balcony, about 25 m above the beach, at an angle of depression of 3°. To the nearest meter, how far out is the boat?
3. A zip line takes passengers on a 200 m ride from high up in the trees to a ground level platform. If the angle of elevation of the zip line is 10°, how high above ground is the tree top start platform? Give your answer to the nearest meter.
4. The angle of depression from the top of an apartment building to the base of a fountain in a nearby park is 57°. If the building is 150 ft tall, how far away, to the nearest foot, is the fountain?

5. A playground slide platform is 6 ft above ground. If the slide is 8 ft long and the end of the slide is 1 ft above ground, what angle does the slide make with the ground? Give your answer to the nearest degree.
6. Benjamin spots a tree directly across the river from where he is standing. He then walks 27 ft upstream and determines that the angle between his previous position and the tree on the other side of the river is 73°. How wide, to the nearest foot, is the river?
7. A rectangle has sides of length 6 in and 10 in. To the nearest degree, what angle does the diagonal make with the longer side?
8. Tommy is flying his kite one afternoon and notices that he has let out the entire 130 ft of string. The angle his string makes with the ground is 48°. How high, to the nearest foot, is his kite at this time?
9. A tree struck by lightning in a storm breaks and falls over to form a triangle with the ground. The tip of the tree makes a 18° angle with the ground 21 ft from the base of the tree. What was the height of the tree to the nearest foot?
10. Upon descent an airplane is 19,000 ft above the ground. The air traffic control tower is 190 ft tall. It is determined that the angle of elevation from the top of the tower to the plane is 15°. To the nearest mile, find the ground distance from the airplane to the tower.
11. Why will the sine and cosine ratios always be less than 1?

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 13.4.

Additional Resources

Video: **Solving Right Triangles - Overview**

Practice: **The Pythagorean Theorem and Trigonometry**

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2.2.2: Trig Function Applications

Use a calculator to find sine, cosine, and tangent in basic applications

Modeling Periodic Behavior

This lesson will explore how real life examples of periodic behavior can be modeled with Trigonometric Functions. In other words, items that are in motion - such as a wheel turning or waves moving - can be modeled and understood through Trigonometry.

Warm-Up

Trigonometric functions like sine and cosine display periodic behavior when graphed. Use the interactive below to see the periodic behavior of the sine function. Later in this section you will see how real world problems can be modeled and solved with functions similar to these.

Work it Out 1

Two students, Manny and Julisa, were having a debate about the how to model the wavelength of their school's radio station frequency. To simplify their debate, they used another model of a wave as shown below.

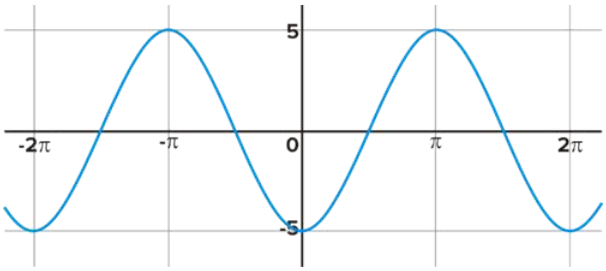


Figure 2.2.2.1

Manny says the equation for the above graph is $f(x) = 5\sin\left(x - \frac{\pi}{2}\right)$ and Julisa says it is $f(x) = 5\cos x$.

a. Without graphing the equations, can you tell which student is correct, and if so, who?
b. How do you know?
c. What would the graph of the incorrect equation look like when compared to the graph above?

Discussion

Recall that the equation for a cosine function is $y = A\cos(w(x - h)) + k$ where A changes the amplitude (the distance between the midpoint and the peak or valley of the graph), h changes the start of a phase (the movement of the graph left and right on the axes), k changes the midline, and w changes the period from the basic function, which occurs between 0 and 2π. We can tell that this graph has an amplitude of + 5, and a period of 1.

But is it a cosine function like Julisa believes, or a sine function with a shift in the phase like Manny believes? If you graph these two functions, and also $y = 5\sin x$ you will see the subtle, but important differences between the three graphs.



Figure 2.2.2.2

A large Ferris wheel is 100 feet in diameter and has 12 passenger cars. The Ferris wheel rotates counterclockwise, and takes 1 minute to complete a full rotation. Riders climb a stairway in order to board from a platform that is 60 feet above the ground, and is horizontal to the center of the Ferris wheel. In this scenario, the **height** of a car is the distance between the car and the platform (not the height of the car from the ground). The **co-height** is the horizontal distance between the car and a vertical line that runs up and down through the center of the Ferris wheel. In other words, if the car was located at an (x, y) coordinate, the co-height is x and the height is y.

1. Let $\theta = 0$ represent the position of Car 1 in the scenario above. Sketch graphs of the height and co-height of Car 1 as functions of θ, the number of radians through which the car has rotated.
2. What is the amplitude of the height and co-height functions for the Ferris wheel?
3. If $X(\theta)$ represents the co-height and $Y(\theta)$ represents the height of the car, what trigonometric functions would represent the co-height and height in terms of θ ?

Discussion

To determine the height of Car 1 as the Ferris wheel rotates, consider that after 15 seconds (which is $\frac{1}{4}$ of the way around or $\frac{\pi}{2}$ radians), the car is at the top of the wheel, which we know is 50 feet above the platform. How high will the car be 15 seconds after that? After a total of 45 seconds? After 1 minute of rotation?

The graph below represents the **co-height** of the car. At 0 seconds, the car is 50 feet away from the center of the wheel. When the car is at π radians, how far will it be from the vertical line that goes through the center of the Ferris wheel? Why are some of the values of the co-height below negative numbers?

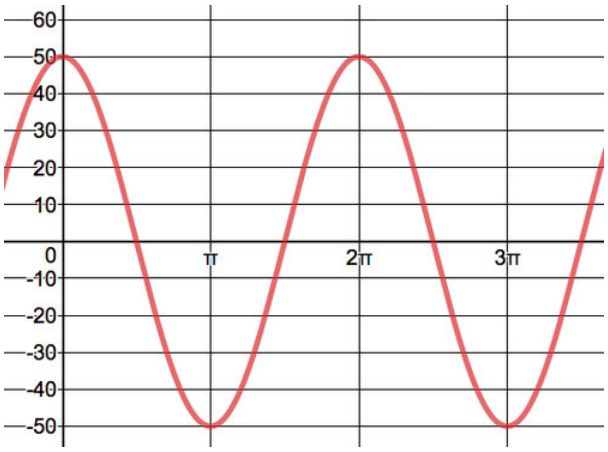


Figure **2.2.2.3**

The amplitude of the function of the co-height is 50 because the car will range between +50 and -50 feet away from a vertical line (like the y-axis) going through the center of the Ferris wheel. Therefore, the co-height can be represented by the function $f(\theta) = 50\cos(\theta)$.

Don't forget: you still need to create a graph and find the function for the height of Car 1.

Example 2.2.2.1

Using the Ferris wheel scenario from above, how would you represent the height of a car as the distance from the ground? How would the co-height be impacted by measuring the height from the ground instead of from the platform? What would graphs of the height and co-height look like?

Solution

The co-height will not change, because the horizontal distance between the car and the vertical line through the center of the Ferris wheel won't be affected by the height of the platform or the position of Car 1 when riders get on.

In this example, since you are measuring the height of Car 1 from the ground, the highest point the car reaches will be 50 feet (the radius of the wheel) plus 60 feet (the height of the platform), or 110 feet. The lowest point the car reaches will be -50 feet (below the center of the wheel) plus 60 feet (the height of the platform), or 10 feet from the ground. The graph for the height as a function of the amount of rotations is below.

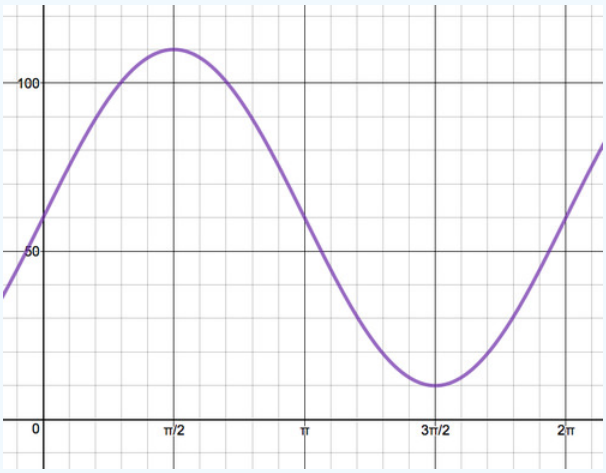


Figure **2.2.2.3**

The above is a graph of the function $Y(\theta) = 50\sin\theta + 60$.

Example 2.2.2.2

A Ferris wheel is 40 feet tall. There is a 25 foot platform to get into the first car, which loads from the bottom of the wheel (not the side as in previous examples). Additionally, the Ferris wheel rotates counterclockwise, making one complete rotation every 2 minutes. Write and graph the functions for the height (from the ground) and co-height of the Ferris wheel in terms of time (in minutes).

Solution

Let t = time. Start by determining the values for A, w, h, and k for both the height and co-height.

Co-Height:

- The wheel is 40 feet in diameter, so it will have maximum and minimum co-heights of 20 feet and -20 feet respective to the midline. Therefore, the Amplitude (A)=20.
- The wheel rotates once every 2 minutes. Because the passengers board from the bottom, which is a -14 clockwise rotation from the x-axis, the Ferris wheel needs to rotate +14 counter-clockwise so that the co-height is at its maximum of +20 feet. This quarter rotation takes 12 of a minute, and therefore the value of h (the phase shift) is 12.
- After 1 minute, the Ferris wheel has made a rotation of \pi . Therefore, the period (w)=\pi .
- The co-height is not affected by the location of the center of the wheel. Therefore, k=0.
- The equation for the co-height is: $X(t) = 20\cos(\pi(x - \frac{1}{2}))$.

Height:

- To determine the height of Car 1 from the ground, you will need to adjust for the height of the platform. The platform adds 25 feet to the bottom of the Ferris wheel, which makes the center 20 feet + 25 feet (or 45 feet) above the ground. This means the height of Car 1, in relation to the ground, is 25 feet at the lowest point and 65 at the highest point, with k=45.
- The equation for the height is: $Y(t) = 20\sin(\pi(x - \frac{1}{2})) + 45$.

The graph of the height as a function of time, with Car 1 starting at the bottom of the wheel and rotating clockwise, looks like this:

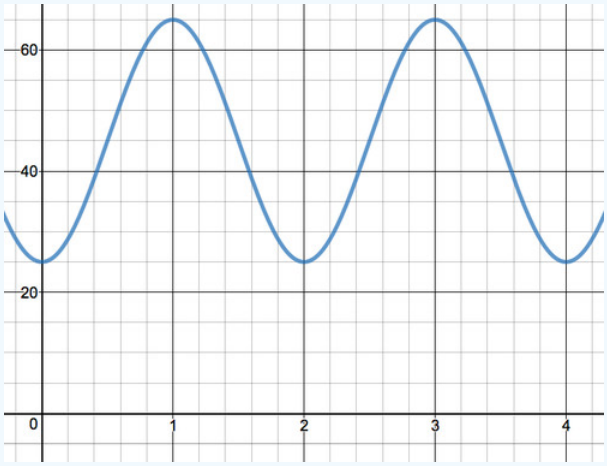
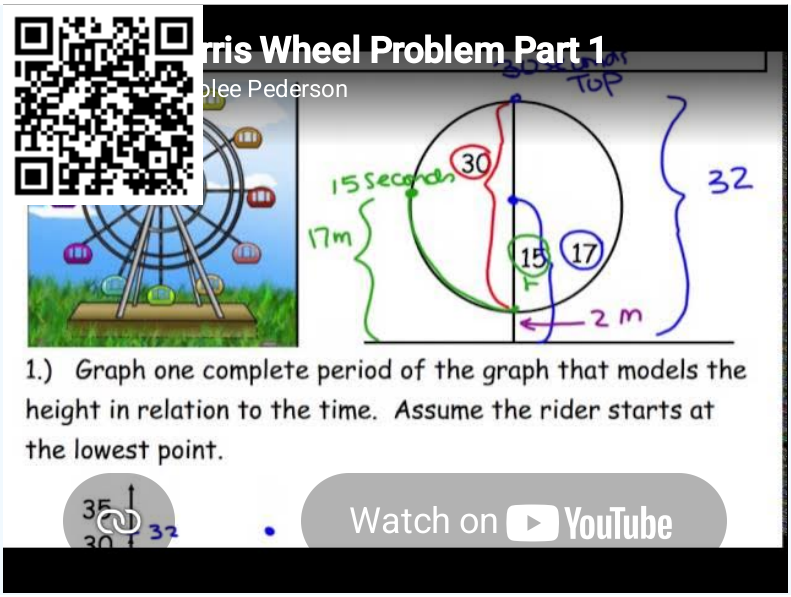


Figure 2.2.2.4

This video demonstrates how to solve a similar Ferris wheel problem.



Work it Out 3

You and two friends are riding the Ferris wheel in Example 2. You are in Car 1, another friend is in Car 3, and the third friend is in Car 5. When you reach the top of the wheel you look down to see your friends somewhere below you. How much higher is your car than Car 3 and Car 5?

Discussion

When you are at the top of the wheel, how high up will you be? And how much of a turn, in radians, will you have made? Consider where the other cars are in the circle; in other words, how much of a turn (again in radians) will Cars 3 and 5 have made?

Given the amplitude (20 feet) and the height of the platform (25 feet), your height will be 45 feet at the top of the wheel, which is π radians from the starting place. At this point, let's say your friend in Car 3 has rotated $120°$ from the starting place, which is 13 of a full turn. How do you use this information and the equation for the height of the Ferris wheel to determine the height of Car 3?

Because you only need to measure the height of the cars, the speed of the Ferris wheel isn't included in the equation, which changes the value of w. Additionally, change the measurement of the turn into radians. What will the equation be without w?

You should get: $Y(t) = 20\sin\left(x - \frac{\pi}{2}\right) + 45$. Next, plug in the part of the turn Car 3 has made (in radians) and solve:

$$X(t) = 20\sin\left(\frac{2\pi}{3} - \frac{\pi}{2}\right) + 45$$
$$X(t) = 20\sin\left(\frac{\pi}{6}\right) + 45$$
$$X(t) = 55$$

This means you are 65 feet - 55 feet = 10 feet higher than your friend in Car 3.

How much higher are you than your friend in Car 5 if the cars are spaced evenly apart?

What would change about the process for solving this problem if instead of there being 12 cars, there were only 8 cars? What about if there were twice as many cars?

Other Models of Periodic Behavior

This lesson modeled periodic behavior by using the rotation of a car around a Ferris wheel to explore the connections between transformed trigonometric functions and their graphs based on real life scenarios. However, rotating Ferris wheels are far from the only "real" things that can be modeled with trigonometric functions, or that have periodic behavior.

For example, high tides and low tides can be modeled and predicted using periodic functions because scientists can determine the height of the water at different times of the day (when the water level is low, the tide is low). Sound waves, blood pressure, and temperature variations can all be modeled as periodic functions, too.

Review

1. For the graphs below, identify the value of the amplitude, period, frequency, phase shift, and midline. Then write the equation for the graph.

a.



Figure 2.2.2.5

b.

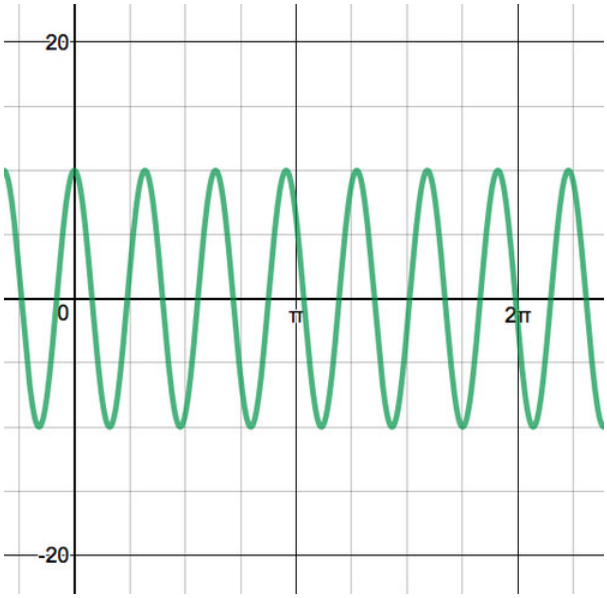


Figure 2.2.2.6

c.

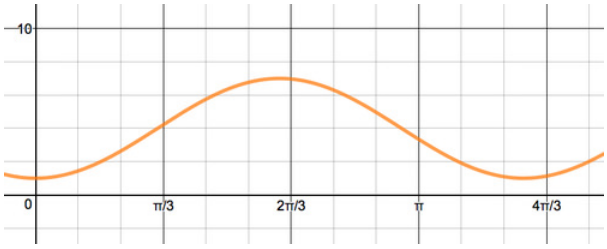


Figure 2.2.2.7

2. The Anansi the Spider ride is a small Ferris wheel at a children's amusement park. There are 8 cars on the ride, spaced every $45°$. The Ferris wheel is 16 feet from top to bottom, and kids load into Car 1 from below on a platform that is 4 feet off the ground. The ride takes 1 minute for a full rotation.
   a. Sketch a function for the height of Car 1 with respect to time as the Ferris wheel rotates for three minutes.
   b. Find a formula for the function you've graphed above.
   c. Determine a different function that will have the same graph you sketched in part a.

3. You are standing on a pier and measuring the height of the water below you at different times each day. At 3 pm, the water level is at 7 feet. By 9 pm the water has dropped down to 1 foot. And by 3 am, the water level is back up at 7 feet.
   a. Sketch the graph given the information you have.
   b. Determine the amplitude, phase shift, period, and midline. [Hint: The tide takes 12 hours to go through a full rotation of 2π, so what would be the period if you're measuring in hours?]
   c. Derive a formula that would model the height of water starting at 12 pm on the first day, and which shows the height of the water over a 24 hour period.
   d. Graph your formula using a graphing calculator or program to confirm your responses for parts a-c are correct.
   e. What would be the height of the water at 6 am?
4. The outside temperature (in Fahrenheit) was collected hourly on a cold winter day over a 24 hour period. At 12 pm, the temperature was $30°$. By 6 pm it had reached $50°$, the warmest recorded temperature during the measurement period. By 6 am, the temperature had dropped to its lowest point of $10°$.
   a. Sketch a graph given the information you have.
   b. Determine the amplitude, phase shift, period, and midline of your graph.
   c. Derive a formula that models the temperature over a 24-hour period.

Additional Resources

Video: **Basic Trigonometry 2**

Practice: **Trig Function Applications**

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2.2.3: Right Triangles and Bearings

Directions based off compass points.

While on a camping trip with your friends, you take an orienteering trip. You end up on a course which results in you hiking $30°$ west of due south. This is represented as $S30°W$ (always start with N or S, then the number of degrees to the east or west of there). You hike until you are 5 miles from where you started. Is it possible to determine how far west you are from where you started?

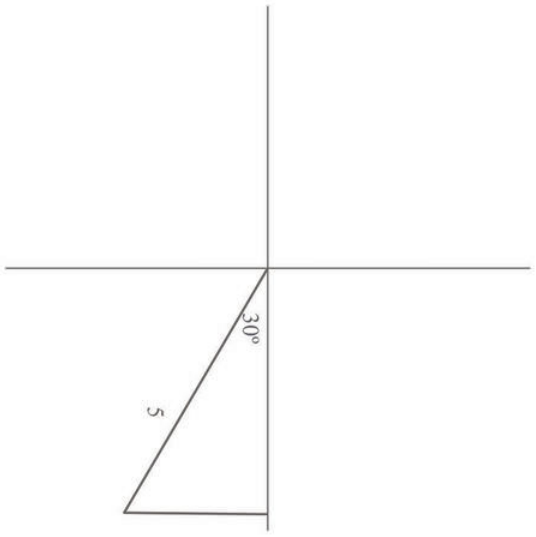
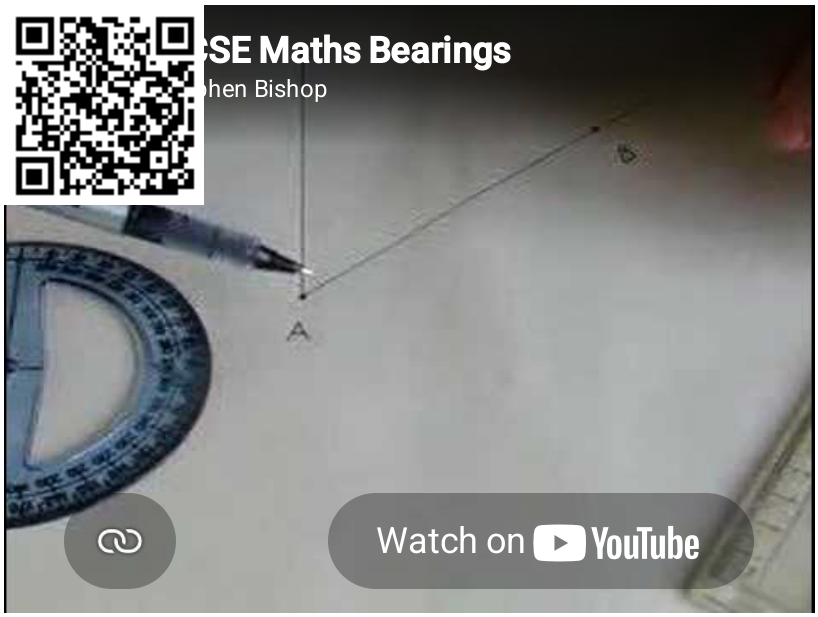


Figure **2.2.3.1**

Bearings

You can use right triangles to find distances using angles given as bearings. In navigation, a **bearing** is the direction from one object to another. In air navigation, bearings are given as angles rotated clockwise from the north.



The graph below shows an angle of 70 degrees:

It is important to keep in mind that angles in navigation problems are measured this way, and not the same way angles are otherwise measured in trigonometry. Further, angles in navigation and surveying may also be given in terms of north, east, south,

and west. For example, $N70°E$ refers to an angle 70 degrees to the east of straight north, while $N70°W$ refers to an angle 70 degrees west of straight north. $N70°E$ is the same as the angle shown in the graph above. $N70°W$ would result in an angle in the second quadrant, like this:

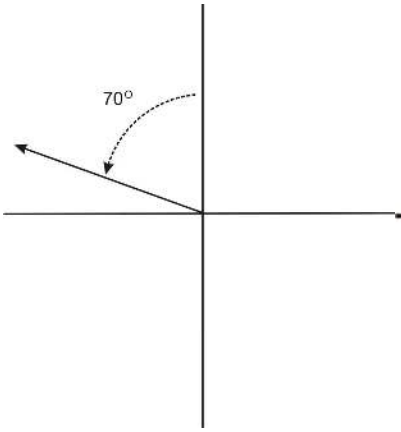


Figure 2.2.3.3

Now, let's look at a problem where we find the distance using right triangles as bearings.

A ship travels on a $N50°E$ course. The ship travels until it is due north of a port which is 10 nautical miles due east of the port from which the ship originated. How far did the ship travel?

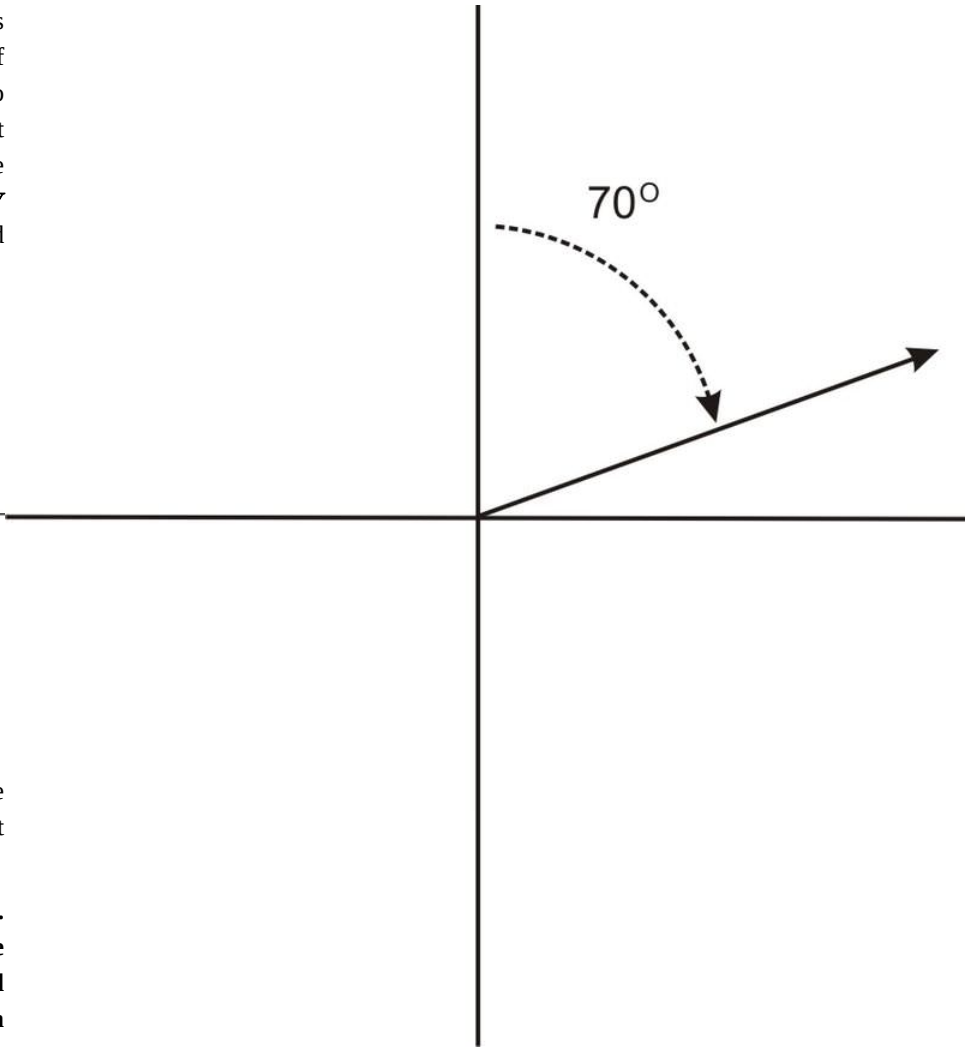


Figure 2.2.3.2

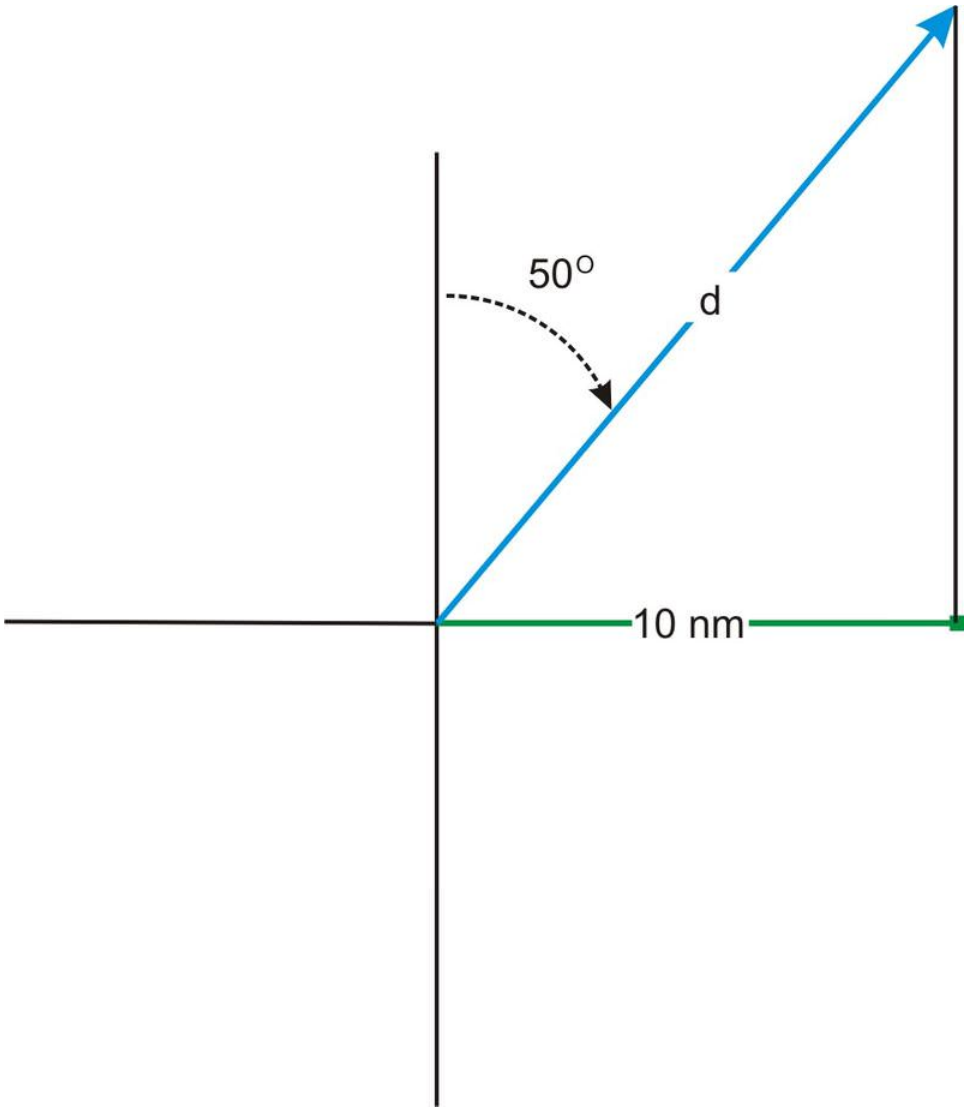


Figure 2.2.3.4

The angle between d and 10 nm is the complement of $50°$, which is $40°$. Therefore we can find d using the cosine function:

$$\cos 40° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{10}{d}$$
$$\cos 40° = \frac{10}{d}$$
$$d \cos 40° = 10$$
$$d = 10 \cos 40° \approx 13.05 \text{ nautical miles}$$

An airplane flies on a course of $S30°E$, for 150 km. How far south is the plane from where it originated?

Construct a triangle using the known information, and then use the cosine function to solve the problem:

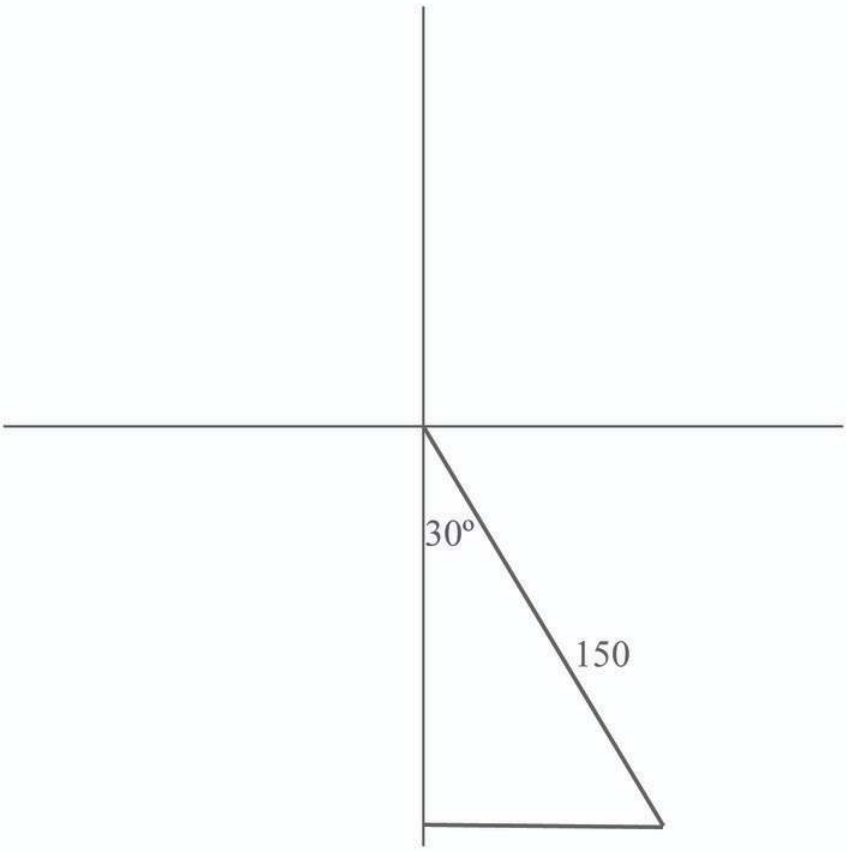


Figure 2.2.3.5

$$\cos 30^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{y}{150}$$
$$\cos 30^\circ = \frac{y}{150}$$
$$150 \cos 30^\circ = y$$
$$y = 150 \cos 30^\circ \approx 130 \text{ km}$$

Jean travels to school each day by walking 200 meters due north, and then turning right and walking 100 meters due eEast. If she had walked in a straight line, what would the angle between her home and the school be if the beginning of the angle is taken from due north? What would be two different ways to describe the direction to take walking there in a straight line, u\sin g what we've learned in this section?

From the triangle given above, we can use the tangent function to determine the angle if she had walked in a straight line.

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{100}{200}$$
$$\tan \theta = \frac{100}{200}$$
$$\theta = 26.57^\circ$$

One way of describing her straight line path is how far east of north she is: $N26.57^\circ E$

Also, since we know the bearings are usually based off of north, her motion can be described as simply a bearing of 26.57°.

Example 2.2.3.1

Earlier, you were given some information about a hiking trip, and asked "Is it possible to determine how far south you are from where you started?"

Solution

The story specified that you hiked for 5 miles from your starting point, on a bearing of $S30°W$. Applying this data with your understanding of how to construct a triangle using bearings, you can draw the following:

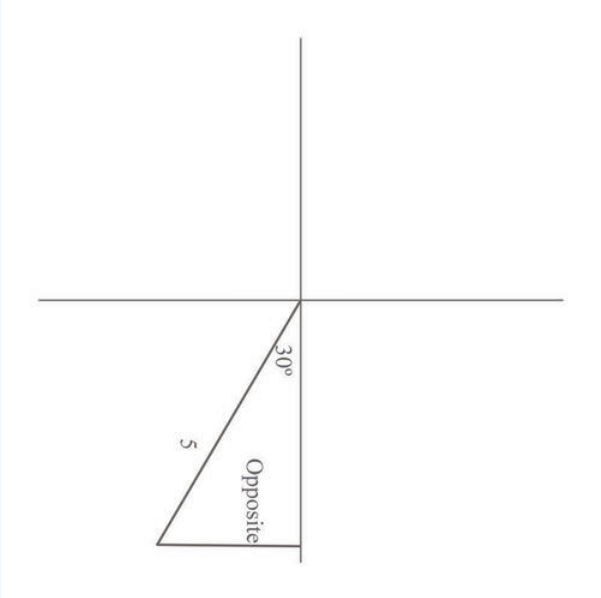


Figure **2.2.3.6**

This shows that the opposite side of the triangle is what's not known. Therefore, you can use the \sin e function to solve the problem:

$$\sin 30° = \frac{\text{opposite}}{5}$$
$$\text{opposite} = 5 \sin 30°$$
$$\text{opposite} = (5)(.5) = 2.5$$

You are 2.5 miles west of where you started.

Example 2.2.3.2

Plot a course or bearing of $240°$ on a rectangular coordinate system.

Solution

This is the same as $S30°W$ (recall that due south is 270 degrees, so 240 degrees is 30 degrees west of that) and can be plotted as:

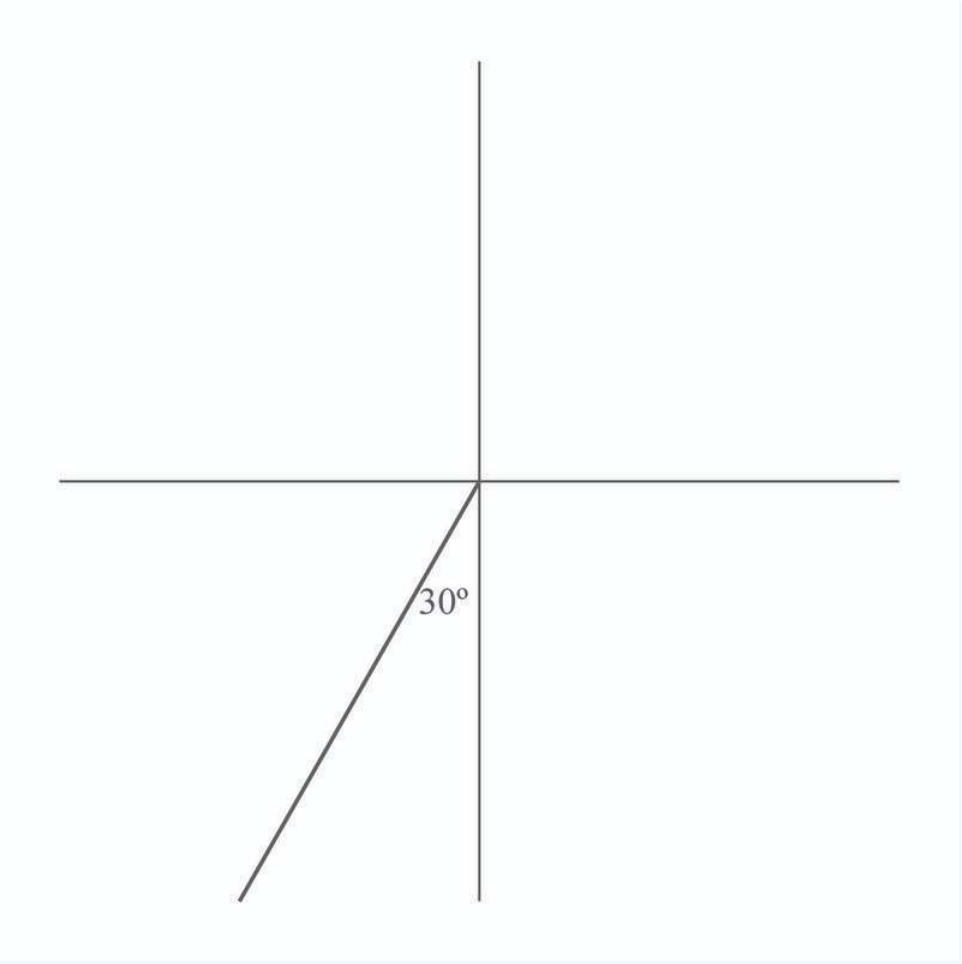


Figure 2.2.3.7

Example 2.2.3.3

Scott is boating on a course of $N15°E$. What course would he need to take to return to where he came from?

Solution

The opposite direction would return him to his starting point. This would be $S15°W$.

Example 2.2.3.4

Adam hikes on a course of $N47°E$ for 7 km. How far east is Adam from where he started?

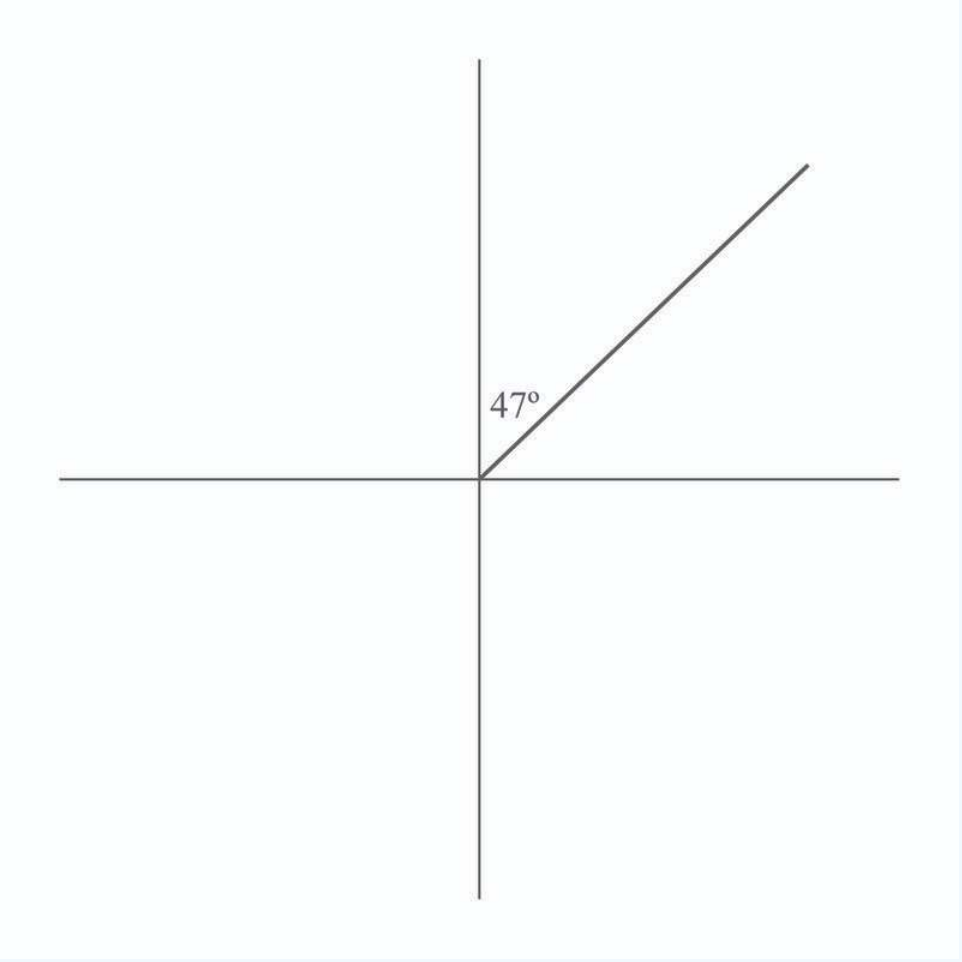


Figure 2.2.3.8

Solution

Find the length of the triangle above (which is how far Adam traveled east) by u\sin g the \sin e function:

$$\sin 47^\circ = \frac{x}{7}$$
$$x = 7 \sin 47^\circ$$
$$x = (7)(.7313)$$
$$x = 5.1191$$

He is 5.1191 km east of where he started.

Review

1. Plot a course of $N40^\circ E$ on a rectangular coordinate system.
2. Plot a course of 60° on a rectangular coordinate system.
3. Plot a course of $S70^\circ W$ on a rectangular coordinate system.
4. Plot a course of $S5^\circ W$ on a rectangular coordinate system.
5. Plot a course of $N42^\circ W$ on a rectangular coordinate system.
6. You are on a course of $N55^\circ E$. What course would you need to take to return to where you came from?
7. You are on a course of $S34^\circ W$. What course would you need to take to return to where you came from?
8. You are on a course of $N72^\circ W$. What course would you need to take to return to where you came from?
9. You are on a course of $S10^\circ E$. What course would you need to take to return to where you came from?
10. You are on a course of $N25^\circ W$. What course would you need to take to return to where you came from?
11. You are on a course of 47° for 5 km. How far east are you from where you started?
12. You are on a course of $S32^\circ E$ for 8 km. How far east are you from where you started?

13. You are on a course of $N15°W$ for 10 km. How far west are you from where you started?
14. You are on a course of $S3°W$ for 12 km. How far west are you from where you started?
15. You are on a course of $S67°E$ for 6 km. How far east are you from where you started?

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.14.

Vocabulary

Term	Definition
Bearing	Bearing is how direction is measured at sea. North is **0°**, east is **90°**, south is **180°**, and west is **270°**.

Additional Resources

Video: Right Triangle Models - Example 1

Practice: Right Triangles and Bearings

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2.2.4: Solve Right Triangles

Using the inverse trigonometric functions to solve for missing information about right triangles.

Inverse Trigonometric Ratios

In mathematics, the word inverse means “undo.” For example, addition and subtraction are inverses of each other because one undoes the other. When we use the inverse trigonometric ratios, we can find acute angle measures as long as we are given two sides.

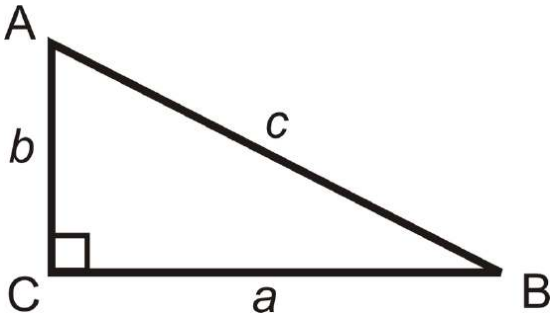


Figure **2.2.4.1**

Inverse Tangent: Labeled $\tan^{-1}$, the “-1” means inverse.

$\tan^{-1}\left(\frac{b}{a}\right) = m\angle B$ and $\tan^{-1}\left(\frac{a}{b}\right) = m\angle A$.

Inverse Sine: Labeled $\sin^{-1}$.

$\sin^{-1}\left(\frac{b}{c}\right) = m\angle B$ and $\sin^{-1}\left(\frac{a}{c}\right) = m\angle A$.

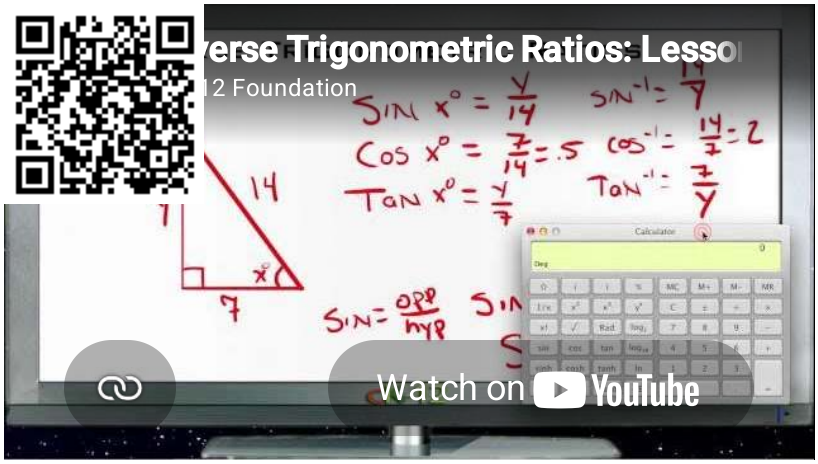
Inverse Cosine: Labeled $\cos^{-1}$.

$\cos^{-1}\left(\frac{a}{c}\right) = m\angle B$ and $\cos^{-1}\left(\frac{b}{c}\right) = m\angle A$.

In most problems, to find the measure of the angles you will need to use your calculator. On most scientific and graphing calculators, the buttons look like [$\sin^{-1}$], [$\cos^{-1}$], and [\($\tan^{-1}$]. You might also have to hit a shift or 2nd button to access these functions.

Now that you know both the trig ratios and the inverse trig ratios you can **solve** a right triangle. To solve a right triangle, you need to find all sides and angles in it. You will usually use sine, cosine, or tangent; inverse sine, inverse cosine, or inverse tangent; or the **Pythagorean Theorem**.

What if you were told the tangent of $\angle Z$ is 0.6494? How could you find the measure of $\angle Z$?



Example 2.2.4.1

Solve the right triangle.

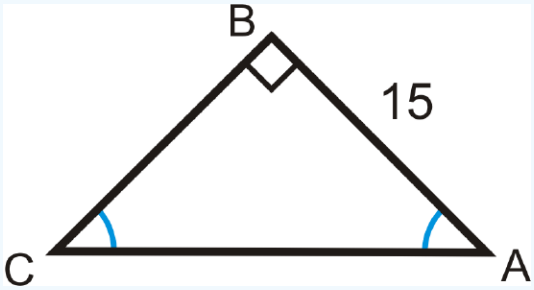


Figure 2.2.4.2

Solution

The two acute angles are congruent, making them both $45°$. This is a 45-45-90 triangle. You can use the trigonometric ratios or the special right triangle ratios.

Trigonometric Ratios

$$\tan 45° = \frac{15}{BC} \qquad \sin 45° = \frac{15}{AC}$$
$$BC = \frac{15}{\tan 45°} = 15 \qquad AC = \frac{15}{\sin 45°} \approx 21.21$$

45-45-90 Triangle Ratios

$$BC = AB = 15,\ AC = 15\sqrt{2} \approx 21.21$$

Example 2.2.4.2

Use the sides of the triangle and your calculator to find the value of $\angle A$. Round your answer to the nearest tenth of a degree.

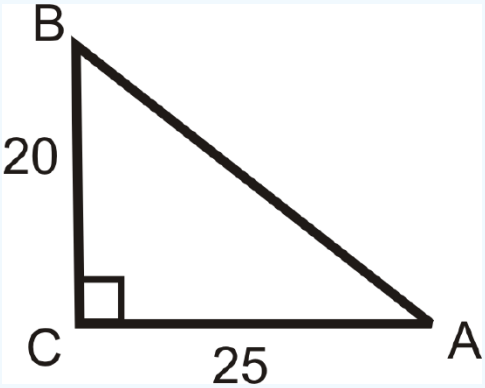


Figure 2.2.4.3

Solution

In reference to $\angle A$, we are given the *opposite* leg and the *adjacent* leg. This means we should use the *tangent* ratio.

$\tan A = \frac{20}{25} = \frac{4}{5}$. So, $\tan^{-1} \frac{4}{5} = m\angle A$. Now, use your calculator.

If you are using a TI-83 or 84, the keystrokes would be: [2nd][TAN]($\frac{4}{5}$) **[ENTER]** and the screen looks like:



Figure 2.2.4.4

$$m\angle A \approx 38.7^\circ$$

Example 2.2.4.3

$\angle A$ is an acute angle in a right triangle. Find $m\angle A$ to the nearest tenth of a degree for $\sin A = 0.68$, $\cos A = 0.85$, and $\tan A = 0.34$.

Solution

$$m\angle A = \sin^{-1} 0.68 \approx 42.8^\circ$$
$$m\angle A = \cos^{-1} 0.85 \approx 31.8^\circ$$
$$m\angle A = \tan^{-1} 0.34 \approx 18.8^\circ$$

Example 2.2.4.4

Solve the right triangle.

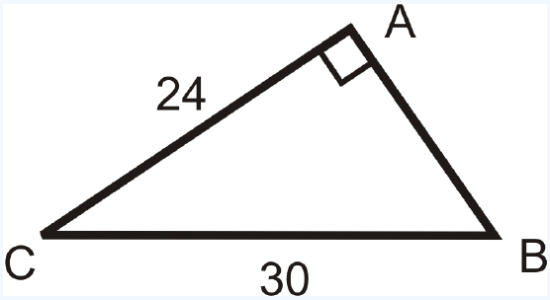


Figure 2.2.4.5

Solution

To solve this right triangle, we need to find AB, $m\angle C$ and $m\angle B$. Use only the values you are given.

$\underline{AB}$: **Use the Pythagorean Theorem.**

$$24^2 + AB^2 = 30^2$$
$$576 + AB^2 = 900$$
$$AB^2 = 324$$
$$AB = \sqrt{324} = 18$$

$\underline{m\angle B}$: **Use the inverse sine ratio.**

$$\sin B = \frac{24}{30} = \frac{4}{5}$$
$$\sin^{-1}(45) \approx 53.1^\circ = m\angle B$$

$\underline{m\angle C}$: **Use the inverse cosine ratio.**

$$\cos C = \frac{24}{30} = \frac{4}{5} \rightarrow \cos^{-1}(\frac{4}{5}) \approx 36.9^\circ = m\angle C$$

Example 2.2.4.5

When would you use sin and when would you use $\sin^{-1}$?

Solution

You would use sin when you are given an angle and you are solving for a missing side. You would use $\sin^{-1}$when you are given sides and you are solving for a missing angle.

Review

Use your calculator to find $m\angle A$ to the nearest tenth of a degree.

1.

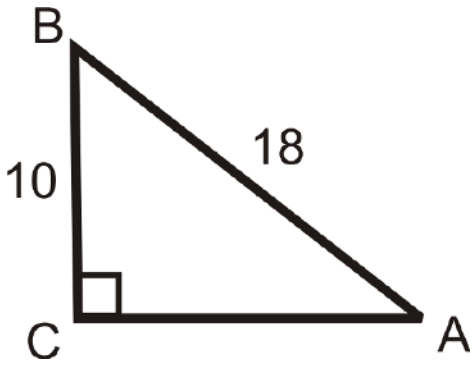


Figure **2.2.4.6**

2.

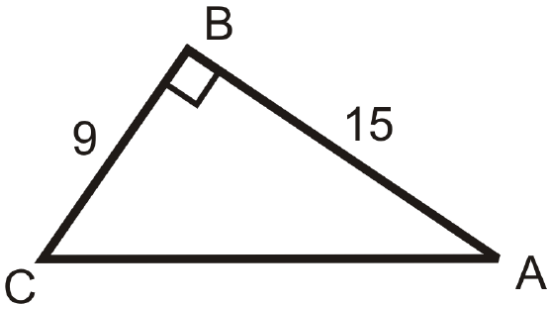


Figure **2.2.4.7**

3.

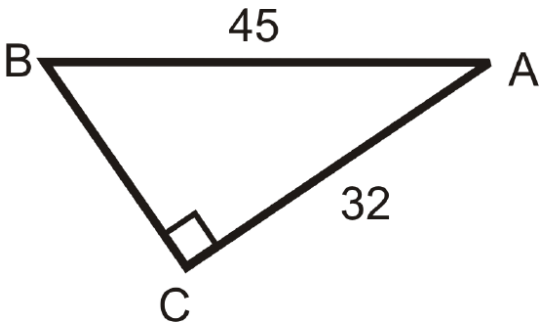


Figure **2.2.4.8**

4.

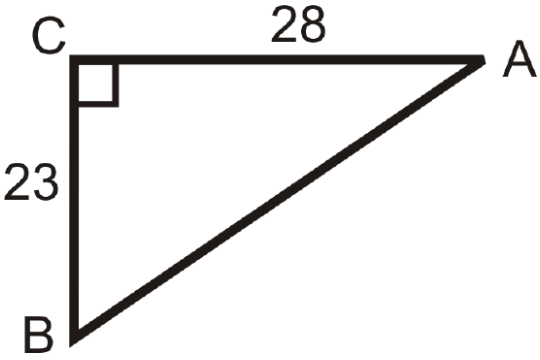


Figure **2.2.4.9**

5.

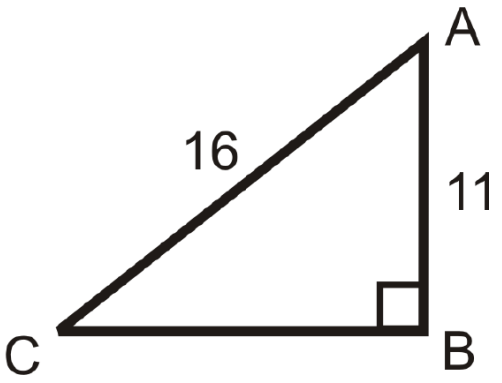


Figure 2.2.4.10

6.

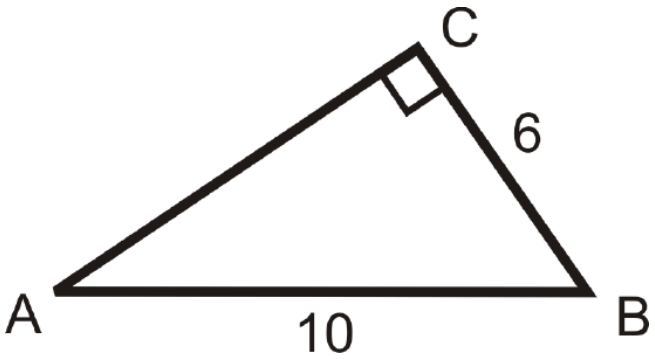


Figure 2.2.4.11

Let $\angle A$ be an acute angle in a right triangle. Find $m\angle A$ to the nearest tenth of a degree.

7. $\sin A = 0.5684$
8. $\cos A = 0.1234$
9. $\tan A = 2.78$
10. $\cos^{-1} 0.9845$
11. $\tan^{-1} 15.93$
12. $\sin^{-1} 0.7851$

Solving the following right triangles. Find all missing sides and angles. Round any decimal answers to the nearest tenth.

13.

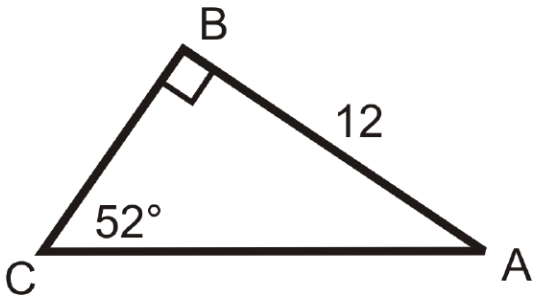


Figure 2.2.4.12

14.

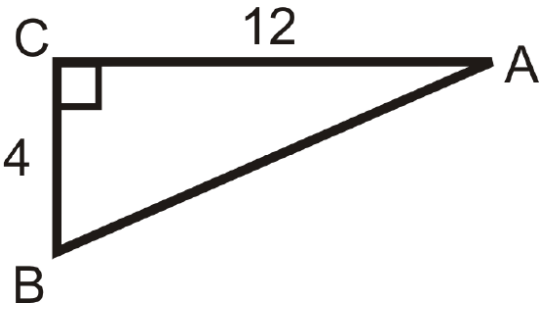


Figure 2.2.4.13

15.

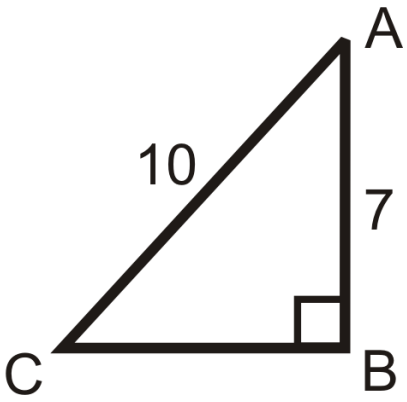


Figure 2.2.4.14

16.

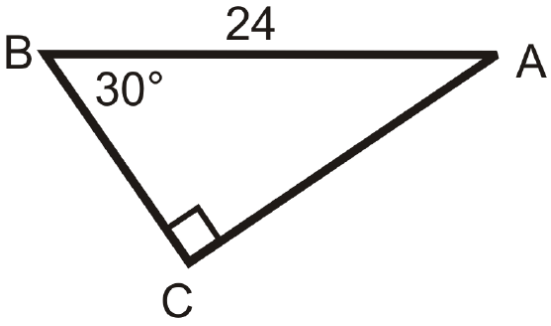


Figure 2.2.4.15

17.

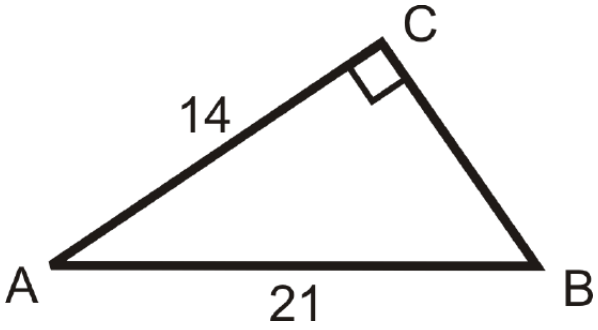


Figure 2.2.4.16

18.

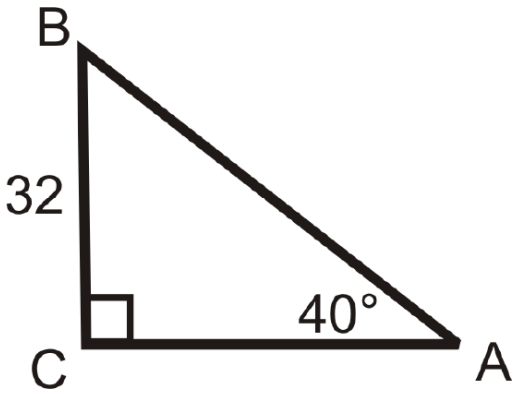


Figure 2.2.4.17

19.

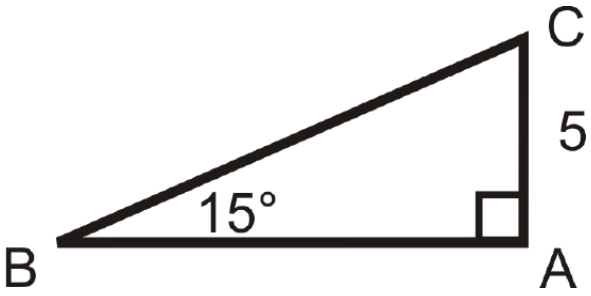


Figure 2.2.4.18

20.

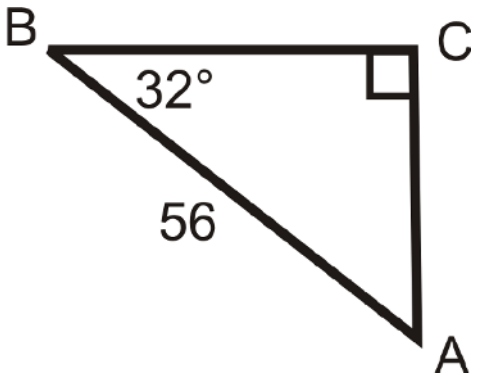


Figure 2.2.4.19

21.

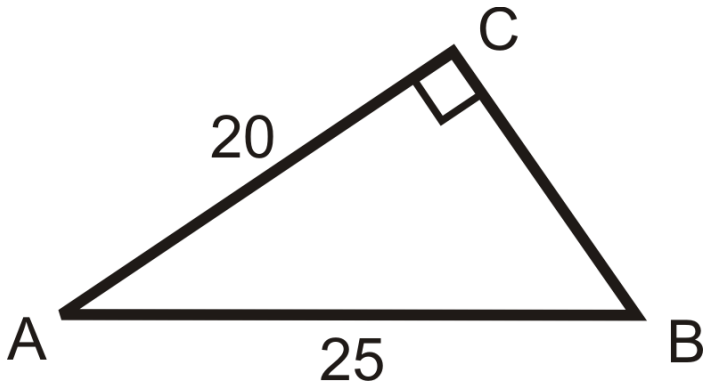


Figure 2.2.4.20

Review (Answers)

To see the Review answers, open this PDF file and look for section 8.10.

Additional Resources

Video: Introduction to Inverse Trigonometric Functions

Activities: Inverse Trigonometric Ratios Discussion Questions

Study Aids: Trigonometric Ratios Study Guide

Practice: Solve Right Triangles

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2.2.5: Applications of Inverse Trigonometric Functions

Angle values in real world examples.

You are mapping out a hiking trip to a new spot in the state park near your home. On your map, you know that you are supposed to go on a course which ends with you having moved 2.5 miles East and 3 miles South. Your map, with the start and end points, is shown here:



Figure **2.2.5.1**

Now you need to calculate the angle that you need to walk with respect to due East. Can you find a way to calculate this angle using inverse trig functions?

Applying Inverse Trig Functions



The following problems are real-world problems that can be solved using the trigonometric functions. In everyday life, indirect measurement is used to obtain answers to problems that are impossible to solve using measurement tools. However, mathematics will come to the rescue in the form of trigonometry to calculate these unknown measurements.

On a cold winter day the sun streams through your living room window and causes a warm, toasty atmosphere. This is due to the angle of inclination of the sun which directly affects the heating and the cooling of buildings. Noon is when the sun is at its maximum height in the sky and at this time, the angle is greater in the summer than in the winter. Because of this, buildings are constructed such that the overhang of the roof can act as an awning to shade the windows for cooling in the summer and yet allow

the sun's rays to provide heat in the winter. In addition to the construction of the building, the angle of inclination of the sun varies according to the latitude of the building's location.

If the latitude of the location is known, then the following formula can be used to calculate the angle of inclination of the sun on any given date of the year:

$\text{Angle of sun} = 90° - \text{latitude} + -23.5° \cdot \cos\left[(N+10)\frac{360}{365}\right]$ where N represents the number of the day of the year that corresponds to the date of the year. Note: This formula is accurate to $\pm\frac{1°}{2}$

Determine the measurement of the sun's angle of inclination for a building located at a latitude of $42°$, March $10^{th}, the \(69^{th}$ day of the year.

$$\begin{aligned}\text{Angle of sun} &= 90° - 42° + -23.5° \cdot \cos\left[(69+10)\frac{360}{365}\right]\\ \text{Angle of sun} &= 48° + -23.5°(0.2093)\\ \text{Angle of sun} &= 48° - 4.92°\\ \text{Angle of sun} &= 43.08°\end{aligned}$$

Determine the measurement of the sun's angle of inclination for a building located at a latitude of $20°$, September 21st.

$$\begin{aligned}\text{Angle of sun} &= 90° - 20° + -23.5° \cdot \cos\left[(264+10)\frac{360}{365}\right]\\ \text{Angle of sun} &= 70° + -23.5°(0.0043)\\ \text{Angle of sun} &= 70.10°\end{aligned}$$

Using inverse trigonometric functions

1. A tower, 28.4 feet high, must be secured with a guy wire anchored 5 feet from the base of the tower. What angle will the guy wire make with the ground?

Draw a picture.

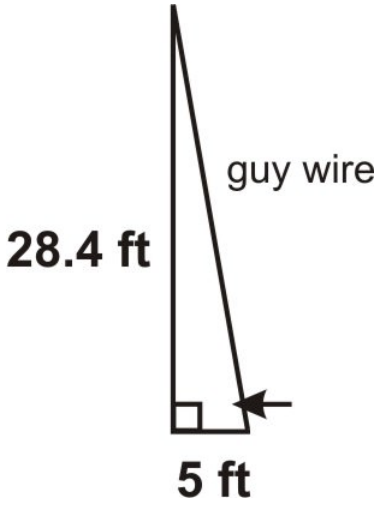


Figure 2.2.5.2

$$\begin{aligned}\tan\theta &= \frac{opp}{adj}\\ \tan\theta &= \frac{28.4}{5}\\ \tan\theta &= 5.68\\ \tan^{-1}(\tan\theta) &= \tan^{-1}(5.68)\\ \theta &= 80.02°\end{aligned}$$

The following problem that involves functions and their inverses will be solved using the property $f(f^{-1}(x)) = f^{-1}(f(x))$. In addition, technology will also be used to complete the solution.

2. In the main concourse of the local arena, there are several viewing screens that are available to watch so that you do not miss any of the action on the ice. The bottom of one screen is 3 feet above eye level and the screen itself is 7 feet high. The angle of vision

(inclination) is formed by looking at both the bottom and top of the screen.

Sketch a picture to represent this problem.

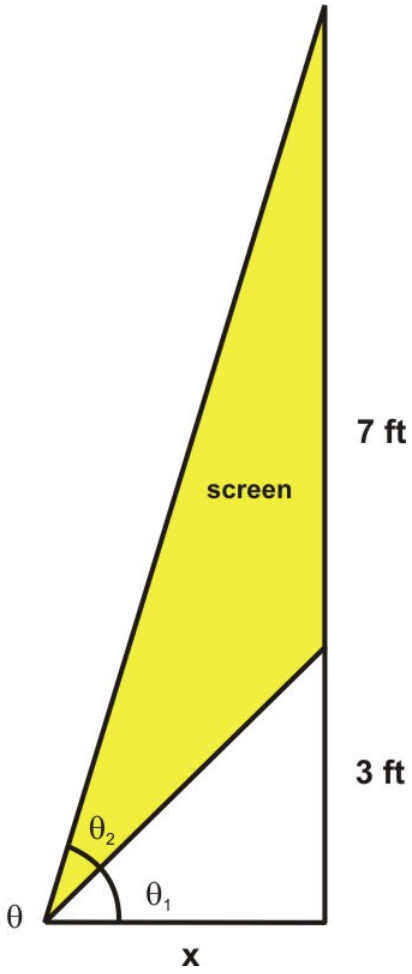


Figure 2.2.5.3

Calculate the measure of the angle of vision that results from looking at the bottom and then the top of the screen. At what distance from the screen does the maximum value for the angle of vision occur?

$$\theta_2 = \tan\theta - \tan\theta_1$$
$$\tan\theta = \frac{10}{x} \text{ and } \tan\theta_1 = \frac{3}{x}$$
$$\theta_2 = \tan^{-1}\left(\frac{10}{x}\right) - \tan^{-1}\left(\frac{3}{x}\right)$$

To determine these values, use a graphing calculator and the trace function to determine when the actual maximum occurs.



Figure 2.2.5.4

From the graph, it can be seen that the maximum occurs when $x \approx 5.59$ ft. and $\theta \approx 32.57°$.

Example 2.2.5.1

Earlier, you were asked to calculate the angle that you need to walk with respect to due East.

Solution

You can set up a triangle that matches the physical situation of this problem. Here is what it should look like:



Figure 2.2.5.5

Using the tangent function, you can solve for the angle you need to find:

$$\theta = \tan^{-1}\left(\frac{3}{-2.5}\right)$$
$$\theta = -50.19$$

Example 2.2.5.2

The intensity of a certain type of polarized light is given by the equation $I = I_0 \sin 2\theta \cos 2\theta$.

Solve for θ.

Solution

$$I = I_0 \sin 2\theta \cos 2\theta$$
$$\frac{I}{I_0} = \frac{I_0}{I_0} \sin 2\theta \cos 2\theta$$
$$\frac{I}{I_0} = \sin 2\theta \cos 2\theta$$
$$\frac{2I}{I_0} = 2 \sin 2\theta \cos 2\theta$$
$$\frac{2I}{I_0} = \sin 4\theta$$
$$\sin^{-1} \frac{2I}{I_0} = 4\theta$$
$$\frac{1}{4} \sin^{-1} \frac{2I}{I_0} = \theta$$

Example 2.2.5.3

The following diagram represents the ends of a water trough. The ends are actually isosceles trapezoids, and the length of the trough from end-to-end is ten feet. Determine the maximum volume of the trough and the value of θ that maximizes that volume.

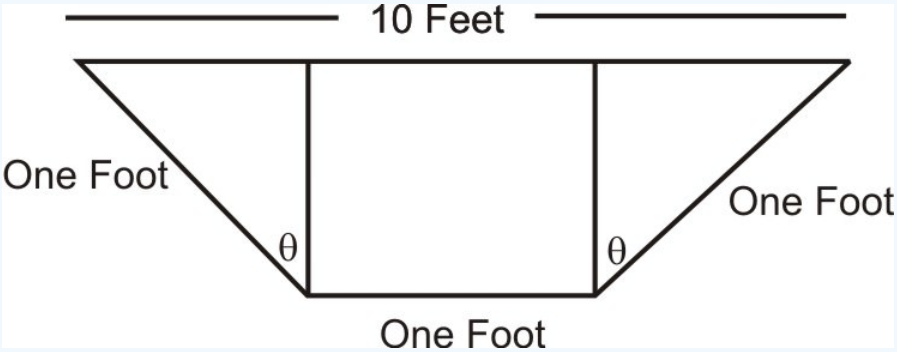


Figure 2.2.5.6

Solution

The volume is 10 feet times the area of the end. The end consists of two congruent right triangles and one rectangle. The area of each right triangle is $\frac{1}{2}(\sin\theta)(\cos\theta)$ and that of the rectangle is $(1)(\cos\theta)$. This means that the volume can be determined by the function $V(\theta) = 10(\cos\theta + \sin\theta\cos\theta)$, and this function can be graphed as follows to find the maximum volume and the angle θ where it occurs.

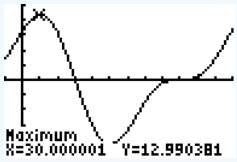


Figure 2.2.5.7

Therefore, the maximum volume is approximately 13 cubic feet and occurs when θ is about $30°$.

Example 2.2.5.4

A boat is docked at the end of a 10 foot pier. The boat leaves the pier and drops anchor 230 feet away 3 feet straight out from shore (which is perpendicular to the pier). What was the bearing of the boat from a line drawn from the end of the pier through the foot of the pier?

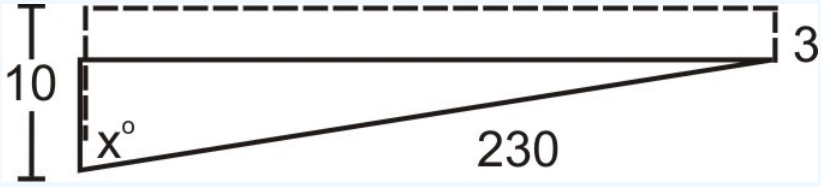


Figure 2.2.5.8

Solution

$$\cos x = \frac{7}{230} \rightarrow x = \cos^{-1}\frac{7}{230}$$
$$x = 88.26°$$

Review

1. The distance from a boat to a lighthouse is 100 feet and the lighthouse is 120 feet tall. What is the angle of depression from the top of the Lighthouse to the boat.

2. You are standing 100 feet from an arch that is 68 feet tall. At what angle do you have to look up to see the top of the arch? Assume you are 5 feet tall.
3. The angle of elevation of the top of a church to a point 100 feet away from the base is $60°$. Find the height of the church.

You are standing looking at a large painting on the wall. The bottom of the painting is 1 foot above your eye level. The painting is 10 feet tall. Assume you are standing x feet from the painting and that angle \theta is formed by the lines of vision to the bottom and to the top of the painting.

4. Draw a picture to represent this situation.
5. Solve for θ in terms of x.
6. If you are standing 10 feet from the painting, what is \theta?
7. If $\theta = 30°$, how far are you standing from the wall (to the nearest foot)?

You are watching a hot-air balloon that was 300 feet from you when it started rising from the ground. Assume the height of the balloon is x and θ is the angle of elevation from the ground where you are standing up to the balloon.

8. Solve for x in terms of θ.
9. Solve for θ in terms of x.
10. What is the angle of elevation when the hot-air balloon is 500 feet above the ground?
11. How high above the ground is the balloon when the angle of elevation is 80^{\circ}?

Recall that if the latitude of the location is known, then the following formula can be used to calculate the angle of inclination of the sun on any given date of the year:

$\text{Angle of sun} = 90° - \text{latitude} + -23.5° \cdot cos[(N + 10)\frac{360}{365}]$ where N represents the number of the day of the year that corresponds to the date of the year.

12. Determine the measurement of the sun's angle of inclination for a building located at a latitude of $30°$, April 12th, the 102th day of the year.
13. Determine the measurement of the sun's angle of inclination for a building located at a latitude of $50°$, August 14th, the 226th day of the year.
14. A tower, 50 feet high, is secured with a guy wire anchored 8 feet from the base of the tower. What angle will the guy wire make with the ground?
15. A 30 foot tall flagpole casts a 12 foot shadow. What is the angle that the sun hits the flagpole?

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.10.

Vocabulary

Term	Definition
inverse trigonometric function	An inverse trigonometric function is a function that reverses a trigonometric function, leaving the argument of the original trigonometric function as a result.

Additional Resources

Video: Height and Distance Word Problem Application of Trigonometry

Practice: Applications of Inverse Trigonometric Functions

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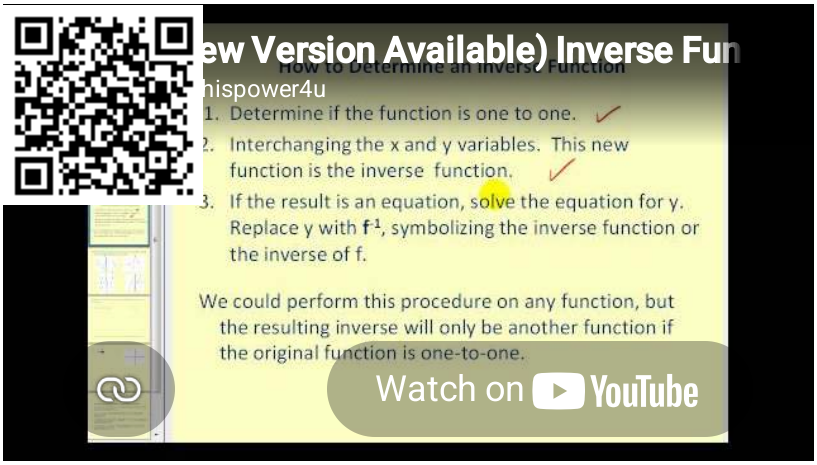
2.2.6: Inverse Trig Functions using Algebra

"Undo" a function by switching x and y values and solving for y.

If you were given a function, such as $f(x) = \frac{2x}{x+7}$, can you tell if the function has an inverse? Is there a way that you could find its inverse through algebraic manipulation?

Finding the Inverse of a Function

An "inverse" is something that undoes a function, giving back the original argument. For example, a function such as $y = \frac{1}{3}x$ has an inverse function of $y = 3x$, since any value placed into the first function will be returned as what it originally was if it is input into the second function. In this case, it is easy to see that to "undo" multiplication by $\frac{1}{3}$, you should multiply by 3. However, in many cases it may not be easy to infer by examination what the inverse of a function is.



To start, let's examine what is required for a function to have an inverse. It is important to remember that each function has an inverse relation and that this inverse relation is a function only if the original function is one-to-one. A function is one-to-one when its graph passes both the vertical and the **horizontal line test**. This means that every vertical and horizontal line will intersect the graph in exactly one place.

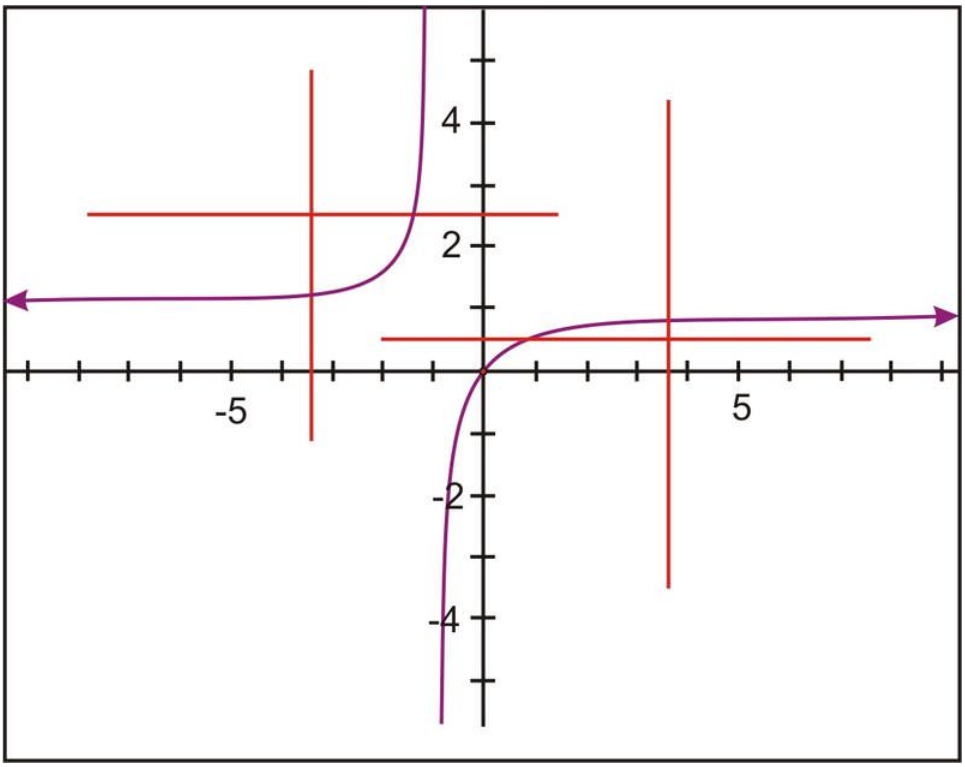


Figure **2.2.6.1**

This is the graph of $f(x) = \frac{x}{x+1}$. The graph suggests that f is one-to-one because it passes both the vertical and the horizontal line tests. To find the inverse of f, switch **the x and y and solve for y.**

First, switch x and y.

$$x = \frac{y}{y+1}$$

Next, multiply both sides by $(y+1)$.

$$(y+1)x = \frac{y}{y+1}(y+1)$$
$$x(y+1) = y$$

Then, apply the distributive property and put all the y terms on one side so you can pull out the y.

$$xy + x = y$$
$$xy - y = -x$$
$$y(x-1) = -x$$

Divide by $(x-1)$ to get y by itself.

$$y = \frac{-x}{x-1}$$

Finally, multiply the right side by $\frac{-1}{-1}$.

$$y = \frac{x}{1-x}$$

Therefore the inverse of f is $f^{-1}(x) = \frac{x}{1-x}$.

The symbol f^{-1} is read "f inverse" and is not the reciprocal of f.

Finding the Inverse of a Function

1. Find the inverse of $f(x) = \frac{1}{x-5}$ algebraically.

To find the inverse algebraically, switch $f(x)$ to y and then switch x and y.

$$
\begin{aligned}
y &= 1x - 5 \\
x &= 1y - 5 \\
x(y-5) &= 1 \\
xy - 5x &= 1 \\
xy &= 5x + 1 \\
y &= \frac{5x+1}{x}
\end{aligned}
$$

2. Find the inverse of $f(x) = 5\sin^{-1}(\frac{2}{x-3})$

$$
\begin{aligned}
f(x) &= 5\sin^{-1}\left(\frac{2}{x-3}\right) \\
x &= 5\sin^{-1}\left(\frac{2}{y-3}\right) \\
\frac{x}{5} &= \sin^{-1}\left(\frac{2}{y-3}\right) \\
\sin\frac{x}{5} &= \left(\frac{2}{y-3}\right) \\
(y-3)\sin\frac{x}{5} &= 2 \\
(y-3) &= \frac{2}{\sin\frac{x}{5}} \\
y &= \frac{2}{\sin\frac{x}{5}} + 3
\end{aligned}
$$

3. Find the inverse of the trigonometric function $f(x) = 4\tan^{-1}(3x+4)$

$$
\begin{aligned}
x &= 4\tan^{-1}(3y+4) \\
\frac{x}{4} &= \tan^{-1}(3y+4) \\
\tan\frac{x}{4} &= 3y+4 \\
\tan\frac{x}{4} - 4 &= 3y \\
\frac{\tan\frac{x}{4} - 4}{3} &= y \\
f^{-1}(x) &= \frac{\tan\frac{x}{4} - 4}{3}
\end{aligned}
$$

Example 2.2.6.1

Earlier, you were asked to find the inverse of a function.

Solution

Since the original function is:

$$f(x) = y = \frac{2x}{x+7}$$

You can first switch all of the "x" and "y" values:

$$x = \frac{2y}{y+7}$$

You can then rearrange the equation and isolate "y":

$$
\begin{aligned}
x(y+7) &= 2y \\
xy + 7x &= 2y \\
xy - 2y &= -7x \\
y(x-2) &= -7x \\
y &= \frac{-7x}{x-2}
\end{aligned}
$$

The inverse function is written as $f^{-1}(x) = \dfrac{-7x}{x-2}$

Example 2.2.6.2

Find the inverse of $f(x) = 2x^3 - 5$

Solution

$$\begin{aligned} f(x) &= 2x^3 - 5 \\ y &= 2x^3 - 5 \\ x &= 2y^3 - 5 \\ x + 5 &= 2y^3 \\ \frac{x+5}{2} &= y^3 \\ \sqrt[3]{\frac{x+5}{2}} &= y \end{aligned}$$

Example 2.2.6.3

Find the inverse of $y = \frac{1}{3}\tan^{-1}(\frac{3}{4}x - 5)$

Solution

$$\begin{aligned} y &= \frac{1}{3}\tan^{-1}\left(\frac{3}{4}x - 5\right) \\ x &= \frac{1}{3}\tan^{-1}\left(\frac{3}{4}y - 5\right) \\ 3x &= \tan^{-1}\left(\frac{3}{4}y - 5\right) \\ \tan(3x) &= \frac{3}{4}y - 5 \\ \tan(3x) + 5 &= \frac{3}{4}y \\ \frac{4(\tan(3x) + 5)}{3} &= y \end{aligned}$$

Example 2.2.6.4

Find the inverse of $g(x) = 2\sin(x-1) + 4$

Solution

$$\begin{aligned} g(x) &= 2\sin(x-1) + 4 \\ y &= 2\sin(x-1) + 4 \\ x &= 2\sin(y-1) + 4 \\ x - 4 &= 2\sin(y-1) \\ \frac{x-4}{2} &= \sin(y-1) \\ 1 + \sin^{-1}\left(\frac{x-4}{2}\right) &= y \end{aligned}$$

Review

Find the inverse of each function.

1. $f(x) = 3x + 5$
2. $g(x) = 0.2x - 7$
3. $h(x) = 0.1x^2$
4. $k(x) = 5x + 6$
5. $f(x) = \sqrt{x-4}$

6. $g(x) = (x)^{\frac{1}{3}} + 1$
7. $h(x) = (x+1)^3$
8. $k(x) = \frac{x^2}{3}$
9. $f(x) = -2 + 4\sin^{-1}(x+7)$
10. $g(x) = 1 + 3\tan^{-1}(2x+1)$
11. $h(x) = 4\cos^{-1}(3x)$
12. $k(x) = -1\tan^{-1}(6x)$
13. $j(x) = 5 + 2\sin^{-1}(x+5)$
14. $m(x) = -2\tan(3x+1)$
15. $p(x) = 5 - 6\sin(\frac{x}{2})$

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.3.

Vocabulary

Term	Definition
Horizontal Line Test	The horizontal line test says that if a horizontal line drawn anywhere through the graph of a function intersects the function in more than one location, then the function is not one-to-one and not invertible.
One to One Function	A function is one to one if its inverse is also a function. A one-to-one function passes both the horizontal and vertical line tests.
Vertical Line Test	The vertical line test says that if a vertical line drawn anywhere through the graph of a relation intersects the relation in more than one location, then the relation is *not* a function.

Additional Resources

Video: Inverse Functions

Practice: Inverse Trig Functions using Algebra

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2.2.7: Angles of Elevation and Depression

Angles going up or down from a horizontal line of sight.

Angles of Elevation and Depression

You have decided to go camping with some friends. While out on a hike, you reach the top of a ridge and look down at the trail behind you. In the distance, you can see your camp. You're thinking about how far you've traveled, and wonder if there is a way to determine it.

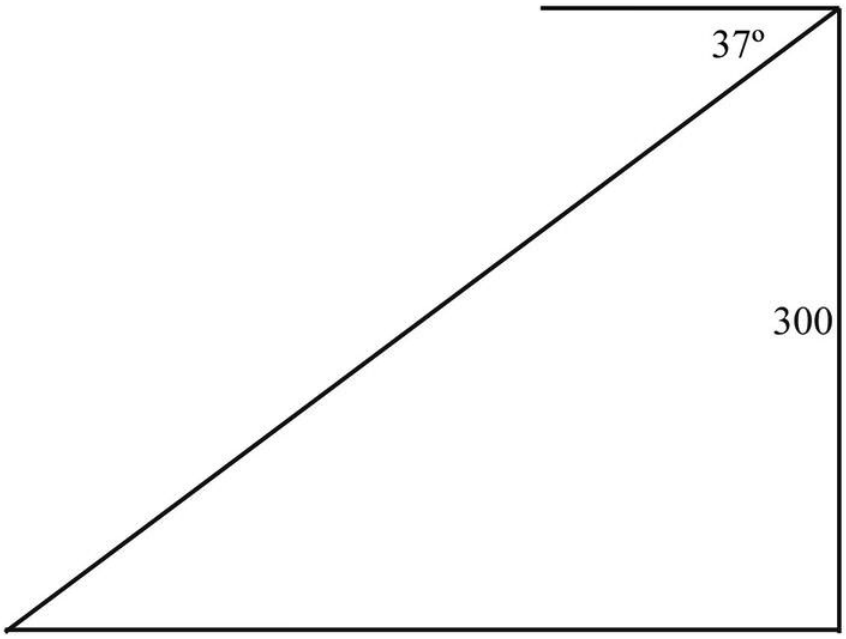
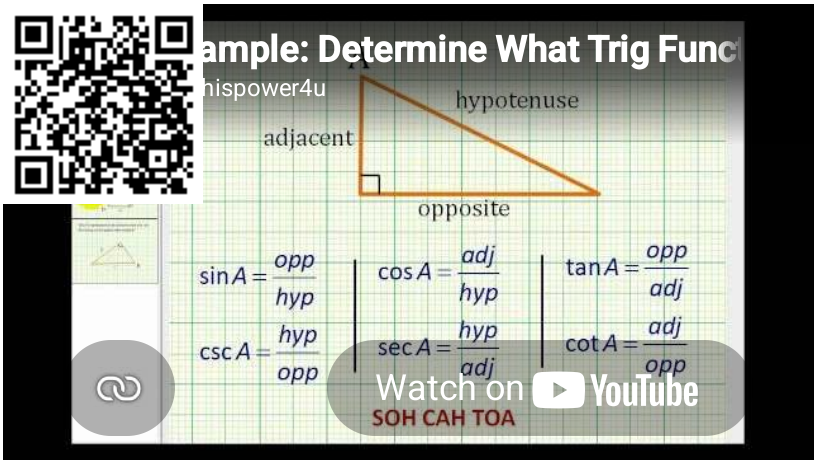


Figure **2.2.7.1**

By using a small device called a clinometer, you're able to measure the angle between your horizontal line of sight and the camp as **37°**, and you know that the hill you just hiked up has a height of 300 m. Is it possible to find out how far away your camp is using this information? (Assume that the trail you hiked is slanted like the side of a triangle.)

Angles of Elevation and Depression

You can use right triangles to find distances, if you know an **angle of elevation** or an **angle of depression**.



The figure below shows each of these kinds of angles.

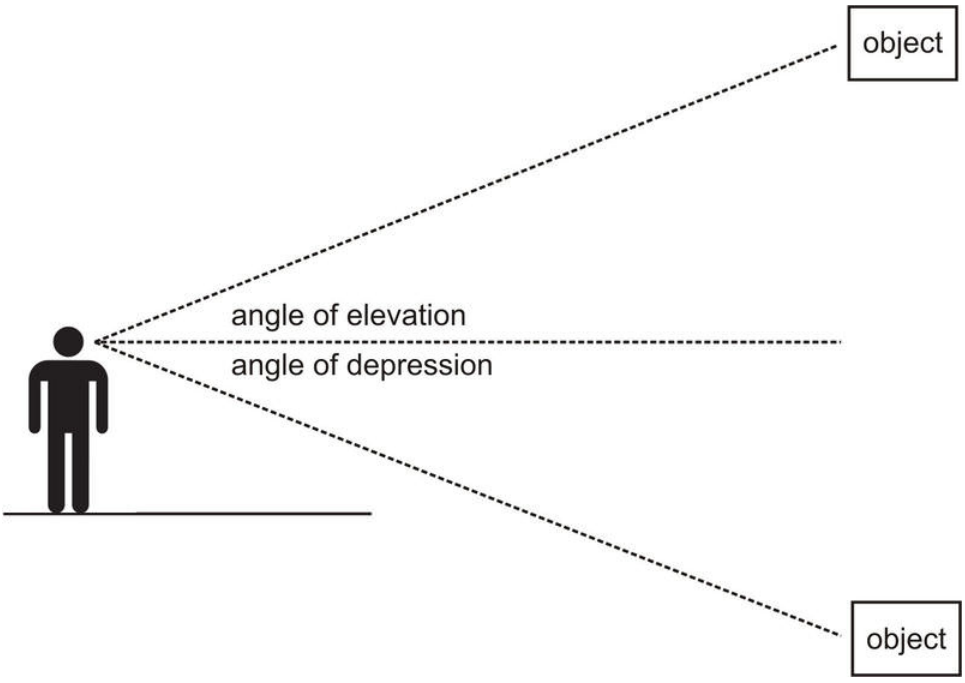


Figure 2.2.7.2

The angle of elevation is the angle between the horizontal line of sight and the line of sight up to an object. For example, if you are standing on the ground looking up at the top of a mountain, you could measure the angle of elevation. The angle of depression is the angle between the horizontal line of sight and the line of sight *down to* an object. For example, if you were standing on top of a hill or a building, looking down at an object, you could measure the angle of depression. You can measure these angles using a clinometer or a theodolite. People tend to use clinometers or theodolites to measure the height of trees and other tall objects. Here we will solve several problems involving these angles and distances.

Finding the angle of elevation

You are standing 20 feet away from a tree, and you measure the angle of elevation to be 38°. How tall is the tree?

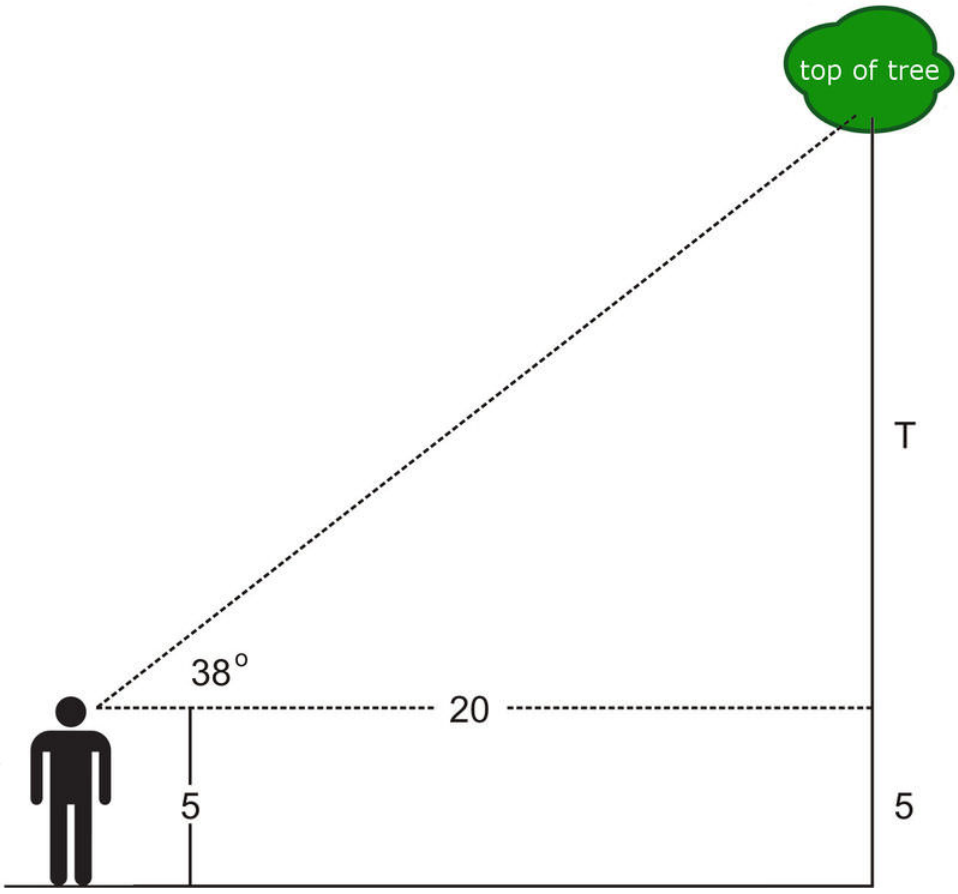


Figure **2.2.7.3**

The solution depends on your height, as you measure the angle of elevation from your line of sight. Assume that you are 5 feet tall.

The figure shows us that once we find the value of T, we have to add 5 feet to this value to find the total height of the triangle. To find T, we should use the tangent value:

$$tan38^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{T}{20}$$

$$tan38^\circ = \frac{T}{20}$$

$$T = 20tan38^\circ \approx 15.63$$

$$\text{Height of tree} \approx 20.63 \text{ ft}$$

You are standing on top of a building, looking at a park in the distance. The angle of depression is 53^{\circ} . If the building you are standing on is 100 feet tall, how far away is the park? Does your height matter?

Finding the angle of depression

If we ignore the height of the person, we solve the following triangle:

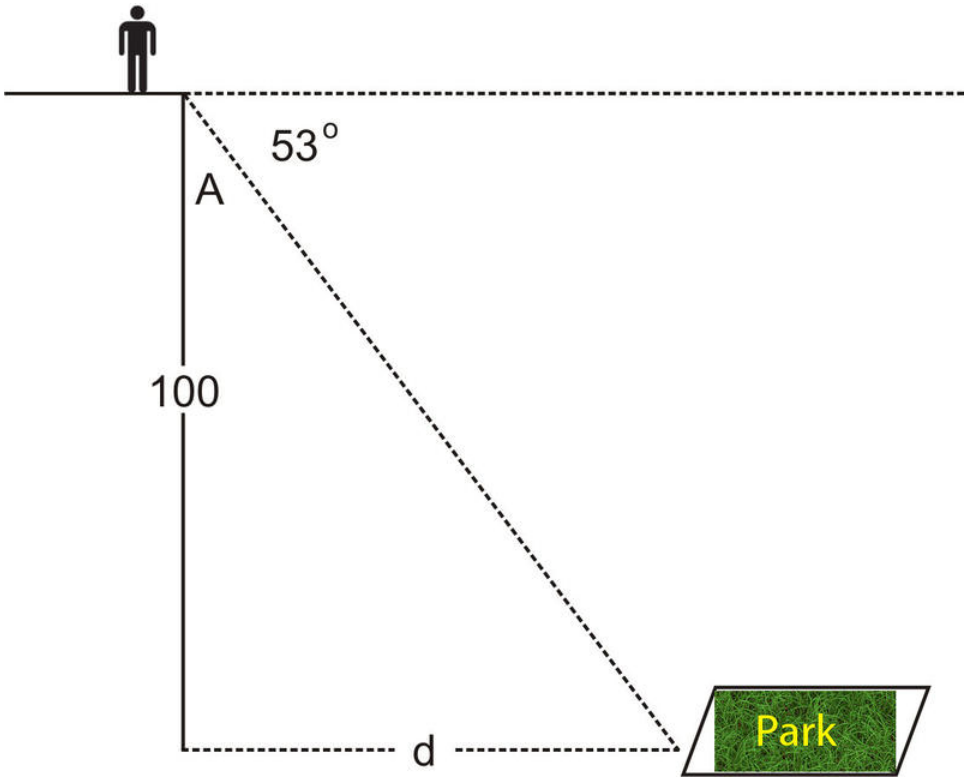


Figure 2.2.7.4

Given the angle of depression is $53°$, $\angle A$ in the figure above is $37°$. We can use the tangent function to find the distance from the building to the park:

$$tan37° = \frac{\text{opposite}}{\text{adjacent}} = \frac{d}{100}$$

$$tan37° = \frac{d}{100}$$

$$d = 100tan37° \approx 75.36 \text{ ft}$$

If we take into account the height if the person, this will change the value of the adjacent side. For example, if the person is 5 feet tall, we have a different triangle:

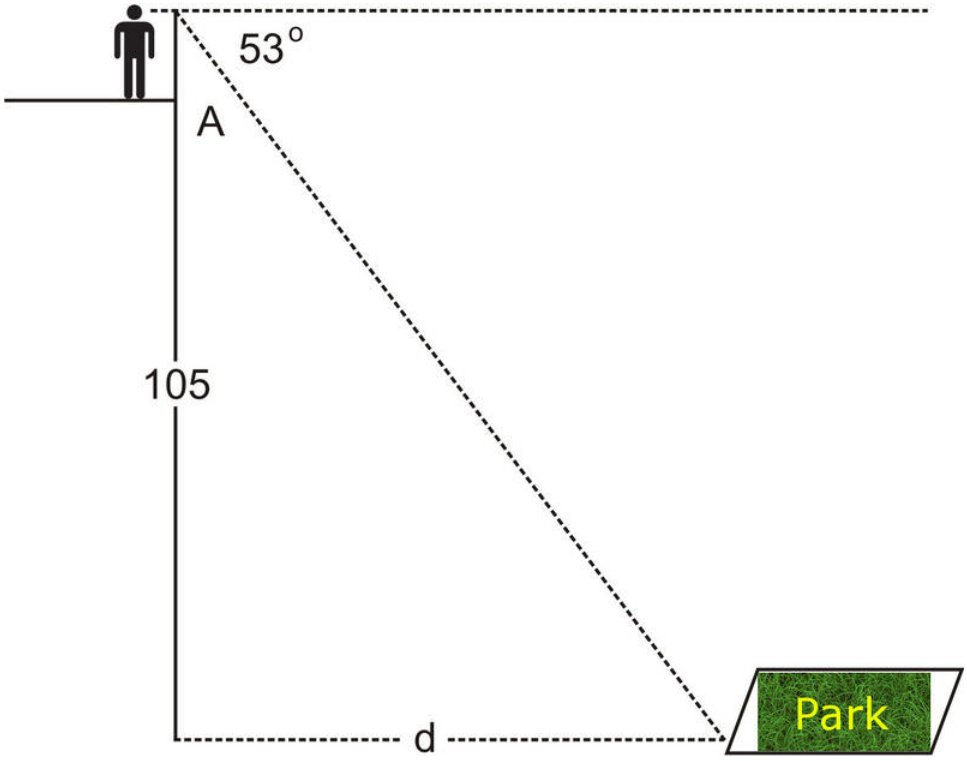


Figure 2.2.7.5

$$\begin{aligned} tan37^{\circ}&=\dfrac{\text{opposite}}{\text{adjacent}}=\dfrac{d}{105} \\ tan37^{\circ} &=\dfrac{d}{105} \\ d&=105{tan37^{\circ} \approx 79.12\text{ ft}\end{aligned}$$

If you are only looking to estimate a distance, then you can ignore the height of the person taking the measurements. However, the height of the person will matter more in situations where the distances or lengths involved are smaller. For example, the height of the person will influence the result more in the tree height problem than in the building problem, as the tree is closer in height to the person than the building is.

Real-World Application: The Horizon

You are on a long trip through the desert. In the distance you can see mountains, and a quick measurement tells you that the angle between the mountaintop and the ground is 13.4°. From your studies, you know that one way to define a mountain is as a pile of land having a height of at least 2,500 meters. If you assume the mountain is the minimum possible height, how far are you away from the center of the mountain?

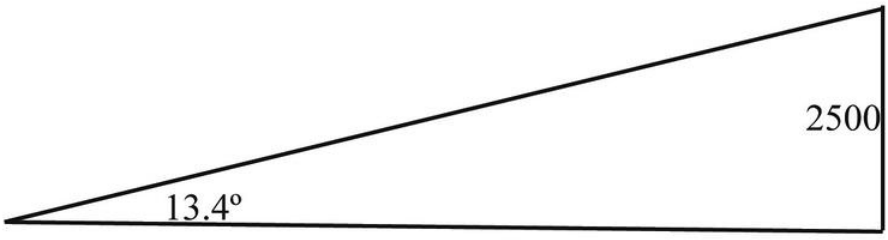


Figure 2.2.7.6

$$tan13.4^{\circ} = \frac{\text{opposite}}{\text{adjacent}} = \frac{2500}{d}$$

$$tan13.4^{\circ} = \frac{2500}{d}$$

$$d = \frac{2500}{tan13.4^{\circ}} \approx 10,494 meters$$

Example 2.2.7.1

Earlier, you were asked if it was possible to find out how far away your camp is using the information given.

Solution

Since you know the angle of depression is $37°$, you can use this information, along with the height of the hill, to create a trigonometric relationship:

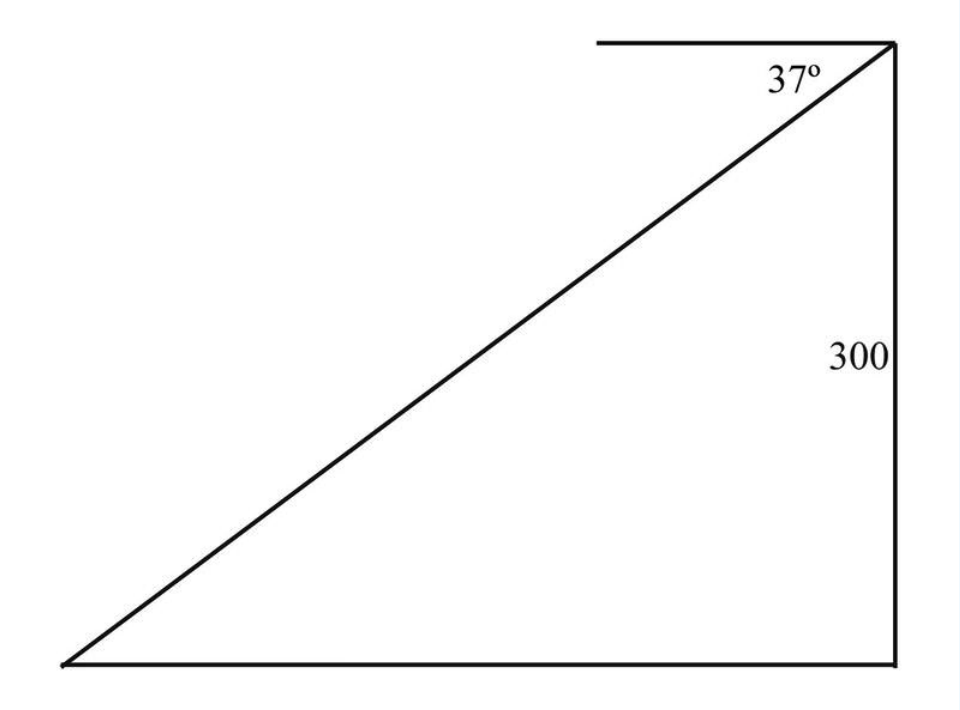


Figure 2.2.7.7

Since the unknown side of the triangle is the hypotenuse, and you know the opposite side, you should use the sine relationship to solve the problem:

$$sin37° = \frac{300}{hypotenuse}$$
$$hypotenuse = \frac{300}{sin37°}$$
$$hypotenuse \approx 498.5$$

You have traveled approximately 498.5 meters up the hill.

Example 2.2.7.2

You are six feet tall and measure the angle between the horizontal and a bird in the sky to be 40^{\circ} . You can see that the shadow of the bird is directly beneath the bird, and 200 feet away from you on the ground. How high is the bird in the sky?

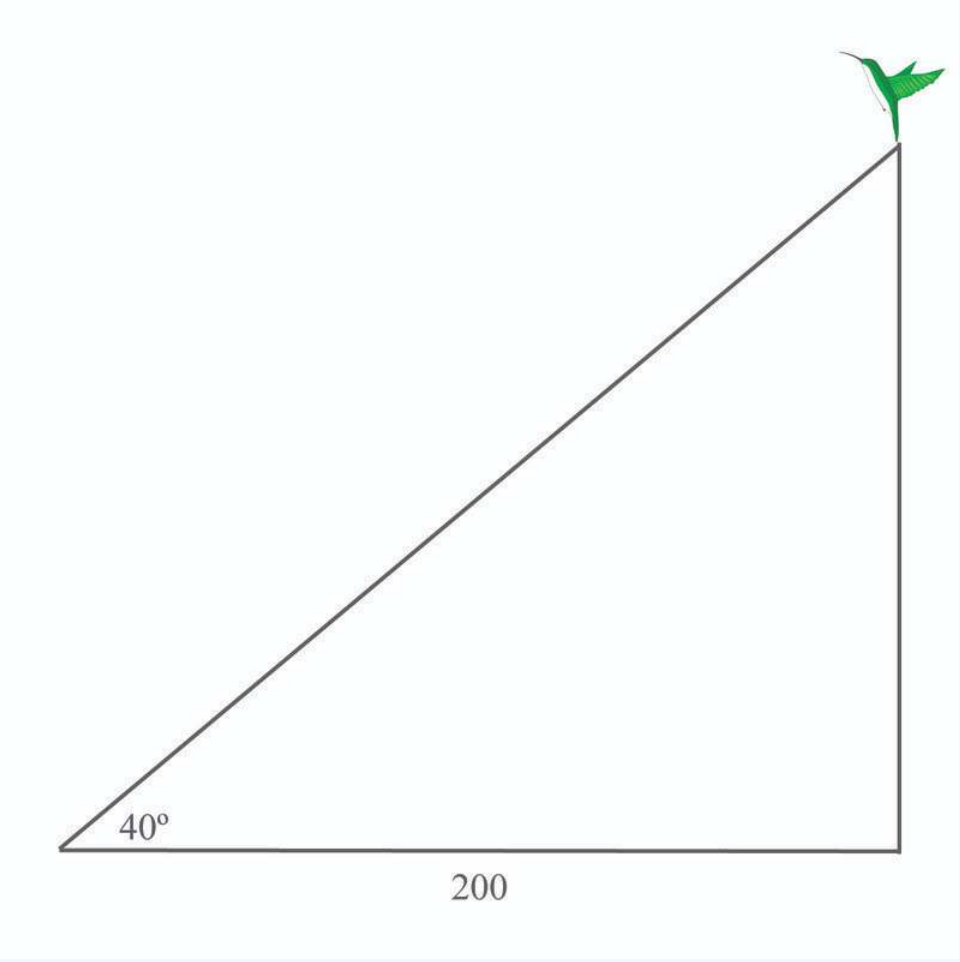


Figure 2.2.7.8

Solution

We can use the tangent function to find out how high the bird is in the sky:

$$\begin{aligned} tan40^\circ &= \frac{height}{200} \\ height &= 200tan40^\circ \\ height &= (200)(.839) \\ height &= 167.8 \end{aligned}$$

We then need to add your height to the solution for the triangle. Since you are six feet tall, the total height of the bird in the sky is 173.8 feet.

Example 2.2.7.3

While out swimming one day you spot a coin at the bottom of the pool. The pool is ten feet deep, and the angle between the top of the water and the coin is 15°. How far away is the coin from you along the bottom of the pool?

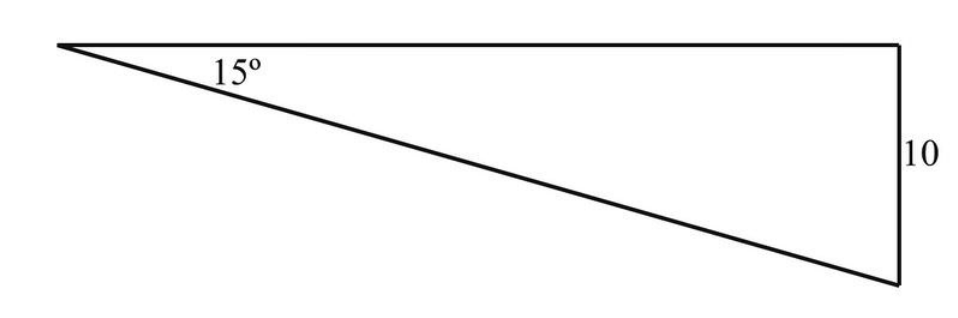


Figure **2.2.7.9**

Solution

Since the distance along the bottom of the pool to the coin is the same as the distance along the top of the pool to the coin, we can use the tangent function to solve for the distance to the coin:

$$tan15^\circ = \frac{\text{opposite}}{\text{adjacent}}$$
$$tan15^\circ = \frac{10}{x}$$
$$x = \frac{10}{tan15^\circ}$$
$$x \approx 37.32^\circ$$

Example 2.2.7.4

You are hiking and come to a cliff at the edge of a ravine. In the distance you can see your campsite at the base of the cliff, on the other side of the ravine. You know that the distance across the ravine is 500 meters, and the angle between your horizontal line of sight and your campsite is 25°. How high is the cliff? (Assume you are five feet tall.)

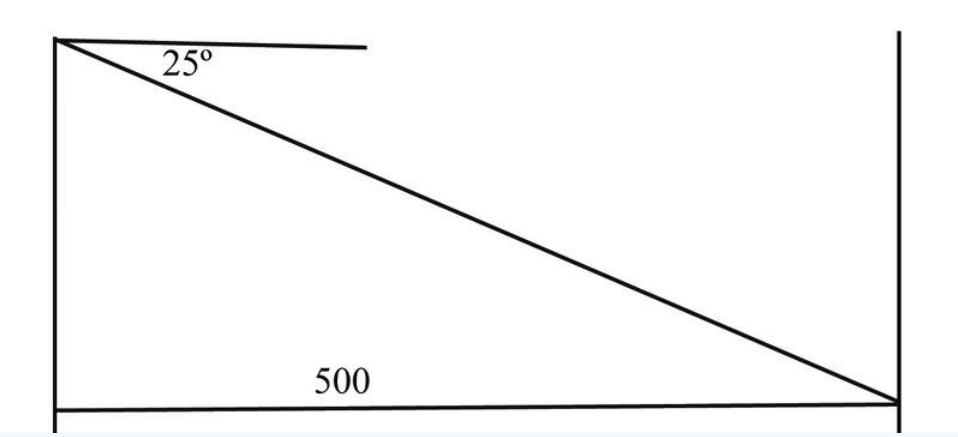


Figure **2.2.7.10**

Solution

Using the information given, we can construct a solution:

$$tan25^\circ = \frac{\text{opposite}}{\text{adjacent}}$$
$$tan25^\circ = \frac{height}{500}$$
$$height = \frac{500}{tan25^\circ}$$
$$height = (500)(.466)$$
$$height = 233 \text{ meters}$$

This is the total height from the bottom of the ravine to your horizontal line of sight. Therefore, to get the height of the ravine, you should take away five feet for your height, which gives an answer of 228 meters.

Review

1. A 70 foot building casts an 50 foot shadow. What is the angle that the sun hits the building?
2. You are standing 10 feet away from a tree, and you measure the angle of elevation to be 65°. How tall is the tree? Assume you are 5 feet tall up to your eyes.
3. Kaitlyn is swimming in the ocean and notices a coral reef below her. The angle of depression is 35° and the depth of the ocean, at that point is 350 feet. How far away is she from the reef?
4. The angle of depression from the top of a building to the base of a car is 60°. If the building is 78 ft tall, how far away is the car?
5. The Leaning Tower of Pisa currently "leans" at a 4° angle and has a vertical height of 55.86 meters. How tall was the tower when it was originally built?
6. The angle of depression from the top of an apartment building to the base of a fountain in a nearby park is 72°. If the building is 78 ft tall, how far away is the fountain?
7. You are standing 15 feet away from a tree, and you measure the angle of elevation to be 35°. How tall is the tree? Assume you are 5 feet tall up to your eyes.
8. Bill spots a tree directly across the river from where he is standing. He then walks 18 ft upstream and determines that the angle between his previous position and the tree on the other side of the river is 55°. How wide is the river?
9. A 50 foot building casts an 50 foot shadow. What is the angle that the sun hits the building?
10. Eric is flying his kite one afternoon and notices that he has let out the entire 100 ft of string. The angle his string makes with the ground is 60°. How high is his kite at this time?
11. A tree struck by lightning in a storm breaks and falls over to form a triangle with the ground. The tip of the tree makes a 36° angle with the ground 25 ft from the base of the tree. What was the height of the tree to the nearest foot?
12. Upon descent an airplane is 15,000 ft above the ground. The air traffic control tower is 200 ft tall. It is determined that the angle of elevation from the top of the tower to the plane is 15°. To the nearest mile, find the ground distance from the airplane to the tower.
13. Tara is trying to determine the angle at which to aim her sprinkler nozzle to water the top of a 10 ft bush in her yard. Assuming the water takes a straight path and the sprinkler is on the ground 4 ft from the tree, at what angle of inclination should she set it?
14. Over 3 miles (horizontal), a road rises 1000 feet (vertical). What is the angle of elevation?
15. Over 4 miles (horizontal), a road rises 1000 feet (vertical). What is the angle of elevation?

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.13.

Vocabulary

Term	Definition
Angle of Depression	The angle of depression is the angle formed by a horizontal line and the line of sight down to an object when the image of an object is located beneath the horizontal line.
Angle of Elevation	The angle of elevation is the angle formed by a horizontal line and the line of sight up to an object when the image of an object is located above the horizontal line.

Additional Resources

Video: Example: Determine What Trig Function Relates Specific Sides of a Right Triangle

Practice: Angles of Elevation and Depression

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2.3: Trig in the Unit Circle

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2.3.1: Trigonometry and the Unit Circle

Determine exact values of trig ratios for common radian measures

The **unit circle** is a circle of radius one, centered at the origin, that summarizes all the 30-60-90 and 45-45-90 triangle relationships that exist. When memorized, it is extremely useful for evaluating expressions like $\cos(135°)$ or $\sin\left(-\frac{5\pi}{3}\right)$. It also helps to produce the parent graphs of sine and cosine.

How can you use the unit circle to evaluate $\cos(135°)$ and $\sin\left(-\frac{5\pi}{3}\right)$?

The Unit Circle

You already know how to translate between degrees and radians and the triangle ratios for 30-60-90 and 45-45-90 right triangles. In order to be ready to completely fill in and memorize a unit circle, two triangles need to be worked out. Start by finding the side lengths of a 30-60-90 triangle and a 45-45-90 triangle each with hypotenuse equal to 1.

$30°$	$60°$	$90°$
x	$x\sqrt{3}$	$2x$
$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1

$45°$	$45°$	$90°$
x	x	$x\sqrt{2}$
$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1

This is enough information to fill out the important points in the first **quadrant** of the unit circle. The values of the x and y coordinates for each of the key points are shown below. Remember that the x and y coordinates come from the lengths of the legs of the special right triangles, as shown specifically for the $30°$ angle. Always remember to measure the angle from the positive portion of the x-axis.

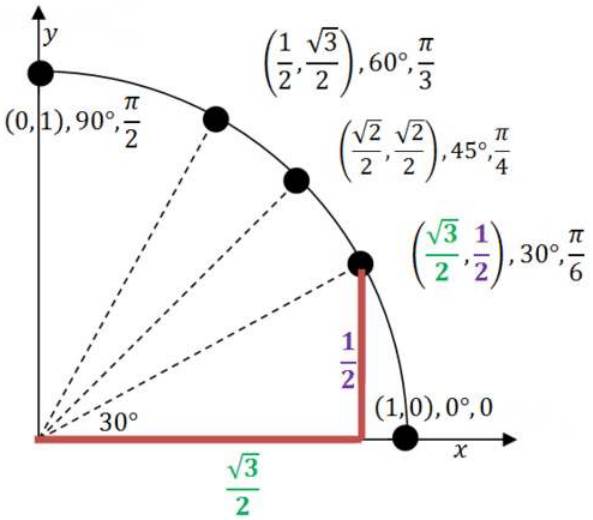


Figure **2.3.1.1**

Knowing the first quadrant well is the key to knowing the entire unit circle. Every other point on the unit circle can be found using logic and this quadrant, so there is no need to memorize the whole circle.

To use your knowledge of the first quadrant of the unit circle to identify the angles and important points of the second quadrant, notice that the heights are mirrored and equal which correspond to the y values. The x values are all negative.

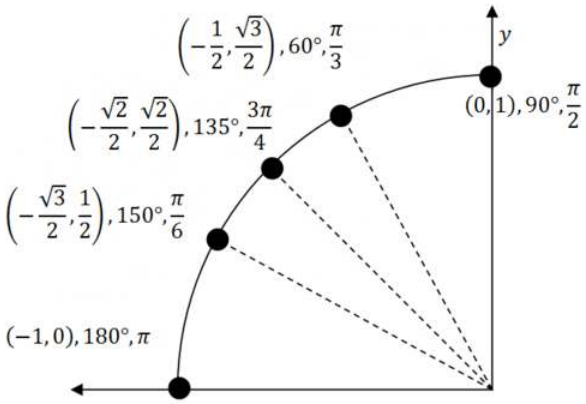


Figure **2.3.1.2**

There is a pattern in the heights of the points in the first quadrant that can help you remember the points.

Notice that the heights of the points in the first quadrant are the y-coordinates: $0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1$

When rewritten, the pattern becomes clear: $\frac{0}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{2}$.

The three points in the middle are the most often mixed up. This pattern illustrates how they increase in size from small 12, to medium $\frac{\sqrt{2}}{2}$, to large $\frac{\sqrt{3}}{2}$. When you fill in the unit circle, look for the heights that are small, medium and large and this will tell you were each value should go. Notice that the heights for these five points in the second quadrant are also $0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1$.

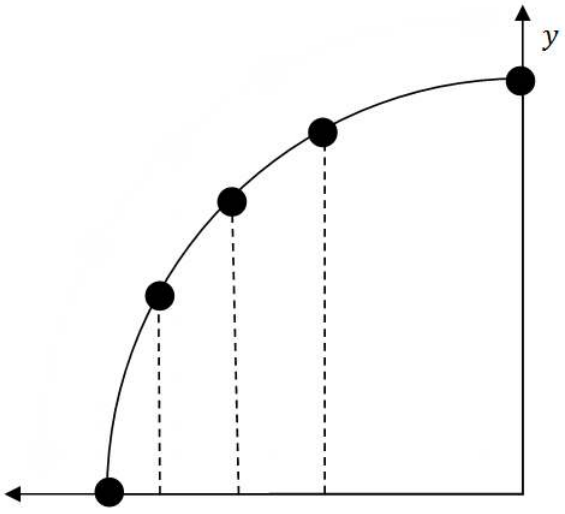
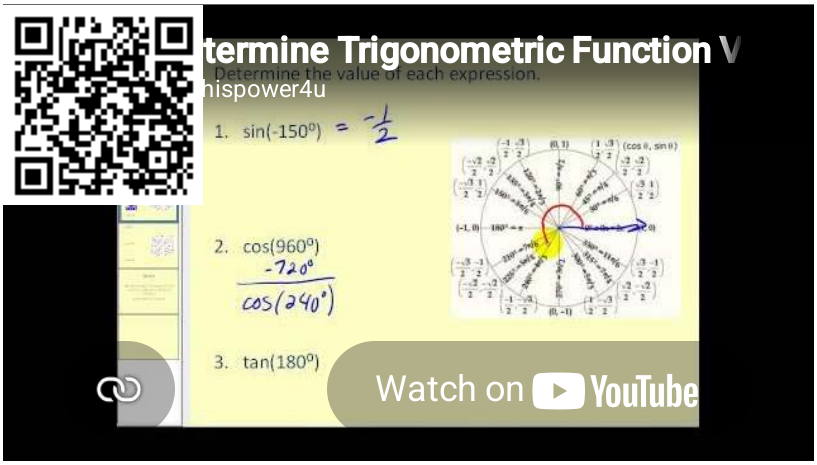


Figure **2.3.1.3**

This technique also works for the widths. This can make memorizing the 16 points of the unit circle a matter of logic and the pattern: $\frac{0}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{2}$.

One last item to note is that **coterminal angles** are sets of angles such as $90°$, $450°$, and $-270°$ that start at the positive x-axis and end at the same terminal side. Since **coterminal** angles end at identical points along the unit circle, trigonometric expressions involving coterminal angles are equivalent: $\sin(90) = \sin(450) = \sin(-270)$.



Example 2.3.1.1

Earlier, you were asked how you can use the unit circle to evaluate $\cos(135)$ and Missing \left or extra \right. The x value of a point along the unit circle corresponds to the cosine of the angle. The y value of a point corresponds to the sine of the angle. When the angles and points are memorized, simply recall the x or y coordinate. If the construction of the unit circle is understood, it becomes easier to determine the coordinates.

Solution

When evaluating $\cos(135°)$ your thought process should be something like this:

You know $135°$ goes with the point $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and cosine is the x portion. So, $\cos(135°) = -\frac{\sqrt{2}}{2}$.

When evaluating $\sin\left(-\frac{5\pi}{3}\right)$ your thought process should be something like this:

You know $-\frac{5\pi}{3}$ goes with the point $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ and sine is the y portion. So, $\sin\left(-\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

Example 2.3.1.2

Evaluate $\cos 60°$ using the unit circle and right triangle trigonometry. What is the connection between the x coordinate of the point and the cosine of the angle?

Solution

The point on the unit circle for $60°$ is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and the point is one unit from the origin. This can be represented as a 30-60-90 triangle.

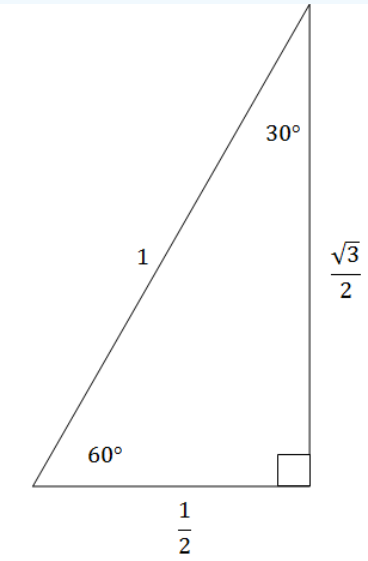


Figure **2.3.1.4**

Since cosine is adjacent over hypotenuse, cosine turns out to be exactly the x coordinate $\frac{1}{2}$.

Example 2.3.1.3

Using knowledge of the first quadrant of the unit circle, identify the angles and important points of the third quadrant.

Solution

Both the x values and y values are negative and their respective coordinates correspond to those of the other quadrants.

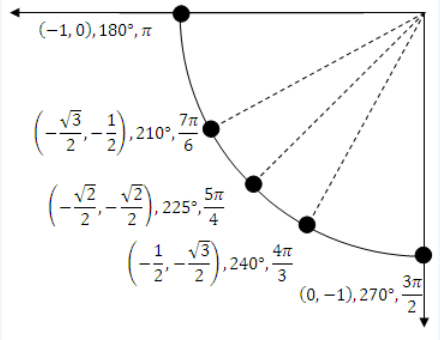


Figure **2.3.1.5**

Example 2.3.1.4

For each of the six trigonometric functions, identify the quadrants where they are positive and the quadrants where they are negative.

Solution

In quadrant I, the hypotenuse, adjacent and opposite side are all positive. Thus all 6 trigonometric functions are positive.

In quadrant II the hypotenuse and opposite sides are positive and the adjacent side is negative. This means that every trigonometric expression involving an adjacent side is negative. Sine and its reciprocal cosecant are the only two trigonometric functions that do not refer to the adjacent side which makes them the only positive ones.

In quadrant III only the hypotenuse is positive. Thus the only trigonometric functions that are positive are tangent and its reciprocal cotangent because these functions refer to both adjacent and opposite sides which will both be negative.

In quadrant IV the hypotenuse and the adjacent sides are positive while the opposite side is negative. This means that only cosine and its reciprocal secant are positive.

A mnemonic device to remember which trigonometric functions are positive and which trigonometric functions are negative is "All Students Take Calculus." All refers to all the trigonometric functions are positive in quadrant I. The letter S refers to sine and its reciprocal cosecant that are positive in quadrant II. The letter T refers to tangent and its reciprocal cotangent that are positive in quadrant III. The letter C refers to cosine and its reciprocal secant that are positive in quadrant IV.

Example 2.3.1.5

Evaluate the following trigonometric expressions using the unit circle.

Solution

1. $\sin \frac{\pi}{2}$

$\sin \frac{\pi}{2} = 1$

2. $\cos 210^\circ$

$\cos 210^\circ = -\frac{\sqrt{3}}{2}$

3. $\tan 315^\circ$

$\tan 315^\circ = -1$

4. $\cot 270^\circ$

$\cot 270^\circ = 0$

5. $\sec \frac{11\pi}{6}$

$\sec \frac{11\pi}{6} = \frac{1}{\cos \frac{11\pi}{6}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

6. $\csc \frac{-5\pi}{4}$

\(\csc\dfrac{-5\pi}{4} =\dfrac{1\sin\dfrac{-5\pi}{4} =−\dfrac{2}{\sqrt{2}}=−\sqrt{2}\)

Review

Name the angle between 0^{\circ} and 360^{\circ} that is coterminal with…

1. -20°
2. 475°
3. -220°
4. 690°
5. -45°

Use your knowledge of the unit circle to help determine whether each of the following trigonometric expressions is positive or negative.

6. $\tan 143^\circ$
7. $\cos \frac{\pi}{3}$
8. $\sin 362^\circ$
9. $\csc \frac{3\pi}{4}$

Use your knowledge of the unit circle to evaluate each of the following trigonometric expressions.

10. $\cos 120^\circ$
11. $\sec \pi 3$
12. $\tan 225^\circ$
13. $\cot 120^\circ$
14. $\sin \frac{11\pi}{6}$
15. $\csc 240^\circ$
16. Find $\sin\theta$ and $\tan\theta$ if $\cos\theta = \frac{\sqrt{3}}{2}$ and $\cot\theta > 0$.
17. Find $\cos\theta$ and $\tan\theta$ if $\sin\theta = -\frac{1}{2}$ and $\sec\theta < 0$.
18. Draw the complete unit circle (all four quadrants) and label the important points.

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.1.

Vocabulary

Term	Definition
Coterminal	Two angles are coterminal if they are drawn in the standard position and both have terminal sides that are at the same location.
Coterminal Angles	A set of coterminal angles are angles with the same terminal side but expressed differently, such as a different number of complete rotations around the unit circle or angles being expressed as positive versus negative angle measurements.
quadrant	A quadrant is one-fourth of the coordinate plane. The four quadrants are numbered using Roman Numerals I, II, III, and IV, starting in the top-right, and increasing counter-clockwise.
unit circle	The unit circle is a circle of radius one, centered at the origin.

Additional Resources

Video: Evaluating Trigonometric Functions (The Unit Circle) - Overview

Practice: Trigonometry and the Unit Circle

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2.3.2: Measuring Rotation

Explore trig ratios of angles greater than 90 degrees

Introduction to Angles of Rotation, Coterminal Angles, and Reference Angles

In which quadrant does the terminal side of the angle $-500°$ lie and what is the reference angle for this angle?

Angles of Rotation

Angles of rotation are formed in the coordinate plane between the positive x-axis **(initial side)** and a ray **(terminal side)**. Positive angle measures represent a counterclockwise rotation while negative angles indicate a clockwise rotation.



Figure **2.3.2.1**

Since the x and y axes are perpendicular, each axis then represents an increment of ninety degrees of rotation. The diagrams below show a variety of angles formed by rotating a ray through the quadrants of the coordinate plane.

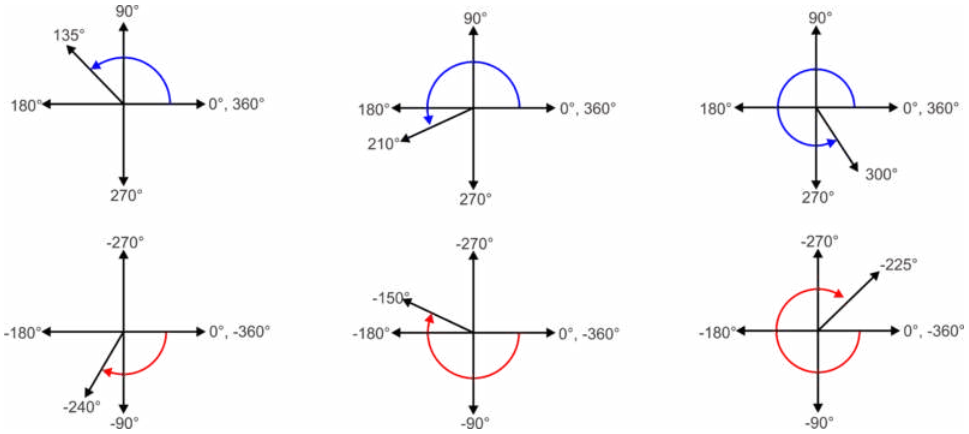


Figure **2.3.2.2**

An angle of rotation can be described infinitely many ways. It can be described by a positive or negative angle of rotation or by making multiple full circle rotations through $360°$. The example below illustrates this concept.

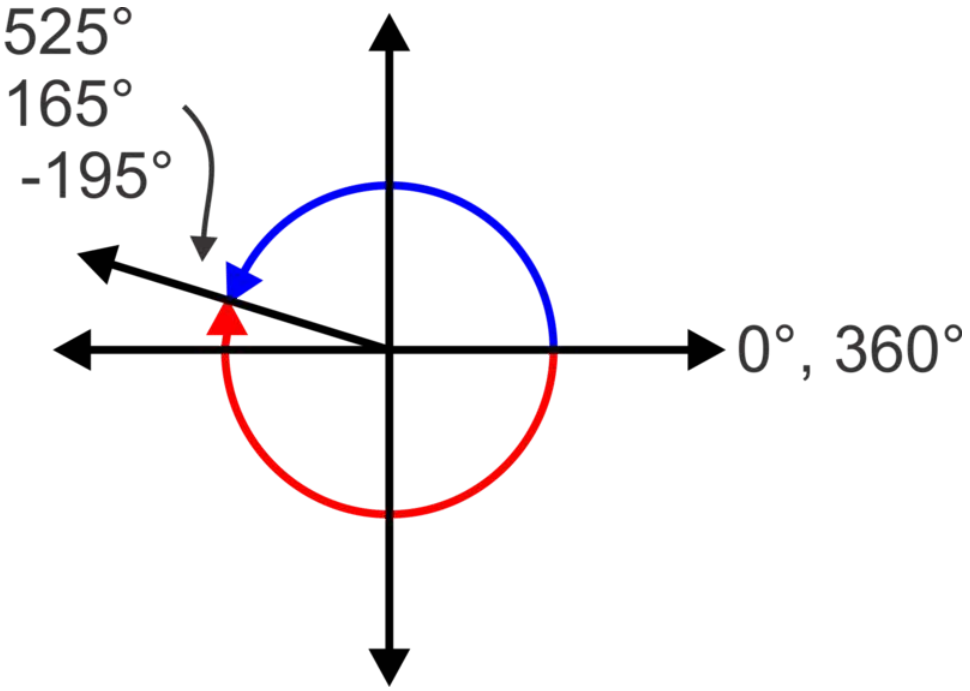
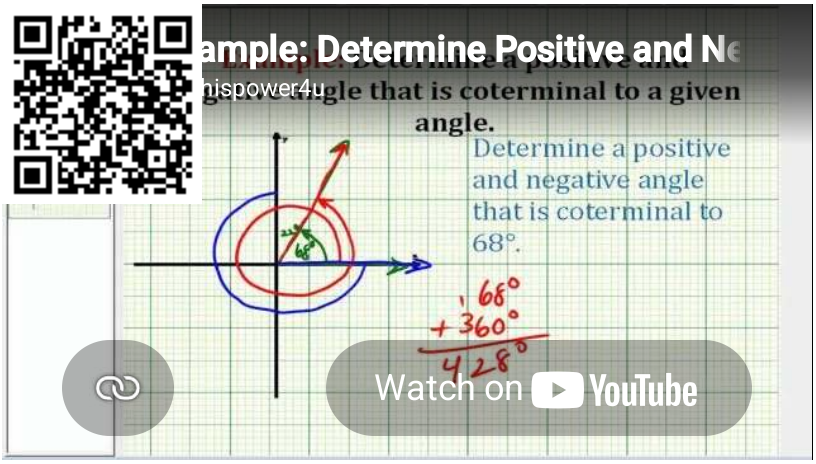


Figure **2.3.2.3**

For the angle $525°$, an entire $360°$ rotation is made and then we keep going another $165°$ to $525°$. Therefore, the resulting angle is equivalent to $525° - 360°$, or $165°$. In other words, the terminal side is in the same location as the terminal side for a $165°$ angle. If we subtract $360°$ again, we get a negative angle, $-195°$. Since they all share the same terminal side, they are called **coterminal angles**.

Let's determine two coterminal angles to $837°$, one positive and one negative.

To find coterminal angles we simply add or subtract $360°$ multiple times to get the angles we desire. $837° - 360° = 477°$, so we have a positive coterminal angle. Now we can subtract $360°$ again to get $477° - 360° = 117°$.



Reference Angle

A **reference angle** is the acute angle between the terminal side of an angle and the x– axis. The diagram below shows the reference angles for terminal sides of angles in each of the four quadrants.

Note: A reference angle is never determined by the angle between the terminal side and the y – axis. This is a common error for students, especially when the terminal side appears to be closer to the y – axis than the x – axis.

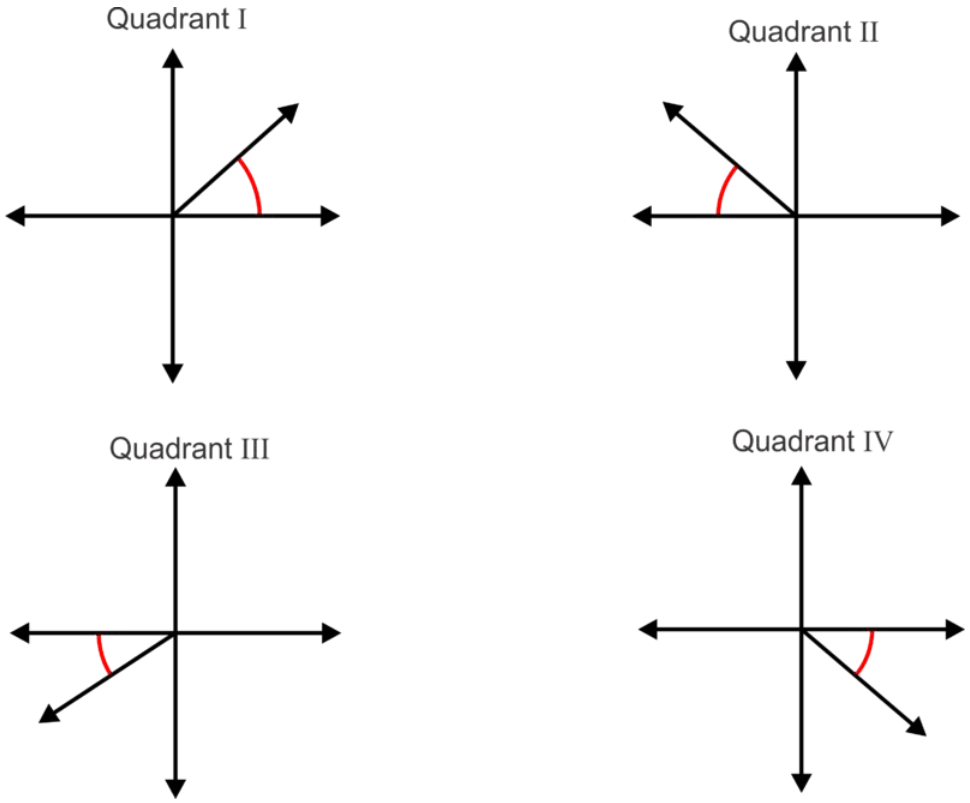


Figure 2.3.2.4

Now, let's determine the quadrant in which -745° lies and hence determine the reference angle.

Since our angle is more than one rotation, we need to add 360° until we get an angle whose absolute value is less than \)360^\circ$: $-745^\circ + 360^\circ = -385^\circ$, again $-385^\circ + 360^\circ = -25^\circ$.

Now we can plot the angle and determine the reference angle:

Note that the reference angle is positive 25°. All reference angles will be positive as they are acute angles (between 0° and 90°).

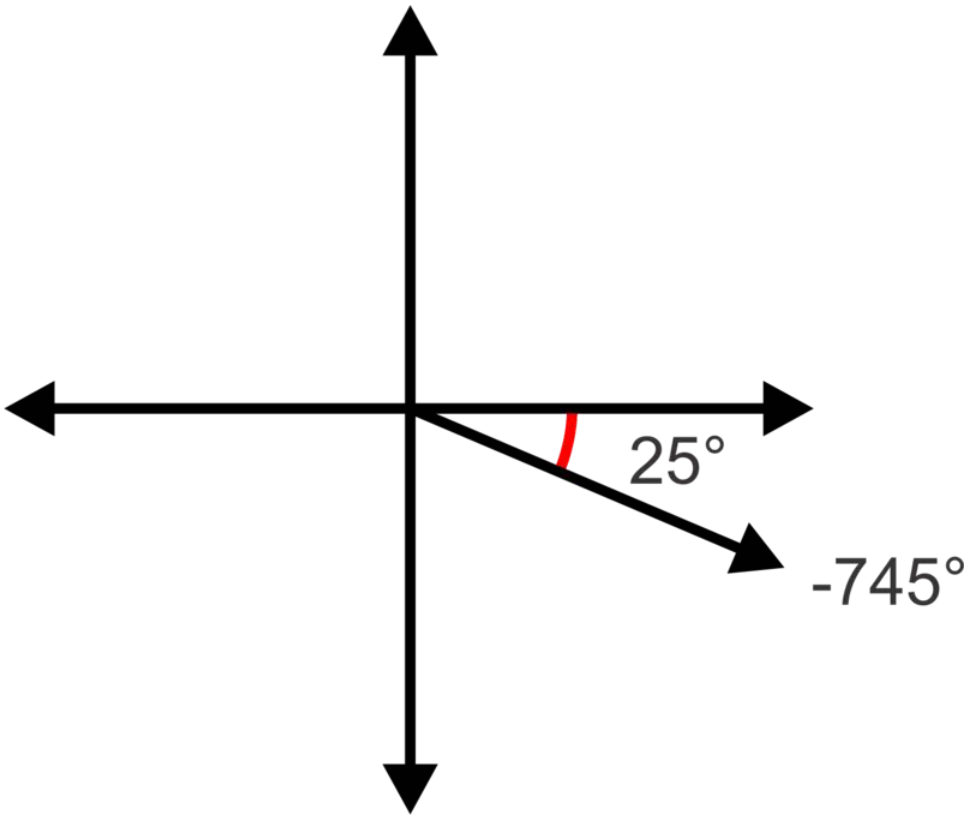


Figure 2.3.2.5

Finally, let's give two coterminal angles to $595°$, one positive and one negative, and find the reference angle.

To find the coterminal angles we can add/subtract $360°$. In this case, our angle is greater than $360°$ so it makes sense to subtract $360°$ to get a positive coterminal angle: $595° - 360° = 235°$. Now subtract again to get a negative angle: $235° - 360° = -125°$.

By plotting any of these angles we can see that the terminal side lies in the third quadrant as shown.

Since the terminal side lies in the third quadrant, we need to find the angle between $180°$ and $235°$, so $235° - 180° = 55°$.

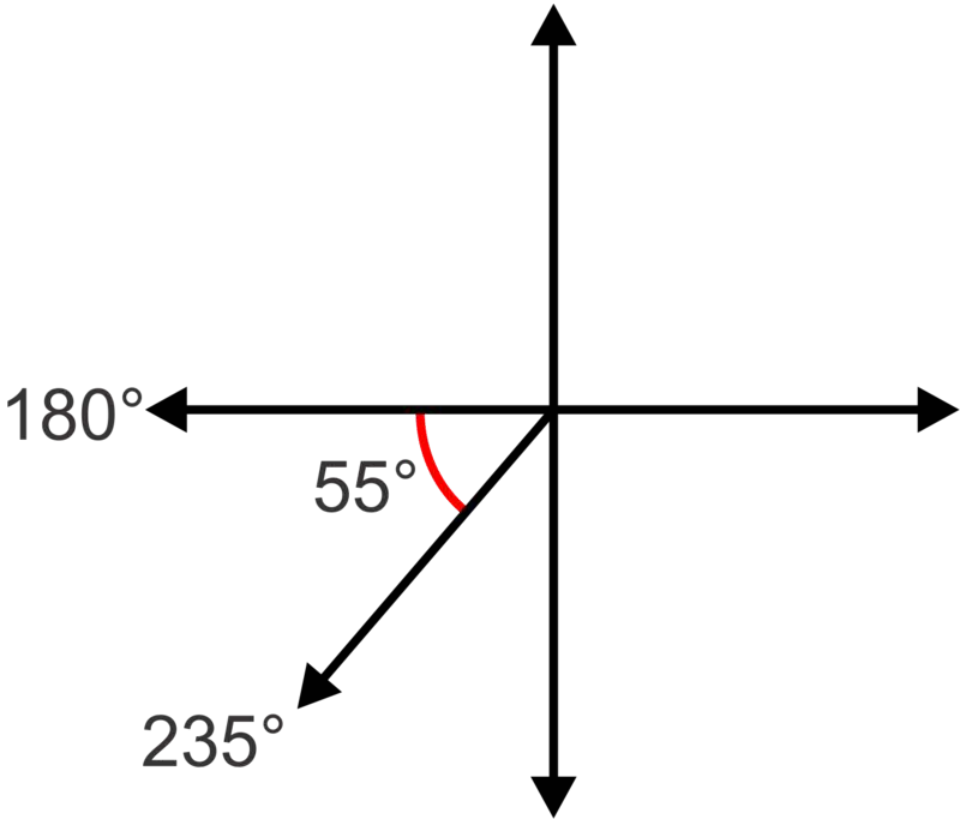
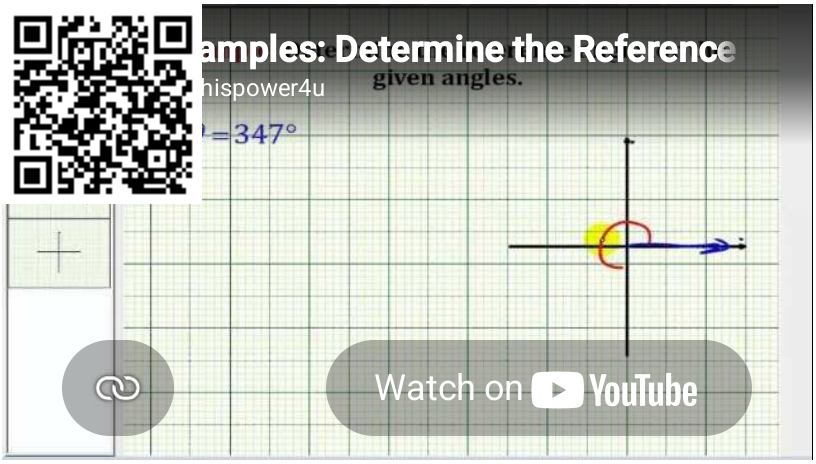


Figure 2.3.2.6



Example 2.3.2.1

Earlier, you were asked to find the reference angle of -500° and find the quadrant in which the terminal side lies.

Solution

Since our angle is more than one rotation, we need to add 360° until we get an angle whose absolute value is less than 360°: $-500^\circ + 360^\circ = -200^\circ$.

If we plot this angle we see that it is -200° clockwise from the origin or 160° counterclockwise. 160° lies in the second quadrant.

Now determine the reference angle: $180^\circ - 160^\circ = 20^\circ$.

Example 2.3.2.2

Find two coterminal angles to $138°$, one positive and one negative.

Solution

$138° + 360° = 498°$ and $138° - 360° = -222°$

Example 2.3.2.3

Find the reference angle for $895°$.

Solution

$895° - 360° = 535°$, $535° - 360° = 175°$. The terminal side lies in the second quadrant, so we need to determine the angle between $175°$ and $180°$, which is $5°$.

Example 2.3.2.4

Find the reference angle for $343°$.

Solution

$343°$ is in the fourth quadrant so we need to find the angle between $343°$ and $360°$ which is $17°$.

Review

Find two coterminal angles to each angle measure, one positive and one negative.

1. $-98°$
2. $475°$
3. $-210°$
4. $47°$
5. $-1022°$
6. $354°$
7. $-7°$

Determine the quadrant in which the terminal side lies and find the reference angle for each of the following angles.

8. $102°$
9. $-400°$
10. $1307°$
11. $-820°$
12. $304°$
13. $251°$
14. $-348°$
15. Explain why the reference angle for an angle between $0°$ and $90°$ is equal to itself.

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 13.5.

Additional Resources

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2.3.3: Angles of Rotation in Standard Positions

Counterclockwise angles beginning at the positive x-axis.

While playing a game with friends, you are using a spinner. You know that the best number to land on is 7. The spinner looks like this:

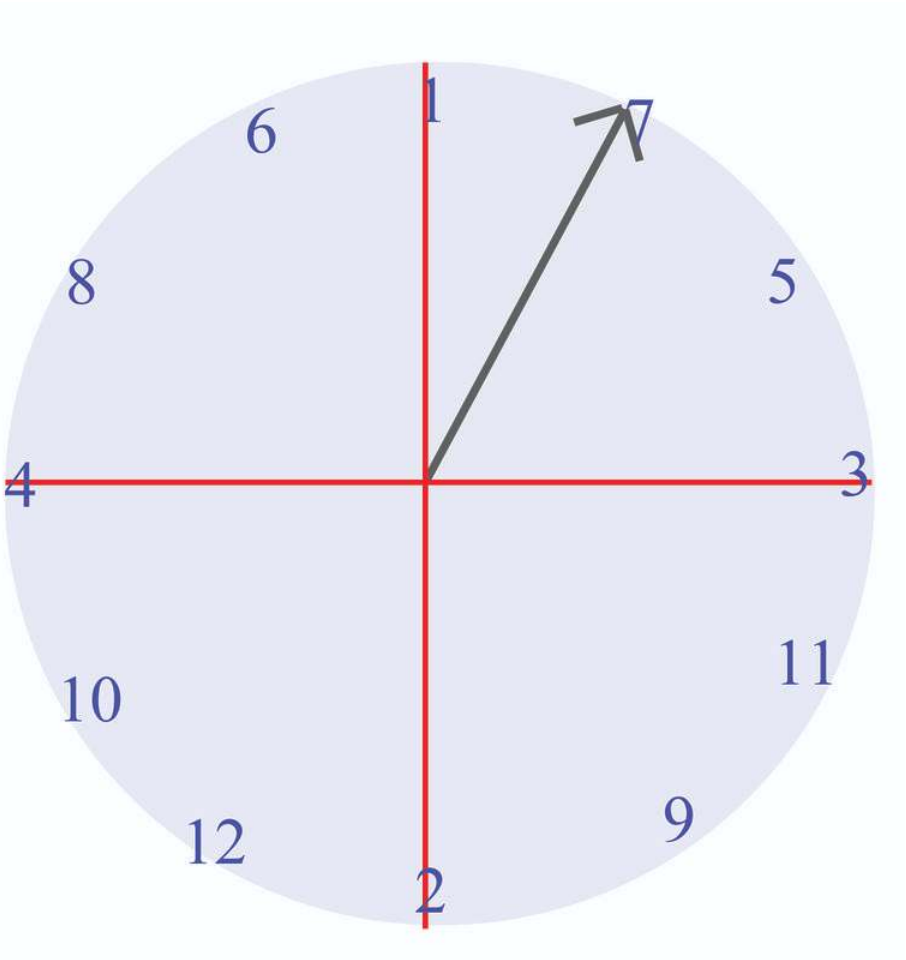
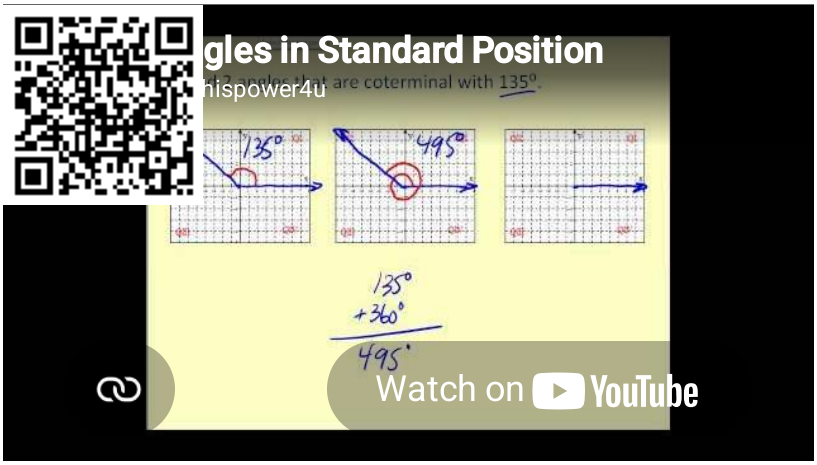


Figure **2.3.3.1**

Can you determine how to represent the angle of the spinner if it lands on the 7?

Angles of Rotation in Standard Positions

Consider our game that is played with a spinner. When you spin the spinner, how far has it gone? You can answer this question in several ways. You could say something like “the spinner spun around 3 times.” This means that the spinner made 3 complete rotations, and then landed back where it started.



We can also measure the rotation in degrees. In the previous lesson we worked with angles in triangles, measured in degrees. You may recall from geometry that a full rotation is 360 degrees, usually written as 360°. Half a rotation is then 180° and a quarter rotation is 90°. Each of these measurements will be important in this Concept. We can use our knowledge of graphing to represent any angle. The figure below shows an angle in what is called **standard position**.

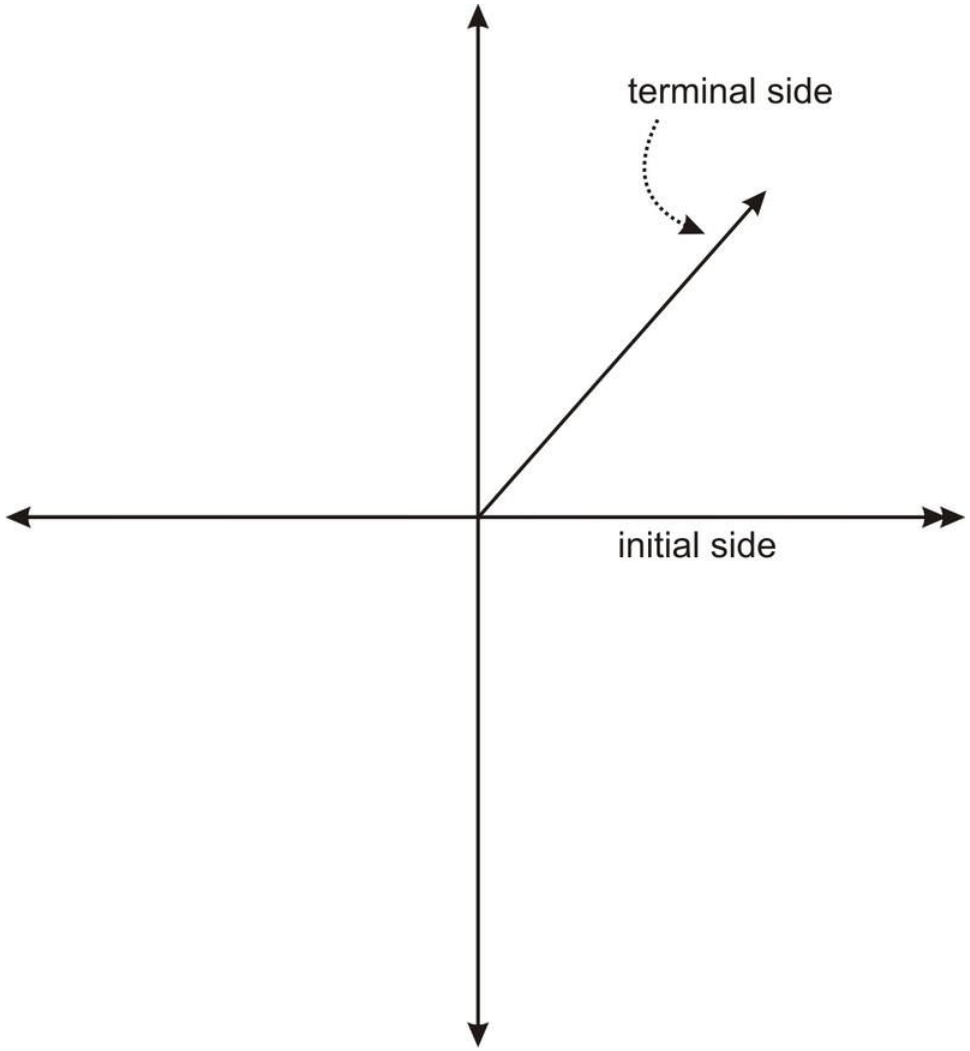


Figure **2.3.3.2**

The initial side of an angle in standard position is always on the positive x−axis. The terminal side always meets the initial side at the origin. Notice that the rotation goes in a **counterclockwise** direction. This means that if we rotate **clockwise**, we will generate a

negative angle. Below are several examples of angles in standard position.

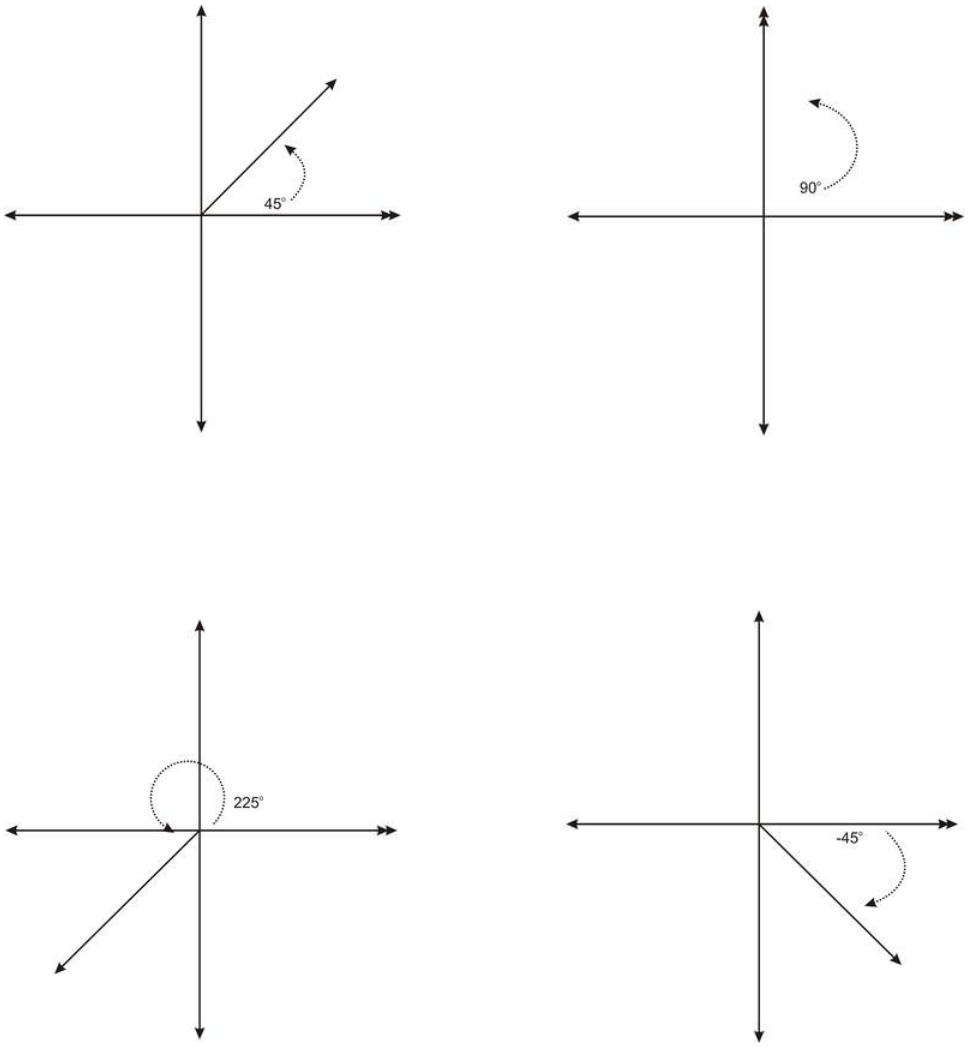


Figure **2.3.3.3**

The 90 degree angle is one of four **quadrantal** angles. A **quadrantal angle** is one whose terminal side lies on an axis. Along with $90°$, $0°$, $180°$ and $270°$ are quadrantal angles.

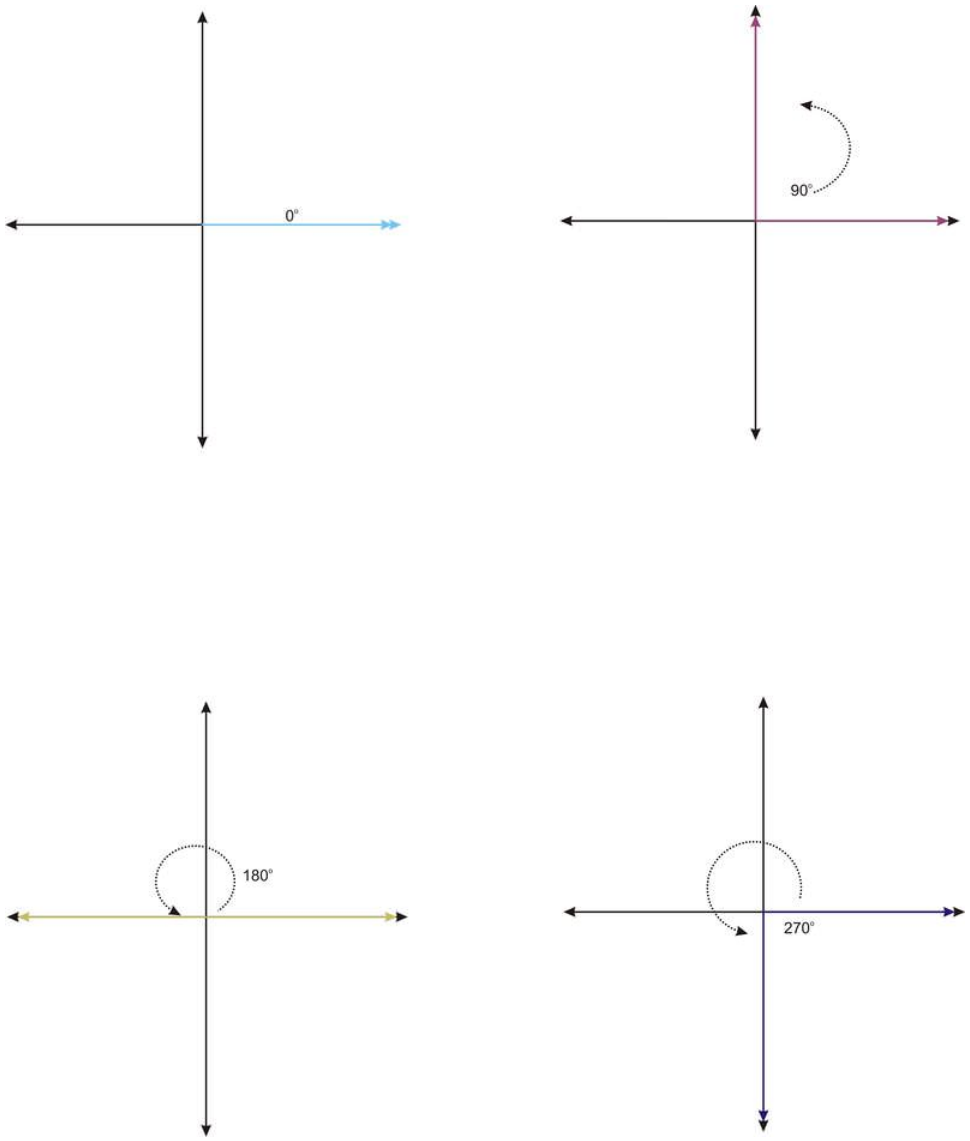


Figure **2.3.3.4**

These angles are referred to as quadrantal because each angle defines a quadrant. Notice that without the arrow indicating the rotation, 270° looks as if it is a -90°, defining the fourth quadrant. Notice also that 360° would look just like 0°.

Finding the Angle of Rotation

Identify what the angle is in this graph:

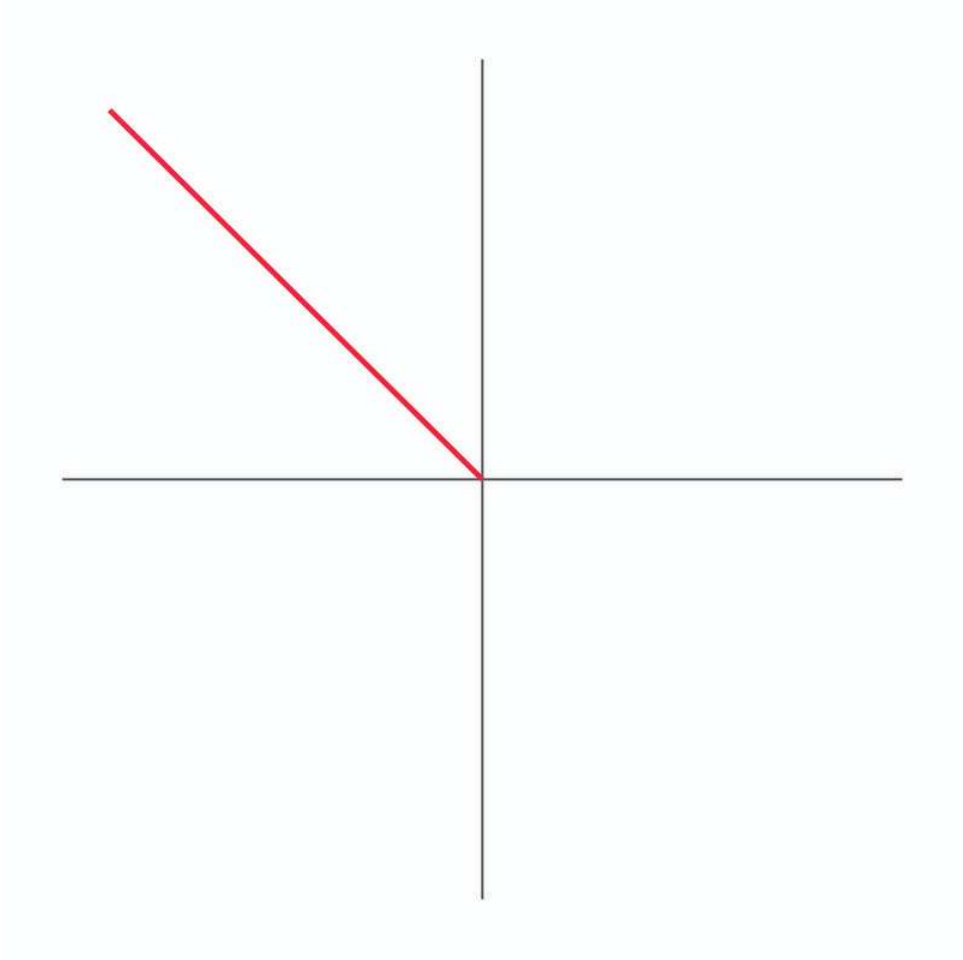


Figure **2.3.3.5**

The angle drawn out is **135°**.

Identifying Angles

Identifying the angles within the following graphs.

1.

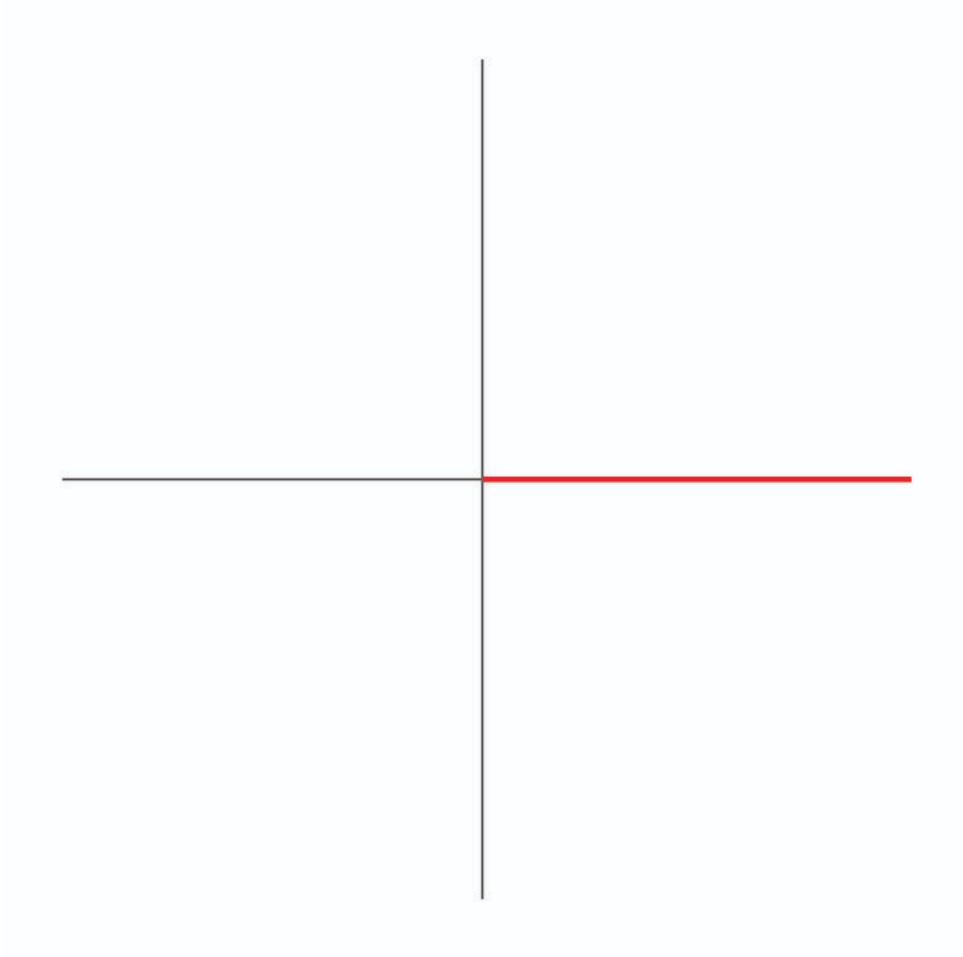


Figure **2.3.3.6**

The angle drawn out is **0°**.

2.

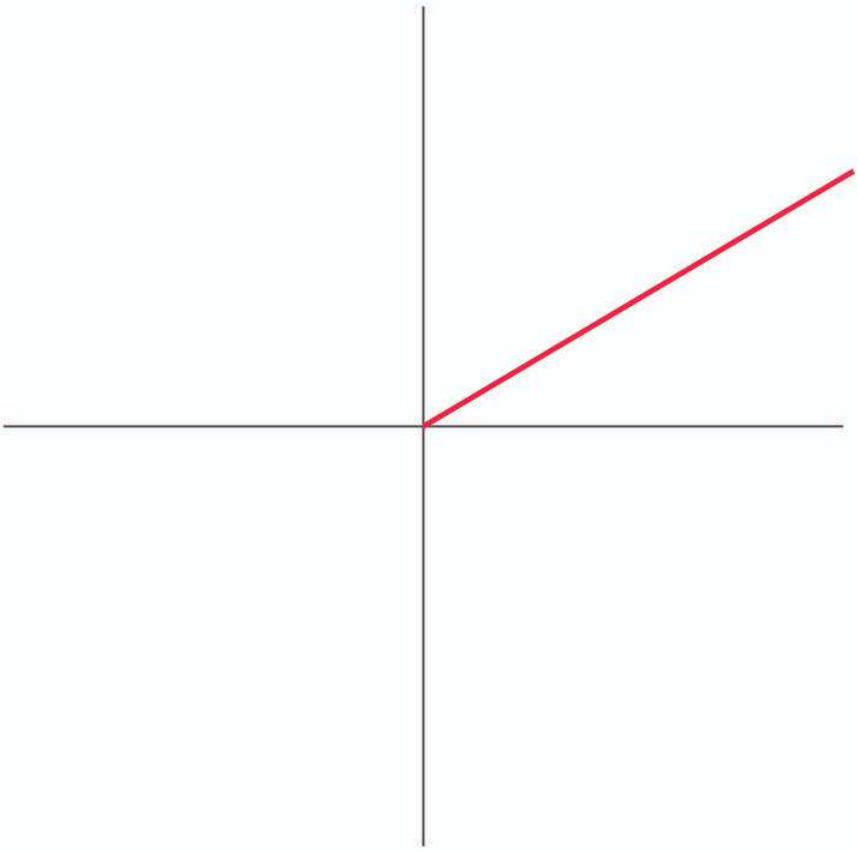


Figure 2.3.3.7

The angle drawn out is $30°$

Example 2.3.3.1

Earlier, you were asked to determine how to represent the angle of the spinner if it lands on the 7.

Solution

Since you know that the angle between the horizontal and vertical directions is $90°$, each number on the spinner takes up $30°$. Therefore, since you are on the 7, you know that you are 23 of the way to the vertical. Therefore, the angle of the spinner when it lands on 7 is $60°$.

Example 2.3.3.2

Identify what the angle is in this graph, using negative angles:

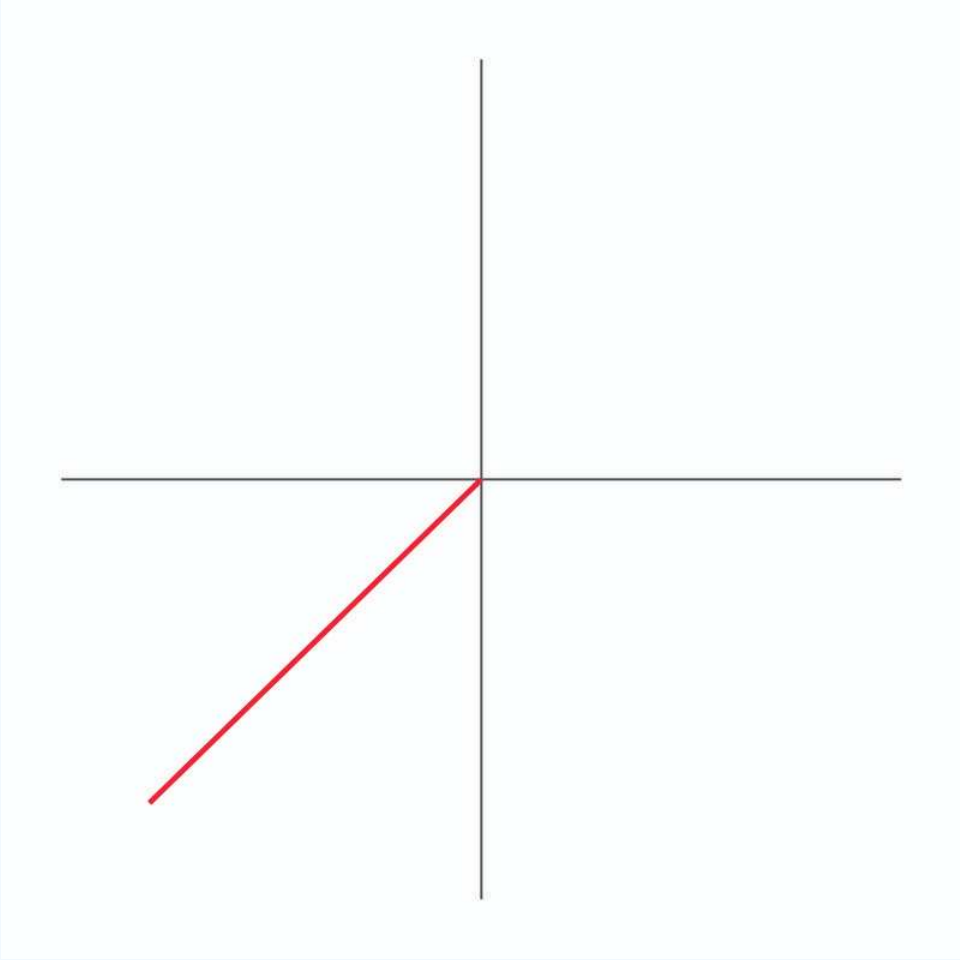


Figure 2.3.3.8

Solution

The angle drawn out is $-135°$.

Example 2.3.3.3

Identify what the angle is in this graph, using negative angles:

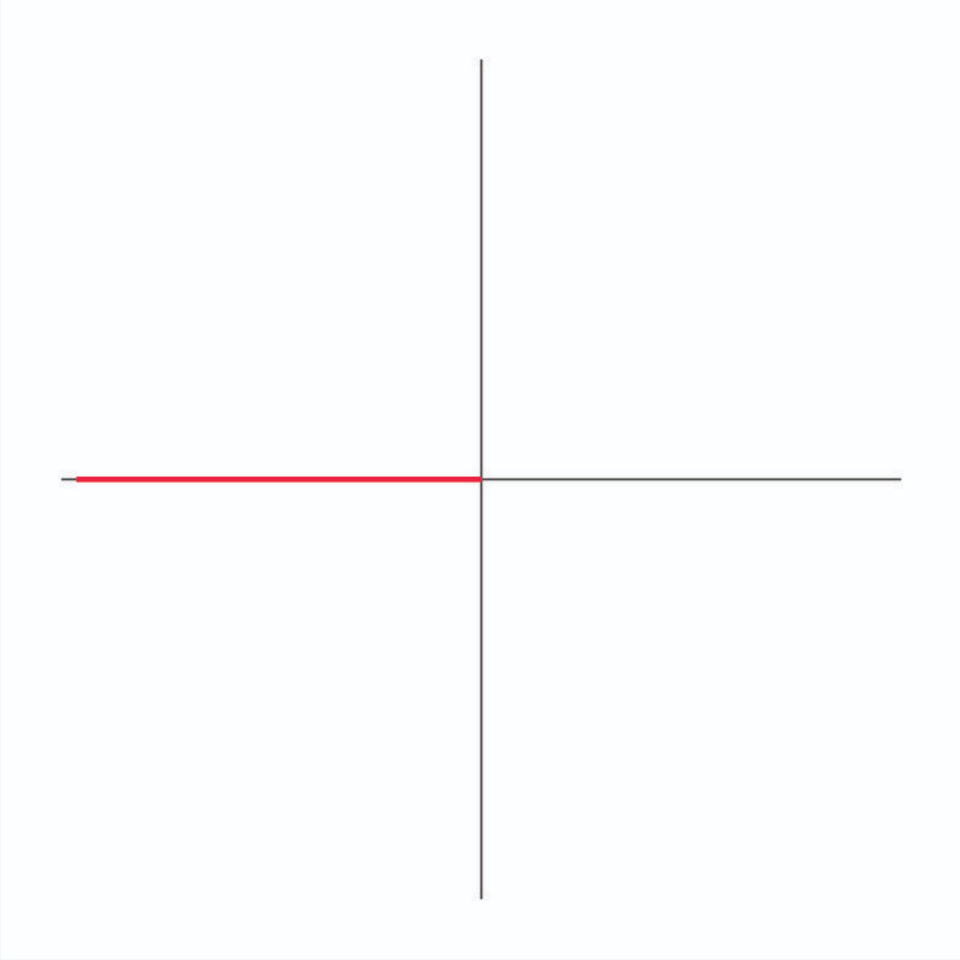


Figure 2.3.3.9

Solution

The angle drawn out is $-180°$.

Example 2.3.3.4

Identify what the angle is in this graph, using negative angles:

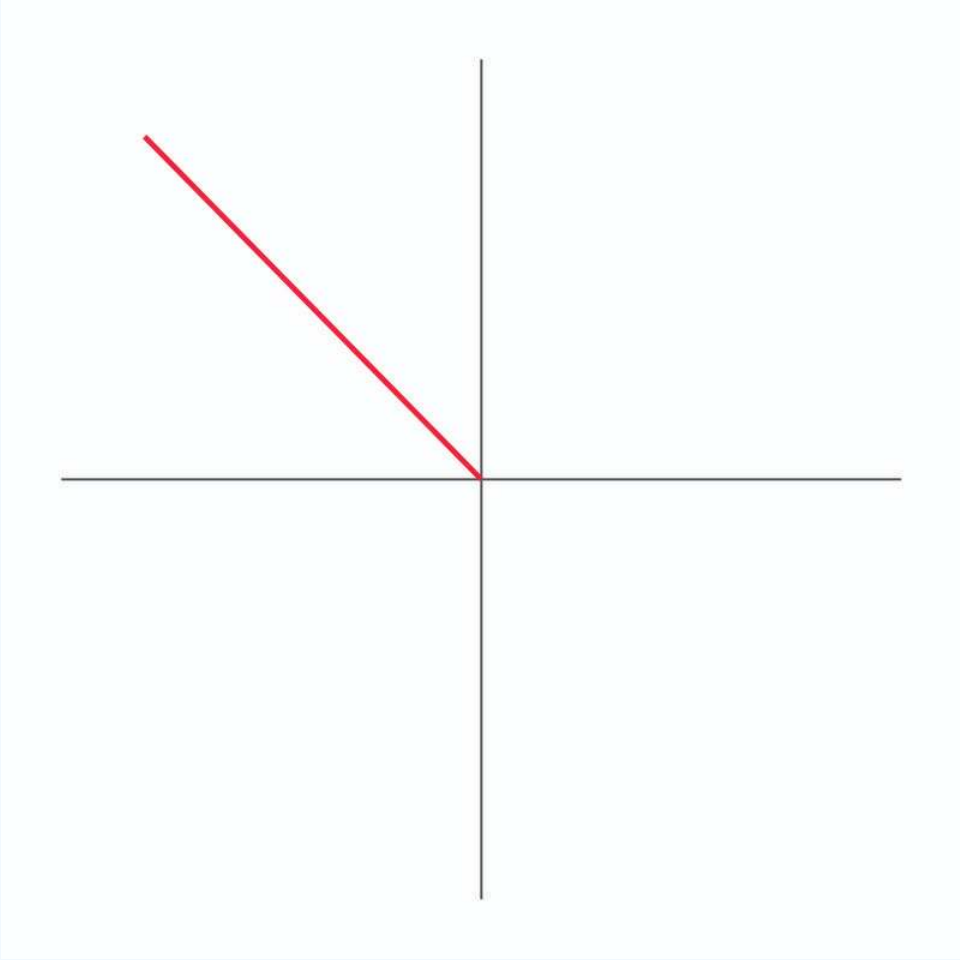


Figure 2.3.3.10

Solution

The angle drawn out is $-225°$.

Review

1. Draw an angle of $90°$.
2. Draw an angle of $45°$.
3. Draw an angle of $-135°$.
4. Draw an angle of $-45°$.
5. Draw an angle of $-270°$.
6. Draw an angle of $315°$.

For each diagram, identify the angle. Write the angle using positive degrees.

7.

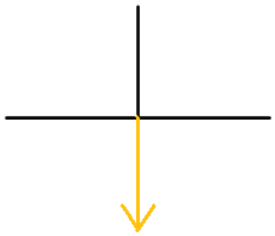


Figure 2.3.3.11

8.

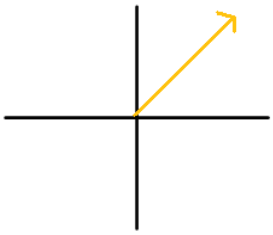


Figure 2.3.3.12

9.

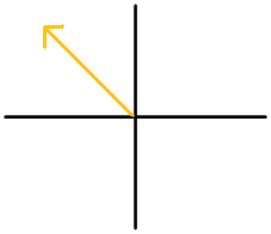


Figure 2.3.3.13

For each diagram, identify the angle. Write the angle using negative degrees.

10.

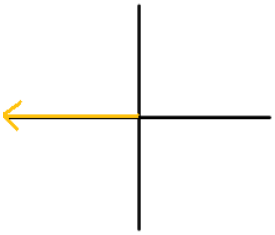


Figure 2.3.3.14

11.

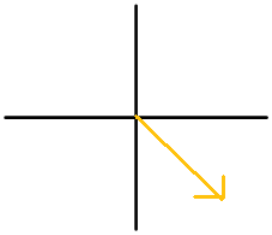


Figure 2.3.3.15

12.

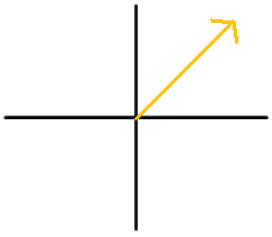


Figure 2.3.3.16

13. Explain how to convert between angles that use positive degrees and angles that use negative degrees.
14. At what angle is the 7 on a standard 12-hour clock? Use positive degrees.
15. At what angle is the 2 on a standard 12-hour clock? Use positive degrees.

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.15.

Vocabulary

Term	Definition
Quadrantal Angle	A quadrantal angle is an angle that has its terminal side on one of the four lines of axis: positive x, negative x, positive y or negative y.
Standard Position	The standard position of an angle measures an angle starting from the positive x-axis and going counter-clockwise. It is the typical method for drawing and measuring an angle.

Additional Resources

Video: Angles in Standard Position

Practice: Angles of Rotation in Standard Positions

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2.3.4: Coterminal Angles

Set of angles with the same terminal or end side.

While playing a game with friends, you use a spinner that looks like this:

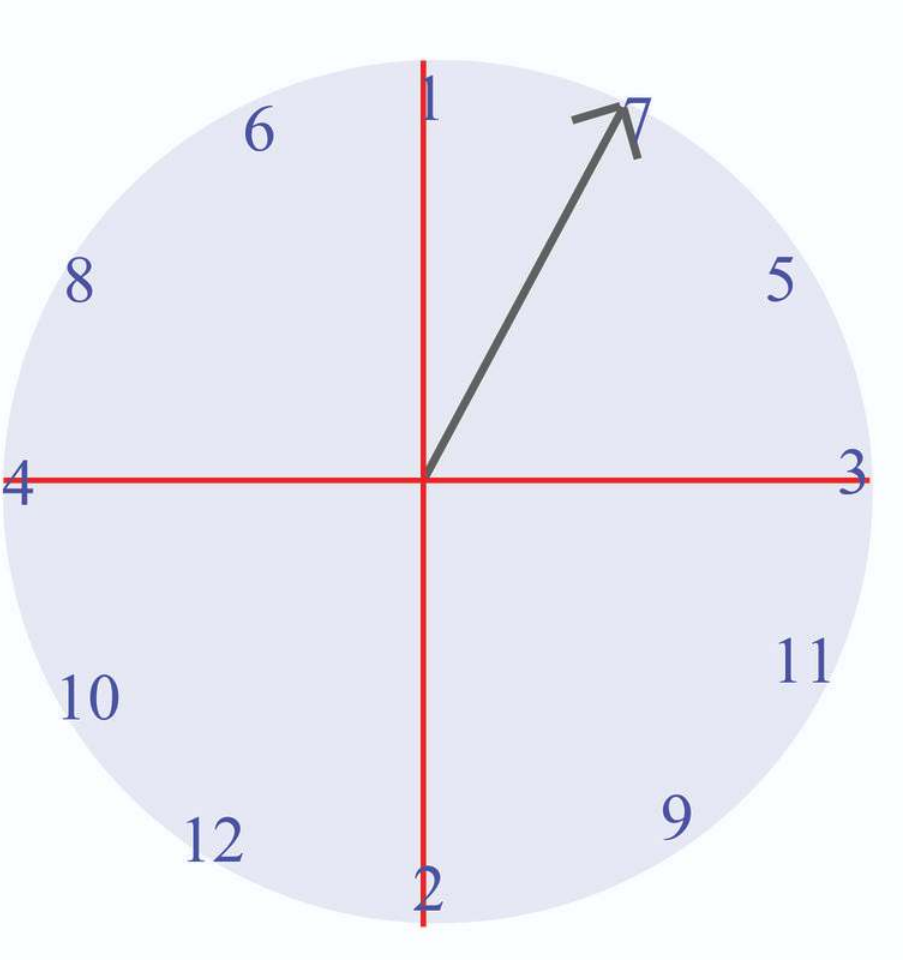


Figure **2.3.4.1**

As you can see, the angle that the spinner makes with the horizontal is $60°$. Is it possible to represent the angle any other way?

Coterminal Angles

Consider the angle $30°$, in standard position.

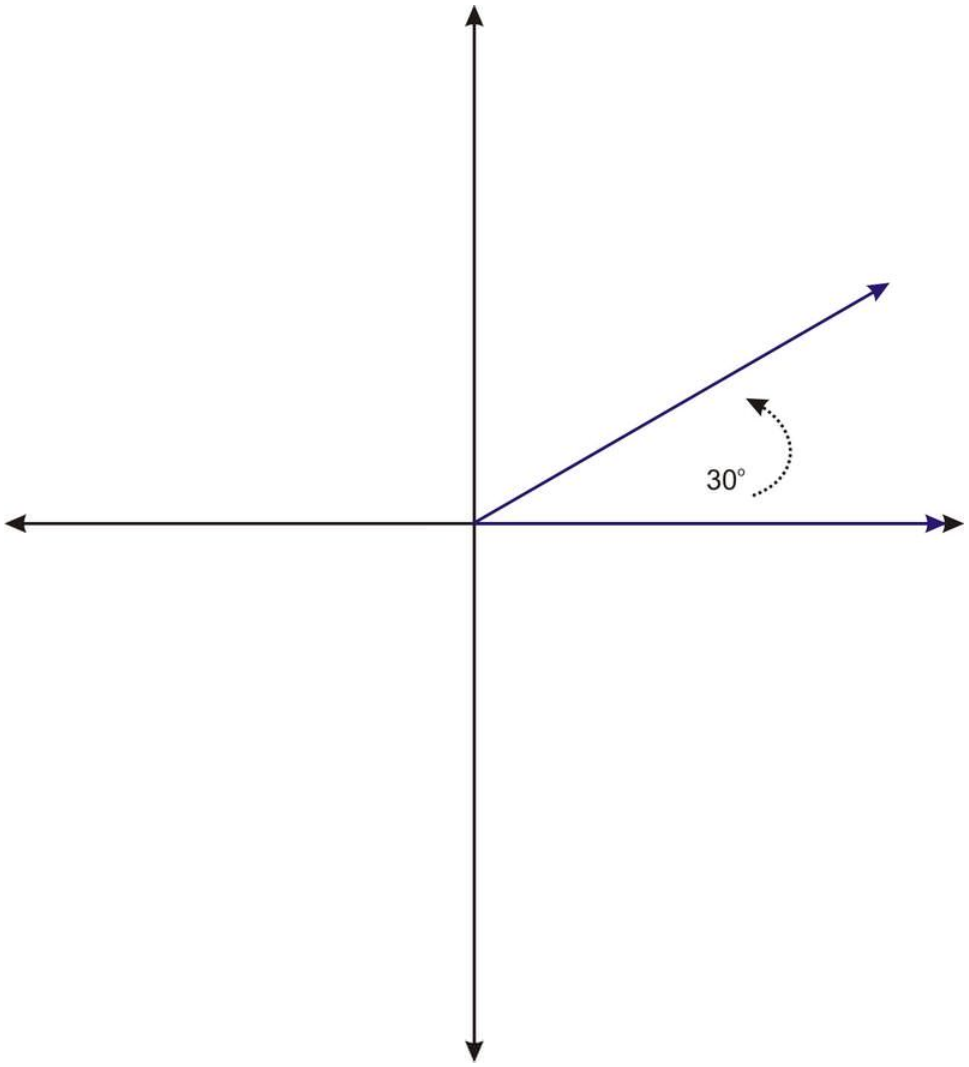


Figure **2.3.4.2**

Now consider the angle **390°**. We can think of this angle as a full rotation (**360°**), plus an additional 30 degrees.

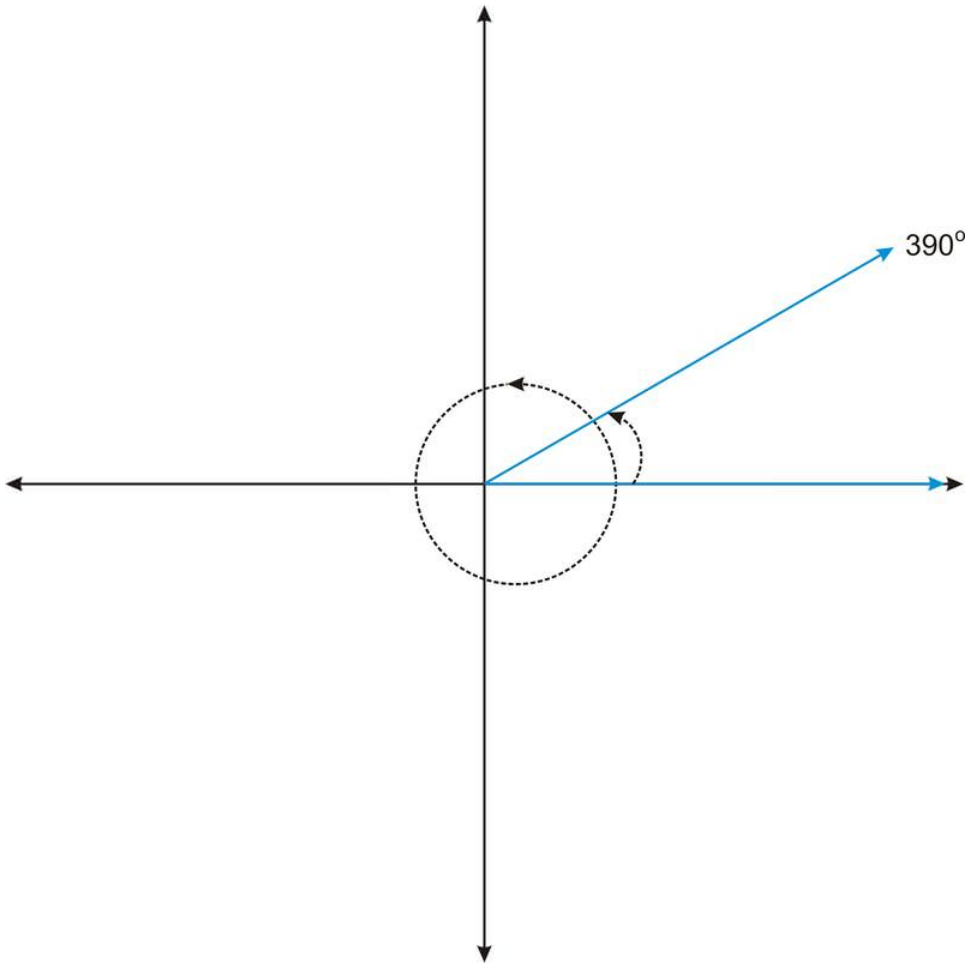
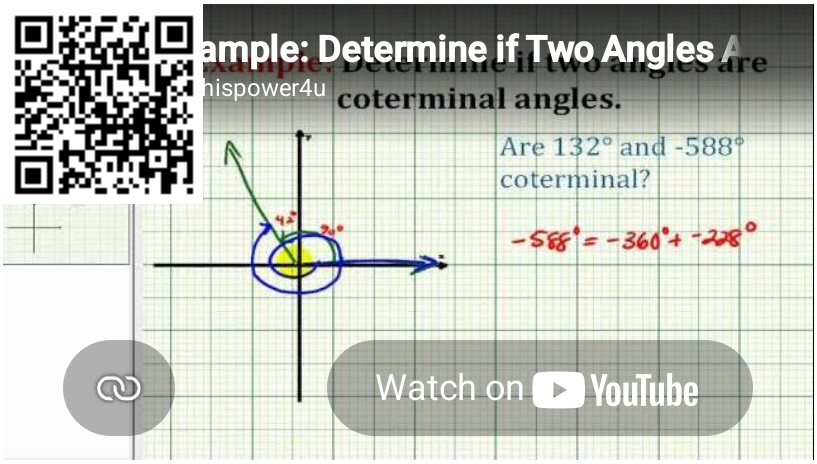


Figure 2.3.4.3

Notice that $390°$ looks the same as $30°$. Formally, we say that the angles share the same terminal side. Therefore we call the angles **co-terminal**. Not only are these two angles co-terminal, but there are infinitely many angles that are co-terminal with these two angles. For example, if we rotate another $360°$, we get the angle $750°$. Or, if we create the angle in the negative direction (clockwise), we get the angle $-330°$. Because we can rotate in either direction, and we can rotate as many times as we want, we can continuously generate angles that are co-terminal with $30°$.



Identifying Co-Terminal Angles

For the following questions, determine if the angle is co-terminal with $45°$.

1. -45°

No, it is not co-terminal with 45°

2. 405°

Yes, 405° is co-terminal with 45°.

3. -315°

Yes, -315° is co-terminal with 45°.

Example 2.3.4.1

Earlier, you were asked if it is possible to represent the angle any other way.

Solution

You can either think of 60° as 420° if you rotate all the way around the circle once and continue the rotation to where the spinner has stopped, or as -300° if you rotate clockwise around the circle instead of counterclockwise to where the spinner has stopped.

Example 2.3.4.2

Find a coterminal angle to 23°

Solution

A coterminal angle would be an angle that is at the same terminal place as 23° but has a different value. In this case, -337° is a coterminal angle.

Example 2.3.4.3

Find a coterminal angle to -90°

Solution

A coterminal angle would be an angle that is at the same terminal place as -90° but has a different value. In this case, 270° is a coterminal angle.

Example 2.3.4.4

Find two coterminal angles to 70° by rotating in the positive direction around the circle.

Solution

Rotating once around the circle gives a coterminal angle of 430°. Rotating again around the circle gives a coterminal angle of 790°.

Review

1. Is 315° co-terminal with -45°?
2. Is 90° co-terminal with -90°?
3. Is 350° co-terminal with -370°?
4. Is 15° co-terminal with 1095°?
5. Is 85° co-terminal with 1880°?

For each diagram, name the angle in 3 ways. At least one way should use negative degrees.

6.

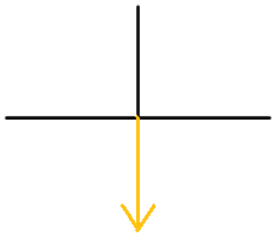


Figure 2.3.4.4

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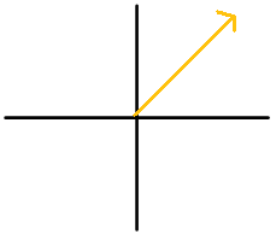


Figure 2.3.4.5

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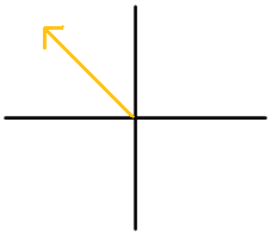


Figure 2.3.4.6

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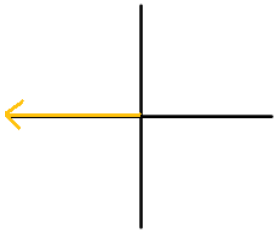


Figure 2.3.4.7

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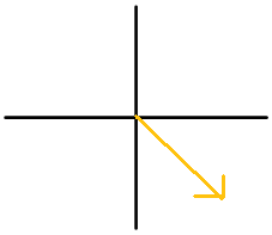


Figure 2.3.4.8

11. Name the angle of the 8 on a standard clock two different ways.
12. Name the angle of the 11 on a standard clock two different ways.
13. Name the angle of the 4 on a standard clock two different ways.
14. Explain how to determine whether or not two angles are co-terminal.
15. How many rotations is $4680°$?

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.16.

Vocabulary

Term	Definition
Coterminal Angles	A set of coterminal angles are angles with the same terminal side but expressed differently, such as a different number of complete rotations around the unit circle or angles being expressed as positive versus negative angle measurements.

Additional Resources

Video: Example: Determine if Two Angles are Coterminal

Practice: Coterminal Angles

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2.3.5: Signs of Trigonometric Functions

x and y values of a function; signs of functions based on quadrants.

Domain, Range, and Signs of Trigonometric Functions

You are doodling in art class one day when you draw a circle. Then you draw a few lines extending outward from the center to the edge of the circle. You draw a triangle with the "x" axis, and realize that you're thinking about your math class again.

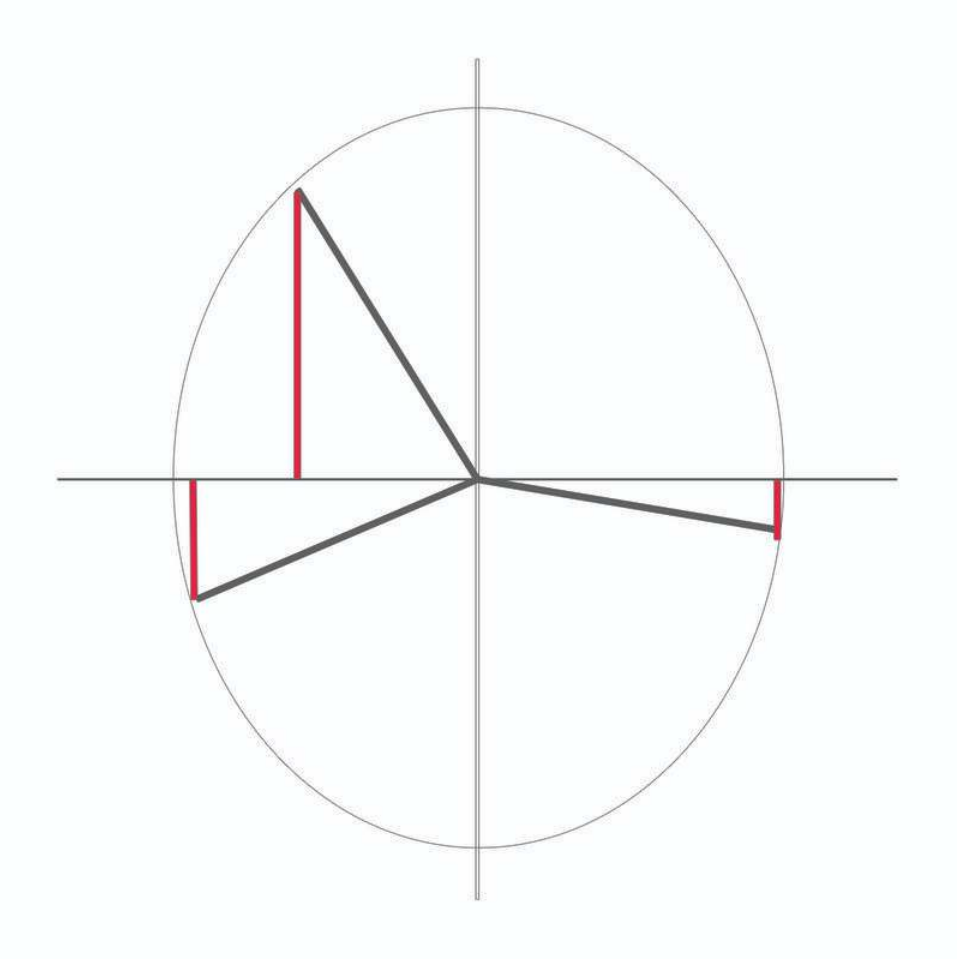


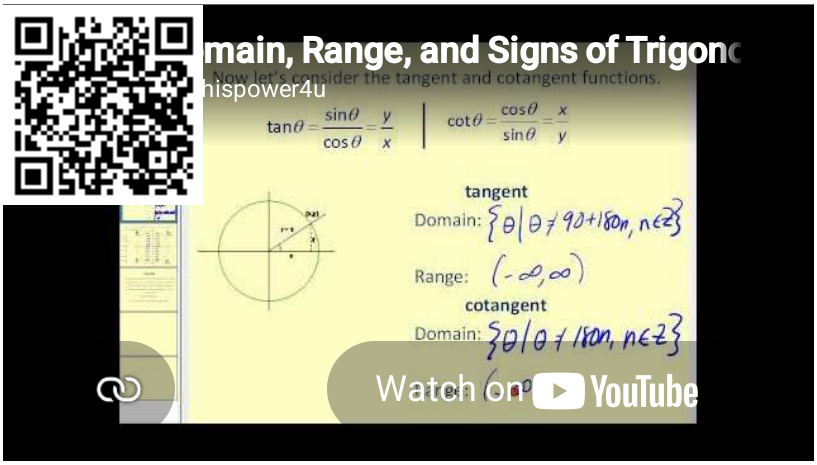
Figure **2.3.5.1**

You notice that the relationship for the sine function involves the length of the side opposite the angle divided by the length of the hypotenuse. But while the hypotenuse is always a positive number, the sign of the opposite side can be different, depending on what quadrant the angle is drawn in.

Can you determine what the sign of the sine function will be in each of the four quadrants, based on the knowledge of the ratio that defines the sine function?

Domain and **Range** of Trigonometric Functions

While the trigonometric functions may seem quite different from other functions you have worked with, they are in fact just like any other function. We can think of a trig function in terms of “input” and “output.” The input is always an angle. The output is a ratio of sides of a triangle. If you think about the trig functions in this way, you can define the domain and range of each function.



Let's first consider the sine and cosine functions. The input of each of these functions is always an angle, and as you learned in the previous sections, these angles can take on any real number value. Therefore the sine and cosine function have the same domain, the set of all real numbers, $\boldsymbol{R}$. We can determine the range of the functions if we think about the fact that the sine of an angle is the $\boldsymbol{y}$−coordinate of the point where the terminal side of the angle intersects the unit circle. The cosine is the $\boldsymbol{x}$−coordinate of that point. Now recall that in the unit circle, we defined the trig functions in terms of a triangle with hypotenuse 1.

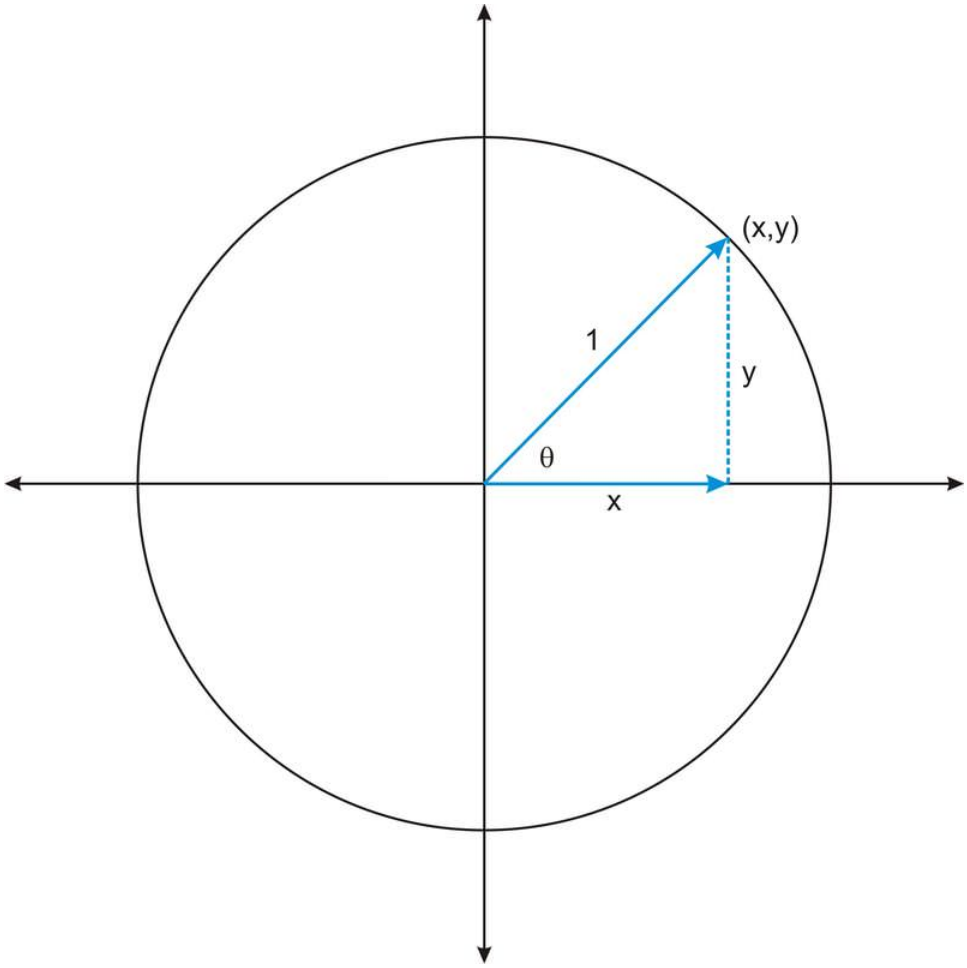


Figure **2.3.5.2**

In this right triangle, $\boldsymbol{x}$ and $\boldsymbol{y}$ are the lengths of the legs of the triangle, which must have lengths less than 1, the length of the hypotenuse. Therefore the ranges of the sine and cosine function do not include values greater than one. The ranges do, however,

contain negative values. Any angle whose terminal side is in the third or fourth quadrant will have a negative y–coordinate, and any angle whose terminal side is in the second or third quadrant will have a negative x–coordinate.

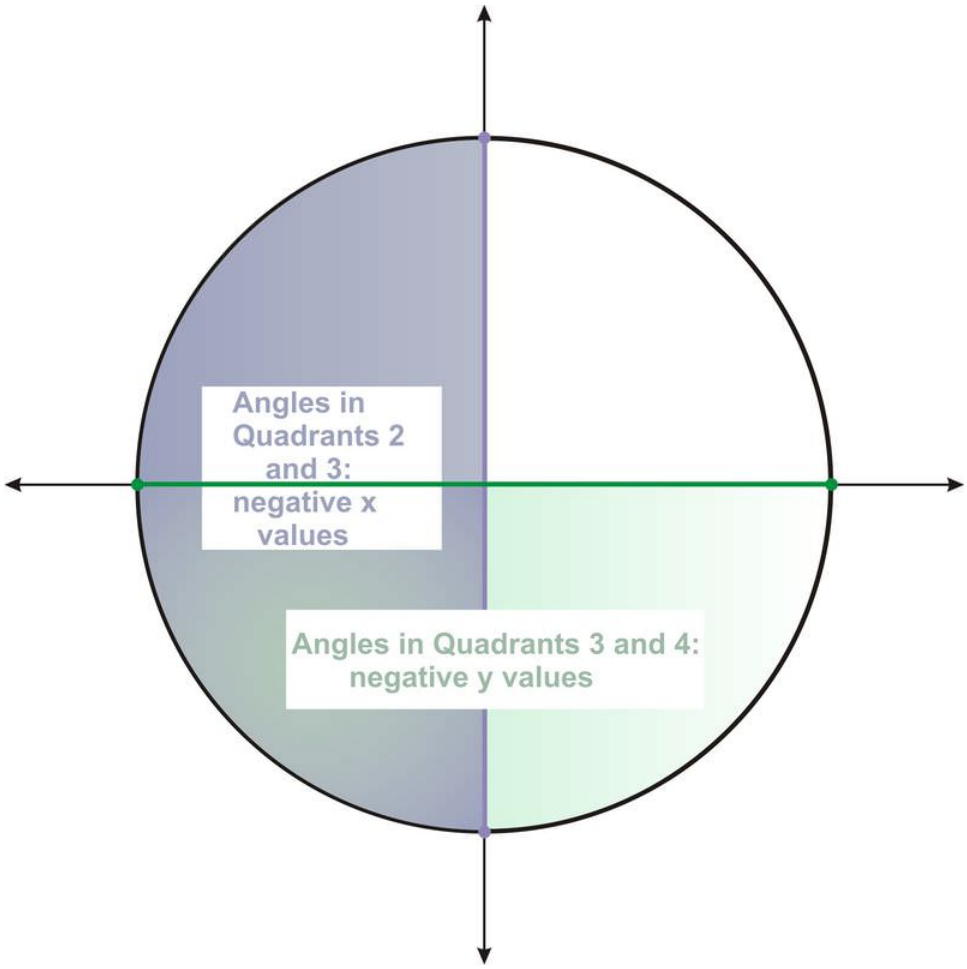


Figure **2.3.5.3**

In either case, the minimum value is -1. For example, $\cos 180^\circ = -1$ and $\sin 270^\circ = -1$. Therefore the sine and cosine function both have range from -1 to 1.

The table below summarizes the domains and ranges of these functions:

	Domain	Range
Sine	$\theta = R$	$-1 \leq y \leq 1$
Cosine	$\theta = R$	$-1 \leq y \leq 1$

Knowing the domain and range of the cosine and sine function can help us determine the domain and range of the secant and cosecant function. First consider the sine and cosecant functions, which as we showed above, are reciprocals. The cosecant function will be defined as long as the sine value is not 0. Therefore the domain of the cosecant function excludes all angles with sine value 0, which are 0°, 180°, 360°, etc.

In Chapter 2 you will analyze the graphs of these functions, which will help you see why the reciprocal relationship results in a particular range for the cosecant function. Here we will state this range, and in the review questions you will explore values of the sine and cosecant function in order to begin to verify this range, as well as the domain and range of the secant function.

Function	Domain	Range
Tangent	$\theta \in R, \theta \neq 90, 270, 450 \ldots$	All reals
Cotangent	$\theta \in R, \theta \neq 0, 180, 360 \ldots$	All reals

Now let's consider the tangent and cotangent functions. The tangent function is defined as $\tan\theta = \frac{y}{x}$. Therefore the domain of this function excludes angles for which the ordered pair has an x−coordinate of 0: $90°$, $270°$, etc. The cotangent function is defined as $\cot\theta = \frac{x}{y}$, so this function's domain will exclude angles for which the ordered pair has a y−coordinate of 0: $0°$, $180°$, $360°$, etc.

Function	Domain	Range
Tangent	$\theta \in R, \theta \neq 90, 270, 450 \ldots$	All reals
Cotangent	$\theta \in R, \theta \neq 0, 180, 360 \ldots$	All reals

Knowing the ranges of these functions tells you the values you should expect when you determine the value of a trig function of an angle. However, for many problems you will need to identify the sign of the function of an angle: Is it positive or negative?

In determining the ranges of the sine and cosine functions above, we began to categorize the signs of these functions in terms of the quadrants in which angles lie. The figure below summarizes the signs for angles in all 4 quadrants.

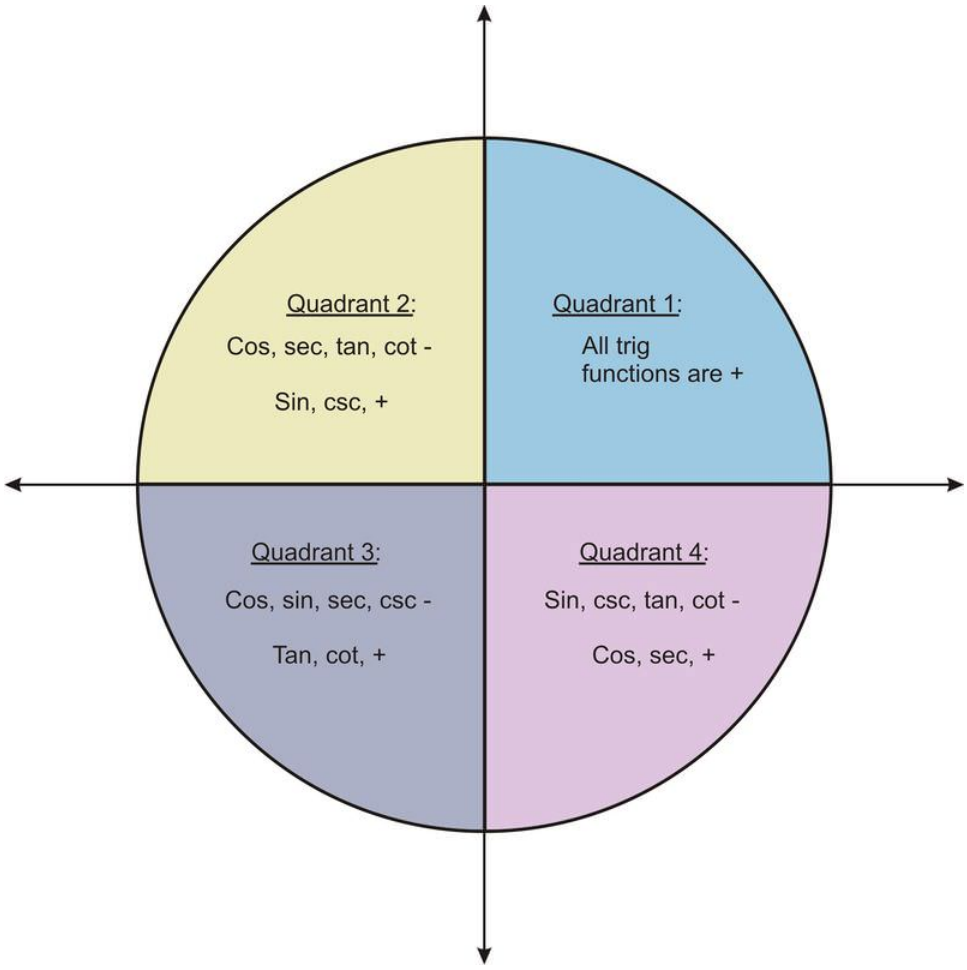


Figure **2.3.5.3**

An easy way to remember this is "All Students Take Calculus". Quadrant I: All values are positive, Quadrant II: Sine is positive, Quadrant III: Tangent is positive, and Quadrant IV: Cosine is positive. This simple memory device will help you remember which

trig functions are positive and where.

Stating the Sign

1. State the sign of $\cos 100^\circ$

The angle 100° is in the second quadrant. Therefore the x−coordinate is negative and so $\cos 100^\circ$ is negative.

2. State the sign of $\csc 220^\circ$

The angle 220° is in the third quadrant. Therefore the y−coordinate is negative. So the sine, and the cosecant are negative.

3. State the sign of $\tan 370^\circ$

The angle 370° is in the first quadrant. Therefore the tangent value is positive.

Example 2.3.5.1

Earlier, you were asked to determine what the sign of the sine function will be in each of the four quadrants.

Solution

Since the sine function is defined to be the length of the opposite side divided by the length of the hypotenuse, the sign of the sine function is the sign of the y -coordinate for whatever quadrant is being considered. In quadrants 1 and 2, the "y" coordinate is positive, so the sine function is positive. In quadrants 3 and 4, the y-coordinate is negative, so the sine function is negative as well.

Example 2.3.5.2

State the sign of $\cos 70^\circ$

Solution

The angle 70° is in the first quadrant. Cosine is defined to be the adjacent side divided by the hypotenuse. Since the hypotenuse of the unit circle is one and the adjacent side is the x-coordinate, the sign of the cosine function is determined by the sign of the x-coordinate. Since 70° is in the first quadrant, the x value is positive. Therefore the cosine value is positive.

Example 2.3.5.3

State the sign of $\sin 130^\circ$

Solution

The angle 130° is in the second quadrant. Sine is defined to be the opposite side divided by the hypotenuse. Since the hypotenuse of the unit circle is one and the opposite side is the y-coordinate, the sign of the sine function is determined by the sign of the y-coordinate. Since 130° is in the second quadrant, the y value is positive. Therefore the sine value is positive.

Example 2.3.5.4

State the sign of $\tan 250^\circ$

Solution

The angle 250° is in the third quadrant. Tangent is defined to be the opposite side divided by the adjacent side. In the third quadrant, the x values are negative, and the y values are negative. A negative divided by a negative equals a positive. Therefore the tangent of 250° is positive.

Review

1. In what quadrants is the sine function positive?
2. In what quadrants is the cotangent function negative?

3. In what quadrants is the cosine function negative?
4. In what quadrants is the tangent function positive?
5. For what angles is the cosecant function undefined?
6. State the sign of $\sin 510°$.
7. State the sign of $\cos 315°$.
8. State the sign of $\tan 135°$.
9. State the sign of $\cot 220°$.
10. State the sign of $\csc 40°$.
11. State the sign of $\cos 330°$.
12. State the sign of $\sin 60°$.
13. State the sign of $\sec -45°$.
14. Explain why the cosecant function is never equal to 0.
15. Using your knowledge of domain and range, make a possible sketch for $y = \sin x$.

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.22.

Vocabulary

Term	Definition
domain	The domain of a function is the set of x-values for which the function is defined.
Range	The range of a function is the set of y-values for which the function is defined.

Additional Resources

Video: Domain, Range, and Signs of Trigonometric Functions

Practice: Signs of Trigonometric Functions

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2.3.6: Trigonometric Functions and Angles of Rotation

Based on the unit circle.

You've been working hard in your math class, and are getting to be quite the expert on trig functions. Then one day your friend, who is a year ahead of you in school, approaches you.

"So, you're doing pretty well in math? And you're good with trig functions?" he asks with a smile.

"Yes," you reply confidently. "I am."

"Alright, then what's the sine of $150°$?" he asks.

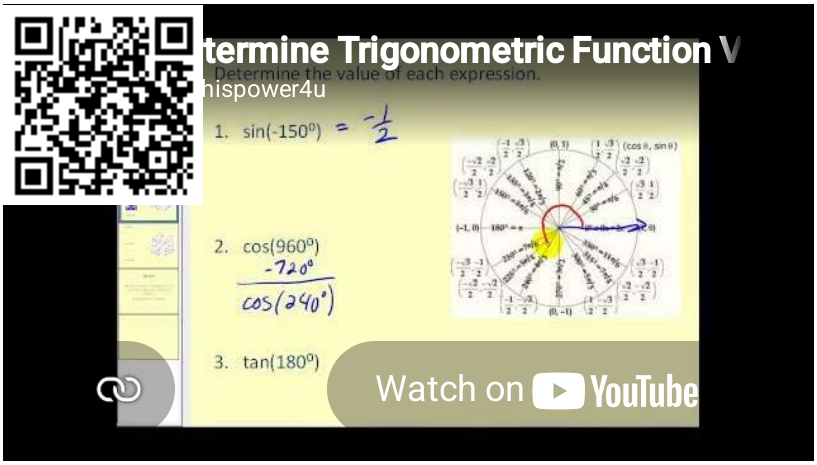
"What? That doesn't make sense. No right triangle has an angle like that, so there's no way to define that function!" you say.

Your friend laughs. "As it turns out, it is quite possible to have trig functions of angles greater than $90°$."

Is your friend just playing a joke on you, or does he mean it? Can you actually calculate $\sin 150°$?

Angles of Rotation and Trigonometric Functions

Just as it is possible to define the six trigonometric functions for angles in right triangles, we can also define the same functions in terms of angles of rotation.



Consider an angle in standard position, whose terminal side intersects a circle of radius r. We can think of the radius as the hypotenuse of a right triangle:

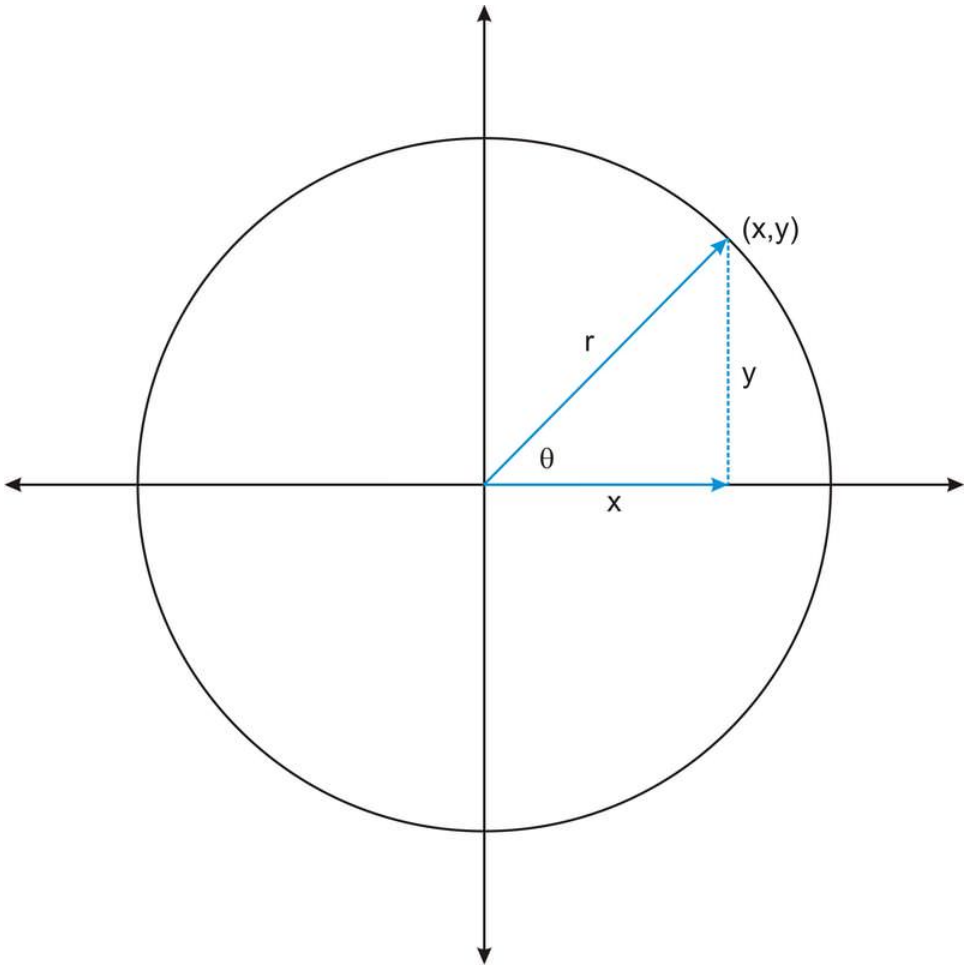


Figure 2.3.6.1

The point (x, y) where the terminal side of the angle intersects the circle tells us the lengths of the two legs of the triangle. Now, we can define the trigonometric functions in terms of x, y, and r:

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$
$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$
$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

And, we can extend these functions to include non-acute angles.

Consider an angle in standard position, such that the point (x, y) on the terminal side of the angle is a point on a circle with radius 1.

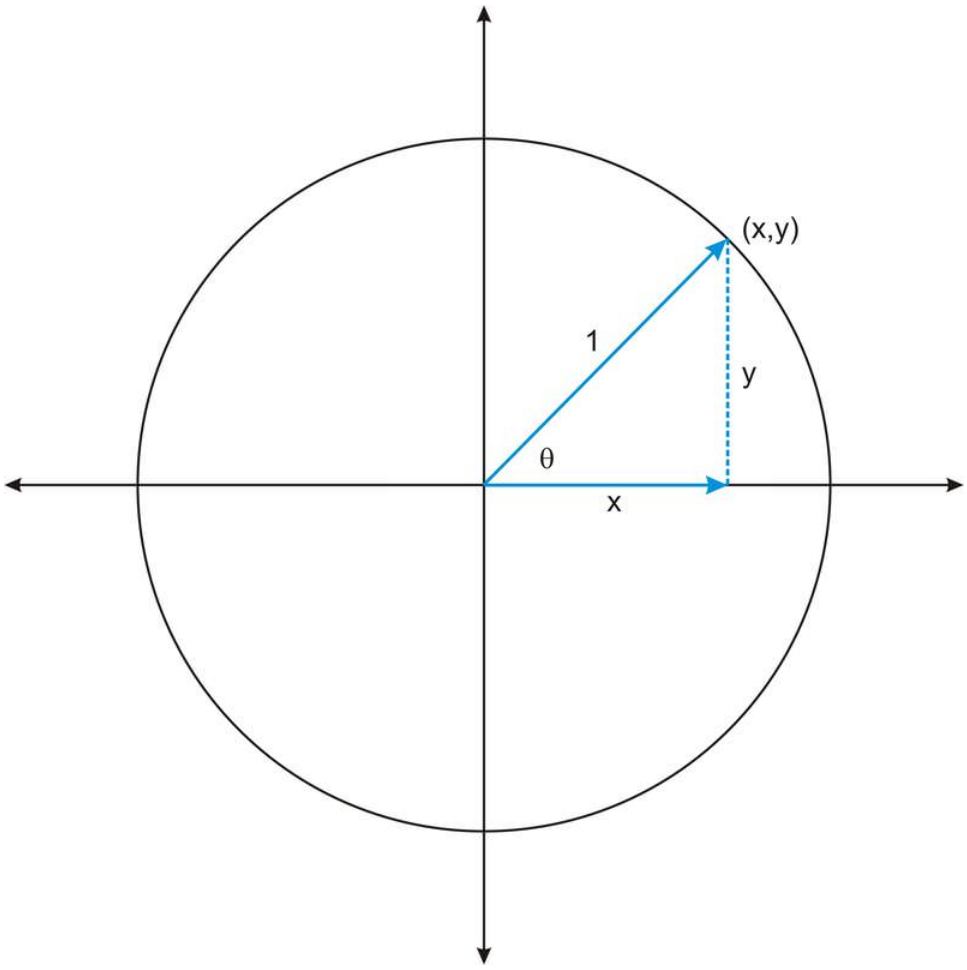


Figure 2.3.6.2

This circle is called the **unit circle**. With $r = 1$, we can define the trigonometric functions in the unit circle:

$$\cos\theta = \frac{x}{r} = \frac{x}{1} = x \qquad \sec\theta = \frac{r}{x} = \frac{1}{x}$$
$$\sin\theta = \frac{y}{r} = \frac{y}{1} = y \qquad \csc\theta = \frac{r}{y} = \frac{1}{y}$$
$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Notice that in the unit circle, the sine and cosine of an angle are the y and x coordinates of the point on the terminal side of the angle. Now we can find the values of the trigonometric functions of any angle of rotation, even the quadrantal angles, which are not angles in triangles.

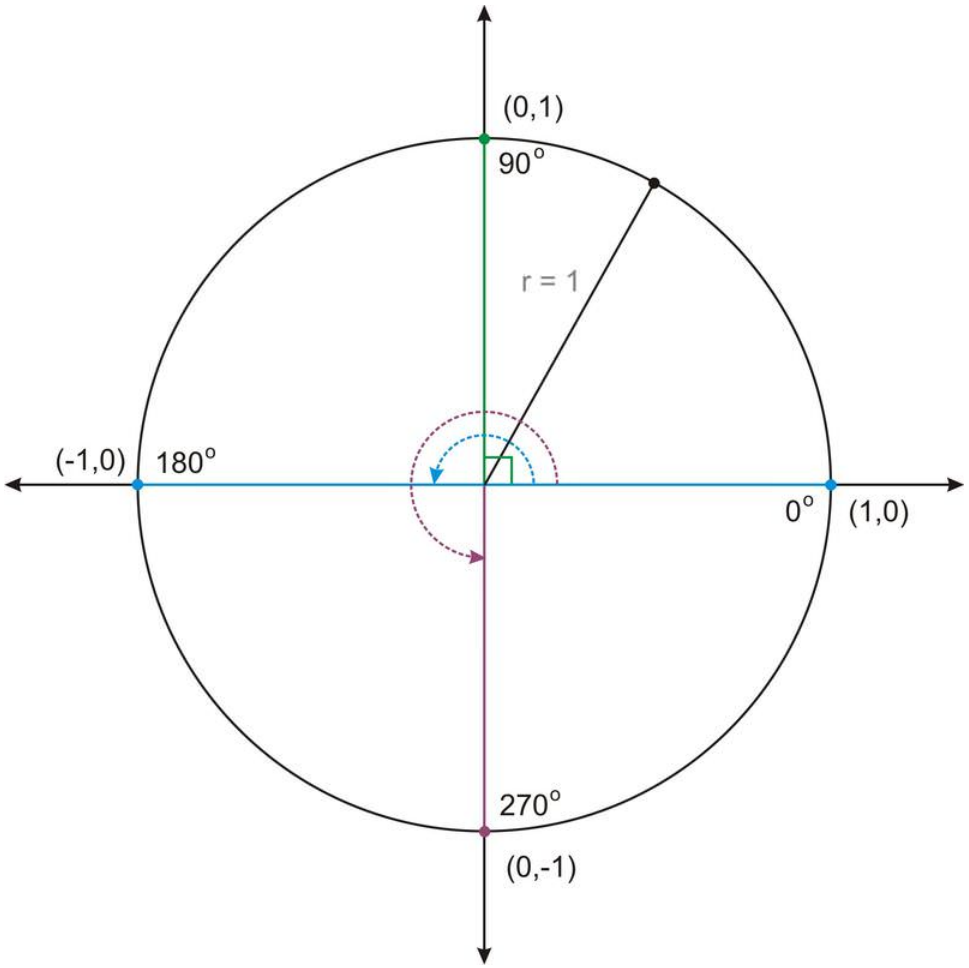


Figure 2.3.6.3

We can use the figure above to determine values of the trig functions for the quadrantal angles. For example, $\sin 90° = y = 1$.

Determining the Value of Trigonometric Functions

1. Determine the values of the six trigonometric functions.

The point $(-3, 4)$ is a point on the terminal side of an angle in standard position. Determine the values of the six trigonometric functions of the angle.

Notice that the angle is more than 90 degrees, and that the terminal side of the angle lies in the second quadrant. This will influence the signs of the trigonometric functions.

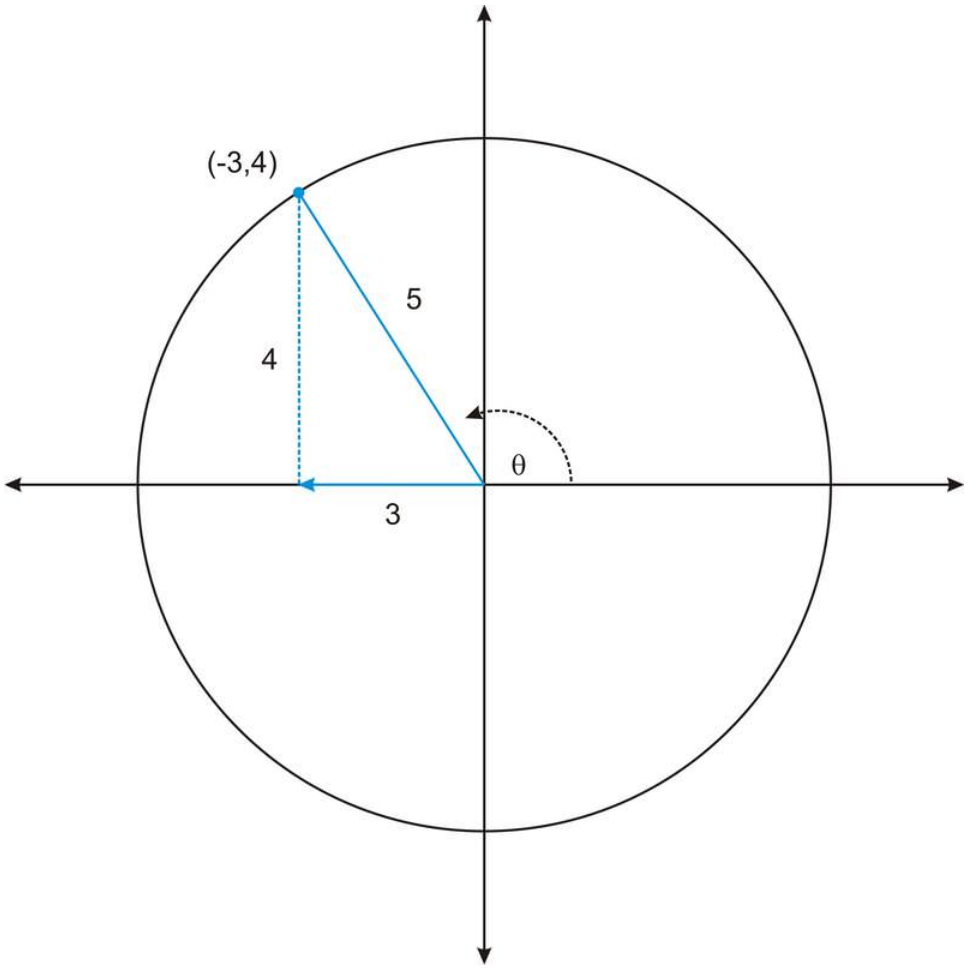


Figure 2.3.6.4

$$\cos\theta = \frac{-3}{5} \qquad \sec\theta = \frac{5}{-3}$$
$$\sin\theta = \frac{4}{5} \qquad \csc\theta = \frac{5}{4}$$
$$\tan\theta = \frac{4}{-3} \qquad \cot\theta = \frac{-3}{4}$$

Notice that the value of r depends on the coordinates of the given point. You can always find the value of r using the Pythagorean Theorem. However, often we look at angles in a circle with radius 1. As you can see, doing this allows us to simplify the definitions of the trig functions.

2. Use the unit circle above to find the value of $\cos 90^\circ$

$\cos 90^\circ = 0$

The ordered pair for this angle is (0, 1). The cosine value is the x coordinate, .

3. Use the unit circle above to find the value of $\cot 180^\circ$

$\cot 180^\circ$ is undefined

The ordered pair for this angle is (-1, 0). The ratio $\frac{x}{y}$ is $\frac{-1}{0}$, which is undefined.

Example 2.3.6.1

Earlier, you were asked if you can actually calculate $\sin 150^\circ$.

Solution

Since you now know that it is possible to apply trigonometric functions to angles greater than $90°$, you can calculate the $\sin 150°$. The easiest way to do this without difficulty is to consider that an angle of $150°$ is in the same position as $30°$, except it's in the second quadrant. This means that it has the same "x" and "y" values as $30°$, except that the "x" value is negative.

Therefore,

$$\sin 150° = \frac{1}{2}$$

Use this figure:

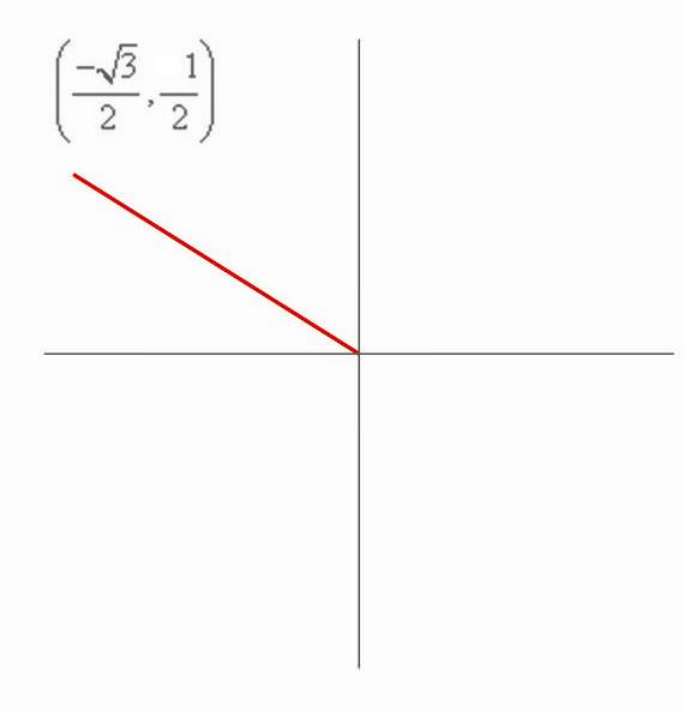


Figure 2.3.6.5

to answer the following examples.

Example 2.3.6.2

Find $\cos\theta$ on the circle above.

Solution

We can see from the "x" and "y" axes that the "x" coordinate is $-\frac{\sqrt{3}}{2}$, the "y" coordinate is $\frac{1}{2}$, and the hypotenuse has a length of 1. This means that the cosine function is:

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\frac{-\sqrt{3}}{2}}{1} = \frac{-\sqrt{3}}{2}$$

Example 2.3.6.3

Find $\cot\theta$ on the circle above.

Solution

We know that $\cot = \frac{1}{\tan} = \frac{1}{\frac{\text{opposite}}{\text{adjacent}}} = \frac{\text{adjacent}}{\text{opposite}}$. The adjacent side to \theta in the circle is $-\frac{\sqrt{3}}{2}$ and the opposite side is $\frac{1}{2}$. Therefore,

$$\cot\theta = \frac{\frac{-\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$$

Example 2.3.6.4

Find $\csc\theta$ on the circle above.

Solution

We know that $\csc = \frac{1}{\sin} = \frac{1}{\frac{\text{opposite}}{\text{hypotenuse}}} = \frac{\text{hypotenuse}}{\text{opposite}}$. The opposite side to θ in the circle is $\frac{1}{2}$ and the hypotenuse is 1.

Therefore,

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{1}{\frac{1}{2}} = 2$$

Review

Find the values of the six trigonometric functions for each angle below.

1. 0°
2. 90°
3. 180°
4. 270°
5. Find the sine of an angle that goes through the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.
6. Find the cosine of an angle that goes through the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.
7. Find the tangent of an angle that goes through the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.
8. Find the secant of an angle that goes through the point $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.
9. Find the cotangent of an angle that goes through the point $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.
10. Find the cosecant of an angle that goes through the point $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.
11. Find the sine of an angle that goes through the point $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.
12. Find the cosine of an angle that goes through the point $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.
13. The sine of an angle in the first quadrant is 0.25. What is the cosine of this angle?
14. The cosine of an angle in the first quadrant is 0.8. What is the sine of this angle?
15. The sine of an angle in the first quadrant is 0.15. What is the cosine of this angle?

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.17.

Vocabulary

Term	Definition
Quadrantal Angle	A quadrantal angle is an angle that has its terminal side on one of the four lines of axis: positive x, negative x, positive y or negative y.

Additional Resources

Video: Determine Trigonometric Function Values Using the Unit Circle

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2.3.7: Reference Angles and Angles in the Unit Circle

Formed between the terminal side of an angle and the closest part of the x-axis.

When you walk into math class one day, your teacher has a surprise for the class. You're going to play series of games related to the things you've been learning about in class. For the first game, your teacher hands each group of students a spinner with an "x" and "y" axis marked. The game is to see how many angles you identify correctly. However, in this game, you are supposed to give what is called the "**reference angle**". You spin your spinner three times. Each picture below shows one of the spins:

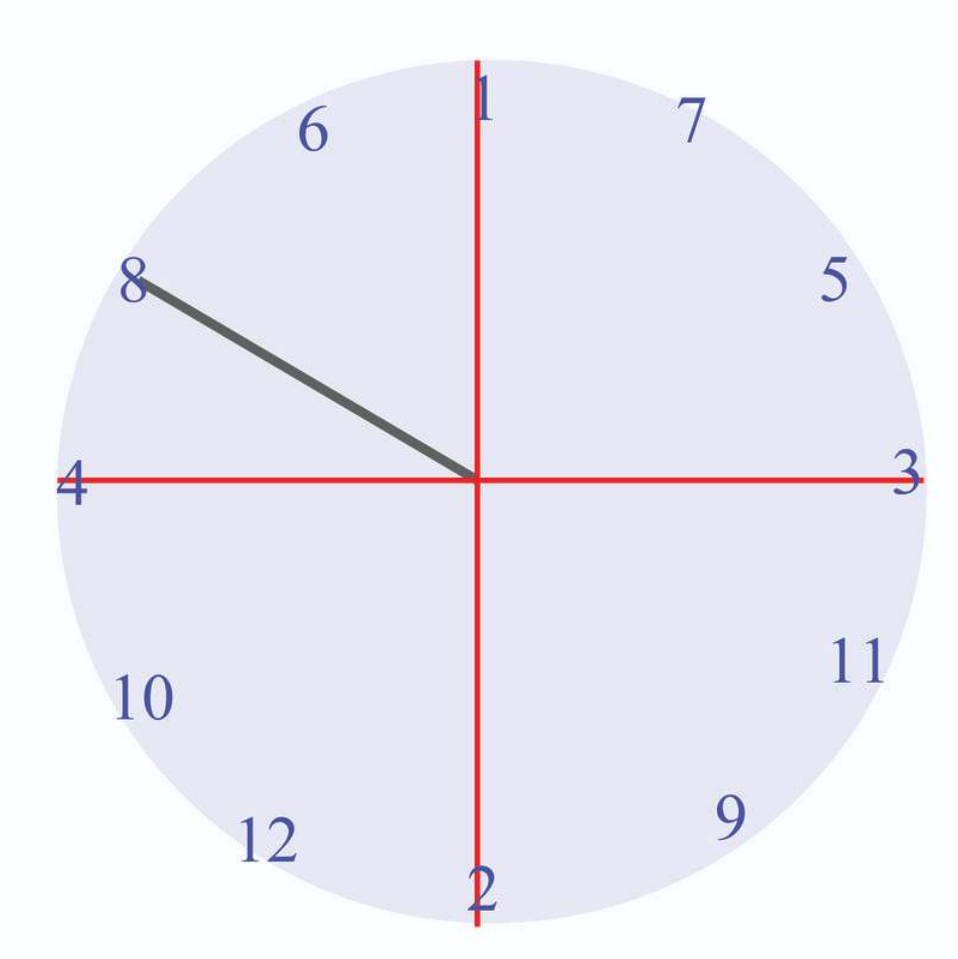


Figure **2.3.7.1**

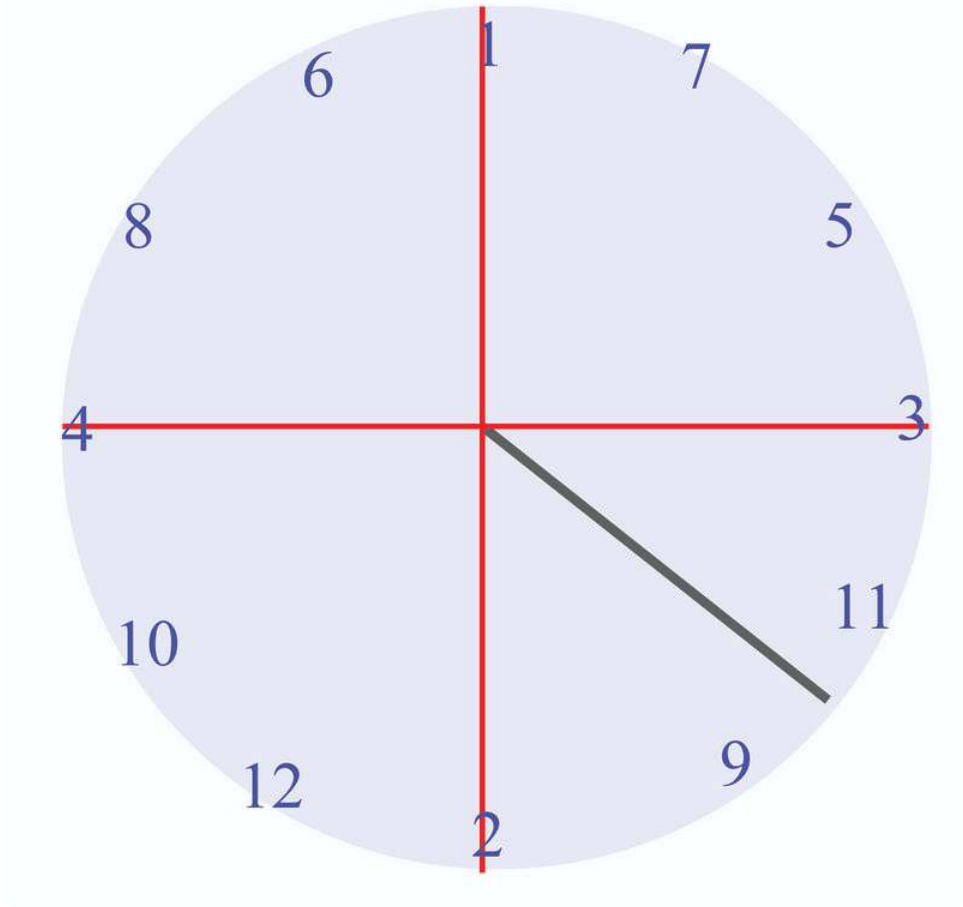


Figure 2.3.7.2

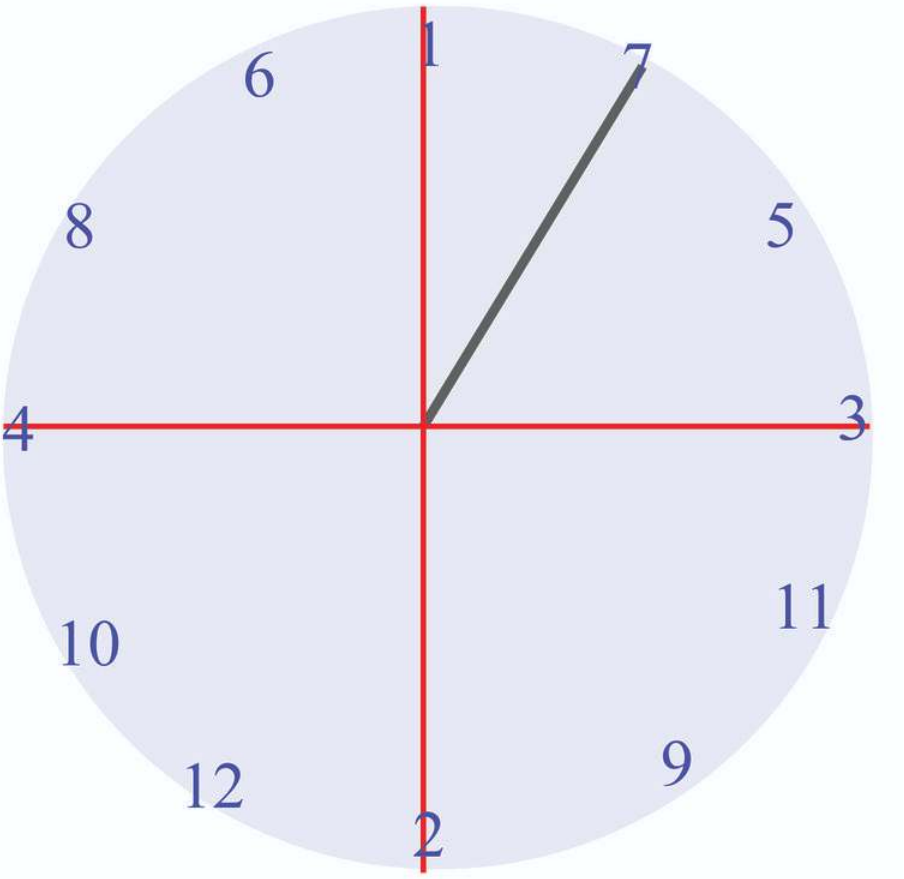


Figure **2.3.7.3**

Can you correctly identify the reference angles for these pictures?

Reference Angles

Consider the angle $150°$. If we graph this angle in standard position, we see that the terminal side of this angle is a reflection of the terminal side of $30°$, across the $y-$axis.

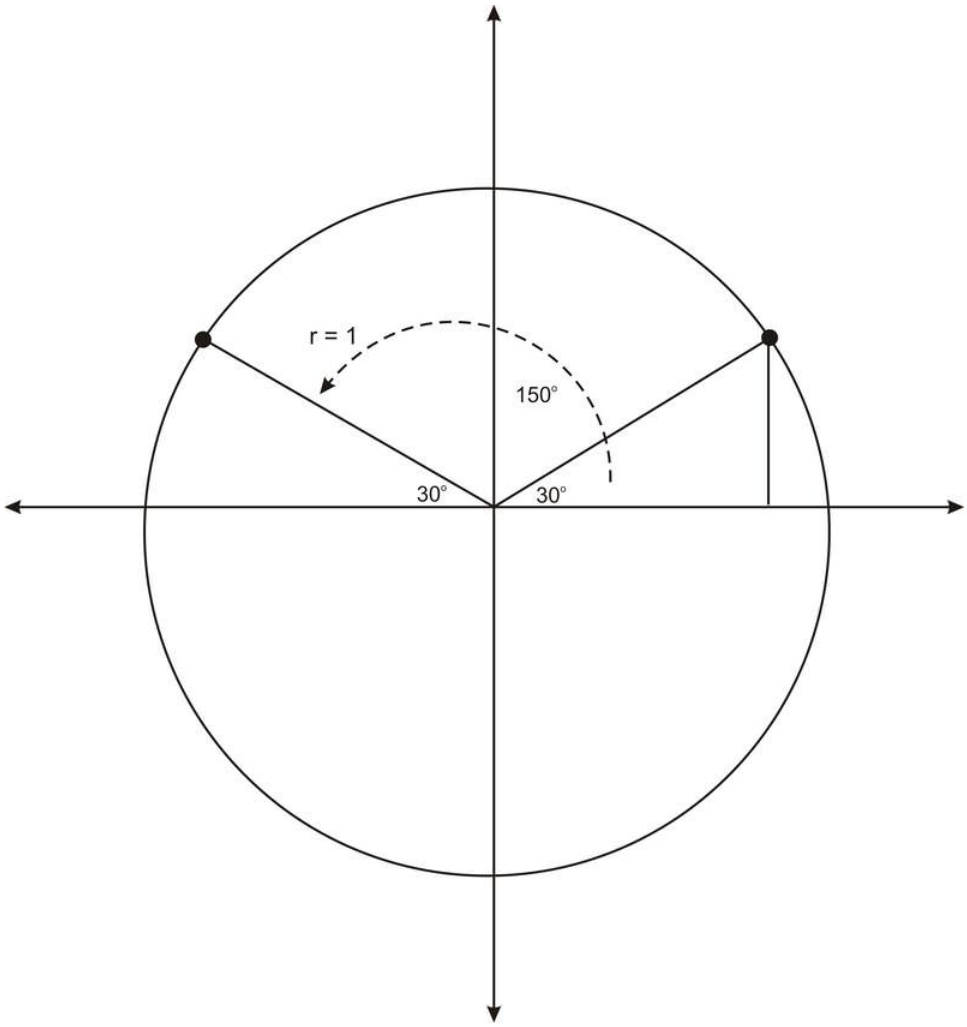


Figure 2.3.7.4

Notice that $150°$ makes a $30°$ angle with the negative x-axis. Therefore we say that $30°$ is the **reference angle** for $150°$. Formally, the **reference angle** of an angle in standard position is the angle formed with the closest portion of the x-axis. Notice that $30°$ is the reference angle for many angles. For example, it is the reference angle for $210°$ and for $-30°$.

In general, identifying the reference angle for an angle will help you determine the values of the trig functions of the angle.



Identifying Reference Angles

Graph each of the following angles and identify their reference angles.

a. $140°$

$140°$ makes a $40°$ angles with the negative x-axis. Therefore the reference angle is $40°$.

b. $240°$

$240°$ makes a $60°$ angle with the negative x-axis. Therefore the reference angle is $60°$

c. $380°$

$380°$ is a full rotation of $360°$, plus an additional $20°$. So this angle is co-terminal with $20°$, and $20°$ is its reference angle.

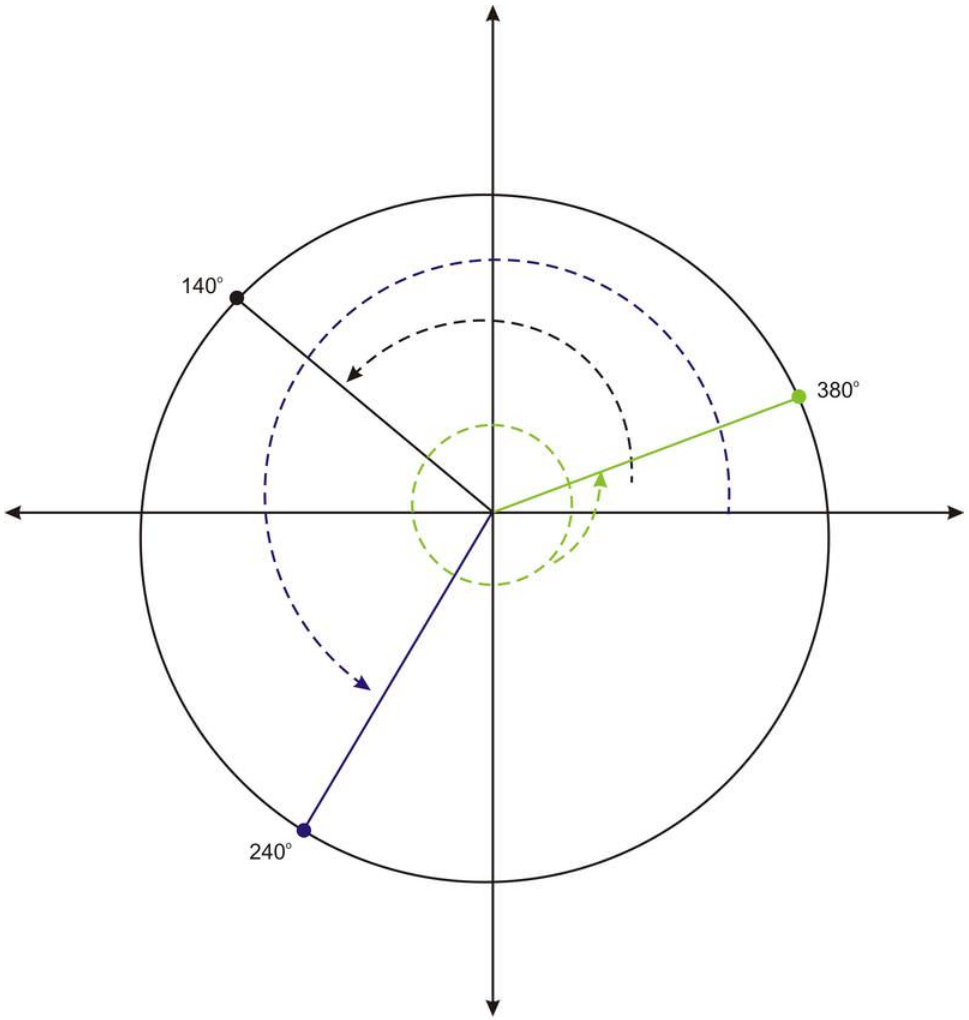


Figure 2.3.7.5

Determining the Value of Trigonometric Functions

1. Find the ordered pair for $240°$ and use it to find the value of $\sin 240°$.

$$\sin 240° = -\frac{\sqrt{3}}{2}$$

As we found in part b under the question above, the reference angle for $240°$ is $60°$. The figure below shows $60°$ and the three other angles in the unit circle that have $60°$ as a reference angle.

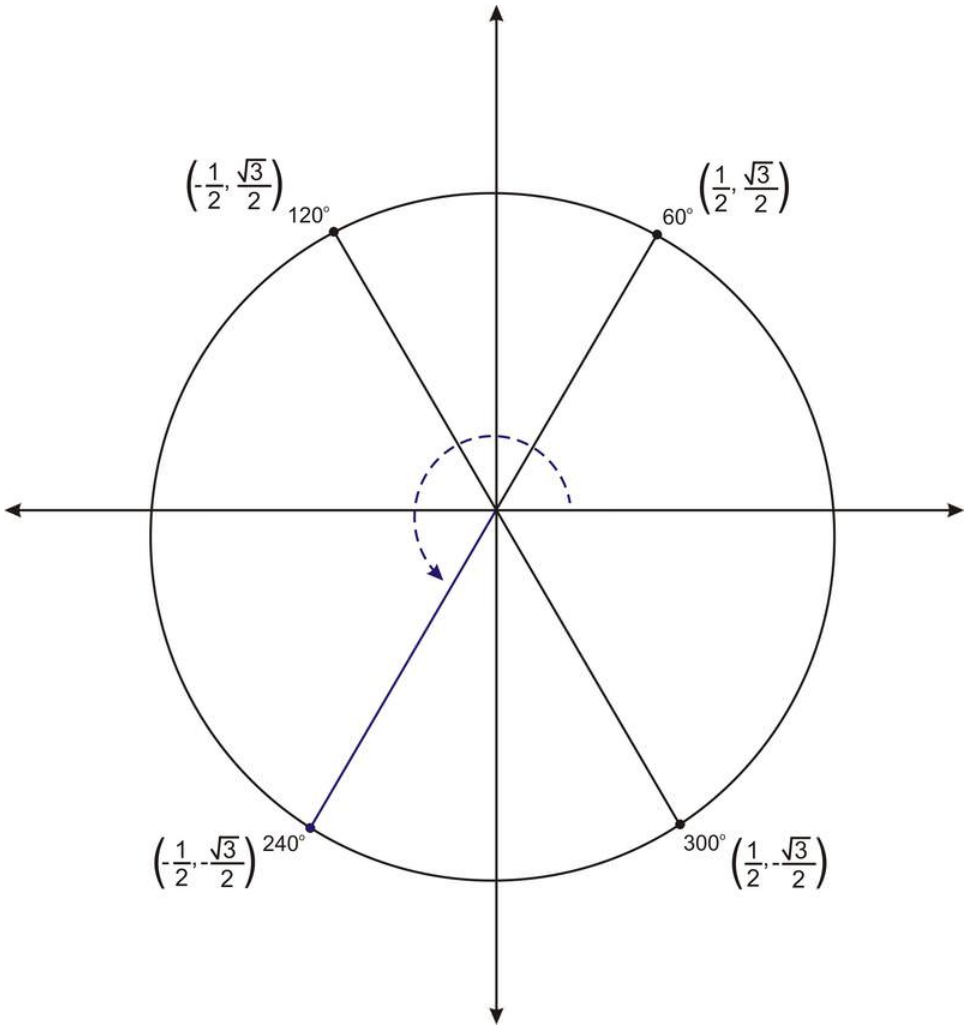


Figure **2.3.7.6**

The terminal side of the angle $240°$ represents a reflection of the terminal side of $60°$ over both axes. So the coordinates of the point are $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. The y−coordinate is the sine value, so $\sin 240° = -\frac{\sqrt{3}}{2}$.

Just as the figure above shows $60°$ and three related angles, we can make similar graphs for $30°$ and $45°$.

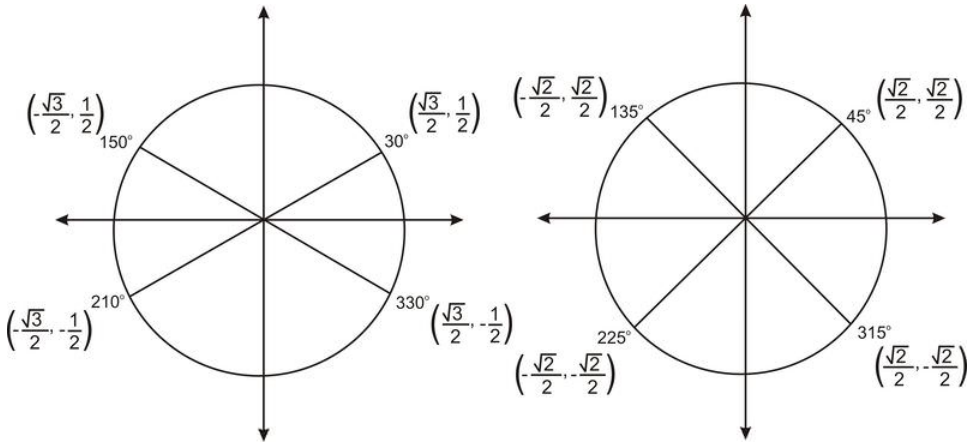


Figure **2.3.7.7**

Knowing these ordered pairs will help you find the value of any of the trig functions for these angles.

2. Find the value of $\cot 300^\circ$

$\cot 300^\circ = -\frac{1}{\sqrt{3}}$

Using the graph above, you will find that the ordered pair is $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. Therefore the cotangent value is $\cot 300^\circ = \frac{x}{y} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}}$

$= \frac{1}{2} \times -\frac{2}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$

We can also use the concept of a reference angle and the ordered pairs we have identified to determine the values of the trig functions for other angles.

Example 2.3.7.1

Earlier, you were asked if you can correctly identify the reference angles in the pictures.

Solution

Since you know how to measure reference angles now, upon examination of the spinners, you know that the first angle is 30°, the second angle is 45°, and the third angle is 60°.

Example 2.3.7.2

Graph 210° and identify its reference angle.

Solution

The graph of 210° looks like this:

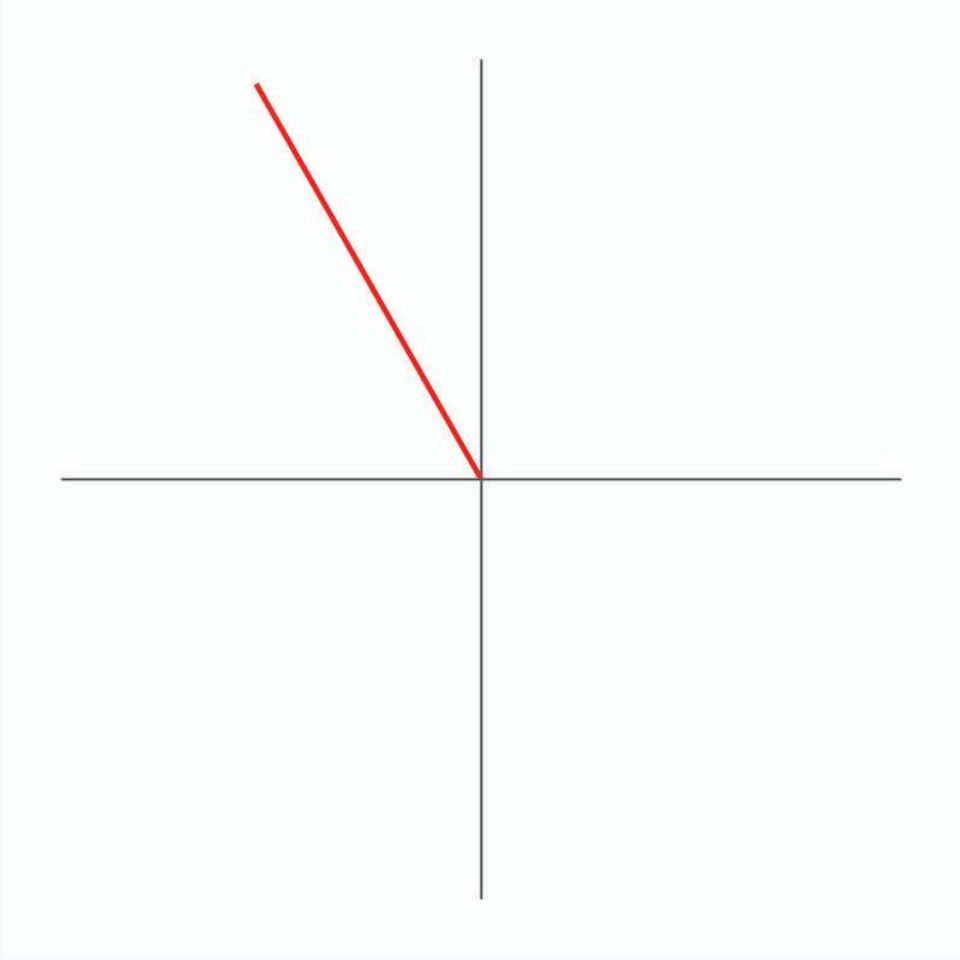


Figure 2.3.7.8

and since the angle makes a $30°$ angle with the negative "x" axis, the reference angle is $30°$.

Example 2.3.7.3

Graph $315°$ and identify its reference angle.

Solution

The graph of $315°$ looks like this:

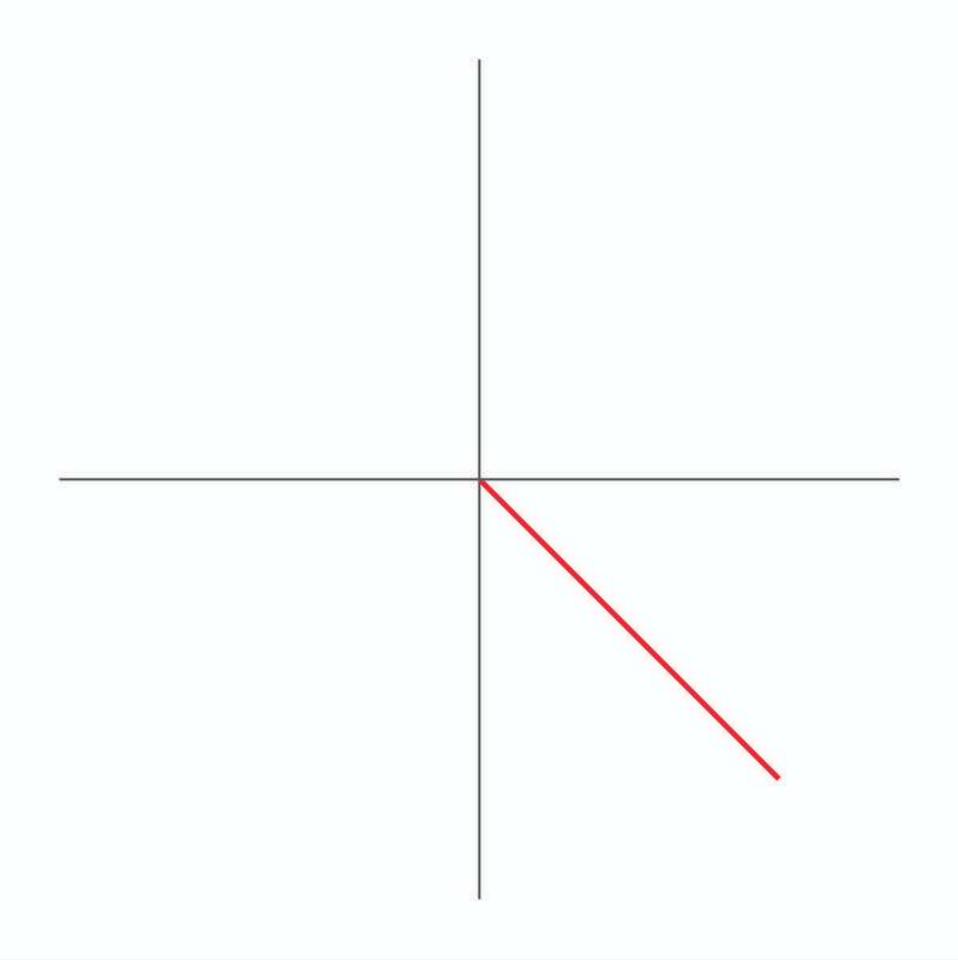


Figure 2.3.7.9

and since the angle makes a $45°$ angle with the positive "x" axis, the reference angle is $45°$.

Example 2.3.7.4

Find the ordered pair for $150°$ and use it to find the value of cos $150°$.

Solution

Since the reference angle is $30°$, we know that the coordinates for the point on the unit circle are $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. This is the same as the value for $30°$, except the "x" coordinate is negative instead of positive. Knowing this,

$$\cos 150° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{-\frac{\sqrt{3}}{2}}{1} = -\frac{\sqrt{3}}{2}$$

Review

1. Graph $100°$ and identify its reference angle.
2. Graph $200°$ and identify its reference angle.
3. Graph $290°$ and identify its reference angle.

Calculate each value using the unit circle and special right triangles.

4. $\sin 120°$
5. $\cos 120°$
6. $\csc 120°$

7. $\cos 135^\circ$
8. $\sin 135^\circ$
9. $\tan 135^\circ$
10. $\sin 210^\circ$
11. $\cos 210^\circ$
12. $\cot 210^\circ$\)
13. $\sin 225^\circ$
14. $\cos 225^\circ$
15. $\sec 225^\circ$

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.18.

Vocabulary

Term	Definition
Reference Angle	A reference angle is the angle formed between the terminal side of the angle and the closest of either the positive or negative x-axis.

Additional Resources

Video: Reference Angles - Overview

Practice: Reference Angles and Angles in the Unit Circle

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2.3.8: Trigonometric Functions of Negative Angles

Angles measured by rotating clockwise from the positive x-axis.

While practicing for the track team, you regularly stop to consider the values of trig functions for the angle you've covered as you run around the circular track at your school. Today, however, is different. To keep things more interesting, your coach has decided to have you and your teammates run the opposite of the usual direction on the track. From your studies at school, you know that this is the equivalent of a "**negative angle**".

You have run $-45°$ around the track, and want to fine the value of the cosine function for this angle. Is it still possible to find the values of trig functions for these new types of angles?

Trigonometric Functions of Negative Angles

Recall that graphing a negative angle means rotating clockwise. The graph below shows $-30°$.

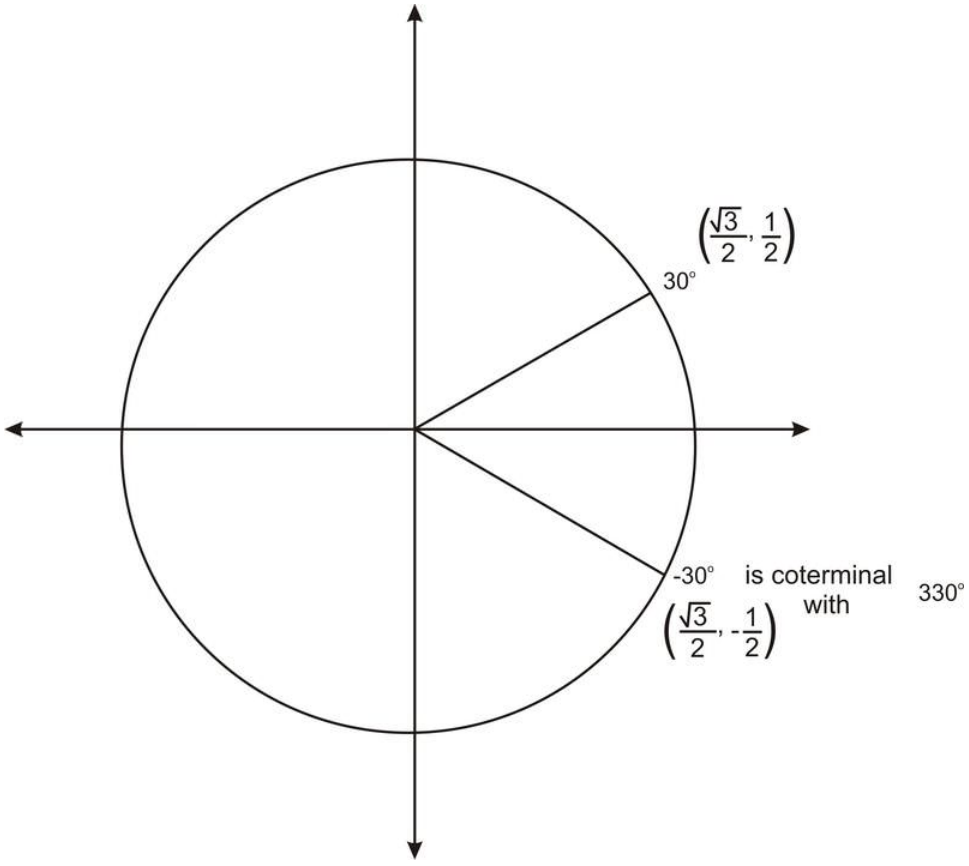
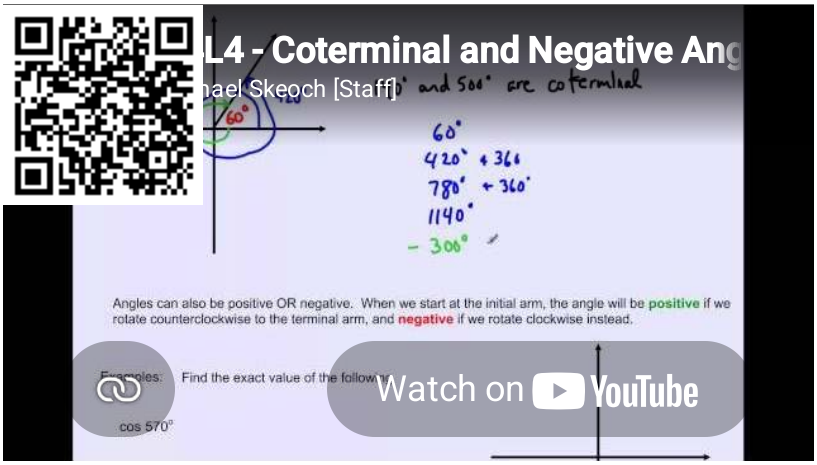


Figure 2.3.8.1

Notice that this angle is coterminal with $330°$. So the ordered pair is $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$. We can use this ordered pair to find the values of any of the trig functions of $-30°$. For example, $\cos(-30°) = x = \frac{\sqrt{3}}{2}$.

In general, if a negative angle has a reference angle of $30°$, $45°$, or $60°$, or if it is a quadrantal angle, we can find its ordered pair, and so we can determine the values of any of the trig functions of the angle.



Finding the Value of Trigonometric Expressions

Find the value of the following expressions:

1. $\sin(-45°)$

$\sin(-45°) = -\frac{\sqrt{2}}{2}$

$-45°$ is in the 4th quadrant, and has a reference angle of $45°$. That is, this angle is coterminal with $315°$. Therefore the ordered pair is $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ and the sine value is $-\frac{\sqrt{2}}{2}$.

2. $\sec(-300°)$

$\sec(-300°) = 2$

The angle $-300°$ is in the 1st quadrant and has a reference angle of $60°$. That is, this angle is coterminal with $60°$. Therefore the ordered pair is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and the secant value is $\frac{1}{x} = \frac{1}{\frac{1}{2}} = 2$.

3. $\cos(-90°)$

$\cos(-90°) = 0$

The angle −90^{\circ}\) is coterminal with $270°$. Therefore the ordered pair is (0, -1) and the cosine value is 0.

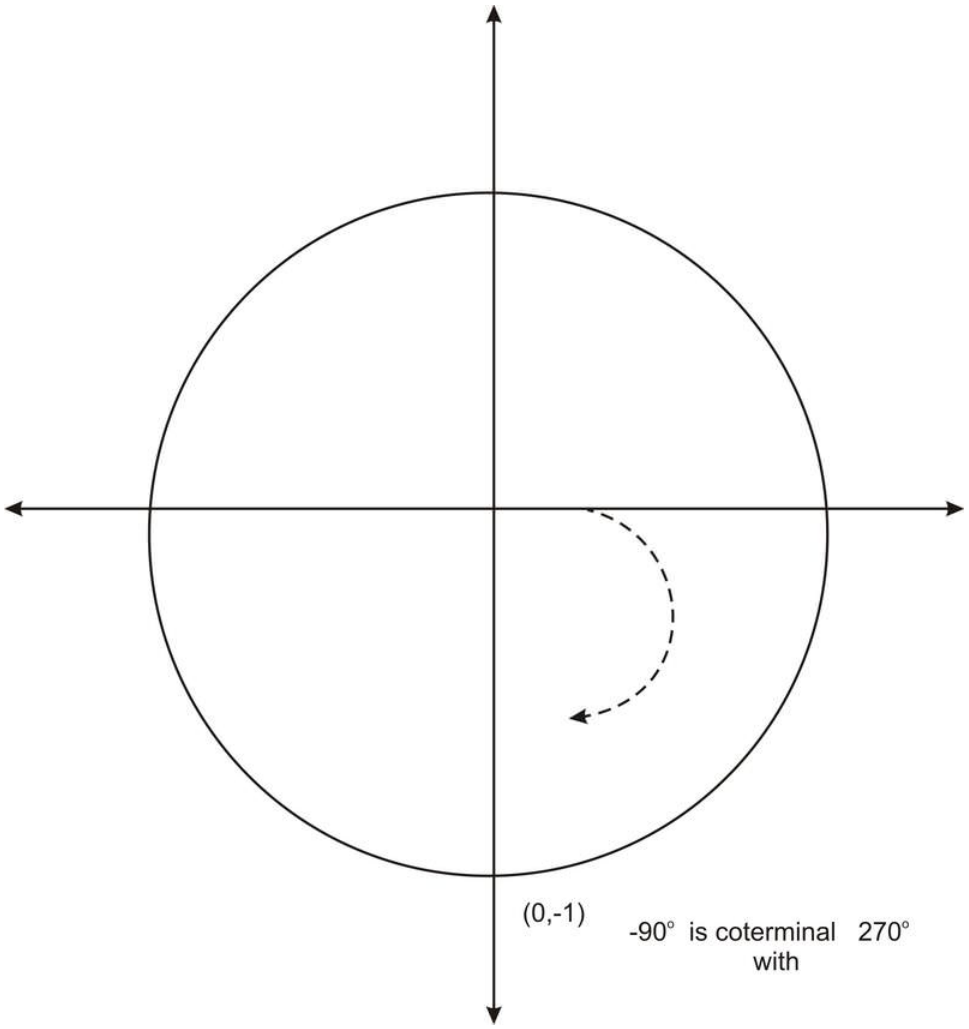


Figure 2.3.8.2

We can also use our knowledge of reference angles and ordered pairs to find the values of trig functions of angles with measure greater than 360 degrees.

Example 2.3.8.1

Earlier, you were asked if it is still possible to find the values of trig functions for the new type of angles.

Solution

What you want to find is the value of the expression: $\cos(-45°)$

$\cos(-45°) = \frac{\sqrt{2}}{2}$

$-45°$ is in the 4th quadrant, and has a reference angle of $45°$. That is, this angle is coterminal with $315°$. Therefore the ordered pair is $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ and the cosine value is $\frac{\sqrt{2}}{2}$.

Example 2.3.8.2

Find the value of the expression: $\cos -180°$

Solution

The angle -180° is coterminal with 180°. Therefore the ordered pair of points is $(-1, 0)$. The cosine is the "x" coordinate, so here it is -1.

Example 2.3.8.3

Find the value of the expression: $\sin -90^\circ$

Solution

The angle -90° is coterminal with 270°. Therefore the ordered pair of points is $(0, -1)$. The sine is the "y" coordinte, so here it is -1.

Example 2.3.8.4

Find the value of the expression: $\tan -270^\circ$

Solution

The angle -270° is coterminal with 90°. Therefore the ordered pair of points is $(0, 1)$. The tangent is the "y" coordinate divided by the "x" coordinate. Since the "x" coordinate is 0, the tangent is undefined.

Review

Calculate each value.

1. $\sin -120^\circ$
2. $\cos -120^\circ$
3. $\tan -120^\circ$
4. $\csc -120^\circ$
5. $\sec -120^\circ$
6. $\cot -120^\circ$
7. $\csc -45^\circ$
8. $\sec -45^\circ$
9. $\tan -45^\circ$
10. $\cos -135^\circ$
11. $\csc -135^\circ$
12. $\sec -135^\circ$
13. $\tan -210^\circ$
14. $\sin -270^\circ$
15. $\cot -90^\circ$

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.19.

Vocabulary

Term	Definition
Negative Angle	A negative angle is an angle measured by rotating clockwise (instead of counterclockwise) from the positive x axis.

Additional Resources

Video: Evaluating Trigonometric Functions of Any Angle - Overview

Practice: Trigonometric Functions of Negative Angles

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2.3.9: Trigonometric Functions of Angles Greater than 360 Degrees

Based on coterminal and reference angles.

While out at the local amusement park with friends, you take a ride on the Go Karts. You ride around a circular track in the carts three and a half times, and then stop at a "pit stop" to rest. While waiting for your Go Kart to get more fuel, you are talking with your friends about the ride. You know that one way of measuring how far something has gone around a circle (or the trig values associated with it) is to use angles. However, you've gone more than one complete circle around the track.

Is it still possible to find out what the values of sine and cosine are for the change in angle you've made?

Angles Greater Than 360°

Consider the angle $390°$. As you learned previously, you can think of this angle as a full 360 degree rotation, plus an additional 30 degrees. Therefore $390°$ is **coterminal** with $30°$. As you saw above with negative angles, this means that $390°$ has the same ordered pair as $30°$, and so it has the same trig values. For example,

$$\cos 390° = \cos 30° = \frac{\sqrt{3}}{2}$$

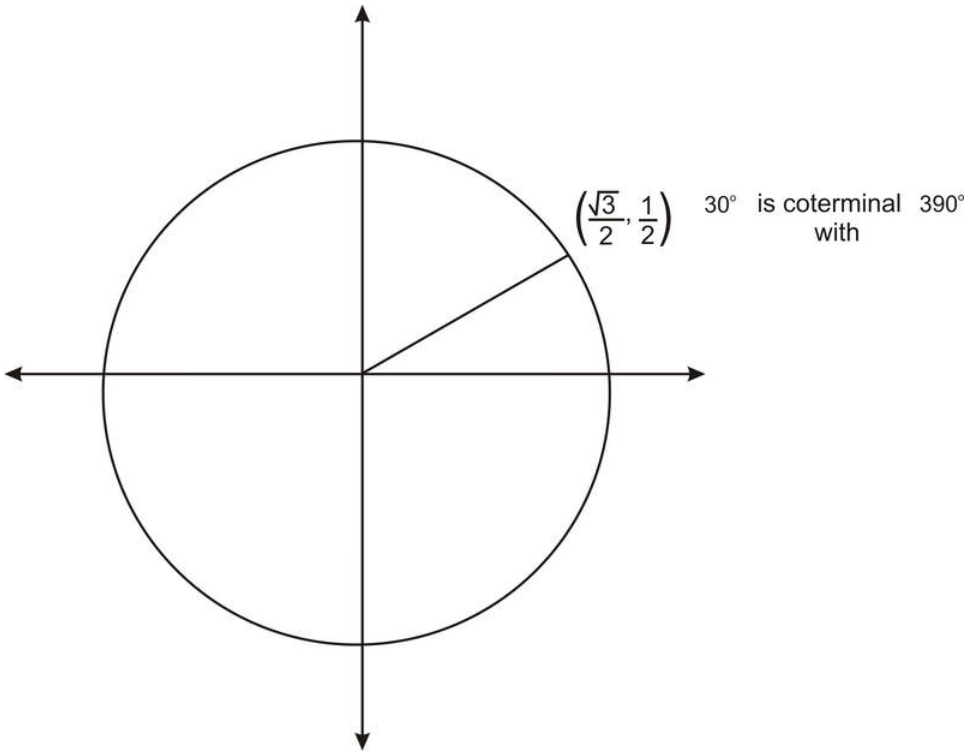
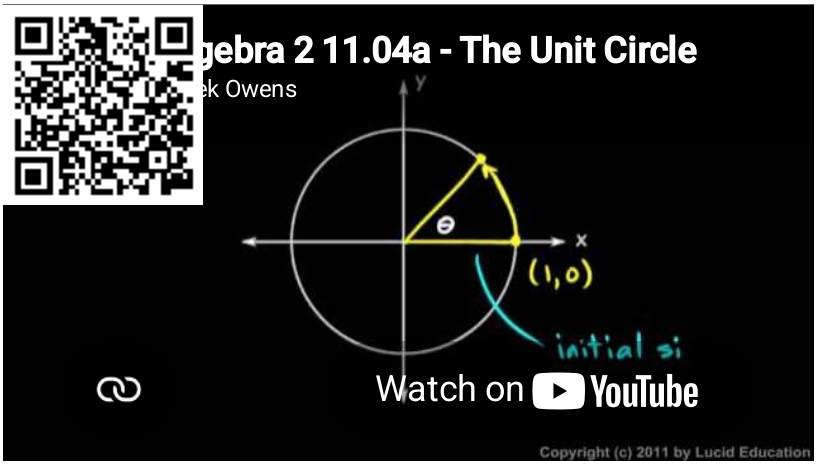


Figure 2.3.9.1



In general, if an angle whose measure is greater than

$360°$ has a reference angle of $30°$, $45°$, or $60°$, or if it is a quadrantal angle, we can find its ordered pair, and so we can find the values of any of the trig functions of the angle. Again, determine the reference angle first.

Let's look at some problems involving angles greater than $360°$.

Find the value of the following expressions:

1. $\sin 420°$

$$\sin 420° = \frac{\sqrt{3}}{2}$$

$420°$ is a full rotation of 360 degrees, plus an additional 60 degrees. Therefore the angle is coterminal with $60°$, and so it shares the same ordered pair, $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. The sine value is the y–coordinate.

2. $\tan 840°$

$$\tan 840° = -\sqrt{3}$$

$840°$ is two full rotations, or 720 degrees, plus an additional 120 degrees:

$$840 = 360 + 360 + 120$$

Therefore $840°$ is coterminal with $120°$, so the ordered pair is $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. The tangent value can be found by the following:

$$\tan 840° = \tan 120° = \frac{y}{x} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \frac{\sqrt{3}}{2} \times -\frac{2}{1} = -\sqrt{3}$$

3. $\cos 540°$

$$\cos 540° = -1$$

$540°$ is a full rotation of 360 degrees, plus an additional 180 degrees. Therefore the angle is coterminal with $180°$, and the ordered pair is $(-1, 0)$. So the cosine value is -1.

Example 2.3.9.1

Earlier, you were asked if it is still possible to find out what the values of sine and cosine are for the change in angle.

Solution

Since you've gone around the track 3.5 times, the total angle you've traveled is $360° \times 3.5 = 1260°$. However, as you learned in this unit, this is equivalent to $180°$. So you can use that value in your computations:

$$\sin 1260^\circ = \sin 180^\circ = 0$$
$$\cos 1260^\circ = \cos 180^\circ = -1$$

Example 2.3.9.2

Find the value of the expression: $\sin 570^\circ$

Solution

Since 570° has the same terminal side as 210°, $\sin 570^\circ = \sin 210^\circ = \frac{\frac{-1}{2}}{1} = \frac{-1}{2}$

Example 2.3.9.3

Find the value of the expression: $\cos 675^\circ$

Solution

Since 675° has the same terminal side as 315°, $\cos 675^\circ = \cos 315^\circ = \frac{\frac{\sqrt{2}}{2}}{1} = \frac{\sqrt{2}}{2}$

Example 2.3.9.4

Find the value of the expression: $\sin 480^\circ$

Solution

Since 480° has the same terminal side as 120°, $\sin 480^\circ = \sin 120^\circ = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2}$

Review

Find the value of each expression.

1. $\sin 405^\circ$
2. $\cos 810^\circ$
3. $\tan 630^\circ$
4. $\cot 900^\circ$
5. $csc 495^\circ$
6. $\sec 510^\circ$
7. $\cos 585^\circ$
8. $\sin 600^\circ$
9. $\cot 495^\circ$
10. $\tan 405^\circ$
11. $\cos 630^\circ$
12. $\sec 810^\circ$
13. $\csc 900^\circ$
14. $\tan 600^\circ$
15. $\sin 585^\circ$
16. $\tan 510^\circ$
17. Explain how to evaluate a trigonometric function for an angle greater than 360°.

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.20.

Additional Resources

Video: The Unit Circle

Practice: Trigonometric Functions of Angles Greater than 360 Degrees

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2.3.10: Exact Values for Inverse Sine, Cosine, and Tangent

Find all angles on unit circle matching exact trig ratios

You are working with a triangular brace in shop class. The brace is a right triangle, and the length of one side of the bracket is $\sqrt{3} \approx 1.732$ and it is connected to the other side at a right angle. The length of the other side is 1. You need to find the angle that the third piece makes with the first piece, labelled below as "C":

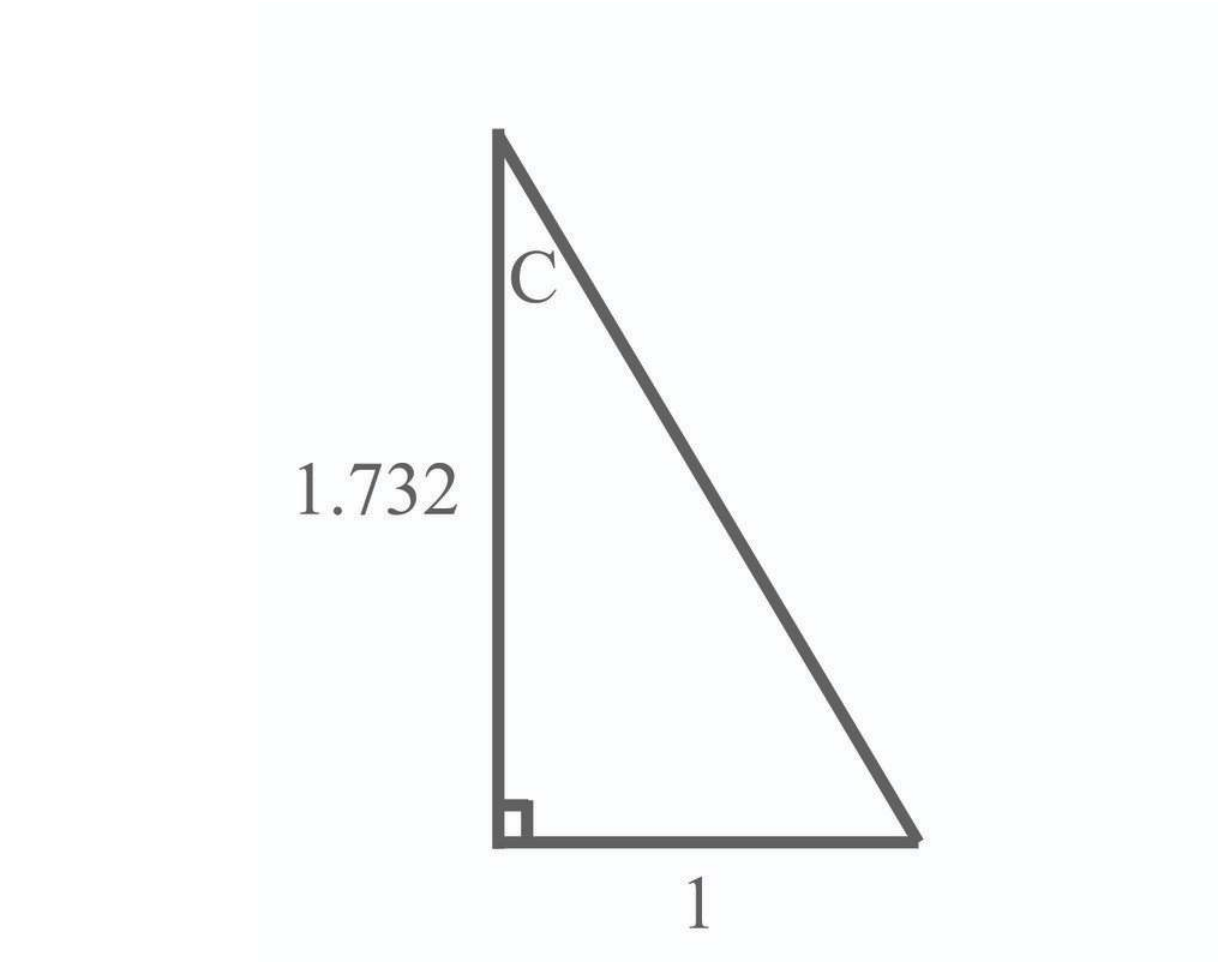


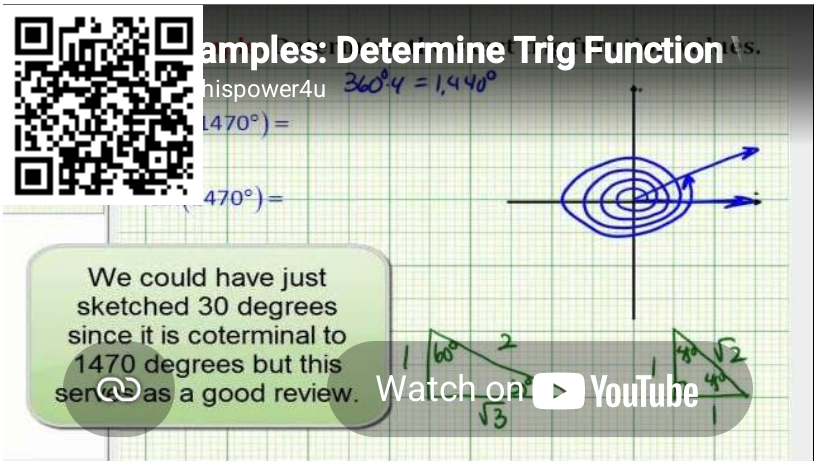
Figure 2.3.10.1

Can you find the angle between the legs of the brace?

By the time you finish reading this lesson, you'll be able to answer this question.

Inverse of Sine, Cosine and Tangent

Inverse trig functions can be useful in a variety of math problems for finding angles that you need to know. In many cases, such as angles involving multiples of $30°$, $60°$ and $90°$, the values of trig functions are often memorized, since they are used so often.



Recall the unit circle and the critical values. With the inverse trigonometric functions, you can find the angle value (in either radians or degrees) when given the ratio and function. Make sure that you find all solutions within the given interval.

Let's take a look at a few example problems.

1. Find the exact value of the expression without a calculator, in $[0, 2\pi)$.

$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

This is a value from the special right triangles and the unit circle.

Recall that $-\frac{\sqrt{3}}{2}$ is from the 30−60−90 triangle. The reference angle for sin and $\frac{\sqrt{3}}{2}$ would be $60°$. Because this is sine and it is negative, it must be in the third or fourth quadrant. The answer is either $\frac{4\pi}{3}$ or $\frac{5\pi}{3}$.

2. Find the exact value of the expression without a calculator, in $[0, 2\pi)$.

$\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

This is a value from the special right triangles and the unit circle.

$-\frac{\sqrt{2}}{2}$ is from an isosceles right triangle. The reference angle is then $45°$. Because this is cosine and negative, the angle must be in either the second or third quadrant. The answer is either $\frac{3\pi}{4}$ or $\frac{5\pi}{4}$.

3. Find the exact value of the expression without a calculator, in $[0, 2\pi)$.

$\tan^{-1}\sqrt{3}$

This is a value from the special right triangles and the unit circle.

$\sqrt{3}$ is also from a 30−60−90 triangle. Tangent is $\sqrt{3}$ for the reference angle $60°$. Tangent is positive in the first and third quadrants, so the answer would be $\frac{\pi}{3}$ or $\frac{4\pi}{3}$.

Example 2.3.10.1

Earlier, you were asked to find the angle between the legs of the brace.

Solution

Using your knowledge of the values of trig functions for angles, you can work backward to find the angle that the brace makes:

$$\tan C = \frac{1}{\sqrt{3}}$$
$$\tan^{-1} C = \tan^{-1}\frac{1}{\sqrt{3}}$$
$$C = 60°$$

Example 2.3.10.2

Find the exact value of the inverse function of $\cos^{-1}(0)$, without a calculator in $[0, 2\pi)$

Solution

$\frac{\pi}{2}, \frac{3\pi}{2}$

Example 2.3.10.3

Find the exact value of the inverse function of $\tan^{-1}(-\sqrt{3})$, without a calculator in $[0, 2\pi)$

Solution

$\frac{2\pi}{3}, \frac{5\pi}{3}$

Example 2.3.10.4

Find the exact value of the inverse function of $\sin^{-1}(-12)$, without a calculator in $[0, 2\pi)$

Solution

$\frac{11\pi}{6}, \frac{7\pi}{6}$

Review

Find the exact value of each expression without a calculator, in $[0, 2\pi)$.

1. $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$
2. $\cos^{-1}\left(\frac{1}{2}\right)$
3. $\sin^{-1}(1)$
4. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
5. $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$
6. $\tan^{-1}(-1)$
7. $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$
8. $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$
9. $\csc^{-1}(\sqrt{2})$
10. $\sec^{-1}(-2)$
11. $\cot^{-1}\left(\frac{\sqrt{3}}{3}\right)$
12. $\sec^{-1}\left(\frac{2\sqrt{3}}{2}\right)$
13. $\csc^{-1}\left(-\frac{2\sqrt{3}}{2}\right)$
14. $\cot^{-1}(-\sqrt{3})$
15. $\cot^{-1}(-1)$

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.2.

Vocabulary

Term	Definition
Trigonometric Inverse	A trigonometric inverse is a function that undoes a trigonometric function to give the original argument of the function. It can also be used to find a missing angle of a triangle from the ratio of two sides of the triangle.

Additional Resources

Video: Examples: Determine Trig Function Values Using Reference Triangles

Practice: Exact Values for Inverse Sine, Cosine, and Tangent

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2.4: Inverse Trigonometric Functions

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2.4.1: Inverse Trig Functions

Solving for an angle given a trigonometric ratio.

Inverse Trig Functions and Solving Right Triangles

A right triangle has legs that measure 2 units and $2\sqrt{3}$ units. What are the measures of the triangle's acute angles?

Inverse of Trigonometric Functions

We have used the trigonometric functions sine, cosine and tangent to find the ratio of particular sides in a right triangle given an angle. In this concept we will use the inverses of these functions, $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$, to find the angle measure when the ratio of the side lengths is known. When we type $\sin 30°$ into our calculator, the calculator goes to a table and finds the trig ratio associated with $30°$, which is 12. When we use an inverse function we tell the calculator to look up the ratio and give us the angle measure. For example: $\sin^{-1}\left(\frac{1}{2}\right) = 30°$. On your calculator you would press $2^{ND}SIN$ to get $\sin^{-1}$ and then type in $\frac{1}{2}$, close the parenthesis and press ENTER. Your calculator screen should read $\sin^{-1}\left(\frac{1}{2}\right)$ when you press ENTER.

Let's find the measure of angle A associated with the following ratios and round answers to the nearest degree.

1. $\sin A = 0.8336$
2. $\tan A = 1.3527$
3. $\cos A = 0.2785$

Using the calculator we get the following:

1. $\sin^{-1}(0.8336) \approx 56°$
2. $\tan^{-1}(1.3527) \approx 54°$
3. $\cos^{-1}(0.2785) \approx 74°$

Now, let's find the measures of the unknown angles in the triangle shown and round answers to the nearest degree.

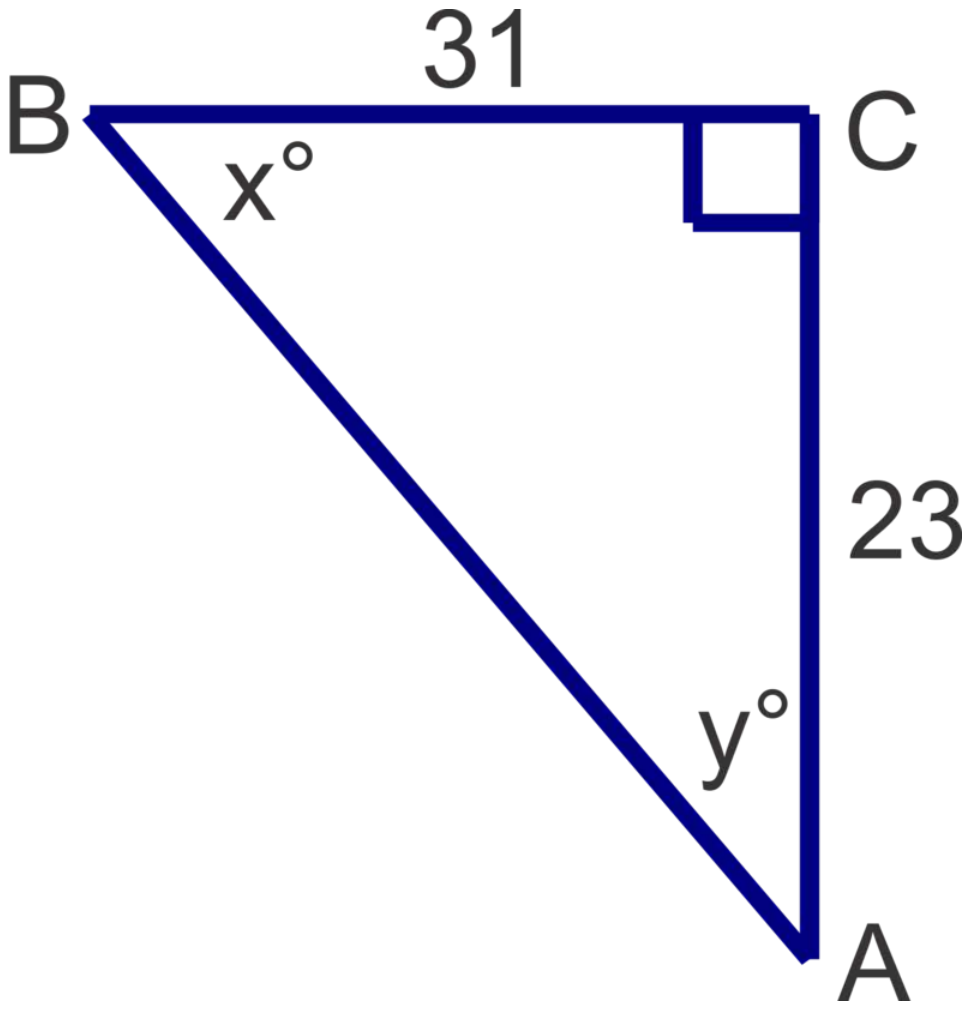


Figure 2.4.1.1

We can solve for either x or y first. If we choose to solve for x first, the 23 is opposite and 31 is adjacent so we will use the tangent ratio.

$x = \tan^{-1}\left(\frac{23}{31}\right) \approx 37°$.

Recall that in a right triangle, the acute angles are always complementary, so $90° - 37° = 53°$, so $y = 53°$. We can also use the side lengths an a trig ratio to solve for y:

$y = \tan^{-1}\left(\frac{31}{23}\right) \approx 53°$.

Finally, let's solve the right triangle shown below and round all answers to the nearest tenth.

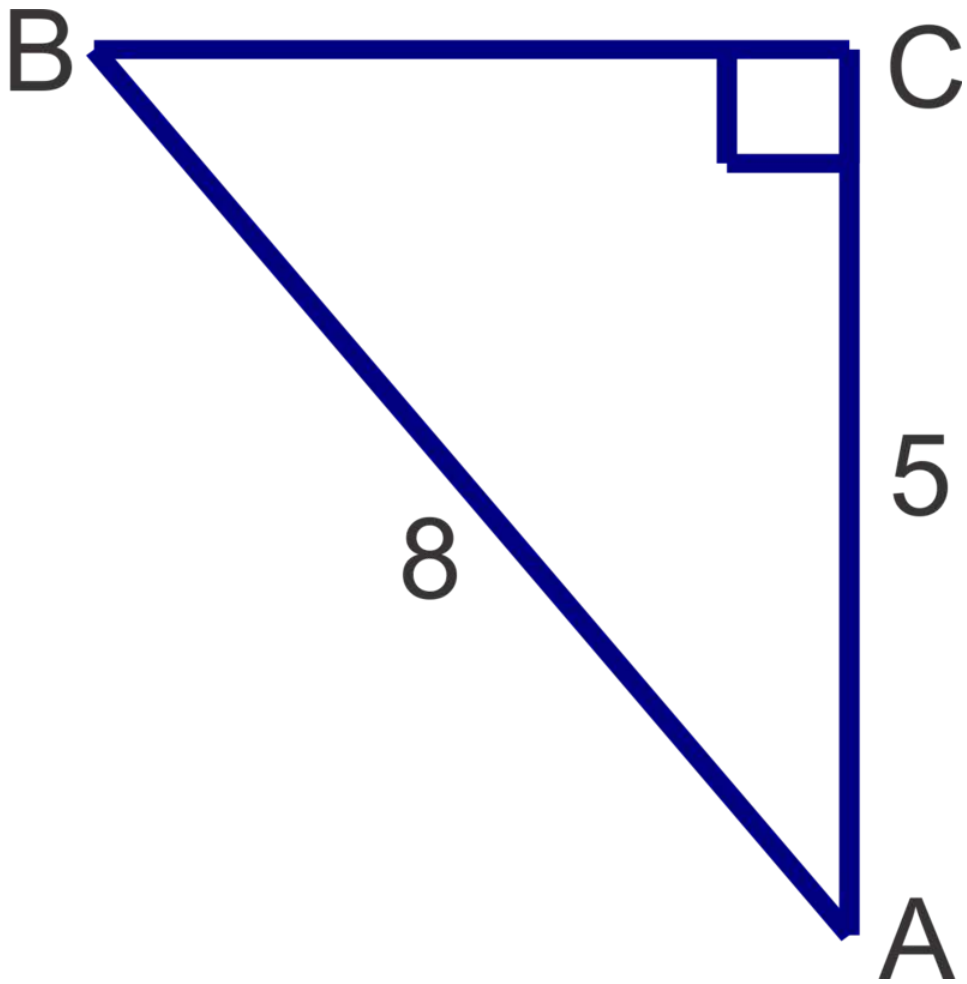


Figure **2.4.1.2**

We can solve for either angle A or angle B first. If we choose to solve for angle B first, then 8 is the hypotenuse and 5 is the opposite side length so we will use the sine ratio.

$$\sin B = \frac{5}{8}$$
$$m\angle B = \sin^{-1}\left(\frac{5}{8}\right) \approx 38.7^\circ$$

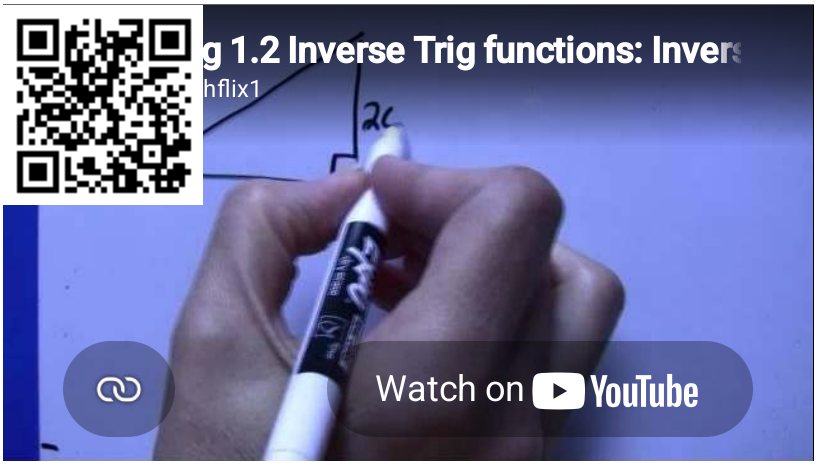
Now we can find A two different ways.

Method 1: We can using trigonometry and the cosine ratio:

$$\cos A = \frac{5}{8}$$
$$m\angle A = \cos^{-1}\left(\frac{5}{8}\right) \approx 51.3^\circ$$

Method 2: We can subtract $m\angle B$ from 90° : $90^\circ - 38.7^\circ = 51.3^\circ$ since the acute angles in a right triangle are always complimentary.

Either method is valid, but be careful with Method 2 because a miscalculation of angle B would make the measure you get for angle A incorrect as well.



Example 2.4.1.1

Earlier, you were asked to find the measures of the triangle's acute angles.

Solution

First, let's find the hypotenuse, then we can solve for either angle.

$$\begin{aligned} 2^2 + (2\sqrt{3})^2 &= c^2 \\ 4 + 12 &= c^2 \\ 16 &= c^2 \\ c &= 4 \end{aligned}$$

One of the acute angles will have a sine of $\frac{2}{4} = \frac{1}{2}$.

$$\sin A = \frac{1}{2}$$

$$m\angle A = \sin^{-1} \frac{1}{2} = 30^\circ$$

Now we can find B by subtracting $m\angle A$ from 90°: $90^\circ - 30^\circ = 60^\circ$ since the acute angles in a right triangle are always complimentary.

Example 2.4.1.2

Find the measure of angle A.

Solution

$$\sin A = 0.2894$$

$$\sin^{-1}(0.2894) \approx 17^\circ$$

\)Example 2.4.1.3

Find the measure of angle A.

Solution

$$\tan A = 2.1432$$

$$\tan^{-1}(2.1432) \approx 65^\circ$$

Example 2.4.1.4

Find the measure of angle A.

Solution

$$\cos A = 0.8911$$
$$\cos^{-1}(0.8911) \approx 27^\circ$$

Example 2.4.1.5

Find the measures of the unknown angles in the triangle shown. Round answers to the nearest degree.

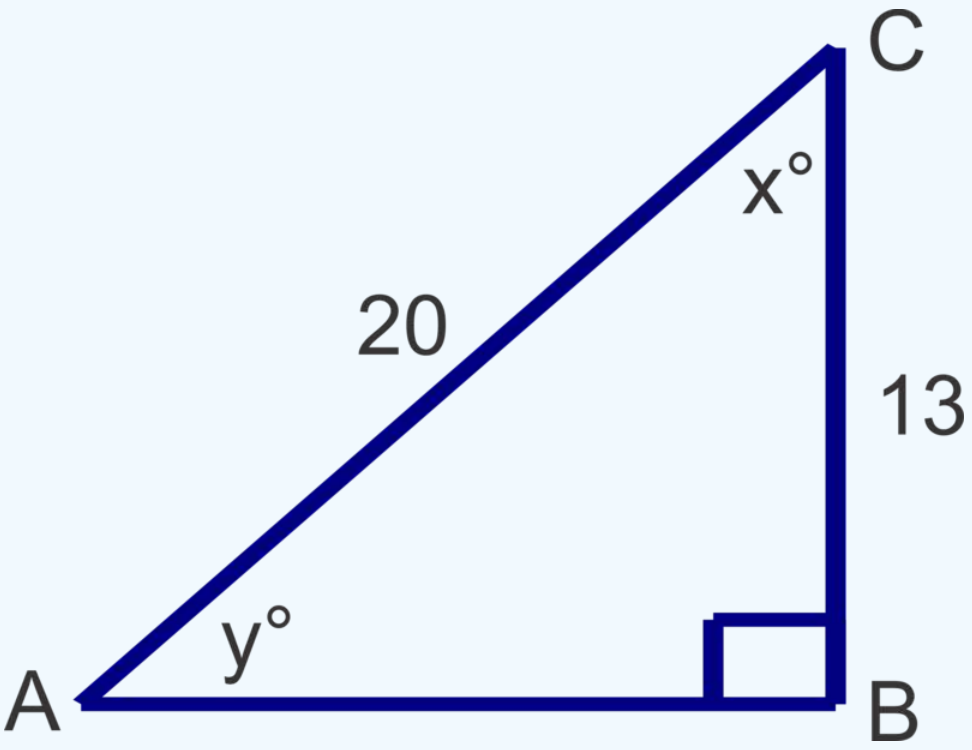


Figure 2.4.1.3

Solution

$$x = \cos^{-1}\left(\frac{13}{20}\right) \approx 49^\circ; \quad y = \sin^{-1}(\frac{13}{20}) \approx 41^\circ$$

Example 2.4.1.6

Solve the triangle. Round side lengths to the nearest tenth and angles to the nearest degree.

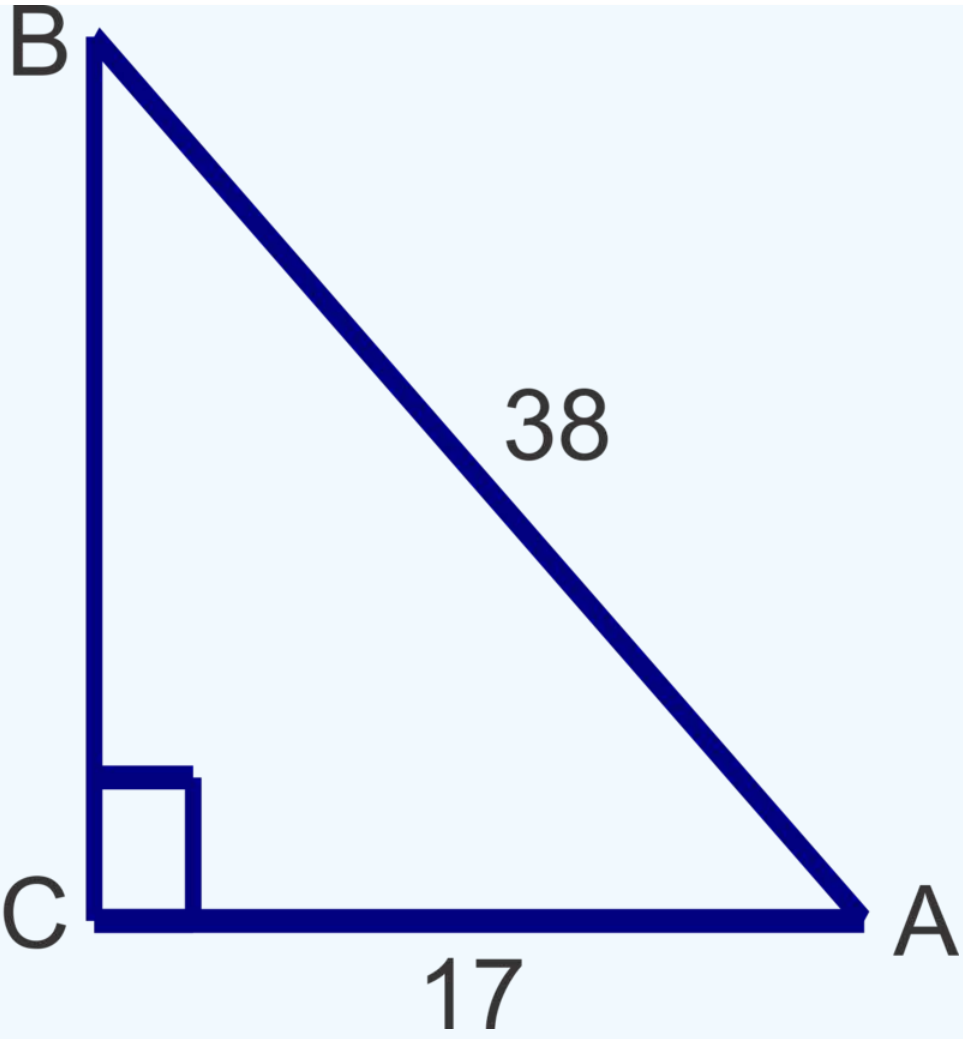


Figure 2.4.1.4

Solution

$$m\angle A = \cos^{-1}(\frac{17}{38}) \approx 63°; m\angle B = \sin^{-1}(\frac{17}{38}) \approx 27°; a = \sqrt{38^2 - 17^2} \approx 34.0$$

Review

Use your calculator to find the measure of $\angle B$. Round answers to the nearest degree.

1. $\tan B = 0.9523$
2. $\sin B = 0.8659$
3. $\cos B = 0.1568$
4. $\sin B = 0.2234$
5. $\cos B = 0.4855$
6. $\tan B = 0.3649$

Find the measures of the unknown acute angles. Round measures to the nearest degree.

7.
8.
9.
10.
11.
12.

Solve the following right triangles. Round angle measures to the nearest degree and side lengths to the nearest tenth.

13.
14.
15.

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 13.3.

Additional Resources

Video: Evaluating Inverse Trigonometric Functions Without Using the Calculator - Overview

Practice: Inverse Trig Functions

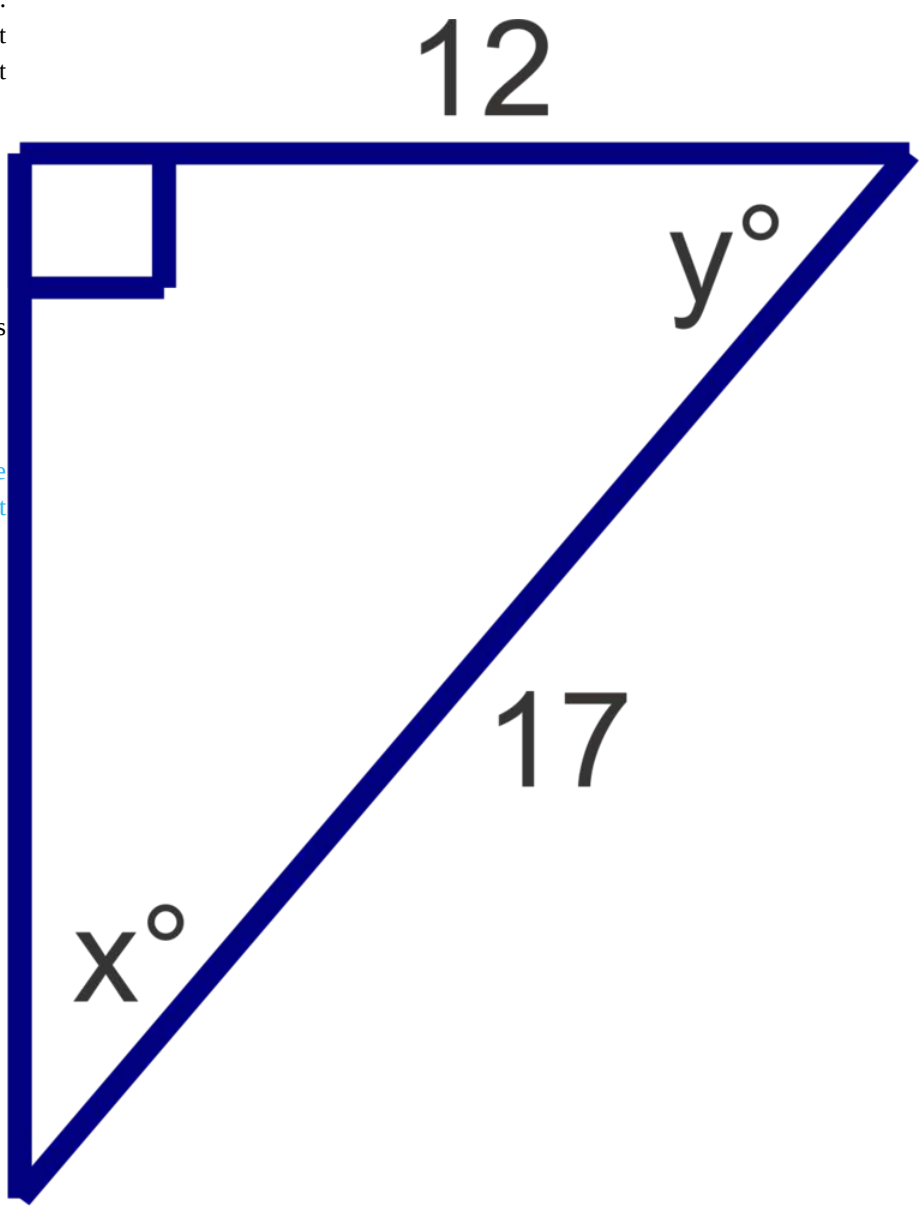


Figure 2.4.1.5

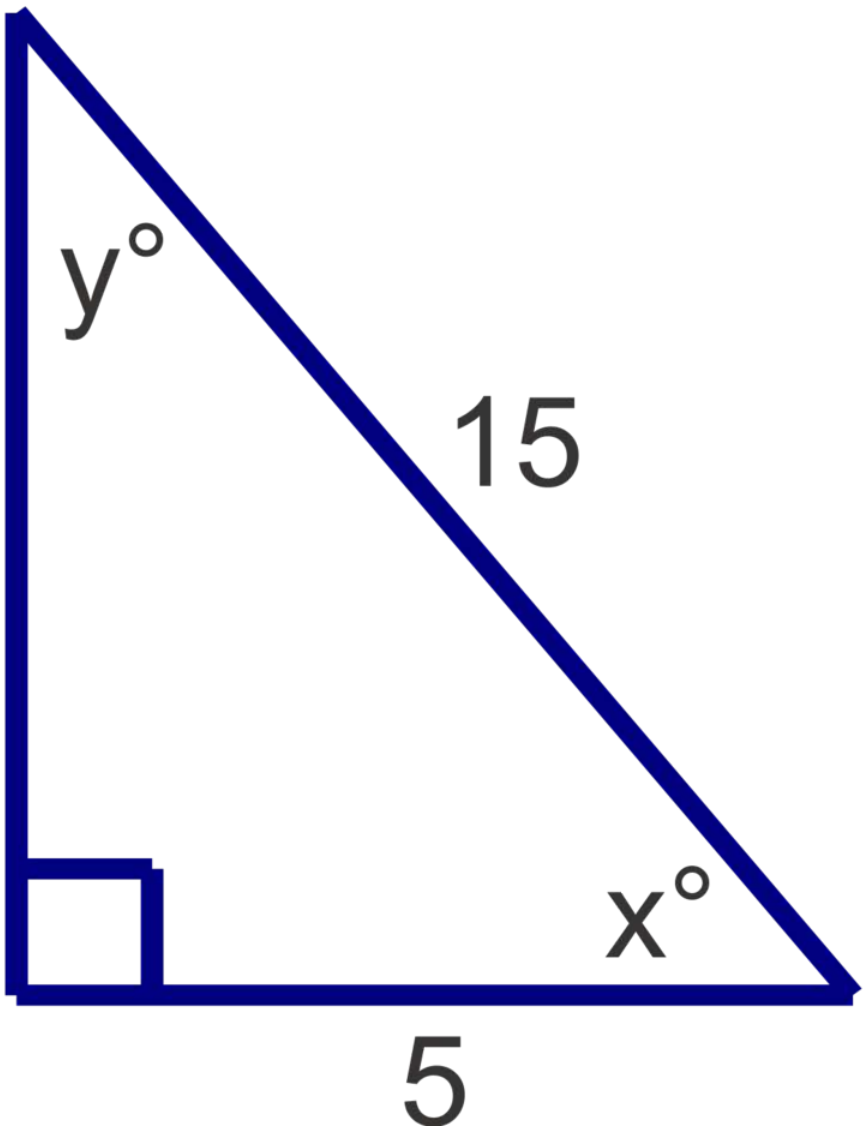


Figure **2.4.1.6**

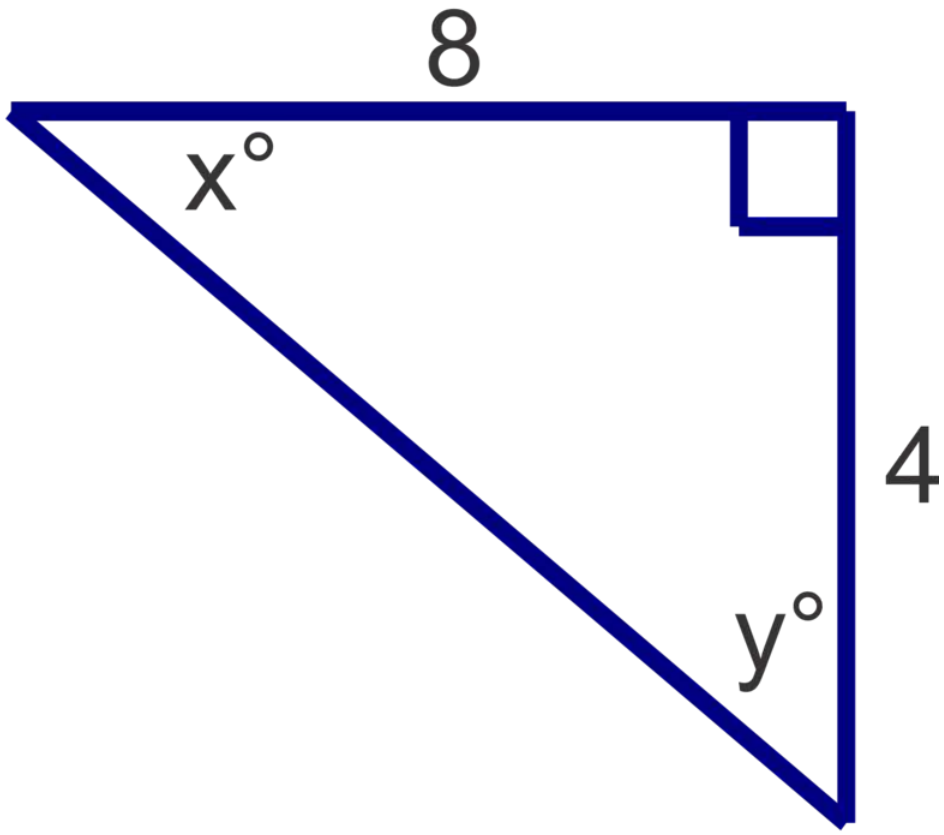


Figure **2.4.1.7**

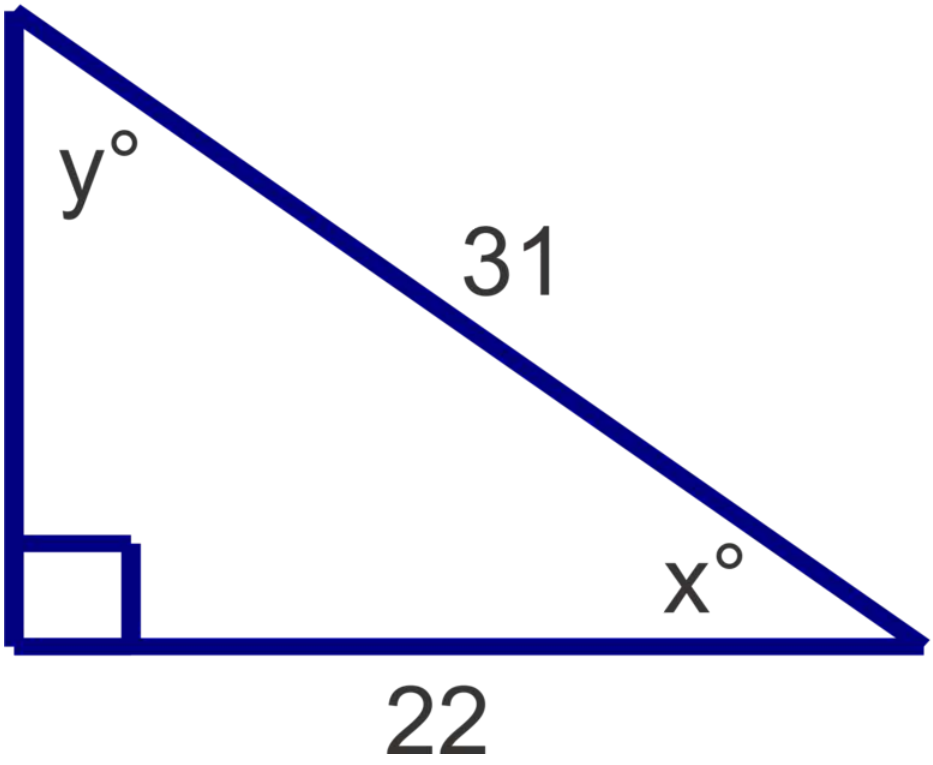


Figure **2.4.1.8**

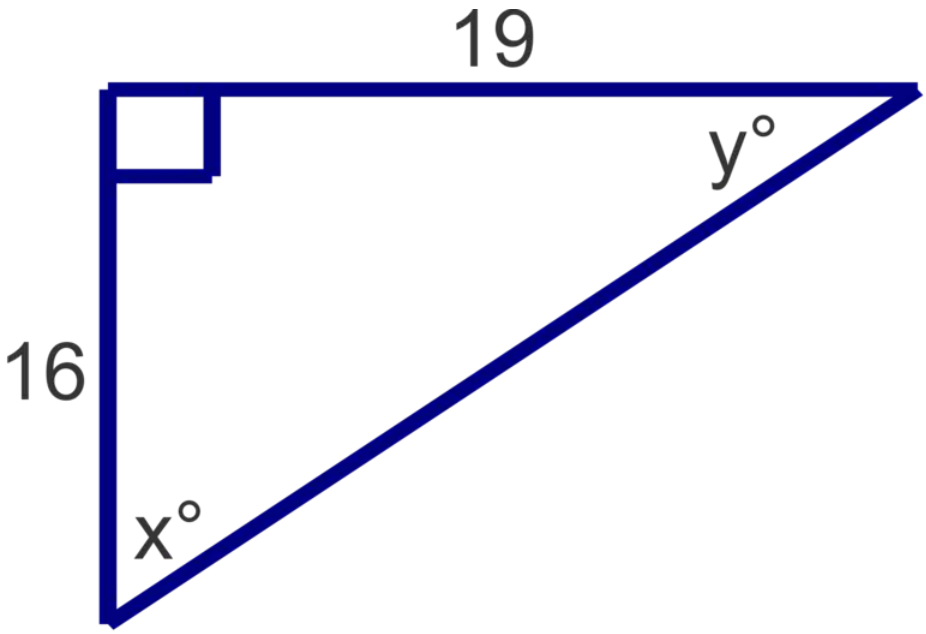


Figure **2.4.1.9**

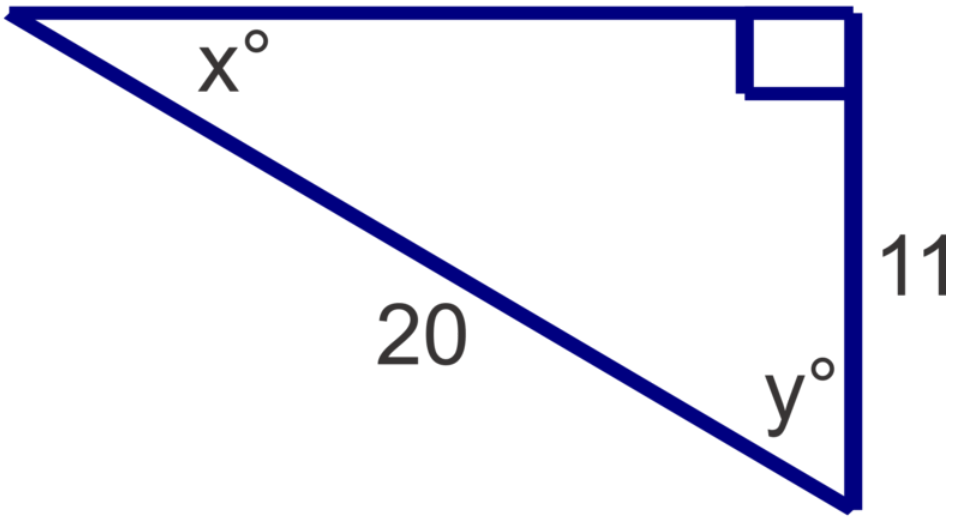


Figure **2.4.1.10**

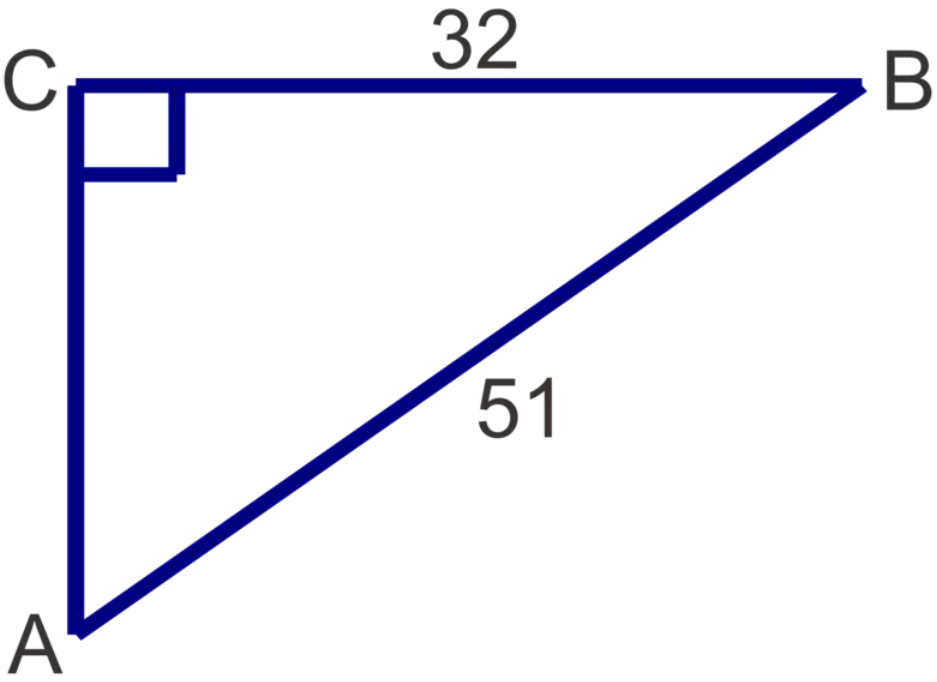


Figure 2.4.1.11

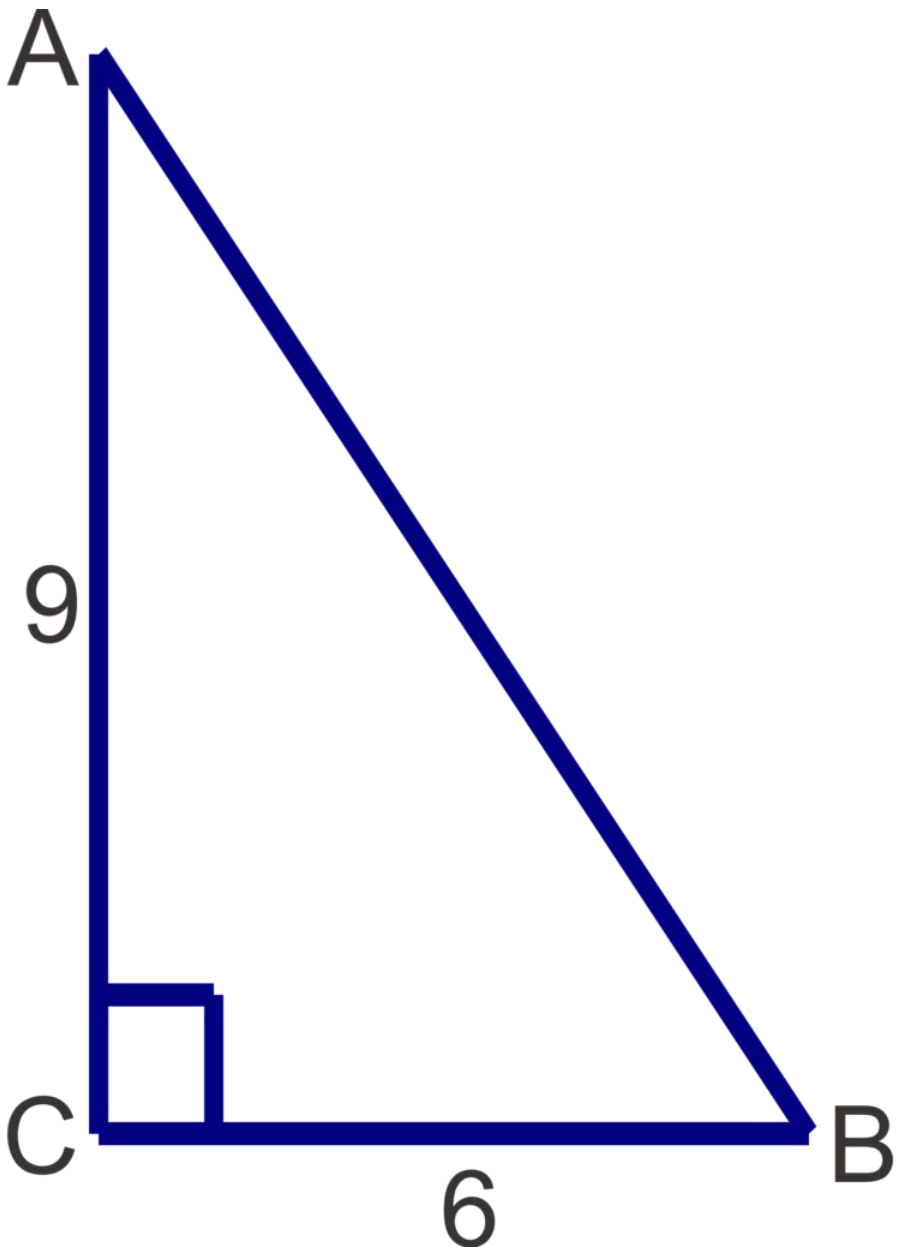


Figure 2.4.1.12

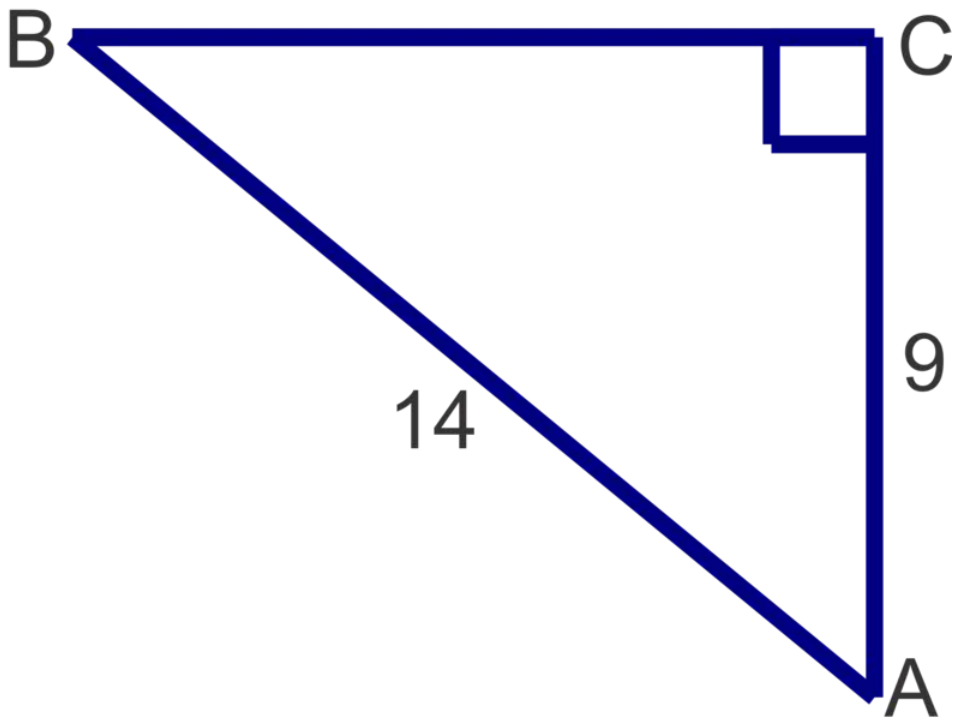


Figure 2.4.1.13

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2.4.2: Inverses by Mapping

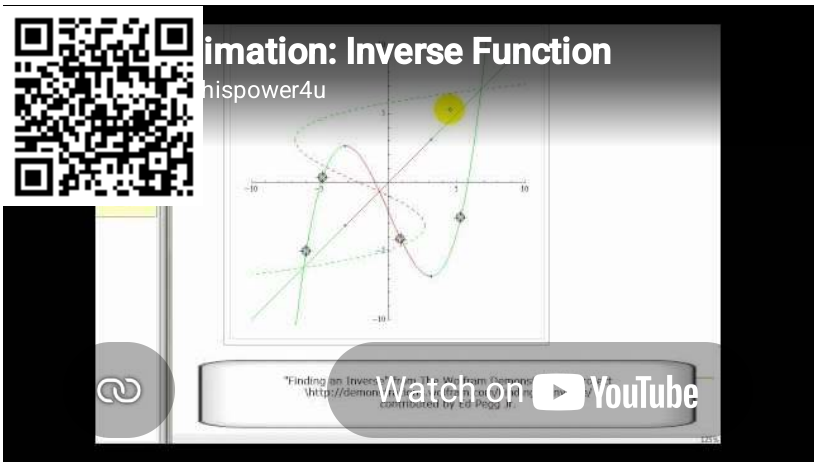
Reflect points from $f(x)$ across the line $y = x$.

Your instructor gives you a function, $f(x) = (x-1)^2 + 3$, where $x \geq 1$, and asks you to find the inverse. You are all set to start manipulating the equation, when your Instructor specifies that you should find the inverse by graphing instead of by algebraic manipulation.

Are you able to do this?

Finding the Inverse by **Mapping**

Determining an **inverse function** algebraically can be both involved and difficult, so it is useful to know how to map f to f^{-1}.



The graph of f can be used to produce the graph of f^{-1} by applying the inverse reflection principle:

The points (a, b) and (b, a) in the coordinate plane are symmetric with respect to the line $y = x$.

The points (a, b) and (b, a) are reflections of each other across the line $y = x$.

Finding the Inverse of a Function

1. Find the inverse of $f(x) = \dfrac{1}{x-5}$ by mapping.

From the last section, we know that the inverse of this function is $y = \dfrac{5x+1}{x}$. To find the inverse by mapping, pick several points on $f(x)$, reflect them using the reflection principle and plot. Note: The coordinates of some of the points are rounded.

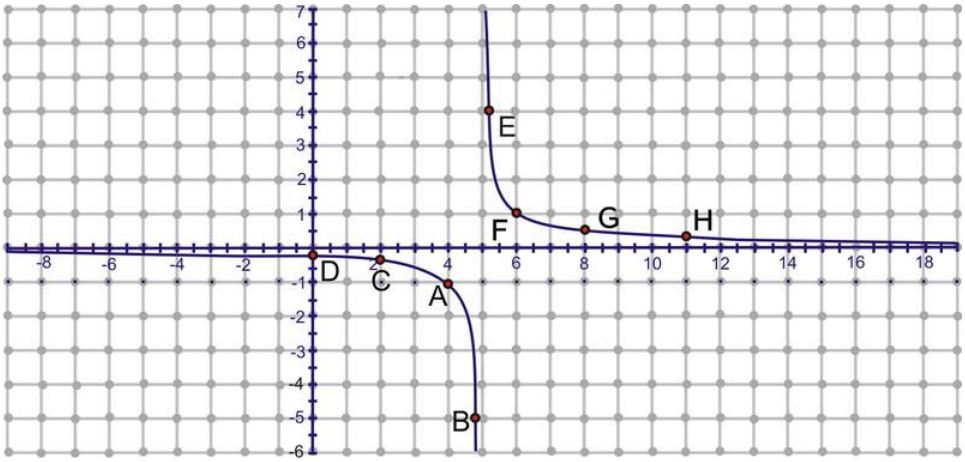


Figure 2.4.2.1

$A : (4, -1)$

B : $(4.8, -5)$

C : $(2, -0.3)$

D : $(0, -0.2)$

E : $(5.3, 3.3)$

F : $(6, 1)$

G : $(8, 0.3)$

H : $(11, 0.2)$

Now, take these eight points, switch the x and y and plot (y, x). Connect them to make the inverse function.

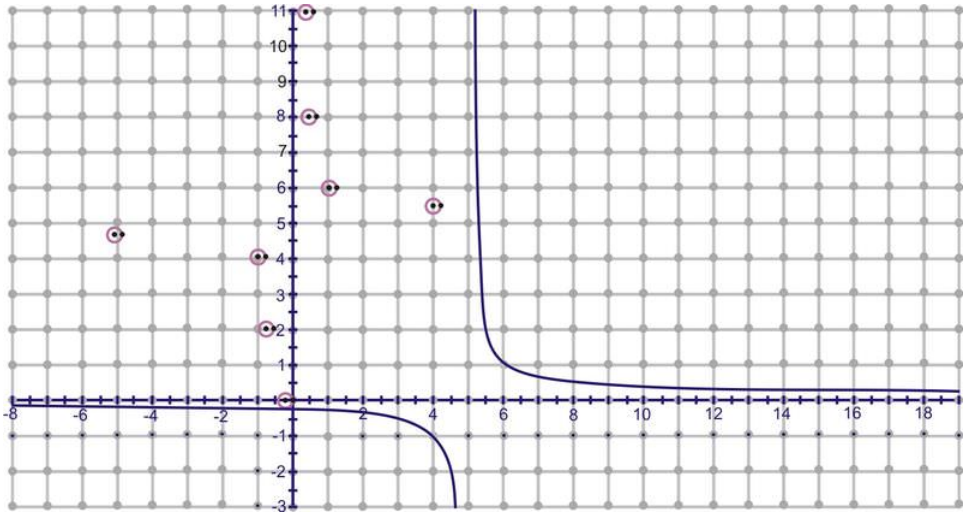


Figure 2.4.2.2

A^{-1} : $(-1, 4)$

B^{-1} : $(-5, 4.8)$

C^{-1} : $(-0.3, 2)$

D^{-1} : $(-0.2, 0)$

E^{-1} : $(3.3, 5.3)$

F^{-1} : $(1, 6)$

G^{-1} : $(0.3, 8)$

H^{-1} : $(0.2, 11)$

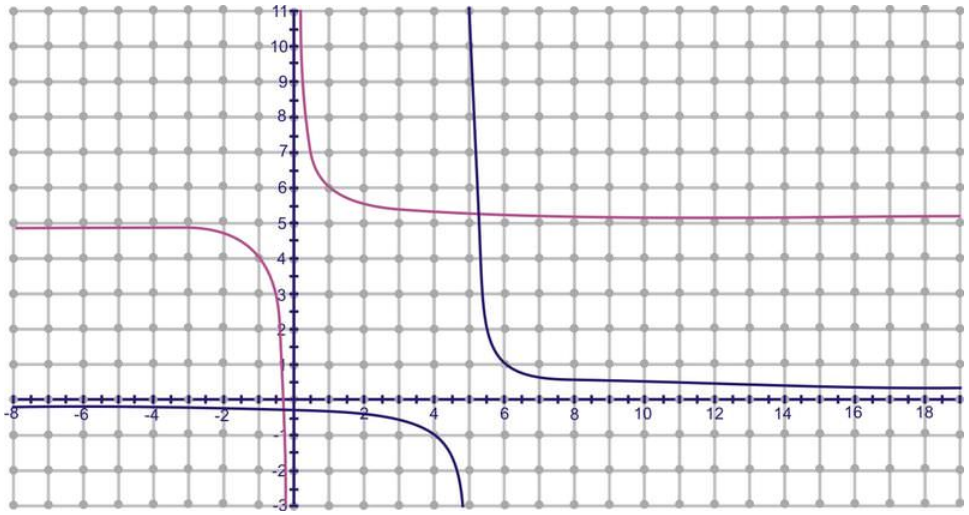


Figure **2.4.2.3**

Not all functions have inverses that are one-to-one. However, the inverse can be modified to a one-to-one function if a "restricted domain" is applied to the inverse function.

2. Find the inverse of $f(x) = x^2 - 4$.

Let's use the graphic approach for this one. The function is graphed in blue and its inverse is red.

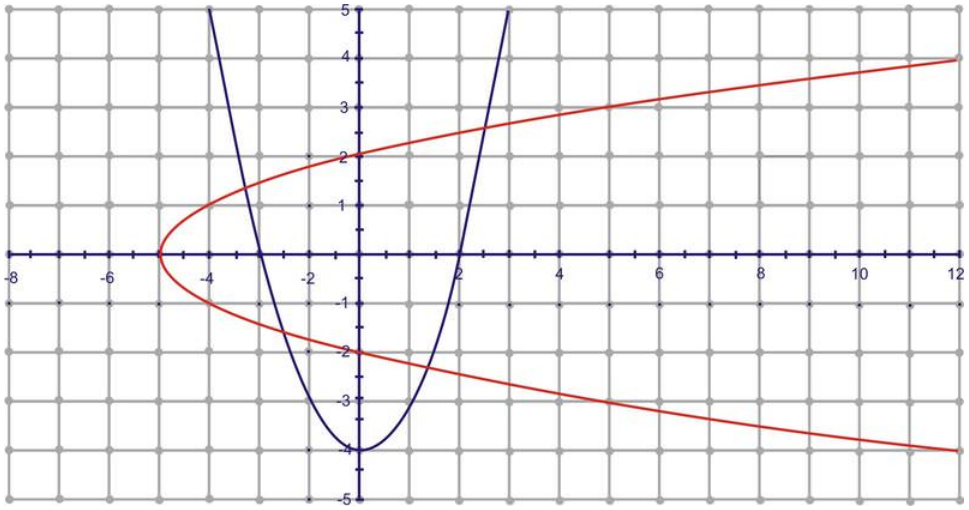


Figure **2.4.2.4**

Clearly, the inverse relation is not a function because it does not pass the vertical line test. This is because all parabolas fail the horizontal line test. To "make" the inverse a function, we restrict the domain of the original function. For parabolas, this is fairly simple. To find the inverse of this function algebraically, we get $f^{-1}(x) = \sqrt{x+4}$. Technically, however, the inverse is $\pm\sqrt{x+4}$ because the square root of any number could be positive or negative. So, the inverse of $f(x) = x^2 - 4$ is both parts of the square root equation, $\sqrt{x+4}$ and $-\sqrt{x+4}$. $\sqrt{x+4}$ will yield the top portion of the horizontal parabola and $-\sqrt{x+4}$ will yield the bottom half. Be careful, because if you just graph $f^{-1}(x) = \sqrt{x+4}$ in your graphing calculator, it will only graph the top portion of the inverse.

This technique of sectioning the inverse is applied to finding the inverse of trigonometric functions because it is periodic.

3. Find the inverse of $f(x) = \dfrac{x-1}{3x+2}$ mapping.

To find the inverse by mapping, pick several points on $f(x)$, reflect them using the reflection principle and plot. Note: The coordinates of some of the points are rounded.

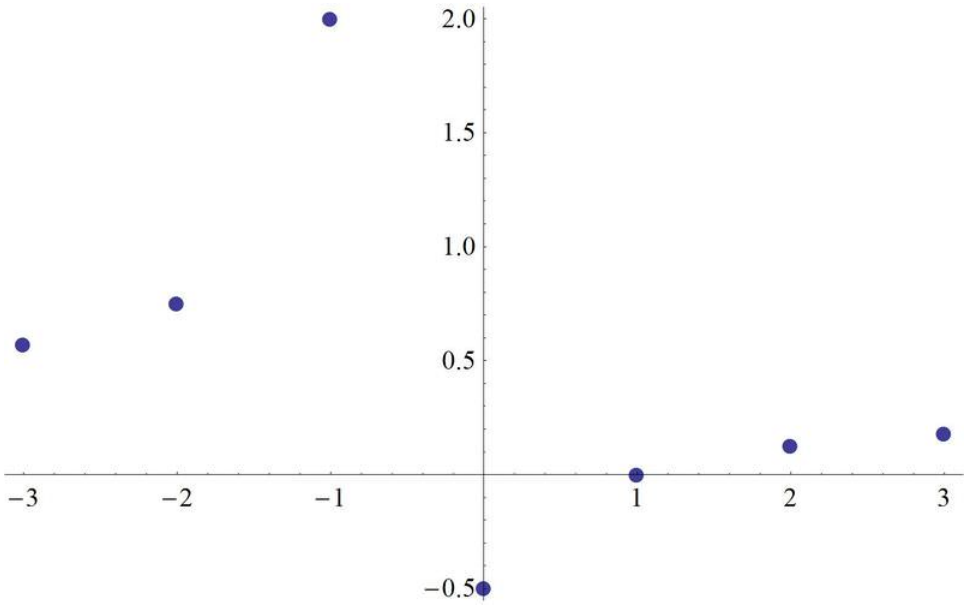


Figure **2.4.2.5**

A : $(0, -.5)$

B : $(-1, 2)$

C : $(1, 0)$

D : $(-2, .75)$

E : $(2, .125)$

F : $(-3, .57)$

G : $(3, .18)$

Now, take these seven points, switch the x and y and plot (y, x). Connect them to make the inverse function.

A^{-1} : $(-.5, 0)\backslash)$

B^{-1} : $(2, -1)$

C^{-1} : $(0, 1)$

D^{-1} : $(.75, -2)$

E^{-1} : $(.125, 2)$

F^{-1} : $(.57, -3)$

G^{-1} : $(.18, 3)$

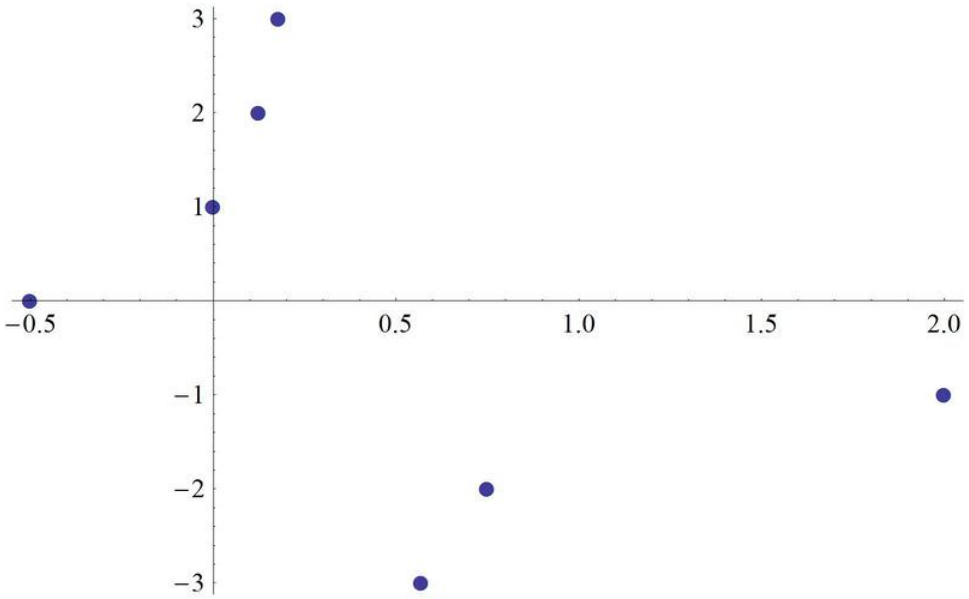


Figure **2.4.2.6**

Not all functions have inverses that are one-to-one. However, the inverse can be modified to a one-to-one function if a "restricted domain" is applied to the inverse function.

Example 2.4.2.1

Earlier, you were asked to find the inverse of a function.

The original equation is $f(x) = (x-1)^2 + 3$.

Solution

Here is a plot of the function:

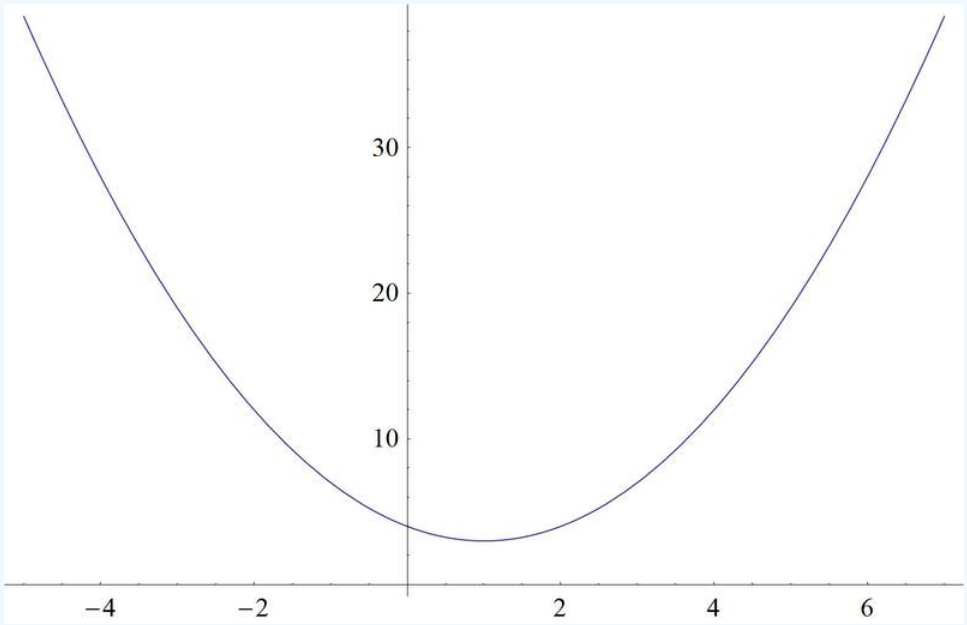


Figure **2.4.2.7**

Notice that the domain of the function under examination has to be restricted to 1 or greater, so that the function will pass the horizontal line test. Some points that are on this graph are:

$A: (1, 3)$

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$B : (2, 4)$

$C : (3, 7)$

To map the inverse function, first take each point and switch the "x" and "y" values:

$A^{-1} : (3, 1)$

$B^{-1} : (4, 2)$

$C^{-1} : (7, 3)$

Then connect these dots, and you can see the graph of the inverse function. The inverse function graph looks like this:

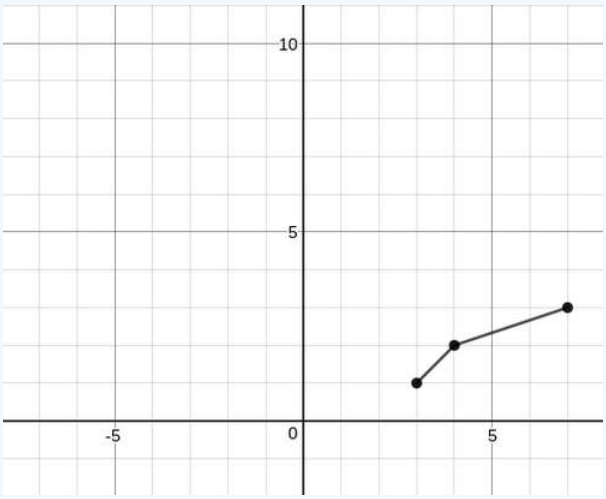


Figure 2.4.2.8

In this case, the range of the function has to be restricted to be 1 or greater, so that the inverse function will pass the vertical line test.

Example 2.4.2.2

Study the following graph and answer these questions:

Is the graphed relation a function?

Does the relation have an inverse that is a function?

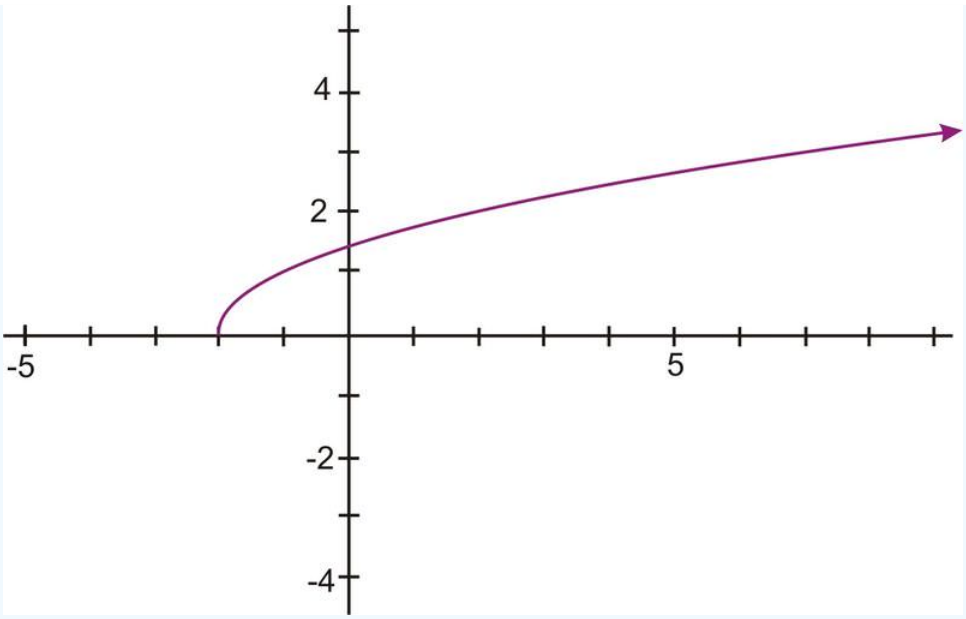


Figure **2.4.2.9**

Solution

The graph represents a one-to-one function. It passes both a vertical and a horizontal line test. The inverse would be a function.

Example 2.4.2.3

Find the inverse of $f(x) = x^2 + 2x - 15$ using the mapping principle.

Solution

By selecting 4-5 points and switching the x and y values, you will get the red graph below.

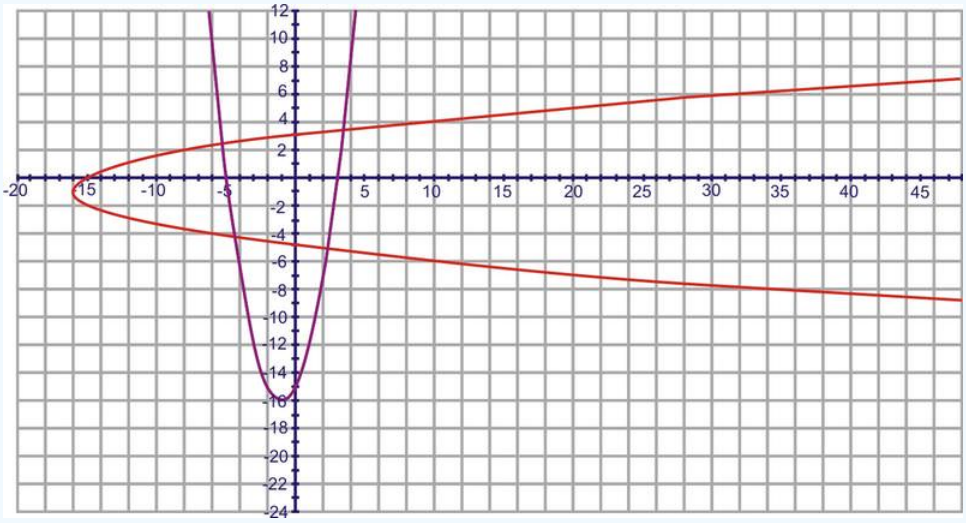


Figure **2.4.2.10**

Example 2.4.2.4

Find the inverse of $y = 1 + 2\sin x$ using the mapping principle.

Solution

By selecting 4-5 points and switching the x and y values, you will get the red graph below.

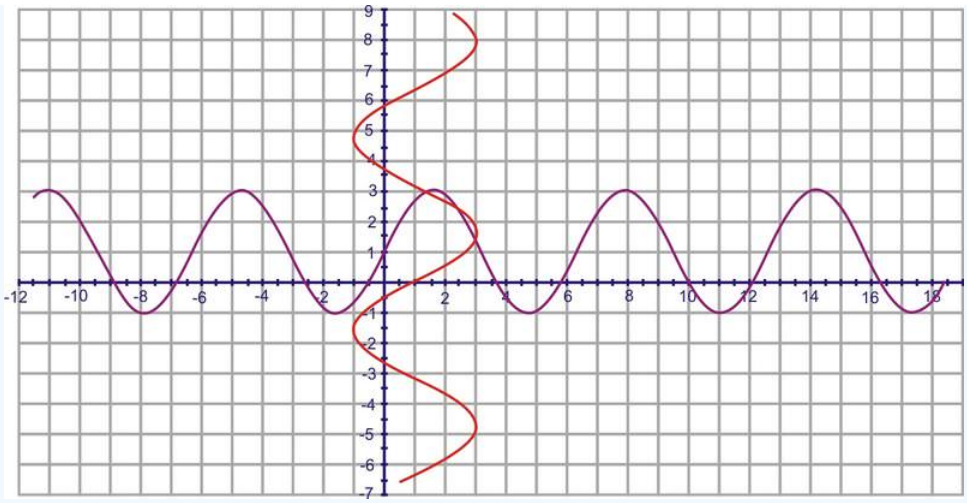


Figure 2.4.2.11

Review

For each of the following graphs answer these questions:

(a) Is the graphed relation a function?

(b) Does the relation have an inverse that is a function?

1.

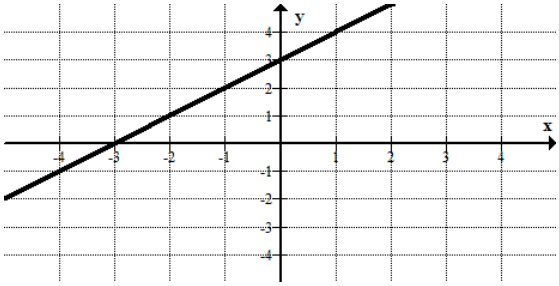


Figure 2.4.2.12

2.

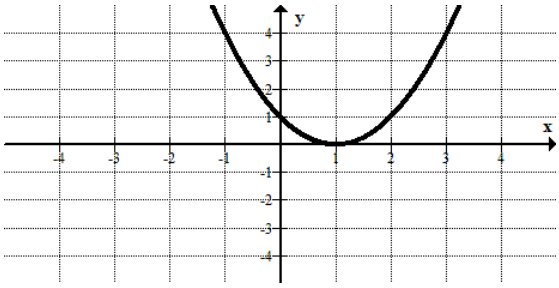


Figure 2.4.2.13

3.

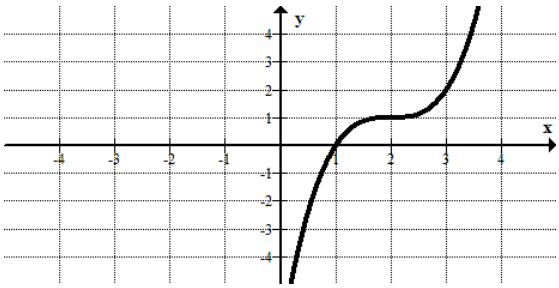


Figure 2.4.2.14

4.

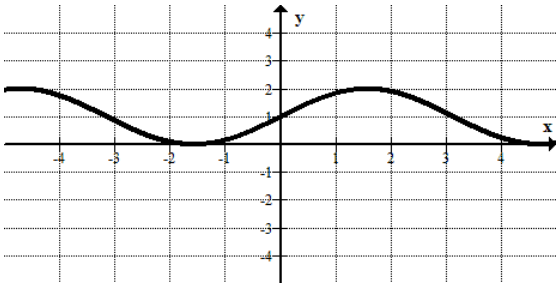


Figure 2.4.2.15

5.

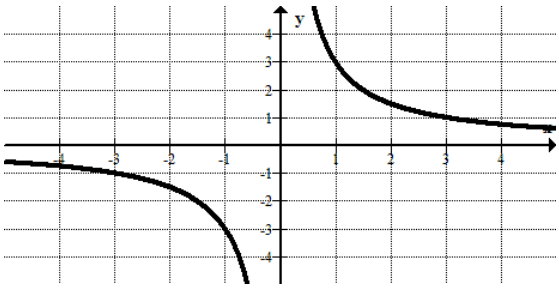


Figure 2.4.2.16

6.

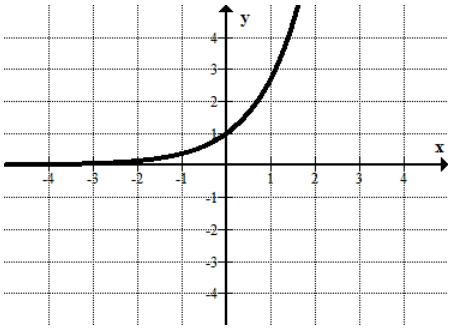


Figure 2.4.2.17

Find the inverse of each function using the mapping principle.

7. $y = x^2 + x - 2$
8. $y = x^3$
9. $y = \sin\left(x - \frac{\pi}{2}\right)$
10. $y = \cos(2x)$
11. $y = \frac{1}{x}$
12. $y = x^2 - 9$
13. $y = -2 + \sin\left(\frac{1}{2}x\right)$
14. What type of points will be in the same place in both a function and its inverse?
15. When you graph a function and its inverse on the same set of axes, where is the line of reflection? Why?

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.4.

Vocabulary

Term	Definition
inverse function	Inverse functions are functions that 'undo' each other. Formally $f(x)$ and $g(x)$ are inverse functions if $f(g(x)) = g(f(x)) = x$.
Mapping	Mapping is a procedure involving the plotting of points on a coordinate grid to see the behavior of a function.

Additional Resources

Video: Animation: Inverse Function

Practice: Inverses by Mapping

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2.4.3: Composition of Trig Functions and Their Inverses

Application of sine, cosine, tangent, or their inverses and then another function.

You've considered trigonometric functions, and you've considered inverse functions, and now it's time consider how to compose trig functions and their inverses. If someone were to ask you to apply the inverse of a trig function to a different trig function, would you be able to do this? For example, can you find $\sin^{-1}\left(\cos\left(\frac{3\pi}{2}\right)\right)$?

Trigonometric Functions and Their Inverses

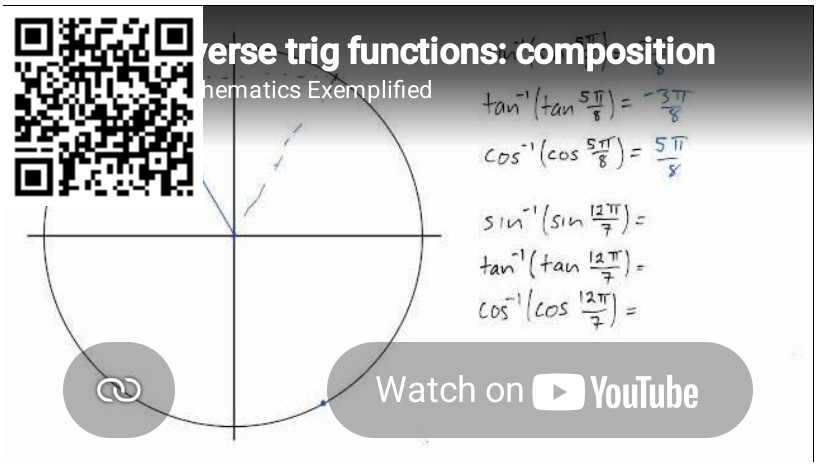
In other sections, you learned that for a function $f(f^{-1}(x)) = x$ for all values of x for which $f^{-1}(x)$ is defined. If this property is applied to the trigonometric functions, the following equations will be true whenever they are defined:

$$\sin(\sin^{-1}(x)) = x \qquad \cos(\cos^{-1}(x)) = x \qquad \tan(\tan^{-1}(x)) = x$$

As well, you learned that $f^{-1}(f(x)) = x$ for all values of x for which $f(x)$ is defined. If this property is applied to the trigonometric functions, the following equations that deal with finding an inverse trig function of a trig function, will only be true for values of x within the restricted domains.

$$\sin^{-1}(\sin(x)) = x \qquad \cos^{-1}(\cos(x)) = x \qquad \tan^{-1}(\tan(x)) = x$$

These equations are better known as composite functions. However, it is not necessary to only have a function and its inverse acting on each other. In fact, it is possible to have **composite function** that are composed of one trigonometric function in conjunction with another different trigonometric function. The composite functions will become algebraic functions and will not display any trigonometry. Let's investigate this phenomenon.



When solving these types of problems, start with the function that is composed inside of the other and work your way out. Use the following problems as a guideline.

1. Find $\sin\left(\sin^{-1}\frac{\sqrt{2}}{2}\right)$.

We know that $\sin^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$, within the defined restricted domain. Then, we need to find $\sin\frac{\pi}{4}$, which is $\frac{\sqrt{2}}{2}$. So, the above properties allow for a short cut. $\sin\left(\sin^{-1}\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}$, think of it like the sine and sine inverse cancel each other out and all that is left is the $\frac{\sqrt{2}}{2}$.

2. Without using technology, find the exact value of each of the following:

a. $\cos\left(\tan^{-1}\sqrt{3}\right)$

$\cos\left(\tan^{-1}\sqrt{3}\right)$: First find $\tan^{-1}\sqrt{3}$, which is $\frac{\pi}{3}$. Then find $\cos\frac{\pi}{3}$. Your final answer is $\frac{1}{2}$. Therefore, $\cos\left(\tan^{-1}\sqrt{3}\right) = \frac{1}{2}$.

b. $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$

$\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right) = \tan\left(-\frac{\pi}{6}\right) = -\sqrt{33}$

c. $\cos\left(\tan^{-1}(-1)\right)$

$\cos\left(\tan^{-1}(-1)\right) = \cos^{-1}\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$

d. $\sin\left(\cos^{-1}\frac{\sqrt{2}}{2}\right)$

$\sin\left(\cos^{-1}\frac{\sqrt{2}}{2}\right) = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

Example 2.4.3.1

Earlier, you were asked to solve $\sin^{-1}\left(\cos\left(\frac{3\pi}{2}\right)\right)$.

Solution

To solve this problem: $\sin^{-1}\left(\cos\left(\frac{3\pi}{2}\right)\right)$, you can work outward.

First find:

$\cos\left(\frac{3\pi}{2}\right) = 0$

Then find:

$\sin^{-1} 0 = 0$

or

$\sin^{-1} 0 = \pi$

Example 2.4.3.2

Find the exact value of $\cos^{-1}\frac{\sqrt{3}}{2}$, without a calculator, over its restricted domain.

Solution

$\frac{\pi}{6}$

Example 2.4.3.3

Evaluate: $\sin\left(\cos^{-1}\frac{5}{13}\right)$

Solution

$$\cos\theta = \frac{5}{13}$$
$$\sin\left(\cos^{-1}\left(\frac{5}{13}\right)\right) = \sin\theta$$
$$\sin\theta = \frac{12}{13}$$

Example 2.4.3.4

Evaluate: $\tan\left(\sin^{-1}\left(-\frac{6}{11}\right)\right)$

Solution

$\tan\left(\sin^{-1}\left(-\frac{6}{11}\right)\right) \rightarrow \sin\theta = -\frac{6}{11}$.

The third side is $b = \sqrt{121 - 36} = \sqrt{85}$.

$\tan\theta = -\frac{6}{\sqrt{85}} = -\frac{6\sqrt{85}}{85}$

Review

Without using technology, find the exact value of each of the following.

1. $\sin\left(\sin^{-1}\frac{1}{2}\right)$
2. $\cos\left(\cos^{-1}\frac{\sqrt{3}}{2}\right)$
3. $\tan\left(\tan^{-1}\sqrt{3}\right)$
4. $\cos\left(\sin^{-1}\frac{1}{2}\right)$
5. $\tan\left(\cos^{-1}1\right)$
6. $\sin\left(\cos^{-1}\frac{\sqrt{2}}{2}\right)$
7. $\sin^{-1}\left(\sin\frac{\pi}{2}\right)$
8. $\cos^{-1}\left(\tan\frac{\pi}{4}\right)$
9. $\tan^{-1}\left(\sin\pi\right)$
10. $\sin^{-1}\left(\cos\frac{\pi}{3}\right)$
11. $\cos^{-1}\left(\sin-\frac{\pi}{4}\right)$
12. $\tan\left(\sin^{-1}0\right)$
13. $\sin\left(\cos^{-1}\frac{\sqrt{3}}{2}\right)$
14. $\tan^{-1}\left(cos\frac{\pi}{2}\right)$
15. $\cos\left(\sin^{-1}\frac{\sqrt{2}}{2}\right)$

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.6.

Vocabulary

Term	Definition
composite function	A composite function is a function $h(x)$ formed by using the output of one function $g(x)$ as the input of another function f(x). Composite functions are written in the form $h(x) = f(g(x))$ or $h = f \circ g$.

Additional Resources

Video: Compositions of Trigonometric Functions - Overview

Practice: Composition of Trig Functions and Their Inverses

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2.4.4: Definition of Inverse Reciprocal Trig Functions

Inverse secant, cosecant, and cotangent functions.

So far you've had to deal with trig functions, reciprocal functions, and inverse functions. Now you'll start to see inverse reciprocal functions. For example, can you compute

$$\sec^{-1}\frac{2}{\sqrt{3}}$$

As it turns out, this can be readily computed.

Inverse Reciprocal Trigonometric Functions

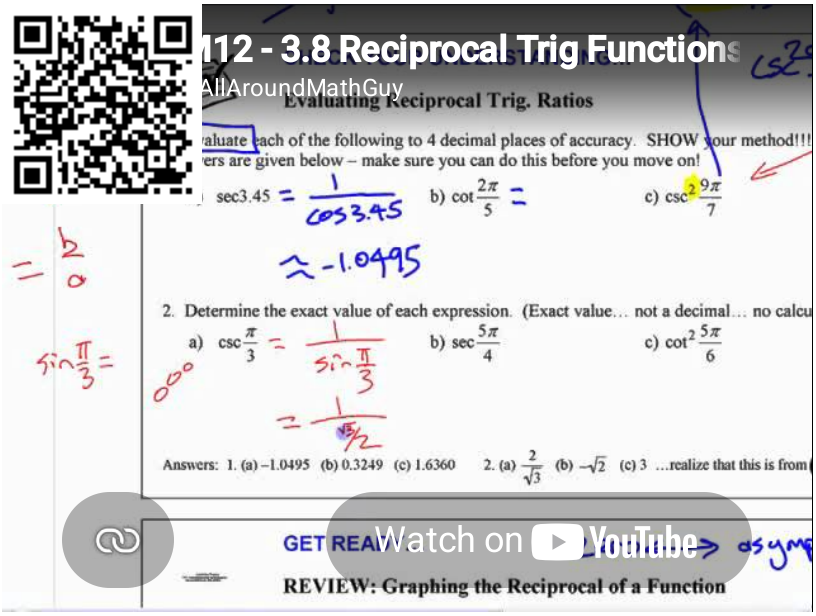
We already know that the cosecant function is the reciprocal of the sine function. This will be used to derive the reciprocal of the inverse sine function.

$$\begin{aligned} y &= \sin^{-1} x \\ x &= \sin y \\ \frac{1}{x} &= \csc y \\ \csc^{-1}\frac{1}{x} &= y \\ \csc^{-1}\frac{1}{x} &= \sin^{-1} x \end{aligned}$$

Because cosecant and secant are inverses, $\sin^{-1}\frac{1}{x} = \csc^{-1} x$ is also true.

The inverse reciprocal identity for cosine and secant can be proven by using the same process as above. However, remember that these inverse functions are defined by using restricted domains and the reciprocals of these inverses must be defined with the intervals of domain and range on which the definitions are valid.

$$\sec^{-1}\frac{1}{x} = \cos^{-1} x \leftrightarrow \cos^{-1}\frac{1}{x} = \sec^{-1} x$$



Tangent and cotangent have a slightly different relationship. Recall that the graph of cotangent differs from tangent by a reflection over the y−axis and a shift of $\frac{\pi}{2}$. As an equation, this can be written as $\cot x = -\tan\left(x - \frac{\pi}{2}\right)$. Taking the inverse of this function will show the inverse reciprocal relationship between arccotangent and arctangent.

$$y = \sin^{-1} x$$
$$x = \sin y$$
$$\frac{1}{x} = \csc y$$
$$\csc^{-1} \frac{1}{x} = y$$
$$\csc^{-1} \frac{1}{x} = \sin^{-1} x$$

Remember that tangent is an odd function, so that $\tan(-x) = -\tan(x)$. Because tangent is odd, its inverse is also odd. So, this tells us that $\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$ and $\tan^{-1} x = \frac{\pi}{2} - \cot^{-1} x$. To graph arcsecant, arccosecant, and arccotangent in your calculator you will use these conversion identities: $\sec^{-1} x = \cos^{-1} \frac{1}{x}$, $\csc^{-1} x = \sin^{-1} \frac{1}{x}$, $\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$. Note: It is also true that $\cot^{-1} x = \tan^{-1} \frac{1}{x}$.

Finding the Inverse

Find the inverse of $\sec^{-1} \sqrt{2}$

Use the inverse reciprocal property. $\sec^{-1} x = \cos^{-1} \frac{1}{x} \rightarrow \sec^{-1} \sqrt{2} = \cos^{-1} \frac{1}{\sqrt{2}}$

Recall that $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. So, $\sec^{-1} \sqrt{2} = \cos^{-1} \frac{\sqrt{2}}{2}$, and we know that $\cos^{-1} \frac{\sqrt{2}}{2} = \frac{\pi}{4}$. Therefore, $\sec^{-1} \sqrt{2} = \frac{\pi}{4}$.

Finding the Exact Value

For each of these problems, first find the reciprocal and then determine the angle from that (without a calculator).

1. $\sec^{-1} \sqrt{2}$

$\sec^{-1} \sqrt{2} = \cos^{-1} \frac{\sqrt{2}}{2}$ From the unit circle, we know that the answer is $\frac{\pi}{4}$.

2. $\cot^{-1}(-\sqrt{3})$

$\cot^{-1}(-\sqrt{3}) = \frac{\pi}{2} - \tan^{-1}(-\sqrt{3})$ From the unit circle, the answer is $\frac{5\pi}{6}$.

3. $\csc^{-1} \frac{2\sqrt{3}}{3}$

$\csc^{-1} \frac{2}{\sqrt{3}} 3 = \sin^{-1} \frac{\sqrt{3}}{2}$ Within our interval, there are is one answer, $\frac{\pi}{3}$.

Using Technology

Make sure that your calculator's MODE is RAD (radians)

1. $\arcsin 0.6384$

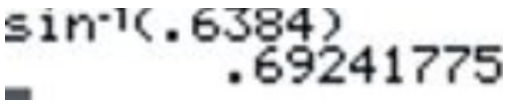


Figure 2.4.4.1

2. $\arccos(-0.8126)$

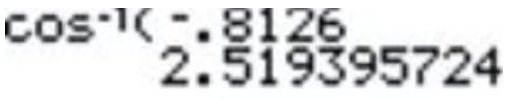


Figure 2.4.4.2

3. $\arctan(-1.9249)$

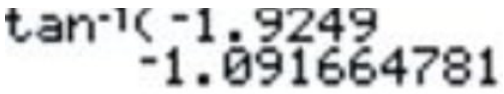


Figure 2.4.4.3

Example 2.4.4.1

Earlier, you were asked to evaluate $\sec^{-1} \frac{2}{\sqrt{3}}$

Solution

You can start with the inverse reciprocal property:

$\sec^{-1} x = \cos^{-1} \frac{1}{x}$

Substituting in values for "x\)" gives:

$\sec^{-1} \frac{2}{\sqrt{3}} = \cos^{-1} \frac{1}{\frac{2}{\sqrt{3}}}$

This can be rewritten as:

$\cos^{-1} \frac{\sqrt{3}}{2}$

And

$\cos^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{6}$

Therefore,

$\sec^{-1} \frac{2}{\sqrt{3}} = \frac{\pi}{6}$

Example 2.4.4.2

Evaluate $\sec^{-1}(-2)$

Solution

$\frac{2\pi}{3}$

Example 2.4.4.3

Evaluate $\cot^{-1}(-1)$

Solution

$-\frac{\pi}{4}$

Example 2.4.4.4

Evaluate $\csc^{-1}(\sqrt{2})$

Solution

$\frac{\pi}{4}$

Review

Using technology, find the value in radian measure, of each of the following.

1. $\sin^{-1}(.345)$
2. $\cos^{-1}(.87)$
3. $\csc^{-1}(4)$
4. $\sec^{-1}(2.32)$
5. $\cot^{-1}(5.2)$

Find the exact value of each expression within the restricted domain, without a calculator.

6. $\sec^{-1}(\frac{2\sqrt{3}}{3})$
7. $\csc^{-1}(1)$
8. $\cot^{-1}(\sqrt{3})$
9. $\csc^{-1}(2)$
10. $\sec^{-1}(\sqrt{2})$
11. $\cot^{-1}(1)$
12. $\cos^{-1}\left(\frac{1}{2}\right)$
13. $\sec^{-1}(2)$
14. $\cot^{-1}\left(\frac{\sqrt{3}}{3}\right)$
15. $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.7.

Vocabulary

Term	Definition
inverse function	Inverse functions are functions that 'undo' each other. Formally $f(x)$ and $g(x)$ are inverse functions if $f(g(x)) = g(f(x)) = x$.
Reciprocal Function	A reciprocal function is a function with the parent function $y = \frac{1}{x}$.

Additional Resources

Video: Reciprocal Trig Functions

Practice: Definition of Inverse Reciprocal Trig Functions

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2.4.5: Composition of Inverse Reciprocal Trig Functions

Application of one of six trigonometric functions or inverses and then another.

Composing functions involves applying one function and then applying another function afterward. In the case of inverse reciprocal functions, you could create compositions of functions such as $\sec^{-1}$, $\csc^{-1}$, and $\cot^{-1}$.

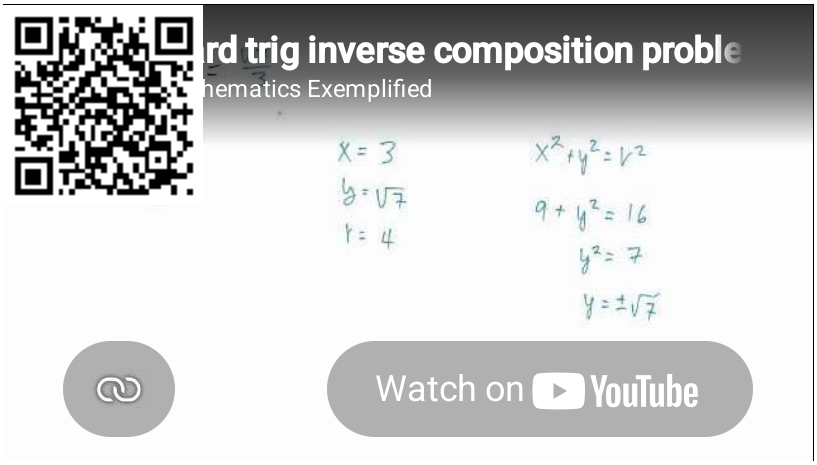
Consider the following problem:

$\csc(\cot^{-1}\sqrt{3})$

Can you solve this problem?

Composition of Inverse Reciprocal Trig Functions

Just as you can apply one function and then another whenever you'd like, you can do the same with inverse reciprocal trig functions. This process is called composition.



Here we'll explore some examples of composition for these inverse reciprocal trig functions by doing some problems.

1. Without a calculator, find $\cos(\cot^{-1}\sqrt{3})$.

First, find $\cot^{-1}\sqrt{3}$, which is also $\tan^{-1}\frac{\sqrt{3}}{3}$. This is $\frac{\pi}{6}$. Now, find $\cos\frac{\pi}{6}$, which is $\frac{\sqrt{3}}{2}$. So, our answer is $\frac{\sqrt{3}}{2}$.

2. Without a calculator, find $\sec^{-1}(csc\frac{\pi}{3})$.

First, $\csc\frac{\pi}{3}=\frac{1}{\sin\frac{\pi}{3}}=\frac{1}{\frac{\sqrt{3}}{2}}=\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$. Then $\sec^{-1}\frac{2\sqrt{3}}{3}=\cos^{-1}\frac{\sqrt{3}}{2}=\frac{\pi}{6}$.

3. Evaluate $\cos\left(\sin^{-1}\frac{3}{5}\right)$.

Even though this problem is not a critical value, it can still be done without a calculator. Recall that sine is the opposite side over the hypotenuse of a triangle. So, 3 is the opposite side and 5 is the hypotenuse. This is a Pythagorean Triple, and thus, the adjacent side is 4. To continue, let $\theta=\sin^{-1}\frac{3}{5}$ or $\sin\theta=\frac{3}{5}$, which means θ is in the Quadrant 1 (from our restricted domain, it cannot also be in Quadrant II). Substituting in θ we get $\cos\left(\sin^{-1}\frac{3}{5}\right)=\cos\theta$ and $\cos\theta=\frac{4}{5}$.

Example 2.4.5.1

Earlier, you were asked to solve $\csc(\cot^{-1}\sqrt{3})$.

Solution

The first step in this problem is to ask yourself "What angle would produce a cotangent of $\sqrt{2}$?"

Since values for "x" and "y" around the unit circle are all fractions, and cotangent is equal to xy, you need to find a pair of equations on the unit circle which, when divided by each other, give $\sqrt{2}$ as the answer.

When looking around the unit circle, you can see that $\cot 30° = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$

Therefore, $\cot^{-1}\sqrt{3} = 30°$

Then you can apply the next function:

$\csc 30° = \frac{hypotenuse}{opposite} = \frac{1}{\frac{1}{2}} = 2$

And so

$\csc\left(\cot^{-1}\sqrt{3}\right) = 2$

Example 2.4.5.2

Find the exact value of $\csc\left(\cos^{-1}\frac{\sqrt{3}}{2}\right)$ without a calculator, over its restricted domains.

Solution

$\csc\left(\cos^{-1}\frac{\sqrt{3}}{2}\right) = \csc\frac{\pi}{6} = 2$

Example 2.4.5.3

Find the exact value of $\sec^{-1}\left(\tan\left(\cot^{-1}1\right)\right)$ without a calculator, over its restricted domains.

Solution

$\sec^{-1}\left(\tan\left(\cot^{-1}1\right)\right) = \sec^{-1}\left(\tan\frac{\pi}{4}\right) = \sec^{-1}1 = 0$

Example 2.4.5.4

Find the exact value of $\tan^{-1}\left(\cos\frac{\pi}{2}\right)$ without a calculator, over its restricted domains.

Solution

$\tan^{-1}\left(\cos\frac{\pi}{2}\right) = \tan^{-1}0 = 0$

Review

Without using technology, find the exact value of each of the following. Use the restricted domain for each function.

1. $\sin\left(\sec^{-1}\sqrt{2}\right)$
2. $\cos\left(\csc^{-1}1\right)$
3. $\tan\left(\cot^{-1}\sqrt{3}\right)$
4. $\cos\left(\csc^{-1}2\right)$
5. $\cot\left(\cos^{-1}1\right)$
6. $\csc\left(\sin^{-1}\frac{\sqrt{2}}{2}\right)$
7. $\sec^{-1}\left(\cos\pi\right)$
8. $\cot^{-1}\left(\tan\frac{\pi}{4}\right)$
9. $\sec^{-1}\left(\csc\frac{\pi}{4}\right)$
10. $\csc^{-1}\left(\sec\frac{\pi}{3}\right)$

11. $\cos^{-1}\left(\cot -\frac{\pi}{4}\right)$
12. $\tan\left(\cot^{-1} 0\right)$
13. $\sin\left(\csc^{-1}\frac{2\sqrt{3}}{3}\right)$
14. $\cot^{-1}\left(\sin\frac{\pi}{2}\right)$
15. $\cos\left(\sec^{-1}\frac{2\sqrt{3}}{3}\right)$

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.8.

Vocabulary

Term	Definition
inverse function	Inverse functions are functions that 'undo' each other. Formally $f(x)$ and $g(x)$ are inverse functions if $f(g(x)) = g(f(x)) = x$.
Reciprocal Function	A reciprocal function is a function with the parent function $y = \frac{1}{x}$.

Additional Resources

Video: Hard Trig Inverse Composition Problems

Practice: Composition of Inverse Reciprocal Trig Functions

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2.4.6: Trig Functions as Algebra Expressions

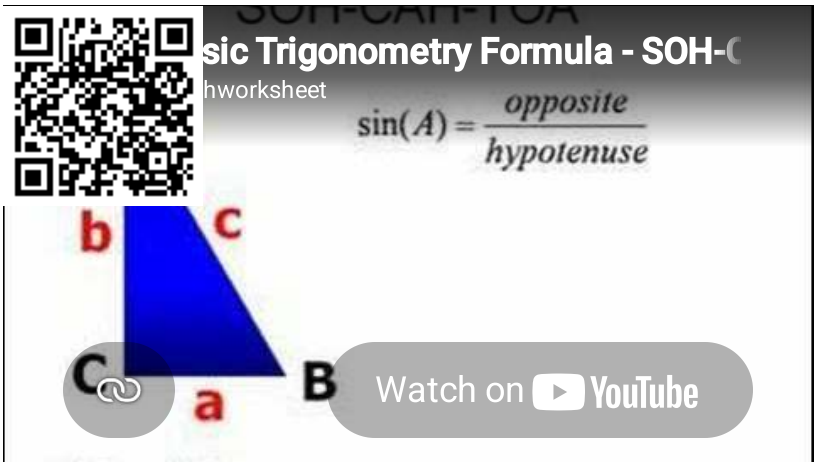
Relationships between the lengths of two sides of a triangle.

Trigonometry in Terms of Algebra

You are babysitting your little cousin while doing your homework. While working on your trig functions, your cousin asks you what you are doing. While trying to explain sine, cosine, and tangent, your cousin is very confused. She doesn't understand what you mean by those words, but really wants to understand what the functions mean. Can you define the trig functions in terms of the relationships of sides for your little cousin?

Solving **Trigonometric Functions**

All of the trigonometric functions can be rewritten in terms of only x, when using one of the inverse trigonometric functions.



Starting with tangent, we draw a triangle where the opposite side (from θ) is defined as x and the adjacent side is 1. The hypotenuse, from the Pythagorean Theorem would be $\sqrt{x^2+1}$. Substituting $\tan^{-1} x$ for θ, we get:

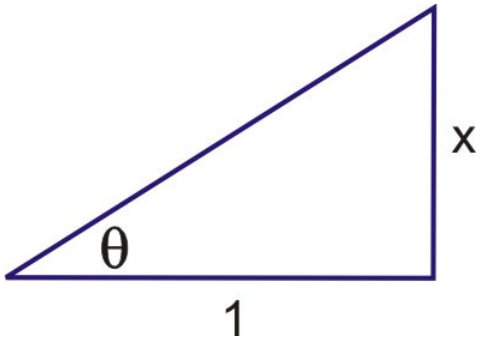


Figure 2.4.6.1

$$\tan\theta = \frac{x}{1}$$

$$\tan\theta = x \quad \text{hypotenuse} = \sqrt{x^2+1}$$

$$\theta = \tan^{-1} x$$

$$\sin\left(\tan^{-1} x\right) = \sin\theta = \frac{x}{\sqrt{x^2+1}} \qquad \csc\left(\tan^{-1} x\right) = \csc\theta = \frac{\sqrt{x^2+1}}{x}$$

$$\cos\left(\tan^{-1} x\right) = \cos\theta = \frac{1}{\sqrt{x^2+1}} \qquad \sec\left(\tan^{-1} x\right) = \sec\theta = \sqrt{x^2+1}$$

$$\tan\left(\tan^{-1} x\right) = \tan\theta = x \qquad \cot\left(\tan^{-1} x\right) = \cot\theta = \frac{1}{x}$$

Simplifying Trigonometric Functions

1. Find $\sin(\tan^{-1} 3x)$.

Instead of using x in the ratios above, use $3x$.

$$\sin(\tan^{-1} 3x) = \sin\theta = \frac{3x}{\sqrt{(3x)^2+1}} = \frac{3x}{\sqrt{9x^2+1}}$$

2. Find $\sec^2(\tan^{-1} x)$.

This problem might be better written as $[\sec(\tan^{-1} x)]^2$. Therefore, all you need to do is square the ratio above.

$$[\sec(\tan^{-1} x)]^2 = (\sqrt{x^2+1})^2 = x^2+1$$

You can also write the all of the trig functions in terms of arcsine and arccosine. However, for each inverse function, there is a different triangle. You will derive these formulas in the exercise for this section.

3. Find $\csc^3(\tan^{-1} 4x)$.

This problem is similar to #1 and #2 above. First, use $4x$ instead of x in the ratios above. Second, the csc^3 is the same as taking the csc function and cubing it.

$$[\csc(\tan^{-1} 4x)]^3 = \left(\frac{\sqrt{(4x)^2+1}}{4x}\right)^3 = \frac{(16x^2+1)^{\frac{3}{2}}}{64x^3}$$

Example 2.4.6.1

Earlier, you were asked if you can define the trig functions in terms of the relationship of sides.

Solution

As it turns out, it's very easy to explain trig functions in terms of ratios. If you look at the unit circle

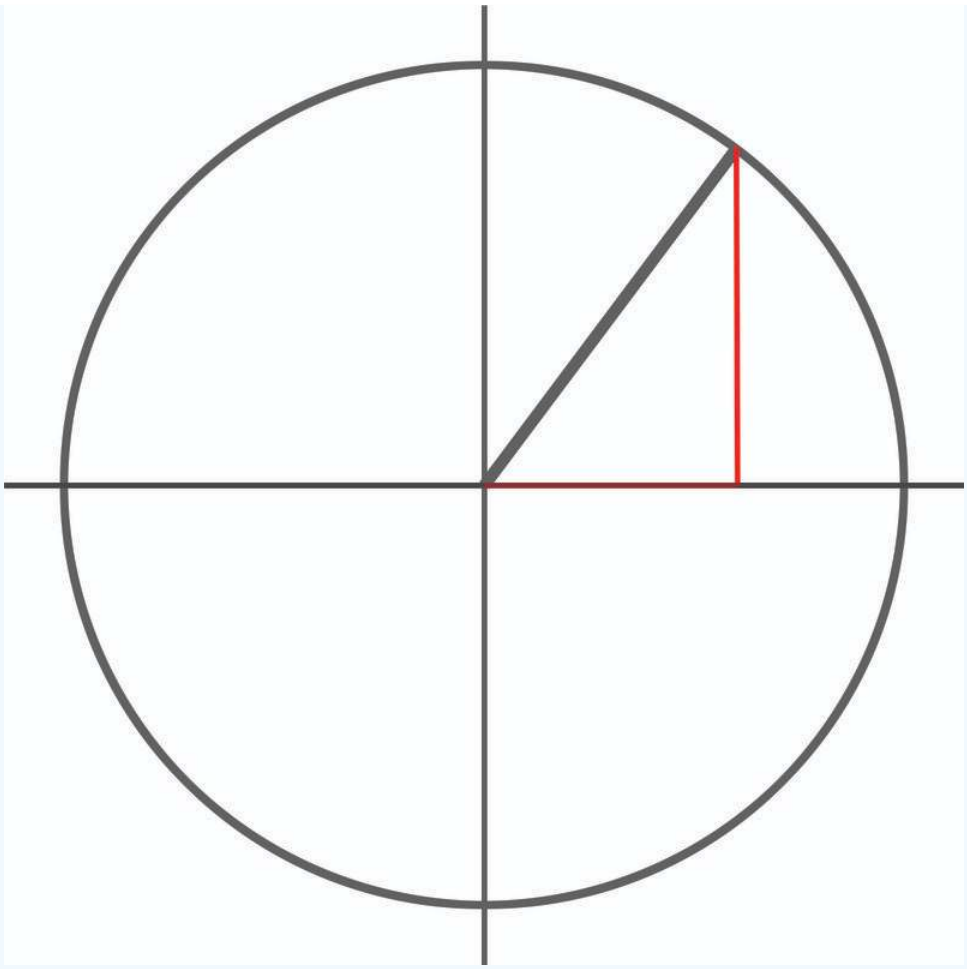


Figure 2.4.6.2

you can see that each trig function can be represented as a ratio of two sides. The value of any trig function can be represented as the length of one of the sides of the triangle (shown with two red sides and the black hypotenuse) divided by the length of one of the other sides. In fact, you should explain to your cousin, the words like "sine", "cosine", and "tangent" are just conveniences in this case to describe relationships that keep coming up over and over again. It would be possible to just describe the trig functions in terms of relationships of one side to another, if you'd like.

Using the sides of a triangle made on the unit circle, if the side opposite the angle is called "x":

$$\sin = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{x}{\sqrt{x^2+1}}$$
$$\cos = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{x^2+1}}$$
$$\tan = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{1}$$

So as you can see, since trig functions are really just relationships between sides, it is possible to work with them in whatever form you want; either in terms of the usual "sine", "cosine" and "tangent", or in terms of algebra.

Example 2.4.6.2

Express $\cos^2(\tan^{-1} x)$ as an algebraic expression involving no trigonometric functions.

Solution

$$\frac{1}{x^2+1}$$

Example 2.4.6.3

Express $\cot(\tan^{-1} x^2)$ as an algebraic expression involving no trigonometric functions.

Solution

$\frac{1}{x^2}$

Example 2.4.6.4

To find trigonometric functions in terms of sine inverse, use the following triangle.

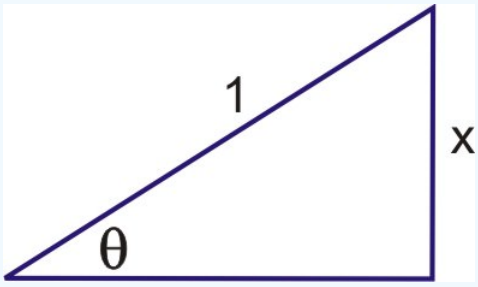


Figure 2.4.6.3

Solution

Determine the sine, cosine and tangent in terms of arcsine. Find $\tan(\sin^{-1} 2x^3)$.

The adjacent side to θ is $\sqrt{1-x^2}$, so the three trig functions are:

$$\sin\left(\sin^{-1} x\right) = \sin\theta = x$$
$$\cos\left(\sin^{-1} x\right) = \cos\theta = \sqrt{1-x^2}$$
$$\tan\left(\sin^{-1} x\right) = \tan\theta = \frac{x}{\sqrt{1-x^2}}$$
$$\tan\left(\sin^{-1}\left(2x^3\right)\right) = \frac{2x^3}{\sqrt{1-\left(2x^3\right)^2}} = \frac{2x^3}{\sqrt{1-4x^6}}$$

Review

Rewrite each expression as an algebraic expression involving no trigonometric functions.

1. $\sin(\tan^{-1} 5x)$
2. $\cos(\tan^{-1} 2x^2)$
3. $\cot(\tan^{-1} 3x^2)$
4. $\sin(\cos^{-1} x)$
5. $\sin(\cos^{-1} 3x)$
6. $\cos(\sin^{-1} 2x^2)$
7. $\csc(\cos^{-1} x)$
8. $\sec(\sin^{-1} x)$
9. $\cos 2(\tan^{-1} 3x^2)$
10. $\sin(\sec^{-1} x)$
11. $\cos(\csc - 1x)$
12. $\sin(\tan^{-1} 2x^3)$
13. $\cos(\sin^{-1} 3x)$
14. $\sin(\sec^{-1} x)$
15. $\cos(\cot^{-1} x)$

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.9.

Vocabulary

Term	Definition
Trigonometric Function	A trigonometric function is a function of an angle that describes the relationship between two sides of a right triangle. Examples of trigonometric functions are sine, cosine, and tangent.
Trigonometric Functions	A trigonometric function is a function of an angle that describes the relationship between two sides of a right triangle. Examples of trigonometric functions are sine, cosine, and tangent.

Additional Resources

Video: Unit Circle Definition of Trig Functions

Practice: Trig Functions as Algebra Expressions

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SECTION OVERVIEW

2.5: Radians

2.5.1: Radian Measure

2.5.2: Conversion between Degrees and Radians

2.5.3: Trig Functions and Radians with Technology

2.5.4: Rotations in Radians

2.5.5: Angular Velocity

2.5.6: Length of an Arc

2.5.7: Area of a Sector

2.5.8: Length of a Chord

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2.5.1: Radian Measure

Angle measure in a circle where the arc length equals the radius.

While working on an experiment in your school science lab, your teacher asks you to turn up a detector by rotating the knob $\frac{\pi}{2}$ radians. You are immediately puzzled, since you don't know what a **radian** measure is or how far to turn the knob.

Measure of Radians

Until now, we have used degrees to measure angles. But, what exactly is a degree? A **degree** is $\frac{1}{360^{th}}$ of a complete rotation around a circle. **Radians** are alternate units used to measure angles in trigonometry. Just as it sounds, a radian is based on the *radius* of a circle. One **radian** (abbreviated rad) is the angle created by bending the radius length around the arc of a circle. Because a radian is based on an actual part of the circle rather than an arbitrary division, it is a much more natural unit of angle measure for upper level mathematics.

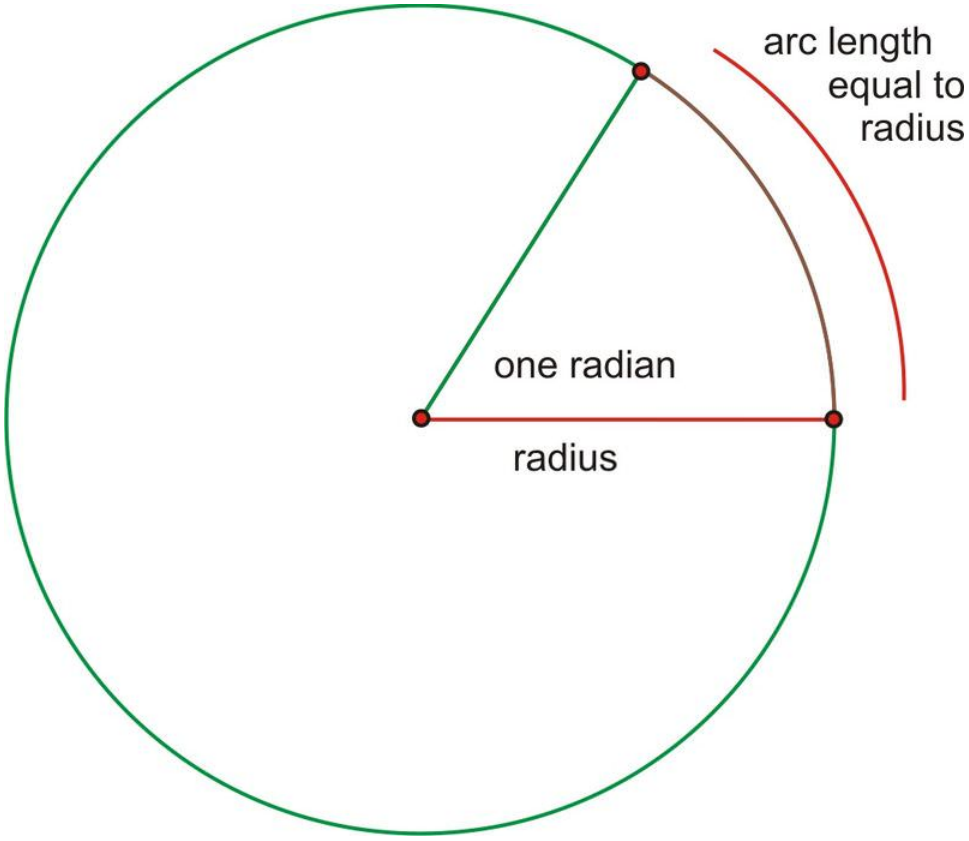


Figure **2.5.1.1**



What if we were to rotate all the way around the circle? Continuing to add radius lengths, we find that it takes a little more than 6 of them to complete the rotation.

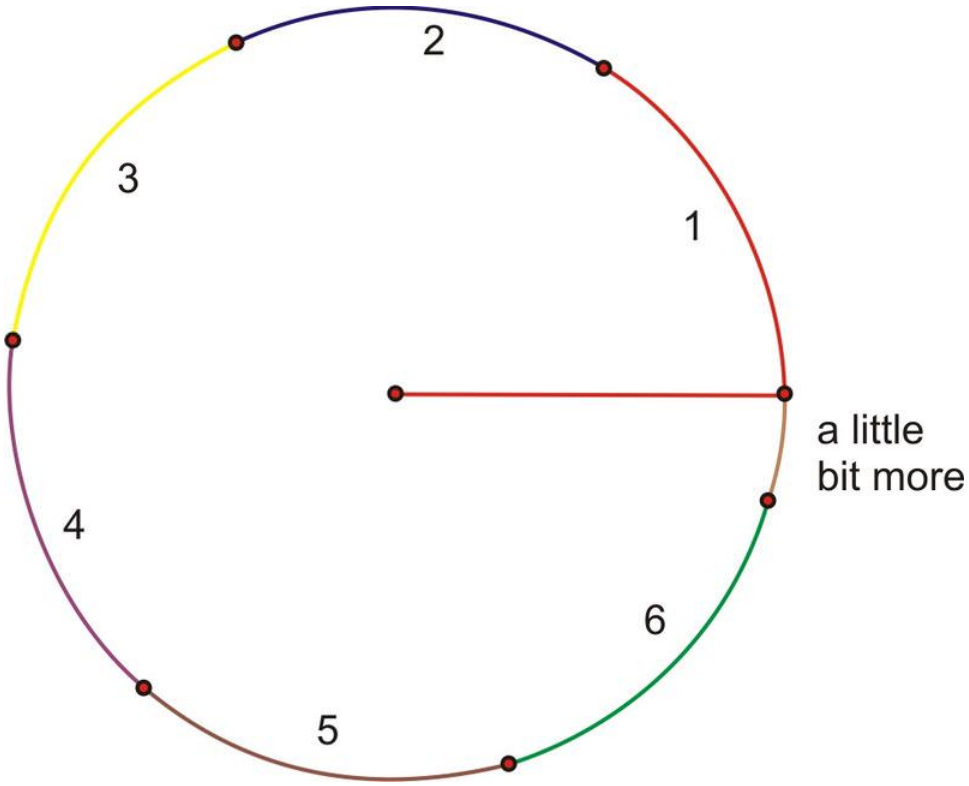


Figure 2.5.1.2

Recall from geometry that the arc length of a complete rotation is the circumference, where the formula is equal to 2π times the length of the radius. 2π is approximately 6.28, so the circumference is a little more than 6 radius lengths. Or, in terms of radian measure, a complete rotation (360 degrees) is 2π radians.

$$360 \text{ degrees} = 2\pi \text{ radians}$$

With this as our starting point, we can find the radian measure of other angles. Half of a rotation, or 180 degrees, must therefore be π radians, and 90 degrees must be $\frac{1}{2}\pi$, written $\frac{\pi}{2}$.

Extending the radian measure past the first quadrant, the quadrantal angles have been determined, except $270°$. Because $270°$ is halfway between $180°$ (π) and $360°$ (2π), it must be 1.5π, usually written $\frac{3\pi}{2}$.

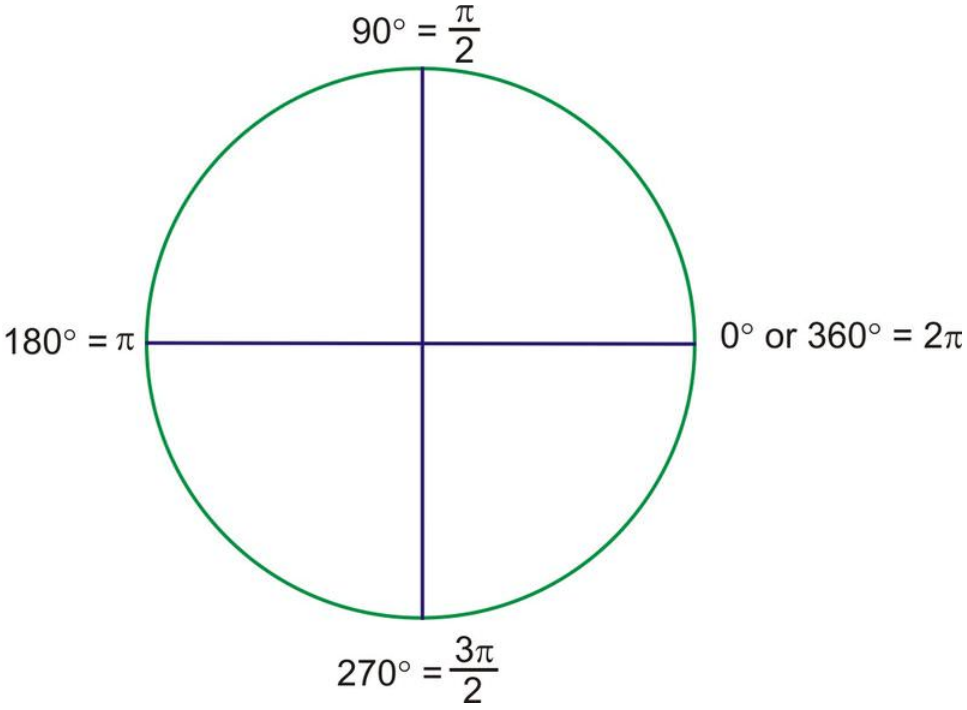


Figure 2.5.1.3

For the $45°$ angles, the radians are all multiples of $\frac{\pi}{4}$.

For example, $135°$ is $3 \cdot 45°$. Therefore, the radian measure should be $3 \cdot \frac{\pi}{4}$, or $\frac{3\pi}{4}$. Here are the rest of the multiples of $45°$,in radians:

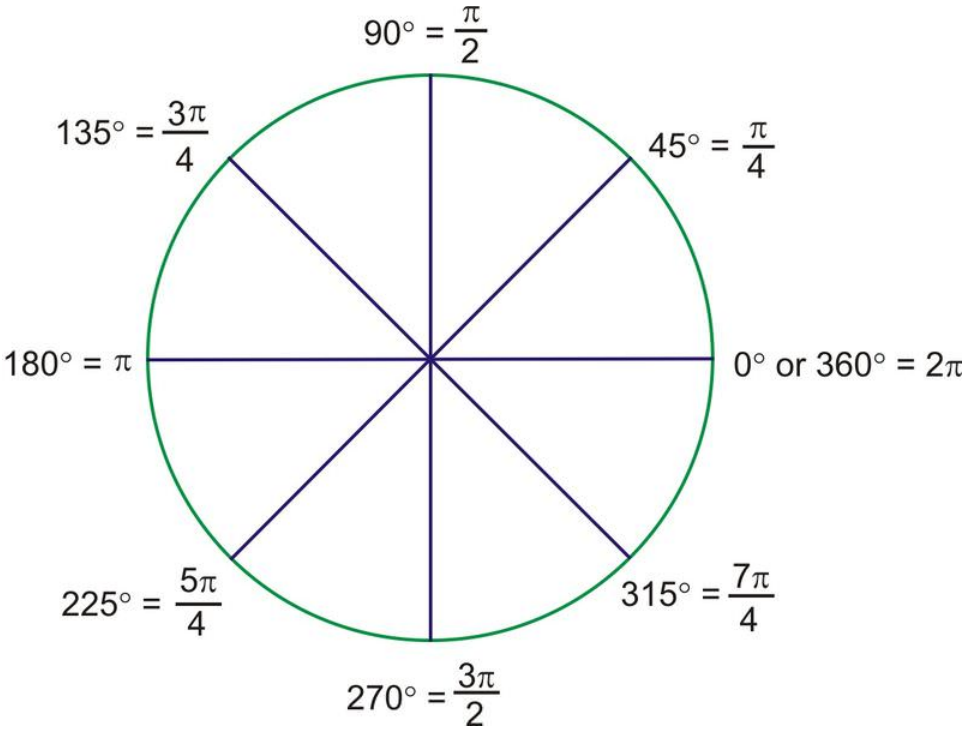


Figure 2.5.1.4

Notice that the additional angles in the drawing all have reference angles of 45 degrees and their radian measures are all multiples of $\frac{\pi}{4}$. All of the even multiples are the quadrantal angles and are reduced, just like any other fraction.

Let's do some problems that involve radian measures.

1. Find the radian measure of these angles.

Angle in Degrees	Angle in Radians
90	$\frac{\pi}{2}$
45	
30	

Because 45 is half of 90, half of $\frac{1}{2}\pi$ is $\frac{1}{4}\pi$. 30 is one-third of a right angle, so multiplying gives:

$\frac{\pi}{2} \times \frac{1}{3} = \frac{\pi}{6}$

and because 60 is twice as large as 30:

$2 \times \frac{\pi}{6} = \frac{2\pi}{6} = \frac{\pi}{3}$

Here is the completed table:

Angle in Degrees	Angle in Radians
90	$\frac{\pi}{2}$
45	$\frac{\pi}{4}$
30	$\frac{\pi}{6}$

There is a formula to convert between radians and degrees that you may already have discovered while doing this example. However, many angles that are commonly used can be found easily from the values in this table. For example, most students find it easy to remember 30 and 60. 30 is π over 6 and 60 is π over 3. Knowing these angles, you can find any of the special angles that have reference angles of 30 and 60 because they will all have the same denominators. The same is true of multiples of $\frac{\pi}{4}$ (45 degrees) and $\frac{\pi}{2}$ (90 degrees).

2. Complete the following radian measures by counting in multiples of $\frac{\pi}{3}$ and $\frac{\pi}{6}$:

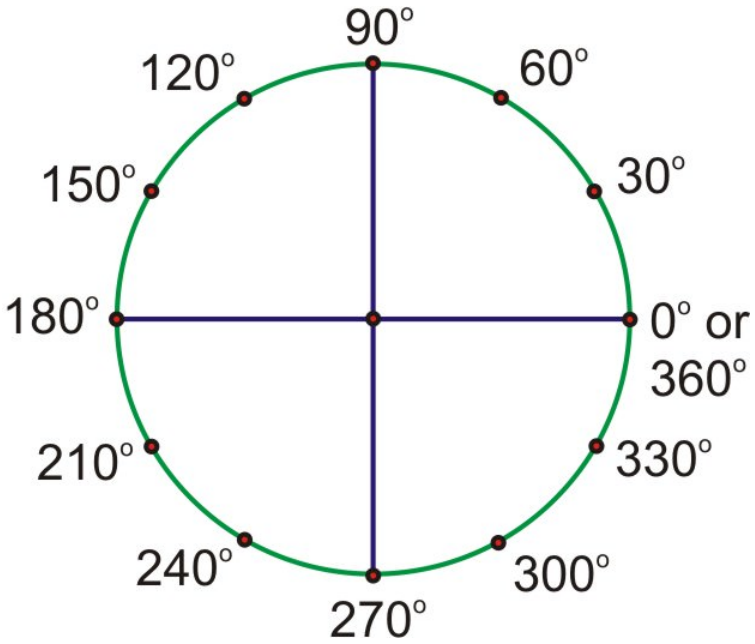


Figure 2.5.1.5

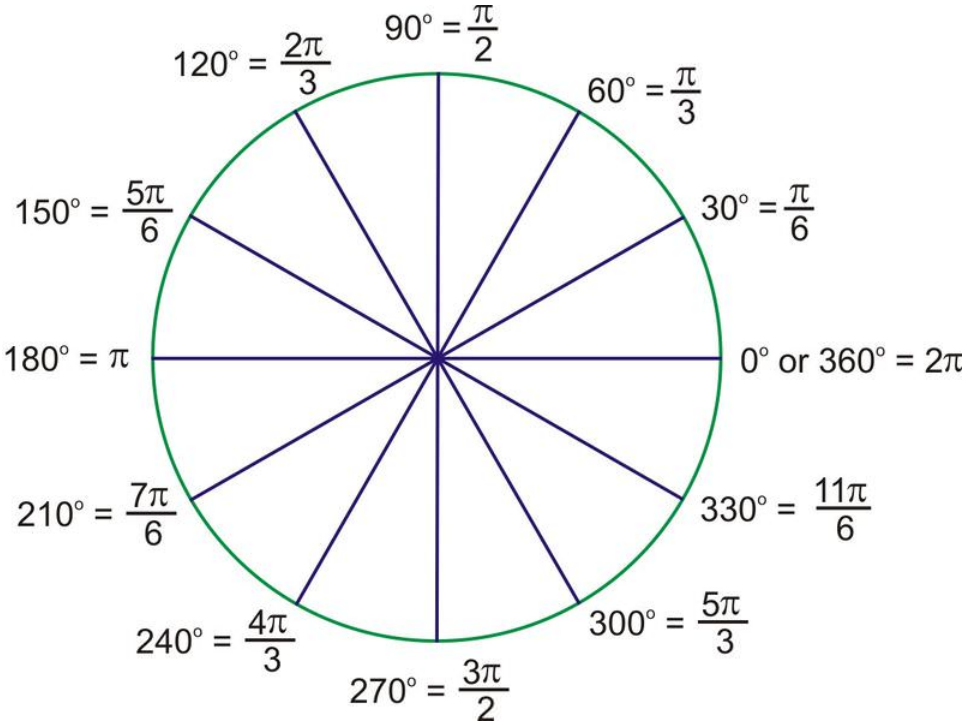


Figure **2.5.1.6**

Notice that all of the angles with 60-degree reference angles are multiples of $\frac{\pi}{3}$, and all of those with 30-degree reference angles are multiples of $\frac{\pi}{6}$. Counting in these terms based on this pattern, rather than converting back to degrees, will help you better understand radians.

3. Find the radian measure of these angles.

Angle in Degrees	Angle in Radians
120	$\frac{2\pi}{3}$
180	
240	
270	
300	

Because 30 is one-third of a right angle, multiplying gives:

$$\frac{\pi}{2} \times 13 = \frac{\pi}{6}$$

adding this to the known value for ninety degrees of $\frac{\pi}{2}$:

$$\frac{\pi}{2} + \frac{\pi}{6} = 3\frac{\pi}{6} + \frac{\pi}{6} = 4\frac{\pi}{6} = \frac{2\pi}{3}$$

Here is the completed table:

Angle in Degrees	Angle in Radians
120	$\frac{2\pi}{3}$
180	π
240	$\frac{4\pi}{3}$
300	$\frac{5\pi}{3}$

Example 2.5.1.1

Earlier, you were given a problem about rotating the knob.

Solution

Since $45^\circ = \frac{\pi}{4}$ rad, then $2 \times \frac{\pi}{4} = \frac{\pi}{2} = 2 \times 45^\circ$. Therefore, a turn of $\frac{\pi}{2}$ is equal to 90°, which is $\frac{1}{4}$ of a complete rotation of the knob.

Example 2.5.1.2

Give the radian measure of 60°

Solution

30 is one-third of a right angle. This means that since $90^\circ = \frac{\pi}{2}$, then $30^\circ = \frac{\pi}{6}$. Therefore, multiplying gives:

$\frac{\pi}{6} \times 2 = \frac{\pi}{3}$

Example 2.5.1.3

Give the radian measure of 75°

Solution

15 is one-sixth of a right triangle. This means that since $90^\circ = \frac{\pi}{2}$, then $15^\circ = \frac{\pi}{12}$. Therefore, multiplying gives:

$\frac{\pi}{12} \times 5 = \frac{5\pi}{12}$

Example 2.5.1.4

Give the radian measure of 180°

Solution

Since $90^\circ = \frac{\pi}{2}$, then $180^\circ = \frac{2\pi}{2} = \pi$

Review

Find the radian measure of each angle.

1. 90°
2. 120°
3. 300°
4. 330°
5. -45°
6. 135°

Find the degree measure of each angle.

7. $\frac{3\pi}{2}$
8. $\frac{5\pi}{4}$
9. $\frac{7\pi}{6}$
10. $\frac{\pi}{6}$
11. $\frac{5\pi}{3}$
12. π
13. Explain why if you are given an angle in degrees and you multiply it by $\frac{\pi}{180}$ you will get the same angle in radians.
14. Explain why if you are given an angle in radians and you multiply it by $\frac{180}{\pi}$ you will get the same angle in degrees.
15. Explain in your own words why it makes sense that there are 2π radians in a circle.

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.1.

Vocabulary

Term	Definition
radian	A radian is a unit of angle that is equal to the angle created at the center of a circle whose arc is equal in length to the radius.

Additional Resources

Video: Angle Measures - Overview

Practice: Radian Measure

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2.5.2: Conversion between Degrees and Radians

Convert between radians and degrees.

Most people are familiar with measuring angles in degrees. It is easy to picture angles like $30°$, $45°$ or $90°$ and the fact that $360°$ makes up an entire circle. Over 2000 years ago the Babylonians used a base 60 number system and divided up a circle into 360 equal parts. This became the standard and it is how most people think of angles today.

However, there are many units with which to measure angles. For example, the gradian was invented along with the metric system and it divides a circle into 400 equal parts. The sizes of these different units are very arbitrary.

A radian is a unit of measuring angles that is based on the properties of circles. This makes it more meaningful than gradians or degrees. How many radians make up a circle?

Radians and Degrees

A **radian** is defined to be the central angle where the subtended **arc length** is the same length as the radius.

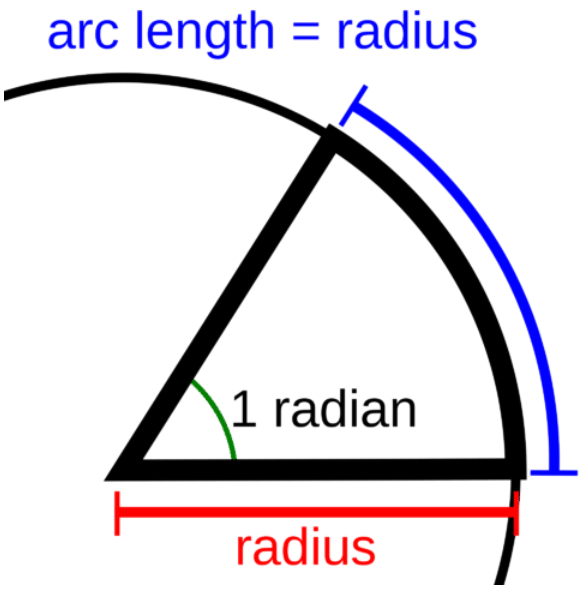


Figure 2.5.2.1

Another way to think about radians is through the circumference of a circle. The circumference of a circle with radius r is $2\pi r$. Just over six radii (exactly 2π radii) would stretch around any circle.

To define a radian in terms of degrees, equate a circle measured in degrees to a circle measured in radians.

$360 \text{ degrees} = 2\pi \text{ radians}$, so $\frac{180}{\pi} \text{ degrees} = 1 \text{ radian}$

Alternatively; $360\text{degrees} = 2\pi$ radians, so 1 degree=$\frac{\pi}{180}$ radians

The conversion factor to convert degrees to radians is: $\frac{\pi}{180°}$

The conversion factor to convert radians to degrees is: $\frac{180°}{\pi}$

If an angle has no units, it is assumed to be in radians.

If you were to convert $150°$ into radians, you would multiply $150°$ by the correct conversion factor. You would get:

$$150° \cdot \frac{\pi}{180°} = \frac{15\pi}{18} = \frac{5\pi}{6} \text{ radians}$$

You can check your work by making sure the degree units appear on both the numerator and denominator.

If you were to convert $\frac{\pi}{6}$ radians into degrees, you would multiply \pi 6 by the correct conversion factor. You would get $\frac{\pi}{6} \cdot \frac{180^\circ}{\pi} = \frac{180^\circ}{6} = 30^\circ$

Notice \pi appears in both the numerator and denominator and \pi \pi =1.



Example 2.5.2.1

Earlier, you were asked how many radians make up a circle.

Solution

Exactly 2π radians describe a circular arc. This is because 2π radii wrap around the circumference of any circle.

Example 2.5.2.2

Convert $(6\pi)^\circ$ into radians.

Solution

Don't be fooled just because this has π. This number is about 19°.

$(6\pi)^\circ \cdot \frac{\pi}{180^\circ} = \frac{6\pi^2}{180} = \frac{\pi^2}{3}$

It is very unusual to ever have a $\frac{\pi}{2}$ term, but it can happen.

Example 2.5.2.3

Convert $\frac{5\pi}{6}$ into degrees.

Solution

$\frac{5\pi}{6} \cdot \frac{180^\circ}{\pi} = \frac{5 \cdot 30^\circ}{1} = 150^\circ$

Example 2.5.2.4

Convert 210° into radians.

Solution

$210^\circ \cdot \frac{\pi}{180^\circ} = \frac{7 \cdot 30 \cdot \pi}{6 \cdot 30} = \frac{7\pi}{6}$

Example 2.5.2.5

Draw a $\frac{\pi}{2}$ angle by first drawing a 2π angle, halving it and halving the result. Recall that $\frac{\pi}{2} = 90°$.

Solution

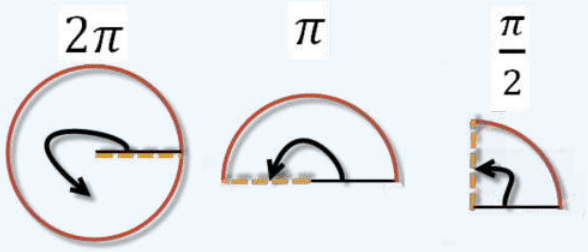


Figure 2.5.2.2

Review

Find the radian measure of each angle.

1. $120°$
2. $300°$
3. $90°$
4. $330°$
5. $270°$
6. $45°$
7. $(5\pi)°$

Find the degree measure of each angle.

8. $\frac{7\pi}{6}$
9. $\frac{5\pi}{4}$
10. $\frac{3\pi}{2}$
11. $\frac{5\pi}{3}$
12. π
13. $\frac{\pi}{6}$
14. 3
15. Explain why if you are given an angle in degrees and you multiply it by $\frac{\pi}{180}$ you will get the same angle in radians.

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.1.

Vocabulary

Term	Definition
subtended arc	A subtended arc is the part of the circle in between the two rays that make the central angle.

Additional Resources

Video: Radian and Degree Conversions - Overview

Practice: Conversion between Degrees and Radians

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2.5.3: Trig Functions and Radians with Technology

Degrees versus radians and calculator modes.

Six Trigonometric Functions and Radians

While working in your math class one day, you are given a sheet of values in radians and asked to find the various trigonometric functions of them, such as sine, cosine, and \tan gent. The first question asks you to find the $\sin \frac{\pi}{6}$. You are about to start converting the measurements in radians into degrees when you wonder if it might be possible to just take the values of the functions directly.

Do you think this is possible? As it turns out, it is indeed possible to apply trig functions to measurements in radians. Here you'll learn to do just that.

Trigonometric Functions and Radians

Even though you are used to performing the trig functions on degrees, they still will work on radians. The only difference is the way the problem looks. If you see $\sin \frac{\pi}{6}$, that is still $\sin 30°$ and the answer is still $\frac{1}{2}$.

Most scientific and graphing calculators have a **MODE** setting that will allow you to either convert between the two, or to find approximations for trig functions using either measure. It is impor\tan t that if you are using your calculator to estimate a trig function that you know which mode you are using. Look at the following screen:

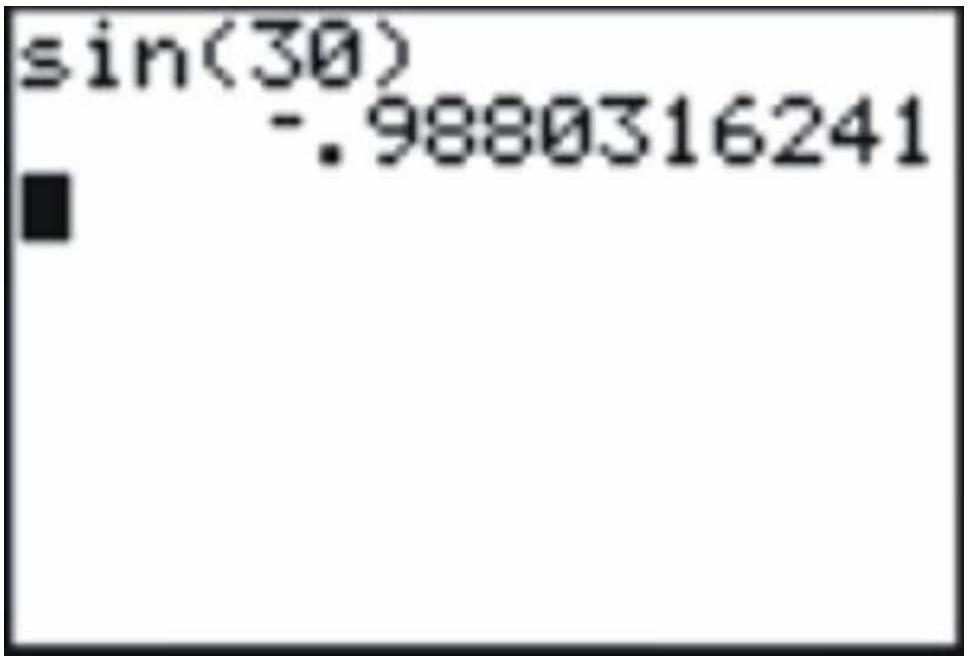


Figure **2.5.3.1**

If you entered this expecting to find the sine of 30 *degrees* you would realize that something is wrong because the answer should be $\frac{1}{2}$. In fact, as you may have suspected, the calculator is interpreting this as 30 ***radians***. In this case, changing the mode to degrees and recalculating will give the expected result.

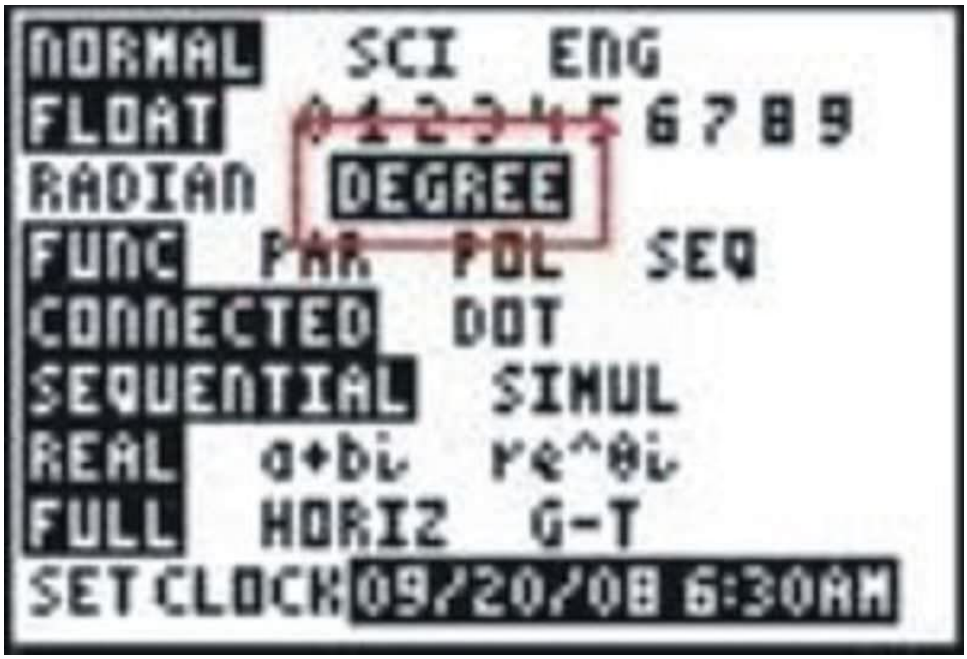


Figure **2.5.3.2**

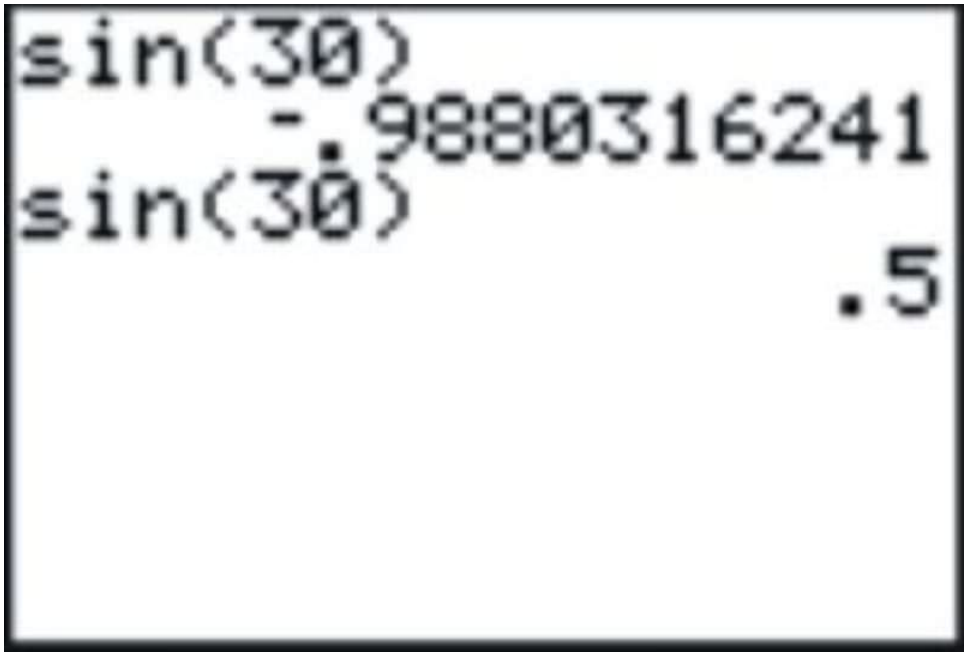
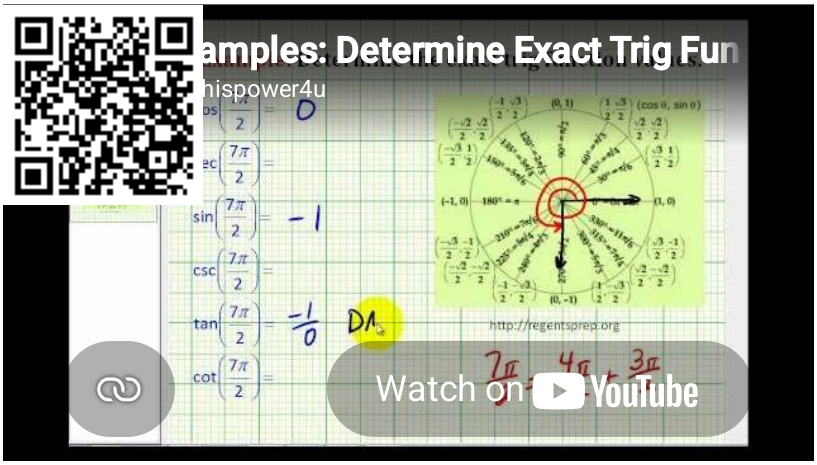


Figure **2.5.3.3**

Scientific calculators will usually have a 3-letter display that shows either **DEG** or **RAD** to tell you which mode the calculator is in.



Let's take a look at a few example problems.

1. Find $\tan \frac{3\pi}{4}$.

If needed, convert $\frac{3\pi}{4}$ to degrees. Doing this, we find that it is $135°$. So, this is $\tan 135°$, which is -1.

2. Find the value of $\cos \frac{11\pi}{6}$.

If needed, convert $\frac{11\pi}{6}$ to degrees. Doing this, we find that it is $330°$. So, this is $\cos 330°$, which is $\frac{\sqrt{3}}{2}$.

3. Convert 1 **radian** to degree measure.

Many students get so used to using π in radian measure that they incorrectly think that 1 radian means 1π radians. While it is more convenient and common to express radian measure in terms of π, don't lose sight of the fact that π radians is a number. It specifies an angle created by a rotation of approximately 3.14 radius lengths. So 1 radian is a rotation created by an arc that is only a single radius in length.

$$\text{radians} \times \frac{180}{\pi} = \text{degrees}$$

So 1 radian would be $\frac{180}{\pi}$ degrees. Using any scientific or graphing calculator will give a reasonable approximation for this degree measure, approximately $57.3°$.

Example 2.5.3.1

Earlier, you were asked to find $\sin \frac{\pi}{6}$.

Solution

As you have learned in this section, the $\sin \frac{\pi}{6}$ is the same as $\sin 30°$, which equals $\frac{1}{2}$. You could find this either by converting $\frac{\pi}{6}$ to degrees, or by using your calculator with angles entered in radians.

Example 2.5.3.2

Using a calculator, find the approximate degree measure (to the nearest tenth) of the angle expressed in radians:

Solution

$\frac{6\pi}{7}$

$154.3°$

Example 2.5.3.3

Using a calculator, find the approximate degree measure (to the nearest tenth) of the angle expressed in radians:

Solution

$\frac{20\pi}{11}$

327.3°

Example 2.5.3.4

Gina wanted to calculate the $\sin 210^\circ$ and got the following answer on her calculator:

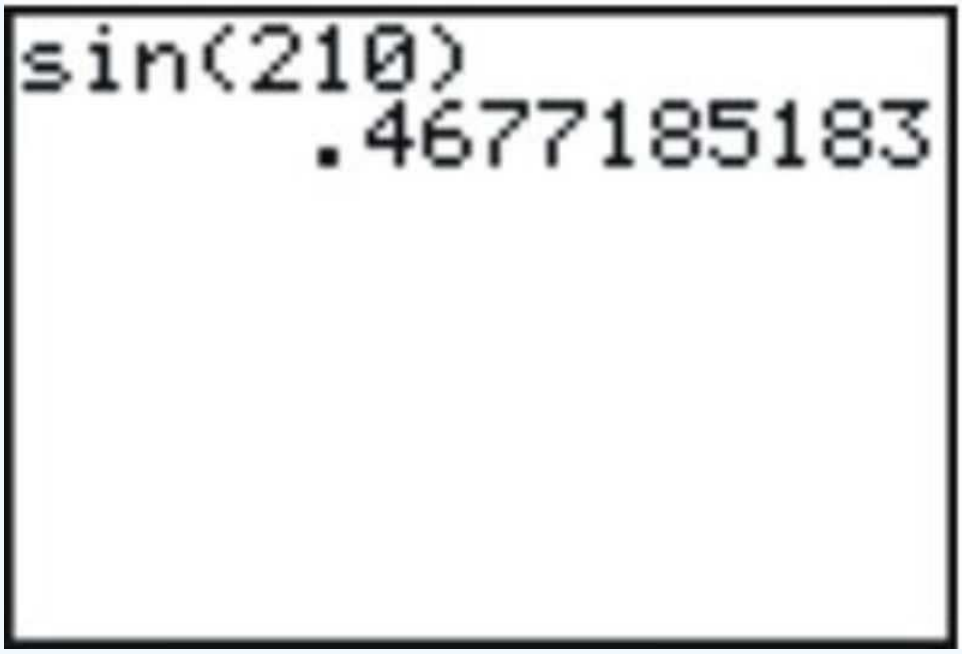


Figure 2.5.3.4

Solution

Fortunately, Kylie saw her answer and told her that it was obviously incorrect.

1. Write the correct answer, in simplest radical form.
2. Explain what she did wrong.

The correct answer is $-\frac{1}{2}$. Her calculator was is the wrong mode and she calculated the sine of 210 *radians*.

Review

Using a calculator, find the approximate degree measure (to the nearest tenth) of the angle expressed in radians.

1. $\frac{4\pi}{7}$
2. $\frac{5\pi}{6}$
3. $\frac{8\pi}{11}$
4. $\frac{5\pi}{3}$
5. $\frac{8\pi}{3}$
6. $\frac{7\pi}{4}$
7. $\frac{12\pi}{5}$

Find the value of each using your calculator.

8. $\sin \frac{3\pi}{2}$
9. $\cos \frac{\pi}{2}$
10. $\tan \frac{\pi}{6}$
11. $\sin \frac{5\pi}{6}$
12. $\tan \frac{4\pi}{3}$
13. $\cot \frac{7\pi}{3}$
14. $\sec \frac{11\pi}{6}$
15. Do you think radians will always be written in terms of \pi ? Is it possible to have, for example, exactly 2 radians?

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.3.

Vocabulary

Term	Definition
radian	A radian is a unit of angle that is equal to the angle created at the center of a circle whose arc is equal in length to the radius.

Additional Resources

Video: Examples: Determine Exact Trig Function Values With the Angle in Radians Using the Unit Circle

Practice: Trig Functions and Radians with Technology

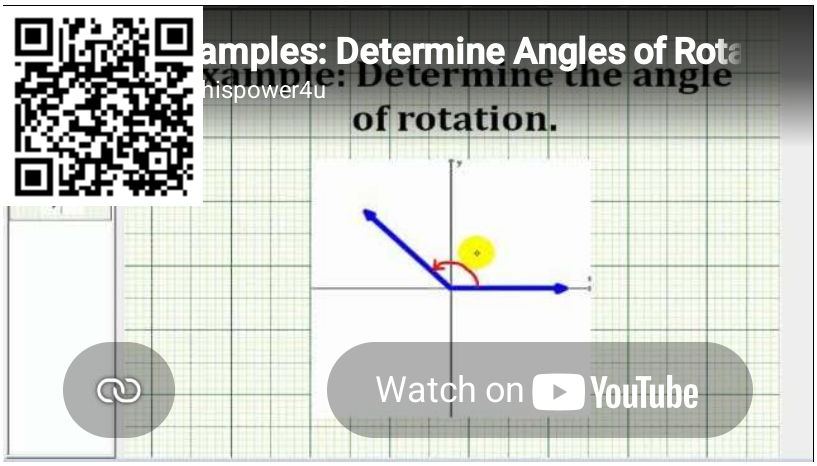
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2.5.4: Rotations in Radians

Rotation of clock hands.

In your math class one morning you finish a quiz early. While you are waiting, you watch the clock as it ticks off five minutes. The time on the clock reads 9:00. Your recent lessons have taught you that one way to measure the position of something on a circle is to use an angle. Suddenly it occurs to you that this can be applied to clocks. Can you determine the angle between the two hands of the clock?

A lot of interesting information about rotations and how to measure them can come from looking at clocks. We are so familiar with clocks in our daily lives that we don't often stop to think about these little devices, with hands continually rotating. Let's take a few minutes in this lesson for a closer look at these examples of rotational motion.



For the following problems, let's express in radians

1. The hands of a clock show 11:20. Express the obtuse angle formed by the hour and minute hands in **radian** measure.

The following diagram shows the location of the hands at the specified time.

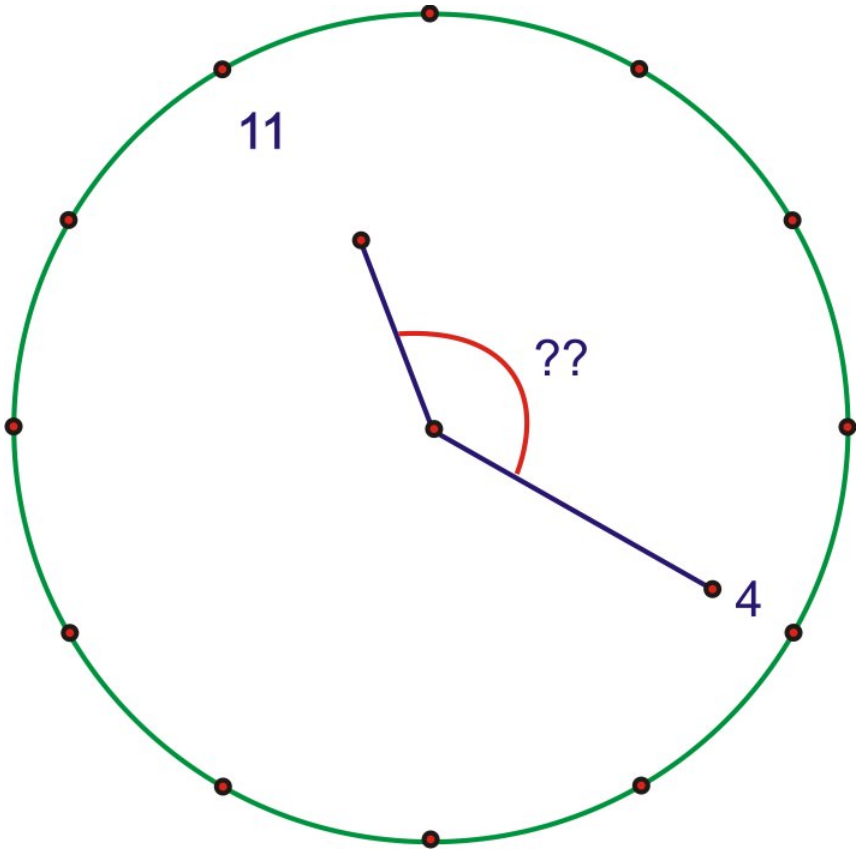


Figure **2.5.4.1**

Because there are 12 increments on a clock, the angle between each hour marking on the clock is $\frac{2\pi}{12} = \frac{\pi}{6}$ (or $30°$). So, the angle between the 12 and the 4 is $4 \times \frac{\pi}{6} = \frac{2\pi}{3}$ (or $120°$). Because the rotation from 12 to 4 is one-third of a complete rotation, it seems reasonable to assume that the hour hand is moving continuously and has therefore moved one-third of the distance between the 11 and the 12. This means that the angle between the hour hand and the 12 is two-thirds of the distance between the 11 and the 12. So, $\frac{2}{3} \times \frac{\pi}{6} = \frac{2\pi}{18} = \frac{\pi}{9}$, and the total measure of the angle is therefore $\frac{\pi}{9} + \frac{2\pi}{3} = \frac{\pi}{9} + \frac{6\pi}{9} = \frac{7\pi}{9}$.

2. Express in radians

The hands of a clock show 4:15. Express the acute angle formed by the hour and minute hands in radian measure.

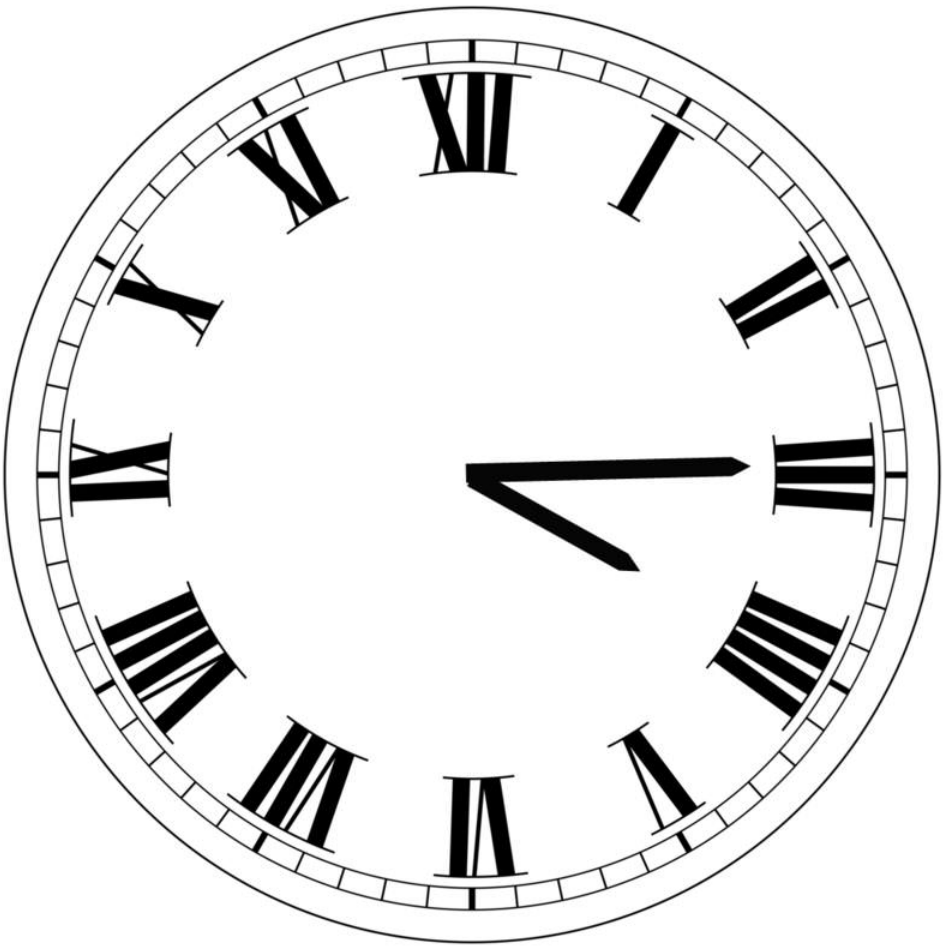


Figure 2.5.4.2

Because there are 12 increments on a clock, the angle between each hour marking on the clock is $\frac{2\pi}{12} = \frac{\pi}{6}$ (or $30°$). So, the angle between the 3 (which is where the minute hand is located when it is 15 minutes after the hour) and the 4 is $\frac{\pi}{6}$ (or $30°$). Further, since the minute hand has moved one quarter of the way around the hour, we can infer that the hour hand has moved one quarter of the way between four and five, which is $\frac{1}{4} \times \frac{\pi}{6} = \frac{\pi}{24}$. Adding these numbers gives: $\frac{\pi}{6} + \frac{\pi}{24} = \frac{4\pi}{24} + \frac{\pi}{24} = \frac{5\pi}{24}$.

3. The hands of a clock show 2:30. Express the acute angle formed by the hour and minute hands in radian measure.

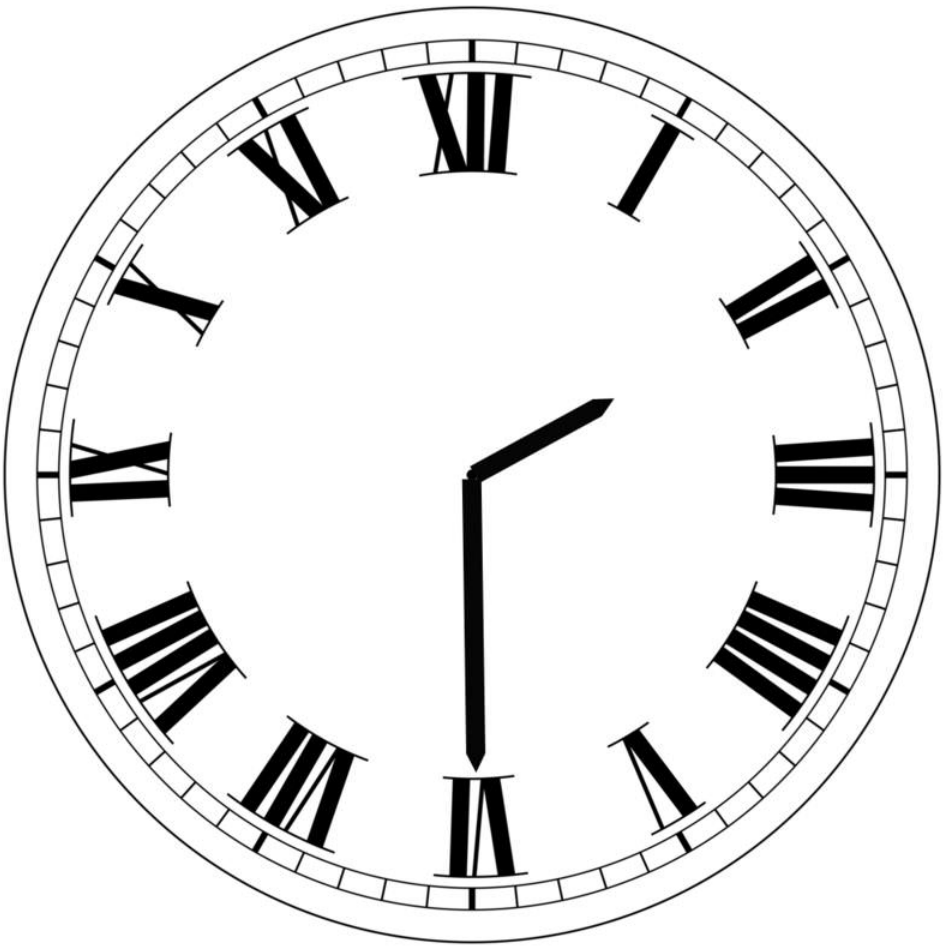


Figure 2.5.4.3

Because there are 12 increments on a clock, the angle between each hour marking on the clock is $\frac{2\pi}{12} = \frac{\pi}{6}$ (or $30°$). Since the hour hand would actually be partway between the 2 and the 3 at 2:30, start by determining the angular distance between the 3 and the 6, then you can add the additional portion between the 2 and the 3:

- The angle between the 3 and the 6 (which is where the minute hand is at 30 minutes after the hour) is $3 \times \frac{\pi}{6} = \frac{3\pi}{6} = \frac{\pi}{2}$ (or $90°$).
- Because the rotation from 12 to 6 is one-half of a complete rotation, it seems reasonable to assume that the hour hand is moving continuously and has therefore moved one-half of the distance between the 2 and the 3. This means that the angle between the hour hand and the 3 is one-half of the distance between the 2 and the 3. So, $\frac{1}{2} \times \frac{\pi}{6} = \frac{\pi}{12}$.
- **The total measure of the angle is therefore** $\frac{\pi}{12} + \frac{\pi}{2} = \frac{\pi}{12} + \frac{6\pi}{12} = \frac{7\pi}{12}$.

Example 2.5.4.1

Earlier, you were asked if you can determine the angle between the two hands of a clock.

Solution

Since you now know that the angle between the hours on a clock is $\frac{\pi}{6} = 30°$, you can use this information to construct an answer. There are three hours between the 9 and the 12 on a clock, so the answer is:

$3 \times \frac{\pi}{6} = \frac{3\pi}{6} = \frac{\pi}{2} = 90°$

So there are $90°$ degrees between the 9 and 12 on the clock.

The following image shows a 24-hour clock.



Figure 2.5.4.4

Example 2.5.4.2

What is the angle between each number of the clock expressed in exact radian measure in terms of π?

Solution

Since there are 2π radians in a circle, and there are 24 separate increments, the answer is $\frac{2\pi}{24} = \frac{\pi}{12}$

Example 2.5.4.3

What is the angle between each number of the clock expressed to the nearest tenth of a radian? What about in degree measure?

Solution

Since there are 2π radians in a circle, the number of radians in each of 24 different divisions is $\frac{2\pi}{24} \approx .3$. In degrees we can do the same by taking the number of degrees in a circle and dividing it by 12: $\frac{360}{24} = 15°$.

Example 2.5.4.4

Estimate the measure of the angle between the hands at the time shown to the nearest whole degree. And then in radian measure in terms of π.

Solution

$20°$. Answers may vary, anything above $15°$ and less than $25°$ is reasonable. In radians, this is $\frac{\pi}{9}$. Again, answers may vary.

Review

Use the clock below to help you find the angle between the hour hand and minute hand at each of the following times. Express your answer in degrees less than $180°$. Then express your answer in radian measure in terms of π.

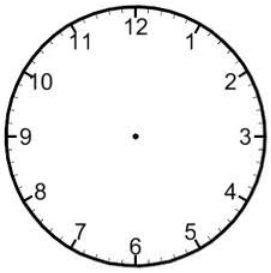


Figure 2.5.4.5

1. 3:30
2. 5:15
3. 4:45
4. 6:30
5. 6:15
6. 2:30
7. 12:30
8. 9:30
9. 10:15
10. 11:30
11. 3:45
12. 2:15
13. 7:15
14. How many times in 12 hours will the hour and minute hands overlap?
15. When is the first time after 12:00 that the hour and minute hands will overlap exactly?

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.4.

Vocabulary

Term	Definition
radian	A radian is a unit of angle that is equal to the angle created at the center of a circle whose arc is equal in length to the radius.

Additional Resources

Video: Examples: Determine Angles of Rotation

Practice: Rotations in Radians

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2.5.5: Angular Velocity

Converting between linear and angular speeds using radius and circumference.

To find a particular song on your MP3/MP4 Player, you may use a scroll wheel. This involves moving your finger around the wheel in a circular motion. Unfortunately for you, the song you want is near the very bottom of your songs list. Since these media players can often hold over 1,000 songs, you have to scroll fast! As you are moving your finger in a circle, you might wonder if you could measure how fast your finger is covering the distance around the circle.

Watching your finger, you realize that your finger is moving around the circle twice every second. If the radius of the scroll wheel is 2 cm, what is the **angular velocity** of your finger as you scroll through your songs list? What is the **linear velocity**?

You may already be familiar with the measurement of speed as the relationship of an object's distance traveled to the time it has been in motion. However, this relationship is for objects that are moving in a straight line. What about objects that are traveling on a circular path?

Do you remember playing on a merry-go-round when you were younger?



Figure **2.5.5.1**

If two people are riding on the outer edge, their velocities should be the same. But, what if one person is close to the center and the other person is on the edge? They are on the same object, but their speed is actually not the same.

Look at the following drawing.

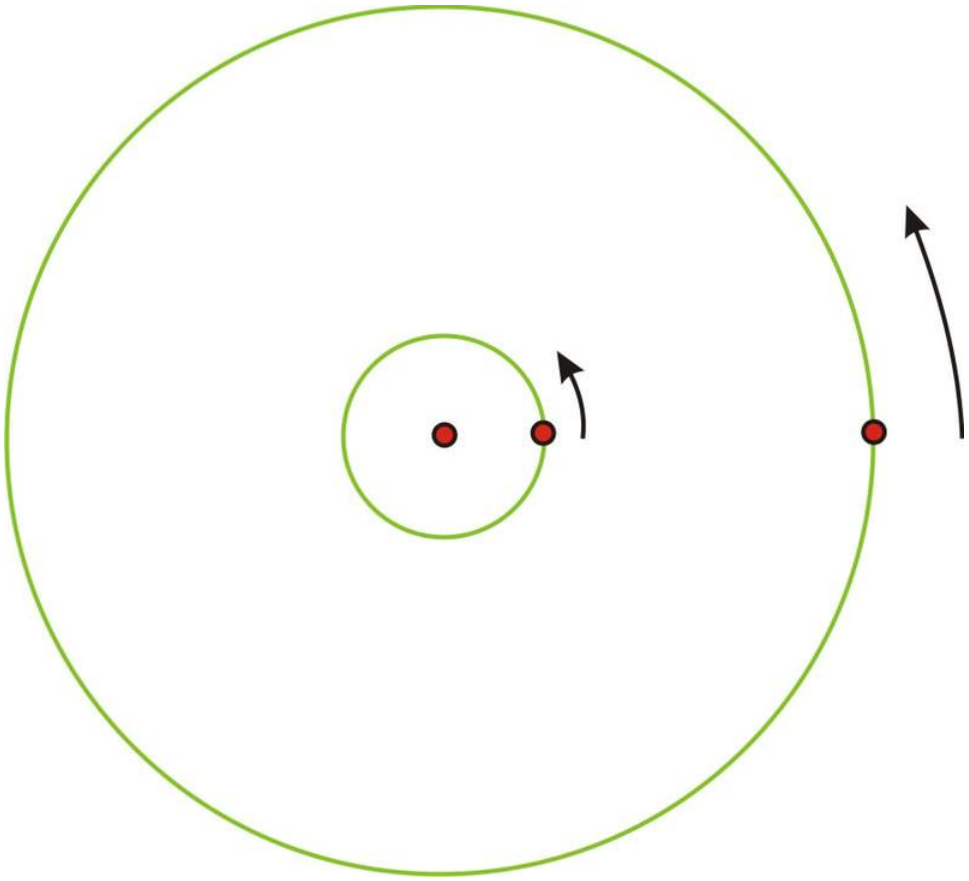


Figure **2.5.5.2**

Imagine the point on the larger circle is the person on the edge of the merry-go-round and the point on the smaller circle is the person towards the middle. If the merry-go-round spins exactly once, then both individuals will also make one complete revolution in the same amount of time.

However, it is obvious that the person in the center did not travel nearly as far. The *circumference* (and of course the radius) of that circle is much smaller and therefore the person who traveled a greater distance in the same amount of time is actually traveling faster, even though they are on the same object. So the person on the edge has a greater *linear velocity* (recall that linear velocity is found using **distance=rate ⋅ time**). If you have ever actually ridden on a merry-go-round, you know this already because it is much more fun to be on the edge than in the center! But, there is something about the two individuals traveling around that is the same. They will both cover the same rotation in the same period of time. This type of speed, measuring the angle of rotation over a given amount of time is called the **angular velocity.**

The formula for angular velocity is:

$$\omega = \frac{\theta}{t}$$

ω is the last letter in the Greek alphabet, omega, and is commonly used as the symbol for angular velocity. \theta is the angle of rotation expressed in **radian** measure, and t is the time to complete the rotation.

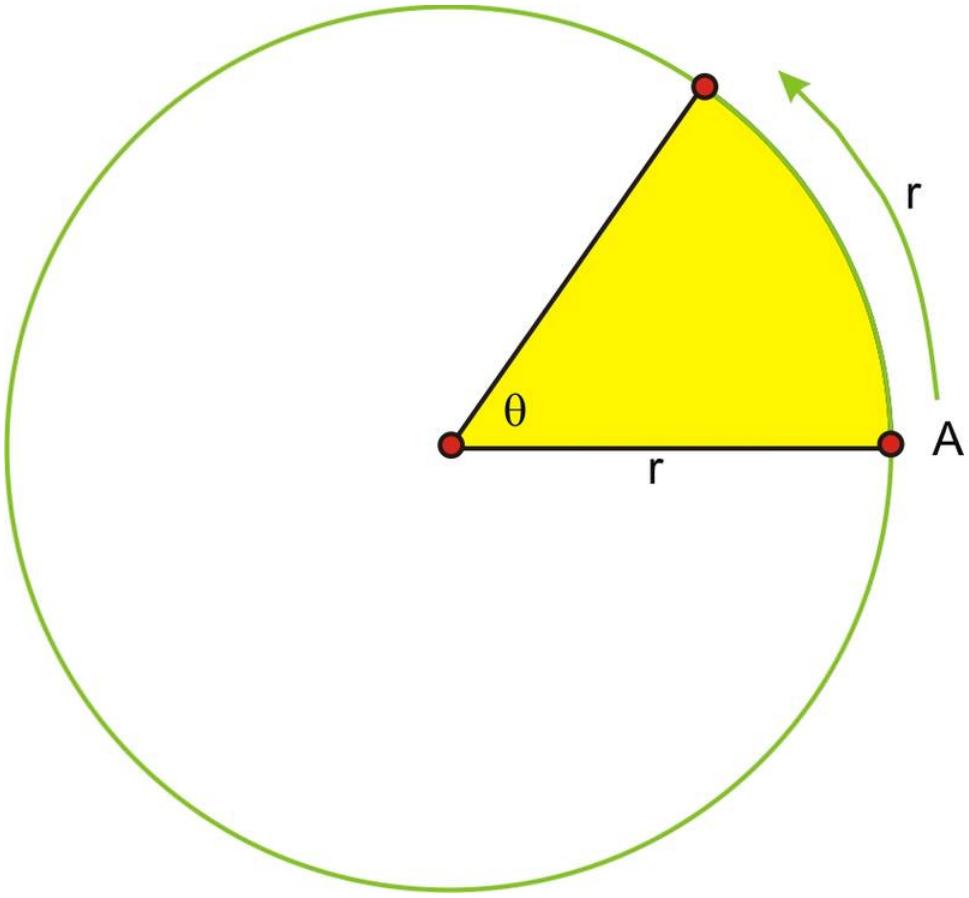


Figure 2.5.5.3

In this drawing, θ is exactly one radian, or the length of the radius bent around the circle. If it took point A exactly 2 seconds to rotate through the angle, the *angular velocity* of A would be:

$$\omega = \frac{\theta}{t}$$
$$\omega = \frac{1}{2} \text{ radians per second}$$

In order to know the ***linear speed*** of the particle, we would have to know the actual distance, that is, the length of the radius. Let's say that the radius is 5 cm.

If linear velocity is v=dt then, $v = \frac{5}{2}$ or 2.5 cm per second.

If the angle were not exactly 1 radian, then the distance traveled by the point on the circle is the length of the arc, $s = r\theta$, or, the radius length times the measure of the angle in radians.

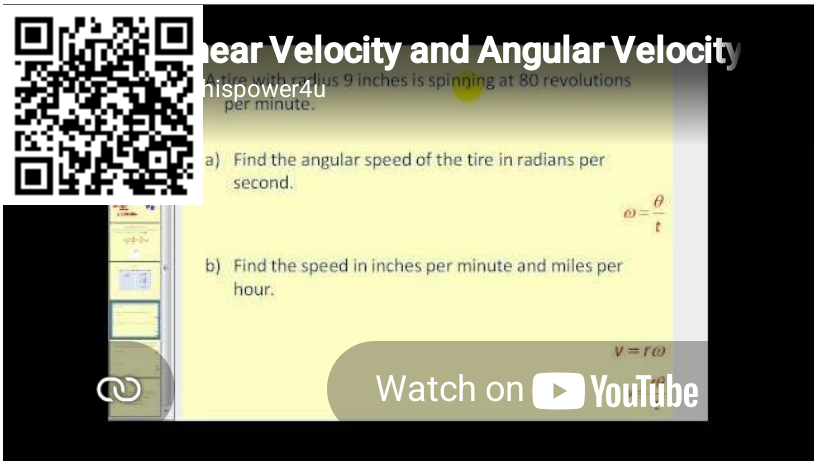
Substituting into the formula for linear velocity gives: $v = \frac{r\theta}{t}$ or $v = r \cdot \frac{\theta}{t}$.

Look back at the formula for angular velocity. Substituting ω gives the following relationship between linear and angular velocity, $v = r\omega$. So, the *linear velocity* is equal to the radius times the *angular velocity.*

Remember in a unit circle, the radius is 1 unit, so in this case the linear velocity is the same as the angular velocity.

$$v = r\omega$$
$$v = 1 \times \omega$$
$$v = \omega$$

Here, the distance traveled around the circle is the same for a given unit of time as the angle of rotation, measured in radians.



Calculating the Linear and Angular Velocity

1. Lindsay and Megan are riding on a Merry-go-round. Megan is standing 2.5 feet from the center and Lindsay is riding on the outside edge 7 feet from the center. It takes them 6 seconds to complete a rotation. Calculate the linear *and* angular velocity of each girl.

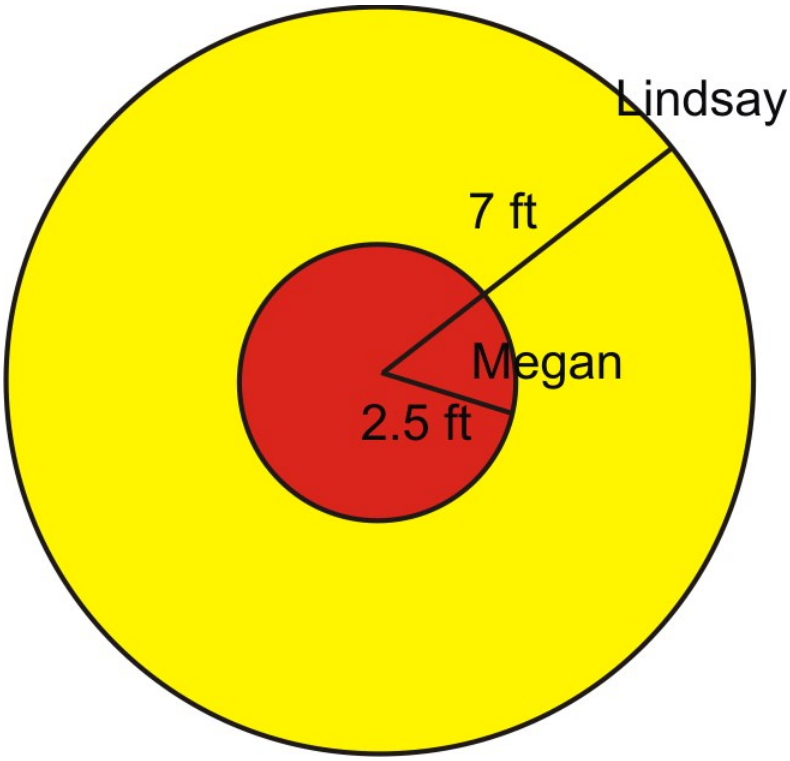


Figure **2.5.5.4**

We are told that it takes 6 seconds to complete a rotation. A complete rotation is the same as 2π radians. So the angular velocity is:

$\omega = \frac{\theta}{t} = \frac{2\pi}{6} = \frac{\pi}{3}$ radians per second, which is slightly more than 1 (about 1.05), radian per second. Because both girls cover the same angle of rotation in the same amount of time, their ***angular speed*** is the same. In this case they rotate through approximately 60 degrees of the circle every second.

As we discussed previously, their linear velocities are different. Using the formula, Megan's linear velocity is:

$$v = r\omega = (2.5)\left(\frac{\pi}{3}\right) \approx 2.6 \text{ ft per sec}$$

Lindsay's linear velocity is:

$$v = r\omega = (7)\left(\frac{\pi}{3}\right) \approx 7.3 \text{ ft per sec}$$

2. A bug is standing near the outside edge of a compact disk (so that his radius from the center of the disc is 6 cm) that is rotating. He notices that he has traveled π radians in two seconds. What is his angular velocity? What is his linear velocity?

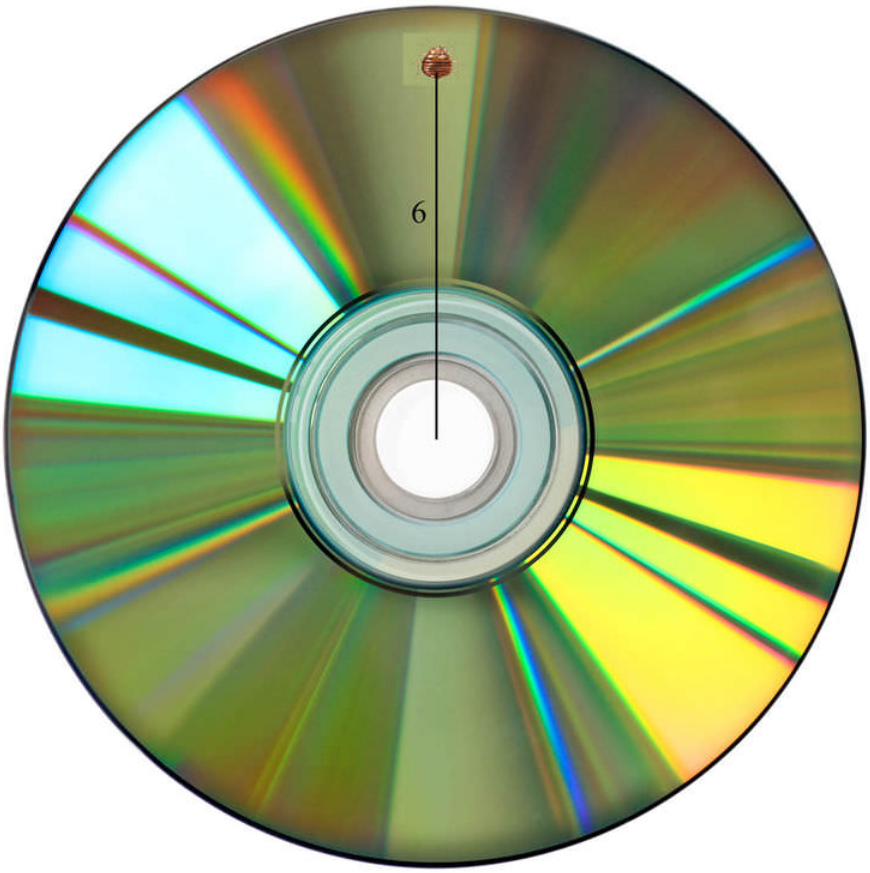


Figure 2.5.5.5

We know that the equation for angular velocity is

$$\omega = \frac{\theta}{t} = \frac{\pi}{2} \text{ radians per second.}$$

We can use the given equation to find his linear velocity:

$$v = r\omega = (6)(\frac{\pi}{2}) \approx 9.42 \text{ cm per sec}$$

Solving for Unknown Values

How long does it take the bug in the previous problem to go through two complete turns?

Since the angular velocity of the bug is $\frac{\pi}{2}$ radians per second, we can use the equation for angular velocity and solve for time:

$$\omega = \frac{\theta}{t}$$
$$t = \theta\omega$$

Since there are 4π radians in two complete turns of the disc, we can use this for the value of \theta :

$$t = \frac{4\pi}{\frac{\pi}{2}} = 4\pi \times \frac{2}{\pi} = 8 \text{ seconds}$$

Example 2.5.5.1

Earlier, you were asked to find the angular and linear velocity of your finger.

Solution

As you found out in this section, the angular velocity is the change in angle divided by the change in time. Since you sweep around the circle twice in a second, this becomes:

$$\omega = \frac{4\pi}{1} = 4\pi \text{ rad/sec}$$

Further, you can find the linear velocity with the equation:

$$v = r\omega = (2)(4\pi) = 8\pi \approx 25.132 \text{ cm/s}$$

Example 2.5.5.2

Doris and Lois go for a ride on a carousel. Doris rides on one of the outside horses and Lois rides on one of the smaller horses near the center. Lois' horse is 3 m from the center of the carousel, and Doris' horse is 7 m farther away from the center than Lois'. When the carousel starts, it takes them 12 seconds to complete a rotation.

Calculate the linear velocity of each girl. Calculate the angular velocity of the horses on the carousel.

Solution

It is actually easier to calculate the angular velocity first. $\omega = \frac{2\pi}{12} = \frac{\pi}{6}$, so the angular velocity is $\frac{\pi}{6}$ rad, or 0.524. Because the linear velocity depends on the radius, each girl has her own.

Lois: $v = r\omega = 3 \cdot \frac{\pi}{6} = \frac{\pi}{2}$ or 1.57 m/sec

Doris: $v = r\omega = 10 \cdot \frac{\pi}{6} = \frac{5\pi}{3}$ or 5.24 m/sec

Example 2.5.5.3

The Large Hadron Collider near Geneva, Switzerland began operation in 2008 and is designed to perform experiments that physicists hope will provide important information about the underlying structure of the universe. The LHC is circular with a circumference of approximately 27,000 m. Protons will be accelerated to a speed that is very close to the speed of light ($\approx 3 \times 10^8$ meters per second).

How long does it take a proton to make a complete rotation around the collider? What is the approximate (to the nearest meter per second) angular speed of a proton traveling around the collider? Approximately how many times would a proton travel around the collider in one full second?

Solution

$$v = \frac{d}{t} \rightarrow 3 \times 10^8 = \frac{27,000}{t} \rightarrow t = \frac{2.7 \times 10^4}{3 \times 10^8} = 0.9 \times 10^{-4} = 9 \times 10^{-5} \text{ or } 0.00009 \text{ seconds}$$

$$\omega = \frac{\theta}{t} = \frac{2\pi}{0.00009} \approx 69,813 \text{ rad/sec}$$

The proton rotates around once in 0.00009 seconds. So, in one second it will rotate around the LHC $1 \div 0.00009 = 11,111.11$ times, or just over 11,111 rotations.

Example 2.5.5.4

Ted is standing 2 meters from the center of a merry go round. If his linear velocity is 6 m/s, what is his angular velocity?

Solution

Since the equation relating linear and angular velocity is given by $v = r\omega$, we can solve for omega: $\omega = \frac{v}{r} = \frac{6}{2} = 3$

Review

Beth and Steve are on a carousel. Beth is 7 ft from the center and Steve is right on the edge, 7 ft further from the center than Beth. Use this information and the following picture to answer questions 1-6.

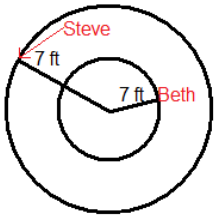


Figure **2.5.5.6**

1. The carousel makes a complete revolution in 12 seconds. How far did Beth go in one revolution? How far did Steve go in one revolution?
2. If the carousel continues making revolutions every 12 seconds, what is the angular velocity of the carousel?
3. What are Beth and Steve's linear velocities?
4. How far away from the center would Beth have to be in order to have a linear velocity of π ft per second.
5. The carousel changes to a new angular velocity of $\frac{\pi}{3}$ radians per second. How long does it take to make a complete revolution now?
6. With the carousel's new velocity, what are Beth and Steve's new linear velocities?
7. Beth and Steve go on another carousel that has an angular velocity of $\frac{\pi}{8}$ radians per second. Beth's linear velocity is 2π feet per second. How far is she standing from the center of the carousel?
8. Steve's linear velocity is only $\frac{\pi}{3}$ feet per second. How far is he standing from the center of the carousel?
9. What is the angular velocity of the minute hand on a clock? (in radians per minute)
10. What is the angular velocity of the hour hand on a clock? (in radians per minute)
11. A certain clock has a radius of 1 ft. What is the linear velocity of the tip of the minute hand?
12. On the same clock, what is the linear velocity of the tip of the hour hand?
13. The tip of the minute hand on another clock has a linear velocity of 2 inches per minute. What is the radius of the clock?
14. What is the angular velocity of the second hand on a clock? (in radians per minute)
15. The tip of the second hand on a clock has a linear velocity of 2 feet per minute. What is the radius of the clock?

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.8.

Vocabulary

Term	Definition
angular speed	Angular speed is the ratio of revolutions that occur per unit of time.
Angular Velocity	The angular velocity of a rotating object is the change in angle of an object divided by the change in time.
dimensional analysis	Dimensional analysis is a process for converting from one unit to another.
linear speed	Linear speed is the ratio of distance per unit of time.
Linear Velocity	The linear velocity of an object is the change in position of an object divided by the change in time.
radian	A radian is a unit of angle that is equal to the angle created at the center of a circle whose arc is equal in length to the radius.

Additional Resources

Video: Linear Velocity and Angular Velocity

Practice: Angular Velocity

Real World: Building Momentum

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2.5.6: Length of an Arc

Radius times the angle in radians.

You have taken your little cousin to the amusement park for the day. While there, she decides she would like a ride on the carousel. After the ride, she excitedly bounces over to you. She is amazed that she went around, but in a way ‘‘didn't go anywhere’’, since she ended up where she started.

‘‘How far did I go when I was halfway around the turn?’’, she asks.

You know that the radius of the carousel is 7 meters. Can you tell your little cousin how far she went in one half of a turn around the ride?

The length of an arc on a circle depends on both the angle of rotation and the radius length of the circle. If you recall from the last lesson, the measure of an angle in radians is defined as the length of the arc cut off by one radius length. What if the radius is 4 cm? Then, the length of the half-circle arc would be π multiplied by the radius length, or 4π cm in length.

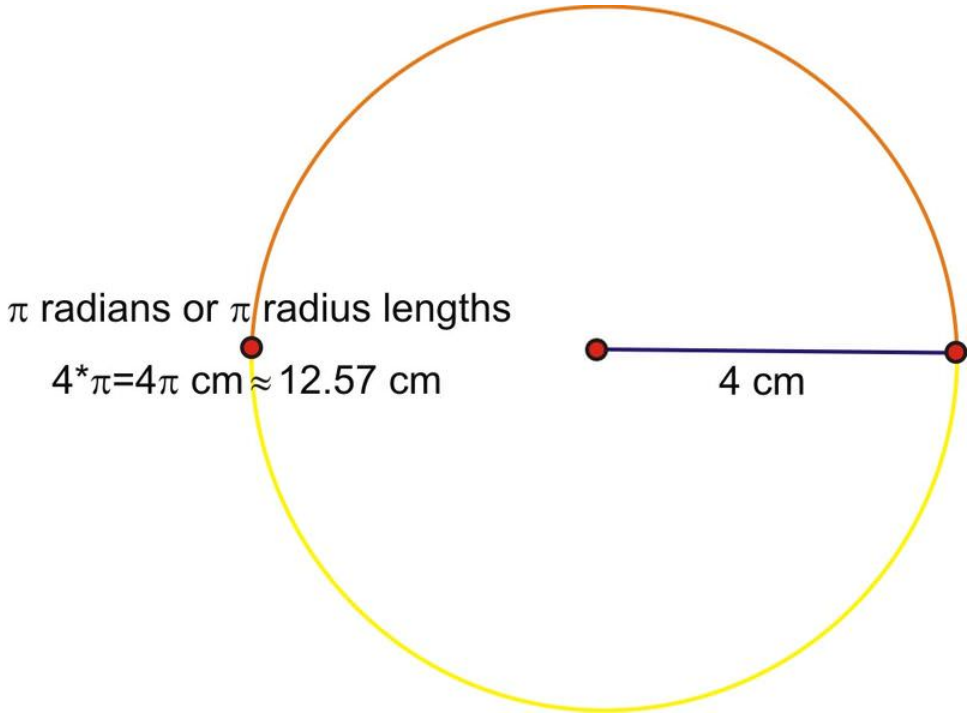


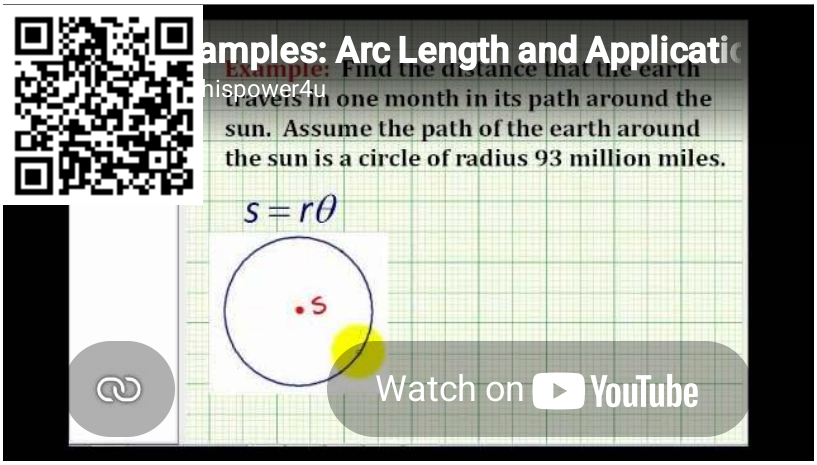
Figure **2.5.6.1**

This results in a formula that can be used to calculate the length of any arc.

$$s = r\theta,$$

where s is the length of the arc, r is the radius, and \theta is the measure of the angle in radians.

Solving this equation for θ will give us a formula for finding the radian measure given the arc length and the radius length.



Let's look at some problems involving arc length.

1. Find the length of the arcs

The free-throw line on an NCAA basketball court is 12 ft wide. In international competition, it is only about 11.81 ft. How much longer is the half circle above the free-throw line on the NCAA court?

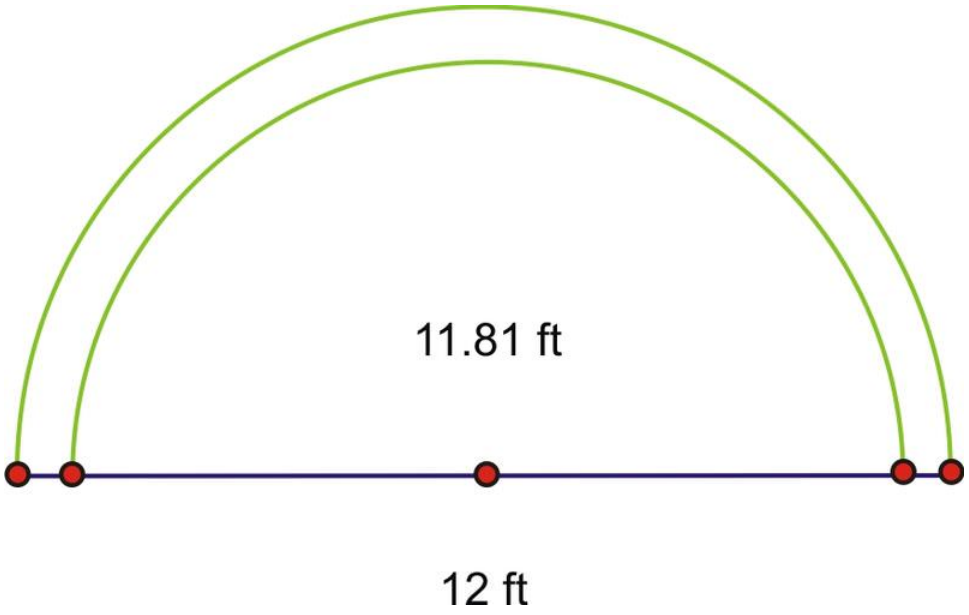


Figure 2.5.6.2

Find both arc lengths.

NCAA	INTERNATIONAL
$s_1 = r\theta$	$s_2 = r\theta$
$s_1 = \frac{12}{2}(\pi)$	$s_2 \approx \frac{11.81}{2}(\pi)$
$s_1 = 6\pi$	$s_2 \approx 5.905\pi$

So the answer is approximately $6\pi - 5.905\pi \approx 0.095\pi$

.095π
.2984513021

Figure 2.5.6.3

This is approximately 0.3 ft, or about 3.6 inches longer.

2. Two connected gears are rotating. The smaller gear has a radius of 4 inches and the larger gear's radius is 7 inches. What is the angle through which the larger gear has rotated when the smaller gear has made one complete rotation?

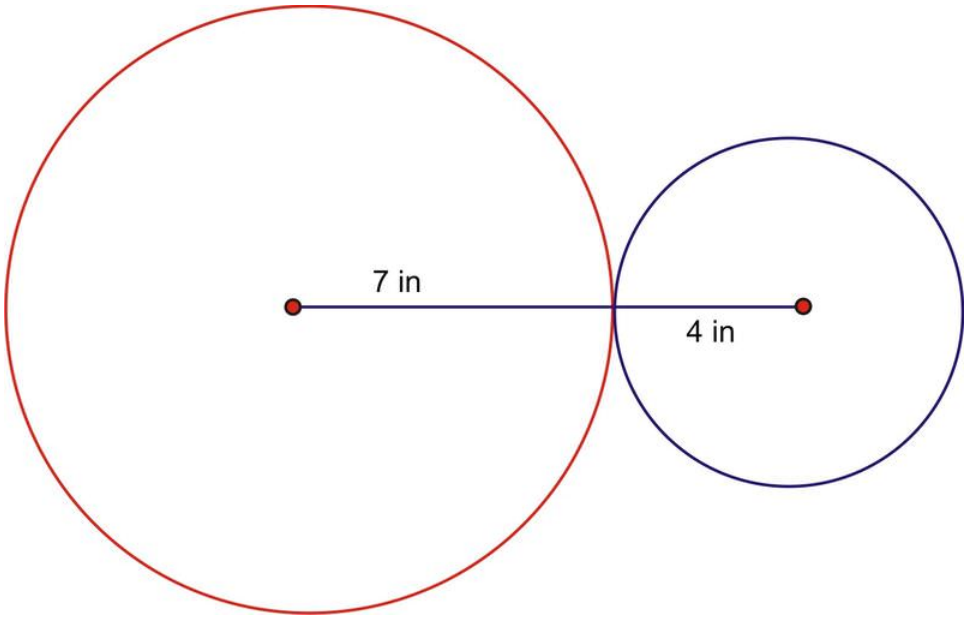


Figure 2.5.6.4

Because the blue gear performs one complete rotation, the length of the arc traveled is:

$$s = r\theta$$
$$s = 4 \times 2\pi$$

So, an 8π arc length on the larger circle would form an angle as follows:

$$\theta = \frac{s}{r}$$
$$\theta = \frac{8\pi}{7}$$
$$\theta \approx 3.6$$

So the angle is approximately 3.6 radians.

$$3.6 \times 180\pi \approx 206^\circ$$

3. The radius of a standard car tire is 27.94 cm. How far does a car go in one revolution of the tire?

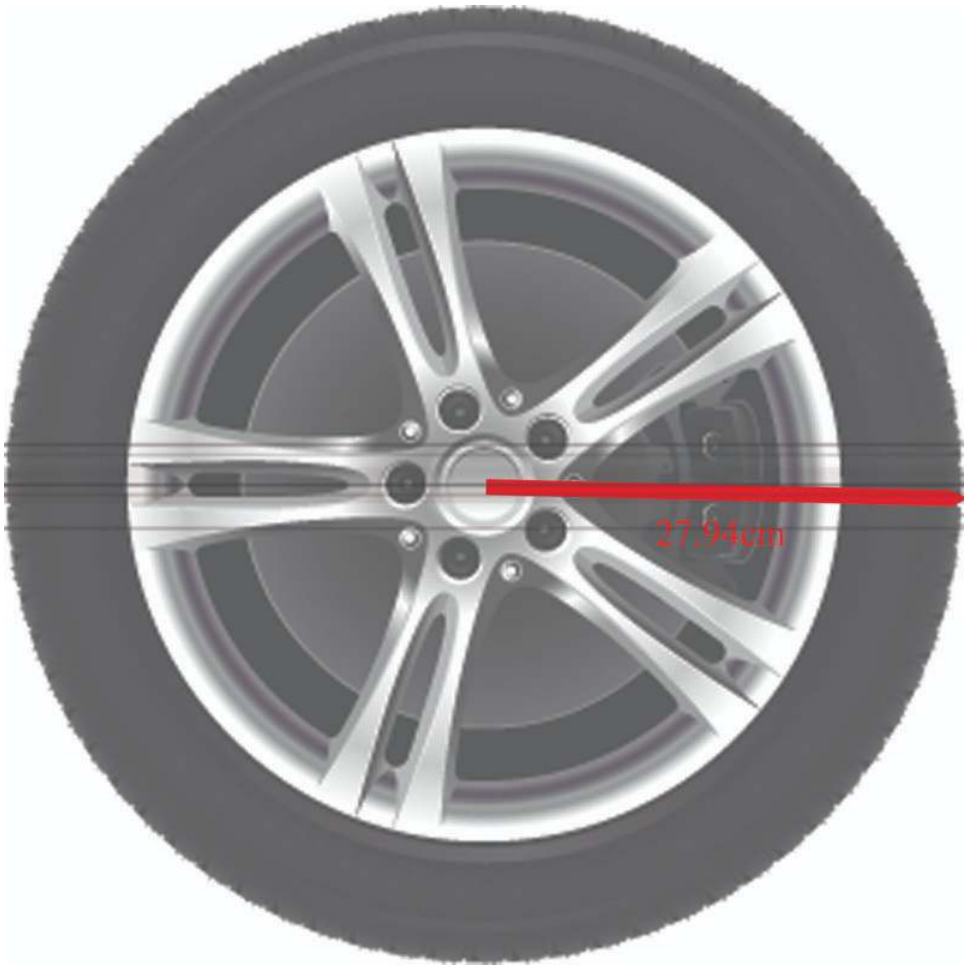


Figure 2.5.6.5

Since the distance traveled by the tire is equal to the distance around the tire, we can use the circumference of the tire to answer the question.

$$s = r\theta$$
$$s = (27.94)(2\pi)$$
$$s = 175.46$$

Example 2.5.6.1

Earlier, you were asked can you tell your little cousin how far she went in one half of a turn around the ride.

Solution

Since you now know that you can measure an arc length using $s = r\theta$, you can use this to find a solution to your cousin's question. Since your cousin wants to know how far she went around when she went $\frac{1}{2}$ of a rotation, and the radius of the ride is 7 meters, you can calculate her arc length:

$$s = r\theta = 7\pi \approx 21.98 \text{ meters}$$

Example 2.5.6.2

You are trying to push your car after it has broken down. Unfortunately, you aren't very strong, and so the car is just rocking back and forth instead of rolling as you push. If the radius of your car's tire is 14 inches, and the change in the tire's angle is $\frac{\pi}{2}$ radians, how far did the tire move?

Solution

Since the distance the tire moved is equal to the length of the arc the tire rolled, you can use the equation s=r\theta to determine how far the tire went:

$$s = r\theta$$
$$s = (14)\left(\frac{\pi}{2}\right)$$
$$s = 7\pi$$
$$s \approx 21.98 \text{ in}$$

Example 2.5.6.3

If an object with a radius of 10 cm spins so that its arc covers 54 cm, what is the change in angle of the object?

Solution

You can again use the equation $s = r\theta$ to solve this problem:

$$s = r\theta$$
$$\theta = \frac{s}{r} = \frac{54}{10} = 5.4$$

The disk moves 5.4 radians, which is a little less than a complete rotation, since a complete rotation is approximately 6.28 radians.

Example 2.5.6.4

If your DVD has a radius of 4.5 inches, how far does a point on the disk spin if the player turns it \dfrac{\pi}{2} radians?

Solution

Using $s = r\theta$,

$$s = r\theta$$
$$s = (4.5)\left(\frac{\pi}{2}\right)$$
$$s = 2.25\pi \approx 7.065$$

A point on the disk turns 7.065 inches.

Review

The radius of a carousel is 8 meters. Use this information to answer questions 1-3.

1. You are half way around the carousel. How far did you travel?
2. You are all the way around the carousel. How far did you travel?
3. You have now traveled all the way around the carousel twice. How far did you travel?

A pizza has a radius of 10 in. Use this information to answer questions 4-6.

4. A slice is removed. The length of the crust of the missing slice is 3 in. What is the central angle of the missing slice?
5. You eat three pieces with a central angle of $\frac{4\pi}{5}$. What is the length of the crust you ate?
6. A large pizza has a radius of 12 in. What is the length of the crust of half of the large pizza?

The diameter of a tire is 35 in. Use this information to answer questions 7-10.

7. What is the length around the whole tire?
8. The tire travels one mile (5280 ft). How many revolutions did the tire make?
9. You roll the tire so it rotates 7π radians. How far did it move?
10. The tire travels half a mile. How many radians did the tire rotate?

Consider a standard 12 hour clock like the one below with a radius of 5 inches. Use this to answer questions 11-15.

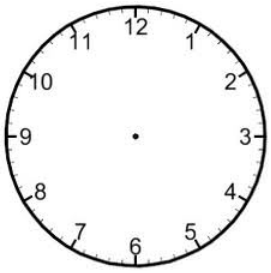


Figure **2.5.6.6**

11. What is the length of the arc between the 3 and the 7?
12. What is the length of the arc between the 3 and the 2?
13. It is 12:30. What is the length of the arc between the minute and hour hands?
14. It is 7:20. What is the length of the arc between the minute and hour hands?
15. It is 1:25. What is the length of the arc between the minute and hour hands?

Review (Answers)

To see the Review, open this PDF file and look for section 2.5.

Vocabulary

Term	Definition
Arc	An arc is a section of the circumference of a circle.

Additional Resources

Video: Arc Length, Area of Sector, Linear Speed, and Angular Speed - Example 5

Practice: Length of an Arc

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2.5.7: Area of a Sector

Half the radius squared times the angle in radians.

While eating lunch with your friends one day you decide to get some pie. The school cafeteria has a sale on pies if you buy the whole pie instead of an individual slice. Since you and your friends plan to eat the whole pie anyway, you are happy to make the purchase.

You decide to cut yourself a piece. When you have taken your section of pie out, one of your friends objects that you have cut out more than your fair share. If the radius of the pie is six inches, and the angle formed by the end of your pie wedge is $30°$, what is the area of your pie piece? Is it more than your fair share if there are five people (including you) splitting the pie?

Area of a **Sector**

One of the most common geometric formulas is the area of a circle:

$$A = \pi r^2$$

In terms of angle rotation, this is the area created by 2π radians.

$$2\pi \text{ rad} = \pi r^2 \text{ area}$$

A half-circle, or π radian rotation would create a section, or **sector** of the circle equal to half the area or:

$$\frac{1}{2}\pi r^2$$

So an angle of 1 radian would define an area of a sector equal to:

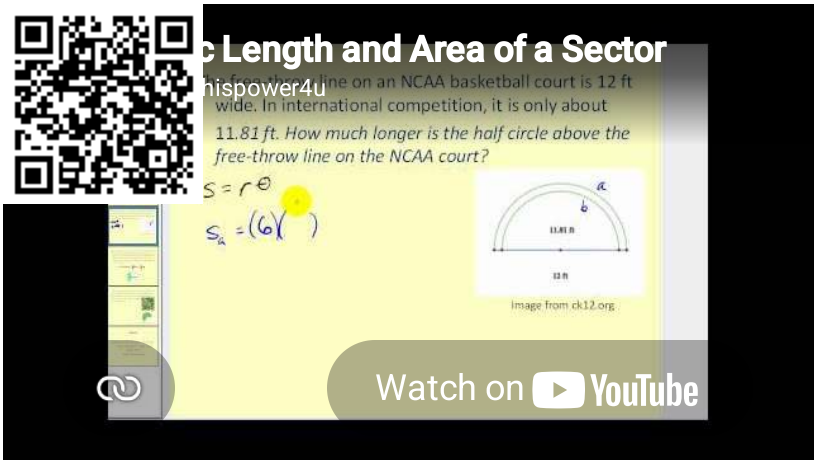
$$\frac{2\pi}{2\pi} = \frac{\pi r^2}{2\pi}$$

Figure 2.5.7.1

$$1 = \frac{1}{2}r^2$$

From this we can determine the area of the sector created by any angle, θ radians, to be:

$$A = \frac{1}{2}r^2\theta$$



Finding the Area

1. Crops are often grown using a technique called center pivot irrigation that results in circular shaped fields.



Figure **2.5.7.2**

Here is a satellite image taken over fields in Kansas that use this type of irrigation system.



Figure **2.5.7.3**

If the irrigation pipe is 450 m in length, what is the area that can be irrigated after a rotation of $\frac{2\pi}{3}$ radians?

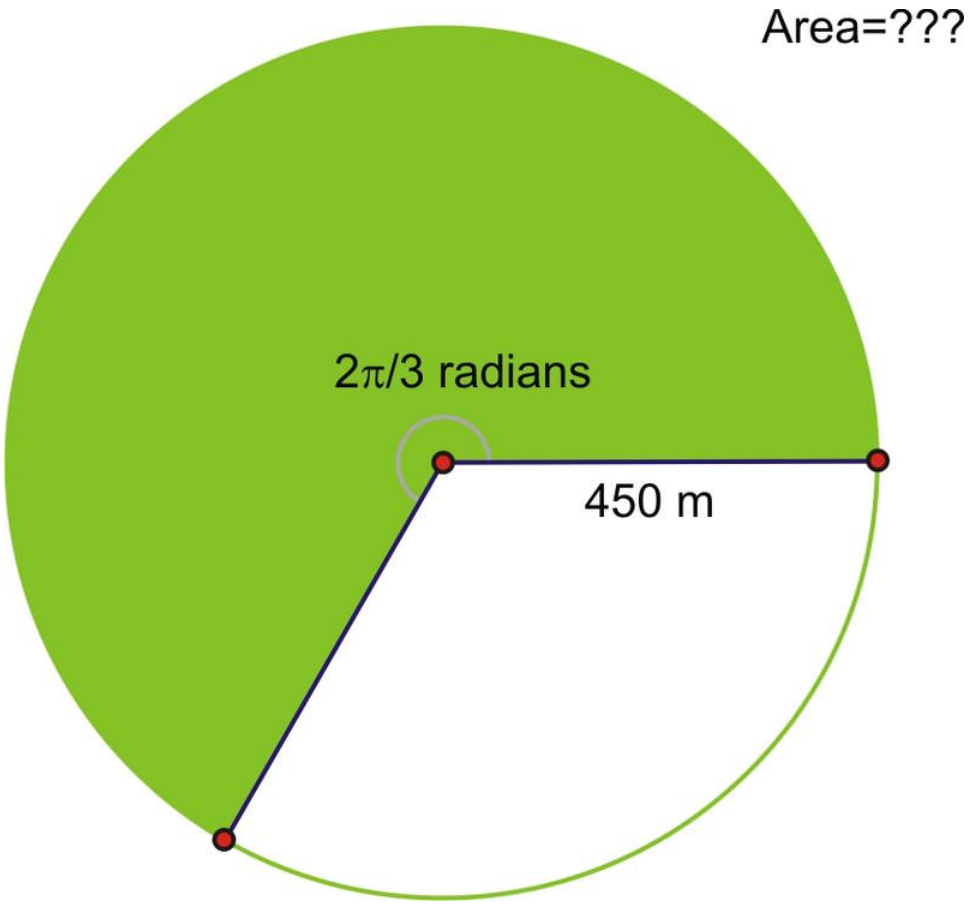


Figure **2.5.7.4**

Using the formula:

$$A = \frac{1}{2} r^2 \theta$$

$$A = \frac{1}{2}(450)^2 \left(\frac{2\pi}{3}\right)$$

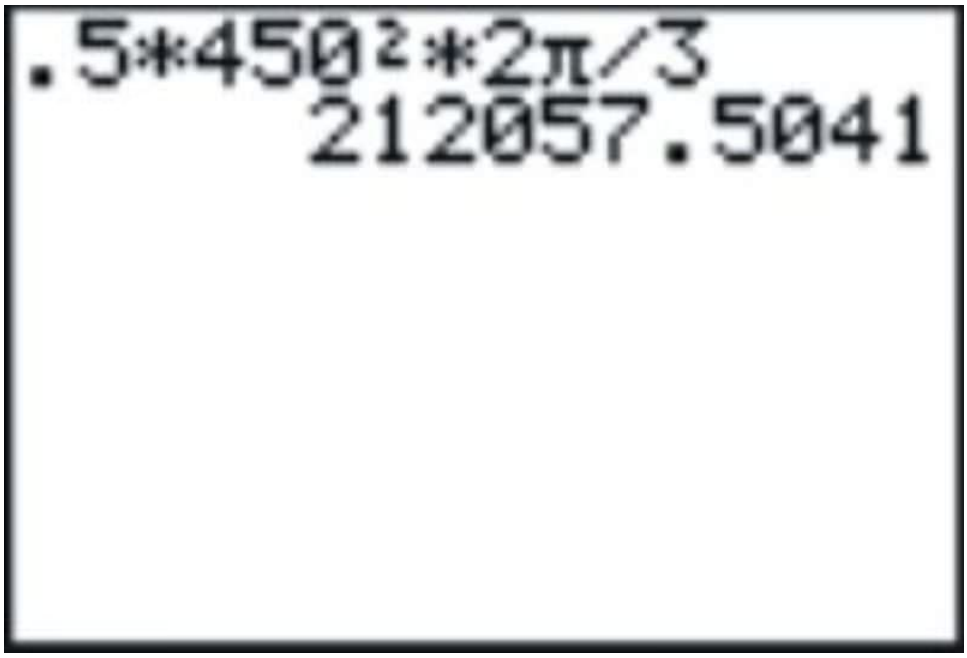


Figure **2.5.7.5**

The area is approximately 212,058 square meters.

2. A doughnut has a hole in the middle with a radius of 1 cm, and the distance from the center of the hole to the outer edge of the doughnut is 3 cm. What is the area of a sector of 14 of the doughnut?

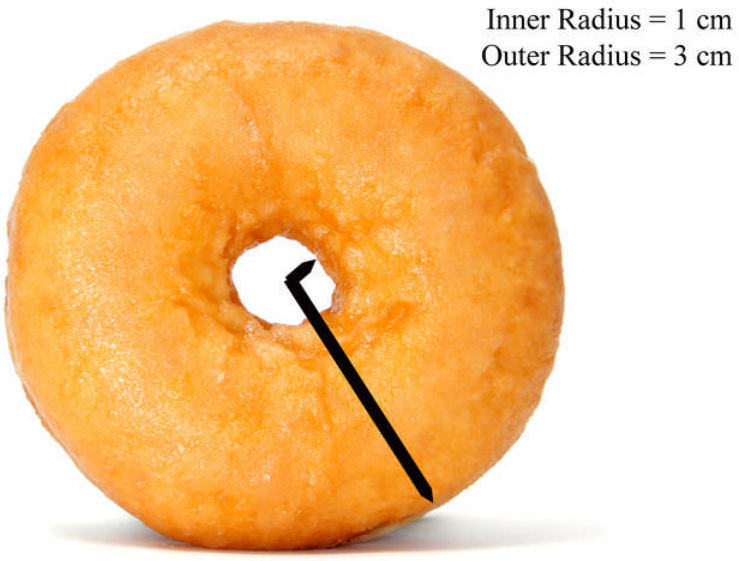


Figure **2.5.7.6**

The formula for the area of a sector is $A = \frac{1}{2}r^2\theta$. Using this formula to find the area of the sector from the center outward gives:

$$A = \frac{1}{2}r^2\theta$$
$$A = \frac{1}{2}3^2\frac{\pi}{2}$$
$$A = \frac{9\pi}{4}$$

Now it is necessary to subtract the area of the sector that is part of the hole, and therefore not part of the doughnut:

$$A = \frac{1}{2}r^2\theta$$
$$A = \frac{1}{2}(1)^2\frac{\pi}{2}$$
$$A = \pi 4$$

Area of the sector of doughnut: $A = \frac{9\pi}{4} - \frac{\pi}{4} = \frac{8\pi}{4} = 2\pi$

3. A driver is traveling around a circular track that has radius of 70 meters. If the angle from the starting line to her current position is $\frac{\pi}{3}$ radians, what is the area of the sector traced out by her car?



Figure 2.5.7.7

The area of a sector is: $A = \frac{1}{2}r^2\theta$. This leads us to:

$$A = \frac{1}{2}r^2\theta$$
$$A = \frac{1}{2}(70)^2\frac{\pi}{3}$$
$$A \approx 73.3 \text{ meters}$$

Example 2.5.7.1

Earlier, you were asked what is the area of your pie piece.

Solution

You now know that the equation for the area of a circle swept out by some angle is:

$$A = \frac{1}{2}r^2\theta$$

Applying this to your pie slice, you have an area of:

$$A = \frac{1}{2} \times 62 \times \frac{\pi}{6} = 3\pi$$

And the total area of the pie is:

$$\pi r^2 = 36\pi$$

To find out what your fair portion of the pie is, multiply the total area by your fraction:

$$\frac{1}{5} \times 36\pi = 7.2\pi$$

And since the piece you took is only 3\pi in size, you are definitely not taking too much!

Example 2.5.7.2

If the radius of a sector is 5 feet, and the sector sweeps out an angle of $43°$, find the area of the sector.

Solution

Since you know that $A = \frac{1}{2}r^2\theta$, you can solve for the area (don't forget to convert the degrees of the angle to radians):

$$A = \frac{1}{2}r^2\theta$$
$$A = \frac{1}{2}(25)(.75)$$
$$A = 9.375$$

Example 2.5.7.3

If a pie wedge has an area of 15 square in inches, and the pie has a radius of 9 inches, find the angle swept out by the sector.

Solution

Since you know that $A = \frac{1}{2}r^2\theta$, you can solve for the angle swept out by the sector (don't forget that the angle will be measured in radians):

$$A = \frac{1}{2}r^2\theta$$
$$15 = \frac{1}{2}(81)(\theta)$$
$$\theta = \frac{(2)(15)}{81}$$
$$\theta = .37$$

Example 2.5.7.4

If you have a piece of round cake that has an area of 20 square inches, and you know the piece sweeps out an angle of $25°$, find the radius of the cake.

Solution

Since you know that $A = \frac{1}{2}r^2\theta$, you can solve for the radius of the cake (don't forget to convert the degrees of the angle to radians):

$$A = \frac{1}{2}r^2\theta$$
$$20 = \frac{1}{2}(r^2)(.436)$$
$$r^2 = \frac{(2)(20)}{.436}$$
$$r^2 = 91.743$$
$$r = 9.58$$

Review

1. If the radius of a sector is 8 inches, and the central angle of the sector is $40°$, find the area of the sector.
2. If the radius of a sector is 12 inches, and the central angle of the sector is $\frac{\pi}{6}$ radians, find the area of the sector.
3. If the radius of a sector is 6 inches, and the central angle of the sector is $140°$, find the area of the sector.
4. If the radius of a sector is 5 inches, and the central angle of the sector is $\frac{5\pi}{3}$ radians, find the area of the sector.
5. If the radius of a sector is 10 inches, and the central angle of the sector is $100°$, find the area of the sector.
6. If a pie wedge has an area of 10 square inches, and the pie has a radius of 6 inches, find the angle swept out by the sector.
7. If a pie wedge has an area of 15 square inches, and the pie has a radius of 4 inches, find the angle swept out by the sector.
8. If a pie wedge has an area of 12 square inches, and the pie has a radius of 3 inches, find the angle swept out by the sector.
9. If you have a piece of round cake that has an area of 20 square inches, and you know the piece sweeps out an angle of $\frac{\pi}{3}$ radians, find the radius of the cake.
10. If you have a piece of round cake that has an area of 100 square inches, and you know the piece sweeps out an angle of $50°$, find the radius of the cake.
11. If you have a piece of round cake that has an area of 35 square inches, and you know the piece sweeps out an angle of $\frac{2\pi}{5}$ radians, find the radius of the cake.

12. If you have a piece of round cake that has an area of 20 square inches, and you know the piece sweeps out an angle of 30°, find the radius of the cake.

A pizza has a radius of 10 in. Use this information to answer questions 13-15.

13. A slice is removed. The length of the crust of the missing slice is 3 in. What is the area of the missing slice?
14. You eat three pieces with a central angle of $\frac{4\pi}{5}$. What is the area of the pizza you ate?
15. A large pizza has a radius of 12 in. What is the area of half of the large pizza?

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.6.

Vocabulary

Term	Definition
Sector	A sector of a circle is a portion of a circle contained between two radii of the circle. Sectors can be measured in degrees.

Additional Resources

Video: Arc Length and Area of a Sector

Practice: Area of a Sector

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2.5.8: Length of a Chord

Twice the radius times the sine of half the angle in radians.

You have been asked to help the younger students at your school with their Physical Education class. While working one afternoon, you are asked to take out a parachute that the students can play with. As the students are playing, one of them walks across a small portion of the parachute instead of under it like she is supposed to. If the chute is shaped like a circle with a radius of 6 meters, and the path the student walked across the chute covered an angle of $50°$, what is the length of the path she walked across the parachute?

You may recall from your Geometry studies that a chord is a segment that begins and ends on a circle.

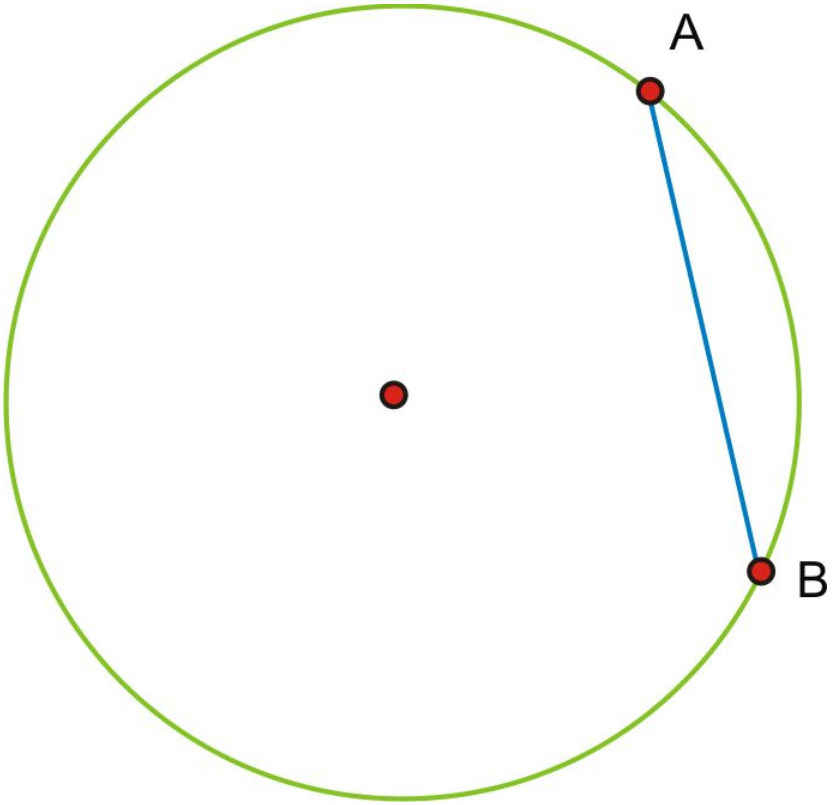


Figure **2.5.8.1**

$\overline{AB}$ is a chord in the circle.

We can calculate the length of any chord if we know the angle measure and the length of the radius. Because each endpoint of the chord is on the circle, the distance from the center to A and B is the same as the radius length.

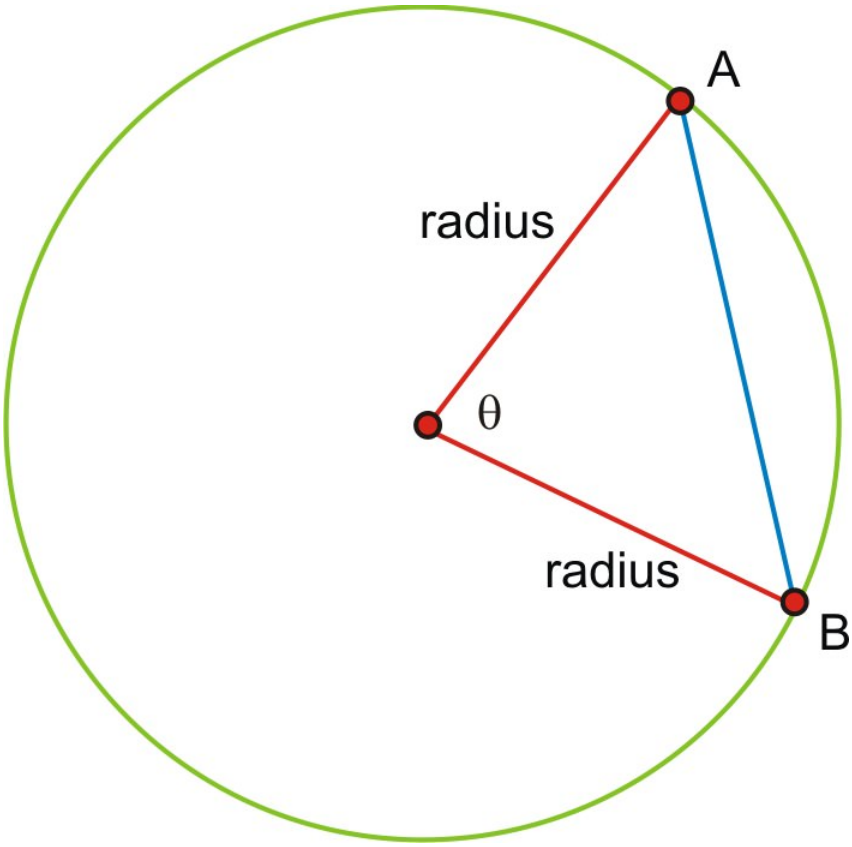


Figure **2.5.8.2**

Next, if we bisect the angle, the angle bisector must be perpendicular to the chord and bisect it (we will leave the proof of this to your Geometry class). This forms a right triangle.

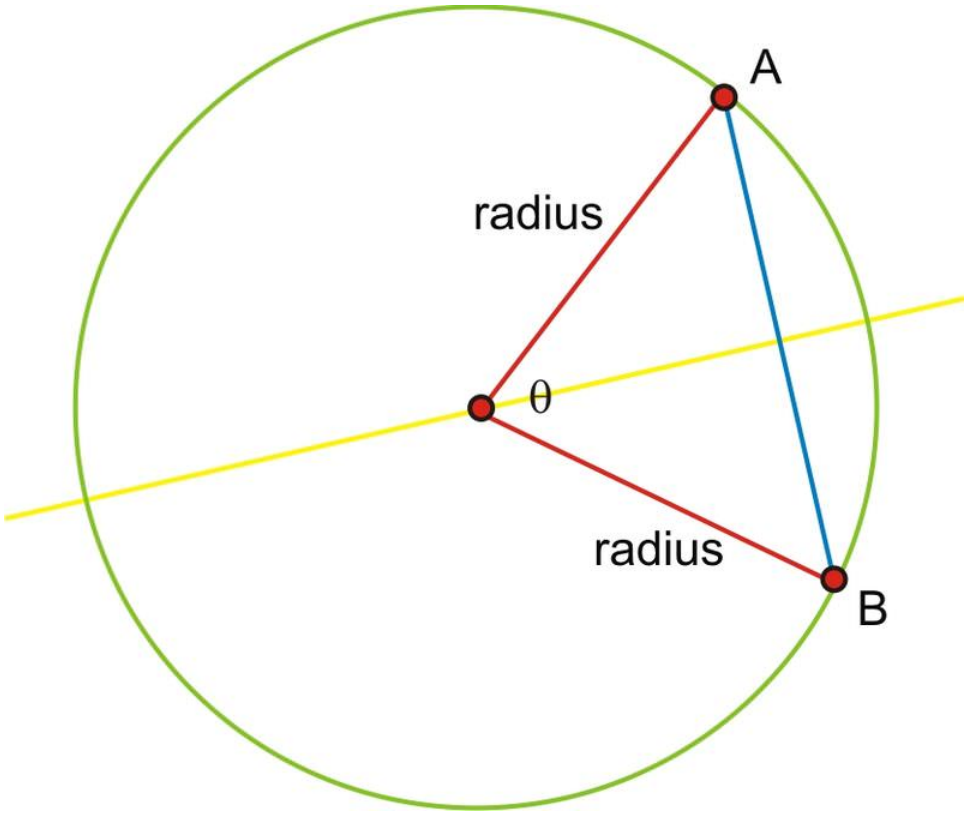


Figure 2.5.8.3

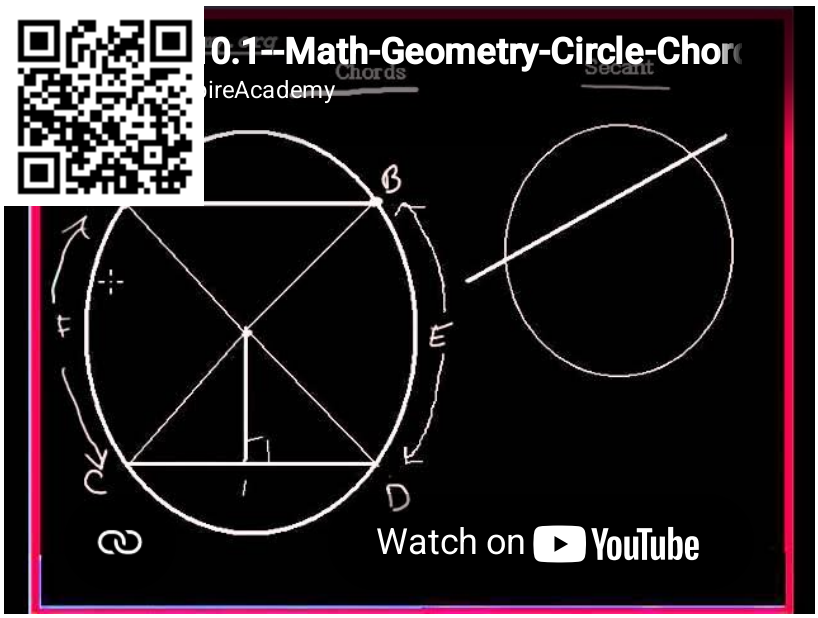
We can now use a simple sine ratio to find half the chord, called c here, and double the result to find the length of the chord.

$$\sin\frac{\theta}{2} = \frac{c}{r}$$

$$c = r \times \sin\frac{\theta}{2}$$

So the length of the chord is:

$$2c = 2r\sin\frac{\theta}{2}$$



Finding the Length of a Chord

1. Find the length of the chord of a circle with radius 8 cm and a central angle of **110°**. Approximate your answer to the nearest mm.

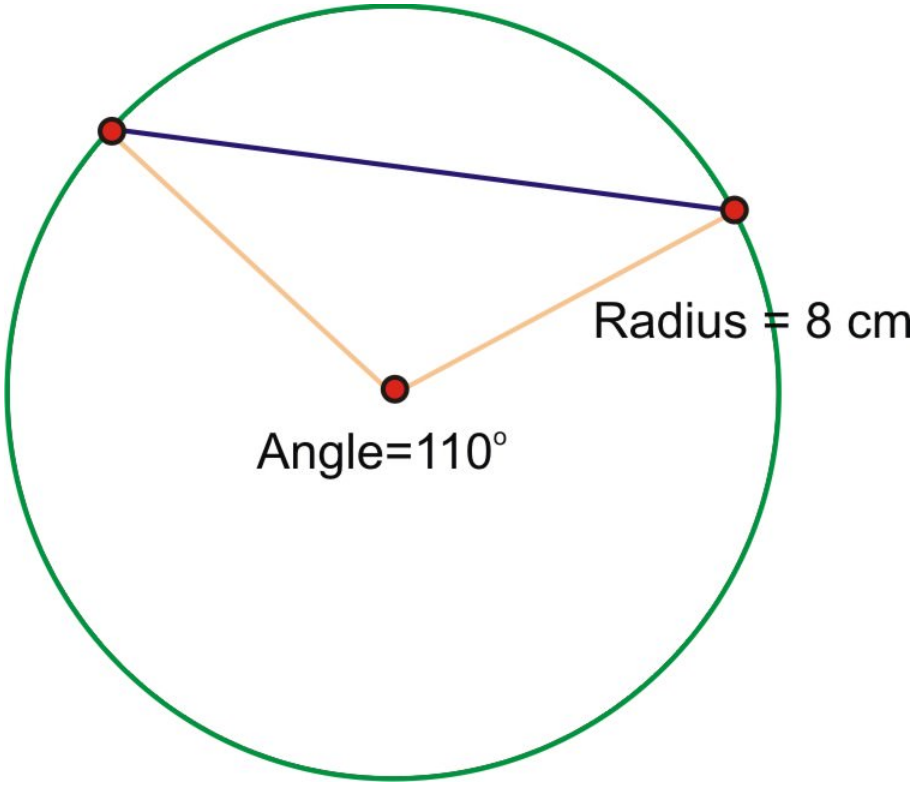


Figure **2.5.8.4**

We must first convert the angle measure to radians:

$$110 \times \frac{\pi}{180} = \frac{11\pi}{18}$$

Using the formula, half of the chord length should be the radius of the circle times the sine of half the angle.

$$\frac{11\pi}{18} \times \frac{1}{2} = \frac{11\pi}{36}$$

$$8 \times \sin \frac{11\pi}{36}$$

Multiply this result by 2.

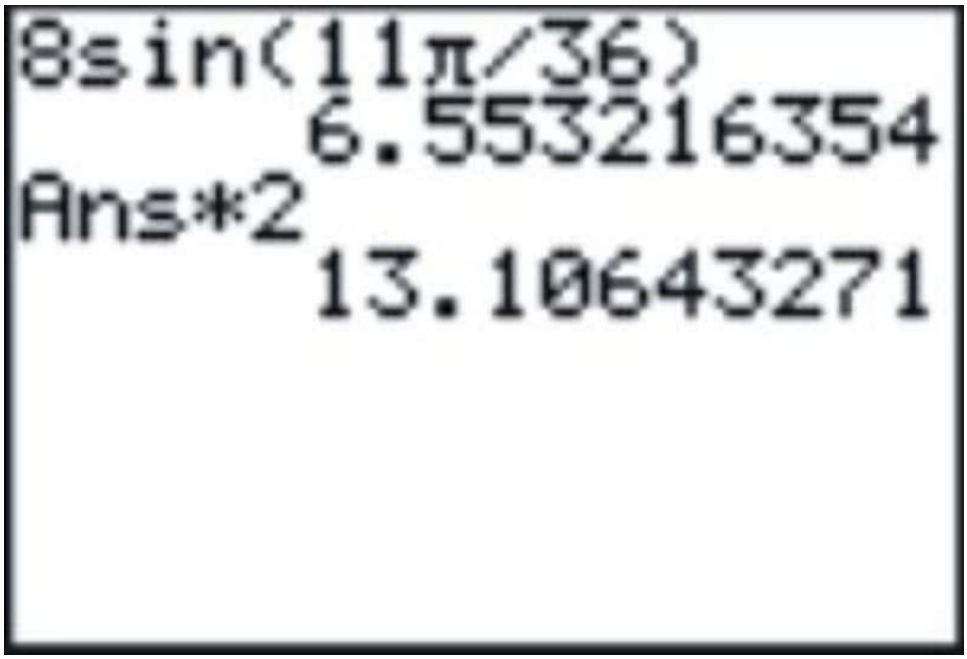


Figure **2.5.8.5**

So, the length of the chord is approximately 13.1 cm.

2. Find the length of the chord of a circle with a radius of 2 m that has a central angle of **90°**.

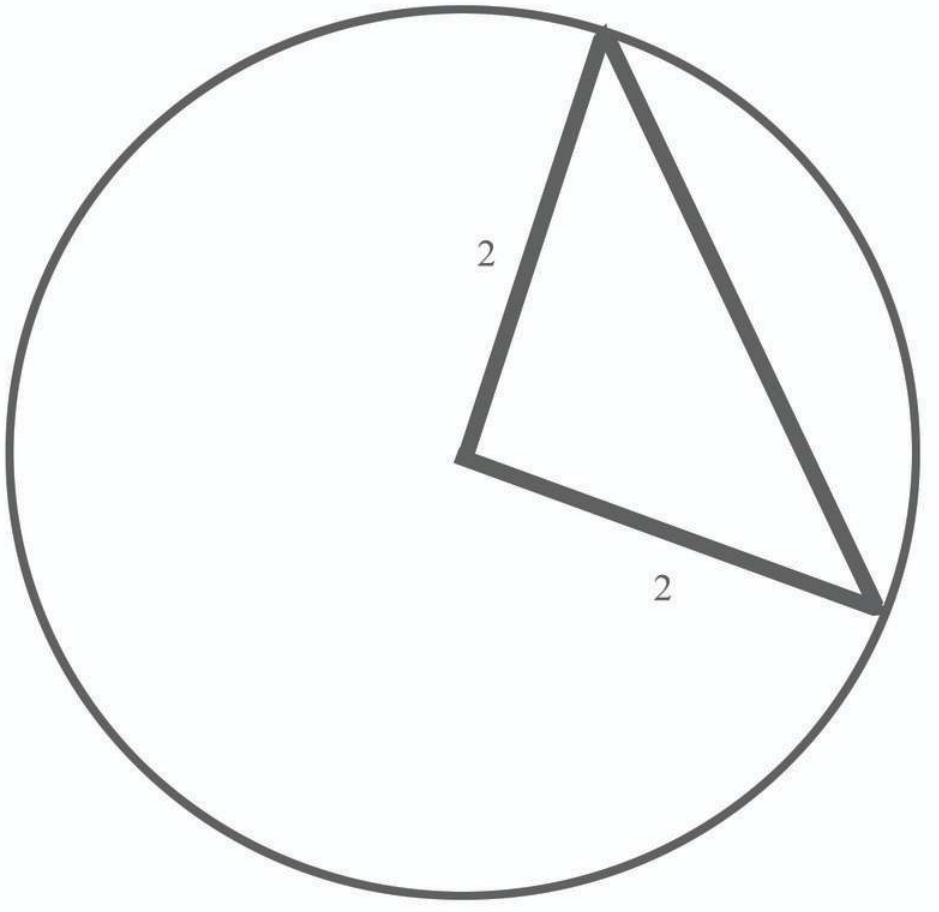


Figure **2.5.8.6**

First convert the angle to radians:

$$90 \times \frac{\pi}{180} = \frac{\pi}{2}$$

Using the formula, half of the chord length should be the radius of the circle times the sine of half the angle.

$$\frac{\pi}{2} \times \frac{1}{2} = \frac{\pi}{4}$$
$$2 \times \sin \frac{\pi}{4}$$

Multiply this result by 2.

So the answer is approximately 2.83 meters.

3. Find the length of the chord of a circle with radius 1 m and a central angle of $170°$.

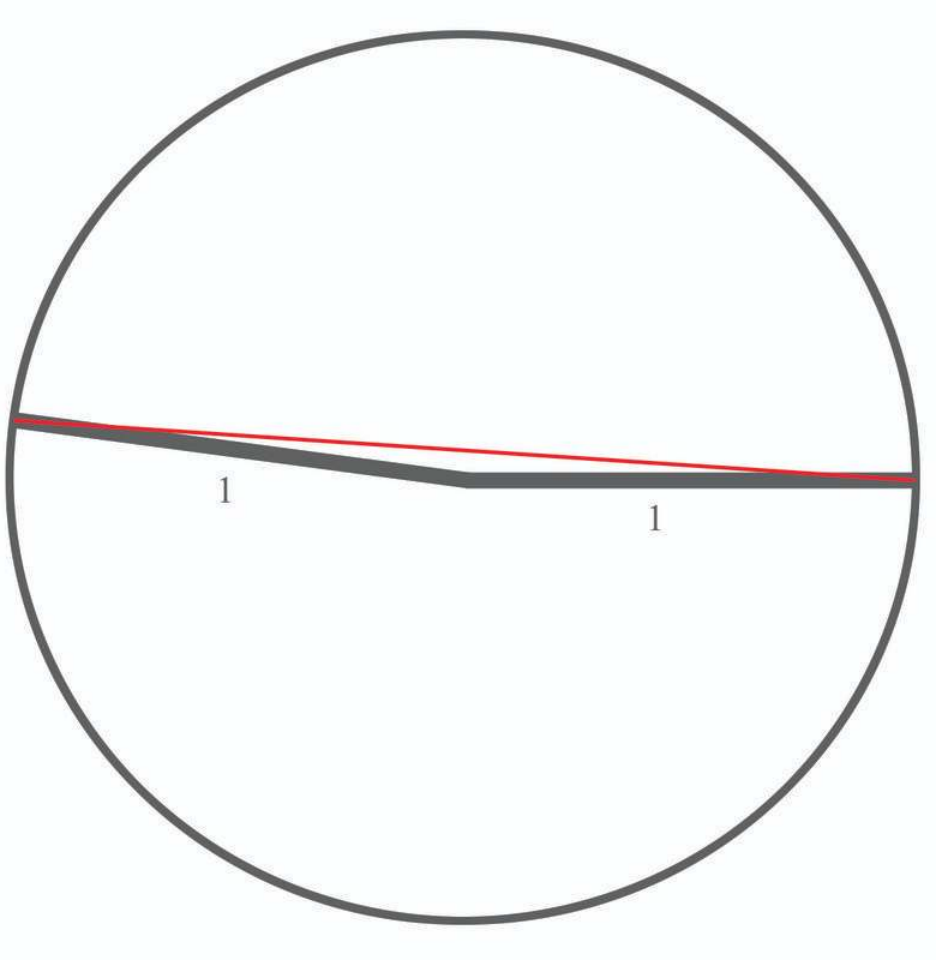


Figure 2.5.8.7

We must first convert the angle measure to radians:

$$170 \times \frac{\pi}{180} = \frac{17\pi}{18}$$

Using the formula, half of the chord length should be the radius of the circle times the sine of half the angle.

$$\frac{17\pi}{18} \times \frac{1}{2} = \frac{17\pi}{36}$$
$$1 \times \sin \frac{17\pi}{36} = .996$$

Multiply this result by 2.

So, the length of the arc is approximately 1.992

Notice that the length of the chord is almost 2 meters, which would be the diameter of the circle. If the angle had been 180 degrees, the chord would have just been the distance all the way across the circle going through the middle, which is the diameter.

Example 2.5.8.1

Earlier, you were asked what is the length of the path she walked across the parachute.

Solution

With the equation for the length of a chord in hand, you can calculate the distance the student ran across the parachute:

First convert the measure in degrees to radians:

$$50 \times \frac{\pi}{180} \approx .27\pi$$

$$2r \sin \frac{\theta}{2} = (2)(6) \sin \frac{.27\pi}{2} = 12 \sin .135\pi \approx 4.94 \text{ meters}$$

Example 2.5.8.2

If you run a piece of string across a doughnut you are eating, and the radius between the endpoints of the string to the center of the doughnut is 4 inches, how long is the string if the angle swept out by the chord is 20°?

Solution

You can use the equation $C = 2r \sin\left(\frac{\theta}{2}\right)$ to solve this problem: (Don't forget to convert angles to radians)

$$C = 2r \sin\left(\frac{\theta}{2}\right)$$
$$C = (2)(4) \sin\left(\frac{.349}{2}\right)$$
$$C = 8(.1736)$$
$$C = 1.388 \text{ inches}$$

Example 2.5.8.3

You are eating dinner one night with your family at the local Italian restaurant. A piece of spaghetti makes a chord across your plate. You know that the length of the spaghetti strand is 5 inches, and the radius of the plate is 7 inches. What is the angle swept out by the chord?

Solution

Since the radius of the plate and the length of the chord are known, you can solve for the angle:

$$C = 2r \sin\left(\frac{\theta}{2}\right)$$
$$\frac{C}{2r} = \sin\left(\frac{\theta}{2}\right)$$
$$\sin^{-1}\left(\frac{c}{2r}\right) = \frac{\theta}{2}$$
$$\sin^{-1}\left(\frac{5}{14}\right) = \frac{\theta}{2}$$
$$.365 = \frac{\theta}{2}$$
$$\theta = .73$$

The angle spanned by the spaghetti is .73 radians.

Example 2.5.8.4

If you draw a chord across a circle and make a chord across it that has a length of 15 inches, sweeping out an angle of π radians, what is the radius of the circle you drew?

Solution

Using the equation for the length of a chord:

$$c = 2r\sin(\frac{\theta}{2})$$
$$15 = (2r)\sin(\frac{\pi}{2})$$
$$r = 7.5$$

As you can see, the radius of the circle is 7.5 inches. This is what you should expect, since the chord sweeps out an angle of π. This means that it sweeps out half of the circle, so that the chord is actually going across the whole diameter of the circle. So if the chord is going across the diameter and has a length of 15 inches, then the radius of the circle should be 7.5 inches.

Review

1. Find the length of the chord of a circle with radius 1 m and a central angle of $100°$.
2. Find the length of the chord of a circle with radius 8 km and a central angle of $130°$.
3. Find the length of the chord of a circle with radius 4 in and a central angle of $45°$.
4. Find the length of the chord of a circle with radius 3 ft and a central angle of $32°$.
5. Find the length of the chord of a circle with radius 2 cm and a central angle of $112°$.
6. Find the length of the chord of a circle with radius 7 in and a central angle of $135°$.

Solve for the missing variable in each circle.

7.

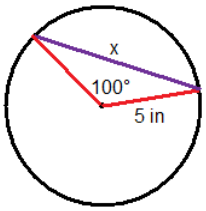


Figure 2.5.8.8

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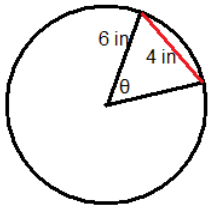


Figure 2.5.8.9

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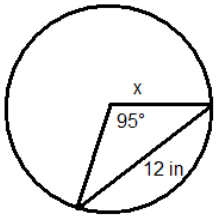


Figure 2.5.8.10

10.

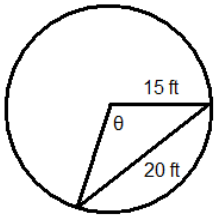


Figure 2.5.8.11

11.

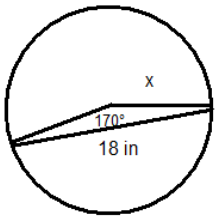


Figure **2.5.8.12**

12.

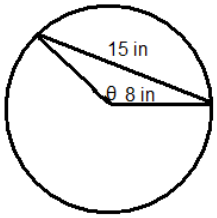


Figure **2.5.8.13**

Use the picture below for questions 13-15.

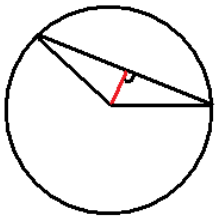


Figure **2.5.8.13**

13. Suppose you knew the length of the chord, the length of the radius, and the central angle of the above circle. Describe one way to find the length of the red segment using the Pythagorean Theorem.
14. Suppose you knew the length of the chord, the length of the radius, and the central angle of the above circle. Describe one way to find the length of the red segment using cosine.
15. What would you need to know in order to find the area of the segment (the portion of the circle between the chord and the edge of the circle)? Describe how to find the area of this region.

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.7.

Vocabulary

Term	Definition
Chord	A chord is a straight line across a circle, intersecting the circle in two places, but not passing through the circle's center.

Additional Resources

Video: Chords

Practice: Length of a Chord

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2.6: Sine and Cosine Graphs

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2.6.1: Sine Graph and Cosine Graph

Graph and stretch trig functions

Your mission, should you choose to accept it, as Agent Trigonometry is to graph the function $y = 2\cos x$. What are the minimum and maximum of your graph?

Graphing Sine and Cosine

In this concept, we will take the unit circle and graph it on the Cartesian plane.

To do this, we are going to "unravel" the unit circle. Recall that for the unit circle the coordinates are $(\cos\theta, \sin\theta)$ where θ is the central angle. To graph, $y = \sin x$ rewrite the coordinates as $(x, \sin x)$ where x is the central angle, in radians. Below we expanded the sine coordinates for $\frac{3\pi}{4}$.

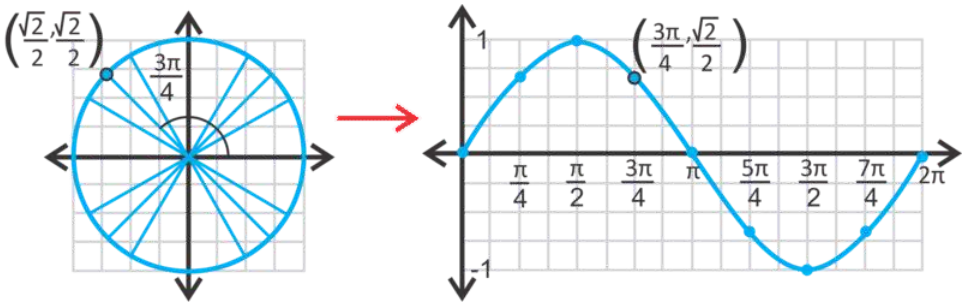


Figure 2.6.1.1

Notice that the curve ranges from 1 to -1. The maximum value is 1, which is at $x = \frac{\pi}{2}$. The minimum value is -1 at $x = \frac{3\pi}{2}$. This "height" of the sine function is called the **amplitude**. The amplitude is the absolute value of average between the highest and lowest points on the curve.

Now, look at the domain. It seems that, if we had continued the curve, it would repeat. This means that the sine curve is **periodic**. Look back at the unit circle, the sine value changes until it reaches 2π. After 2π, the sine values repeat. Therefore, the curve above will repeat every 2π units, making the **period** 2π. The domain is all real numbers.

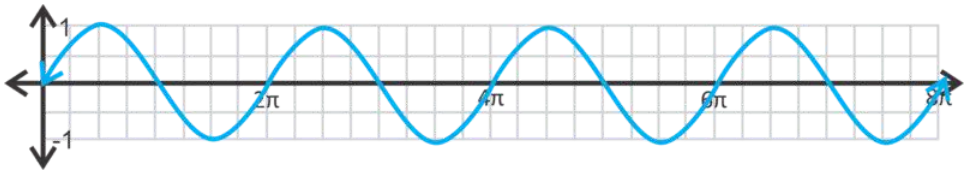


Figure 2.6.1.2

Similarly, when we expand the cosine curve, $y = \cos x$, from the unit circle, we have:

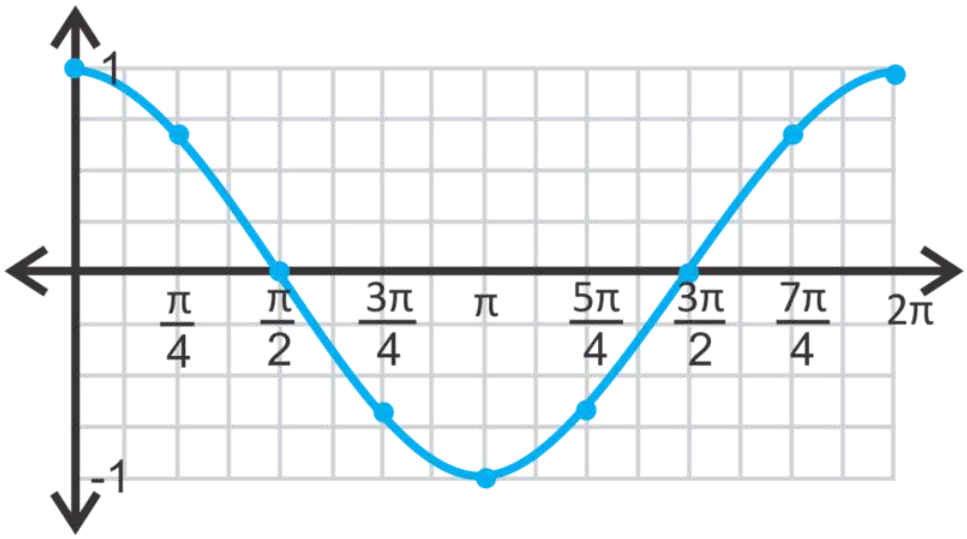


Figure **2.6.1.3**

Notice that the range is also between 1 and -1 and the domain will be all real numbers. The cosine curve is also periodic, with a period of 2π. If we draw the graph past 2π, it would look like:

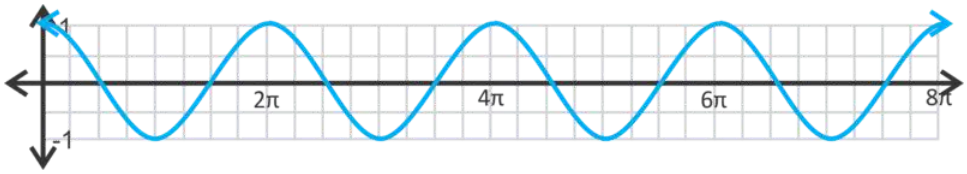


Figure **2.6.1.4**

Comparing $y = \sin x$ and $y = \cos x$ (below), we see that the curves are almost identical, except that the sine curve starts at $y = 0$ and the cosine curve starts at $y = 1$.

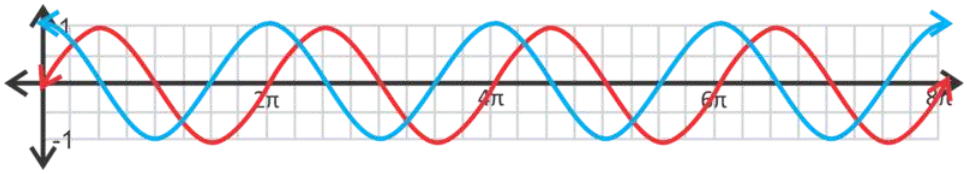


Figure **2.6.1.5**

If we shift either curve $\frac{\pi}{2}$ units to the left or right, they will overlap. Any horizontal shift of a trigonometric function is called a **phase shift**.

Let's identify the highlighted points on $y = \sin x$ and $y = \cos x$ below.

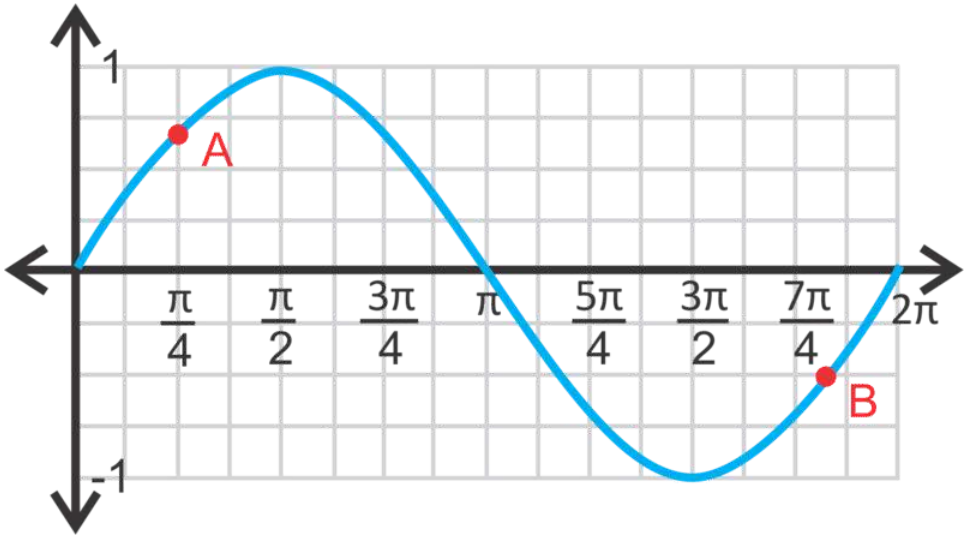


Figure **2.6.1.6**

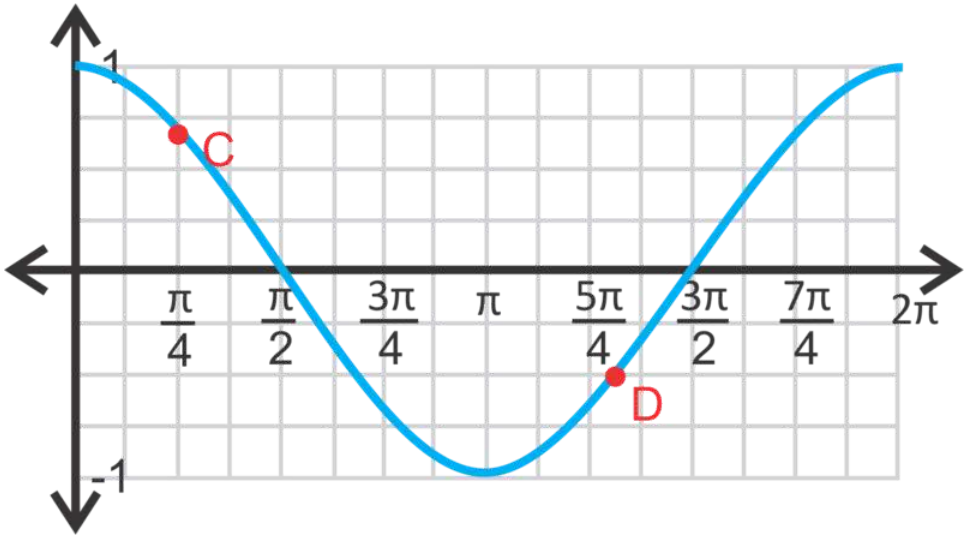


Figure **2.6.1.7**

For each point, think about what the sine or cosine value is at those values. For point A, $\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}$, therefore the point is $\left(\frac{\pi}{4},\frac{\sqrt{2}}{2}\right)$. For point B, we have to work backwards because it is not exactly on a vertical line, but it is on a horizontal one. When is $\sin x=\frac{-1}{2}$? When $x=\frac{7\pi}{6}$ or $\frac{11\pi}{6}$. By looking at point B's location, we know it is the second option. Therefore, the point is $\left(\frac{11\pi}{6},\frac{1}{2}\right)$.

For the cosine curve, point C is the same as point A because the sine and cosine for $\frac{\pi}{4}$ is the same. As for point D, we use the same logic as we did for point B. When does$\cos x=-\frac{1}{2}$? When $x=\frac{2\pi}{3}$ or $\frac{4\pi}{3}$. Again, looking at the location of point D, we know it is the second option. The point is $\left(\frac{4\pi}{3},\frac{1}{2}\right)$.

Amplitude

In addition to graphing $y=\sin x$ and $y=\cos x$, we can stretch the graphs by placing a number in front of the sine or cosine, such as $y=a\sin x$ or $y=a\cos x$. $|a|$ is the amplitude of the curve.

Let's graph $y=3\sin x$ over two periods.

Start with the basic sine curve. Recall that one period of the parent graph, $y=\sin x$, is 2π. Therefore, two periods will be 4π. The 3 indicates that the range will now be from 3 to -3 and the curve will be stretched so that the maximum is 3 and the minimum is -3.

The red curve is $y = 3 \sin x$.

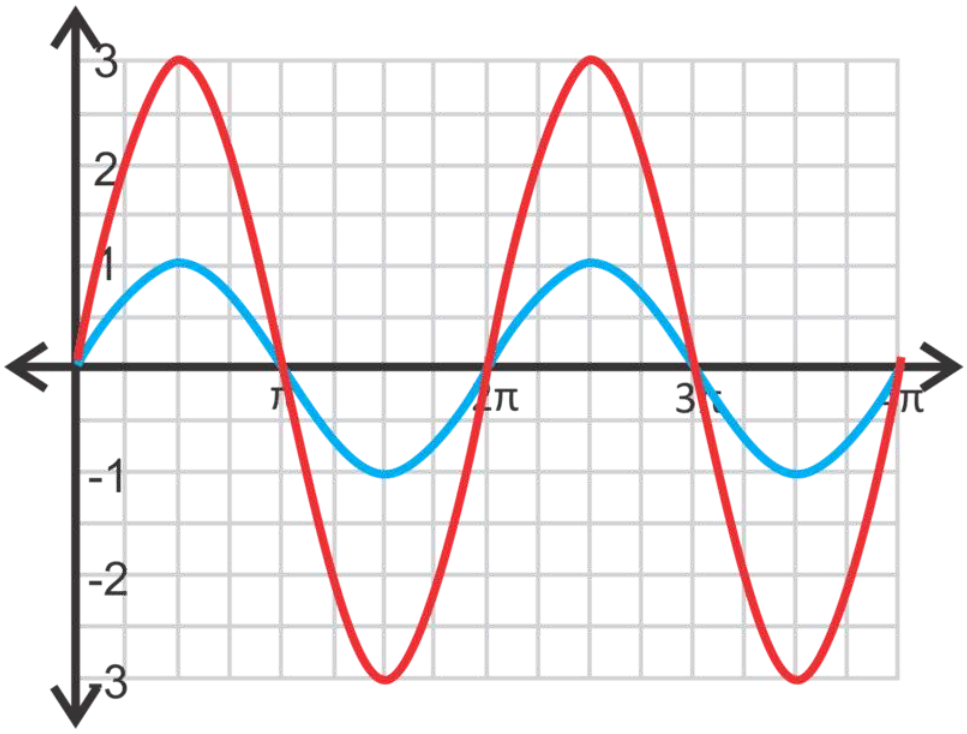


Figure **2.6.1.8**

Notice that the x-intercepts are the same as the parent graph. Typically, when we graph a trigonometric function, we always show two full periods of the function to indicate that it does repeat.

Now, let's graph $y = \frac{1}{2} \cos x$ over two periods.

Now, the amplitude will be 12 and the function will be “smooshed” rather than stretched.

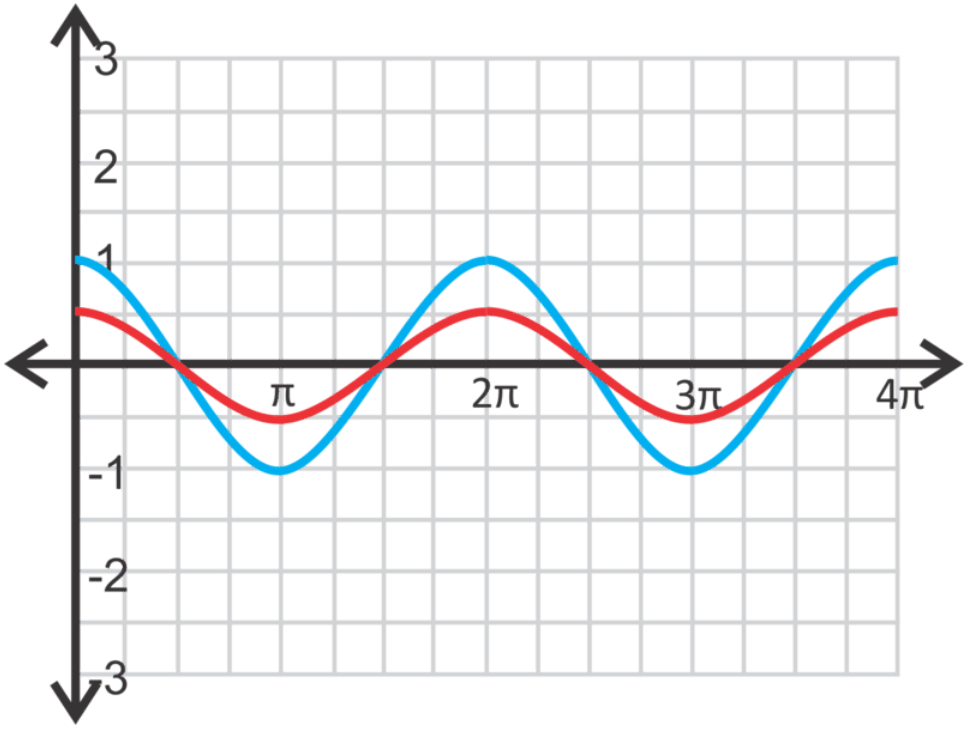


Figure **2.6.1.9**

Finally, let's graph $y = -\sin x$ over two periods.

The last two problems dealt with changing a and a was positive. Now, a is negative. Just like with other functions, when the leading coefficient is negative, the function is reflected over the x-axis. $y = -\sin x$ is in red.

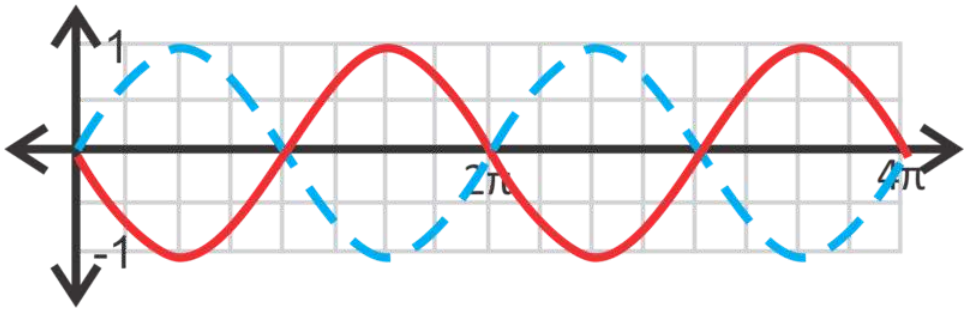


Figure 2.6.1.10

Example 2.6.1.1

Earlier, you were asked to find the minimum and maximum of the graph $y = 2\cos x$.

Solution

The 2 in front of the cosine function indicates that the range will now be from 2 to –2 and the curve will be stretched so that the maximum is 2 and the minimum is –2.

Example 2.6.1.2

Is the point $\left(\frac{5\pi}{6}, \frac{1}{2}\right)$ on $y = \sin x$? How do you know?

Solution

Substitute in the point for x and y and see if the equation holds true.

$\frac{1}{2} = \sin\left(\frac{5\pi}{6}\right)$

This is true, so $\left(\frac{5\pi}{6}, \frac{1}{2}\right)$ is on the graph.

Example 2.6.1.3

$y = 6\cos x$

Solution

Stretch the cosine curve so that the maximum is 6 and the minimum is -6.

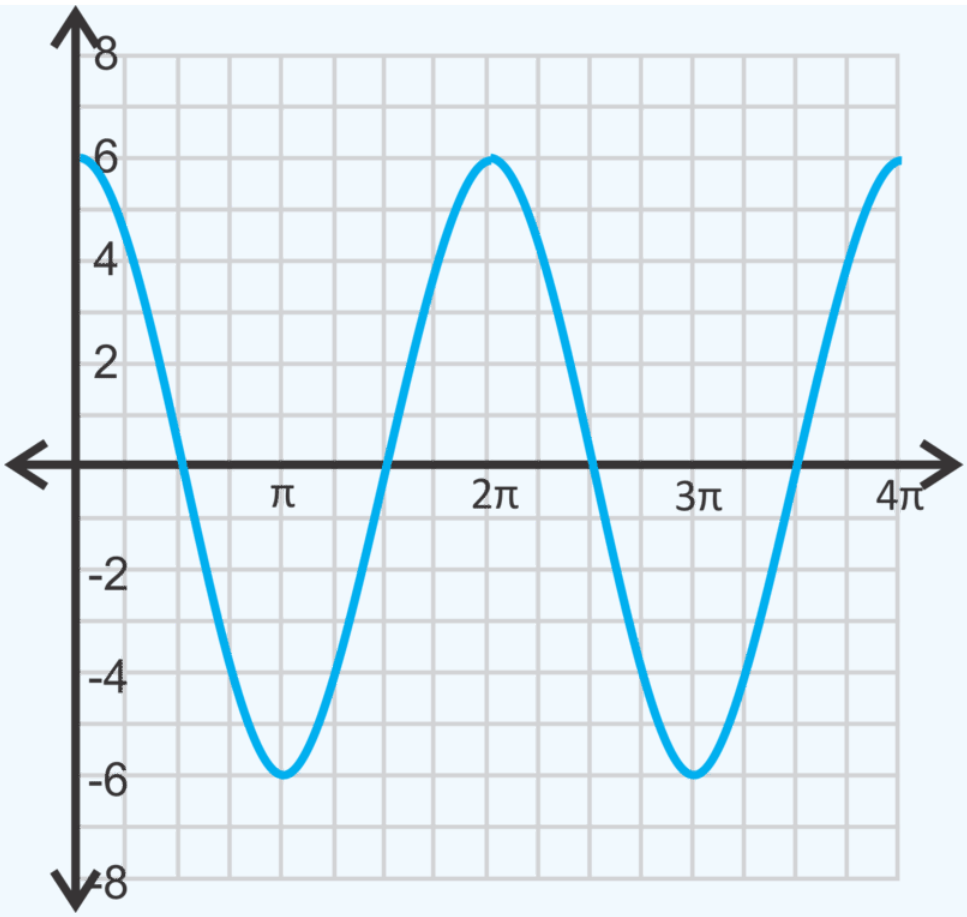


Figure **2.6.1.11**

Example 2.6.1.4

$y = -3\cos x$

Solution

The graph is reflected over the x-axis and stretched so that the amplitude is 3.

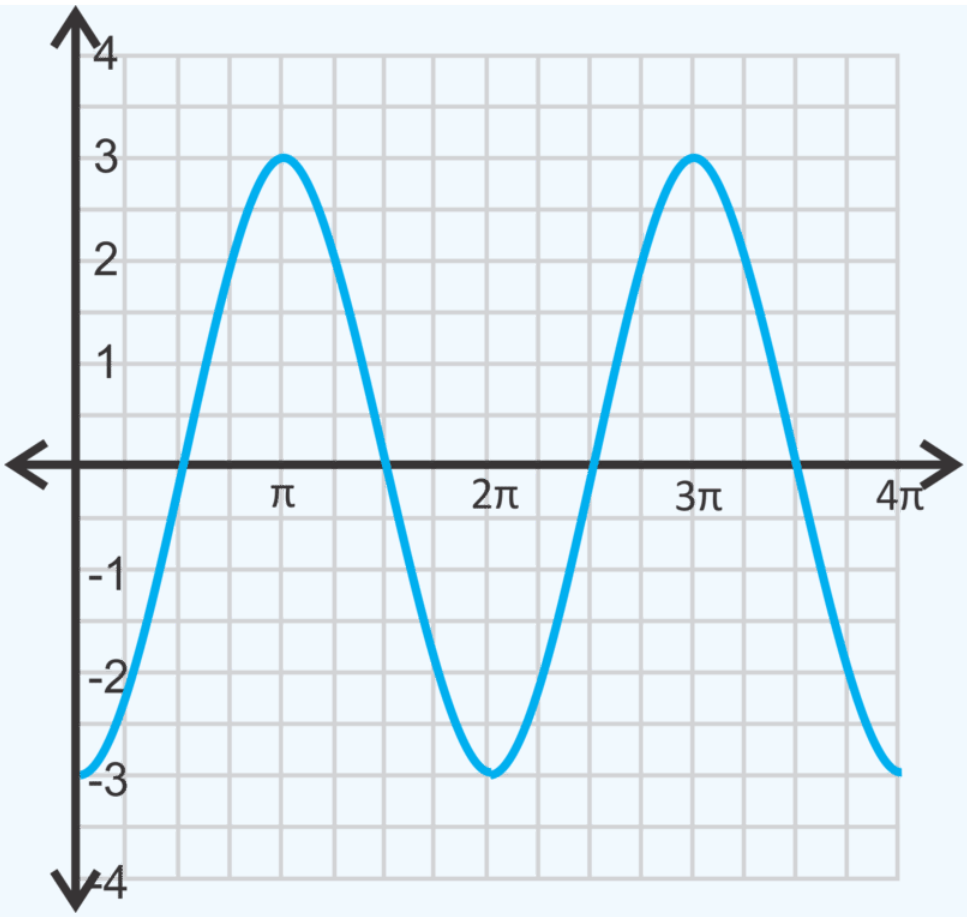


Figure **2.6.1.12**

Example 2.6.1.5

$y = \frac{3}{2}\sin x$

Solution

The fraction is equivalent to 1.5, making 1.5 the amplitude.

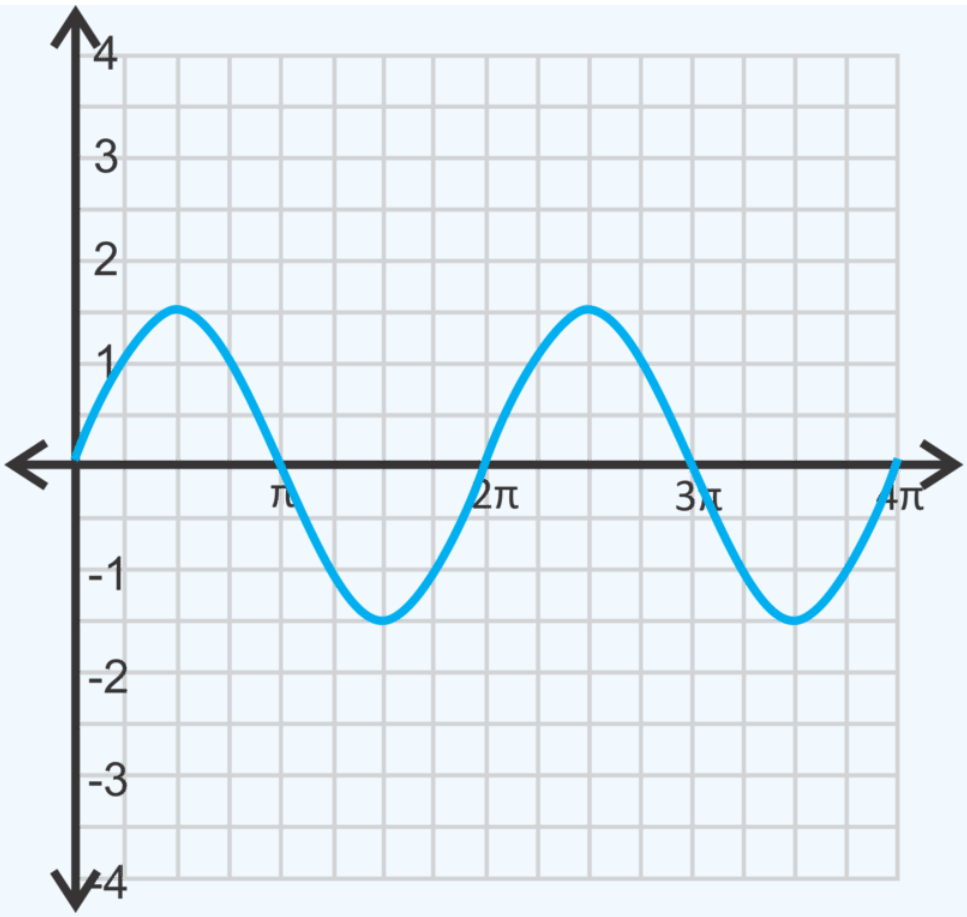


Figure **2.6.1.13**

Review

1. Determine the exact value of each point on $y = \sin x$ or $y = \cos x$.

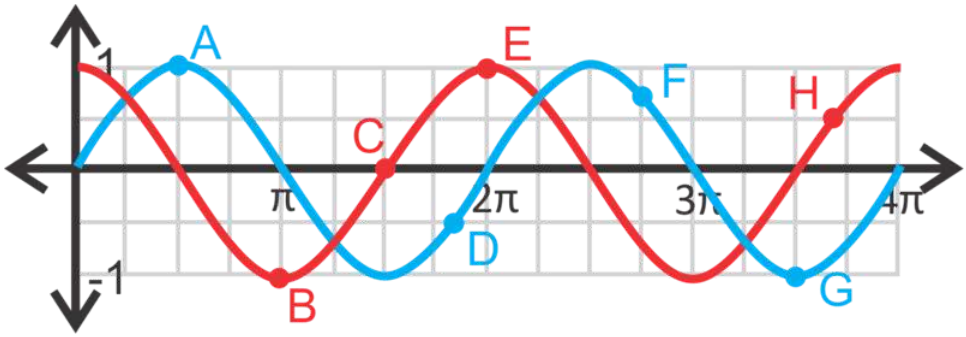


Figure **2.6.1.14**

2. List all the points in the interval $[0, 4\pi]$ where $\sin x = \cos x$. Use the graph from #1 to help you.
3. Draw from $y = \sin x$ from $[0, 2\pi]$. Find $f\left(\frac{\pi}{3}\right)$ and $f\left(\frac{5\pi}{3}\right)$. Plot these values on the curve.

For questions 4-12, graph the sine or cosine curve over two periods.

4. $y = 2\sin x$
5. $y = -5\cos x$
6. $y = \frac{1}{4}\cos x$
7. $y = -\frac{2}{3}\sin x$
8. $y = 4\sin x$

9. $y = -1.5 \cos x$
10. $y = \frac{5}{3} \cos x$
11. $y = 10 \sin x$
12. $y = -7.2 \sin x$
13. Graph $y = \sin x$ and $y = \cos x$ on the same set of axes. How many units would you have to shift the sine curve (to the left or right) so that it perfectly overlaps the cosine curve?
14. Graph $y = \sin x$ and $y = -\cos x$ on the same set of axes. How many units would you have to shift the sine curve (to the left or right) so that it perfectly overlaps $y = -\cos x$?

Write the equation for each sine or cosine curve below. $a > 0$ for both questions.

15.

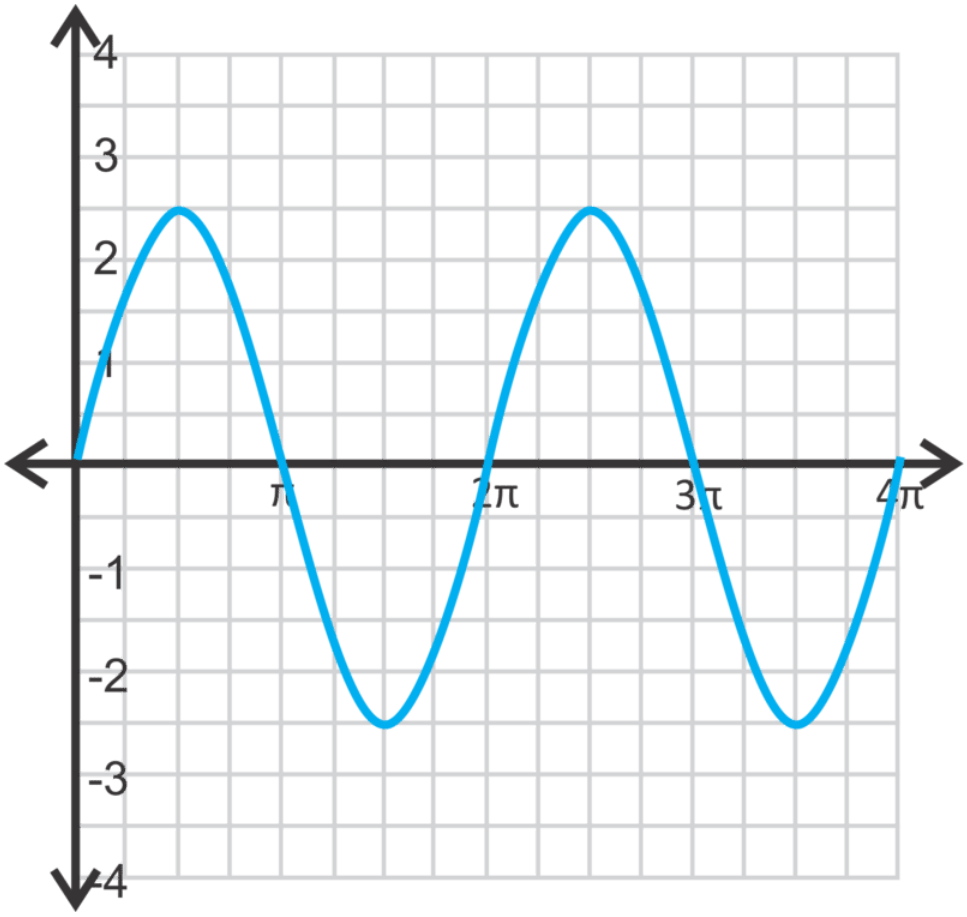


Figure **2.6.1.15**

16.

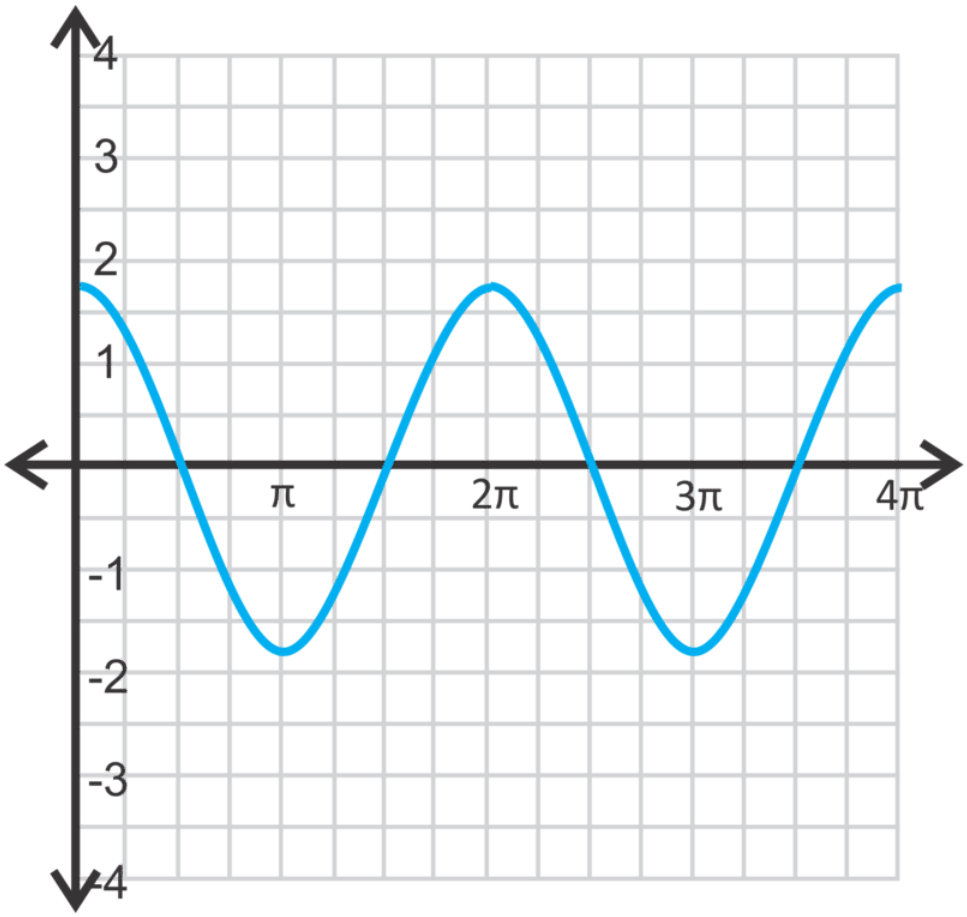


Figure **2.6.1.16**

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 14.1.

Additional Resources

Video: Graphs of Sine and Cosine Functions - Overview

Practice: Sine Graph and Cosine Graph

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2.6.2: Translating Sine and Cosine Functions

Graph shifted trig functions

Explore how changes to the equations of sine and cosine functions affect the graphs of the functions through transformations (stretches and shifts).

Warm-Up

Sine and cosine are functions that display periodic behavior when graphed. Use the interactive below to change amplitude of sine and cosine and alter their graphs. Later in this section you will see how to alter these graphs via transformation.

Work it Out 1

You have seen linear, quadratic and even exponential and logarithmic functions undergo transformations to stretch and shift them. How might the graph of a periodic function behave under these types of transformations? Explore this question using the interactive below.

The general equation for a sine curve is: $y = A\sin(w(x-h)) + k$. In this equation, A represents the **amplitude**. How does changing the value of the amplitude impact the graph of a sine function? Begin by sketching a graph of the sine function $y = \sin(x)$. Then pick several different values for A including positive and negative values, numbers close 0, and 0 itself, and graph those equations as well. What happens to your graph as the amplitude changes?

Discussion

These are the graphs of $y = \sin(x)$ (red) and $y = 2\sin(x)$ (green). What impact did changing A from 1 to 2 have on the graph? Predict what the graph would look like when $A = -2$, and then graph the function. How would you compare the graphs of $y = 2\sin(x)$ and $y = -2\sin(x)$? In other words, does the amplitude change when A is a positive number vs. a negative number? What generalizations can you make about the relationship between the absolute value of A and the amplitude?

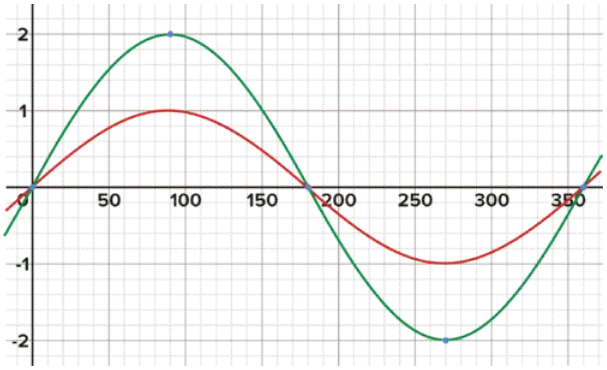


Figure 2.6.2.1

What generalizations can you make about the relationship between the absolute value of A and the amplitude?

Example 2.6.2.1

Determine the equation for the graph below. The scale of the x-axis is in degrees.

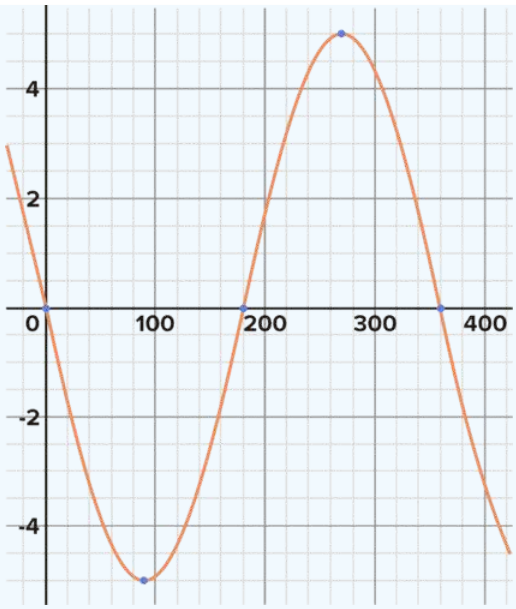


Figure 2.6.2.2

Solution

The graph is the inverse of a basic sine function, which means A is a negative number. The amplitude of the graph is at $y = -5$ and $y = +5$, or in other words, the minimum and maximum values on the graph are 5 away from the midline. Therefore, the equation is $y = -5\sin(x)$.

Work it Out 2

Just as you did in the previous Active Learning, start by graphing $y = \sin(x)$. Then experiment with different values for h including positive and negative values, and values close to 0. How do these different values for h impact the graph?

Discussion

Recall that the expression for a sine curve is $y = A\sin(w(x - h)) + k$. Therefore, when h has a positive value, it will be subtracted from x. The h represents the **phase shift** of the function. How is the graph impacted differently by positive and negatives phase shifts? Can you create an equation with a non-zero phase shift that would look the same when graphed as the equation $y = \sin(x)$?

✔ Example 2.6.2.2

Graph $y = \cos\left(x - \frac{\pi}{4}\right)$

Solution

This function will be shifted $\frac{\pi}{4}$ units to the right. The easiest way to sketch the curve is to start with the parent graph and then move it to the right the correct number of units.

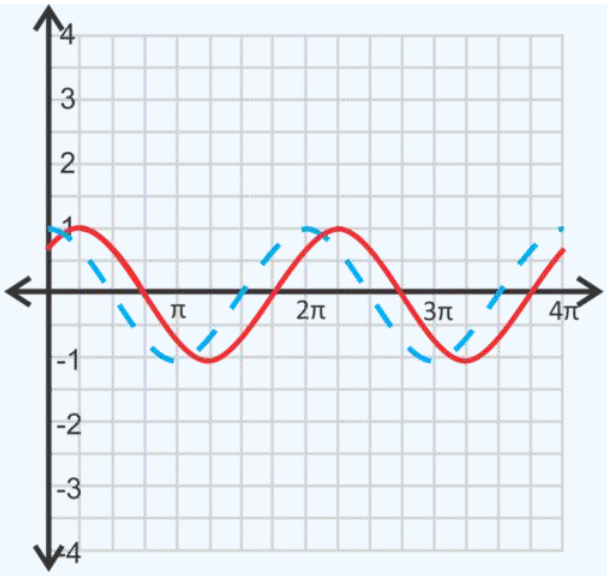


Figure **2.6.2.3**

Work it Out 3

Use the same process as above to explore how a sine curve is impacted when you change the value of k, which is also called the **midline** or the **sinusoidal axis**. Pick several different values for k, including positive and negative values, to see how k affects the graph.

Discussion

The parent graph of $y = \sin(x)$ is shifted up or down on the graph as k changes. In other words, the graph is translated k units. If $k = 1$, does the graph shift up or down? What about if $k = -1$?

Work it Out 4

How is the parent graph of the sine curve impacted by changing the value of w, the **frequency**? What happens if w is a positive number v. a negative number? Predict what happens if w (the frequency) is changed to both positive and negative values, as well as values between 0 and 1, and values greater than 1.

Discussion

The frequency describes the number of cycles of the graph between $0°$ and $360°$, or between π and 2π. When the frequency is 2, there are two cycles within this range.

Work it Out 5: Transforming the Graph of Sine Functions

Experiment with these PLIX to see the relationship between the sine function and its graph.

CK-12 INTERACTIVES

Example 2.6.2.3

Graphing Translated Sine Functions

Without using a graphing calculator, sketch the graph of a sine function where $A = 3$, $\frac{\pi}{2}$, $w = 2$, and the midline is 4.

Solution

Start by writing an equation with the values given above:

$y = 3\sin\left(2\left(x - \frac{\pi}{2}\right)\right) + 4$

It will be helpful to start with the parent graph and translate it one step at a time. This is the graph of the function $y = \sin(x)$

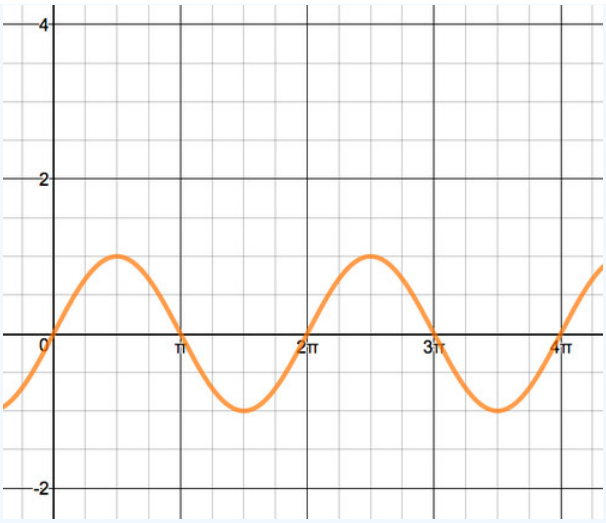


Figure 2.6.2.4

The graph of y = sin x

Now adjust the phase shift of the function, which shifts the graph to the left by $\frac{\pi}{2}$ or $90°$. The blue graph below is of the function $y = \sin\left(x - \frac{\pi}{2}\right)$.

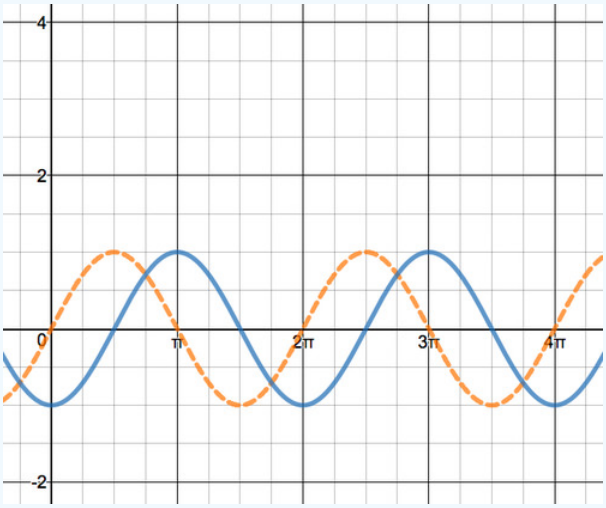


Figure 2.6.2.5

The graphs y = sin x (orange), and $y = \sin\left(x - \frac{\pi}{2}\right)$ (blue)

Now adjust the midline of the function. The location of the midline is impacted by the value of *k*. In this example, $k = 4$, which shifts the graph up by 4 units on the y-axis. The purple graph below is of the function $y = \sin(x - \frac{\pi}{2}) + 4$.

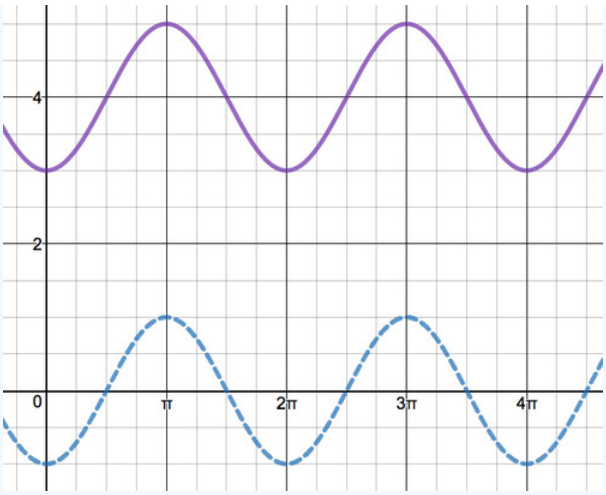


Figure **2.6.2.6**

The graphs $y = \sin\left(x - \frac{\pi}{2}\right)$ (blue), and $y = \sin\left(x - \frac{\pi}{2}\right) + 4$ (purple)

Next adjust the amplitude of the function. The amplitude of the purple graph is 1, but the new equation has an amplitude of 3, so the *distance* between the midline (which is now at $y = 4$) and the maximum and minimum y-values should be 3. Therefore, the maximum and minimum values should be at $y = 1$ and $y = 7$. The green graph below is for the equation: $y = 3\sin\left(x - \frac{\pi}{2}\right) + 4$.

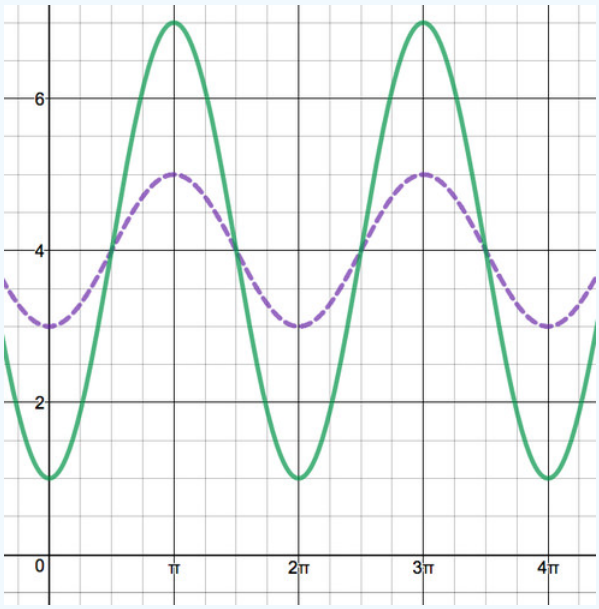


Figure **2.6.2.7**

The graphs $y = \sin\left(x - \frac{\pi}{2}\right) + 4$ (purple) and $y = 3\sin\left(x - \frac{\pi}{2}\right) + 4$ (green)

Finally, you need to adjust the frequency of the graph. Because the value of $w = 2$, the graph needs to go through 2 complete cycles between $0°$ and $360°$ or between 0 and 2π. The red graph below is of the function $y = 3\sin\left(2\left(x - \frac{\pi}{2}\right)\right) + 4$.

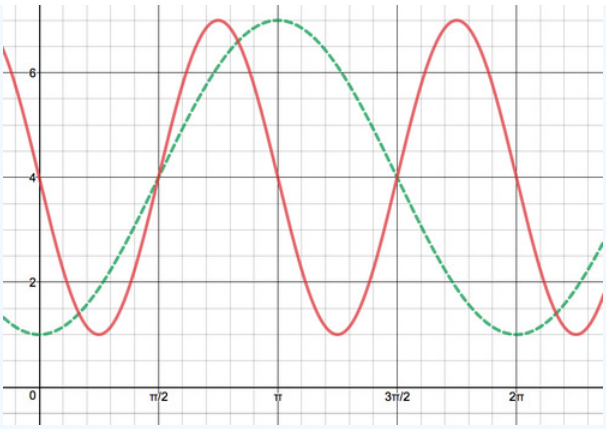


Figure 2.6.2.8

The red graph is of the function $y = 3\sin\left(2\left(x - \frac{\pi}{2}\right)\right) + 4$.

Example 2.6.2.4

Determining a Trigonometric Equation from a Transformed Parent Graph

Find the equation of the cosine curve below.

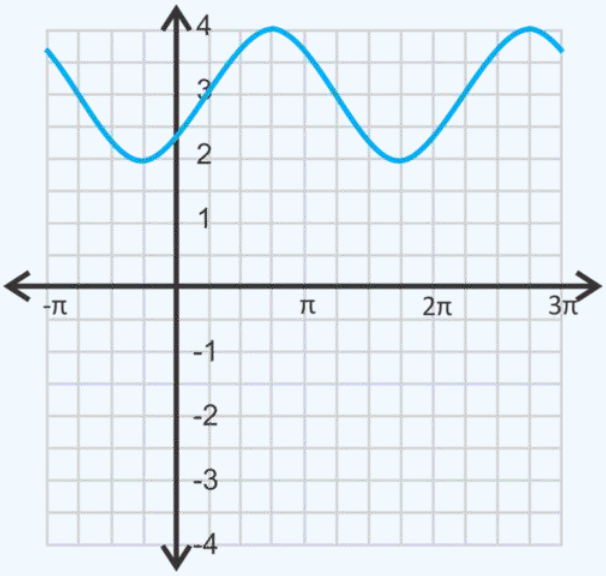


Figure 2.6.2.9

Solution

The parent graph is in green (below). It moves up 3 units (red) and then to the right 3\pi 4 units (blue). Therefore, the equation is $y = \cos\left(x - \frac{3\pi}{4}\right) + 3$.

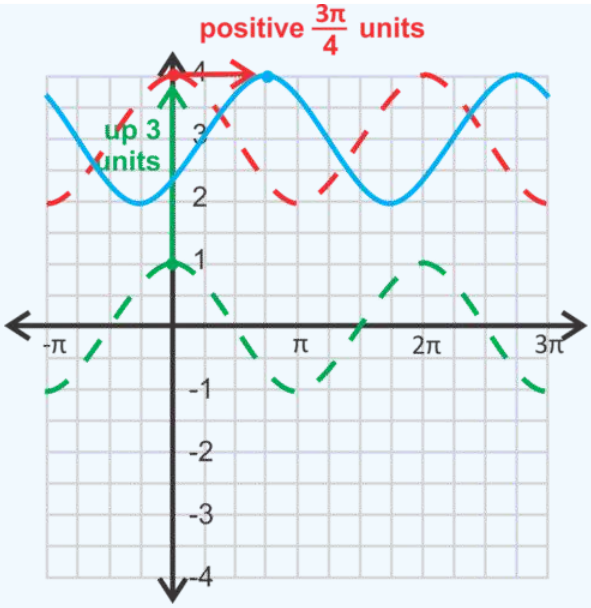


Figure 2.6.2.10

If you moved the cosine curve backward, then the equation would be $y = \cos\left(x + \frac{5\pi}{4}\right) + 3$.

Review

For questions 1-4, match the equation with its graph.

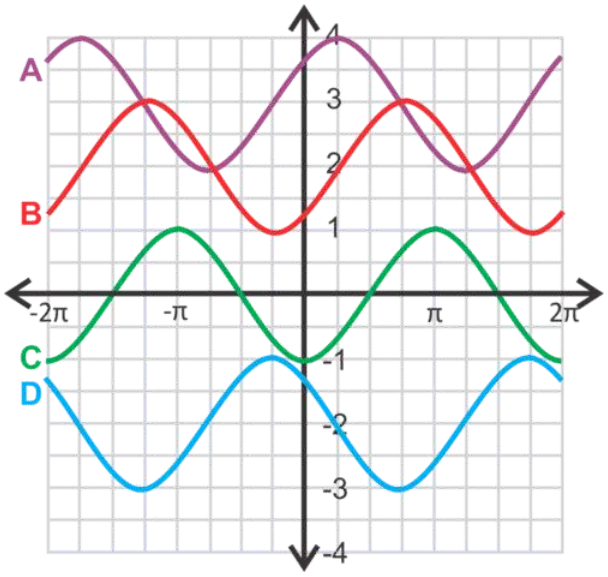


Figure 2.6.2.11

Match the Equation with the Graph.

1. $y = \sin\left(x - \frac{\pi}{2}\right)$
2. $y = \cos\left(x - \frac{\pi}{4}\right) + 3$
3. $y = \cos\left(x + \frac{\pi}{4}\right) - 2$
4. $y = \sin\left(x - \frac{\pi}{4}\right) + 2$

Which graph above also represents these equations in #5 and #6?

5. $y = \cos(x - \pi)$
6. $y = \sin\left(x + \frac{3\pi}{4}\right) - 2$

7. Write another sine equation for graph A.
8. **Writing:** How many sine (or cosine) equations can be generated for one curve? Why?

For questions 9 - 14, graph the following equations from $[-2\pi, 2\pi]$.

9. $y = 2\sin(x + \pi 4)$
10. $y = 1 + \cos x$
11. $y = \cos(x + \pi) - 2$
12. $y = \sin\left(x - \frac{\pi}{6}\right)$
13. $y = \cos(3(x - 1)) - 3$
14. **Critical Thinking:** Is there a difference between $y = \sin x + 1$ and $y = \sin(x + 1)$? Explain your answer.

Additional Resources

Practice: Translating Sine and Cosine Functions

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2.6.3: Vertical Translations

Shift up or down the y-axis.

Your knowledge of transformations, specifically **vertical shift**, apply directly to sinusoidal functions. In practice, sketching shifted sine and cosine functions requires greater attention to detail and more careful labeling than other functions. Can you describe the following transformation in words?

$f(x) = \sin x \rightarrow g(x) = -3 \sin x - 4$

In what order do the reflection, stretch and shift occur? Is there a difference?

Vertical Shift of Sinusoidal Functions

The general form of a sinusoidal function is:

$f(x) = \pm a \cdot \sin(b(x + c)) + d$

Recall that a controls **amplitude** and the $\pm$ controls reflection. Here you will see how d controls the vertical shift.

The most straightforward way to think about vertical shift of sinusoidal functions is to focus on the **sinusoidal axis**, the horizontal line running through the middle of the sine or cosine wave. At the start of the problem identify the vertical shift and immediately draw the new sinusoidal axis. Then proceed to graph amplitude and reflection about that axis as opposed to the x axis.

The graphs of the following three functions are shown below:

$$f(x) = \sin x + 3$$
$$g(x) = \sin x - 2$$
$$h(x) = \sin x + 12$$

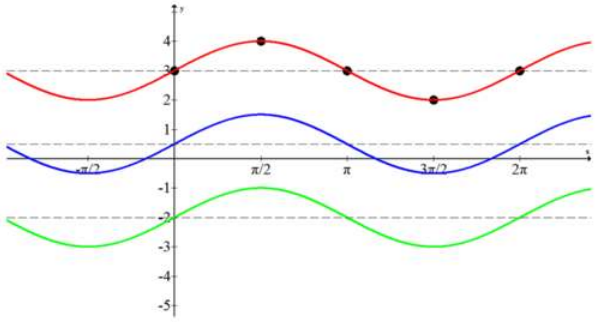
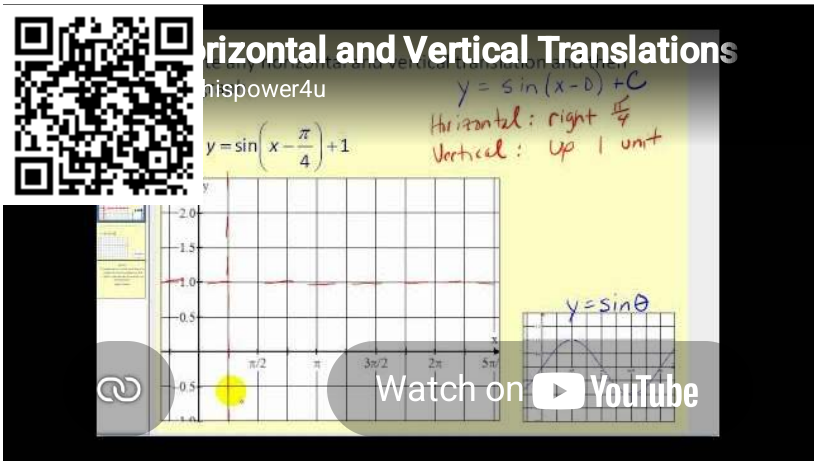


Figure **2.6.3.1**

To draw these graphs, the new sinusoidal axis for each graph is drawn first. Then, a complete sine wave for each one is drawn. Note the five important points that separate each quadrant to help to get a clear sense of the graph. There are no reflections in these graphs and they all have an amplitude of 1. Right now every cycle starts at 0 and ends at 2π but this will not always be the case.

Watch the portions of the following video focused on vertical translations:



Example 2.6.3.1

Earlier, you were asked which order vertical shift and reflection should be performed in and if it matters. The following transformation can be described as follows.

$f(x) = \sin x \rightarrow g(x) = -3\sin x - 4$

Solution

Describe the stretching and reflecting first and then the vertical shift. This is the most logical way to discuss the transformation verbally because then the numbers like 3 and -4 can be explicitly identified in the graph.

The order in describing the transformation matters. When describing vertical transformations it is most intuitive to simply describe the transformations in the same order as the order of operations.

Example 2.6.3.2

Identify the equation of the following transformed cosine graph.

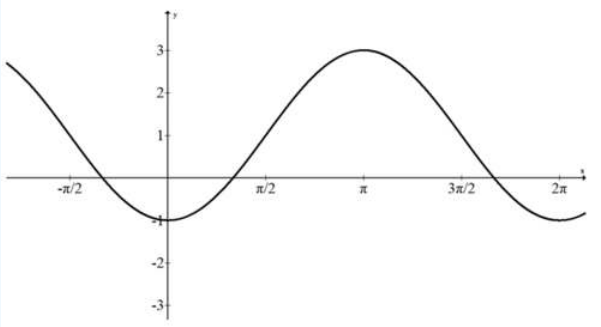


Figure 2.6.3.2

Solution

Since there is no sinusoidal axis given, you must determine the vertical shift, stretch and reflection. The peak occurs at $(\pi, 3)$ and the trough occurs at (0, -1) so the horizontal line directly between +3 and -1 is $y = 1$. Since the sinusoidal axis has been shifted up by one unit $d = 1$. From this height, the graph goes two above and two below which means that the amplitude is 2. Since this cosine graph starts its cycle at (0, -1) which is a lowest point, it is a negative cosine. The function is $f(x) = -2\cos x + 1$.

Example 2.6.3.3

Transform the following sine graph in two ways. First, transform the sine graph by shifting it vertically up 1 unit and then stretching it vertically by a factor of 2 units. Second, transform the sine graph by stretching it vertically by a factor of 2 units and then shifting it vertically up 1 unit.

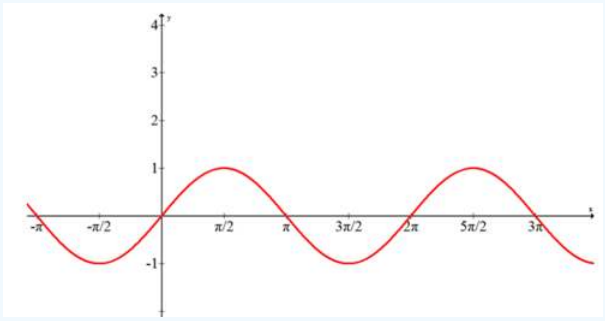


Figure 2.6.3.3

Solution

When doing ordered transformations it is good to show where you start and where you end up so that you can effectively compare and contrast the outcomes. See how both transformations start with a regular sine wave. The two columns represent the sequence of transformations that produce different outcomes.

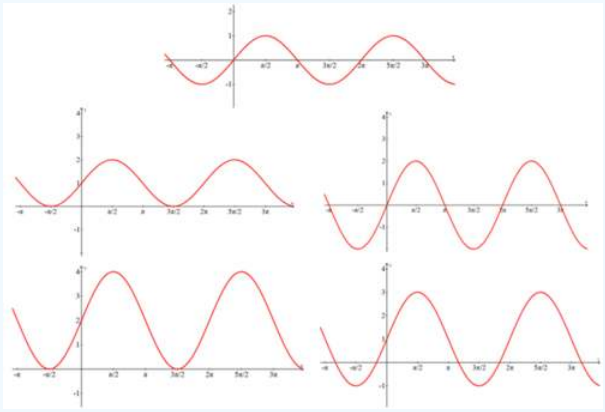


Figure 2.6.3.4

Example 2.6.3.4

What equation models the following graph?

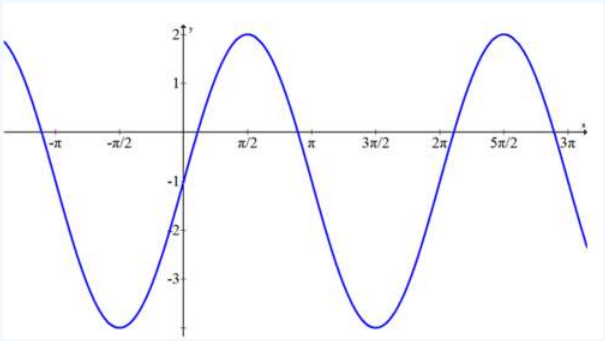


Figure 2.6.3.5

Solution

$f(x) = 3 \cdot \sin x - 1$

Example 2.6.3.5

Graph the following function: $f(x) = -2 \cdot \cos x + 1$.

Solution

First draw the horizontal sinusoidal axis and identify the five main points for the cosine wave. Be careful to note that the amplitude is 2 and the cosine wave starts and ends at a low point because of the negative sign.

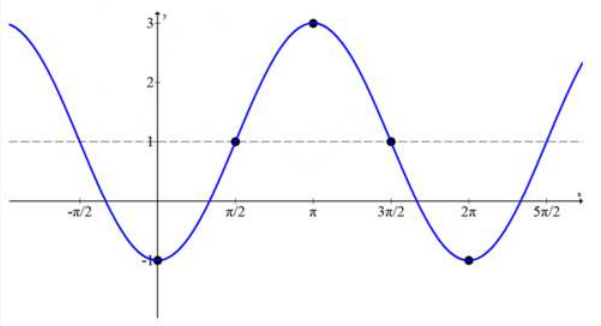


Figure **2.6.3.6**

Review

Graph each of the following functions that have undergone a vertical stretch, reflection, and/or a vertical shift.

1. $f(x) = -2 \sin x + 4$
2. $g(x) = \frac{1}{2} \cos x - 1$
3. $h(x) = 3 \sin x + 2$
4. $j(x) = -1.5 \cos x + \frac{1}{2}$
5. $k(x) = \frac{2}{3} \sin x - 3$

Find the minimum and maximum values of each of the following functions.

6. $f(x) = -3 \sin x + 1$
7. $g(x) = 2 \cos x - 4$
8. $h(x) = \frac{1}{2} \sin x + 1$
9. $j(x) = -\cos x + 5$
10. $k(x) = \sin(x) - 1$

Give the equation of each function graphed below.

11.

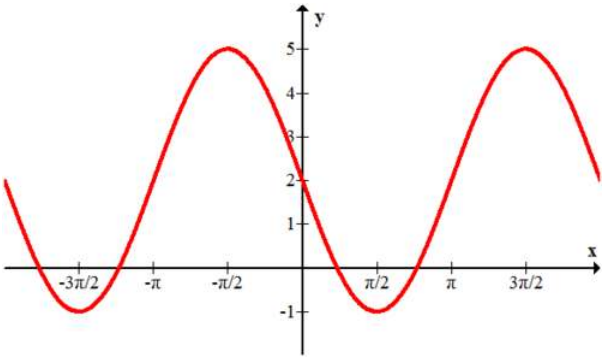


Figure **2.6.3.7**

12.

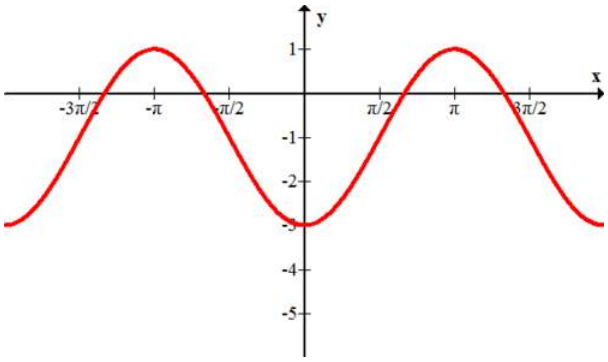


Figure 2.6.3.8

13.

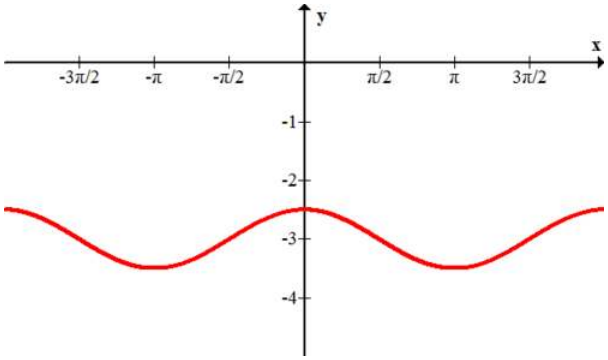


Figure 2.6.3.9

14.

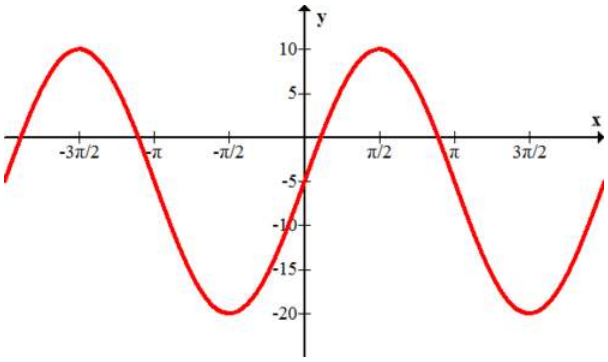


Figure 2.6.3.10

15.

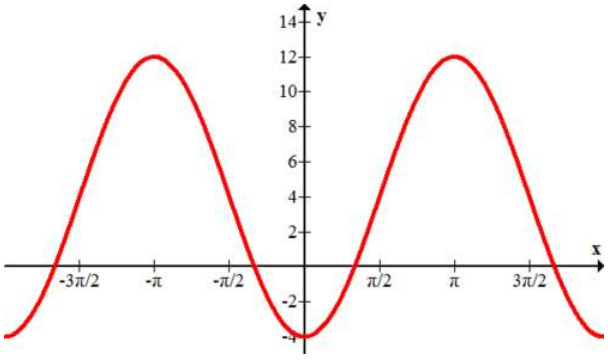


Figure 2.6.3.11

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.4.

Additional Resources

Practice: Vertical Translations

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2.6.4: Horizontal Translations or Phase Shifts

Shift right or left along the x-axis.

A **periodic function** that does not start at the **sinusoidal axis** or at a maximum or a minimum has been shifted horizontally. This horizontal movement allows for different starting points since a sine wave does not have a beginning or an end.

What are five other ways of writing the function $f(x) = 2 \cdot \sin x$?

Phase Shift of **sinusoidal Functions**

The general **sinusoidal function** is:

$$f(x) = \pm a \cdot \sin(b(x + c)) + d$$

The constant c controls the phase shift. **Phase shift** is the **horizontal shift** left or right for periodic functions. If $c = \frac{\pi}{2}$ then the sine wave is shifted left by $\frac{\pi}{2}$. If $c = -3$ then the sine wave is shifted right by 3. This is the opposite direction than you might expect, but it is consistent with the rules of transformations for all functions.

To graph a function such as $f(x) = 3 \cdot \cos\left(x - \frac{\pi}{2}\right) + 1$, first find the start and end of one period. Then sketch only that portion of the sinusoidal axis. Finally, plot the 5 important points for a cosine graph while keeping the **amplitude** in mind. The graph is shown below:

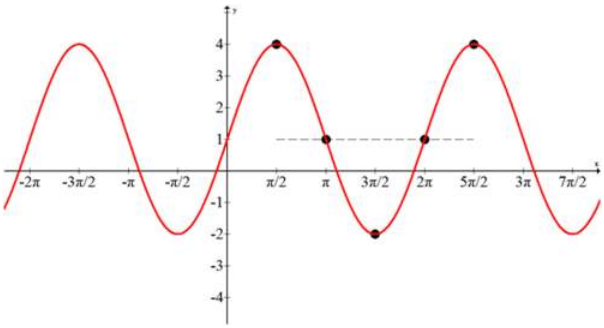
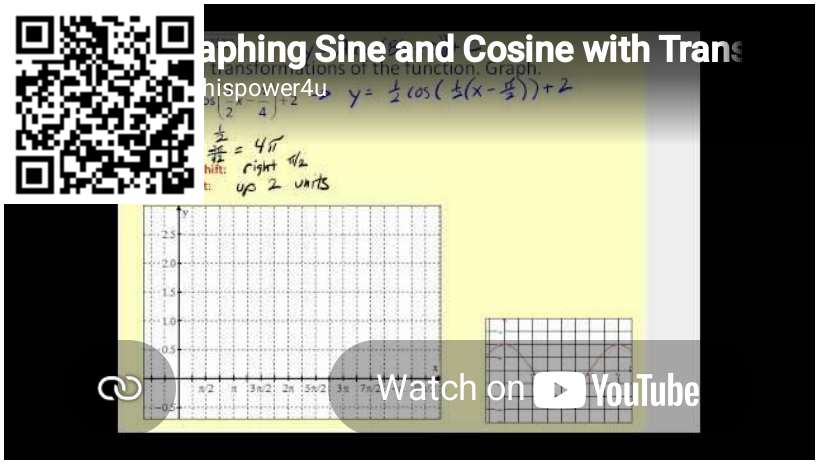


Figure **2.6.4.1**

Generally$b$ is always written to be positive. If you run into a situation where b is negative, use your knowledge of even and odd functions to rewrite the function.

$$\cos(-x) = \cos(x)$$
$$\sin(-x) = -\sin(x)$$



Example 2.6.4.1

Earlier, you were asked to write $f(x) = 2 \cdot \sin x$ in five different ways.

Solution

The function $f(x) = 2 \cdot \sin x$ can be rewritten an infinite number of ways.

$$2 \cdot \sin x = -2 \cdot \cos\left(x + \frac{\pi}{2}\right) = 2 \cdot \cos\left(x - \frac{\pi}{2}\right) = -2 \cdot \sin(x - \pi) = 2 \cdot \sin(x - 8\pi)$$

It all depends on where you choose start and whether you see a positive or negative sine or cosine graph.

Example 2.6.4.2

Given the following graph, identify equivalent sine and cosine algebraic models.

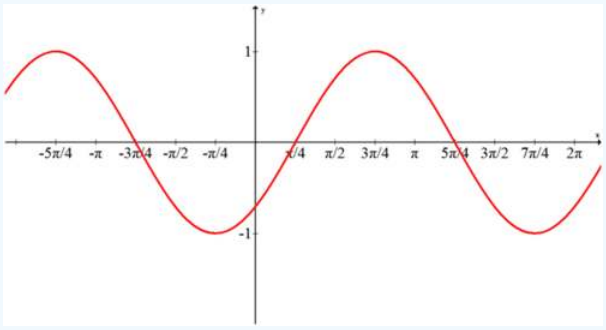


Figure 2.6.4.2

Solution

Either this is a sine function shifted right by $\frac{\pi}{4}$ or a cosine graph shifted left $\frac{5\pi}{4}$.

$$f(x) = \sin(x - \pi 4) = \cos\left(x + \frac{5\pi}{4}\right)$$

Example 2.6.4.3

At $t = 5$ minutes William steps up 2 feet to sit at the lowest point of the Ferris wheel that has a diameter of 80 feet. A full hour later he finally is let off the wheel after making only a single revolution. During that hour he wondered how to model his height over time in a graph and equation.

Solution

Since the period is 60 which works extremely well with the $360°$ in a circle, this problem will be shown in degrees.

Time (minutes)	Height (feet)
5	2
20	42
35	82
50	42
65	2

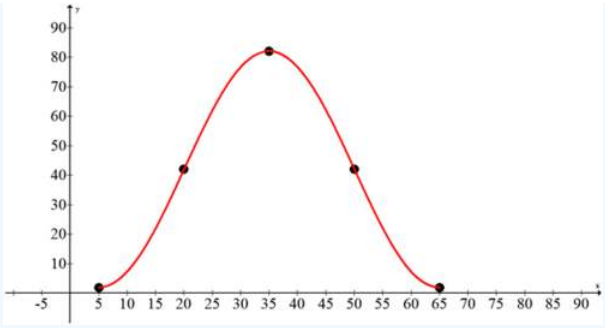


Figure **2.6.4.3**

William chooses to see a negative cosine in the graph. He identifies the amplitude to be 40 feet. The vertical shift of the sinusoidal axis is 42 feet. The horizontal shift is 5 minutes to the right.

The period is 60 (not 65) minutes which implies $b = 6$ when graphed in degrees.

$$60 = \frac{360}{b}$$

Thus one equation would be:R5

$$f(x) = -40 \cdot \cos(6(x - 5)) + 42$$

Example 2.6.4.4

Tide tables report the times and depths of low and high tides. Here is part of tide report from Salem, Massachusetts dated September 19, 2006.

10:15 AM	9 ft.	High Tide
4:15 PM	1 ft.	Low Tide
10:15 PM	9 ft.	High Tide

Find an equation that predicts the height based on the time. Choose when t=0 carefully.

Solution

There are two logical places to set $t = 0$. The first is at midnight the night before and the second is at 10:15 AM. The first option illustrates a phase shift that is the focus of this concept, but the second option produces a simpler equation. Set $t = 0$ to be at midnight and choose units to be in minutes.

Time (hours : minutes)	**Time (minutes)**	**Tide (feet)**
10:15	615	9
16:15	975	1
22:15	1335	9
	$\frac{615 + 975}{2} = 795$	5
	$\frac{1335 + 975}{2} = 1155$	5

These numbers seem to indicate a positive cosine curve. The amplitude is 4 and the vertical shift is 5. The horizontal shift is 615 and the period is 720.

$720 = \frac{2\pi}{b} \rightarrow b = \frac{\pi}{360}$

Thus one equation is:

$$f(x) = 4 \cdot \cos\left(\frac{\pi}{360}(x - 615)\right) + 5$$

Example 2.6.4.5

Use the equation from Example 4 to find out when the tide will be at exactly 8 ft on September 19th.

Solution

This problem gives you the y and asks you to find the x. Later you will learn how to solve this algebraically, but for now use the power of the intersect button on your calculator to intersect the function with the line y=8. Remember to find all the x values between 0 and 1440 to account for the entire 24 hours.

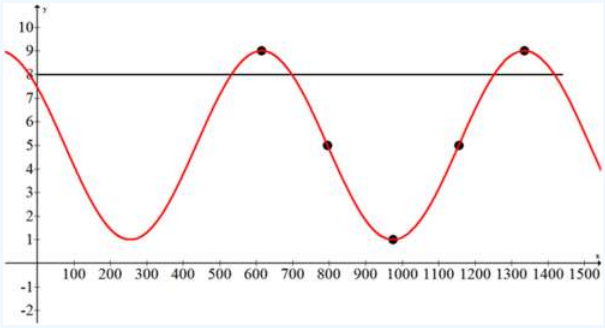


Figure 2.6.4.4

There are four times within the 24 hours when the height is exactly 8 feet. You can convert these times to hours and minutes if you prefer.

$t \approx 532.18(8:52),\ 697.82(11:34),\ 1252.18(20:52),\ 1417.82(23:38)$

Review

Graph each of the following functions.

1. $f(x) = 2\cos\left(x - \frac{\pi}{2}\right) - 1$
2. $g(x) = -\sin(x - \pi) + 3$
3. $h(x) = 3\cos(2(x - \pi))$
4. $k(x) = -2\sin(2x - \pi) + 1$
5. $j(x) = -\cos\left(x + \frac{\pi}{2}\right)$

Give one possible sine equation for each of the graphs below.

6.

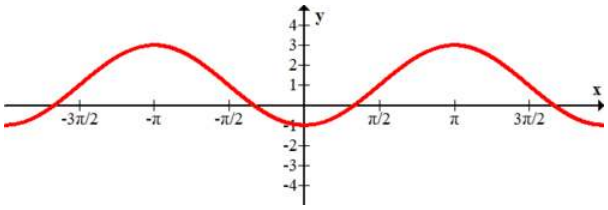


Figure 2.6.4.5

7.

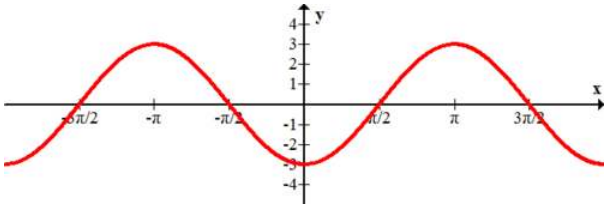


Figure 2.6.4.6

8.

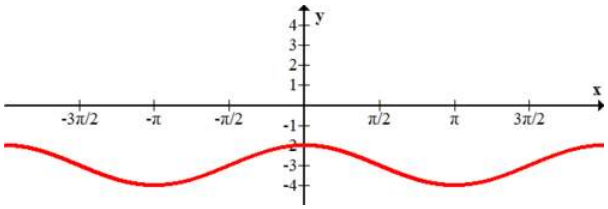


Figure **2.6.4.7**

Give one possible cosine function for each of the graphs below.

9.

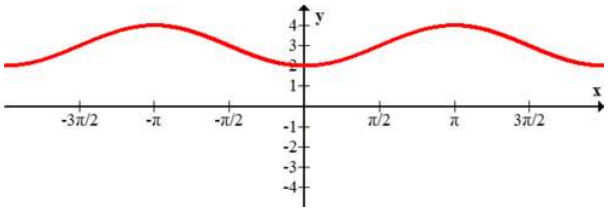


Figure **2.6.4.8**

10.

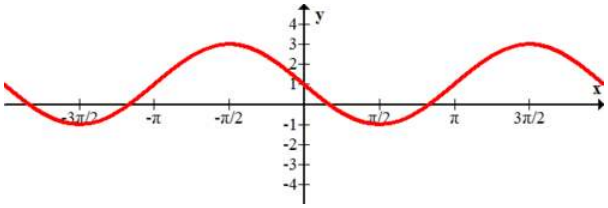


Figure **2.6.4.9**

11.

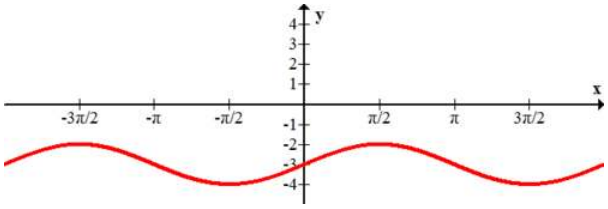


Figure **2.6.4.10**

The temperature over a certain 24 hour period can be modeled with a sinusoidal function. At 3:00, the temperature for the period reaches a low of **22°**F. At 15:00, the temperature for the period reaches a high of **40°**F.

12. Find an equation that predicts the temperature based on the time in minutes. Choose t=0 to be midnight.

13. Use the equation from #12 to predict the temperature at 4:00 PM.

14. Use the equation from #12 to predict the temperature at 8:00 AM.

15. Use the equation from #12 to predict the time(s) it will be **32°**F.

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.6.

Vocabulary

Term	Definition
amplitude	the maximum distance the particles of the medium move from their resting positions when a wave passes through.
Horizontal shift	A horizontal shift is the result of adding a constant term to the function inside the parentheses. A positive term results in a shift to the left and a negative term in a shift to the right.
periodic function	A periodic function is a function with a predictable repeating pattern. sine waves and cosine waves are periodic functions.
sinusoidal axis	The sinusoidal axis is the neutral horizontal line that lies between the crests and the troughs of the graph of a sine or cosine function.
sinusoidal function	A sinusoidal function is a sine or cosine wave.
sinusoidal functions	A sinusoidal function is a sine or cosine wave.

Additional Resources

Video: Horizontal and Vertical Translations of sine and cosine

Practice: Horizontal Translations or Phase Shifts

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2.6.5: Amplitude, Period, and Frequency

Vertical and horizontal properties of sine and cosine waves.

You are working in science lab one afternoon when your teacher asks you to do a little more advanced work with her on sound. Excited to help, you readily agree. She gives you a device that graphs sound waves as they come in through a microphone. She then gives you a "baseline" graph of what the sound wave's graph would look like:

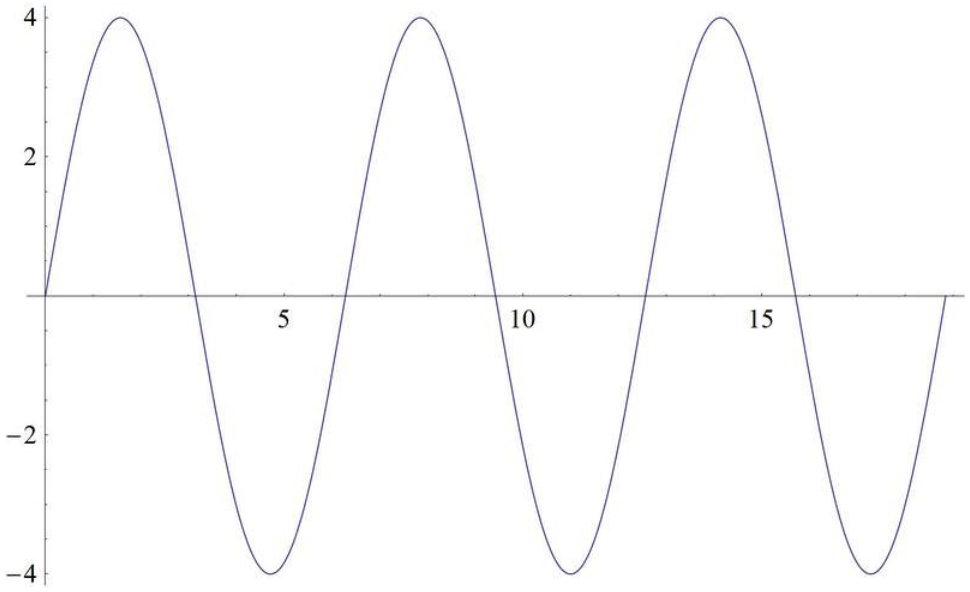


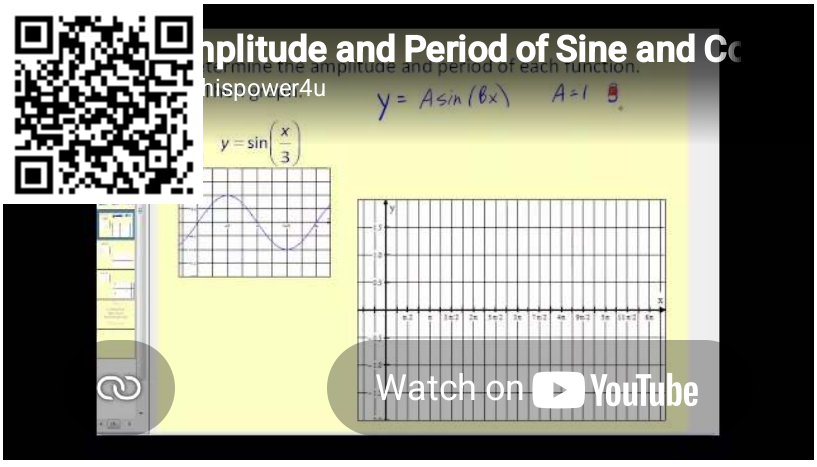
Figure **2.6.5.1**

She then asks you to plot the sound wave she's about to generate. However, she tells you that the sound wave will be twice as loud and twice as high in pitch as the baseline sound wave she gave you.

Can you determine how large the graph needs to be to plot the new sound wave? What about the spacing of numbers on the "x" axis?

Amplitude and **Period**

In other lessons you have dealt with how find the amplitude of a wave, or the period of a wave. Here we'll take a few minutes to work problems that involve both the amplitude and period, giving us two variables to work with when thinking about sinusoidal equations.



Finding the Period, Amplitude, and **Frequency**

1. Find the period, amplitude and frequency of $y = 2\cos\frac{1}{2}x$ and sketch a graph from 0 to 2π.

This is a cosine graph that has been stretched both vertically and horizontally. It will now reach up to 2 and down to -2. The frequency is $\frac{1}{2}$ and to see a complete period we would need to graph the interval $[0, 4\pi]$. Since we are only going out to 2π, we will only see half of a wave. A complete cosine wave looks like this:

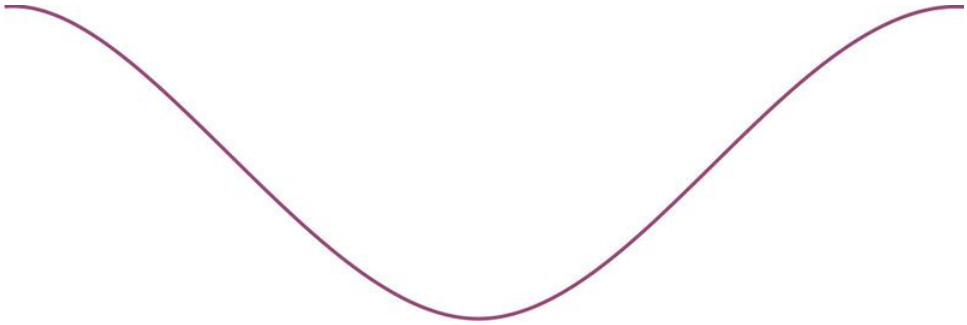


Figure **2.6.5.2**

So, half of it is this:

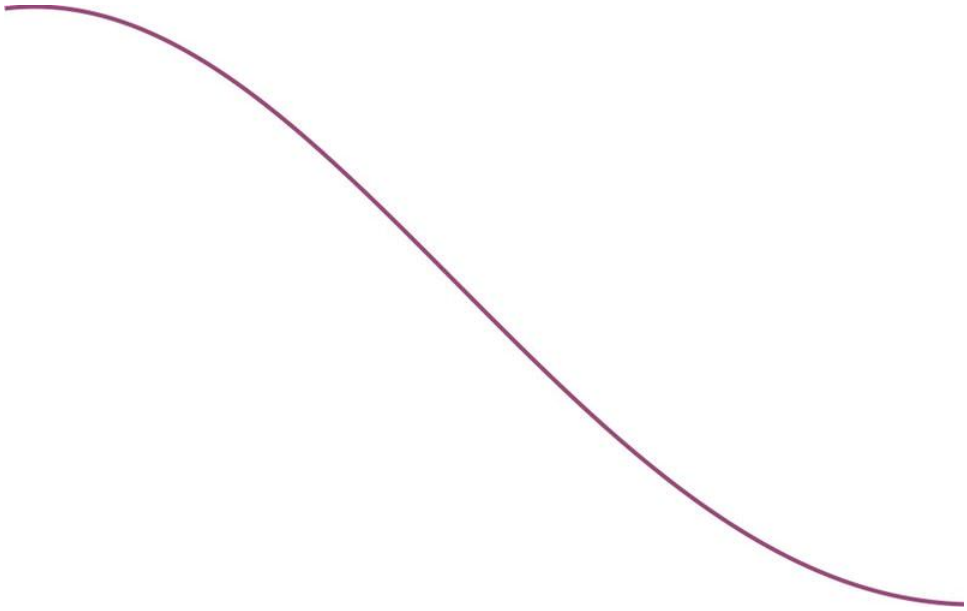


Figure **2.6.5.3**

This means that this half needs to be stretched out so it finishes at 2π, which means that at π the graph should cross the x−axis:

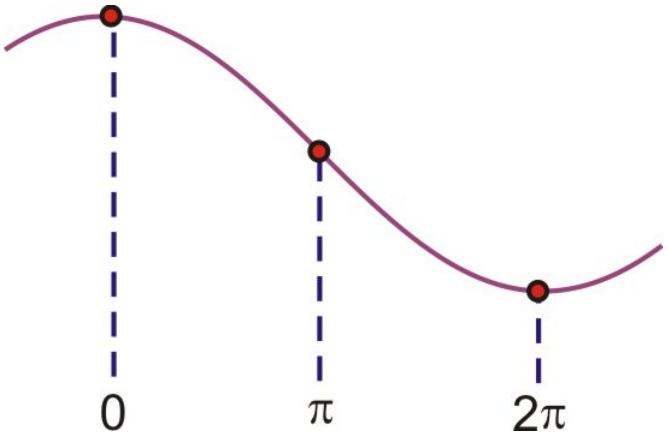


Figure 2.6.5.4

The final sketch would look like this:

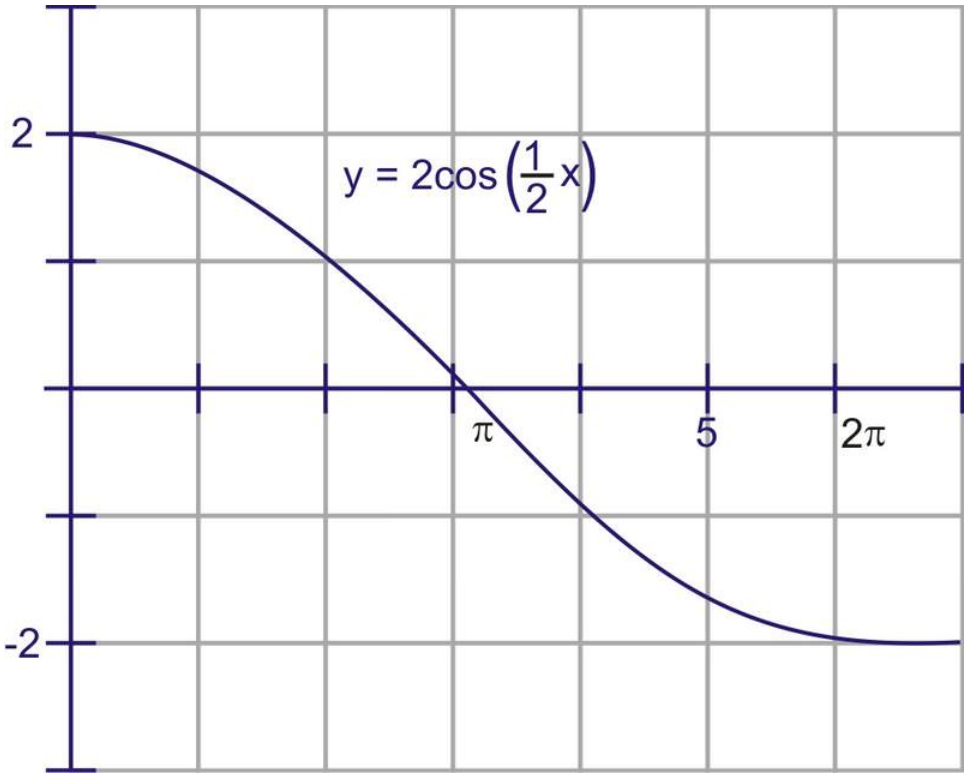


Figure 2.6.5.5

amplitude=2, frequency=$\frac{1}{2}$, period$= \frac{2\pi}{\frac{1}{2}} = 4\pi$

2. Identify the period, amplitude, frequency, and equation of the following sinusoid:

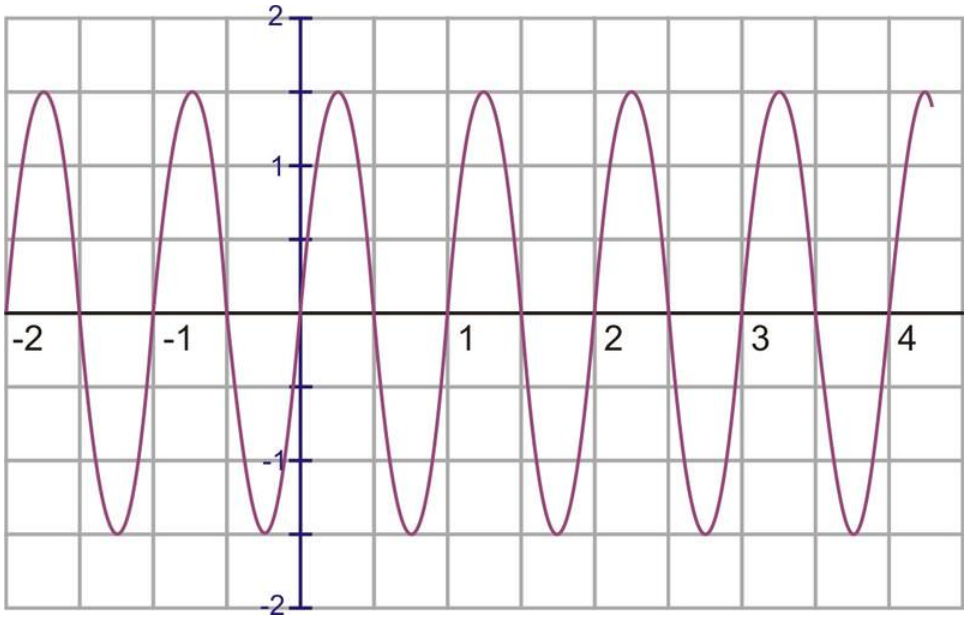


Figure 2.6.5.6

The amplitude is 1.5. Notice that the units on the x−axis are not labeled in terms of π. This appears to be a sine wave because the y−intercept is 0.

One wave appears to complete in 1 unit (*not* 1π *units!*), so the period is 1. If one wave is completed in 1 unit, how many waves will be in 2π units? In previous problems, you were given the frequency and asked to find the period using the following relationship:

$$p = \frac{2\pi}{B}$$

Where B is the frequency and p is the period. With just a little bit of algebra, we can transform this formula and solve it for B:

$$p = \frac{2\pi}{B} \rightarrow Bp = 2\pi \rightarrow B = \frac{2\pi}{p}$$

Therefore, the frequency is:

$$B = \frac{2\pi}{1} = 2\pi$$

If we were to graph this out to 2π we would see 2π (or a little more than 6) complete waves.

Replacing these values in the equation gives: $f(x) = 1.5 \sin 2\pi x$.

3. Find the period, amplitude and frequency of $y = 3 \sin 2x$ and sketch a graph from 0 to 6π.

This is a sine graph that has been stretched both vertically and horizontally. It will now reach up to 3 and down to -3. The frequency is 2 and so we will see the wave repeat twice over the interval from 0 to 2π.

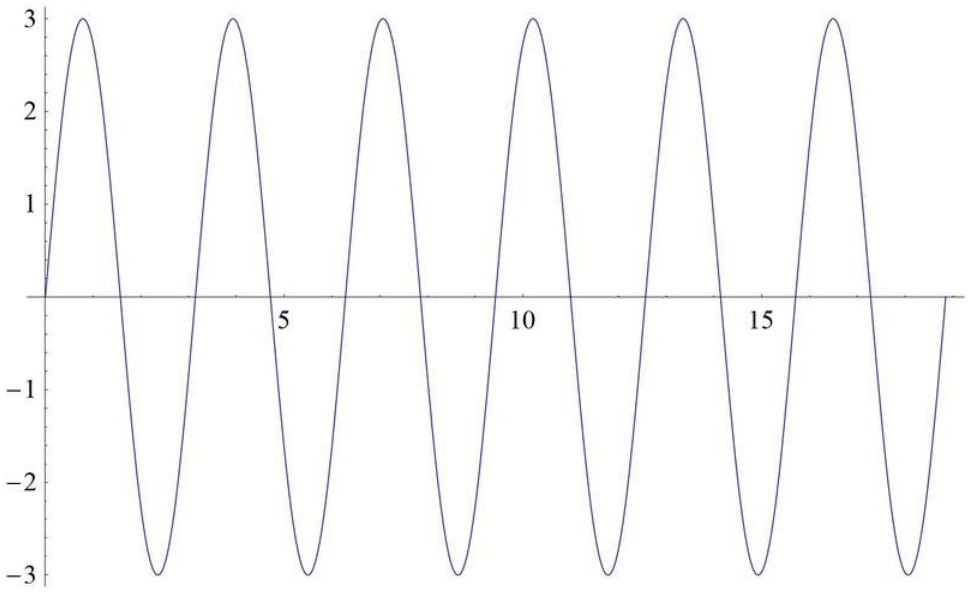


Figure 2.6.5.7

amplitude=3, frequency=2, $\text{period } = \frac{2\pi}{2} = \pi$

Example 2.6.5.1

Earlier, you were asked can you determine how large the graphs needs to be to plot the new sound wave.

Solution

You know that the amplitude of the wave is the maximum height it makes above zero. You also know that the frequency is the number of cycles in a second. The scale of the graph you make should be able to take into account a maximum height of the wave that has been doubled, as well as a frequency that is twice as high. Your graph should look like this:

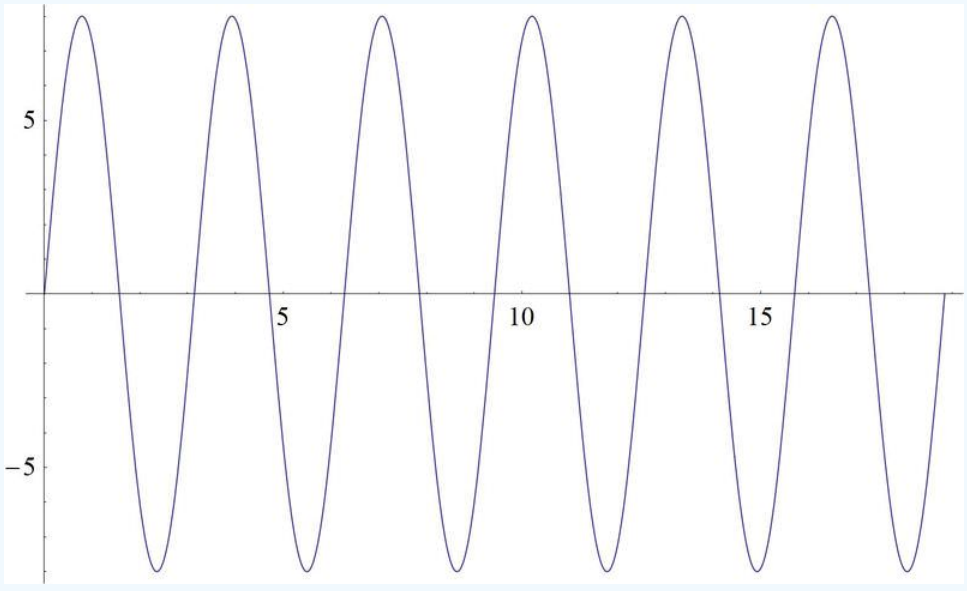


Figure 2.6.5.8

Example 2.6.5.2

Identify the amplitude, period, and frequency of $y = \cos 2x$

Solution

period: π, amplitude: 1, frequency: 2

Example 2.6.5.3

Identify the amplitude, period, and frequency of $y = 3\sin x$

Solution

period: 2π, amplitude: 3, frequency: 1

Example 2.6.5.4

Identify the amplitude, period, and frequency of $y = 2\sin \pi x$

Solution

period: 2, amplitude: 2, frequency: π

Review

Find the period, amplitude, and frequency of the following functions.

1. $y = 2\sin(3x)$
2. $y = 5\cos\left(\frac{3}{4}x\right)$
3. $y = 3\cos(2x)$
4. $y = -2\sin\left(\frac{1}{2}x\right)$
5. $y = -\sin(2x)$
6. $y = \frac{1}{2}\cos(4x)$

Identify the equation of each of the following graphs.

7.

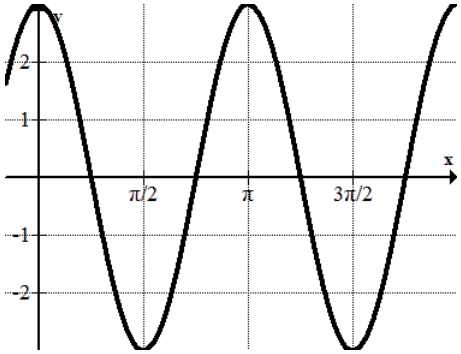


Figure 2.6.5.9

8.

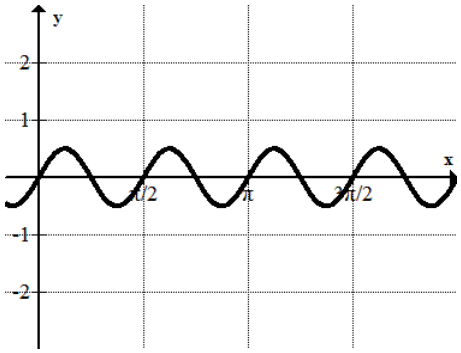


Figure 2.6.5.10

9.

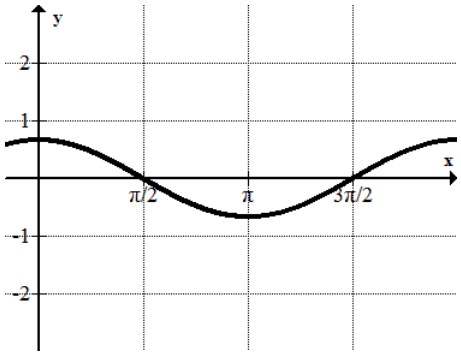


Figure **2.6.5.11**

10.

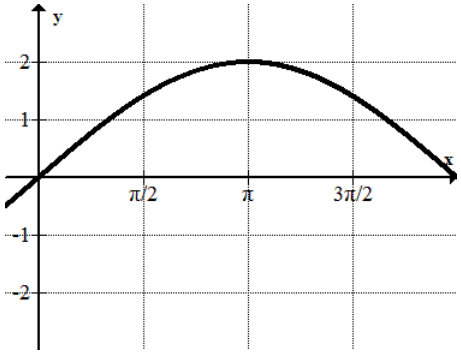


Figure **2.6.5.9**

Graph each of the following functions from 0 to 2π.

11. $y = 2\cos(4x)$
12. $y = 3\sin\left(\frac{5}{4}x\right)$
13. $y = -\cos(2x)$
14. $y = -2\sin\left(\frac{1}{2}x\right)$
15. $y = 4\sec(3x)$
16. $y = \frac{1}{2}\cos(3x)$
17. $y = 4\tan(3x)$
18. $y = \frac{1}{2}\csc(3x)$

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.16.

Additional Resources

Practice: **Amplitude, Period, and Frequency**

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2.6.6: Amplitude

Measure of a wave's height from the center axis.

The **amplitude** of the sine and cosine functions is the vertical distance between the **sinusoidal axis** and the maximum or minimum value of the function. In relation to sound waves, amplitude is a measure of how loud something is.

What is the most common mistake made when graphing the amplitude of a sine wave?

Amplitude of **sinusoidal Functions**

The general form a **sinusoidal function** is:

$f(x) = \pm a \cdot \sin(b(x + c)) + d$

The cosine function can just as easily be substituted and for many problems it will be easier to use a cosine equation. since both the sine and cosine waves are identical except for a horizontal shift, it all depends on where you see the wave starting.

The coefficient a is the amplitude. When there is no number present, then the amplitude is 1. The best way to define amplitude is through a picture. Below is the graph of the function $f(x) = 3 \cdot \sin x$, which has an amplitude of 3.

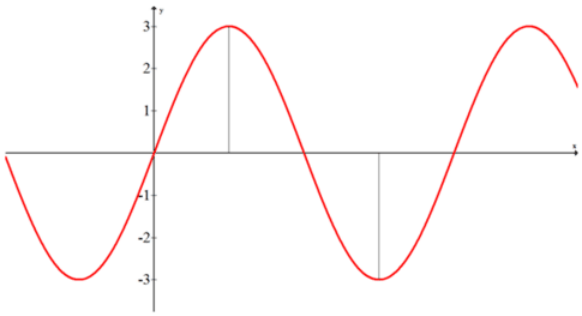
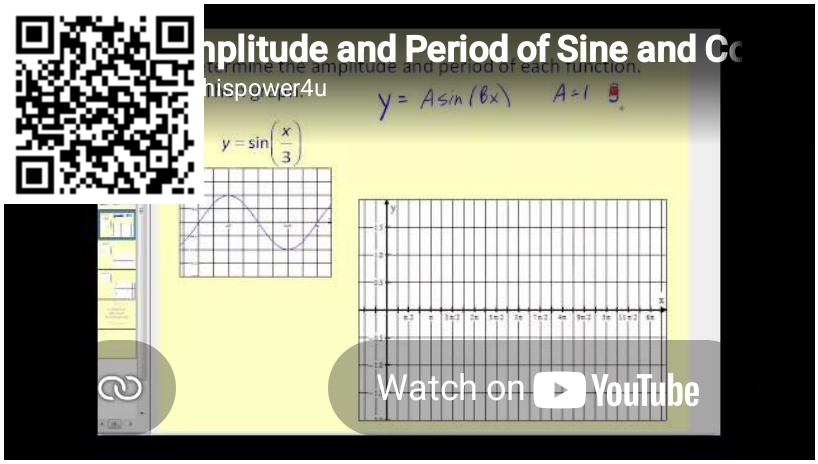


Figure **2.6.6.1**

Notice that the amplitude is 3, not 6. This corresponds to the absolute value of the maximum and minimum values of the function. If the function had been $f(x) = -3 \cdot \sin x$, then the whole graph would be reflected across the x axis.

Also notice that the x axis on the graph above is unlabeled. This is to show that amplitude is a vertical distance. The **sinusoidal axis** is the neutral horizontal line that lies between the **crests** and the **troughs** (or peaks and valleys if you prefer). For this function, the sinusoidal axis was just the x axis, but if the whole graph were shifted up, the sinusoidal axis would no longer be the x axis. Instead, it would still be the horizontal line directly between the crests and troughs which is also the average of the maximum and minimum values.

Watch the portion of this video discussing amplitude:



Example 2.6.6.1

Earlier, you were asked about the most common mistake made when graphing the amplitude of a singe wave. The most common mistake is doubling or halving the amplitude unnecessarily.

Solution

Earlier, you were asked about the most common mistake made when graphing the amplitude of a singe wave. The most common mistake is doubling or halving the amplitude unnecessarily.

Example 2.6.6.2

Graph the following function by plotting main points: $f(x) = -2 \cdot \cos x$.

Solution

The amplitude is 2, which means the maximum values will be at 2 and the minimum values will be at -2. Normally with a basic cosine curve the points corresponding to 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, 2π fall above, on or below the line in the following sequence: above, on, below, on, above. The negative sign in the equation switches above with below and below with above. The whole graph is reflected across the x-axis.

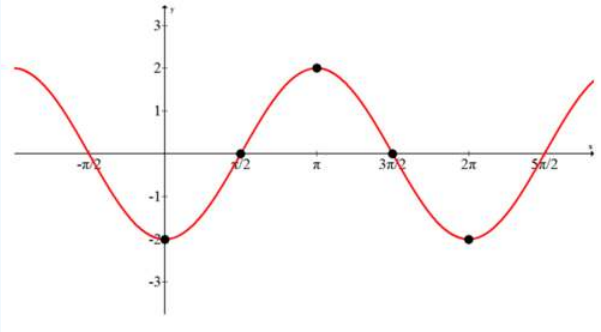


Figure 2.6.6.2

Example 2.6.6.3

Write a cosine equation for each of the following functions.

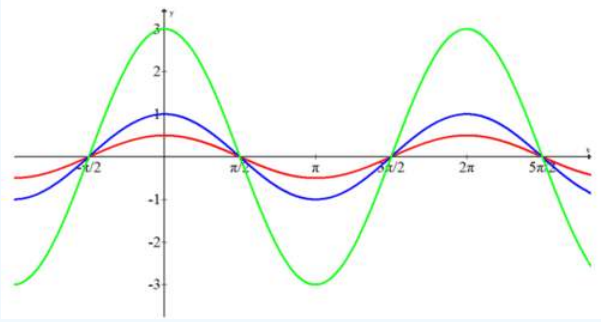


Figure 2.6.6.3

Solution

The amplitudes of the three functions are 3, 1 and $\frac{1}{2}$ and none of them are reflected across the x-axis.

$$f(x) = 3 \cdot \cos x$$
$$h(x) = \cos x$$
$$g(x) = \frac{1}{2} \cdot \cos x$$

Note that amplitude itself is always positive.

Example 2.6.6.4

A Ferris wheel with radius 25 feet sits next to a platform. The ride starts at the platform and travels down to start. Model the height versus time of the ride.

Solution

since no information is given about the time, simply label the x axis as time. At time zero the height is zero. Initially the height will decrease as the ride goes below the platform. Eventually, the wheel will find the minimum and start to increase again all the way until it reaches a maximum.

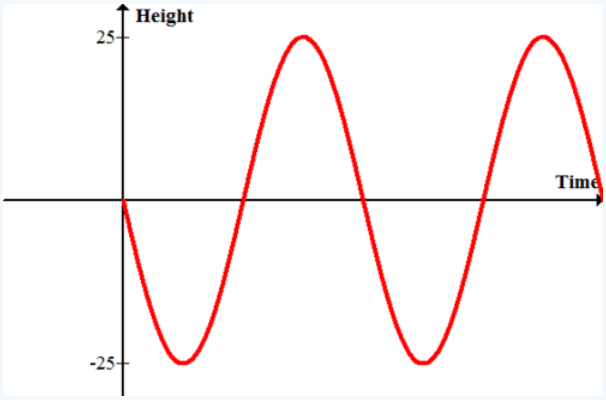


Figure 2.6.6.4

Example 2.6.6.5

Find the amplitude of the function $f(x) = -3\cos x$ and use the language of transformations to describe how the graph is related to the parent function $y = \cos x$.

Solution

The new function is reflected across the x axis and vertically stretched by a factor of 3.

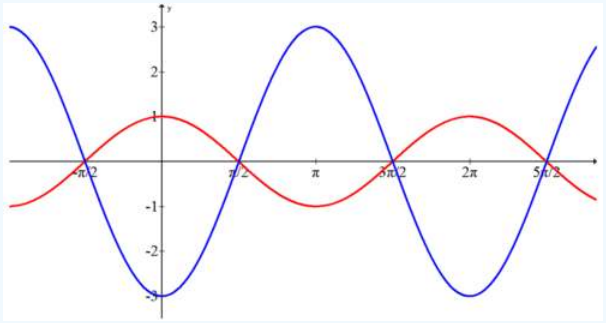


Figure 2.6.6.5

Review

1. Explain how to find the amplitude of a sinusoidal function from its equation.
2. Explain how to find the amplitude of a sinusoidal function from its graph.

Find the amplitude of each of the following functions.

3. $g(x) = -5\cos x$

4.

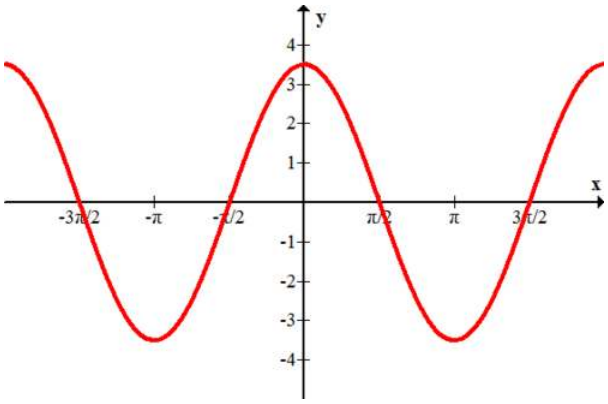


Figure 2.6.6.6

5. $f(x) = \frac{1}{2}\sin x$

6.

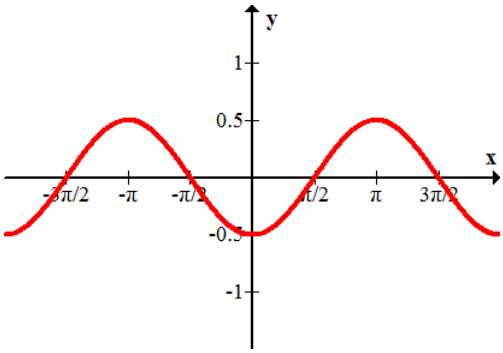


Figure 2.6.6.7

7. $j(x) = 3.12\cos x$

8.

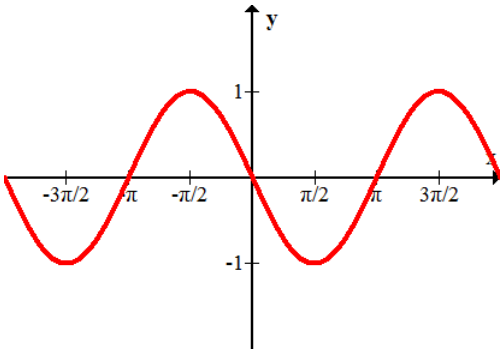


Figure 2.6.6.8

Sketch each of the following functions.

9. $f(x) = 3\sin x$

10. $g(x) = -4\cos x$

11. $h(x) = \pi\sin x$

12. $k(x) = -1.2\cos x$

13. $p(x) = \frac{2}{3}\cos x$

14. $m(x) = -\frac{1}{2}\sin x$

15. Preview: $r(x) = 3\sin x + 2$

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.3.

Additional Resources

Video: Amplitude and Period of sine and cosine

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2.6.7: Period and Frequency

Horizontal distance traveled before y values repeat; number of complete waves in 2π.

The transformation rules about horizontal stretching and shrinking directly apply to sine and cosine graphs. If a sine graph is horizontally stretched by a factor of $\frac{1}{2}$ that is the same as a horizontal compression by a factor of 2.

How does the equation change when a sine or cosine graph is stretched by a factor of 3?

Period and Frequency of **sinusoidal Functions**

The general equation for a **sinusoidal function** is:

$$f(x) = \pm a \cdot \sin(b(x + c)) + d$$

The $\pm$ controls the reflection across the x-axis. The coefficient a controls the **amplitude**. The constant d controls the **vertical shift**. Here you will see that the coefficient b controls the horizontal stretch.

Period

Horizontal stretch is measured for sinusoidal functions as their periods. This is why this function family is also called the periodic function family. The **period** of a sinusoid is the length of a complete cycle. For basic sine and cosine functions, the period is 2π. This length can be measured in multiple ways. In word problems and in other tricky circumstances, it may be most useful to measure from peak to peak.

The following three waves have different periods. To rank each wave by period from shortest to longest, look at he distance between each peak.

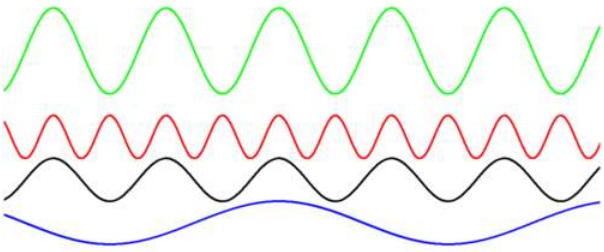


Figure **2.6.7.1**

The red wave has the shortest period.

The green and black waves have equal periods. Even though the green wave has greater amplitude than the black wave, they both have the same period.

The blue wave has the longest period.

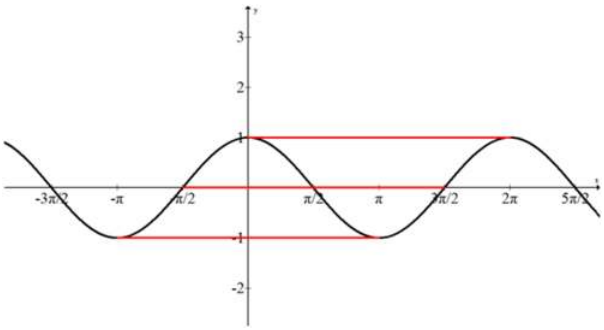


Figure **2.6.7.2**

The ability to measure the period of a function in multiple ways allows different equations to model an identical graph. In the image above, the top red line would represent a regular cosine wave. The center red line would represent a regular sine wave with a

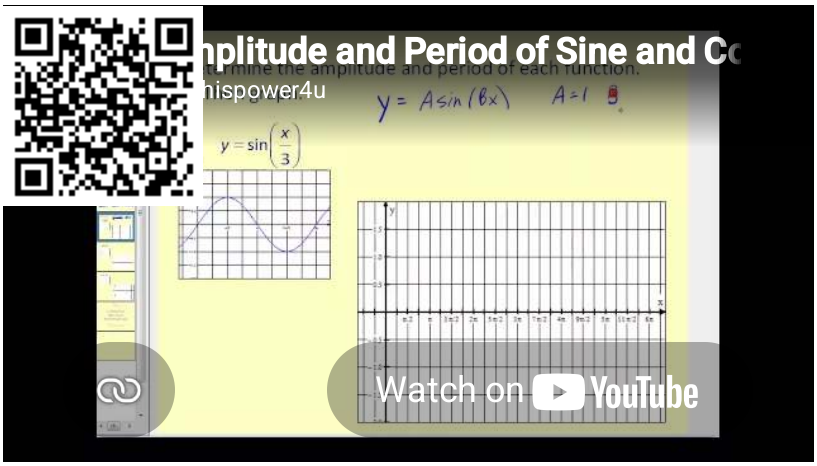
horizontal shift. The bottom red line would represent a negative cosine wave with a horizontal shift. This flexibility in perspective means that many of the examples, guided practice and practice problems may have multiple solutions. For now, try to always choose the function that has a period starting at x=0.

Frequency

Frequency is a different way of measuring horizontal stretch. For sound, frequency is known as pitch. With sinusoidal functions, **frequency** is the number of cycles that occur in 2π. A shorter period means more cycles can fit in 2\pi and thus a higher frequency. Period and frequency are inversely related by the equation:

$$period = \frac{2\pi}{frequency}$$

The equation of a basic sine function is $f(x) = \sin x$. In this case b, the frequency, is equal to 1 which means one cycle occurs in 2π. If $b = \frac{1}{2}$, the period is $\frac{2\pi}{\frac{1}{2}}$ which means the period is 4π and the graph is stretched.



Example 2.6.7.1

Earlier, you were asked how an equation changes when a sine or cosine graph is stretched by a factor of 3.

Solution

If a sine graph is horizontally stretched by a factor of 3 then the general equation has $b = \frac{1}{3}$. This is because b is the frequency and counts the number (or fraction) of a period that fits in a normal period of 2π. Graphically, the sine wave will make a complete cycle in 6π. Similarly, a cosine graph will have $b = \frac{1}{3}$ and will have a period of 6π.

Example 2.6.7.2

Identify the amplitude, vertical shift, period and frequency of the following function. Then graph the function.

Solution

$f(x) = 2\sin\left(\frac{x}{3}\right) + 1$

$a = 2, b = \frac{1}{3}, d = 1$

The amplitude is 2, the vertical shift is 1, and the frequency is $\frac{1}{3}$. The period would be $\frac{2\pi}{\frac{1}{3}}$, or 6π.

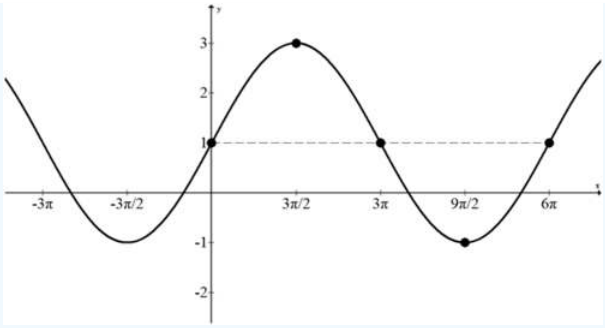


Figure 2.6.7.3

Often the most challenging part of graphing periodic functions is labeling the axes. since the period is 6π, start by drawing the sinusoidal axis shifted appropriately. Then divide the 6π into four parts so that the 5 guiding points of the sine graph can be plotted with the amplitude and reflection in mind. The very last thing to do is to draw and extend the curve.

Example 2.6.7.3

A measuring stick on a dock measures high tide to be 18 feet and low tide to be 6 feet. It takes about 6 hours for the tide to switch between low and high tides. Determine a graphical and algebraic model for the tides knowing that at $t = 0$ there is a high tide.

Solution

Usually the best course of action for word problems is to identify information, plot points, sketch and then finally come up with an equation.

From the given information you can deduce the following points. Notice how the sinusoidal axis can be assumed to be the average of the high and low tides.

Time (hours)	Water level (feet)
0	18
6	6
12	18
3	$\frac{18+6}{2} = \frac{1}{2}$
9	12

By plotting those points and filling in the sinusoidal axis you can observe a cosine graph.

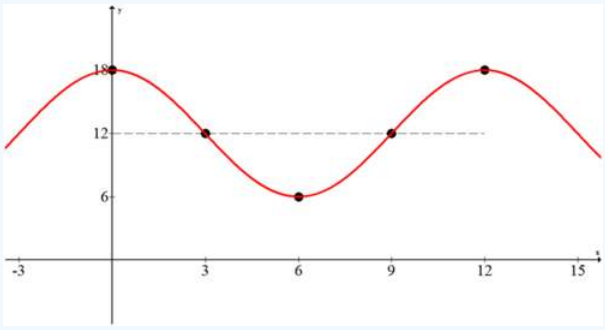


Figure 2.6.7.4

The amplitude is 6 so $a = 6$. There is no vertical reflection. Since the period is 12 you can determine the frequency b:

$12 = \frac{2\pi}{b} \rightarrow b = \frac{\pi}{6}$

The vertical shift is 12 so $d = 12$. Thus you have all the pieces to make an algebraic model:

$f(x) = 6 \cdot \cos\left(\frac{\pi}{6}x\right) + 12$

Example 2.6.7.4

Graph the following function: $g(x) = -\cos(8x) + 2$.

Solution

The labeling is the most important and challenging part of this problem. The amplitude is 1. The shape is a negative cosine. The vertical shift is 2. The period is $\frac{2\pi}{8} = \frac{\pi}{4}$.

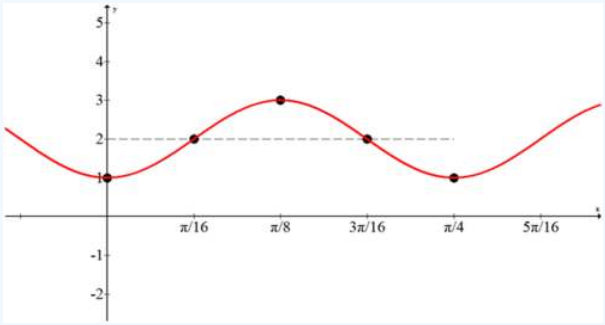


Figure 2.6.7.5

Example 2.6.7.5

Given the following graph, identify the amplitude, period, and frequency and create an algebraic model.

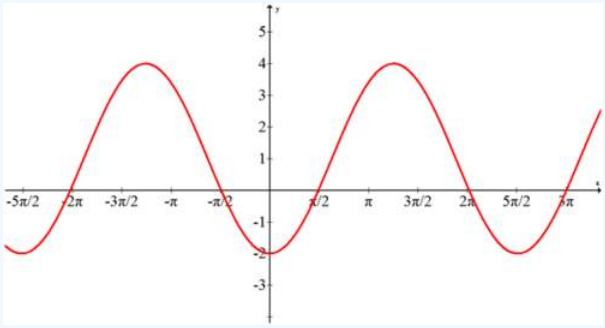


Figure 2.6.7.6

Solution

The amplitude is 3. The shape is a negative cosine. The period is $\frac{5\pi}{2}$ which implies that $b = \frac{4}{5}$. The vertical shift is 1. $f(x) = -3 \cdot \cos\left(\frac{4}{5}x\right) + 1$.

Review

Find the frequency and period of each function below.

1. $f(x) = \sin(4x) + 1$
2. $g(x) = -3\cos(2x)$
3. $h(x) = \cos\left(\frac{1}{2}x\right) + 2$
4. $k(x) = -2\sin\left(\frac{3}{4}x\right) + 1$
5. $j(x) = 4\cos(3x) - 1$

Graph each of the following functions.

6. $f(x) = 3\sin(2x) + 1$
7. $g(x) = 2.5\cos(\pi x) - 4$
8. $h(x) = -\sin(4x) - 3$
9. $k(x) = \frac{1}{2}\cos(2x)$
10. $j(x) = -2\sin\left(\frac{3}{4}x\right) - 1$

Create an algebraic model for each of the following graphs.

11.

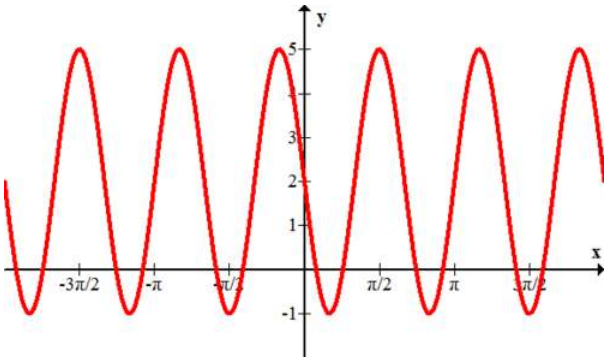


Figure **2.6.7.7**

12.

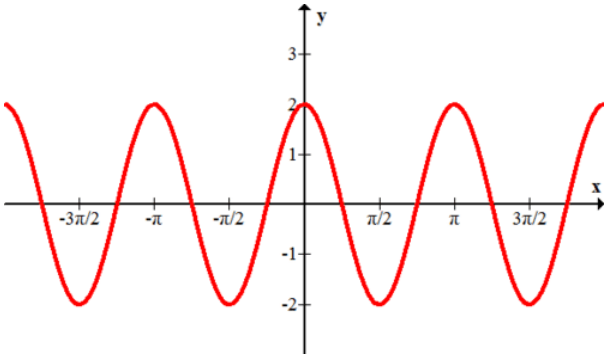


Figure **2.6.7.8**

13.

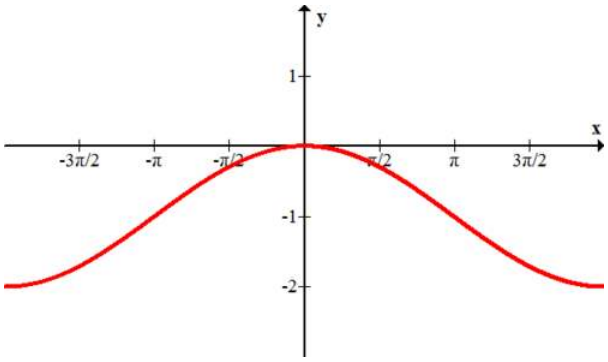


Figure **2.6.7.9**

14. At time 0 it is high tide and the water at a certain location is 10 feet high. At low tide 6 hours later, the water is 2 feet high. Given that tides can be modeled by sinusoidal functions, find a graph that models this scenario.
15. Find the equation that models the scenario in the previous problem.

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.5.

Vocabulary

Term	Definition
amplitude	the maximum distance the particles of the medium move from their resting positions when a wave passes through.
sinusoidal function	A sinusoidal function is a sine or cosine wave.
sinusoidal functions	A sinusoidal function is a sine or cosine wave.
Vertical shift	A vertical shift is the result of adding a constant term to the value of a function. A positive term results in an upward shift, and a negative term in a downward shift.

Additional Resources

Video: Transformation of Trigonometric Graphs: Amplitude, Period, and Phase Shift

Practice: Period and Frequency

Real World: Like Clockwork

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2.6.8: Trigonometric Identities and Equations

Based on amplitude, frequency, and horizontal and vertical translations.

Your math teacher has decided to give you a quiz to see if you recognize how to combine changes to graphs of sine and cosine functions. You recall that you've learned about shifting graphs, as well as stretching/dilating them. But now your teacher wants to see if you know how to combine both of these effects into one graph. She gives you the equation:

$$f(x) = 3 + 7\sin\left(4\left(x + \frac{\pi}{2}\right)\right)$$

and asks you to plot the equation, and then identify what each part of the above equation does to change the graph.

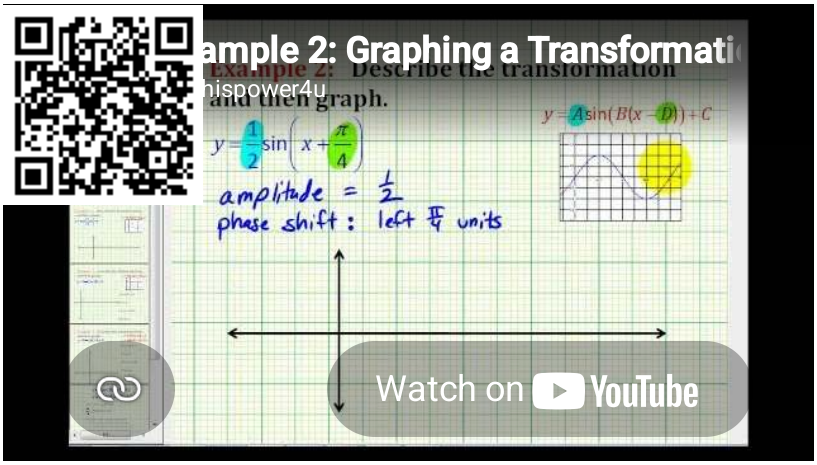
Trigonometric Identities and Equations

The general equations to translate and dilate sine and cosine waves both horizontally and vertically are: $y = D \pm A\cos(B(x \pm C))$ or $y = D \pm A\sin(B(x \pm C))$, where A is the amplitude, B is the frequency, C is the horizontal translation, and D is the vertical translation.

Recall the relationship between period, p, and frequency, B.

$p = \frac{2\pi}{B}$ and $B = \frac{2\pi}{p}$

With this knowledge, we should be able to sketch any sine or cosine function as well as write an equation given its graph.



Let's take a look at some problems involving trigonometric identities and equations.

1. Given the function: $f(x) = 1 + 2\sin(2(x + \pi))$

a. Identify the period, amplitude, and frequency.

From the equation, the amplitude is 2 and the frequency is also 2. To find the period we use:

$$p = \frac{2\pi}{B} \rightarrow p = \frac{2\pi}{2} = \pi$$

So, there are two complete waves from $[0, 2\pi]$ and each individual wave requires π radians to complete.

b. Explain any vertical or horizontal translations present in the equation.

$D = 1$ and $C = -\pi$, so this graph has been translated 1 unit up, and \pi units to the left.

c. Sketch the graph from -2π to 2π.

To sketch the graph, start with the graph of $y = \sin(x)$

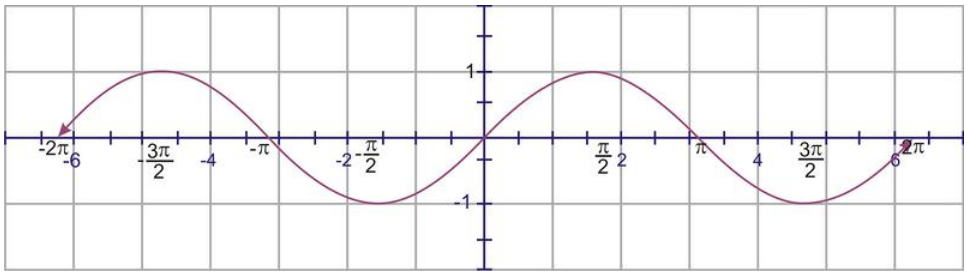


Figure **2.6.8.1**

Translate the graph π units to the left (the C value).

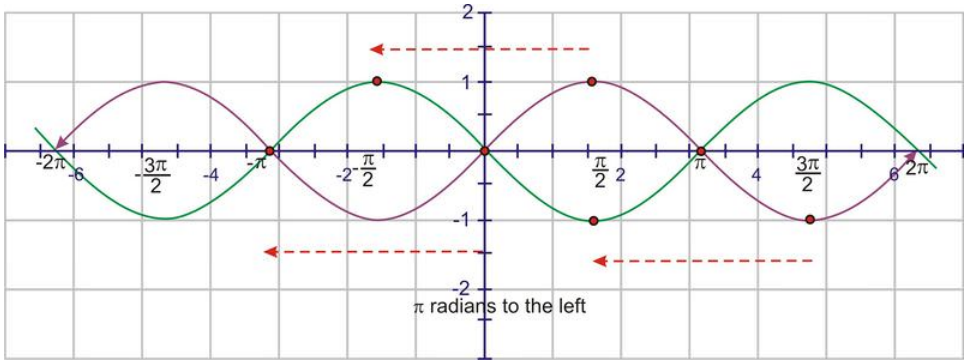


Figure **2.6.8.2**

Next, move the graph 1 unit up (D value)

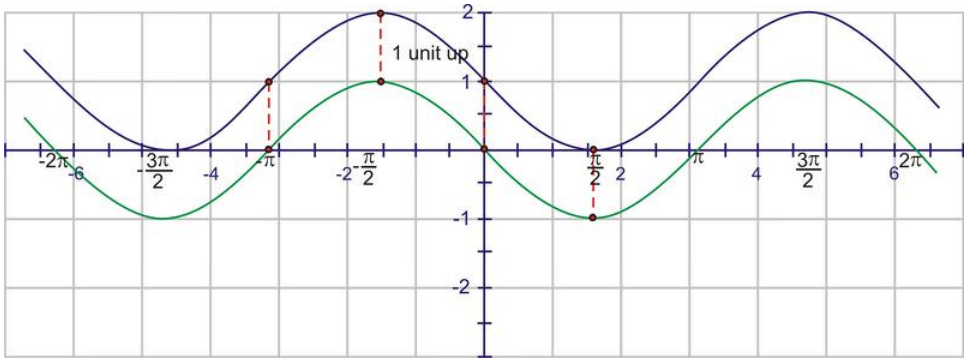


Figure **2.6.8.3**

Now we can add the dilations. Remember that the "starting point" of the wave is $-\pi$ because of the horizontal translation. A normal sine wave takes 2π units to complete a cycle, but this wave completes one cycle in π units. The first wave will complete at 0, then we will see a second wave from 0 to π and a third from π to 2π. Start by placing points at these values:

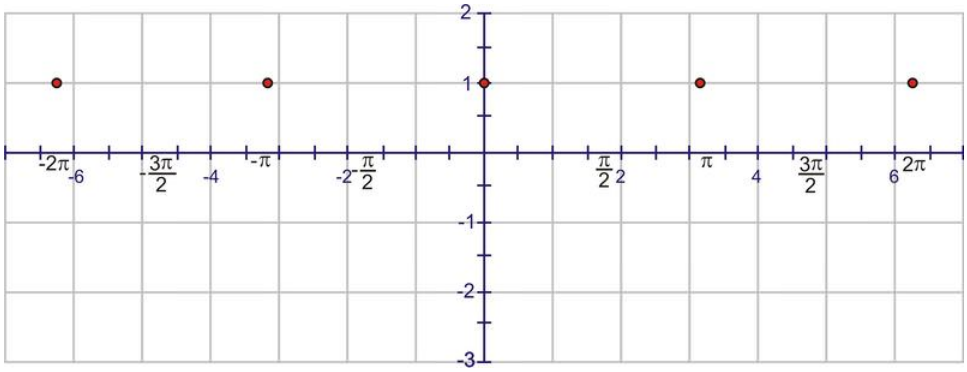


Figure **2.6.8.4**

Using symmetry, each interval needs to cross the line $y = 1$ through the center of the wave.

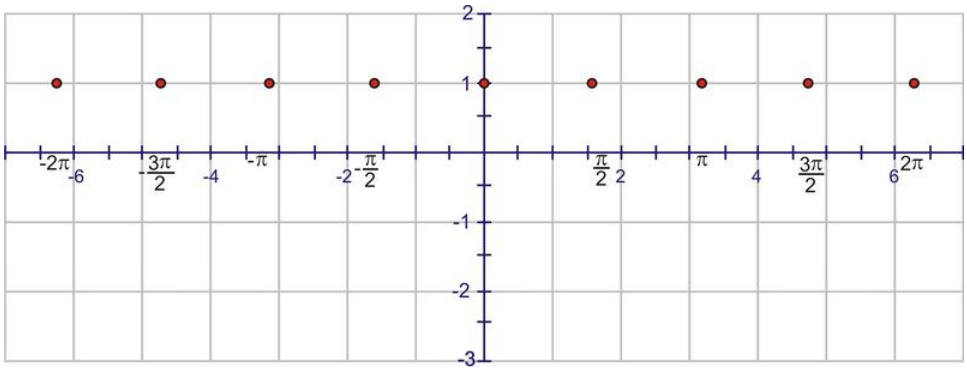


Figure **2.6.8.5**

One sine wave contains a “mountain” and a “valley”. The mountain “peak” and the valley low point must occur halfway between the points above.

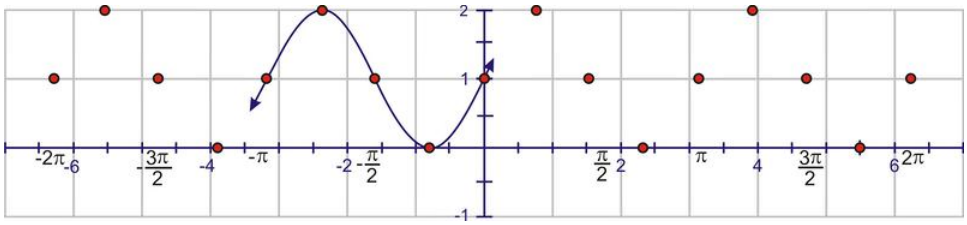


Figure **2.6.8.6**

Extend the curve through the domain.

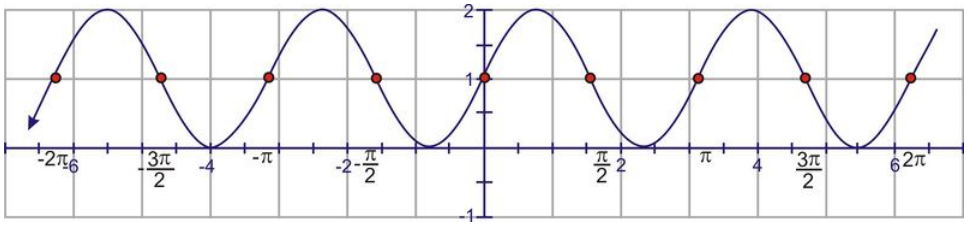


Figure **2.6.8.7**

Finally, extend the minimum and maximum points to match the amplitude of 2.

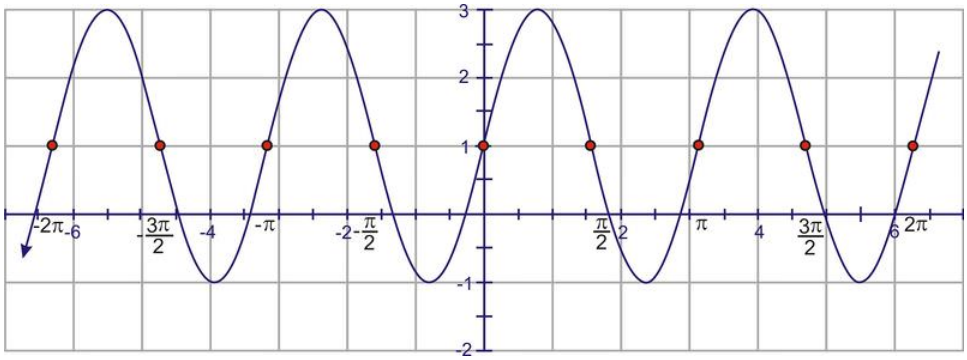


Figure **2.6.8.8**

2. Given the function: $f(x) = 3 + 3\cos\left(\frac{1}{2}\left(x - \frac{\pi}{2}\right)\right)$

a. Identify the period, amplitude, and frequency.

From the equation, the amplitude is 3 and the frequency is $\frac{1}{2}$. To find the period we use:

$$\text{period} = \frac{2\pi}{\frac{1}{2}} = 4\pi$$

So, there is only one half of a cosine curve from 0 to 2π and each individual wave requires 4π radians to complete.

b. Explain any vertical or horizontal translations present in the equation.

$D = 3$ and $C = \frac{\pi}{2}$, so this graph has been translated 3 units up, and $\frac{\pi}{2}$ units to the right.

c. Sketch the graph from -2π to 2π.

To sketch the graph, start with the graph of $y = \cos(x)$

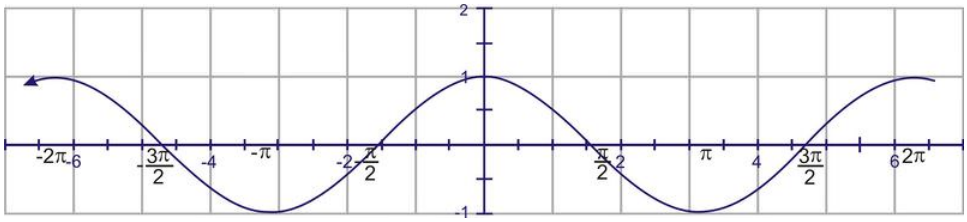


Figure 2.6.8.9

Adjust the amplitude so the cosine wave reaches up to 3 and down to negative three. This affects the maximum points, but the points on the x−axis remain the same. These points are sometimes called **nodes.**

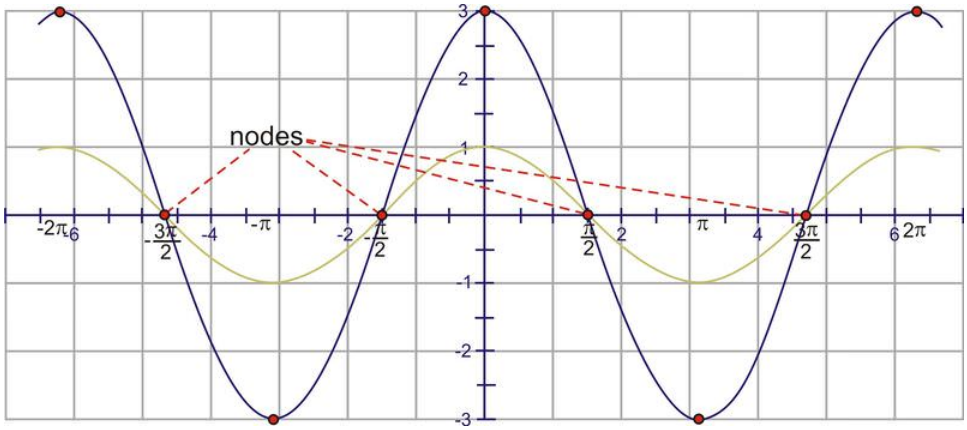


Figure 2.6.8.10

According to the period, we should see one of these shapes every 4π units. Because the interval specified is $[-2\pi, 2\pi]$ and the cosine curve "starts" at the y−axis, at $(0, 3)$ and at 2π the value is -3. Conversely, at -2π, the function is also -3.

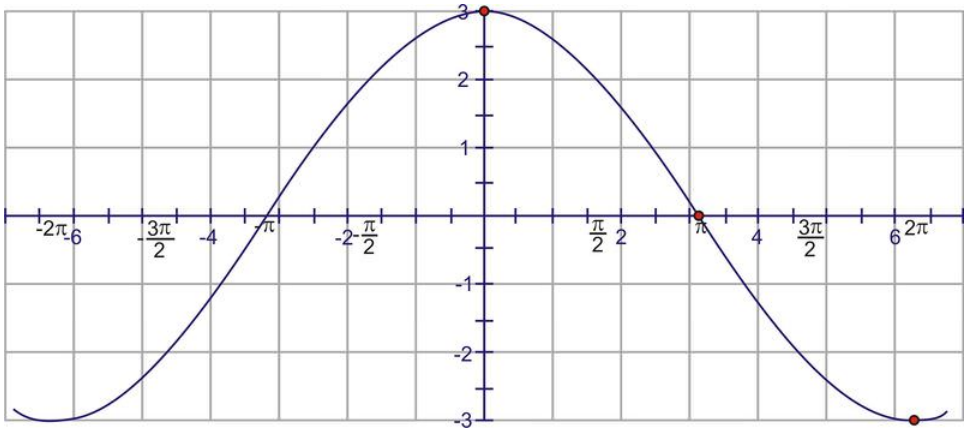


Figure 2.6.8.11

Now, shift the graph $\frac{\pi}{2}$ units to the right.

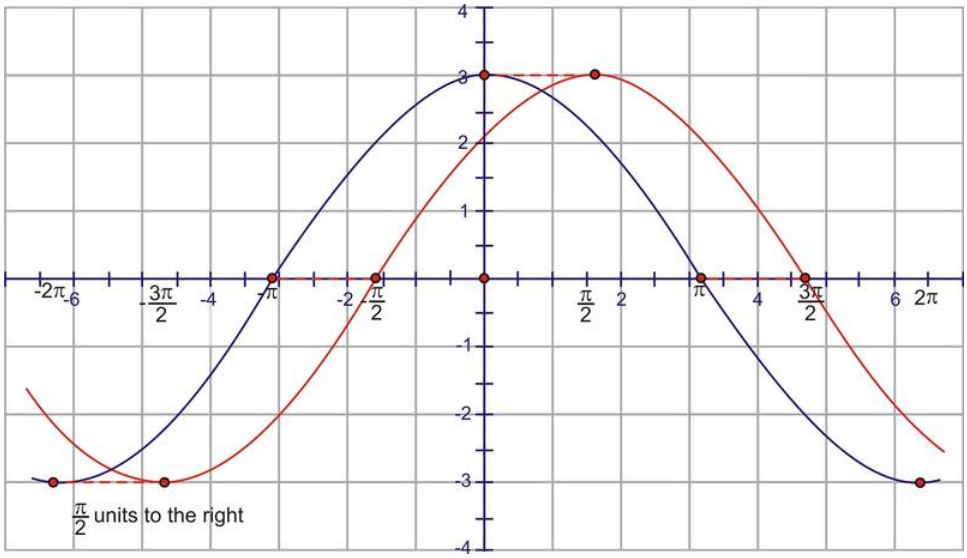


Figure 2.6.8.12

Finally, we need to adjust for the vertical shift by moving it up 3 units.

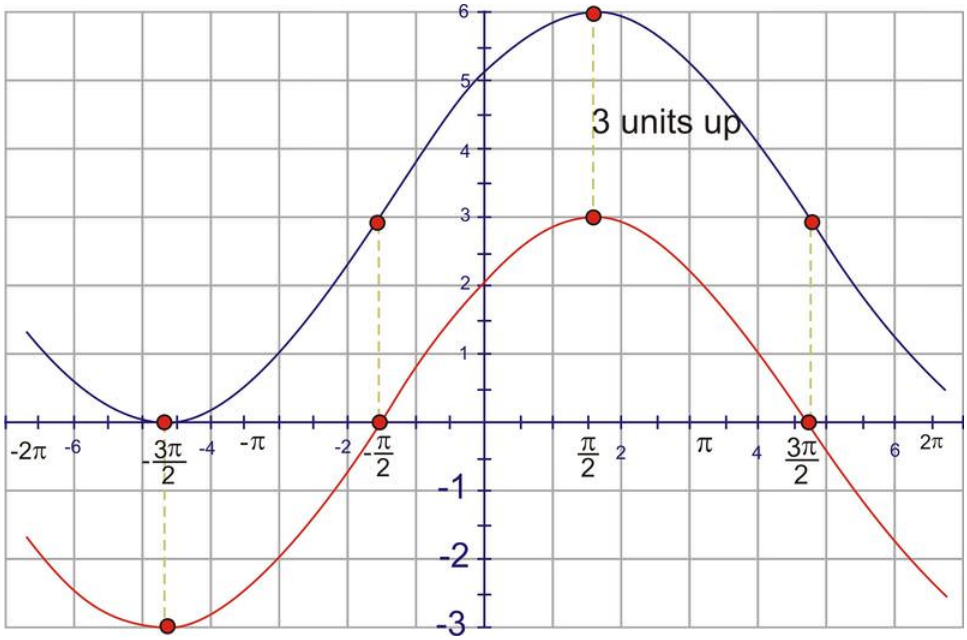


Figure 2.6.8.13

3. Find the equation of the sinusoid graphed here.

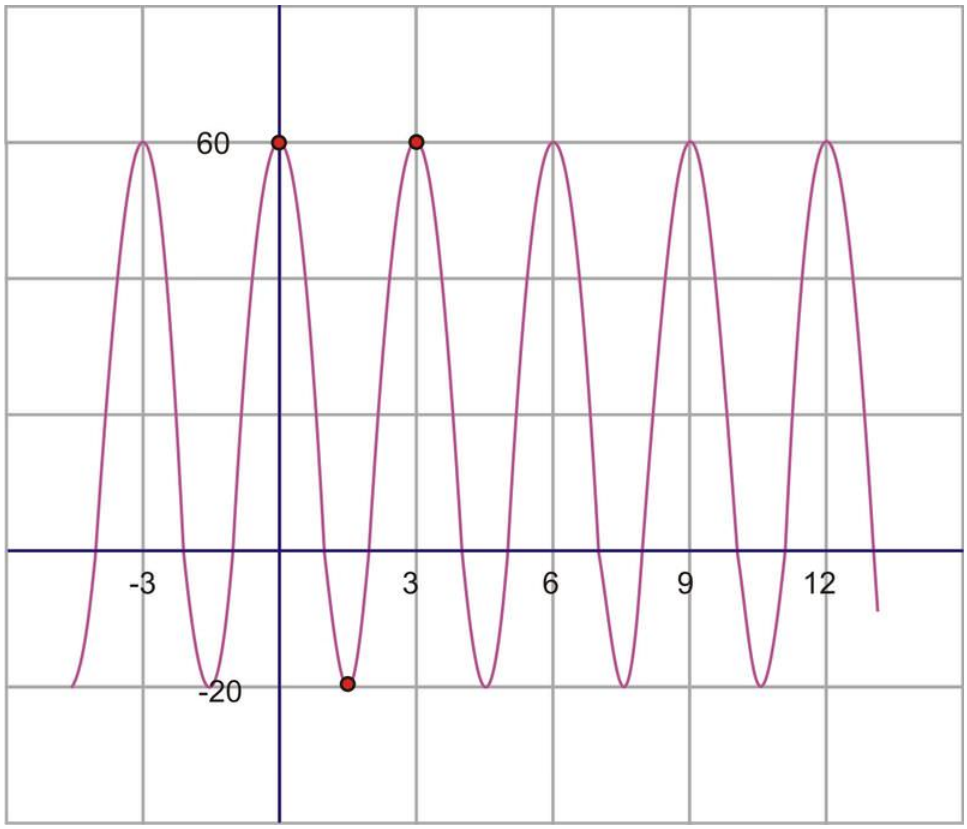


Figure 2.6.8.14

First of all, remember that either sine or cosine could be used to model these graphs. However, it is usually easier to use cosine because the horizontal shift is easier to locate in most cases. Therefore, the model that we will be using is $y = D \pm A\cos(B(x \pm C))$.

First, if we think of the graph as a cosine function, it has a horizontal translation of zero. The maximum point is also the y−intercept of the graph, so there is no need to shift the graph horizontally and therefore, $C = 0$. The amplitude is the height from the center of the wave. If you can't find the center of the wave by sight, you can calculate it. The center should be halfway between the highest and the lowest points, which is really the **average** of the maximum and minimum. This value will actually be the vertical shift, or D value.

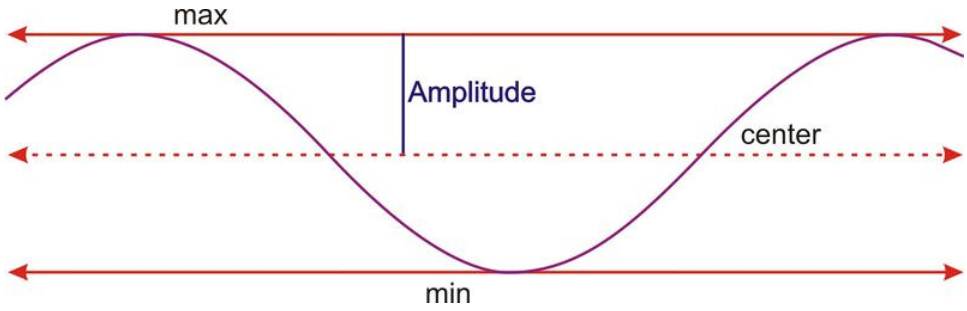


Figure 2.6.8.15

$$D = \text{center} = \frac{60 + -20}{2} = \frac{40}{2} = 20$$

The amplitude is the height from the center line, or vertical shift, to either the minimum or the maximum. So, $A = 60 - 20 = 40$.

The last value to find is the frequency. In order to do so, we must first find the period. The period is the distance required for one complete wave. To find this value, look at the horizontal distance between two consecutive maximum points.

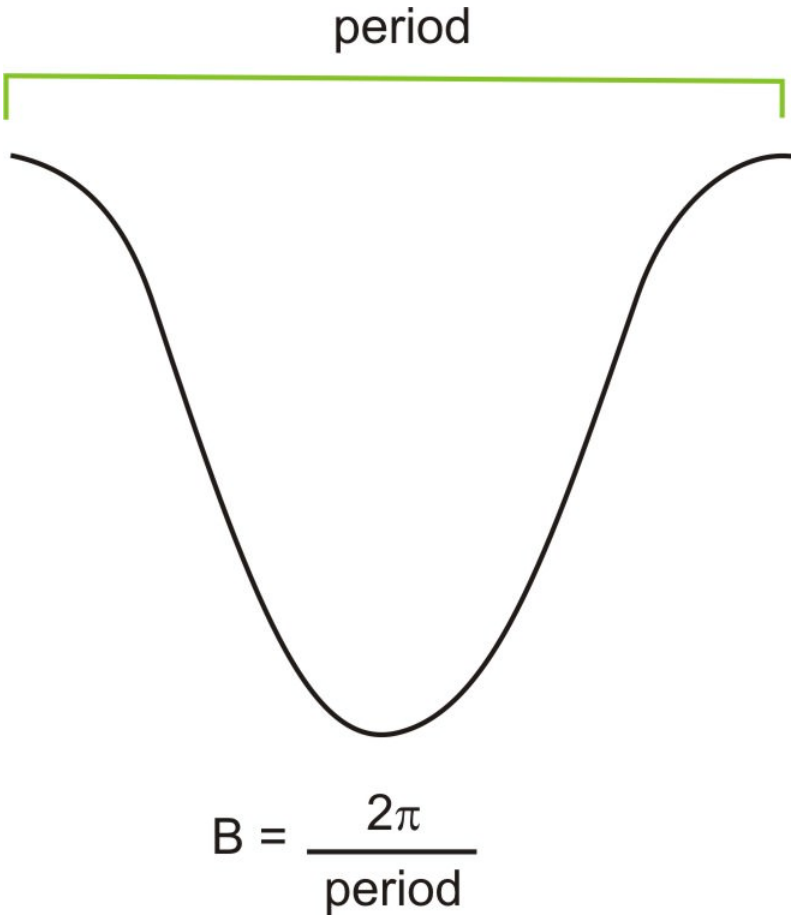


Figure 2.6.8.16

On our graph, from maximum to maximum is 3.

Therefore, the period is 3, so the frequency is $B = \frac{2\pi}{3}$.

We have now calculated each of the four parameters necessary to write the equation. Replacing them in the equation gives:

$$y = 20 + 40 \cos \frac{2\pi}{3} x$$

If we had chosen to model this curve with a sine function instead, the amplitude, period and frequency, as well as the vertical shift would all be the same. The only difference would be the horizontal shift. The sine wave starts in the middle of an upward sloped section of the curve as shown by the red circle.

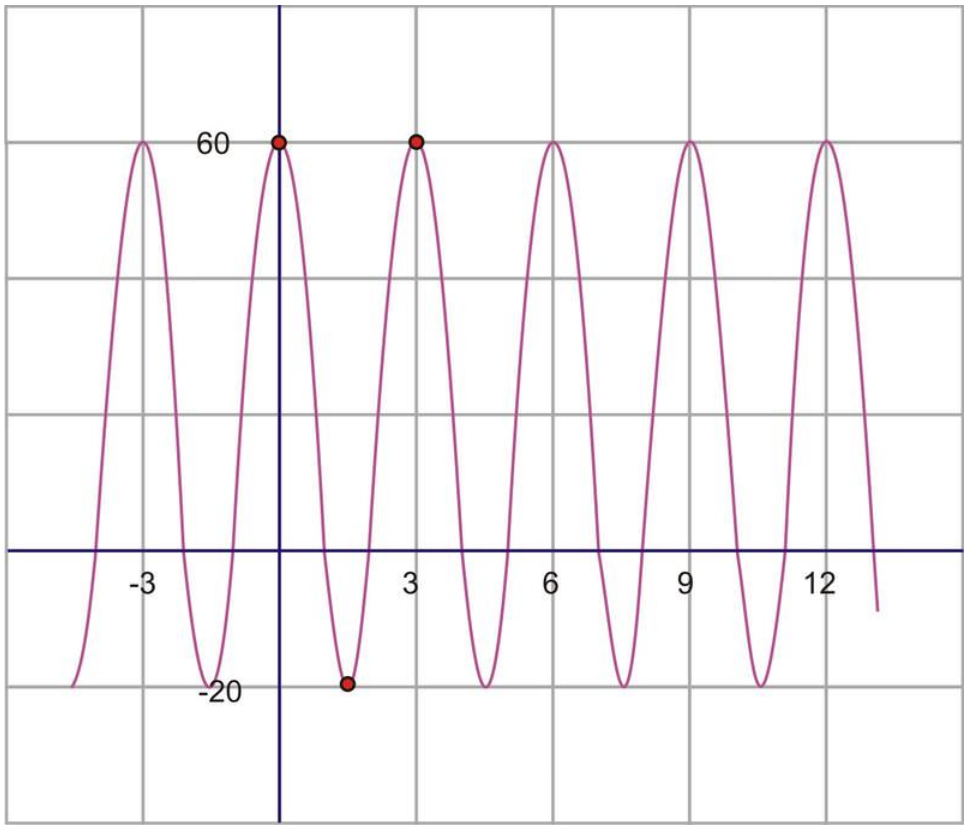


Figure **2.6.8.17**

This point intersects with the vertical translation line and is a third of the distance back to -3. So, in this case, the sine wave has been translated 1 unit *to the left*. The equation using a sine function instead would have been: $y = 20 + 40 \sin\left(\frac{2\pi}{3}(x+1)\right)$

Example 2.6.8.1

Earlier, you were asked to plot $f(x) = 3 + 7\sin\left(4\left(x + \frac{\pi}{2}\right)\right)$.

Solution

With your advanced knowledge of sinusoidal equations, you can identify in the equation:

$$(f(x) = 3 + 7\sin\left(4\left(x + \frac{\pi}{2}\right)\right)$$

The vertical shift of the graph is 3 units up. The amplitude of the graph is 7. The horizontal shift of the graph is $\frac{\pi}{2}$ units to the left. The frequency is 4.

Since the frequency is 4, the period can be calculated:

$$p = \frac{2\pi}{f}$$
$$p = \frac{2\pi}{4}$$
$$p = \frac{\pi}{2}$$

This means that the graph takes \pi 2 units to make one complete cycle.

The graph of this equation looks like this:

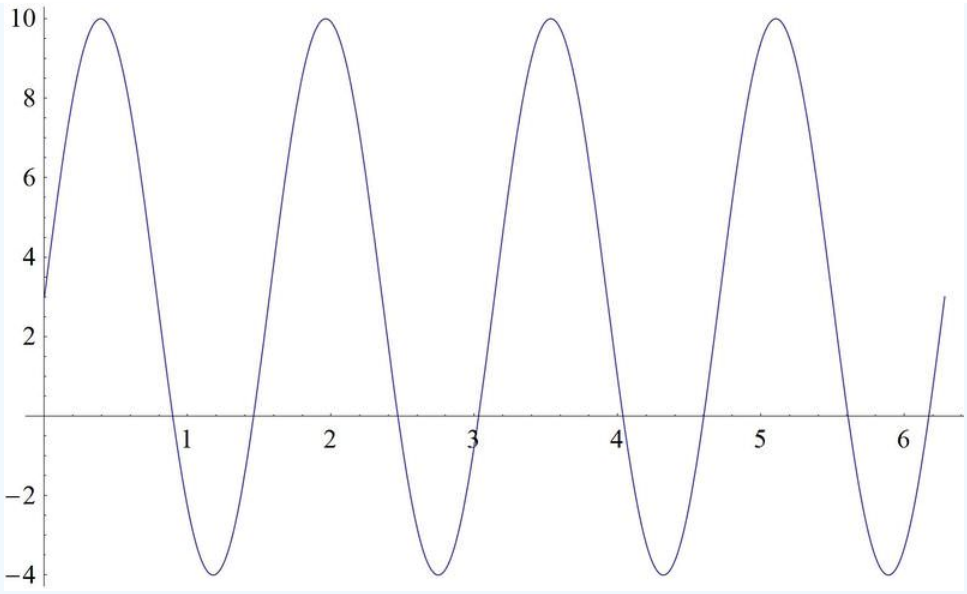


Figure 2.6.8.18

Example 2.6.8.2

Identify the amplitude, period, frequency, maximum and minimum points, vertical shift, and horizontal shift of $y = 2 + 3\sin(2(x - 1))$.

Solution

This is a sine wave that has been translated 1 unit to the right and 2 units up. The amplitude is 3 and the frequency is 2. The period of the graph is π. The function reaches a maximum point of 5 and a minimum of -1.

Example 2.6.8.3

Identify the amplitude, period, frequency, maximum and minimum points, vertical shift, and horizontal shift of $y = -1 + \sin\left(\pi\left(x + \frac{\pi}{3}\right)\right)$.

Solution

This is a sine wave that has been translated 1 unit down and $\frac{\pi}{3}$ radians to the left. The amplitude is 1 and the period is 2. The frequency of the graph is π. The function reaches a maximum point of 0 and a minimum of -2.

Example 2.6.8.4

Identify the amplitude, period, frequency, maximum and minimum points, vertical shift, and horizontal shift of $y = \cos(40(x - 120)) + 5$.

Solution

This is a cosine wave that has been translated 5 units up and 120 radians to the right. The amplitude is 1 and the frequency is 40. The period of the graph is $\frac{\pi}{20}$. The function reaches a maximum point of 6 and a minimum of 4.

Review

For each equation below, identify the period, amplitude, frequency, and any vertical/horizontal translations.

1. $y = 2 - 4\cos\left(\frac{2}{3}(x - 3)\right)$
2. $y = 3 + \frac{1}{2}\sin\left(\frac{1}{2}(x - \pi)\right)$

3. $y = 1 + 5\cos(4(x + \frac{\pi}{2}))$
4. $y = 4 - \cos(2(x + 1))$
5. $y = 3 + 2\sin(x - 4)$

Graph each of the following equations from -2π to 2π.

6. $y = 1 - 3\sin\left(\frac{1}{3}(x - \pi)\right)$
7. $y = 5 + 12\sin\left(\frac{1}{2}(x - 2)\right)$
8. $y = 2 + \cos\left(4\left(x + \frac{\pi}{2}\right)\right)$
9. $y = 4 + 2\cos(2(x + 3))$
10. $y = 2 - 3\sin\left(x - \frac{3\pi}{2}\right)$

Find the equation of each sinusoid.

11.

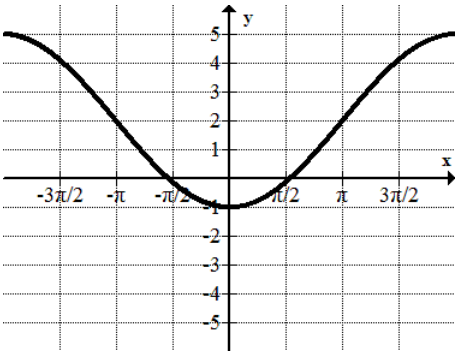


Figure 2.6.8.19

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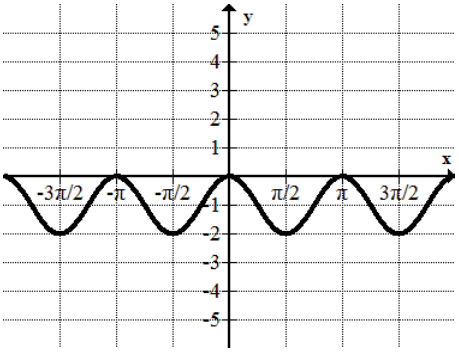


Figure 2.6.8.20

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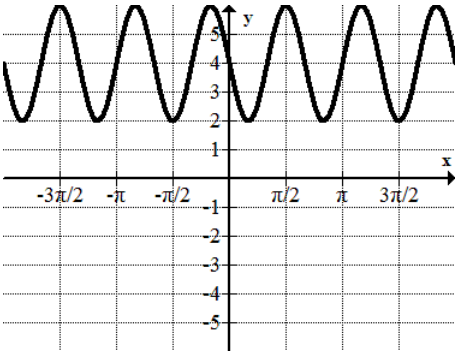


Figure 2.6.8.21

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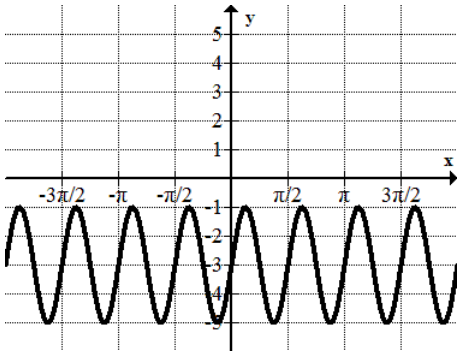


Figure 2.6.8.22

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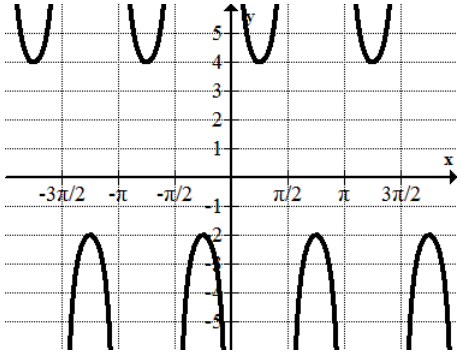


Figure 2.6.8.23

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.17.

Vocabulary

Term	Definition
Trigonometric General Equations	The trigonometric general equations are $y = D \pm A\cos(B(x \pm C))$ or $y = D \pm A\sin(B(x \pm C))$, where A is the amplitude, B is the frequency, C is the horizontal translation, and D is the vertical translation.

Additional Resources

Video: Example: Graphing a Transformation of Sine and cosine

Practice: Trigonometric Identities and Equations

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2.6.9: General Sinusoidal Graphs

sine and cosine waves and their relationship to the unit circle.

The **cosine function is the x coordinates** of the **unit circle** and the **sine function is the y coordinates**. since the unit circle has radius one and is centered at the origin, both sine and cosine oscillate between positive and negative one.

What happens when the circle is not centered at the origin and does not have a radius of 1?

Graphs of sinusoidal Functions

The **sinusoidal function** family refers to either sine or cosine waves since they are the same except for a horizontal shift. This function family is also called the periodic function family because the function repeats after a given period of time.

Consider a Ferris wheel that spins evenly with a radius of 1 unit. It starts at (1, 0) or an angle of 0 radians and spins counterclockwise at a rate of one cycle per 2π minutes (so you can use time is equal to radians).

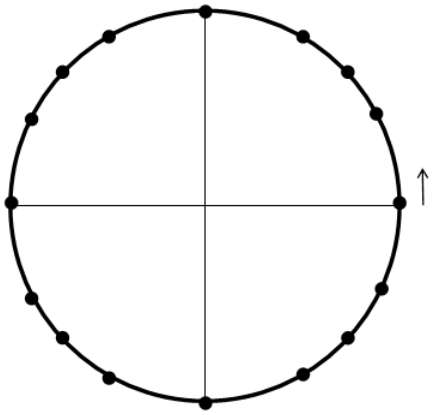


Figure **2.6.9.1**

The 16 points around the circle are chosen because they correspond to the key points of the unit circle. Their heights (y-values) and widths (x-values) are already known and can be filled in.

First consider the height at each of the points as you travel around half of the circle from the starting location. Keep track of your work in a table.

Angle (radians)	**Height (units)**
0	0
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2} \approx 0.707$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2} \approx 0.866$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2} \approx 0.866$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2} \approx 0.707$
$\frac{5\pi}{6}$	$\frac{1}{2}$
π	0

Notice the symmetry of the height around $\frac{\pi}{2}$ and see the rest of the table in the examples. Once the table is finished, you can plot these points on a regular coordinate plane where the x axis is the angle and the y axis is the height. This is the first part of the graph of the sine function.

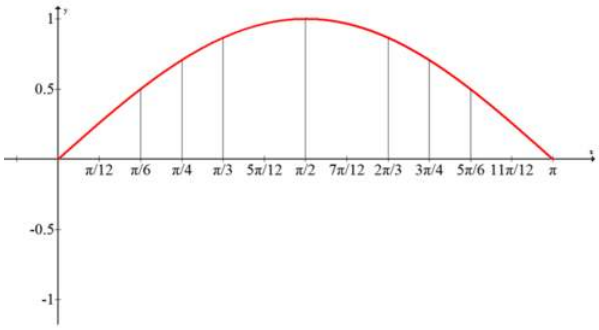


Figure **2.6.9.2**

To finish the graph of the sine function, finish the table for heights of the points in quadrants III and IV and draw an entire cycle (known as a period) of the sine function.

Angle (radians)	**Height (units)**
π	0
$\frac{7\pi}{6}$	$-\frac{1}{2}$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2} \approx -0.707$
$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2} \approx -0.866$
$\frac{3\pi}{2}$	-1
$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2} \approx -0.866$
$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2} \approx -0.707$
$\frac{11\pi}{6}$	$-\frac{1}{2}$
2π	0

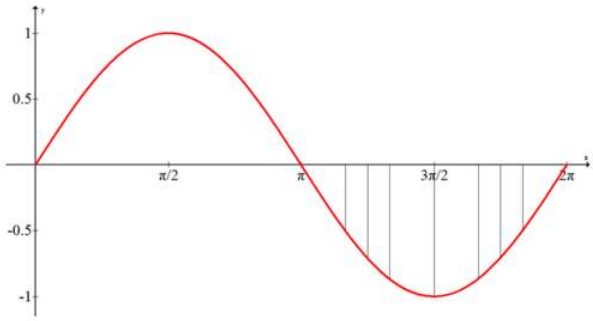
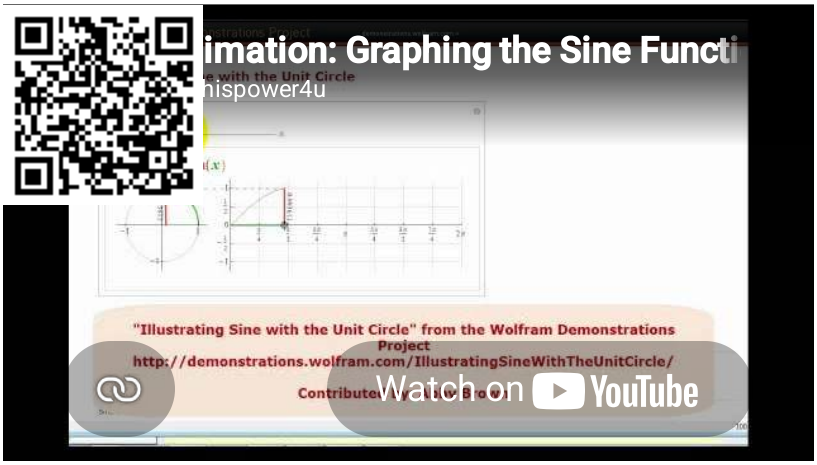


Figure **2.6.9.3**



Similar to sine, you can use your knowledge of the angles 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, 2π on the unit circle to get a complete cycle of the cosine graph. While the sine function uses the y-coordinates, the cosine function is the x-coordinates of the unit circle and measures width. By referring to a unit circle or your memory, you can fill out a much shorter table than before.

Angle (radians)	Width (units)
0	1
$\frac{\pi}{2}$	0
π	-1
$\frac{3\pi}{2}$	0
2π	1

First plot these five points and then connect them with a smooth curve. This will produce the cosine graph.

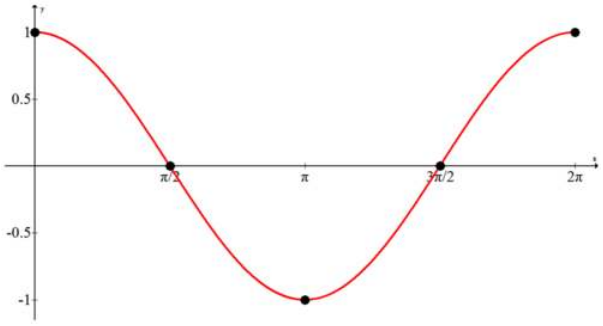
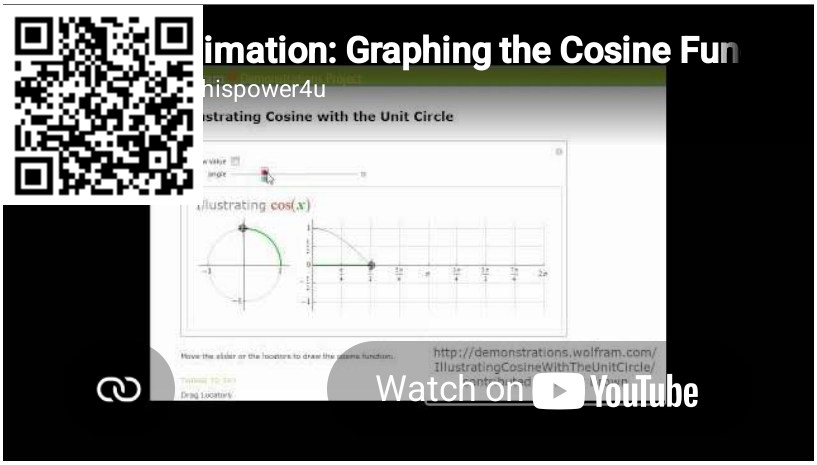


Figure **2.6.9.4**

Determining these five main points is the key to graphing sine or cosine graphs even when the graph is shifted or stretched.



Example 2.6.9.1

Earlier, you were asked what happens when the circle is not centered at the origin and does not have a radius of 1.

Solution

The unit circle produces the parent function sine and cosine graphs. When the unit circle is shifted up or down, made wider or narrower, or spun faster or slower in either direction, then the graphs of the sine and cosine functions will be transformed using basic function transformation rules.

Example 2.6.9.2

What happens on either side of the sine and cosine graphs? Can you explain why?

Solution

The graphs of the sine (blue) and cosine (red) functions repeat forever in both directions.

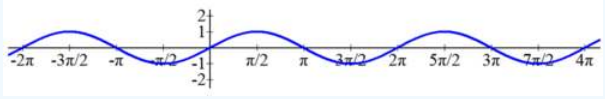


Figure 2.6.9.5

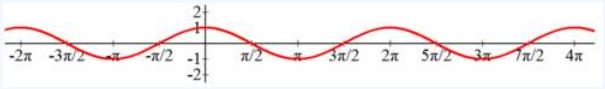


Figure 2.6.9.6

If you think about the example with the Ferris wheel, the ride will keep on spinning and has been spinning forever. This is why the same cycle of the graph repeats over and over.

Example 2.6.9.3

How are the sine and cosine graphs the same and how are they different?

Solution

The sine graph is the same as the cosine graph offset by $\frac{\pi}{2}$. Besides the shift, both curves are identical due to the perfect symmetry of circles.

Example 2.6.9.4

Where are two maximums and two minimums of the sine graph?

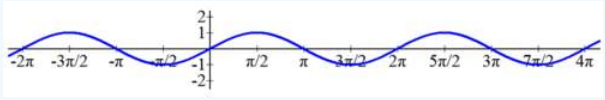


Figure **2.6.9.7**

Solution

One maximum of the sine graph occurs at $\left(\frac{\pi}{2}, 1\right)$. One minimum occurs at $\left(\frac{3\pi}{2}, -1\right)$. This is one cycle of the sine graph. Since it completes a cycle every 2π, when you add 2π to an x-coordinate you will be on the same point of the cycle giving you another maximum or minimum.

$\left(\frac{5\pi}{2}, 1\right)$ is another maximum. $\left(\frac{7\pi}{2}, -1\right)$ is another minimum.

Example 2.6.9.5

In the interval $[-2\pi, 4\pi)$ where does cosine have **zeroes**?

Solution

Observe where the cosine curve has x-coordinates equal to zero. Note that 4\pi is excluded from the interval. The values are $-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$.

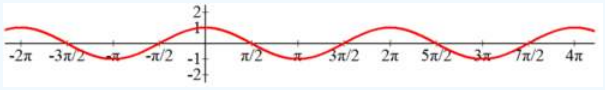


Figure **2.6.9.8**

Review

1. Sketch $p(x) = \sin x$ from memory.
2. Sketch $j(x) = \cos x$ from memory.
3. Where do the maximums of the cosine graph occur?
4. Where do the minimums of the cosine graph occur?
5. Find all the zeroes of the sine function on the interval $\left[-\pi, \frac{5\pi}{2}\right]$.
6. Find all the zeroes of the cosine function on the interval $\left(-\frac{\pi}{2}, \frac{7\pi}{2}\right]$.
7. Preview: using your knowledge of function **transformations** and the cosine graph, predict what the graph of $f(x) = 2\cos x$ will look like.
8. Preview: using your knowledge of function transformations and the cosine graph, predict what the graph $g(x) = \cos x + 2$ of will look like.
9. Preview: using your knowledge of function transformations and the cosine graph, predict what the graph of $h(x) = \cos(x - \pi)$ will look like.
10. Preview: using your knowledge of function transformations and the cosine graph, predict what the graph of $k(x) = -\cos x$ will look like.

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.2.

Vocabulary

Term	Definition
cosine	The cosine of an angle in a right triangle is a value found by dividing the length of the side adjacent the given angle by the length of the hypotenuse.
sine	The sine of an angle in a right triangle is a value found by dividing the length of the side opposite the given angle by the length of the hypotenuse.
sinusoidal function family	The sinusoidal function family refers to either sine or cosine waves since they are the same except for a horizontal shift. This function family is also called the periodic function family because the function repeats after a given period of time.
Transformations	Transformations are used to change the graph of a parent function into the graph of a more complex function.
unit circle	The unit circle is a circle of radius one, centered at the origin.
Zeroes	The zeroes of a function $f(x)$ are the values of x that cause $f(x)$ to be equal to zero.

Additional Resources

Video: Transformation of Trigonometric Graphs: Amplitude, Period, and Phase Shift

Real World: Banding Together

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2.7.1: Graphing Tangent, Cotangent, Secant and Cosecant

2.7.2: Tangent Graphs

2.7.3: Tangent and Cotangent Graphs

2.7.4: Sine and Cosecant Graphs

2.7.5: Cosine and Secant Graphs

2.7.6: Graph Inverse Trigonometric Functions

2.7.1: Graphing Tangent, Cotangent, Secant and Cosecant

Tangent, cotangent, cosecant, and secant graphs. Concept Map

If you already know the relationship between the equation and graph of sine and cosine functions then the other four functions can be found by identifying zeroes, **asymptotes** and key points. Are the four new functions **transformations** of the sine and cosine functions?

Graphing Other Trigonometric Functions

Secant and **Cosecant**

Since secant is the inverse of cosine the graphs are very closely related.

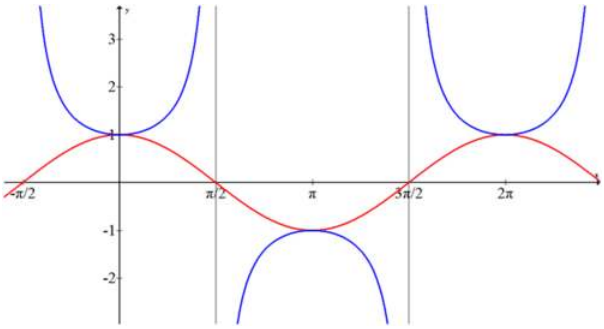
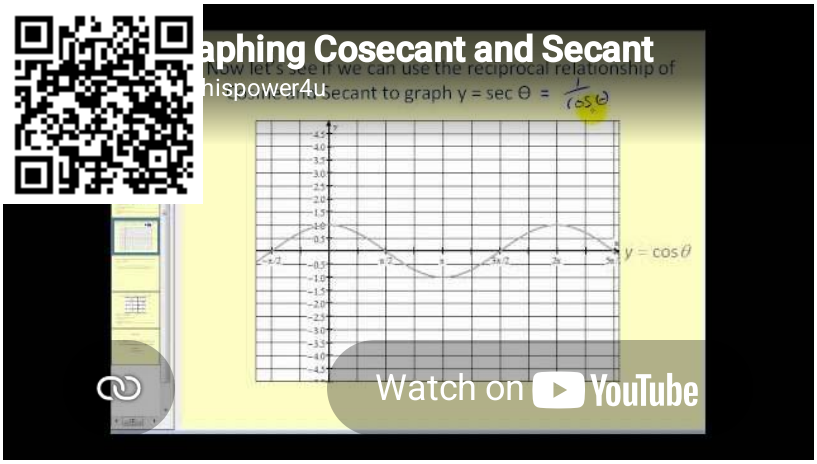


Figure **2.7.1.1**

Notice wherever cosine is zero, secant has a **vertical asymptote** and where $\cos x = 1$ then $\sec x = 1$ as well. These two logical pieces allow you to graph any secant function of the form:

$f(x) = \pm a \cdot \sec(b(x + c)) + d$

The method is to graph it as you would a cosine and then insert asymptotes and the secant curves so they touch the cosine curve at its maximum and minimum values. This technique is identical to graphing cosecant graphs. Simply use the sine graph to find the location and asymptotes.



Tangent and **Cotangent**

The tangent and cotangent graphs are more difficult because they are a ratio of the sine and cosine functions.

- $\tan x = \dfrac{\sin x}{\cos x}$
- $\cot x = \dfrac{\cos x}{\sin x}$

The way to think through the graph of $f(x) = \tan x$ is to first determine its asymptotes. The asymptotes occur when the denominator, cosine, is zero. This happens at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}\ldots$ The next thing to plot is the zeros which occur when the numerator, sine,

is zero. This happens at $0, \pm\pi, \pm, 2\pi \ldots$ From the unit circle and basic right triangle trigonometry, you already know some values of $\tan x$:

- $\tan \frac{\pi}{4} = 1$
- $\tan \left(-\frac{\pi}{4}\right) = -1$

By plotting all this information, you get a very good sense as to what the graph of tangent looks like and you can fill in the rest.

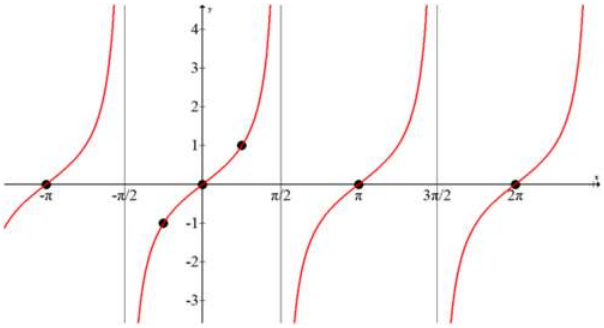


Figure 2.7.1.1

Notice that the period of tangent is π not 2π, because it has a shorter cycle.

The graph of cotangent can be found using identical logic as tangent. You know $\cot x = \frac{1}{\tan x}$. This means that the graph of cotangent will have zeros wherever tangent has asymptotes and asymptotes wherever tangent has zeroes. You also know that where tangent is 1, cotangent is also 1. Thus the graph of cotangent is:

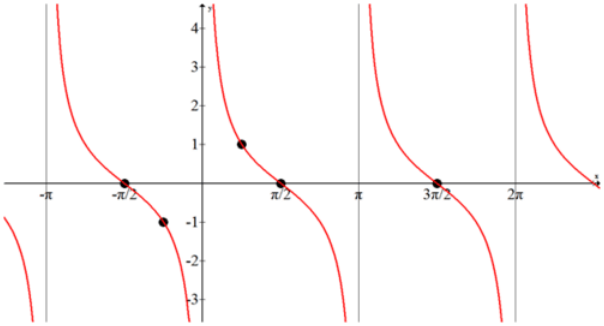
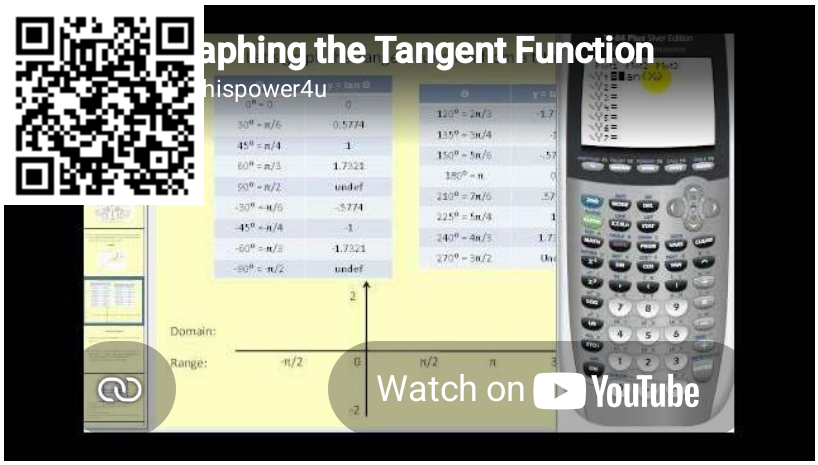
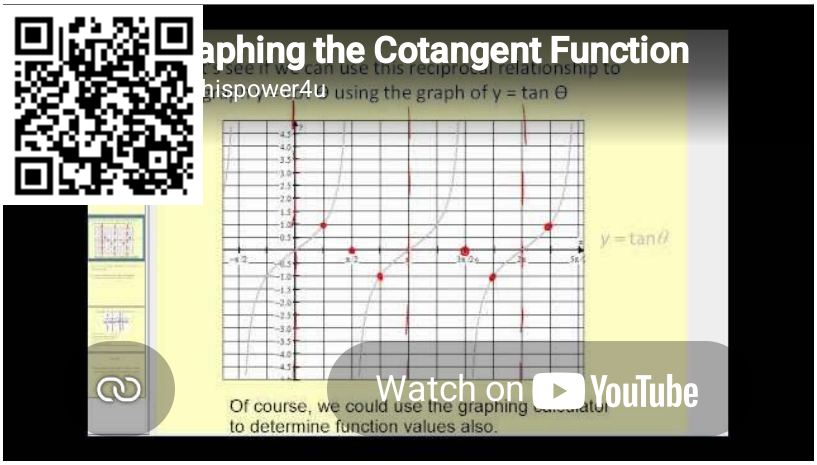


Figure 2.7.1.2





Example 2.7.1.1

Earlier, you were asked if the four new functions are transformations of sine and cosine.

Solution

The four new functions are not purely transformations of the sine and cosine functions. However, secant and cosecant are transformations of each other as are tangent and cotangent.

Example 2.7.1.2

Graph the function $f(x) = -2 \cdot \csc(\pi(x-1)) + 1$.

Solution

Graph the function as if it were a sine function. Then insert asymptotes wherever the sine function crosses the sinusoidal axis. Lastly add in the cosecant curves.

The amplitude is 2. The shape is negative sine. The function is shifted up one unit and to the right one unit.

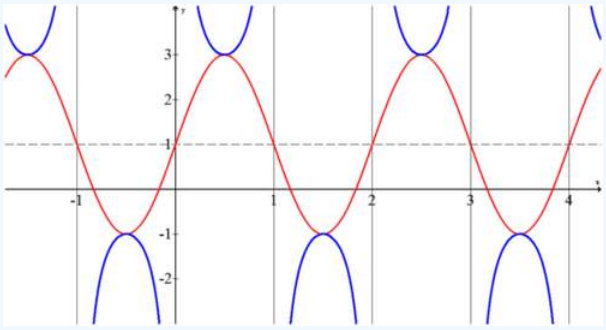


Figure 2.7.1.3

Note that only the blue portion of the graph represents the given function.

Example 2.7.1.3

How do you write a tangent function as a cotangent function?

Solution

There are two main ways to go between a tangent function and a cotangent function. The first method was discussed in Example A: $f(x) = \tan x = \frac{1}{\cot x}$.

The second approach involves two transformations. Start by reflecting across the x or the y axis. Notice that this produces an identical result. Next shift the function to the right or left by $\frac{\pi}{2}$. Again this produces an identical result.

$f(x) = \tan x = -\cot\left(x - \frac{\pi}{2}\right)$.

Example 2.7.1.4

Find the equation of the function in the following graph.

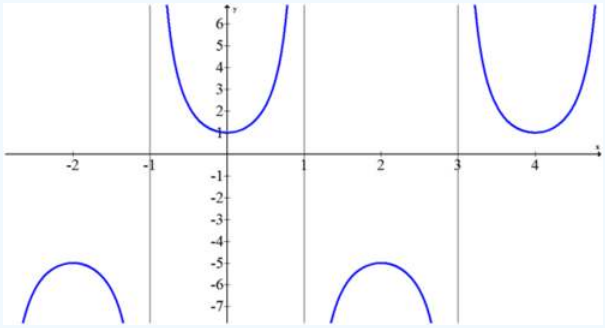


Figure 2.7.1.4

Solution

If you connect the relative maximums and minimums of the function, it produces a shifted cosine curve that is easier to work with.

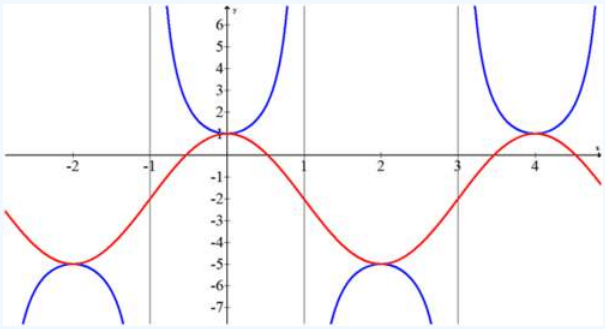


Figure 2.7.1.5

The amplitude is 3. The vertical shift is 2 down. The period is 4 which implies that $b = \frac{\pi}{2}$. The shape is positive cosine and if you choose to start at x=0 there is no phase shift.

$f(x) = 3 \cdot \csc\left(\frac{\pi}{2}x\right) - 2$

Example 2.7.1.5

Where are the asymptotes for tangent and why do they occur?

Solution

Since $\tan x = \frac{\sin x}{\cos x}$ the asymptotes occur whenever $\cos x = 0$ which is $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

Review

1. What function can you use to help you make a sketch of $f(x) = \sec x$? Why?

2. What function can you use to help you make a sketch of $g(x) = \csc x$? Why?

Make a sketch of each of the following from memory.

3. $f(x) = \sec x$

4. $g(x) = \csc x$
5. $h(x) = \tan x$
6. $k(x) = \cot x$

Graph each of the following.

7. $f(x) = 2\csc(x) + 1$
8. $g(x) = 2\csc(\frac{\pi}{2}x) + 1$
9. $h(x) = 2\csc\left(\frac{\pi}{2}(x-3)\right) + 1$
10. $j(x) = \cot\left(\frac{\pi}{2}x\right) + 3$
11. $k(x) = -\sec\left(\frac{\pi}{3}(x+1)\right) - 4$
12. $m(x) = -\tan(x) + 1$
13. $p(x) = -2\tan\left(x - \frac{\pi}{2}\right) + 1$
14. Find two ways to write $\sec x$ in terms of other trigonometric functions.
15. Find two ways to write $\csc x$ in terms of other trigonometric functions.

Review (Answers)

To see the Review answers, open this PDF file and look for \sec tion 5.7.

Vocabulary

Term	Definition
Asymptotes	An asymptote is a line on the graph of a function representing a value toward which the function may approach, but does not reach (with certain exceptions).
Cosecant	The cosecant of an angle in a right triangle is a relationship found by dividing the length of the hypotenuse by the length of the side opposite to the given angle. This is the reciprocal of the sine function.
Cotangent	The cotangent of an angle in a right triangle is a relationship found by dividing the length of the side adjacent to the given angle by the length of the side opposite to the given angle. This is the reciprocal of the tangent function.
Secant	The secant of an angle in a right triangle is the value found by dividing length of the hypotenuse by the length of the side adjacent the given angle. The secant ratio is the reciprocal of the cosine ratio.
Transformations	Transformations are used to change the graph of a parent function into the graph of a more complex function.
Vertical Asymptote	A vertical asymptote is a vertical line marking a specific value toward which the graph of a function may approach, but will never reach.

Additional Resources

Video: Graphs of Cosecant and Secant Functions - Overview

Practice: Graphing Tangent, Cotangent, Secant and Cosecant

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2.7.2: Tangent Graphs

Adjust the length of the curve, or the distance before the y values repeat, from 2π.

Your mission, should you choose to accept it, as Agent Trigonometry is to find the period and the zeros of the function $y = \frac{1}{2}\tan 4x$.

Graph of a Tangent Function

The graph of the **tangent function** is very different from the sine and cosine functions. First, recall that the tangent ratio is $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$. In radians, the coordinate for the tangent function would be $(\theta, \tan\theta)$

x	θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{3\pi}{6}$	π
y	$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	und.	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0

After π, the y-values repeat, making the tangent function periodic with a period of π.

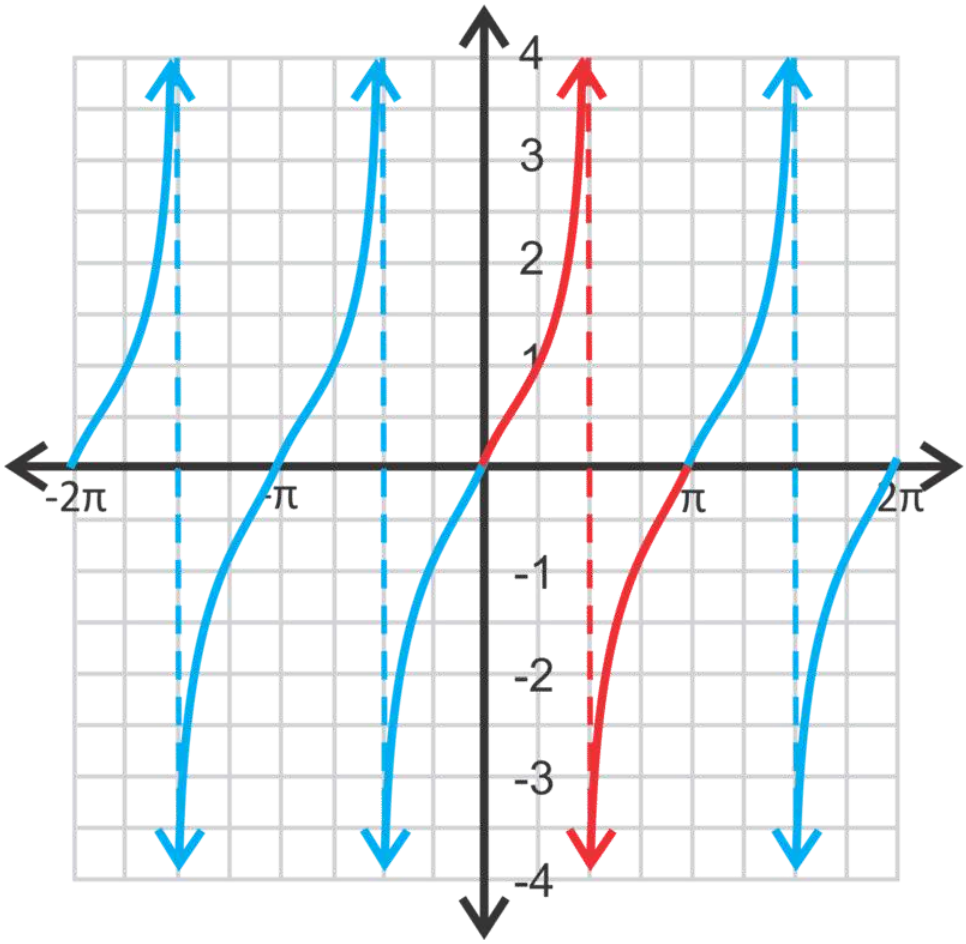


Figure **2.7.2.1**

The red portion of the graph represents the coordinates in the table above. Repeating this portion, we get the entire tangent graph. Notice that there are vertical asymptotes at $x = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}$ and $\frac{3\pi}{2}$. If we were to extend the graph out in either direction, there would continue to be vertical asymptotes at the odd multiples of \pi 2. Therefore, the domain is all real numbers, $x \neq n\pi \pm \frac{\pi}{2}$, where n is an integer. The range would be all real numbers. Just like with sine and cosine functions, you can change the amplitude, phase shift, and vertical shift.

The standard form of the equation is $y = a\tan b(x - h) + k$ where a, b, h, and k are the same as they are for the other trigonometric functions. For simplicity, we will not address phase shifts (k) in this concept.

Let's graph $y = 3\tan x + 1$ from $[-2\pi, 2\pi]$ and state the domain and range.

First, the amplitude is 3, which means each y-value will be tripled. Then, we will shift the function up one unit.

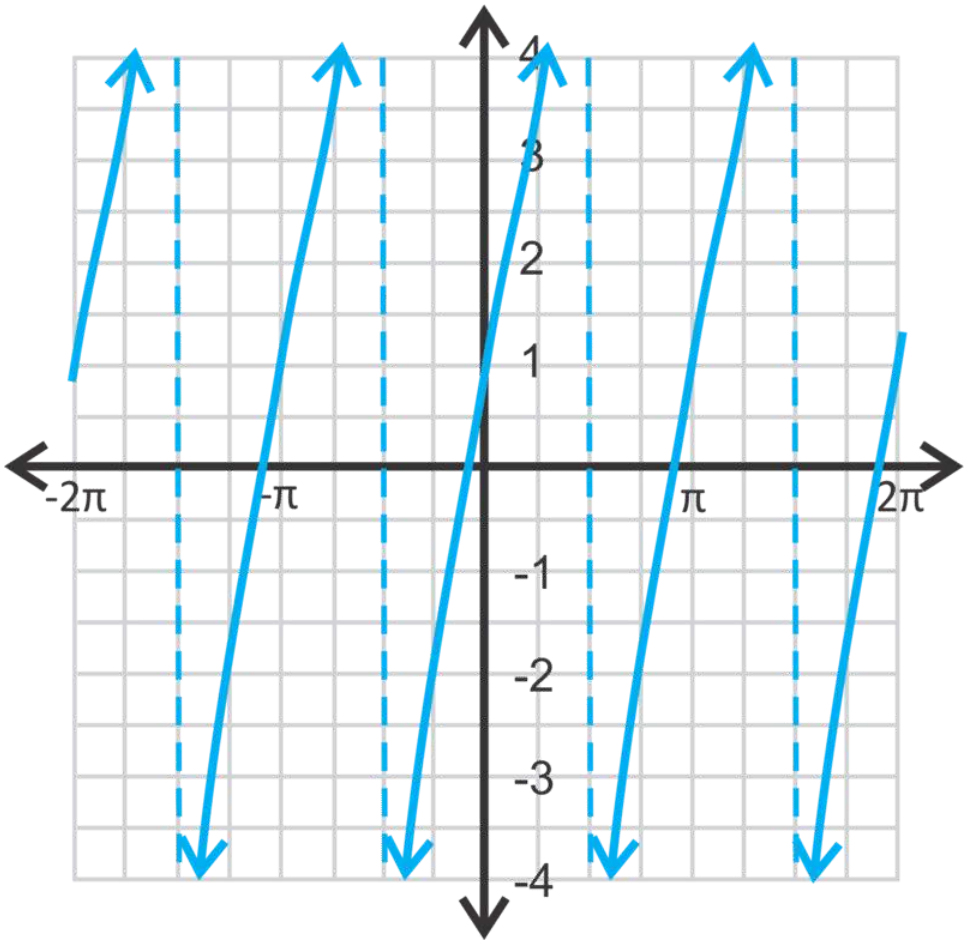


Figure 2.7.2.2

Notice that the vertical asymptotes did not change. The **period** of this function is still π. Therefore, if we were to change the period of a tangent function, we would use a different formula than what we used for sine and cosine. To change the period of a tangent function, use the formula $\frac{\pi}{|b|}$.

The domain will be all real numbers, except where the asymptotes occur. Therefore, the domain of this function will be $x \in R$, $x \notin n\pi \pm \frac{\pi}{2}$. The range is all real numbers.

Now, let's graph $y = -\tan 2\pi$ from $[0, 2\pi]$, state the domain and range, and find all zeros within this domain.

The period of this tangent function will be $\frac{\pi}{2}$ and the curves will be reflected over the x-axis.

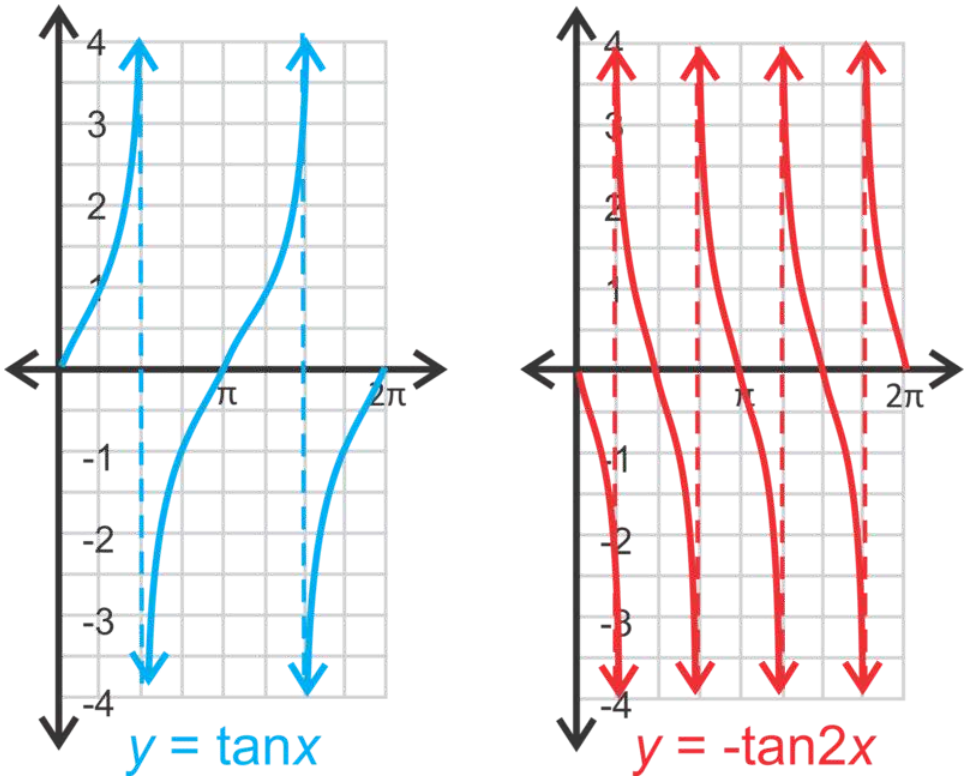


Figure **2.7.2.3**

The domain is all real numbers, $x \neq \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{\pi}{4} \pm \frac{\pi}{2}n$ where n is any integer. The range is all real numbers. To find the zeros, set $y = 0$.

$$0 = -\tan 2x$$
$$0 = \tan 2x$$
$$2x = \tan^{-1} 0 = 0, \pi, 2\pi, 3\pi, 4\pi$$
$$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$$

Finally, let's graph $y = \frac{1}{4}\tan\frac{1}{4}x$ from $[0, 4\pi]$ and state the domain and range.

This function has a period of $\frac{\pi}{\frac{1}{4}} = 4\pi$. The domain is all real numbers, except $2\pi,\ 6\pi,\ 10\pi,\ 2\pi \pm 4\pi n$, where n is any integer. The range is all real numbers.

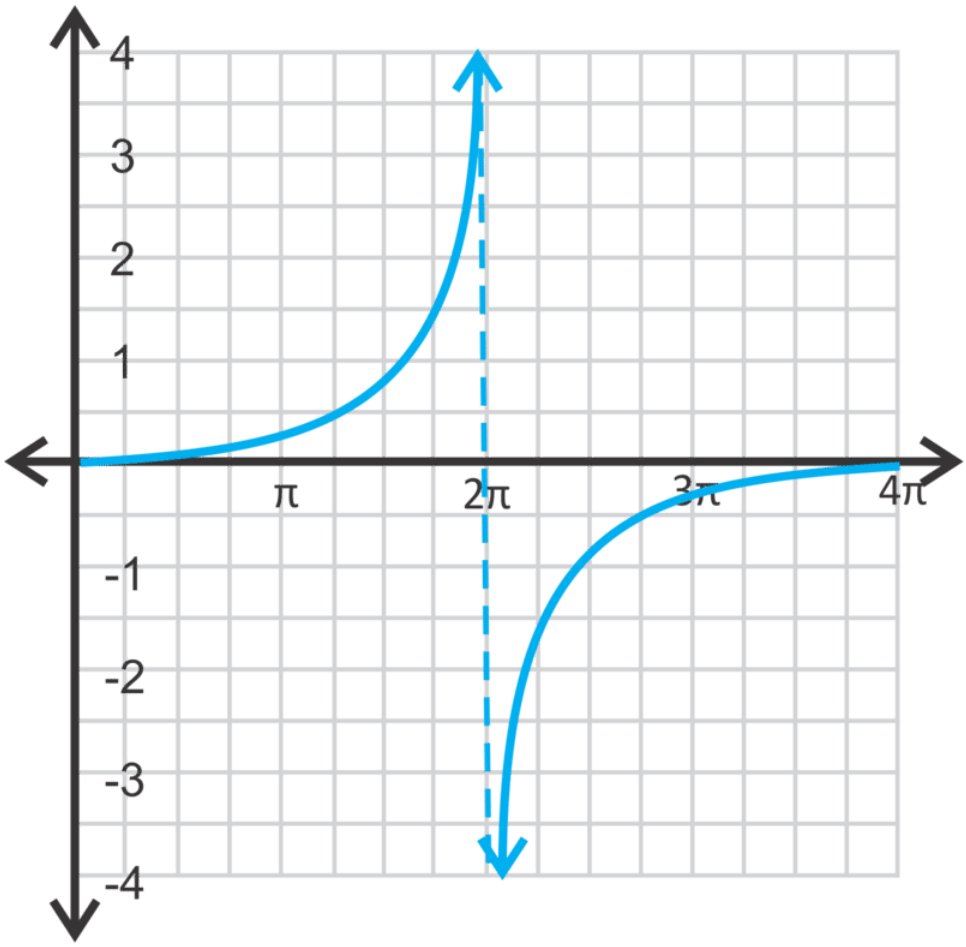


Figure **2.7.2.4**

Example 2.7.2.1

Earlier, you were asked to find the period and zeros of the function $y = \frac{1}{2}\tan 4x$.

Solution

The period is $\frac{\pi}{4}$.

The zeros are where y is zero.

$$0 = \frac{1}{2}\tan 4x$$
$$0 = \tan 4x$$
$$4x = \tan^{-1} 0 = 0, \pi, 2\pi, 3\pi$$
$$x = \frac{1}{4}(0, \pi, 2\pi, 3\pi)$$
$$x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3}{4}\pi$$

Example 2.7.2.2

Find the period of the function $y = -4\tan\frac{3}{2}x$.

Solution

The period is $\frac{\pi}{\frac{3}{2}} = \pi \cdot \frac{2}{3} = \frac{2\pi}{3}$.

Example 2.7.2.3

Find the zeros of the function from Example 2, from $[0, 2\pi]$.

Solution

The zeros are where y is zero.

$$0 = -4\tan\frac{3}{2}x$$
$$0 = \tan\frac{3}{2}x$$
$$\frac{3}{2}x = \tan^{-1}0 = 0, \pi, 2\pi, 3\pi$$
$$x = \frac{2}{3}(0, \pi, 2\pi, 3\pi)$$
$$x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$$

Example 2.7.2.4

Find the equation of the tangent function with an amplitude of 8 and a period of 6π.

Solution

The general equation is $y = a\tan bx$. We know that $a = 8$. Let's use the period to solve for the frequency, or b.

$$\frac{\pi}{b} = 6\pi$$
$$b = \frac{\pi}{6\pi} = \frac{1}{6}$$

The equation is $y = 8\tan\frac{1}{6}x$.

Review

Graph the following tangent functions over $[0, 4\pi]$. Determine the period, domain, and range.

1. $y = 2\tan x$
2. $y = -\frac{1}{3}\tan x$
3. $y = -\tan 3x$
4. $y = 4\tan 2x$
5. $y = \frac{1}{2}\tan 4x$
6. $y = -\tan\frac{1}{2}x$
7. $y = 4 + \tan x$
8. $y = -3 + \tan 3x$
9. $y = 1 + \frac{2}{3}\tan\frac{1}{2}x$
10. Find the zeros of the function from #1.
11. Find the zeros of the function from #3.
12. Find the zeros of the function from #5.

Write the equation of the tangent function, in the form $y = a\tan bx$, with the given amplitude and period.

13. Amplitude: 3 Period: $\frac{3\pi}{2}$
14. Amplitude: $\frac{1}{4}$ Period: 2π
15. Amplitude: -2.5 Period: 8
16. **Challenge** Graph $y = 2\tan\frac{1}{3}\left(x + \frac{\pi}{4}\right) - 1$ over $[0, 6\pi]$. Determine the domain and period.

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 14.5.

Additional Resource

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2.7.3: Tangent and Cotangent Graphs

Involve asymptotes spaced π radians apart.

What if your instructor gave you a set of graphs like these:

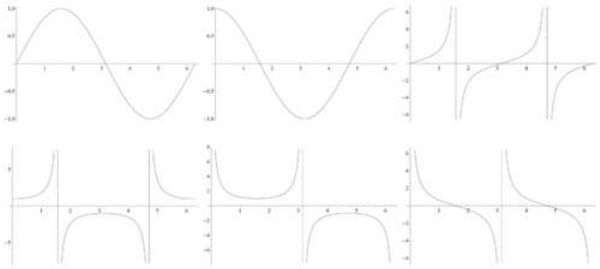


Figure **2.7.3.1**

and asked you to identify which were the graphs of the tangent and cotangent functions?

Tangent and cotangent Graphs

The name of the tangent function comes from the tangent line of a circle. This is a line that is perpendicular to the radius at a point on the circle so that the line touches the circle at exactly one point.

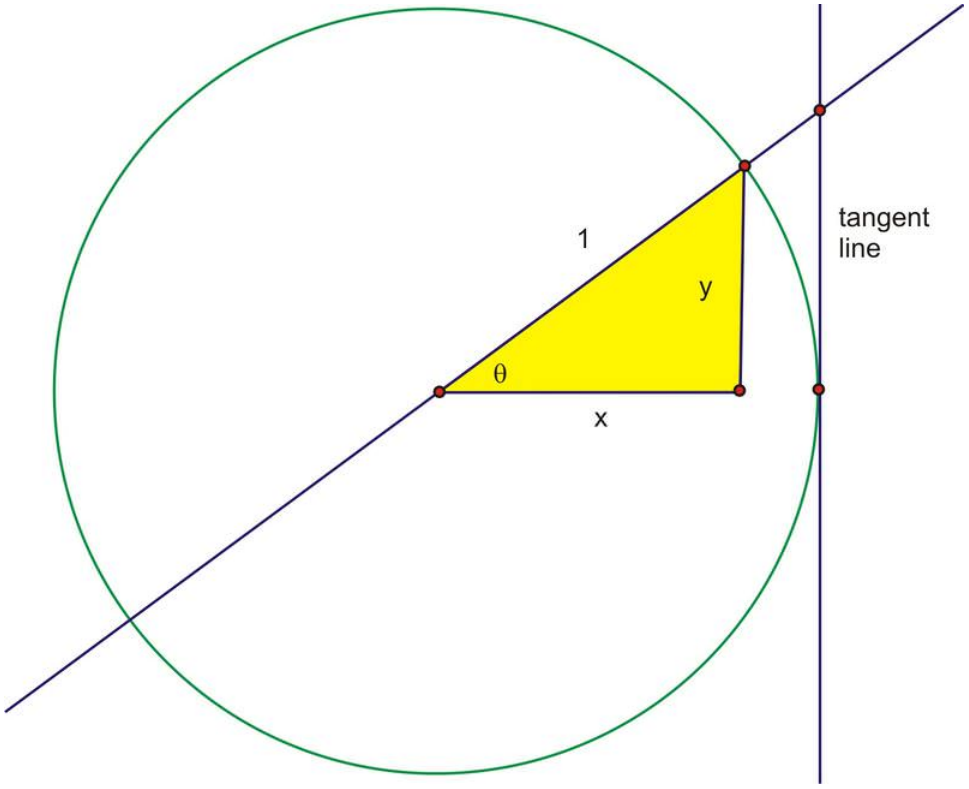


Figure **2.7.3.2**

If we extend angle θ through the unit circle so that it intersects with the tangent line, $\tan\theta$ will be equal to the length of the red segment. Below, this segment is labeled the "tangent segment".

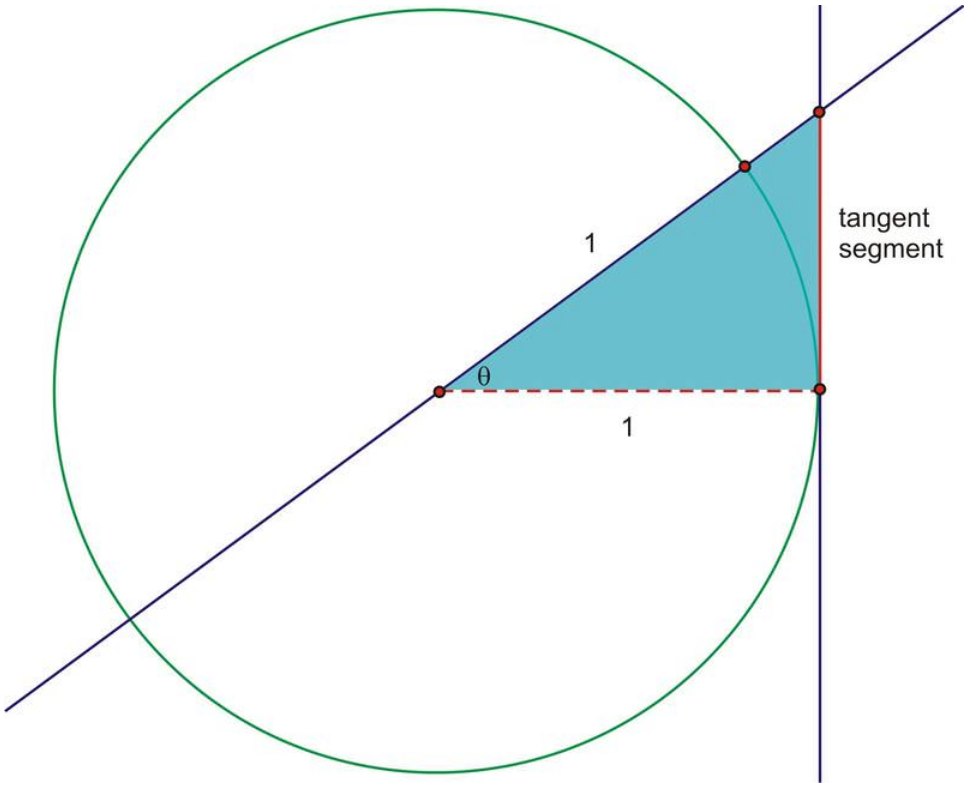


Figure 2.7.3.3

Why? The dashed segment is 1 because it is the radius of the unit circle. Recall that in general, $\tan\theta = \frac{y}{x}$. So here, $\tan\theta = \frac{\text{tangent segment}}{1} = \text{tangent segment}$

As the value of θ increases, the value of $\tan\theta$ changes. As we rotate through the first quadrant, the value of $\tan\theta$ will increase very slowly at first and then more rapidly.

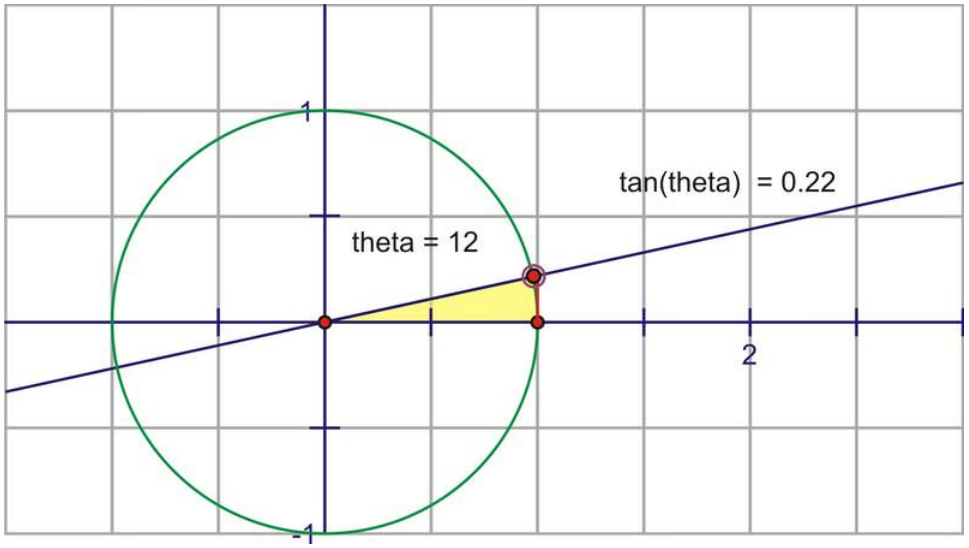


Figure 2.7.3.4

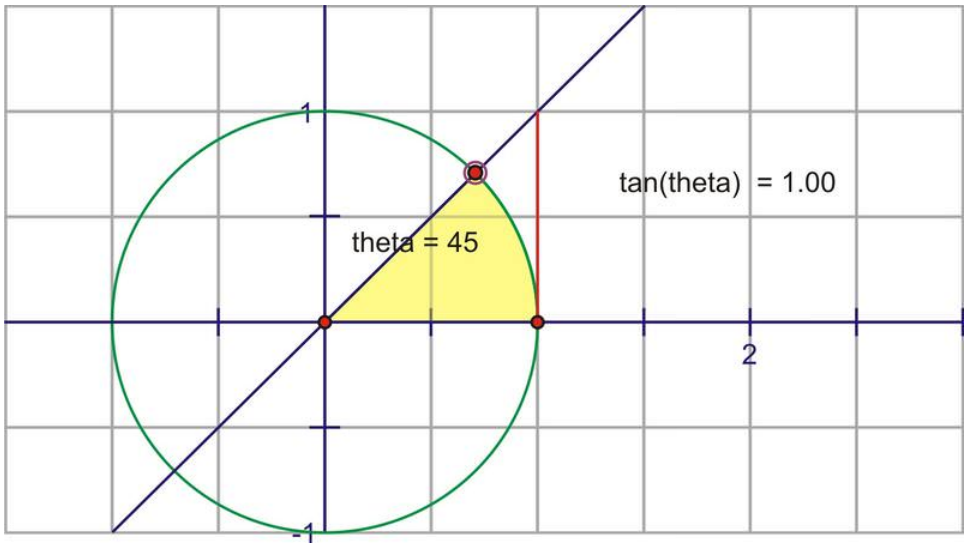


Figure **2.7.3.5**

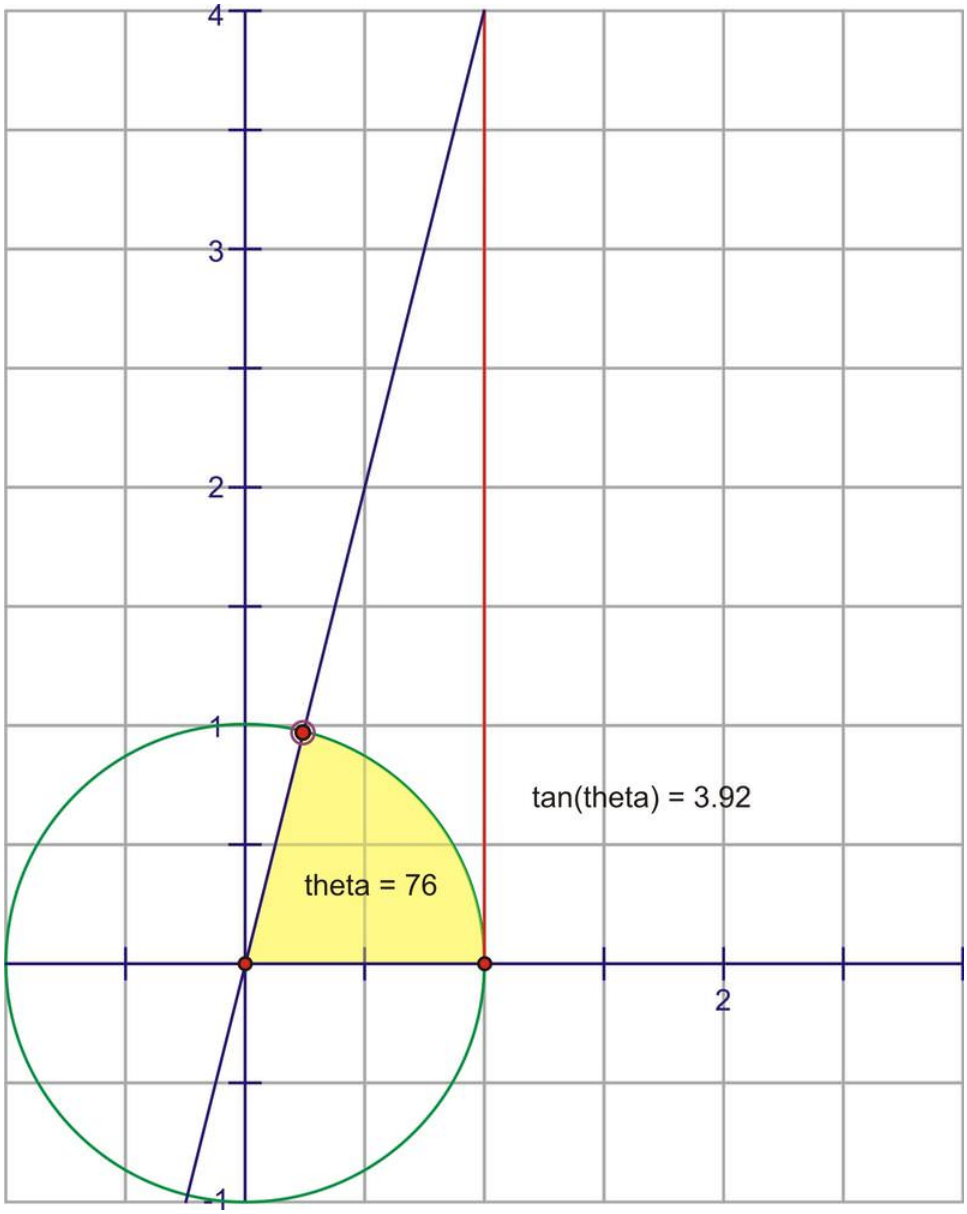


Figure **2.7.3.6**

As we get ***very*** close to the y−axis the segment gets infinitely large, until when the angle really hits 90°, at which point the extension of the angle and the tangent line will actually be parallel and therefore never intersect.

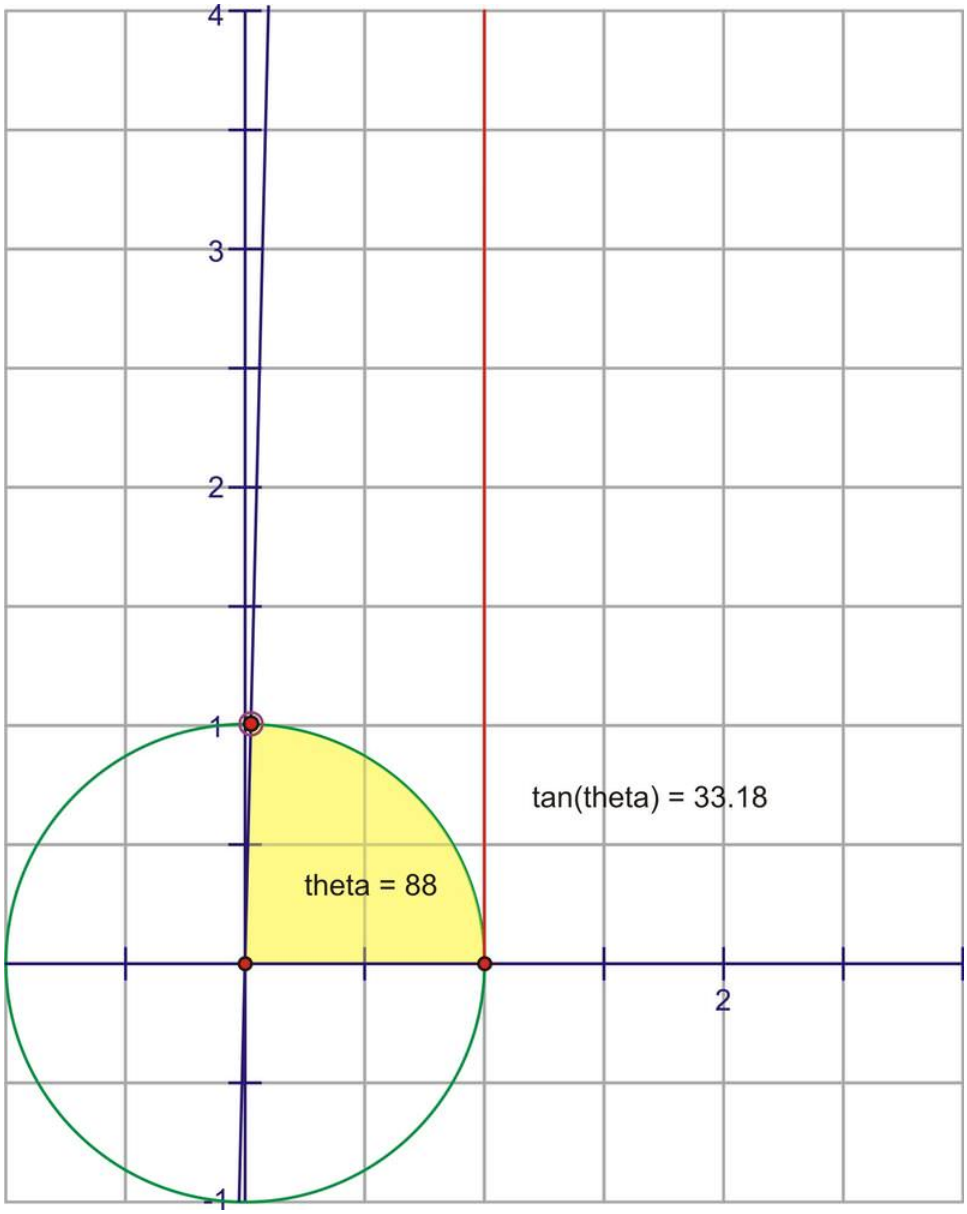


Figure **2.7.3.7**

This means there is no finite length of the tangent segment, or the tangent segment is *infinitely large*.

Let's translate this portion of the graph onto the coordinate plane. Plot $(\theta, \tan\theta)$ as (x, y).

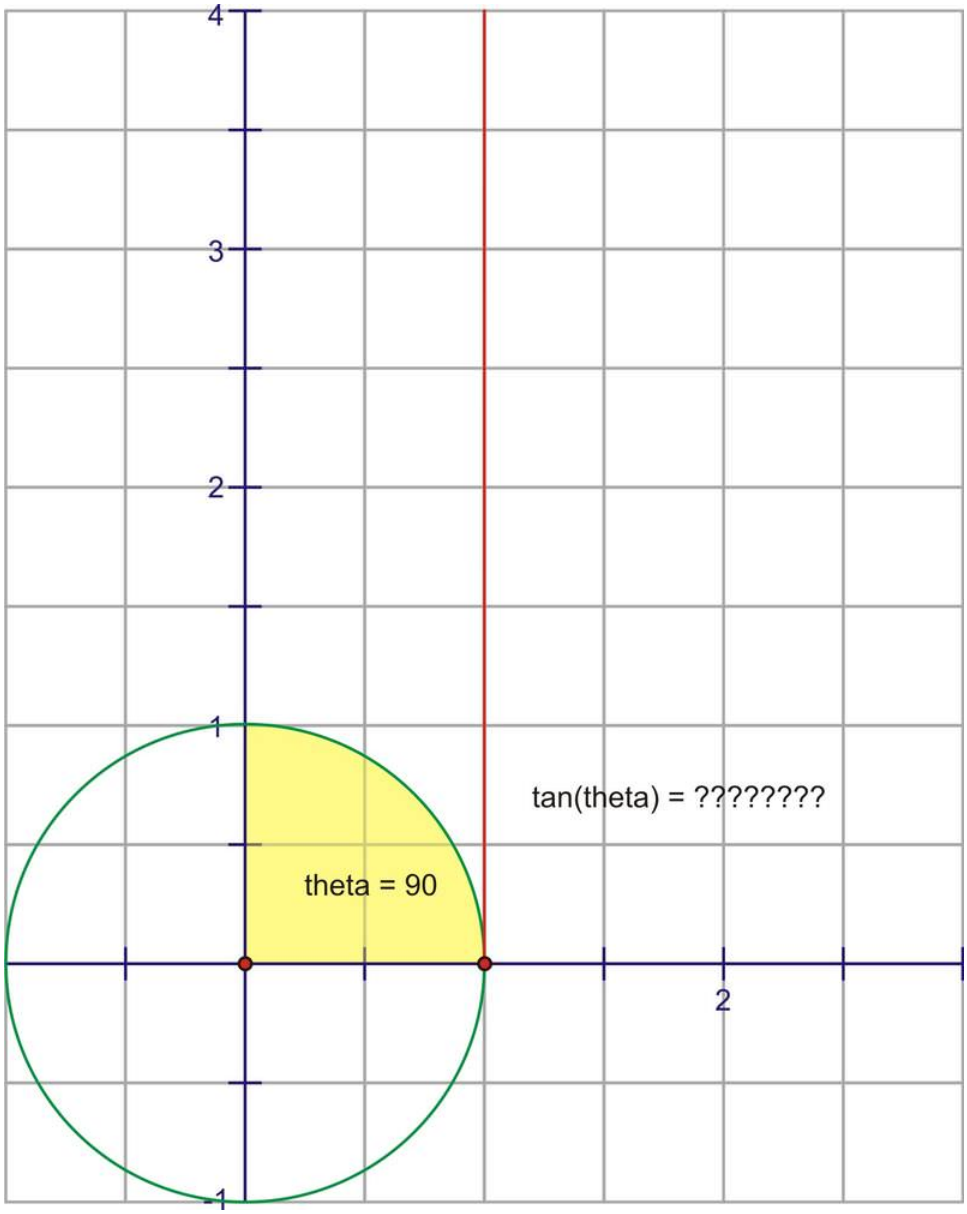


Figure 2.7.3.8

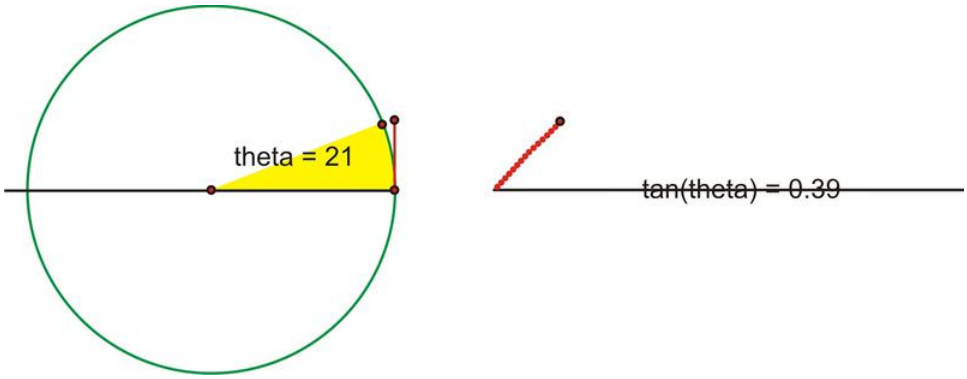


Figure 2.7.3.9

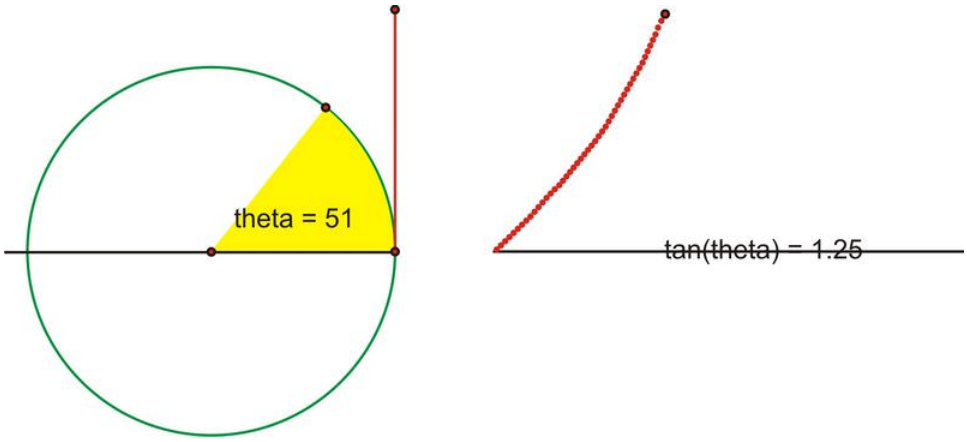


Figure 2.7.3.10

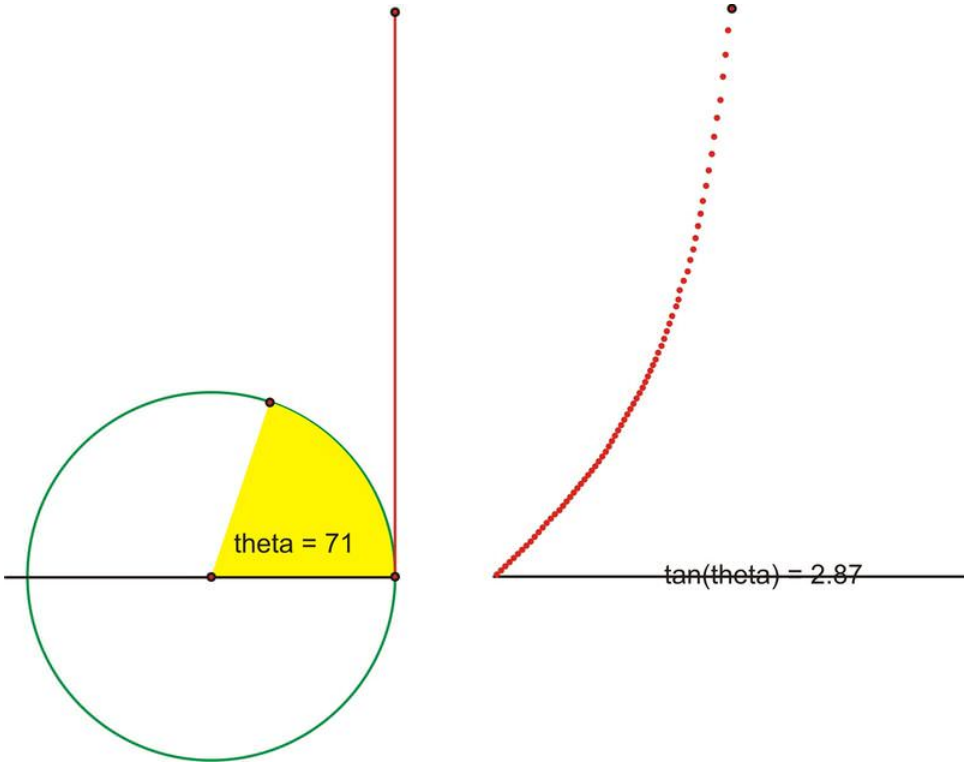


Figure 2.7.3.11

In fact as we get infinitely close to $90°$, the tangent value increases without bound, until when we actually reach $90°$, at which point the tangent is undefined. Recall there are some angles ($90°$ and $270°$, for example) for which the tangent is not defined. Therefore, at these points, there are going to be vertical asymptotes.

Rotating past $90°$, the intersection of the extension of the angle and the tangent line is actually below the x−axis. This fits nicely with what we know about the tangent for a 2nd quadrant angle being negative. At first, it will have very large negative values, but as the angle rotates, the segment gets shorter, reaches 0, then crosses back into the positive numbers as the angle enters the 3rd quadrant. The segment will again get infinitely large as it approaches $270°$. After being undefined at $270°$, the angle crosses into the 4th quadrant and once again changes from being infinitely negative, to approaching zero as we complete a full rotation.

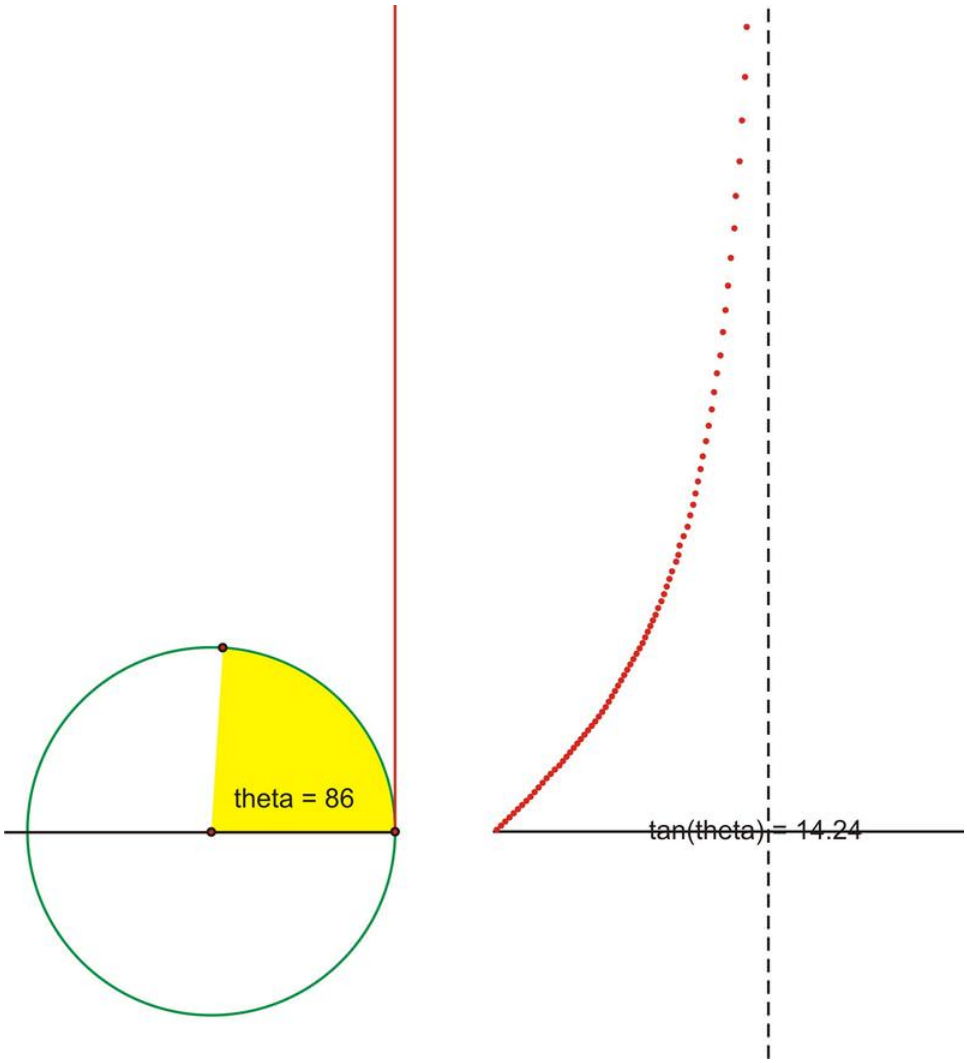


Figure **2.7.3.12**

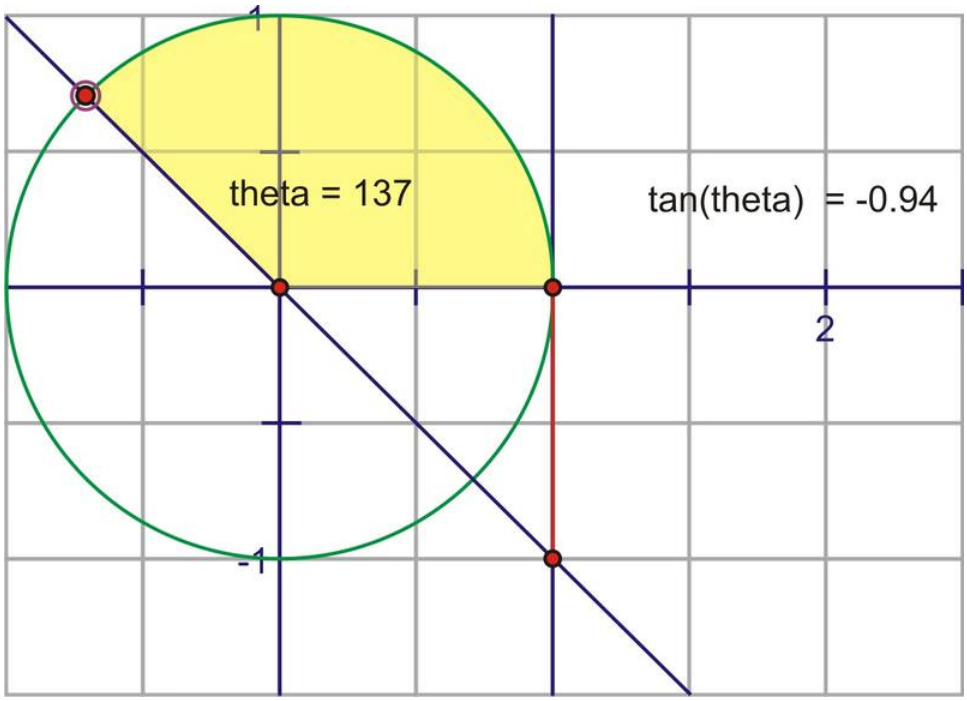


Figure **2.7.3.13**

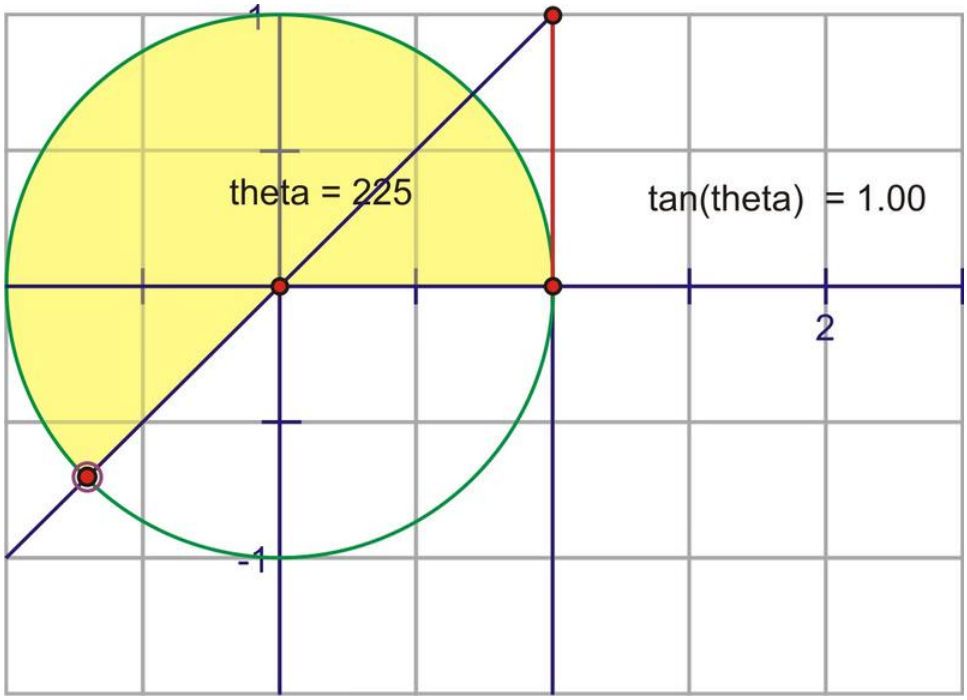


Figure **2.7.3.14**

The graph $y = \tan x$ over several rotations would look like this:

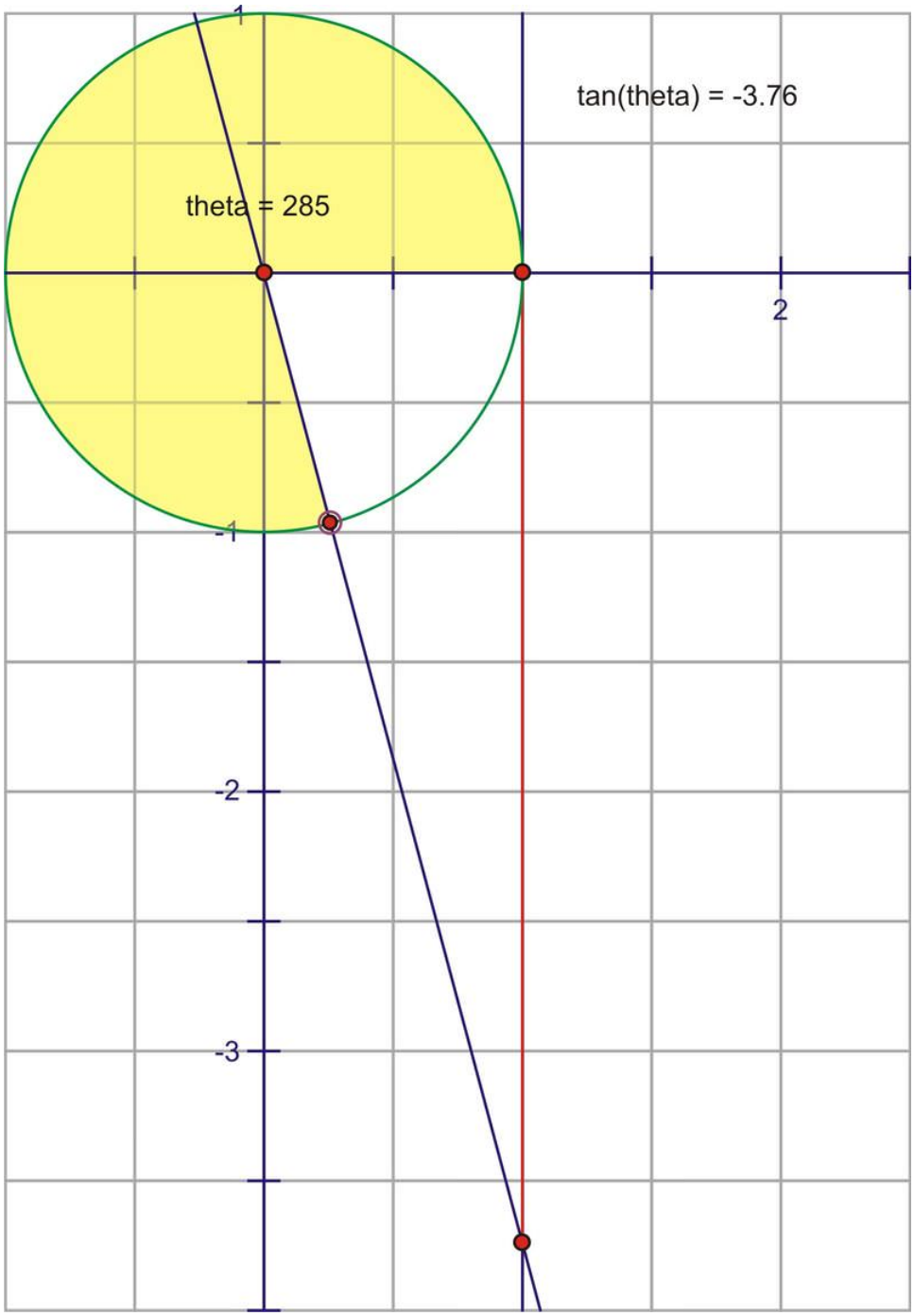


Figure **2.7.3.15**

Notice the x−axis is measured in radians. Our asymptotes occur every π radians, starting at $\frac{\pi}{2}$. The period of the graph is therefore π radians. The domain is all reals except for the asymptotes at $\frac{\pi}{2}$, $\frac{3\pi}{2}$, $-\frac{\pi}{2}$, etc. and the range is all real numbers.

cotangent is the reciprocal of tangent, $\frac{x}{y}$, so it would make sense that where ever the tangent had an asymptote, now the cotangent will be zero. The opposite of this is also true. When the tangent is zero, now the cotangent will have an asymptote. The shape of the curve is generally the same, so the graph looks like this:

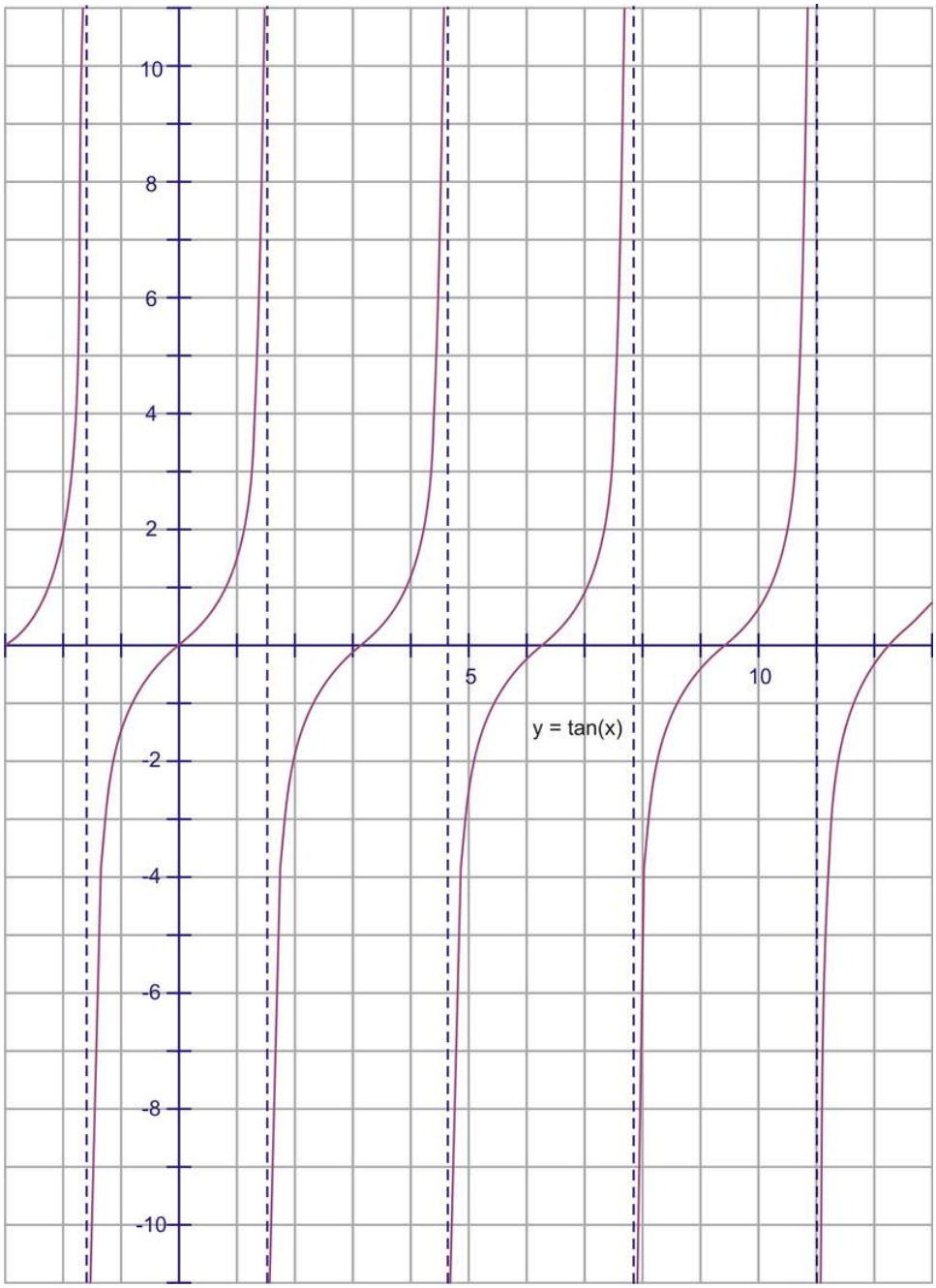


Figure **2.7.3.16**

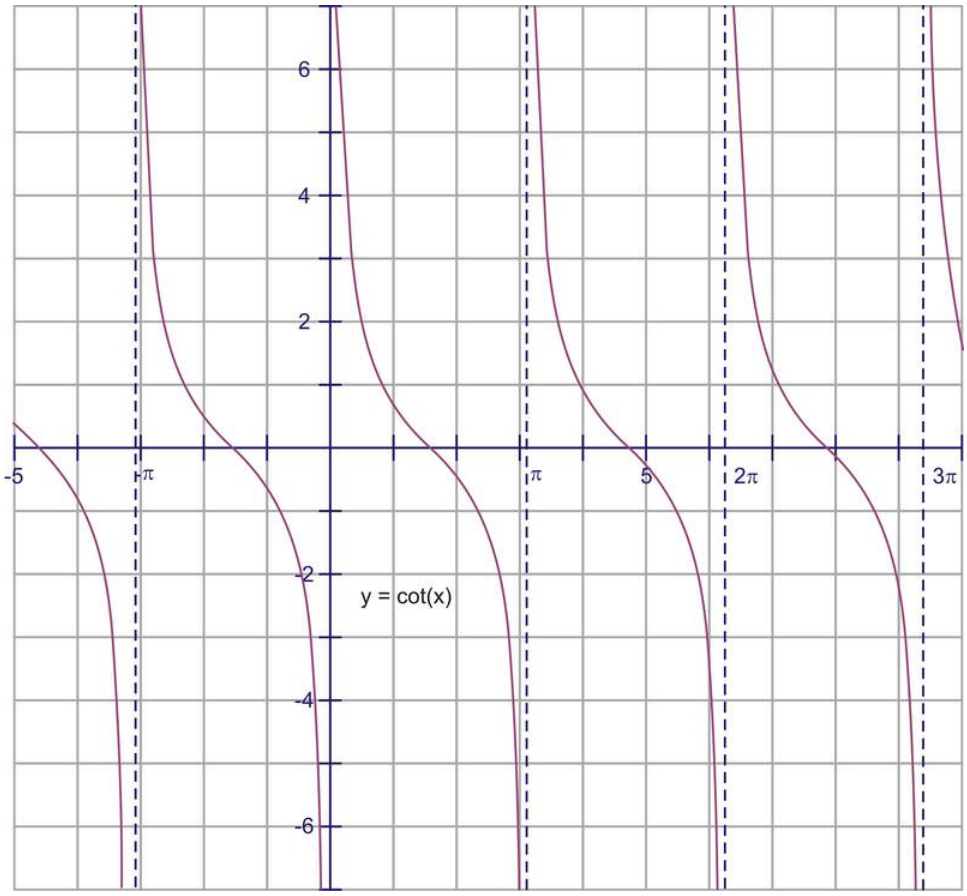
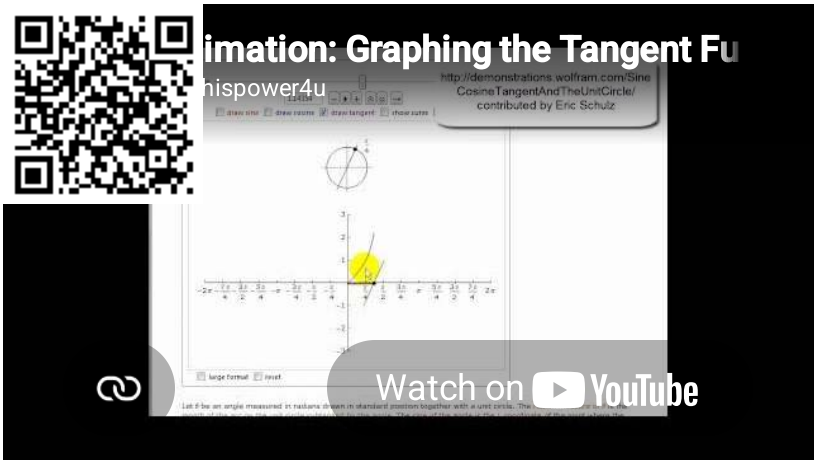


Figure **2.7.3.17**

When you overlap the two functions, notice that the graphs consistently intersect at 1 and -1. These are the angles that have 45° as reference angles, which always have tangents and cotangents equal to 1 or -1. It makes sense that 1 and -1 are the only values for which a function and it's reciprocal are the same. Keep this in mind as we look at cosecant and secant compared to their reciprocals of sine and cosine.

The cotangent function has a domain of all real angles except multiples of $\pi ... -2\pi, -\pi, 0, \pi, 2\pi ...$ The range is all real numbers.



Sketching Graphs

1. Sketch the graph of $g(x) = -2 + \cot \frac{1}{3}x$ over the interval $[0, 6\pi]$

Starting with $y = \cot x$, $g(x)$ would be shifted down two and frequency is $\frac{1}{3}$, which means the period would be 3π, instead of 9π. So, in our interval of $[0, 6\pi]$ there would be two complete repetitions. The red graph is $y = \cot x$.

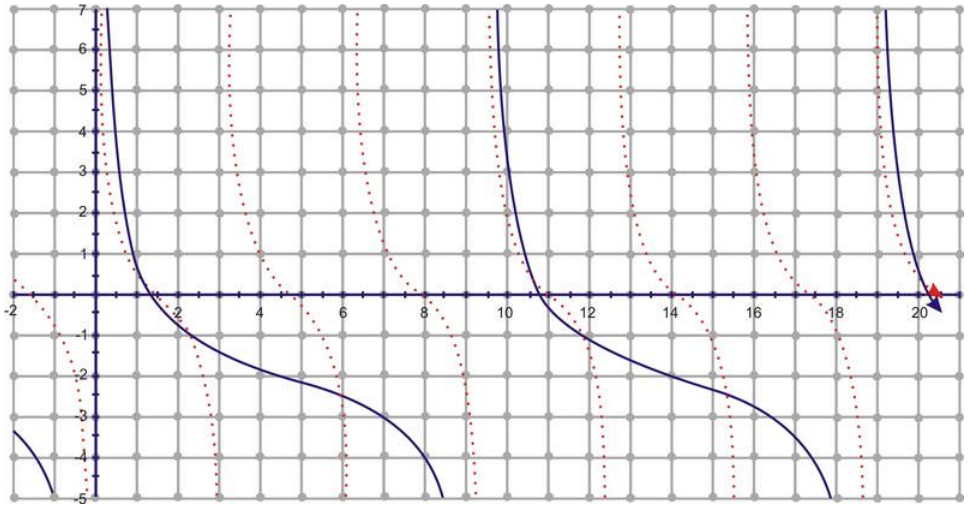


Figure 2.7.3.18

2. Sketch the graph of $y = -3\tan\left(x - \frac{\pi}{4}\right)$ over the interval $[-\pi, 2\pi]$

If you compare this graph to $y = \tan x$, it will be stretched and flipped. It will also have a phase shift of $\frac{\pi}{4}$ to the right. The red graph is $y = \tan x$.

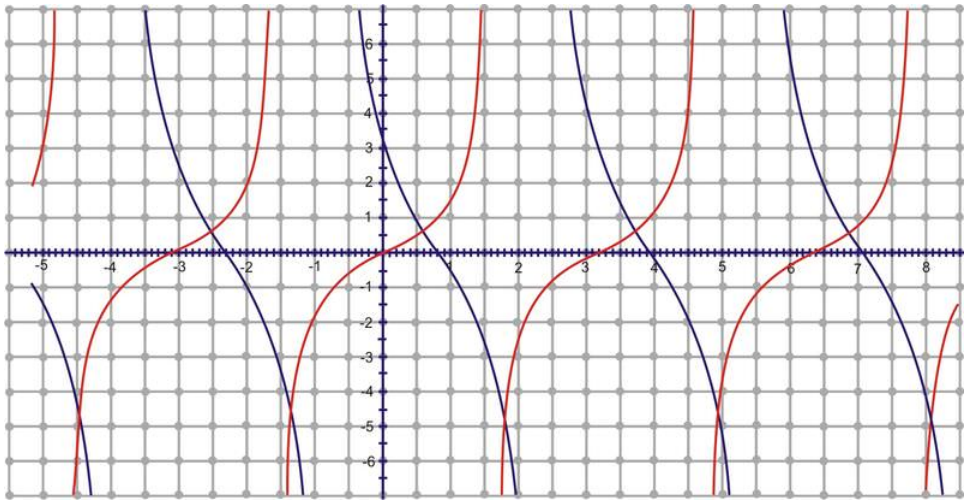


Figure 2.7.3.19

3. Sketch the graph of $h(x) = 4\tan\left(x + \frac{\pi}{2}\right) + 3$ over the interval $[0, 2\pi]$

The constant in front of the tangent function will cause the graph to be stretched. It will also have a phase shift of $\frac{\pi}{2}$ to the left. Finally, the graph will be shifted up three. Here you can see both graphs, where the red graph is $y = \tan x$.

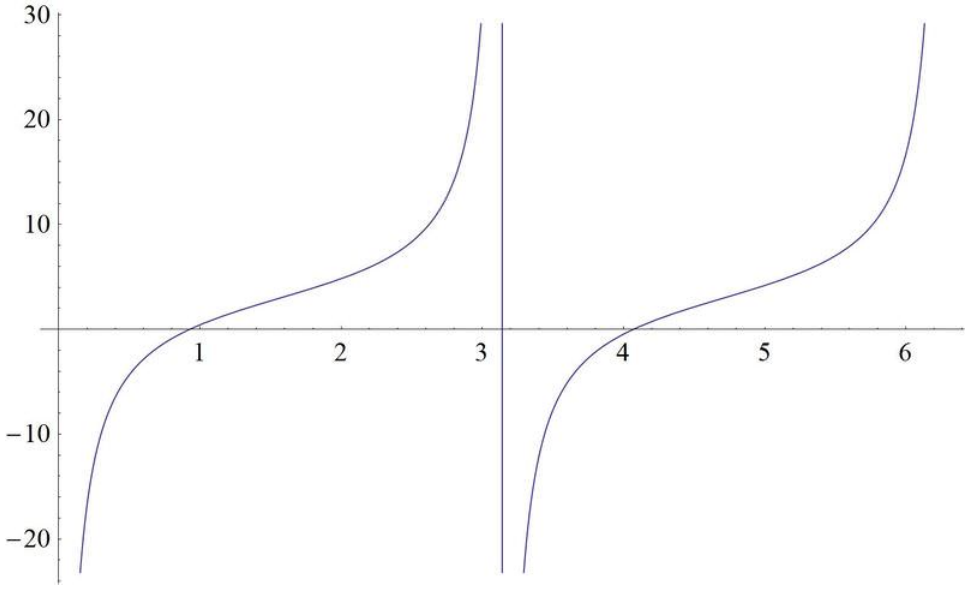


Figure **2.7.3.20**

Example 2.7.3.1

Earlier, you were asked to identify which graphs are tangent and cotangent.

Solution

As you can tell after completing this section, when presented with the graphs:

1.

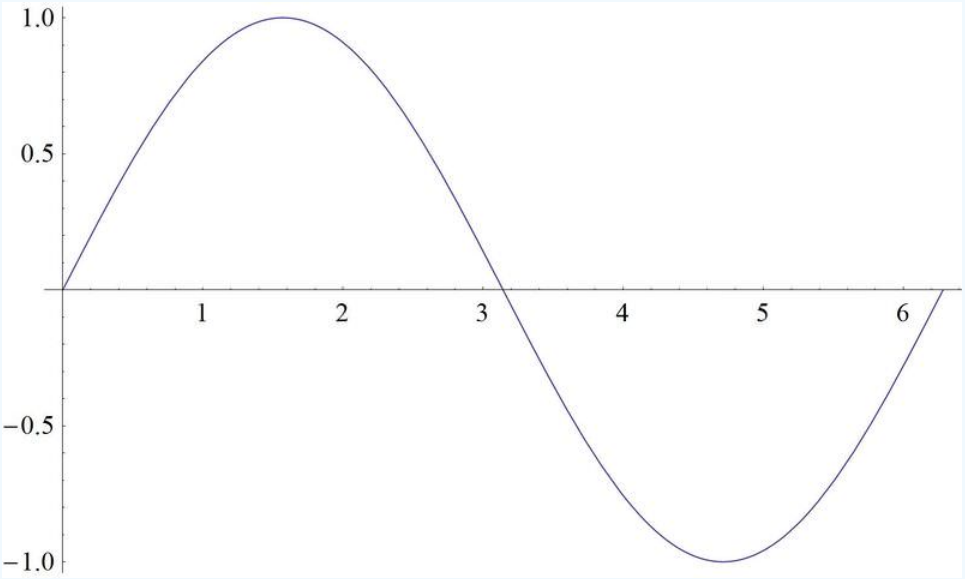


Figure **2.7.3.21**

2.

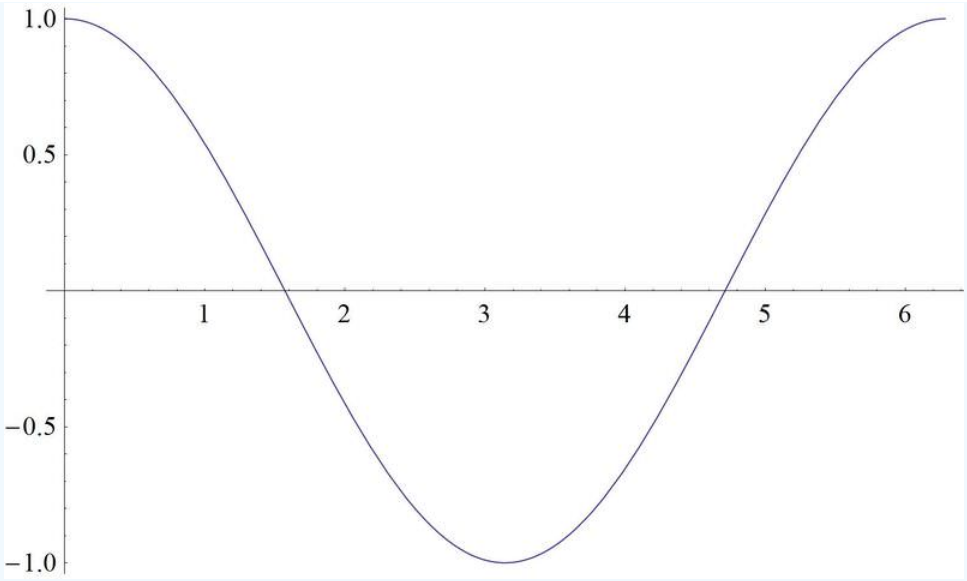


Figure **2.7.3.22**

3.

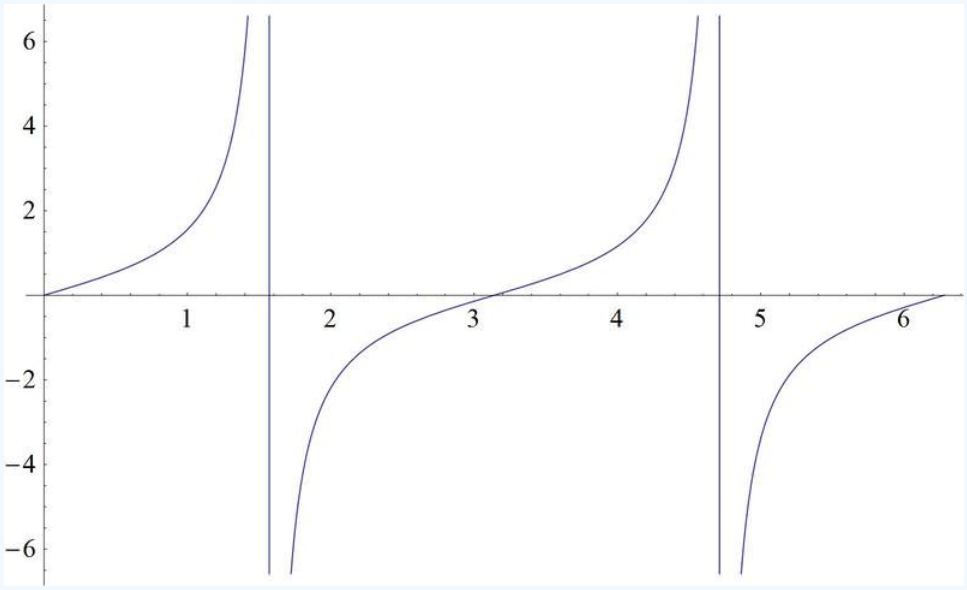


Figure **2.7.3.23**

4.

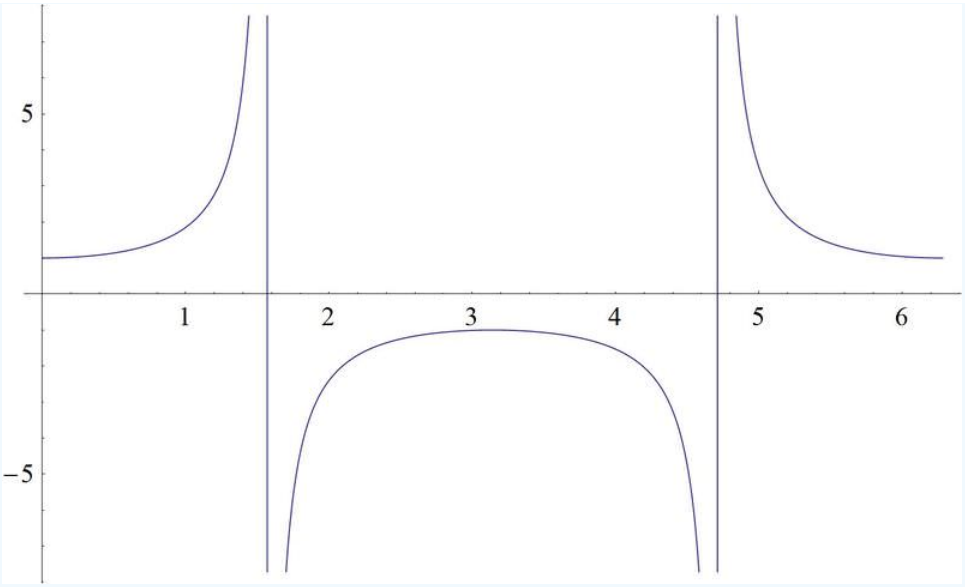


Figure 2.7.3.24

5.

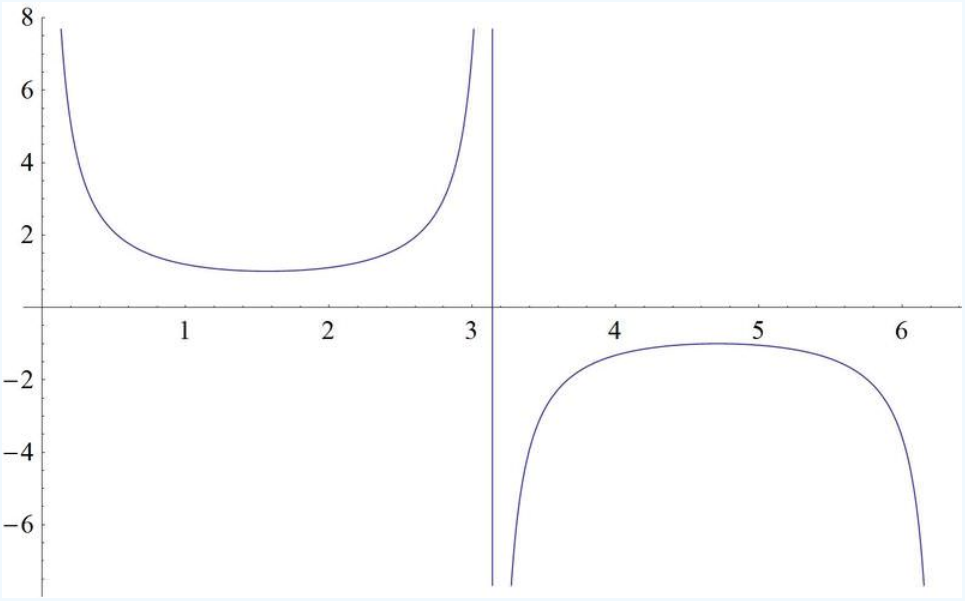


Figure 2.7.3.25

6.

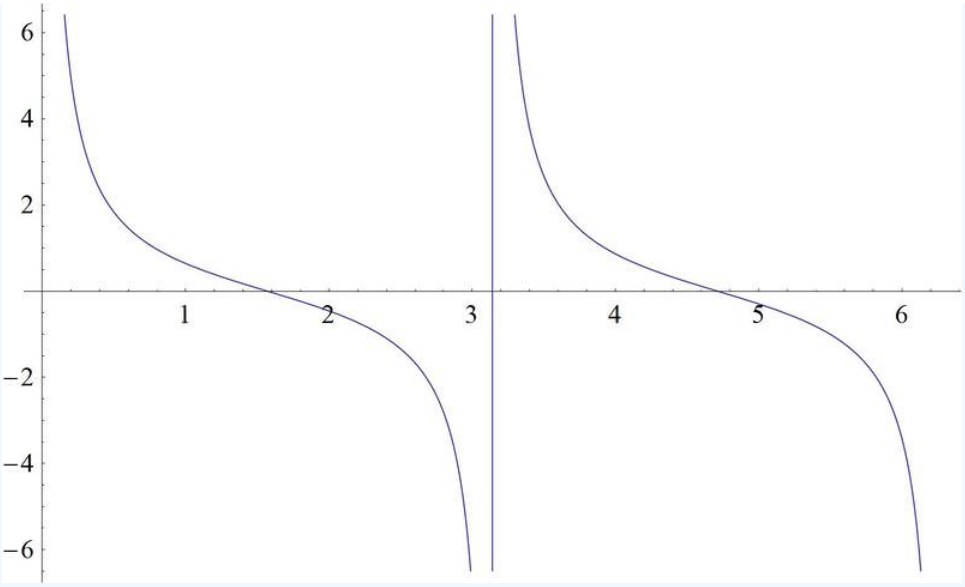


Figure **2.7.3.26**

The tangent and cotangent graphs are the third and sixth graphs.

Example 2.7.3.2

Graph $y = -1 + \frac{1}{3}\cot 2x$

Solution

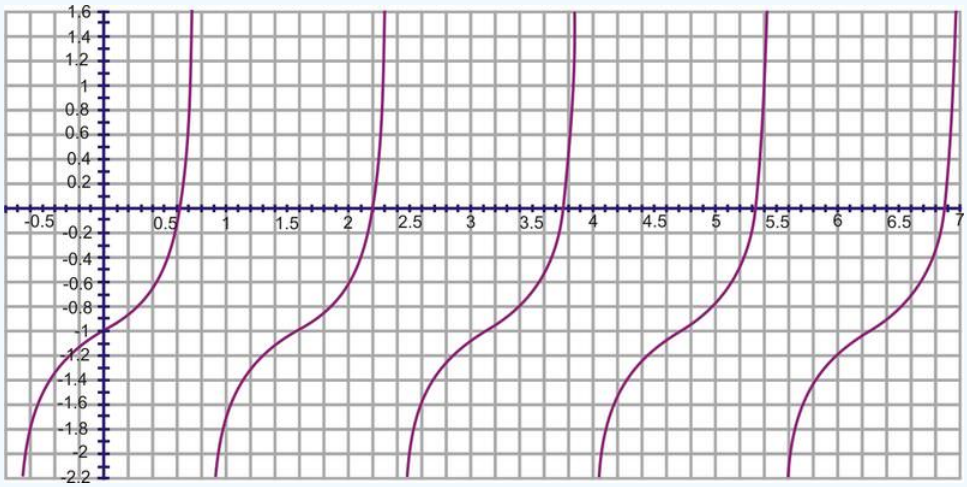


Figure **2.7.3.27**

Example 2.7.3.4

Graph $f(x) = 4 + \tan(0.5(x - \pi))$

Solution

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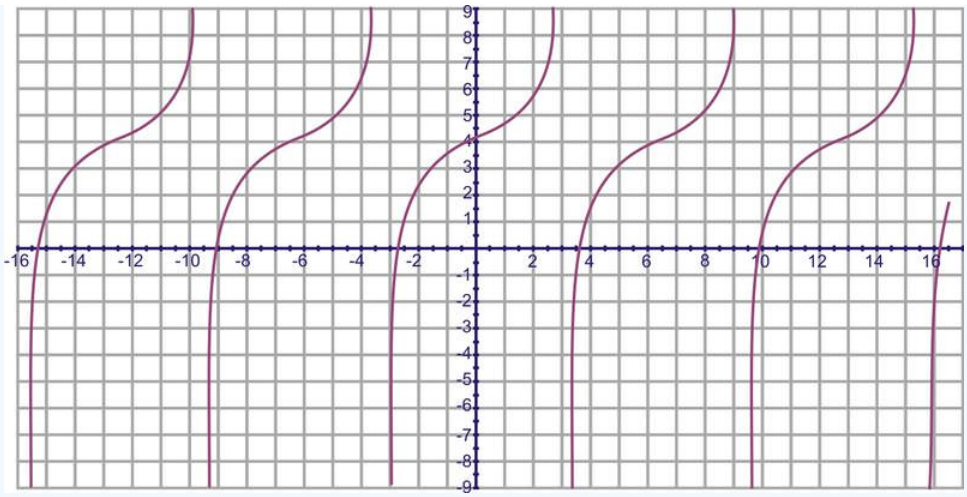


Figure 2.7.3.28

Example 2.7.3.5

Graph $y = -2\tan 2x$

Solution

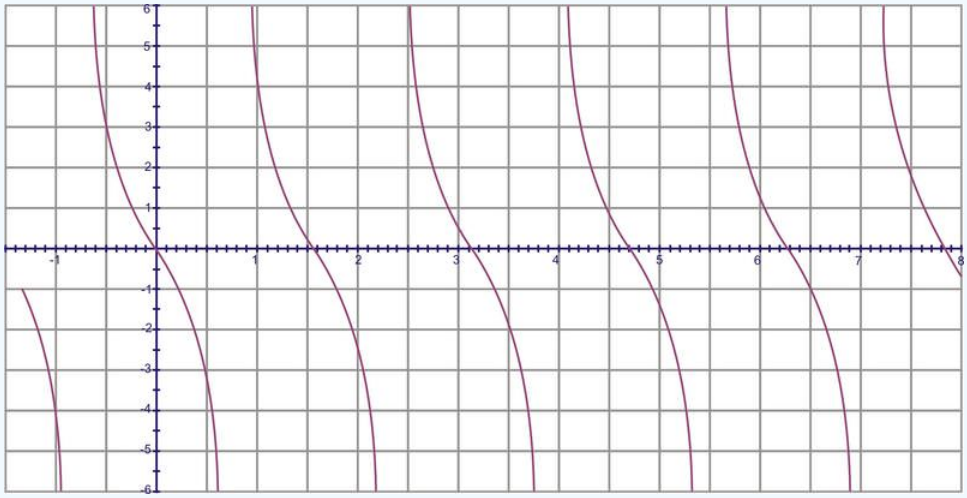


Figure 2.7.3.29

Review

Graph each of the following functions

1. $f(x) = \tan(x)$
2. $h(x) = \tan(2x)$
3. $k(x) = \tan(2x + \pi)$
4. $m(x) = -\tan(2x + \pi)$
5. $g(x) = -\tan(2x + \pi) + 3$
6. $f(x) = \cot(x)$
7. $h(x) = \cot(2x)$
8. $k(x) = \cot(2x + \pi)$
9. $m(x) = 3\cot(2x + \pi)$
10. $g(x) = -2 + 3\cot(2x + \pi)$
11. $h(x) = \tan\left(\frac{x}{2}\right)$
12. $k(x) = \tan\left(\frac{x}{2} + \frac{\pi}{4}\right)$

13. $m(x) = 3\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)$
14. $g(x) = 3\tan\left(\frac{x}{2} + \frac{\pi}{4}\right) - 1$
15. $h(x) = \cot\left(\frac{x}{2}\right)$
16. $k(x) = \cot\left(\frac{x}{2} + \frac{3\pi}{2}\right)$
17. $m(x) = -3\cot\left(\frac{x}{2} + \frac{3\pi}{2}\right)$
18. $g(x) = 2 - 3\cot\left(\frac{x}{2} + \frac{3\pi}{2}\right)$

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.11.

Vocabulary

Term	Definition
Circular Function	A circular function is a function measured by examining the angle of rotation around the coordinate plane.

Additional Resources

Video: Animation: Graphing the Tangent Function Using the Unit Circle

Practice: Tangent and cotangent Graphs

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2.7.4: Sine and Cosecant Graphs

Wave based on y value and radius of a circle; graph based on the reciprocal.

Sine and Cosecant Graphs

Imagine for a moment that you have a clock that has only one hand - that rotates counterclockwise! However, the hand is very slim all the way until the tip, where there is a ball on the end. In fact, the hand is so slim you won't notice it. You only notice the ball on the end of the rotating hand. This hand is rotating faster than normal. Here is a picture of the clock:

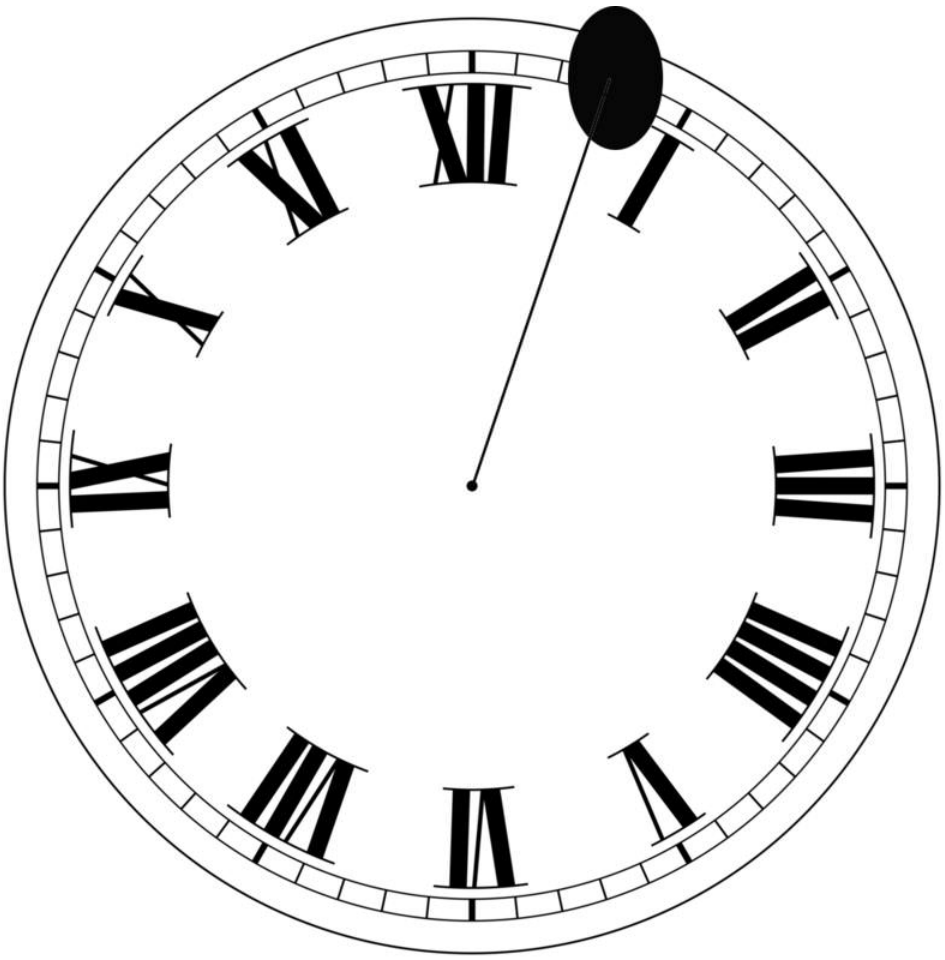


Figure 2.7.4.1

Consider what it would be like if you put a light next to the clock and let the shadow of the hands fall on the far wall.

What pattern would that shadow trace out? If you think about it, you might realize that the shadow would make an up and down motion, over and over as the hand of the clock rotated. Now imagine that instead of a wall, there was a large piece of paper for the shadow to fall on. And wherever the shadow fell, there would be a mark on the paper. Finally, imagine moving the paper as the clock rotates. Can you imagine sort of pattern this would trace out?

Sine and Cosecant Graphs

By now, you have become very familiar with the specific values of sine, cosine, and tangents for certain angles of rotation around the coordinate grid. In mathematics, we can often learn a lot by looking at how one quantity changes as we consistently vary another. We will be looking at the sine value **as a function** of the angle of rotation around the coordinate plane. We refer to any such function as a **circular function**, because they can be defined using the unit circle. Recall from earlier sections that the sine of an angle in standard position is the ratio of $\frac{y}{r}$, where y is the $y-$coordinate of any point on the circle and r is the distance from the origin to that point.

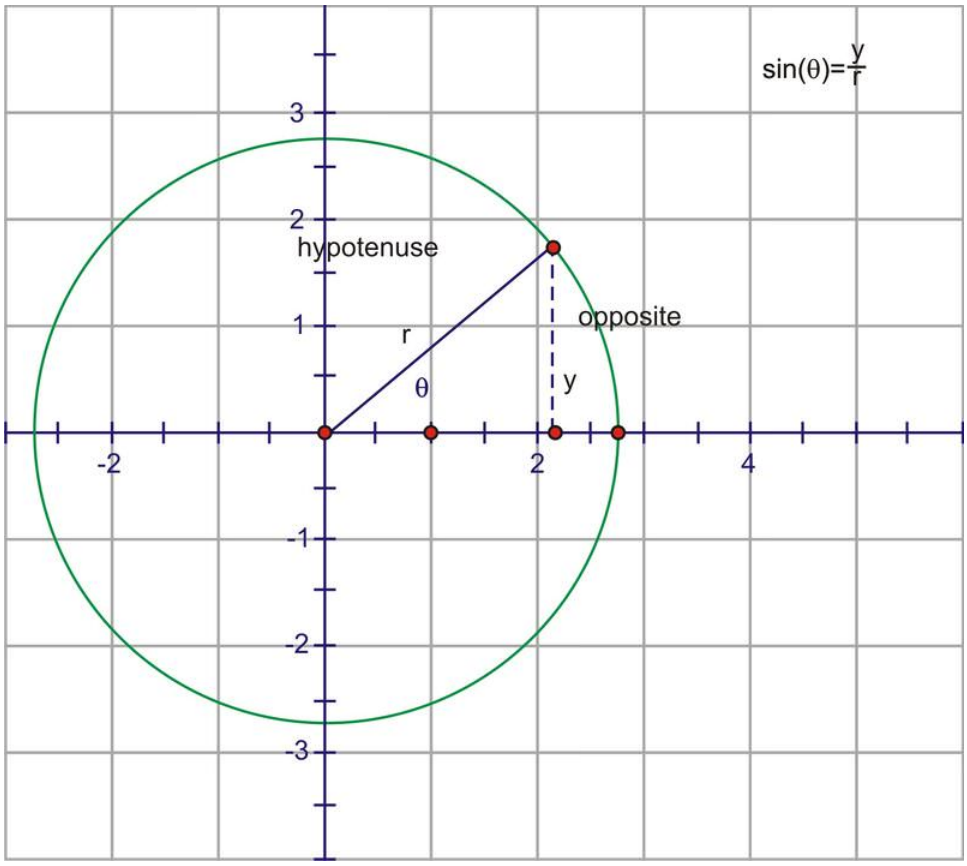


Figure **2.7.4.2**

Because the ratios are the same for a given angle, regardless of the length of the radius r, we can use the **unit circle** as a basis for all calculations.

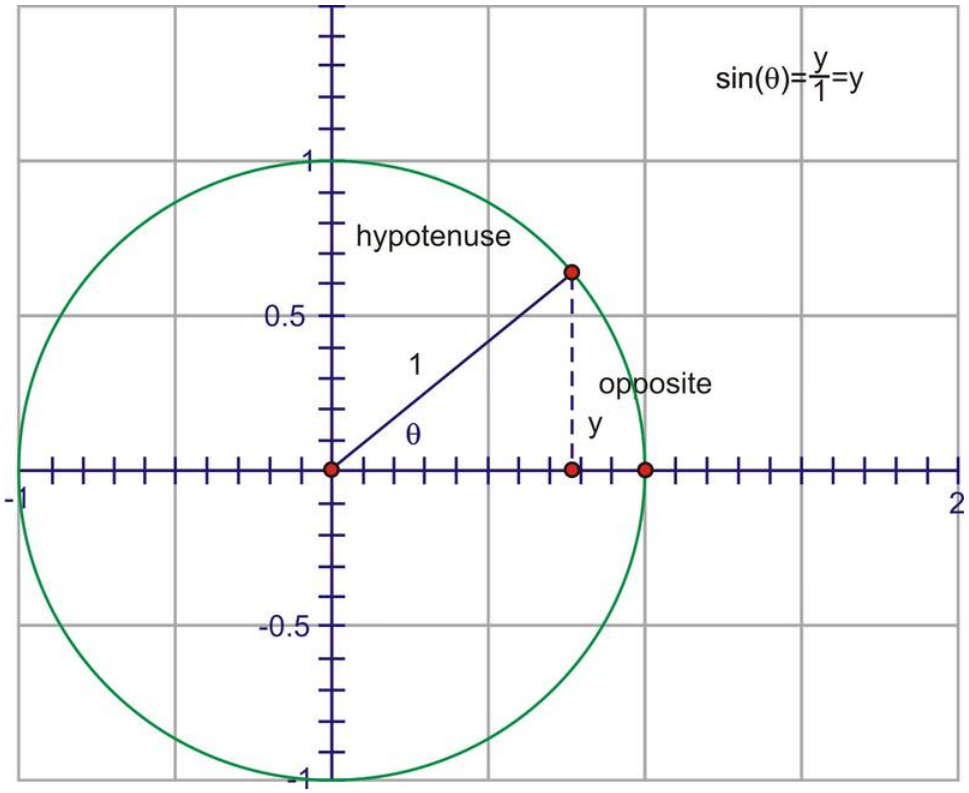


Figure 2.7.4.3

The denominator is now 1, so we have the simpler expression, $\sin x = y$. The advantage to this is that we can use the y−coordinate of the point on the unit circle to trace the value of $\sin \theta$ through a complete rotation. Imagine if we start at 0 and then rotate counter-clockwise through gradually increasing angles. Since the y−coordinate is the sine value, watch the height of the point as you rotate.

Through Quadrant I that height gets larger, starting at 0, increasing quickly at first, then slower until the angle reaches $90°$, at which point, the height is at its maximum value, 1.

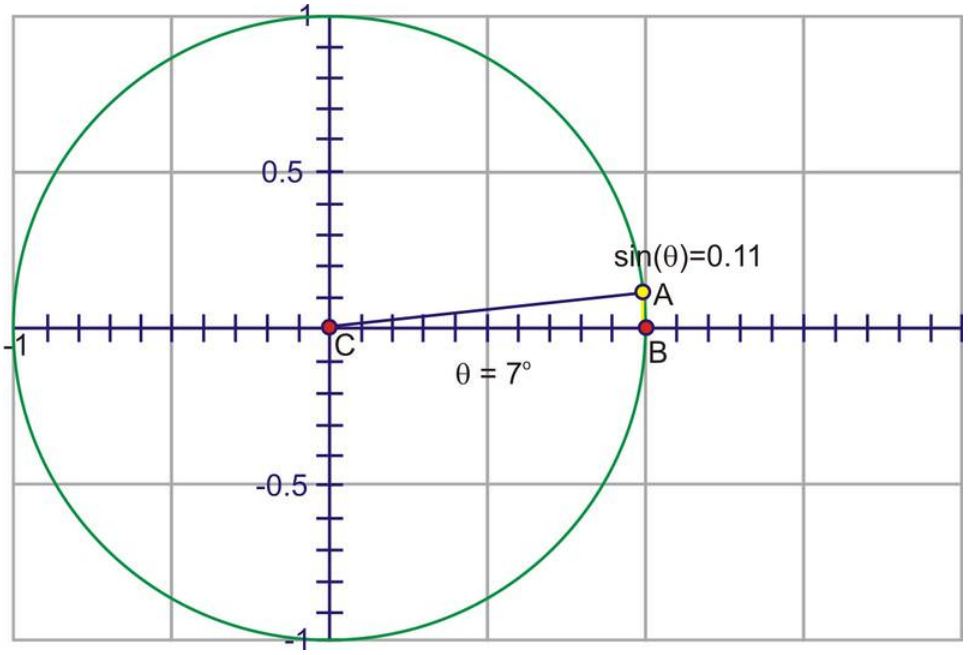


Figure 2.7.4.4

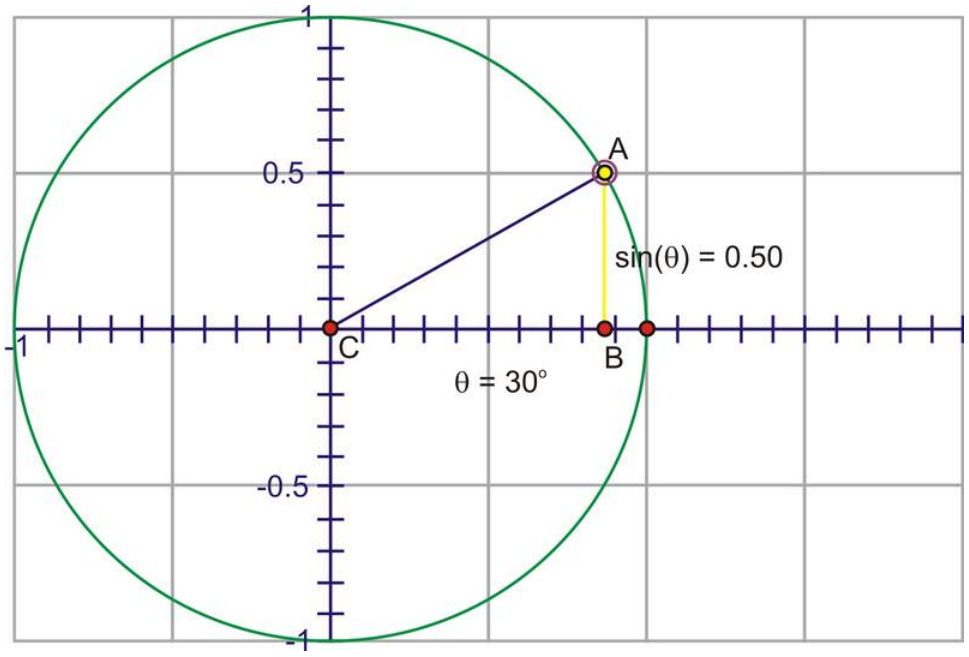


Figure **2.7.4.5**

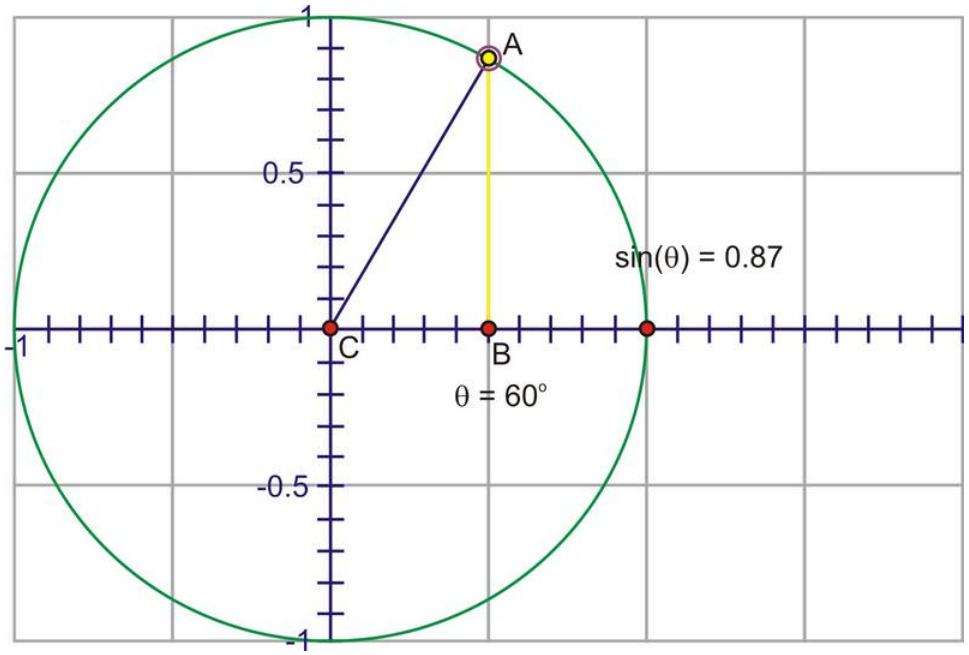


Figure **2.7.4.6**

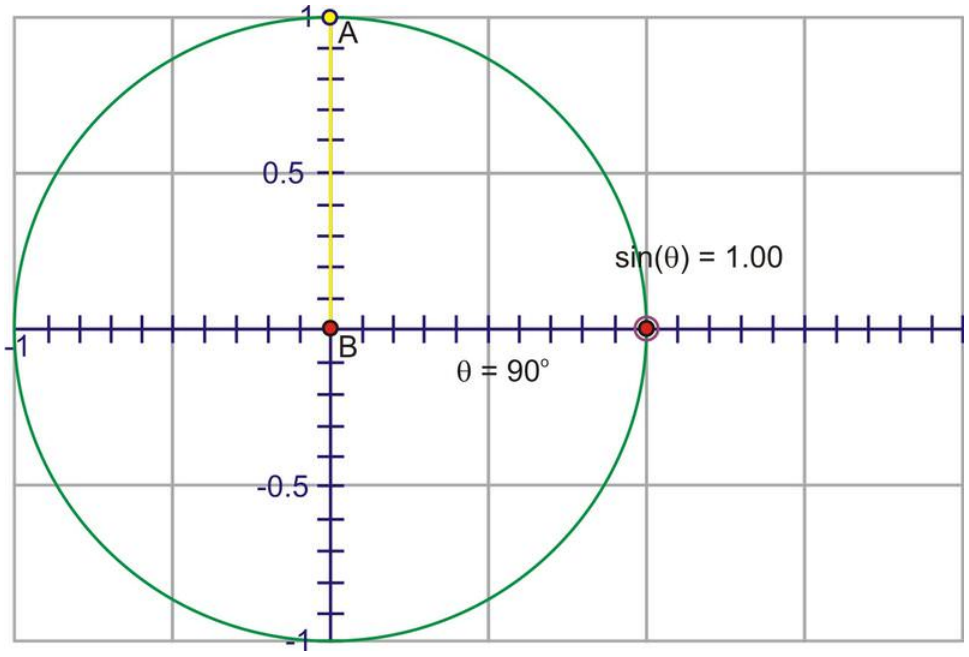


Figure **2.7.4.7**

As you rotate into the second quadrant, the height starts to decrease towards zero.

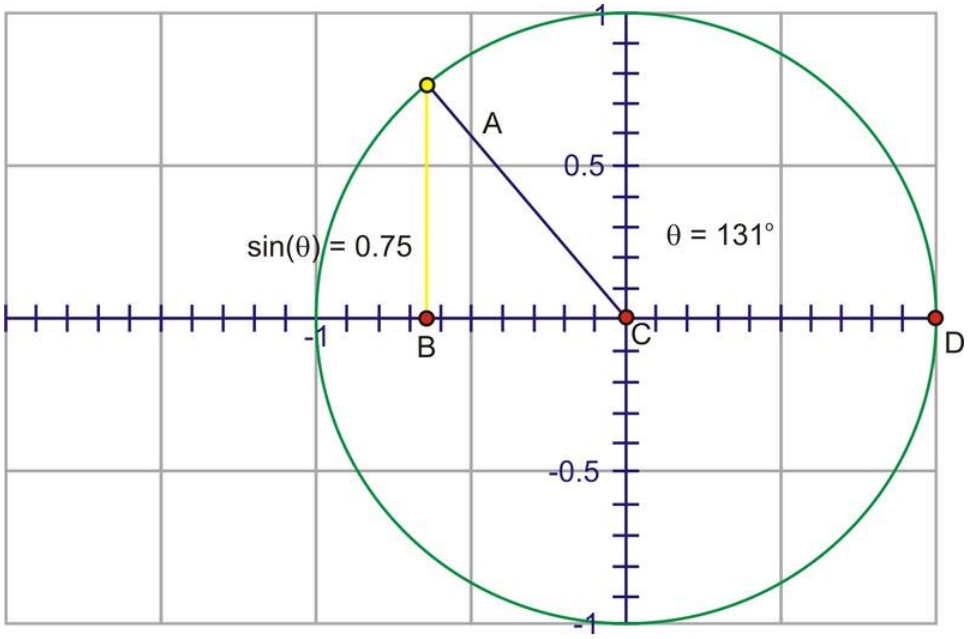


Figure **2.7.4.8**

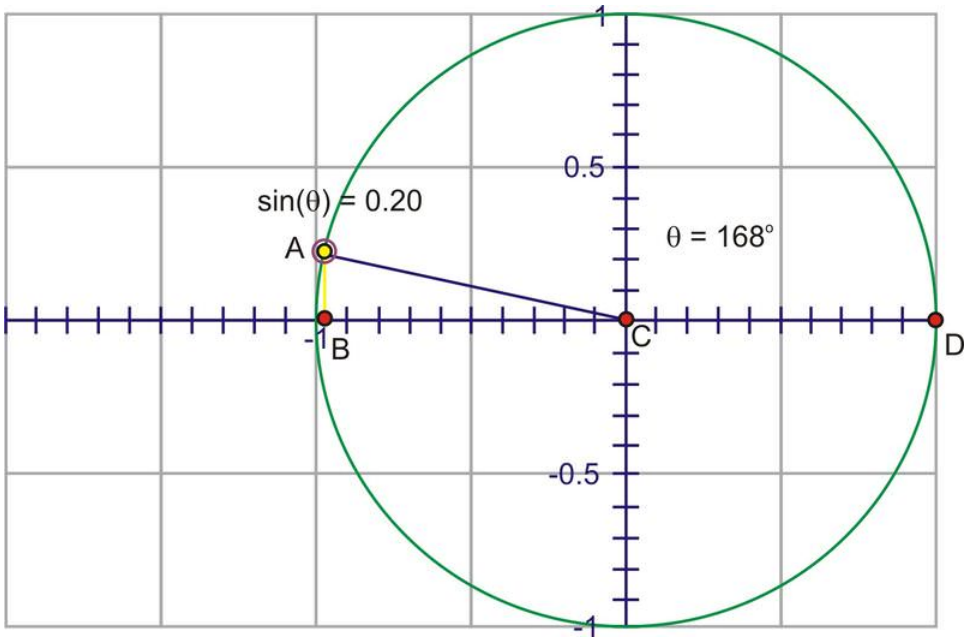


Figure 2.7.4.9

When you start to rotate into the third and fourth quadrants, the length of the segment increases, but this time in a negative direction, growing to -1 at 270° and heading back toward 0 at 360°.

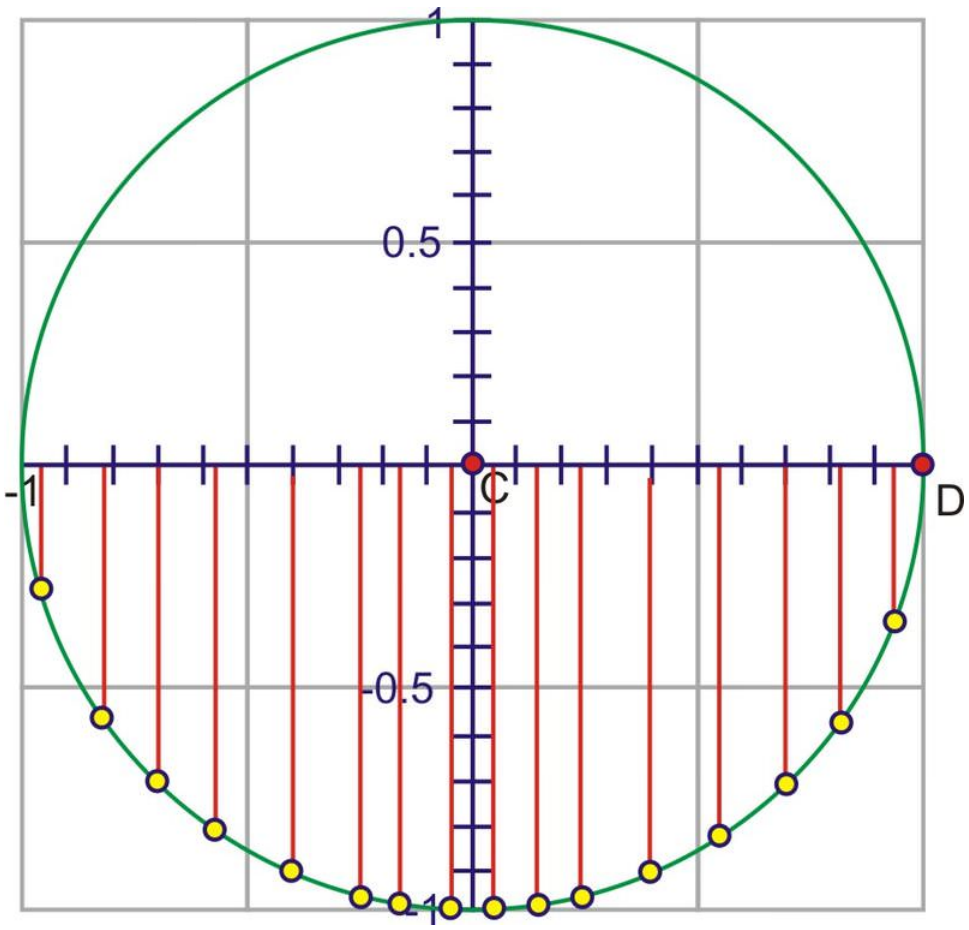
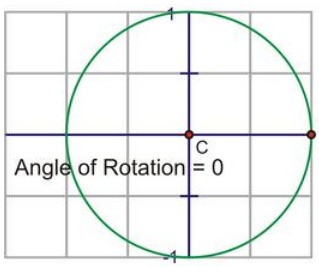


Figure 2.7.4.10

After one complete rotation, even though the angle continues to increase, the sine values will repeat themselves. The same would have been true if we chose to rotate clockwise to investigate negative angles, and this is why the sine function is a periodic function. The period is 2π because that is the angle measure before the sine of the angle will repeat its values.

Let's translate this circular motion into a graph of the sine value vs. the angle of rotation. The following sequence of pictures demonstrates the connection. These pictures plot $(\theta, \sin\theta)$ on the coordinate plane as (x, y).



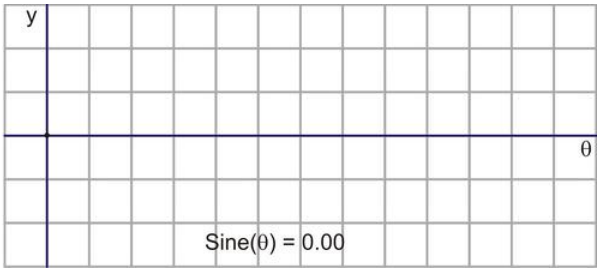
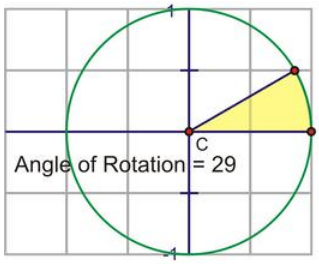


Figure **2.7.4.11**



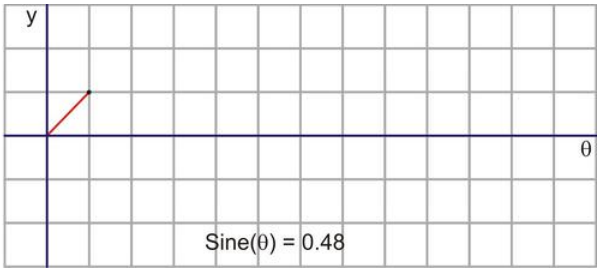
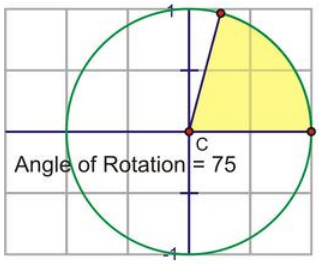


Figure **2.7.4.12**



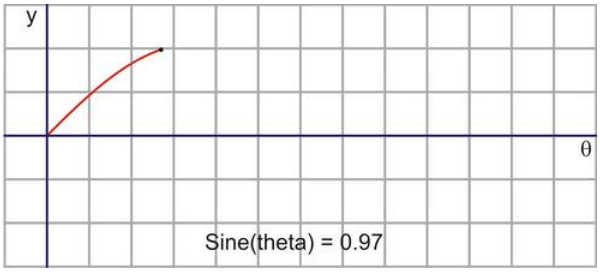
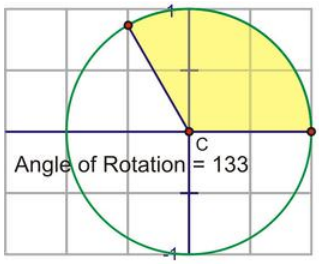


Figure **2.7.4.13**



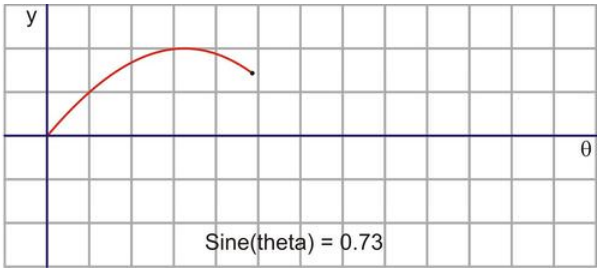
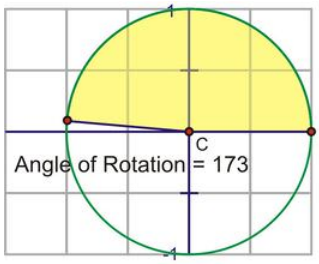


Figure **2.7.4.14**



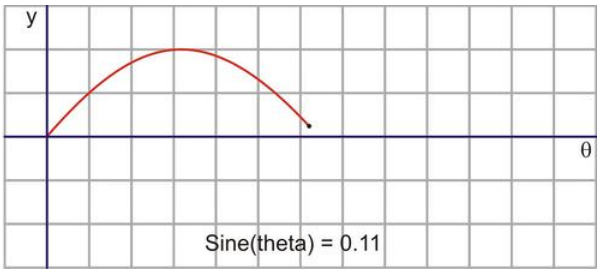


Figure **2.7.4.15**

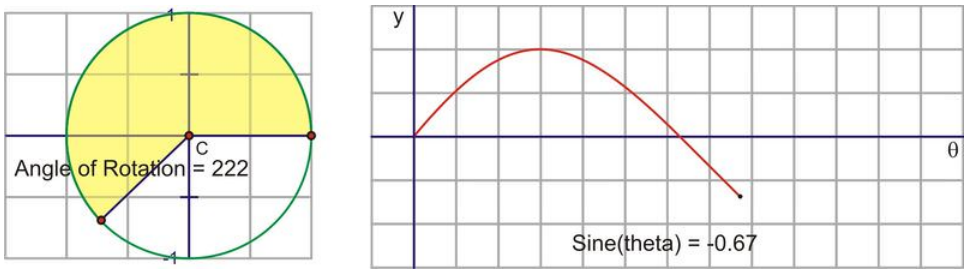


Figure 2.7.4.16

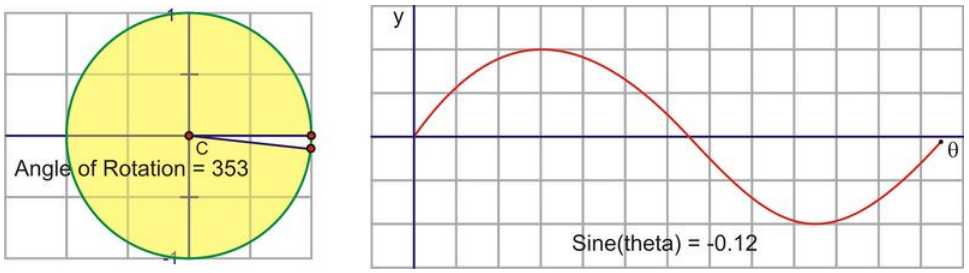


Figure 2.7.4.17

After we rotate around the circle once, the values start repeating. Therefore, the sine curve, or "wave," also continues to repeat. The easiest way to sketch a sine curve is to plot the points for the quadrant angles. The value of $\sin\theta$ goes from 0 to 1 to 0 to -1 and back to 0. Graphed along a horizontal axis, it would look like this:

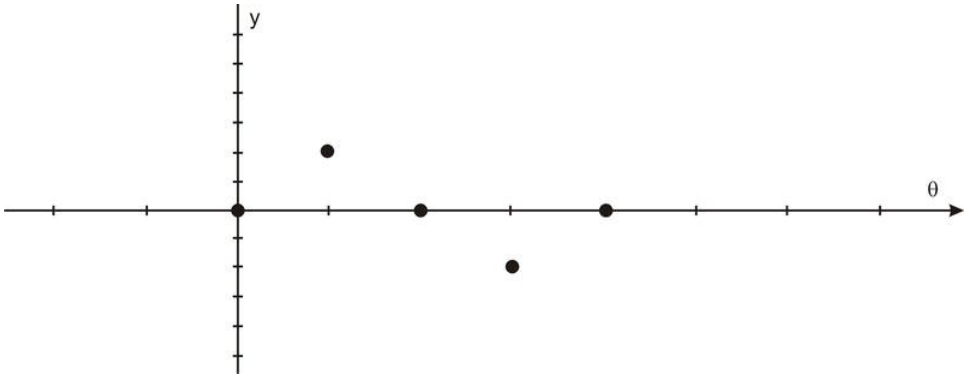


Figure 2.7.4.18

Filling in the gaps in between and allowing for multiple rotations as well as negative angles results in the graph of $y = \sin x$ where x is any angle of rotation, in radians.

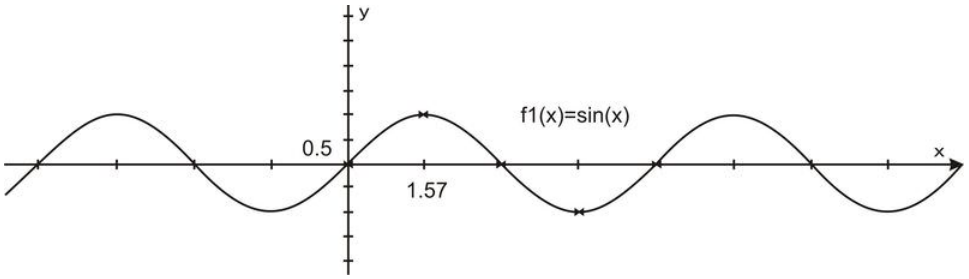


Figure 2.7.4.19

As we have already mentioned, $\sin x$ has a period of 2π. You should also note that the y−values never go above 1 or below -1, so the **range** of a sine curve is $\{-1 \leq y \leq 1\}$. Because angles can be any value and will continue to rotate around the circle infinitely, there is no restriction on the angle x, so the **domain** of $\sin x$ is all reals.

Cosecant is the reciprocal of sine, or 1y. Therefore, whenever the sine is zero, the cosecant is going to have a vertical asymptote because it will be undefined. It also has the same sign as the sine function in the same quadrants. Here is the graph.

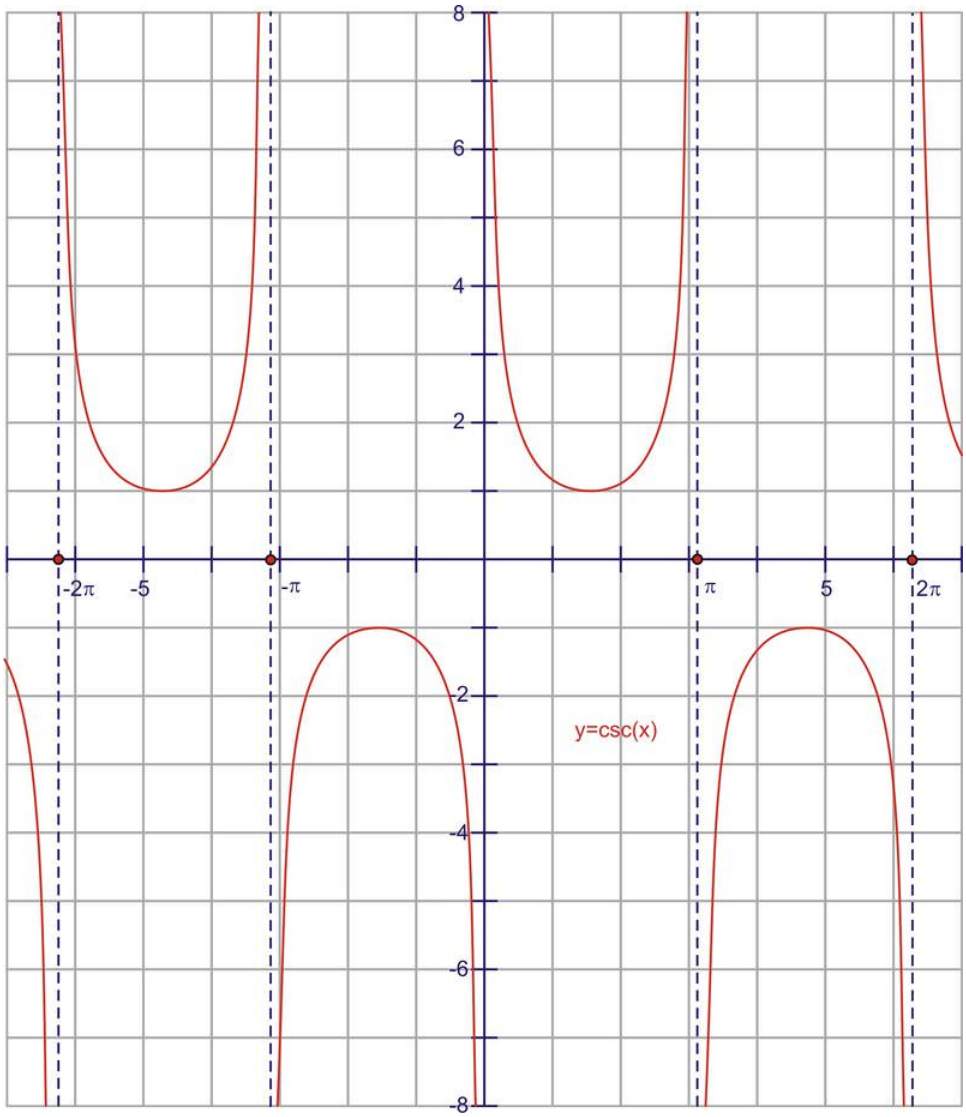


Figure 2.7.4.20

The period of the function is 2π, just like sine. The domain of the function is all real numbers, except multiples of $\pi \{... -2\pi, -\pi, 0, \pi, 2\pi ...\}$. The range is all real numbers greater than or equal to 1, as well as all real numbers less than or equal to -1. Notice that the range is everything *except* where sine is defined (other than the points at the top and bottom of the sine curve

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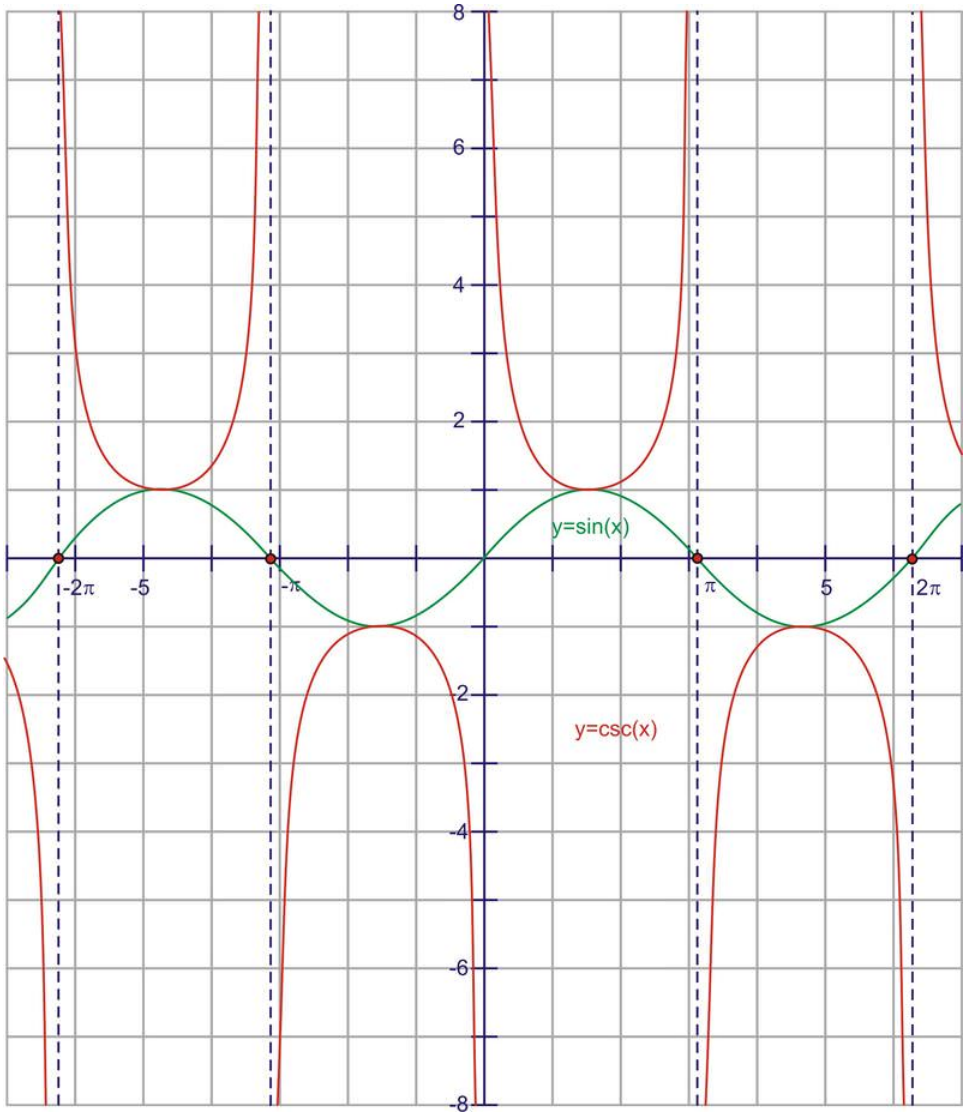
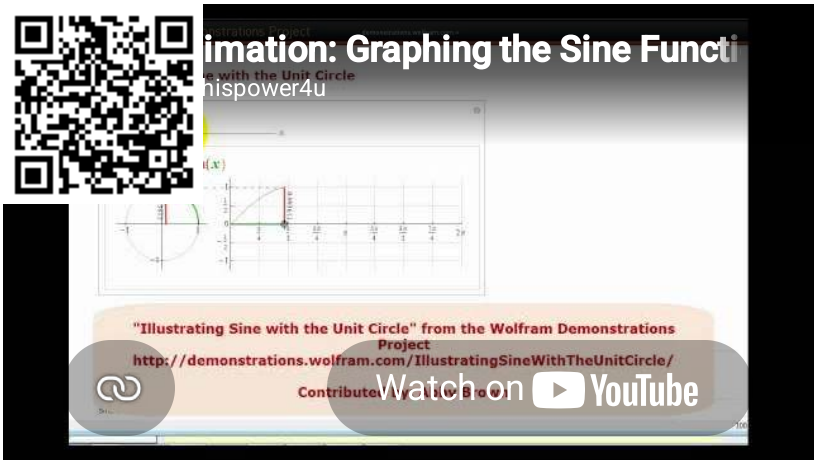


Figure **2.7.4.21**

Notice again the reciprocal relationships at 0 and the asymptotes. Also look at the intersection points of the graphs at 1 and -1. Many students are reminded of parabolas when they look at the half-period of the cosecant graph. While they are similar in that they each have a local minimum or maximum and they have the same beginning and ending behavior, the comparisons end there. Parabolas are not restricted by asymptotes, whereas the cosecant curve is.



Graphing Functions

Graph the following functions:

1. $g(x) = \frac{1}{2}\sin(3x)$.

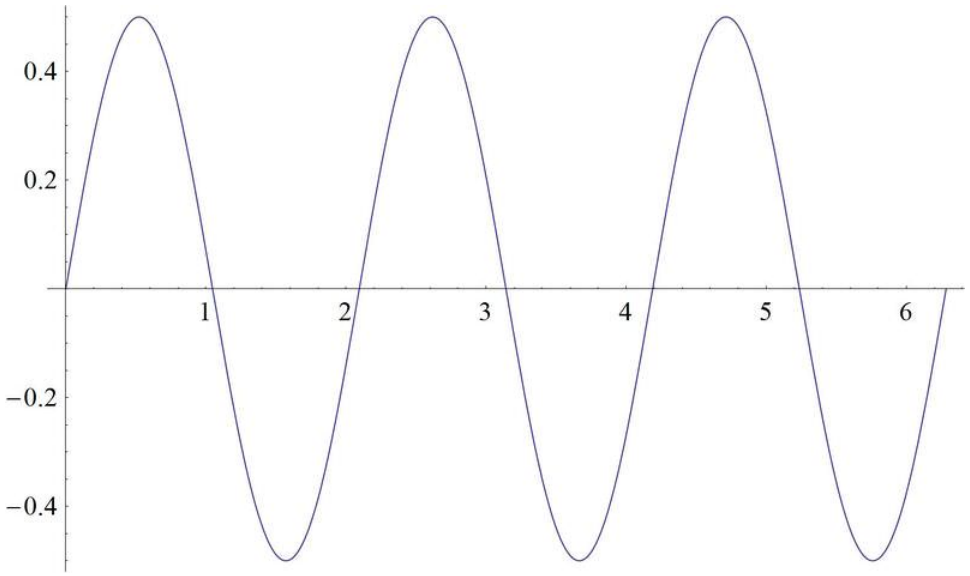


Figure 2.7.4.22

As you can see from the graph, the $\frac{1}{2}$ in front of the function reduces the function's height, while the 3 inside the argument of the function makes the function "squished" along the "x" axis.

2. $f(x) = \frac{1}{3}\csc\left(\frac{1}{2}x\right)$.

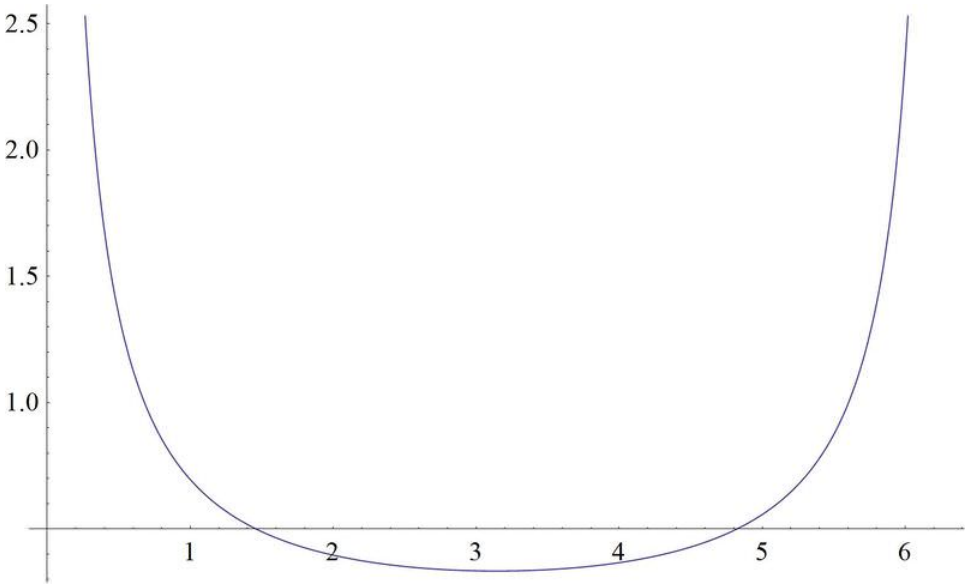


Figure 2.7.4.23

3. $f(x) = 5\sin\left(2\left(x + \frac{\pi}{3}\right)\right)$.

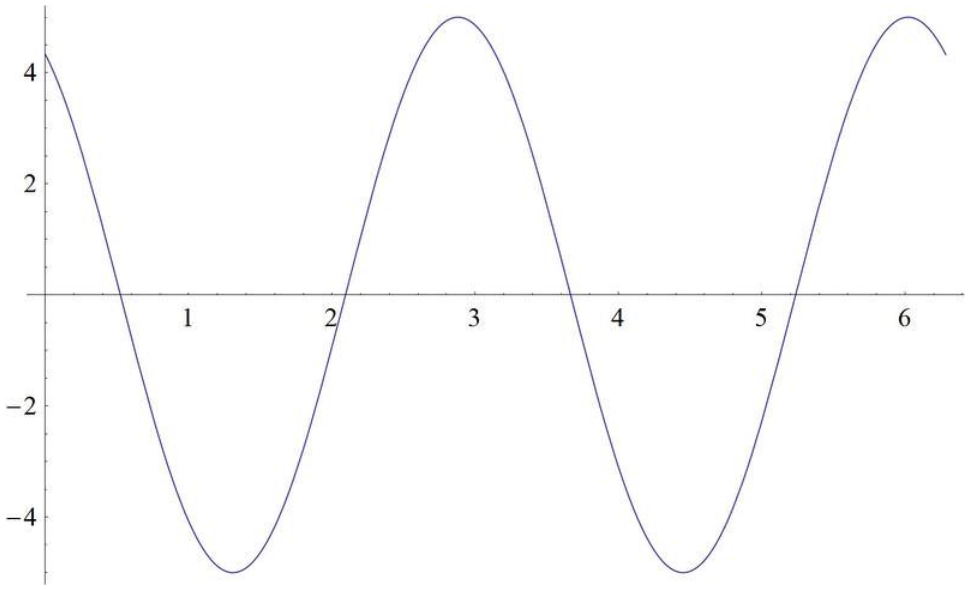


Figure **2.7.4.23**

Example 2.7.4.1

Earlier, you were asked what pattern the shadow would trace out.

Solution

As you have seen in this lesson, the shadow of a light applied vertically to a rotating clock hand would trace out a sine graph. The graph would begin at zero when the hand is lying flat along the positive "x" axis. It would then increase until the hand was vertical. It would then decrease until the rotating hand was pointing straight down. Finally, the graph would increase again to zero when the hand is returning to the positive "x" axis.

Example 2.7.4.2

Graph $g(x) = 5 \csc\left(\frac{1}{4}(x + \pi)\right)$.

Solution

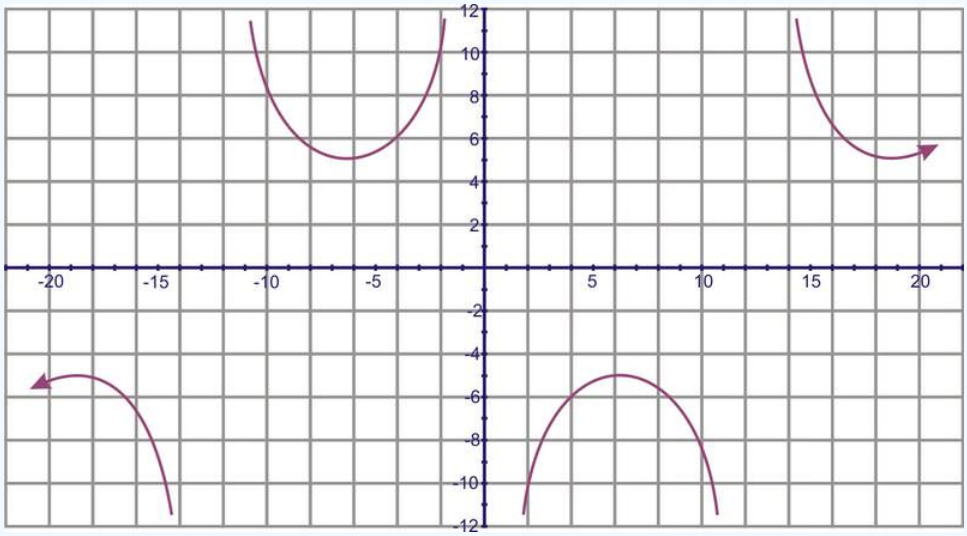


Figure **2.7.4.24**

Example 2.7.4.3

Determine the function creating this graph:

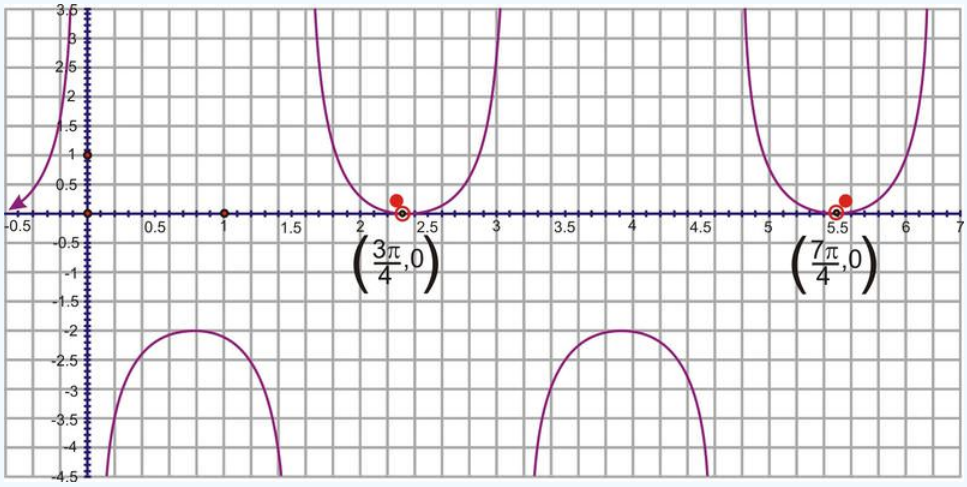


Figure 2.7.4.25

Solution

This could be either a secant or cosecant function. We will use a cosecant model. First, the vertical shift is -1. The period is the difference between the two given x−values, $\frac{7\pi}{4}-\frac{3\pi}{4}=\pi$, so the frequency is $\frac{2\pi}{\pi}=2$. The horizontal shift incorporates the frequency, so in $y=\csc x$ the corresponding x−value to $\left(\frac{3\pi}{4},0\right)$ is $\left(\frac{\pi}{2},1\right)$. The difference between the x−values is $\frac{3\pi}{4}-\frac{\pi}{2}=3\frac{\pi}{4}-\frac{2\pi}{4}=\frac{\pi}{4}$ and then multiply it by the frequency, $2\cdot\frac{\pi}{4}=\frac{\pi}{2}$. The equation is $y=-1+\csc\left(2\left(x-\frac{\pi}{2}\right)\right)$.

Example 2.7.4.4

Graph $h(x)=3\sin\left(\frac{1}{2}\left(x+\frac{\pi}{2}\right)\right)$.

Solution

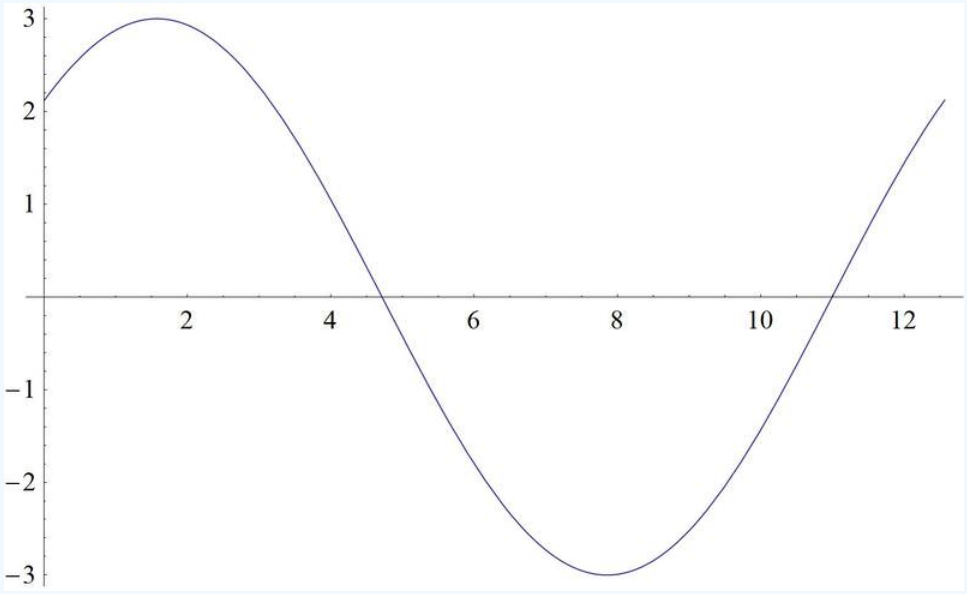


Figure 2.7.4.26

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Review

Graph each of the following functions.

1. $f(x) = \sin(x)$.
2. $h(x) = \sin(2x)$.
3. $k(x) = \sin(2x + \pi)$.
4. $m(x) = 2\sin(2x + \pi)$.
5. $g(x) = 2\sin(2x + \pi) + 2$.
6. $f(x) = \csc(x)$.
7. $h(x) = \csc(2x)$.
8. $k(x) = \csc(2x + \pi)$.
9. $m(x) = 2\csc(2x + \pi)$.
10. $g(x) = 2\csc(2x + \pi) + 2$.
11. $h(x) = \sin(3x)$.
12. $k(x) = \sin\left(3x + \frac{\pi}{2}\right)$.
13. $m(x) = 3\sin\left(3x + \frac{\pi}{2}\right)$.
14. $g(x) = 3\sin\left(3x + \frac{\pi}{2}\right) + 3$.
15. $h(x) = \csc(3x)$
16. $k(x) = \csc\left(3x + \frac{3\pi}{2}\right)$.
17. $m(x) = 4\csc\left(3x + \frac{3\pi}{2}\right)$.
18. $g(x) = 4\csc\left(3x + \frac{3\pi}{2}\right) - 3$.

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.9.

Vocabulary

Term	Definition
Circular Function	A circular function is a function measured by examining the angle of rotation around the coordinate plane.

Additional Resources

Video: Animation: Graphing the Sine Function U\sin g the Unit Circle

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2.7.5: Cosine and Secant Graphs

Wave based on x value and radius of a circle; graph based on the reciprocal.

Imagine for a moment that you have a clock that has only one hand - that rotates counterclockwise!. However, the hand is very slim all the way until the tip, where there is a ball on the end. In fact, the hand is so slim you won't notice it. You only notice the ball on the end of the rotating hand. This hand is rotating faster than normal.

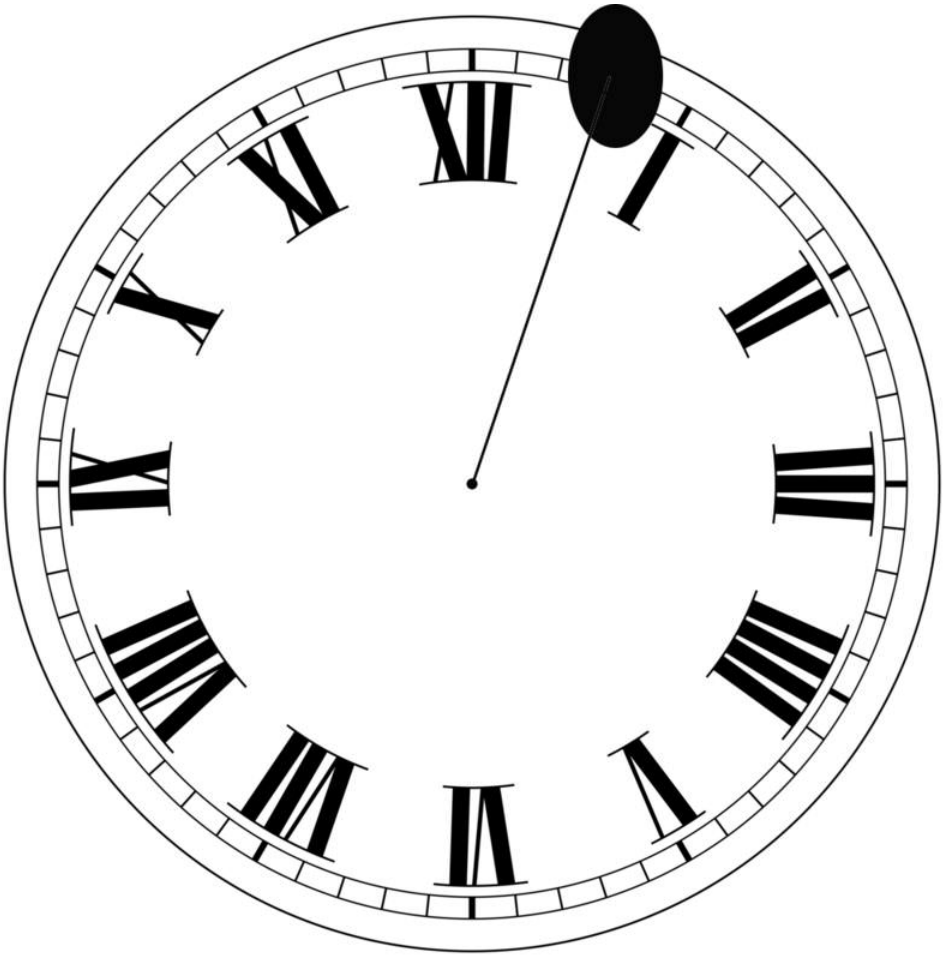


Figure **2.7.5.1**

Consider what it would be like if you put a light above the clock and let the shadow of the hands fall on the wall under the clock. What pattern would that shadow trace out? If you think about it, you might realize that the shadow would make an left and right motion, over and over as the hand of the clock rotated. Now imagine that instead of a wall, there was a large piece of paper for the shadow to fall on. And wherever the shadow fell, there would be a mark on the paper. Finally, imagine moving the paper as the clock rotates. Can you imagine sort of pattern this would trace out?

Cosine and Secant Graphs

If you have read other Trigonometry sections in this course, you may have learned that sine and cosine are very closely related. The cosine of an angle is the same as the sine of its complementary angle. So, it should not be a surprise that sine and cosine waves are very similar in that they are both periodic with a period of 2π, a range from -1 to 1, and a domain of all real angles.

The cosine of an angle is the ratio of $\frac{x}{r}$, so in the unit circle, the cosine is the x−coordinate of the point of rotation. If we trace the x−coordinate through a rotation, notice the change in the distance of $\cos x$ starts at one. The x−coordinate at $0°$ is 1 and the x−coordinate for $90°$ is 0, so the cosine value is decreasing from 1 to 0 through the 1st quadrant.

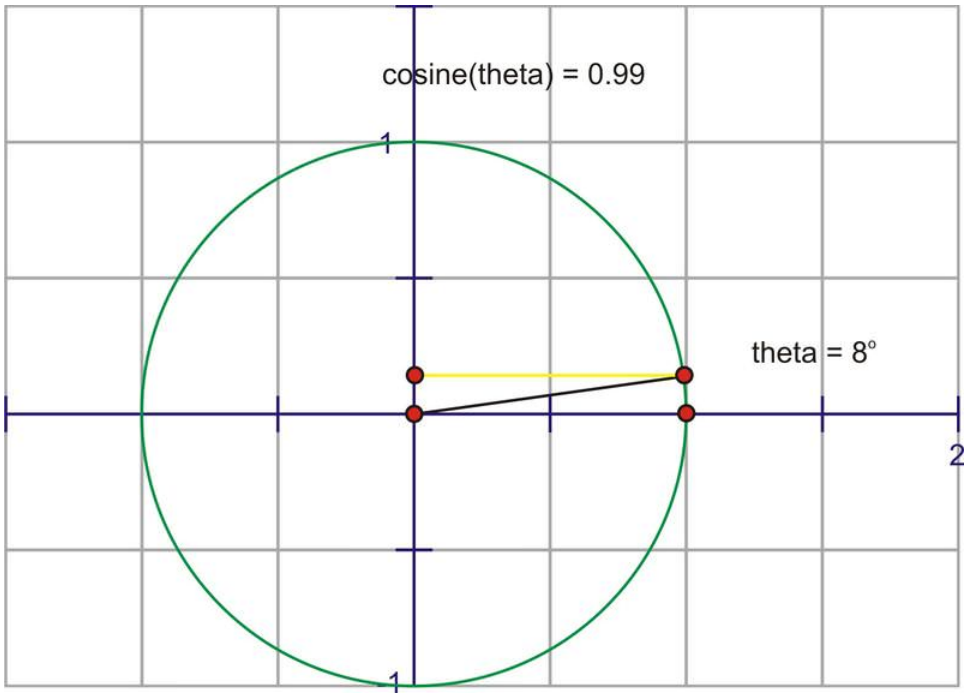


Figure **2.7.5.2**

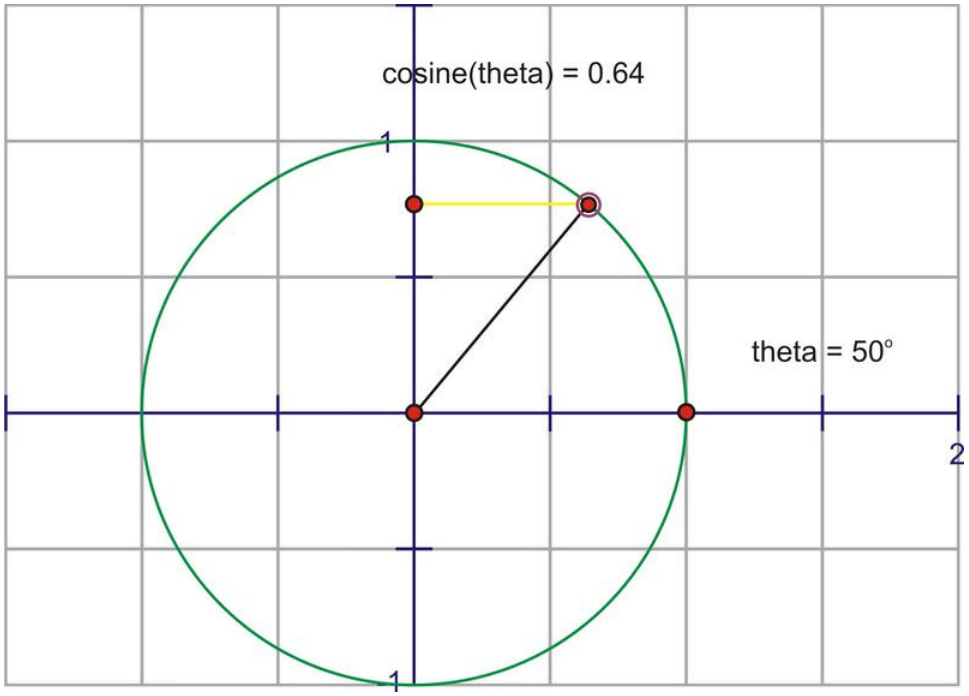


Figure **2.7.5.3**

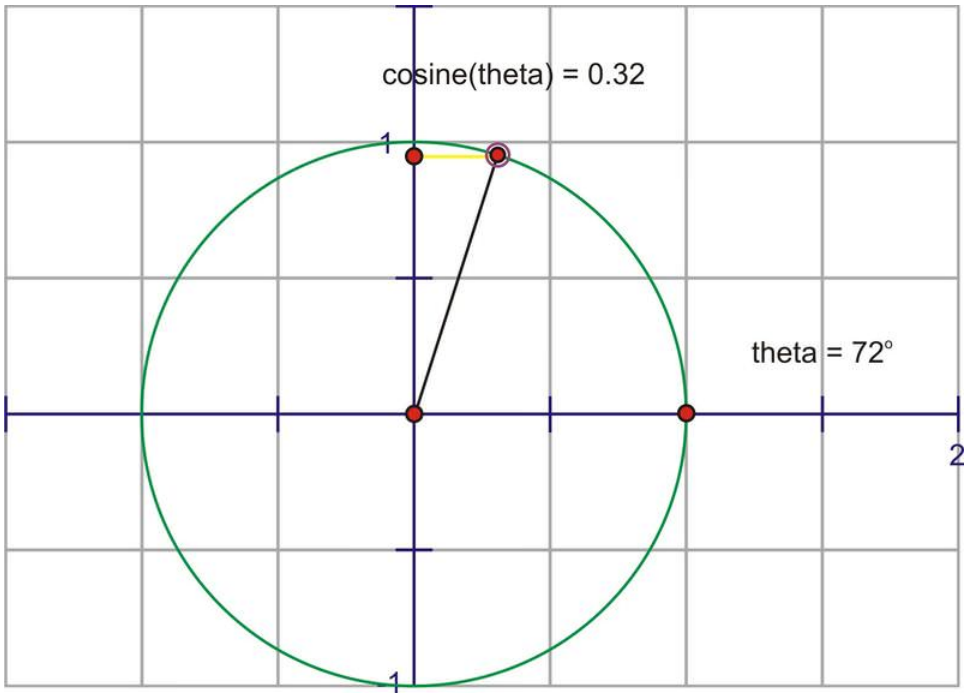


Figure 2.7.5.4

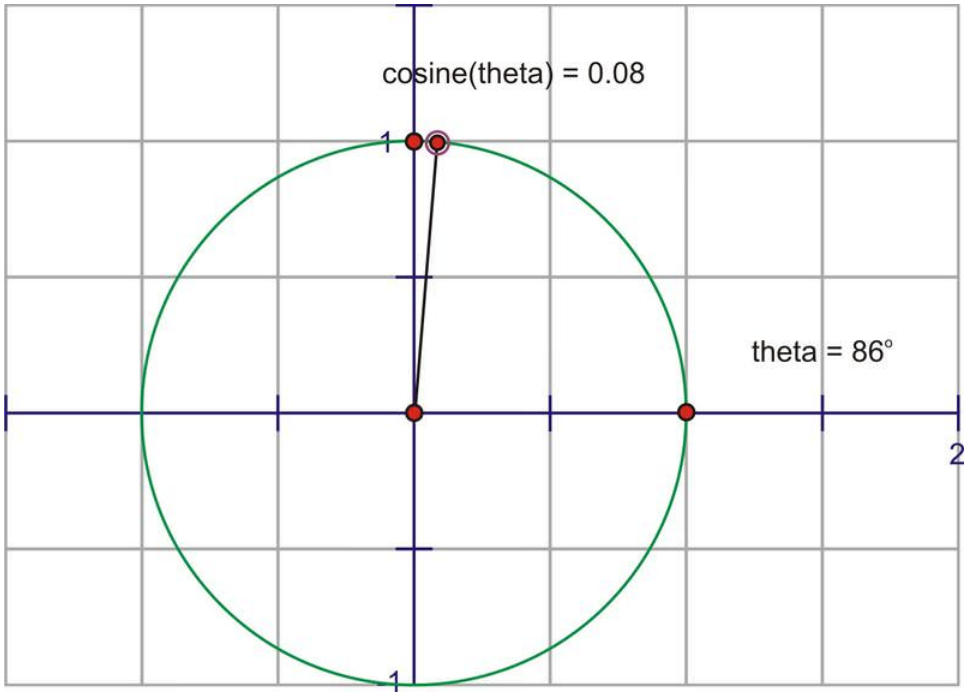
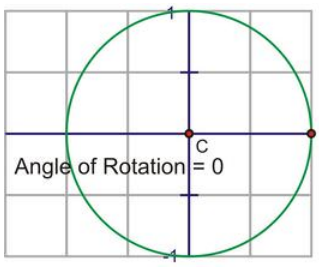


Figure 2.7.5.5

Here is a sequence of rotations. Compare the x− coordinate of the point of rotation with the height of the point as it traces along the horizontal. These pictures plot $(\theta, \cos\theta)$ on the coordinate plane as (x, y).



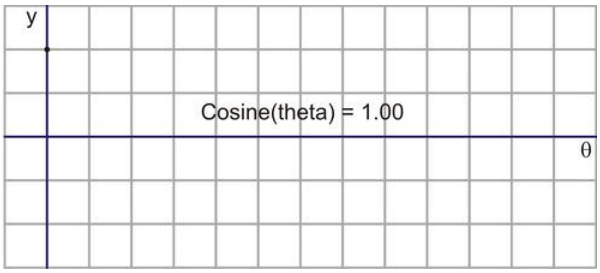
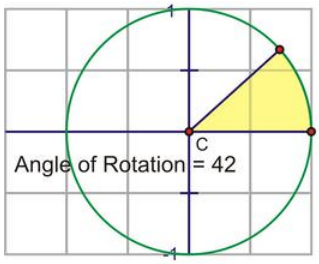


Figure **2.7.5.6**



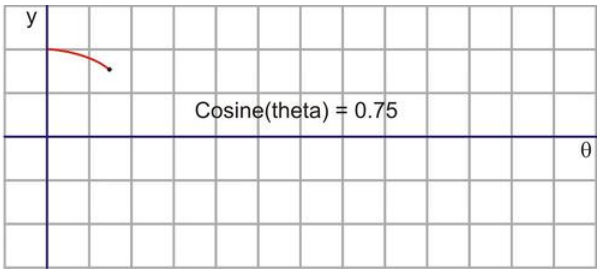
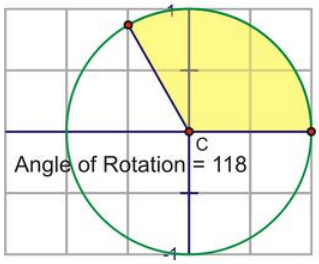


Figure **2.7.5.7**



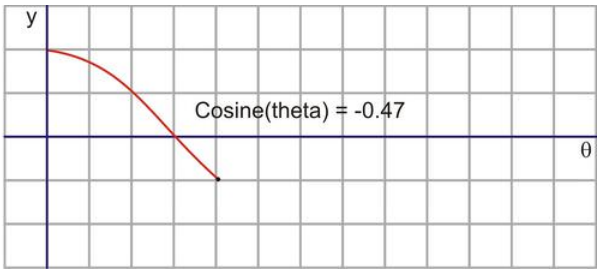
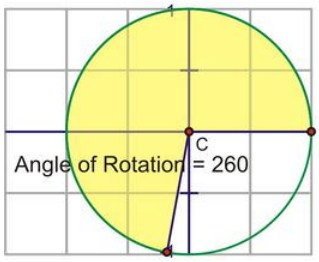


Figure **2.7.5.8**



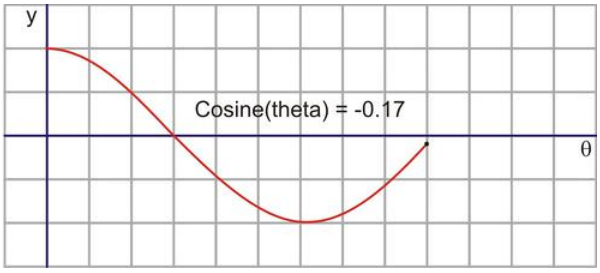
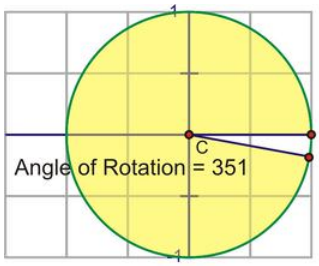


Figure **2.7.5.9**



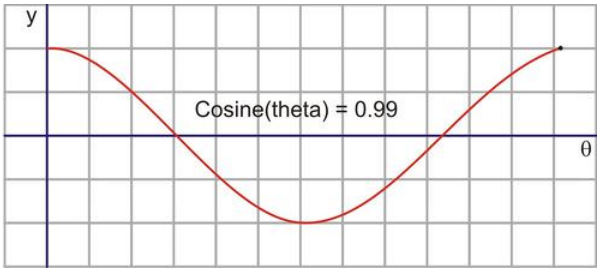


Figure **2.7.5.10**

Plotting the quadrant angles and filling in the in-between values shows the graph of $y = \cos x$

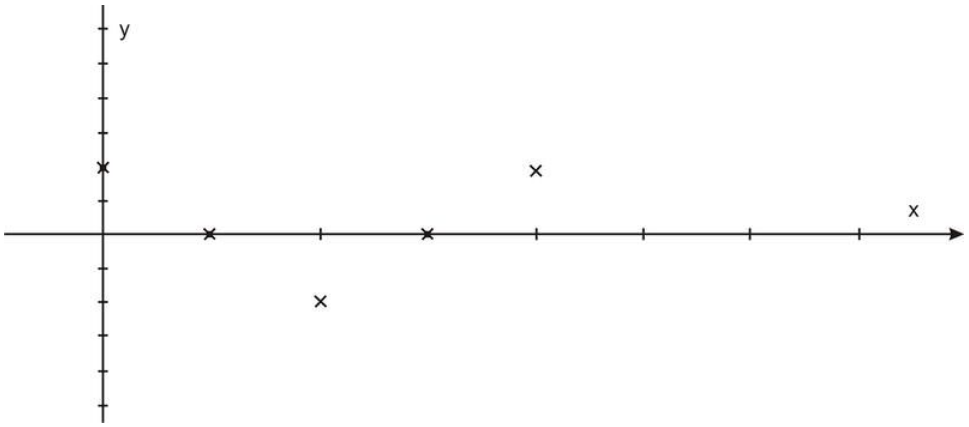


Figure 2.7.5.11

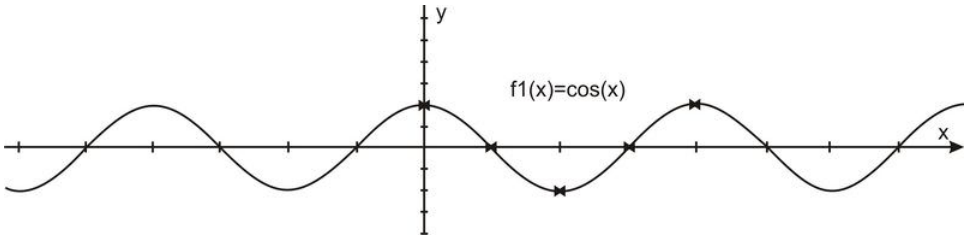


Figure 2.7.5.12

The graph of $y = \cos x$ has a period of 2π. The **range** of a cosine curve is $-1 \leq y \leq 1$ and the **domain** of $\cos x$ is all reals. If you've studied the sine function, you may notice that the shape of the curve is exactly the same, but *shifted* by $\frac{\pi}{2}$.

Secant is the reciprocal of cosine, or $\frac{1}{x}$. Therefore, whenever the cosine is zero, the secant is going to have a vertical asymptote because it will be undefined. It also has the same sign as the cosine function in the same quadrants. Here is the graph.

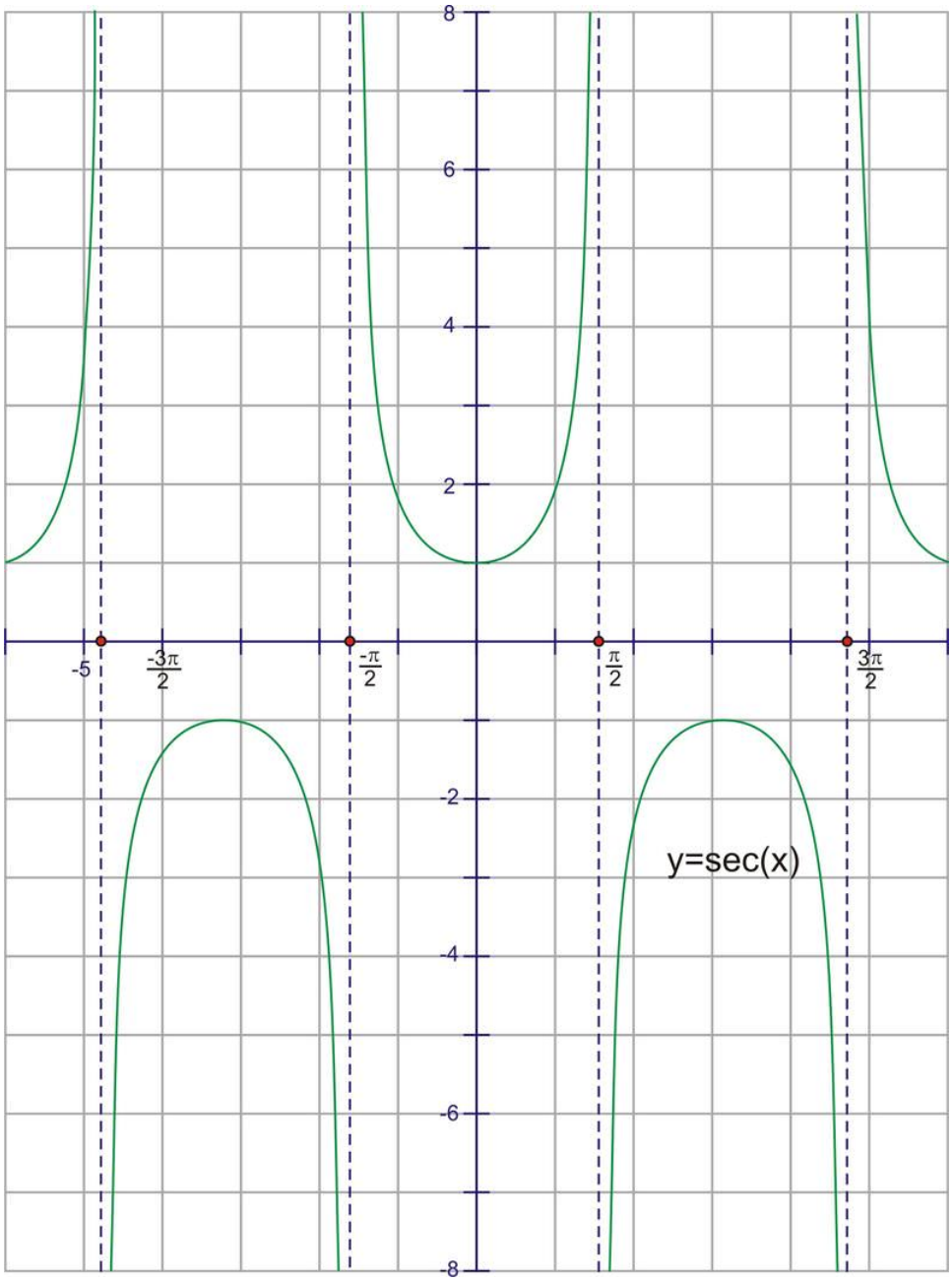


Figure **2.7.5.13**

The period of the function is 2π, just like cosine. The domain of the function is all real numbers, except multiples of π starting at $\frac{\pi}{2}$. $\left\{\ldots, -\frac{\pi}{2}, \frac{\pi}{2}, 0, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots\right\}$. The range is all real numbers greater than or equal to 1 as well as all real numbers less than or equal to -1. Notice that the range is everything *except* where cosine is defined (other than the tops and bottoms of the cosine curve).

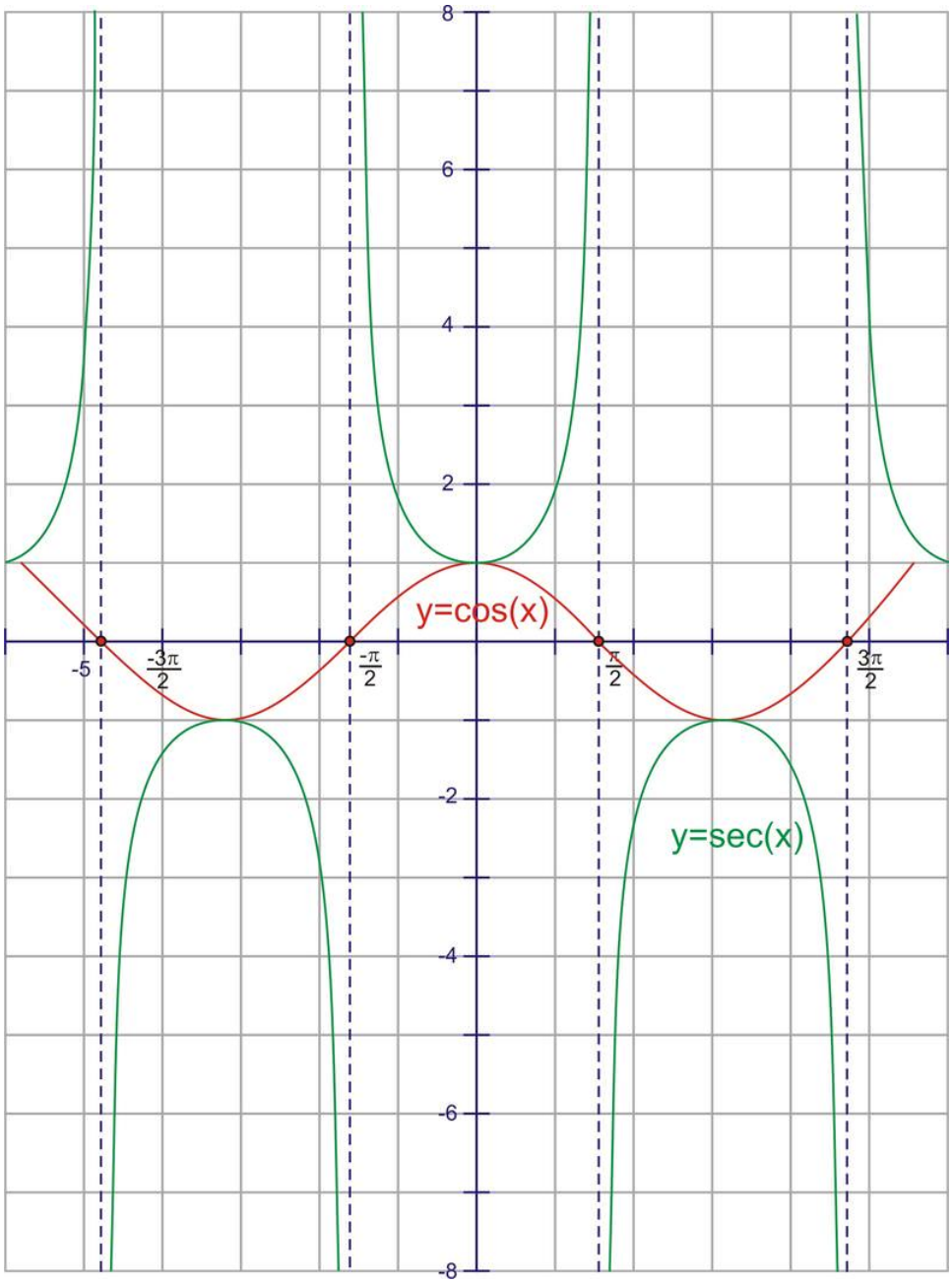
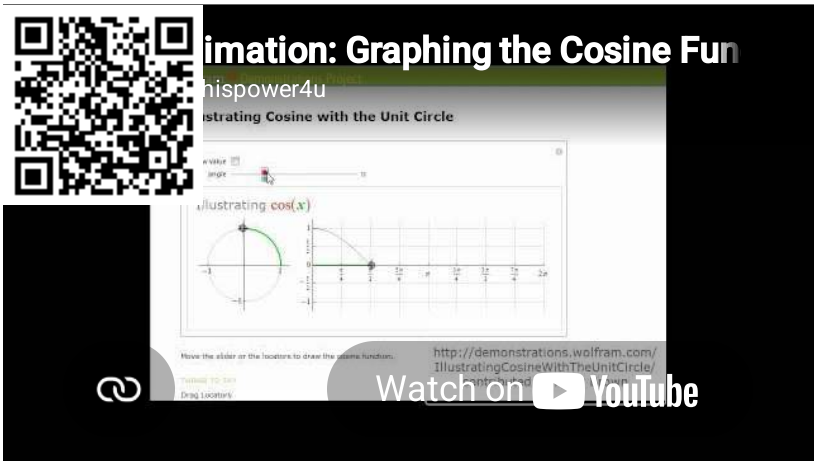


Figure **2.7.5.14**

Notice again the reciprocal relationships at 0 and the asymptotes. Also look at the intersection points of the graphs at 1 and -1. Again, this graph looks parabolic, but it is not.



Sketch the graph

Sketch a graph of $h(x) = 5 + \frac{1}{2}\sec 4x$ over the interval $[0, 2\pi]$.

If you compare this example to $f(x) = \sec x$, it will be translated 5 units up, with an amplitude of $\frac{1}{2}$ and a frequency of 4. This means in our interval of 0 to 2π, there will be 4 secant curves.

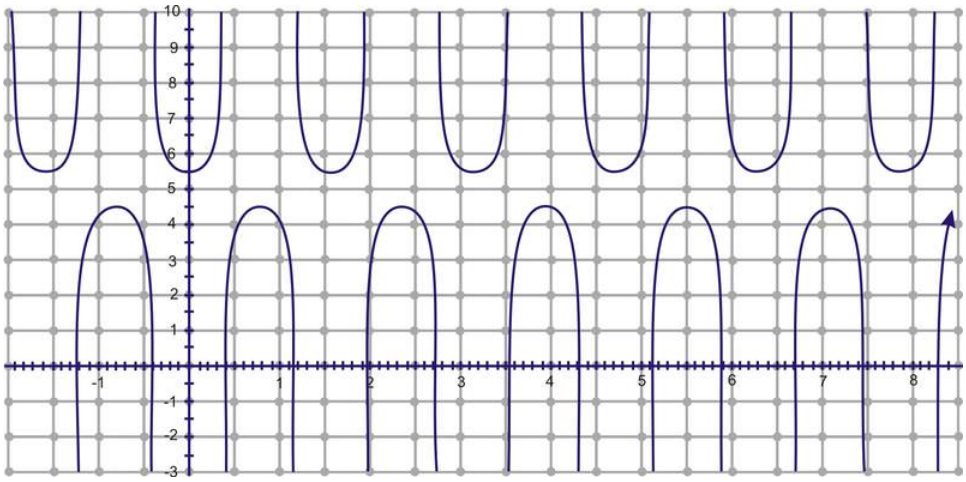


Figure 2.7.5.15

Find the equation for the graph below.

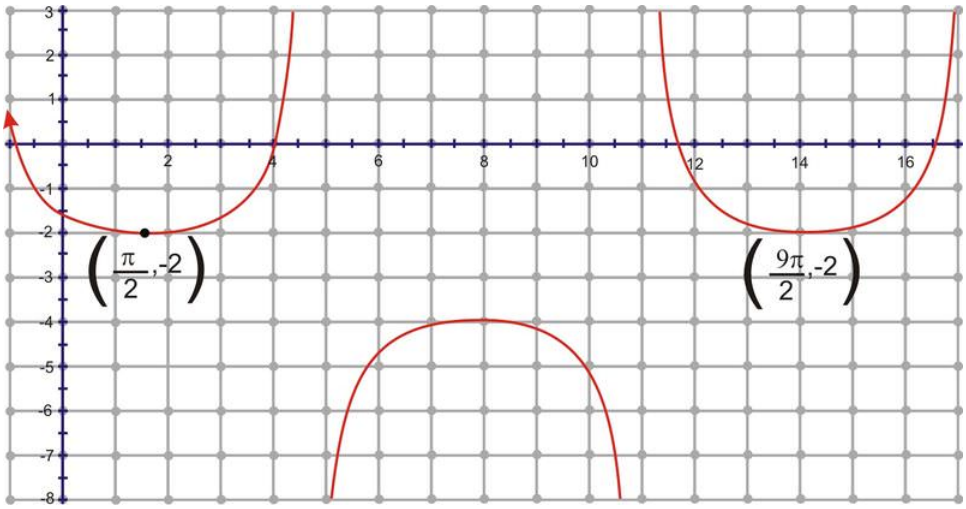


Figure 2.7.5.16

First of all, this could be either a secant or cosecant function. Let's say this is a secant function. Secant usually intersects the y−axis at (0,1) at a minimum. Now, that corresponding minimum is $\left(\frac{\pi}{2}, -2\right)$. Because there is no amplitude change, we can say that the vertical shift is the difference between the two y−values, -3. It looks like there is a phase shift and a period change. From minimum to minimum is one period, which is $\frac{9\pi}{2} - \frac{\pi}{2} = \frac{8\pi}{2} = 4\pi$ and $B = \frac{2\pi}{4\pi} = \frac{1}{2}$. Lastly, we need to find the horizontal shift. Since secant usually intersects the y−axis at $(0,1)$ at a minimum, and now the corresponding minimum is $\left(\frac{\pi}{2}, -2\right)$, we can say that the horizontal shift is the difference between the two x−values, $\frac{\pi}{2}$. Therefore, our equation is $f(x) = -3 + \sec\left(\frac{1}{2}\left(x - \frac{\pi}{2}\right)\right)$.

Graph the function $h(x) = 2 - 3\cos 4x$

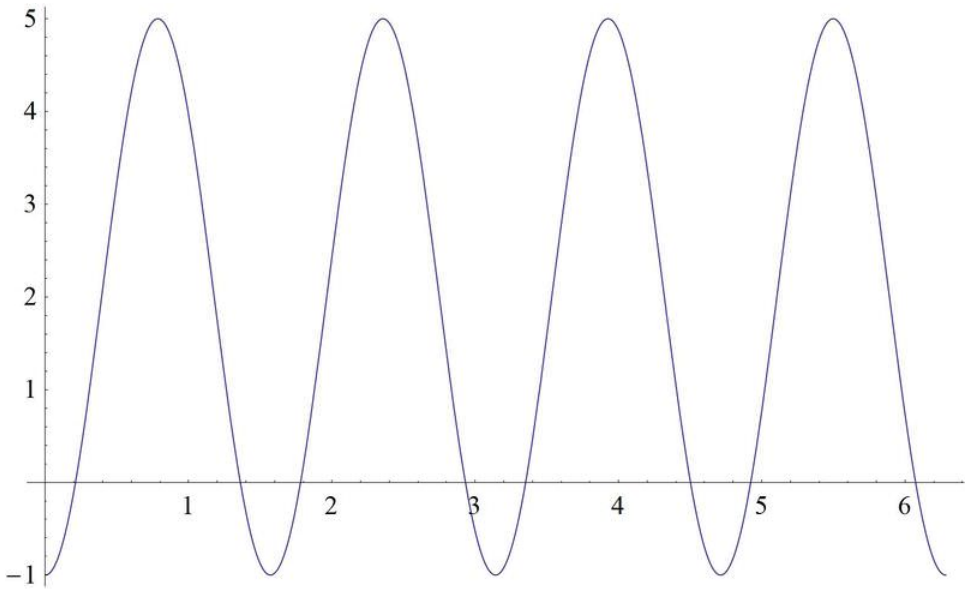


Figure **2.7.5.17**

Example 2.7.5.1

Earlier, you were asked what would the shadow trace out.

Solution

As you have learned in this section, a light shining down on the rotating hand would create a shadow in the pattern of a cosine function, starting at a maximum value as the hand is lying along the "x" axis, going through zero to a maximum negative value when the hand is lying along the negative "y" axis. It would then begin to increase until it returned to a maximum value when the rotating hand was again lying along the positive "x" axis.

Example 2.7.5.2

Graph $y = -2 + \frac{1}{2}\sec(4(x-1))$.

Solution

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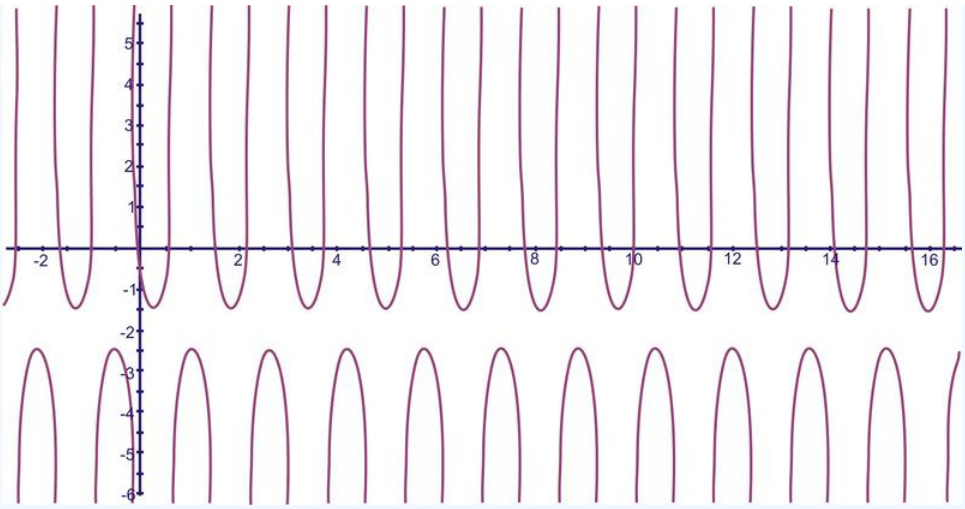


Figure 2.7.5.18

Example 2.7.5.3

Determine the function creating this graph:

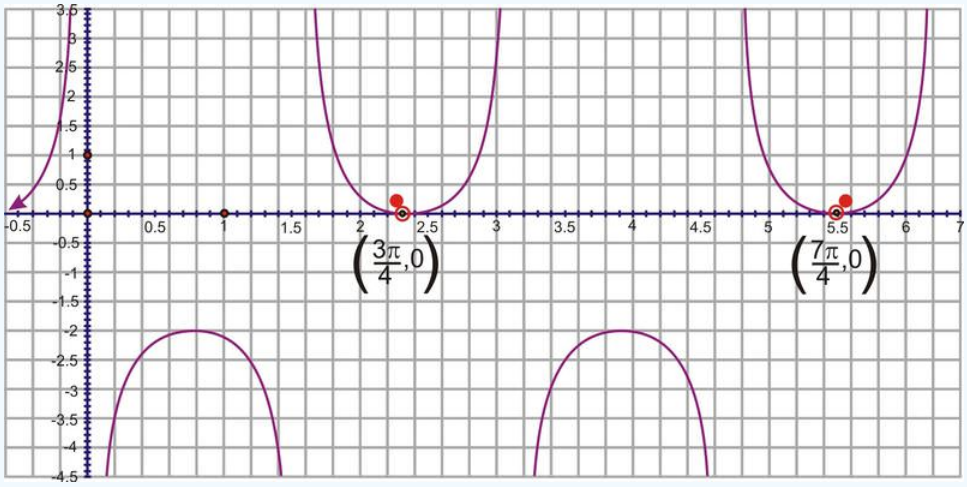


Figure 2.7.5.19

Solution

This could be either a secant or cosecant function. We will use a cosecant model. First, the vertical shift is -1. The period is the difference between the two given x−values, $7\frac{\pi}{4}-\frac{3\pi}{4}=\pi$, so the frequency is $\frac{2\pi}{\pi}=2$. The horizontal shift incorporates the frequency, so in $y=\csc x$ the corresponding x−value to $\left(\frac{3\pi}{4},0\right)$ is $\left(\frac{\pi}{2},1\right)$. The difference between the x−values is $\frac{3\pi}{4}-\frac{\pi}{2}$ $=\frac{3\pi}{4}-\frac{2\pi}{4}=\frac{\pi}{4}$ and then multiply it by the frequency, $2\cdot\frac{\pi}{4}=\frac{\pi}{2}$. The equation is $y=-1+\csc\left(2\left(x-\frac{\pi}{2}\right)\right)$.

Example 2.7.5.4

Graph $h(x)=\frac{1}{3}\cos 2x$

Solution

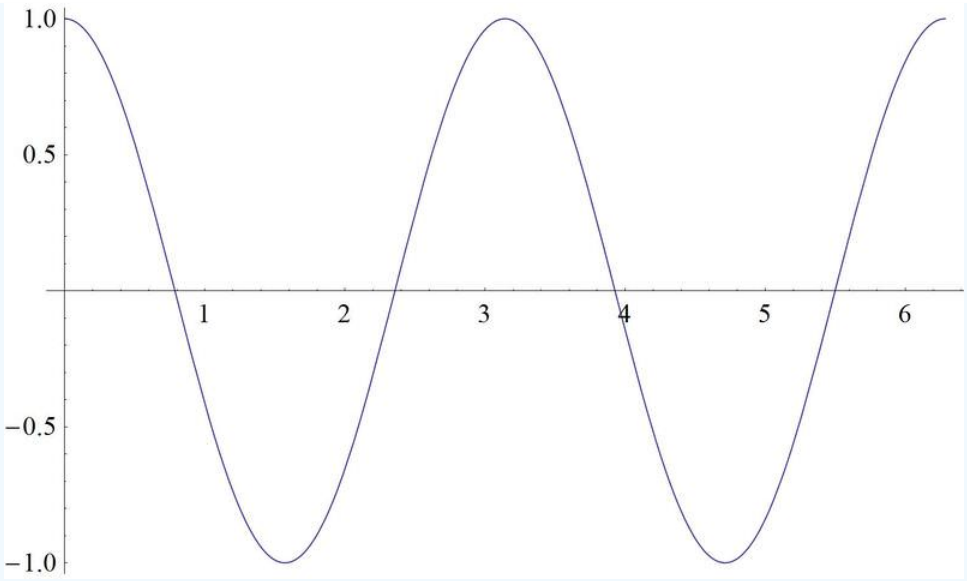


Figure 2.7.5.20

Review

Graph each of the following functions.

1. $f(x) = \cos(x)$.
2. $h(x) = \cos(2x)$.
3. $k(x) = \cos(2x + \pi)$.
4. $m(x) = -2\cos(2x + \pi)$.
5. $g(x) = -2\cos(2x + \pi) + 1$.
6. $f(x) = \sec(x)$.
7. $h(x) = \sec(3x)$.
8. $k(x) = \sec(3x + \pi)$.
9. $m(x) = 2\sec(3x + \pi)$.
10. $g(x) = 3 + 2\sec(3x + \pi)$.
11. $h(x) = \cos\left(\frac{x}{2}\right)$.
12. $k(x) = \cos\left(\frac{x}{2} + \frac{\pi}{2}\right)$.
13. $m(x) = 2\cos\left(\frac{x}{2} + \frac{\pi}{2}\right)$.
14. $g(x) = 2\cos\left(\frac{x}{2} + \frac{\pi}{2}\right) - 3$.
15. $h(x) = \sec\left(\frac{x}{4}\right)$.
16. $k(x) = \sec\left(\frac{x}{4} + \frac{3\pi}{2}\right)$.
17. $m(x) = -3\sec\left(\frac{x}{4} + \frac{3\pi}{2}\right)$.
18. $g(x) = 2 - 3\sec\left(\frac{x}{4} + \frac{3\pi}{2}\right)$.

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.10.

Vocabulary

Term	Definition
Circular Function	A circular function is a function measured by examining the angle of rotation around the coordinate plane.

Additional Resources

Video: Animation: Graphing the Cosine Function Using the Unit Circle

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2.7.6: Graph Inverse Trigonometric Functions

Domain and range of inverse functions.

In order for inverses of functions to be functions, the original function must pass the **horizontal line test**. Though none of the trigonometric functions pass the horizontal line test, you can restrict their domains so that they can pass. Then the inverses are produced just like with normal functions. Once you have the basic inverse functions, the normal transformation rules apply.

Why is $\sin^{-1}(\sin 370°) \neq 370°$? Don't the arcsin and sin just cancel out?

Graphs of Inverse Trigonometric Functions

Since none of the six trigonometric functions pass the horizontal line test, you must **restrict their domains** before finding inverses of these functions. This is just like the way $y = \sqrt{x}$ is the inverse of $y = x^2$ when you restrict the **domain** to $x \geq 0$.

Consider the sine graph:

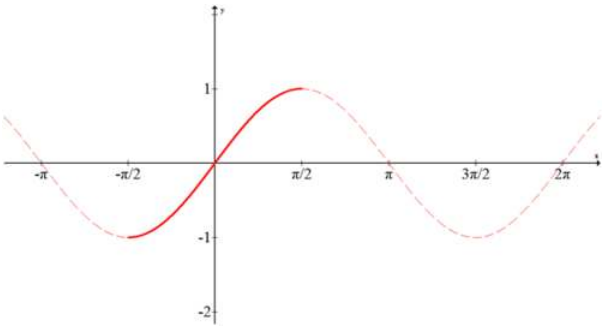


Figure **2.7.6.1**

As a general rule, the restrictions to the domain are either the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ or $[0, \pi]$ to keep things simple. In this case sine is restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, as shown above. To find the inverse, reflect the bold portion across the line $y = x$. The blue curve below shows $f(x) = \sin^{-1} x$.

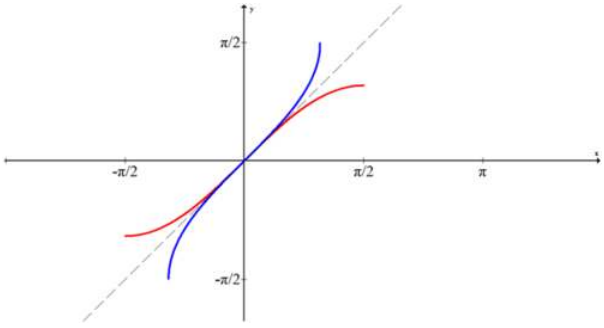


Figure **2.7.6.2**

The result of this inversion is that arcsine will only ever produce angles between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

The blue curve below shows $f(x) = \cos^{-1} x$?

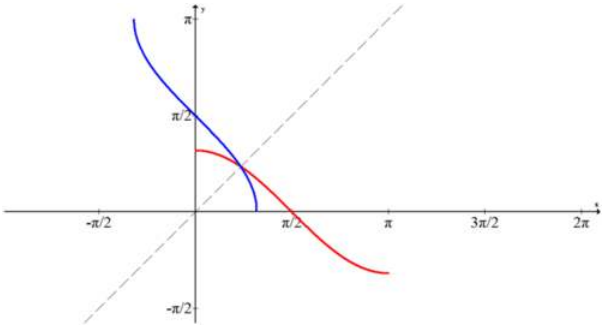
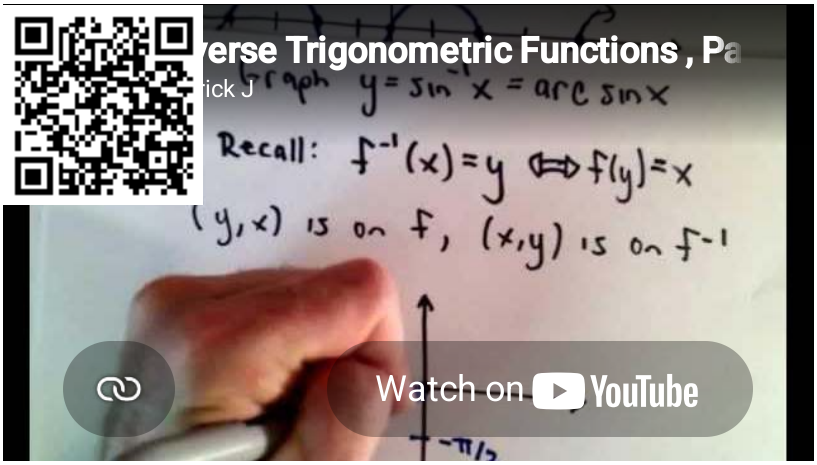


Figure **2.7.6.3**

The portion of cosine that fits the horizontal line test is the interval $[0, \pi]$. To find the inverse, that portion is reflected across the line $y = x$.



Example 2.7.6.1

Earlier, you were asked why sine and arcsine don't always just cancel out.

Solution

Since arcsine only produces angles between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ or $-90°$ to $+90°$ the result of $\sin^{-1}(\sin 370°)$ is $10°$ which is **coterminal** to $370°$.

Example 2.7.6.2

Graph the function $f(x) = 2\cos^{-1}(x - 1)$.

Solution

Since $f(x) = \cos^{-1} x$ was graphed earlier, now you just need to shift it right one unit and stretch it vertically by a factor of 2. It intersected the x axis at 1 before and now it will intersect at 2. It reached a height of π before and now it will reach a height of 2π.

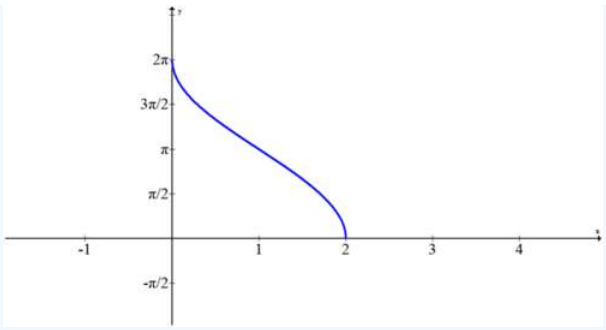


Figure **2.7.6.4**

Example 2.7.6.3

Evaluate the following expression with and without a calculator using right triangles and your knowledge of inverse trigonometric functions.

$$\cot\left(\csc^{-1}\left(-\frac{13}{5}\right)\right)$$

Solution

In order to be able to effectively calculate this out it is best to write the expression explicitly only in terms of functions that your calculator does have. Be aware that some calculators have both $\sin^{-1} x$ and $(\sin x)^{-1}$.

The hardest part of this question is seeing the csc as a function (which produces an angle) on a ratio of a hypotenuse of 13 and an opposite side of -5. The sine of the inverse ratio must produce the same angle, so you can substitute it.

- $\csc^{-1}\left(-\frac{13}{5}\right) = \sin^{-1}(-\frac{5}{13})$
- $\cot(\theta) = \frac{1}{\tan\theta}$

$$\cot\left(\csc^{-1}\left(-\frac{13}{5}\right)\right) = \frac{1}{\tan\left(\sin^{-1}\left(-\frac{5}{13}\right)\right)} = -\frac{12}{5}$$

Start with your knowledge that $\csc^{-1}\left(-\frac{13}{5}\right)$ describes an angle in the third or the fourth quadrant because those are the two quadrants where **cosecant** is negative. Since $\csc^{-1}\theta$ has range $-\frac{\pi}{2}, \frac{\pi}{2}$, it only produces angles in quadrant I or quadrant IV. This triangle must then be in the fourth quadrant. All you need to do is draw the triangle and identify the **cotangent** ratio.

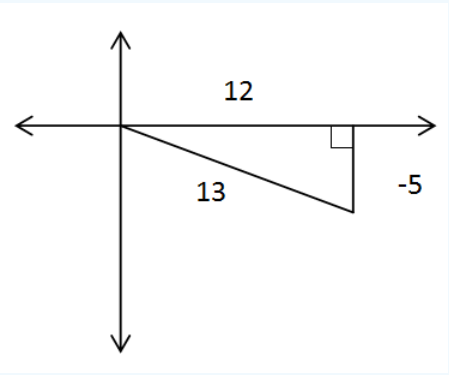


Figure **2.7.6.5**

Cotangent is adjacent over opposite.

$$\cot\left(\csc^{-1}\left(-\frac{13}{5}\right)\right) = -\frac{12}{5}$$

Example 2.7.6.4

What is the graph of $y = \tan^{-1} x$?

Solution

Graph the portion of tangent that fits the horizontal line test and reflect across the line y=x. Note that the graph of arctan is in blue.

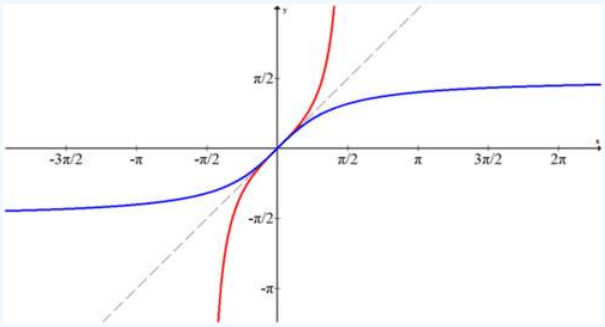


Figure 2.7.6.6

Example 2.7.6.5

What is the graph of $y = \csc^{-1} x$

Solution

Graph the portion of cosecant that fits the horizontal line test and reflect across the line y=x.

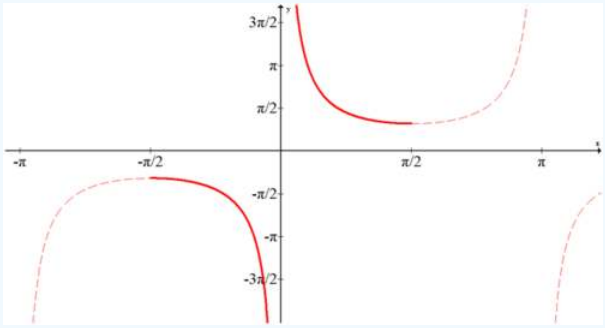


Figure 2.7.6.7

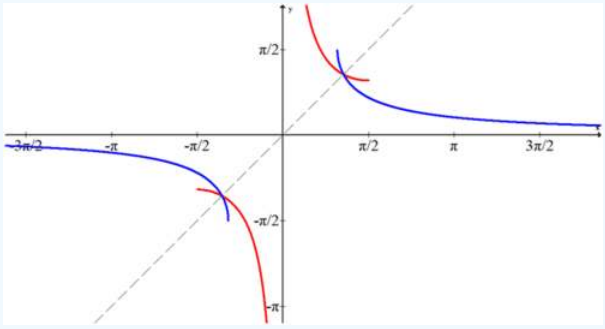


Figure 2.7.6.8

Note that $f(x) = \csc^{-1} x$ is in blue.

Review

1. Graph $f(x) = \cot^{-1} x$.
2. Graph $g(x) = \sec^{-1} x$.

Name each of the following graphs.

3.

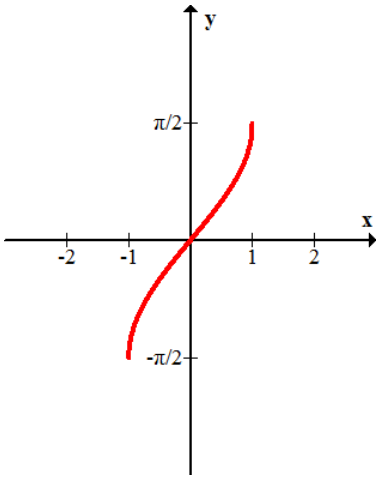


Figure **2.7.6.8**

4.

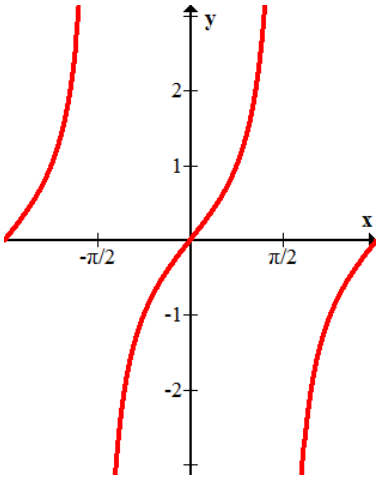


Figure **2.7.6.9**

5.

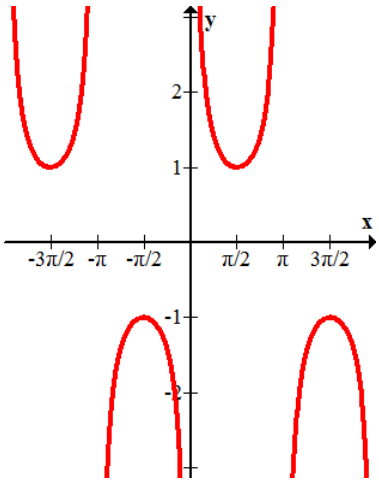


Figure **2.7.6.10**

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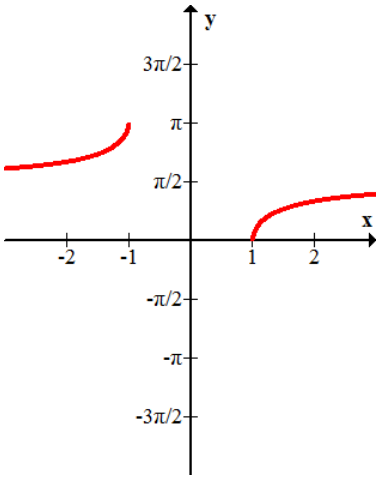


Figure **2.7.6.11**

7.

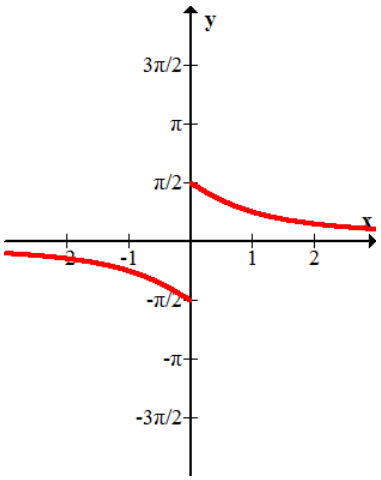


Figure **2.7.6.12**

Graph each of the following functions using your knowledge of function **transformations**.

8. $h(x) = 3\sin^{-1}(x+1)$
9. $k(x) = 2\sin^{-1}(x) + \frac{\pi}{2}$
10. $m(x) = -\cos^{-1}(x-2)$
11. $j(x) = \cot^{-1}(x) + \pi$
12. $p(x) = -2\tan^{-1}(x-1)$
13. $q(x) = \csc^{-1}(x-2)$
14. $r(x) = -\sec^{-1}(x) + 4$
15. $t(x) = \csc^{-1}(x+1) - \frac{3\pi}{2}$
16. $v(x) = 2\sec^{-1}(x+2) + \frac{\pi}{2}$
17. $w(x) = -\cot^{-1}(x) - \frac{\pi}{2}$

Evaluate each expression.

18. $\sec\left(\tan^{-1}\left[\frac{3}{4}\right]\right)$
19. $\cot\left(\csc^{-1}\left[\frac{13}{12}\right]\right)$
20. $\csc\left(\tan^{-1}\left[\frac{4}{3}\right]\right)$

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.8.

Vocabulary

Term	Definition
Cosecant	The cosecant of an angle in a right triangle is a relationship found by dividing the length of the hypotenuse by the length of the side opposite to the given angle. This is the reciprocal of the sine function.
Cotangent	The cotangent of an angle in a right triangle is a relationship found by dividing the length of the side adjacent to the given angle by the length of the side opposite to the given angle. This is the reciprocal of the tangent function.
Coterminal	Two angles are coterminal if they are drawn in the standard position and both have terminal sides that are at the same location.
domain	The domain of a function is the set of x-values for which the function is defined.
Horizontal Line Test	The horizontal line test says that if a horizontal line drawn anywhere through the graph of a function intersects the function in more than one location, then the function is not one-to-one and not invertible.
restricted domain	Restricted domain refers to the fact that when creating an inverse you sometimes must cut off the domain of most of the function, saving the largest possible portion so that when the inverse is created it is also a function.
Secant	The secant of an angle in a right triangle is the value found by dividing length of the hypotenuse by the length of the side adjacent the given angle. The secant ratio is the reciprocal of the cosine ratio.
Transformations	Transformations are used to change the graph of a parent function into the graph of a more complex function.

Additional Resources

Practice: Graph Inverse Trigonometric Functions

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3.5.1: Sum to Product Formulas for Sine and Cosine

3.5.2: Product to Sum Formulas for Sine and Cosine

3.5.3: Triple-Angle Formulas and Linear Combinations

Thumbnail: The sine and cosine function in the unit circle. (Public Domain; Stephan Kulla via Wikipedia)

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3.1.1: Fundamental Trigonometric Identities

Prove equations are true using Reciprocal, Tangent, and other identities.

Basic Trigonometric Identities

The basic trigonometric identities are ones that can be logically deduced from the definitions and graphs of the six trigonometric functions. Previously, some of these identities have been used in a casual way, but now they will be formalized and added to the toolbox of trigonometric identities.

How can you use the trigonometric identities to simplify the following expression?

$$\left[\frac{\sin(\frac{\pi}{2}-\theta)}{\sin(-\theta)}\right]^{-1}$$

Trigonometric Identities

An **identity** is a mathematical sentence involving the symbol "=" that is always true for variables within the domains of the expressions on either side.

Reciprocal Identities

The **reciprocal identities** refer to the connections between the trigonometric functions like sine and cosecant. Sine is opposite over hypotenuse and cosecant is hypotenuse over opposite. This logic produces the following six identities.

- $\sin\theta = \frac{1}{\csc\theta}$
- $\cos\theta = \frac{1}{\sec\theta}$
- $\tan\theta = \frac{1}{\cot\theta}$
- $\cot\theta = \frac{1}{\tan\theta}$
- $\sec\theta = \frac{1}{\cos\theta}$
- $\csc\theta = \frac{1}{\sin\theta}$

Quotient Identities

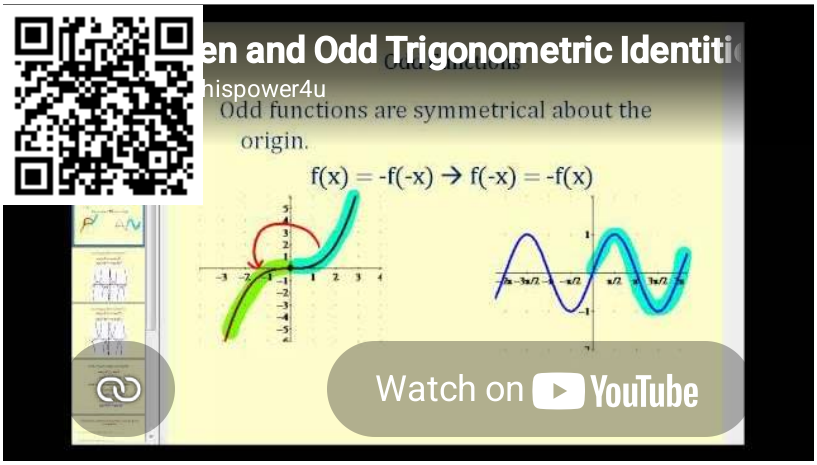
The **quotient identities** follow from the definition of sine, cosine and tangent.

- $\tan\theta = \sin\theta\cos\theta$
- $\cot\theta = \cos\theta\sin\theta$

Odd/**Even** Identities

The **odd-even identities** follow from the fact that only cosine and its reciprocal secant are even and the rest of the trigonometric functions are odd.

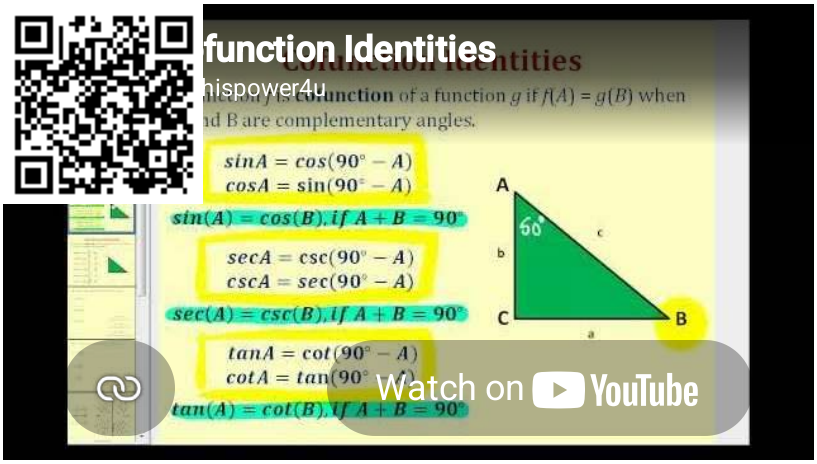
- $\sin(-\theta) = -\sin\theta$
- $\cos(-\theta) = \cos\theta$
- $\tan(-\theta) = -\tan\theta$
- $\cot(-\theta) = -\cot\theta$
- $\sec(-\theta) = \sec\theta$
- $\csc(-\theta) = -\csc\theta$



Cofunction Identities

The **cofunction identities** make the connection between trigonometric functions and their "co" counterparts like sine and cosine. Graphically, all of the cofunctions are reflections and horizontal shifts of each other.

- $\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$
- $\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$
- $\tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta$
- $\cot\left(\frac{\pi}{2}-\theta\right)=\tan\theta$
- $\sec\left(\frac{\pi}{2}-\theta\right)=\csc\theta$
- $\csc\left(\frac{\pi}{2}-\theta\right)=\sec\theta$



Example 3.1.1.1

Earlier, you were asked how you could simplify the trigonometric expression:

$$\left[\frac{\sin\left(\frac{\pi}{2}-\theta\right)}{\sin(-\theta)}\right]^{-1}$$

Solution

It can be simplified to be equivalent to negative tangent as shown below:

$$\left[\frac{\sin(\frac{\pi}{2}-\theta)}{\sin(-\theta)}\right]^{-1} = \frac{\sin(-\theta)}{\sin(\frac{\pi}{2}-\theta)}$$
$$= -\sin\theta\cos\theta$$
$$= -\tan\theta$$

Example 3.1.1.2

If $\sin\theta = 0.87$, find $\cos\left(\theta - \frac{\pi}{2}\right)$.

Solution

While it is possible to use a calculator to find \theta , using identities works very well too.

First you should factor out the negative from the argument. Next you should note that cosine is even and apply the odd-even identity to discard the negative in the argument. Lastly recognize the cofunction identity.

$$\cos\left(\theta - \frac{\pi}{2}\right) = \cos\left(-\left(\frac{\pi}{2} - \theta\right)\right) = \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta = 0.87$$

Example 3.1.1.3

If $\cos\left(\theta - \frac{\pi}{2}\right) = 0.68$ then determine $\csc(-\theta)$.

Solution

You need to show that $\cos\left(\theta - \frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2} - \theta\right)$.

$$0.68 = \cos\left(\theta - \frac{\pi}{2}\right)$$
$$= \cos\left(\frac{\pi}{2} - \theta\right)$$
$$= \sin(\theta)$$

Then, $\csc(-\theta) = -\csc\theta$

$$= -\frac{1}{\sin\theta}$$
$$= -(0.68)^{-1} \approx -1.47$$

Example 3.1.1.4

Use identities to prove the following: $\cot(-\beta)\cot\left(\frac{\pi}{2} - \beta\right)\sin(-\beta) = \cos\left(\beta - \frac{\pi}{2}\right)$.

Solution

When doing trigonometric proofs, it is vital that you start on one side and only work with that side until you derive what is on the other side. Sometimes it may be helpful to work from both sides and find where the two sides meet, but this work is not considered a **proof**. You will have to rewrite your steps so they follow from only one side. In this case, work with the left side and keep rewriting it until you have $\cos\left(\beta - \frac{\pi}{2}\right)$.

$$\cot(-\beta)\cot\left(\frac{\pi}{2} - \beta\right)\sin(-\beta)$$
$$= -\cot\beta\tan\beta \cdot -\sin\beta$$
$$= -1 \cdot -sin\beta$$
$$= sin\beta$$
$$= \cos\left(\frac{\pi}{2} - \beta\right)$$
$$= \cos\left(-\left(\beta - \frac{\pi}{2}\right)\right)$$
$$= \cos\left(\beta - \frac{\pi}{2}\right)$$

Example 3.1.1.5

Prove the following trigonometric identity by working with only one side.

$$\cos x \sin x \tan x \cot x \sec x \csc x = 1$$

Solution

$$\cos x \sin x \tan x \cot x \sec x \csc x = \cos x \sin x \tan x \cdot \frac{1}{\tan x} \cdot \frac{1}{\cos x} \cdot \frac{1}{\sin x} \quad = 1$$

Review

1. Prove the quotient identity for cotangent using sine and cosine.
2. Explain why $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ using graphs and transformations.
3. Explain why $\sec\theta = \frac{1}{\cos\theta}$.
4. Prove that $\tan\theta \cdot cot\theta = 1$.
5. Prove that $\sin\theta \cdot \csc\theta = 1$.
6. Prove that $\sin\theta \cdot sec\theta = \tan\theta$.
7. Prove that $\cos\theta \cdot \csc\theta = \cot\theta$.
8. If $\sin\theta = 0.81$, what is $\sin(-\theta)$?
9. If $\cos\theta = 0.5$, what is $\cos(-\theta)$?
10. If $\cos\theta = 0.25$, what is $\sec(-\theta)$?
11. If $\csc\theta = 0.7$, what is $\sin(-\theta)$?
12. How can you tell from a graph if a function is even or odd?
13. Prove $\tan x \cdot \sec x \csc x \cdot \cot x = \tan x$.
14. Prove $\sin 2x \cdot \sec x \tan x \cdot \csc x = 1$.
15. Prove $\cos x \cdot \tan x = \sin x$.

Review (Answers)

To see the Review answers, open this PDF file and look for section 6.1.

Vocabulary

Term	Definition
cofunction	Cofunctions are functions that are identical except for a reflection and horizontal shift. Examples include: sine and cosine, tangent and cotangent, secant and cosecant.
even	An even function is a function with a graph that is symmetric with respect to the y-axis and has the property that $f(-x) = f(x)$.
identity	An identity is a mathematical sentence involving the symbol "=" that is always true for variables within the domains of the expressions on either side.
Odd Function	An odd function is a function with the property that $f(-x) = -f(x)$. Odd functions have rotational symmetry about the origin.
proof	A proof is a series of true statements leading to the acceptance of truth of a more complex statement.

Additional Resources

Video: Trigonometric Identities - Overview

Practice: Fundamental Trigonometric Identities

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3.1.2: Quotient Identities

Tangent equals sine divided by cosine.

You are working in math class one day when your friend leans over and asks you what you got for the sine and cosine of a particular angle.

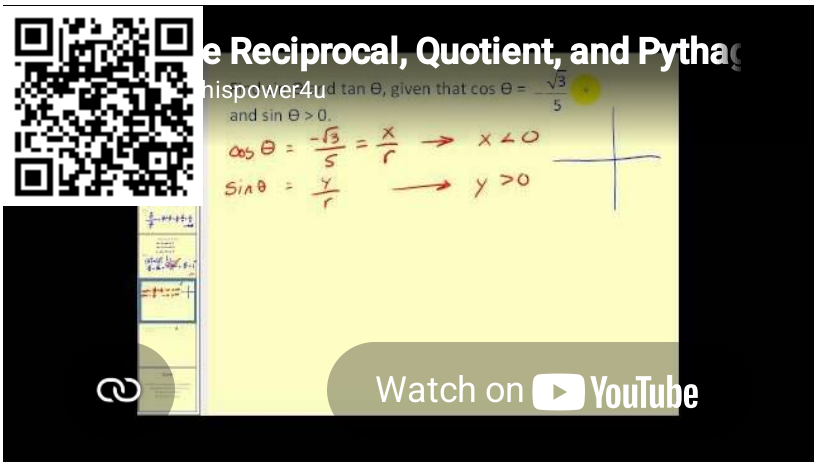
"I got $\frac{1}{2}$ for the sine, and $\frac{\sqrt{3}}{2}$ for the cosine. Why?" you ask.

"It looks like I'm supposed to calculate the tangent function for the same angle you just did, but I can't remember the relationship for tangent. What should I do?" he says.

Do you know how you can help your friend find the answer, even if both you and he don't remember the relationship for tangent?

Quotient Identities

The definitions of the trig functions led us to the reciprocal identities, which can be seen in the Concept about that topic. They also lead us to another set of identities, the quotient identities.



Consider first the sine, cosine, and tangent functions. For angles of rotation (not necessarily in the unit circle) these functions are defined as follows:

$$\sin\theta = \frac{y}{r}$$
$$\cos\theta = \frac{x}{r}$$
$$\tan\theta = \frac{y}{x}$$

Given these definitions, we can show that $\tan\theta = \frac{\sin\theta}{\cos\theta}$, as long as $\cos\theta \neq 0$:

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{r} \times \frac{r}{x} = \frac{y}{x} = \tan\theta.$$

The equation $\tan\theta = \frac{\sin\theta}{\cos\theta}$ is therefore an identity that we can use to find the value of the tangent function, given the value of the sine and cosine.

Let's take a look at some problems involving quotient identities.

1. Find the value of $\tan\theta$?

If $\cos\theta = \frac{5}{13}$ and $\sin\theta = \frac{12}{13}$, what is the value of $\tan\theta$?

$\tan\theta = \frac{12}{5}$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{12}{13}}{\frac{5}{13}} = \frac{12}{13} \times \frac{13}{5} = \frac{12}{5}$$

2. Show that $\cot\theta = \frac{\cos\theta}{\sin\theta}$

$$\cos\theta \sin\theta = \frac{\frac{x}{r}}{\frac{y}{r}} = \frac{x}{r} \times \frac{r}{y} = \frac{x}{y} = \cot\theta$$

3. What is the value of $\cot\theta$?

If $\cos\theta = \frac{7}{25}$ and $\sin\theta = \frac{24}{25}$, what is the value of $\cot\theta$?

$\cot\theta = \frac{7}{24}$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\frac{7}{25}}{\frac{24}{25}} = \frac{7}{25} \times \frac{25}{24} = \frac{7}{24}$$

Example 3.1.2.1

Earlier, you were asked if you can help your friend find the answer.

Solution

Since you now know that:

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

you can use this knowledge to help your friend with the sine and cosine values you measured for yourself earlier:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

Example 3.1.2.2

If $\cos\theta = \frac{17}{145}$ and $\sin\theta = \frac{144}{145}$, what is the value of $\tan\theta$?

Solution

$\tan\theta = \frac{144}{17}$. We can see this from the relationship for the tangent function:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{144}{145}}{\frac{17}{145}} = \frac{144}{145} \times \frac{145}{17} = \frac{144}{17}$$

Example 3.1.2.3

If $\sin\theta = \frac{63}{65}$ and $\cos\theta = \frac{16}{65}$, what is the value of $\tan\theta$?

Solution

$\tan\theta = \frac{63}{16}$. We can see this from the relationship for the tangent function:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{63}{65}}{\frac{16}{65}} = \frac{63}{65} \times \frac{65}{16} = \frac{63}{16}$$

Example 3.1.2.4

If $\tan\theta = \frac{40}{9}$ and $\cos\theta = \frac{9}{41}$, what is the value of $\sin\theta$?

Solution

$\sin\theta = \frac{40}{41}$. We can see this from the relationship for the tangent function:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$\sin\theta = (\tan\theta)(\cos\theta)$$
$$\sin\theta = \frac{40}{9} \times \frac{9}{41}$$
$$\sin\theta = \frac{40}{41}$$

Review

Fill in each blank with a trigonometric function.

1. $\tan\theta = \frac{\sin\theta}{?}$
2. $\cos\theta = \frac{\sin\theta}{?}$
3. $\cot\theta = \frac{?}{\sin\theta}$
4. $\cos\theta = (\cot\theta) \cdot (?)$
5. If $\cos\theta = \frac{5}{13}$ and $\sin\theta = \frac{1}{13}$, what is the value of $\tan\theta$?
6. If $\sin\theta = \frac{3}{5}$ and $\cos\theta = \frac{4}{5}$, what is the value of $\tan\theta$?
7. If $\cos\theta = \frac{7}{25}$ and $\sin\theta = \frac{24}{25}$, what is the value of $\tan\theta$?
8. If $\sin\theta = \frac{12}{37}$ and $\cos\theta = \frac{35}{37}$, what is the value of $\tan\theta$?
9. If $\cos\theta = \frac{20}{29}$ and $\sin\theta = \frac{21}{29}$, what is the value of$\tan\theta$?
10. If $\sin\theta = \frac{39}{89}$ and $\cos\theta = \frac{80}{89}$, what is the value of $\tan\theta$?
11. If $\cos\theta = \frac{48}{73}$ and $\sin\theta = \frac{55}{73}$, what is the value of $\tan\theta$?
12. If$\sin\theta = \frac{65}{97}$ and $\cos\theta = \frac{72}{97}$, what is the value of $\tan\theta$?
13. If $\cos\theta = \frac{1}{2}$ and $\cot\theta = \frac{\sqrt{3}}{3}$, what is the value of $\sin\theta$?
14. If $\tan\theta = 0$ and $\cos\theta = -1$, what is the value of $\sin\theta$?
15. If $\cot\theta = -1$ and $\sin\theta = -\frac{\sqrt{2}}{2}$, what is the value of $\cos\theta$?

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.23.

Vocabulary

Term	Definition
Quotient Identity	The quotient identity is an identity relating the tangent of an angle to the sine of the angle divided by the cosine of the angle.

Additional Resources

Video: The Reciprocal, Quotient, and Pythagorean Identities

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3.1.3: Reciprocal Identities

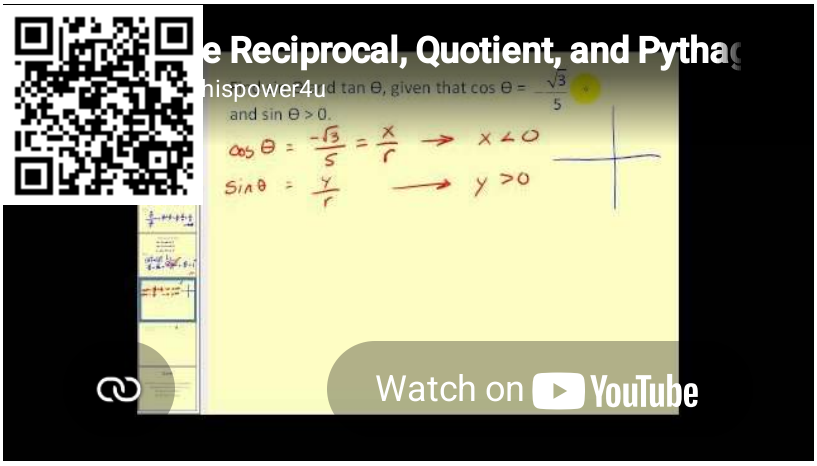
Relationship between sine / cosine / tangent and cosecant / secant / cotangent.

You are already familiar with the trig identities of sine, cosine, and tangent. As you know, any fraction also has an inverse, which is found by reversing the positions of the numerator and denominator.

Can you list what the ratios would be for the three trig functions (sine, cosine, and tangent) with the numerators and denominators reversed?

Reciprocal Identities

A **reciprocal** of a fraction ab is the fraction ba. That is, we find the reciprocal of a fraction by interchanging the numerator and the denominator, or flipping the fraction. The six trig functions can be grouped in pairs as reciprocals.



First, consider the definition of the sine function for angles of rotation: $\sin\theta = \frac{y}{r}$. Now consider the cosecant function: $\csc\theta = \frac{r}{y}$. In the unit circle, these values are $\sin\theta = \frac{y}{1} = y$ and $\csc\theta = \frac{1}{y}$. These two functions, by definition, are reciprocals. Therefore the sine value of an angle is always the reciprocal of the cosecant value, and vice versa. For example, if $\sin\theta = \frac{1}{2}$, then $\csc\theta = \frac{2}{1} = 2$.

Analogously, the cosine function and the secant function are reciprocals, and the tangent and cotangent function are reciprocals:

$$\sec\theta = \frac{1}{\cos\theta} \quad \text{or } \cos\theta = \frac{1}{\sec\theta}$$
$$\cot\theta = \frac{1}{\tan\theta} \quad \text{or } \tan\theta = \frac{1}{\cot\theta}$$

Using Reciprocal Identities

Find the value of the following expressions using a reciprocal identity.

1. $\cos\theta = .3$, $\sec\theta = ?$

$\sec\theta = \frac{10}{3}$

These functions are reciprocals, so if $\cos\theta = .3$, then $\sec\theta = 1.3$. It is easier to find the reciprocal if we express the values as fractions: $\cos\theta = .3 = \frac{3}{10} \Rightarrow \sec\theta = 103$.

2. $\cot\theta = \frac{4}{3}$, $\tan\theta = ?$

These functions are reciprocals, and the reciprocal of $\frac{4}{3}$ is $\frac{3}{4}$.

We can also use the reciprocal relationships to determine the **domain** and **range** of functions.

3. $\sin\theta = \frac{1}{2}$, $\csc\theta = ?$

These functions are reciprocals, and the reciprocal of $\frac{1}{2}$ is 2.

Example 3.1.3.1

Earlier, you were asked to list the ratios for the three trigonometric functions with the numerators and denominators reversed.

Solution

Since the three regular trig functions are defined as:

$$\sin = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\cos = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\tan = \frac{\text{opposite}}{\text{adjacent}}$$

then the three functions - called "reciprocal functions" are:

$$\csc = \frac{\text{hypotenuse}}{\text{opposite}}$$
$$\sec = \frac{\text{hypotenuse}}{\text{adjacent}}$$
$$\cot = \frac{\text{adjacent}}{\text{opposite}}$$

Example 3.1.3.2

State the reciprocal function of cosecant.

Solution

The reciprocal function of cosecant is sine.

Example 3.1.3.3

Find the value of the expression using a reciprocal identity.

$\sec\theta = \frac{2}{\pi}$, $\cos\theta = ?$

Solution

These functions are reciprocals, and the reciprocal of $\frac{2}{\pi}$ is $\frac{\pi}{2}$.

Example 3.1.3.4

Find the value of the expression using a reciprocal identity.

$\csc\theta = 4$, $\cos\theta = ?$

Solution

These functions are reciprocals, and the reciprocal of 4 is $\frac{1}{4}$.

Review

1. State the reciprocal function of secant.
2. State the reciprocal function of cotangent.
3. State the reciprocal function of sine.

Find the value of the expression using a reciprocal identity.

4. $\sin\theta = \frac{1}{2}$, $\csc\theta = ?$
5. $\cos\theta = \frac{-\sqrt{3}}{2}$, $\sec\theta = ?$

6. $\tan\theta = 1$, $\cot\theta = ?$
7. $\sec\theta = \sqrt{2}$, $\cos\theta = ?$
8. $\csc\theta = 2$, $\sin\theta = ?$
9. $\cot\theta = -1$, $\tan\theta = ?$
10. $\sin\theta = \frac{\sqrt{3}}{2}$, $\csc\theta = ?$
11. $\cos\theta = 0$, $\sec\theta = ?$
12. $\tan\theta =$ undefined, $\cot\theta = ?$
13. $\csc\theta = \frac{2\sqrt{3}}{3}$, $\sin\theta = ?$
14. $\sin\theta = \frac{-1}{2}$ and $\tan\theta = \frac{\sqrt{3}}{3}$,$\cos\theta = ?$
15. $\cos\theta = \frac{\sqrt{2}}{2}$ and $\tan\theta = 1$, $\sin\theta = ?$

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.21.

Vocabulary

Term	Definition
domain	The domain of a function is the set of x-values for which the function is defined.
Range	The range of a function is the set of y values for which the function is defined.
Reciprocal Trig Function	A reciprocal trigonometric function is a function that is the reciprocal of a typical trigonometric function. For example, since $\sin x = \frac{\text{opposite}}{\text{hypotenuse}}$, the reciprocal function is $\csc x = \frac{\text{hypotenuse}}{\text{opposite}}$

Additional Resources

Video: The Reciprocal, Quotient, and Pythagorean Identities

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3.1.4: Pythagorean Identities

Sine squared plus cosine squared equals one.

he **Pythagorean Theorem** works on right triangles. If you consider the x coordinate of a point along the unit circle to be the cosine and the y coordinate of the point to be the sine and the distance to the origin to be 1 then the Pythagorean Theorem immediately yields the **identity**:

$$y^2 + x^2 = 1$$
$$\sin^2 x + \cos^2 x = 1$$

An observant student may guess that other Pythagorean identities exist with the rest of the trigonometric functions. Is $\tan^2 x + \cot^2 x = 1$ a legitimate identity?

The proof of the **Pythagorean identity** for sine and cosine is essentially just drawing a right triangle in a unit circle, identifying the cosine as the x coordinate, the sine as the y coordinate and 1 as the hypotenuse.

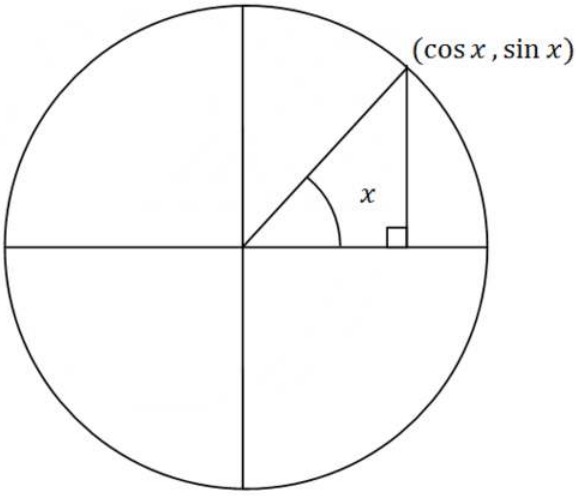


Figure 3.1.4.1

$\cos^2 x + \sin^2 x = 1$

or

$\sin^2 x + \cos^2 x = 1$

The two other Pythagorean identities are:

- $1 + \cot^2 x = \csc^2 x$
- $\tan^2 x + 1 = \sec^2 x$

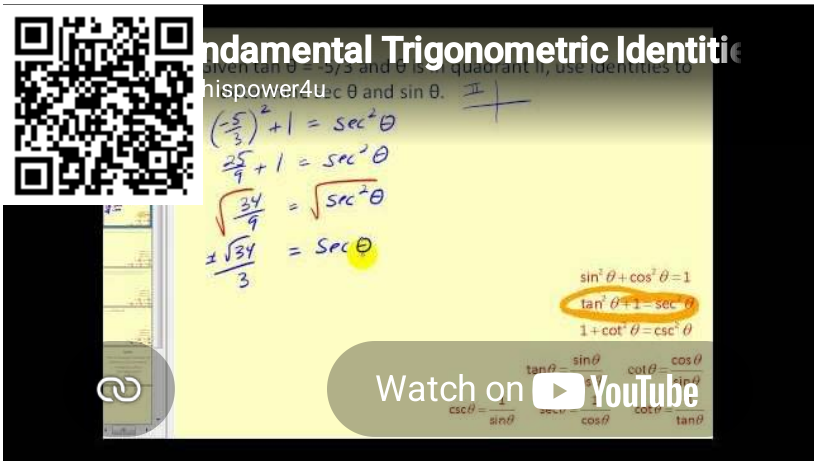
To derive these two Pythagorean identities, divide the original Pythagorean identity by $\sin^2 x$ and $\cos^2 x$ respectively.

To derive the Pythagorean identity $1 + \cot^2 x = \csc^2 x$ divide through by $\sin^2 x$ and simplify.

$$\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$$
$$1 + \cot^2 x = \csc^2 x$$

Similarly, to derive the Pythagorean identity $\tan^2 x + 1 = \sec^2 x$, divide through by $\cos^2 x$ and simplify.

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$
$$\tan^2 x + 1 = \sec^2 x$$



Example 3.1.4.1

Earlier, you were asked if $\tan^2 x + \cot^2 x = 1$ is a legitimate identity. Cofunctions are not always connected directly through a Pythagorean identity.

$\tan^2 x + \cot^2 x \neq 1$

Solution

Visually, the right triangle connecting tangent and secant can also be observed in the unit circle. Most people do not know that tangent is named "tangent" because it refers to the distance of the line tangent from the point on the unit circle to the x axis. Look at the picture below and think about why it makes sense that $\tan x$ and $\sec x$ are as marked. $\tan x = \frac{opp}{adj}$. Since the adjacent side is equal to 1 (the radius of the circle), $\tan x$ simply equals the opposite side. Similar logic can explain the placement of $\sec x$.

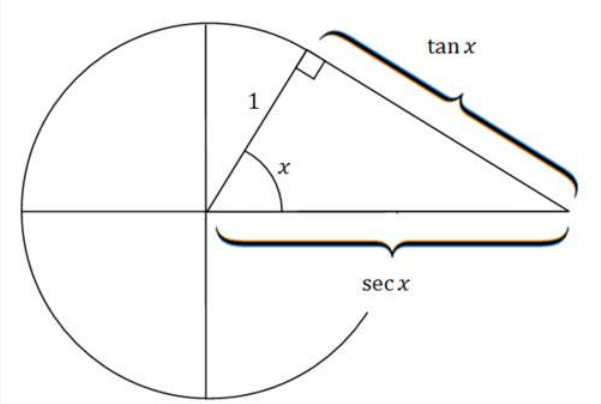


Figure 3.1.4.2

Example 3.1.4.2

Simplify the following expression: $\frac{\sin x(\csc x - \sin x)}{1 - \sin x}$.

Solution

$$\begin{aligned}\frac{\sin x(\csc x - \sin x)}{1 - \sin x} &= \frac{\sin x \cdot \csc x - \sin^2 x}{1 - \sin x} \\ &= \frac{1 - \sin^2 x}{1 - \sin x} \\ &= \frac{(1 - \sin x)(1 + \sin x)}{1 - \sin x} \\ &= 1 + \sin x\end{aligned}$$

Note that factoring the Pythagorean identity is one of the most powerful and common applications.

Example 3.1.4.3

Prove the following trigonometric identity: $(\sec^2 x + \csc^2 x) - (\tan^2 x + \cot^2 x) = 2$

Solution

Group the terms and apply a different form of the second two Pythagorean identities which are $1 + \cot^2 x = \csc^2 x$ and $\tan^2 x + 1 = \sec^2 x$.

$$\begin{aligned}(\sec^2 x + \csc^2 x) - (\tan^2 x + \cot^2 x) &= \sec^2 x - \tan^2 x + \csc^2 x - \cot^2 x \\ &= 1 + 1 \\ &= 2\end{aligned}$$

Example 3.1.4.4

Simplify the following expression. Note: $\sec^2 x = \dfrac{1}{\cos^2 x}$

Solution

$(\sec^2 x)(1 - \sin^2 x) - (\dfrac{\sin x}{\csc x} + \dfrac{\cos x}{\sec x})$

$$\begin{aligned}(\sec^2 x)(1 - \sin^2 x) - (\frac{\sin x}{\csc x} + \frac{\cos x}{\sec x}) &= \sec^2 x \cdot \cos^2 x - (\sin^2 x + \cos^2 x) \\ &= 1 - 1 \\ &= 0\end{aligned}$$

Example 3.1.4.5

Simplify the following expression.

$(\cos t - \sin t)^2 + (cost + \sin t)^2$

Solution

Note that initially, the expression is not the same as the Pythagorean identity.

$$\begin{aligned}(\cos t - \sin t)^2 + (\cos t + \sin t)^2 &= \cos^2 t - 2\cos t \sin t + sin^2 t + cos^2 t + 2\cos t \sin t + \sin^2 t \\ &= 1 - 2\cos t \sin t + 1 + 2\cos t \sin t \\ &= 2\end{aligned}$$

Review

1. Prove each of the following:
2. $(1 - \cos^2 x)(1 + \cot^2 x) = 1$
3. $\cos x(1 - \sin^2 x) = cos^3 x$
4. $\sin^2 x = (1 - \cos x)(1 + \cos x)$
5. $\sin x = \dfrac{\sin^2 x + \cos^2 x}{\csc x}$
6. $sin^4 x - cos^4 x = \sin^2 x - \cos^2 x$
7. $\sin^2 x cos^3 x = (\sin^2 x - \sin^4 x)(\cos x)$

Simplify each expression as much as possible.

7. $\tan^3 x \csc^3 x$
8. $\dfrac{\csc^2 x - 1}{sec^2 x}$
9. $\dfrac{1 - \sin^2 x}{1 + \sin x}$
10. $\sqrt{1 - \cos^2 x}$
11. $\dfrac{\sin^2 x - sin^4 x}{\cos^2 x}$

12. $(1+\tan^2 x)(\sec^2 x)$
13. $\frac{\sin^2 x+\tan^2 x+\cos^2 x}{\sec x}$
14. $\frac{1+\tan^2 x}{\csc^2 x}$
15. $\frac{1-\sin^2 x}{\cos x}$

Review (Answers)

To see the Review answers, open this PDF file and look for section 6.2.

Vocabulary

Term	Definition
identity	An identity is a mathematical sentence involving the symbol "=" that is always true for variables within the domains of the expressions on either side.
Pythagorean Identity	The Pythagorean identity is a relationship showing that the sine of an angle squared plus the cosine of an angle squared is equal to one.
Pythagorean Theorem	The Pythagorean Theorem is a mathematical relationship between the sides of a right triangle, given by $a^2+b^2=c^2$, where a and b are legs of the triangle and c is the hypotenuse of the triangle.

Additional Resources

Video: The Reciprocal, Quotient, and Pythagorean Identities

Practice: Pythagorean Identities

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3.1.5: Even and Odd Identities

Functions symmetric with respect to the y-axis or about the origin.

You and your friend are in math class together. You enjoy talking a lot outside of class about all of the interesting topics you cover in class. Lately you've been covering trig functions and the unit circle. As it turns out, trig functions of certain angles are pretty easy to remember. However, you and your friend are wishing there was an easy way to "shortcut" calculations so that if you knew a trig function for an angle you could relate it to the trig function for another angle; in effect giving you more reward for knowing the first trig function.

You're examining some notes and starting writing down trig functions at random. You eventually write down:

$\cos\left(\frac{\pi}{18}\right)$

Is there any way that if you knew how to compute this, you'd automatically know the answer for a different angle?

An **even function** is a function where the value of the function acting on an argument is the same as the value of the function when acting on the negative of the argument. Or, in short:

$f(x) = f(-x)$

So, for example, if $f(x)$ is some function that is even, then $f(2)$ has the same answer as $f(-2)$. $f(5)$ has the same answer as $f(-5)$, and so on.

In contrast, an **odd function** is a function where the negative of the function's answer is the same as the function acting on the negative argument. In math terms, this is:

$-f(x) = f(-x)$

If a function were negative, then $f(-2) = -f(2)$, $f(-5) = -f(5)$, and so on.

Functions are even or odd depending on how the end behavior of the graphical representation looks. For example, $y = x^2$ is considered an even function because the ends of the parabola both point in the same direction and the parabola is symmetric about the y−axis. $y = x^3$ is considered an odd function for the opposite reason. The ends of a cubic function point in opposite directions and therefore the parabola is not symmetric about the y−axis. What about the trig functions? They do not have exponents to give us the even or odd clue (when the degree is even, a function is even, when the degree is odd, a function is odd).

Even Function	Odd Function
$y = (-x)^2 = x^2$	$y = (-x)^3 = -x^3$

Let's consider sine. Start with $\sin(-x)$. Will it equal $\sin x$ or $-\sin x$? Plug in a couple of values to see.

$$\sin(-30^\circ) = \sin 330^\circ = -\frac{1}{2} = -\sin 30^\circ$$
$$\sin(-135^\circ) = \sin 225^\circ = -\frac{\sqrt{2}}{2} = -\sin 135^\circ$$

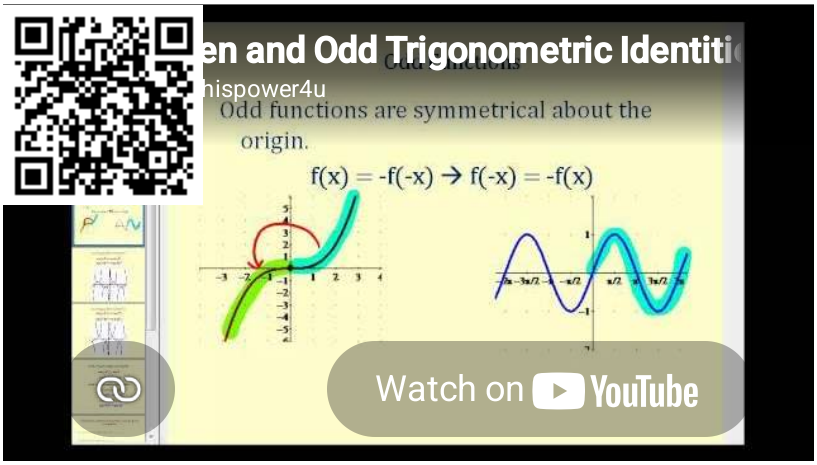
From this we see that sine is **odd**. Therefore, $\sin(-x) = -\sin x$, for any value of x. For cosine, we will plug in a couple of values to determine if it's even or odd.

$$\cos(-30^\circ) = \cos 330^\circ = \frac{\sqrt{3}}{2} = \cos 30^\circ$$
$$\cos(-135^\circ) = \cos 225^\circ = -\frac{\sqrt{2}}{2} = \cos 135^\circ$$

This tells us that the cosine is **even**. Therefore, $\cos(-x) = \cos x$, for any value of x. The other four trigonometric functions are as follows:

$$\tan(-x) = -\tan x$$
$$\csc(-x) = -\csc x$$
$$\sec(-x) = \sec x$$
$$\cot(-x) = -\cot x$$

Notice that cosecant is odd like sine and secant is even like cosine.



Finding Even and Odd Identities

1. Find $\sin x$

If $\cos(-x) = \frac{3}{4}$ and $\tan(-x) = -\frac{\sqrt{7}}{3}$, find $\sin x$.

We know that sine is odd. Cosine is even, so $\cos x = \frac{3}{4}$. Tangent is odd, so $\tan x = \frac{\sqrt{7}}{3}$. Therefore, sine is positive and $\sin x = \frac{\sqrt{7}}{4}$.

2. Find sin(-x)

If $\sin(x) = .25$, find $\sin(-x)$

Since sine is an odd function, $\sin(-\theta) = -\sin(\theta)$.

Therefore, $\sin(-x) = -\sin(x) = -.25$

3. Find cos(-x)

If $\cos(x) = .75$, find $\cos(-x)$

Since cosine is an even function, $\cos(x) = \cos(-x)$.

Therefore, $\cos(-x) = .75$

Example 3.1.5.1

Earlier, you were asked to compute $\cos\left(\frac{\pi}{18}\right)$.

Solution

Since you now know that cosine is an even function, you get to know the cosine of the negative of an angle automatically if you know the cosine of the positive of the angle.

Therefore, since $\cos\left(\frac{\pi}{18}\right) = .9848$, you automatically know that $\cos\left(-\frac{\pi}{18}\right) = \cos\left(\frac{17\pi}{18}\right) = .9848$.

Example 3.1.5.2

What two angles have a value for cosine of $\frac{\sqrt{3}}{2}$?

Solution

On the unit circle, the angles $30°$ and $330°$ both have $\frac{\sqrt{3}}{2}$ as their value for cosine. $330°$ can be rewritten as $-30°$

Example 3.1.5.3

If $\cos\theta = \frac{\sqrt{3}}{2}$, find $sec(-\theta)$

Solution

There are 2 ways to think about this problem. Since $\cos\theta = \cos -\theta$, you could say $\sec(-\theta) = \frac{1}{\cos(-\theta)} = \frac{1}{cos(\theta)}$ Or you could leave the cosine function the way it is and say that $\sec(-\theta) = \sec(\theta) = \frac{1}{\cos\theta}$. But either way, the answer is $\frac{2}{\sqrt{3}}$

Example 3.1.5.4

If $cot\theta = -\sqrt{3}$ find $cot -\theta$

Solution

Since $\cot(-\theta) = -cot(\theta)$, if $\cot\theta = -\sqrt{3}$ then $-\cot(-\theta) = -\sqrt{3}$. Therefore, $\cot(-\theta) = \sqrt{3}$.

Review

Identify whether each function is even or odd.

1. $y = \sin(x)$
2. $y = \cos(x)$
3. $y = \cot(x)$
4. $y = x^4$
5. $y = x$
6. If $\sin(x) = .3$, what is $\sin(-x)$?
7. If $\cos(x) = .5$, what is $\cos(-x)$?
8. If $\tan(x) = .1$, what is $\tan(-x)$?
9. If $\cot(x) = .3$, what is $\cot(-x)$?
10. If $\csc(x) = .3$, what is $\csc(-x)$?
11. If $\sec(x) = 2$, what is $\sec(-x)$?
12. If $\sin(x) = -.2$, what is $\sin(-x)$?
13. If $\cos(x) = -.25$, what is $\sec(-x)$?
14. If $\csc(x) = 4$, what is $\sin(-x)$?
15. If $\tan(x) = -.2$, what is $\cot(-x)$?
16. If $\sin(x) = -.5$ and $\cos(x) = -\frac{\sqrt{3}}{2}$, what is $\cot(-x)$?
17. If $\cos(x) = -.5$ and $\sin(x) = \frac{\sqrt{3}}{2}$, what is $\tan(-x)$?
18. If $\cos(x) = -\frac{\sqrt{2}}{2}$ and $\tan(x) = -1$, what is $\sin(-x)$?

Review (Answers)

To see the Review answers, open this PDF file and look for section 3.1.

Vocabulary

Term	Definition
Even Function	An even function is a function with a graph that is symmetric with respect to the y-axis and has the property that $f(-x) = f(x)$.
Odd Function	An odd function is a function with the property that $f(-x) = -f(x)$. Odd functions have rotational symmetry about the origin.

Additional Resources

Video: Even and Odd Trigonometric Identities

Practice: Even and Odd Identities

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3.1.6: Cofunction Identities

A cofunction identity is a relationship between one trig function of an angle and another trig function of the complement of that angle.

Cofunction Identities and Reflection

While toying with a triangular puzzle piece, you start practicing your math skills to see what you can find out about it. You realize one of the interior angles of the puzzle piece is $30°$, and decide to compute the trig functions associated with this angle. You immediately want to compute the cosine of the angle, but can only remember the values of your sine functions.

Is there a way to use this knowledge of sine functions to help you in your computation of the cosine function for $30°$?

In a right triangle, you can apply what are called "cofunction identities". These are called cofunction identities because the functions have common values. These identities are summarized below.

$$\sin\theta = \cos(90° - \theta) \quad \cos\theta = \sin(90° - \theta)$$
$$\tan\theta = \cot(90° - \theta) \quad \cot\theta = \tan(90° - \theta)$$

Let's take a look at some problems involving cofunction identities and reflection.

1. Find the value of $\cos 120°$.

Because this angle has a reference angle of $60°$, the answer is $\cos 120° = -12$.

2. Find the value of $cos(-120°)$.

Because this angle has a reference angle of $60°$, the answer is $\cos(-120°) = \cos 240° = -\frac{1}{2}$.

3. Find the value of $\sin 135°$.

Because this angle has a reference angle of $45°$, the answer is $\sin 135° = \frac{\sqrt{2}}{2}$

Example 3.1.6.1

Earlier, you were asked if there is a way to use your knowledge of sine functions to help you in your computation of the cosine function.

Solution

Since you now know the cofunction relationships, you can use your knowledge of sine functions to help you with the cosine computation:

$$\cos 30° = \sin(90° - 30°) = \sin(60°) = \frac{\sqrt{3}}{2}$$

Example 3.1.6.2

Find the value of $\sin 45°$ using a **cofunction identity**.

Solution

The sine of $45°$ is equal to $\cos(90° - 45°) = \cos 45° = \frac{\sqrt{2}}{2}$.

Example 3.1.6.3

Find the value of $\cos 45°$ using a cofunction identity.

Solution

The cosine of $45°$ is equal to $\sin(90° - 45°) = \sin 45° = \frac{\sqrt{2}}{2}$.

Example 3.1.6.4

Find the value of $\cos 60^\circ$ using a cofunction identity.

Solution

The cosine of 60° is equal to $\sin(90^\circ - 60^\circ) = \sin 30^\circ = .5$.

Review

1. Find a value for θ for which $\sin\theta = \cos 15^\circ$ is true.
2. Find a value for θ for which $\cos\theta = \sin 55^\circ$ is true.
3. Find a value for θ for which $\tan\theta = \cot 80^\circ$ is true.
4. Find a value for θ for which $\cot\theta = \tan 30^\circ$ is true.
5. Use cofunction identities to help you write the expression $\tan 255^\circ$ as the function of an acute angle of measure less than 45°.
6. Use cofunction identities to help you write the expression $\sin 120^\circ$ as the function of an acute angle of measure less than 45°.
7. Use cofunction identities to help you write the expression $\cos 310^\circ$ as the function of an acute angle of measure less than 45°.
8. Use cofunction identities to help you write the expression $\cot 260^\circ$ as the function of an acute angle of measure less than 45°.
9. Use cofunction identities to help you write the expression $\cos 280^\circ$ as the function of an acute angle of measure less than 45°.
10. Use cofunction identities to help you write the expression $\tan 60^\circ$ as the function of an acute angle of measure less than 45°.
11. Use cofunction identities to help you write the expression $\sin 100^\circ$ as the function of an acute angle of measure less than 45°.
12. Use cofunction identities to help you write the expression $\cos 70^\circ$ as the function of an acute angle of measure less than 45°.
13. Use cofunction identities to help you write the expression $\cot 240^\circ$ as the function of an acute angle of measure less than 45°.
14. Use a right triangle to prove that $\sin\theta = \cos(90^\circ - \theta)$.
15. Use the sine and cosine cofunction identities to prove that $\tan(90^\circ - \theta) = \cot\theta$.

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.24.

Vocabulary

Term	Definition
Cofunction Identity	A cofunction identity is a relationship between one trig function of an angle and another trig function of the complement of that angle.

Additional Resources

Video: Cofunctions

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SECTION OVERVIEW

3.2: Basic Trig Identity Applications

3.2.1: Trig Identities to Find Exact Trigonometric Values

3.2.2: Simplifying Trigonometric Expressions

3.2.3: Proofs of Trigonometric Identities

3.2.4: Simpler Form of Trigonometric Equations

3.2.5: Solving Trigonometric Equations

3.2.6: Solving Trigonometric Equations Using Basic Algebra

3.2.7: Trigonometric Equations Using the Quadratic Formula

3.2.8: Trigonometric Equations Using Factoring

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3.2.1: Trig Identities to Find Exact Trigonometric Values

Pythagorean, Tangent, and Reciprocal Identities used to find values of functions.

Using Trig Identities to Find Exact Trig Values

You are given the following information about θ

$\sin\theta = \frac{2}{3},\ \frac{\pi}{2} < \theta < \pi$

What are $\cos\theta$ and $\tan\theta$?

Trigonometric Identities

You can use the Pythagorean, Tangent and Reciprocal Identities to find all six trigonometric values for certain angles. Let's walk through a few problems so that you understand how to do this.

Let's solve the following problems using trigonometric identities.

1. Given that $\cos\theta = \frac{3}{5}$ and $0 < \theta < \frac{\pi}{2}$, find $\sin\theta$.

Use the Pythagorean Identity to find sin\theta .

$$\sin^2\theta + \cos^2\theta = 1$$
$$\sin^2\theta + \left(\frac{3}{5}\right)^2 = 1$$
$$\sin^2\theta = 1 - \frac{9}{25}$$
$$\sin^2\theta = \frac{16}{25}$$
$$\sin\theta = \pm\frac{4}{5}$$

Because θ is in the first quadrant, we know that sine will be positive. $\sin\theta = \frac{4}{5}$

2. Find $\tan\theta$ from #1 above.

Use the Tangent Identity to find $\tan\theta$.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{4}{5}}{\frac{3}{5}} = \frac{4}{3}$$

3. Find the other three trigonometric functions of θ from #1.

To find secant, cosecant, and cotangent use the Reciprocal Identities.

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{4}{5}} = \frac{5}{4} \quad \sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{3}{5}} = \frac{5}{3} \quad \cot\theta = \frac{1}{\tan\theta} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$$

Example 3.2.1.1

Earlier, you were asked to find $\cos\theta$ and $\tan\theta$ of $\sin\theta = \frac{2}{3},\ \frac{\pi}{2} < \theta < \pi$.

Solution

First, use the Pythagorean Identity to find $\cos\theta$.

$$\sin^2\theta + \cos^2\theta = 1$$
$$\left(\frac{2}{3}\right)^2 + \cos^2\theta = 1$$
$$\cos^2\theta = 1 - \frac{4}{9}$$
$$\cos^2\theta = \frac{5}{9}$$
$$\cos\theta = \pm\frac{\sqrt{5}}{3}$$

However, because θ is restricted to the second quadrant, the cosine must be negative. Therefore, $\cos\theta = -\frac{\sqrt{5}}{3}$.

Now use the Tangent Identity to find tan\theta .

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{2}{3}}{-\frac{\sqrt{5}}{3}} = -2\sqrt{5} = \frac{-2\sqrt{5}}{5}$$

Find the values of the other five trigonometric functions.

Example 3.2.1.2

$\tan\theta = -\frac{5}{12}, \frac{\pi}{2} < \theta < \pi$

Solution

First, we know that \theta is in the second quadrant, making sine positive and cosine negative. For this problem, we will use the Pythagorean Identity $1 + \tan^2\theta = \sec^2\theta$ to find secant.

$$1 + \left(-\frac{5}{12}\right)^2 = \sec^2\theta$$
$$1 + \frac{25}{144} = \sec^2\theta$$
$$\frac{169}{144} = \sec^2\theta$$
$$\pm\frac{13}{12} = \sec\theta$$
$$-\frac{13}{12} = \sec\theta$$

If $\sec\theta = -\frac{13}{12}$, then $\cos\theta = -\frac{12}{13}$. $\sin\theta = \frac{5}{13}$ because the numerator value of tangent is the sine and it has the same denominator value as cosine. $\csc\theta = \frac{13}{5}$ and $\cot\theta = -\frac{12}{5}$ from the Reciprocal Identities.

Example 3.2.1.3

$\csc\theta = -8, \pi < \theta < \frac{3\pi}{2}$

Solution

\theta is in the third quadrant, so both sine and cosine are negative. The reciprocal of $\csc\theta = -8$, will give us $\sin\theta = -\frac{1}{8}$. Now, use the Pythagorean Identity $sin^2\theta + \cos^2\theta = 1$ to find cosine.

$$\left(-\frac{1}{8}\right)^2 + \cos^2\theta = 1$$
$$\cos^2\theta = 1 - \frac{1}{64}$$
$$\cos^2\theta = \frac{63}{64}$$
$$\cos\theta = \pm\frac{3\sqrt{7}}{8}$$
$$\cos\theta = -\frac{3\sqrt{7}}{8}$$

$\sec\theta = -\frac{8}{3\sqrt{7}} = -\frac{8\sqrt{7}}{21}$, $\tan\theta = \frac{1}{3\sqrt{7}} = \frac{\sqrt{7}}{21}$, and $\cot\theta = 3\sqrt{7}$

Review

1. In which quadrants is the sine value positive? Negative?
2. In which quadrants is the cosine value positive? Negative?
3. In which quadrants is the tangent value positive? Negative?

Find the values of the other five trigonometric functions of θ.

4. $\sin\theta = \frac{8}{17}$, $0 < \theta < \frac{\pi}{2}$
5. $\cos\theta = -\frac{5}{6}$, $\frac{\pi}{2} < \theta < \pi$
6. $\tan\theta = \frac{\sqrt{3}}{4}$, $0 < \theta < \frac{\pi}{2}$
7. $\sec\theta = -\frac{41}{9}$, $\pi < \theta < \frac{3\pi}{2}$
8. $\sin\theta = -\frac{11}{14}$, $\frac{3\pi}{2} < \theta < 2\pi$
9. $\cos\theta = \frac{\sqrt{2}}{2}$, $0 < \theta < \frac{\pi}{2}$
10. $\cot\theta = \sqrt{5}$, $\pi < \theta < \frac{3\pi}{2}$
11. $\csc\theta = 4$, $\frac{\pi}{2} < \theta < \pi$
12. $\tan\theta = -\frac{7}{10}$, $\frac{3\pi}{2} < \theta < 2\pi$
13. Aside from using the identities, how else can you find the values of the other five trigonometric functions?
14. Given that $\cos\theta = \frac{6}{11}$ and θ is in the 2nd quadrant, what is $\sin(-\theta)$?
15. Given that $\tan\theta = -\frac{5}{8}$ and θ is in the 4th quadrant, what is $\sec(-\theta)$?

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 14.7.

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3.2.2: Simplifying Trigonometric Expressions

Convert to sine/cosine and use basic trig identities to simplify.

How could you write the trigonometric function $\cos\theta + \cos\theta(\tan^2\theta)$ more simply?

Simplifying Trigonometric Expressions

Now that you are more familiar with trig identities, we can use them to simplify expressions. Remember, that you can use any of the following identities.

Reciprocal Identities: $\csc\theta = \frac{1}{\sin\theta}$, $\sec\theta = \frac{1}{\cos\theta}$, and $\cot\theta = \frac{1}{\tan\theta}$

Tangent and Cotangent Identities: $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Pythagorean Identities: $\sin^2\theta + \cos^2\theta = 1$, $1 + \tan^2\theta = \sec^2\theta$, and $1 + \cot^2\theta = \csc^2\theta$

Cofunction Identities: $\sin(\frac{\pi}{2} - \theta) = \cos\theta$, $\cos(\frac{\pi}{2} - \theta) = \sin\theta$, and $\tan(\frac{\pi}{2} - \theta) = \cot\theta$

Negative Angle Identities: $\sin(-\theta) = -sin\theta$, $\cos(-\theta) = \cos\theta$, and $\tan(-\theta) = -\tan\theta$

Let's simplify the following expressions.

1. $\frac{\sec x}{\sec x - 1}$

When simplifying trigonometric expressions, one approach is to change everything into sine or cosine. First, we can change secant to cosine using the Reciprocal Identity.

$$\frac{\sec x}{\sec x - 1} \rightarrow \frac{\frac{1}{\cos x}}{\frac{1}{\cos x} - 1}$$

Now, combine the denominator into one fraction by multiplying 1 by $\frac{\cos x}{\cos x}$.

$$\frac{\frac{1}{\cos x}}{\frac{1}{\cos x} - 1} \rightarrow \frac{\frac{1}{\cos x}}{\frac{1}{\cos x} - \frac{\cos x}{\cos x}} \rightarrow \frac{\frac{1}{\cos x}}{\frac{1 - \cos x}{\cos x}}$$

Change this problem into a division problem and simplify.

$$\frac{\frac{1}{\cos x}}{\frac{1 - \cos x}{\cos x}} \rightarrow \frac{1}{\cos x} \div \frac{1 - \cos x}{\cos x}$$

$$\frac{1}{\cancel{\cos x}} \cdot \frac{\cancel{\cos x}}{1 - \cos x}$$

$$\frac{1}{1 - \cos x}$$

2. $\frac{\sin^4 x - \cos^4 x}{\sin^2 x - \cos^2 x}$

With this problem, we need to factor the numerator and denominator and see if anything cancels. The rules of factoring a quadratic and the special quadratic formulas can be used in this scenario.

$$\frac{\sin^4 x - \cos^4 x}{\sin^2 x - \cos^2 x} \rightarrow \frac{\cancel{(\sin^2 x - \cos^2 x)}\,(\sin^2 x + \cos^2 x)}{\cancel{(\sin^2 x - \cos^2 x)}} \rightarrow \sin^2 x + \cos^2 x \rightarrow 1$$

In the last step, we simplified to the left hand side of the Pythagorean Identity. Therefore, this expression simplifies to 1.

3. $\sec\theta\tan^2\theta + \sec\theta$

First, pull out the GCF.

$$\sec\theta\tan^2\theta + \sec\theta \rightarrow \sec\theta(\tan^2\theta + 1)$$

Now, $\tan^2\theta + 1 = \sec^2\theta$ from the Pythagorean Identities, so simplify further.

$$\sec\theta(\tan^2\theta + 1) \rightarrow \sec\theta \cdot \sec^2\theta \rightarrow \sec^3\theta$$

Example 3.2.2.1

Earlier, you were asked to simplify the trigonometric function $\cos\theta + \cos\theta(\tan^2\theta)$.

Solution

Notice that the terms in the expression $\cos\theta + \cos\theta(\tan^2\theta)$ have a common factor of $\cos\theta$, so start by factoring this common term out.

$$\cos\theta + \cos\theta(\tan^2\theta)$$
$$\cos\theta(1 + \tan^2\theta)$$

Now, use the trigonometric identity $1 + \tan^2\theta = \sec^2\theta$, substitute, and simplify.

$$\begin{aligned}\cos\theta(1 + tan^2\theta) &= \cos\theta(\sec^2\theta)\\ &= \cos\theta\left(\frac{1}{\cos^2\theta}\right)\\ &= \frac{1}{\cos\theta}\\ &= \sec\theta\end{aligned}$$

Simplify the following trigonometric expressions.

Example 3.2.2.2

$\cos\left(\frac{\pi}{2} - x\right)\cot x$

Solution

Use the Cotangent Identity and the Cofunction Identity $\cos(\frac{\pi}{2} - \theta) = \sin\theta$.

$$\cos(\tfrac{\pi}{2} - x)\cot x \rightarrow \cancel{\sin x} \cdot \frac{\cos x}{\cancel{\sin x}} \rightarrow \cos x$$

Example 3.2.2.3

$\frac{\sin(-x)\cos x}{\tan x}$

Solution

Use the Negative Angle Identity and the Tangent Identity.

$$\frac{\sin(-x)\cos x}{\tan x} \rightarrow \frac{-\sin x\cos x}{\frac{\sin x}{\cos x}} \rightarrow -\cancel{\sin x}\cos x \cdot \frac{\cos x}{\sin x} \rightarrow -\cos^2 x$$

Example 3.2.2.4

$\frac{\cot x\cos x}{\tan(-x)\sin\left(\frac{\pi}{2} - x\right)}$

Solution

In this problem, you will use several identities.

$$\frac{\cot x\cos x}{\tan(-x)\sin\left(\frac{\pi}{2} - x\right)} \rightarrow \frac{\frac{\cos x}{\sin x}\cdot\cos x}{-\frac{\sin x}{\cancel{\cos x}}\cdot\cancel{\cos x}} \rightarrow \frac{\frac{\cos^2 x}{\sin x}}{-\sin x} \rightarrow \frac{\cos^2 x}{\sin x}\cdot -\frac{1}{\sin x} \rightarrow -\frac{\cos^2 x}{\sin^2 x} \rightarrow -\cot^2 x$$

Review

Simplify the following expressions.

1. $\cot x \sin x$
2. $\cos^2 x \tan(-x)$
3. $\frac{cos(-x)}{\sin(-x)}$
4. $\sec x \cos(-x) - \sin^2 x$
5. $\sin x(1 + cot^2 x)$
6. $1 - \sin^2\left(\frac{\pi}{2} - x\right)$
7. $1 - \cos^2\left(\frac{\pi}{2} - x\right)$
8. $\frac{\tan\left(\frac{\pi}{2} - x\right)\sec x}{1 - \csc^2 x}$
9. $\frac{\cos^2 x \tan^2 x - 1}{\cos^2 x}$
10. $\cot^2 x + \sin^2 x + \cos^2(-x)$
11. $\frac{\sec x \sin x + \cos\left(\frac{\pi}{2} - x\right)}{1 + \cos x}$
12. $\frac{\cos(-x)}{1 + \sin(-x)}$
13. $\frac{\sin^2(-x)}{\tan^2 x}$
14. $\tan\left(\frac{\pi}{2} - x\right)\cot x - \csc^2 x$
15. $\frac{\csc x(1 - \cos^2 x)}{\sin x \cos x}$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 14.8.

Additional Resources

Practice: Simplifying Trigonometric Expressions

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3.2.3: Proofs of Trigonometric Identities

Convert to sine/cosine, use basic identities, and simplify sides of the equation.

Verify that $\frac{\sin^2 x}{\tan^2 x} = 1 - \sin^2 x$.

Verifying Trigonometric Identities

Now that you are comfortable simplifying expressions, we will extend the idea to verifying entire identities. Here are a few helpful hints to verify an identity:

- Change everything into terms of sine and cosine.
- Use the identities when you can.
- Start with simplifying the left-hand side of the equation, then, once you get stuck, simplify the right-hand side. As long as the two sides end up with the same final expression, the identity is true.

Let's verify the following identities.

1. $\frac{\cot^2 x}{\csc x} = \csc x - \sin x$

Rather than have an equal sign between the two sides of the equation, we will draw a vertical line so that it is easier to see what we do to each side of the equation. Start with changing everything into sine and cosine.

$$\begin{array}{l|l} \frac{\cot^2 x}{\csc x} & \csc x - \sin x \\ \frac{\frac{\cos^2 x}{\sin^2 x}}{\frac{1}{\sin x}} & \frac{1}{\sin x} \\ \frac{\cos^2 x}{\sin x} & \end{array}$$

Now, it looks like we are at an impasse with the left-hand side. Let's combine the right-hand side by giving them same denominator.

$$\begin{array}{|l} \frac{1}{\sin x} - \frac{\sin^2 x}{\sin x} \\ \frac{1 - \sin^2 x}{\sin x} \\ \frac{\cos^2 x}{\sin x} \end{array}$$

The two sides reduce to the same expression, so we can conclude this is a valid identity. In the last step, we used the Pythagorean Identity, $\sin^2\theta + \cos^2\theta = 1$, and isolated the $\cos^2 x = 1 - \sin^2 x$.

There are usually more than one way to verify a trig identity. When proving this identity in the first step, rather than changing the cotangent to $\frac{\cos^2 x}{\sin^2 x}$, we could have also substituted the identity $\cot^2 x = \csc^2 x - 1$.

2. $\frac{\sin x}{1 - \cos x} = \frac{1 + \cos x}{\sin x}$

Multiply the left-hand side of the equation by $\frac{1 + \cos x}{1 + \cos x}$.

$$\begin{aligned} \frac{\sin x}{1 - \cos x} &= \frac{1 + \cos x}{\sin x} \\ \frac{1 + \cos x}{1 + \cos x} \cdot \frac{\sin x}{1 - \cos x} &= \\ \frac{\sin(1 + \cos x)}{1 - \cos^2 x} &= \\ \frac{\sin(1 + \cos x)}{\sin^2 x} &= \\ \frac{1 + \cos x}{\sin x} &= \end{aligned}$$

The two sides are the same, so we are done.

3. $\sec(-x) = \sec x$

Change secant to cosine.

$$\sec(-x) = \frac{1}{cos(-x)}$$

From the Negative Angle Identities, we know that $\cos(-x) = \cos x$.

$$= \frac{1}{\cos x}$$
$$= \sec x$$

Example 3.2.3.1

Earlier, you were asked to verify that $\sin^2 x \tan^2 x = 1 - \sin^2 x$.

Solution

Start by simplifying the left-hand side of the equation.

$$\sin^2 x \tan^2 x = \frac{\sin^2 x}{\frac{\sin^2 x}{\cos^2 x}} = \cos^2 x$$

Now simplify the right-hand side of the equation. By manipulating the Trigonometric Identity,

$\sin^2 x + \cos^2 x = 1$, we get $\cos^2 x = 1 - \sin^2 x$.

$\cos^2 x = \cos^2 x$ and the equation is verified.

Verify the following identities.

Example 3.2.3.2

$\cos x \sec x = 1$

Solution

Change secant to cosine.

$$\cos x \sec x = \cos x \cdot \frac{1}{\cos x}$$
$$= 1$$

Example 3.2.3.3

$2 - sec^2 x = 1 - \tan^2 x$

Solution

Use the identity $1 + \tan^2 \theta = \sec^2 \theta$.

$$2 - \sec^2 x = 2 - (1 + \tan^2 x)$$
$$= 2 - 1 - \tan^2 x$$
$$= 1 - \tan^2 x$$

Example 3.2.3.4

$\frac{\cos(-x)}{1 + \sin(-x)} = \sec x + \tan x$

Solution

Here, start with the Negative Angle Identities and multiply the top and bottom by $\frac{1+\sin x}{1+\sin x}$ to make the denominator a monomial.

$$\begin{aligned}\frac{\cos(-x)}{1+\sin(-x)} &= \frac{\cos x}{1-\sin x}\cdot\frac{1+\sin x}{1+\sin x}\\ &= \frac{\cos x(1+\sin x)}{1-\sin^2 x}\\ &= \frac{\cos x(1+\sin x)}{\cos^2 x}\\ &= \frac{1+\sin x}{\cos x}\\ &= \frac{1}{\cos x}+\frac{\sin x}{\cos x}\\ &= \sec x+\tan x\end{aligned}$$

Review

Verify the following identities.

1. $\cot(-x) = -\cot x$
2. $\csc(-x) = -\csc x$
3. $\tan x \csc x \cos x = 1$
4. $\sin x + \cos x \cot x = \csc x$
5. $\csc\left(\frac{\pi}{2}-x\right) = \sec x$
6. $\tan\left(\frac{\pi}{2}-x\right) = \tan x$
7. $\frac{\csc x}{\sin x} - \frac{\cot x}{\tan x} = 1$
8. $\frac{\tan^2 x}{\tan^2 x+1} = \sin^2 x$
9. $(\sin x - \cos x)^2 + (\sin x + \cos x)^2 = 2$
10. $\sin x - \sin x \cos^2 x = \sin^3 x$
11. $\tan^2 x + 1 + \tan x \sec x = \frac{1+\sin x}{\cos^2 x}$
12. $\cos^2 x = \csc x \cos x \tan x + \cot x$
13. $\frac{1}{1-\sin x} - \frac{1}{1+\sin x} = 2\tan x \sec x$
14. $\csc^4 x - \cot^4 x = \csc^2 x + \cot^2 x$
15. $(\sin x - \tan x)(\cos x - \cot x) = (\sin x - 1)(\cos x - 1)$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 14.9.

Additional Resources

Video: Verifying Trigonometric Identities - Example 2

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3.2.4: Simpler Form of Trigonometric Equations

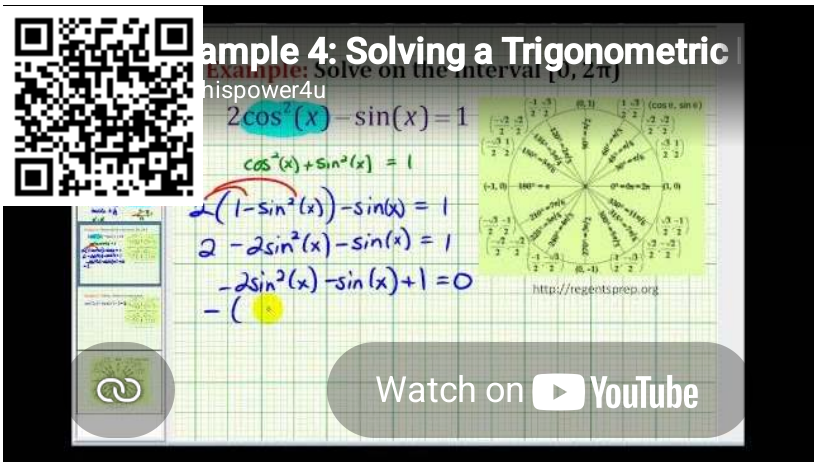
Breakdown of complex expressions using trigonometric identities.

Sometimes things are simpler than they look. For example, trigonometric identities can sometimes be reduced to simpler forms by applying other rules. For example, can you find a way to simplify $\frac{\cos^3\theta = 3\cos\theta + \cos^3\theta}{4}$?

Trigonometric Equations

By this time in your school career you have probably seen trigonometric functions represented in many ways: ratios between the side lengths of right triangles, as functions of coordinates as one travels along the unit circle and as abstract functions with graphs. Now it is time to make use of the properties of the trigonometric functions to gain knowledge of the connections between the functions themselves. The patterns of these connections can be applied to simplify trigonometric expressions and to solve trigonometric equations.

In order to do this, look for parts of the complex trigonometric expression that might be reduced to fewer trig functions if one of the identities you already know is applied to the expression. As you apply identities, some complex trig expressions have parts that can be cancelled out, others can be reduced to fewer trig functions. Observe how this is accomplished in the examples below.



Simplifying Expressions

1. Simplify the following expression using the basic trigonometric identities: $\frac{1+\tan^2 x}{\csc 2x}$

$$\frac{1+\tan^2 x}{\csc^2 x} \ldots (1+\tan^2 x = \sec^2 x) \text{ Pythagorean Identity}$$

$$\frac{\sec^2 x}{\csc^2 x} \ldots \left(\sec^2 x = \frac{1}{\cos^2 x} \text{ and } \csc^2 x = \frac{1}{\sin^2 x}\right) \text{ Reciprocal Identity}$$

$$\frac{\frac{1}{\cos^2 x}}{\frac{1}{\sin^2 x}} = \left(\frac{1}{\cos^2 x}\right) \div \left(\frac{1}{\sin^2 x}\right)$$

$$\left(\frac{1}{\cos^2 x}\right) \cdot \left(\frac{1}{\sin^2 x}\right) = \frac{\sin^2 x}{\cos^2 x}$$

$$= \tan^2 x \rightarrow \text{ Quotient Identity}$$

2. Simplify the following expression using the basic trigonometric identities: $\frac{\sin^2 x + \tan^2 x + \cos^2 x}{\sec x}$

$$\frac{\sin^2 x + \tan^2 x + \cos^2 x}{\sec x} \ldots (\sin^2 x + \cos^2 x = 1) \text{ Pythagorean Identity}$$

$$\frac{1+\tan^2 x}{\sec x} \ldots (1+\tan^2 x = \sec^2 x) \text{ Pythagorean Identity}$$

$$\frac{\sec^2 x}{\sec x} = \sec x$$

3. Simplify the following expression using the basic trigonometric identities: $\cos x - \cos^3 x$

$$\cos x - \cos^3 x$$
$$\cos x \left(1 - \cos^2 x\right) \quad \text{... Factor out } \cos x \text{ and } \sin^2 x = 1 - \cos^2 x$$
$$\cos x \left(\sin^2 x\right)$$

Example 3.2.4.1

Earlier, you were asked to simplify $\cos^3 \theta = \frac{3\cos\theta + \cos^3\theta}{4}$.

Solution

The easiest way to start is to recognize the triple angle identity:

$$\cos^3\theta = \cos^3\theta - 3\sin^2\theta\cos\theta$$

Substituting this into the original equation gives:

$$\cos^3\theta = \frac{3\cos\theta + (\cos^3\theta - 3\sin^2\theta cos\theta)}{4}$$

Notice that you can then multiply by four and subtract a $\cos^3\theta$ term:

$$3\cos^3\theta = 3\cos\theta - 3\sin^2\theta cos\theta$$

And finally pulling out a three and dividing:

$$\cos^3\theta = \cos\theta - \sin^2\theta cos\theta$$

Then pulling out a $\cos\theta$ and dividing:

$$\cos^2\theta = 1 - \sin^2\theta$$

Example 3.2.4.2

Simplify $\tan^3(x)\csc^3(x)$.

Solution

$$\begin{aligned}\tan^3(x)\csc^3(x) &= \sin^3(x)\cos^3(x) \times \frac{1}{\sin^3(x)} \\ &= \frac{1}{\cos^3(x)} \\ &= sec^3(x)\end{aligned}$$

Example 3.2.4.3

Show that $\cot^2(x) + 1 = \csc^2(x)$.

Solution

Start with $\sin^2(x) + \cos^2(x) = 1$, and divide everything through by $sin^2(x)$:

$$\begin{aligned}\sin^2(x) + \cos^2(x) &= 1 \\ &= \frac{\sin^2(x)}{\sin^2(x)} + \frac{\cos^2(x)}{\sin^2(x)} \\ &= \frac{1}{\sin^2(x)} \\ &= 1 + \cot^2(x) = \csc^2(x)\end{aligned}$$

Example 3.2.4.4

Simplify $\frac{\csc^2(x)-1}{\csc^2(x)}$.

Solution

$$\frac{\csc^2(x)-1}{\csc^2(x)}$$

Using $\cot^2(x)+1=\csc^2(x)$ that was proven in #2, you can find the relationship: $cot^2(x)=\csc^2(x)-1$, you can substitute into the above expression to get:

$$\frac{\cot^2(x)}{\csc^2(x)}=\frac{\frac{\cos^2(x)}{\sin^2(x)}}{\frac{1}{\sin^2(x)}}$$
$$=\cos^2(x)$$

Review

Simplify each trigonometric expression as much as possible.

1. $\sin(x)\cot(x)$
2. $\cos(x)tan(x)$
3. $\frac{1+\tan(x)}{1+\cot(x)}$
4. $\frac{1-\sin^2(x)}{1+\sin(x)}$
5. $\frac{\sin^2(x)}{1+\cos(x)}$
6. $(1+\tan^2(x))(\sec^2(x))$
7. $\sin(x)(\tan(x)+\cot(x))$
8. $\frac{\sec(x)}{\sin(x)}-\frac{\sin(x)}{\cos(x)}$
9. $\frac{\sin(x)}{\cot^2(x)}-\frac{\sin(x)}{\cos^2(x)}$
10. $\frac{1+\sin(x)}{cos(x)}-\sec(x)$
11. $\frac{\sin^2(x)-\sin^4(x)}{\cos^2(x)}$
12. $\frac{\tan(x)}{\csc^2(x)}+\frac{\tan(x)}{\sec^2(x)}$
13. $\sqrt{1-\cos^2(x)}$
14. $(1-\sin^2(x))(\cos(x))$
15. $(\sec^2(x)+\csc^2(x))-(\tan^2(x)+\cot^2(x))$

Review (Answers)

To see the Review answers, open this PDF file and look for section 3.3.

Vocabulary

Term	Definition
Trigonometric Identity	A trigonometric identity is an equation that relates two or more trigonometric functions.

Additional Resources

Video: Example: Solving a Trigonometric Equation Using a Trig Substitution and Factoring

Practice: Simpler Form of Trigonometric Equations

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3.2.5: Solving Trigonometric Equations

Identities and solving equations on an interval or with no solutions.

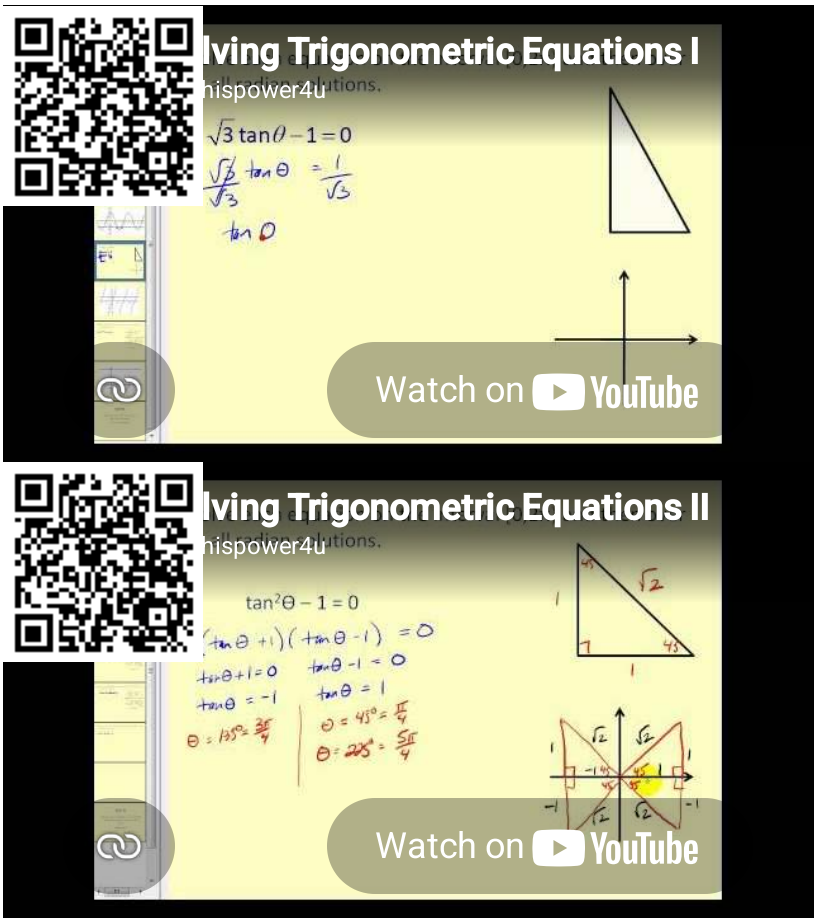
Solving a trigonometric equation is just like solving a regular equation. You will use factoring and other algebraic techniques to get the variable on one side. The biggest difference with trigonometric equations is the opportunity for there to be an infinite number of solutions that must be described with a pattern. The equation $\cos x = 1$ has many solutions including 0 and 2π. How would you describe all of them?

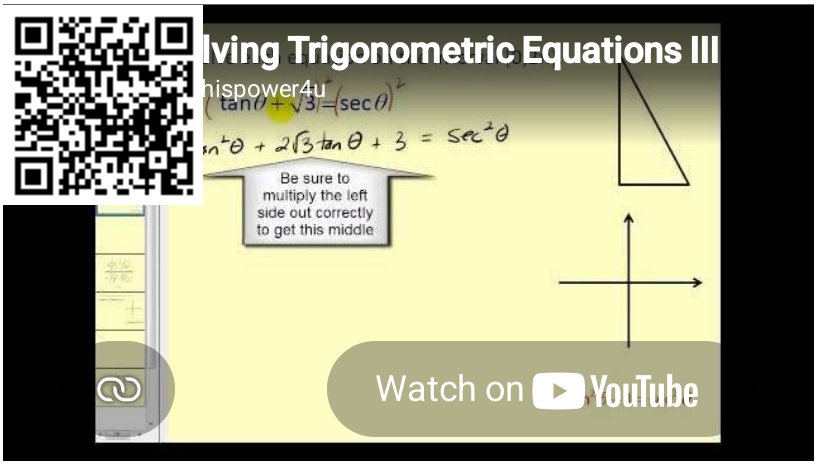
Solving Trigonometric Equations

The identities you have learned are helpful in solving **trigonometric equations**. The goal of solving an equation hasn't changed. Do whatever it takes to get the variable alone on one side of the equation. Factoring, especially with the **Pythagorean identity**, is critical.

Try to give exact (non-rounded) answers when solving trigonometric equations. If you are working with a calculator, keep in mind that while some newer calculators can provide exact answers like $\frac{\sqrt{3}}{2}$, most calculators will produce a decimal of 0.866... If you see a decimal like 0.866..., try squaring it. The result might be a nice fraction like $\frac{3}{4}$. Then you can logically conclude that the original decimal must be the square root of $\frac{3}{4}$ or $\frac{\sqrt{3}}{2}$.

When solving, if the two sides of the equation are always equal, then the equation is an identity. If the two sides of an equation are never equal, as with \sin x =3, then the equation has no solution.





Example 3.2.5.1

Earlier, you were asked how you could describe the many solutions of $\cos x = 1$. When you type $\cos^{-1} 1$ on your calculator, it will yield only one solution which is 0. In order to describe all the solutions you must use logic and the graph to figure out that cosine also has a height of 1 at $-2\pi,\ 2\pi,\ -4\pi,\ 4\pi \ldots$ Luckily all these values are sequences in a clear pattern so you can describe them all in general with the following notation:

Solution

$x = 0 \pm n \cdot 2\pi$ where n is an integer, or $x = \pm n \cdot 2\pi$ where n is an integer.

Example 3.2.5.2

Solve the following equation algebraically and confirm graphically on the interval $[-2\pi,\ 2\pi]$.

Solution

$\cos 2x = \sin x$

$$\begin{aligned} \cos 2x &= \sin x \\ 1 - 2\sin^2 x &= \sin x \\ 0 &= 2\sin^2 x + \sin x - 1 \\ 0 &= (2\sin x - 1)(\sin x + 1) \end{aligned}$$

Solving the first part set equal to zero within the interval yields:

$$\begin{aligned} 0 &= 2\sin x - 1 \\ \frac{1}{2} &= \sin x \\ x &= \frac{\pi}{6},\ \frac{5\pi}{6},\ -\frac{11\pi}{6},\ -\frac{7\pi}{6} \end{aligned}$$

Solving the second part set equal to zero yields:

$$\begin{aligned} 0 &= \sin x + 1 \\ -1 &= \sin x \\ x &= -\frac{\pi}{2},\ \frac{3\pi}{2} \end{aligned}$$

These are the six solutions that will appear as intersections of the two graphs $f(x) = \cos 2x$ and $g(x) = \sin x$.

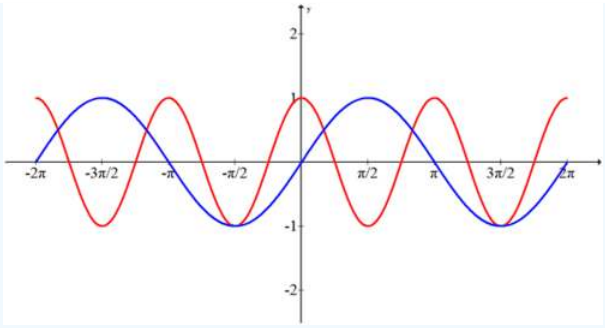


Figure 3.2.5.1

Example 3.2.5.3

Determine the **general solution** to the following equation.

$\cot x - 1 = 0$

Solution

$$\cot x - 1 = 0$$
$$\cot x = 1$$

One solution is $x = \frac{\pi}{4}$. However, since this question asks for the general solution, you need to find every possible solution. You have to know that cotangent has a period of π which means if you add or subtract π from $\frac{\pi}{4}$ then it will also yield a height of 1. To capture all these other possible x values you should use this notation.

$x = \frac{\pi}{4} \pm n \cdot \pi$ where n is a integer

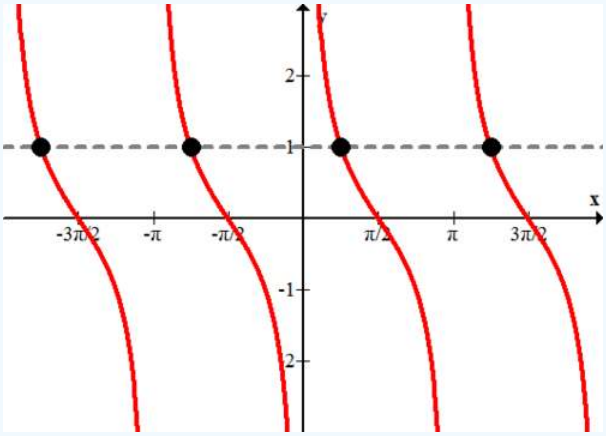


Figure 3.2.5.2

Notice that trigonometric equations may have an infinite number of solutions that repeat in a certain pattern because they are periodic functions. When you see these directions remember to find all the solutions by using notation like in this example.

Example 3.2.5.4

Solve the following equation.

$4\cos^2 x - 1 = 3 - 4\sin^2 x$

Solution

$$4\cos^2 x - 1 = 3 - 4\sin^2 x$$
$$4\cos^2 x + 4\sin^2 x = 3 + 1$$
$$4(\cos^2 x + \sin^2 x) = 4$$
$$4 = 4$$

This equation is always true which means the right side is always equal to the left side. This is an identity.

Example 3.2.5.5

Solve the following equation exactly.

$2\cos^2 x + 3\cos x - 2 = 0$

Solution

Start by factoring:

$$2\cos^2 x + 3\cos x - 2 = 0$$
$$(2\cos x - 1)(\cos x + 2) = 0$$

Note that $\cos x \neq -2$ which means only one equation needs to be solved for solutions.

$$2\cos x - 1 = 0$$
$$\cos x = 12$$
$$x = \frac{\pi}{3}, -\frac{\pi}{3}$$

These are the solutions within the interval $-\pi$ to π. Since this represents one full period of cosine, the rest of the solutions are just multiples of 2π added and subtracted to these two values.

$x = \pm\frac{\pi}{3} \pm n \cdot 2\pi$ where n is an integer

Review

Solve each equation on the interval $[0,\ 2\pi)$.

1. $3\cos^2 \frac{x}{2} = 3$
2. $4\sin^2 x = 8\sin^2 \frac{x}{2}$

Find approximate solutions to each equation on the interval $[0,\ 2\pi)$.

3. $3\cos^2 x + 10\cos x + 2 = 0$
4. $\sin^2 x + 3\sin x = 5$
5. $\tan^2 x + \tan x = 3$
6. $\cot^2 x + 5\tan x + 14 = 0$
7. $\sin^2 x + \cos^2 x = 1$

Solve each equation on the interval [0,\;360^{\circ})\).

8. $2\sin(x - \frac{\pi}{2}) = 1$
9. $4\cos(x - \pi) = 4$

Solve each equation on the interval $[2\pi,\ 4\pi)$.

10. $\cos^2 x + 2\cos x + 1 = 0$
11. $3\sin x = 2\cos^2 x$
12. $\tan x \sin^2 x = \tan x$
13. $\sin^2 x + 1 = 2\sin x$
14. $\sec^2 x = 4$
15. $\sin^2 x - 4 = \cos^2 x - \cos 2x - 4$

Review (Answers)

To see the Review answers, open this PDF file and look for section 6.5.

Additional Resources

Video: Solving Trigonometric Equations - Overview

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3.2.6: Solving Trigonometric Equations Using Basic Algebra

Substitute in potential solutions or solve using inverse trig functions.

As Agent Trigonometry, you are given this clue: $2\sin x - \sqrt{2} = 0$. If $0 \le x < 2\pi$, what is/are the value(s) of x.

Solving Trigonometric Equations

We have already verified trigonometric identities, which are true for every real value of x. In this concept, we will solve trigonometric equations. An equation is only true for some values of x.

Let's verify that $\csc x - 2 = 0$ when $x = \frac{5\pi}{6}$.

Substitute in $x = \frac{5\pi}{6}$ to see if the equations holds true.

$$\csc\left(\frac{5\pi}{6}\right) - 2 = 0$$
$$\frac{1}{\sin\left(\frac{5\pi}{6}\right)} - 2 = 0$$
$$\frac{1}{\frac{1}{2}} - 2 = 0$$
$$2 - 2 = 0$$

This is a true statement, so $x = \frac{5\pi}{6}$ is a solution to the equation.

Now, let's solve $2\cos x + 1 = 0$.

To solve this equation, we need to isolate cosx and then use inverse to find the values of x when the equation is valid.

$$2\cos x + 1 = 0$$
$$2\cos x = -1$$
$$\cos x = -\frac{1}{2}$$

So, when is the $\cos x = -\frac{1}{2}$? Between $0 \le x < 2\pi$, $x = \frac{2\pi}{3}$ and $\frac{4\pi}{3}$. But, the trig functions are periodic, so there are more solutions than just these two. You can write the general solutions as $x = \frac{2\pi}{3} \pm 2\pi n$ and $x = \frac{4\pi}{3} \pm 2\pi n$, where n is any integer. You can check your answer graphically by graphing $y = \cos x$ and $y = -\frac{1}{2}$ on the same set of axes. Where the two lines intersect are the solutions.

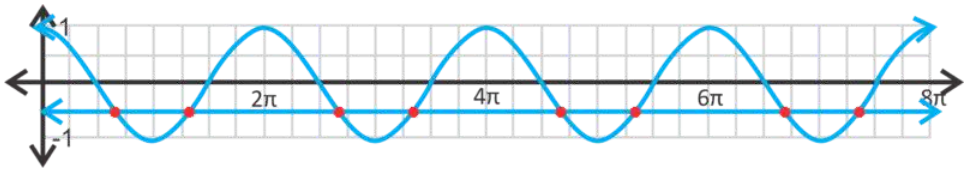


Figure 3.2.6.1

Finally, let's solve $5\tan(x + 2) - 1 = 0$, where $0 \le x < 2\pi$.

In this problem, we have an interval where we want to find x. Therefore, at the end of the problem, we will need to add or subtract π, the period of tangent, to find the correct solutions within our interval.

$$5\tan(x + 2) - 1 = 0$$
$$5\tan(x + 2) = 1$$
$$\tan(x + 2) = \frac{1}{5}$$

Using the tan−1 button on your calculator, we get that $\tan^{-1}\left(\frac{1}{5}\right) = 0.1974$. Therefore, we have:

$$x + 2 = 0.1974$$
$$x = -1.8026$$

This answer is not within our interval. To find the solutions in the interval, add π a couple of times until we have found all of the solutions in $[0, 2\pi]$.

$$x = -1.8026 + \pi = 1.3390$$
$$= 1.3390 + \pi = 4.4806$$

The two solutions are $x = 1.3390$ and 4.4806.

Example 3.2.6.1

Earlier, you were asked to find the value of x from the equation $2\sin x - \sqrt{2} = 0$.

Solution

To solve this equation, we need to isolate $\sin x$ and then use inverse to find the values of x when the equation is valid.

$$2\sin x - \sqrt{2} = 0$$
$$2\sin x = \sqrt{2}$$
$$\sin x = \frac{\sqrt{2}}{2}$$

So now we need to find the values of x for which $\sin x = \frac{\sqrt{2}}{2}$. We know from the special triangles that this value of sine holds true for a $45°$ angle, but is that the only value of x for which it is true?

We are told that $0 \le x < 2\pi$. Recall that the sine is positive in both the first and second quadrants, so $\sin x = \frac{\sqrt{2}}{2}$ when x also is $135°$.

Example 3.2.6.2

Determine if $x = \frac{\pi}{3}$ is a solution for $2\sin x = \sqrt{3}$.

Solution

$2\sin\frac{\pi}{3} = \sqrt{3} \rightarrow 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$

Yes, $x = \frac{\pi}{3}$ is a solution.

Example 3.2.6.3

Solve the following trig equation in the interval $0 \le x < 2\pi$.

$3\cos^2 x - 5 = 0$

Solution

$$9\cos^2 x - 5 = 0$$
$$9\cos^2 x = 5$$
$$\cos^2 x = \frac{5}{9}$$
$$\cos x = \pm\frac{\sqrt{5}}{3}$$

The $\cos x = \frac{\sqrt{5}}{3}$ at $x = 0.243\text{rad}$ (use your graphing calculator). To find the other value where cosine is positive, subtract 0.243 from 2π, $x = 2\pi - 0.243 = 6.037$ rad.

The $\cos x = -\frac{\sqrt{5}}{3}$ at $x = 2.412\text{rad}$, which is in the 2nd quadrant. To find the other value where cosine is negative (the 3rd quadrant), use the reference angle, 0.243, and add it to π. $x = \pi + 0.243 = 3.383$ rad.

Example 3.2.6.4

Solve the following trig equation in the interval $0 \le x < 2\pi$.

$3\sec(x - 1) + 2 = 0$

Solution

Here, we will find the solution within the given range, $0 \le x < 2\pi$.

$$\begin{aligned} 3\sec(x-1)+2 &= 0 \\ 3\sec(x-1) &= -2 \\ \sec(x-1) &= -\frac{2}{3} \\ \cos(x-1) &= -\frac{3}{2} \end{aligned}$$

At this point, we can stop. The range of the cosine function is from 1 to -1. $-\frac{3}{2}$ is outside of this range, so there is no solution to this equation.

Review

Determine if the following values for x. are solutions to the equation $5 + 6\csc x = 17$.

1. $x = -\frac{7\pi}{6}$
2. $x = \frac{11\pi}{6}$
3. $x = \frac{5\pi}{6}$

Solve the following trigonometric equations. If no solutions exist, write *no solution*.

4. $1 - \cos x = 0$
5. $3\tan x - \sqrt{3} = 0$
6. $4\cos x = 2\cos x + 1$
7. $5\sin x - 2 = 2\sin x + 4$
8. $\sec x - 4 = -\sec x$
9. $\tan^2(x-2) = 3$

Sole the following trigonometric equations within the interval $0 \le x < 2\pi$. If no solutions exist, write *no solution*.

10. $\cos x = \sin x$
11. $-\sqrt{3}\csc x = 2$
12. $6\sin(x-2) = 14$
13. $7\cos x - 4 = 1$
14. $5 + 4\cot^2 x = 17$
15. $2\sin^2 x - 7 = -6$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 14.10.

Additional Resources

Practice: Solving Trigonometric Equations Using Basic Algebra

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3.2.7: Trigonometric Equations Using the Quadratic Formula

The quadratic formula with a trigonometric function in place of the variable.

Solving equations is a fundamental part of mathematics. Being able to find which values of a variable fit an equation allows us to determine all sorts of interesting behavior, both in math and in the sciences. Solving trig equations for angles that satisfy the equation is one application of mathematical methods for solving equations. Suppose someone gave you the following equation:

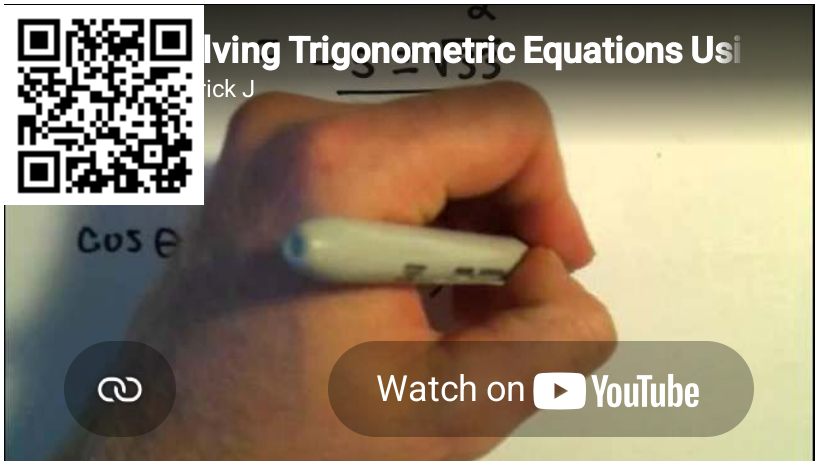
$3\sin^2\theta + 8\sin\theta - 3 = 0$

Quadratic Functions with Trigonometric Equations

When solving quadratic equations that do not factor, the quadratic formula is often used.

Remember that the **quadratic equation** is:

$ax^2 + bx + c = 0$ (where a, b, and c are constants)



In this situation, you can use the quadratic formula to find out what values of "x" satisfy the equation.

The same method can be applied when solving trigonometric equations that do not factor. The values for a is the numerical coefficient of the function's squared term, b is the numerical coefficient of the function term that is to the first power and c is a constant. The formula will result in two answers and both will have to be evaluated within the designated interval.

Solving for Unknown Values

1. Solve $3\cot^2 x - 3\cot x = 1$ for exact values of x over the interval $[0, 2\pi]$.

$$3\cot^2 x - 3\cot x = 1$$
$$3\cot^2 x - 3\cot x - 1 = 0$$

The equation will not factor. Use the quadratic formula for $\cot x$, $a = 3$, $b = -3$, $c = -1$.

$$\cot x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$\cot x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(3)(-1)}}{2(3)}$$
$$\cot x = \frac{3 \pm \sqrt{9 + 12}}{6}$$
$$\cot x = \frac{3 + \sqrt{21}}{6} \qquad \text{or} \qquad \cot x = \frac{3 - \sqrt{21}}{6}$$
$$\cot x = \frac{3 + 4.5826}{6} \qquad \cot x = \frac{3 - 4.5826}{6}$$
$$\cot x = 1.2638 \qquad \cot x = -0.2638$$
$$\tan x = \frac{1}{1.2638} \qquad \tan x = \frac{1}{-0.2638}$$
$$x = 0.6694, 3.81099 \qquad x = 1.8287, 4.9703$$

2. Solve $-5\cos^2 x + 9\sin x + 3 = 0$ for values of x over the interval $[0, 2\pi]$.

Change $\cos^2 x$ to $1 - \sin^2 x$ from the Pythagorean Identity.

$$\begin{aligned} -5\cos^2 x + 9\sin x + 3 &= 0 \\ -5(1 - \sin^2 x) + 9\sin x + 3 &= 0 \\ -5 + 5\sin^2 x + 9\sin x + 3 &= 0 \\ 5\sin^2 x + 9\sin x - 2 &= 0 \end{aligned}$$

$$\sin x = \frac{-9 \pm \sqrt{9^2 - 4(5)(-2)}}{2(5)}$$

$$\sin x = \frac{-9 \pm \sqrt{81 + 40}}{10}$$

$$\sin x = \frac{-9 \pm \sqrt{121}}{10}$$

$$\sin x = \frac{-9 + 11}{10} \text{ and } \sin x = \frac{-9 - 11}{10}$$

$$\sin x = \frac{1}{5} \text{ and } -2$$

$$\sin^{-1}(0.2) \text{ and } \sin^{-1}(-2)$$

$x \approx .201\text{rad}$ and $\pi - .201 \approx 2.941$

This is the only solutions for x since -2 is not in the range of values.

3. Solve $3\sin^2 x - 6\sin x - 2 = 0$ for values of x over the interval $[0, 2\pi]$.

$$3\sin^2 x - 6\sin x - 2 = 0$$

$$\sin x = \frac{6 \pm \sqrt{(-6)^2 - 4(3)(-2)}}{2(3)}$$

$$\sin x = \frac{6 \pm \sqrt{36 - 24}}{6}$$

$$\sin x = \frac{6 \pm \sqrt{12}}{6}$$

$$\sin x = \frac{6 + 3.46}{10} \text{ and } \sin x = \frac{6 - 3.46}{10}$$

$$\sin x = .946 \text{ and } .254$$

$$\sin^{-1}(0.946) \text{ and } \sin^{-1}(0.254)$$

$x \approx 71.08 \text{ deg}$ and $\approx 14.71 \text{ deg}$

Example 3.2.7.1

Earlier, you were asked to solve an equation.

The original equation to solve was:

3sin2\theta +8sin\theta −3=0

Solution

Using the quadratic formula, with a=3,b=8,c=−3, we get:

$$\sin\theta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-8 \pm \sqrt{64 - (4)(3)(-3)}}{6} = \frac{-8 \pm \sqrt{100}}{6} = \frac{-8 \pm 10}{6} = \frac{1}{3} \text{or} -3$$

The solution of -3 is ignored because sine can't take that value, however:

$\sin^{-1}\frac{1}{3} = 19.471°$

Example 3.2.7.2

Solve $\sin^2 x - 2\sin x - 3 = 0$ for x over $[0, \pi]$.

Solution

You can factor this one like a quadratic.

$$\begin{aligned} \sin^2 x - 2\sin x - 3 &= 0 \\ (\sin x - 3)(\sin x + 1) &= 0 \\ \sin x - 3 &= 0 \\ \sin x &= 3 \qquad \text{or } \sin x + 1 = 0 \\ x &= \sin^{-1}(3) \qquad x = \frac{3\pi}{2} \end{aligned}$$

For this problem the only solution is $\frac{3\pi}{2}$ because sine cannot be 3 (it is not in the range).

Example 3.2.7.3

Solve $\tan^2 x + \tan x - 2 = 0$ for values of x over the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Solution

$\tan^2 x + \tan x - 2 = 0$

$$\begin{aligned} -1 \pm \sqrt{1^2 - 4(1)(-2)} &= \tan x \\ 2 \\ \frac{-1 \pm \sqrt{1+8}}{2} &= \tan x \\ \frac{-1 \pm 3}{2} &= \tan x \\ \tan x &= -2 \quad \text{or} \quad 1 \end{aligned}$$

$\tan x = 1$ when $x = \frac{\pi}{4}$, in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\tan x = -2$ when $x = -1.107\text{rad}$

Example 3.2.7.4

Solve the trigonometric equation such that $5\cos^2\theta - 6\sin\theta = 0$ over the interval $[0, 2\pi]$.

Solution

$5\cos^2\theta - 6\sin\theta = 0$ over the interval $[0, 2\pi]$.

$$\begin{aligned} 5\left(1 - \sin^2 x\right) - 6\sin x &= 0 \\ -5\sin^2 x - 6\sin x + 5 &= 0 \\ 5\sin^2 x + 6\sin x - 5 &= 0 \\ -6 \pm \sqrt{6^2 - 4(5)(-5)} &= \sin x \\ 2(5) \\ \frac{-6 \pm \sqrt{36+100}}{10} &= \sin x \\ \frac{-6 \pm \sqrt{136}}{10} &= \sin x \\ \frac{-6 \pm 2\sqrt{34}}{10} &= \sin x \\ \frac{-3 \pm \sqrt{34}}{5} &= \sin x \end{aligned}$$

$x = \sin^{-1}\left(\frac{-3+\sqrt{34}}{5}\right)$ or $\sin^{-1}\left(\frac{-3-\sqrt{34}}{5}\right)$ $x = 0.6018$ rad or 2.5398 rad from the first expression, the second expression will not yield any answers because it is out the range of sine.

Review

Solve each equation using the quadratic formula.

1. $3x^2 + 10x + 2 = 0$
2. $5x^2 + 10x + 2 = 0$
3. $2x^2 + 6x - 5 = 0$

Use the quadratic formula to solve each quadratic equation over the interval $[0, 2\pi)$.

4. $3\cos^2(x) + 10\cos(x) + 2 = 0$
5. $5\sin^2(x) + 10\sin(x) + 2 = 0$
6. $2\sin^2(x) + 6\sin(x) - 5 = 0$
7. $6\cos^2(x) - 5\cos(x) - 21 = 0$
8. $9\tan^2(x) - 42\tan(x) + 49 = 0$
9. $\sin^2(x) + 3\sin(x) = 5$
10. $3\cos^2(x) - 4\sin(x) = 0$
11. $-2\cos^2(x) + 4\sin(x) = 0$
12. $\tan^2(x) + \tan(x) = 3$
13. $\cot^2(x) + 5\tan(x) + 14 = 0$
14. $\sin^2(x) + \sin(x) = 1$
15. What type of sine or cosine equations have no solution?

Review (Answers)

To see the Review answers, open this PDF file and look for section 3.5.

Vocabulary

Term	Definition
Quadratic Equation	A quadratic equation is an equation that can be written in the form $= ax^2 + bx + c = 0$, where a, b, and c are real constants and $a \neq 0$.

Additional Resources

Video: Solving Trigonometric Equations Using the Quadratic Formula

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3.2.8: Trigonometric Equations Using Factoring

Factoring and the Quadratic Formula.

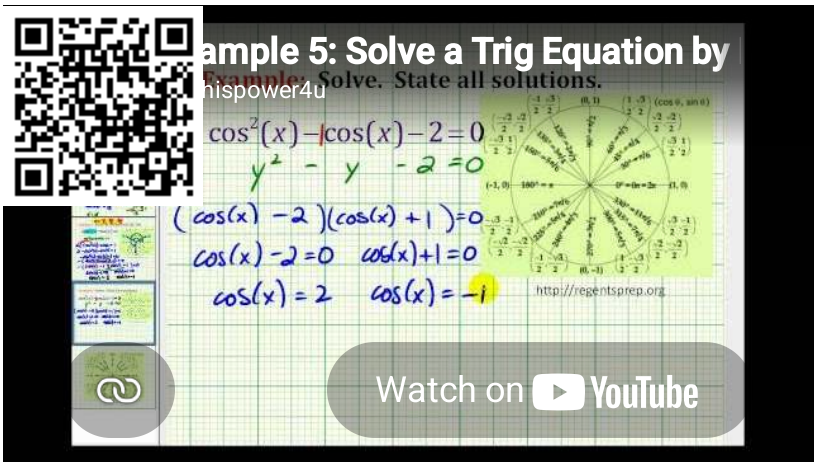
Solving trig equations is an important process in mathematics. Quite often you'll see powers of trigonometric functions and be asked to solve for the values of the variable which make the equation true. For example, suppose you were given the trig equation

$$2\sin x\cos x = \cos x$$

Trigonometric Equations Using **Factoring**

You have no doubt had experience with factoring. You have probably factored equations when looking for the possible values of some variable, such as "x". It might interest you to find out that you can use the same factoring method for more than just a variable that is a number. You can factor trigonometric equations to find the possible values the function can take to satisfy an equation.

Algebraic skills like factoring and substitution that are used to solve various equations are very useful when solving trigonometric equations. As with algebraic expressions, one must be careful to avoid dividing by zero during these maneuvers.



Solving for Unknown Values

1. Solve $2\sin^2 x - 3\sin x + 1 = 0$ for $0 < x \le 2\pi$.

$$\begin{aligned} x^2\sin^2 x - 3\sin x + 1 &= 0 \qquad \text{Factor this like a quadratic equation} \\ (2\sin x - 1)(\sin x - 1) &= 0 \\ \downarrow & \qquad\quad \searrow \\ 2\sin x - 1 &= 0 \qquad \text{or} \quad \sin x - 1 = 0 \\ 2\sin x &= 1 \qquad\quad \sin x = 1 \\ \sin x &= \frac{1}{2} \qquad\quad x = \frac{\pi}{2} \\ x = \frac{\pi}{6} \text{ and } x &= \frac{5\pi}{6} \end{aligned}$$

2. Solve $2\tan x\sin x + 2\sin x = \tan x + 1$ for all values of x.

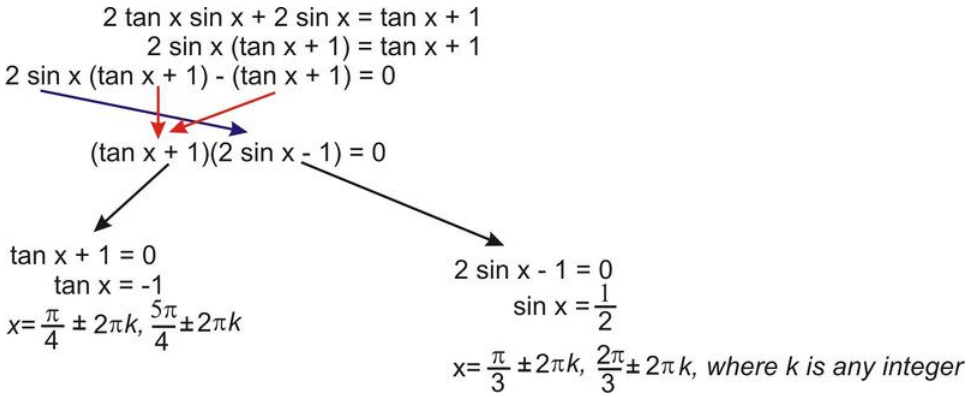


Figure 3.2.8.1

Pull out $\sin x$

There is a common factor of $(\tan x + 1)$

Think of the $-(\tan x + 1)$ as $(-1)(\tan x + 1)$, which is why there is a -1 behind the $2 \sin x$.

3. Solve $2 \sin^2 x + 3 \sin x - 2 = 0$ for all x, $[0, \pi]$.

$$2 \sin^2 x + 3 \sin x - 2 = 0 \rightarrow \text{ Factor like a quadratic}$$
$$(2 \sin x - 1)(\sin x + 2) = 0$$
$$2 \sin x - 1 = 0 \qquad \sin x + 2 = 0$$
$$\sin x = \frac{1}{2} \qquad \sin x = -2$$
$$x = \frac{\pi}{6} \text{ and } x = \frac{5\pi}{6} \text{ There is no solution because the range of } \sin x \text{ is } [-1, 1] .$$

Some trigonometric equations have no solutions. This means that there is no replacement for the variable that will result in a true expression.

Example 3.2.8.1

Earlier, you were asked to solve this:

$2 \sin x \cos x = \cos x$

Solution

Subtract $\cos x$ from both sides and factor it out of the equation:

$$2 \sin x \cos x - \cos x = 0$$
$$\cos x (2 \sin x - 1) = 0$$

Now set each factor equal to zero and solve. The first is $\cos x$:

$$\cos x = 0$$
$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

And now for the other term:

$$2 \sin x - 1 = 0$$
$$\sin x = \frac{1}{2}$$
$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

Example 3.2.8.2

Solve the trigonometric equation $4 \sin x \cos x + 2 \cos x - 2 \sin x - 1 = 0$ such that $0 \leq x < 2\pi$.

Solution

Use factoring by grouping.

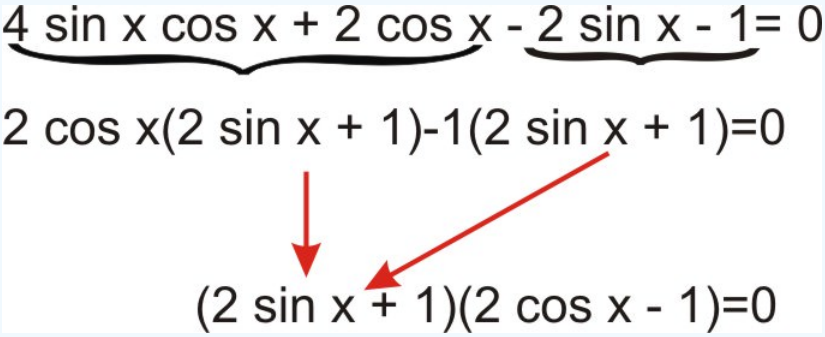


Figure 3.2.8.2

$$\begin{aligned} 2\sin x + 1 &= 0 \quad \text{or} \quad 2\cos x - 1 = 0 \\ 2\sin x &= -1 \qquad\qquad 2\cos x = 1 \\ \sin x &= -\frac{1}{2} \qquad\qquad \cos x = \frac{1}{2} \\ x &= \frac{7\pi}{6}, \frac{11\pi}{6} \qquad\qquad x = \frac{\pi}{3}, \frac{5\pi}{3} \end{aligned}$$

Example 3.2.8.3

Solve $\tan^2 x = 3\tan x$ for x over $[0, \pi]$.

Solution

$$\begin{aligned} \tan^2 x &= 3\tan x \\ \tan^2 x - 3\tan x &= 0 \\ \tan x(\tan x - 3) &= 0 \\ \tan x = 0 \quad &\text{or} \quad \tan x = 3 \\ x = 0, \pi \quad &\qquad\quad x = 1.25 \end{aligned}$$

Example 3.2.8.4

Find all the solutions for the trigonometric equation $2\sin^2 \frac{x}{4} - 3\cos \frac{x}{4} = 0$ over the interval $[0, 2\pi)$.

Solution

$2\sin^2 \frac{x}{4} - 3\cos \frac{x}{4} = 0$

$$\begin{aligned} 2\left(1 - \cos^2 \frac{x}{4}\right) - 3\cos \frac{x}{4} &= 0 \\ 2 - 2\cos^2 \frac{x}{4} - 3\cos \frac{x}{4} &= 0 \\ 2\cos^2 \frac{x}{4} + 3\cos \frac{x}{4} - 2 &= 0 \\ \left(2\cos \frac{x}{4} - 1\right)\left(\cos \frac{x}{4} + 2\right) &= 0 \\ \swarrow \qquad\qquad &\qquad \searrow \\ 2\cos \frac{x}{4} - 1 = 0 \quad &\text{or} \quad \cos \frac{x}{4} + 2 = 0 \\ 2\cos \frac{x}{4} = 1 \quad &\qquad\quad \cos \frac{x}{4} = -2 \\ \cos \frac{x}{4} = \frac{1}{2} \quad &\qquad\quad \\ \frac{x}{4} = \frac{\pi}{3} \quad &\text{or} \quad \frac{5\pi}{3} \\ x = \frac{4\pi}{3} \quad &\text{or} \quad \frac{20\pi}{3} \end{aligned}$$

$\frac{20\pi}{3}$ is eliminated as a solution because it is outside of the range and $\cos x^4 = -2$ will not generate any solutions because -2 is outside of the range of cosine. Therefore, the only solution is $\frac{4\pi}{3}$.

Review

Solve each equation for x over the interval $[0, 2\pi)$.

1. $\cos^2(x) + 2\cos(x) + 1 = 0$
2. $1 - 2\sin(x) + \sin^2(x) = 0$
3. $2\cos(x)\sin(x) - \cos(x) = 0$
4. $\sin(x)\tan^2(x) - \sin(x) = 0$
5. $\sec^2(x) = 4$
6. $\sin^2(x) - 2\sin(x) = 0$
7. $3\sin(x) = 2\cos^2(x)$
8. $2\sin^2(x) + 3\sin(x) = 2$
9. $\tan(x)\sin^2(x) = \tan(x)$
10. $2\sin^2(x) + \sin(x) = 1$
11. $2\cos(x)\tan(x) - \tan(x) = 0$
12. $\sin^2(x) + \sin(x) = 2$
13. $\tan(x)(2\cos^2(x) + 3\cos(x) - 2) = 0$
14. $\sin^2(x) + 1 = 2\sin(x)$
15. $2\cos^2(x) - 3\cos(x) = 2$

Review (Answers)

To see the Review answers, open this PDF file and look for section 3.4.

Vocabulary

Term	Definition
Factoring	Factoring is the process of dividing a number or expression into a product of smaller numbers or expressions.

Additional Resources

Video: Example: Solve a Trig Equation by Factoring

Practice: Trigonometric Equations Using Factoring

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SECTION OVERVIEW

3.3: Sum and Difference Identities

3.3.1: Sum and Difference Formulas

3.3.2: Simplifying Trigonometric Expressions using Sum and Difference Formulas

3.3.3: Sine Sum and Difference Formulas

3.3.4: Cosine Sum and Difference Formulas

3.3.5: Tangent Sum and Difference Formulas

3.3.6: Solving Trigonometric Equations using Sum and Difference Formulas

3.3.7: Finding Exact Trigonometric Values Using Sum and Difference Formulas

3.3.8: Applications of Sum and Difference Formulas

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3.3.1: Sum and Difference Formulas

Sine, cosine, or tangent of two angles that are added or subtracted.

With your knowledge of special angles like the sine and cosine of $30°$ and $45°$, you can find the sine and cosine of $15°$, the difference of $45°$ and $30°$, and $75°$, the sum of $45°$ and $30°$. Using what you know about the unit circle and the sum and difference identities, how do you determine $\sin 15°$ and $\sin 75°$?

Sum and Difference Identities

First look at the derivation of the **cosine difference identity**:

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

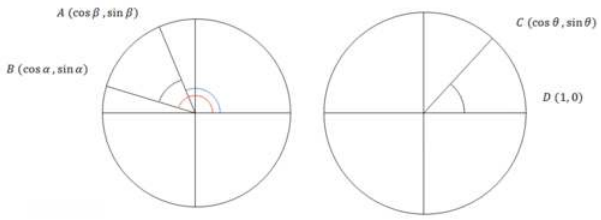


Figure **3.3.1.1**

Start by drawing two arbitrary angles α and β. In the image above α is the angle in red and β is the angle in blue. The difference $\alpha - \beta$ is noted in black as θ. The reason why there are two pictures is because the image on the right has the same angle θ in a rotated position. This will be useful to work with because the length of $\overline{AB}$ will be the same as the length of $\overline{CD}$.

$$\begin{aligned}
\overline{AB} &= \overline{CD} \\
\sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2} &= \sqrt{(\cos\theta - 1)^2 + (\sin\theta - 0)^2} \\
(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 &= (\cos\theta - 1)^2 + (\sin\theta - 0)^2 \\
(\cos\alpha)^2 - 2\cos\alpha\cos\beta + (\cos\beta)^2 + (\sin\alpha)^2 - 2\sin\alpha\sin\beta + (\sin\beta)^2 &= (\cos\theta - 1)^2 + (\sin\theta)^2 \\
2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta &= (\cos\theta)^2 - 2\cos\theta + 1 + (\sin\theta)^2 \\
2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta &= 1 - 2\cos\theta + 1 \\
-2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta &= -2\cos\theta \\
\cos\alpha\cos\beta + \sin\alpha\sin\beta &= \cos\theta \\
&= \cos(\alpha - \beta)
\end{aligned}$$

You can use this identity to prove the **cosine of a sum identity**. First, start with the cosine of a difference and make a substitution. Then use the odd-even identity.

$\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)$

Let $\gamma = -\beta$

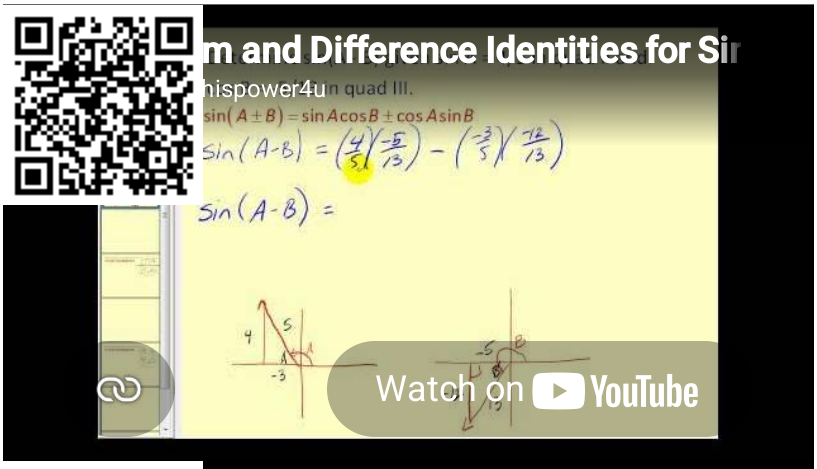
$$\begin{aligned}
\cos\alpha\cos(-\gamma) + \sin\alpha\sin(-\gamma) &= \cos(\alpha + \gamma) \\
\cos\alpha\cos\gamma - \sin\alpha\sin\gamma &= \cos(\alpha + \gamma)
\end{aligned}$$

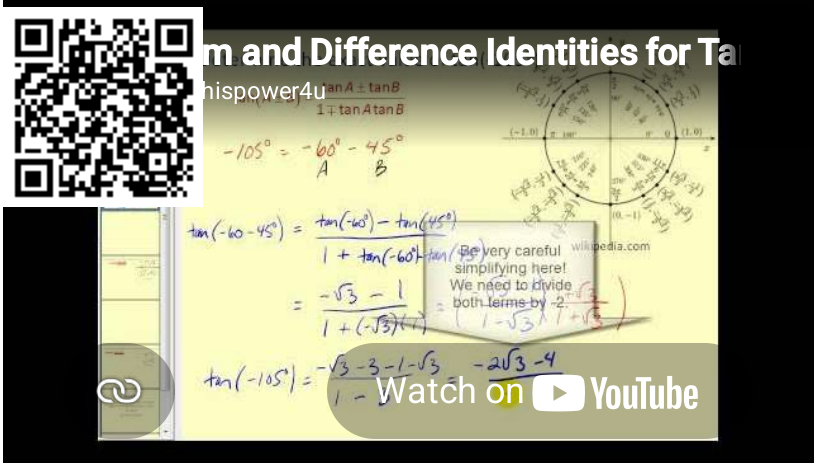
The proofs for sine and tangent are left to the videos and examples. They are listed here for your reference. Cotangent, secant and cosecant are excluded because you can use reciprocal identities to get those once you have sine, cosine and tangent.

Summary

- $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
- $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
- $\tan(\alpha \pm \beta) = \sin(\alpha \pm \beta)\cos(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

The order of the plus or minus signs is important because for cosine of a sum, the negative sign is used on the other side of the identity. This is the opposite of sine of a sum, where a positive sign is used on the other side of the identity.





Example 3.3.1.1

Earlier, you were asked to evaluate $\sin 15^\circ$ and $\sin 75^\circ$ exactly without a calculator. To do this you need to use the sine of a difference and sine of a sum.

Solution

$$\sin(45^\circ - 30^\circ) = \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

$$\sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + cos45^\circ sin30^\circ = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}$$

Example 3.3.1.2

Find the exact value of $\tan 15^\circ$ without using a calculator.

Solution

$$\tan 15^\circ = \tan(45^\circ - 30^\circ) = \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ} \tan 30^\circ = \frac{1 - \frac{\sqrt{3}}{3}}{1 + 1 \cdot \frac{\sqrt{3}}{3}} = \frac{3 - \sqrt{3}}{3 + \sqrt{3}}$$

A final solution will not have a radical in the denominator. In this case multiplying through by the conjugate of the denominator will eliminate the radical.

$$= \frac{(3-\sqrt{3}) \cdot (3-\sqrt{3})}{(3+\sqrt{3}) \cdot (3-\sqrt{3})}$$
$$= \frac{(3-\sqrt{3})^2}{9-3}$$
$$= \frac{(3-\sqrt{3})^2}{6}$$

Example 3.3.1.3

Evaluate the expression exactly without using a calculator.

$\cos 50^\circ \cos 5^\circ + \sin 50^\circ \sin 5^\circ$

Solution

Once you know the general form of the sum and difference identities then you will recognize this as cosine of a difference.

$$\cos 50^\circ \cos 5^\circ + \sin 50^\circ \sin 5^\circ = \cos(50^\circ - 5^\circ) = \cos 45^\circ = \frac{\sqrt{2}}{2}$$

Example 3.3.1.4

Use a sum or difference identity to find an exact value of $\cot\left(\frac{5\pi}{12}\right)$.

Solution

Start with the definition of cotangent as the inverse of tangent.

$$\cot\left(\frac{5\pi}{12}\right) = \frac{1}{\tan\left(\frac{5\pi}{12}\right)}$$
$$= \frac{1}{\tan\left(\frac{9\pi}{12} - \frac{4\pi}{12}\right)}$$
$$= \frac{1}{\tan(135^\circ - 60^\circ)}$$
$$= \frac{1 + \tan 135^\circ \tan 60^\circ}{\tan 135^\circ - \tan 60^\circ}$$
$$= \frac{1 + (-1) \cdot \sqrt{3}}{(-1) - \sqrt{3}}$$
$$= \frac{(1-\sqrt{3})}{(-1-\sqrt{3})}$$
$$= \frac{(1-\sqrt{3})^2}{(-1+\sqrt{3}) \cdot (1-\sqrt{3})}$$
$$= \frac{(1-\sqrt{3})^2}{-(1-3)}$$
$$= \frac{(1-\sqrt{3})^2}{2}$$

Example 3.3.1.5

Prove the following identity:

$$\frac{\sin(x-y)}{\sin(x+y)} = \frac{\tan x - \tan y}{\tan x + \tan y}$$

Solution

Here are the steps:

$$\frac{\sin(x-y)}{\sin(x+y)} = \frac{\tan x - \tan y}{\tan x + \tan y}$$

$$\frac{\sin x \cos y - \cos x \sin y}{\sin x \cos y + \cos x \sin y} =$$

$$\frac{\sin x \cos y - \cos x \sin y}{\sin x \cos y + \cos x \sin y} \cdot \frac{\left(\frac{1}{\cos x \cdot \cos y}\right)}{\left(\frac{1}{\cos x \cdot \cos y}\right)} =$$

$$\frac{\left(\frac{\sin x \cancel{\cos y}}{\cos x \cdot \cancel{\cos y}}\right) - \left(\frac{\cancel{\cos x} \sin y}{\cancel{\cos x} \cdot \cos y}\right)}{\left(\frac{\sin x \cancel{\cos y}}{\cos x \cdot \cancel{\cos y}}\right) + \left(\frac{\cancel{\cos x} \sin y}{\cancel{\cos x} \cdot \cos y}\right)} =$$

$$\frac{\tan x - \tan y}{\tan x + \tan y} =$$

Review

Find the exact value for each expression by using a sum or difference identity.

1. $\cos 75^\circ$
2. $\cos 105^\circ$
3. $\cos 165^\circ$
4. $\sin 105^\circ$
5. $\sec 105^\circ$
6. $\tan 75^\circ$
7. Prove the sine of a sum identity.
8. Prove the tangent of a sum identity.
9. Prove the tangent of a difference identity.
10. Evaluate without a calculator: $\cos 50^\circ \cos 10^\circ - \sin 50^\circ \sin 10^\circ$.
11. Evaluate without a calculator: $\sin 35^\circ \cos 5^\circ - \cos 35^\circ \sin 5^\circ$.
12. Evaluate without a calculator: $\sin 55^\circ \cos 5^\circ + \cos 55^\circ \sin 5^\circ$.
13. If $\cos\alpha \cos\beta = \sin\alpha \sin\beta$, then what does $\cos(\alpha + \beta)$ equal?
14. Prove that $\tan\left(x + \frac{\pi}{4}\right) = \frac{1 + \tan x}{1 - \tan x}$.
15. Prove that $\sin(x + \pi) = -\sin x$.

Review (Answers)

To see the Review answers, open this PDF file and look for section 6.3.

Vocabulary

Term	Definition
$\mp$	The symbol $\mp$ is shorthand for “minus or plus.”
$\pm$	The symbol $\pm$ is shorthand for “plus or minus.”
identity	An identity is a mathematical sentence involving the symbol “$=$” that is always true for variables within the domains of the expressions on either side.

Additional Resources

Practice: Sum and Difference Formulas

Real World: A Tendency to Project

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3.3.2: Simplifying Trigonometric Expressions using Sum and Difference Formulas

Simplify sine, cosine, and tangent of angles that are added or subtracted.

As Agent Trigonometry you are given this clue: $\sin\left(\frac{\pi}{2}-x\right)$. How could you simplify this expression to make solving your case easier?

Simplifying Trigonometric Expressions

We can also use the sum and difference formulas to simplify trigonometric expressions.

The $\sin a=-\frac{3}{5}$ and $\cos b=\frac{12}{13}$. a is in the 3rd quadrant and b is in the 1st. Let's find $\sin(a+b)$.

First, we need to find $\cos a$ and $\sin b$. Using the Pythagorean Theorem, missing lengths are 4 and 5, respectively. So, $\cos a=-\frac{4}{5}$ because it is in the 3rd quadrant and $\sin b=\frac{5}{13}$. Now, use the appropriate formulas.

$$\begin{aligned}\sin(a+b)&=\sin a\cos b+\cos a\sin b\\&=-\frac{3}{5}\cdot\frac{12}{13}+-\frac{4}{5}\cdot\frac{5}{13}=-\frac{56}{65}\end{aligned}$$

Now, using the information from the previous problem above, let's find $\tan(a+b)$.

From the cosine and sine of a and b, we know that $\tan a=\frac{3}{4}$ and $\tan b=\frac{5}{12}$.

$$\begin{aligned}\tan(a+b)&=\frac{\tan a+\tan b}{1-\tan a\tan b}\\&=\frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\cdot\frac{5}{12}}\\&=\frac{\frac{14}{12}}{\frac{11}{16}}\\&=\frac{56}{33}\end{aligned}$$

Finally, let's simplify $\cos(\pi-x)$.

Expand this using the difference formula and then simplify.

$$\begin{aligned}\cos(\pi-x)&=\cos\pi\cos x+\sin\pi\sin x\\&=-1\cdot\cos x+0\cdot\sin x\\&=-\cos x\end{aligned}$$

Example 3.3.2.1

Earlier, you were asked to simplify $\sin\left(\frac{\pi}{2}-x\right)$.

Solution

You can expand the expression using the difference formula and then simplify.

$$\begin{aligned}\sin\left(\frac{\pi}{2}-x\right)&=\sin\frac{\pi}{2}\cos x-\cos\frac{\pi}{2}\sin x\\&=1\cdot cosx-0\cdot\sin x\\&=\cos x\end{aligned}$$

Example 3.3.2.2

Using the information from the first problem above (where we found $\sin(a+b)$), find $\cos(a-b)$.

Solution

$$\cos(a-b) = \cos a \cos b + \sin a \sin b = -\frac{4}{5} \cdot \frac{12}{13} + -\frac{3}{5} \cdot \frac{5}{13}$$
$$= -\frac{63}{65}$$

Example 3.3.2.3

Simplify $\tan(x+\pi)$.

Solution

$$\tan(x+\pi) = \frac{\tan x + \tan \pi}{-\tan x \tan \pi}$$
$$= \frac{\tan x + 0}{1 - \tan 0}$$
$$= \tan x$$

Review

$\sin a = -\frac{8}{17}$, $\pi \leq a < \frac{3\pi}{2}$ and $\sin b = -\frac{1}{2}$, $\frac{3\pi}{2} \leq b < 2\pi$. Find the exact trig values of:

1. $\sin(a+b)$
2. $\cos(a+b)$
3. $\sin(a-b)$
4. $\tan(a+b)$
5. $\cos(a-b)$
6. $\tan(a-b)$

Simplify the following expressions.

7. $\sin(2\pi - x)$
8. $\sin\left(\frac{\pi}{2} + x\right)$
9. $\cos(x+\pi)$
10. $\cos\left(\frac{3\pi}{2} - x\right)$
11. $\tan(x+2\pi)$
12. $\tan(x-\pi)$
13. $\sin(\pi 6 - x)$
14. $\tan\left(\frac{\pi}{4} + x\right)$
15. $\cos\left(x - \frac{\pi}{3}\right)$

Determine if the following trig statements are true or false.

16. $\sin(\pi - x) = \sin(x - \pi)$
17. $\cos(\pi - x) = \cos(x - \pi)$
18. $\tan(\pi - x) = \tan(x - \pi)$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 14.13.

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3.3.3: Sine Sum and Difference Formulas

sine of a sum or difference related to a set of cosine and sine functions.

You've gotten quite good at knowing the values of trig functions. So much so that you and your friends play a game before class everyday to see who can get the most trig functions of different angles correct. However, your friend Jane keeps getting the trig functions of more angles right. You're amazed by her memory, until she smiles one day and tells you that she's been fooling you all this time.

"What you do you mean?" you say.

"I have a trick that lets me calculate more functions in my mind by breaking them down into sums of angles." she replies.

You're really surprised by this. And all this time you thought she just had an amazing memory!

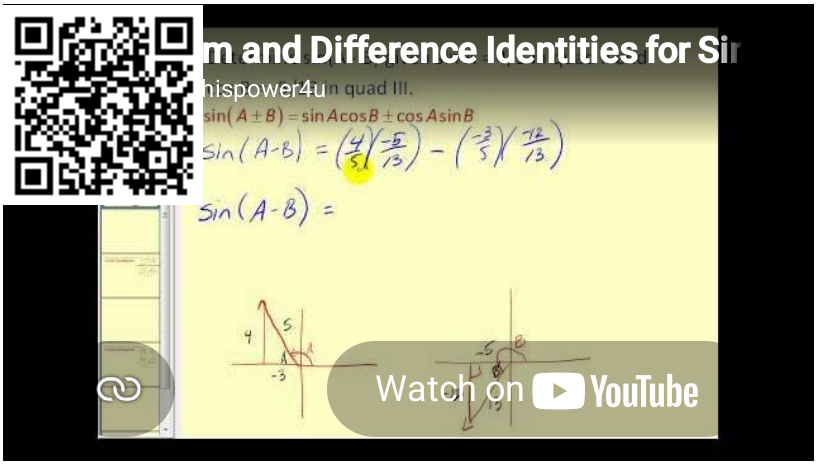
"Here, let me show you," she says. She takes a piece of paper out and writes down:

$\sin \frac{7\pi}{12}$

"This looks like an unusual value to remember for a trig function. So I have a special rule that helps me to evaluate it by breaking it into a sum of different numbers."

sine Sum and Difference Formulas

Our goal here is to figure out a formula that lets you break down a the sine of a sum of two angles (or a difference of two angles) into a simpler formula that lets you use the sine of only one argument in each term.



To find

$\sin(a+b)$:

$$\begin{aligned}
\sin(a+b) &= \cos\left[\frac{\pi}{2} - (a+b)\right] && \text{Set } \theta = a+b \\
&= \cos\left[\left(\frac{\pi}{2} - a\right) - b\right] && \text{Distribute the negative} \\
&= \cos\left(\frac{\pi}{2} - a\right)\cos b + \sin\left(\frac{\pi}{2} - a\right)\sin b && \text{Difference Formula for cosines} \\
&= \sin a \cos b + \cos a \sin b && \text{Co-function Identities}
\end{aligned}$$

In conclusion, $\sin(a+b) = \sin a \cos b + \cos a \sin b$, which is the *sum* formula for sine.

To obtain the identity for $\sin(a-b)$:

$$\begin{aligned}
\sin(a-b) &= \sin[a + (-b)] \\
&= \sin a \cos(-b) + \cos a \sin(-b) && \text{Use the sine sum formula} \\
\sin(a-b) &= \sin a \cos b - \cos a \sin b && \text{Use } \cos(-b) = \cos b \text{ , and } \sin(-b) = -\sin b
\end{aligned}$$

In conclusion, $\sin(a-b) = \sin a \cos b - \cos a \sin b$, so, this is the *difference* formula for sine.

Using the sine Sum and Difference Formula

1. Find the exact value of $\sin\frac{5\pi}{12}$

Recall that there are multiple angles that add or subtract to equal any angle. Choose whichever formula that you feel more comfortable with.

$$\begin{aligned}\sin\frac{5\pi}{12} &= \sin\left(\frac{3\pi}{12}+\frac{2\pi}{12}\right)\\ &= \sin\frac{3\pi}{12}\cos\frac{2\pi}{12}+\cos\frac{3\pi}{12}\sin\frac{2\pi}{12}\\ \sin\frac{5\pi}{12} &= \frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\times\frac{1}{2}\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

2. Given $\sin\alpha=\frac{12}{13}$, where α is in Quadrant II, and $\sin\beta=\frac{3}{5}$, where β is in Quadrant I, find the exact value of $\sin(\alpha+\beta)$.

To find the exact value of $\sin(\alpha+\beta)$, here we use $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$. The values of $\sin\alpha$ and $\sin\beta$ are known, however the values of $\cos\alpha$ and $\cos\beta$ need to be found.

Use $\sin^2\alpha+\cos^2\alpha=1$, to find the values of each of the missing cosine values.

For $\cos\alpha$: $\sin^2\alpha+\cos^2\alpha=1$, substituting $\sin\alpha=\frac{12}{13}$ transforms to $\left(\frac{12}{13}\right)^2+\cos^2\alpha=\frac{144}{169}+\cos^2\alpha=1$ or $\cos^2\alpha=\frac{25}{169}$, $\cos\alpha=\pm\frac{5}{13}$, however, since α is in Quadrant II, the cosine is negative, $\cos\alpha=-\frac{5}{13}$.

For $\cos\beta$ use $\sin^2\beta+\cos^2\beta=1$ and substitute $\sin\beta=\frac{3}{5}$, $\left(\frac{3}{5}\right)^2+\cos^2\beta=\frac{9}{25}+\cos^2\beta=1$ or $\cos^2\beta=\frac{16}{25}$ and $\cos\beta=\pm\frac{4}{5}$ and since β is in Quadrant I, $\cos\beta=\frac{4}{5}$

Now the sum formula for the sine of two angles can be found:

$$\begin{aligned}\sin(\alpha+\beta) &= \frac{12}{13}\times\frac{4}{5}+\left(-\frac{5}{13}\right)\times\frac{3}{5} \text{ or } \frac{48}{65}-\frac{15}{65}\\ \sin(\alpha+\beta) &= \frac{33}{65}\end{aligned}$$

3. Find the exact value of $\sin 15^\circ$

Recall that there are multiple angles that add or subtract to equal any angle. Choose whichever formula that you feel more comfortable with.

$$\begin{aligned}\sin 15^\circ &= \sin(45^\circ-30^\circ)\\ &= \sin 45^\circ\cos 30^\circ+\cos 45^\circ\sin 30^\circ\\ \sin 15^\circ &= (.707)\times(.866)+(.707)\times(.5)\\ &= (.612262)\times(.3535)\\ &= .2164\end{aligned}$$

Example 3.3.3.1

Earlier, you were given a problem about Using the **sine sum formula**.

Solution

With the sine sum formula, you can break the sine into easier to calculate quantities:

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$$\begin{aligned}\sin\frac{7\pi}{12} &= \sin\left(\frac{4\pi}{12}+\frac{3\pi}{12}\right)\\ &= \sin\left(\frac{\pi}{3}+\frac{\pi}{4}\right)\\ &= \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)+\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)\\ &= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)+\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\\ &= \frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

Example 3.3.3.2

Find the exact value for $\sin 345^\circ$

Solution

$$\begin{aligned}\sin 345^\circ &= \sin(300^\circ+45^\circ) = \sin 300^\circ\cos 45^\circ+\cos 300^\circ\sin 45^\circ\\ &= -\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}+\frac{1}{2}\cdot\frac{\sqrt{2}}{2} = -\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4} = \frac{\sqrt{2}-\sqrt{6}}{4}\end{aligned}$$

Example 3.3.3.3

Find the exact value for $\sin\frac{17\pi}{12}$

Solution

$$\begin{aligned}\sin\frac{17\pi}{12} &= \sin\left(\frac{9\pi}{12}+\frac{8\pi}{12}\right) = \sin\left(\frac{3\pi}{4}+\frac{2\pi}{3}\right) = \sin\frac{3\pi}{4}\cos\frac{2\pi}{3}+\cos\frac{3\pi}{4}\sin\frac{2\pi}{3}\\ &= \frac{\sqrt{2}}{2}\cdot\left(-\frac{1}{2}\right)+-\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} = -\frac{\sqrt{2}}{4}-\frac{\sqrt{6}}{4} = \frac{-\sqrt{2}-\sqrt{6}}{4}\end{aligned}$$

Example 3.3.3.4

If $\sin y = -\frac{5}{13}$, y is in quad III, and $\sin z = \frac{4}{5}$, z is in quad II find $\sin(y+z)$

Solution

If $\sin y = -\frac{5}{13}$ and in Quadrant III, then cosine is also negative. By the Pythagorean Theorem, the second leg is $12(5^2+b^2=13^2)$, so $\cos y = -\frac{12}{13}$. If the $\sin z = \frac{4}{5}$ and in Quadrant II, then the cosine is also negative. By the Pythagorean Theorem, the second leg is $3(4^2+b^2=5^2)$, so $\cos = -\frac{3}{5}$. To find $\sin(y+z)$, plug this information into the sine sum formula.

$$\begin{aligned}\sin(y+z) &= \sin y\cos z+\cos y\sin z\\ &= -\frac{5}{13}\cdot-\frac{3}{5}+-\frac{12}{13}\cdot\frac{4}{5}\\ &= \frac{15}{65}-\frac{48}{65}\\ &= -\frac{33}{65}\end{aligned}$$

Review

Find the exact value for each sine expression.

1. $\sin 75^\circ$
2. $\sin 105^\circ$
3. $\sin 165^\circ$
4. $\sin^2 55^\circ$
5. $\sin -15^\circ$

Write each expression as the sine of an angle.

6. $\sin 46^\circ \cos^2 0^\circ + \cos 46^\circ \sin^2 0^\circ$
7. $\sin 3x \cos^2 x - \cos 3x \sin^2 x$
8. $\sin 54^\circ \cos 12^\circ + \cos 54^\circ \sin 12^\circ$
9. $\sin^2 9^\circ \cos 10^\circ - \cos^2 9^\circ \sin 10^\circ$
10. $\sin 4y \cos 3y + \cos 4y \sin^2 y$
11. Prove that $\sin\left(x - \frac{\pi}{2}\right) = -\cos(x)$
12. Suppose that x, y, and z are the three angles of a triangle. Prove that $\sin(x + y) = \sin(z)$
13. Prove that $\sin\left(\frac{\pi}{2} - x\right) = \cos(x)$
14. Prove that $\sin(x + \pi) = -\sin(x)$
15. Prove that $\sin(x - y) + \sin(x + y) = 2\sin(x)\cos(y)$

Review (Answers)

To see the Review answers, open this PDF file and look for section 3.7.

Vocabulary

Term	Definition
sine Difference Formula	The sine difference formula relates the sine of a difference of two arguments to a set of sine and cosines functions, each containing one argument.
sine Sum Formula	The sine sum formula relates the sine of a sum of two arguments to a set of sine and cosines functions, each containing one argument.

Additional Resources

Video: Sum and Difference Identities for sine

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3.3.4: Cosine Sum and Difference Formulas

cosine of a sum or difference related to a set of cosine and sine functions.

While playing a board game with friends, you are using a spinner like this one:



Figure **3.3.4.1**

When you tap the spinner with your hand, it rotates $110°$. However, at that moment, someone taps the game board and the spinner moves back a little to $80°$. One of your friends, who is a grade above you in math, starts talking to you about trig functions.

"Do you think you can calculate the cosine of the difference between those angles?" he asks.

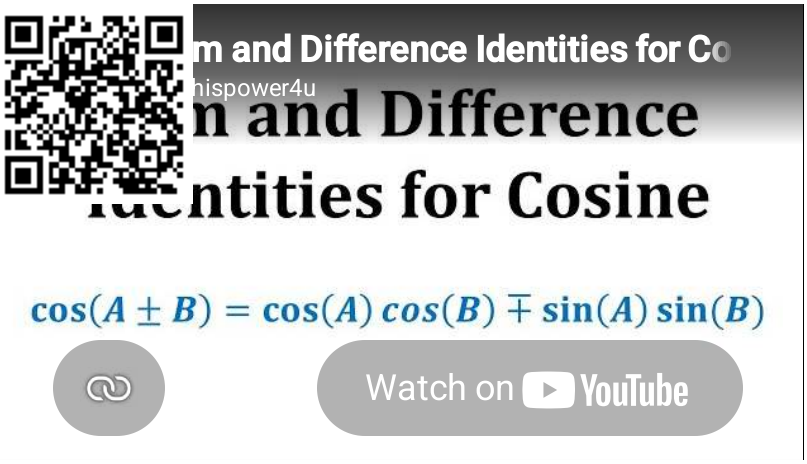
"Hmm," you reply. "Sure. I think it's just $\cos(110° - 80°) = \cos 30°$."

Your friend smiles. "Are you sure?" he asks.

You realize you aren't sure at all. Can you solve this problem? Read this lesson, and by the end you'll be able to calculate the cosine of the difference of the angles.

Formulas for the Sum and Difference of cosines

When thinking about how to calculate values for trig functions, it is natural to consider what the value is for the trig function of a difference of two angles.



For example, is $\cos 15^\circ = \cos(45^\circ - 30^\circ)$? Upon appearance, yes, it is. This section explores how to find an expression that would equal $\cos(45^\circ - 30^\circ)$. To simplify this, let the two given angles be a and b where $0 < b < a < 2\pi$.

Begin with the unit circle and place the angles a and b in standard position as shown in Figure A. Point Pt1 lies on the terminal side of b, so its coordinates are $(\cos b,\ \sin b)$ and Point Pt2 lies on the terminal side of a so its coordinates are $(\cos a, \sin a)$. Place the $a - b$ in standard position, as shown in Figure B. The point A has coordinates $(1, 0)$ and the Pt3 is on the terminal side of the angle $a - b$, so its coordinates are $(\cos[a - b], \sin[a - b])$.

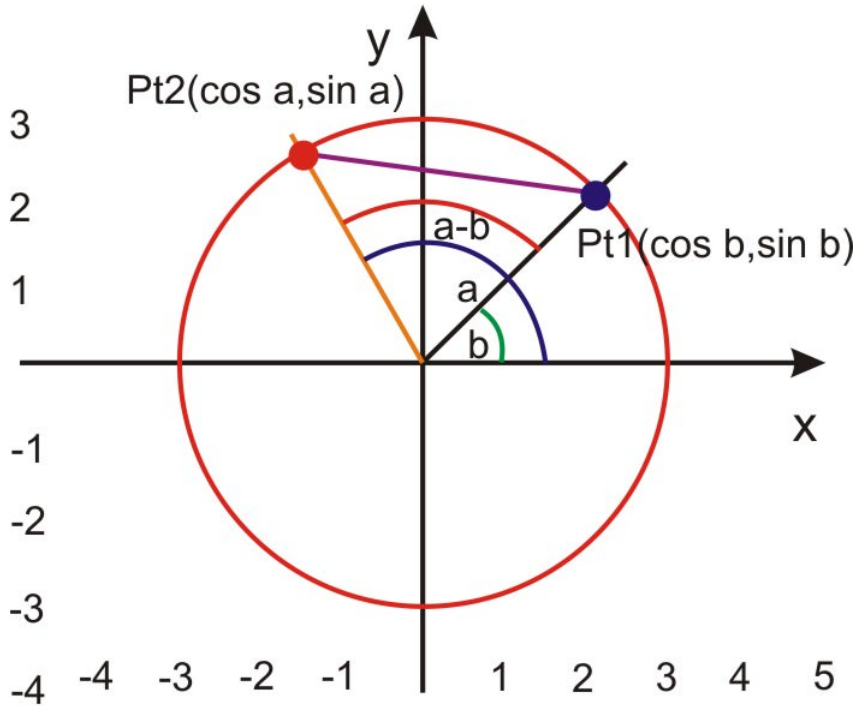


Figure A

Figure 3.3.4.2

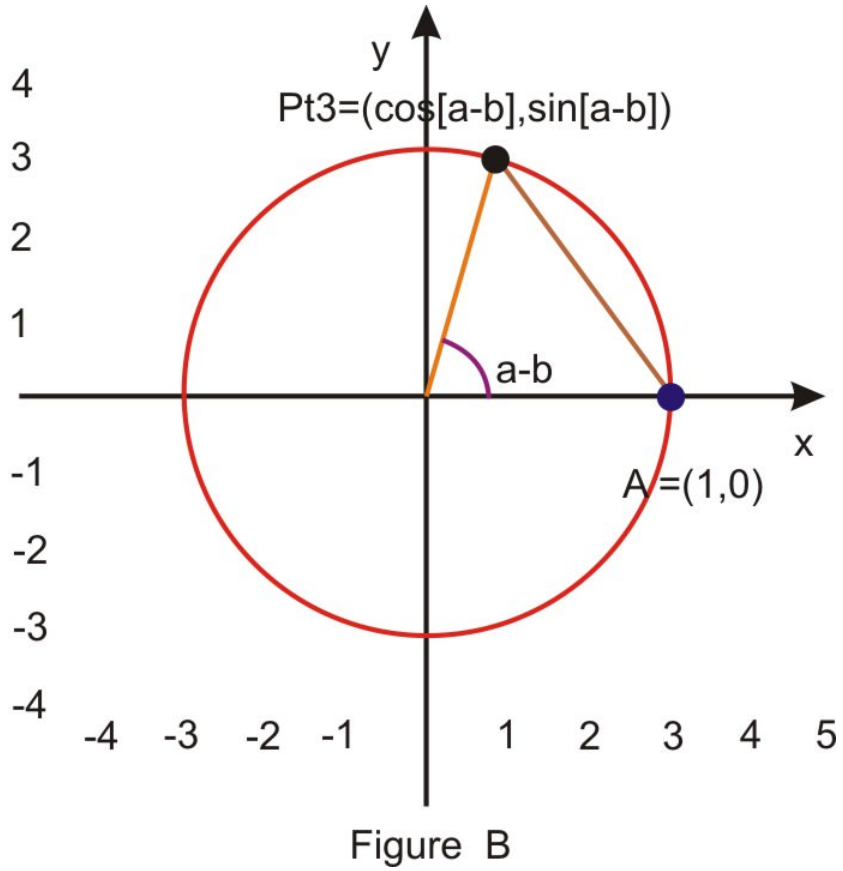


Figure **3.3.4.3**

Triangles OP_1P_2 in figure A and Triangle OAP_3 in figure B are congruent. (Two sides and the included angle, $a-b$, are equal). Therefore the unknown side of each triangle must also be equal. That is: $d(A, P_3) = d(P_1,\ P_2)$

Applying the distance formula to the triangles in Figures A and B and setting them equal to each other:

$$\sqrt{[\cos(a-b)-1]^2+[\sin(a-b)-0]^2} = \sqrt{(\cos a-\cos b)^2+(\sin a-\sin b)^2}$$

Square both sides to eliminate the square root.

$$[\cos(a-b)-1]^2+[\sin(a-b)-0]^2 = (\cos a-\cos b)^2+(\sin a-\sin b)^2$$

FOIL all four squared expressions and simplify.

$$\cos^2(a-b)-2\cos(a-b)+1+\sin^2(a-b) = \cos^2 a-2\cos a\cos b+\cos^2 b+\sin^2 a-2$$

$$\underbrace{\sin^2(a-b)+\cos^2(a-b)}-2\cos(a-b)+1 = \underbrace{\sin^2 a+\cos^2 a}-2\cos a\cos b+\underbrace{\sin^2 b+\cos^2 b}-2\sin a\sin b$$

$$2\cos(a-b)+1 = 1-2\cos a\cos b+1-2\sin a\sin b$$
$$2-2\cos(a-b) = 2-2\cos a\cos b-2\sin a\sin b$$
$$-2\cos(a-b) = -2\cos a\cos b-2\sin a\sin b$$
$$\cos(a-b) = \cos a\cos b+\sin a\sin b$$

In $\cos(a-b) = \cos a\cos b + \sin a\sin b$, the *difference* formula for cosine, you can substitute $a-(-b) = a+b$ to obtain: $\cos(a+b) = \cos[a-(-b)]$ or $\cos a\cos(-b) + \sin a\sin(-b)$. since $\cos(-b) = \cos b$ and $\sin(-b) = -\sin b$, then $\cos(a+b) = \cos a\cos b - \sin a\sin b$, which is the *sum* formula for cosine.

using the **cosine Difference Formula**

1. Find an equivalent form of $\cos\left(\frac{\pi}{2}-\theta\right)$ using the cosine difference formula.

$$\cos\left(\frac{\pi}{2}-\theta\right)=\cos\frac{\pi}{2}\cos\theta+\sin\frac{\pi}{2}\sin\theta$$
$$\cos\left(\frac{\pi}{2}-\theta\right)=0\times\cos\theta+1\times\sin\theta,\ \text{substitute}\ \cos\frac{\pi}{2}=0\ \text{and}\ \sin\frac{\pi}{2}=1$$
$$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$$

We know that is a true identity because of our understanding of the sine and cosine curves, which are a phase shift of $\frac{\pi}{2}$ off from each other.

The cosine formulas can also be used to find exact values of cosine that we weren't able to find before, such as $15^\circ=(45^\circ-30^\circ)$, $75^\circ=(45^\circ+30^\circ)$, among others.

2. Find the exact value of $\cos 15^\circ$

Use the difference formula where $a=45^\circ$ and $b=30^\circ$.

$$\cos(45^\circ-30^\circ)=\cos 45^\circ\cos 30^\circ+\sin 45^\circ\sin 30^\circ$$
$$\cos 15^\circ=\frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\times\frac{1}{2}$$
$$\cos 15^\circ=\frac{\sqrt{6}+\sqrt{2}}{4}$$

3. Find the exact value of $\cos\frac{5\pi}{12}$, in radians.

$\cos\frac{5\pi}{12}=\cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right)$, notice that $\frac{\pi}{4}=\frac{3\pi}{12}$ and $\frac{\pi}{6}=\frac{2\pi}{12}$

$$\cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right)=\cos\frac{\pi}{4}\cos\frac{\pi}{6}-\sin\frac{\pi}{4}\sin\frac{\pi}{6}$$
$$\cos\frac{\pi}{4}\cos\frac{\pi}{6}-\sin\frac{\pi}{4}\sin\frac{\pi}{6}=\frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\times\frac{1}{2}$$
$$=\frac{\sqrt{6}-\sqrt{2}}{4}$$

Example 3.3.4.1

Earlier, you were asked if you can calculate the cosine of the difference between the two angles.

Solution

It would seem that your friend was having some fun with you, since he figured you didn't know the cosine difference formula. But now, with this formula in hand, you can readily solve for the difference of the two angles:

$$\cos(110^\circ-80^\circ)=(\cos 110^\circ)(\cos 80^\circ)+(\sin 110^\circ)(\sin 80^\circ)$$
$$=(-.342)(.174)+(.9397)(.9848)$$
$$=-.0595+.9254=.8659$$

Therefore,

$\cos(110^\circ-80^\circ)=.8659$

Example 3.3.4.2

Find the exact value for $\cos\frac{5\pi}{12}$

Solution

$$\cos\frac{5\pi}{12}=\cos\left(\frac{2\pi}{12}+\frac{3\pi}{12}\right)$$
$$=\cos\left(\frac{\pi}{6}+\frac{\pi}{4}\right)=\cos\frac{\pi}{6}\cos\frac{\pi}{4}-\sin\frac{\pi}{6}\sin\frac{\pi}{4}=\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}-12\cdot\frac{\sqrt{2}}{2}=\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}=\sqrt{6}-\frac{\sqrt{2}}{4}$$

Example 3.3.4.3

Find the exact value for $\cos\frac{7\pi}{12}$

Solution

$$\begin{aligned}\cos\frac{7\pi}{12} &= \cos\left(\frac{4\pi}{12}+\frac{3\pi}{12}\right)\\ &= \cos\left(\frac{\pi}{3}+\frac{\pi}{4}\right)\\ &= \cos\frac{\pi}{3}\cos\frac{\pi}{4}-\sin\frac{\pi}{3}\sin\frac{\pi}{4}\\ &= 12\cdot\frac{\sqrt{2}}{2}-\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}\\ &= \frac{\sqrt{2}}{4}-\frac{\sqrt{6}}{4}\\ &= \frac{\sqrt{2}-\sqrt{6}}{4}\end{aligned}$$

Example 3.3.4.4

Find the exact value for $\cos 345^\circ$

Solution

$$\begin{aligned}\cos 345^\circ &= \cos(315^\circ+30^\circ)\\ &= \cos 315^\circ\cos 30^\circ-\sin 315^\circ\sin 30^\circ\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-\left(-\frac{\sqrt{2}}{2}\right)\cdot\frac{1}{2}\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

Review

Find the exact value for each cosine expression.

1. $\cos 75^\circ$
2. $\cos 105^\circ$
3. $\cos 165^\circ$
4. $\cos 255^\circ$
5. $\cos -15^\circ$

Write each expression as the cosine of an angle.

6. $\cos 96^\circ\cos 20^\circ+\sin 96^\circ\sin 20^\circ$
7. $\cos 4x\cos 3x-\sin 4x\sin 3x$
8. $\cos 37^\circ\cos 12^\circ+\sin 37^\circ\sin 12^\circ$
9. $\cos 59^\circ\cos 10^\circ-\sin 59^\circ\sin 10^\circ$
10. $\cos 5y\cos 2y+\sin 5y\sin 2y$
11. Prove that $\cos\left(x-\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}(\cos(x)+\sin(x))$
12. If $\cos(x)\cos(y)=\sin(x)\sin(y)$, then what does $\cos(x+y)$ equal?
13. Prove that $\cos\left(x-\frac{\pi}{2}\right)=\sin(x)$
14. Use the fact that $\cos\left(\frac{\pi}{2}-x\right)=\sin(x)$ (shown in examples), to show that $\sin\left(\frac{\pi}{2}-x\right)=\cos(x)$.
15. Prove that $\cos(x-y)+\cos(x+y)=2\cos(x)\cos(y)$.

Review (Answers)

To see the Review answers, open this PDF file and look for section 3.6.

Additional Resources

Video: Sum and Difference Identities for cosine

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3.3.5: Tangent Sum and Difference Formulas

tangent of a sum or difference related to a set of tangent functions.

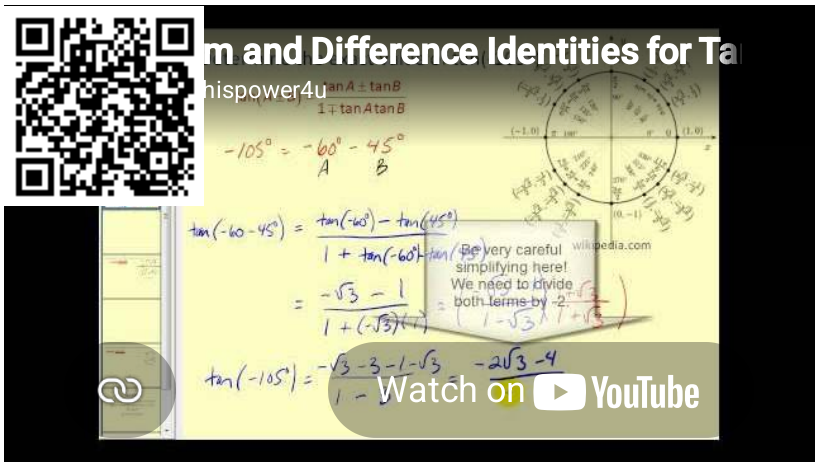
Suppose you were given two angles and asked to find the tangent of the difference of them. For example, can you compute:

$$\tan(120^\circ - 40^\circ)$$

Would you just subtract the angles and then take the tangent of the result? Or is something more complicated required to solve this problem? Keep reading, and by the end of this lesson, you'll be able to calculate trig functions like the one above.

tangent Sum and Difference Formulas

In this lesson, we want to find a formula that will make computing the tangent of a sum of arguments or a difference of arguments easier. As first, it may seem that you should just add (or subtract) the arguments and take the tangent of the result. However, it's not quite that easy.



To find the sum formula for tangent:

$$\begin{aligned}
\tan(a+b) &= \frac{\sin(a+b)}{\cos(a+b)} && \text{Using } \tan\theta = \frac{\sin\theta}{\cos\theta} \\
&= \frac{\sin a\cos b + \sin b\cos a}{\cos a\cos b - \sin a\sin b} && \text{Substituting the sum formulas for sine and cosine} \\
&= \frac{\dfrac{\sin a\cos b + \sin b\cos a}{\cos a\cos b}}{\dfrac{\cos a\cos b - \sin a\sin b}{\cos a\cos b}} && \text{Divide both the numerator and the denominator } \cos a\cos b \\
&= \frac{\dfrac{\sin a\cos b}{\cos a\cos b} + \dfrac{\sin b\cos a}{\cos a\cos b}}{\dfrac{\cos a\cos b}{\cos a\cos b} - \dfrac{\sin a\sin b}{\cos a\cos b}} && \text{Reduce each of the fractions} \\
&= \frac{\dfrac{\sin a}{\cos a} + \dfrac{\sin b}{\cos b}}{1 - \dfrac{\sin a\sin b}{\cos a\cos b}} && \text{Substitute } \frac{\sin\theta}{\cos\theta} = \tan\theta \\
\tan(a+b) &= \frac{\tan a + \tan b}{1 - \tan a\tan b} && \text{Sum formula for tangent}
\end{aligned}$$

In conclusion, $\tan(a+b) = \dfrac{\tan a + \tan b}{1 - \tan a\tan b}$. Substituting $-b$ for b in the above results in the difference formula for tangent:

$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a\tan b}$$

Using the **tangent Difference Formula**

1. Find the exact value of $\tan 285^\circ$.

Use the difference formula for tangent, with $285^\circ = 330^\circ - 45^\circ$

$$\begin{aligned}\tan(330^\circ - 45^\circ) &= \frac{\tan 330^\circ - \tan 45^\circ}{1 + \tan 330^\circ \tan 45^\circ} \\ &= \frac{-\frac{\sqrt{3}}{3} - 1}{1 - \frac{\sqrt{3}}{3} \cdot 1} = \frac{-3 - \sqrt{3}}{3 - \sqrt{3}} \\ &= \frac{-3 - \sqrt{3}}{3 - \sqrt{3}} \cdot \frac{3 + \sqrt{3}}{3 + \sqrt{3}} \\ &= \frac{-9 - 6\sqrt{3} - 3}{9 - 3} \\ &= \frac{-12 - 6\sqrt{3}}{6} \\ &= -2 - \sqrt{3}\end{aligned}$$

To verify this on the calculator, $\tan 285^\circ = -3.732$ and $-2 - \sqrt{3} = -3.732$.

2. Verify the tangent difference formula by finding $\tan \frac{6\pi}{6}$, since this should be equal to $\tan \pi = 0$.

Use the difference formula for tangent, with $\tan \frac{6\pi}{6} = \tan\left(\frac{7\pi}{6} - \frac{\pi}{6}\right)$

$$\begin{aligned}\tan\left(\frac{7\pi}{6} - \frac{\pi}{6}\right) &= \frac{\tan \frac{7\pi}{6} - \tan \frac{\pi}{6}}{1 + \tan \frac{7\pi}{6} \tan \frac{\pi}{6}} \\ &= \frac{\frac{\sqrt{2}}{6} - \frac{\sqrt{2}}{6}}{1 - \frac{\sqrt{2}}{6} \cdot \frac{\sqrt{2}}{6}} = \frac{0}{1 - \frac{2}{36}} \\ &= \frac{0}{\frac{34}{36}} \\ &= 0\end{aligned}$$

3. Find the exact value of $\tan 165^\circ$.

Use the difference formula for tangent, with $165^\circ = 225^\circ - 60^\circ$

$$\begin{aligned}\tan(225^\circ - 60^\circ) &= \frac{\tan 225^\circ - \tan 60^\circ}{1 + \tan 225^\circ \tan 60^\circ} \\ &= \frac{1 - \sqrt{3}}{1 - 1 \cdot \sqrt{3}} = 1\end{aligned}$$

Example 3.3.5.1

Earlier, you were asked to find $\tan(120^\circ - 40^\circ)$.

Solution

You can use the tangent difference formula:

$$\tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$$

to help solve this. Substituting in known quantities:

$$\tan(120^\circ - 40^\circ) = \frac{\tan 120^\circ - \tan 40^\circ}{1 + (\tan 120^\circ)(\tan 40^\circ)} = \frac{-1.732 - .839}{1 + (-1.732)(.839)} = \frac{-2.571}{-.453148} = 5.674$$

Example 3.3.5.2

Find the exact value for $\tan 75^\circ$

Solution

$$\begin{aligned}\tan 75^\circ &= \tan(45^\circ + 30^\circ)\\ &= \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ}\\ &= \frac{1 + \frac{\sqrt{3}}{3}}{1 - 1 \cdot \frac{\sqrt{3}}{3}} = \frac{\frac{3+\sqrt{3}}{3}}{\frac{3-\sqrt{3}}{3}}\\ &= \frac{3+\sqrt{3}}{3-\sqrt{3}} \cdot \frac{3+\sqrt{3}}{3+\sqrt{3}}\\ &= \frac{9+6\sqrt{3}+3}{9-3} = \frac{12+6\sqrt{3}}{6}\\ &= 2+\sqrt{3}\end{aligned}$$

Example 3.3.5.3

Simplify $\tan(\pi + \theta)$

Solution

$$\tan(\pi+\theta) = \frac{\tan\pi + \tan\theta}{1 - \tan\pi\tan\theta} = \frac{\tan\theta}{1} = \tan\theta$$

Example 3.3.5.4

Find the exact value for $\tan 15^\circ$

Solution

$$\begin{aligned}\tan 15^\circ &= \tan(45^\circ - 30^\circ)\\ &= \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ}\\ &= \frac{1 - \frac{\sqrt{3}}{3}}{1 + 1 \cdot \frac{\sqrt{3}}{3}} = \frac{\frac{3-\sqrt{3}}{3}}{\frac{3+\sqrt{3}}{3}}\\ &= \frac{3-\sqrt{3}}{3+\sqrt{3}} \cdot \frac{3-\sqrt{3}}{3-\sqrt{3}}\\ &= \frac{9+6\sqrt{3}+3}{9-3} = \frac{12+6\sqrt{3}}{6}\\ &= 2+\sqrt{3}\end{aligned}$$

Review

Find the exact value for each tangent expression.

1. $\tan\frac{5\pi}{12}$
2. $\tan\frac{11\pi}{12}$
3. $\tan -165^\circ$
4. $\tan 255^\circ$
5. $\tan -15^\circ$

Write each expression as the tangent of an angle.

6. $\frac{\tan 15^\circ + \tan 42^\circ}{1 - \tan 15^\circ \tan 42^\circ}$
7. $\frac{\tan 65^\circ - \tan 12^\circ}{1 + \tan 65^\circ \tan 12^\circ}$
8. $\frac{\tan 10^\circ + \tan 50^\circ}{1 - \tan 10^\circ \tan 50^\circ}$
9. $\frac{\tan 2y + \tan 4}{1 - \tan 2\tan 4y}$
10. $\frac{\tan x - \tan 3x}{1 + \tan x \tan 3x}$

11. $\frac{\tan 2x - \tan y}{1 + \tan 2x \tan y}$
12. Prove that $\tan\left(x + \frac{\pi}{4}\right) = \frac{1 + \tan(x)}{1 - \tan(x)}$
13. Prove that $\tan\left(x - \frac{\pi}{2}\right) = -\cot(x)$
14. Prove that $\tan\left(\frac{\pi}{2} - x\right) = \cot(x)$
15. Prove that $\tan(x + y)\tan(x - y) = \frac{\tan^2(x) - \tan^2(y)}{1 - \tan^2(x)\tan^2(y)}$

Review (Answers)

To see the Review answers, open this PDF file and look for section 3.8.

Vocabulary

Term	Definition
tangent Difference Formula	The tangent difference formula relates the tangent of a difference of two arguments to a set of tangent functions, each containing one argument.
tangent Sum Formula	The tangent sum formula relates the tangent of a sum of two arguments to a set of tangent functions, each containing one argument.

Additional Resources

Video: Sum and Difference Identities for tangent

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3.3.6: Solving Trigonometric Equations using Sum and Difference Formulas

Solve sine, cosine, and tangent of angles that are added or subtracted.

As Agent Trigonometry, you are given a piece of the puzzle: $\sin\left(\frac{\pi}{2}-x\right)=-1$. What is the value of x?

Solving Trigonometric Functions

We can use the sum and difference formulas to solve trigonometric equations. For this concept, we will only find solutions in the interval $0 \le x < 2\pi$.

Let's solve the following functions using the sum and difference formulas.

1. $\cos(x-\pi)=\frac{\sqrt{2}}{2}$

Use the formula to simplify the left-hand side and then solve for x.

$$\begin{aligned}\cos(x-\pi)&=\frac{\sqrt{2}}{2}\\ \cos x\cos\pi+\sin x\sin\pi&=\frac{\sqrt{2}}{2}\\ -\cos x&=\frac{\sqrt{2}}{2}\\ \cos x&=-\frac{\sqrt{2}}{2}\end{aligned}$$

The cosine negative in the 2nd and 3rd quadrants. $x=\frac{3\pi}{4}$ and $\frac{5\pi}{4}$.

2. $\sin\left(x+\frac{\pi}{4}\right)+1=\sin\left(\frac{\pi}{4}-x\right)$

$$\begin{aligned}\sin\left(x+\frac{\pi}{4}\right)+1&=\sin\left(\frac{\pi}{4}-x\right)\\ \sin x\cos\frac{\pi}{4}+\cos x\sin\frac{\pi}{4}+1&=\sin\frac{\pi}{4}\cos x-\cos\frac{\pi}{4}\sin x\\ \sin x\cdot\frac{\sqrt{2}}{2}+\cos x\cdot\frac{\sqrt{2}}{2}+1&=\frac{\sqrt{2}}{2}\cdot\cos x-\frac{\sqrt{2}}{2}\cdot\sin x\\ \sqrt{2}\sin x&=-1\\ \sin x&=-\frac{1}{\sqrt{2}}=-\frac{\sqrt{2}}{2}\end{aligned}$$

In the interval, $x=\frac{5\pi}{4}$ and $\frac{7\pi}{4}$.

3. $2\sin\left(x+\frac{\pi}{3}\right)=\tan\frac{\pi}{3}$

$$\begin{aligned}2\sin\left(x+\frac{\pi}{3}\right)&=\tan\frac{\pi}{3}\\ 2\left(\sin x\cos\frac{\pi}{3}+\cos x\sin\frac{\pi}{3}\right)&=\sqrt{3}\\ 2\sin x\cdot\frac{1}{2}+2\cos x\cdot\frac{\sqrt{3}}{2}&=\sqrt{3}\\ \sin x+\sqrt{3}\cos x&=\sqrt{3}\\ \sin x&=\sqrt{3}(1-\cos x)\\ \sin^2 x&=3\left(1-2\cos x+\cos^2 x\right) && \text{square both sides}\\ 1-\cos^2 x&=3-6\cos x+3\cos^2 x && \text{substitute } \sin^2 x=1-\cos^2 x\\ 0&=4\cos^2 x-6\cos x+2\\ 0&=2\cos^2 x-3\cos x+1\end{aligned}$$

At this point, we can factor the equation to be $(2\cos x-1)(\cos x-1)=0$. $\cos x=\frac{1}{2}$, and 1, so $x=0,\ \frac{\pi}{3},\ \frac{5\pi}{3}$. Be careful with these answers. When we check these solutions it turns out that $\frac{5\pi}{3}$ does not work.

$$2\sin\left(\frac{5\pi}{3}+\frac{\pi}{3}\right)=\tan\frac{\pi}{3}$$
$$2\sin 2\pi=\sqrt{3}$$
$$0\neq\sqrt{3}$$

Therefore, $\frac{5\pi}{3}$ is an extraneous solution.

Example 3.3.6.1

Earlier, you were asked to find the value of x from the equation $\sin\left(\frac{\pi}{2}-x\right)=-1$.

Solution

First, simplify the expression $\sin\left(\frac{\pi}{2}-x\right)$ as:

$$\begin{aligned}\sin\left(\frac{\pi}{2}-x\right)&=\sin\frac{\pi}{2}\cos x-\cos\frac{\pi}{2}\sin x\\&=1\cdot\cos x-0\cdot\sin x\\&=\cos x\end{aligned}$$

So what you're now looking for is the value of x where $\cos x=-1$.

The cosine of 180° is equal to −1.

Solve the following equations in the interval $0\leq x<2\pi$.

Example 3.3.6.2

$\cos(2\pi-x)=\frac{1}{2}$

Solution

$$\begin{aligned}\cos(2\pi-x)&=\frac{1}{2}\\\cos 2\pi\cos x+\sin 2\pi\sin x&=\frac{1}{2}\\\cos x&=\frac{1}{2}\\x&=\frac{\pi}{3}\text{ and }\frac{5\pi}{3}\end{aligned}$$

Example 3.3.6.3

$\sin\left(\frac{\pi}{6}-x\right)+1=\sin\left(x+\frac{\pi}{6}\right)$

Solution

$$\begin{aligned}\sin\left(\frac{\pi}{6}-x\right)+1&=\sin\left(x+\frac{\pi}{6}\right)\\\sin\frac{\pi}{6}\cos x-\cos\frac{\pi}{6}\sin x+1&=\sin x\cos\frac{\pi}{6}+\cos x\sin\frac{\pi}{6}\\\frac{1}{2}\cos x-\frac{\sqrt{3}}{2}\sin x+1&=\frac{\sqrt{3}}{2}\sin x+\frac{1}{2}\cos x\\1&=\sqrt{3}\sin x\\\frac{1}{\sqrt{3}}&=\sin x\\x&=\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)=0.6155\text{ and }2.5261\text{ rad}\end{aligned}$$

Example 3.3.6.4

$\cos\left(\frac{\pi}{2}+x\right)=\tan\frac{\pi}{4}$

Solution

$$\begin{aligned}\cos\left(\frac{\pi}{2}+x\right)&=\tan\frac{\pi}{4}\\ \cos\frac{\pi}{2}\cos x-\sin\frac{\pi}{2}\sin x&=1\\ -\sin x&=1\\ \sin x&=-1\\ x&=\frac{3\pi}{2}\end{aligned}$$

Review

Solve the following trig equations in the interval $0\le x<2\pi$.

1. $\sin(x-\pi)=-\frac{\sqrt{2}}{2}$
2. $\cos(2\pi+x)=-1$
3. $\tan\left(x+\frac{\pi}{4}\right)=1$
4. $\sin\left(\frac{\pi}{2}-x\right)=\frac{1}{2}$
5. $\sin\left(x+3\frac{\pi}{4}\right)+\sin\left(x-\frac{3\pi}{4}\right)=1$
6. $\sin\left(x+\frac{\pi}{6}\right)=-\sin\left(x-\frac{\pi}{6}\right)$
7. $\cos\left(x+\frac{\pi}{6}\right)=\cos\left(x-\frac{\pi}{6}\right)+1$
8. $\cos\left(x+\frac{\pi}{3}\right)+\cos\left(x-\frac{\pi}{3}\right)=1$
9. $\tan(x+\pi)+2\sin(x+\pi)=0$
10. $\tan(x+\pi)+\cos\left(x+\frac{\pi}{2}\right)=0$
11. $\tan\left(x+\frac{\pi}{4}\right)=\tan\left(x-\frac{\pi}{4}\right)$
12. $\sin\left(x-\frac{5\pi}{3}\right)-\sin\left(x-\frac{2\pi}{3}\right)=0$
13. $4\sin(x+\pi)-2=2\cos\left(x+\frac{\pi}{2}\right)$
14. $1+2\cos(x-\pi)+\cos x=0$
15. **Real Life Application** The height, h (in feet), of two people in different seats on a Ferris wheel can be modeled by $h_1=50\cos 3t+46$ and $h_2=50\cos 3\left(t-\frac{3\pi}{4}\right)+46$ where t is the time (in minutes). When are the two people at the same height?

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 14.14.

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3.3.7: Finding Exact Trigonometric Values Using Sum and Difference Formulas

Convert angles to sum or difference of 30, 45, and 60 degrees to solve.

You measure an angle with your protractor to be $165°$. How could you find the exact sine of this angle without Using a calculator?

Sum and Difference Formulas

You know that $\sin 30° = \frac{1}{2}$, $\cos 135° = -\frac{\sqrt{2}}{2}$, $\tan 300° = -\sqrt{3}$, etc... from the special right triangles. In this concept, we will learn how to find the exact values of the trig functions for angles other than these multiples of 30^{\circ} ,45^{\circ} , and 60^{\circ} . Using the Sum and Difference Formulas, we can find these exact trig values.

Sum and Difference Formulas

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$
$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$
$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$$

Let's find the following exact values Using the Sum and Difference Formulas.

1. $\sin 75°$

This is an example of where we can use the sine sum formula from above, $\sin(a + b) = \sin a \cos b + \cos a \sin b$, where $a = 45°$ and $b = 30°$.

$$\begin{aligned}\sin 75° &= \sin(45° + 30°) \\ &= \sin 45° \cos 30° + \cos 45° \sin 30° \\ &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \\ &= \frac{\sqrt{6} + \sqrt{2}}{4}\end{aligned}$$

In general, $\sin(a + b) \neq \sin a + \sin b$ and similar statements can be made for the other sum and difference formulas.

2. $\cos \frac{11\pi}{12}$

For this problem, we could use either the sum or difference cosine formula, $\frac{11\pi}{12} = \frac{2\pi}{3} + \frac{\pi}{4}$ or $\frac{11\pi}{12} = \frac{7\pi}{6} - \frac{\pi}{4}$. Let's use the sum formula.

$$\begin{aligned}\cos \frac{11\pi}{12} &= \cos\left(\frac{2\pi}{3} + \frac{\pi}{4}\right) \\ &= \cos \frac{2\pi}{3} \cos \frac{\pi}{4} - \sin \frac{2\pi}{3} \sin \frac{\pi}{4} \\ &= -\frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} \\ &= -\frac{\sqrt{2} + \sqrt{6}}{4}\end{aligned}$$

3. $\tan\left(-\frac{\pi}{12}\right)$

This angle is the difference between $\frac{\pi}{4}$ and $\frac{\pi}{3}$.

$$\begin{aligned}\tan\left(\frac{\pi}{4} - \frac{\pi}{3}\right) &= \frac{\tan \frac{\pi}{4} - \tan \frac{\pi}{3}}{1 + \tan \frac{\pi}{4} \tan \frac{\pi}{3}} \\ &= \frac{1 - \sqrt{3}}{1 + \sqrt{3}}\end{aligned}$$

This angle is also the same as $\frac{23\pi}{12}$. You could have also used this value and done $\tan\left(\frac{\pi}{4} + \frac{5\pi}{3}\right)$ and arrived at the same answer.

Example 3.3.7.1

Earlier, you were asked to find the exact value of $\sin 165^\circ$ without Using the calculator.

Solution

We can use the sine sum formula, $\sin(a+b) = \sin a \cos b + \cos a \sin b$, where $a = 120^\circ$ and $b = 45^\circ$.

$$\begin{aligned}\sin 165^\circ &= \sin(120^\circ + 45^\circ) \\ &= \sin 120^\circ \cos 45^\circ + \cos 120^\circ \sin 45^\circ \\ &= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{-1}{2} \cdot \frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

Example 3.3.7.2

Find the exact value of $\cos 15^\circ$.

Solution

$$\begin{aligned}\cos 15^\circ &= \cos(45^\circ - 30^\circ) \\ &= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ \\ &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \\ &= \frac{\sqrt{6} + \sqrt{2}}{4}\end{aligned}$$

Example 3.3.7.3

Find the exact value of $\tan 255^\circ$.

Solution

$$\begin{aligned}\tan(210^\circ + 45^\circ) &= \frac{\tan 210^\circ + \tan 45^\circ}{1 - \tan 210^\circ \tan 45^\circ} \\ &= \frac{\frac{\sqrt{3}}{3} + 1}{1 - \frac{\sqrt{3}}{3}} = \frac{\frac{\sqrt{3}+3}{3}}{\frac{3-\sqrt{3}}{3}} = \frac{\sqrt{3}+3}{3-\sqrt{3}}\end{aligned}$$

Review

Find the exact value of the following trig functions.

1. $\sin 15^\circ$
2. $\cos \frac{5\pi}{12}$
3. $\tan 345^\circ$
4. $\cos(-255^\circ)$
5. $\sin \frac{13\pi}{12}$
6. $\sin \frac{17\pi}{12}$
7. $\cos 15^\circ$
8. $\tan(-15^\circ)$
9. $\sin 345^\circ$
10. Now, use $\sin 15^\circ$ from #1, and find $\sin 345^\circ$. Do you arrive at the same answer? Why or why not?
11. Using $\cos 15^\circ$ from #7, find $\cos 165^\circ$. What is another way you could find $\cos 165^\circ$?
12. Describe any patterns you see between the sine, cosine, and tangent of these "new" angles.
13. Using your calculator, find the $\sin 142^\circ$. Now, use the sum formula and your calculator to find the $\sin 142^\circ$ Using 83° and 59°.

14. Use the sine difference formula to find $\sin 142^\circ$ with any two angles you choose. Do you arrive at the same answer? Why or why not?
15. **Challenge** Using $\sin(a+b) = \sin a \cos b + \cos a \sin b$ and $\cos(a+b) = \cos a \cos b - \sin a \sin b$, show that $\tan(a+b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$.

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 14.12.

Additional Resources

Video: Finding Exact Values of Trigonometric Functions Using Sum and Difference Formulas - Overview

Practice: Finding Exact Trigonometric Values Using Sum and Difference Formulas

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3.3.8: Applications of Sum and Difference Formulas

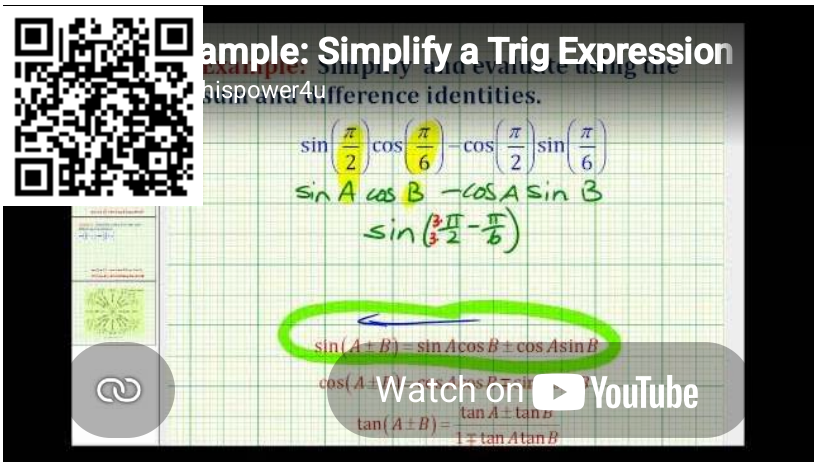
Sine, cosine, and tangent sum and difference formulas.

You are quite likely familiar with the values of trig functions for a variety of angles. Angles such as $30°$, $60°$, and $90°$ are common. However, if you were asked to find the value of a trig function for a more rarely used angle, could you do so? Or what if you were asked to find the value of a trig function for a sum of angles? For example, if you were asked to find $\sin\left(\frac{3\pi}{2}+\frac{\pi}{4}\right)$ could you?

Read on, and in this section, you'll get practice with simplifying trig functions of angles using the sum and difference formulas.

Sum and Difference Formulas

Quite frequently one of the main obstacles to solving a problem in trigonometry is the inability to transform the problem into a form that makes it easier to solve. Sum and difference formulas can be very valuable in helping with this.



Here we'll get some extra practice putting the sum and difference formulas to good use. If you haven't gone through them yet, you might want to review the sections on the Sum and Difference Formulas for sine, cosine, and tangent.

Solve using the Sum Formula

Verify the identity $\frac{\cos(x-y)}{\sin x \sin y} = \cot x \cot y + 1$

$$\begin{aligned}
&\text{Verify the identity } \frac{\cos(x-y)}{\sin x \sin y} = \cot x \cot y + 1 \\
\cot x \cot y + 1 &= \frac{\cos(x-y)}{\sin x \sin y} \\
&= \frac{\cos x \cos y}{\sin x \sin y} + \frac{\sin x \sin y}{\sin x \sin y} && \text{Expand using the cosine difference} \\
&= \frac{\cos x \cos y}{\sin x \sin y} + 1 \\
\cot x \cot y + 1 &= \cot x \cot y + 1 && \text{cotangent equals cosine over sine}
\end{aligned}$$

Solve using the Difference Formula

Solve $3\sin(x-\pi) = 3$ in the interval $[0, 2\pi)$.

First, get $\sin(x-\pi)$ by itself, by dividing both sides by 3.

$$\begin{aligned}
\frac{3\sin(x-\pi)}{3} &= \frac{3}{3} \\
\sin(x-\pi) &= 1
\end{aligned}$$

Now, expand the left side using the sine difference formula.

$$\begin{aligned}
\sin x \cos \pi - \cos x \sin \pi &= 1 \\
\sin x(-1) - \cos x(0) &= 1 \\
-\sin x &= 1 \\
\sin x &= -1
\end{aligned}$$

The $\sin x = -1$ when x is $\frac{3\pi}{2}$.

Solve using the Sum Formula

Find all the solutions for $2\cos^2\left(x+\frac{\pi}{2}\right)=1$ in the interval $[0, 2\pi)$.

Get the $\cos^2\left(x+\frac{\pi}{2}\right)$ by itself and then take the square root.

$$\begin{aligned} 2\cos^2\left(x+\frac{\pi}{2}\right) &= 1 \\ \cos^2\left(x+\frac{\pi}{2}\right) &= \frac{1}{2} \\ \cos\left(x+\frac{\pi}{2}\right) &= \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \end{aligned}$$

Now, use the cosine sum formula to expand and solve.

$$\begin{aligned} \cos x\cos\frac{\pi}{2} - \sin x\sin\frac{\pi}{2} &= \frac{\sqrt{2}}{2} \\ \cos x(0) - \sin x(1) &= \frac{\sqrt{2}}{2} \\ -\sin x &= \frac{\sqrt{2}}{2} \\ \sin x &= -\frac{\sqrt{2}}{2} \end{aligned}$$

The $\sin x = -\frac{\sqrt{2}}{2}$ is in Quadrants III and IV, so $x = \frac{5\pi}{4}$ and $\frac{7\pi}{4}$.

Example 3.3.8.1

Earlier, you were asked to find $\sin\left(\frac{3\pi}{2}+\frac{\pi}{4}\right)$, use the sine sum formula:

Solution

$$\begin{aligned} \sin(a+b) &= \sin(a)\cos(b) + \cos(a)\sin(b) \\ \sin\left(\frac{3\pi}{2}+\frac{\pi}{4}\right) &= \sin\left(\frac{3\pi}{2}\right)\times\cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{3\pi}{2}\right)\times\sin\left(\frac{\pi}{4}\right) \\ &= (-1)\left(\frac{\sqrt{2}}{2}\right) + (0)\left(\frac{\sqrt{2}}{2}\right) \\ &= -\frac{\sqrt{2}}{2} \end{aligned}$$

Example 3.3.8.2

Find all solutions to $2\cos^2\left(x+\frac{\pi}{2}\right)=1$, when x is between $[0, 2\pi)$.

Solution

To find all the solutions, between $[0, 2\pi)$, we need to expand using the sum formula and isolate the $\cos x$.

$$\begin{aligned} 2\cos^2\left(x+\frac{\pi}{2}\right) &= 1 \\ \cos^2\left(x+\frac{\pi}{2}\right) &= \frac{1}{2} \\ \cos\left(x+\frac{\pi}{2}\right) &= \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2} \\ \cos x\cos\frac{\pi}{2} - \sin x\sin\frac{\pi}{2} &= \pm\frac{\sqrt{2}}{2} \\ \cos x\cdot 0 - \sin x\cdot 1 &= \pm\frac{\sqrt{2}}{2} \\ -\sin x &= \pm\frac{\sqrt{2}}{2} \\ \sin x &= \pm\frac{\sqrt{2}}{2} \end{aligned}$$

This is true when $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, or $\frac{7\pi}{4}$

Example 3.3.8.3

Solve for all values of x between $[0, 2\pi)$ for $2\tan^2\left(x + \frac{\pi}{6}\right) + 1 = 7$.

Solution

First, solve for $\tan\left(x + \frac{\pi}{6}\right)$.

$$\begin{aligned} 2\tan^2\left(x + \frac{\pi}{6}\right) + 1 &= 7 \\ 2\tan^2\left(x + \frac{\pi}{6}\right) &= 6 \\ \tan^2\left(x + \frac{\pi}{6}\right) &= 3 \\ \tan\left(x + \frac{\pi}{6}\right) &= \pm\sqrt{3} \end{aligned}$$

Now, use the tangent sum formula to expand for when \tan (x+\dfrac{\pi}{6})=\sqrt{3}.

$$\begin{aligned} \frac{\tan x + \tan\frac{\pi}{6}}{1 - \tan x \tan\frac{\pi}{6}} &= \sqrt{3} \\ \tan x + \tan\frac{\pi}{6} &= \sqrt{3}\left(1 - \tan x \tan\frac{\pi}{6}\right) \\ \tan x + \frac{\sqrt{3}}{3} &= \sqrt{3} - \sqrt{3}\tan x \cdot \frac{\sqrt{3}}{3} \\ \tan x + \frac{\sqrt{3}}{3} &= \sqrt{3} - \tan x \\ 2\tan x &= \frac{2\sqrt{3}}{3} \\ \tan x &= \frac{\sqrt{3}}{3} \end{aligned}$$

This is true when $x = \frac{\pi}{6}$ or $\frac{7\pi}{6}$.

If the tangent sum formula to expand for when $\tan(x + \frac{\pi}{6}) = -\sqrt{3}$, we get no solution as shown.

$$\begin{aligned} \frac{\tan x + \tan\frac{\pi}{6}}{1 - \tan x \tan\frac{\pi}{6}} &= -\sqrt{3} \\ \tan x + \tan\frac{\pi}{6} &= -\sqrt{3}\left(1 - \tan x \tan\frac{\pi}{6}\right) \\ \tan x + \frac{\sqrt{3}}{3} &= -\sqrt{3} + \sqrt{3}\tan x \cdot \frac{\sqrt{3}}{3} \\ \tan x + \frac{\sqrt{3}}{3} &= -\sqrt{3} + \tan x \\ \frac{\sqrt{3}}{3} &= -\sqrt{3} \end{aligned}$$

Therefore, the tangent sum formula cannot be used in this case. However, since we know that $\tan(x + \frac{\pi}{6}) = -\sqrt{3}$ when $x + \frac{\pi}{6} = \frac{5\pi}{6}$ or $\frac{11\pi}{6}$, we can solve for x as follows.

$$\begin{aligned} x + \frac{\pi}{6} &= 5\frac{\pi}{6} \\ x &= \frac{4\pi}{6} \\ x &= \frac{2\pi}{3} \\ x + \frac{\pi}{6} &= \frac{11\pi}{6} \\ x &= 10\frac{\pi}{6} \\ x &= \frac{5\pi}{3} \end{aligned}$$

Therefore, all of the solutions are $x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$

Example 3.3.8.4

Find all solutions to $\sin\left(x + \frac{\pi}{6}\right) = \sin\left(x - \frac{\pi}{4}\right)$, when x is between $[0, 2\pi)$.

Solution

To solve, expand each side:

$$\sin\left(x + \frac{\pi}{6}\right) = \sin x \cos\frac{\pi}{6} + \cos x \sin\frac{\pi}{6} = \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x$$
$$\sin\left(x - \frac{\pi}{4}\right) = \sin x \cos\frac{\pi}{4} - \cos x \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}\sin x - \frac{\sqrt{2}}{2}\cos x$$

Set the two sides equal to each other:

$$\begin{aligned}
\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x &= \frac{\sqrt{2}}{2}\sin x - \frac{\sqrt{2}}{2}\cos x \\
\sqrt{3}\sin x + \cos x &= \sqrt{2}\sin x - \sqrt{2}\cos x \\
\sqrt{3}\sin x - \sqrt{2}\sin x &= -\cos x - \sqrt{2}\cos x \\
\sin x(\sqrt{3} - \sqrt{2}) &= \cos x(-1 - \sqrt{2}) \\
\frac{\sin x}{\cos x} &= \frac{-1 - \sqrt{2}}{\sqrt{3} - \sqrt{2}} \\
\tan x &= \frac{-1 - \sqrt{2}}{\sqrt{3} - \sqrt{2}} \cdot \frac{\sqrt{3} + \sqrt{2}}{\sqrt{3} + \sqrt{2}} \\
&= \frac{-\sqrt{3} - \sqrt{2} + \sqrt{6} - 2}{3 - 2} \\
&= -2 + \sqrt{6} - \sqrt{3} - \sqrt{2}
\end{aligned}$$

As a decimal, this is -2.69677, so $\tan^{-1}(-2.69677) = x$, $x = 290.35°$ and $110.35°$.

Review

Prove each identity.

1. $\cos(3x) + \cos(x) = 2\cos(2x)\cos(x)$
2. $\cos(3x) = \cos 3(x) - 3\sin 2(x)\cos(x)$
3. $\sin(3x) = 3\cos 2(x)\sin(x) - \sin 3(x)$
4. $\sin(4x) + \sin(2x) = 2\sin(3x)\cos(x)$
5. $\tan(5x)\tan(3x) = \frac{\tan^2(4x) - \tan^2(x)}{1 - \tan^2(4x)\tan^2(x)}$
6. $\cos\left(\left(\frac{\pi}{2} - x\right) - y\right) = \sin(x + y)$

Use sum and difference formulas to help you graph each function.

7. $y = \cos(3)\cos(x) + \sin(3)\sin(x)$
8. $y = \cos(x)\cos\left(\frac{\pi}{2}\right) + \sin(x)\sin\left(\frac{\pi}{2}\right)$
9. $y = \sin(x)\cos\left(\frac{\pi}{2}\right) + \cos(x)\sin\left(\frac{\pi}{2}\right)$
10. $y = \sin(x)\cos\left(\frac{3\pi}{2}\right) - \cos(3)\sin\left(\frac{\pi}{2}\right)$
11. $y = \cos(4x)\cos(2x) - \sin(4x)\sin(2x)$
12. $y = \cos(x)\cos(x) - \sin(x)\sin(x)$

Solve each equation on the interval $[0, 2\pi)$.

13. $2\sin\left(x - \frac{\pi}{2}\right) = 1$
14. $4\cos(x - \pi) = 4$
15. $2\sin(x - \pi) = \sqrt{2}$

Review (Answers)

To see the Review answers, open this PDF file and look for section 3.9.

Vocabulary

Term	Definition
Difference Formula	Trigonometric function difference formulas exist for each of the primary trigonometric functions. For example, the cosine difference formula is $\cos(A-B)=\cos A\cos B+\sin A\sin B$.
Sum Formula	A sum formula is a formula to help simplify a trigonometric function of the sum of two angles, such as \sin (a+b).

Additional Resources

Video: Example: Simplify a Trig Expression using the Sum and Difference Identities

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SECTION OVERVIEW

3.4: Double and Half Angle Identities

3.4.1: Double and Half Angle Formulas

3.4.2: Double Angle Identities

3.4.3: Simplifying Trigonometric Expressions with Double-Angle Identities

3.4.4: Solving Equations with Double-Angle Identities

3.4.5: Half Angle Formulas

3.4.6: Trigonometric Equations Using Half Angle Formulas

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3.4.1: Double and Half Angle Formulas

Sine, cosine, and tangent of angles other than multiples of 30, 45, and 60 degrees.

You want to find the exact value of $\tan \frac{3\pi}{8}$. How could you find this value without u\sin g a calculator?

Double Angle and Half Angle Formulas

In this concept, we will learn how to find the exact values of the trig functions for angles that are half or double of other angles. Here we will introduce the Double-Angle $(2a)$ and Half-Angle $\left(\frac{a}{2}\right)$ Formulas.

Double-Angle and Half-Angle Formulas

$$\begin{aligned}
\cos 2a &= \cos^2 a - \sin^2 a & \sin 2a &= 2\sin a \cos a \\
&= 2\cos^2 a - 1 & \tan 2a &= \frac{2\tan a}{1-\tan^2 a} \\
&= 1 - \sin^2 a & & \\
\sin \frac{a}{2} &= \pm\sqrt{\frac{1-\cos a}{2}} & \tan \frac{a}{2} &= \frac{1-\cos a}{\sin a} \\
\cos \frac{a}{2} &= \pm\sqrt{\frac{1+\cos a}{2}} & &= \frac{\sin a}{1+\cos a}
\end{aligned}$$

The signs of $\sin \frac{a}{2}$ and $\cos \frac{a}{2}$ depend on which quadrant $\frac{a}{2}$ lies in. For $\cos 2a$ and $\tan \frac{a}{2}$ any formula can be used to solve for the exact value.

Let's find the exact value of $\cos \frac{\pi}{8}$.

$\frac{\pi}{8}$ is half of $\frac{\pi}{4}$ and in the first quadrant.

$$\begin{aligned}
\cos\left(\frac{1}{2}\cdot\frac{\pi}{4}\right) &= \sqrt{\frac{1+\cos\frac{\pi}{4}}{2}} \\
&= \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} \\
&= \sqrt{\frac{1}{2}\cdot\frac{2+\sqrt{2}}{2}} \\
&= \frac{\sqrt{2+\sqrt{2}}}{2}
\end{aligned}$$

Now, let's find the exact value of $\sin 2a$ if $\cos a = -\frac{4}{5}$ and $\frac{3\pi}{2} \le a < 2\pi$.

To use the sine double-angle formula, we also need to find $\sin a$, which would be $\frac{3}{5}$ because a is in the 4th quadrant.

$$\begin{aligned}
\sin 2a &= 2\sin a \cos a \\
&= 2\cdot\frac{3}{5}\cdot -\frac{4}{5} \\
&= -\frac{24}{25}
\end{aligned}$$

Finally, let's find the exact value of $\tan 2a$ for a from the previous problem.

Use $\tan a = \frac{\sin a \cos a}{-\frac{4}{5}} = -\frac{3}{4}$ to solve for $\tan 2a$.

$$\tan 2a = \frac{2\cdot -\frac{3}{4}}{1-\left(-\frac{3}{4}\right)^2} = \frac{-\frac{3}{2}}{\frac{7}{16}} = -\frac{3}{2}\cdot\frac{16}{7} = -\frac{24}{7}$$

Example 3.4.1.1

Earlier, you were asked to find the value of $\tan\frac{3\pi}{8}$ without a calculator.

Solution

$\frac{3\pi}{8}=\frac{1}{2}\cdot\frac{3\pi}{4}$ so we can use the formula $\tan\frac{a}{2}=\frac{\sin a}{1+\cos a}$ for $a=\frac{3\pi}{4}$

$$\tan\frac{3\pi}{8}=\frac{\sin\frac{3\pi}{4}}{1+\cos\frac{3\pi}{4}}$$

$$=\frac{\frac{\sqrt{2}}{2}}{1+\frac{-\sqrt{2}}{2}}$$

If we simplify this expression, we get $\sqrt{2}+1$.

Example 3.4.1.2

Find the exact value of $\cos\left(-\frac{5\pi}{8}\right)$.

Solution

$-\frac{5\pi}{8}$ is in the 3rd quadrant.

$$-\frac{5\pi}{8}=\frac{1}{2}\left(-\frac{5\pi}{4}\right)\rightarrow\cos\frac{1}{2}\left(-\frac{5\pi}{4}\right)=-\sqrt{\frac{1+\cos\left(-\frac{5\pi}{4}\right)}{2}}$$

$$=-\frac{1-\frac{\sqrt{2}}{2}}{2}=\sqrt{\frac{1}{2}\cdot\frac{2-\sqrt{2}}{2}}=\frac{\sqrt{2-\sqrt{2}}}{2}$$

Example 3.4.1.3

Given the function $\cos a=\frac{4}{7}$ and $0\le a<\frac{\pi}{2}$, find $\sin 2a$.

Solution

First, find $\sin a$. $4^2+y^2=7^2\rightarrow y=\sqrt{33}$, so $\sin a=\frac{\sqrt{33}}{7}$

$$\sin 2a=2\cdot\frac{\sqrt{33}}{7}\cdot\frac{4}{7}=\frac{8\sqrt{33}}{49}$$

Example 3.4.1.4

Given the function $\cos a=\frac{4}{7}$ and $0\le a<\frac{\pi}{2}$, find $\tan\frac{a}{2}$.

Solution

You can use either $\tan\frac{a}{2}$ formula.

$$\tan\frac{a}{2}=\frac{1-\frac{4}{7}}{\frac{\sqrt{33}}{7}}=\frac{3}{7}\cdot\frac{7}{\sqrt{33}}=\frac{3}{\sqrt{33}}=\frac{\sqrt{33}}{11}$$

Review

Find the exact value of the following angles.

1. $\sin 105^\circ$
2. $\tan \frac{\pi}{8}$
3. $\cos \frac{5\pi}{12}$
4. $\cos 165^\circ$
5. $\sin \frac{3\pi}{8}$
6. $\tan \left(-\frac{\pi}{12}\right)$
7. $\sin \frac{11\pi}{8}$
8. $\cos \frac{19\pi}{12}$

The $\cos a = \frac{5}{13}$ and $\frac{3\pi}{2} \leq a < 2\pi$. Find:

9. $\sin 2a$
10. $\cos \frac{a}{2}$
11. $\tan \frac{a}{2}$
12. $\cos 2a$

The $\sin a = \frac{8}{11}$ and $\frac{\pi}{2} \leq a < \pi$. Find:

13. $\tan 2a$
14. $\sin \frac{a}{2}$
15. $\cos \frac{a}{2}$
16. $\sin 2a$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 14.15.

Additional Resources

Practice: Double and Half Angle Formulas

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3.4.2: Double Angle Identities

Simplifying trigonometric functions with twice a given angle.

These identities are significantly more involved and less intuitive than previous identities. By practicing and working with these advanced identities, your toolbox and fluency substituting and proving on your own will increase. Each **identity** in this concept is named aptly. Double angles work on finding $\sin 80^\circ$ if you already know $\sin 40^\circ$. Half angles allow you to find $\sin 15^\circ$ if you already know $\sin 30^\circ$. Power reducing identities allow you to find $\sin^2 15^\circ$ if you know the sine and cosine of 30°.

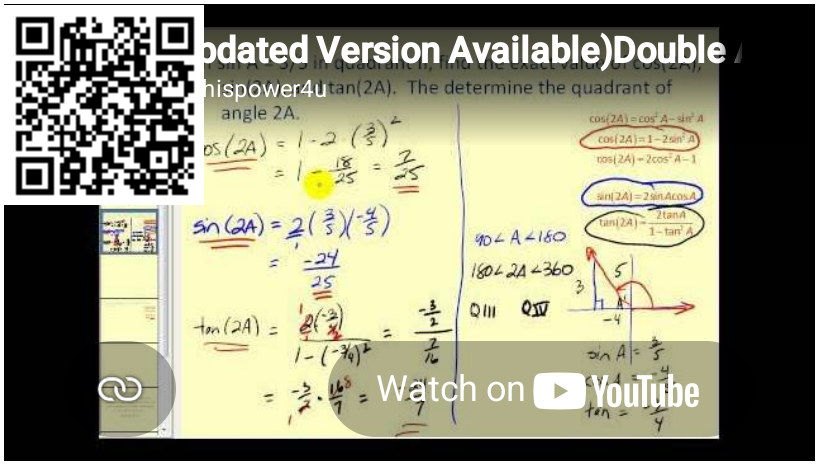
What is $\sin^2 15^\circ$?

Double Angle, Half Angle, and Power Reducing Identities

Double Angle Identities

The **double angle identities** are proved by applying the sum and difference identities. They are left as review problems. These are the double angle identities.

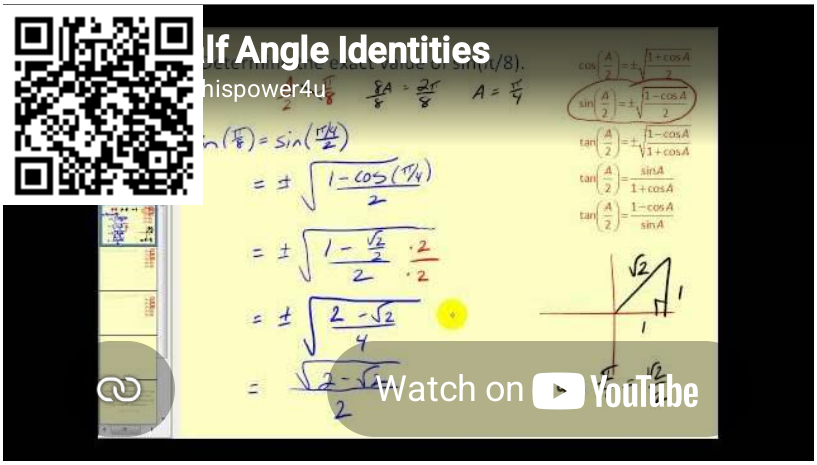
- $\sin 2x = 2 \sin x \cos x$
- $\cos 2x = \cos^2 x - \sin^2 x$
- $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$



Half Angle Identities

The **half angle identities** are a rewritten version of the power reducing identities. The proofs are left as review problems.

- $\sin \dfrac{x}{2} = \pm \sqrt{\dfrac{1 - \cos x}{2}}$
- $\cos \dfrac{x}{2} = \pm \sqrt{\dfrac{1 + \cos x}{2}}$
- $\tan \dfrac{x}{2} = \pm \sqrt{\dfrac{1 - \cos x}{1 + \cos x}}$



Power Reducing Identities

The **power reducing identities** allow you to write a trigonometric function that is squared in terms of smaller powers. The proofs are left as examples and review problems.

- $\sin^2 x = \dfrac{1-\cos 2x}{2}$
- $\cos^2 x = \dfrac{1+\cos 2x}{2}$
- $\tan^2 x = \dfrac{1-\cos 2x}{1+\cos 2x}$

Power reducing identities are most useful when you are asked to rewrite expressions such as \sin 4x as an expression without powers greater than one. While $\sin x \cdot \sin x \cdot \sin x \cdot \sin x$ does technically simplify this expression as necessary, you should try to get the terms to sum together not multiply together.

$$\begin{aligned}\sin^4 x = (\sin^2 x)^2 &= \left(\frac{1-\cos 2x}{2}\right)^2 \\ &= \frac{1-2\cos 2x+\cos^2 2x}{4} \\ &= \frac{1}{4}\left(1-2\cos 2x+\frac{1+\cos 4x}{2}\right)\end{aligned}$$

Example 3.4.2.1

Earlier, you were asked to find $\sin^2 15°$. In order to fully identify $\sin^2 15°$ you need to use the power reducing formula.

Solution

$$\begin{aligned}\sin^2 x &= \frac{1-\cos 2x}{2} = \frac{1}{2}-\frac{\sqrt{3}}{4} \\ \sin^2 15° &= \frac{1-\cos 30°}{2} \\ &= \frac{2-\sqrt{3}}{4}\end{aligned}$$

Example 3.4.2.2

Write the following expression with only $\sin x$ and $\cos x$: $\sin 2x + \cos 3x$.

Solution

$$\begin{aligned}\sin 2x+\cos 3x &= 2\sin x\cos x+\cos(2x+x) \\ &= 2\sin x\cos x+\cos 2x\cos x-\sin 2x\sin x \\ &= 2\sin x\cos x+(\cos^2 x-\sin^2 x)\cos x-(2\sin x\cos x)\sin x \\ &= 2\sin x\cos x+\cos^3 x-\sin^2 x\cos x-2\sin^2 x\cos x \\ &= 2\sin x\cos x+\cos^3 x-3\sin^2 x\cos x\end{aligned}$$

Example 3.4.2.3

Use half angles to find an exact value of \tan 22.5^{\circ} without using a calculator.

Solution

$$\tan\frac{x}{2}=\pm\sqrt{\frac{1-\cos x}{1+\cos x}}$$

$$\tan 22.5^\circ=\tan\frac{45^\circ}{2}=\pm\sqrt{\frac{1-\cos 45^\circ}{1+\cos 45^\circ}}=\pm\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}}}=\pm\sqrt{\frac{\frac{2}{2}-\frac{\sqrt{2}}{2}}{\frac{2}{2}+\frac{\sqrt{2}}{2}}}=\pm\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}}$$

$$=\pm\sqrt{\frac{(2-\sqrt{2})^2}{2}}$$

Example 3.4.2.4

Prove the **power reducing identity** for sine.

$$\sin^2 x=\frac{1-\cos 2x}{2}$$

Solution

Using the double angle identity for cosine:

$$\cos 2x=\cos^2 x-\sin^2 x$$
$$\cos 2x=(1-\sin^2 x)-\sin^2 x$$
$$\cos 2x=1-2\sin^2 x$$

This expression is an equivalent expression to the double angle identity and is often considered an alternate form.

Example 3.4.2.5

Simplify the following identity: $\sin^4 x-\cos^4 x$.

Solution

Here are the steps:

$$\sin^4 x-\cos^4 x=(\sin^2 x-\cos^2 x)(\sin^2 x+\cos^2 x)$$
$$=-(\cos^2 x-\sin^2 x)$$
$$=-\cos^2 x$$

Review

Prove the following identities.

1. $\sin 2x=2\sin x\cos x$
2. $\cos 2x=\cos^2 x-\sin^2 x$
3. $\tan 2x=\frac{2\tan x}{1-\tan^2 x}$
4. $\cos^2 x=\frac{1+\cos 2x}{2}$
5. $\tan^2 x=\frac{1-\cos 2x}{1+\cos 2x}$
6. $\sin\frac{x}{2}=\pm\sqrt{\frac{1-\cos x}{2}}$
7. $\cos\frac{x}{2}=\pm\sqrt{\frac{1+\cos x}{2}}$
8. $\tan\frac{x}{2}=\pm\sqrt{\frac{1-\cos x}{1+\cos x}}$
9. $\csc 2x=\frac{1}{2}\csc x\sec x$

10. $\cot 2x = \dfrac{\cot^2 x - 1}{2 \cot x}$

Find the value of each expression using half angle identities.

11. $\tan 15^\circ$
12. $\tan 22.5^\circ$
13. $\sec 22.5^\circ$
14. Show that $\tan \dfrac{x}{2} = \dfrac{1 - \cos x}{\sin x}$.
15. Using your knowledge from the answer to question 14, show that $\tan \dfrac{x}{2} = \dfrac{\sin x}{1 + \cos x}$.

Review (Answers)

To see the Review answers, open this PDF file and look for section 6.4.

Vocabulary

Term	Definition
Half Angle Identity	A half angle identity relates a trigonometric function of one half of an argument to a set of trigonometric functions, each containing the original argument.
identity	An identity is a mathematical sentence involving the symbol "=" that is always true for variables within the domains of the expressions on either side.
power reducing identity	A power reducing identity relates the power of a trigonometric function containing a given argument to a set of trigonometric functions, each containing the original argument.

Additional Resources

Video: Multiple Angle Formulas - Overview

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3.4.3: Simplifying Trigonometric Expressions with Double-Angle Identities

Simplify sine, cosine, and tangent of angles multiplied or divided by 2.

As Agent Trigonometry, you are given the following cryptic clue. How could you simplify this clue?

$$\frac{\frac{\tan 2x}{\tan x}}{1+\tan x}$$

Simplifying Trigonometric Expressions

We can also use the double-angle and half-angle formulas to simplify trigonometric expressions.

Let's simplify $\frac{\cos 2x}{\sin x \cos x}$.

Use $\cos 2a = \cos^2 a - \sin^2 a$ and then factor.

$$\begin{aligned}\frac{\cos 2x}{\sin x \cos x} &= \frac{\cos^2 x - \sin^2 x}{\sin x + \cos x} \\ &= \frac{(\cos x - \sin x)\,\cancel{(\cos x + \sin x)}}{\cancel{\sin x + \cos x}} \\ &= \cos x - \sin x\end{aligned}$$

Now, let's find the formula for $\sin 3x$.

You will need to use the sum formula and the double-angle formula. \sin 3x=\sin (2x+x)

$$\begin{aligned}\sin 3x &= \sin(2x + x) \\ &= \sin 2x \cos x + \cos 2x \sin x \\ &= 2 \sin x \cos x \cos x + \sin x(2 \cos^2 x - 1) \\ &= 2 \sin x \cos^2 x + 2 \sin x \cos^2 x - \sin x \\ &= 4 \sin x \cos^2 x - \sin x \\ &= \sin x(4 \cos^2 x - 1)\end{aligned}$$

Finally, let's verify the identity $\cos x + 2 \sin^2 \frac{x}{2} = 1$.

Simplify the left-hand side use the half-angle formula.

$$\begin{aligned}&\cos x + 2 \sin^2 \frac{x}{2} \\ &\cos x + 2 \left(\sqrt{\frac{1 - \cos x}{2}}\right)^2 \\ &\cos x + 2 \cdot \frac{1 - \cos x}{2} \\ &\cos x + 1 - \cos x \\ &1\end{aligned}$$

Example 3.4.3.1

Earlier, you were asked to simplify $\frac{\frac{\tan 2x}{\tan x}}{1+\tan x}$.

Solution

Use $\tan 2a = \frac{2 \tan a}{1 - \tan^2 a}$ and then factor.

$$\begin{aligned}\frac{\frac{\tan 2x}{\tan x}}{1+\tan x} &= \frac{2 \tan x}{1 - \tan^2 x} \cdot \frac{1 + \tan x}{\tan x} \\ &= \frac{2 \tan x}{(1 + \tan x)(1 - \tan x)} \cdot \frac{1 + \tan x}{\tan x} = \frac{2}{1 - \tan x}\end{aligned}$$

Example 3.4.3.2

Simplify $\frac{\sin 2x}{\sin x}$.

Solution

$$\frac{\sin 2x}{\sin x} = \frac{2\sin x \cos x}{\sin x} = 2\cos x$$

Example 3.4.3.3

Verify $\cos x + 2\cos^2 \frac{x}{2} = 1 + 2\cos x$.

Solution

$$\begin{aligned} \cos x + 2\cos^2 \frac{x}{2} &= 1 + 2\cos x \\ \cos x + 2\sqrt{\frac{1+\cos x}{2}} &= \\ \cos x + 1 + \cos x &= \\ 1 + 2\cos x &= \end{aligned}$$

Review

Simplify the following expressions.

1. $\sqrt{2+2\cos x}\left(\cos \frac{x}{2}\right)$
2. $\frac{\cos 2x}{\cos^2 x}$
3. $\tan 2x(1+\tan x)$
4. $\cos 2x - 3\sin^2 x$
5. $\frac{1+\cos 2x}{\cot x}$
6. $(1+\cos x)^2 \tan \frac{x}{2}$

Verify the following identities.

7. $\cot \frac{x}{2} = \frac{\sin x}{1-\cos x}$
8. $\frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$
9. $\frac{\sin 2x}{1+\cos 2x} = \tan x$
10. $(\sin x + \cos x)^2 = 1 + \sin 2x$
11. $\sin x \tan \frac{x}{2} + 2\cos x = 2\cos^2 \frac{x}{2}$
12. $\cot x + \tan x = 2\csc 2x$
13. $\cos 3x = 4\cos^3 x - 3\cos x$
14. $\cos 3x = \cos^3 x - 3\sin^2 x \cos x$
15. $\sin 2x - \tan x = \tan x \cos 2x$
16. $\cos^4 x - \sin^4 x = \cos 2x$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 14.16.

Additional Resources

Practice: Simplifying Trigonometric Expressions with Double-Angle Identities

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3.4.4: Solving Equations with Double-Angle Identities

Solve sine, cosine, and tangent of angles multiplied or divided by 2.

Trig Riddle: I am an angle x such that $0 \le x < 2\pi$. I satisfy the equation $\sin 2x - \sin x = 0$. What angle am I?

Solve Trigonometric Equations

We can use the half and double angle formulas to solve trigonometric equations.

Let's solve the following trigonometric equations.

1. Solve $\tan 2x + \tan x = 0$ when $0 \le x < 2\pi$.

Change $\tan 2x$ and simplify.

$$\begin{aligned} \tan 2x + \tan x &= 0 \\ \frac{2\tan x}{1-\tan^2 x} + \tan x &= 0 \\ 2\tan x + \tan x(1-\tan^2 x) &= 0 \rightarrow \text{Multiply everything by } 1-\tan^2 x \text{ to eliminate denominator.} \\ 2\tan x + \tan x - \tan^3 x &= 0 \\ 3\tan x - \tan^3 x &= 0 \\ \tan x(3-\tan^2 x) &= 0 \end{aligned}$$

Set each factor equal to zero and solve.

$$\begin{aligned} \tan x &= 0 \quad \text{and} \quad 3-\tan^2 x = 0 \\ x &= 0 \text{ and } \pi \quad -\tan^2 x = -3 \\ & \quad \tan^2 x = 3 \\ & \quad \tan x = \pm\sqrt{3} \\ & \quad x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3} \end{aligned}$$

2. Solve $2\cos\frac{x}{2} + 1 = 0$ when $0 \le x < 2\pi$.

In this case, you do not have to use the half-angle formula. Solve for $\frac{x}{2}$.

$$\begin{aligned} 2\cos\frac{x}{2} + 1 &= 0 \\ 2\cos\frac{x}{2} &= -1 \\ \cos\frac{x}{2} &= -\frac{1}{2} \end{aligned}$$

Now, let's find $\cos a = -\frac{1}{2}$ and then solve for x by dividing by 2.

$$\begin{aligned} \frac{x}{2} &= \frac{2\pi}{3}, \frac{4\pi}{3} \\ &= \frac{4\pi}{3}, \frac{8\pi}{3} \end{aligned}$$

Now, the second solution is not in our range, so the only solution is $x = \frac{4\pi}{3}$.

3. Solve $4\sin x\cos x = \sqrt{3}$ for $0 \le x < 2\pi$.

Pull a 2 out of the left-hand side and use the $\sin 2x$ formula.

$$\begin{aligned} 4\sin x\cos x &= \sqrt{3} \\ 2\cdot 2\sin x\cos x &= \sqrt{3} \\ 2\cdot \sin 2x &= \sqrt{3} \\ \sin 2x &= \frac{\sqrt{3}}{2} \\ 2x &= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3} \\ x &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6} \end{aligned} \tag{3.4.4.1}$$

Example 3.4.4.1

Earlier, you were asked to find the angle x, where $0 \le x < 2\pi$, such that x satisfies the equation $\sin 2x - \sin x = 0$.

Solution

Use the double angle formula and simplify.

$$\begin{aligned} \sin 2x - \sin x &= 0 \\ 2\sin x \cos x - \sin x &= 0 \\ \sin x(2\cos x - 1) &= 0 \\ \sin x = 0 \text{ OR } \cos x &= \frac{1}{2} \end{aligned}$$

Under the constraint $0 \le x < 2\pi$, $\sin x = 0$ when $x = 0$ or when $x = \pi$. Under this same constraint, $\cos x = \frac{1}{2}$ when $x = \frac{\pi}{3}$ or when $x = \frac{5\pi}{3}$.

Example 3.4.4.2

Solve the following equation for $0 \le x < 2\pi$.

$\sin \frac{x}{2} = -1$

Solution

$$\begin{aligned} \sin \frac{x}{2} &= -1 \\ \frac{x}{2} &= \frac{3\pi}{2} \\ x &= 3\pi \end{aligned}$$

From this we can see that there are no solutions within our interval.

Example 3.4.4.3

Solve the following equation for $0 \le x < 2\pi$.

$\cos 2x - \cos x = 0$

Solution

$$\begin{aligned} \cos 2x - \cos x &= 0 \\ 2\cos 2x - \cos x - 1 &= 0 \\ (2\cos x - 1)(\cos x + 1) &= 0 \end{aligned}$$

Set each factor equal to zero and solve.

$$\begin{aligned} 2\cos x - 1 &= 0 & & \\ 2\cos x &= 1 & & \cos x + 1 = 0 \\ \cos x &= \frac{1}{2} & \text{and} \quad & \cos x = -1 \\ x &= \frac{\pi}{3}, \frac{5\pi}{3} & & x = \pi \end{aligned}$$

Review

Solve the following equations for $0 \le x < 2\pi$.

1. $\cos x - \cos \frac{1}{2}x = 0$
2. $\sin 2x \cos x = \sin x$
3. $\cos 3x - \cos^3 x = 3\sin^2 x \cos x$
4. $\tan 2x - \tan x = 0$
5. $\cos 2x - \cos x = 0$
6. $2\cos^2 \frac{x}{2} = 1$

7. $\tan \frac{x}{2} = 4$
8. $\cos \frac{x}{2} = 1 + \cos x$
9. $\sin 2x + \sin x = 0$
10. $\cos^2 x - \cos 2x = 0$
11. $\frac{\cos 2x}{\cos^2 x} = 1$
12. $\cos 2x - 1 = \sin^2 x$
13. $\cos 2x = \cos x$
14. $\sin 2x - \cos 2x = 1$
15. $\sin^2 x - 2 = \cos 2x$
16. $\cot x + \tan x = 2 \csc 2x$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 14.17.

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3.4.5: Half Angle Formulas

Derivation of sine and cosine formulas for half a given angle

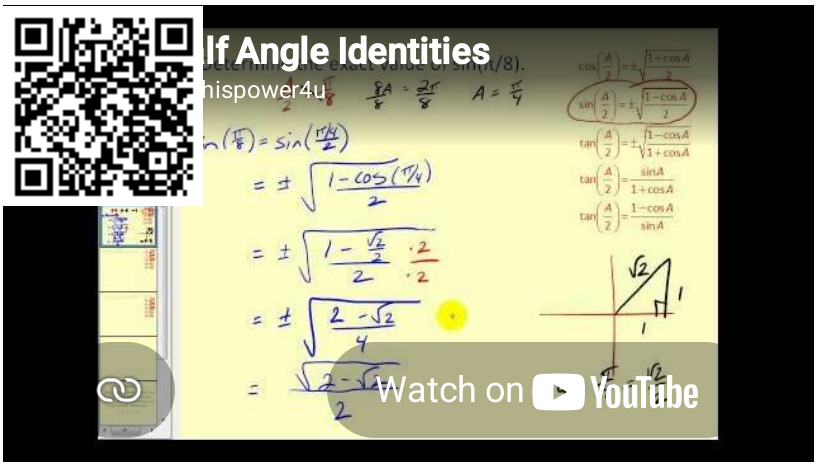
After all of your experience with trig functions, you are feeling pretty good. You know the values of trig functions for a lot of common angles, such as $30°$, $60°$ etc. And for other angles, you regularly use your calculator. Suppose someone gave you an equation like this:

$\cos 75°$

Could you solve it without the calculator? You might notice that this is half of $150°$. This might give you a hint!

Half Angle Formulas

Here we'll attempt to derive and use formulas for trig functions of angles that are half of some particular value.



To do this, we'll start with the double angle formula for cosine: $\cos 2\theta = 1 - 2\sin^2\theta$. Set $\theta = \frac{\alpha}{2}$, so the equation above becomes $\cos 2\frac{\alpha}{2} = 1 - 2\sin^2\frac{\alpha}{2}$.

Solving this for $\sin\frac{\alpha}{2}$, we get:

$$\begin{aligned}\cos 2\frac{\alpha}{2} &= 1 - 2\sin 2\frac{\alpha}{2}\\ \cos\alpha &= 1 - 2\sin 2\frac{\alpha}{2}\\ 2\sin^2\frac{\alpha}{2} &= 1 - \cos\alpha\\ \sin^2\frac{\alpha}{2} &= \frac{1-\cos\alpha}{2}\\ \sin\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{2}}\end{aligned}$$

$\sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}}$ if $\frac{\alpha}{2}$ is located in either the first or second quadrant.

$\sin\frac{\alpha}{2} = -\sqrt{\frac{1-\cos\alpha}{2}}$ if $\frac{\alpha}{2}$ is located in the third or fourth quadrant.

This formula shows how to find the sine of half of some particular angle.

One of the other formulas that was derived for the cosine of a double angle is:

$\cos 2\theta = 2\cos^2\theta - 1$. Set $\theta = \frac{\alpha}{2}$, so the equation becomes $\cos 2\frac{\alpha}{2} = -1 + 2\cos^2\frac{\alpha}{2}$. Solving this for $\cos\frac{\alpha}{2}$, we get:

$$\cos 2\frac{\alpha}{2} = 2\cos^2\frac{\alpha}{2} - 1$$
$$\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1$$
$$2\cos^2\frac{\alpha}{2} = 1 + \cos\alpha$$
$$\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2}$$
$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$\cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}}$ if $\frac{\alpha}{2}$ is located in either the first or fourth quadrant.

$\cos\frac{\alpha}{2} = -\sqrt{\frac{1+\cos\alpha}{2}}$ if $\frac{\alpha}{2}$ is located in either the second or fourth quadrant.

This formula shows how to find the cosine of half of some particular angle.

Let's see some examples of these two formulas (sine and cosine of half angles) in action.

1. Determine the exact value of $\sin 15°$.

Using the **half angle identity**, $\alpha = 30°$, and $15°$ is located in the first quadrant. Therefore, $\sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}}$.

$$\sin 15° = \sqrt{\frac{1-\cos 30°}{2}}$$
$$= \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}}$$

Plugging this into a calculator, $\sqrt{\frac{2-\sqrt{3}}{4}} \approx 0.2588$. Using the sine function on your calculator will validate that this answer is correct.

2. Use the half angle identity to find exact value of $\sin 112.5°$

Since $\sin\frac{225°}{2} = \sin 112.5°$, use the half angle formula for sine, where $\alpha = 225°$. In this example, the angle $112.5°$ is a second quadrant angle, and the SIN of a second quadrant angle is positive.

$$\sin 112.5° = \sin\frac{225°}{2}$$
$$= \pm\sqrt{\frac{1-\cos 225°}{2}}$$
$$= +\sqrt{\frac{1-\left(-\frac{\sqrt{2}}{2}\right)}{2}}$$
$$= \sqrt{\frac{\frac{2}{2}+\frac{\sqrt{2}}{2}}{2}}$$
$$= \sqrt{\frac{2+\sqrt{2}}{4}}$$

3. Use the half angle formula for the cosine function to prove that the following expression is an identity: $2\cos^2\frac{x}{2} - \cos x = 1$

Use the formula $\cos\frac{\alpha}{2} = \frac{1+\cos\alpha}{2}$ and substitute it on the left-hand side of the expression.

$$2\left(\sqrt{\frac{1+\cos\theta}{2}}\right)^2 - \cos\theta = 1$$
$$2\left(\frac{1+\cos\theta}{2}\right) - \cos\theta = 1$$
$$1 + \cos\theta - \cos\theta = 1$$
$$1 = 1$$

Example 3.4.5.1

Earlier, you were asked you to find $\cos 75°$. If you use the half angle formula, then $\alpha = 150°$

Substituting this into the half angle formula:

Solution

$$\sin \frac{150°}{2} = \sqrt{\frac{1-\cos\alpha}{2}} = \sqrt{\frac{1-\cos 150°}{2}} = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2+\sqrt{3}}{4}} = \frac{\sqrt{2+\sqrt{3}}}{2}$$

Example 3.4.5.2

Prove the identity: $\tan \frac{b}{2} = \frac{\sec b}{\sec b \csc b + \csc b}$

Solution

Step 1: Change right side into sine and cosine.

$$\begin{aligned}
\tan \frac{b}{2} &= \frac{\sec b}{\sec b \csc b + \csc b} \\
&= \frac{1}{\cos b} \div \csc b(\sec b + 1) \\
&= \frac{1}{\cos b} \div \frac{1}{\sin b}\left(\frac{1}{\cos b} + 1\right) \\
&= \frac{1}{\cos b} \div \frac{1}{\sin b}\left(\frac{1+\cos b}{\cos b}\right) \\
&= \frac{1}{\cos b} \div \frac{1+\cos b}{\sin b \cos b} \\
&= \frac{1}{\cos b} \cdot \frac{\sin b \cos b}{1+\cos b} \\
&= \frac{\sin b}{1+\cos b}
\end{aligned}$$

Step 2: At the last step above, we have simplified the right side as much as possible, now we simplify the left side, using the half angle formula.

$$\begin{aligned}
\sqrt{\frac{1-\cos b}{1+\cos b}} &= \frac{\sin b}{1+\cos b} \\
\frac{1-\cos b}{1+\cos b} &= \frac{\sin^2 b}{(1+\cos b)^2} \\
(1-\cos b)(1+\cos b)^2 &= \sin^2 b(1+\cos b) \\
(1-\cos b)(1+\cos b) &= \sin^2 b \\
1-\cos^2 b &= \sin^2 b
\end{aligned}$$

Example 3.4.5.3

Verify the identity: $\cot \frac{c}{2} = \frac{\sin c}{1-\cos c}$

Solution

Step 1: change cotangent to cosine over sine, then cross-multiply.

$$\begin{aligned}\cot\frac{c}{2} &= \frac{\sin c}{1-\cos c}\\ &= \frac{\cos\frac{c}{2}}{\sin\frac{c}{2}} = \sqrt{\frac{1+\cos c}{1-\cos c}}\\ \sqrt{\frac{1+\cos c}{1-\cos c}} &= \frac{\sin c}{1-\cos c}\\ \frac{1+\cos c}{1-\cos c} &= \frac{\sin^2 c}{(1-\cos c)^2}\\ (1+\cos c)(1-\cos c)^2 &= \sin^2 c(1-\cos c)\\ (1+\cos c)(1-\cos c) &= \sin^2 c\\ 1-\cos^2 c &= \sin^2 c\end{aligned}$$

Example 3.4.5.4

Prove that $\sin x \tan\frac{x}{2} + 2\cos x = 2\cos^2\frac{x}{2}$

Solution

$$\begin{aligned}\sin x \tan\frac{x}{2} + 2\cos x &= \sin x\left(\frac{1-\cos x}{\sin x}\right) + 2\cos x\\ \sin x \tan\frac{x}{2} + 2\cos x &= 1-\cos x + 2\cos x\\ \sin x \tan\frac{x}{2} + 2\cos x &= 1+\cos x\\ \sin x \tan\frac{x}{2} + 2\cos x &= 2\cos^2\frac{x}{2}\end{aligned}$$

Review

Use half angle identities to find the exact values of each expression.

1. $\sin 22.5^\circ$
2. $\sin 75^\circ$
3. $\sin 67.5^\circ$
4. $\sin 157.5^\circ$
5. $\cos 22.5^\circ$
6. $\cos 75^\circ$
7. $\cos 157.5^\circ$
8. $\cos 67.5^\circ$
9. Use the two half angle identities presented in this section to prove that $\tan\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$.
10. Use the result of the previous problem to show that $\tan\left(\frac{x}{2}\right) = \frac{1-\cos x}{\sin x}$.
11. Use the result of the previous problem to show that $\tan\left(\frac{x}{2}\right) = \frac{\sin x}{1+\cos x}$.

Use half angle identities to help you find all solutions to the following equations in the interval $[0, 2\pi)$.

12. $\sin^2 x = \cos^2\left(\frac{x}{2}\right)$
13. $\tan\left(\frac{x}{2}\right) = \frac{1-\cos x}{1+\cos x}$
14. $\cos^2 x = \sin^2\left(\frac{x}{2}\right)$
15. $\sin^2\left(\frac{x}{2}\right) = 2\cos^2 x - 1$

Review (Answers)

To see the Review answers, open this PDF file and look for section 3.11.

Vocabulary

Term	Definition
Half Angle Identity	A half angle identity relates a trigonometric function of one half of an argument to a set of trigonometric functions, each containing the original argument.

Additional Resources

Video: Half-Angle Formulas - Overview

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3.4.6: Trigonometric Equations Using Half Angle Formulas

Simplifying all six trigonometric functions with half a given angle.

As you've seen many times, the ability to find the values of trig functions for a variety of angles is a critical component to a course in Trigonometry. If you were given an angle as the argument of a trig function that was half of an angle you were familiar with, could you solve the trig function?

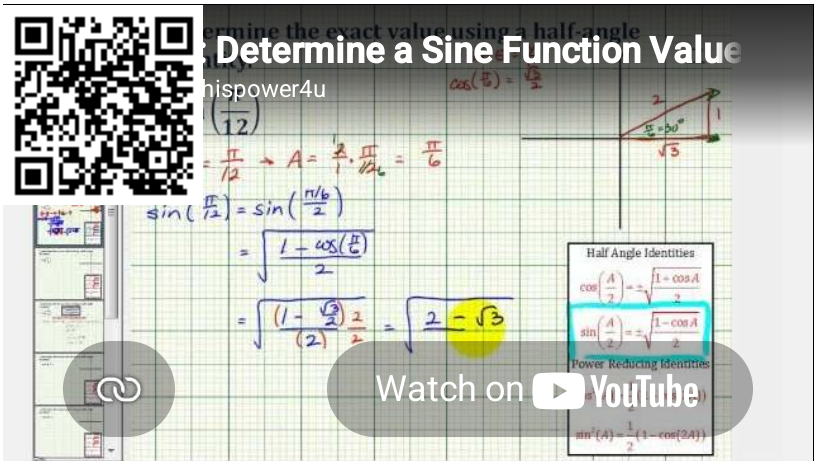
For example, if you were asked to find

$\sin 22.5^\circ$

would you be able to do it? Keep reading, and in this section you'll learn how to do this.

using Half Angle Formulas on Trigonometric Equations

It is easy to remember the values of trigonometric functions for certain common values of θ. However, sometimes there will be fractional values of known trig functions, such as wanting to know the sine of half of the angle that you are familiar with. In situations like that, a **half angle identity** can prove valuable to help compute the value of the trig function.



In addition, half angle identities can be used to simplify problems to solve for certain angles that satisfy an expression. To do this, first remember the half angle identities for sine and cosine:

$\sin \alpha 2 = \sqrt{\frac{1-\cos\alpha}{2}}$ if $\frac{\alpha}{2}$ is located in either the first or second quadrant.

$\sin \alpha 2 = -\sqrt{\frac{1-\cos\alpha}{2}}$ if $\frac{\alpha}{2}$ is located in the third or fourth quadrant.

$\cos \alpha 2 = \sqrt{\frac{1+\cos\alpha}{2}}$ if $\frac{\alpha}{2}$ is located in either the first or fourth quadrant.

$\cos \alpha 2 = -\sqrt{\frac{1+\cos\alpha}{2}}$ if $\frac{\alpha}{2}$ is located in either the second or fourth quadrant.

When attempting to solve equations using a half angle identity, look for a place to substitute using one of the above identities. This can help simplify the equation to be solved.

Let's look at some problems that use the half angle formula.

1. Solve the trigonometric equation $\sin^2\theta = 2\sin^2\frac{\theta}{2}$ over the interval $[0, 2\pi)$.

$$\begin{aligned} \sin^2\theta &= 2\sin^2\frac{\theta}{2} \\ \sin^2\theta &= 2\left(\frac{1-\cos\theta}{2}\right) && \text{Half angle identity} \\ 1-\cos^2\theta &= 1-\cos\theta && \text{Pythagorean identity} \\ \cos\theta - \cos^2\theta &= 0 \\ \cos\theta(1-\cos\theta) &= 0 \end{aligned}$$

Then $\cos\theta = 0$ or $1 - \cos\theta = 0$, which is $\cos\theta$.

$\theta = 0,\ \frac{\pi}{2},\ \frac{3\pi}{2},\ \text{or } 2\pi.$

2. Solve $2\cos^2\frac{x}{2} = 1$ for $0 \le x < 2\pi$

To solve $2\cos^2\frac{x}{2} = 1$, first we need to isolate cosine, then use the half angle formula.

$$\begin{aligned} 2\cos^2\frac{x}{2} &= 1 \\ \cos^2\frac{x}{2} &= \frac{1}{2} \\ \frac{1+\cos x}{2} &= \frac{1}{2} \\ 1+\cos x &= 1 \\ \cos x &= 0 \end{aligned}$$

$\cos x = 0$ when $x = \frac{\pi}{2},\ \frac{3\pi}{2}$

3. Solve $\tan\frac{a}{2} = 4$ for $0° \le a < 360°$

To solve $\tan\frac{a}{2} = 4$, first isolate tangent, then use the half angle formula.

$$\begin{aligned} \tan\frac{a}{2} &= 4 \\ \sqrt{\frac{1-\cos a}{1+\cos a}} &= 4 \\ \frac{1-\cos a}{1+\cos a} &= 16 \\ 16 + 16\cos a &= 1 - \cos a \\ 17\cos a &= -15 \\ \cos a &= -\frac{15}{17} \end{aligned}$$

using your graphing calculator, $\cos a = -\frac{15}{17}$ when $a = 152°,\ 208°$

Example 3.4.6.1

Earlier, you were asked to solve sin $22.5°$.

Solution

Knowing the half angle formulas, you can compute $\sin 22.5°$ easily:

$$\begin{aligned} \sin 22.5° &= \sin\left(\frac{45°}{2}\right) \\ &= \sqrt{\frac{1-\cos 45°}{2}} \\ &= \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} \\ &= \sqrt{\frac{\frac{2-\sqrt{2}}{2}}{2}} \\ &= \sqrt{\frac{2-\sqrt{2}}{4}} \\ &= \frac{\sqrt{2-\sqrt{2}}}{2} \end{aligned}$$

Example 3.4.6.2

Find the exact value of $\cos 112.5°$

Solution

$$\begin{aligned}\sin 105^\circ &= \sin\frac{210^\circ}{2}\\ &= \sqrt{\frac{1-\cos 210^\circ}{2}}\\ &= \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}\\ &= \sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}}\\ &= \sqrt{\frac{2-\sqrt{3}}{4}}\\ &= \frac{\sqrt{2-\sqrt{3}}}{2}\end{aligned}$$

Example 3.4.6.3

Find the exact value of $\sin 105^\circ$

Solution

$$\begin{aligned}\sin 105^\circ &= \sin\frac{210^\circ}{2}\\ &= \sqrt{\frac{1-\cos 210^\circ}{2}}\\ &= \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}\\ &= \sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}}\\ &= \sqrt{\frac{2-\sqrt{3}}{4}}\\ &= \frac{\sqrt{2-\sqrt{3}}}{2}\end{aligned}$$

Example 3.4.6.4

Find the exact value of $\tan\frac{7\pi}{8}$.

Solution

$$\begin{aligned}\tan\frac{7\pi}{8} &= \tan\frac{1}{2}\cdot\frac{7\pi}{4}\\ &= \frac{1-\cos\frac{7\pi}{4}}{\sin\frac{7\pi}{4}}\\ &= \frac{1-\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}}\\ &= \frac{\frac{2-\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}}\\ &= -\frac{2-\sqrt{2}}{\sqrt{2}} = \frac{-2\sqrt{2}+2}{2}\\ &= -\sqrt{2}+1\end{aligned}$$

Review

Use half angle identities to find the exact value of each expression.

1. $\tan 15^\circ$
2. $\tan 22.5^\circ$
3. $\cot 75^\circ$
4. $\tan 67.5^\circ$
5. $\tan 157.5^\circ$
6. $\tan 112.5^\circ$
7. $\cos 105^\circ$
8. $\sin 112.5^\circ$
9. $\sec 15^\circ$
10. $\csc 22.5^\circ$
11. $\csc 75^\circ$
12. $\sec 67.5^\circ$
13. $\cot 157.5^\circ$

Use half angle identities to help solve each of the following equations on the interval $[0, 2\pi)$.

14. $3\cos^2\left(\frac{x}{2}\right) = 3$
15. $4\sin^2 x = 8\sin^2\left(\frac{x}{2}\right)$

Review (Answers)

To see the Review answers, open this PDF file and look for section 3.12.

Vocabulary

Term	Definition
Half Angle Identity	A half angle identity relates a trigonometric function of one half of an argument to a set of trigonometric functions, each containing the original argument.

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SECTION OVERVIEW

3.5: Sum to Product and Triple Angle Formulas

3.5.1: Sum to Product Formulas for Sine and Cosine

3.5.2: Product to Sum Formulas for Sine and Cosine

3.5.3: Triple-Angle Formulas and Linear Combinations

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3.5.1: Sum to Product Formulas for Sine and Cosine

Relation of the sum or difference of two trigonometric functions to a product.

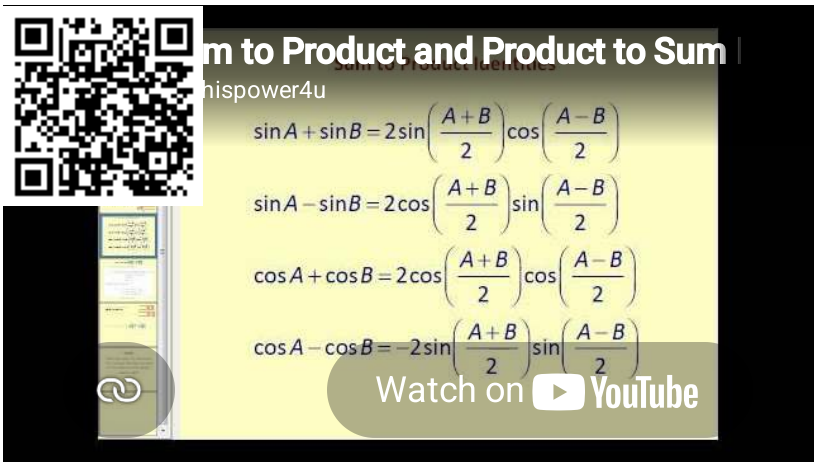
Can you solve problems that involve the sum of sines or cosines? For example, consider the equation:

$\cos 10t + \cos 3t$

You could just compute each expression separately and add their values at the end. However, there is an easier way to do this. You can simplify the equation first, and then solve.

Sine and Cosine Sum to Product Formulas

In some problems, the product of two trigonometric functions is more conveniently found by the sum of two trigonometric functions by use of identities.



Here is an example:

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2} \times \cos\frac{\alpha-\beta}{2}$$

This can be verified by using the sum and difference formulas:

$$\begin{aligned}
&2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\
&= 2\left[\sin\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)\cos\left(\frac{\alpha}{2}-\frac{\beta}{2}\right)\right] \\
&= 2\left[\left(\sin\frac{\alpha}{2}\cos\frac{\beta}{2}+\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\right)\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2}+\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right)\right] \\
&= 2\left[\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\cos^2\frac{\beta}{2}+\sin^2\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\beta}{2}+\sin\frac{\beta}{2}\cos^2\frac{\alpha}{2}\cos\frac{\beta}{2}+\sin\frac{\alpha}{2}\sin^2\frac{\beta}{2}\cos\frac{\alpha}{2}\right] \\
&= 2\left[\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\left(\sin^2\frac{\beta}{2}+\cos^2\frac{\beta}{2}\right)+\sin\frac{\beta}{2}\cos\frac{\beta}{2}\left(\sin^2\frac{\alpha}{2}+\cos^2\frac{\alpha}{2}\right)\right] \\
&= 2\left[\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}+\sin\frac{\beta}{2}\cos\frac{\beta}{2}\right] \\
&= 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}+2\sin\frac{\beta}{2}\cos\frac{\beta}{2} \\
&= \sin\left(2\cdot\frac{\alpha}{2}\right)+\sin\left(2\cdot\frac{\beta}{2}\right) \\
&= \sin\alpha+\sin\beta
\end{aligned}$$

The following variations can be derived similarly:

$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2} \times \cos\frac{\alpha+\beta}{2}$$
$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2} \times \cos\frac{\alpha-\beta}{2}$$
$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2} \times \sin\frac{\alpha-\beta}{2}$$

Here are some problems using this type of transformation from a sum of terms to a product of terms.

1. Change $\sin 5x - \sin 9x$ into a product.

Use the formula $\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2} \times \cos\frac{\alpha+\beta}{2}$.

$$\begin{aligned}\sin 5x - \sin 9x &= 2\sin\frac{5x-9x}{2}\cos\frac{5x+9x}{2}\\ &= 2\sin(-2x)\cos 7x\\ &= -2\sin 2x\cos 7x\end{aligned}$$

2. Change $\cos(-3x) + \cos 8x$ into a product.

Use the formula $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2} \times \cos\frac{\alpha-\beta}{2}$

$$\begin{aligned}\cos(-3x) + \cos(8x) &= 2\cos\frac{-3x+8x}{2}\cos\frac{-3x-8x}{2}\\ &= 2\cos(2.5x)\cos(-5.5x)\\ &= 2\cos(2.5x)\cos(5.5x)\end{aligned}$$

3. Change $2\sin 7x\cos 4x$ to a sum.

This is the reverse of what was done in the previous two examples. Looking at the four formulas above, take the one that has sine and cosine as a product, $\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2} \times \cos\frac{\alpha-\beta}{2}$. Therefore, $7x = \frac{\alpha+\beta}{2}$ and $4x = \frac{\alpha-\beta}{2}$.

$$\begin{aligned}7x &= \frac{\alpha+\beta}{2} & 4x &= \frac{\alpha-\beta}{2}\\ & & \text{and} & \\ 14x &= \alpha+\beta & 8x &= \alpha-\beta\\ \alpha &= 14x-\beta & 8x &= [14x-\beta]-\beta\\ & & \text{so} & \\ \alpha &= 14x-3x & -6x &= -2\beta\\ \alpha &= 11x & 3x &= \beta\end{aligned}$$

So, this translates to $\sin(11x) + \sin(3x)$. A shortcut for this problem, would be to notice that the sum of $7x$ and $4x$ is $11x$ and the difference is $3x$.

Example 3.5.1.1

Earlier, you were asked to solve

$\cos 10t + \cos 3t$

Solution

You can easily transform this equation into a product of two trig functions using:

$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2} \times \cos\frac{\alpha-\beta}{2}$

Substituting the known quantities:

$\cos 10t + \cos 3t = 2\cos\frac{13t}{2} \times \cos\frac{7t}{2} = 2\cos(6.5t)\cos(3.5t)$

Example 3.5.1.2

Express the sum as a product: $\sin 9x + \sin 5x$

Solution

Using the sum-to-product formula:

$$\begin{aligned}&\sin 9x + \sin 5x\\ &2\left(\sin\left(\frac{9x+5x}{2}\right)\cos\left(\frac{9x-5x}{2}\right)\right)\\ &2\sin 7x\cos 2x\end{aligned}$$

Example 3.5.1.3

Express the difference as a product: $\cos 4y - \cos 3y$

Solution

Using the difference-to-product formula:

$$\begin{aligned} &\cos 4y - \cos 3y \\ &-2\sin\left(\frac{4y+3y}{2}\right)\sin\left(\frac{4y-3y}{2}\right) \\ &-2\sin\frac{7y}{2}\sin\frac{y}{2} \end{aligned}$$

Example 3.5.1.4

Verify the identity (using sum-to-product formula): $\dfrac{\cos 3a - \cos 5a}{\sin 3a - \sin 5a} = -\tan 4a$

Solution

Using the difference-to-product formulas:

$$\begin{aligned} &\frac{\cos 3a - \cos 5a}{\sin 3a - \sin 5a} = -\tan 4a \\ &\frac{-2\sin\left(\frac{3a+5a}{2}\right)\sin\left(\frac{3a-5a}{2}\right)}{2\sin\left(\frac{3a-5a}{2}\right)\cos\left(\frac{3a+5a}{2}\right)} \\ &-\frac{\sin 4a}{\cos 4a} \\ &-\tan 4a \end{aligned}$$

Review

Change each sum or difference into a product.

1. $\sin 3x + \sin 2x$
2. $\cos 2x + \cos 5x$
3. $\sin(-x) - \sin 4x$
4. $\cos 12x + \cos 3x$
5. $\sin 8x - \sin 4x$
6. $\sin x + \sin \dfrac{1}{2}x$
7. $\cos 3x - \cos(-3x)$

Change each product into a sum or difference.

8. $-2\sin 3.5x \sin 2.5x$
9. $2\cos 3.5x \sin 0.5x$
10. $2\cos 3.5x \cos 5.5x$
11. $2\sin 6x \cos 2x$
12. $-2\sin 3x \sin x$
13. $2\sin 4x \cos x$
14. Show that $\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2} = \dfrac{1}{2}(\cos A + \cos B)$.
15. Let $u = \dfrac{A+B}{2}$ and $v = \dfrac{A-B}{2}$. Show that $\cos u \cos v = \dfrac{1}{2}(\cos(u+v) + \cos(u-v))$.

Review (Answers)

To see the Review answers, open this PDF file and look for section 3.13.

Vocabulary

Term	Definition
Sum to Product Formula	A sum to product formula relates the sum or difference of two trigonometric functions to the product of two trigonometric functions.

Additional Resources

Video: Sum-to-Product Formulas - Overview

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3.5.2: Product to Sum Formulas for Sine and Cosine

Relation of the product of two trigonometric functions to a sum or difference.

Let's say you are in class one day, working on calculating the values of trig functions, when your instructor gives you an equation like this:

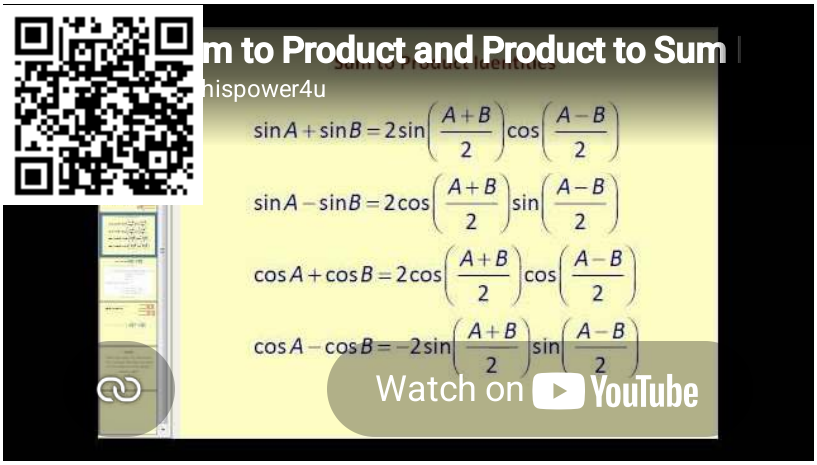
$\sin 75^\circ \sin 15^\circ$

Can you solve this sort of equation? You might want to just calculate each term separately and then compute the result. However, there is another way. You can transform this product of trig functions into a sum of trig functions.

Read on, and by the end of this lesson, you'll know how to solve this problem by changing it into a sum of trig functions.

Product to Sum Formulas for sine and cosine

Here we'll begin by deriving formulas for how to convert the product of two trig functions into a sum or difference of trig functions.



There are two formulas for transforming a product of sine or cosine into a sum or difference. First, let's look at the product of the sine of two angles. To do this, we need to start with the cosine of the difference of two angles.

$$\cos(a-b)=\cos a\cos b+\sin a\sin b \text{ and } \cos(a+b)=\cos a\cos b-\sin a\sin b$$
$$\cos(a-b)-\cos(a+b)=\cos a\cos b+\sin a\sin b-(\cos a\cos b-\sin a\sin b)$$
$$\cos(a-b)-\cos(a+b)=\cos a\cos b+\sin a\sin b-\cos a\cos b+\sin a\sin b$$
$$\cos(a-b)-\cos(a+b)=2\sin a\sin b$$
$$\frac{1}{2}[\cos(a-b)-\cos(a+b)]=\sin a\sin b$$

The following product to sum formulas can be derived using the same method:

$$\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$
$$\sin\alpha\cos\beta=\frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$
$$\cos\alpha\sin\beta=\frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$$

Using the **Product to Sum Formula**

1. Change $\cos 2x\cos 5y$ to a sum.

Use the formula $\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$. Set $\alpha=2x$ and $\beta=5y$.

$$\cos 2x\cos 5y=\frac{1}{2}[\cos(2x-5y)+\cos(2x+5y)]$$

2. Change $\frac{\sin 11z+\sin z}{2}$ to a product.

Use the formula $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$. Therefore, $\alpha+\beta = 11z$ and $\alpha-\beta = z$. Solve the second equation for \alpha and plug that into the first.

$$\begin{aligned} \alpha = z+\beta \rightarrow (z+\beta)+\beta &= 11z \qquad \text{and} \qquad \alpha = z+5z = 6z \\ z+2\beta &= 11z \\ 2\beta &= 10z \\ \beta &= 5z \end{aligned}$$

$\frac{\sin 11z+\sin z}{2} = \sin 6z\cos 5z$. Again, the sum of $6z$ and $5z$ is $11z$ and the difference is z.

3. Solve $\cos 5x+\cos x = \cos 2x$.

Use the formula $\cos\alpha+\cos\beta = 2\cos\frac{\alpha+\beta}{2}\times\cos\frac{\alpha-\beta}{2}$.

$$\begin{aligned} \cos 5x+\cos x &= \cos 2x \\ 2\cos 3x\cos 2x &= \cos 2x \\ 2\cos 3x\cos 2x-\cos 2x &= 0 \\ \cos 2x(2\cos 3x-1) &= 0 \end{aligned}$$

$$\begin{aligned} \cos 2x &= 0 \qquad\qquad 2\cos 3x-1 = 0 \\ & \qquad\qquad\quad 2\cos 3x = 1 \\ 2x &= \frac{\pi}{2},\frac{3\pi}{2} \qquad \text{and} \qquad \cos 3x = \frac{1}{2} \\ x &= \frac{\pi}{4},\frac{3\pi}{4} \qquad\qquad 3x = \frac{\pi}{3},\frac{5\pi}{3},\frac{7\pi}{3},\frac{11\pi}{3},\frac{13\pi}{3},\frac{17\pi}{3} \\ & \qquad\qquad\quad x = \frac{\pi}{9},\frac{5\pi}{9},\frac{7\pi}{9},\frac{11\pi}{9},\frac{13\pi}{9},\frac{17\pi}{9} \end{aligned}$$

Example 3.5.2.1

Earlier, you were asked to solve sin $75^\circ\sin 15^\circ$.

Solution

Changing $\sin 75^\circ\sin 15^\circ$ to a product of trig functions can be accomplished using

$\sin a\sin b = \frac{1}{2}[\cos(a-b)-\cos(a+b)]$

Substituting in known values gives:

$$\sin 75^\circ\sin 15^\circ = \frac{1}{2}[\cos(60^\circ)-\cos(90^\circ)] = \frac{1}{2}\left[\frac{1}{2}-0\right] = \frac{1}{4}$$

Example 3.5.2.2

Express the product as a sum: $\sin(6\theta)\sin(4\theta)$

Solution

Using the product-to-sum formula:

$$\begin{aligned} & \sin 6\theta\sin 4\theta \\ & \frac{1}{2}(\cos(6\theta-4\theta)-\cos(6\theta+4\theta)) \\ & \frac{1}{2}(\cos 2\theta-\cos 10\theta) \end{aligned}$$

Example 3.5.2.3

Express the product as a sum: $\sin(5\theta)\cos(2\theta)$

Solution

Using the product-to-sum formula:

$$\sin 5\theta \cos 2\theta$$
$$\frac{1}{2}(\sin(5\theta + 2\theta) - \sin(5\theta - 2\theta))$$
$$\frac{1}{2}(\sin 7\theta - \sin 3\theta)$$

Example 3.5.2.4

Express the product as a sum: $\cos(10\theta)\sin(3\theta)$

Solution

Using the product-to-sum formula:

$$\cos 10\theta \sin 3\theta$$
$$\frac{1}{2}(\sin(10\theta + 3\theta) - \sin(10\theta - 3\theta))$$
$$\frac{1}{2}(\sin 13\theta - \sin 7\theta)$$

Review

Express each product as a sum or difference.

1. $\sin(5\theta)\sin(3\theta)$
2. $\sin(6\theta)\cos(\theta)$
3. $\cos(4\theta)\sin(3\theta)$
4. $\cos(\theta)\cos(4\theta)$
5. $\sin(2\theta)\sin(2\theta)$
6. $\cos(6\theta)\sin(8\theta)$
7. $\sin(7\theta)\cos(4\theta)$
8. $\cos(11\theta)\cos(2\theta)$

Express each sum or difference as a product.

9. $\frac{\sin 8\theta + \sin 6\theta}{2}$
10. $\frac{\sin 6\theta - \sin 2\theta}{2}$
11. $\frac{\cos 12\theta + \cos 6\theta}{2}$
12. $\frac{\cos 12\theta - \cos 4\theta}{2}$
13. $\frac{\sin 10\theta + \sin 4\theta}{2}$
14. $\frac{\sin 8\theta - \sin 2\theta}{2}$
15. $\frac{\cos 8\theta - \cos 4\theta}{2}$

Review (Answers)

To see the Review answers, open this PDF file and look for section 3.14.

Vocabulary

Term	Definition
Product to Sum Formula	A product to sum formula relates the product of two trigonometric functions to the sum of two trigonometric functions.

Additional Resources

Video: Product-to-Sum Formulas - Example 1

Practice: Product to Sum Formulas for sine and cosine

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3.5.3: Triple-Angle Formulas and Linear Combinations

Combination of the sum and double angle formulas; set of terms added or subtracted with a constant multiplier.

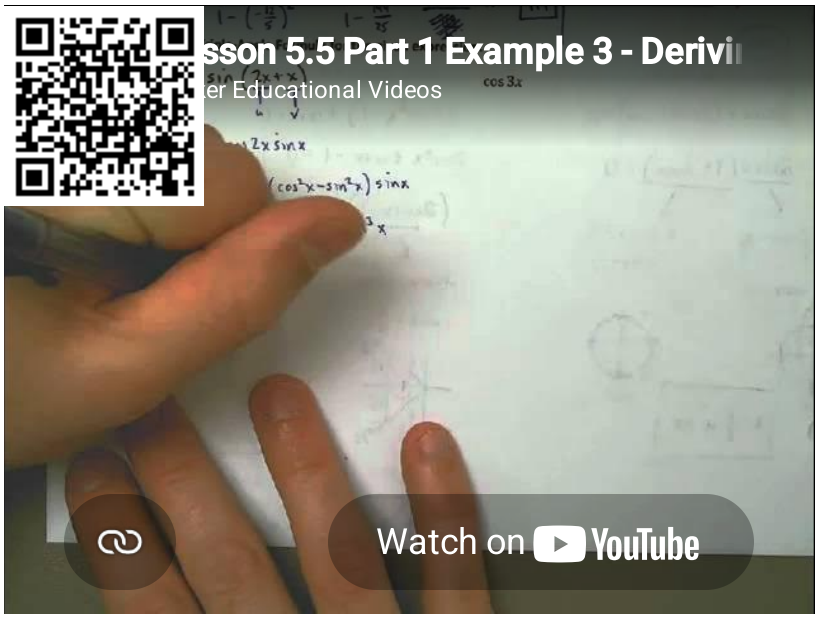
In other lessons you've dealt with double angle formulas. This was useful for finding the value of an angle that was double your well known value. Now consider the idea of a "triple angle formula". If someone gave you a problem like this:

$\sin 135^\circ$

Could you compute its value?

Triple Angle Formulas and Linear Combinations

Double angle formulas are great for computing the value of a trig function in certain cases. However, sometimes different multiples than two times and angle are desired. For example, it might be desirable to have three times the value of an angle to use as the argument of a trig function.



By combining the sum formula and the double angle formula, formulas for triple angles and more can be found.

Here, we take an equation which takes a linear combination of sine and cosine and converts it into a simpler cosine function.

$A\cos x + B\sin x = C\cos(x - D)$, where $C = \sqrt{A^2 + B^2}$, $\cos D = \frac{A}{C}$ and $\sin D = \frac{B}{C}$.

You can also use the TI-83 to solve trigonometric equations. It is sometimes easier than solving the equation algebraically. Just be careful with the directions and make sure your final answer is in the form that is called for. You calculator cannot put radians in terms of \pi .

Finding Formulas

Find the formula for $\sin 3x$

Use both the double angle formula and the sum formula.

$$\begin{aligned}\sin 3x &= \sin(2x + x)\\ &= \sin(2x)\cos x + \cos(2x)\sin x\\ &= (2\sin x\cos x)\cos x + (\cos 2x - \sin 2x)\sin x\\ &= 2\sin x\cos 2x + \cos 2x\sin x - \sin 3x\\ &= 3\sin x\cos 2x - \sin 3x\\ &= 3\sin x(1 - \sin 2x) - \sin 3x\\ &= 3\sin x - 4\sin 3x\end{aligned}$$

Performing Transformations

Transform $3\cos 2x - 4\sin 2x$ into the form $C\cos(2x - D)$

$A = 3$ and $B = -4$, so $C = \sqrt{3^2 + (-4)^2} = 5$. Therefore $\cos D = \frac{3}{5}$ and $\sin D = -\frac{4}{5}$ which makes the reference angle is $-53.1°$ or -0.927 radians. Since cosine is positive and sine is negative, the angle must be a fourth quadrant angle. D must therefore be $306.9°$ or 5.36 radians. The final answer is $3\cos 2x - 4\sin 2x = 5\cos(2x - 5.36)$.

Solving for Unknown Values

Solve $\sin x = 2\cos x$ such that $0 \le x \le 2\pi$ using a graphing calculator.

Solution: In $y =$, graph $y_1 = \sin x$ and $y_2 = 2\cos x$.

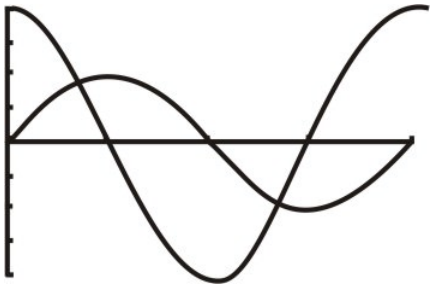
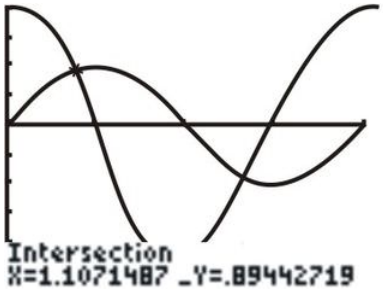


Figure 3.5.3.1

Next, use **CALC** to find the intersection points of the graphs.



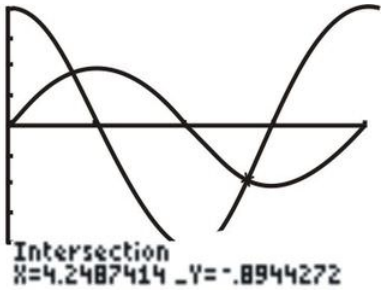


Figure 3.5.3.2

Example 3.5.3.1

Earlier, you were asked to solve sin $135°$.

Solution

Using the triple angle formula we learned in this lesson for the sine function, we can break the angle down into three times a well known angle:

$\sin 3x = 3\sin x - 4\sin^3 x$

we can solve this problem.

$$\begin{aligned}\sin(3 \times 45°) &= 3\sin 45° - 4\sin^3 45° \\ &= 3\frac{\sqrt{2}}{2} - 4\left(\frac{\sqrt{2}}{2}\right)^3 \\ &= 3\frac{\sqrt{2}}{2} - \left(\frac{4(2)^{2/3}}{8}\right) \\ &= \frac{3\sqrt{2} - 2\sqrt{2}}{2} \\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

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Example 3.5.3.2

Transform $5\cos x - 5\sin x$ to the form $C\cos(x - D)$

Solution

If $5\cos x - 5\sin x$, then $A = 5$ and $B = -5$. By the Pythagorean Theorem, $C = 5\sqrt{2}$ and $\cos D = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. So, because B is negative, D is in Quadrant IV. Therefore, $D = \frac{7\pi}{4}$. Our final answer is $5\sqrt{2}\cos\left(x - \frac{7\pi}{4}\right)$.

Example 3.5.3.3

Transform $-15\cos 3x - 8\sin 3x$ to the form $C\cos(x - D)$

Solution

If $-15\cos 3x - 8\sin 3x$, then $A = -15$ and $B = -8$. By the Pythagorean Theorem, $C = 17$. Because A and B are both negative, D is in Quadrant III, which means $D = \cos^{-1}\left(\frac{15}{17}\right) = 0.49 + \pi = 3.63$ rad. Our final answer is $17\cos 3(x - 3.63)$.

Example 3.5.3.4

Derive a formula for $\tan 4x$.

Solution

$$\begin{aligned}
\tan 4x &= \tan(2x + 2x) \\
&= \frac{\tan 2x + \tan 2x}{1 - \tan 2x \tan 2x} \\
&= \frac{2\tan 2x}{1 - \tan^2 2x} \\
&= \frac{2 \cdot \frac{2\tan x}{1 - \tan^2 x}}{1 - \left(\frac{2\tan x}{1 - \tan^2 x}\right)^2} \\
&= \frac{4\tan x}{1 - \tan^2 x} \div \frac{(1 - \tan^2 x)^2 - 4\tan^2 x}{(1 - \tan^2 x)^2} \\
&= \frac{4\tan x}{1 - \tan^2 x} \div \frac{1 - 2\tan^2 x + \tan^4 x - 4\tan^2 x}{(1 - \tan^2 x)^2} \\
&= \frac{4\tan x}{1 - \tan^2 x} \cdot \frac{(1 - \tan^2 x)^2}{1 - 6\tan^2 x + \tan^4 x} \\
&= \frac{4\tan x - 4\tan^3 x}{1 - 6\tan^2 x + \tan^4 x}
\end{aligned}$$

Review

Transform each expression to the form $C\cos(x - D)$.

1. $3\cos x - 2\sin x$
2. $2\cos x - \sin x$
3. $-4\cos x + 5\sin x$
4. $7\cos x - 6\sin x$
5. $11\cos x + 9\sin x$
6. $14\cos x + 2\sin x$
7. $-2\cos x - 4\sin x$

Derive a formula for each expression.

8. $\sin 4x$
9. $\cos 6x$

10. $\cos 4x$
11. $\csc 2x$
12. $\cot 2x$

Find all solutions to each equation in the interval $[0, 2\pi)$.

13. $\cos x + \cos 3x = 0$
14. $\sin 2x = \cos 3x$
15. $\cos 2x + \cos 4x = 0$

Review (Answers)

To see the Review answers, open this PDF file and look for section 3.15.

Vocabulary

Term	Definition
Linear Combination	A linear combination is a set of terms that are added or subtracted from each other with a multiplicative constant in front of each term.
Triple Angle Identity	A triple angle identity (also referred to as a triple angle formula) relates a trigonometric function of three times an argument to a set of trigonometric functions, each containing the original argument. Examples include: the Triple Angle Formula for sine $\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$, the Triple Angle Formula for cosine $\cos(3\theta) = -3\cos\theta + 4\cos^3\theta$, and the Triple Angle Formula for tangent $\tan(3\theta) = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

Additional Resources

Video: **Deriving a Triple Angle Formula**

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CHAPTER OVERVIEW

Unit 4: Non-Right Triangle Trigonometry

4.1: Law of Sines and Law of Cosines

4.1.1: Laws of Sines and Cosines

4.1.2: Law of Sines

4.1.3: Angle-Angle-Side Triangles

4.1.4: Angle-Side-Angle Triangles

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4.1.9: Identify Accurate Drawings of Triangles

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4.1.1: Laws of Sines and Cosines

Solving for parts of non-right triangles using trigonometry.

Law of Sines: If ΔABC has sides of length, a, b, and c, then $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

Looking at a triangle, the lengths a,b, and c are opposite the angles of the same letter.

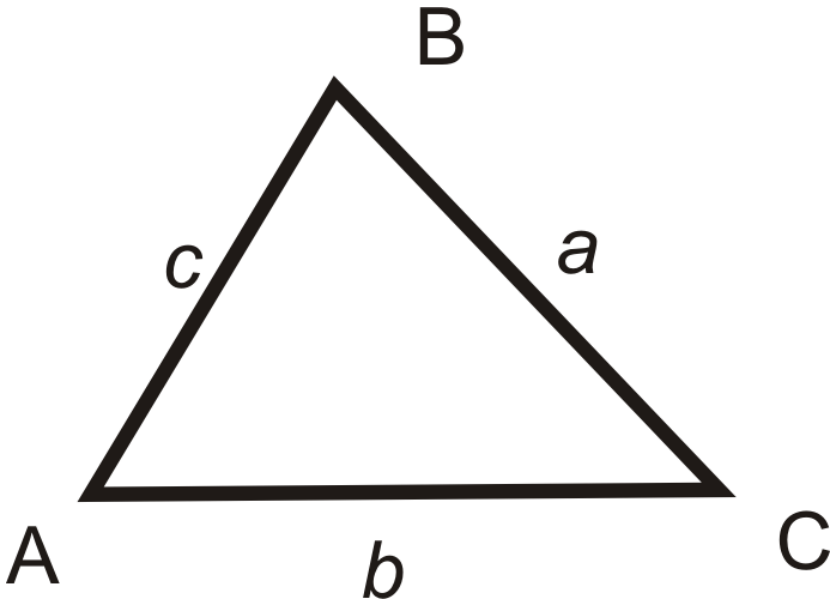
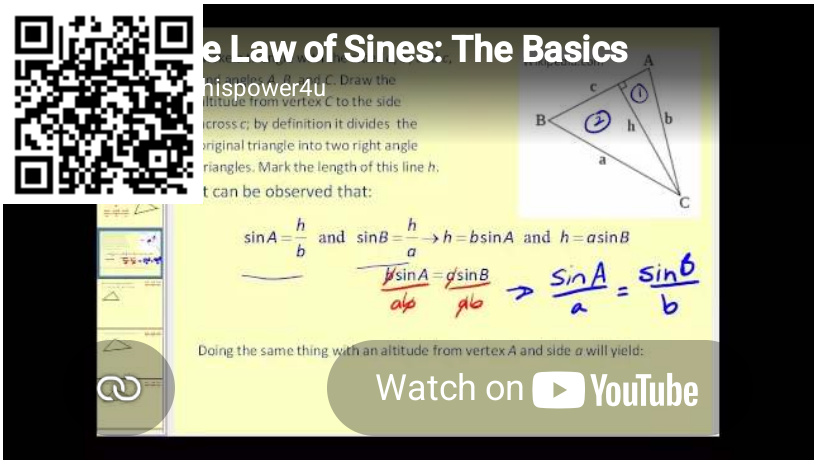


Figure 4.1.1.1

Use Law of Sines when given:

- An angle and its opposite side.
- Any two angles and one side.

Two sides and the non-included angle.



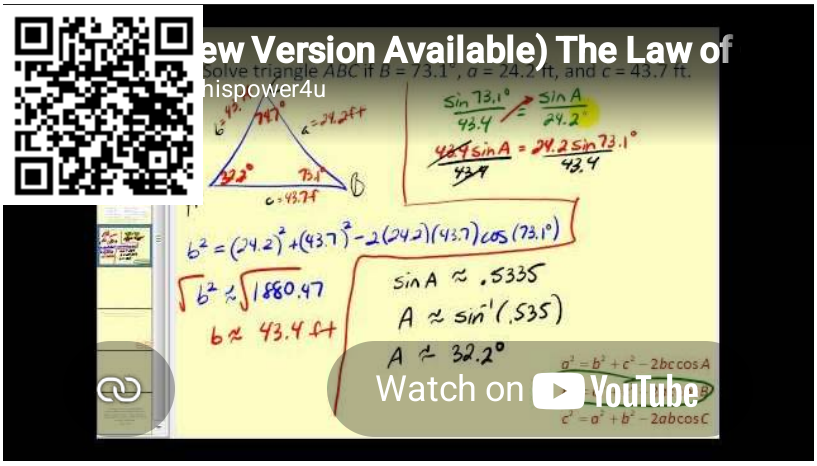
Law of Cosines: If ΔABC has sides of length a,b, and c, then:

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Even though there are three formulas, they are all very similar. First, notice that whatever angle is in the cosine, the opposite side is on the other side of the equal sign.

Use Law of Cosines when given:

- Two sides and the included angle.
- All three sides.



Using the Law of Sines

1. Solve the triangle using the Law of Sines. Round decimal answers to the nearest tenth.

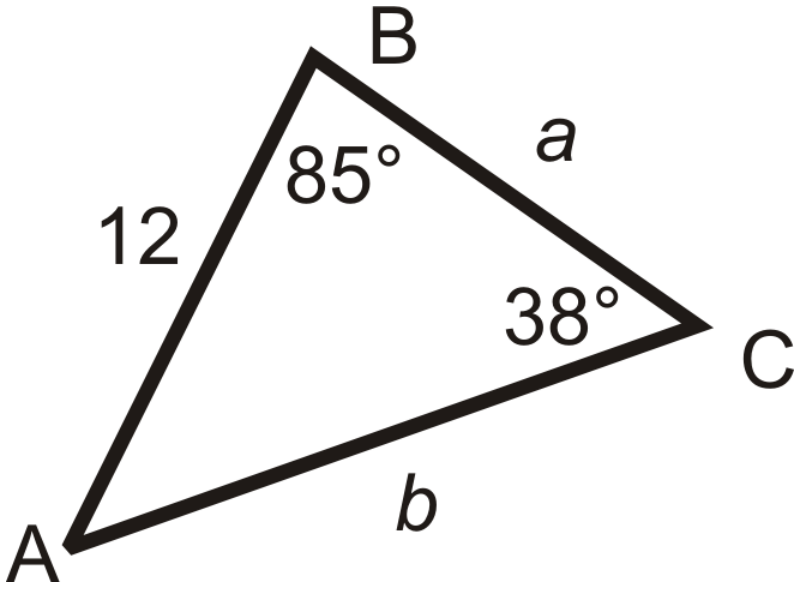


Figure 4.1.1.2

First, to find $m\angle A$, we can use the Triangle Sum Theorem.

$$m\angle A + 85^\circ + 38^\circ = 180^\circ$$
$$m\angle A = 57^\circ$$

Now, use the Law of Sines to set up ratios for a and b.

$$\frac{\sin 57^\circ}{a} = \frac{\sin 85^\circ}{b} = \frac{\sin 38^\circ}{12}$$

$$\frac{\sin 57^\circ}{a} = \frac{\sin 38^\circ}{12} \qquad \frac{\sin 85^\circ}{b} = \frac{\sin 38^\circ}{12}$$
$$a \cdot \sin 38^\circ = 12 \cdot \sin 57^\circ \qquad b \cdot \sin 38^\circ = 12 \cdot \sin 85^\circ$$
$$a = \frac{12 \cdot \sin 57^\circ}{\sin 38^\circ} \approx 16.4 \qquad b = \frac{12 \cdot \sin 85^\circ}{\sin 38^\circ} \approx 19.4$$

2. Solve the triangle using the Law of Sines. Round decimal answers to the nearest tenth.

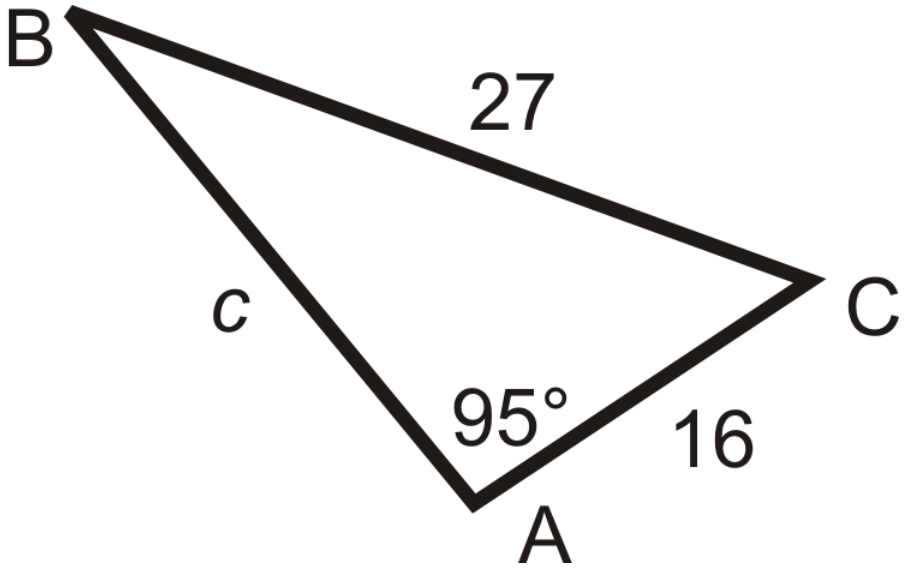


Figure 4.1.1.3

Set up the ratio for $\angle B$ using Law of Sines.

$$\frac{\sin 95^\circ}{27} = \frac{\sin B}{16}$$
$$27 \cdot \sin B = 16 \cdot \sin 95^\circ$$
$$\sin B = \frac{16 \cdot \sin 95^\circ}{27} \rightarrow \sin^{-1}\left(\frac{16 \cdot \sin 95^\circ}{27}\right) = 36.2^\circ$$

To find $m\angle C$ use the Triangle Sum Theorem.

$$m\angle C + 95^\circ + 36.2^\circ = 180^\circ \rightarrow m\angle C = 48.8^\circ$$

To find c, use the Law of Sines again. $\frac{\sin 95^\circ}{27} = \frac{\sin 48.8^\circ}{c}$

$$c \cdot \sin 95^\circ = 27 \cdot \sin 48.8^\circ$$
$$c = 27 \cdot \sin 48.8^\circ \sin 95^\circ \approx 20.4$$

Using the Law of Cosines

Solve the triangle using Law of Cosines. Round your answers to the nearest hundredth.

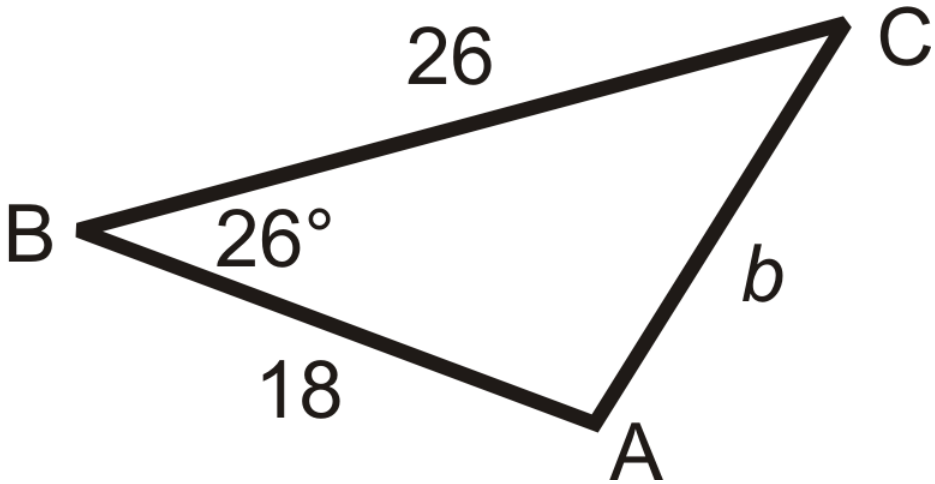


Figure 4.1.1.4

Use the second equation to solve for $\angle B$.

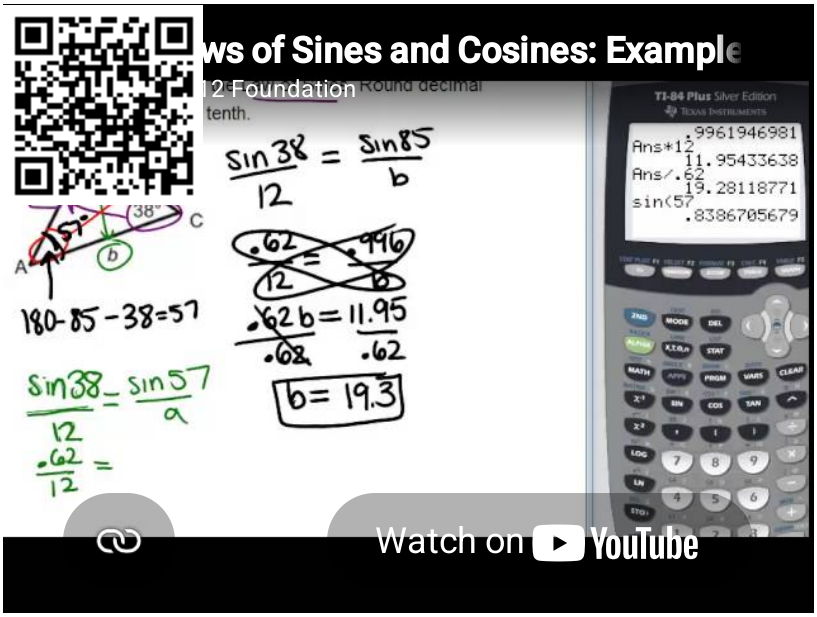
$$b^2 = 26^2 + 18^2 - 2(26)(18)\cos 26^\circ$$
$$b^2 = 1000 - 936\cos 26^\circ$$
$$b^2 = 158.7288$$
$$b \approx 12.60$$

To find $m\angle A$ or $m\angle C$, you can use either the Law of Sines or Law of Cosines. Let's use the Law of Sines.

$$\frac{\sin 26^\circ}{12.60} = \frac{\sin A}{18}$$
$$12.60 \cdot \sin A = 18 \cdot \sin 26^\circ$$
$$\sin A = 18 \cdot \sin 26^\circ 12.60$$

$\sin^{-1}(18 \cdot \sin 26^\circ 12.60) \approx 38.77^\circ$ To find $m\angle C$, use the Triangle Sum Theorem.

$$26^\circ + 38.77^\circ + m\angle C = 180^\circ$$
$$m\angle C = 115.23^\circ$$



Examples

Find the following angles in the triangle below. Round your answers to the nearest hundredth.

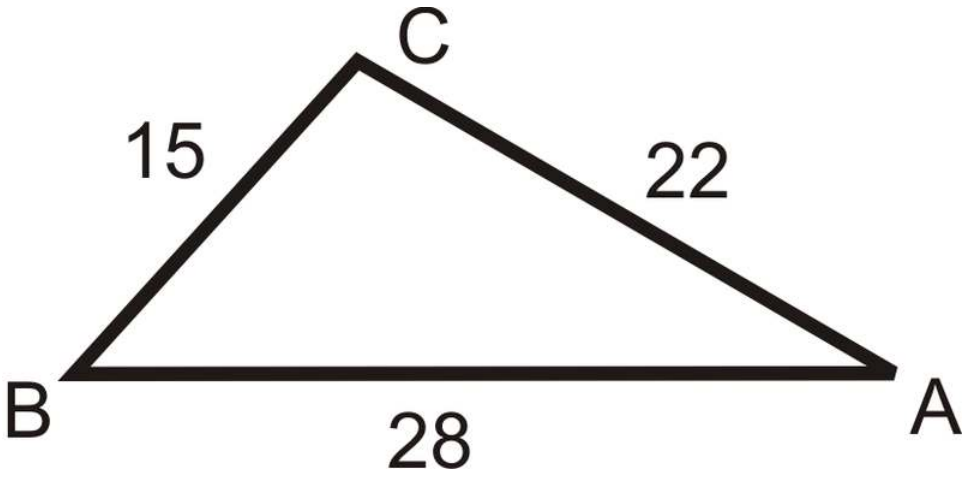


Figure 4.1.1.5

Example 4.1.1.1

$m\angle A$

Solution

When you are given only the sides, you have to use the Law of Cosines to find one angle and then you can use the Law of Sines to find another.

$$\begin{aligned} 15^2 &= 22^2 + 28^2 - 2(22)(28)\cos A \\ 225 &= 1268 - 1232\cos A \\ -1043 &= -1232\cos A \\ \frac{-1043}{-1232} &= \cos A \rightarrow \cos^{-1}\left(\frac{1043}{1232}\right) \approx 32.16^\circ \end{aligned}$$

Example 4.1.1.2

$m\angle B$

Solution

Now that we have an angle and its opposite side, we can use the Law of Sines.

$$\begin{aligned} \frac{\sin 32.16^\circ}{15} &= \frac{\sin B}{22} \\ 15 \cdot \sin B &= 22 \cdot \sin 32.16^\circ \\ \sin B &= \frac{22 \cdot \sin 32.16^\circ}{15} \end{aligned}$$

$\sin^{-1}\left(\frac{22 \cdot \sin 32.16^\circ}{15}\right) \approx 51.32^\circ$.

Example 4.1.1.3

$m\angle C$

Solution

To find $m\angle C$, use the Triangle Sum Theorem.

$$\begin{aligned} 32.16^\circ + 51.32^\circ + m\angle C &= 180^\circ \\ m\angle C &= 96.52^\circ \end{aligned}$$

Review

Use the Law of Sines or Cosines to solve ΔABC. If you are not given a picture, draw one. Round all decimal answers to the nearest tenth.

1.

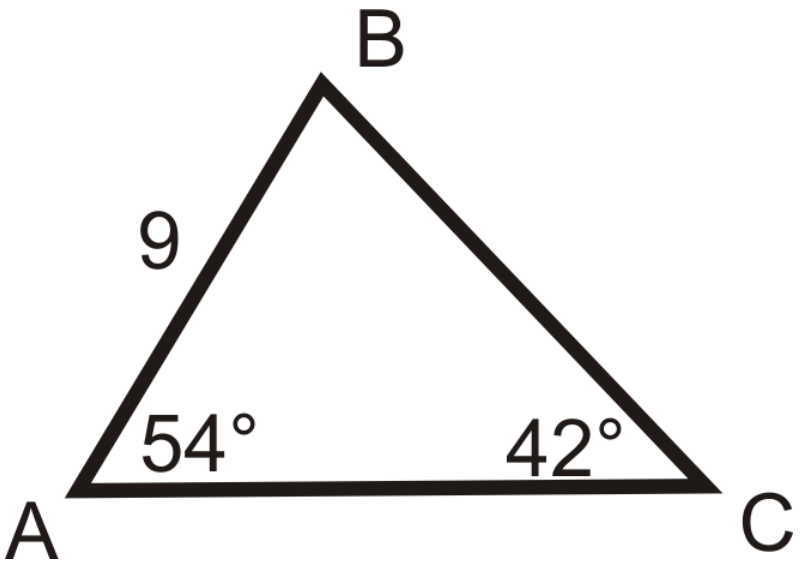


Figure **4.1.1.6**

2.

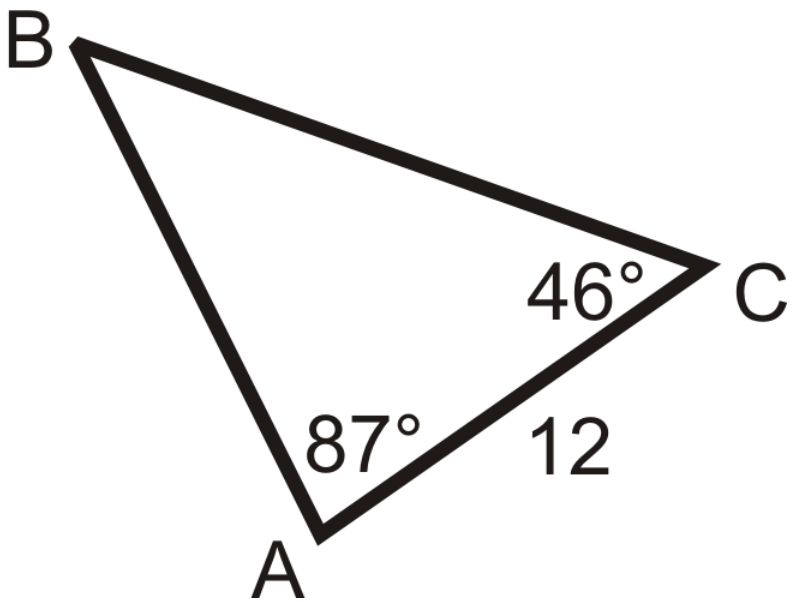


Figure **4.1.1.7**

3.

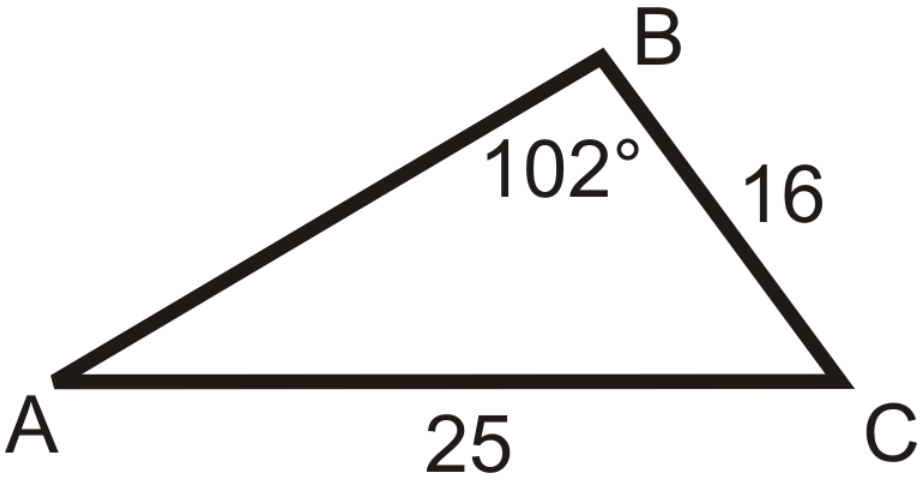


Figure 4.1.1.8

4.

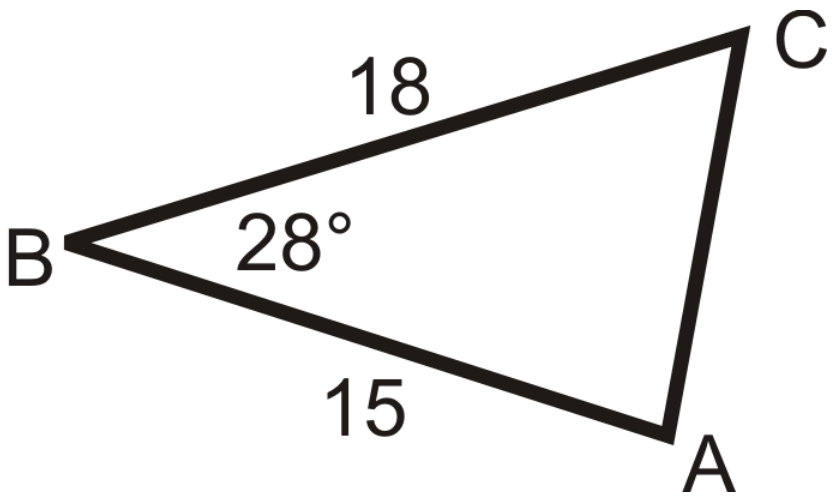


Figure 4.1.1.9

5.

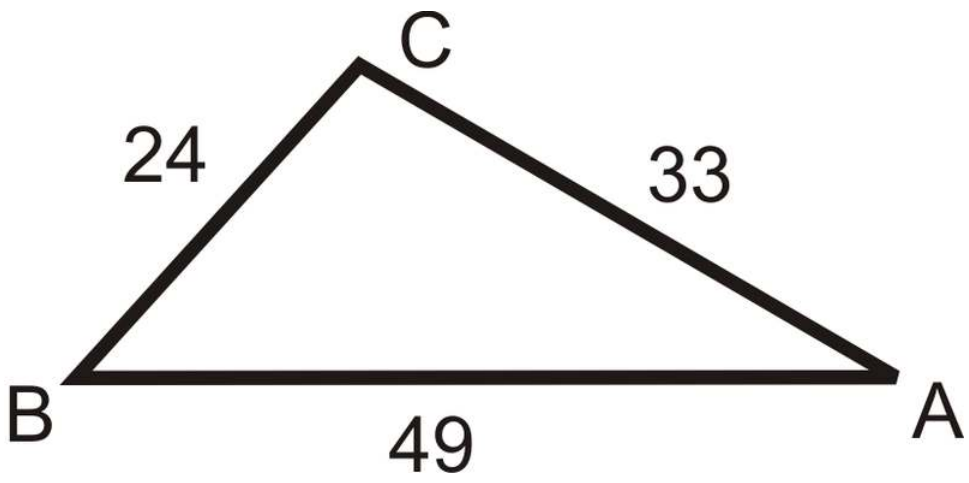


Figure 4.1.1.10

6.

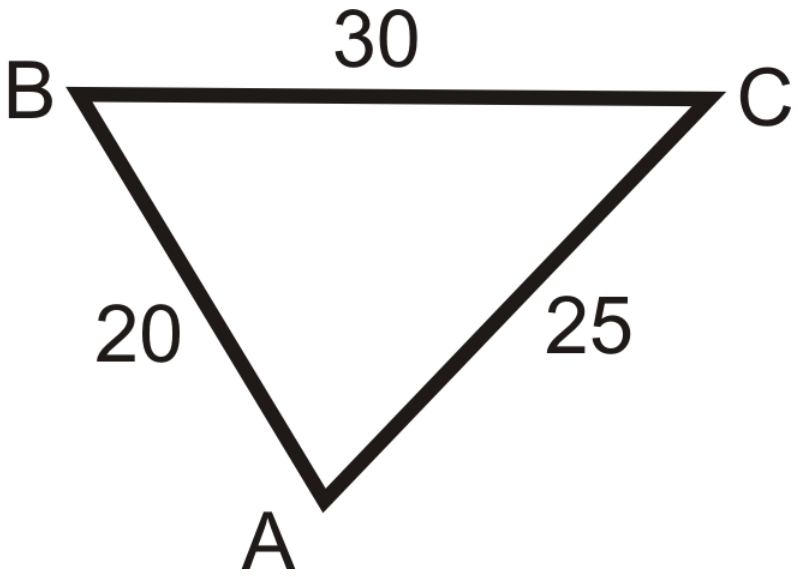


Figure 4.1.1.11

7.

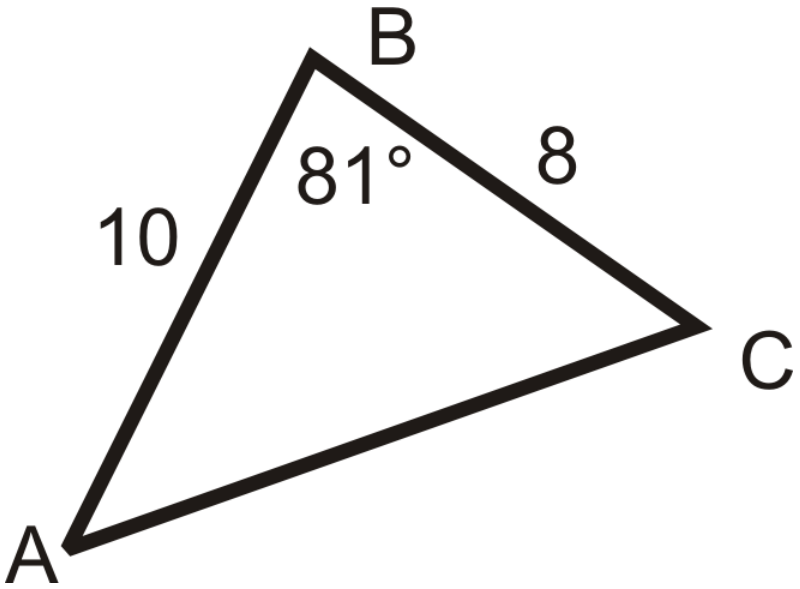


Figure 4.1.1.12

8.

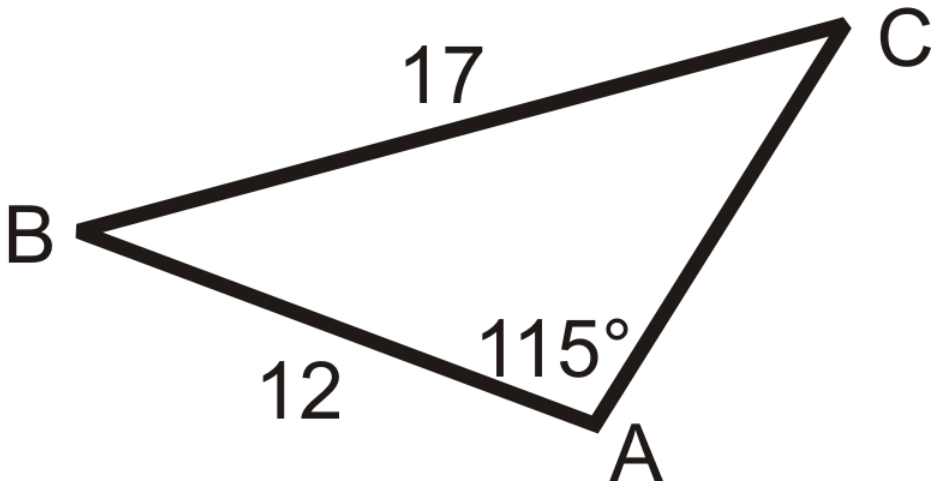


Figure 4.1.1.13

9.

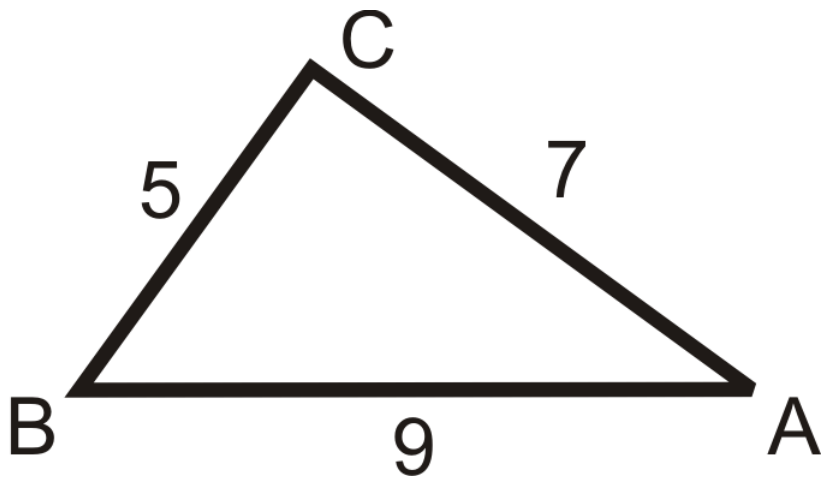


Figure 4.1.1.14

10. $m\angle A = 74°$, $m\angle B = 11°$, $BC = 16$
11. $m\angle A = 64°$, $AB = 29$, $AC = 34$
12. $m\angle C = 133°$, $m\angle B = 25°$, $AB = 48$

Use the Law of Sines to solve ΔABC below.

13. $m\angle A = 20°$, $AB = 12$, $BC = 5$

Recall that when we learned how to prove that triangles were congruent we determined that SSA (two sides and an angle not included) did not determine a unique triangle. When we are using the Law of Sines to solve a triangle and we are given two sides and the angle not included, we may have two possible triangles. Problem 14 illustrates this.

14. Let's say we have ΔABC as we did in problem 13. In problem 13 you were given two sides and the not included angle. This time, you have two angles and the side between them (ASA). Solve the triangle given that $m\angle A = 20°$, $m\angle C = 125°$, $AC = 8.4$
15. Does the triangle that you found in problem 14 meet the requirements of the given information in problem 13? How are the two different $m\angle C$ related? Draw the two possible triangles overlapping to visualize this relationship.

Review (Answers)

To view the Review answers, open this PDF file and look for section 8.10.

Additional Resources

Video: The Law of Sines: The Basics

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4.1.2: Law of Sines

Proportion based on ratios of sides and sines of the opposite angles for non-right triangles.

When given a right triangle, you can use basic trigonometry to solve for missing information. When given SSS or SAS, you can use the Law of Cosines to solve for the missing information. But what happens when you are given two sides of a triangle and an angle that is not included? There are many ways to show two triangles are congruent, but **SSA** is not one of them. Why not?

The Law of Sines

When given two sides and an angle that is not included between the two sides, you can use the Law of Sines. The **Law of Sines** states that in every triangle the ratio of each side to the \sin e of its corresponding angle is always the same. Essentially, it clarifies the general concept that opposite the largest angle is always the longest side.

a\sin A=b\sin B=c\sin C

Here is a proof of the Law of Sines:

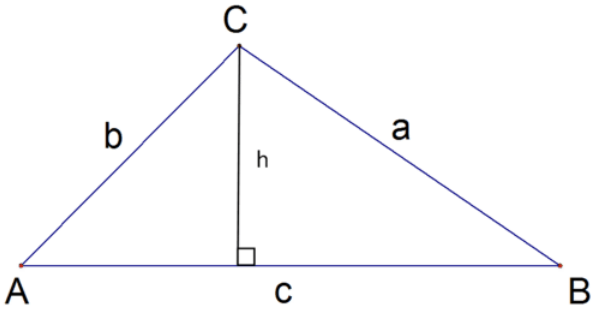


Figure **4.1.2.1**

Looking at the right triangle formed on the left:

$$\sin A = \frac{h}{b}$$
$$h = b \sin A$$

Looking at the right triangle formed on the right:

$$\sin B = \frac{h}{a}$$
$$h = a \sin B$$

Equating the heights which must be identical:

$$a \sin B = b \sin A$$
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

The best way to use the Law of Sines is to draw an extremely consistent picture each and every time even if that means redrawing and relabeling a picture. The reason why the consistency is important is because sometimes given SSA information defines zero, one or even two possible triangles.

Always draw the given angle in the bottom left with the two given sides above.

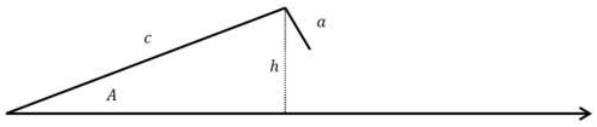


Figure **4.1.2.2**

In this image side a is deliberately too short, but in most problems you will not know this. You will need to compare a to the height.

$$\sin A = \frac{h}{c}$$
$$h = c \sin A$$

This is commonly referred to testing the **ambiguous** case. There are four different tests to determine the number of triangles that exist given the measurements.

Case 1: $a < h$

Simply put, side a is not long enough to reach the opposite side and constructing the triangle is impossible. Zero triangles exist.

Case 2: $a = h$

Side a just barely reaches the opposite side forming a 90^{\circ} angle.

Case 3: $h < a < c$

In this case side a can swing toward the interior of the triangle or the exterior of the triangle- there are two possible triangles. This is called the ambiguous case because the given information does not uniquely identify one triangle. To solve for both triangles, use the Law of Sines to solve for angle C_1 first and then use the supplement to determine C_2.

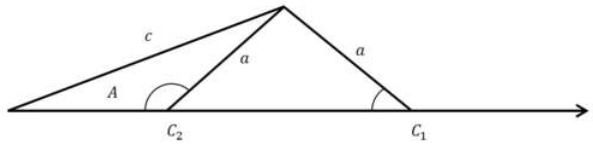


Figure **4.1.2.3**

Case 4: $c \leq a$

In this case, side a can only swing towards the exterior of the triangle, only producing C_1.

For the case of SSA, you should always check how many triangles there are before starting to find measures. Take the following triangle:

$\angle A = 40°$, $c = 13$, and $a = 2$.

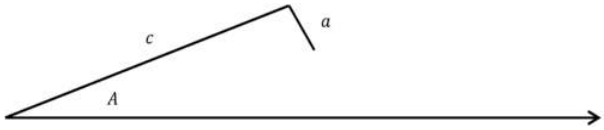
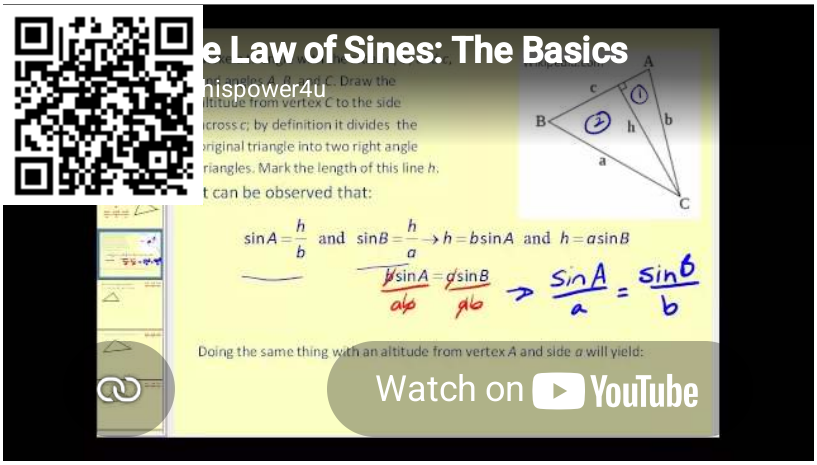


Figure **4.1.2.4**

Before trying find $\angle C$, you need to check that a triangle is possible and if there is more than one solution. Use the equation from above,

$$\sin 40° = \frac{h}{13}$$
$$h = 13 \sin 40° \approx 8.356$$

Because a<h (2<8.356), this information does not form a proper triangle.



Example 4.1.2.1

Earlier, you were asked why SSA is not a method to show that two triangles are congruent.

Solution

SSA is not a method from Geometry that shows two triangles are congruent because it does not always define a unique triangle. Sometimes there is no triangle, one triangle, or two triangles.

Example 4.1.2.2

$\angle A = 17°$, $c = 14$, and $a = 4.0932\ldots$ If possible, find $\angle C$.

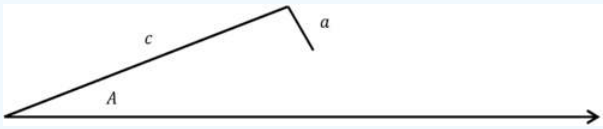


Figure 4.1.2.5

Solution

Check that a triangle is possible:

$$\sin 17° = \frac{h}{14}$$
$$h = 14\sin 17° \approx 4.0932$$

Since $a = h$, this information forms exactly one triangle and angle C must be $90°$.

Example 4.1.2.3

$\angle A = 22°$, $c = 11$ and $a = 9$. If possible, find $\angle C$.

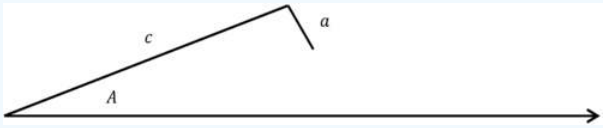


Figure 4.1.2.6

Solution

Check that a triangle is possible:

$$\sin 22° = \frac{h}{11}$$
$$h = 11\sin 22° \approx 4.12$$

Since $h < a < c$, there must be two possible angles for angle C.

Apply the Law of Sines:

$$\frac{9}{\sin 22^\circ} = \frac{11}{\sin C_1}$$
$$9 \sin C_1 = 11 \sin 22^\circ$$
$$\sin C_1 = \frac{11 \sin 22^\circ}{9}$$
$$C_1 = \sin^{-1}\left(\frac{11 \sin 22^\circ}{9}\right) \approx 27.24^\circ$$
$$C_2 = 180 - C_1 \approx 152.75^\circ$$

Example 4.1.2.4

Given \Delta ABC where A=12^{\circ}, $B = 50^\circ$, \(a=14 find b.

Solution

$$\frac{14}{\sin 12^\circ} = \frac{b}{\sin 50^\circ}$$
$$b = \frac{14 \sin 50^\circ}{\sin 12^\circ} \approx 51.58$$

Example 4.1.2.5

Given ΔABC where $A = 70^\circ$, $b = 8$, $a = 3$, find $\angle B$ if possible.

Solution

$$\sin 70^\circ = \frac{h}{8}$$
$$h = 8 \sin 70^\circ \approx 7.51 \ldots$$

Because $a < h$, this triangle is impossible.

Review

For 1-3, draw a picture of the triangle and state how many triangles could be formed with the given values.

1. $A = 30^\circ$, $a = 13$, $b = 15$
2. $A = 22^\circ$, $a = 21$, $b = 12$
3. $A = 42^\circ$, $a = 36$, $b = 37$

For 4-7, find all possible measures of \angle B (if any exist) for each of the following triangle values.

4. $A = 86^\circ$, $a = 15$, $b = 11$
5. $A = 30^\circ$, $a = 24$, $b = 43$
6. $A = 48^\circ$, $a = 34$, $b = 39$
7. $A = 80^\circ$, $a = 22$, $b = 20$

For 8-12, find the length of b for each of the following triangle values.

8. $A = 94^\circ$, $a = 31$, $B = 34^\circ$
9. $A = 112^\circ$, $a = 12$, $B = 15^\circ$
10. $A = 78^\circ$, $a = 20$, $B = 16^\circ$
11. $A = 54^\circ$, $a = 15$, $B = 112^\circ$
12. $A = 39^\circ$, $a = 9$, $B = 98^\circ$
13. In ΔABC, $b = 10$ and $\angle A = 39^\circ$. What's a possible value for a that would produce two triangles?
14. In ΔABC, $b = 10$ and $\angle A = 39^\circ$. What's a possible value for a that would produce no triangles?
15. In ΔABC, $b = 10$ and $\angle A = 39^\circ$. What's a possible value for a that would produce one triangle?

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.6.

Vocabulary

Term	Definition
ambiguous	Ambiguous means that the given information is not specific. In the context of Geometry or Trigonometry, it means that the given data may not uniquely identify one shape.
SSA	SSA means side, side, angle and refers to the fact that two sides and the non-included angle of a triangle are known in a problem.

Additional Resource

Video: Solving Multivariable Linear Systems - Overview

Practice: Law of Sines

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4.1.3: Angle-Angle-Side Triangles

Law of sines: proportion based on ratios of sides and sines of the opposite angles.

You and a friend decide to go fly kites on a breezy Saturday afternoon. While sitting down to make your kites, you are working on make the best shape possible to catch the breeze. While your friend decides to go with a diamond shaped kite, you try out making a triangle shaped one. While trying to glue the kite together, you make the first and second piece lock together with a $70°$ angle. The angle between the first and third pieces is $40°$. Finally, you also have measured the length of the second piece and found that it is 22 inches long.

Your kite looks like this:

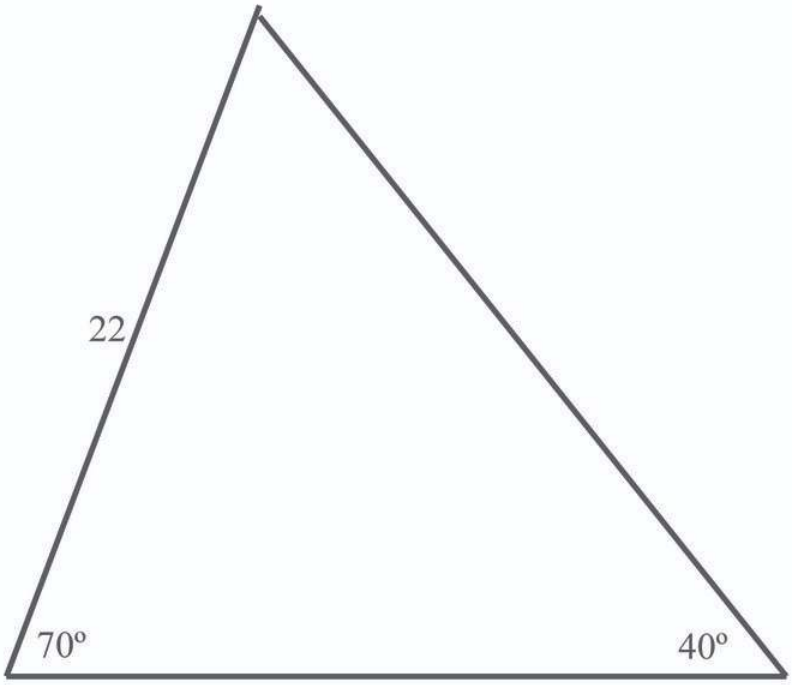
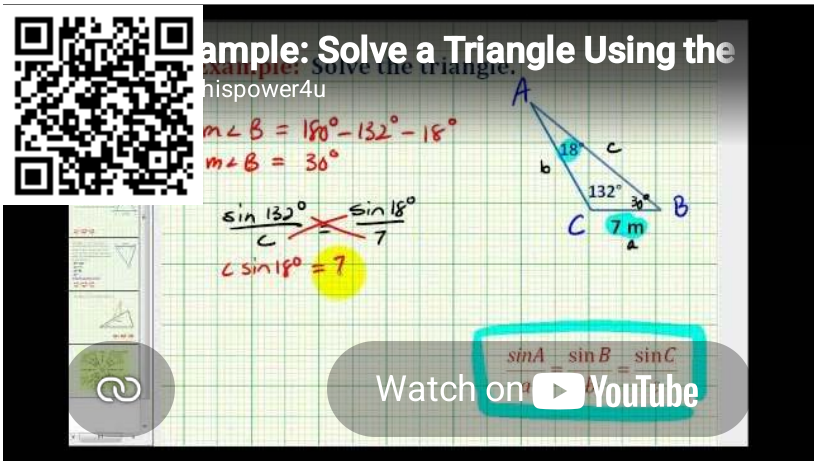


Figure 4.1.3.1

Is there a way to find out, using math, what the length of the third side will be?

AAS Triangles

The Law of sines states: $\frac{\sin A}{a} = \frac{\sin B}{b}$. This is a ratio between the sine of an angle in a triangle and the length of the side opposite that angle to the sine of a different angle in that triangle and the length of the side opposing that second angle.



The Law of sines allows us to find many quantities of interest in triangles by comparing sides and interior angles as a ratio. One case where we can to use the Law of sines is when we know two of the angles in a triangle and a non-included side (AAS).

Using the Law of sines

1. using ΔGMN, $\angle G = 42°$, $\angle N = 73°$ and $g = 12$. Find n.

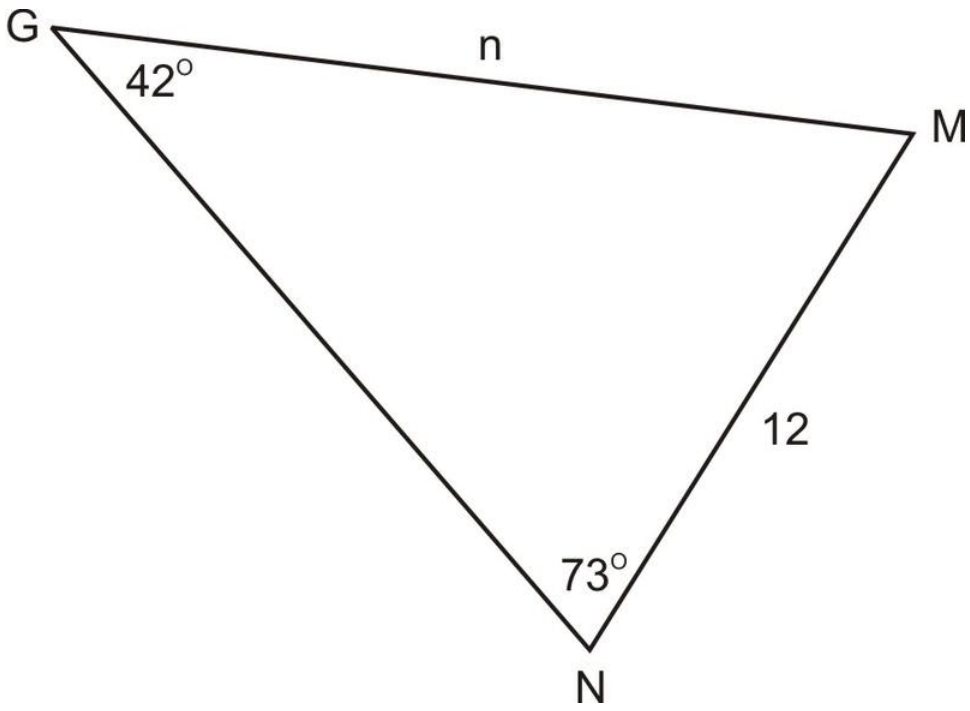


Figure 4.1.3.2

since we know two angles and one non-included side (g), we can find the other non-included side (n).

$$\frac{\sin 73°}{n} = \frac{\sin 42°}{12}$$
$$n \sin 42° = 12 \sin 73°$$
$$n = \frac{12 \sin 73°}{\sin 42°}$$
$$n \approx 17.15$$

2. Continuing on from #1, find $\angle M$ and m.

$\angle M$ is simply $180° - 42° - 73° = 65°$. To find side m, you can now use either the Law of sines or Law of Cosines. Considering that the Law of sines is a bit simpler and new, let's use it. It does not matter which side and opposite angle you use in the ratio with $\angle M$

and m.

Option 1: $\angle G$ and g

$$\frac{\sin 65^\circ}{m} = \frac{\sin 42^\circ}{12}$$
$$m \sin 42^\circ = 12 \sin 65^\circ$$
$$m = \frac{12 \sin 65^\circ}{\sin 42^\circ}$$
$$m \approx 16.25$$

Option 2: $\angle N$ and n

$$\frac{\sin 65^\circ}{m} = \frac{\sin 73^\circ}{17.15}$$
$$m \sin 73^\circ = 17.15 \sin 65^\circ$$
$$m = \frac{17.15 \sin 65^\circ}{\sin 73^\circ}$$
$$m \approx 16.25$$

3. A business group wants to build a golf course on a plot of land that was once a farm. The deed to the land is old and information about the land is incomplete. If AB is 5382 feet, BC is 3862 feet, $\angle AEB$ is 101°, $\angle BDC$ is 74°, $\angle EAB$ is 41° and $\angle DCB$ is 32°, what are the lengths of the sides of each triangular piece of land? What is the total area of the land?

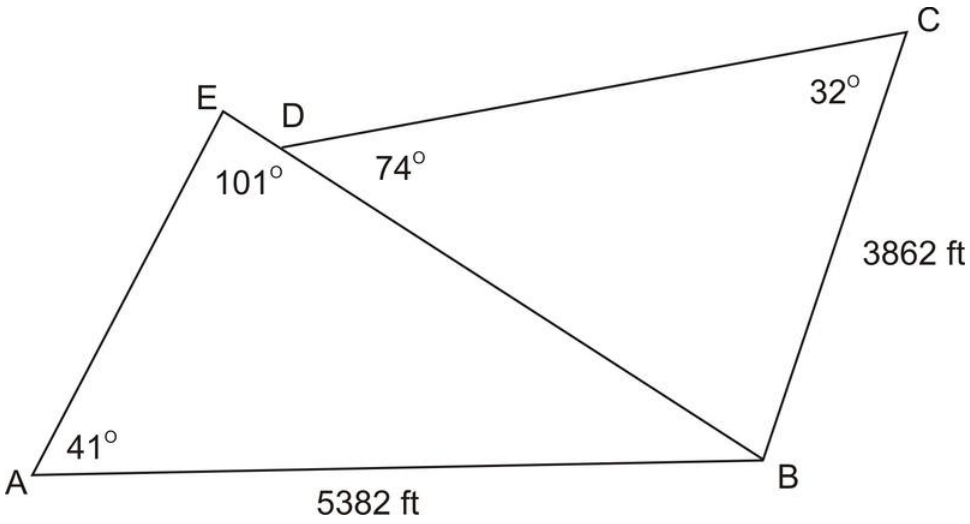


Figure 4.1.3.3

Before we can figure out the area of the land, we need to figure out the length of each side. In ΔABE, we know two angles and a non-included side. This is the AAS case. First, we will find the third angle in ΔABE by using the Triangle Sum Theorem. Then, we can use the Law of sines to find both AE and EB.

$$\frac{\sin 101}{5382} = \frac{\sin 38}{AE}$$
$$AE(\sin 101) = 5382(\sin 38)$$
$$AE = \frac{5382(\sin 38)}{\sin 101}$$
$$AE = 3375.5 \text{ feet}$$

$$\frac{\sin 101}{5382} = \frac{\sin 41}{EB}$$
$$EB(\sin 101) = 5382(\sin 41)$$
$$EB = \frac{5382(\sin 41)}{\sin 101}$$
$$EB \approx 3597.0 \text{ feet}$$

Next, we need to find the missing side lengths in ΔDCB. In this triangle, we again know two angles and a non-included side (AAS), which means we can use the Law of sines. First, let's find $\angle DBC = 180 - (74 + 32) = 74^\circ$. since both $\angle BDC$ and $\angle DBC$ measure 74°, ΔDCB is an isosceles triangle. This means that since BC is 3862 feet, DC is also 3862 feet. All we have left to find now is DB.

$$\frac{\sin 74}{3862} = \frac{\sin 32}{DB}$$
$$DB(\sin 74) = 3862(\sin 32)$$
$$DB = \frac{3862(\sin 32)}{\sin 74}$$
$$DB \approx 2129.0 \text{ feet}$$

Finally, we need to calculate the area of each triangle and then add the two areas together to get the total area. From the last section, we learned two area formulas, $K = \frac{1}{2}bc\sin A$ and Heron's Formula. In this case, since we have enough information to use either formula, we will use $K = \frac{1}{2}bc\sin A$ since it is less computationally intense.

First, we will find the area of ΔABE.

ΔABE:

$$K = 12(3375.5)(5382)\sin 41$$
$$K = 5,959,292.8 \text{ ft}^2$$

ΔDBC:

$$K = \frac{1}{2}(3862)(3862)\sin 32$$
$$K = 3,951,884.6 \text{ ft}^2$$

The total area is $5,959,292.8 + 3,951,884.6 = 9,911,177.4 \text{ ft}^2$.

Example 4.1.3.1

Earlier, you were asked to find the length of the third side of the triangle.

Solution

since you know two angles and one non-included side of the kite, you can find the other non-included side using the Law of sines. Set up a ratio using the angles and side you know and the side you don't know.

$$\frac{\sin 70^\circ}{x} = \frac{\sin 40^\circ}{22}$$
$$x = \frac{22\sin 70^\circ}{\sin 40^\circ}$$
$$x \approx 32.146$$

The length of the dowel rod on the unknown side will be approximately 32 inches.

Example 4.1.3.2

Find side "d" in the triangle below with the following information: $e = 214.9$, $D = 39.7^\circ$, $E = 41.3^\circ$

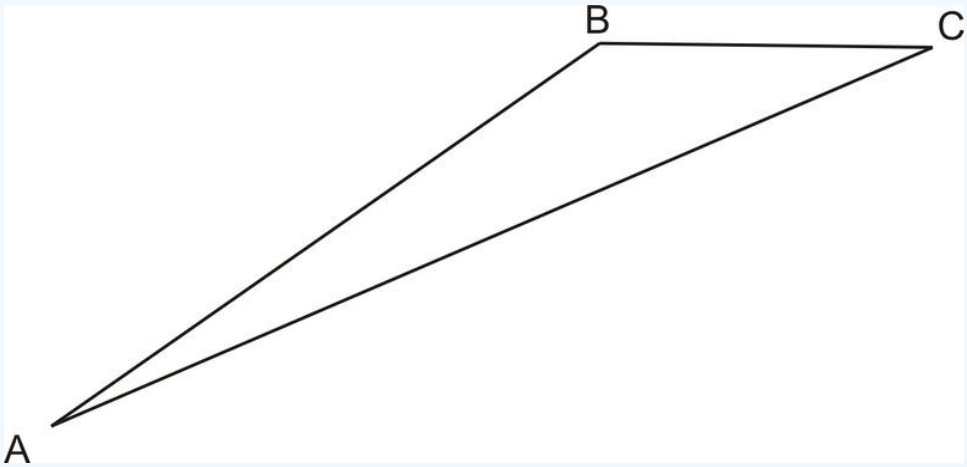


Figure 4.1.3.4

Solution

$\frac{\sin 41.3^\circ}{214.9} = \frac{\sin 39.7^\circ}{d}$, $d = 208.0$

Example 4.1.3.3

Find side "o" in the triangle below with the following information: $M = 31^\circ$, $O = 9^\circ$, $m = 15$

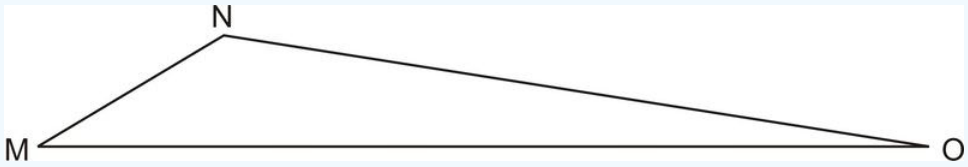


Figure 4.1.3.5

Solution

$\frac{\sin 9^\circ}{o} = \frac{\sin 31^\circ}{15}$, $o = 4.6$

Example 4.1.3.4

Find side "q" in the triangle below with the following information: $Q = 127^\circ$, $R = 21.8^\circ$,$r = 3.62$

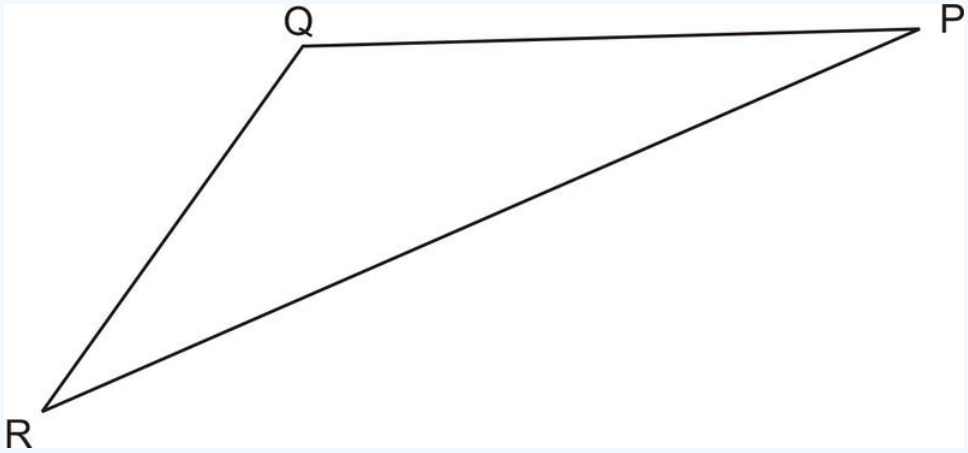


Figure 4.1.3.6

Solution

$\frac{\sin 127^\circ}{q} = \frac{\sin 21.8^\circ}{3.62}$, $q = 7.8$

Review

In ΔABC, $m\angle A = 50^\circ$, $m\angle B = 34^\circ$, and $a = 6$.

1. Find the length of b.
2. Find the length of c.

In ΔKMS, $m\angle K = 42^\circ$, $m\angle M = 26^\circ$, and $k = 14$.

3. Find the length of m.
4. Find the length of s.

In ΔDEF, $m\angle D = 52^\circ$, $m\angle E = 78^\circ$, and $d = 23$.

5. Find the length of e.
6. Find the length of f.

In ΔPQR, $m\angle P = 2^\circ$, $m\angle Q = 79^\circ$, and $p = 20$.

7. Find the length of q.

8. Find the length of r.

In ΔDOG, $m\angle D = 50°$, $m\angle G = 59°$, and $o = 12$.

9. Find the length of d.
10. Find the length of g.

In ΔCAT, $m\angle C = 82°$, $m\angle T = 4°$, and a=8\).

11. Find the length of c\).
12. Find the length of t\).

In ΔYOS, $m\angle Y = 65°$, $m\angle O = 72°$, and $s = 15$.

13. Find the length of o.
14. Find the length of y.

In ΔHCO, $m\angle H = 87°$, $m\angle C = 14°$, and $o = 19$.

15. Find the length of h.
16. Find the length of c.

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.7.

Vocabulary

Term	Definition
Angle Angle Side Triangle	An 'angle angle side triangle' is a triangle where two of the angles and the non-included side are known quantities.

Additional Resources

Video: Example: Solving a Triangle using the Law of sines Given Two Angles and One Side

Practice: Angle-Angle-Side Triangles

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4.1.4: Angle-Side-Angle Triangles

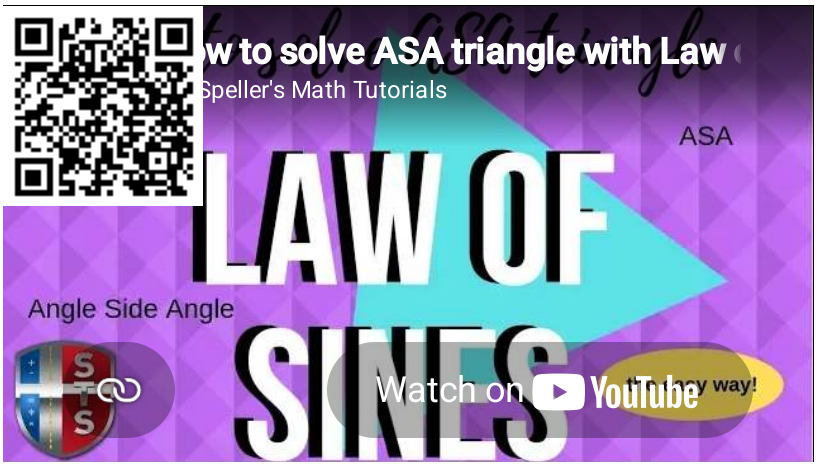
Law of sines given two angles and the included side.

You're eating lunch in the cafeteria one afternoon while working on your math homework. Lately you seem to notice the triangular shapes in everything. At home, at school, with your friends. It seems like triangles are everywhere. And you find yourself trying to apply what you are learning in math class to all of the triangles around you. And today is no exception. As you start to take a bite of your chip, you suddenly recognize that familiar shape - the triangle.

You estimate the length of one of the sides of the chip to be 3 cm. You also can tell that the angle adjacent to the 3 cm side is $50°$ and the angle adjacent on the other side of the 3 cm edge is $60°$. Can you find the lengths of the other two sides using techniques from your math class?

ASA Triangles

The Law of Sines states: $\frac{\sin A}{a} = \frac{\sin B}{b}$. This is a ratio between the sine of an angle in a triangle and the length of the side opposite that angle to the sine of a different angle in that triangle and the length of the side opposing that second angle.



One case where we use the Law of Sines is when we know two angles in a triangle and the **included** side (ASA). For instance, in ΔTRI:

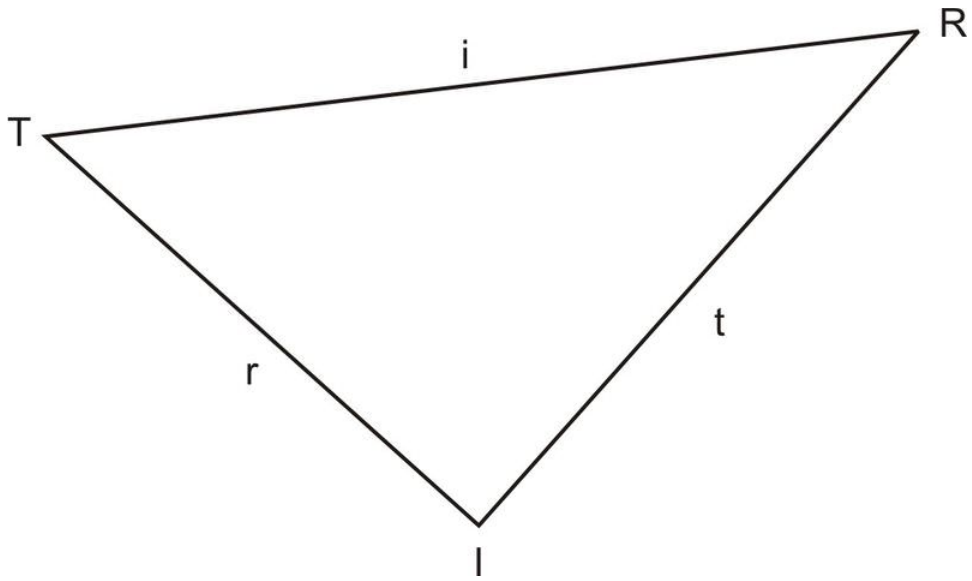


Figure 4.1.4.1

$\angle T$, $\angle R$, and i are known

$\angle T$, $\angle I$, and r are known

$\angle R$, $\angle I$, and t are known

In this case, the Law of Sines allows us to find either of the non-included sides.

Using the Law of Sines

1. In the triangle above, ΔTRI, $\angle T = 83°$, $\angle R = 24°$, and $i = 18.5$. Find the measure of t.

Since we know two angles and the included side, we can find either of the non-included sides using the Law of Sines. Since we already know two of the angles in the triangle, we can find the third angle using the fact that the sum of all of the angles in a triangle must equal $180°$.

$$\begin{aligned} \angle I &= 180 - (83 + 24) \\ \angle I &= 180 - 107 \\ \angle I &= 73° \end{aligned}$$

Now that we know $\angle I = 73°$, we can use the Law of Sines to find t.

$$\begin{aligned} \frac{\sin 73}{18.5} &= \frac{\sin 83}{t} \\ t(\sin 73) &= 18.5(\sin 83) \\ t &= \frac{18.5(\sin 83)}{\sin 73} \\ t &\approx 19.2 \end{aligned}$$

Notice how we wait until the last step to input the values into the calculator. This is so our answer is as accurate as possible.

2. In order to avoid a large and dangerous snowstorm on a flight from Chicago to Buffalo, pilot John starts out $27°$ off of the normal flight path. After flying 412 miles in this direction, he turns the plane toward Buffalo. The angle formed by the first flight course and the second flight course is $88°$. For the pilot, two issues are pressing:

1. What is the total distance of the modified flight path?
2. How much further did he travel than if he had stayed on course?

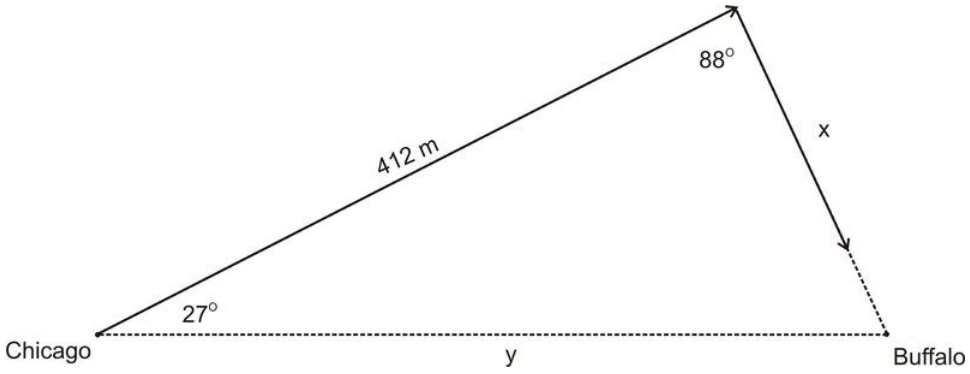


Figure 4.1.4.2

In order to find the total distance of the modified flight path, we need to know side x. To find side x, we will need to use the Law of Sines. Since we know two angles and the included side, this is an ASA case. Remember that in the ASA case, we need to first find the third angle in the triangle.

$$\begin{aligned} \text{Missing Angle} &= 180 - (27 + 88) = 65° && \text{The sum of angles in a triangle is } 180 \\ \frac{\sin 65}{412} &= \frac{\sin 27}{x} && \text{Law of Sines} \\ x(\sin 65) &= 412(\sin 27) && \text{Cross multiply} \\ x &= \frac{412(\sin 27)}{\sin 65} && \text{Divide by } \sin 65 \\ x &\approx 206.4 \text{ miles} \end{aligned}$$

The total distance of the modified flight path is $412 + 206.4 = 618.4$ miles.

To find how much farther John had to travel, we need to know the distance of the original flight path, y. We can use the Law of Sines again to find y.

$$\begin{aligned} \frac{\sin 65}{412} &= \frac{\sin 88}{y} && \text{Law of Sines} \\ y(\sin 65) &= 412(\sin 88) && \text{Cross multiply} \\ y &= \frac{412(\sin 88)}{\sin 65} && \text{Divide by } \sin 65 \\ y &\approx 454.3 \text{ miles} \end{aligned}$$

John had to travel $618.4 - 454.3 = 164.1$ miles farther.

2. In the triangle shown here:

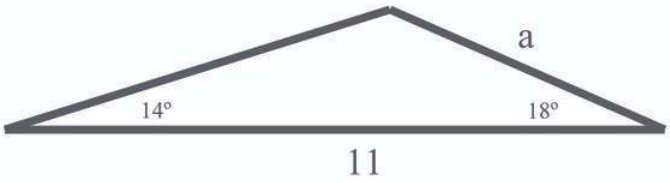


Figure 4.1.4.3

The sides given are $\angle A = 14°$, $\angle B = 18°$, and $c = 11$. Find the length of side "a".

Since we know two angles and the included side, we can find either of the non-included sides using the Law of Sines. Since we already know two of the angles in the triangle, we can find the third angle using the fact that the sum of all of the angles in a triangle must equal $180°$.

$$\begin{aligned} \angle C &= 180 - (18 + 14) \\ \angle C &= 180 - 32 \\ \angle C &= 148° \end{aligned}$$

Use the Law of Sines to find the length of side "a":

$$\begin{aligned} \frac{\sin 14}{a} &= \frac{\sin 148}{11} \\ 11(\sin 14) &= a(\sin 148) \\ a &= \frac{11(\sin 14)}{\sin 148} \\ a &\approx 5.02 \end{aligned}$$

Example 4.1.4.1

Earlier, you were asked to find the other two lengths of the triangle.

Solution

You can use the Law of Sines to find the length of either of the other 2 sides. However, first it is good to note that since the sum of the interior angles of a triangle must equal $180°$, the third angle in the triangle must measure $180° - 50° - 60° = 70°$.

Now to set up the ratios:

$$\frac{\sin 60}{a} = \frac{\sin 70}{3}$$
$$3(\sin 60) = a(\sin 70)$$
$$a = \frac{3(\sin 60)}{\sin 70}$$
$$a \approx 2.7647$$

The length of one of the other 2 sides is approximately 2.7647 centimeters.

To find the length of the last side:

$$\frac{\sin 50}{b} = \frac{\sin 70}{3}$$
$$3(\sin 50) = b(\sin 70)$$
$$b = \frac{3(\sin 50)}{\sin 70}$$
$$b \approx 2.46$$

The length of the other one of the 2 unknown sides is approximately 2.46 centimeters.

Example 4.1.4.2

Find side "a" in the triangle below using the following information: $b = 16,\ A = 11.7°,\ C = 23.8°$

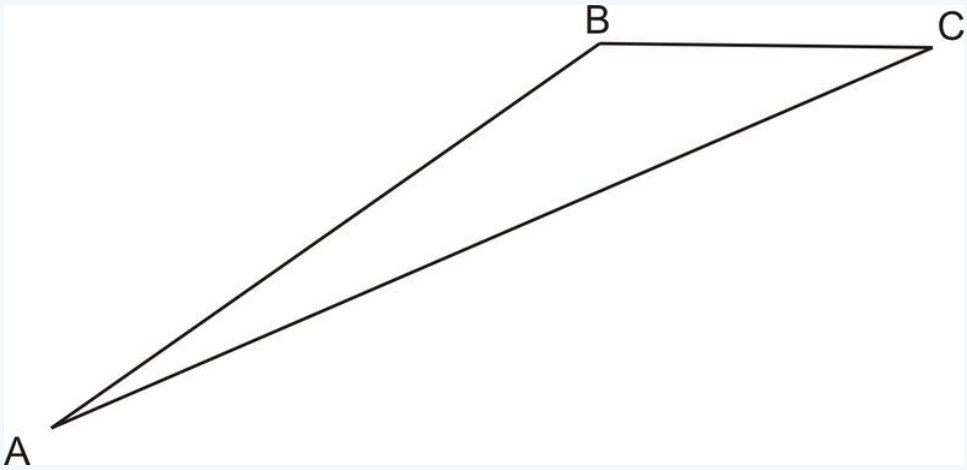


Figure 4.1.4.4

Solution

$\frac{\sin 11.7°}{a} = \frac{\sin 144.5°}{16},\ a = 5.6$

Example 4.1.4.3

Find side "a" in the triangle below using the following information: $k = 6.3,\ J = 16.2°,\ L = 40.3°$

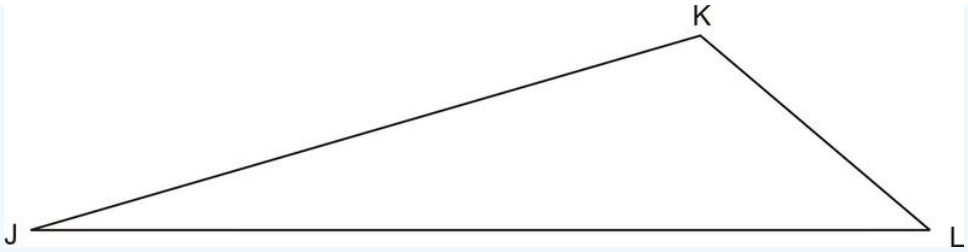


Figure 4.1.4.5

Solution

$\frac{\sin 40.3^\circ}{l} = \frac{\sin 123.5^\circ}{6.3}$, $l = 4.9$

Example 4.1.4.4

Even though ASA and AAS triangles represent two different cases of the Law of Sines, what do they both have in common?

Solution

Student answers will vary but they should notice that in both cases you know or can find an angle and the side across from it.

Review

In ΔABC, $m\angle A = 40^\circ$, $m\angle B = 67^\circ$, and $c = 6$.

1. Find $m\angle C$.
2. Find the length of a.
3. Find the length of b.

In ΔDEF, $m\angle D = 36^\circ$, $m\angle E = 101^\circ$, and $f = 11$.

4. Find $m\angle F$.
5. Find the length of d.
6. Find the length of e.

In ΔBIG, $m\angle B = 56^\circ$, $m\angle I = 71^\circ$, and $g = 23$.

7. Find $m\angle G$.
8. Find the length of b.
9. Find the length of i.

In ΔAPL, $m\angle A = 79^\circ$, $m\angle P = 40^\circ$, and $l = 15$.

10. Find $m\angle L$.
11. Find the length of a.
12. Find the length of p.

In ΔSAU, $m\angle S = 5^\circ$, $m\angle A = 99^\circ$, and $u = 21$.

13. Find $m\angle U$.
14. Find the length of s.
15. Find the length of a.

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.8.

Vocabulary

Term	Definition
Angle Side Angle Triangle	The term 'angle-side-angle triangle' refers to a triangle with known measures of two angles and the length of the side between them.

Additional Resources

Video: Law of Sines - Solving ASA Triangle

Practice: Angle-Side-Angle Triangles

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4.1.5: Side-Side-Angle- The Ambiguous Case

Evaluate triangles given two sides and one angle not between them

Possible Triangles with Side-Side-Angle

Your team has just won the flag in a flag football tournament at your school. As a reward, you get to take home the flag and keep it until the next game, when the other team will try to win it back. The flag looks like this:

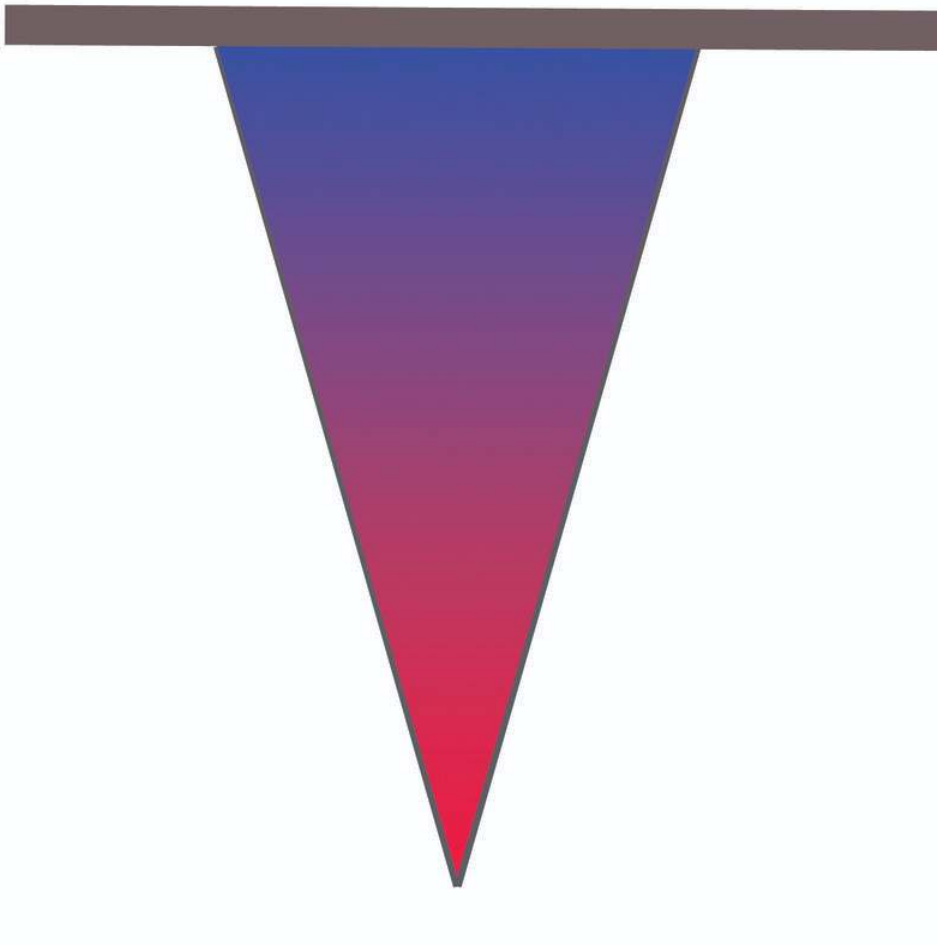
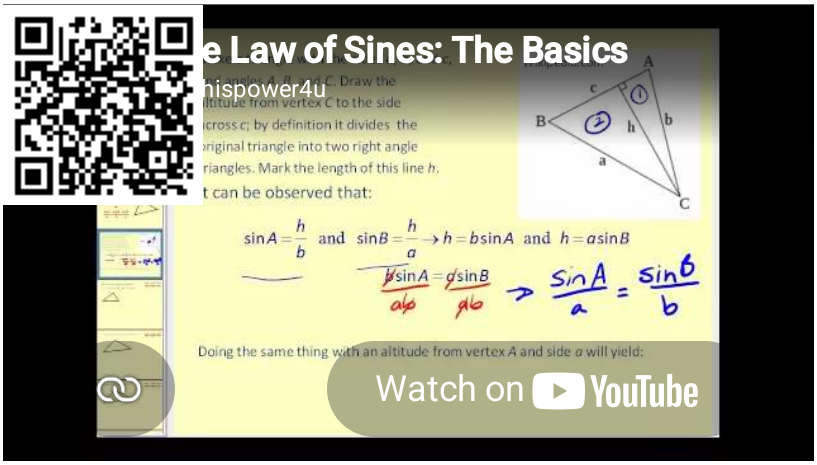


Figure **4.1.5.1**

It makes an isosceles triangle. You start to wonder how many different possible triangles there are for different lengths of sides. For example, if you make an oblique triangle that has a given angle greater than ninety degrees, how many ways are there to do this? Can you determine how many different possible triangles there are if the triangle is an isosceles triangle?

SSA Triangles

In Geometry, you learned that two sides and a non-included angle do not necessarily define a unique triangle.



Consider the following cases given a, b and $\angle A$:

Case 1: No triangle exists ($a < b$)

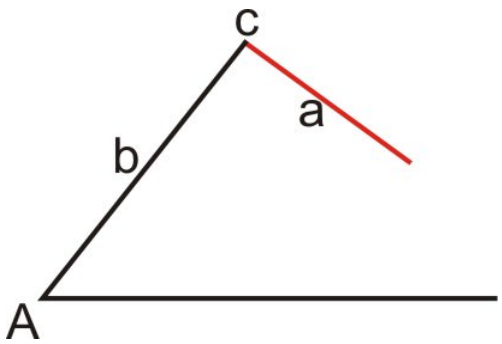


Figure **4.1.5.2**

In this case $a < b$ and side a is too short to reach the base of the triangle. Since no triangle exists, there is no solution.

Case 2: One triangle exists ($a < b$)

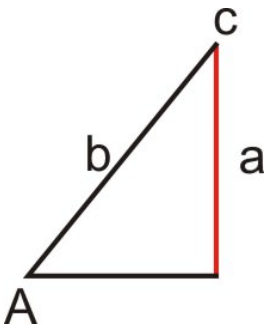


Figure **4.1.5.3**

In this case, $a < b$ and side a is perpendicular to the base of the triangle. Since this situation yields exactly one triangle, there is exactly one solution.

Case 3: Two triangles exist ($a < b$)

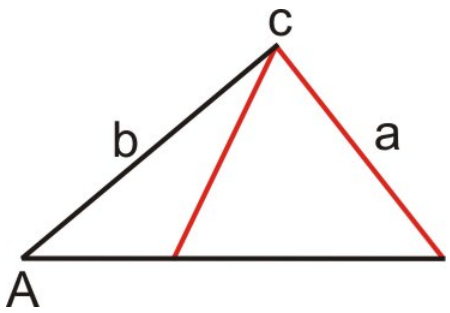


Figure **4.1.5.4**

In this case, $a < b$ and side a meets the base at exactly two points. Since two triangles exist, there are two solutions.

Case 4: One triangle exists ($a = b$)

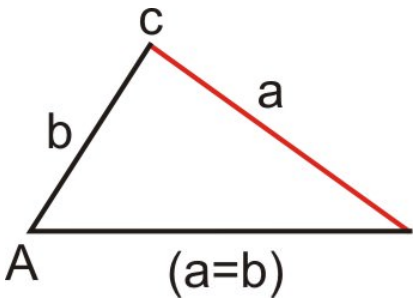


Figure **4.1.5.5**

In this case $a = b$ and side a meets the base at exactly one point. Since there is exactly one triangle, there is one solution.

Case 5: One triangle exists ($a > b$)

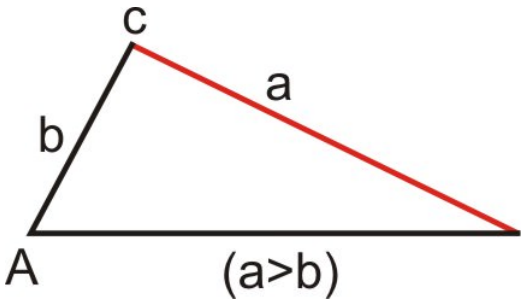


Figure **4.1.5.6**

In this case, $a > b$ and side a meets the base at exactly one point. Since there is exactly one triangle, there is one solution.

Case 3 is referred to as the Ambiguous Case because there are two possible triangles and two possible solutions. One way to check to see how many possible solutions (if any) a triangle will have is to compare sides a and b. If you are faced with the first situation, where $a < b$, we can still tell how many solutions there will be by using a and $b \sin A$.

	If:	Then:
a.	$a < b$	No solution, one solution, two solutions
i.	$a < b \sin A$	No solution
ii.	$a = b \sin A$	One solution
iii.	$a > b \sin A$	Two solutions
b.	$a = b$	One solution
c.	$a > b$	One solution

Identifying Triangles

For the following problems, determine if the sides and angles given determine no, one or two triangles.

1. The set contains an angle, its opposite side and the side between them.

$a = 5, b = 8, A = 62.19°$

$5 < 8, 8 \sin 62.19° = 7.076$. So $5 < 7.076$, which means there is no solution.

2. The set contains an angle, its opposite side and the side between them.

$c = 14, b = 10, B = 15.45°$

Even though a, b and $\angle A$ is not used in this example, follow the same pattern from the table by multiplying the non-opposite side (of the angle) by the angle.

$10 < 14, 14 \sin 15.45° = 3.73$. So $10 > 3.73$, which means there are two solutions.

3. The set contains an angle, its opposite side and the side between them.

$d = 16, g = 11, D = 44.94°$

Even though a, b and $\angle A$ is not used in this example, follow the same pattern from the table by multiplying the non-opposite side (of the angle) by the angle.

$16 > 11$, there is one solution.

Example 4.1.5.1

Earlier, you were given a problem about a triangle.

Solution

As you now know, when two sides of a triangle with an included side are known, and the lengths of the two sides are equal, there is one possible solution. Since an isosceles triangle meets these criteria, there is only one possible solution.

Example 4.1.5.2

Determine how many solutions there would be for a triangle based on the given information and by calculating $b \sin A$ and comparing it with a. Sketch an approximate diagram for each problem in the box labeled "diagram."

Solution

$A = 32.5°, a = 26, b = 37$

$A = 32.5°, a = 26, b = 37$

$26 > 19.9$

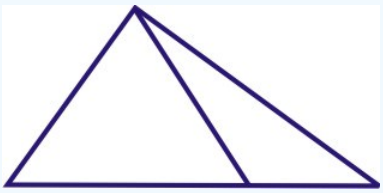


Figure 4.1.5.7

2 solutions

Example 4.1.5.3

Determine how many solutions there would be for a triangle based on the given information and by calculating $b \sin A$ and comparing it with a. Sketch an approximate diagram for each problem in the box labeled "diagram."

Solution

$A = 42.3°$, $a = 16$, $b = 26$

$A = 42.3°$, $a = 16$, $b = 26$

$16 < 17.5$

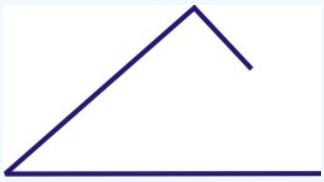


Figure 4.1.5.8

0 solutions

Example 4.1.5.1

Determine how many solutions there would be for a triangle based on the given information and by calculating $b \sin A$ and comparing it with a. Sketch an approximate diagram for each problem in the box labeled "diagram."

Solution

$A = 47.8°$, $a = 13.48$, $b = 18.2$

$A = 47.8°$, $a = 13.48$, $b = 18.2$

$13.48 = 13.48$

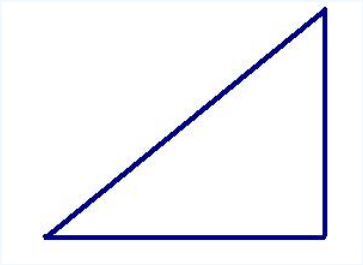


Figure 4.1.5.9

1 solution

Review

Determine if the sides and angle given determine no, one or two triangles. The set contains an angle, its opposite side and another side of the triangle.

1. $a = 6$, $b = 6$, $A = 45°$
2. $a = 4$, $b = 7$, $A = 115°$
3. $a = 5$, $b = 2$, $A = 68°$
4. $a = 7$, $b = 6$, $A = 34°$
5. $a = 5$, $b = 3$, $A = 89°$
6. $a = 4$, $b = 4$, $A = 123°$
7. $a = 6$, $b = 8$, $A = 57°$
8. $a = 4$, $b = 9$, $A = 24°$
9. $a = 12$, $b = 11$, $A = 42°$
10. $a = 15$, $b = 17$, $A = 96°$
11. $a = 9$, $b = 10$, $A = 22°$
12. In ΔABC, $a = 4$, $b = 5$, and $m\angle A = 32°$. Find the possible value(s) of c.
13. In ΔDEF, $d = 7$, $e = 5$, and $m\angle D = 67°$. Find the possible value(s) of f.

14. In ΔKQD, $m\angle K = 20°$, $k = 24$, and $d = 31$. Find $m\angle D$.
15. In ΔMRS, $m\angle M = 70°$, $m = 44$, and $r = 25$. Find $m\angle R$.

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.9.

Vocabulary

Term	Definition
Side Side Angle Triangle	A side side angle triangle is a triangle where the length of two sides and one of the angles that is not between the two sides are known quantities.

Additional Resources

Video: Determining the Amount of Triangles Resulting from an SSA Triangle

Practice: Side-Side-Angle: The Ambiguous Case

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4.1.6: Law of Cosines

Find unknown side given two sides and included angle.

Sides of an Oblique Triangle

You are playing a game called ‘‘Over the Line’’, where you stand at one corner of a triangle and hit a ball. The field looks like this:

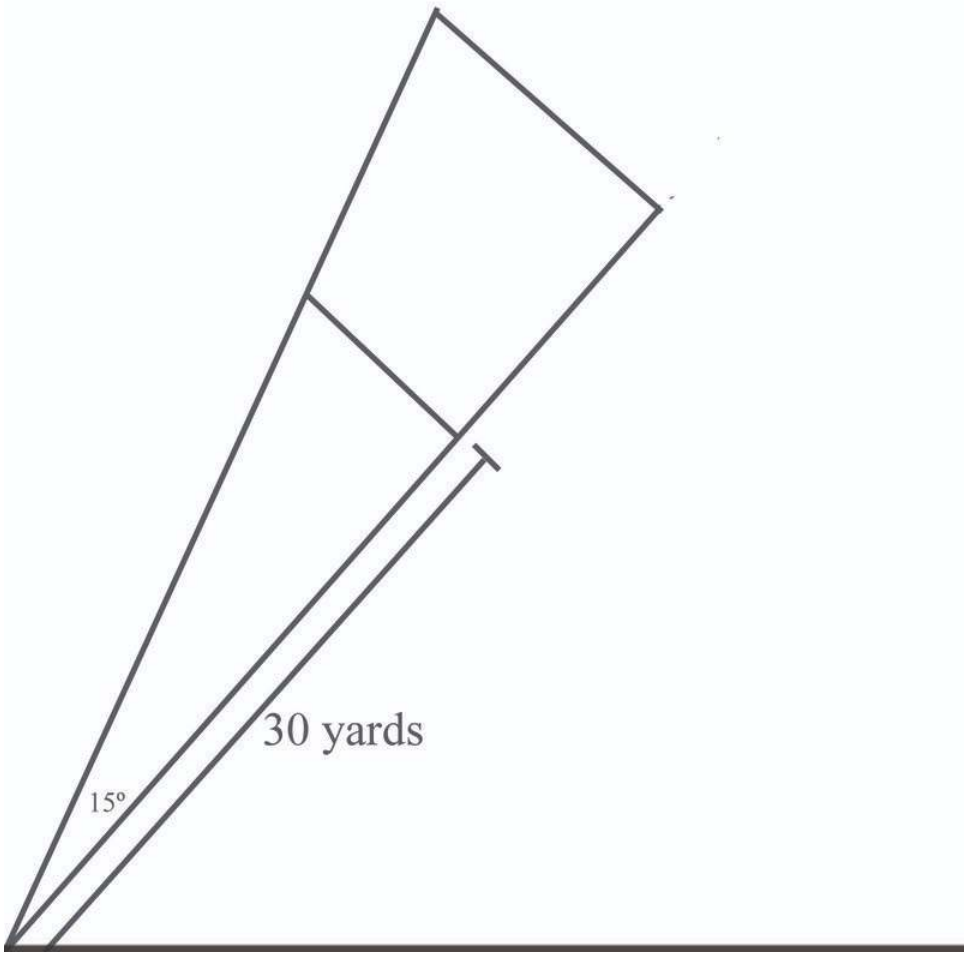


Figure **4.1.6.1**

Points are scored by hitting the ball so that it lands beyond the first line in the triangle, but before the second line.

Given that the angle on the left hand side of the triangle is **15°**, and the length of the sides of the triangle going to the first scoring line are 30 yards, can you calculate the length of the line you have to hit the ball past to score?

Finding the Sides of an **Oblique Triangle**

This lesson takes ideas that have only been applied to right triangles and interprets them so that they can be used for any type of triangle.



First, the laws of sines and cosines take the Pythagorean Theorem and ratios and apply them to any triangle.

The **Law of Cosines** is a generalization of the Pythagorean Theorem, where the angle C is the angle between the two given sides of a triangle:

$$c^2 = a^2 + b^2 - 2(a)(b)\cos C$$

You'll notice that if this were a right triangle, $\cos C = \cos 90^\circ = 0$, and so the third term would disappear, leaving the familiar Pythagorean Theorem.

One case where we can use the Law of Cosines is when we know two sides and the **included angle** in a triangle (SAS) and want to find the third side.

Solving for Unknown Values

1. Using ΔDEF, $\angle E = 12^\circ$, $d = 18$, and $f = 16.8$. Find e.

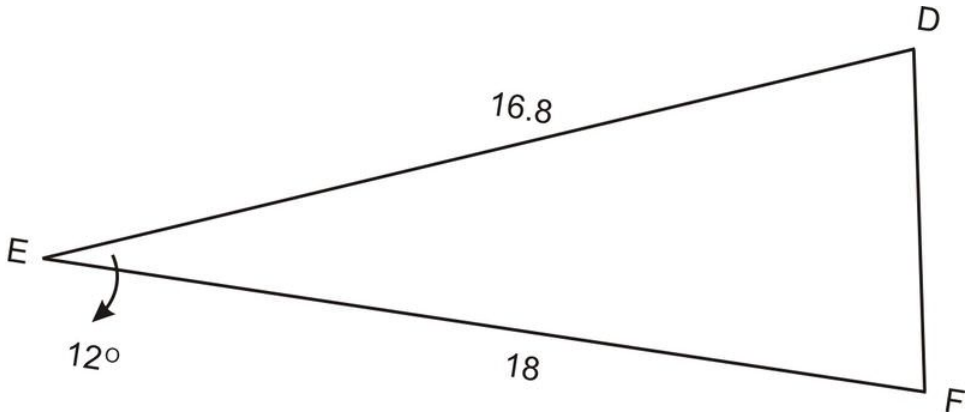


Figure 4.1.6.2

Since ΔDEF isn't a right triangle, we cannot use the Pythagorean Theorem or trigonometry functions to find the third side. However, we can use our newly derived Law of Cosines.

$e^2 = 18^2 + 16.8^2 - 2(18)(16.8)\cos 12$	Law of Cosines
$e^2 = 324 + 282.24 - 2(18)(16.8)\cos 12$	Simplify squares
$e^2 = 324 + 282.24 - 591.5836689$	Multiply
$e^2 = 14.6563311$	Add and subtract from left to right
$e \approx 3.8$	Square root

∗Note that the negative answer is thrown out as having no geometric meaning in this case.

2. An architect is designing a kitchen for a client. When designing a kitchen, the architect must pay special attention to the placement of the stove, sink, and refrigerator. In order for a kitchen to be utilized effectively, these three amenities must form a triangle with each other. This is known as the "work triangle." By design, the three parts of the work triangle must be *no less than 3*

feet apart and no more than 7 feet apart. Based on the dimensions of the current kitchen, the architect has determined that the sink will be 3.6 feet away from the stove and 5.7 feet away from the refrigerator. If the sink forms a 103° angle with the stove and the refrigerator, will the distance between the stove and the refrigerator remain within the confines of the work triangle?

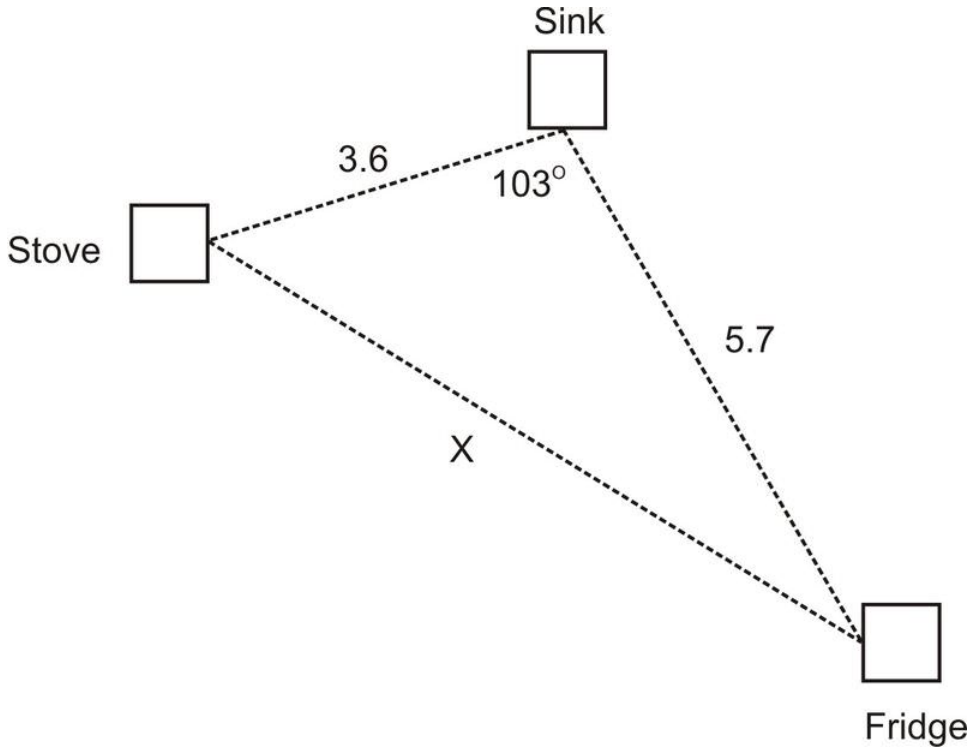


Figure **4.1.6.3**

In order to find the distance from the sink to the refrigerator, we need to find side x. To find side x, we will use the Law of Cosines because we are dealing with an obtuse triangle (and thus have no right angles to work with). We know the length two sides: the sink to the stove and the sink to the refrigerator. We also know the included angle (the angle between the two known lengths) is 103°. This means we have the SAS case and can apply the Law of Cosines.

$x^2 = 3.6^2 + 5.7^2 - 2(3.6)(5.7)\cos 103$	Law of Cosines
$x^2 = 12.96 + 32.49 - 2(3.6)(5.7)\cos 103$	Simplify squares
$x^2 = 12.96 + 32.49 + 9.23199127$	Multiply
$x^2 = 54.68199127$	Evaluate
$x \approx 7.4$	Square root

No, this triangle does not conform to the definition of a work triangle. The sink and the refrigerator are too far apart by 0.4 feet.Solve for j.

3. Using ΔJKL, $\angle J = 2^\circ$, $l = 25$, and $k = 27$. Find j.

Since ΔJKL isn't a right triangle, we cannot use the Pythagorean Theorem or trigonometry functions to find the third side. However, we can use our newly derived Law of Cosines.

$j^2 = 25^2 + 27^2 - 2(25)(27)\cos 2$	Law of Cosines
$j^2 = 625 + 729 - 2(25)(27)\cos 2$	Simplify squares
$j^2 = 625 + 729 - 1349.18$	Multiply
$j^2 = 4.82$	Add and subtract from left to right
$j \approx 2.20$	Square root

∗Note that the negative answer is thrown out as having no geometric meaning in this case.

Example 4.1.6.1

Earlier, you were asked to calculate the length of the line you have to hit the ball past to score.

Solution

Since you know that the length of each of the other 2 sides is 30 yards, and the angle is $15°$, you can use the Law of Cosines to find the length of the third side.

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$
$$c^2 = 30^2 + 30^2 - (2)(30)(30)\cos 15°$$
$$c^2 = 900 + 900 - 1738.67$$
$$c^2 = 1800 - 1738.67$$
$$c^2 = 61.33 c \approx 7.83$$

Example 4.1.6.2

Find side "a" in this triangle, where $\angle A = 50°$, $b = 8$, $c = 11$

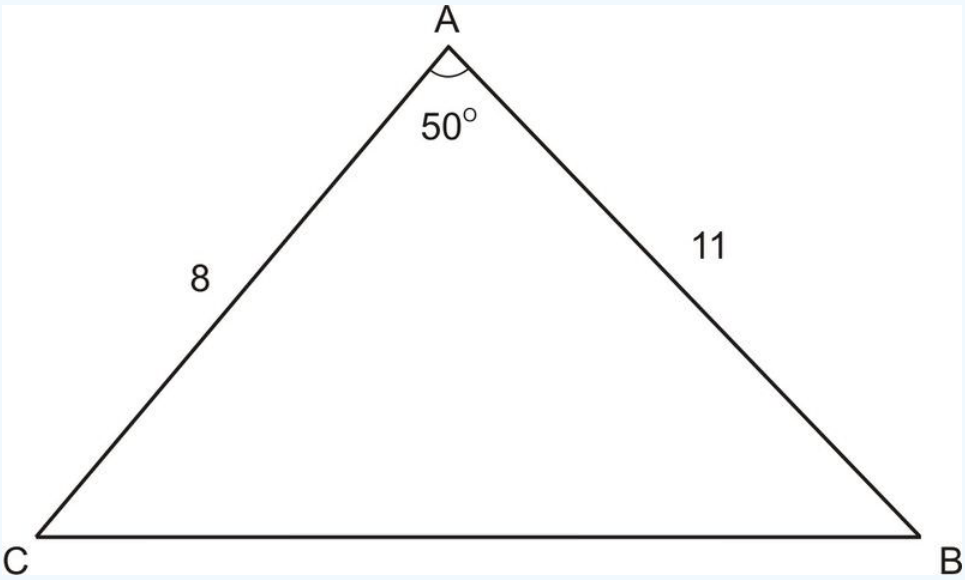


Figure 4.1.6.4

Solution

$a^2 = 8^2 + 11^2 - 2 \cdot 8 \cdot 11 \cdot \cos 50°$, $a \approx 8.5$

Example 4.1.6.3

Find side "l" in this triangle where $\angle L = 79.5°$, $m = 22.4$, $p = 13.17$

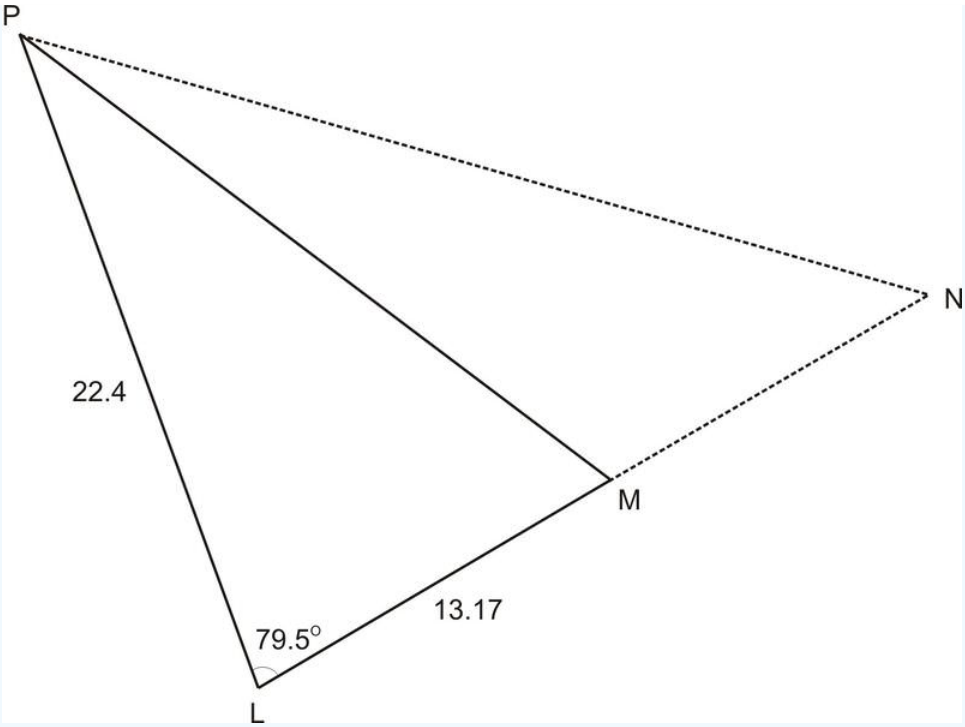


Figure 4.1.6.5

Solution

$l^2 = 22.4^2 + 13.17^2 - 2 \cdot 22.4 \cdot 13.17 \cdot \cos 79.5°$, $l \approx 23.8$

Example 4.1.6.4

Find side "b" in this triangle where $\angle B = 67.2°$, $d = 43$, $e = 39$

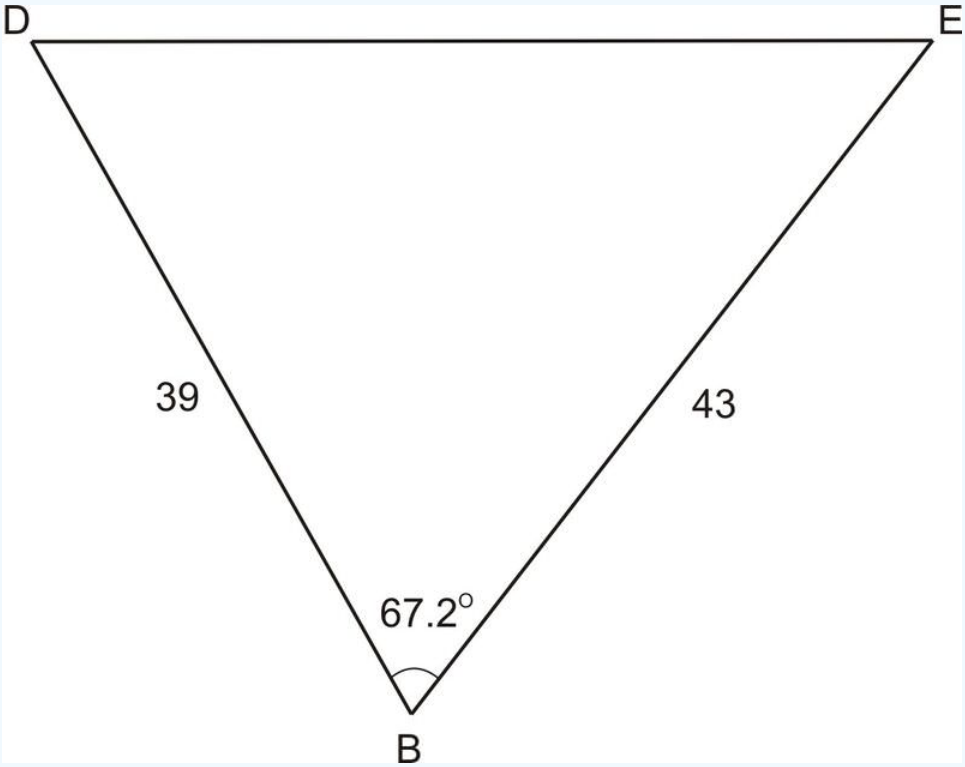


Figure 4.1.6.6

Solution

$b^2 = 39^2 + 43^2 - 2 \cdot 39 \cdot 43 \cdot \cos 67.2°$, $b \approx 45.5$

Review

1. State the Law of Cosines.

For each triangle below, state the values of a, b, and c.

2.

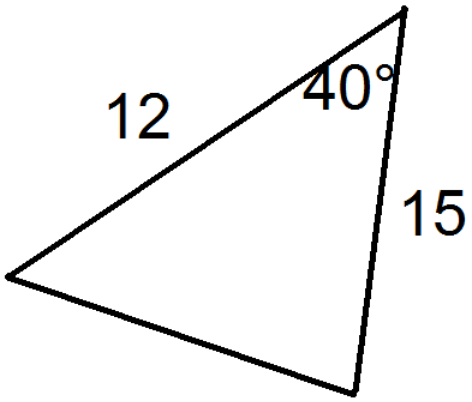


Figure 4.1.6.7

3.

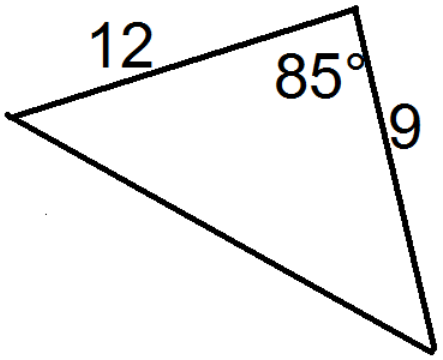


Figure 4.1.6.8

4.

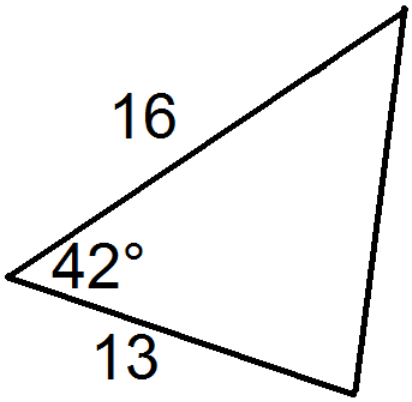


Figure 4.1.6.9

5.

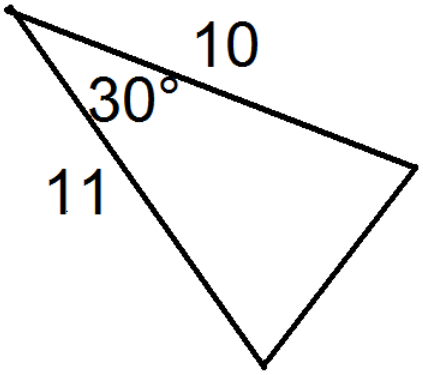


Figure **4.1.6.10**

6.

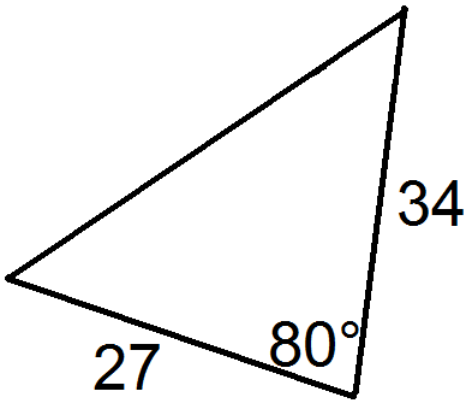


Figure **4.1.6.11**

7.

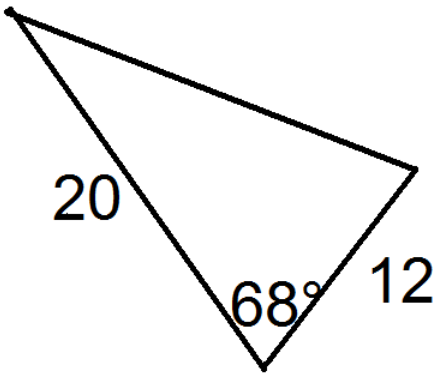


Figure **4.1.6.12**

8.

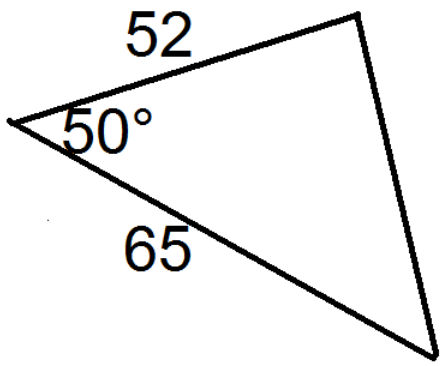


Figure **4.1.6.13**

Now, for each triangle, solve for the missing side using the Law of Cosines.

9.

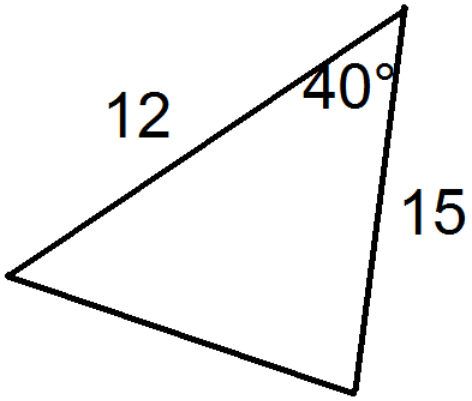


Figure **4.1.6.14**

10.

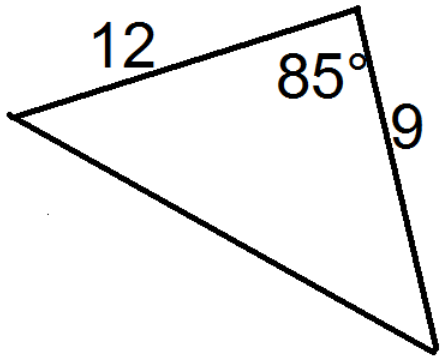


Figure **4.1.6.15**

11.

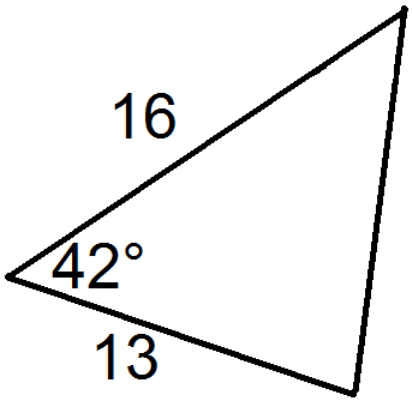


Figure **4.1.6.16**

12.

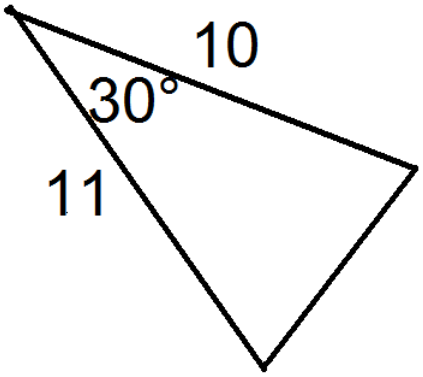


Figure **4.1.6.17**

13.

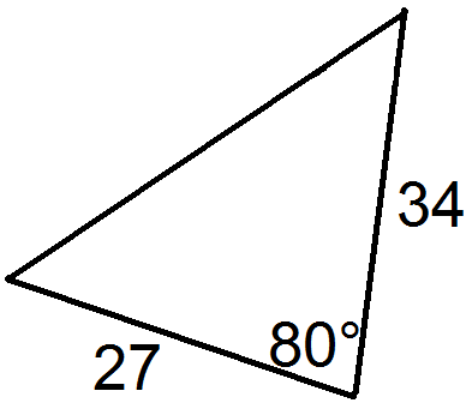


Figure **4.1.6.18**

14.

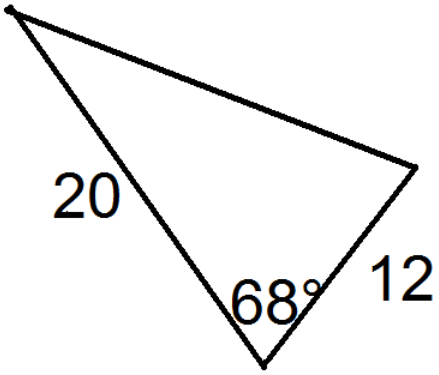


Figure **4.1.6.19**

15.

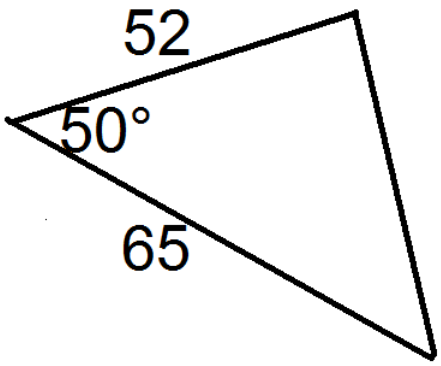


Figure **4.1.6.20**

16. Prove that the Law of Cosines is equivalent to the Pythagorean Theorem for all right triangles.

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.1.

Vocabulary

Term	Definition
Included Angle	The included angle in a triangle is the angle between two known sides.
law of cosines	The law of cosines is a rule relating the sides of a triangle to the cosine of one of its angles. The law of cosines states that $c^2 = a^2 + b^2 - 2ab\cos C$, where C is the angle across from side c.
Oblique Triangle	An oblique triangle is a triangle without a right angle as one of its internal angles.
Side Angle Side Triangle	A side angle side triangle is a triangle where two of the sides and the angle between them are known quantities.

Additional Resources

Video: Example: Application of the Law of Cosine

Practice: Law of Cosines

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4.1.7: Trigonometry Word Problems

Contextual use of triangle properties, ratios, theorems, and laws.

Angle of Depression and **Angle of Elevation**

One application of the **trigonometric ratios** is to find lengths that you cannot measure. Very frequently, angles of depression and elevation are used in these types of problems.

Angle of Depression: The angle measured down from the horizon or a horizontal line.



Figure **4.1.7.1**

Angle of Elevation: The angle measured up from the horizon or a horizontal line.

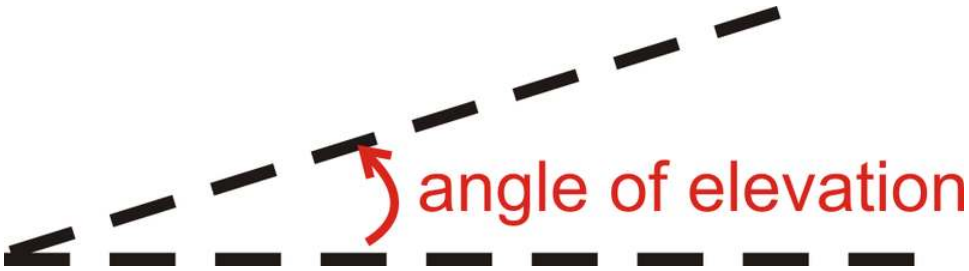
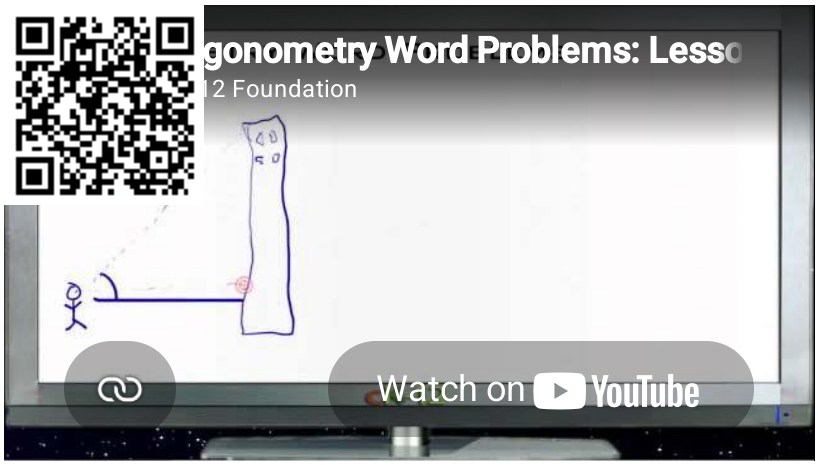


Figure **4.1.7.2**

What if you placed a ladder 10 feet from a haymow whose floor is 20 feet from the ground? How tall would the ladder need to be to reach the haymow's floor if it forms a $30°$ angle with the ground?



Example 4.1.7.1

A math student is standing 25 feet from the base of the Washington Monument. The angle of elevation from her horizontal line of sight is $87.4°$. If her “eye height” is 5 ft, how tall is the monument?



Figure 4.1.7.3

Solution

We can find the height of the monument by using the **tangent** ratio.

$$\tan 87.4^\circ = \frac{h}{25}$$
$$h = 25 \cdot \tan 87.4^\circ = 550.54$$

Adding 5 ft, the total height of the Washington Monument is 555.54 ft.

Example 4.1.7.2

A 25 foot tall flagpole casts a 42 foot shadow. What is the angle that the sun hits the flagpole?

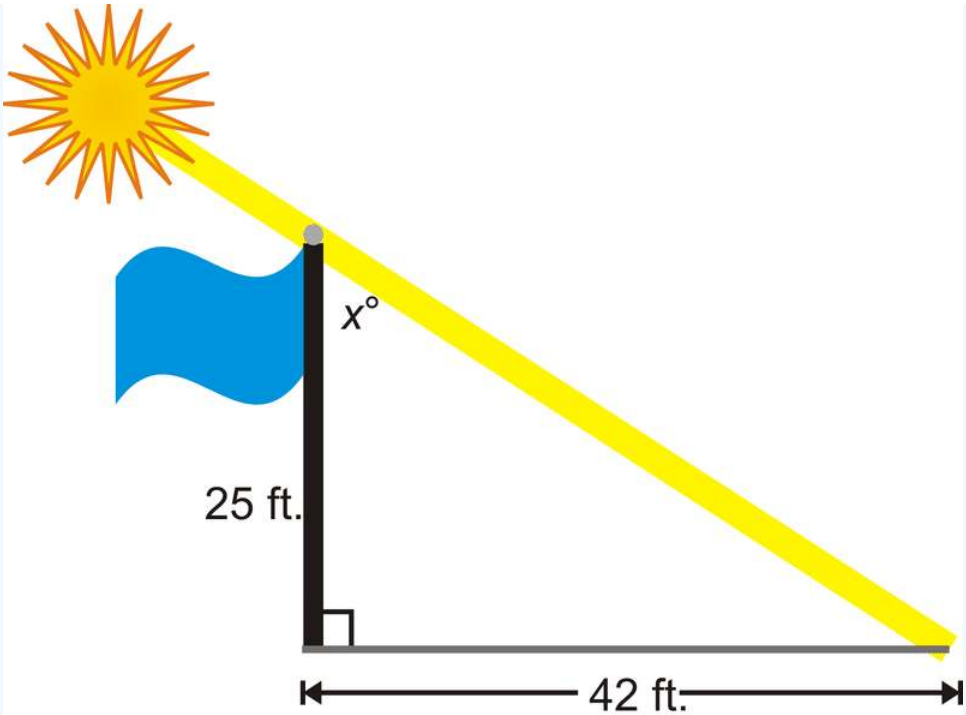


Figure **4.1.7.3**

Solution

Draw a picture. The angle that the sun hits the flagpole is $x°$. We need to use the inverse tangent ratio.

$$\tan x = \frac{42}{25}$$

$$\tan^{-1} \frac{42}{25} \approx 59.2° = x$$

Example 4.1.7.3

Elise is standing on top of a 50 foot building and sees her friend, Molly. If Molly is 30 feet away from the base of the building, what is the angle of depression from Elise to Molly? Elise's eye height is 4.5 feet.

Solution

Because of parallel lines, the angle of depression is equal to the angle at Molly, or $x°$. We can use the inverse tangent ratio.

$$\tan^{-1} \left(\frac{54.5}{30} \right) = 61.2° = x$$

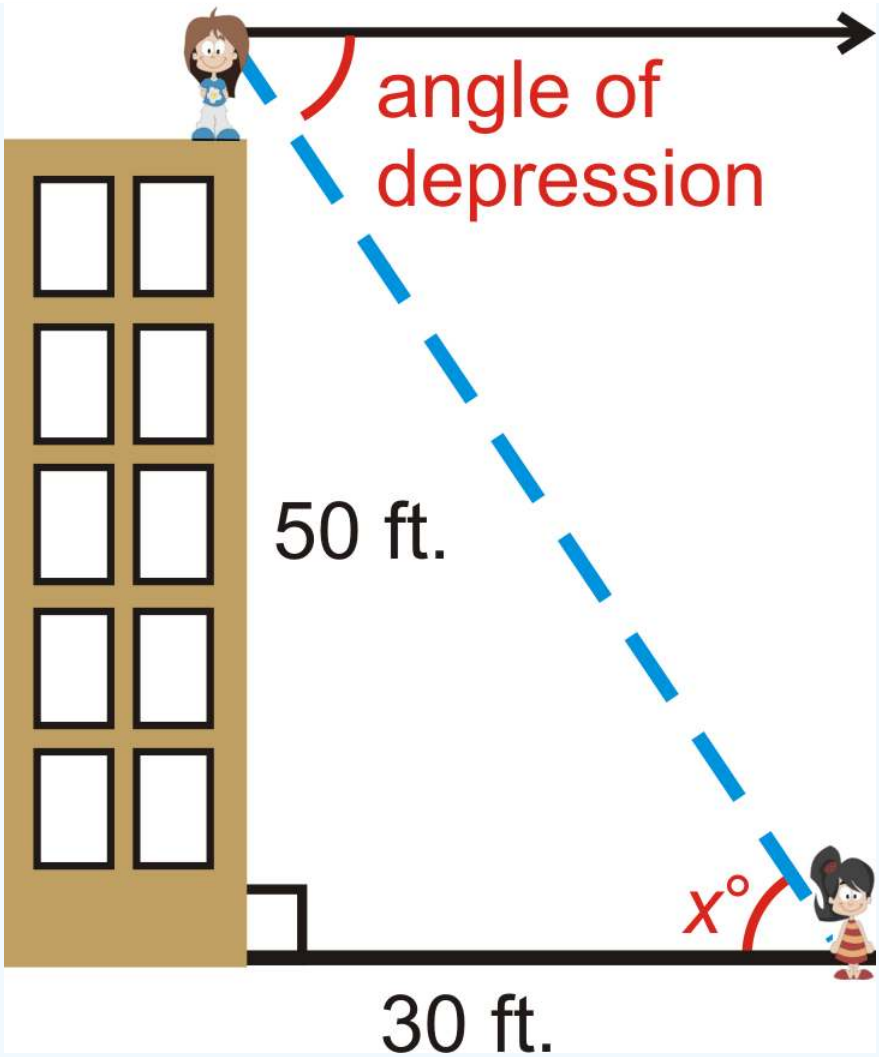


Figure 4.1.7.4

Example 4.1.7.4

Mark is flying a kite and realizes that 300 feet of string are out. The angle of the string with the ground is $42.5°$. How high is Mark's kite above the ground?

Solution

It might help to draw a picture. Then write and solve a trig equation.

$$\begin{aligned} \sin 42.5° &= \frac{x}{300} \\ 300 \cdot \sin 42.5° &= x \\ x &\approx 202.7 \end{aligned}$$

The kite is about 202.7 feet off of the ground.

Example 4.1.7.5

A 20 foot ladder rests against a wall. The base of the ladder is 7 feet from the wall. What angle does the ladder make with the ground?

Solution

It might help to draw a picture.

$$\cos x = \frac{7}{20}$$
$$x = \cos^{-1} \frac{7}{20}$$
$$x \approx 69.5^\circ$$

Review

1. Kristin is swimming in the ocean and notices a coral reef below her. The angle of depression is 35° and the depth of the ocean, at that point is 250 feet. How far away is she from the reef?

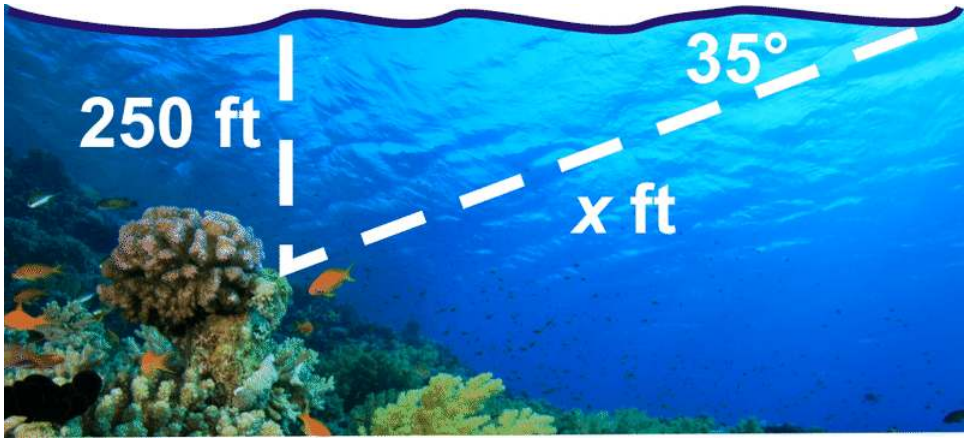


Figure 4.1.7.5

2. The Leaning Tower of Piza currently "leans" at a 4° angle and has a vertical height of 55.86 meters. How tall was the tower when it was originally built?

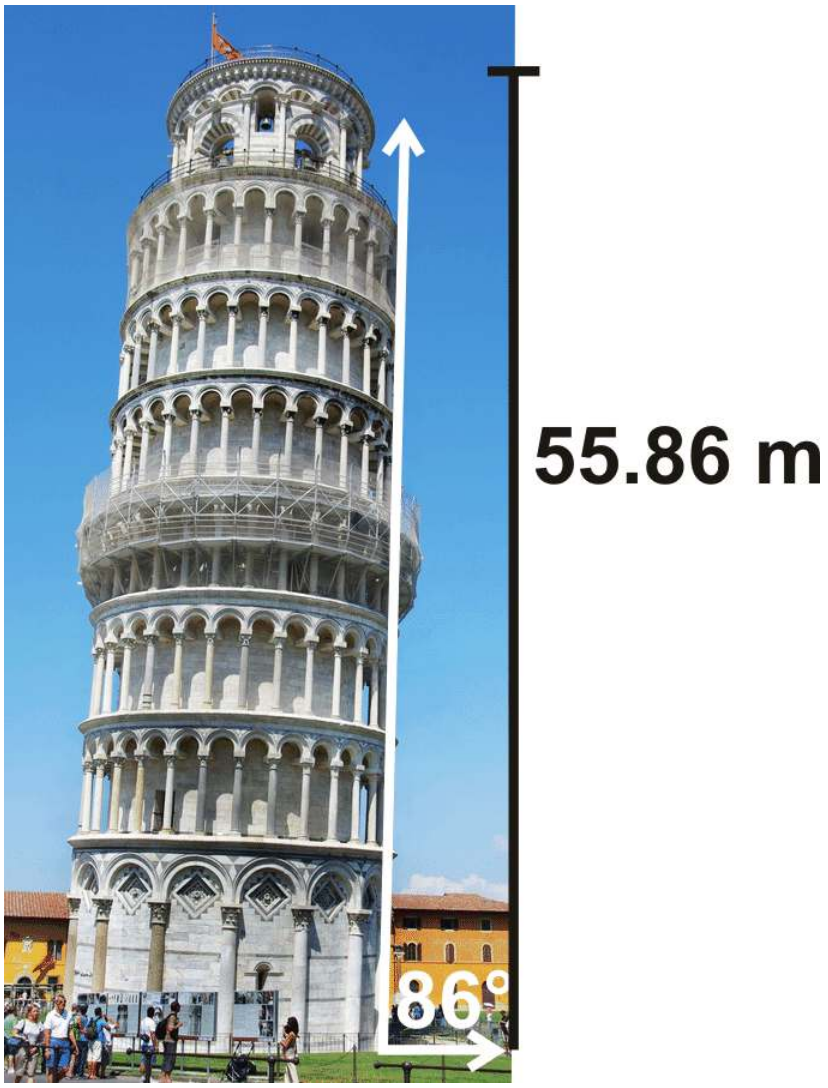


Figure **4.1.7.6**

Use what you know about right triangles to solve for the missing angle. If needed, draw a picture. Round all answers to the nearest tenth of a degree.

3. A 75 foot building casts an 82 foot shadow. What is the angle that the sun hits the building?
4. Over 2 miles (horizontal), a road rises 300 feet (vertical). What is the angle of elevation?
5. A boat is sailing and spots a shipwreck 650 feet below the water. A diver jumps from the boat and swims 935 feet to reach the wreck. What is the angle of depression from the boat to the shipwreck?
6. Standing 100 feet from the base of a building, Sam measures the angle to the top of the building from his eye height to be $50°$. If his eyes are 6 feet above the ground, how tall is the building?
7. Over 4 miles (horizontal), a road rises 200 feet (vertical). What is the angle of elevation?
8. A 90 foot building casts an 110 foot shadow. What is the angle that the sun hits the building?
9. Luke is flying a kite and realizes that 400 feet of string are out. The angle of the string with the ground is $50°$. How high is Luke's kite above the ground?
10. An 18 foot ladder rests against a wall. The base of the ladder is 10 feet from the wall. What angle does the ladder make with the ground?

Review (Answers)

To see the Review answers, open this PDF file and look for section 8.9.

Resources



Vocabulary

Term	Definition
Angle of Depression	The angle of depression is the angle formed by a horizontal line and the line of sight down to an object when the image of an object is located beneath the horizontal line.
Angle of Elevation	The angle of elevation is the angle formed by a horizontal line and the line of sight up to an object when the image of an object is located above the horizontal line.
ASA	ASA, angle-side-angle, refers to two known angles in a triangle with one known side between the known angles.
law of cosines	The law of cosines is a rule relating the sides of a triangle to the cosine of one of its angles. The law of cosines states that c2=a2+b2−2abcosC, where C is the angle across from side c.
law of sines	The law of sines is a rule applied to triangles stating that the ratio of the sine of an angle to the side opposite that angle is equal to the ratio of the sine of another angle in the triangle to the side opposite that angle.
Tangent	The tangent of an angle in a right triangle is a value found by dividing the length of the side opposite the given angle by the length of the side adjacent to the given angle.
Trigonometric Ratios	Ratios that help us to understand the relationships between sides and angles of right triangles.

Additional Resources

Video: Trigonometry Word Problems Principles - Basic

Activities: Trigonometry Word Problems Discussion Questions

Practice: Trigonometry Word Problems

Real World: Measuring Mountains

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4.1.8: Determination of Unknown Angles Using Law of Cosines

Find unknown angle given lengths of all 3 sides

Sarine draws a triangle. She measures the length of the sides and records her measurements as follows. What is the measure of angle C of the triangle?

$$a = 3$$
$$b = 4$$
$$c = 5$$

Law of Cosines with SSS

The Law of Cosines, $a^2 + b^2 - 2ab\cos C$, can be rearranged to facilitate the calculation of the measure of angle C when a, b and c are all known lengths.

$$a^2 + b^2 - 2ab\cos C = c^2$$
$$a^2 + b^2 - c^2 = 2ab\cos C$$
$$\frac{a^2 + b^2 - c^2}{2ab} = \cos C$$

which can be further manipulated to $C = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right)$.

Let's find the measure of the largest angle in the triangle with side lengths 12, 18 and 21.

First, we must determine which angle will be the largest. Recall from Geometry that the longest side is opposite the largest angle. The longest side is 21 so we will let $c = 21$ since C is the angle we are trying to find. Let $a = 12$ and $b = 18$ and use the formula to solve for C as shown. It doesn't matter which sides we assign to a and b. They are interchangeable in the formula.

$$m\angle C = \cos^{-1}\left(\frac{12^2 + 18^2 - 21^2}{2(12)(18)}\right) \approx 86^\circ$$

Note: Be careful to put parenthesis around the entire numerator and entire denominator on the calculator to ensure the proper order of operations. Your calculator screen should look like this:

$$\cos^{-1}((12^2 + 18^2 - 21^2)/(2(12)(18)))$$

Now let's find the value of x, to the nearest degree.

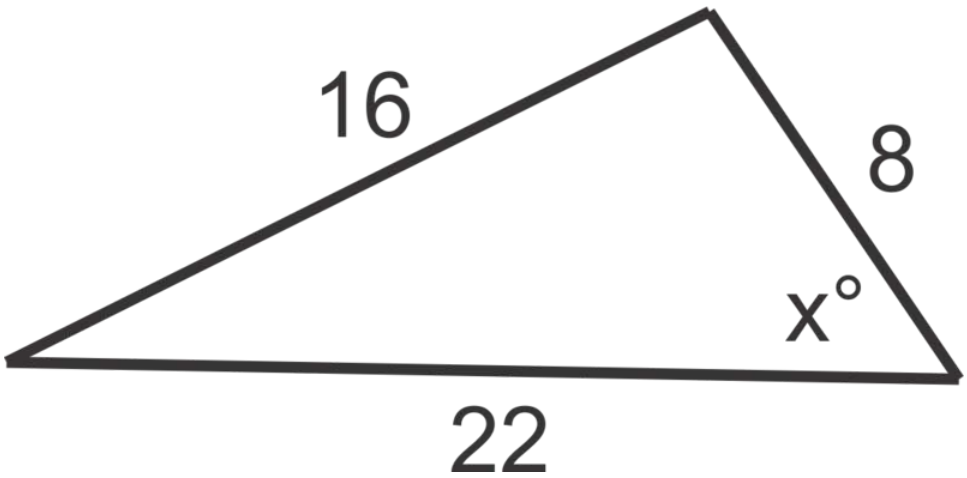


Figure 4.1.8.1

The angle with measure x° will be angle C so $c = 16$, $a = 22$ and $b = 8$. Remember, a and b are interchangeable in the formula. Now we can replace the variables with the known measures and solve.

$$\cos^{-1}\left(\frac{22^2 + 8^2 - 16^2}{2(22)(8)}\right) \approx 34^\circ$$

Finally, let's find the $m\angle A$, if $a = 10$, $b = 15$ and $c = 21$.

First, let's rearrange the formula to reflect the sides given and requested angle:

$\cos A = \left(\frac{b^2 + c^2 - a^2}{2(b)(c)}\right)$, now plug in our values

$m\angle A = \cos^{-1}\left(\frac{15^2 + 21^2 - 10^2}{2(15)(21)}\right) \approx 26^\circ$

Example 4.1.8.1

Earlier, you were asked to find the measure of angle C of the triangle that has sides $a = 3$, $b = 4$, and $c = 5$.

Solution

We can use the manipulated Law of Cosines to solve for C.

$$C = \cos^{-1}\frac{3^2 + 4^2 - 5^2}{2(3)(4)}$$
$$C = \cos^{-1}\frac{9 + 16 - 25}{24}$$
$$C = \cos^{-1}\frac{0}{24} = \cos^{-1} 0$$
$$C = 90^\circ$$

Therefore, the triangle is a right triangle.

Example 4.1.8.2

Find the measure of x in the diagram:

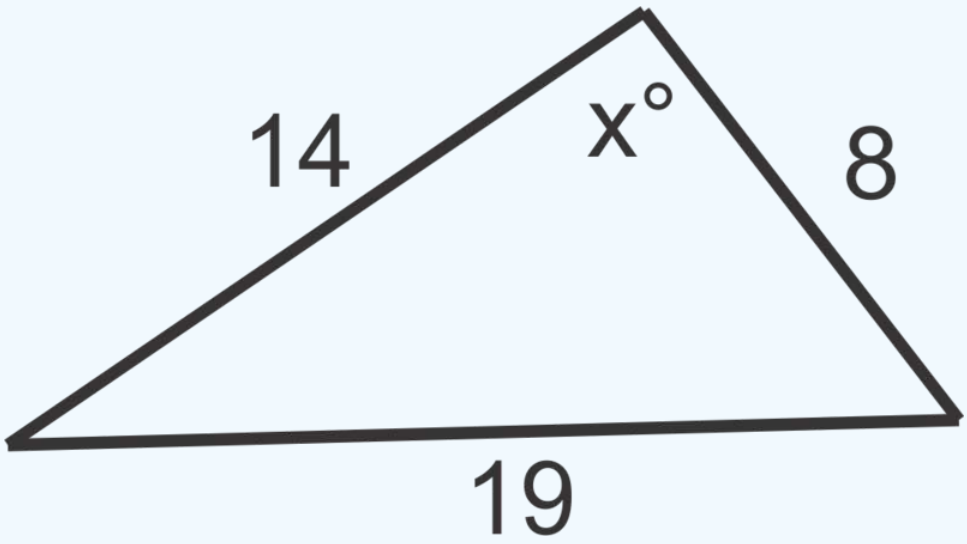


Figure 4.1.8.2

Solution

$$\cos^{-1}\left(\frac{14^2 + 8^2 - 19^2}{2(14)(8)}\right) \approx 117^\circ$$

Example 4.1.8.3

Find the measure of the smallest angle in the triangle with side lengths 47, 54 and 72.

Solution

The smallest angle will be opposite the side with length 47, so this will be our c in the equation.

$$\cos^{-1}\left(\frac{54^2+72^2-47^2}{2(54)(72)}\right)\approx 41^\circ$$

Example 4.1.8.4

Find $m\angle B$, if $a=68$, $b=56$ and $c=25$.

Solution

Rearrange the formula to solve for $m\angle B$,

$$\cos B=\left(\frac{a^2+c^2-b^2}{2(a)(c)}\right);\ \cos^{-1}\left(\frac{68^2+25^2-56^2}{2(68)(25)}\right)\approx 52^\circ$$

Review

Use the Law of Cosines to find the value of x, to the nearest degree, in problems 1 through 6.

1.

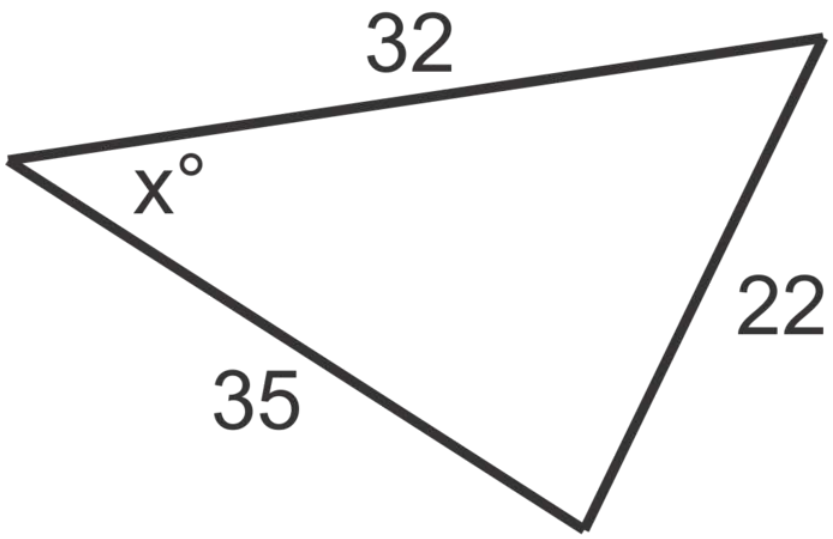


Figure 4.1.8.3

2.

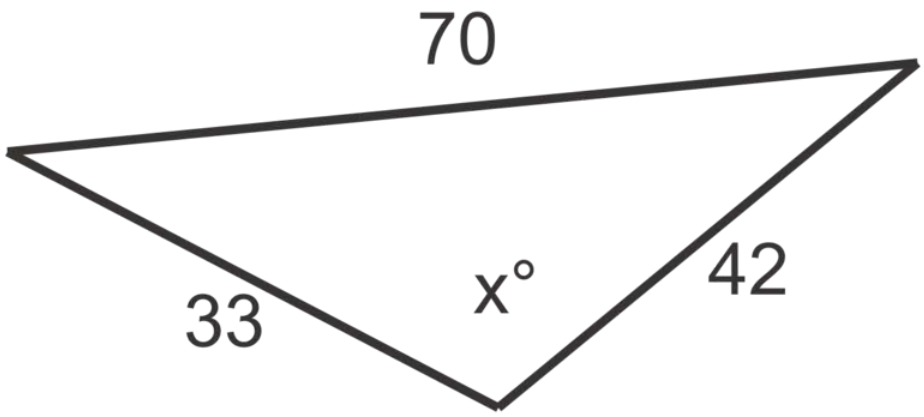


Figure 4.1.8.4

3.

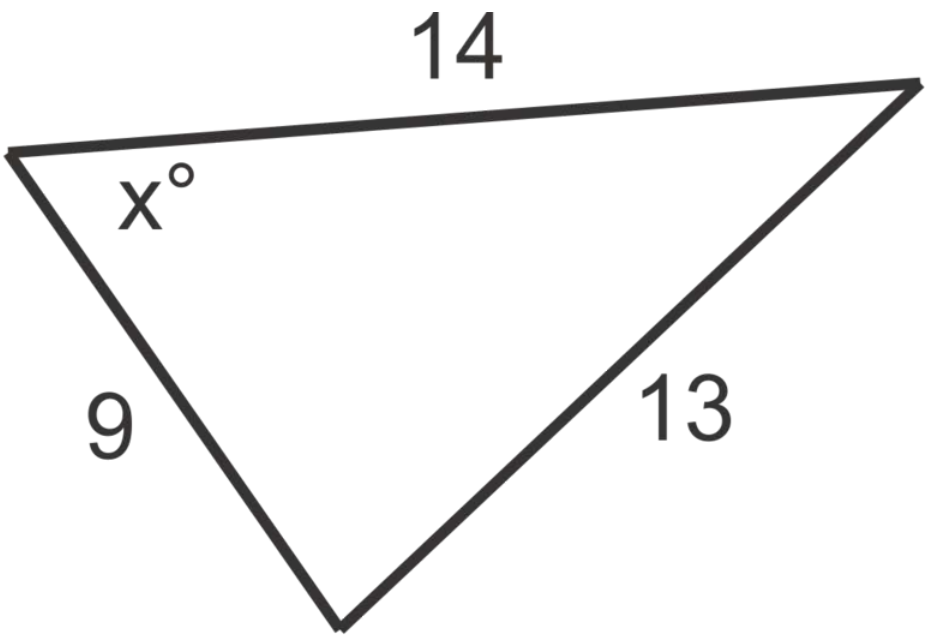


Figure **4.1.8.5**

4.

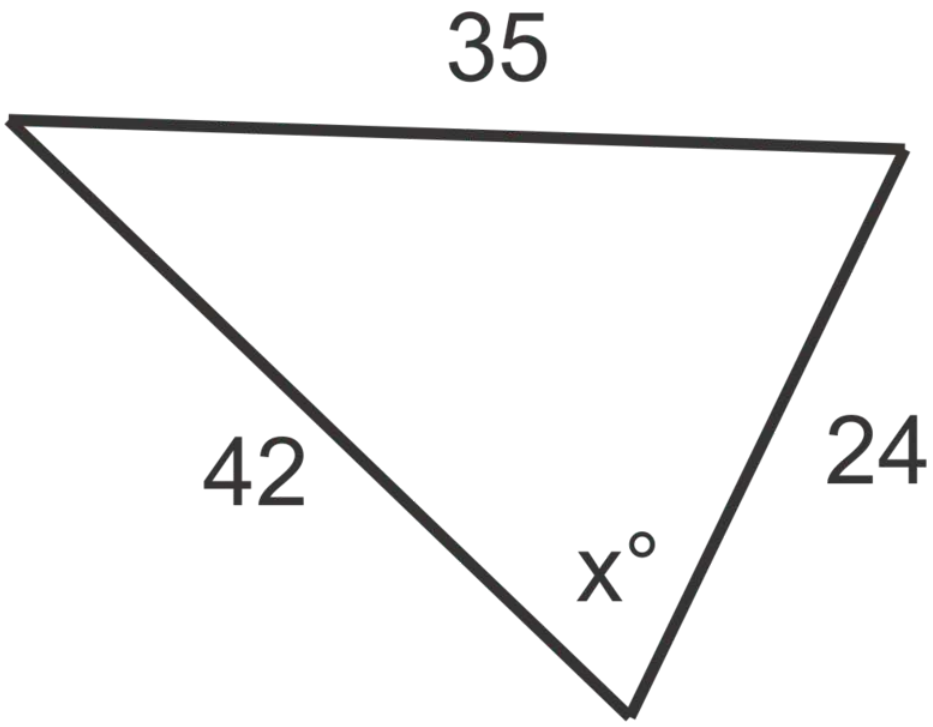


Figure **4.1.8.6**

5.

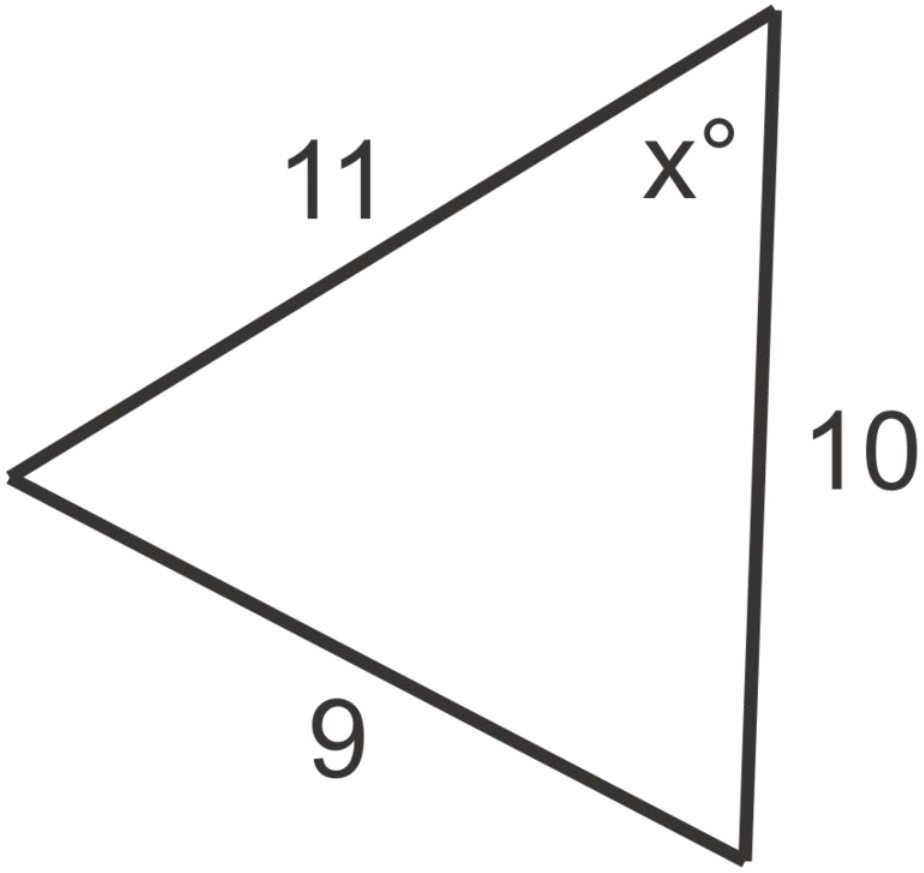


Figure 4.1.8.7

6.

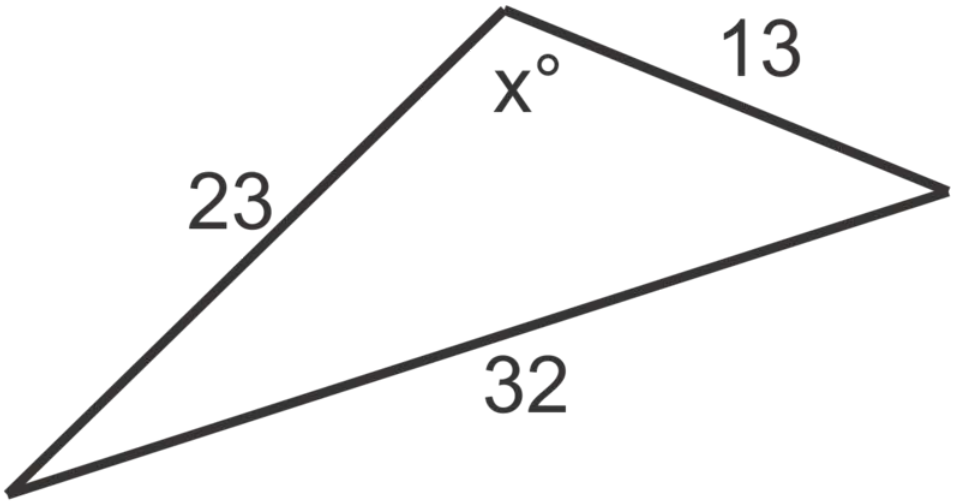


Figure 4.1.8.8

7. Find the measure of the smallest angle in the triangle with side lengths 150, 165 and 200 meters.

8. Find the measure of the largest angle in the triangle with side length 59, 83 and 100 yards.
9. Find the $m\angle C$ if $a = 6$, $b = 9$ and $c = 13$.
10. Find the $m\angle B$ if $a = 15$, $b = 8$ and $c = 9$.
11. Find the $m\angle A$ if $a = 24$, $b = 20$ and $c = 14$.
12. A triangular plot of land is bordered by a road, a fence and a creek. If the stretch along the road is 100 meters, the length of the fence is 115 meters and the side along the creek is 90 meters, at what angle do the fence and road meet?

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 13.16.

Additional Resources

Video: Example: Determine the Measure of an Angle of a Triangle Given the Length of Three Sides

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4.1.9: Identify Accurate Drawings of Triangles

Whether a triangle is correct based on law of cosines.

Your friend is creating a new board game that involves several different triangle shaped pieces. However, the game requires accurate measurements of several different pieces that all have to fit together. She brings some of the pieces to you and asks if you can verify that her measurements of the pieces' side lengths and angles are correct.

You take out the first piece. According to your friend, the piece has sides of length 4 in, 5 in and 7 in, and the angle between the side of the length 4 and the side of length 5 is $78°$. She's very confident in the lengths of the sides, but not quite sure if she measured the angle correctly. Is there a way to determine if your friend's game piece has the correct measurements, or did she make a mistake?

Analyzing Triangles

Our extension of the analysis of triangles draws us naturally to oblique triangles.

The **Law of Cosines** can be used to verify that drawings of oblique triangles are accurate. In a right triangle, we might use the Pythagorean Theorem to verify that all three sides are the correct length, or we might use trigonometric ratios to verify an angle measurement. However, when dealing with an obtuse or acute triangle, we must rely on the Law of Cosines.



For the following problems, let's use the Law of Cosines

1. In ΔABC at the right, $a = 32$, $b = 20$, And $c = 16$. Is the drawing accurate if it labels $\angle C$ as $35.2°$? If not, what should $\angle C$ measure?

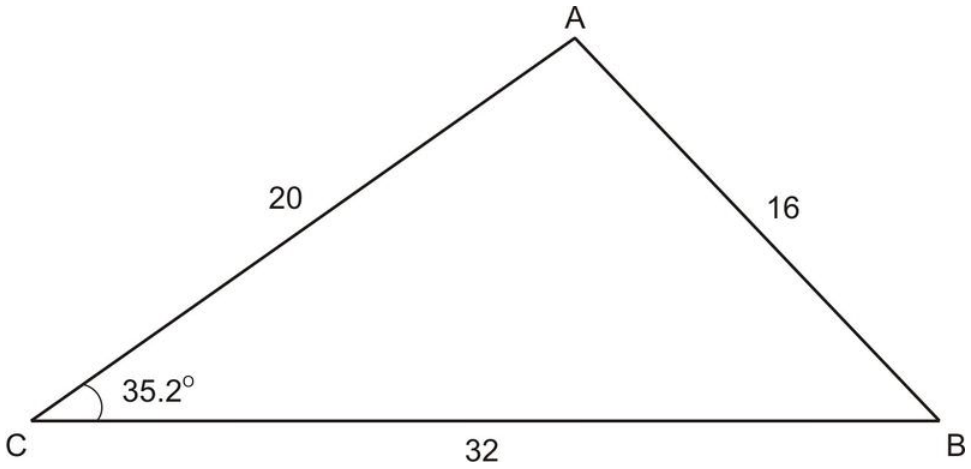


Figure 4.1.9.1

We will use the Law of Cosines to check whether or not $\angle C$ is $35.2°$.

$$
\begin{aligned}
16^2 &= 20^2 + 32^2 - 2(20)(32)\cos 35.2 && \text{Law of Cosines}\\
256 &= 400 + 1024 - 2(20)(32)\cos 35.2 && \text{Simply squares}\\
256 &= 400 + 1024 - 1045.94547 && \text{Multiply}\\
256 &\neq 378.05453 && \text{Add and subtract}
\end{aligned}
$$

Since $256 \neq 378.05453$, we know that $\angle C$ is not $35.2°$. Using the Law of Cosines, we can figure out the correct measurement of $\angle C$.

$$
\begin{aligned}
16^2 &= 20^2 + 32^2 - 2(20)(32)\cos C && \text{Law of Cosines}\\
256 &= 400 + 1024 - 2(20)(32)\cos C && \text{Simplify Squares}\\
256 &= 400 + 1024 - 1280\cos C && \text{Multiply}\\
256 &= 1424 - 1280\cos C && \text{Add}\\
-1168 &= -1280\cos C && \text{Subtract}\\
0.9125 &= \cos C && \text{Divide}\\
24.1° &\approx \angle C && \cos^{-1}(0.9125)
\end{aligned}
$$

For some situations, it will be necessary to utilize not only the Law of Cosines, but also the Pythagorean Theorem and trigonometric ratios to verify that a triangle or quadrilateral has been drawn accurately.

2. A builder received plans for the construction of a second-story addition on a house. The diagram shows how the architect wants the roof framed, while the length of the house is 20 ft. The builder decides to add a perpendicular support beam from the peak of the roof to the base. He estimates that new beam should be 8.3 feet high, but he wants to double-check before he begins construction. Is the builder's estimate of 8.3 feet for the new beam correct? If not, how far off is he?

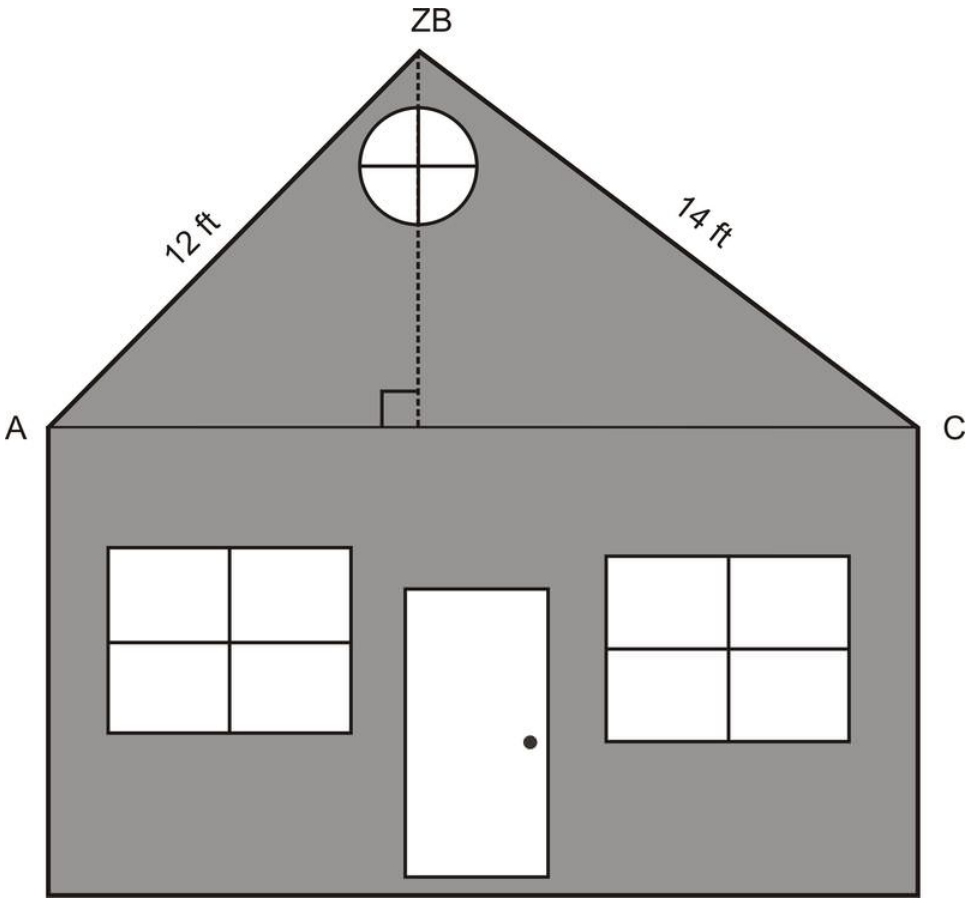


Figure 4.1.9.2

If we knew either $\angle A$ or $\angle C$, we could use trigonometric ratios to find the height of the support beam. However, neither of these angle measures are given to us. Since we know all three sides of ΔABC, we can use the Law of Cosines to find one of these angles. We will find $\angle A$.

$$
\begin{aligned}
14^2 &= 12^2 + 20^2 - 2(12)(20)\cos A && \text{Law of Cosines} \\
196 &= 144 + 400 - 480\cos A && \text{Simplify} \\
196 &= 544 - 480\cos A && \text{Add} \\
-348 &= -480\cos A && \text{Subtract} \\
0.725 &= \cos A && \text{Divide} \\
43.5^\circ &\approx \angle A && \cos^{-1}(0.725)
\end{aligned}
$$

Now that we know $\angle A$, we can use it to find the length of BD.

$$
\begin{aligned}
\sin 43.5 &= \frac{x}{12} \\
12\sin 43.5 &= x \\
8.3 &\approx x
\end{aligned}
$$

Yes, the builder's estimate of 8.3 feet for the support beam is accurate.

In ΔCIR, $c = 63$, $i = 52$, and $r = 41.9$. Find the measure of all three angles.

$$
\begin{aligned}
63^2 &= 52^2 + 41.9^2 - 2 \cdot 52 \cdot 41.9 \cdot \cos C \\
52^2 &= 63^2 + 41.9^2 - 2 \cdot 63 \cdot 41.9 \cdot \cos I \\
180^\circ - 83.5^\circ - 55.1^\circ &= 41.4^\circ \\
\angle C &\approx 83.5^\circ \\
\angle I &\approx 55.1^\circ \\
\angle R &\approx 41.4^\circ
\end{aligned}
$$

Example 4.1.9.1

Earlier, you were asked if there was a way to determine if your friend's game piece has the correct measurements.

Solution

Since your friend is certain of the lengths of the sides of the triangle, you should use those as the known quantities in the Law of Cosines and solve for the angle:

$$
\begin{aligned}
7^2 &= 5^2 + 4^2 + (2)(5)(4)\cos\theta \\
49 &= 25 + 16 + 40\cos\theta \\
49 - 25 - 16 &= 40\cos\theta \\
\frac{8}{40} &= \cos\theta \\
\cos^{-1}\frac{8}{40} &= \theta \\
\theta &= 78.46
\end{aligned}
$$

So as it turns out, your friend is rather close. Her measurements were probably slight inaccurate because of her round off from the protractor.

Example 4.1.9.2

Find AD using the Pythagorean Theorem, Law of Cosines, trig functions, or any combination of the three.

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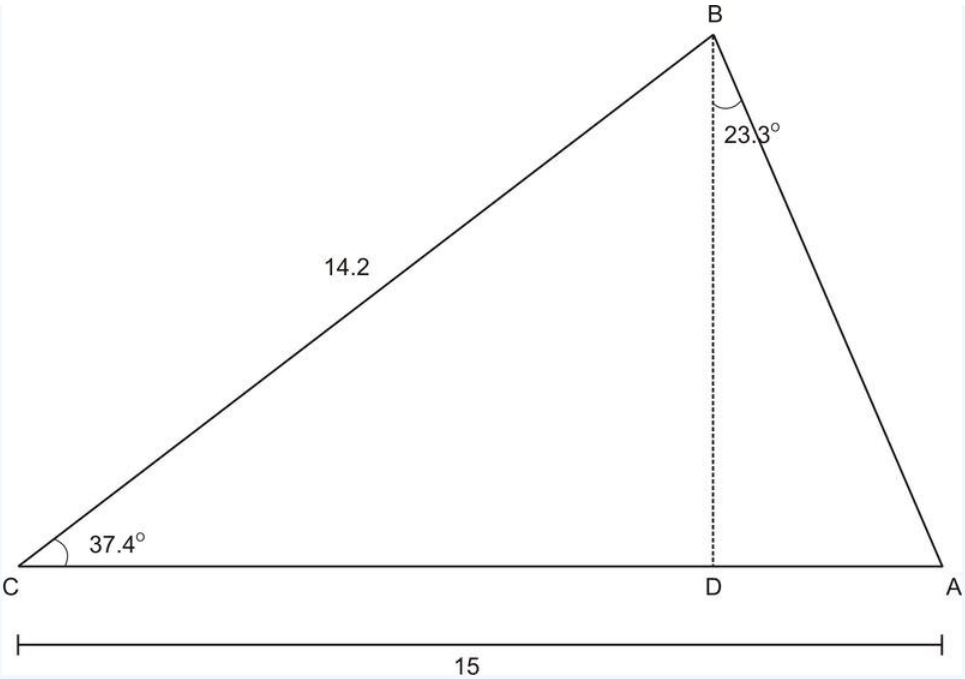


Figure 4.1.9.3

Solution

First, find AB.

$AB^2 = 14.2^2 + 15^2 - 2 \cdot 14.2 \cdot 15 \cdot \cos 37.4°$, $AB = 9.4$. $\sin 23.3° = \frac{AD}{9.4}$, $AD = 3.7$.

Example 4.1.9.3

Find HK using the Pythagorean Theorem, Law of Cosines, trig functions, or any combination of the three if $JK = 3.6$, $KI = 5.2$, $JI = 1.9$, $HI = 6.7$, and $\angle KJI = 96.3°$.

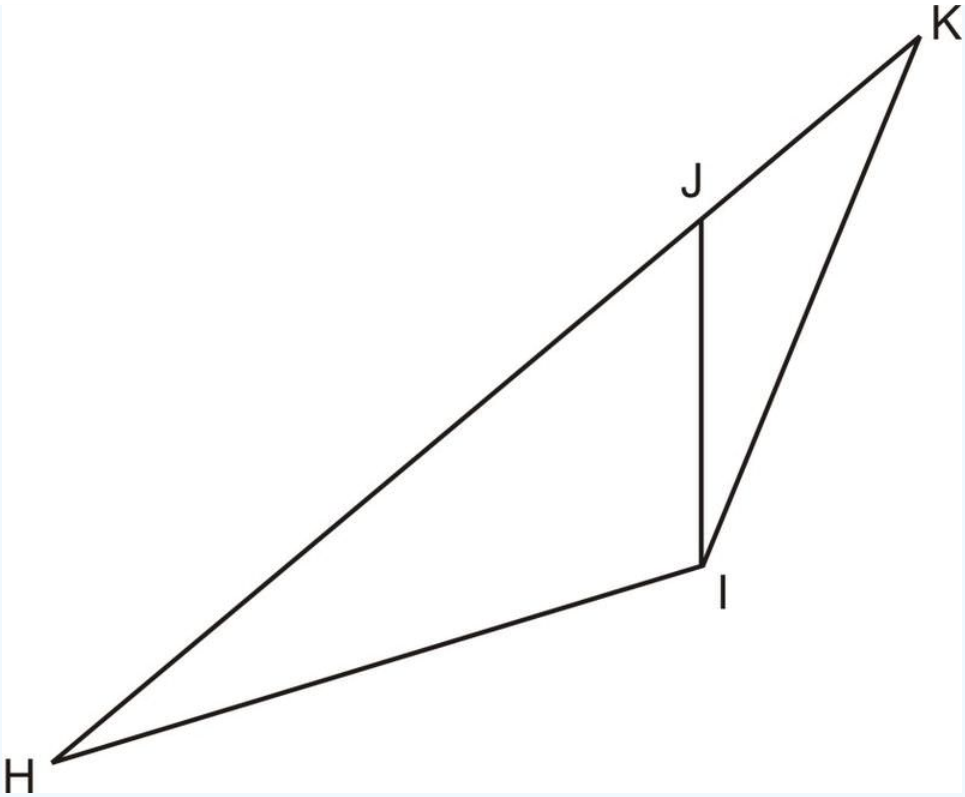


Figure 4.1.9.4

Solution

$\angle HJI = 180° - 96.3° = 83.7°$ (these two angles are a linear pair). $6.7^2 = HJ^2 + 1.9^2 - 2 \cdot HJ \cdot 1.9 \cdot \cos 83.7°$. This simplifies to the quadratic equation $HJ^2 - 0.417HJ - 41.28$. Using the quadratic formula, we can determine that $HJ \approx 6.64$. So, since $HJ + JK = HK$, $6.64 + 3.6 \approx HK \approx 10.24$.

Example 4.1.9.4

Use the Law of Cosines to determine whether or not the following triangle is drawn accurately. If not, determine how far the measurement of side "d" is from the correct value.

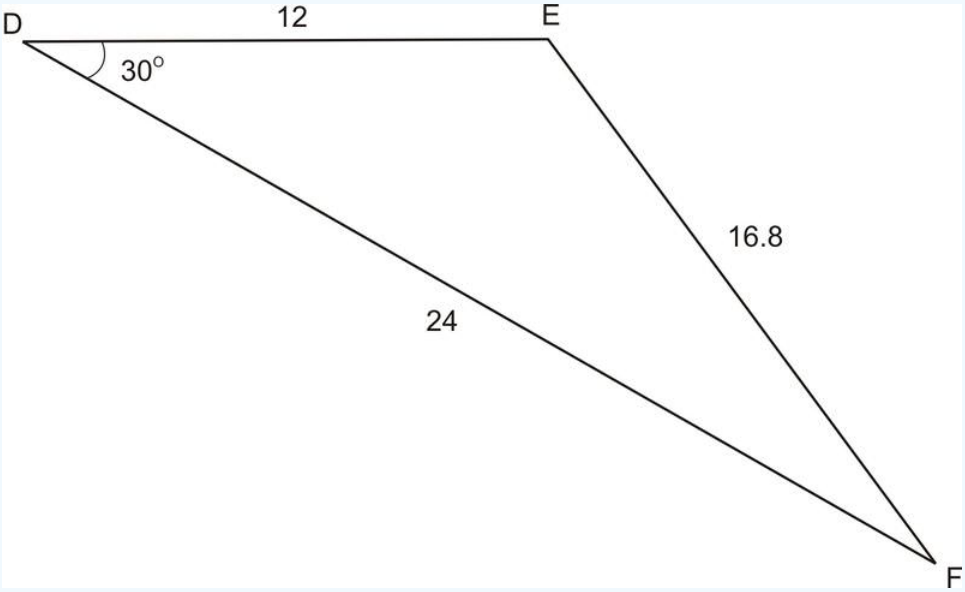


Figure 4.1.9.5

Solution

To determine this, use the Law of Cosines and solve for d to determine if the picture is accurate. $d^2 = 12^2 + 24^2 - 2 \cdot 12 \cdot 24 \cdot \cos 30°$, $d = 14.9$, which means d in the picture is off by 1.9.

Review

1. If you know the lengths of all three sides of a triangle and the measure of one angle, how can you determine if the triangle is drawn accurately?

Determine whether or not each triangle is labelled correctly.

2.

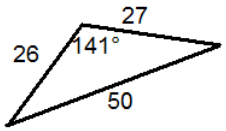


Figure 4.1.9.6

3.

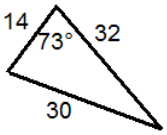


Figure 4.1.9.7

4.

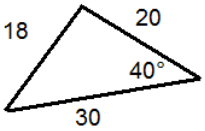


Figure 4.1.9.8

5.

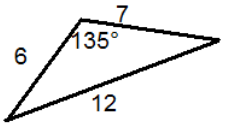


Figure 4.1.9.9

6.

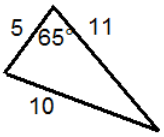


Figure 4.1.9.10

7.

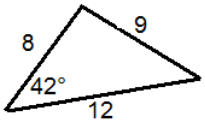


Figure 4.1.9.11

Determine whether or not each described triangle is possible. Assume angles have been rounded to the nearest degree.

8. In ΔBCD, $b = 4$, $c = 4$, $d = 5$, and $m\angle B = 51°$.

9. In ΔABC, $a = 7$, $b = 4$, $c = 9$, and $m\angle B = 34°$.
10. In ΔBCD, $b = 3$, $c = 2$, $d = 7$, and $m\angle D = 138°$.
11. In ΔABC, $a = 8$, $b = 6$, $c = 13.97$, and $m\angle C = 172°$.
12. In ΔABC, $a = 4$, $b = 4$, $c = 9$, and $m\angle B = 170°$.
13. In ΔBCD, $b = 3$, $c = 5$, $d = 4$, and $m\angle C = 90°$.
14. In ΔABC, $a = 8$, $b = 3$, $c = 6$, and $m\angle A = 122°$.
15. If you use the Law of Cosines to solve for $m\angle C$ in ΔABC where $a = 3$, $b = 7$, and $c = 12$, you will an error. Explain why.

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.3.

Vocabulary

Term	Definition
law of cosines	The law of cosines is a rule relating the sides of a triangle to the cosine of one of its angles. The law of cosines states that $c^2 = a^2 + b^2 - 2ab\cos C$, where C is the angle across from side c.

Additional Resources

Video: The Law of Cosines: Applications

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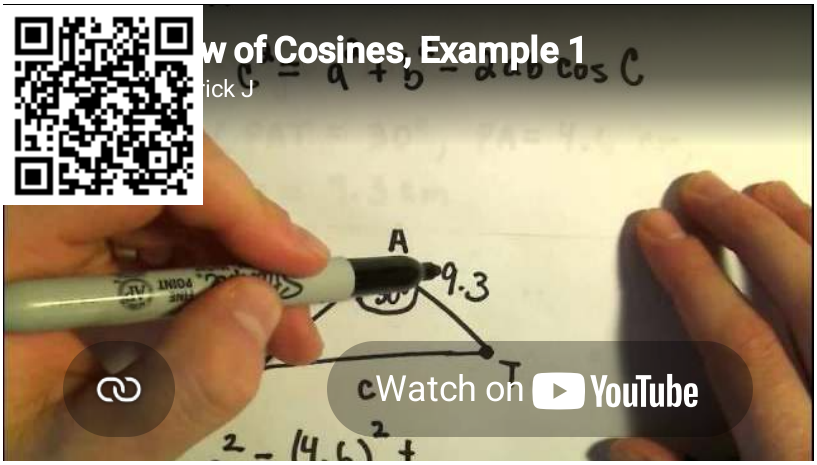
4.1.10: Applications of the Law of Cosines

Relationship between the three sides and an angle for non-right triangles.

While helping your mom bake one day, the two of you get an unusual idea. You want to cut the cake into pieces, and then frost over the surface of each piece. You start by cutting out a slice of the cake, but you don't quite cut the slice correctly. It ends up being an oblique triangle, with a 5 inch side, a 6 inch side, and an angle of $70°$ between the sides you measured. Can you help your mom determine the length of the third side, so she can figure out how much frosting to put out?

Law of Cosines

The Law of Cosines is a fantastic extension of the Pythagorean Theorem to oblique triangles. In this section, we show some interesting ways to utilize this formula to analyze real world situations.



Let's take a look at a few problems where we use the Law of Cosines.

1. In a game of pool, a player must put the eight ball into the bottom left pocket of the table. Currently, the eight ball is 6.8 feet away from the bottom left pocket. However, due to the position of the cue ball, she must bank the shot off of the right side bumper. If the eight ball is 2.1 feet away from the spot on the bumper she needs to hit and forms a $168°$ angle with the pocket and the spot on the bumper, at what angle does the ball need to leave the bumper?

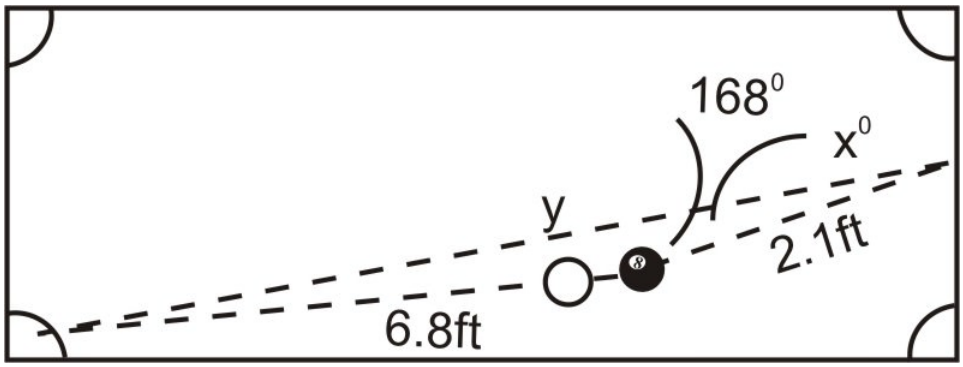


Figure 4.1.10.1

Note: This is actually a trick shot performed by spinning the eight ball, and the eight ball will not actually travel in straight-line trajectories. However, to simplify the problem, assume that it travels in straight lines.

In the scenario above, we have the SAS case, which means that we need to use the Law of Cosines to begin solving this problem. The Law of Cosines will allow us to find the distance from the spot on the bumper to the pocket (y). Once we know y, we can use the Law of Sines to find the angle (X).

$$y^2 = 6.8^2 + 2.1^2 - 2(6.8)(2.1)\cos 168°$$
$$y^2 = 78.59$$
$$y = 8.86 \text{ feet}$$

2. The distance from the spot on the bumper to the pocket is 8.86 feet. We can now use this distance and the Law of Sines to find angle X. Since we are finding an angle, we are faced with the SSA case, which means we could have no solution, one solution, or two solutions. However, since we know all three sides this problem will yield only one solution.

$$\begin{aligned}\frac{\sin 168^\circ}{8.86} &= \frac{\sin X}{6.8}\\ \frac{6.8\sin 168^\circ}{8.86} &= \sin X\\ 0.1596 &\approx \sin B\\ \angle B &= 8.77^\circ\end{aligned}$$

In the previous example, we looked at how we can use the Law of Sines and the Law of Cosines together to solve a problem involving the SSA case. In this section, we will look at situations where we can use not only the Law of Sines and the Law of Cosines, but also the Pythagorean Theorem and trigonometric ratios. We will also look at another real-world application involving the SSA case.

3. Three scientists are out setting up equipment to gather data on a local mountain. Person 1 is 131.5 yards away from Person 2, who is 67.8 yards away from Person 3. Person 1 is 72.6 yards away from the mountain. The mountains forms a 103° angle with Person 1 and Person 3, while Person 2 forms a 92.7° angle with Person 1 and Person 3. Find the angle formed by Person 3 with Person 1 and the mountain.

In the triangle formed by the three people, we know two sides and the **included angle** (SAS). We can use the Law of Cosines to find the remaining side of this triangle, which we will call x. Once we know x, we will two sides and the non-included angle (SSA) in the triangle formed by Person 1, Person 2, and the mountain. We will then be able to use the Law of Sines to calculate the angle formed by Person 3 with Person 1 and the mountain, which we will refer to as Y.

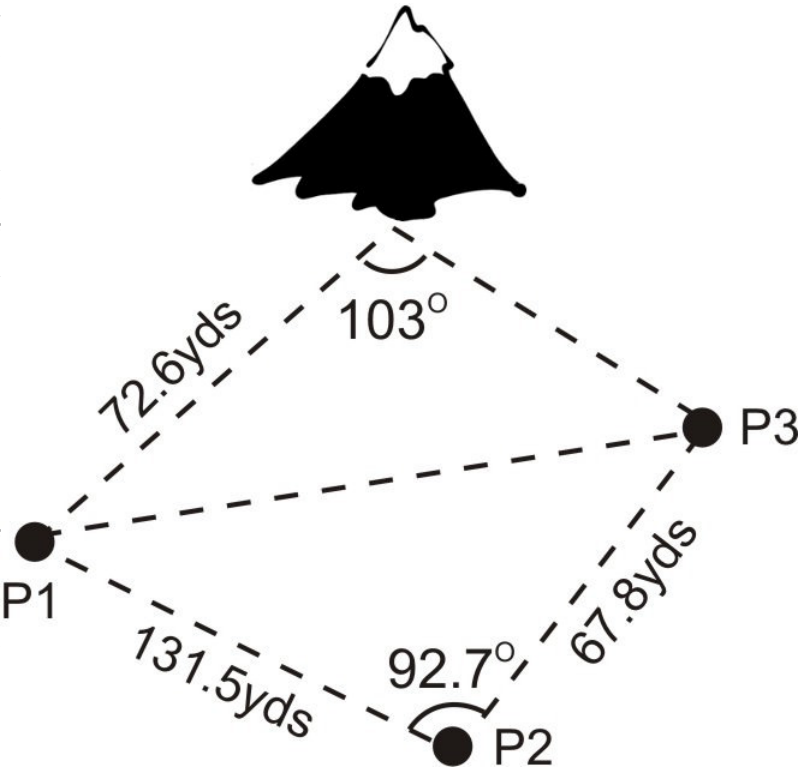


Figure 4.1.10.2

To find x:

$$\begin{aligned}x^2 &= 131.5^2 + 67.8^2 - 2(131.5)(67.8)\cos 92.7\\ x^2 &= 22729.06397\\ x &= 150.8 \text{ yds}\end{aligned}$$

Now that we know $x = 150.8$, we can use the Law of Sines to find Y. Since this is the SSA case, we need to check to see if we will have no solution, one solution, or two solutions. Since $150.8 > 72.6$, we know that we will have only one solution to this problem.

$$\begin{aligned}\frac{\sin 103}{150.8} &= \frac{\sin Y}{72.6}\\ \frac{72.6\sin 103}{150.8} &= \sin Y\\ 0.4690932805 &= \sin Y\\ 28.0 &\approx \angle Y\end{aligned}$$

4. Katie is constructing a kite shaped like a triangle.

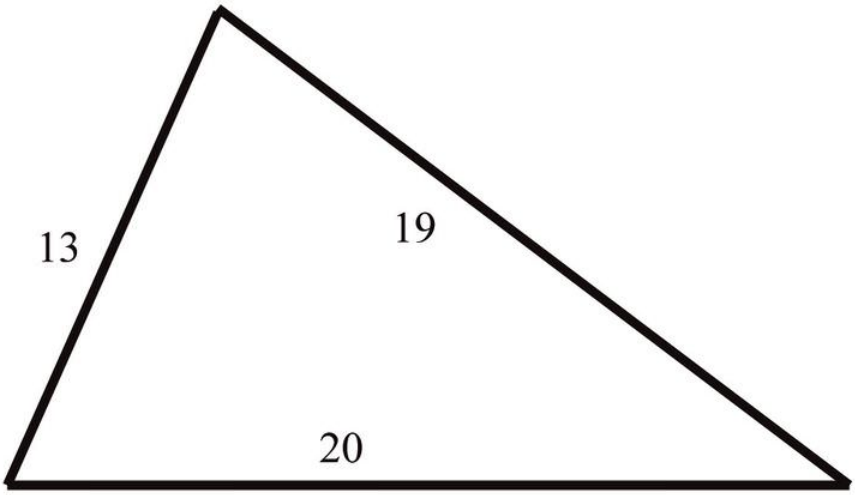


Figure **4.1.10.3**

She knows that the lengths of the sides are a = 13 inches, b = 20 inches, and c = 19 inches. What is the measure of the angle between sides "a" and "b"?

Since she knows the length of each of the sides of the triangle, she can use the Law of Cosines to find the angle desired:

$$\begin{aligned} c^2 &= a^2 + b^2 - 2(a)(b)\cos C \\ 19^2 &= 13^2 + 20^2 - (2)(13)(20)\cos C \\ 361 &= 169 + 400 - 520\cos C \\ -208 &= -520\cos C \\ \cos C &= 0.4 \\ C &\approx 66.42^\circ \end{aligned}$$

Example 4.1.10.1

Earlier, you were asked to determine the length of the third side.

Solution

You can use the Law of Cosines to help your mom find out the length of the third side on the piece of cake:

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos C \\ c^2 &= 5^2 + 6^2 + (2)(5)(6)\cos 70^\circ \\ c^2 &= 25 + 36 + 60(.342) \\ c^2 &= 81.52 \\ c &\approx 9.03 \end{aligned}$$

The piece of cake is just a little over 9 inches long.

Example 4.1.10.2

You are cutting a triangle out for school that looks like this:

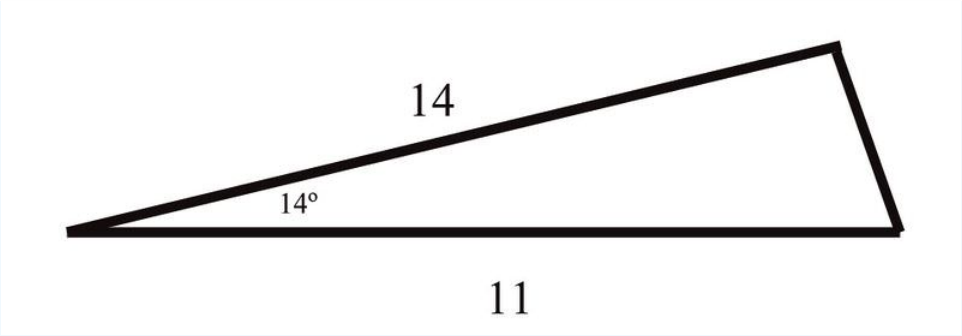


Figure 4.1.10.4

Find side c (which is the side opposite the $14°$ angle) and $\angle B$(which is the angle opposite the side that has a length of 14).

Solution

You know that two of the sides have lengths of 11 and 14 inches, and that the angle between them is $14°$. You can use this to find the length of the third side:

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos\theta \\ c^2 &= 121 + 196 - (2)(11)(14)(.97) \\ c^2 &= 121 + 196 - 307.384 \\ c^2 &= 9.16 \\ c &= 3.03 \end{aligned}$$

And with this you can use the Law of Sines to solve for the unknown angle:

$$\begin{aligned} \frac{\sin 14°}{3.03} &= \frac{\sin B}{11} \\ \sin B &= \frac{11\sin 14°}{3.03} \\ \sin B &= .878 \\ B &= \sin^{-1}(.0307) = 61.43° \end{aligned}$$

Example 4.1.10.3

While hiking one day you walk for 2 miles in one direction. You then turn $110°$ to the left and walk for 3 more miles. Your path looks like this:

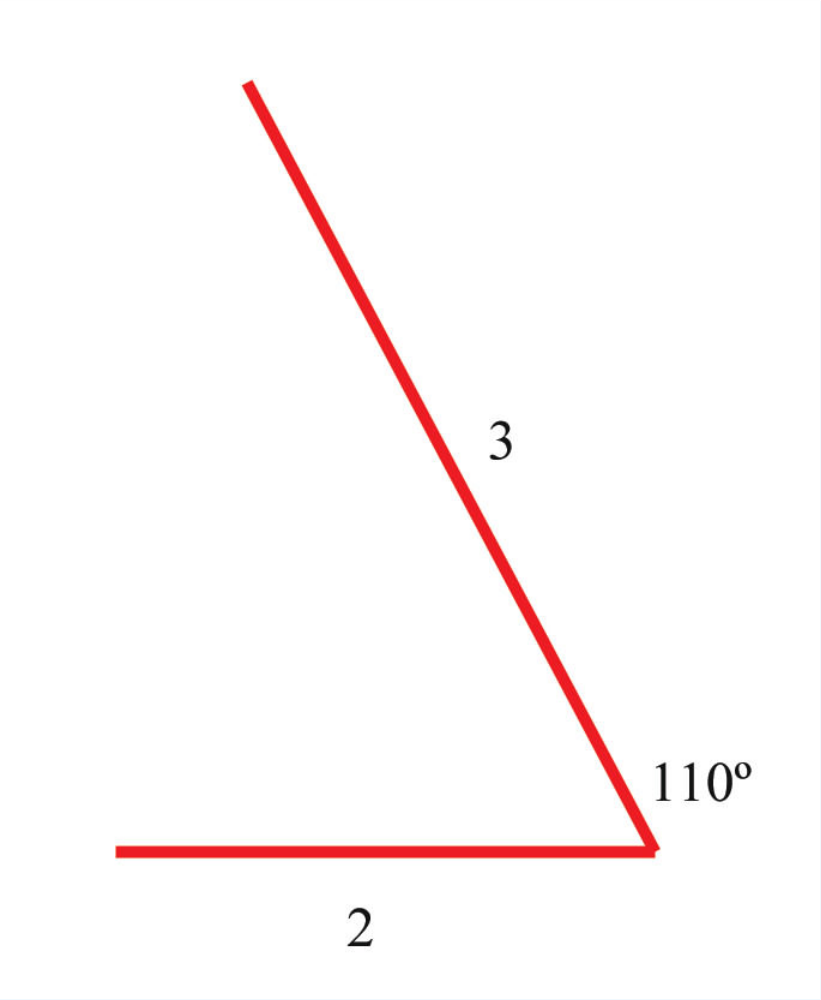


Figure 4.1.10.5

Solution

When you turn to the left again to complete the triangle that is your hiking path for the day, how far will you have to walk to complete the third side? What angle should you turn before you start walking back home?

Since you know the lengths of two of the legs of the triangle, along with the angle between them, you can use the Law of Cosines to find out how far you'll have to walk along the third leg:

$$\begin{aligned} c^2 &= a^2 + b^2 + 2ab\cos 70^\circ \\ c^2 &= 4 + 1 + (2)(2)(1)(.342) \\ c^2 &= 6.368 \\ c &= \sqrt{6.368} \approx 2.52 \end{aligned}$$

Now you have enough information to solve for the interior angle of the triangle that is supplementary to the angle you need to turn:

$$\begin{aligned} \frac{\sin A}{a} &= \frac{\sin B}{b} \\ \frac{\sin 70^\circ}{2.52} &= \frac{\sin B}{2} \\ \sin B &= \frac{2\sin 70^\circ}{2.52} = \frac{1.879}{2.52} = .746 \\ B &= \sin^{-1}(.746) = 48.25^\circ \end{aligned}$$

The angle 48.25° is the interior angle of the triangle. So you should turn $90^\circ + (90^\circ - 48.25^\circ) = 90^\circ + 41.75^\circ = 131.75^\circ$ to the left before starting home.

Example 4.1.10.4

A support at a construction site is being used to hold up a board so that it makes a triangle, like this:

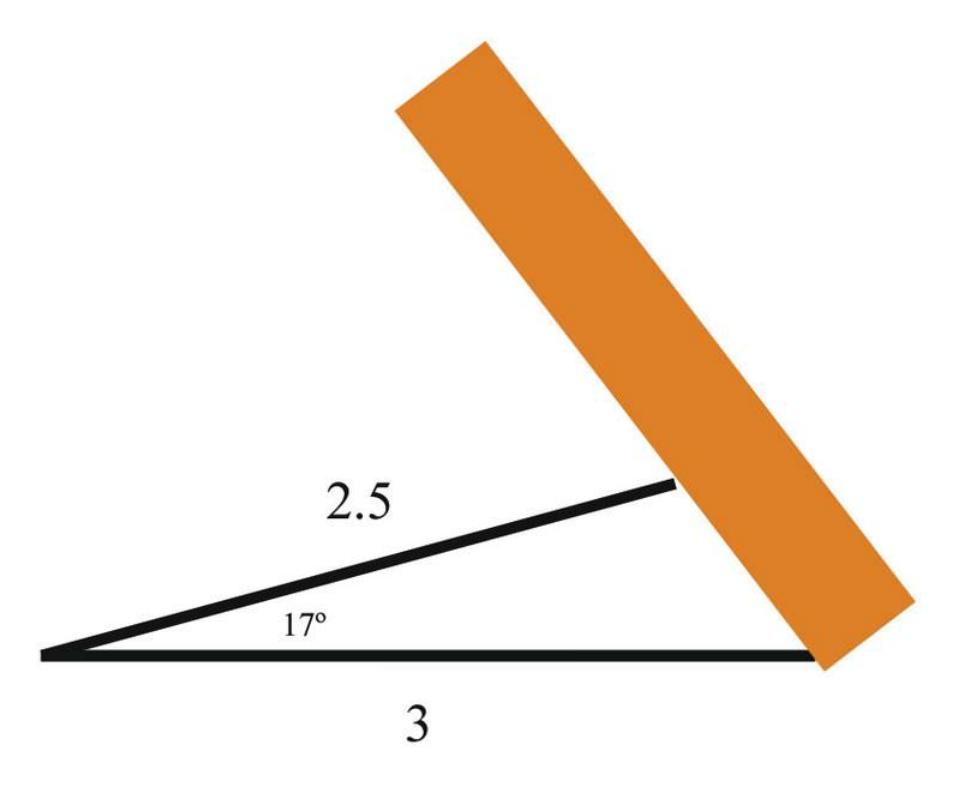


Figure 4.1.10.6

Solution

If the angle between the support and the ground is $17°$, the length of the support is 2.5 meters, and the distance between where the board touches the ground and the bottom of the support is 3 meters, how far along the board is the support touching? What is the angle between the board and the ground?

You should use the Law of Cosines first to solve for the distance from the ground to where the support meets the board:

$$\begin{aligned} c^2 &= a^2 + b^2 + 2ab\cos 17° \\ c^2 &= 6.25 + 9 + (2)(2.5)(3)\cos 17° \\ c^2 &= 6.25 + 9 + (2)(2.5)(3)(.956) \\ c^2 &= 26.722 \\ c &\approx 5.17 \end{aligned}$$

And now you can use the Law of Sines:

$$\begin{aligned} \frac{\sin A}{a} &= \frac{\sin B}{b} \\ \frac{\sin 17°}{5.17} &= \frac{\sin B}{2.5} \\ \sin B &= \frac{2.5\sin 17°}{5.17} = .1414 \\ B &= \sin^{-1}(.1414) = 8.129° \end{aligned}$$

Review

In ΔABC, $a = 12$, $b = 15$, and $c = 20$.

1. Find $m\angle A$.
2. Find $m\angle B$.
3. Find $m\angle C$.

In ΔDEF, $d = 25$, $e = 13$, and $f = 16$.

4. Find $m\angle D$.
5. Find $m\angle E$.
6. Find $m\angle F$.

In ΔKBP, $k = 19$, $\angle B = 61°$, and $p = 12$.

7. Find the length of b.
8. Find $m\angle K$.
9. Find $m\angle P$.
10. While hiking one day you walk for 5 miles due east, then turn to the left and walk 3 more miles $30°$ west of north. At this point you want to return home. How far are you from home if you were to walk in a straight line?
11. A parallelogram has sides of 20 and 31 ft, and an angle of $46°$. Find the length of the longer diagonal of the parallelogram.
12. Dirk wants to find the length of a long building from one side (point A) to the other (point B). He stands outside of the building (at point C), where he is 500 ft from point A and 220 ft from point B. The angle at C is $94°$. Find the length of the building.

Determine whether or not each triangle is possible.

13. $a = 12$, $b = 15$, $c = 10$
14. $a = 1$, $b = 5$, $c = 4$
15. $\angle A = 32°$, $a = 8$, $b = 10$

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.11.

Vocabulary

Term	Definition
Included Angle	The included angle in a triangle is the angle between two known sides.
law of cosines	The law of cosines is a rule relating the sides of a triangle to the cosine of one of its angles. The law of cosines states that $c^2 = a^2 + b^2 - 2ab\cos C$, where C is the angle across from side c.
SAS	SAS means side, angle, side, and refers to the fact that two sides and the included angle of a triangle are known.
SSS	SSS means side, side, side and refers to the fact that all three sides of a triangle are known in a problem.

Additional Resources

Video: Solving triangles u\sin g the Law of Cosines - Example 1

Practice: Applications of the Law of Cosines

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SECTION OVERVIEW

4.2: Area of Non-Right Triangles

4.2.1: Area Formula for Non-Right Triangles

4.2.2: Introduction to the Triangle Area Formula

4.2.3: Determination of Unknown Triangle Measures Given Area

4.2.4: Heron's Formula

4.2.5: General Solutions of Triangles

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4.2.1: Area Formula for Non-Right Triangles

Area equals half the product of two sides and the sine of the included angle.

Alternate Formula for the Area of a Triangle

You are studying the Gulf of Mexico in your Geography class. Your Instructor brings up the idea of the Bermuda Triangle. This is a place where, according to some, many planes get lost. Here is an illustration of it:

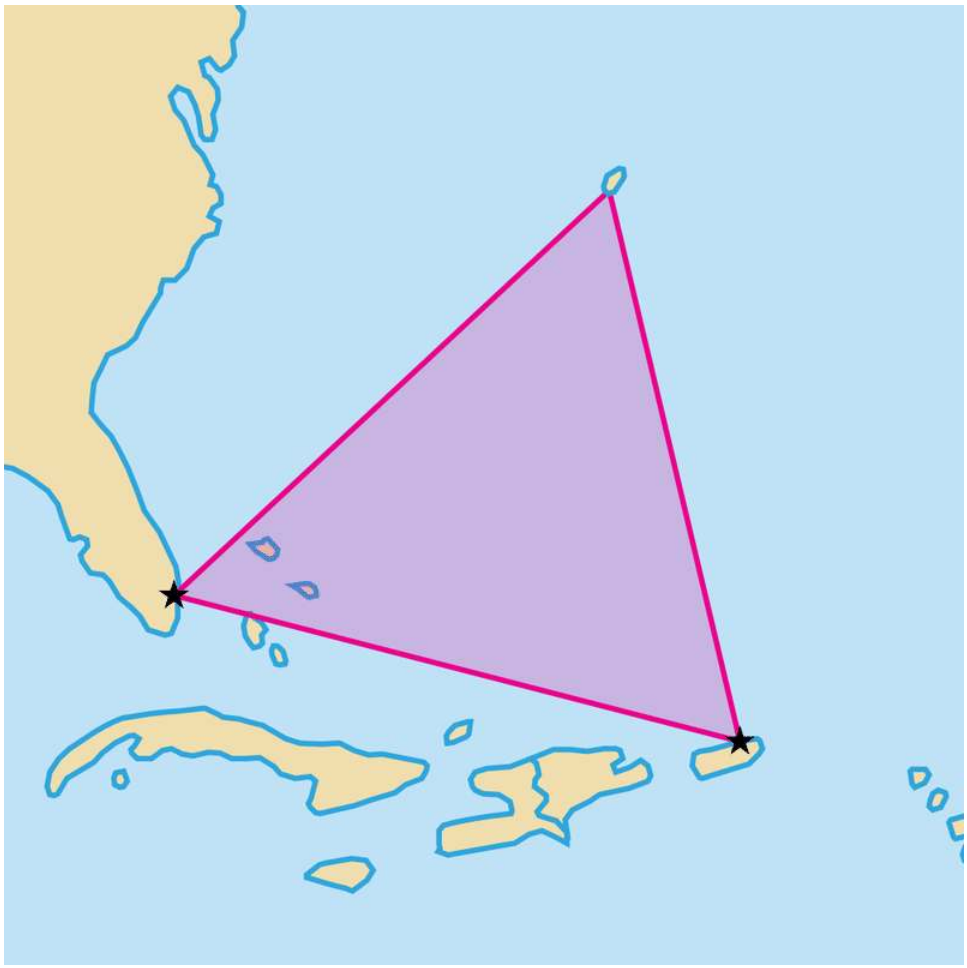


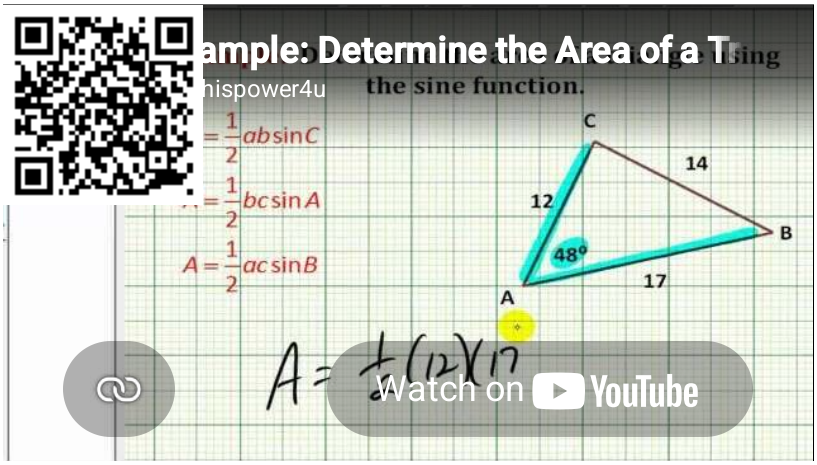
Figure 4.2.1.1

The first thing this makes you think of is your math class, since that class is your favorite. You would like to know just how big the Bermuda Triangle is. Unfortunately, the Bermuda Triangle isn't a right triangle. However, you do know that the lengths of one of the sides is 950 miles, the other side is 975 miles, and the angle between them is $60°$.

Is there any way to use this information to help you find out just how big the Bermuda Triangle is?

Area of a Triangle

In Geometry, you learned that the area of a triangle is $A = \frac{1}{2}bh$, where b is the base and h is the height, or altitude.



Now that you know the trig ratios, this formula can be changed around, using **sine**.

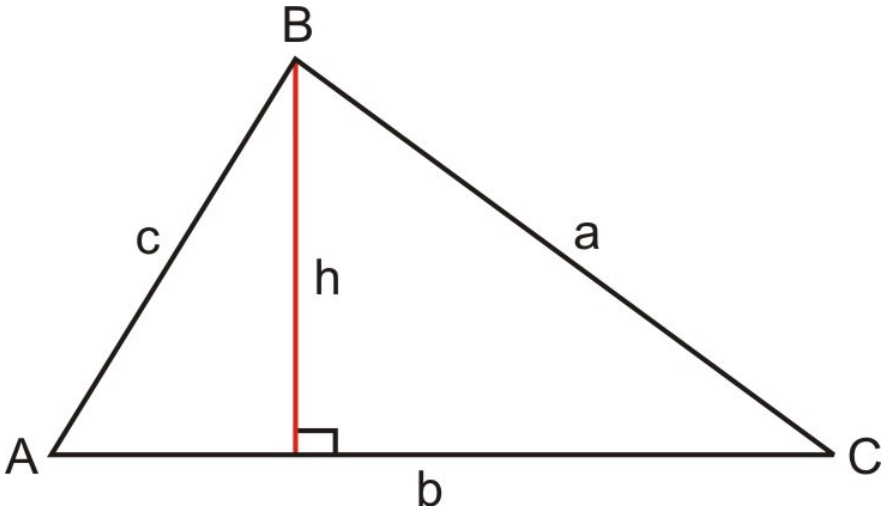


Figure 4.2.1.2

Looking at the triangle above, you can use sine to determine h, $\sin C = \frac{h}{a}$. So, solving this equation for h, we have $a \sin C = h$. Substituting this for h, we now have a new formula for area.

$$A = \frac{1}{2}ab\sin C$$

What this means is you do not need the height to find the area anymore. All you now need is two sides and the angle between the two sides, called the **included angle**.

Finding Area

1.

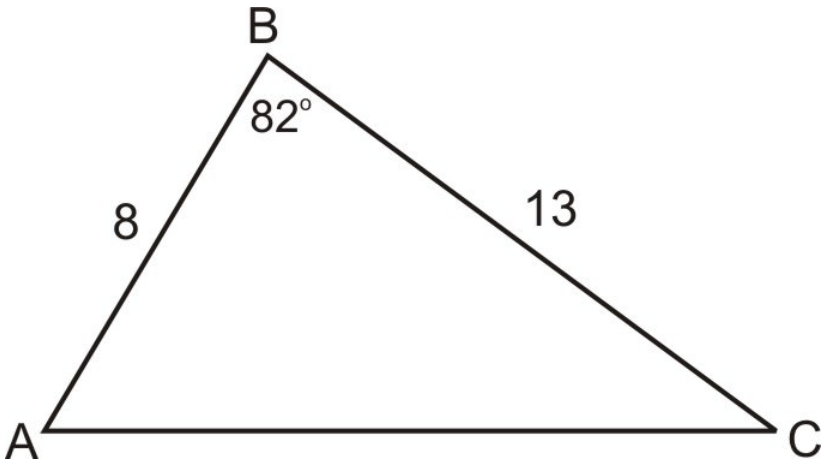


Figure 4.2.1.3

Using the formula, $A = \frac{1}{2}ab\sin C$, we have

$$\begin{aligned} A &= \frac{1}{2} \cdot 8 \cdot 13 \cdot \sin 82^\circ \\ &= 4 \cdot 13 \cdot 0.990 \\ &= 51.494 \end{aligned}$$

2.

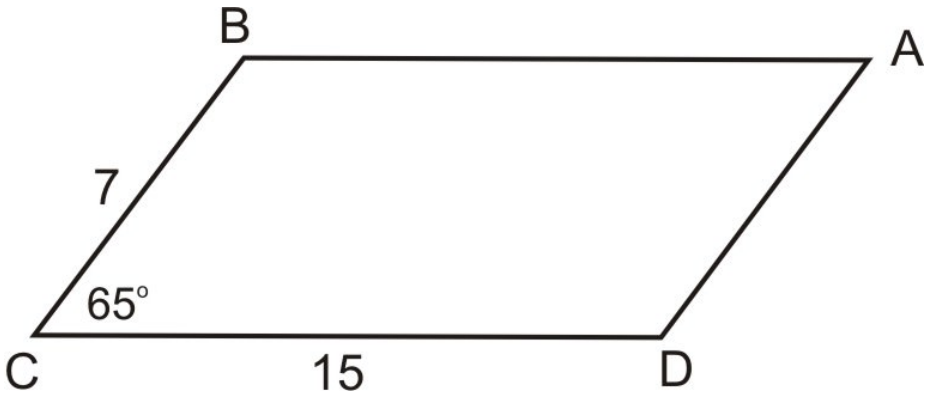


Figure 4.2.1.4

Recall that a parallelogram can be split into two triangles. So the formula for a parallelogram, using the new formula, would be: $A = 2 \cdot \frac{1}{2}ab\sin C$ or $A = ab\sin C$.

$$\begin{aligned} A &= 7 \cdot 15 \cdot \sin 65^\circ \\ &= 95.162 \end{aligned}$$

3.

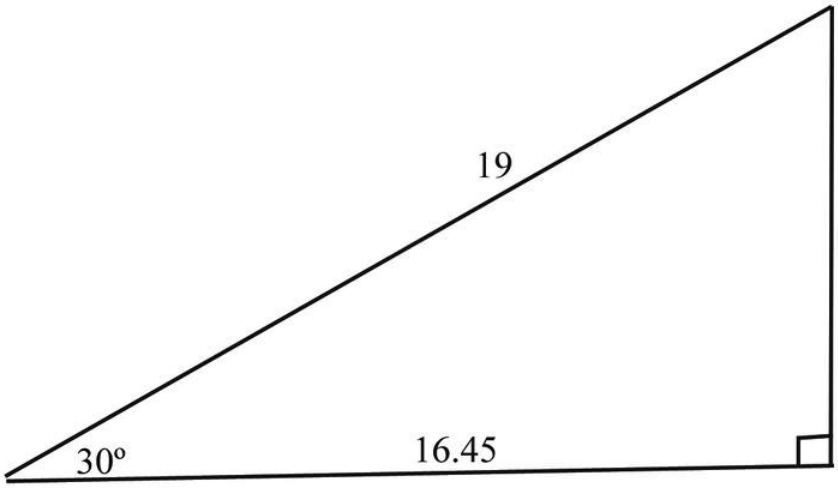


Figure 4.2.1.5

Using the formula, $A = \frac{1}{2}ab\sin C$, we have

$$\begin{aligned} A &= \frac{1}{2} \cdot 16.45 \cdot 19 \cdot \sin 30^\circ \\ &= 8.225 \cdot 19 \cdot 0.5 \\ &= 78.14 \end{aligned}$$

Example 4.2.1.1

Earlier, you were asked how big the Bermuda Triangle is.

Solution

Now that you know the equation for the area of a triangle in terms of two of the sides and the included angle, we can use that to solve for the area of the Bermuda Triangle:

$$\begin{aligned} A &= \frac{1}{2}ab\sin\theta \\ A &= \frac{1}{2}(950)(975)\sin 60^\circ \\ A &= \frac{1}{2}(950)(975)(.866) \\ A &= 401066.25 \end{aligned}$$

The area of the triangle is approximately 401,006 square miles.

Example 4.2.1.2

Find the area of the triangle.

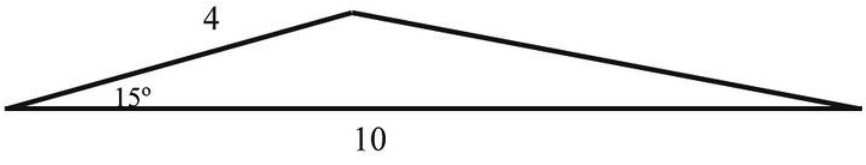


Figure 4.2.1.6

Solution

Using the formula, $A = \frac{1}{2}ab\sin C$, we have

$$\begin{aligned} A &= \frac{1}{2} \cdot 4 \cdot 10 \cdot \sin 15^\circ \\ &= 2 \cdot 10 \cdot 0.2589 \\ &= 5.178 \end{aligned}$$

Example 4.2.1.3

Find the area of the triangle.

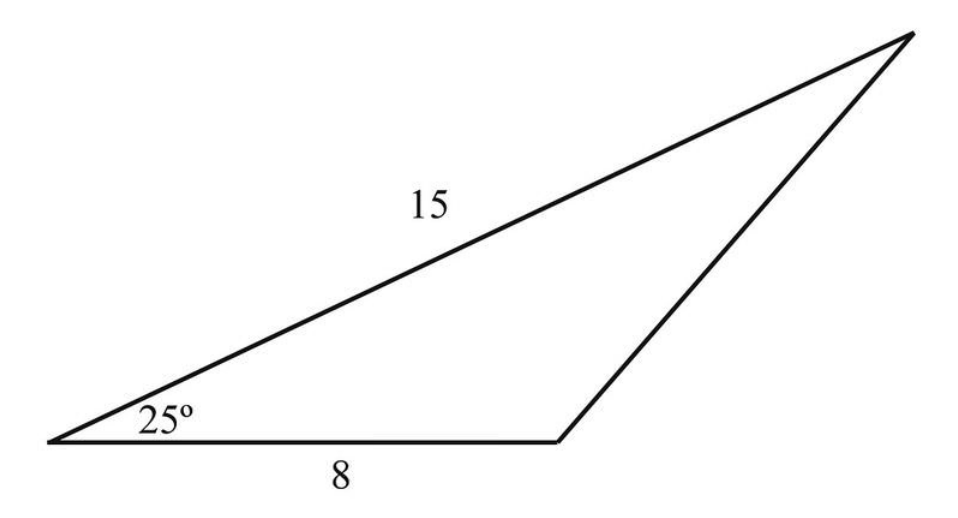


Figure 4.2.1.7

Solution

Using the formula, $A = \frac{1}{2}ab\sin C$, we have

$$\begin{aligned} A &= \frac{1}{2} \cdot 8 \cdot 15 \cdot \sin 25^\circ \\ &= 4 \cdot 15 \cdot 0.4226 \\ &= 25.356 \end{aligned}$$

Example 4.2.1.4

Find the area of the triangle.

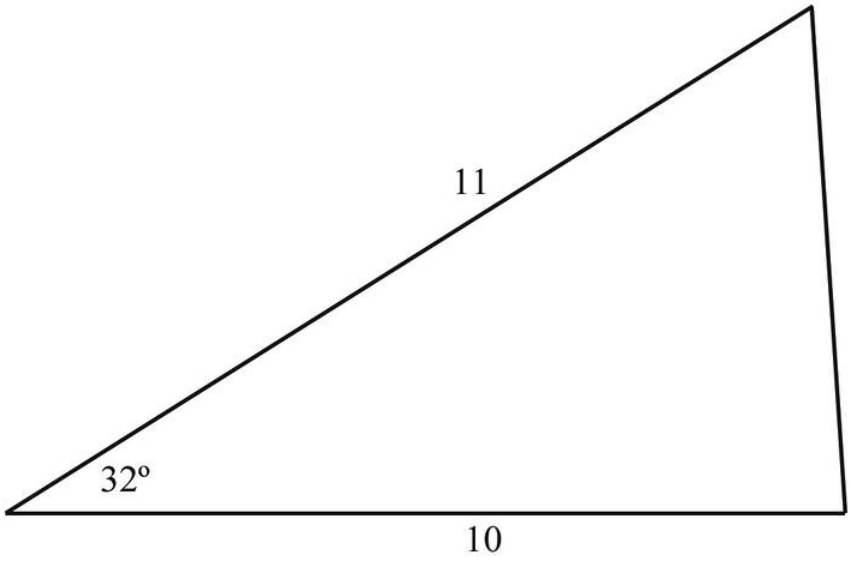


Figure 4.2.1.8

Solution

Using the formula, $A = \frac{1}{2}ab\sin C$, we have

$$\begin{aligned} A &= \frac{1}{2} \cdot 10 \cdot 11 \cdot \sin 32^\circ \\ &= 5 \cdot 11 \cdot 0.53 \\ &= 29.15 \end{aligned}$$

Review

Use the following picture for questions 1 and 2.

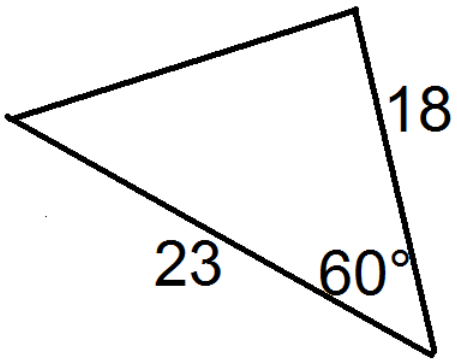


Figure 4.2.1.9

1. Find the values of a, b, and C needed for the formula to find the area of the triangle.
2. Now find the area of the triangle.

Use the following picture for questions 3 and 4.

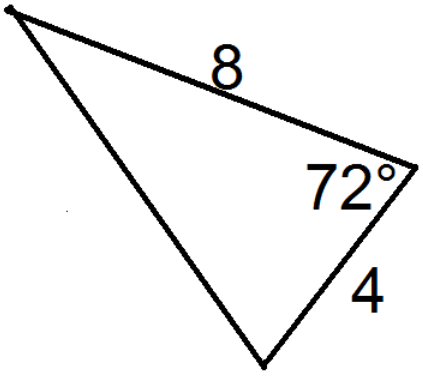


Figure 4.2.1.10

3. Find the values of a, b, and C needed for the formula to find the area of the triangle.
4. Now find the area of the triangle.

Find the area of each triangle below.

5.

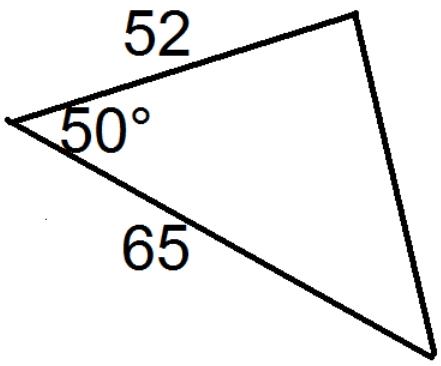


Figure 4.2.1.11

6.

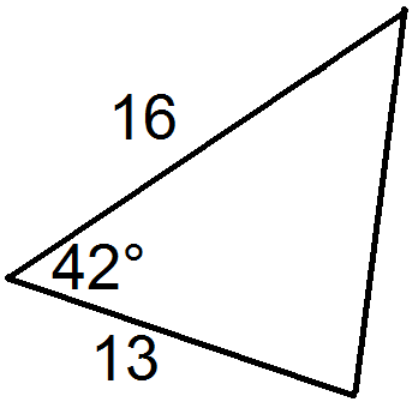


Figure 4.2.1.12

7.

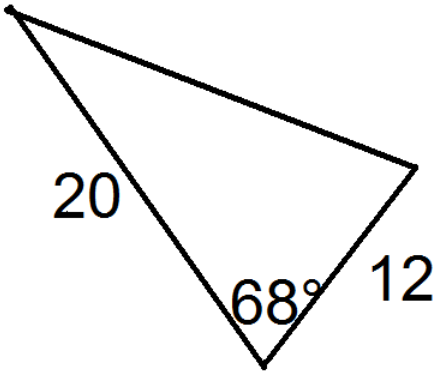


Figure 4.2.1.13

8.

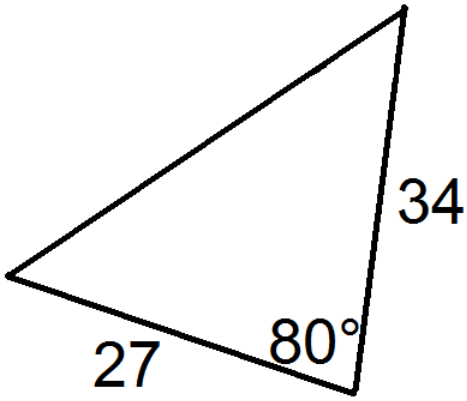


Figure 4.2.1.14

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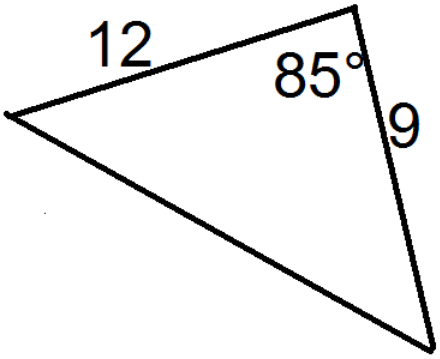


Figure 4.2.1.15

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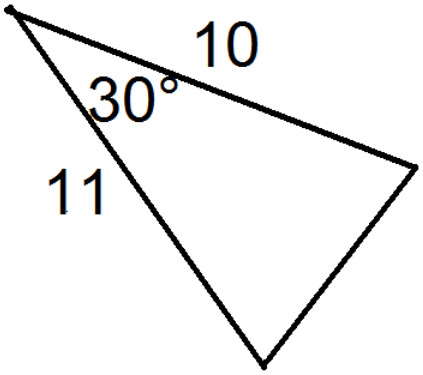


Figure **4.2.1.16**

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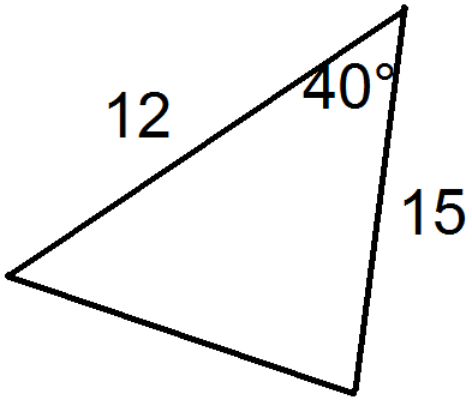


Figure **4.2.1.17**

Find the area of each parallelogram.

12.

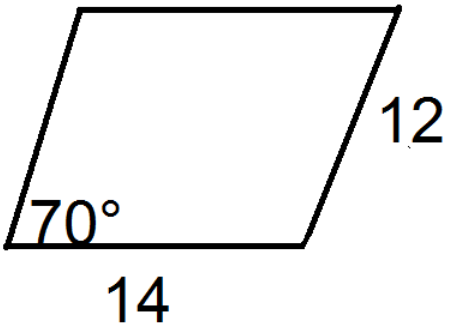
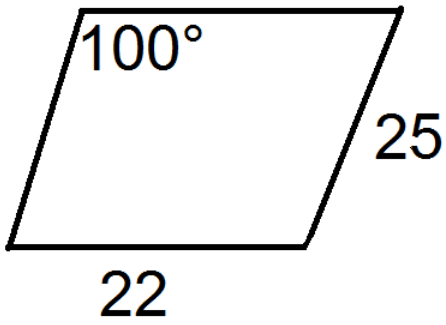


Figure **4.2.1.18**



13.

Figure **4.2.1.19**

14. Describe another way you could have found the area of the parallelogram in the previous problem.
15. When you first learned about sine, you learned how it worked for right triangles. Explain why this method for calculating area uses sine, but works for non-right triangles.

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.12.

Vocabulary

Term	Definition
Included Angle	The included angle in a triangle is the angle between two known sides.
sine	The sine of an angle in a right triangle is a value found by dividing the length of the side opposite the given angle by the length of the hypotenuse.

Additional Resources

Video: Example: Determine the Area of a Triangle Using the Sine Function

Practice: Area Formula for Non-Right Triangles

Real World: An Unusual Area?

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4.2.2: Introduction to the Triangle Area Formula

Find areas of non-right triangles using the sine ratio

While in the lunch room with your friends one day, you're discussing different ways you can use the things you've learned in math class. You tell your friends that you've been learning a lot about triangles, such as how to find their area. One of your friends looks down at your plate and starts to smile.

"Alright," he says. "If you're so good at things involving triangles, I dare you to find something simple. Tell me the area of your slice of pizza." He points down at the pizza on your plate.

The pizza is shaped like a triangle. But unfortunately its not a right triangle. The outer edge is 5 inches long, and the long sides are 7 inches long. The angle between the edge and the long side of the slice is $69°$. Is there any way to tell the area of your pizza slice?

Deriving the Triangle Area Formula

We can use the area formula from Geometry, $A = \frac{1}{2}bh$, as well as the sine function, to derive a new formula that can be used when the height, or altitude, of a triangle is unknown.

In ΔABC below, BD is altitude from B to AC. We will refer to the length of BD as h since it also represents the height of the triangle. Also, we will refer to the area of the triangle as K to avoid confusing the area with $\angle A$.

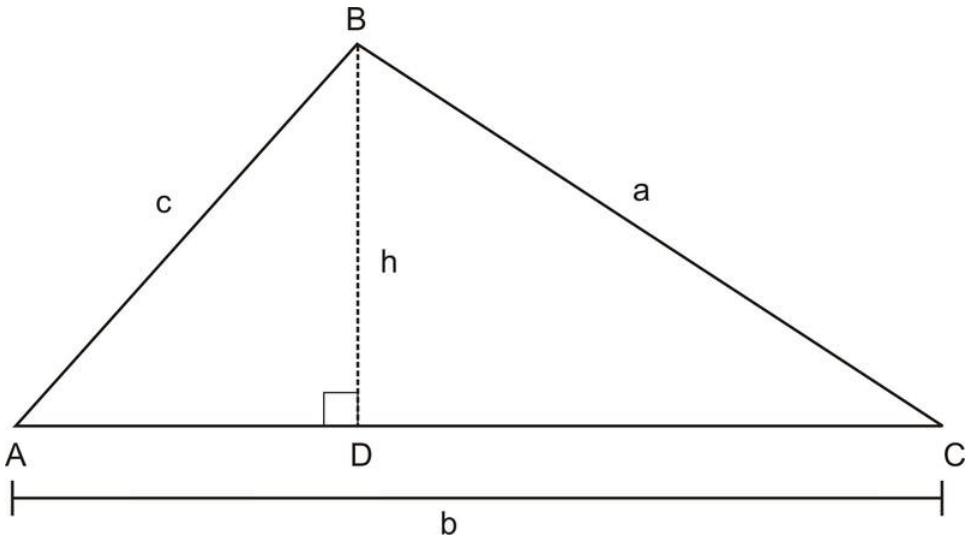


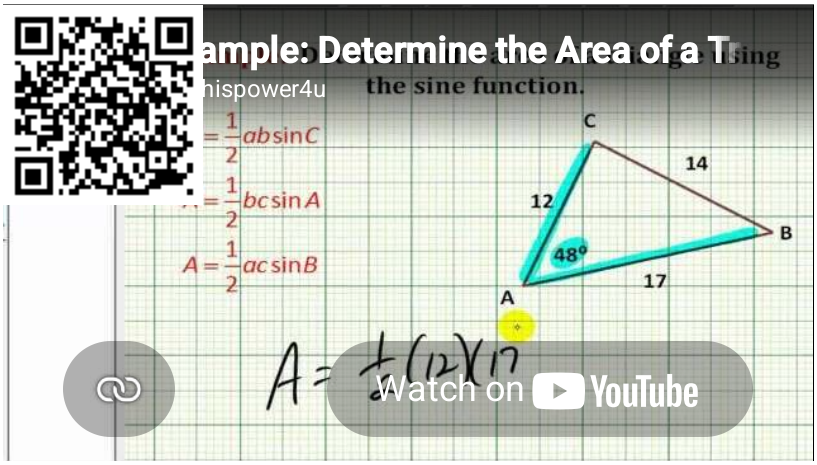
Figure 4.2.2.1

$$k = \frac{1}{2}bh \qquad \text{Area of a triangle}$$
$$k = \frac{1}{2}b(c\sin A) \qquad \sin A = \frac{h}{c} \text{ therefore } c\sin A = h$$
$$k = \frac{1}{2}bc\sin A \qquad \text{Simplify}$$

We can use a similar method to derive all three forms of the area formula, regardless of the angle:

$$K = \frac{1}{2}bc\sin A$$
$$K = \frac{1}{2}ac\sin B$$
$$K = \frac{1}{2}ab\sin C$$

The formula $K = \frac{1}{2}bc\sin A$ requires us to know two sides and the included angle (SAS) in a triangle. Once we know these three things, we can easily calculate the area of an **oblique triangle**.



Finding the Area of a Triangle

1. In ΔABC, $\angle C = 62°$, $b = 23.9$, and $a = 31.6$. Find the area of the triangle.

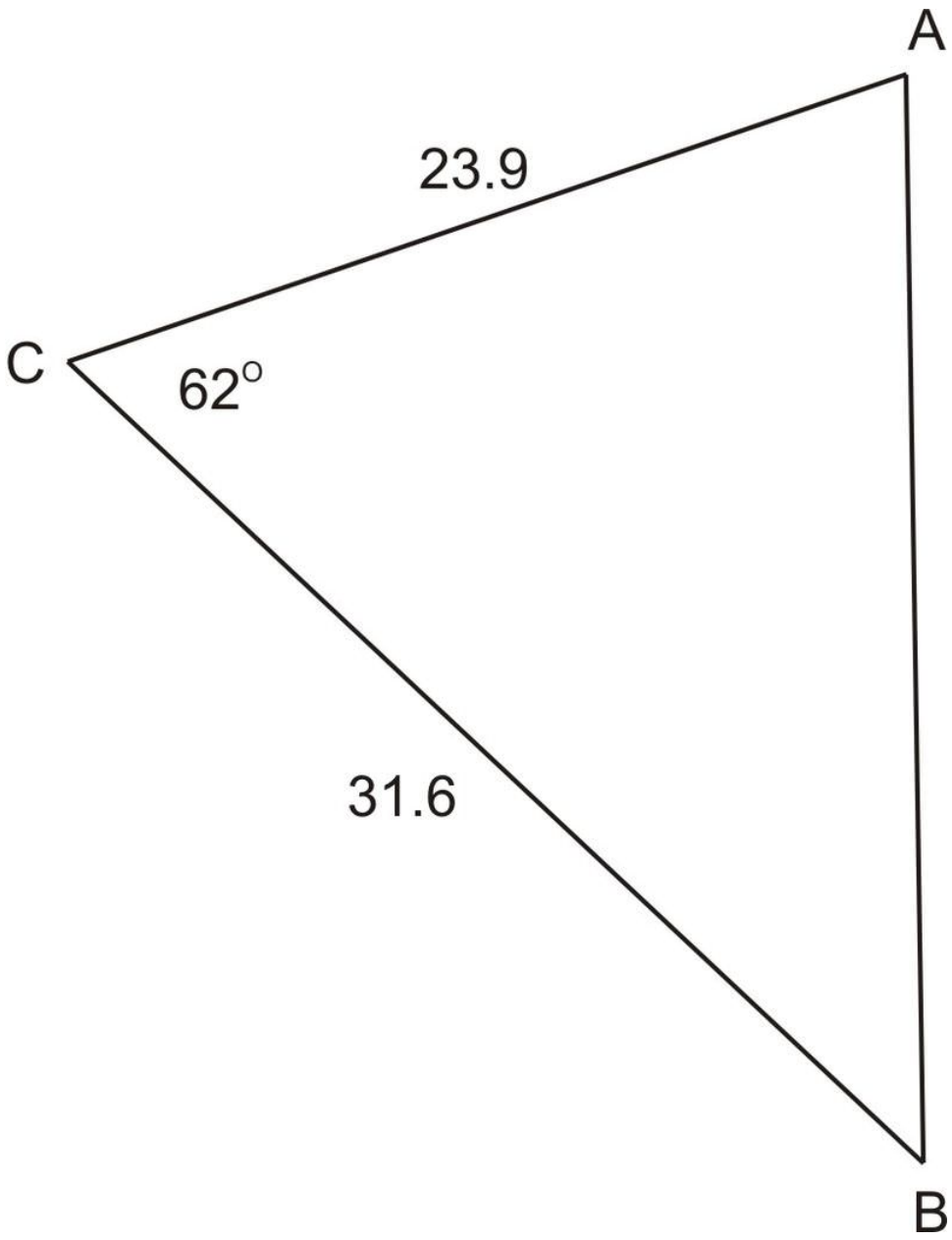


Figure 4.2.2.2

Using our new formula, $K = \frac{1}{2}ab\sin C$, plug in what is known and solve for the area.

$$K = \frac{1}{2}(31.6)(23.9)\sin 62$$
$$K \approx 333.4$$

2. The Pyramid Hotel recently installed a triangular pool. One side of the pool is 24 feet, another side is 26 feet, and the angle in between the two sides is $87°$. If the hotel manager needs to order a cover for the pool, and the cost is $\$35$ per square foot, how much can he expect to spend?

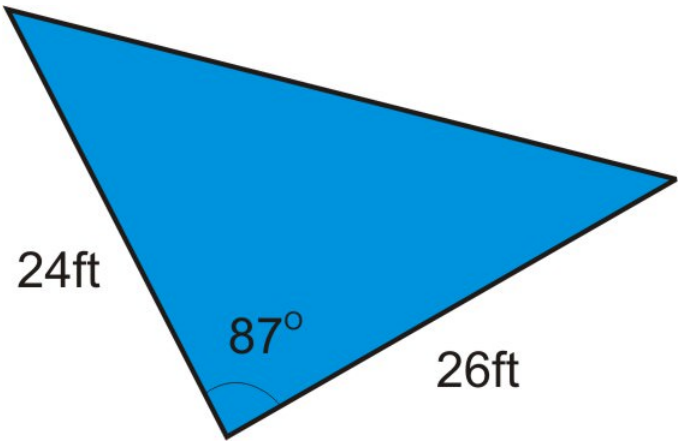


Figure **4.2.2.3**

In order to find the cost of the cover, we first need to know the area of the cover. Once we know how many square feet the cover is, we can calculate the cost. In the illustration above, you can see that we know two of the sides and the included angle. This means we can use the formula $K = \frac{1}{2}bc\sin A$.

$$K = \frac{1}{2}(24)(26)\sin 87$$
$$K \approx 311.6$$
$$311.6 \text{ sq.ft.} \times \$35/\text{ sq.ft.} = \$10,905.03$$

The cost of the cover will be $\$10,905.03$.

3. In ΔGHI, $\angle I = 15°$, $g = 14.2$, and $h = 7.9$. Find the area of the triangle.

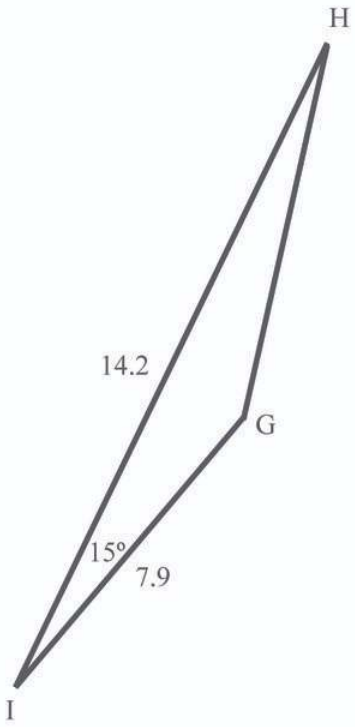


Figure 4.2.2.4

Using our new formula, $K = \frac{1}{2}ab\sin C$, which is the same as $K = \frac{1}{2}gh\sin I$, plug in what is known and solve for the area.

$$K = \frac{1}{2}(14.2)(7.9)\sin 15$$
$$K \approx 14.52$$

Example 4.2.2.1

Earlier, you were asked if there was any way to tell the area of your pizza slice.

Solution

Use the formula $K = \frac{1}{2}bc\sin A$

where in this case, one of the sides is equal to 5, the other is equal to 7, and the angle is $69°$.

$$K = \frac{1}{2}(5)(7)\sin 69° = 16.34 \text{ in}^2$$

Example 4.2.2.2

A farmer needs to replant a triangular section of crops that died unexpectedly. One side of the triangle measures 186 yards, another measures 205 yards, and the angle formed by these two sides is $148°$.

What is the area of the section of crops that needs to be replanted?

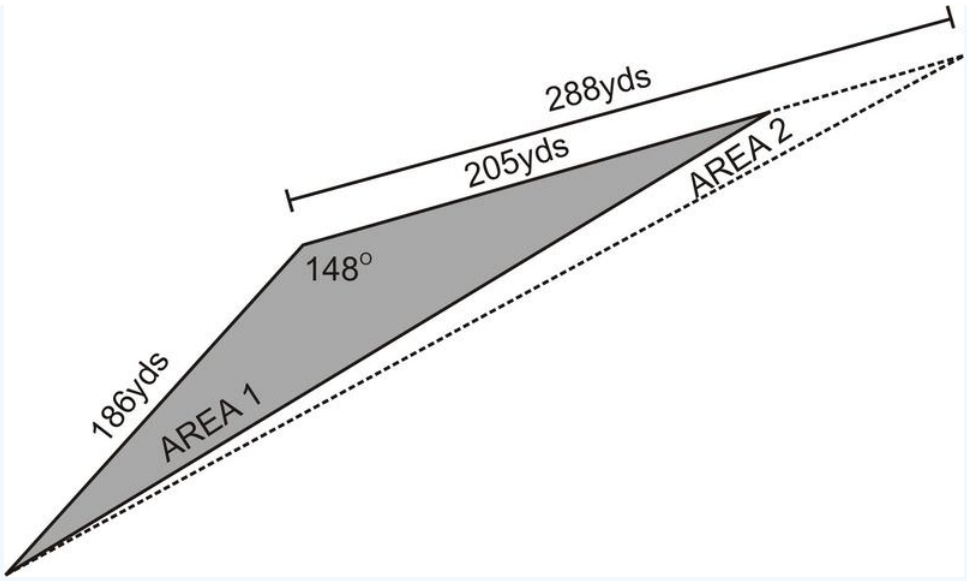


Figure 4.2.2.5

Solution

Use $K = \frac{1}{2}bc\sin A$, $K = \frac{1}{2}(186)(205)\sin 148°$. So, the area that needs to be replaced is 10102.9 square yards.

Example 4.2.2.3

The farmer goes out a few days later to discover that more crops have died. The side that used to measure 205 yards now measures 288 yards. How much has the area that needs to be replanted increased by?

Solution

$K = \frac{1}{2}(186)(288)\sin 148° = 14193.4$, the area has increased by 4090.5 yards.

Example 4.2.2.4

Find the perimeter of the quadrilateral at the left If the area of $\Delta DEG = 56.5$ and the area of $\Delta EGF = 84.7$.

Solution

You need to use the $K = \frac{1}{2}bc\sin A$ formula to find DE and GF.

$$56.5 = \frac{1}{2}(13.6)DE\sin 39° \rightarrow DE = 13.2 \qquad 84.7 = \frac{1}{2}(13.6)EF\sin 60° \rightarrow EF = 14.4$$

Second, you need to find sides DG and GF using the Law of Cosines.

$$DG^2 = 13.2^2 + 13.6^2 - 2 \cdot 13.2 \cdot 13.6 \cdot \cos 39° \rightarrow DG = 8.95$$
$$GF^2 = 14.4^2 + 13.6^2 - 2 \cdot 14.4 \cdot 13.6 \cdot \cos 60° \rightarrow GF = 14.0$$

The perimeter of the quadrilateral is 50.55.

Review

Find the area of each triangle.

1. ΔABC if $a = 13$, $b = 15$, and $m\angle C = 71°$.
2. ΔABC if $b = 8$, $c = 4$, and $m\angle A = 67°$.
3. ΔABC if $b = 34$, $c = 29$, and $m\angle A = 138°$.
4. ΔABC if $a = 3$, $b = 7$, and $m\angle C = 80°$.
5. ΔABC if $a = 4.8$, $c = 3.7$, and $m\angle B = 43°$.
6. ΔABC if $a = 12$, $b = 5$, and $m\angle C = 20°$.

7. ΔABC if $a = 3$, $b = 10$, and $m\angle C = 50°$.
8. ΔABC if $a = 5$, $b = 9$, and $m\angle C = 14°$.
9. ΔABC if $a = 5$, $b = 7$, and $c = 11$.
10. ΔABC if $a = 7$, $b = 8$, and $c = 9$.
11. ΔABC if $a = 12$, $b = 14$, and $c = 4$.
12. A farmer measures the three sides of a triangular field and gets 114, 165, and 257 feet. What is the measure of the largest angle of the triangle?
13. Using the information from the previous problem, what is the area of the field?

Another field is a quadrilateral where three sides measure 30, 50, and 60 yards, and two angles measure $130°$ and $140°$, as shown below.

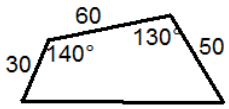


Figure **4.2.2.6**

14. Find the area of the quadrilateral. Hint: divide the quadrilateral into two triangles and find the area of each.
15. Find the length of the fourth side.
16. Find the measures of the other two angles.

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.4.

Vocabulary

Term	Definition
Oblique Triangle	An oblique triangle is a triangle without a right angle as one of its internal angles.
SAS Triangle	An SAS triangle is a triangle where two sides and the angle in between them are known quantities.

Additional Resources

Video: Finding the Area of an Oblique Triangle using the Law of Sines - Example 1

Practice: Introduction to the Triangle Area Formula

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4.2.3: Determination of Unknown Triangle Measures Given Area

Solve for missing triangle parts using area formulas and the law of cosines/sines.

You are working on creating a mobile for your art class. A mobile is a piece of art that has a rod with different shapes hanging from it, so they can spin.

To create your project, you need to cut a set of triangles that have a variety of sizes. You are about to start cutting triangles, when your friend, who is helping you with the project, comes over. She tells you that each piece needs to have a rod through the side of it to balance the shape in a certain way. She wants you to make a piece that looks like this:

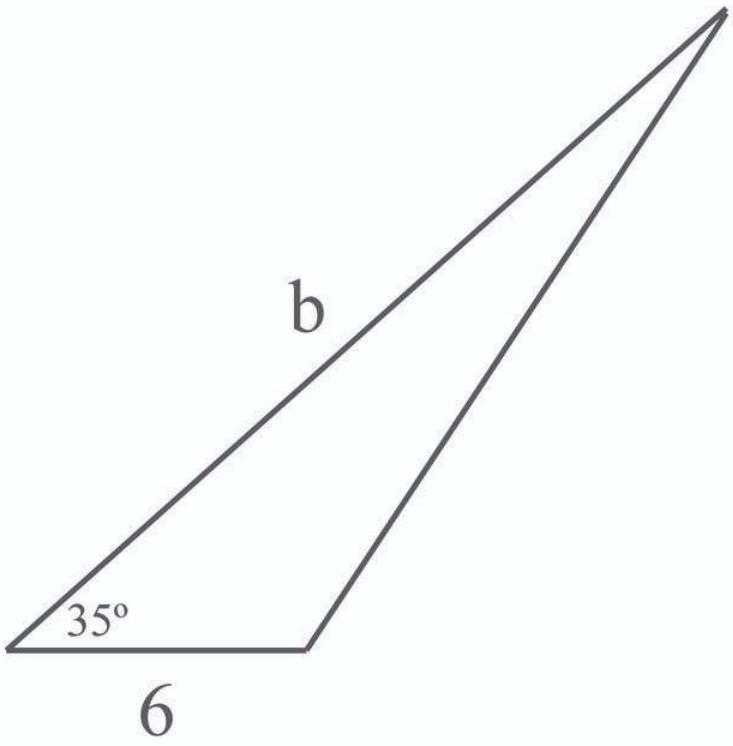


Figure **4.2.3.1**

You have already cut a triangle by cutting a piece out of construction paper. You know that one side of your triangle is 6 inches long, but you don't know the length of the other two sides. Can you use the information you have to find the length of b in the mobile piece above? (The area of the triangle is $25\ \text{in}^2$, and the interior angle between the six inch side and the side you want to know is $35°$).

Finding the Measures of a Triangle Given the Area

In this section, we will look at situations where we know the area but need to find another part of the triangle, as well as an application involving a quadrilateral. All of this will involve the use of the **Law of Cosines**, **Law of Sines**, and the Alternate Formula for the Area of a Triangle.

1. The jib sail on a sailboat came untied and the rope securing it was lost. If the area of the jib sail is 56.1 square feet, use the figure and information below to find the length of the rope.

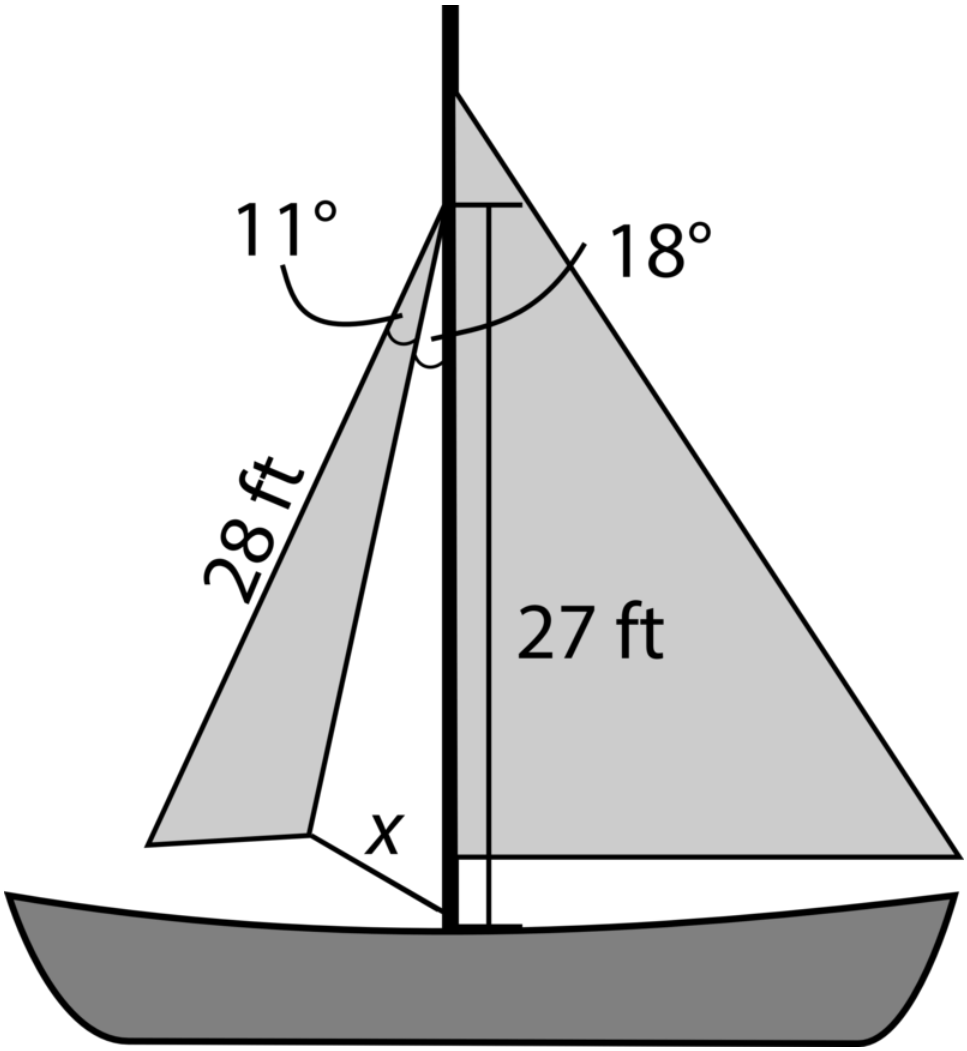


Figure 4.2.3.2

Since we know the area, one of the sides, and one angle of the jib sail, we can use the formula $K = \frac{1}{2}bc\sin A$ to find the side of the jib sail that is attached to the mast. We will call this side y.

$$56.1 = \frac{1}{2}28(y)\sin 11$$
$$56.1 = 2.671325935y$$
$$21.0 = y$$

Now that we know side y, we know two sides and the included angle in the triangle formed by the mast, the rope, and the jib sail. We can now use the Law of Cosines to calculate the length of the rope.

$$x^2 = 21^2 + 27^2 - 2(21)(27)\cos 18$$
$$x^2 = 91.50191052$$
$$x \approx 9.6 \text{ ft}$$

The length of the rope is approximately 9.6 feet.

2. In quadrilateral $QUAD$ below, the area of $\Delta QUD = 5.64$, the area of $\Delta UAD = 6.39$, $\angle QUD = 31°$, $\angle DUA = 40°$, and $UD = 7.8$. Find the perimeter of $QUAD$.

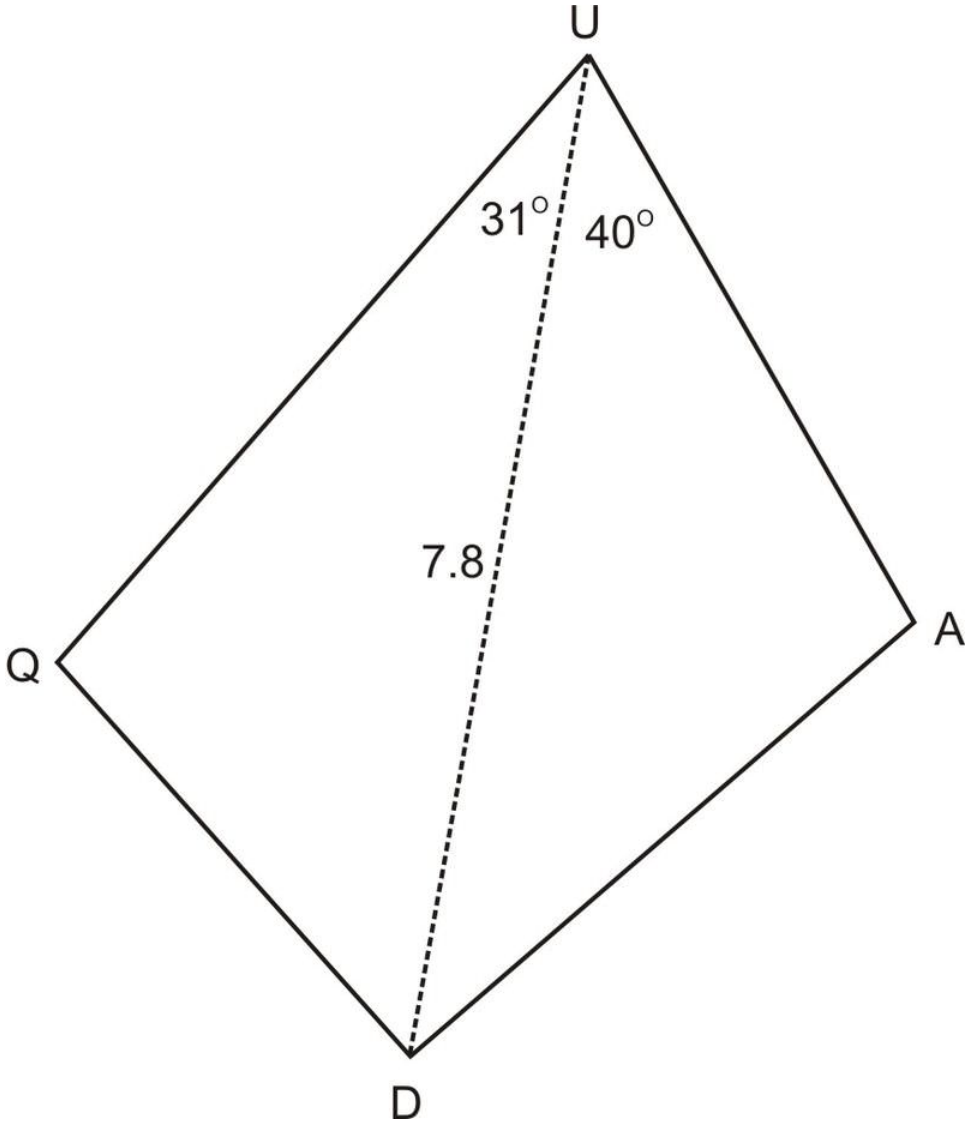


Figure 4.2.3.3

In order to find the perimeter of $QUAD$, we need to know sides QU, QD, UA, and AD. Since we know the area, one side, and one angle in each of the triangles, we can use $K = \frac{1}{2}bc\sin A$ to figure out QU and UA.

$$5.64 = \frac{1}{2}(7.8)(QU)\sin 31 \qquad 6.39 = \frac{1}{2}(7.8)UA\sin 40$$
$$2.8 \approx QU \qquad 2.5 \approx UA$$

Now that we know QU and UA, we know two sides and the included angle in each triangle (SAS). This means that we can use the Law of Cosines to find the other two sides, QD and DA. First we will find QD and DA.

$$QD^2 = 2.8^2 + 7.8^2 - 2(2.8)(7.8)\cos 31 \qquad DA^2 = 2.5^2 + 7.8^2 - 2(2.5)(7.8)\cos 40$$
$$QD^2 = 31.23893231 \qquad DA^2 = 37.21426672$$
$$QD \approx 5.6 \qquad DA \approx 6.1$$

Finally, we can calculate the perimeter since we have found all four sides of the quadrilateral.

$$pQUAD = 2.8 + 5.6 + 6.1 + 2.5 = 17$$

3. In ΔABC, BD is an altitude from B to AC. The area of $\Delta ABC = 232.96$, $AB = 16.2$, and $AD = 14.4$. Find DC.

First, find BD by using the Pythagorean Theorem. $BD = \sqrt{16.2^2 - 14.4^2} = 7.42$. Then, using the area and formula ($A = \frac{1}{2}bh$), you can find AC.

$$232.96 = \frac{1}{2}(7.42)AC \rightarrow AC = 62.78.\ DC = 62.78 - 14.4 = 48.38.$$

Example 4.2.3.1

Earlier, you were asked to find the length of b in the mobile piece.

Solution

Since you know that the mobile piece is six inches on one side, and that the area of the triangle is $25\ \text{in}^2$, you can use the formula $K = \frac{1}{2}ab\sin C$ to find the length of the other side:

$$\begin{aligned} K &= \frac{1}{2}ab\sin C \\ 25 &= \frac{1}{2}(6)(b)\sin 35^\circ \\ 25 &= 1.72b \\ b &= \frac{25}{1.72} \\ b &= 14.53\ \text{in} \end{aligned}$$

Example 4.2.3.2

Find "h" in the triangle below: $\text{Area} = 1618.98$, $b = 36.3$

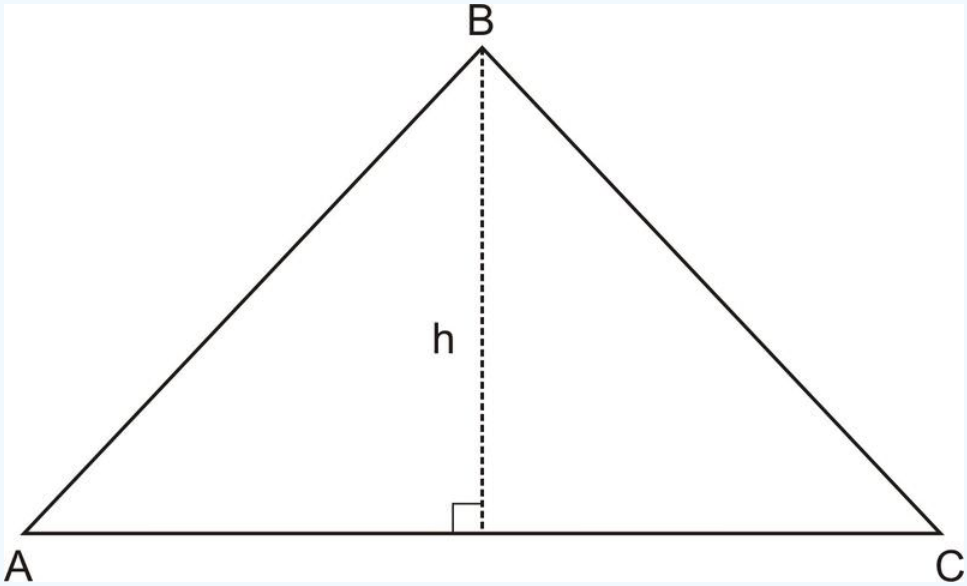


Figure 4.2.3.4

Solution

Since we know the area, one of the sides (18.15), and one angle of the triangle (45°), we can use the formula $K = \frac{1}{2}bc\sin A$ to find the other side of the triangle. We can then use the Pythagorean Theorem to find the height of the triangle.

This gives a result of:

$h = 89.2$

Example 4.2.3.3

Find $\angle A$ in the triangle below: $\text{Area} = 387.6$, $b = 25.6$, $c = 32.9$

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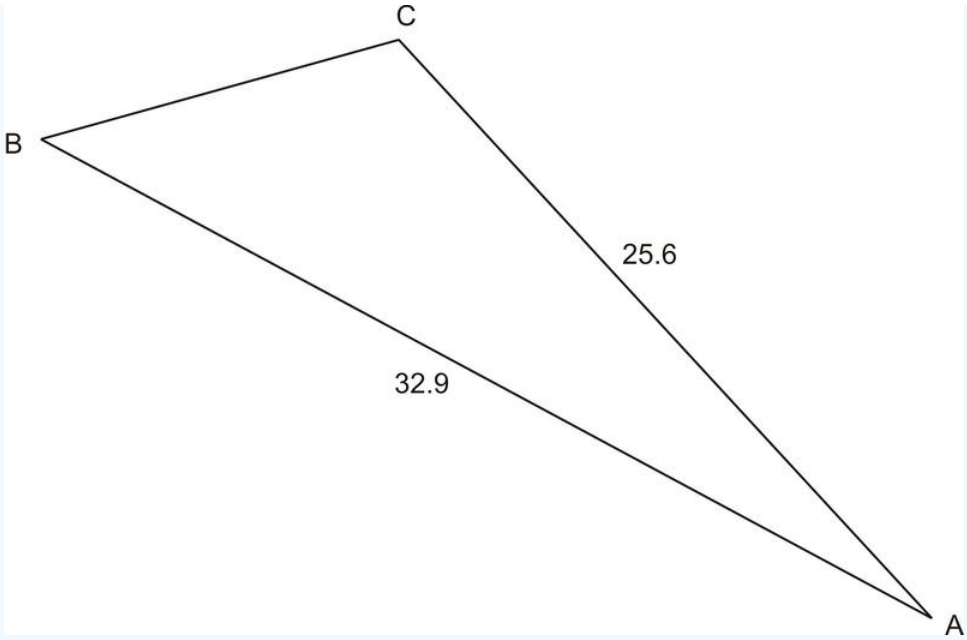


Figure 4.2.3.5

Solution

Since we know the area and the lengths of two of the sides of the triangle, we can use the formula $K = \frac{1}{2}bc\sin A$ to solve for the included angle, which gives:

$$\angle A = 67^\circ$$

Example 4.2.3.4

Find the area of ΔABC below: $\text{Area}\Delta ABD = 16.96$, $AD = 3.2$, $\angle DBC = 49.6^\circ$

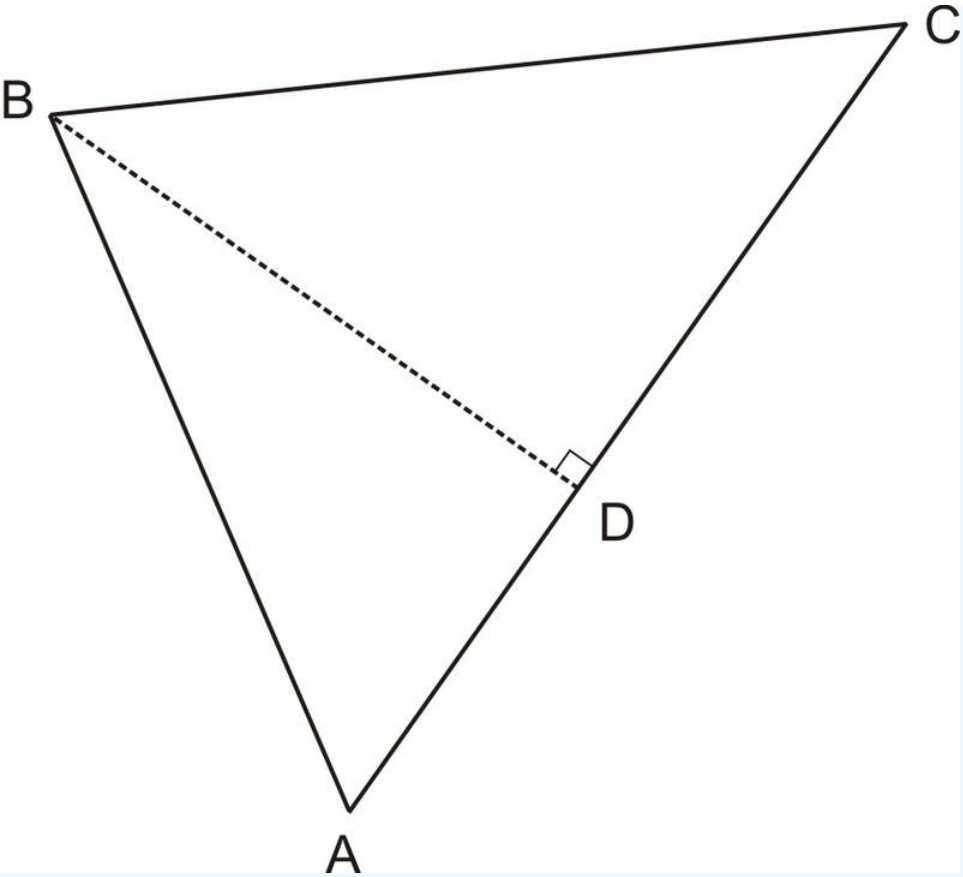


Figure 4.2.3.6

Solution

$$\text{Area of } \Delta ABC = 83.0$$

Review

1. The area of the triangle below is $138\ \text{in}^2$. Solve for x, the height.

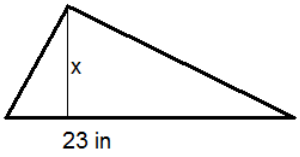


Figure 4.2.3.7

2. The area of the triangle below is $250\ \text{cm}^2$. A height is given on the diagram. Solve for x.

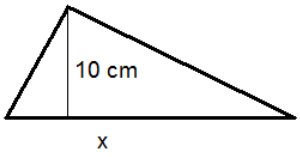


Figure 4.2.3.8

Use the triangle below for questions 3-5. The area of the large triangle is $65\ \text{cm}^2$.

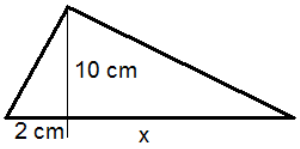


Figure 4.2.3.9

3. Solve for x.
4. Find the perimeter of the large triangle.
5. Find the measure of all three angles of the large triangle.

Use the triangle below for questions 6-8. The area of the triangle is $244\ cm^2$.

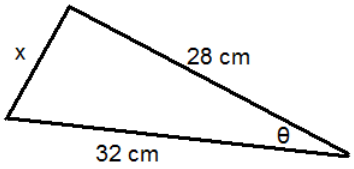


Figure 4.2.3.10

6. Solve for θ.
7. Solve for x.
8. Find the perimeter of the triangle.

Use the triangle below for questions 9-11. The area of the triangle is $299.8\ in^2$.

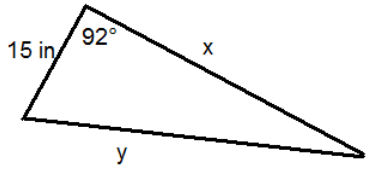


Figure 4.2.3.11

9. Solve for x.
10. Solve for y.
11. Find the measure of the other two angles of the triangle.

Use the triangle below for questions 12-15. The area of the large triangle is $84\ in^2$.

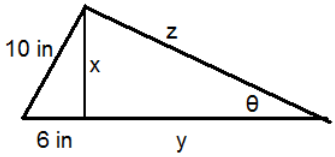


Figure 4.2.3.12

12. Solve for x.
13. Solve for y.
14. Solve for z.
15. Solve for θ.

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.6.

Vocabulary

Term	Definition
law of cosines	The law of cosines is a rule relating the sides of a triangle to the cosine of one of its angles. The law of cosines states that $c^2 = a^2 + b^2 - 2ab\cos C$, where C is the angle across from side c.
law of sines	The law of sines is a rule applied to triangles stating that the ratio of the sine of an angle to the side opposite that angle is equal to the ratio of the sine of another angle in the triangle to the side opposite that angle.

Additional Resource

Practice: Determination of Unknown Triangle Measures Given Area

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4.2.4: Heron's Formula

Area formula based on lengths of sides of a triangle and half its perimeter.

Sarine draws a triangle and measures its sides as 2 inches, 5 inches and 6 inches. What is the area of her triangle?

Heron's Formula

Heron's Formula, named after Hero of Alexandria 2000 years ago, can be used to find the area of a triangle given the three side lengths. The formula requires the semi-perimeter, s, or $\frac{1}{2}(a+b+c)$, where a, b and c are the lengths of the sides of the triangle.

Heron's Formula:

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

Let's use Heron's formula to find the area of a triangle with side lengths 13 cm, 16 cm and 23 cm.

First, find the semi-perimeter or s: $s = \frac{1}{2}(13+16+23) = 26$. Next, substitute our values into the formula as shown and evaluate:

$$A = \sqrt{26(26-13)(26-16)(26-23)} = \sqrt{26(13)(10)(3)} = \sqrt{10140} \approx 101 \text{ cm}^2$$

Now, let's answer the following questions.

1. Alena is planning a garden in her yard. She is using three pieces of wood as a border. If the pieces of wood have lengths 4 ft, 6 ft and 3 ft, what is the area of her garden?

The garden will be triangular with side lengths 4 ft, 6 ft and 3 ft. Find the semi-perimeter and then use Heron's formula to find the area.

$$s = \frac{1}{2}(4+6+3) = \frac{13}{2}$$
$$A = \sqrt{\frac{13}{2}\left(\frac{13}{2}-4\right)\left(\frac{13}{2}-6\right)\left(\frac{13}{2}-3\right)} = \sqrt{\frac{13}{2}\left(\frac{5}{2}\right)\left(\frac{1}{2}\right)\left(\frac{7}{2}\right)} = \sqrt{\frac{455}{16}} \approx 28 \text{ ft}^2$$

2. Caroline wants to measure the height of a radio tower. From some distance away from the tower, the angle of elevation from her spot to the top of the tower is $65°$. Caroline walks 100 m further away from the tower and measures the angle of elevation to the top of tower to be $48°$. How tall is the tower?

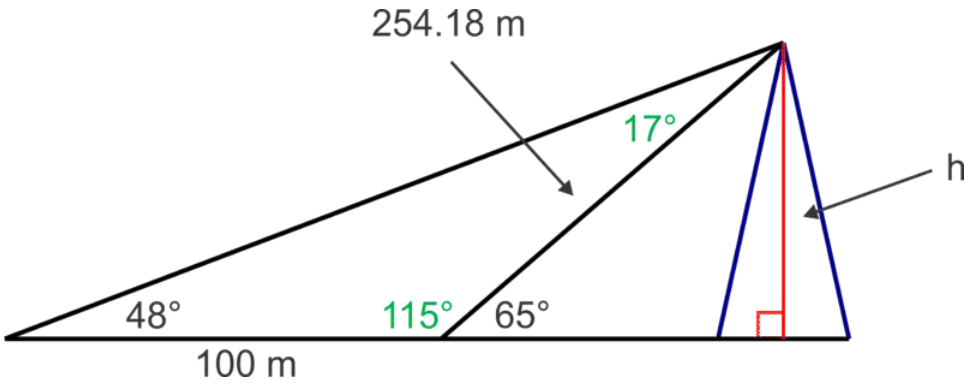


Figure 4.2.4.1

First, make a diagram to illustrate the situation.

We can use angle properties (linear pair and triangle sum) to find the angles shown in green in the diagram.

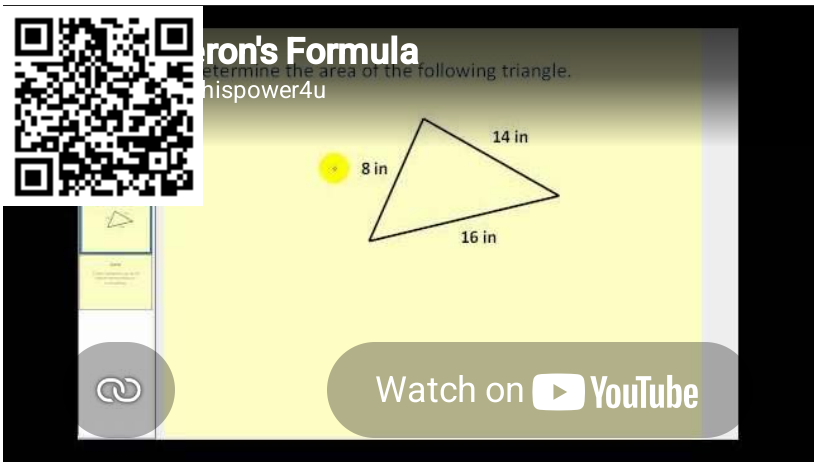
$180° - 65° = 115°$ and $180° - 48° - 115° = 17°$

Next, we can use the Law of Sines in the obtuse triangle to find the hypotenuse in the right triangle:

$$\frac{\sin 17°}{100} = \frac{\sin 48°}{x}$$
$$x = \frac{100 \sin 48°}{\sin 17°} \approx 254.18$$

Finally we can use the sine ratio in the right triangle to find the height of the tower:

$\sin 65^\circ = \frac{h}{254.18}$, $h = 254.18 \sin 65^\circ \approx 230.37$ m



Example 4.2.4.1

Earlier, you were asked to find the area of the triangle with sides of length 2 inches, 5 inches, and 6 inches.

Solution

First, find the semi-perimeter or s: $s = \frac{1}{2}(2+5+6) = 6.5$. Next, substitute our values into Heron's formula and evaluate:

$A = \sqrt{6.5(6.5-2)(6.5-5)(6.5-6)} = \sqrt{6.5(4.5)(1.5)(0.5)} = \sqrt{21.94} \approx 4.7$ in.^{2}

Use the most appropriate rule or formula (Law of Sines, Law of Cosines, area formula with sine or Heron's formula) to answer the following questions.

Example 4.2.4.2

Find the area of a triangle with side lengths 50 m, 45 m and 25 m.

Solution

Heron's Formula:

$s = \frac{1}{2}(50+45+25) = 60$, $A = \sqrt{60(60-50)(60-45)(60-25)} \approx 561$ m^2.

Example 4.2.4.3

Matthew is planning to fertilize his grass. Each bag of fertilizer claims to cover 500 sq ft of grass. His property of land is approximately in the shape of a triangle. He measures two sides of his yard to be 75 ft and 100 ft and the angle between them is 72°. How many bags of fertilizer must he buy?

Solution

Area formula with sine: $\frac{1}{2}(75)(100)\sin 72^\circ \approx 3566$ ft^2, Number of bags $\frac{3566}{500} \approx 7.132 \approx 8$ bags. We round up because 7 bags is not quite enough.

Example 4.2.4.4

A pair of adjacent sides in a parallelogram are 3 in and 7 in and the angle between them is 62°, find the length of the diagonals.

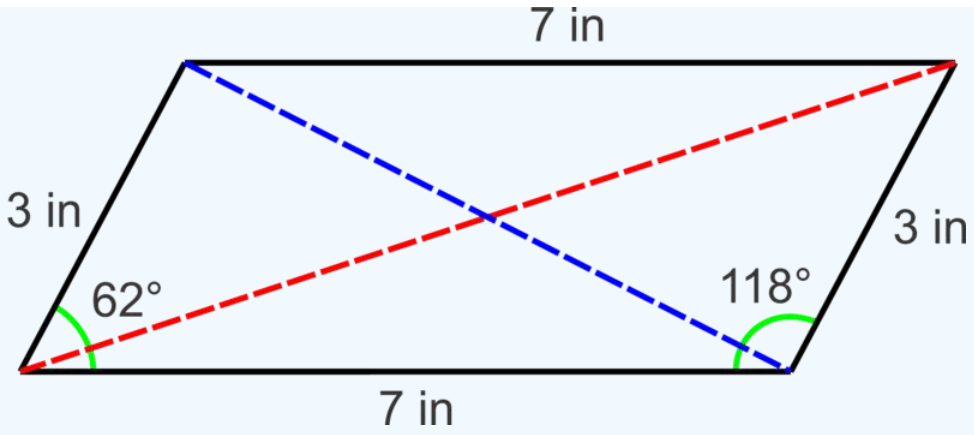


Figure 4.2.4.2

Solution

Law of Cosines to find the blue diagonal:

$$c^2 = 3^2 + 7^2 - 2(3)(7)\cos 62^\circ$$
$$c = \sqrt{38.28} \approx 6.19$$

So, 6.19 in

To find the green diagonal we can use the Law of Cosines with the adjacent angle: $180^\circ - 62^\circ - 118^\circ$:

$$c^2 = 7^2 + 3^2 - 2(7)(3)\cos 118^\circ$$
$$c = \sqrt{77.72} \approx 8.82$$

So, 8.82 in

Review

Use the Law of Sines, Law of Cosine, area of triangle with sine or Heron's Formula to solve the real world application problems.

1. Two observers, Rachel and Luis, are standing on the shore, 0.5 miles apart. They each measure the angle between the shoreline and a sailboat out on the water at the same time. If Rachel's angle is 63° and Luis' angle is 56°, find the distance between Luis and the sailboat to the nearest hundredth of a mile.
2. Two pedestrians walk from opposite ends of a city block to a point on the other side of the street. The angle formed by their paths is 125°. One pedestrian walks 300 ft and the other walks 320 ft. How long is the city block to the nearest foot?
3. Two sides and the included angle of a parallelogram have measures 3.2 cm, 4.8 cm and 54.3° respectively. Find the lengths of the diagonals to the nearest tenth of a centimeter.
4. A bridge is supported by triangular braces. If the sides of each brace have lengths 63 ft, 46 ft and 40 ft, find the measure of the largest angle to the nearest degree.
5. Find the triangular area, to the nearest square meter, enclosed by three pieces of fencing 123 m, 150 m and 155 m long.
6. Find the area, to the nearest square inch, of a parallelogram with sides of length 12 in and 15 in and included angle of 78°.
7. A person at point A looks due east and spots a UFO with an angle of elevation of 40°. At the same time, another person, 1 mi due west of A looks due east and sights the same UFO with an angle of elevation of 25°. Find the distance between A and the UFO. How far is the UFO above the ground? Give answers to the nearest hundredth of a mile.
8. Find the area of a triangular playground, to the nearest square meter, with sides of length 10 m, 15 m and 16 m.
9. A yard is bounded on two sides with fences of length 80 ft and 60 ft. If these fences meet at a 75° angle, how many feet of fencing are required to completely enclosed a triangular region?
10. When a boy stands on the bank of a river and looks across to the other bank, the angle of depression is 12°. If he climbs to the top of a 10 ft tree and looks across to other bank, the angle of depression is 15°. What is the distance from the first position of the boy to the other bank of the river? How wide is the river? Give your answers to the nearest foot.

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 13.17.

Additional Resources

Video: Heron's Area Formula - Example 1

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4.2.5: General Solutions of Triangles

Application of pythagorean theorem, trigonometric functions, and laws of sines/cosines.

While talking with your little sister one day, the conversation turns to shapes. Your sister is only in junior high school, so while she knows some things about right triangles, such as the Pythagorean Theorem, she doesn't know anything about other types of triangles. You show her an example of an oblique triangle by drawing this on a piece of paper:

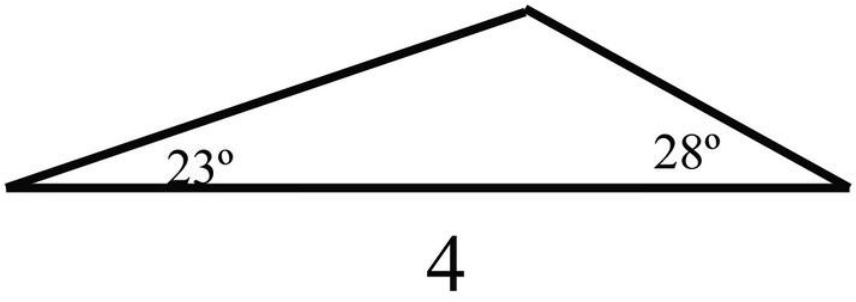


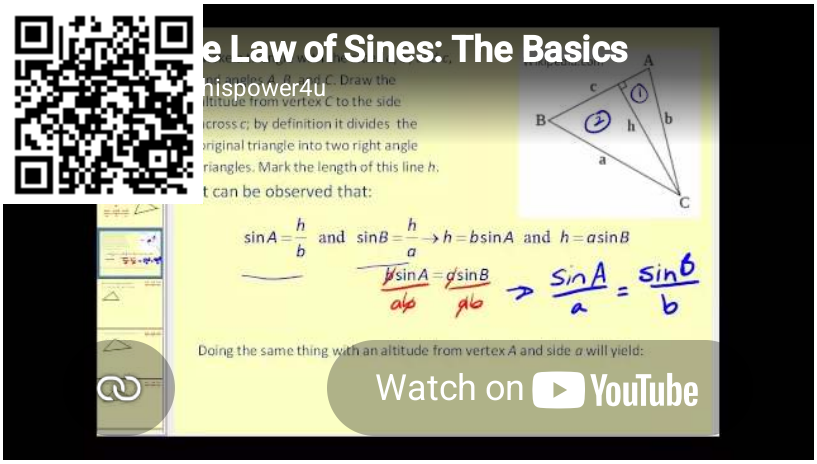
Figure 4.2.5.1

Fascinated, she tells you that she knows how to calculate the area of a triangle using the familiar formula $\frac{1}{2}bh$ and the lengths of sides if the triangle is a right triangle, but that she can't use the formulas on the triangle you just drew.

"Do you know how to find the lengths of sides of the triangle and the area?" she asks.

Finding Solutions for Triangles

Finding the sides, angles, and area for right triangles is often learned in Algebra and/or Geometry. However, it is common to learn how to determine this information in non-right triangles in Trigonometry.



Below is a chart summarizing

common

triangle techniques. This chart describes the type of triangle (either right or oblique), the given information, the appropriate technique to use, and what we can find using each technique.

Type of Triangle:	Given Information:	Technique:	What we can find:
Right	Two sides	Pythagorean Theorem	Third side
Right	One angle and one side	Trigonometric ratios	Either of the other two sides
Right	Two sides	Trigonometric ratios	Either of the other two angles
Oblique	2 angles and a non-included side (AAS)	**Law of Sines**	The other non-included side
Oblique	2 angles and the included side (ASA)	Law of Sines	Either of the non-included sides
Oblique	2 sides and the angle opposite one of those sides (SSA) – Ambiguous case	Law of Sines	The angle opposite the other side (can yield no, one, or two solutions)
Oblique	2 sides and the included angle (SAS)	**Law of Cosines**	The third side
Oblique	3 sides	Law of Cosines	Any of the three
			angles

Solving Triangles

1. In ΔABC, $a = 12$, $b = 13$, $c = 8$. Solve the triangle.

Since we are given all three sides in the triangle, we can use the Law of Cosines. Before we can solve the triangle, it is important to know what information we are missing. In this case, we do not know any of the angles, so we are solving for $\angle A$, $\angle B$, and $\angle C$. We will begin by finding $\angle A$:

$$\begin{aligned} 12^2 &= 8^2 + 13^2 - 2(8)(13)\cos A \\ 144 &= 233 - 208\cos A \\ -89 &= -208\cos A \\ 0.4278846154 &= \cos A \\ 64.7 &\approx \angle A \end{aligned}$$

Now, we will find \angle B by using the Law of Cosines. Keep in mind that you can now also use the Law of Sines to find $\angle B$. Use whatever method you feel more comfortable with.

$$\begin{aligned} 13^2 &= 8^2 + 12^2 - 2(8)(12)\cos B \\ 169 &= 208 - 192\cos B \\ -39 &= -192\cos B \\ 0.2031 &= \cos B \\ 78.3^\circ &\approx \angle B \end{aligned}$$

We can now quickly find $\angle C$ by using the Triangle Sum Theorem,

$180^\circ - 64.7^\circ - 78.3^\circ = 37^\circ$

2. In triangle DEF, $d = 43$, $e = 37$, and $\angle F = 124^\circ$. Solve the triangle.

In this triangle, we have the SAS case because we know two sides and the included angle. This means that we can use the Law of Cosines to solve the triangle. In order to solve this triangle, we need to find side f, $\angle D$, and $\angle E$. First, we will need to find side f using the Law of Cosines.

$$\begin{aligned} f^2 &= 43^2 + 37^2 - 2(43)(37)\cos 124 \\ f^2 &= 4997.351819 \\ f &\approx 70.7 \end{aligned}$$

Now that we know f, we know all three sides of the triangle. This means that we can use the Law of Cosines to find either $\angle D$ or $\angle E$. We will find $\angle D$ first.

$$43^2 = 70.7^2 + 37^2 - 2(70.7)(37)\cos D$$
$$1849 = 6367.49 - 5231.8\cos D$$
$$-4518.49 = -5231.8\cos D$$
$$0.863658779 = \cos D$$
$$30.3^\circ \approx \angle D$$

To find $\angle E$, we need only to use the Triangle Sum Theorem, $\angle E = 180 - (124 + 30.3) = 25.7^\circ$.

3. In triangle ABC, $A = 43^\circ$, $B = 82^\circ$, and $c = 10.3$. Solve the triangle.

This is an example of the ASA case, which means that we can use the Law of Sines to solve the triangle. In order to use the Law of Sines, we must first know $\angle C$, which we can find using the Triangle Sum Theorem, $\angle C = 180^\circ - (43^\circ + 82^\circ) = 55^\circ$.

Now that we know $\angle C$, we can use the Law of Sines to find either side a or side b.

$$\frac{\sin 55}{10.3} = \frac{\sin 43}{a} \qquad \frac{\sin 55}{10.3} = \frac{\sin 82}{b}$$
$$a = \frac{10.3\sin 43}{\sin 55} \qquad b = \frac{10.3\sin 82}{\sin 55}$$
$$a = 8.6 \qquad b = 12.5$$

Example 4.2.5.1

Earlier, you were asked how you might help your sister find the lengths of the sides and the area of a non-right triangle.

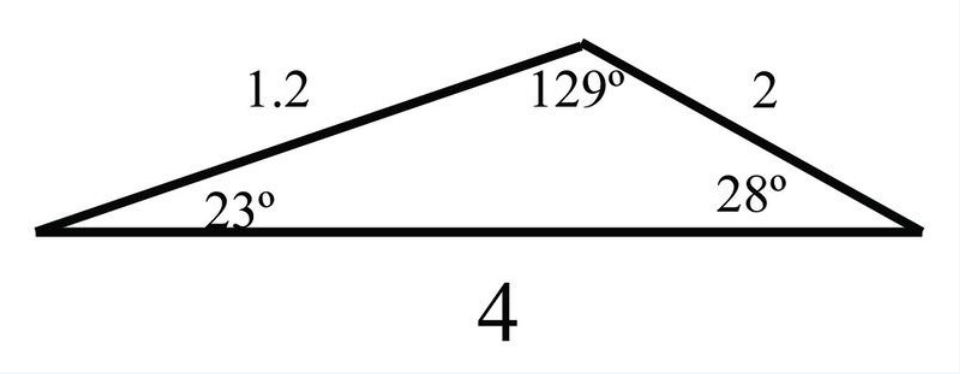


Figure 4.2.5.2

Solution

Since you know that two of the angles are 23° and 28°, the third angle in the triangle must be $180^\circ - 23^\circ - 28^\circ = 129^\circ$. Using these angles and the knowledge that one of the sides has a length of 4, you can solve for the lengths of the other two sides using the Law of Sines:

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$
$$\frac{\sin 23^\circ}{a} = \frac{\sin 129^\circ}{4}$$
$$a = \frac{4\sin 23^\circ}{\sin 129^\circ} = \frac{1.56}{.777}$$
$$a \approx 2$$

And repeating the process for the third side:

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$
$$\frac{\sin 23^\circ}{2} = \frac{\sin 28^\circ}{c}$$
$$c = \frac{2\sin 28^\circ}{\sin 23^\circ} = \frac{.939}{.781}$$
$$c \approx 1.2$$

Now you know all three angles and all three sides. You can use Heron's formula or the alternative formula for the area of a triangle to find the area:

$$K = \frac{1}{2}bc\sin A$$
$$K = \frac{1}{2}(4)(1.2)\sin 23^\circ$$
$$K = \frac{1}{2}(4)(1.2)(.391)$$
$$K \approx .9384$$

Example 4.2.5.2

Using the information provided, decide which case you are given (SSS, SAS, AAS, ASA, or SSA), and whether you would use the Law of Sines or the Law of Cosines to find the requested side or angle. Make an approximate drawing of the triangle and label the given information. Also, state how many solutions (if any) the triangle would have. If a triangle has no solution or two solutions, explain why.

Solution

$A = 69^\circ$, $B = 12^\circ$, $a = 22.3$, find b

AAS, Law of Sines, one solution

Example 4.2.5.3

Using the information provided, decide which case you are given (SSS, SAS, AAS, ASA, or SSA), and whether you would use the Law of Sines or the Law of Cosines to find the requested side or angle. Make an approximate drawing of the triangle and label the given information. Also, state how many solutions (if any) the triangle would have. If a triangle has no solution or two solutions, explain why.

Solution

$a = 1.4$, $b = 2.3$, $C = 58^\circ$, find c.

SAS, Law of Cosines, one solution

Example 4.2.5.4

Using the information provided, decide which case you are given (SSS, SAS, AAS, ASA, or SSA), and whether you would use the Law of Sines or the Law of Cosines to find the requested side or angle. Make an approximate drawing of the triangle and label the given information. Also, state how many solutions (if any) the triangle would have. If a triangle has no solution or two solutions, explain why.

Solution

$a = 3.3$, $b = 6.1$, $c = 4.8$, find A.

SSS, Law of Cosines, one solution

Review

Using the information provided, decide which case you are given (SSS, SAS, AAS, ASA, or SSA), and whether you would use the Law of Sines or the Law of Cosines to find the requested side or angle. Make an approximate drawing of the triangle and label the given information. Also, state how many solutions (if any) the triangle would have.

1. $a = 3$, $b = 4$, $C = 71^\circ$, find c.
2. $a = 8$, $b = 7$, $c = 9$, find A.
3. $A = 135^\circ$, $B = 12^\circ$, $c = 100$, find a.
4. $a = 12$, $b = 10$, $A = 80^\circ$, find c.
5. $A = 50^\circ$, $B = 87^\circ$, $a = 13$, find b.
6. In ΔABC, $a = 15$, $b = 19$, $c = 20$. Solve the triangle.
7. In ΔDEF, $d = 12$, $E = 39^\circ$, $f = 17$. Solve the triangle.

8. In ΔPQR, $P = 115°$, $Q = 30°$, $q = 10$. Solve the triangle.
9. In ΔMNL, $m = 5$, $n = 9$, $L = 20°$. Solve the triangle.
10. In ΔSEV, $S = 50°$, $E = 44°$, $s = 12$. Solve the triangle.
11. In ΔKTS, $k = 6$, $t = 15$, $S = 68°$. Solve the triangle.
12. In ΔWRS, $w = 3$, $r = 5$, $s = 6$. Solve the triangle.
13. In ΔDLP, $D = 52°$, $L = 110°$, $p = 8$. Solve the triangle.
14. In ΔXYZ, $x = 10$, $y = 12$, $z = 9$. Solve the triangle.
15. In ΔAMF, $A = 99°$, $m = 1$, $f = 16$. Solve the triangle.

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.12.

Vocabulary

Term	Definition
law of cosines	The law of cosines is a rule relating the sides of a triangle to the cosine of one of its angles. The law of cosines states that $c^2 = a^2 + b^2 - 2ab\cos C$, where C is the angle across from side c.
law of sines	The law of sines is a rule applied to triangles stating that the ratio of the sine of an angle to the side opposite that angle is equal to the ratio of the sine of another angle in the triangle to the side opposite that angle.

Additional Resources

Video: The Law of Sines: The Basics

Practice: General Solutions of Triangles

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CHAPTER OVERVIEW

Unit 5: Polar System and Complex Numbers

5.1: The Polar Coordinate System

5.1.1: Plots of Polar Coordinates

5.1.2: Distance Between Two Polar Coordinates

5.1.3: Graph Polar Equations

5.1.4: Transformations of Polar Graphs

5.1.5: Special Polar Equations and Graphs

5.2: Polar Curves and Rectangular Conversions

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5.3.8: Geometry of Complex Roots

5.3.9: Polar Form of a Complex Number

5.3.10: DeMoivre's Theorem and nth Roots

5.3.11: DeMoivre's Theorem

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SECTION OVERVIEW

5.1: The Polar Coordinate System

5.1.1: Plots of Polar Coordinates

5.1.2: Distance Between Two Polar Coordinates

5.1.3: Graph Polar Equations

5.1.4: Transformations of Polar Graphs

5.1.5: Special Polar Equations and Graphs

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5.1.1: Plots of Polar Coordinates

Points as distance from the origin and angle from an axis.

While playing a game of darts with your friend, you decide to see if you can plot the coordinates of where your darts land. The dartboard looks like this

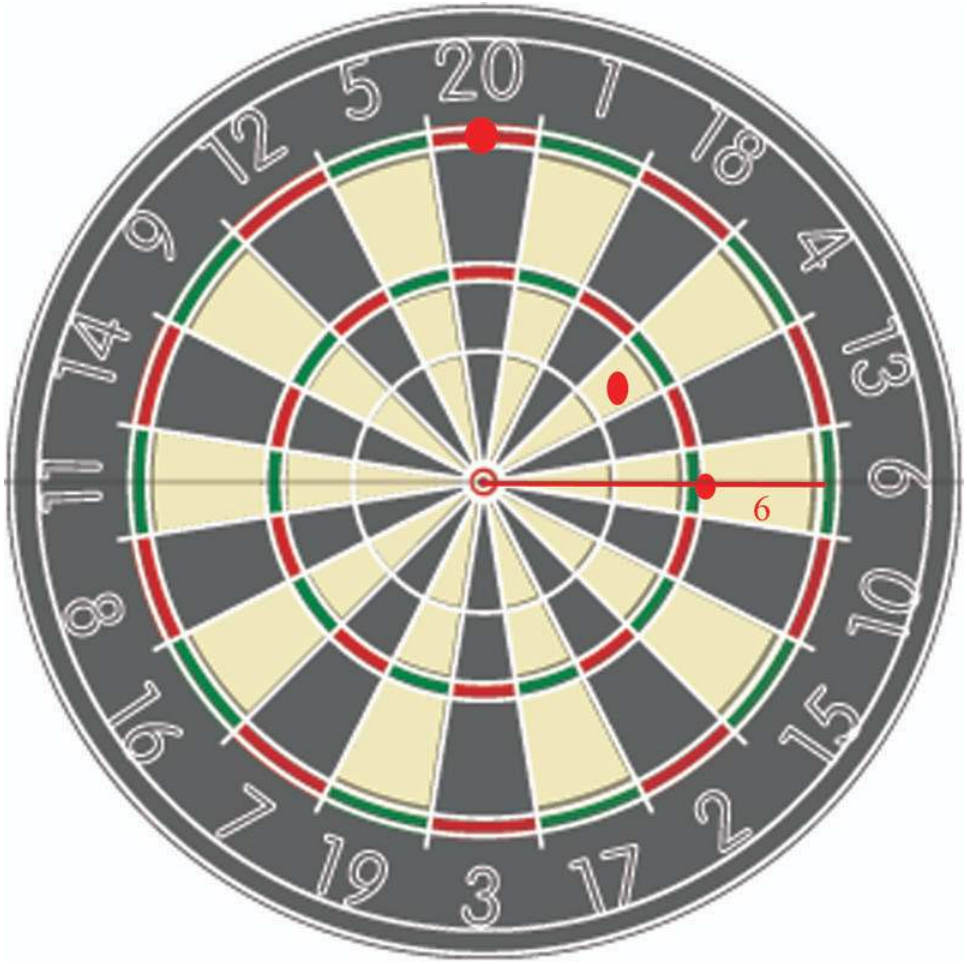
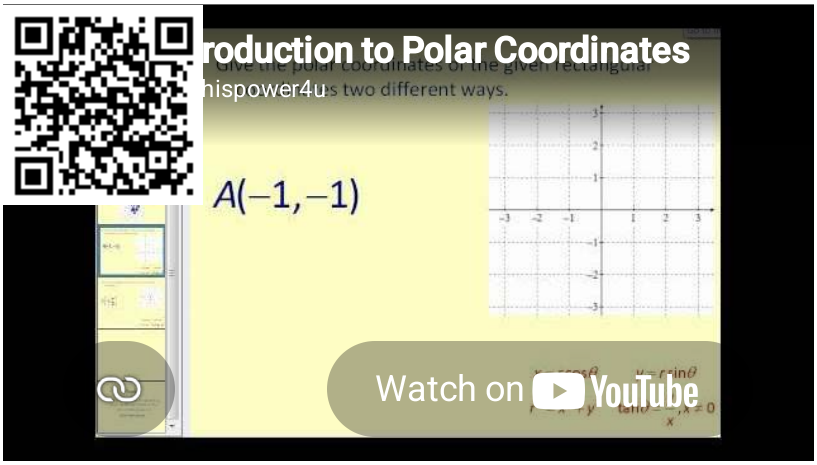


Figure **5.1.1.1**

While trying to set up a rectangular coordinate system, your friend tells you that it would be easier to plot the positions of your darts using a ‘‘polar coordinate system’’. Can you do this?

Plots of Polar Coordinates

The graph paper that you have used for plotting points and sketching graphs has been rectangular grid paper. All points were plotted in a rectangular form (x, y) by referring to a set of perpendicular x– and y– axes. In this section you will discover an alternative to graphing on rectangular grid paper – graphing on circular grid paper.



Look at the two options below:

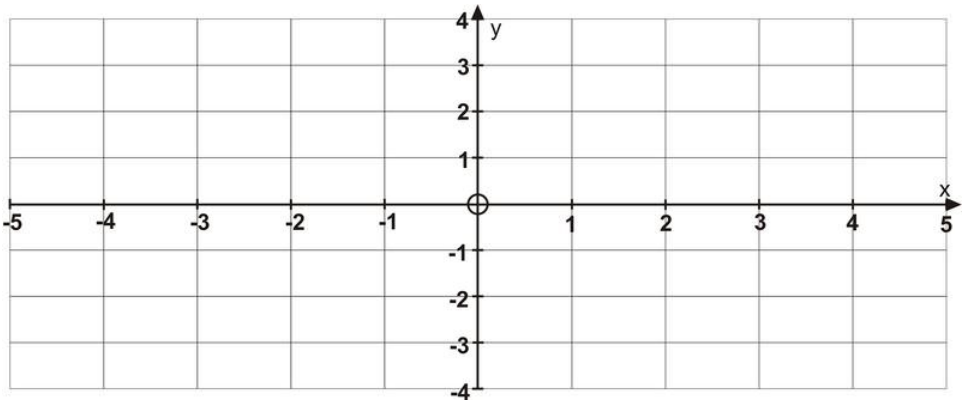


Figure **5.1.1.2**

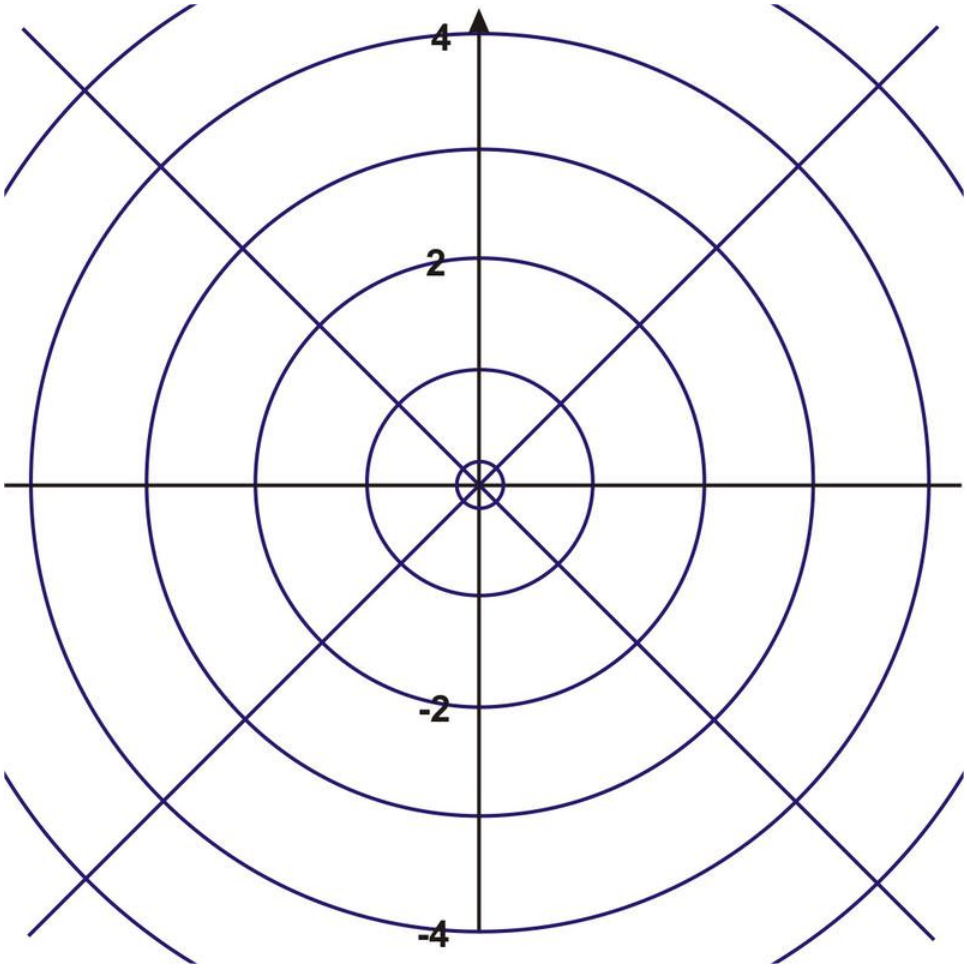


Figure 5.1.1.3

You are all familiar with the rectangular grid paper shown above. However, the circular paper lends itself to new discoveries. The paper consists of a series of concentric circles-circles that share a common center. The common center O, is known as the pole or origin and the polar axis is the horizontal line r that is drawn from the pole in a positive direction. The point P that is plotted is described as a directed distance r from the pole and by the angle that $\overline{OP}$ makes with the polar axis. The coordinates of P are (r, θ).

These coordinates are the result of assuming that the angle is rotated counterclockwise. If the angle were rotated clockwise then the coordinates of P would be $(r, -\theta)$. These values for P are called **polar coordinates** and are of the form $P(r, \theta)$ where r is the absolute value of the distance from the pole to P and θ is the angle formed by the polar axis and the terminal arm $\overline{OP}$.

Plotting Points

Plot the point $A(5, -255°)$ and the point $B(3, 60°)$.

To plot A, move from the pole to the circle that has $r = 5$ and then rotate $255°$ **clockwise** from the polar axis and plot the point on the circle. Label it A.

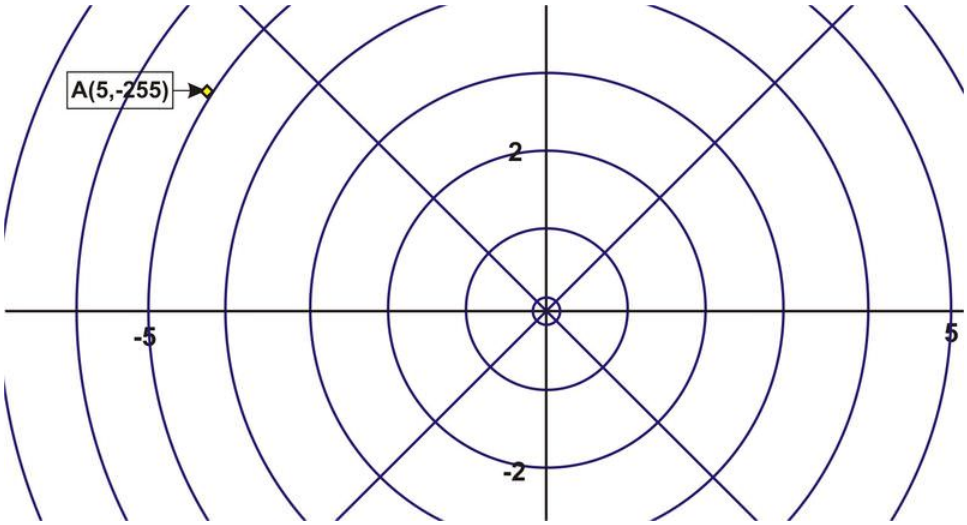


Figure 5.1.1.4

To plot B, move from the pole to the circle that has $r = 3$ and then rotate $60°$ **counter clockwise** from the polar axis and plot the point on the circle. Label it B.

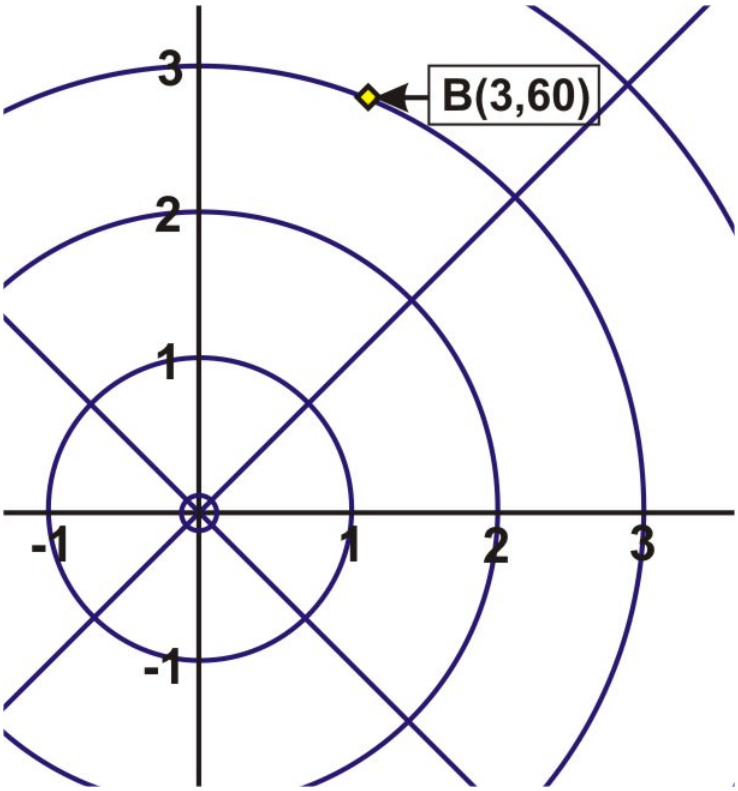


Figure 5.1.1.5

Determining Pairs of Polar Coordinates

Determine four pairs of polar coordinates that represent the following point $P(r, \theta)$ such that $-360° \leq \theta \leq 360°$.

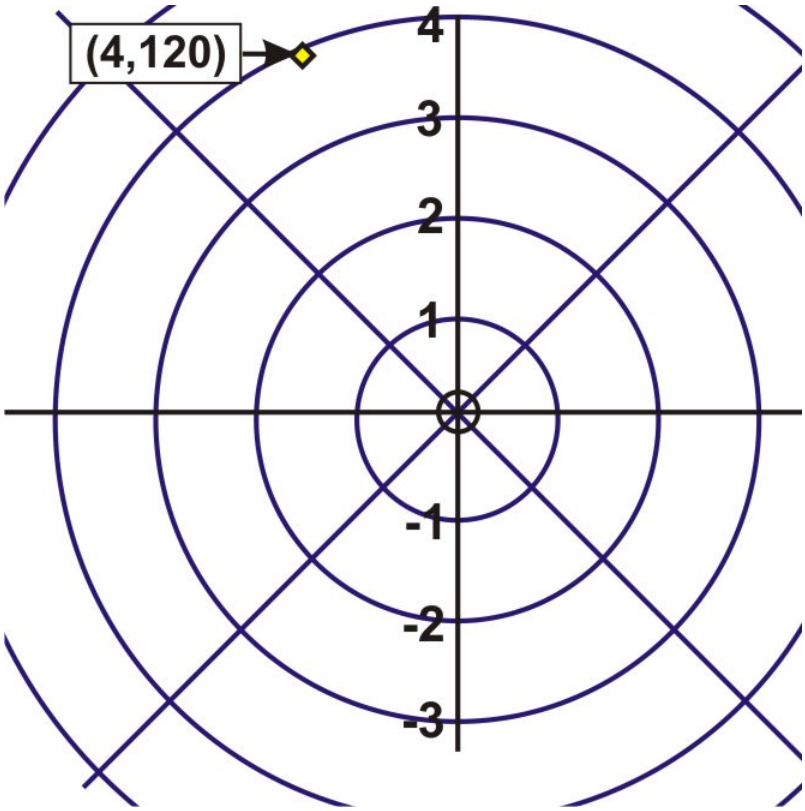


Figure **5.1.1.6**

Pair 1 $\rightarrow (4, 120°)$. Pair 2 $\rightarrow (4, -240°)$ comes from using $k = -1$ and $(r, \theta + 360°k)$,(4,120^{\circ}+360(−1)). Pair 3 $\rightarrow (-4, 300°)$ comes from using $k = 0$ and $(-r, \theta + [2k + 1]180°), (-4, 120° + [2(0) + 1]180°)$. Pair 4 $\rightarrow (-4, -60°)$ comes from using $k = -1$ and $(-r, \theta + [2k + 1]180°), (-4, 120° + [2(-1) + 1]180°)$.

Plotting Polar Coordinates

Plot the following coordinates in polar form and give their description in polar terms: $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(-1, 1)$.

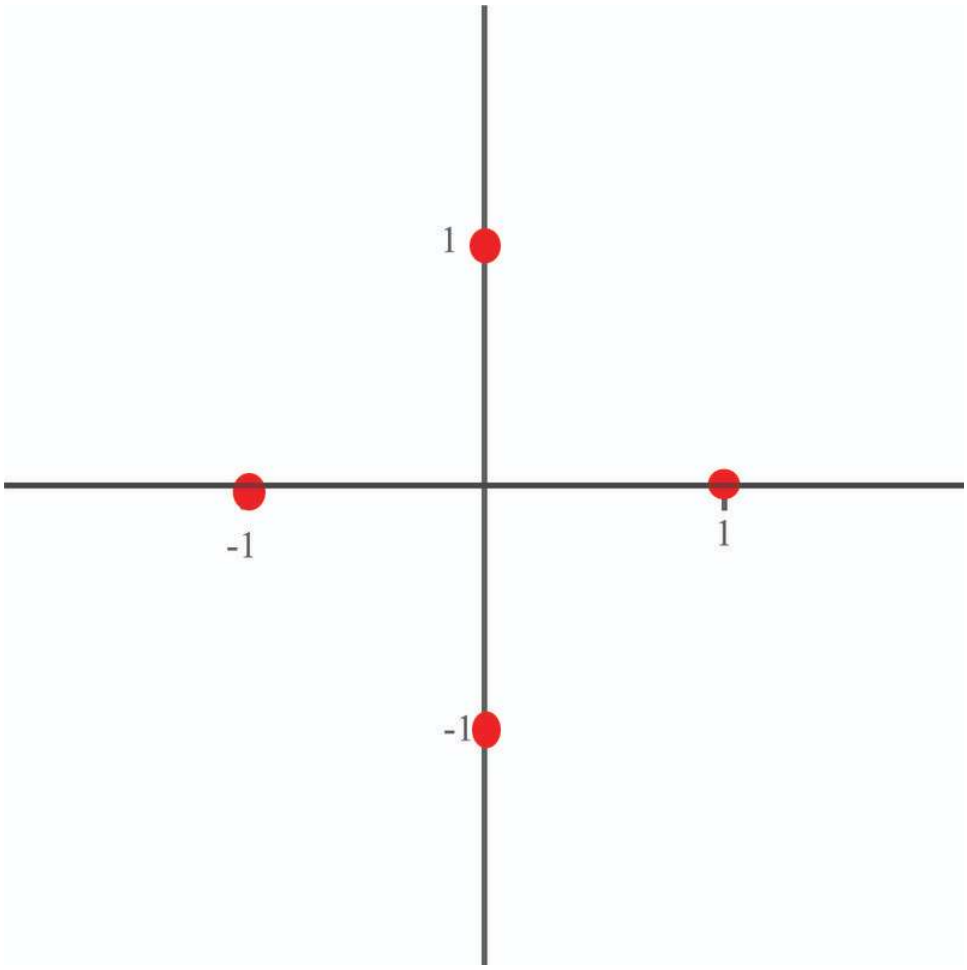


Figure 5.1.1.7

The points plotted are shown above. Since each point is 1 unit away from the origin, we know that the radius of each point in polar form will be equal to 1.

The first point lies on the positive 'x' axis, so the angle in polar coordinates is $0°$. The second point lies on the positive 'y' axis, so the angle in polar coordinates is $90°$. The third point lies on the negative 'x' axis, so the angle in polar coordinates is $180°$. The fourth point lies on the negative 'y' axis, so the angle in polar coordinates is $270°$.

Example 5.1.1.1

Earlier, you were asked to plot the positions of your darts using a polar coordinate system.

Since you have the positions of the darts on the board with both the distance from the origin and the angle they make with the horizontal, you can describe them using polar coordinates.

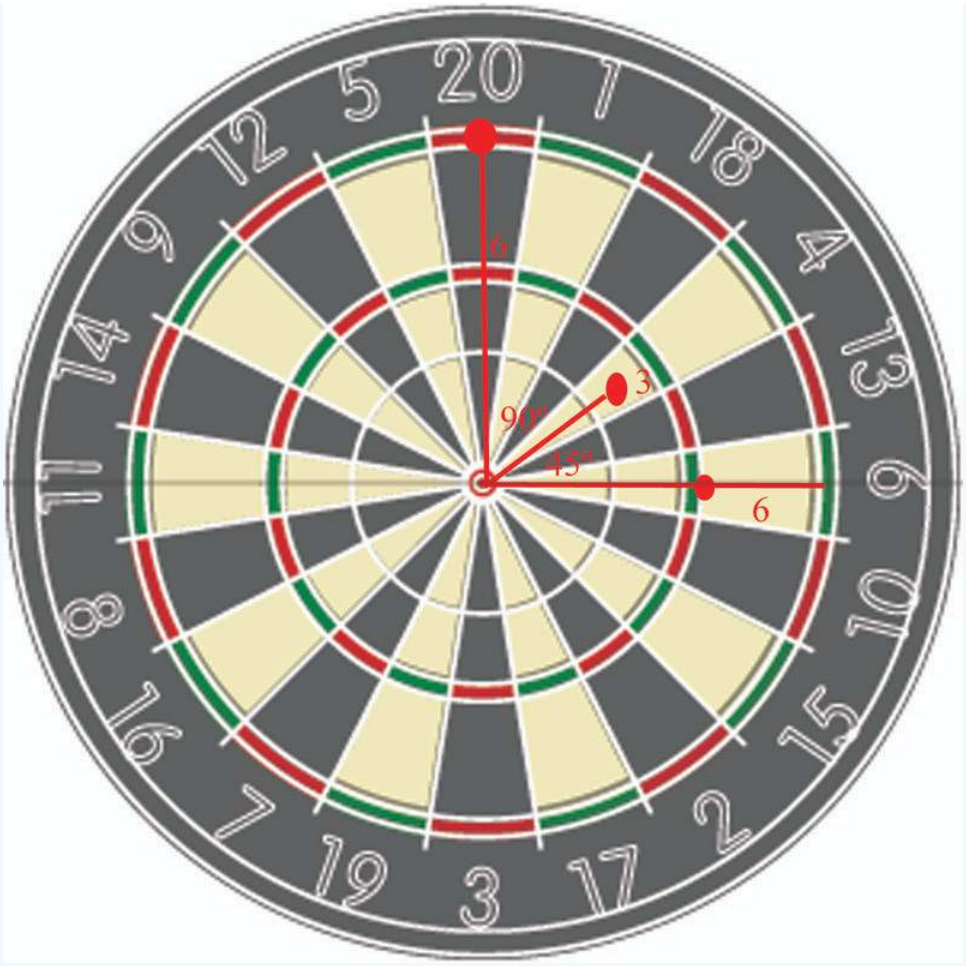


Figure 5.1.1.8

Solution

As you can see, the positions of the darts are:

$(3, 45°)$, $(6, 90°)$

and

$(4, 0°)$

Example 5.1.1.2

Plot the point $M\,(2.5, 210°)$.

Solution

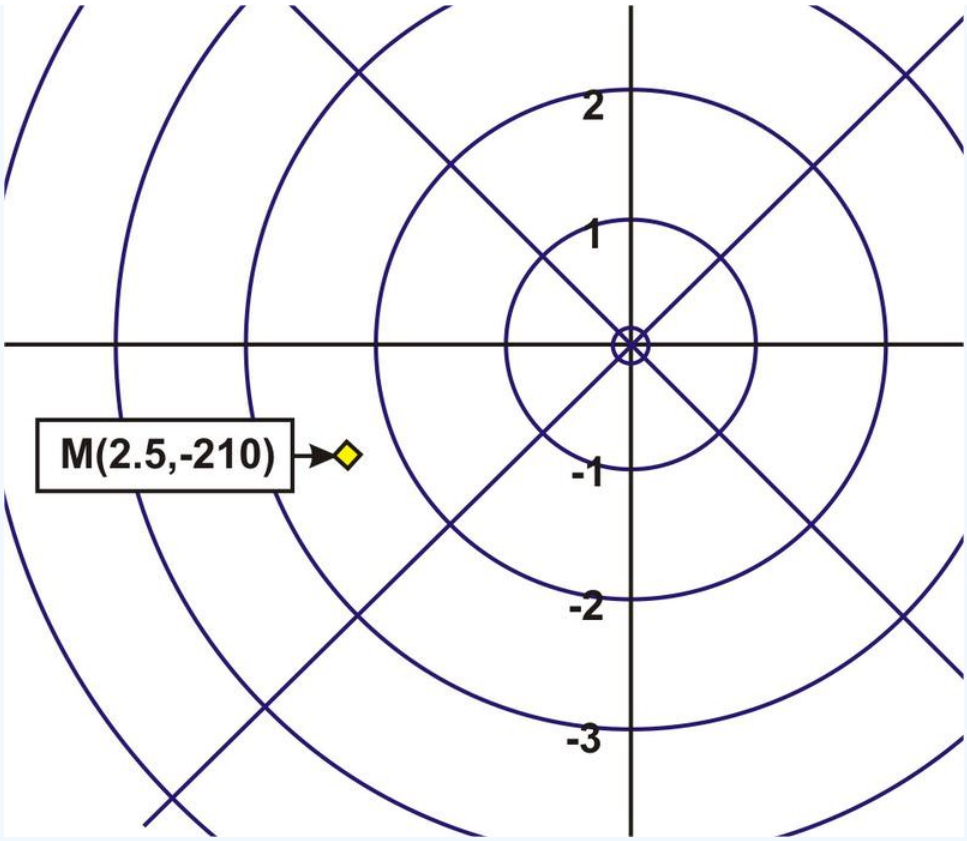


Figure 5.1.1.9

Example 5.1.1.3

Plot the point $S\left(-3.5, \frac{5\pi}{6}\right)$.

Solution

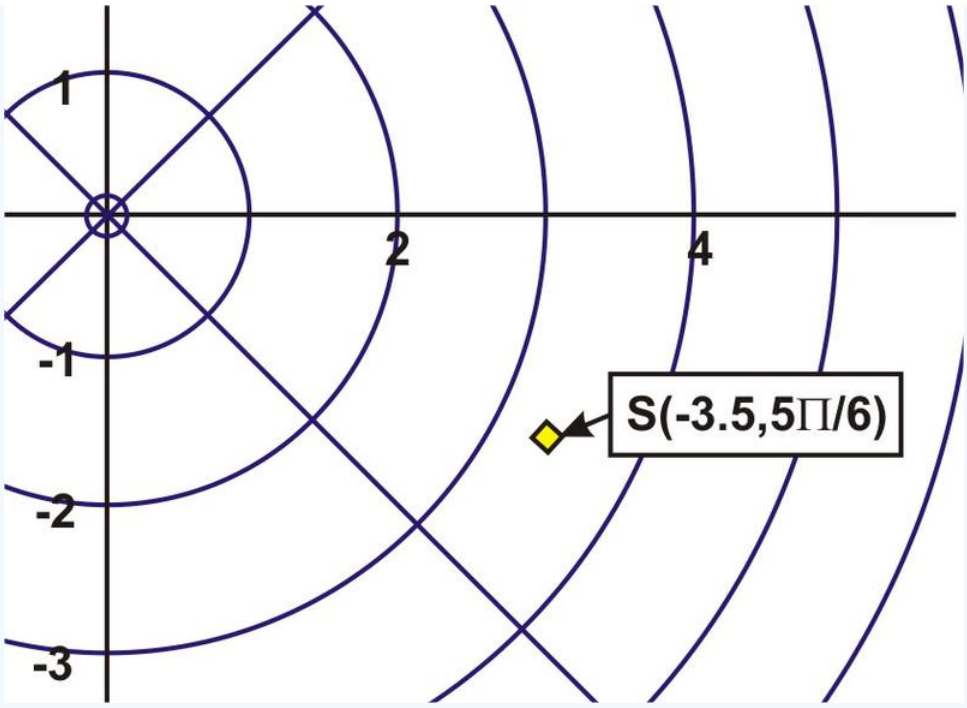


Figure 5.1.1.10

Example 5.1.1.4

Plot the point $A\left(1, \frac{3\pi}{4}\right)$.

Solution

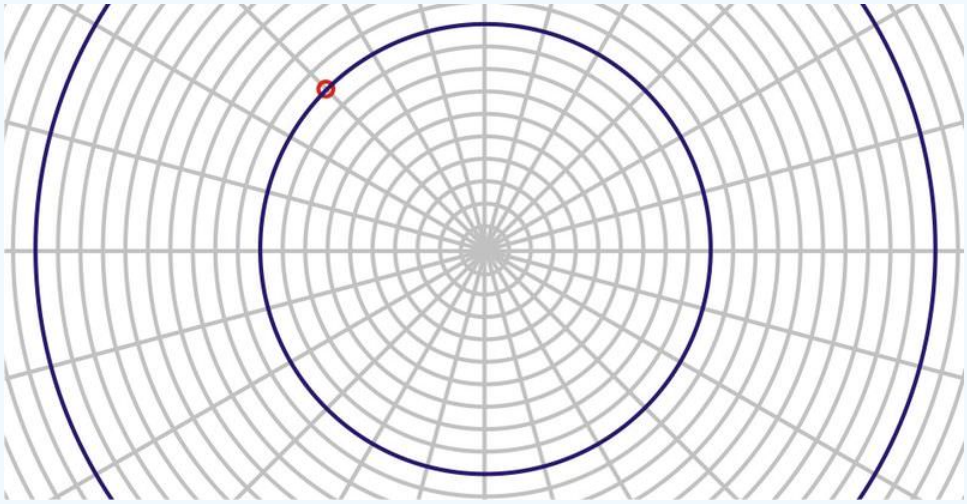


Figure 5.1.1.11

Review

Plot the following points on a polar coordinate grid.

1. $(3, 150°)$
2. $(2, 90°)$
3. $(5, 60°)$
4. $(4, 120°)$
5. $(3, 210°)$

6. $(-2, 120°)$
7. $(4, -90°)$
8. $(-5, -30°)$
9. $(2, -150°)$
10. $(-3, 300°)$

Give three alternate sets of coordinates for the given point within the range $-360° \leq \theta \leq 360°$.

11. $(3, 60°)$
12. $(2, 210°)$
13. $(4, 330°)$
14. Find the length of the arc between the points $(2, 30°)$ and $(2, 90°)$.
15. Find the area of the sector created by the origin and the points $(4, 30°)$ and $(4, 90°)$.

Review (Answers)

To see the Review answers, open this PDF file and look for section 6.1.

Additional Resources

Video: Polar Coordinates 1

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5.1.2: Distance Between Two Polar Coordinates

Application of the law of cosines.

Distance Between Two Polar Coordinates

When playing a game of darts with your friend, the darts you throw land in a pattern like this

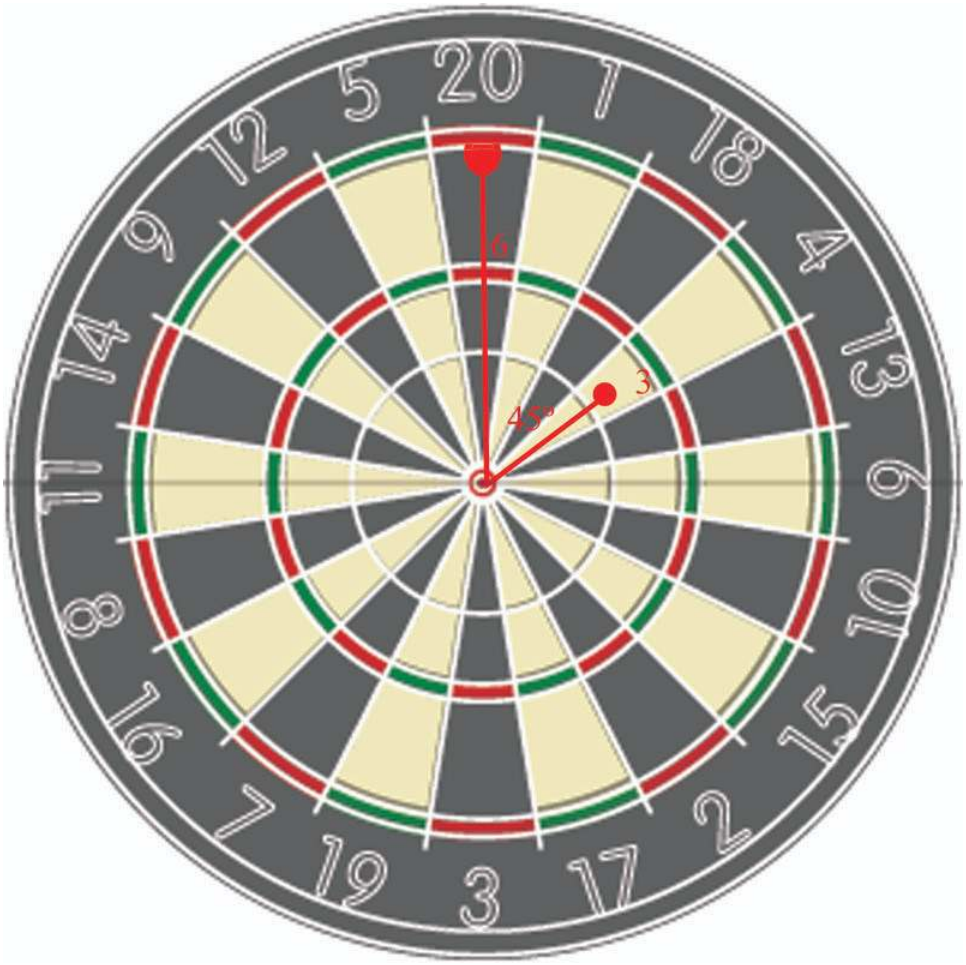
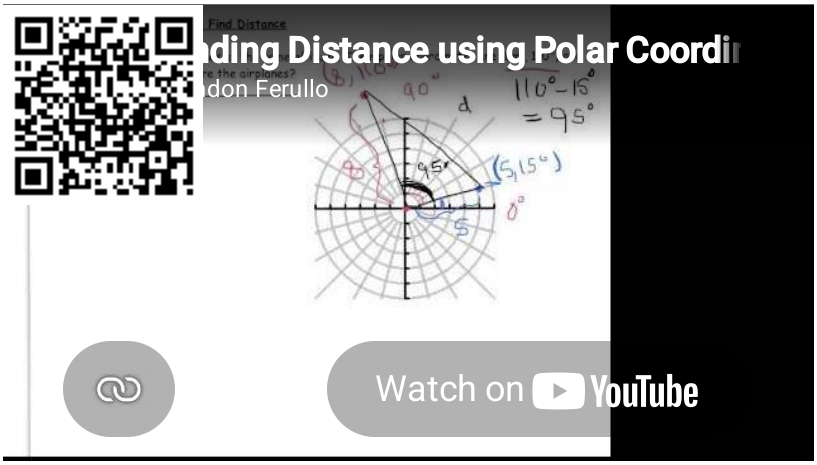


Figure **5.1.2.1**

You and your friend decide to find out how far it is between the two darts you threw. If you know the positions of each of the darts in polar coordinates, can you somehow find a formula to let you determine the distance between the two darts?

Finding the Distance Between Two Polar Coordinates

Just like the Distance Formula for x and y coordinates, there is a way to find the distance between two polar coordinates. One way that we know how to find distance, or length, is the Law of Cosines, $a^2 = b^2 + c^2 - 2bc\cos A$ or $a = \sqrt{b^2 + c^2 - 2bc\cos A}$. If we have two points (r_1, θ_1) and (r_2, θ_2), we can easily substitute r1 for b and r2 for c. As for A, it needs to be the angle between the two radii, or $(\theta_2 - \theta_1)$. Finally, a is now distance and you have $d = r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)$.



Finding the Distance

1. Find the distance between $(3, 60°)$ and $(5, 145°)$.

After graphing these two points, we have a triangle. Using the new Polar Distance Formula, we have $d = \sqrt{3^2 + 5^2 - 2(3)(5)\cos 85°} \approx 5.6$.

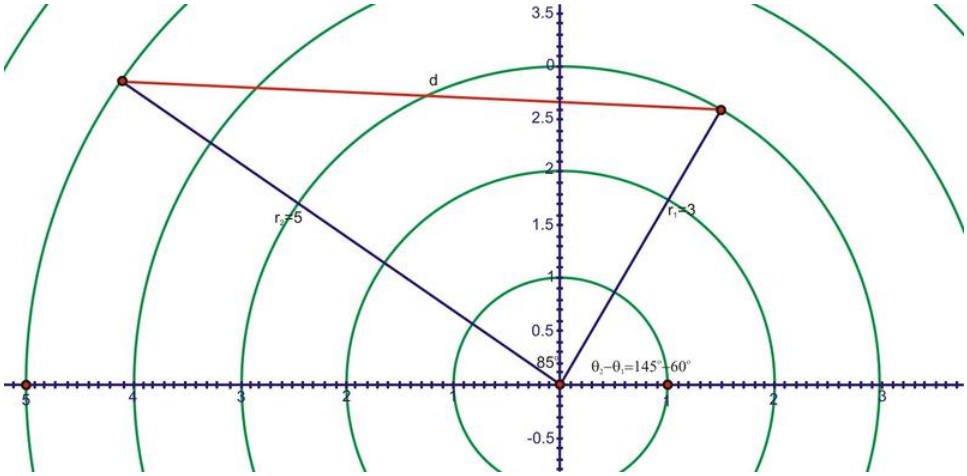


Figure 5.1.2.2

2. Find the distance between $(9, -45°)$ and $(-4, 70°)$.

This one is a little trickier than the last #1 because we have negatives. The first point would be plotted in the fourth quadrant and is equivalent to $(9, 315°)$. The second point would be $(4, 70°)$ reflected across the pole, or $(4, 250°)$. Use these two values of \theta for the formula. Also, the radii should always be positive when put into the formula. That being said, the distance is $d = 9^2 + 4^2 - 2(9)(4)\cos(315 - 250)° \approx 8.16$.

3. Find the distance between $(2, 10°)$ and $(7, 10°)$.

This problem is straightforward from looking at the relationship between the points. The two points lie at the same angle, so the straight line distance between them is 7−2=5. However, we can confirm this using the distance formula:

$d = \sqrt{2^2 + 7^2 - 2(2)(7)\cos 0°} = \sqrt{4 + 49 - 28} = \sqrt{25} = 5$.

Example 5.1.2.1

Earlier, you were asked to find the distance between the two darts.

Using the Distance Formula for points in a polar plot, it is possible to determine the distance between the 2 darts:

Solution

$$d = \sqrt{3^2 + 6^2 - 2(4)(6)\cos 45^\circ}$$
$$d = \sqrt{9 + 36 - 48(.707)}$$
$$d = \sqrt{45 - 33.936}$$
$$d \approx 11.064$$

Example 5.1.2.2

Given P1 and P2, calculate the distance between the points.

$P_1(1, 30^\circ)$ and $P_2(6, 135^\circ)$

Solution

Use $P_1P_2 = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}$.

$$P_1P_2 = 1^2 + 6^2 - 2(1)(6)\cos(135^\circ - 30^\circ)$$
$$P_1P_2 \approx 6.33 \text{ units}$$

Example 5.1.2.3

Given P_1 and P_2, calculate the distance between the points.

$P_1(2, -65^\circ)$ and $P_2(9, 85^\circ)$

Solution

Use $P_1P_2 = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}$.

$P_1P_2 = \sqrt{2^2 + 9^2 - 2(2)(9)\cos 150^\circ} = 10.78$

Example 5.1.2.4

Given (P_1 and P_2, calculate the distance between the points.

$P_1(-3, 142^\circ)$ and $P_2(10, -88^\circ)$

Solution

3. Use $P_1P_2 = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}$.

$$P_1P_2 = \sqrt{3^2 + 10^2 - 2(3)(10)\cos(322 - 272)^\circ}$$
$$= 8.39$$

Review

Find the distance between each set of points.

1. $(1, 150^\circ)$ and $(2, 130^\circ)$
2. $(4, 90^\circ)$ and $(5, 210^\circ)$
3. $(6, 60^\circ)$ and $(2, 90^\circ)$
4. $(2, 120^\circ)$ and $(1, 150^\circ)$
5. $(7, 210^\circ)$ and $(4, 300^\circ)$
6. $(-4, 120^\circ)$ and $(2, 100^\circ)$
7. $(3, -90^\circ)$ and $(5, 150^\circ)$
8. $(-4, -30^\circ)$ and $(3, 250^\circ)$
9. $(7, -150^\circ)$ and $(4, 130^\circ)$
10. $(-2, 300^\circ)$ and $(2, 10^\circ)$
11. Find the length of the arc between the points $(3, 40^\circ)$ and $(3, 150^\circ)$.
12. Find the length of the arc between the points $(1, 10^\circ)$ and $(1, 70^\circ)$.

13. Find the area of the sector created by the origin and the points $(4, 20°)$ and $(4, 110°)$.
14. Find the area of the sector created by the origin and the points $(2, 100°)$ and $(2, 180°)$.
15. Find the area of the sector created by the origin and the points $(5, 110°)$ and $(5, 160°)$.

Review (Answers)

To see the Review answers, open this PDF file and look for section 6.2.

Additional Resources

Practice: Distance Between Two Polar Coordinates

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5.1.3: Graph Polar Equations

Points and graphs based on equations involving radius and angle measurements.

In the **rectangular coordinate system**, points are identified by their distances from the x and y axes. In the **polar coordinate system,** points are identified by their angle on the **unit circle** and their distance from the origin. You can use basic right triangle trigonometry to translate back and forth between the two representations of the same point. How are lines and other functions affected by this new coordinate system?

Polar and **Rectangular Coordinates**

Rectangular coordinates are the ordinary (x, y) coordinates that you are used to.

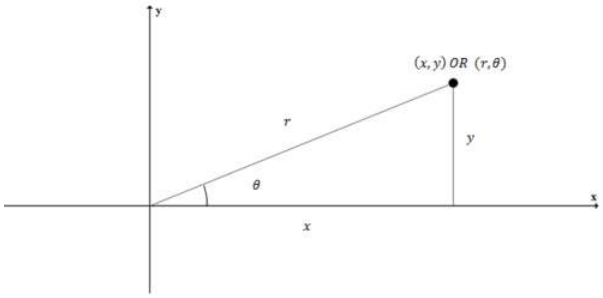


Figure 5.1.3.1

Polar coordinates represent the same point, but describe the point by its distance from the origin (r) and its angle on the unit circle (θ). To translate back and forth between polar and rectangular coordinates you should use the basic trig relationships:

$$\sin\theta = \frac{y}{r} \rightarrow r \cdot \sin\theta = y$$
$$\cos\theta = \frac{x}{r} \rightarrow r \cdot \cos\theta = x$$
$$\tan\theta = \frac{y}{x} \rightarrow \theta = \tan^{-1}\frac{y}{x}$$

You can also express the relationship between x, y and r using the Pythagorean Theorem.

$x^2 + y^2 = r^2$

Note that coordinates in polar form are not unique. This is because there are an infinite number of **coterminal angles** that point towards any given $(x,\ y)$ coordinate.

For example, the point (3, 4) can be written in polar coordinates in at least three different ways. To find θ, use the third equation from above and to find r use the pythagorean theorem.

$$\tan\theta = \frac{4}{3}$$
$$\theta = \tan^{-1}\left(\frac{4}{3}\right) \approx 53.1^\circ$$
$$r^2 = 3^2 + 4^2$$
$$r = 5$$

Three equivalent polar coordinates for the point (3, 4) are:

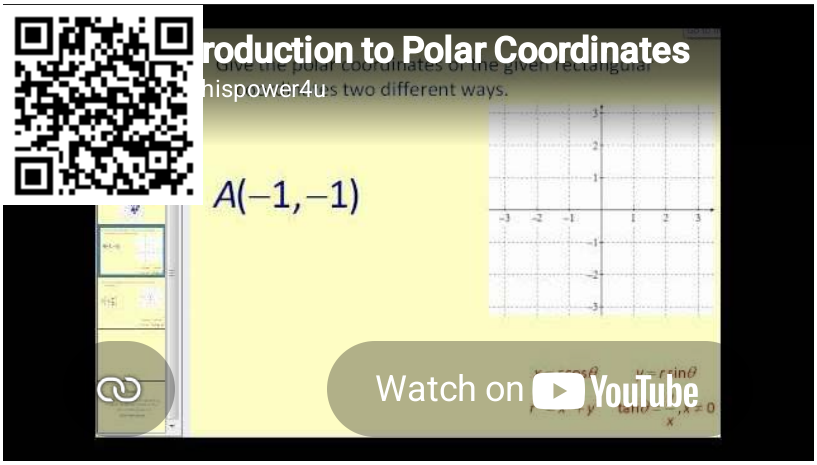
$(5, 53.1^\circ),\ (5, 413.1^\circ),\ (-5, 233.1^\circ)$

Notice how the third coordinate points in the opposite direction and has a seemingly negative radius. This means go in the opposite direction of the angle.

Once you can translate back and forth between points, use the same substitutions to change equations too. A polar equation is written with the radius as a function of the angle. This means an equation in polar form should be written in the form $r =$ ____.

To write an equation in polar form, use the conversion equations to substitute. For example, to convert $y = -x + 1$ to polar form make substitutions for y and x. Then, solve for r.

$$r \cdot \sin\theta = -r \cdot \cos\theta + 1$$
$$r \cdot \sin\theta + r \cdot \cos\theta = 1$$
$$r(\sin\theta + \cos\theta) = 1$$
$$r = \frac{1}{\sin\theta + \cos\theta}$$



Example 5.1.3.1

Earlier, you were asked how lines can be represented in the polar coordinate system.

Solution

The general way to express a line $y = mx + b$ in polar form is $r = \frac{b}{\sin\theta - m \cdot cos\theta}$.

Example 5.1.3.2

Express the following equation using rectangular coordinates: $r = \frac{8}{1 + 2\cos\theta}$.

Solution

Use the fact that $r = \pm\sqrt{x^2 + y^2}$ and $r\cos\theta = x$

$$r + 2r \cdot \cos\theta = 8$$
$$\pm\sqrt{x^2 + y^2} + 2x = 8$$
$$\pm\sqrt{x^2 + y^2} = 8 - 2x$$
$$x^2 + y^2 = 64 - 32x + 4x^2$$
$$-3x^2 + 32x + y^2 - 64 = 0$$

This is the equation of a hyperbola.

Example 5.1.3.3

Sketch the following polar equation: $r = 3$.

Solution

Since theta is not in the equation, it can vary freely. This simple equation produces a perfect circle of radius 3 centered at the origin.

You can show this equation is equivalent to $x^2 + y^2 = 9$

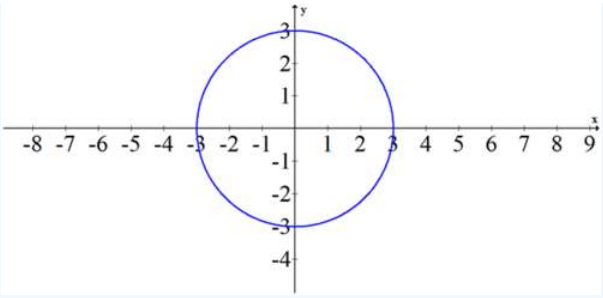


Figure 5.1.3.2

Example 5.1.3.4

Sketch the following polar equation: $r = \theta$ with $\theta : 0 \leq \theta \leq 2\pi$.

Solution

The equation r=\theta is an example of a polar equation that cannot be easily expressed in rectangular form. In order to sketch the graph, identify a few key points: $(0,0)$, $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$, (π, π),$\left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$, $(2\pi, 2\pi)$. You should see that the shape is very recognizable as a spiral.

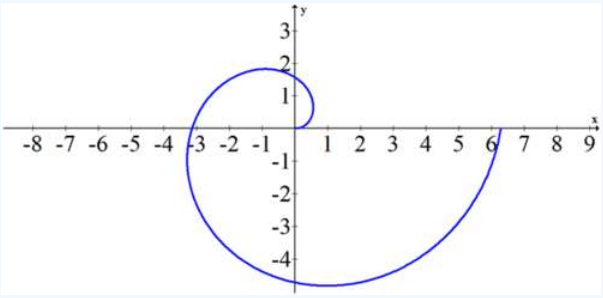


Figure 5.1.3.3

Example 5.1.3.5

Translate the following polar expression into rectangular coordinates and then graph.

$r = 2 \cdot \sec\left(\theta - \frac{\pi}{2}\right)$

Solution

Simplify the polar equation first before converting to rectangular coordinates.

$$r = 2 \cdot \sec\left(\theta - \frac{\pi}{2}\right)$$
$$r \cdot \cos\left(\theta - \frac{\pi}{2}\right) = 2$$
$$r \cdot cos(\frac{\pi}{2} - \theta) = 2$$
$$r \cdot \sin\theta = 2$$
$$y = 2$$

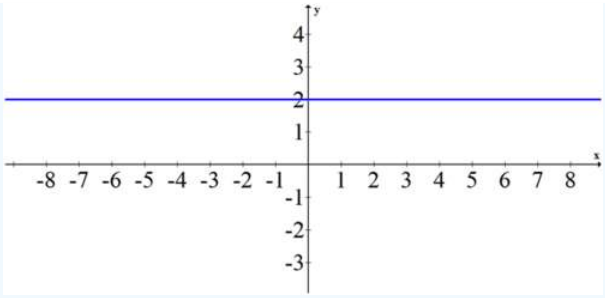


Figure 5.1.3.4

Review

Plot the following polar coordinates.

1. $\left(3, \frac{5\pi}{6}\right)$
2. $\left(2, \frac{\pi}{2}\right)$
3. $\left(4, \frac{-7\pi}{6}\right)$
4. $\left(-2, \frac{5\pi}{3}\right)$

Give two alternate sets of coordinates for each point.

5. $(2, 60°)$
6. $(5, 330°)$
7. $(2, 210°)$

Graph each equation.

8. $r = 4$
9. $\theta = \frac{\pi}{4}$
10. $r = 2\theta$ with $\theta : 0 \leq \theta \leq 2\pi$.

Convert each point to rectangular form.

11. $\left(4, \frac{2\pi}{3}\right)$
12. $\left(3, \frac{\pi}{4}\right)$
13. $(5, \frac{\pi}{3})$

Convert each point to polar form using radians where $0 \leq \theta < 2\pi$.

14. (1, 3)
15. (1, -4)
16. (2, 6)

Convert each equation to polar form.

17. $x = 3$
18. $2x + 4y = 2$

Review (Answers)

To see the Review answers, open this PDF file and look for section 10.1.

Vocabulary

Term	Definition
Coterminal Angles	A set of coterminal angles are angles with the same terminal side but expressed differently, such as a different number of complete rotations around the unit circle or angles being expressed as positive versus negative angle measurements.
polar coordinate system	The polar coordinate system is a special coordinate system in which the location of each point is determined by its distance from the pole and its angle with respect to the polar axis.
polar coordinates	Polar coordinates describe locations on a grid using the polar coordinate system. The location of each point is determined by its distance from the pole and its angle with respect to the polar axis.
rectangular coordinates	A point is written using rectangular coordinates if it is written in terms of x and y and can be graphed on the Cartesian plane.
unit circle	The unit circle is a circle of radius one, centered at the origin.

Additional Resources

Video: Graphing Polar Equations - Overview

Practice: Graph Polar Equations

Real World: Listen Up

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5.1.4: Transformations of Polar Graphs

Alteration of graph based on changing constants and/or function of a polar equation.

While playing around with your calculator one day, you create a polar plot that looks like this.

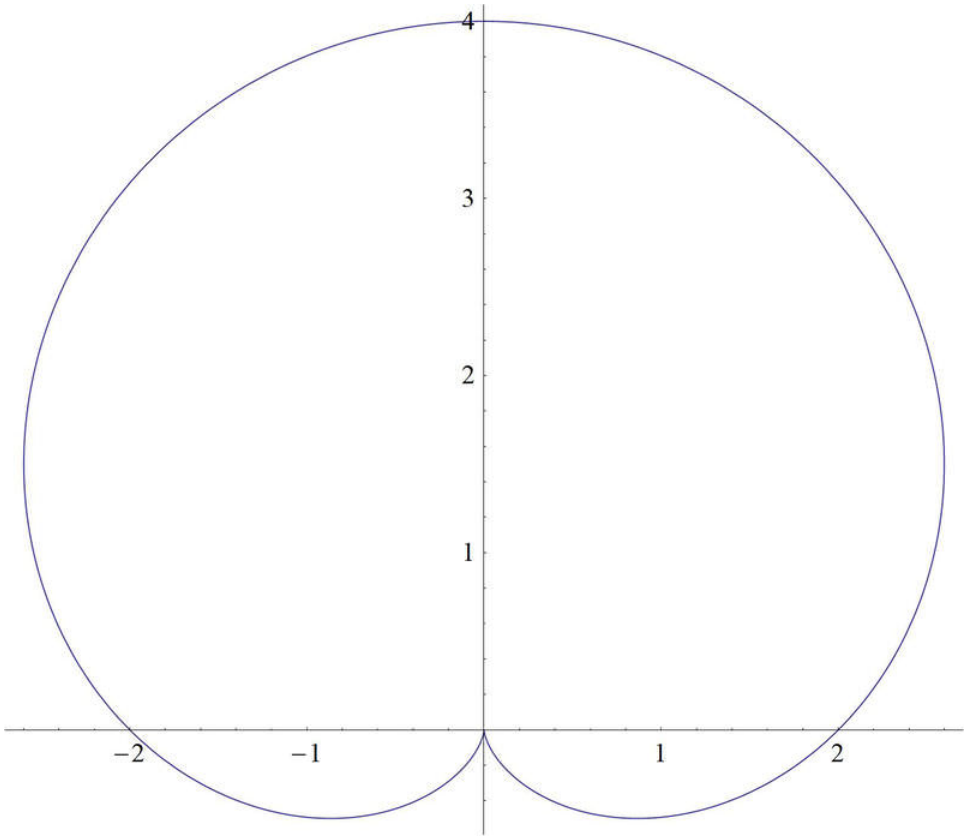


Figure 5.1.4.1

Your teacher tells you that this is a polar plot with an equation $2 + 2\sin\theta$. You decide you'd like to rotate the graph, so that it is actually on its side, like this

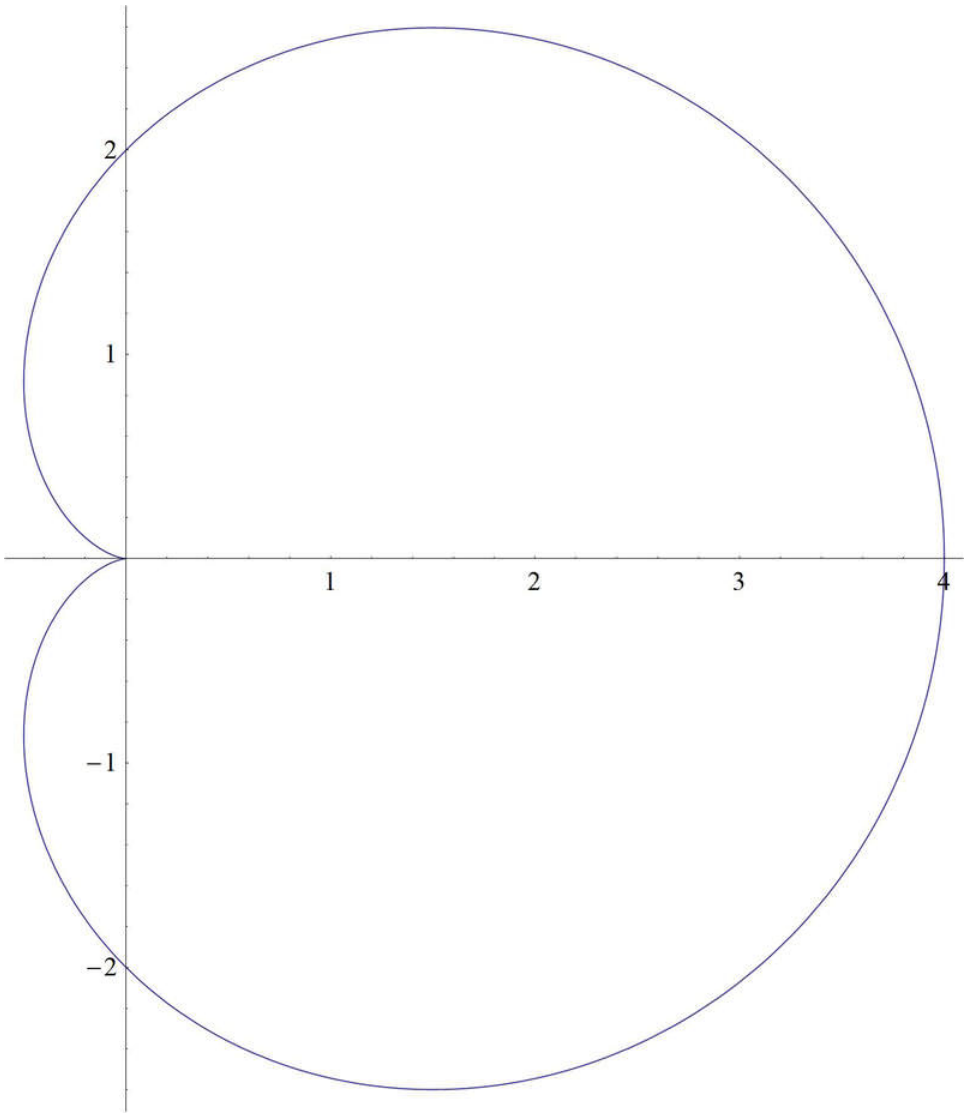
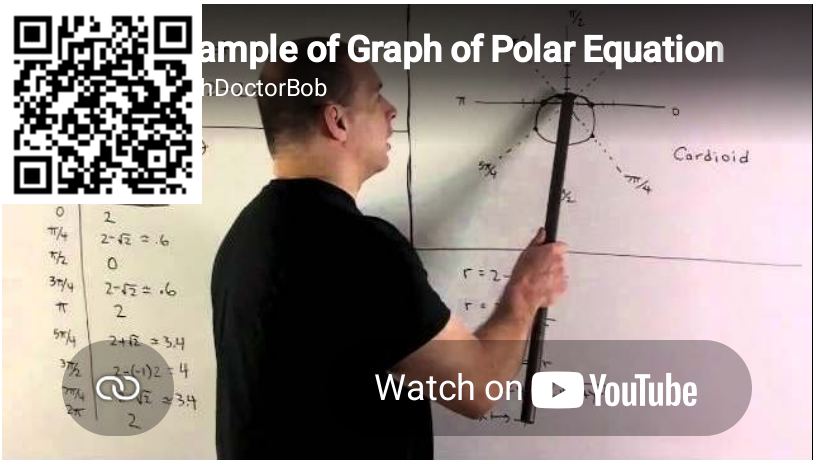


Figure **5.1.4.2**

Transforming Polar Graphs

Just as in graphing on a rectangular grid, you can also graph polar equations on a polar grid.



These equations may be simple or complex. To begin, you should try something simple like $r = k$ or $\theta = k$ where k is a constant. The solution for $r = 1.5$ is simply all ordered pairs such that $r = 1.5$ and θ is any real number. The same is true for the solution of $\theta = 30°$. The ordered pairs will be any real number for r and θ will equal $30°$. Here are the graphs for each of these polar equations.

Graphing Equations

1. On a polar plane, graph the equation $r = 1.5$

The solution is all ordered pairs of (r, θ) such that r is always 1.5. This means that it doesn't matter what θ is, so the graph is a circle with radius 1.5 and centered at the origin.

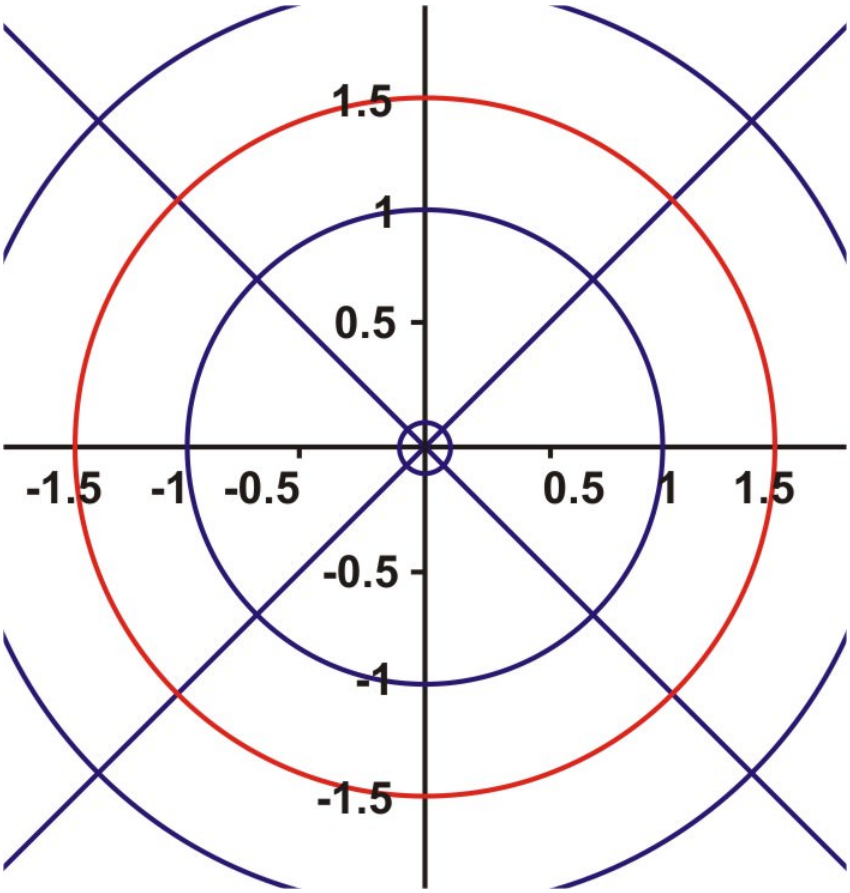


Figure 5.1.4.3

2. On a polar plane, graph the equation $\theta = 30°$

For this problem, the r value, or radius, is arbitrary. θ must equal $30°$, so the result is a straight line, with an angle of elevation of $30°$.

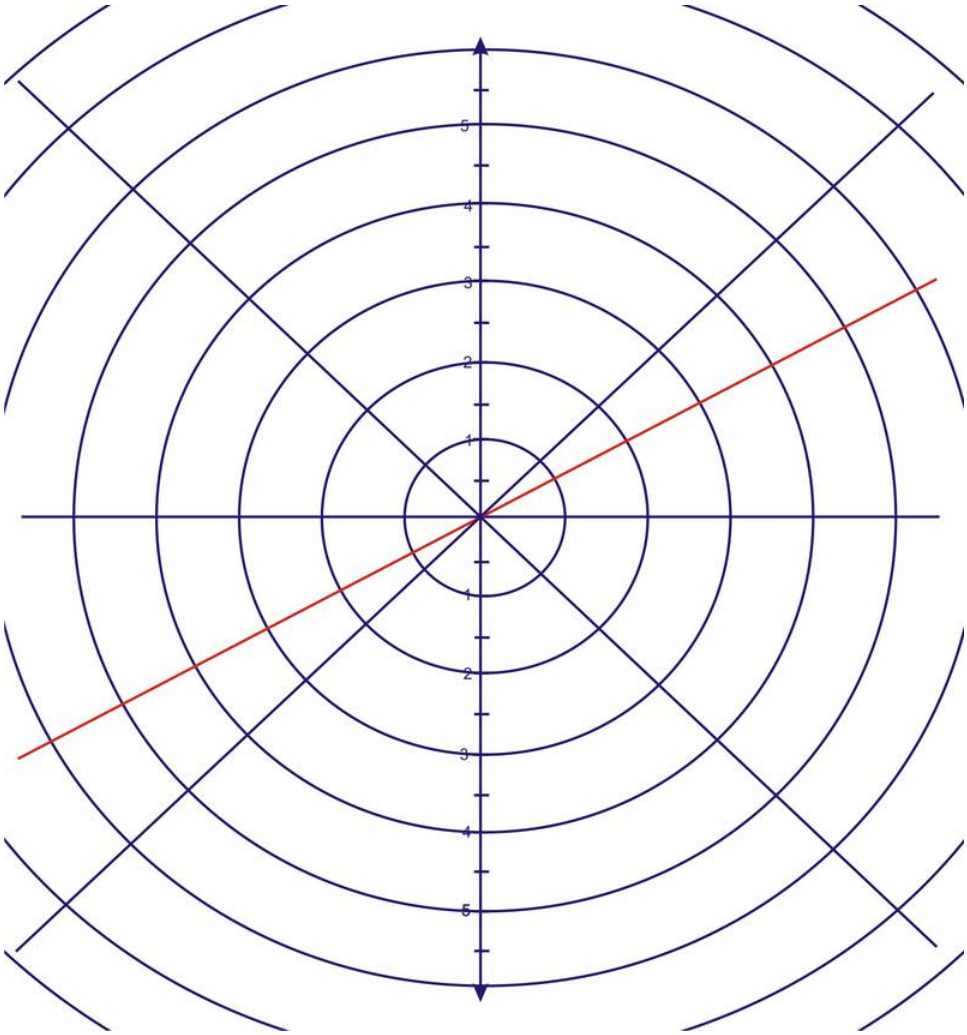


Figure 5.1.4.4

To begin graphing more complicated polar equations, we will make a table of values for $y = \sin\theta$ or in this case $r = \sin\theta$. When the table has been completed, the graph will be drawn on a polar plane by using the coordinates (r, θ).

3. Graph the following polar equations on the same polar grid and compare the graphs.

$$r = 5 + 5\sin\theta \quad r = 5 - 5\sin\theta$$
$$r = 5(1 + \sin\theta) \quad r = 5(1 - \sin\theta)$$

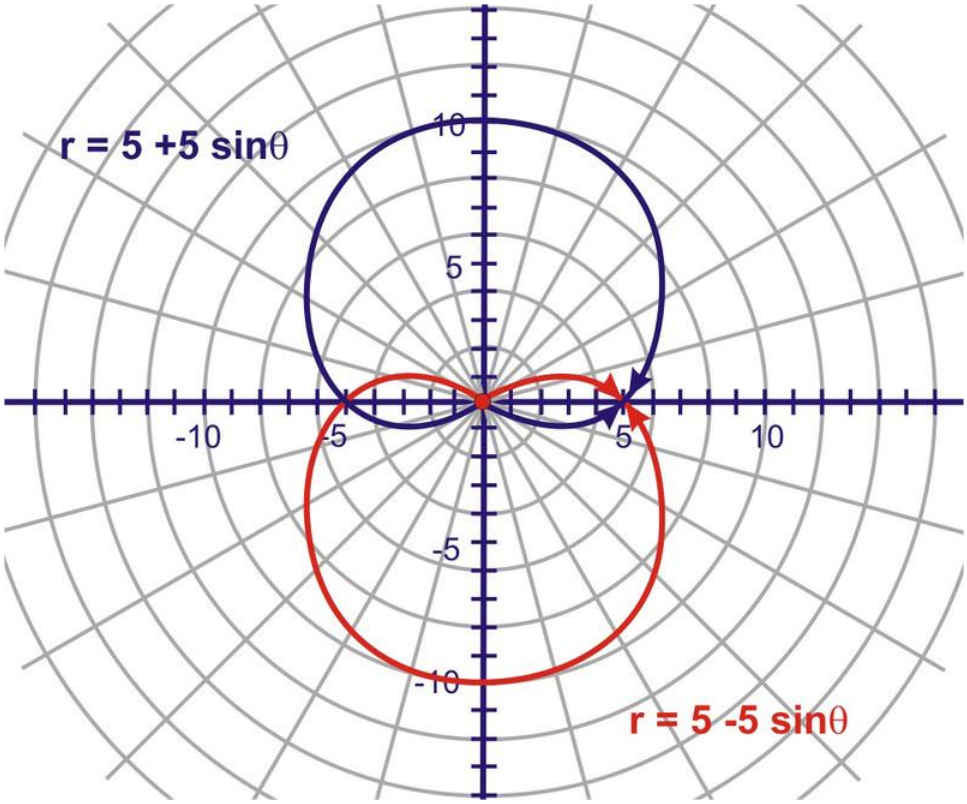


Figure **5.1.4.5**

The cardioid is symmetrical about the positive y−axis and the point of indentation is at the pole. The result of changing + to - is a reflection in the x−axis. The cardioid is symmetrical about the negative y−axis and the point of indentation is at the pole.

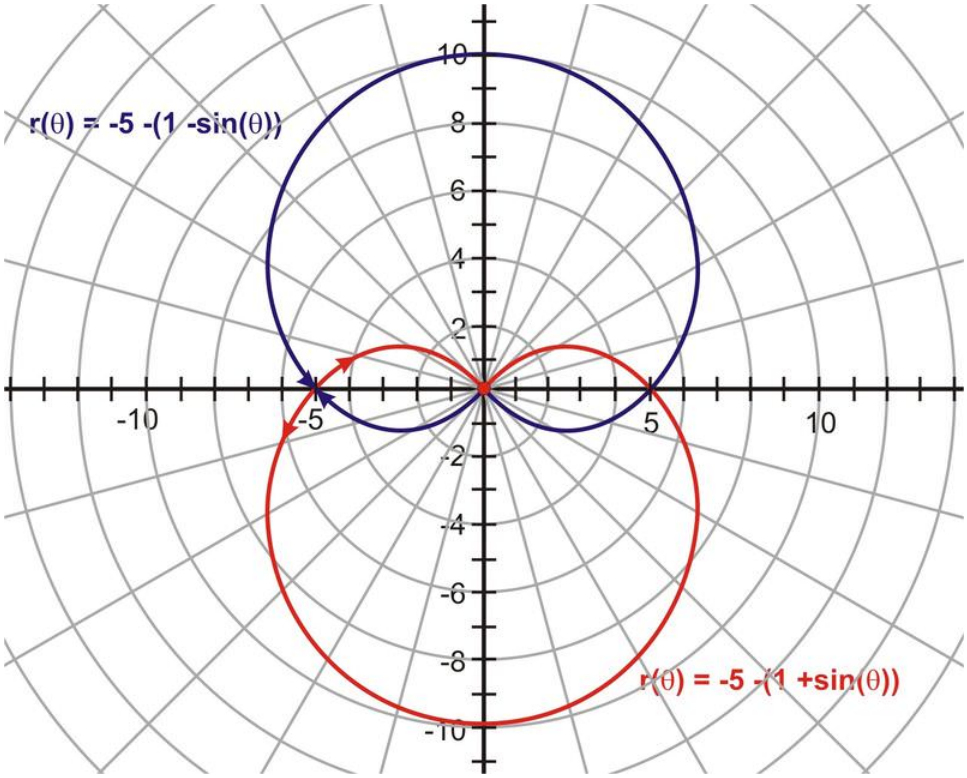


Figure
5.1.4.6

Changing the value of "a" to a negative did not change the graph of the cardioid.

It is also possible to create a sinusoidal curve called a limaçon. It has $r = a \pm b\sin\theta$ or $r = a \pm b\cos\theta$ as its polar equation. Not all limaçons have the inner loop as a part of the shape. Some may curve to a point, have a simple indentation (known as a dimple) or curve outward. The shape of the limaçon depends upon the ratio of ab where a is a constant and b is the coefficient of the trigonometric function.

As we've seen with cardioids, it is possible to create transformations of graphs of limaçons by changing values of constants in the equation of the shape.

Example 5.1.4.1

Earlier, you were asked to graph a polar equation.

Solution

As you've seen in this section, transformations to the graph of a cardioid can be accomplished by 2 different ways. In this case, you want to rotate the graph so that it is around the "x" axis instead of the "y" axis. To accomplish this, you change the function from a sine function to a cosine function:

$r = 2 + 2\cos\theta$

Example 5.1.4.2

Graph the curve $r = -3 - 3\cos\theta$

Solution

$r = -3 - 3\cos\theta$.

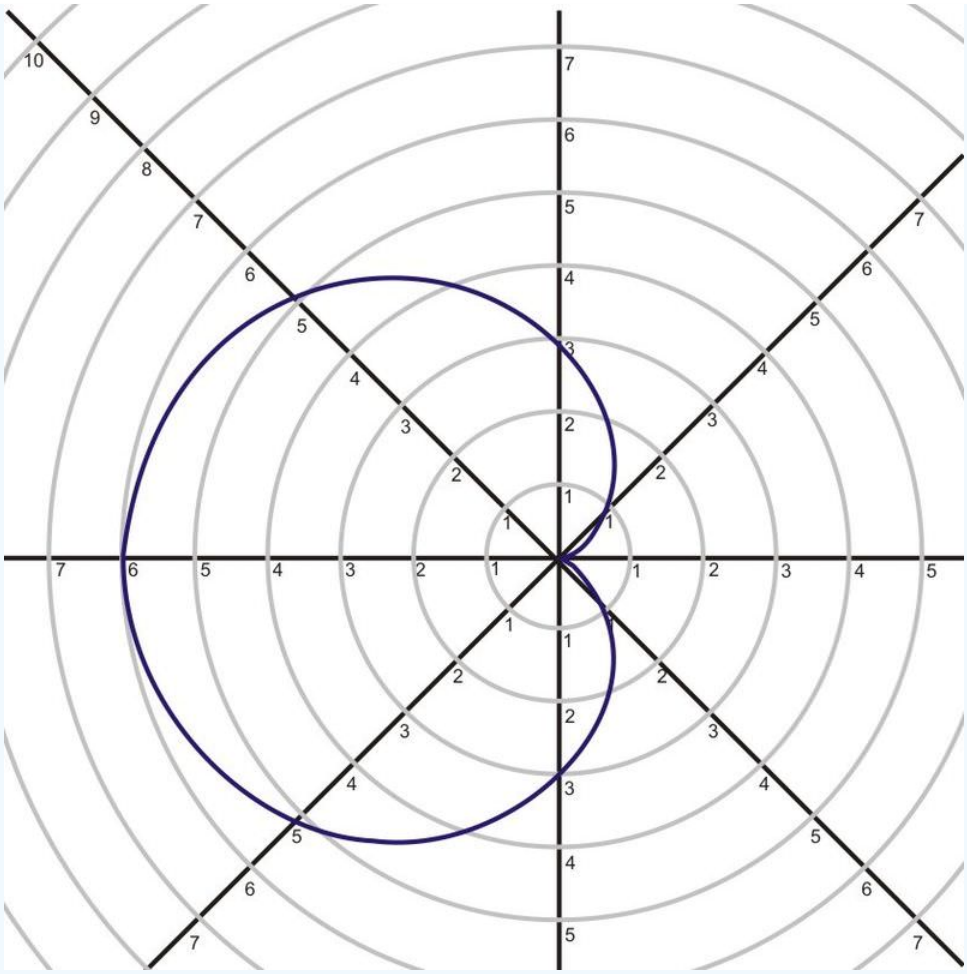


Figure 5.1.4.7

Example 5.1.4.3

Graph the curve $r = 2 + 4\sin\theta$

Solution

$r = 2 + 4\sin\theta$

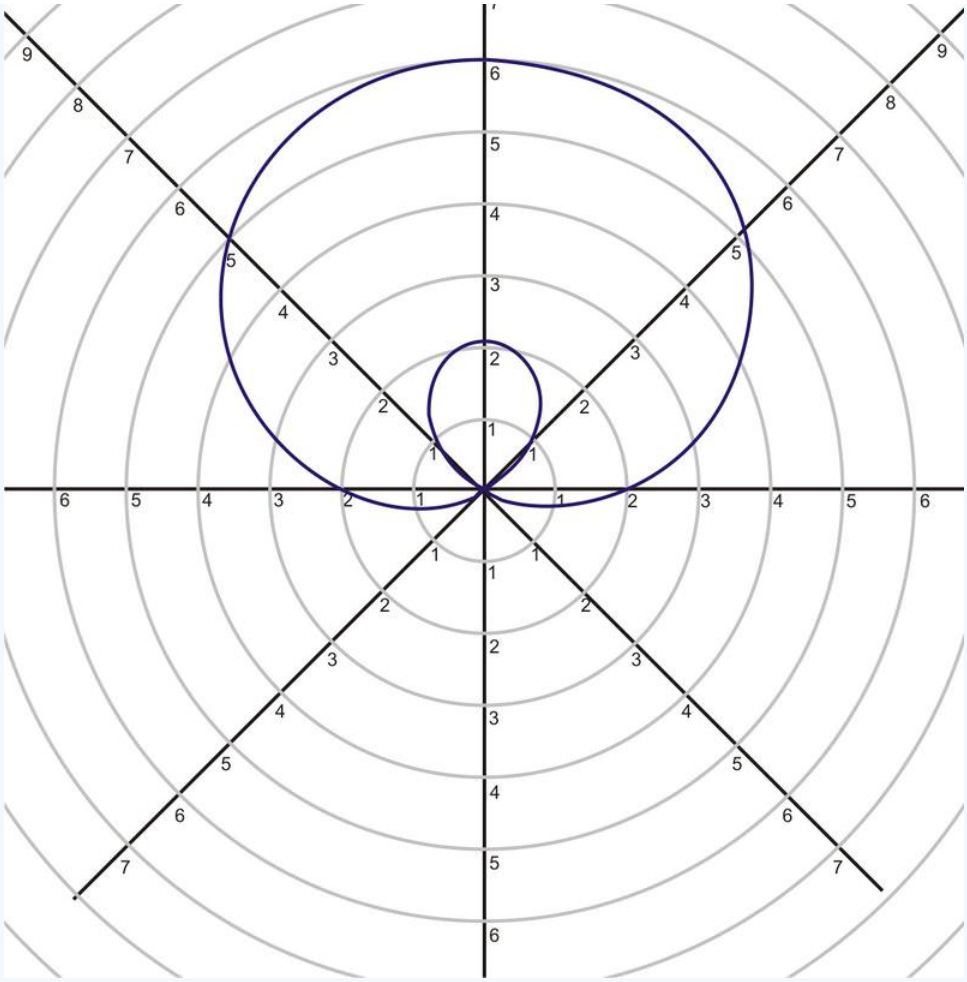


Figure 5.1.4.8

Example 5.1.4.4

Graph the curve $r = 4$.

Solution

$r = 4$

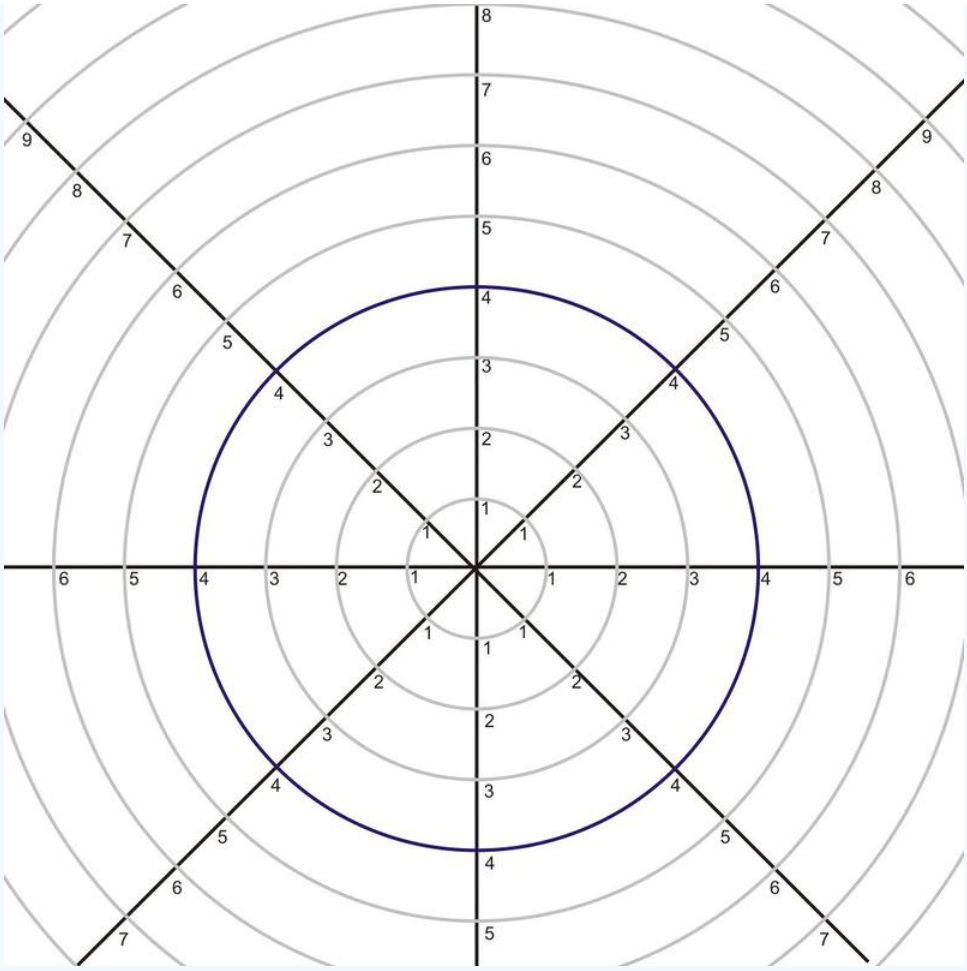


Figure 5.1.4.9

Review

Graph each equation.

1. $r = 4$
2. $\theta = 60°$
3. $r = 2$
4. $\theta = 110°$

Graph each function using your calculator and sketch on your paper.

5. $r = 3 + 3\sin(\theta)$
6. $r = 2 + 4\sin(\theta)$
7. $r = 1 - 5\sin(\theta)$
8. $r = 2 - 2\sin(\theta)$
9. $r = 3 + 6\sin(\theta)$
10. $r = -3 + 6\sin(\theta)$
11. Analyze the connections between the equations and their graphs above. Make a hypothesis about how to graph $r = a + b\sin(\theta)$ for positive or negative values of a and b where $b \geq a$.

Graph each function using your calculator and sketch on your paper.

12. $r = 3 + 3\cos(\theta)$
13. $r = 2 + 4\cos(\theta)$
14. $r = 1 - 5\cos(\theta)$

15. $r = 2 - 2\cos(\theta)$
16. $r = 3 + 6\cos(\theta)$
17. $r = -3 + 6\cos(\theta)$
18. Analyze the connections between the equations and their graphs above. Make a hypothesis about how to graph $r = a + b\cos(\theta)$ for positive or negative values of a and b where $b \geq a$.

Review (Answers)

To see the Review answers, open this PDF file and look for section 6.3.

Additional Resources

Video: Example of Graph of Polar Equation

Practice: Transformations of Polar Graphs

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5.1.5: Special Polar Equations and Graphs

Archimedean spirals, cardioids, and roses.

Max is in charge of the sound system for his big sister's wedding reception. He can't get into the hall until the morning of the wedding, so he's going to have to set up his microphones and speakers in record time. He does, however, have a floor plan of the hall. How can he make sure that the pick-up areas of the microphones and the broadcast area of the speakers don't overlap and cause feedback? He'll have time for some minimal trial and error, but he needs to have a general idea where all the microphones and speakers will be during the reception. Can he use polar coordinates to help him place his equipment?

More Polar Equations and Graphs

Why do people continue to use polar coordinates when modern computers are powerful and fast enough to solve extremely complicated problems in rectangular form? One reason is that many polar graphs are beautiful and intriguing. Polar graphs can help people see patterns that they might otherwise overlook. Artists have even used polar graphs as the basis of their designs.

One of the simplest equations that forms a special polar curve is $r = a\theta$, where a is any real number and \theta ranges from zero to infinity. Equations of this form create a shape known as an **Archimedean spiral**. As θ increases, the graph continues to spiral out like a perfect snail's shell. The following graphs demonstrate how changing the value of a alters the spiral. Note that each curve will continue to spiral forever.

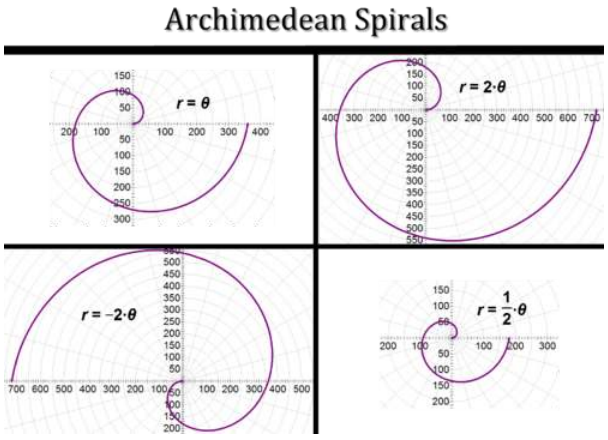


Figure 5.1.5.1

Another important polar curve is the **cardioid**. People who work with acoustics know that the cardioid is an accurate model for both the pick-up range of certain types of microphones and the broadcast range for certain kinds of speakers. Cardioids get their name from their heart-like shapes. Equations of the form $r = 1 + a\cos\theta$ produce cardioid curves. You can change the orientation of a cardioid, or of any other polar equation with cosine in its standard form, by replacing cosine with sine, negative cosine, or negative sine.

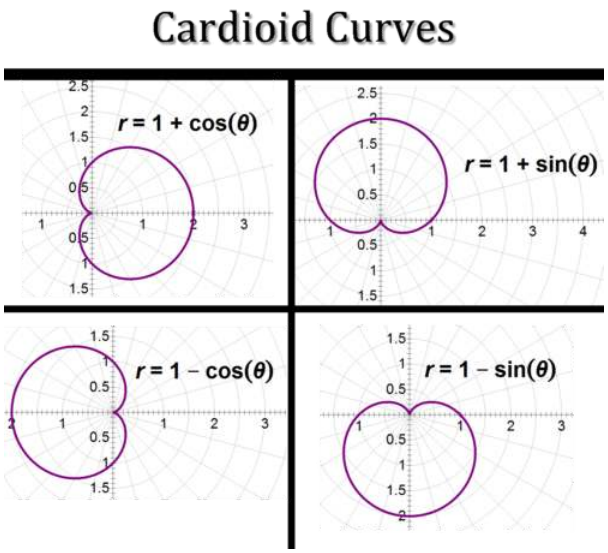


Figure 5.1.5.2

Rose curves are another interesting set of polar curves. For these equations of the form $r = a\cos n\theta$, where n is a natural number, the plots resemble flowers. When n is odd, the flowers have n petals, and when n is even, the flowers have $2n$ petals.

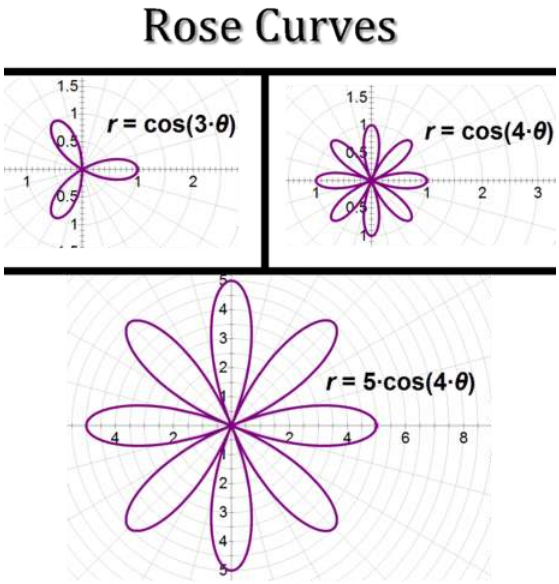


Figure 5.1.5.3

You can use your graphing calculator or other technology to help you graph all of these polar curves.

With this knew information, what is the shape of the graph of the polar equation $6r = 5 + 5\sin\theta$?

First, isolate r to get $r = \frac{5}{6} + \frac{5}{6}\sin\theta$. This graph appears most similar to the cardioid curve, which is $r = 1 + a\cos\theta$. However, the changes to the form mean that the graph will be rotated by $\frac{\pi}{2}$ and slightly smaller than the standard cardioid. Now, graph the equation to test the predictions.

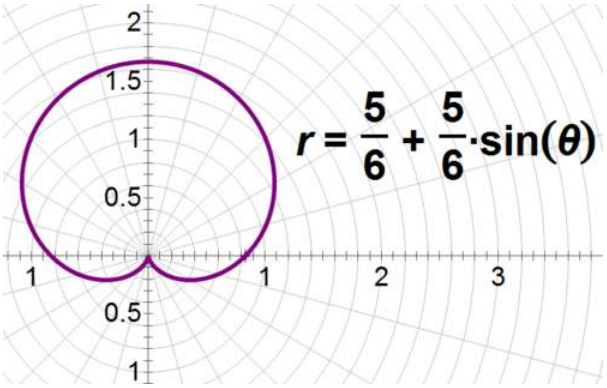


Figure 5.1.5.4

Now, let's describe the graph of $r = -8\sin(32\theta)$, then graph it.

This graph matches the format for a rose curve: $r = a\cos n\theta$. Since n is even the final graph should have 64 petals. The – sine means that the graph will be rotated $-\frac{\pi}{2}$ radians from its starting position.

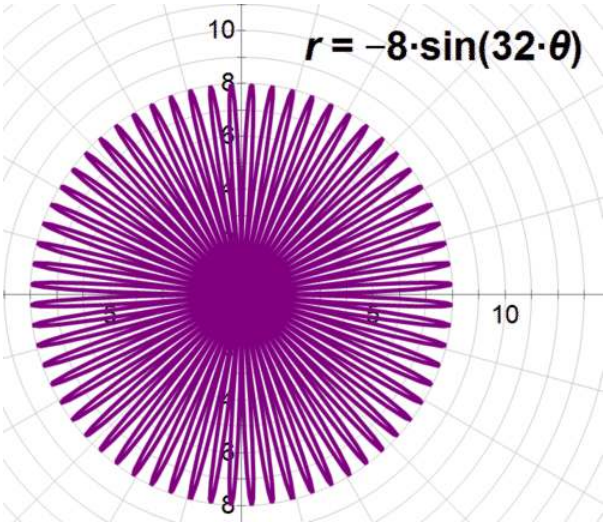
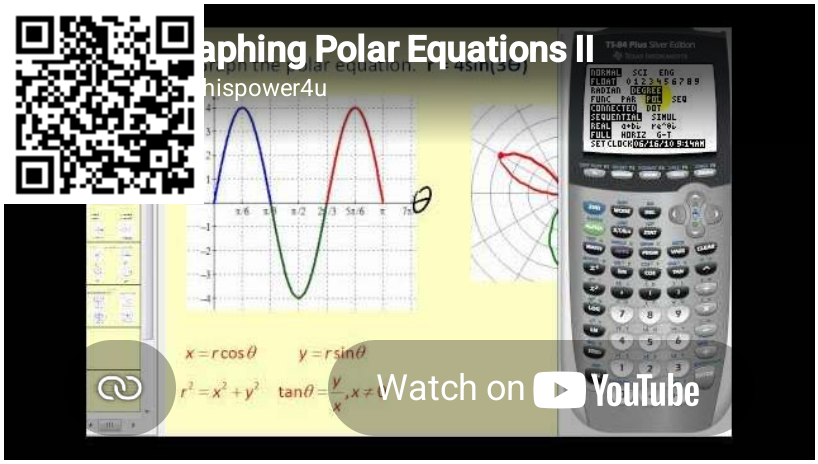


Figure 5.1.5.5



Example 5.1.5.1

Earlier, you were asked how Max can set up the sound system for his sister's wedding so that the speakers don't overlap. Max can use cardioids to help him set up the sound system without causing feedback. He researches his microphones and speakers online and finds out that the pick-up pattern of his microphones can be graphed using the equation $r = \frac{1}{2} + \frac{1}{2}\cos\theta$. Since he

has multiple microphones to place, he can graph the curve and use it with his floor plan to ensure that the mikes don't overlap and that he places the speakers in the dead zones behind the microphones, where they won't pick up any sound.

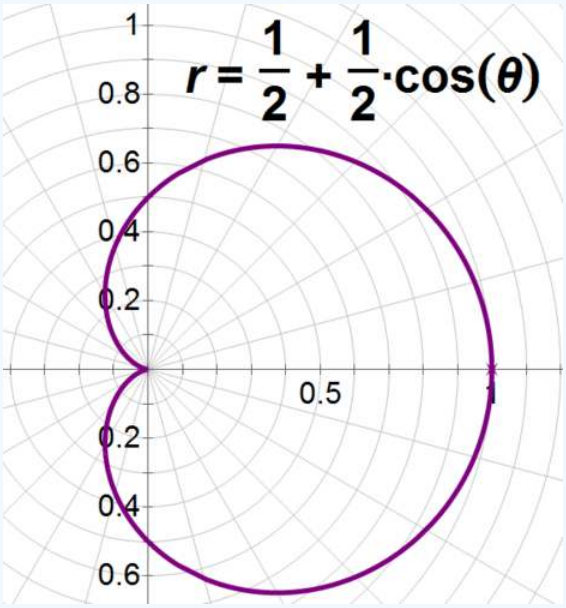


Figure 5.1.5.6

Solution

Since the microphones pick up along the polar axis between 0 and 1, he'll want to position speakers in the dead zone, where $r < 0$, and $\theta = 0$. He can also position speakers at other places in the dead zone, but cardioid microphones are least likely to pick up sound when it's \pi radians from their optimal pick-up areas.

Example 5.1.5.2

Describe what the graph $r = -3\theta$ will look like. Then, change the equation to rotate it by π radians. Graph the original and rotated graphs.

Solution

The graph will make an Archimedean spiral three times larger than the normal one. To rotate the graph, change the -3 to 3.

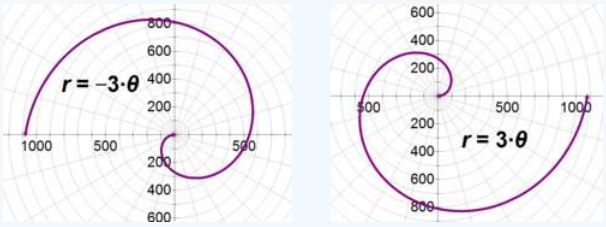


Figure 5.1.5.7

For each of the following examples, identify the polar curve given by the equation and then graph it.

Example 5.1.5.3

$6\cos 4\theta - 3r = 0$

Solution

First, solve for r. Then, identify the equation.

$$\begin{aligned} 6\cos 4\theta - 3r &= 0 \\ 3r &= 6\cos 4\theta \\ r &= 2\cos 4\theta \end{aligned}$$

The graph will be of a rose curve with 8 petals.

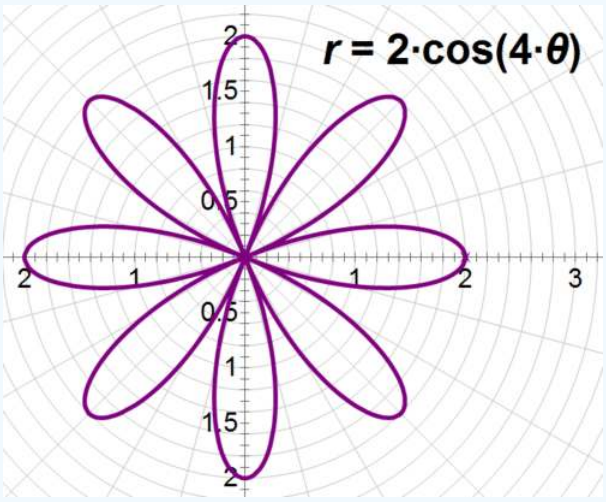


Figure 5.1.5.8

Example 5.1.5.4

$5r = 3\theta$

Solution

First, solve for r. Then, identify the equation.

$$5r = 3\theta$$
$$r = \frac{3}{5}\theta$$

The graph will be Archimedean spiral.

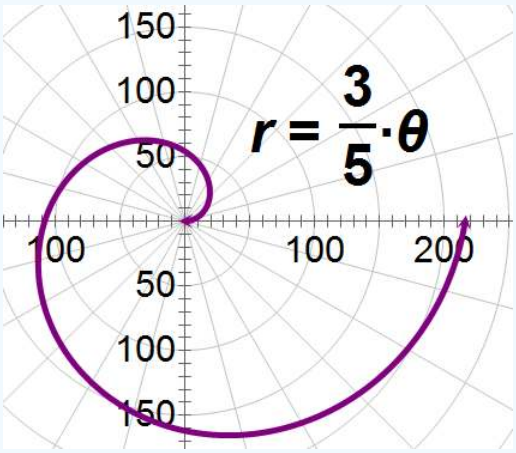


Figure 5.1.5.9

Example 5.1.5.5

$5r - 25 + 45\cos 3\theta = 0$

Solution

First, solve for r. Then, identify the equation.

$$5r - 25 + 45\cos 3\theta = 0$$
$$5r = 25 - 45\cos 3\theta$$
$$r = 5 - 9\cos 3\theta$$

At first glance, this equation looks similar to a cardioid. However, it also has $n\theta$ like a rose curve. You'll have to graph it to get a good sense for what the combination does - it's a rose curve within a rose curve!

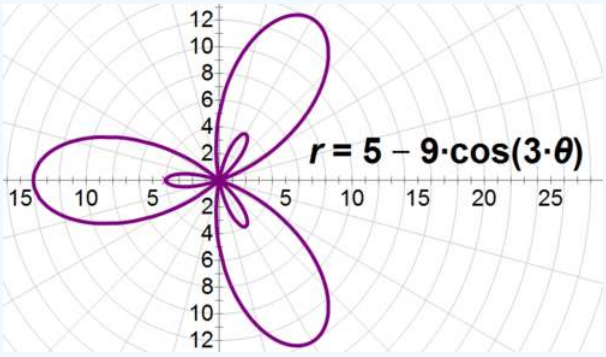


Figure 5.1.5.10

However, take out the three, graph $r = 5 - 9\cos\theta$, and you'll get a curve similar to a cardioid curve. This equation illustrates how small changes can produce complicated, beautiful, polar graphs.

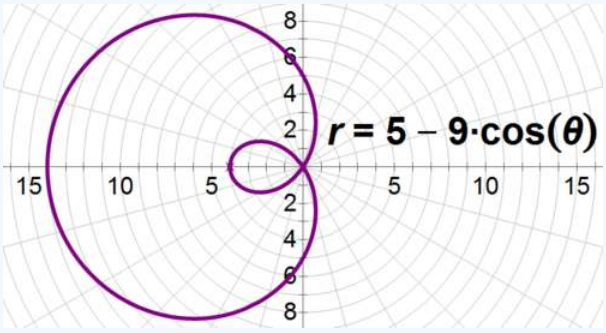


Figure 5.1.5.10

Review

For #1-3, describe the family of equations that produces that curve.

1. Rose curve
2. Archimedean spiral
3. Cardioid curve

For #4-15, identify the polar curve and then graph it.

4. $\frac{r}{2} = \theta$
5. $r = 1 - 4\cos\theta$
6. $3r = 4\cos(8\theta)$
7. $r = \frac{1}{2} + \frac{1}{2}\cos(\theta)$
8. $r = -\theta$
9. $\frac{r}{4} = 4\cos(4\theta)$
10. $r = -4\theta$
11. $r = -2$ $sin(8\theta)$
12. $r = 1 -$ $sin\theta$
13. $1 = \cos\theta + r$
14. $r = 5$
15. $r = 5\theta$

Review (Answers)

To see the Review answers, open this PDF file and look for section 10.12.

Vocabulary

Term	Definition
Archimedean spiral	An Archimedean spiral is a pattern that resembles a snail shell. It's formed by equations in the r=a\theta family.
cardioid	A cardioid curve is a polar graph formed by variations on the equation $r = 1 + a\cos\theta$, where a is a real number. Cardioid curves are heart-shaped. They're especially important for people who work in acoustics and sound design, \since they model the performance of many microphones and speakers.
rose	A rose curve is a polar curve that has captivated artists and designers. It's formed by equations in the r=a\cos n\theta family. The coefficient n is a natural number that determines the number of petals on the graph. When n is odd, the graph has n petals. When n is even, the graph has 2n petals.

Additional Resources

Real World: Spirals in the Sea

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SECTION OVERVIEW

5.2: Polar Curves and Rectangular Conversions

5.2.1: Polar and Rectangular Conversions

5.2.2: Rectangular to Polar Conversions

5.2.3: Rectangular to Polar Form for Equations

5.2.4: Intersections of Polar Curves

5.2.5: Equivalent Polar Curves

5.2.6: Systems of Polar Equations

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5.2.1: Polar and Rectangular Conversions

Convert from polar to cartesian coordinates.

You will see during this lesson that points can be converted from **rectangular form** to **polar form** with a little algebra and trigonometry.

Can the equation of a shape be converted also? How about a circle, for instance?

Polar and Cartesian Transformation

Polar Form to Rectangular Form

Sometimes a problem will be given with coordinates in polar form but rectangular form may be needed.

To transform the polar point $\left(4, \frac{3\pi}{4}\right)$ into **rectangular coordinates**: first identify (r, θ).

$r = 4$ and $\theta = \frac{3\pi}{4}$.

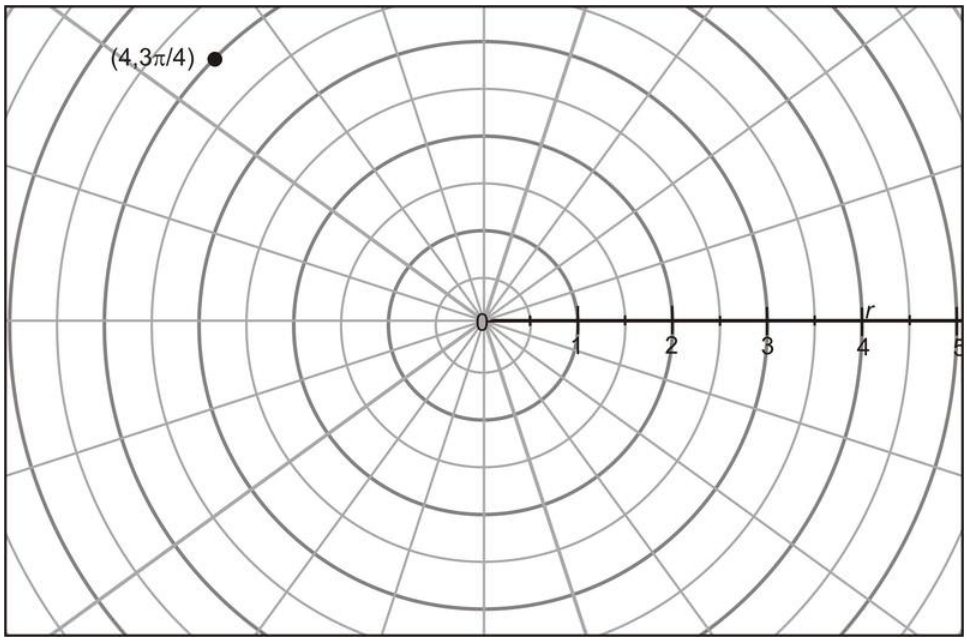


Figure **5.2.1.1**

Second, draw a vertical line from the point to the polar axis (the horizontal axis). The distance from the pole to where the line you just drew intersects the polar axis is the x value, and the length of the line segment from the point to the polar axis is the y value.

These distances can be calculated using trigonometry:

$x = r\cos\theta$ and $y = r\sin\theta$

$x = 4\cos\frac{3\pi}{4}$ and $y = 4\sin\frac{3\pi}{4}$ or $x = -2\sqrt{2}\ y = 2\sqrt{2}$

$\left(4, \frac{3\pi}{4}\right)$ in **polar coordinates** is equivalent to $(-2\sqrt{2}, 2\sqrt{2})$ in rectangular coordinates.

Rectangular Form to Polar Form

Going from rectangular coordinates to polar coordinates is also possible, but it takes a bit more work. Suppose we want to find the polar coordinates of the rectangular point (2,2). To begin doing this operation, the distance that the point (2,2) is from the origin (the radius, *r*) can be found by

$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{2^2 + 2^2}$$
$$r = \sqrt{8} = 2\sqrt{2}$$

The angle that the line segment between the point and the origin can be found by

$$\begin{aligned}\tan\theta &= \frac{y}{x}\\ \tan\theta &= \frac{2}{2}\\ \tan\theta &= 1\\ \theta &= \tan^{-1} 1\\ \theta &= \frac{\pi}{4}\end{aligned}$$

Since this point is in the first **quadrant** (both the *x* and *y* coordinate are positive) the angle must be 45° or $\frac{\pi}{4}$ radians. It is also possible that when $tan\theta = 1$ the angle can be in the third quadrant, or $\frac{5\pi}{4}$ radians. But this angle will not satisfy the conditions of the problem, since a third quadrant angle must have both *x* and *y* negative.

Note: when using $\tan\theta = \frac{y}{x}$ to find the measure of \theta you should consider, at first, the quotient $\tan\theta = \left|\frac{y}{x}\right|$ and find the first quadrant angle that satisfies this condition. This angle will be called the reference angle, denoted θ_{ref}. Find the actual angle by analyzing which quadrant the angle must be given the signs of *x* and *y*.

Example 5.2.1.1

Earlier, you were asked if the equation of a circle could be converted from rectangular form to polar form.

Solution

Equation of a circle: $x^2 + y^2 = k^2$ is the equation of a circle with a radius of *k* in rectangular coordinates.

The equation of a circle is extremely simple in polar form. In fact, a circle on a polar graph is analogous to a horizontal line on a rectangular graph!

You can transform this equation to polar form by substituting the polar values for *x*, *y*. Recall $x = r\cos\theta$ and $y = r\sin\theta$.

$(r\cos\theta)^2 + (r\sin\theta)^2 = k^2$

square the terms: $r^2\cos^2\theta + r^2\sin^2\theta = k^2$,

factor the r^2 from both terms on the left: $r^2(\cos^2\theta + \sin^2\theta) = k^2$

recall the identity: $\cos^2\theta + \sin^2\theta = 1$

$r^2 = k^2$

Therefore: $r = \pm k$ is an equation for a circle in polar units.

When r is equal to a constant, the polar graph is a circle.

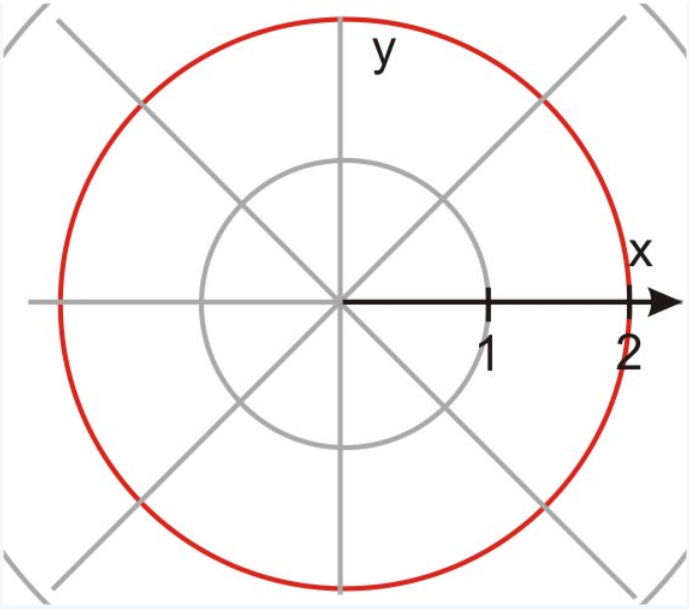


Figure 5.2.1.2

Example 5.2.1.2

Transform the polar coordinates $\left(2, \frac{11\pi}{6}\right)$ to rectangular form.

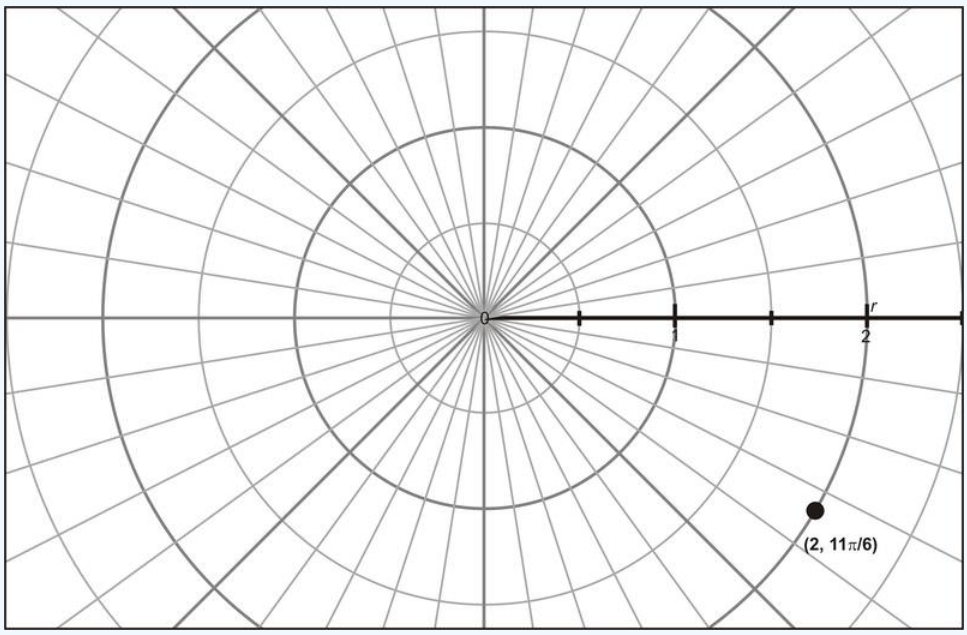


Figure 5.2.1.3

Solution

$r = 2$ and $\theta = \frac{11\pi}{6}$

$x = r\cos\theta$ and $y = r\sin\theta$

$x = 2\cos\frac{11\pi}{6}$ and $y = 2\sin\frac{11\pi}{6}$ or $x = 3\sqrt{2}\ y = -1$

$\left(2, \frac{11\pi}{6}\right)$ is equivalent to $(3\sqrt{2}, -1)$ or in decimal form, approximately $(4.342, -1)$.

Example 5.2.1.3

Find the polar coordinates for $(3, -3\sqrt{3})$.

Solution

$x = 3$ and $y = -3\sqrt{3}$

Draw a right triangle in standard form. Find the distance the point is from the origin and the angle the line segment that represents this distance makes with the +x axis:

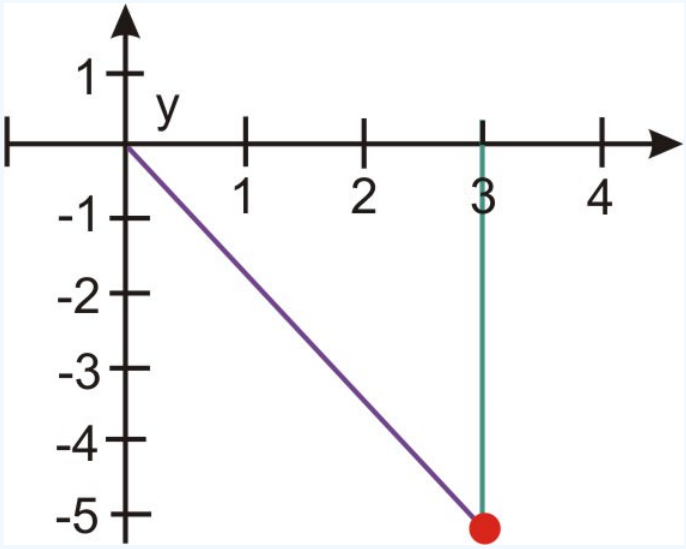


Figure 5.2.1.4

$$\begin{aligned} r &= \sqrt{3^2 + (-3\sqrt{3})^2} \\ &= \sqrt{9 + 27} \\ &= \sqrt{36} \\ &= 6 \end{aligned}$$

And for the angle,

$$\begin{aligned} \tan\theta_{ref} &= \left| \frac{(-3\sqrt{3})}{3} \right| \\ \tan\theta_{ref} &= \sqrt{3} \\ \theta_{ref} &= \tan^{-1}\sqrt{3} \\ \theta_{ref} &= \frac{\pi}{3} \end{aligned}$$

So, $\theta_{ref} = \frac{\pi}{3}$ and we can look at the signs of *x* and *y* -- (+, -) -- to see that $\theta = \frac{5\pi}{3}$ since it is a 4th quadrant angle.

The rectangular point $(3, -3\sqrt{3})$ is equivalent to the polar point $\left(6, \frac{5\pi}{3}\right)$.

Recall that when solving for θ, we used

$$\tan\theta = \left| \frac{(-3\sqrt{3})}{3} \right| \text{ or } \tan\theta = \sqrt{3}$$

We found $\theta = \frac{5\pi}{3}$. BUT, θ could also be $\theta = \frac{2\pi}{3}$. You must examine the signs of each coordinate to see that the angle must be in the fourth quadrant in rectangular units or between $3\frac{\pi}{2}$ and 2π in polar units. Of the two possible angles for θ, only $\frac{5\pi}{3}$ is valid. Note that when you use $\tan^{-1}$ on a calculator you will always get an answer in the range $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

Example 5.2.1.4

Convert the following rectangular coordinates to polar coordinates.

a. $(3, 3\sqrt{3})$

$(6, 66°)$

b. $(-2, 2)$

$(2\sqrt{2}, 225°)$

Convert the following polar coordinates to rectangular coordinates.

c. $\left(4, \frac{2\pi}{3}\right)$

$(-2, \sqrt{3})$

d. $(-1, \frac{5\pi}{6})$

$(\frac{\sqrt{3}}{2}, -\frac{1}{2})$

Example 5.2.1.5

Express the equation in rectangular form: $r = 6\cos\theta$.

Solution

$r^2 = 6r\cos\theta$: multiply both sides by r

$x^2 + y^2 = 6x$: Using $x^2 + y^2 = r^2$ and $x = r\cos\theta$

$\therefore x^2 + y^2 = 6x$ is the equation in rectangular form.

Example 5.2.1.6

Express the equation in rectangular form: $r = 6$.

Solution

This one is easy:

$r = 6$ is the polar form of the equation for a circle

$r^2 = 6^2$: square both sides

$x^2 + y^2 = 36$: Using $x^2 + y^2 = r^2$ and simplifying

$\therefore x^2 + y^2 = 36$ is the equation in rectangular form.

Convert the rectangular coordinate to a polar coordinate

Review

1. How is the point with polar coordinates $(5, \pi)$ represented in rectangular coordinates?

Plot each point below in polar coordinates (r, θ). Then write the rectangular coordinates (x, y) for the point.

2. $(3, 60°)$
3. $(-10, \frac{\pi}{3})$
4. $(15, \pi)$

The rectangular coordinates (x, y) are given. For each question: a) Find two pairs of polar coordinates (r, θ), one with $r > 0$ and the other with $r < 0$. b) Express \theta in radians, and round to the nearest hundredth.

5. $(5, -5)$

6. $(0, 10)$
7. $(-8, 6)$

Transform each polar equation to an equation using rectangular coordinates. Identify the graph, and give a rough sketch or description of the sketch.

8. $\theta = \frac{\pi}{10}$
9. $r = 8$
10. $r \sin \theta = 7$
11. $r \cos \theta = -3$

Transform each rectangular equation to an equation using polar coordinates. Identify the graph, and give a rough sketch or description of the sketch.

12. $x^2 + y^2 - 2x = 0$
13. $y = \sqrt{3}x$
14. $y = -5$
15. $xy = 15$

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.2.

Vocabulary

Term	Definition
Tangent	The tangent of an angle in a right triangle is a value found by dividing the length of the side opposite the given angle by the length of the side adjacent to the given angle.
cosine	The cosine of an angle in a right triangle is a value found by dividing the length of the side adjacent the given angle by the length of the hypotenuse.
polar coordinates	Polar coordinates describe locations on a grid using the polar coordinate system. The location of each point is determined by its distance from the pole and its angle with respect to the polar axis.
polar form	The polar form of a point or a curve is given in terms of r and θ and is graphed on the polar plane.
quadrant	A quadrant is one-fourth of the coordinate plane. The four quadrants are numbered using Roman Numerals I, II, III, and IV, starting in the top-right, and increasing counter-clockwise.
Quadrants	A quadrant is one-fourth of the coordinate plane. The four quadrants are numbered using Roman Numerals I, II, III, and IV, starting in the top-right, and increasing counter-clockwise.
rectangular coordinates	A point is written using rectangular coordinates if it is written in terms of x and y and can be graphed on the Cartesian plane.
rectangular form	The rectangular form of a point or a curve is given in terms of x and y and is graphed on the Cartesian plane.
sine	The sine of an angle in a right triangle is a value found by dividing the length of the side opposite the given angle by the length of the hypotenuse.

Additional Resources

Video: Polar Coordinates - Example^2

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5.2.2: Rectangular to Polar Conversions

Convert from cartesian to polar coordinates

Trig Riddle: I am the point $(1, -3)$. What are my polar coordinates?

Trigonometric Ratios of Points

Any point in the coordinate plane can be represented by its angle of rotation and radius, or distance from the origin. The point is said to lie on the terminal side of the angle. We can find the measure of the reference angle using right triangle trigonometry. When the point is identified in this manner we call the coordinates Polar coordinates. They are written as (r, θ), where r is the radius and θ is the angle of rotation. The angle of rotation can be given in degrees or radians.

Let's find the angle of rotation (in degrees) and radius (distance from the origin) of the point $(-3, 6)$.

First, make a sketch, plot the point and drop a perpendicular to the x-axis to make a right triangle.

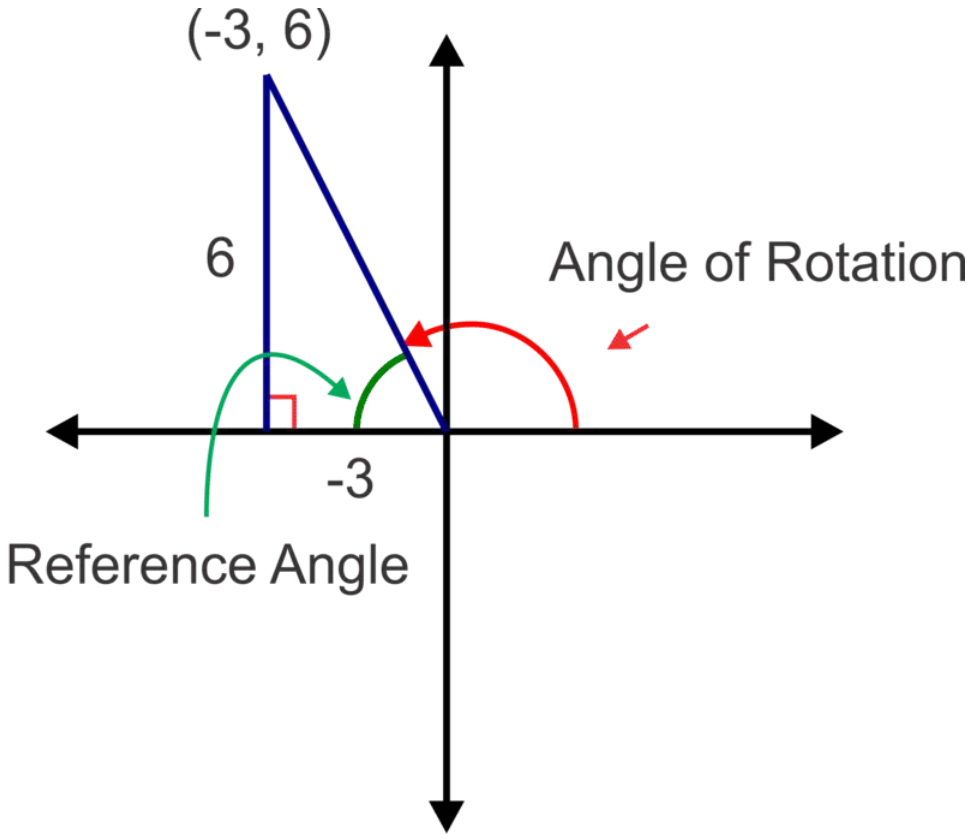


Figure 5.2.2.1

From the sketch, we can see that $\tan^{-1}(-63) = 63.4°$ is the reference angle so the angle of rotation is $180° - 63.4° = 116.6°$.

The radius or distance from the origin is the hypotenuse of the right triangle.

$$r^2 = (-3)2 + (6)2$$
$$r^2 = 45$$
$$r = \sqrt{45} = 3\sqrt{5}$$

Using this information, we can write the point $(-3, 6)$ in Polar coordinate form as $(3\sqrt{5}, 116.6°)$

Now, let's write the Cartesian coordinates, $(3, -4)$, in Polar form and give the angle in degrees.

Again, start with a sketch.

We can find the reference angle again using tangent: $\tan^{-1}\left(\frac{-4}{3}\right) = -53.1°$. So the angle of rotation is $360° - 53.1° = 306.9°$

Now find the radius:

$$r^2 = 3^2 + (-4)^2$$
$$r^2 = 25$$
$$r = \sqrt{25} = 5$$

The Polar coordinates are thus $(5, 306.9°)$

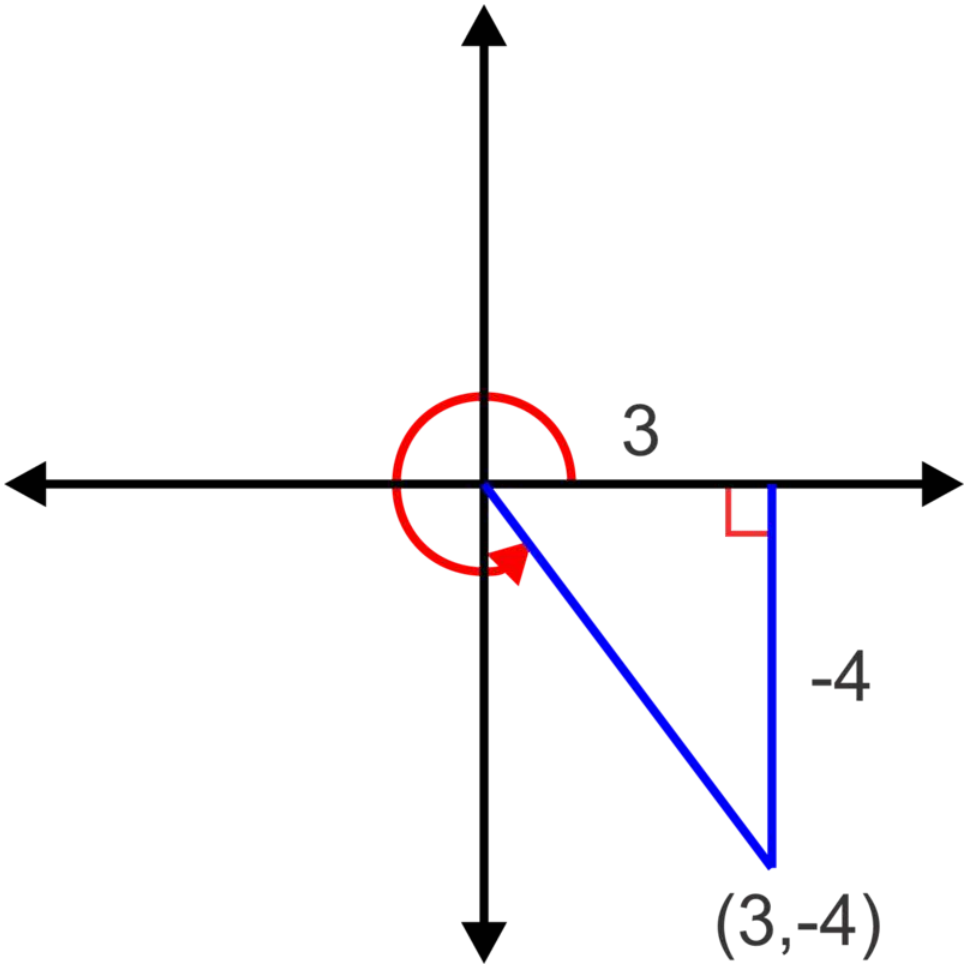


Figure 5.2.2.2

Note: You may have noticed that there is a pattern that gives us a short cut for finding the Polar coordinates for any Cartesian coordinates, (x, y):

The reference angle can be found using, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ and then the angle of rotation can be found by placing the reference angle in the appropriate quadrant and giving a positive angle of rotation from the positive x – axis ($0° \leq \theta < 360°$ or $0 \leq \theta < 2\pi$). The radius is always $r = \sqrt{x^2 + y^2}$ and should be given in reduced radical form.

Finally, given the point (−9,−5) on the terminal side of an angle, let's find the Polar coordinates (in radians) of the point and the six trigonometric ratios for the angle.

Make sure your calculator is in radian mode. Using the shortcut, we can find the Polar coordinates:

$\tan^{-1}\left(\frac{5}{9}\right) = 0.51$. Since x and y are both negative, the point lies in the third quadrant which makes the angle of rotation $\pi + 0.51 = 3.65$. The radius will be $r = \sqrt{9^2 + 5^2} = \sqrt{106}$. The Polar coordinates are $(\sqrt{106}, 3.65)$. As for the six trigonometric ratios, a diagram will help us:

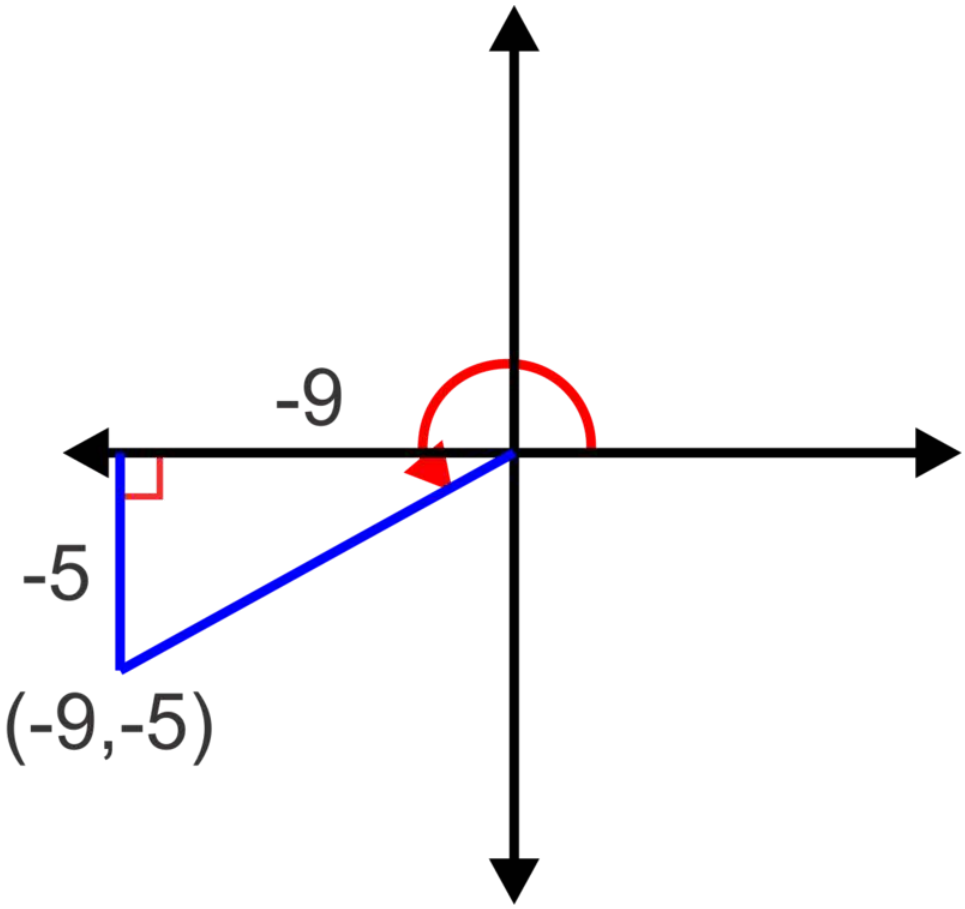


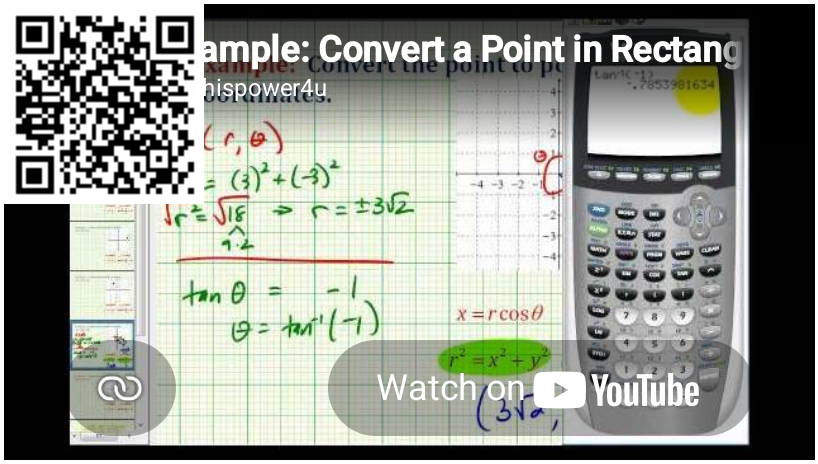
Figure 5.2.2.3

We already know that $\tan 3.65 = \frac{5}{9}$, so $\cot 3.65 = \frac{9}{5}$.

Now we can use the hypotenuse, $\sqrt{106}$ to find the other ratios:

$\sin 3.65 = \frac{-5}{\sqrt{106}} = -\frac{5\sqrt{106}}{106}$ and $\csc 3.65 = \frac{\sqrt{-106}}{5}$.

$\cos 3.65 = \frac{-9}{\sqrt{106}} = -\frac{9\sqrt{106}}{106}$ and $\sec 3.65 = -\frac{\sqrt{106}}{9}$



Example 5.2.2.1

Earlier, you were asked to find the polar coordinates of the point $(1, -3)$.

Solution

First, make a sketch, plot the point and drop a perpendicular to the x-axis to make a right triangle.

From the sketch, we can see that $\tan^{-1}(-31) = 71.6^\circ$ is the reference angle. The point $(1, -3)$ is in the second quadrant, so the angle of rotation is $180^\circ - 71.6^\circ = 108.4^\circ$.

The radius or distance from the origin is the hypotenuse of the right triangle.

$$r^2 = (1)^2 + (-3)^2$$
$$r^2 = 10$$
$$r = \sqrt{10}$$

Therefore, my polar coordinates are $(\sqrt{10}, 108.4^\circ)$.

Example 5.2.2.2

Find the angle of rotation (in degrees) and radius (distance from the origin) of the point $(7, 24)$.

Solution

$r = \sqrt{7^2 + 24^2} = 25$, $\theta = \tan^{-1}\left(\frac{24}{7}\right) \approx 73.7^\circ$

Example 5.2.2.3

Write the Cartesian coordinates, $(-8, -15)$, in Polar form(in radians) and find the six trigonometric ratios for the angle.

Solution

$r = \sqrt{(-8)^2 + (-15)^2} = 17$ and $\theta = \tan^{-1}\left(\frac{-15}{-8}\right) \approx 1.08$ so the polar coordinates are $(17, 1.08)$.

The six trigonometric ratios are:

$$\sin 1.08 = -\frac{15}{17} \quad \csc 1.08 = -\frac{17}{15}$$
$$\cos 1.08 = -\frac{8}{17} \quad \sec 1.08 = -\frac{8}{15}$$
$$\tan 1.08 = \frac{15}{8} \quad \cot 1.08 = -\frac{8}{15}$$

Example 5.2.2.4

Given the point $(12, -4)$ on the terminal side of an angle, find the Polar coordinates (in degrees) of the point and the six trigonometric ratios for the angle.

Solution

$r = \sqrt{12^2 + (-4)^2} = 4\sqrt{10}$ and $\theta = \tan^{-1}\left(\frac{-4}{12}\right) \approx 341.6^\circ$ so the polar coordinates are $(4\sqrt{10}, 341.6^\circ)$.

The six trigonometric ratios are:

$$\sin 341.6^\circ = -\frac{\sqrt{10}}{10} \quad \csc 341.6^\circ = -\sqrt{10}$$
$$\cos 341.6^\circ = \frac{3\sqrt{10}}{10} \quad \csc 341.6^\circ = \frac{\sqrt{10}}{3}$$
$$\tan 341.6^\circ = -\frac{1}{3} \quad \tan 341.6^\circ = -3$$

Review

Angle measures should be rounded to the dearest degree or hundredth of a radian or given exactly if possible. All values of r should be given in reduced radical form.

Write the following Cartesian coordinate pairs in Polar form. Use degrees for problems 1 and 2 and radians for problems 3-5.

1. $(16, -30)$
2. $(5, 5)$
3. $(-5, -12)$
4. $(-9, 40)$
5. $(-4, 8)$

Given the points on the terminal side of an angle, find the Polar coordinates (in degrees) of the point and the six trigonometric ratios for the angles.

6. $(-6, 8)$
7. $(0, -15)$
8. $(10, -8)$
9. $(4\sqrt{3}, 4)$
10. $(-6, 6)$

Given the points on the terminal side of an angle, find the Polar coordinates (in radians) of the point and the six trigonometric ratios for the angles.

11. $(-9, 0)$
12. $(13, -13)$
13. $(2, 3)$
14. $(-7, -7\sqrt{3})$
15. $(-8, -4)$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 13.10.

Additional Resources

Video: Example: Convert a Point in Rectangular Coordinates to Polar Coordinates Using Radians

Practice: Rectangular to Polar Conversions

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5.2.3: Rectangular to Polar Form for Equations

Convert equations by substituting x and y with expressions of distance and angles.

You are working diligently in your math class when your teacher gives you an equation to graph:

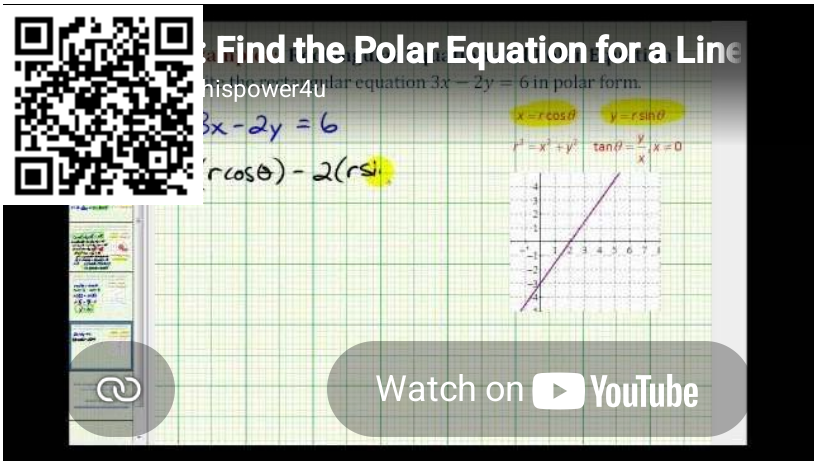
$(x+1)^2-(y+2)^2=7$

As you start to consider how to rearrange this equation, you are told that the goal of the class is to convert the equation to polar form instead of rectangular form.

Can you find a way to do this?

Rectangular Equations to Polar Form

Interestingly, a rectangular coordinate system isn't the only way to plot values. A polar system can be useful. However, it will often be the case that there are one or more equations that need to be converted from rectangular to polar form. To write a rectangular equation in polar form, the conversion equations of $x=r\cos\theta$ and $y=r\sin\theta$ are used.



If the graph of the polar equation is the same as the graph of the rectangular equation, then the conversion has been determined correctly.

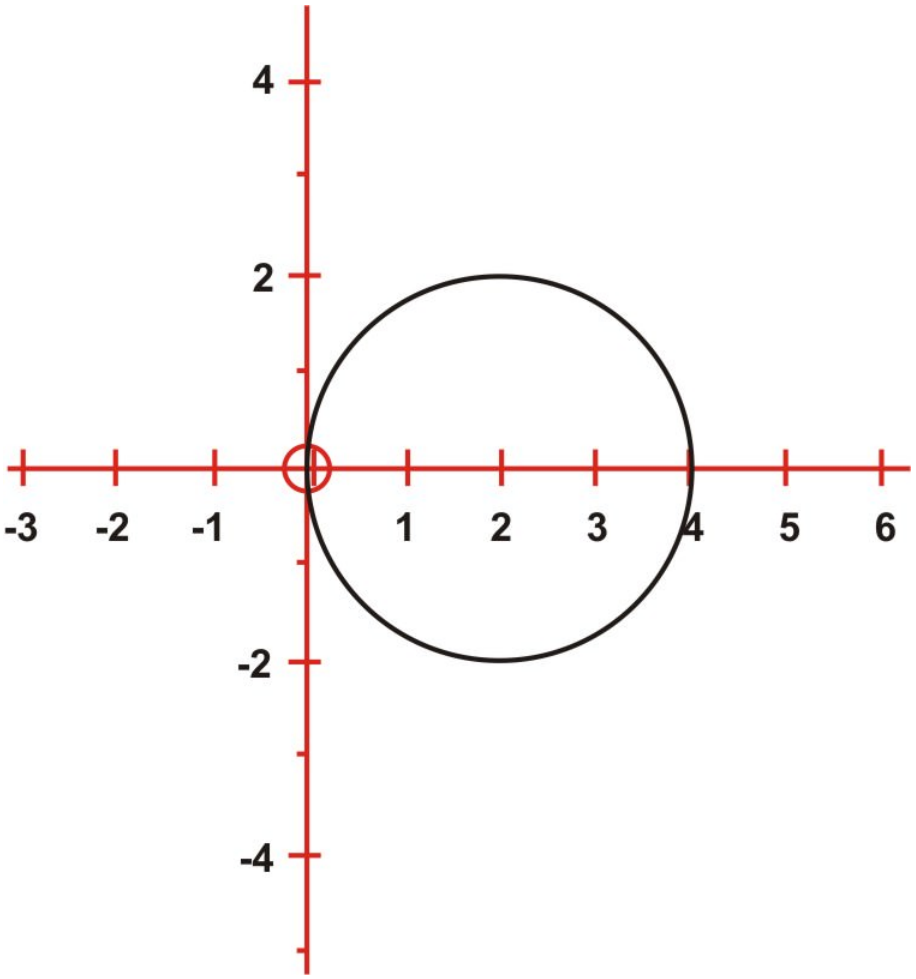


Figure 5.2.3.1

$(x-2)^2+y^2=4$

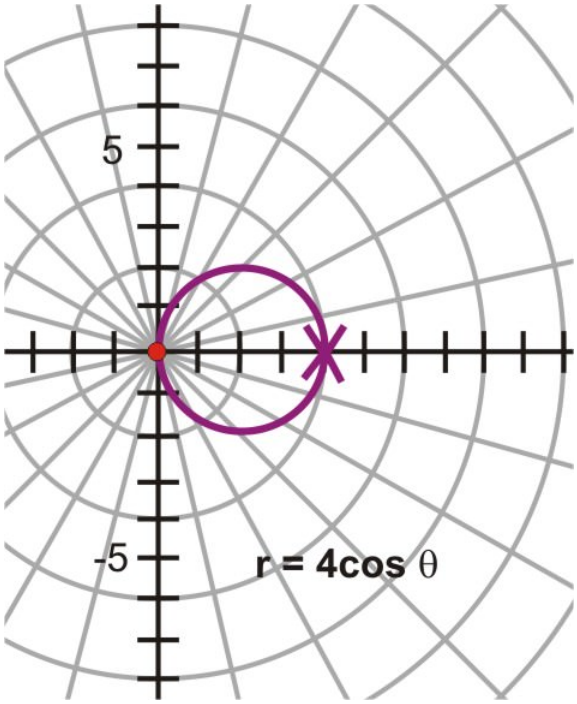


Figure 5.2.3.2

The rectangular equation $(x-2)^2+y^2=4$ represents a circle with center $(2,0)$ and a radius of 2 units. The polar equation $r=4\cos\theta$ is a circle with center $(2,0)$ and a radius of 2 units.

Converting Equations

1. Write the rectangular equation $x^2+y^2=2x$ in polar form.

Remember $r=\sqrt{x^2+y^2}$, $r^2=x^2+y^2$ and $x=r\cos\theta$.

$$\begin{aligned} x^2+y^2&=2x \\ r^2&=2(r\cos\theta) && \text{Pythagorean Theorem} \\ r^2&=2r\cos\theta \\ r&=2\cos\theta and x=r\cos\theta && \text{Divide each side by r} \end{aligned}$$

2. Write $(x-2)^2+y^2=4$ in polar form.

Remember $x=r\cos\theta$ and $y=r\sin\theta$.

$$\begin{aligned} (x-2)^2+y^2&=4 \\ (r\cos\theta-2)^2+(r\sin\theta)^2&=4 && x=r\cos\theta \text{ and} y=r\sin\theta \\ r^2\cos^2\theta-4r\cos\theta+4+r^2\sin^2\theta&=4 && \text{expand the terms} \\ r^2\cos^2\theta-4r\cos\theta+r^2\sin^2\theta&=0 && \text{subtract 4 from each side} \\ r^2\cos^2\theta+r^2\sin^2\theta&=4r\cos\theta && \text{isolate the squared terms} \\ r^2\left(\cos^2\theta+\sin^2\theta\right)&=4r\cos\theta && \text{factor } r^2 \text{ -a common factor} \\ r^2&=4r\cos\theta && \text{Pythagorean Identity} \\ r&=4\cos\theta && \text{Divide each side by } r \end{aligned}$$

3. Write the rectangular equation $(x+4)^2+(y-1)^2=17$ in polar form.

$$\begin{aligned} (x+4)^2+(y-1)^2&=17 \\ (r\cos\theta+4)^2+(r\sin\theta-1)^2&=17 && x=r\cos\theta \text{ and } y=r\sin\theta \\ r^2\cos^2\theta+8r\cos\theta+16+r^2\sin^2\theta-2r\sin\theta+1&=17 && \text{expand the terms} \\ r^2\cos^2\theta+8r\cos\theta-2r\sin\theta+r^2\sin^2\theta&=0 && \text{subtract 17 from each side} \\ r^2\cos^2\theta+r^2\sin^2\theta&=-8r\cos\theta+2r\sin\theta && \text{isolate the squared terms} \\ r^2\left(\cos^2\theta+\sin^2\theta\right)&=-2r(4\cos\theta-\sin\theta) && \text{factor } r^2\text{-a common factor} \\ r^2&=-2r(4\cos\theta-\sin\theta) && \text{Pythagorean Identity} \\ r&=-2(4\cos\theta-\sin\theta) && \text{Divide each side by } r \end{aligned}$$

Example 5.2.3.1

Earlier, you were asked to convert an equation to polar form.

Solution

The original equation to convert is:

$(x+1)^2-(y+2)^2=7$

You can substitute $x=r\cos\theta$ and $y=r\sin\theta$ into the equation, and then simplify:

$$\begin{aligned} (r\cos\theta+1)^2-(r\sin\theta+2)^2&=7 \\ \left(r^2\cos^2\theta+2r\cos\theta+1\right)-\left(r^2\sin^2\theta+4r\sin\theta+4\right)&=7 \\ r^2\left(\cos^2\theta-\sin^2\theta\right)+2r(\cos\theta-2\sin\theta)-3&=7 \\ r^2\left(\cos^2\theta-\sin^2\theta\right)+2r(\cos\theta-2\sin\theta)&=10 \end{aligned}$$

Example 5.2.3.2

Write the rectangular equation $(x-4)^2+(y-3)^2=25$ in polar form.

Solution

$$
\begin{aligned}
(x-4)^2+(y-3)^2 &= 25 \\
x^2-8x+16+y^2-6y+9 &= 25 \\
x^2-8x+y^2-6y+25 &= 25 \\
x^2-8x+y^2-6y &= 0 \\
x^2+y^2-8x-6y &= 0 \\
r^2-8(r\cos\theta)-6(r\sin\theta) &= 0 \\
r^2-8r\cos\theta-6r\sin\theta &= 0 \\
r(r-8\cos\theta-6\sin\theta) &= 0 \\
r=0 \text{ or } r-8\cos\theta-6\sin\theta &= 0 \\
r=0 \text{ or } r &= 8\cos\theta+6\sin\theta
\end{aligned}
$$

From graphing r−8\cos \theta−6\sin \theta=0, we see that the additional solutions are 0 and 8.

Example 5.2.3.3

Write the rectangular equation $3x-2y=1$ in polar form.

Solution

$$
\begin{aligned}
3x-2y &= 1 \\
3r\cos\theta-2r\sin\theta &= 1 \\
r(3\cos\theta-2\sin\theta) &= 1 \\
r &= \frac{1}{3\cos\theta-2\sin\theta}
\end{aligned}
$$

Example 5.2.3.4

Write the rectangular equation $x^2+y^2-4x+2y=0$ in polar form.

Solution

$$
\begin{aligned}
x^2+y^2-4x+2y &= 0 \\
r^2\cos^2\theta+r^2\sin^2\theta-4r\cos\theta+2r\sin\theta &= 0 \\
r^2\left(\sin^2\theta+\cos^2\theta\right)-4r\cos\theta+2r\sin\theta &= 0 \\
r(r-4\cos\theta+2\sin\theta) &= 0 \\
r=0 \text{ or } r-4\cos\theta+2\sin\theta &= 0 \\
r=0 \text{ or } r &= 4\cos\theta-2\sin\theta
\end{aligned}
$$

Review

Write each rectangular equation in polar form.

1. $x=3$
2. $y=4$
3. $x^2+y^2=4$
4. $x^2+y^2=9$
5. $(x-1)^2+y^2=1$
6. $(x-2)^2+(y-3)^2=13$
7. $(x-1)^2+(y-3)^2=10$
8. $(x+2)^2+(y+2)^2=8$
9. $(x+5)^2+(y-1)^2=26$
10. $x^2+(y-6)^2=36$
11. $x^2+(y+2)^2=4$
12. $2x+5y=11$
13. $4x-7y=10$
14. $x+5y=8$
15. $3x-4y=15$

Review (Answers)

To see the Review answers, open this PDF file and look for section 6.6.

Additional Resources

Video: Example: Find the Polar Equation for a Line

Practice: Rectangular to Polar Form for Equations

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5.2.4: Intersections of Polar Curves

Polar curves are graphs that form spiral patterns

Have you ever walked a labyrinth? These spirals can be created in stone or sometimes even with hedges of hemlock. They say if you ask a meditative question and then walk the labyrinth, a solution will come to you by the time that you complete it. Well, here is a math question. How would you graph a spiral pattern? Do you have any idea how you would graph from that fixed center point out to the edge? Think about this question. Need a hint? You will need to use something called a "polar curve".

Here are some additional resources that provide a more in-depth exploration of this topic.

- Intersections of Polar Curves
- Equivalent Polar Curves

Additional Resources

Video: Graphing Polar Equations on the TI84 Graphing Calculator

Practice: Intersections of Polar Curves

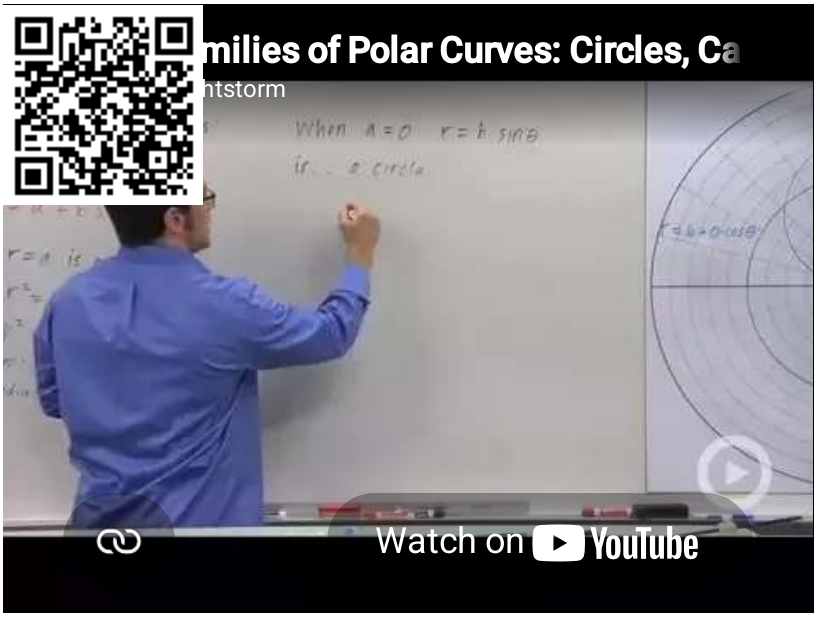
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5.2.5: Equivalent Polar Curves

While working on a problem in math class, you get a solution with a certain equation. In this case, your solution is $3 + 2\cos(\theta)$. Your friend comes over and tells you that he thinks he has solved the problem. However, when he shows you his paper, his equation looks different from yours. His solution is $-3 + 2\cos(-\theta)$. Is there a way you can determine if the two equations are equivalent?

The expression “same only different” comes into play in this section. We will graph two distinct polar equations that will produce two equivalent graphs. Use your graphing calculator and create these curves as the equations are presented.

In some other section, graphs were generated of a limaçon, a dimpled limaçon, a looped limaçon and a cardioid. All of these were of the form $r = a \pm b\sin\theta$ or $r = a \pm b\cos\theta$. The easiest way to see what polar equations produce equivalent curves is to use either a graphing calculator or a software program to generate the graphs of various polar equations.



Comparing Graphs of Polar Equations

Plot the following polar equations and compare the graphs.

$$r = 1 + 2\sin\theta$$
$$r = -1 + 2\sin\theta$$

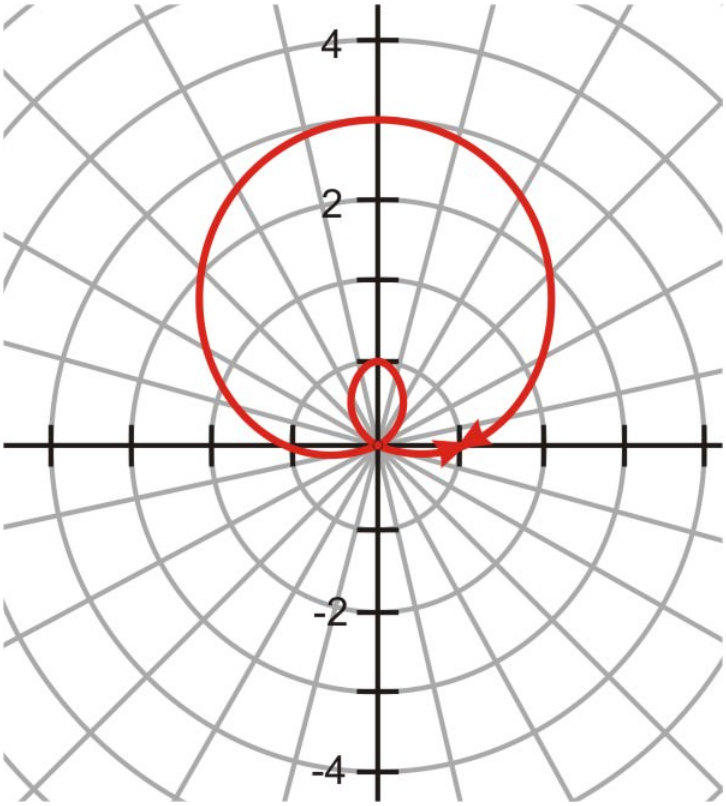


Figure 5.2.5.1

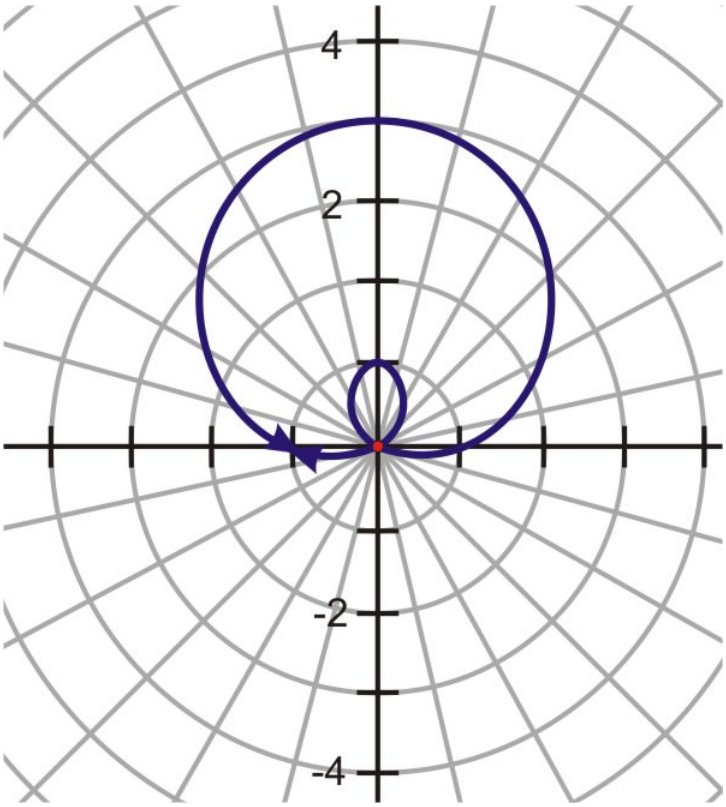


Figure 5.2.5.2

By looking at the graphs, the result is the same. So, even though a is different in both, they have the same graph. We can assume that the sign of a does not matter.

$r = 4\cos\theta$
$r = 4\cos(-\theta)$

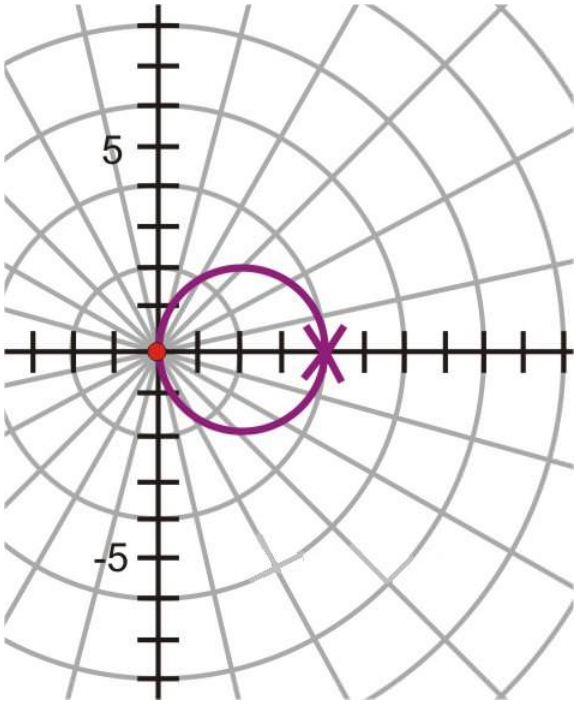


Figure 5.2.5.3

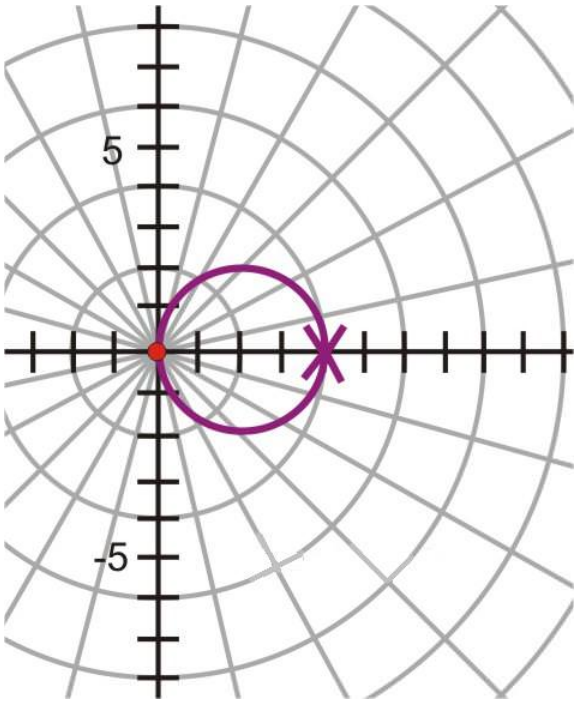


Figure 5.2.5.4

These functions also result in the same graph. Here, θ differed by a negative. So we can assume that the sign of θ does not change the appearance of the graph.

Describing Graphs

1. Graph the equations $x^2 + y^2 = 16$ and $r = 4$. Describe the graphs.

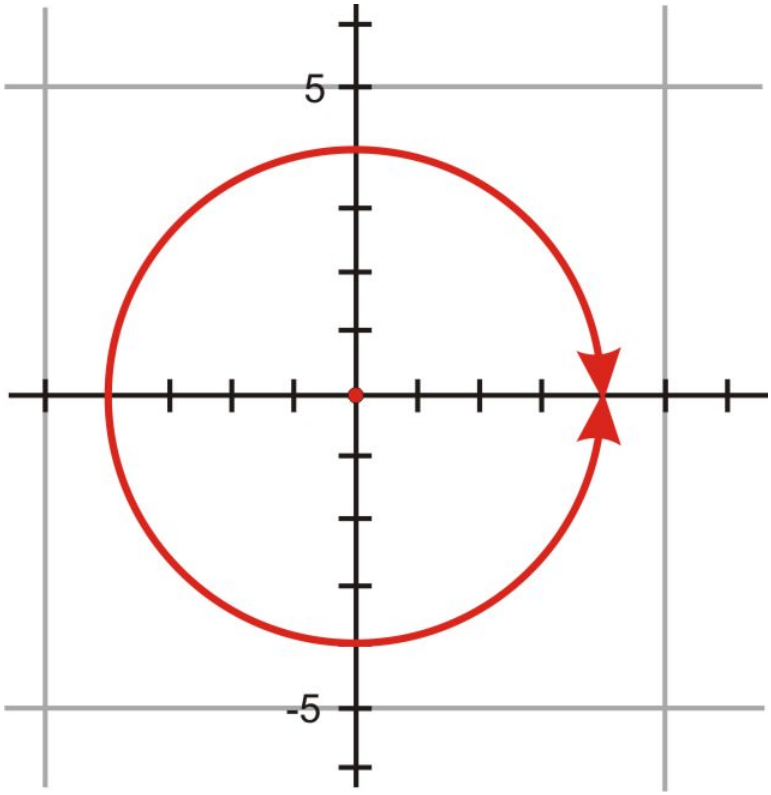


Figure 5.2.5.5

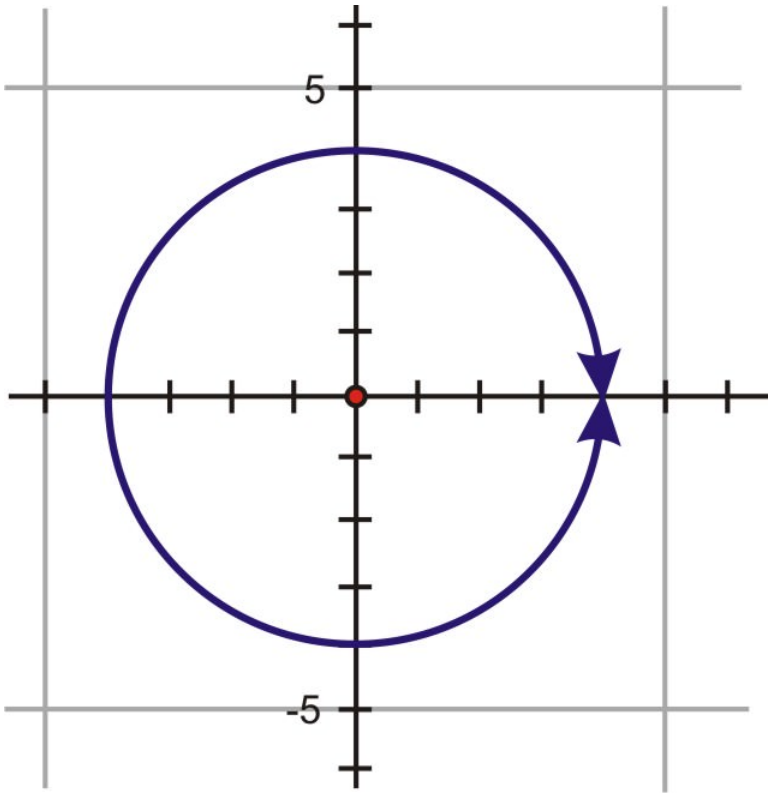


Figure 5.2.5.5

Both equations, one in rectangular form and one in polar form, are circles with a radius of 4 and center at the origin.

2. Graph the equations $(x-2)^2+(y+2)^2=8$ and $r=4\cos\theta-4\sin\theta$. Describe the graphs.

There is not a visual representation shown here, but on your calculator you should see that the graphs are circles centered at $(2, -2)$ with a radius $2\sqrt{2} \approx 2.8$.

Example 5.2.5.1

Earlier, you were asked if there is a way you can determine if the two equations are equivalent.

Solution

As you learned in this section, we can compare graphs of equations to see if the equations are the same or not.

A graph of $3 + 2\cos(\theta)$ looks like this:

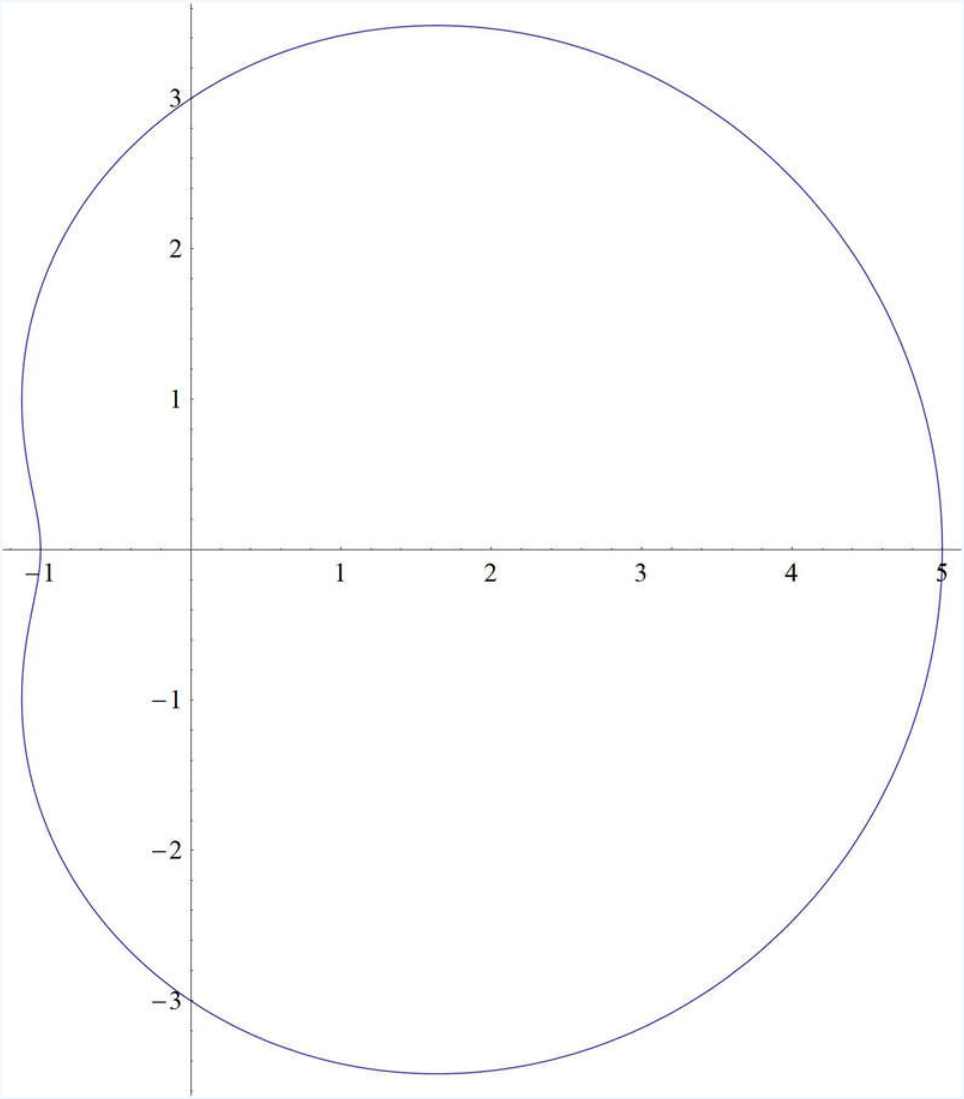


Figure 5.2.5.6

And a graph of $-3 + 2\cos(-\theta)$ looks like this:

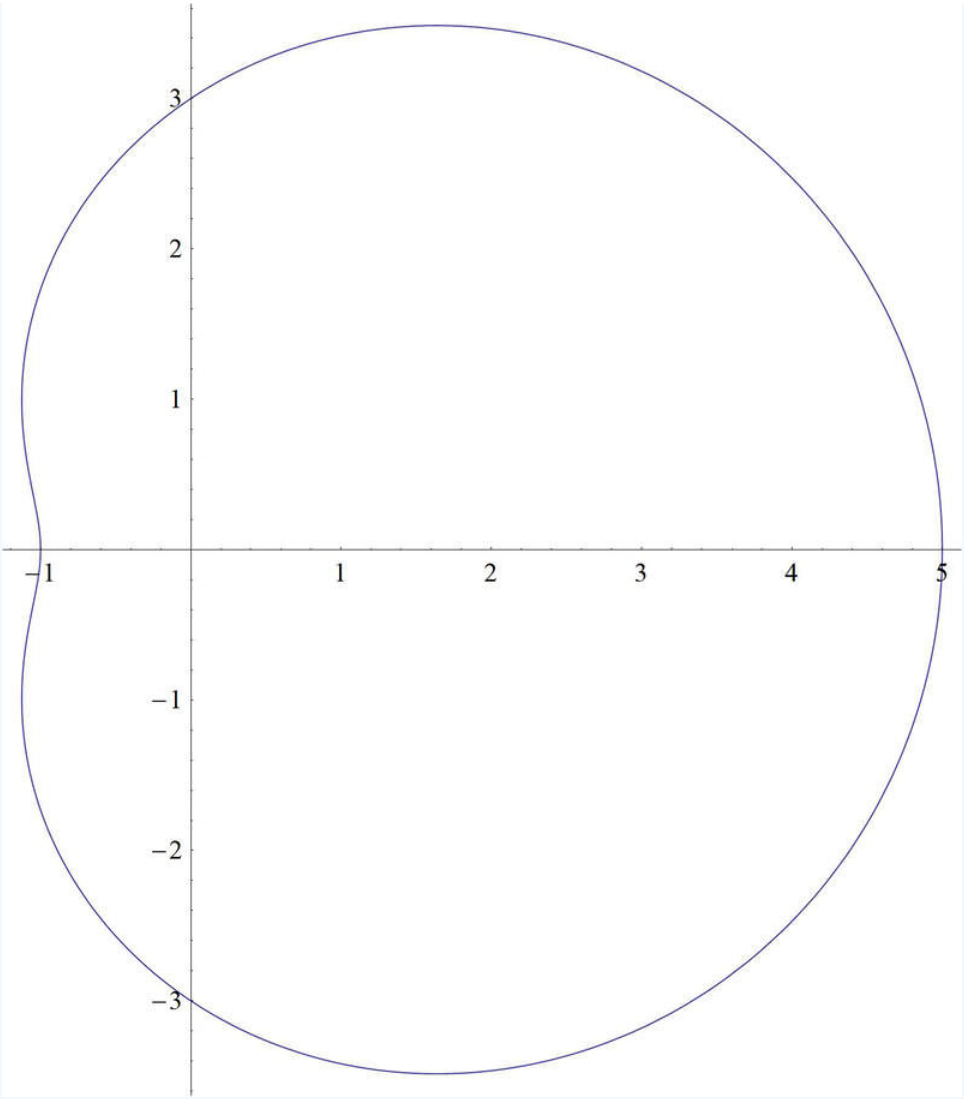


Figure 5.2.5.7

As you can see from the plots, your friend is correct. Your graph and his are the same, therefore the equations are equivalent.

Example 5.2.5.2

Write the rectangular equation $x^2 + y^2 = 6x$ in polar form and graph both equations. Should they be equivalent?

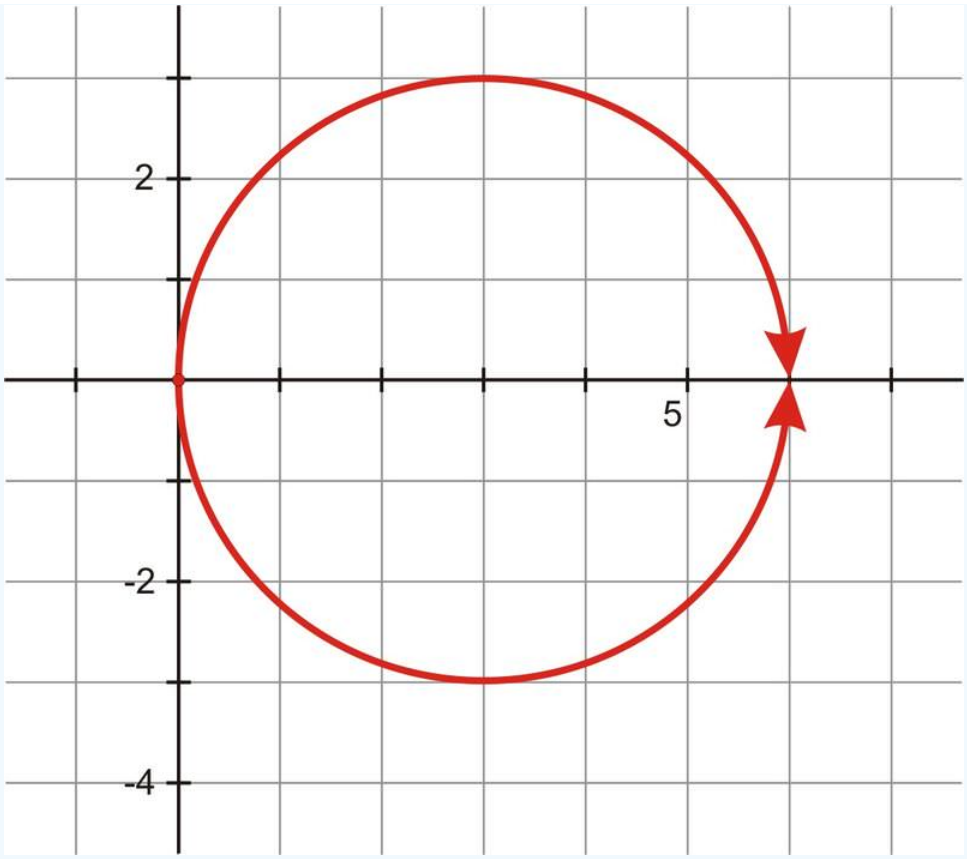


Figure 5.2.5.8

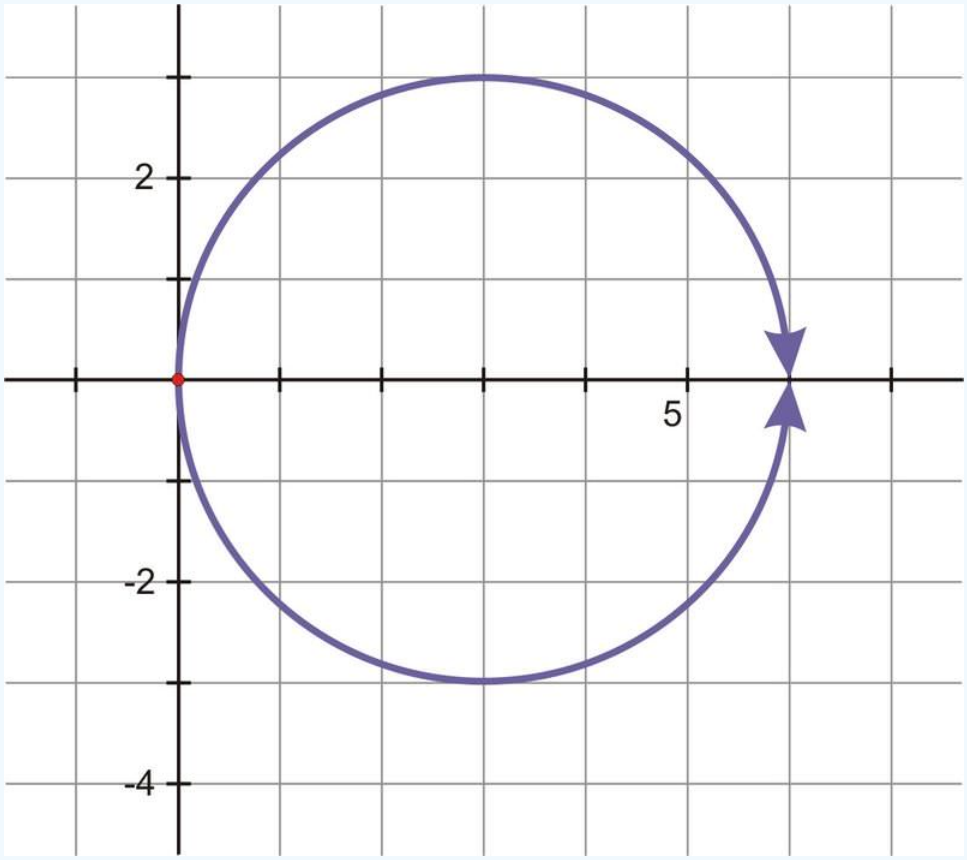


Figure 5.2.5.9

Solution

$$x^2+y^2=6x$$
$$r^2=6(r\cos\theta) \qquad r^2=x^2+y^2 \quad \text{and} \quad x=y\cos\theta$$
$$r=6\cos\theta \qquad \text{divide by } r$$

Both equations produced a circle with center $(3,0)$ and a radius of 3.

Example 5.2.5.3

Determine if $r=-2+\sin\theta$ and $r=2-\sin\theta$ are equivalent *without* graphing.

Solution

$r=-2+\sin\theta$ and $r=2-\sin\theta$ are not equivalent because the sine has the opposite sign. $r=-2+\sin\theta$ will be primarily above the horizontal axis and $r=2-\sin\theta$ will be mostly below. However, the two do have the same pole axis intercepts.

Example 5.2.5.4

Determine if $r=-3+4\cos(-\pi)$ and $r=3+4\cos\pi$ are equivalent *without* graphing.

Solution

$r=-3+4\cos(-\pi)$ and $r=3+4\cos\pi$ are equivalent because the sign of "a" does not matter, nor does the sign of θ.

Review

For each equation in rectangular form given below, write the equivalent equation in polar form.

1. $x^2+y^2=4$
2. $x^2+y^2=6y$
3. $(x-1)^2+y^2=1$
4. $(x-4)^2+(y-1)^2=17$
5. $x^2+y^2=9$

For each equation below in polar form, write another equation in polar form that will produce the same graph.

6. $r=4+3\sin\theta$
7. $r=2-\sin\theta$
8. $r=2+2\cos\theta$
9. $r=3-\cos\theta$
10. $r=2+\sin\theta$

Determine whether each of the following sets of equations produce equivalent graphs *without* graphing.

11. $r=3-\sin\theta$ and $r=3+\sin\theta$
12. $r=1+2\sin\theta$ and $r=-1+2\sin\theta$
13. $r=3\sin\theta$ and $r=3\sin(-\theta)$
14. $r=2\cos\theta$ and $r=2\cos(-\theta)$
15. $r=1+3\cos\theta$ and $r=1-3\cos\theta$

Review (Answers)

To see the Review answers, open this PDF file and look for section 6.8.

Additional Resources

Practice: Equivalent Polar Curves

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5.2.6: Systems of Polar Equations

Points of intersection or locations where two polar equations have the same solution.

You likely recall that when you graph multiple equations on the same line, you usually end up with locations on the graph where they intersect (unless you are graphing parallel lines!).

The same is true when graphing equations in polar form, and/or on a polar graph. When you graph the intersection of multiple polar equations, you treat them just as you would rectangular equations, graph both and find the areas that are true for both equations.

Systems of Polar Equations

Polar equations can be graphed using **polar coordinates**. Graphing two polar equations on the same set of axes may result in having point(s) of intersection.

All points on a polar graph are coordinates that make the equation valid. The coordinates of point(s) of intersection when substituted into each equation will make both of the equations valid.

One method to find point(s) of intersection for two polar graphs is by setting the equations equal to each other.

Call the first equation r_1 and the second equation r_2.

Points of intersection are when $r_1 = r_2$, so set the equations equal and then solve the resulting trigonometric equation.

Example 5.2.6.1

Find the intersection of $r_1 = 3\sin\theta$ and $r_2 = \sqrt{3}\cos\theta$.

Solution

Set the equations equal to each other: $3\sin\theta = \sqrt{3}\cos\theta$

divide both sides of the equation by cos θ and 3: $\frac{3\sin\theta}{3\cos\theta} = \frac{\sqrt{3}\cos\theta}{3\cos\theta}$

Simplify: $\frac{\sin\theta}{\cos\theta} = \frac{\sqrt{3}}{3}$

Use the identity: $\frac{\sin\theta}{\cos\theta} = \tan\theta$

$\tan\theta = \frac{\sqrt{3}}{3}$

$\theta = \frac{\pi}{6}$ or $\frac{7\pi}{6}$

substitute $\frac{\pi}{6}$ in either equation to obtain $r = 1.5$

substitute $\frac{7\pi}{6}$ in either equation to obtain -1.5

NOTE: the coordinates $\left(1.5, \frac{\pi}{6}\right)$ and $\left(-1.5, \frac{7\pi}{6}\right)$ represent the SAME polar point so there is only one solution to this equation.

Are we done? If we look at the graphs of r_1 and r_2, we can see that there is another point of intersection:

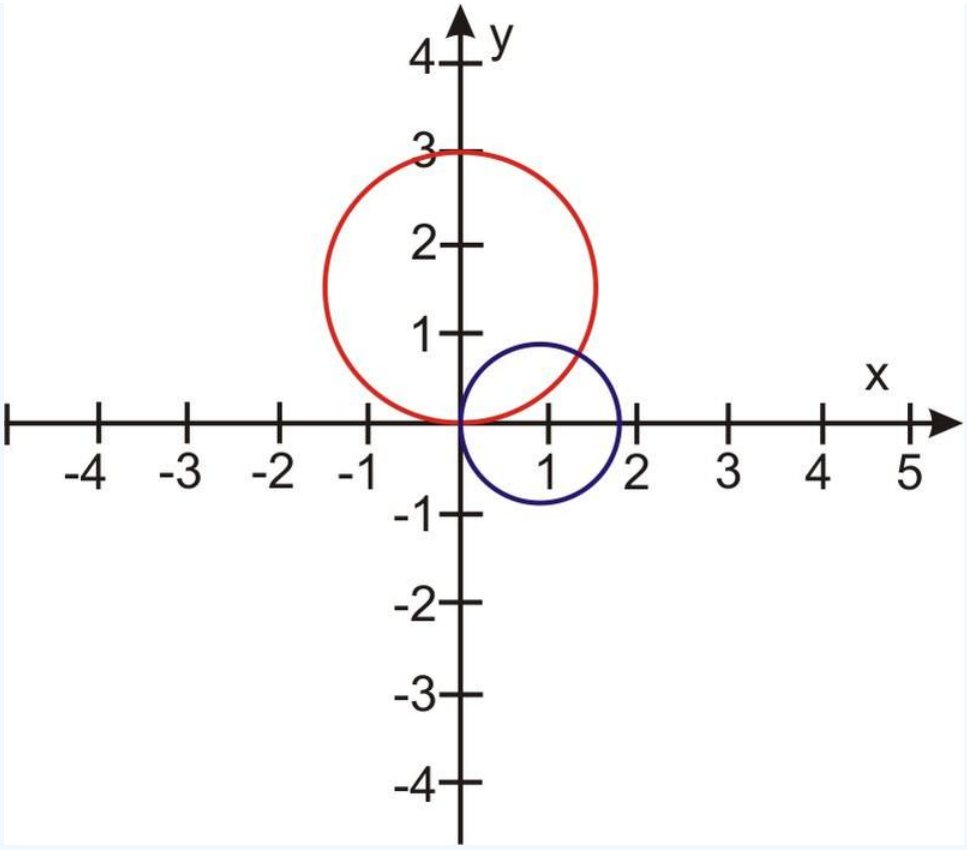


Figure 5.2.6.1

when $\theta = 0$, $r_1 = 3 \sin \theta = 3 \sin (0) = 0$

That means $r_1 = 3 \sin \theta$ goes through the **pole** $(0, 0)$.

For r_2 : when $\theta = \frac{\pi}{2}$, $r_2 = 0$ that is $r_2 = \sqrt{3} \cos \theta$ goes through the point $\left(0, \frac{\pi}{2}\right)$.

Therefore, both graphs go through the pole and the pole is a point of intersection.

The pole was NOT revealed as a point of intersection using the first step! (Why? Hint: How many ways are there to represent the pole in polar coordinates?) This shows us that after you use algebraic methods to find intersections at points other than the pole, you should also check for intersections at the pole.

Example 5.2.6.2

Find the point(s) of intersection for the two graphs: $r_1 = 1$ and $r_2 = 2 \sin 2\theta$.

Solution

Set $r_1 = r_2$ and solve:

$$1 = 2 \sin 2\theta$$
$$\frac{1}{2} = \sin 2\theta$$

Use a substitution $\alpha = 2\theta$ to solve

$$\frac{1}{2} = \sin \alpha$$
$$\alpha = \frac{\pi}{6}, \frac{5\pi}{6}$$

Since $\alpha = 2\theta$, solving for θ gives us

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}$$

But, recall that θ has the range $0 \leq \theta \leq 2\pi$. Since we solved with $0 \leq \alpha \leq 2\pi$ we actually need to consider values of θ with $0 \leq \theta \leq 4\pi$. Why? Recall that $\sin(2\theta)$ has two cycles between 0 and 2π, and so we add two more solutions,

$$\alpha = \frac{13\pi}{6}, \frac{17\pi}{6}$$

and since $\alpha = 2\theta$,

$$\theta = \frac{13\pi}{12}, \frac{17\pi}{12}$$

Finally, we need to consider solutions when $r = -1$ because $r = 1$ and $r = -1$ are the same polar equation. So, solving

$$-\frac{1}{2} = \sin\alpha$$
$$\alpha = \frac{7\pi}{6}, \frac{11\pi}{6}$$

Again, using $\alpha = 2\theta$ and adding solutions for the repetition gives us four more solutions,

$$\theta = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{21\pi}{12}$$

So in total, there are eight solutions to this set of equations.

NOTE: Recall that solving trigonometric equations where the angle is \theta requires looking at all potential values between 0 and 2π. When the angle is 2θ as it is in this case, be sure to look for all potential values between 0 and 4π. When the angle is 3θ as it is in this case, be sure to look for all potential values between 0 and 6π., and so on.

Since r_1 cannot equal 0, the pole is not on its graph and not a point of intersection.

The graph reveals eight points of intersection which were found earlier.

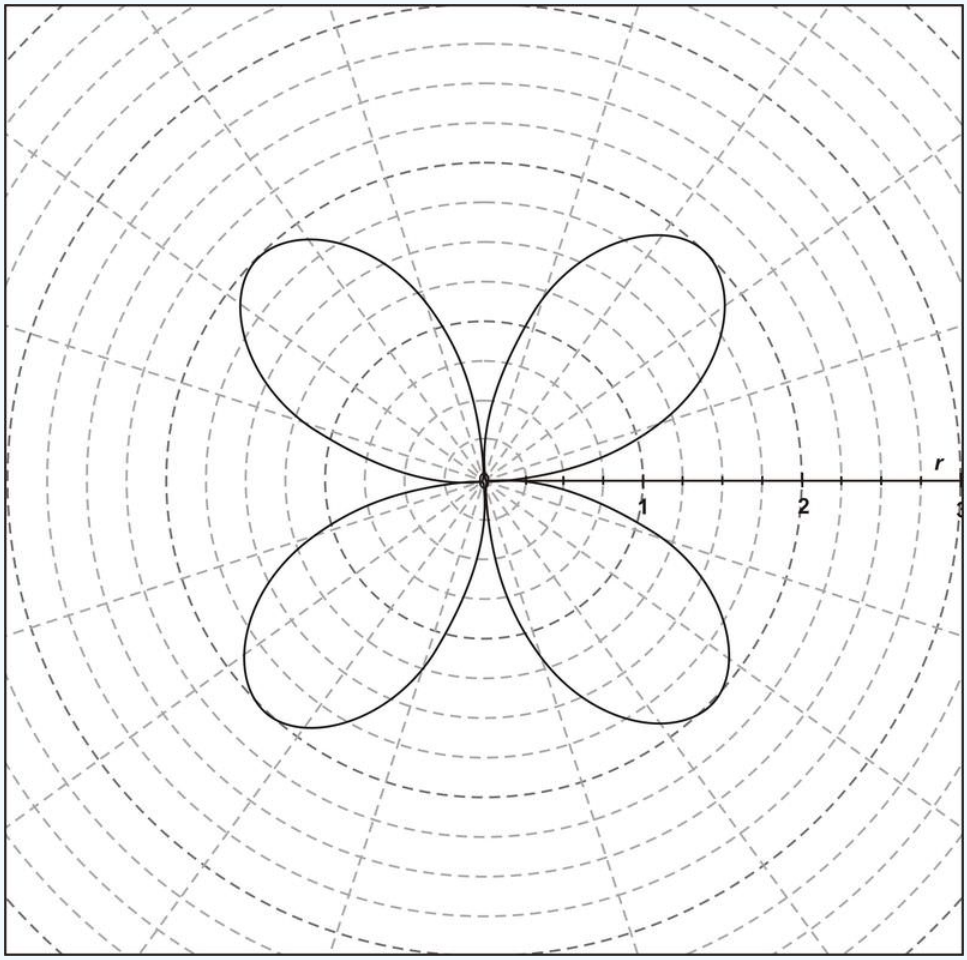


Figure 5.2.6.2

Example 5.2.6.3

Find point(s) of intersection, if any exist, for the following pair of equations: $r_1 = 2$ and $r_2 = \sec\theta$.

Here we will use a table of values for each function, solving by **quadrant**. Recall that the period of $\sec\theta$ is 2π.

Solution

For the first quadrant:

θ (angle)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r_1 (distance)	2	2	2	2	2
r_2	1	1.15	1.4	2	und

For the second quadrant:

θ (angle)	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r_1 (distance)	2	2	2	2
r_2	-2	-1.4	-1.15	-1

For the third quadrant:

θ (angle)	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$
r_1 (distance)	2	2	2	2
r_2	-1.15	-1.4	-2	und

For the fourth quadrant:

θ (angle)	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
r_1 (distance)	2	2	2	2
r_2	2	1.4	1.15	1

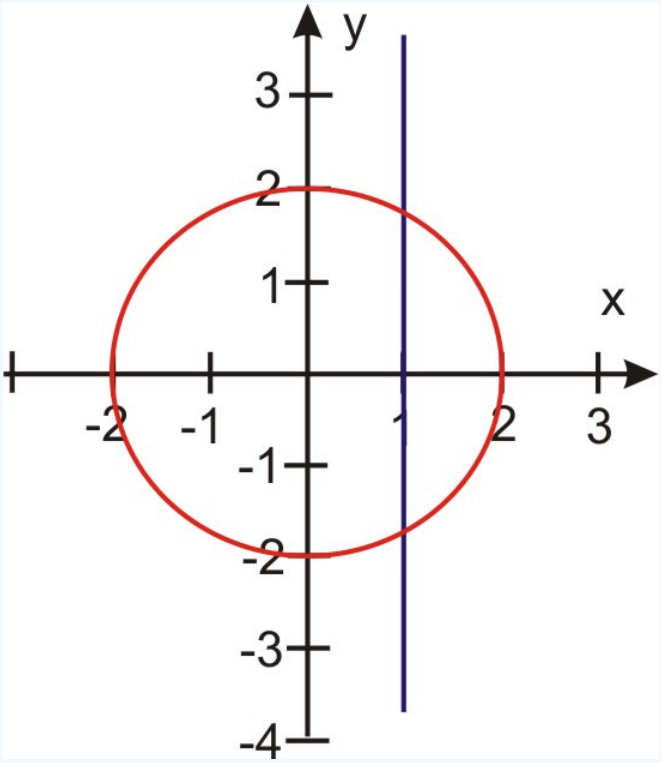


Figure **5.2.6.3**

Note that the 3rd and 4th quadrant repeat 1st and 2nd quadrant values.

Observe in the table of values that $\left(2, \frac{\pi}{3}\right)$ and $\left(2, \frac{5\pi}{3}\right)$ are the points of intersection. Look at the curves- the first equation yields a circle while the second yields a line. The maximum number of intersecting points for a line and a circle is 2. The two points have been found.

Example 5.2.6.4

Find the point(s) of intersection for this pair of Polar equations: $r = 2 + 4\sin\theta$ and $\theta = 60°$.

Solution

The equation $\theta = 60°$ is a line making a $60°$ angle with the r axis.

Make a table of values for $r = 2 + 4\sin\theta$

For the first quadrant:

θ (angle)	0	30	45	60	90
R (distance)	2	4	4.83	5.46	6

For the second quadrant:

θ (angle)	120	135	150	180
R (distance)	5.46	4.83	4	2

For the third quadrant:

θ (angle)	210	225	240	270
R (distance)	0	-.83	-1.46	-2

For the fourth quadrant:

θ (angle)	300	315	330	360
R (distance)	-1.46	-.83	0	2

Notice there are two solutions in the table., $(60, 5.46)$ and $(240, -1.46) = (60, 1.46)$. Recall that when $r < 0$, you plot a point (r, θ), by rotating $180°$ (*or* π).

Finally, we need to check the pole: $r = 2 + 4\sin\theta$ passes through the pole for $\theta = 330°$, and $\theta = 60°$ also passes through the pole. Thus, the third point of intersection is $(0, 0)$.

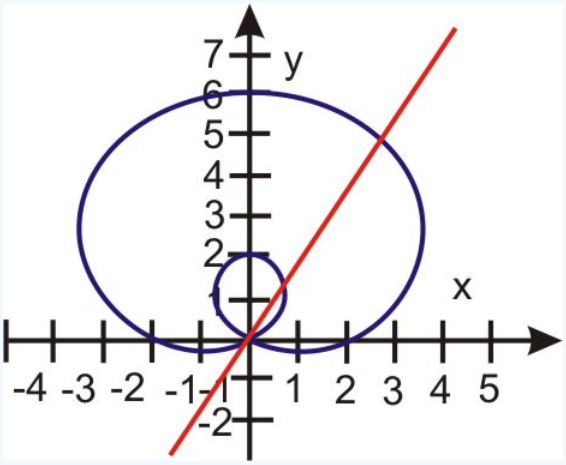


Figure 5.2.6.4

Example 5.2.6.5

Find the point(s) of intersection for this pair of Polar equations: $r_1 = 2\cos\theta$ and $r_2 = 1$.

Solution

Make a table:

For the first quadrant:

θ (angle)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r_1 (distance)	2	$\sqrt{3}$	$\sqrt{2}$	1	0
r_2	1	1	1	1	1

For the second quadrant:

θ (angle)	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r_1 (distance)	-1	$-\sqrt{2}$	$-\sqrt{3}$	-2
r_2	1	1	1	1

For the third quadrant:

θ (angle)	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$
r_1 (distance)	$-\sqrt{3}$	$-\sqrt{2}$	-1	0
r_2	1	1	1	1

For the fourth quadrant:

θ (angle)	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
r_1 (distance)	1	$\sqrt{2}$	$\sqrt{3}$	2
r_2	1	1	1	1

So the unique solutions are at $\theta = \frac{\pi}{3}, \frac{4\pi}{3}$. There are also two repeated solutions in this set (can you find them?).

Here is a graph showing the two solutions:

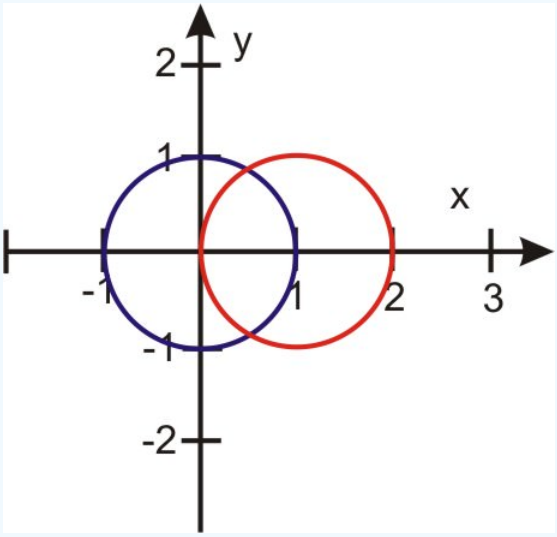
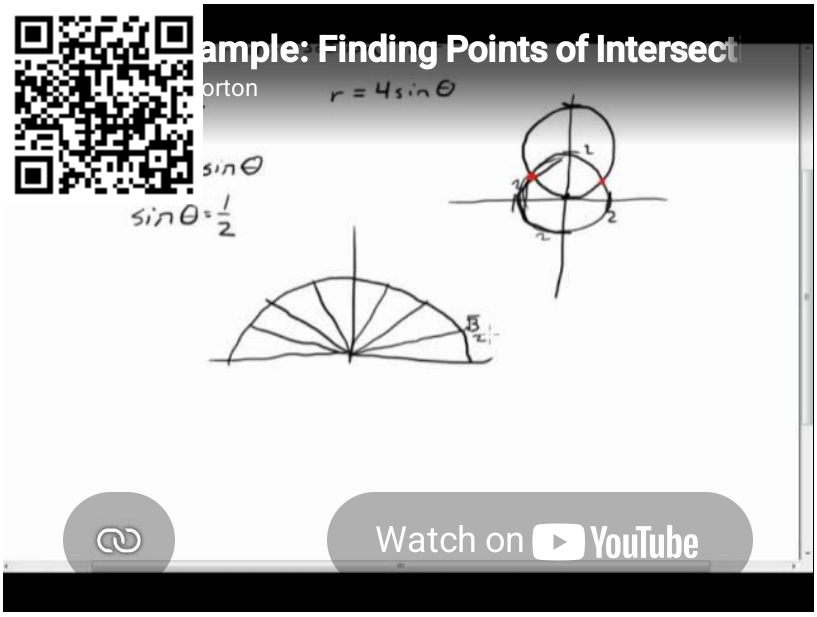
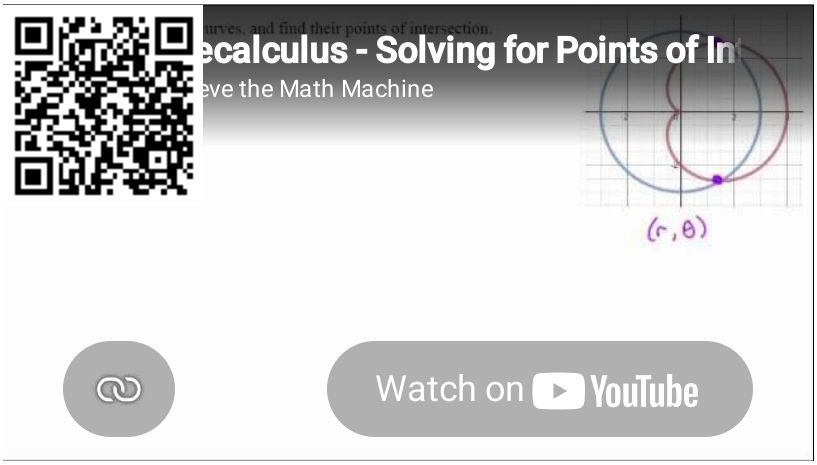


Figure 5.2.6.5





Review

The graphs of $r = 1$ and $r = 2\cos\theta$ are shown below.

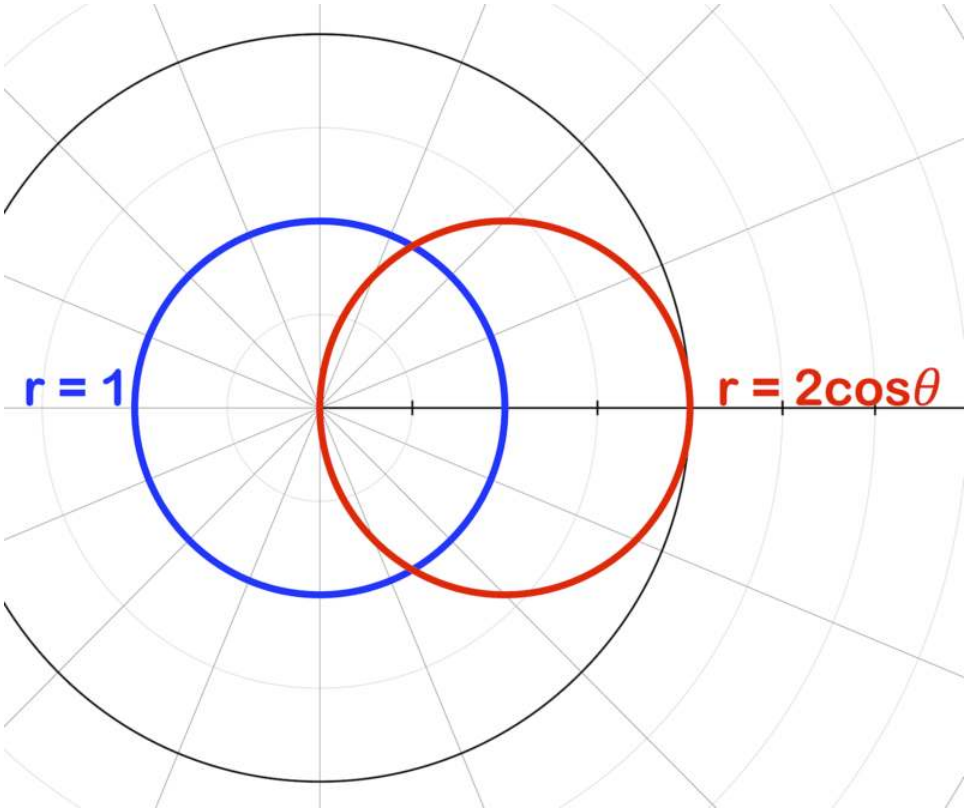


Figure **5.2.6.6**

1. How many times do they intersect?
2. In which quadrants do they intersect?
3. At what points do the intersections occur?

Based on the image below and the following information: the intersection of the graphs of $r = \cos\theta$ and $r = 1 - \cos\theta$

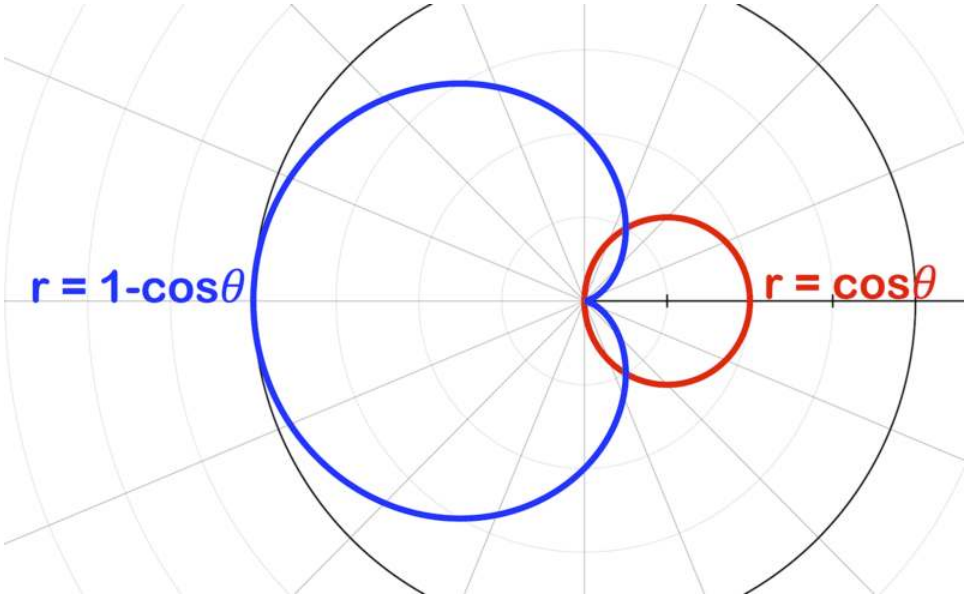


Figure **5.2.6.7**

4. Identify how many times they intersect?
5. At what points do the intersections occur?

Find the points of intersection of the following pairs of curves.

6. $r = 2; r = 2\cos\theta$
7. $r = \sin 2\theta; r = 2\sin\theta$
8. $r = 2 + 2\sin\theta; r = 2 - 2\cos\theta$
9. $r = 3\cos\theta; r = 2 - \cos\theta$

Find the point(s) of intersection for each system of equations. Graph to verify your solution.

10. $r_1 = csc\theta; r_2 = 2\sin\theta$
11. $r_1 = \cos\theta; r_2 = 1 + \sin\theta$
12. $r_1 = \sin\theta; r_2 = \sin 2\theta$
13. $r_1 = -4\sin\theta; r_2 = -4\cos\theta$
14. $r_1 = 1 - 2\sin\theta; r_2 = \sqrt{9\cos(\theta)}$
15. $r_1 = 1 - \cos\theta; r_2 = 4\cos(3\theta)$

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.3.

Vocabulary

Term	Definition
Conic	Conic sections are those curves that can be created by the intersection of a double cone and a plane. They include circles, ellipses, parabolas, and hyperbolas.
Points of intersection	Points of intersection are locations where two different equations have the same solutions.
polar coordinates	Polar coordinates describe locations on a grid using the polar coordinate system. The location of each point is determined by its distance from the pole and its angle with respect to the polar axis.
pole	The pole is the center point on a polar graph.
quadrant	A quadrant is one-fourth of the coordinate plane. The four quadrants are numbered using Roman Numerals I, II, III, and IV, starting in the top-right, and increasing counter-clockwise.

Additional Resources

Video: **Example: Finding Points of Intersection on a Polar Graph**

Practice: **Systems of Polar Equations**

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SECTION OVERVIEW

5.3: Complex Numbers

5.3.1: Imaginary Numbers

5.3.2: The Complex Numbers

5.3.3: Quadratic Formula and Complex Sums

5.3.4: Products and Quotients of Complex Numbers

5.3.5: Product and Quotient Theorems

5.3.6: Trigonometric Form of Complex Numbers

5.3.7: Equations Using DeMoivre's Theorem

5.3.8: Geometry of Complex Roots

5.3.9: Polar Form of a Complex Number

5.3.10: DeMoivre's Theorem and nth Roots

5.3.11: DeMoivre's Theorem

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5.3.1: Imaginary Numbers

The imaginary number 'i' is equal to the square root of negative 1.

From Quadratics to Complex Numbers

This section introduces imaginary and complex numbers in the context of solutions to quadratic equations that don't have real-number solutions.

Warm-Up

The locations where quadratic functions cross the x−axis represent the solutions to the function. Use the interactive below to graph different parabolas. Later in this section you will see how complex numbers can be used to represent solutions to quadratic functions that do not cross the x−axis twice.

Work it Out 1

Start with the quadratic equation: $y = x^2 - 9$.

1. Use the interactive below to graph the equation.
2. Recall that the solutions to an equation are the x-values that make $y = 0$. Estimate the location(s) on the graph that represent these (x, y) points. Round to the nearest whole number.
3. Substitute the x-coordinates in place of x in the original equation, $y = x^2 - 9$. Explain how to use the result to determine if your estimates were accurate.

Discussion

What are the y-coordinates of any point on the x-axis? What are the x-coordinates of the locations where the graph crosses the x-axis?

Recall that the solutions to a quadratic equation are the values for x that make y equal to zero. If you check your answers by substituting them back into the equation, do they make $y = 0$?

Work it Out 2

Start with the quadratic equation: $y = 2x^2 - 2x + 2$.

1. Use the interactive to graph the equation.
2. Recall that the solutions to an equation are the x-values that make $y = 0$. Estimate the location on the graph that represents these (x, y) points. Explain any difficulties you encounter.

Discussion

Why were you not able to visualize where along the curve there is an x-value that makes $y = 0$?

Since the graph isn't helpful, how else might you find the solution(s) to the equation: $0 = 2x^2 - 2x + 2$?

Work it Out 3: **The Quadratic Formula and the Discriminant**

Using the interactive from Active Learning 1, consider the graphs of each of these quadratic equations. They are given in the form: $y = ax^2 + bx + c$. Pay particular attention to the number of places the graph intersects the x-axis in each equation.

1. $y = 0.5x^2 - x - 4$
2. $y = x^2 - 2x + 1$
3. $y = x^2 + 2x + 4$

Remember, the Quadratic Formula is: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The binomial term below the radical sign, $b^2 - 4ac$, is called the **discriminant**.

Calculate the discriminant for each of the three equations that you graphed.

Next, compare the value of the discriminant and the graph of each quadratic equation, paying particular attention to whether the discriminant is less than, equal to, or greater than zero. What do you notice? Based on your findings, how many **solutions** do you

think there are for a quadratic equation where the discriminant is greater than, less than, or equal to zero?

Discussion

The graph below is of the first equation: $y = \frac{1}{2}x^2 - x - 4$.

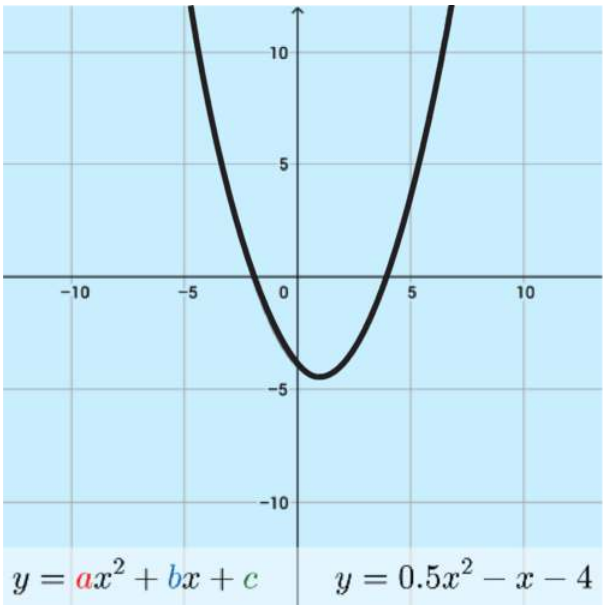


Figure 5.3.1.1

You can calculate the discriminant from this equation as follows:

$$\begin{aligned} \text{Given: } a = \frac{1}{2}, b = -1, c = -4 \\ &= (-1)^2 - 4\left(\frac{1}{2}\right)(-4) \\ &= 1 - (-8) \\ &= 9 \end{aligned}$$

When the discriminant of a quadratic equation is positive, the equation will have two, distinct, real number solutions. In this case, the solutions are $x = -2$ and $x = 4$.

Determine the discriminant of the other two equations. What generalization can you make relating the number of real solutions to the value of the discriminant being positive, negative, or zero?

Example 5.3.1.1

The Discriminant and Solutions to Quadratic Equations

Calculate the discriminant of the equation $y = 2x^2 - 5x + 4$, and use that information to make a statement about the solutions.

Solution

The discriminant can be calculated as follows:

$$\begin{aligned} \text{Given: } b = -5, a = 2, c = 4 \\ &= b^2 - 4(a)(c) \\ &= (-5)^2 - 4(2)(4) \\ &= -7 \end{aligned}$$

Because the discriminant is less than zero, this equation has no real solutions.

Consider the equation $y = 2x^2 - 5x + 4$ from this example. Because the discriminant is less than 0, this equation has no real solutions. However, it *does* have two non-real solutions. In other words, the solutions to this quadratic equation include an imaginary component. Numbers with a real component and an imaginary component are called **complex**.

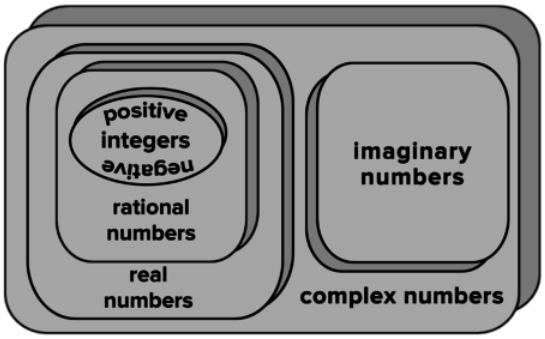


Figure 5.3.1.2

When humans first created the concept of numbers, they only had the counting numbers (whole numbers greater than zero) 1, 2, 3... and so on, because numbers were intended to count physical objects. For a long time, it was accepted that the square root of a negative number did not exist.

In order for a solution to exist to the equation x2=-1, mathematicians invented a solution. This solution is called the **imaginary number** and is noted by the letter i. The imaginary numbers, 1i,2i,3i... are called imaginary because they cannot be found on a traditional number line of real numbers.

A **pure imaginary number** is a number in the form bi where b is a nonzero real number. Examples include $4i$, $i\sqrt{2}$, and $-6i$.

A **complex number** has both real and imaginary parts, and is written in the form a\pm bi. Interestingly, *any* number can be written as a complex number, since either the real or imaginary component can either be zero or have a zero coefficient.

Examples of Complex Numbers

1. The set of Complex Numbers is a 'superset' of all other number sets. That means every other set of numbers is part of the Complex Number set.
2. Complex numbers appear in the form $a+bi$, where a and b are Real Numbers, and $i=\sqrt{-1}$.
3. *ANY* number may be written as a Complex Number:

$$\begin{aligned}\text{The Real Number 4 in Complex form} &= 4+0i\\ \text{The Imaginary Number } 5i \text{ in Complex form} &= 0+5i\\ \text{The Complex Number } 4+5i \text{ in Complex form} &= 4+5i\end{aligned}$$

Example 5.3.1.2

Given that $\sqrt{-1}=i$, what is i^2 equivalent to? Explain why this is remarkable.

Solution

Before trying to answer this question, think about how you'd answer this question for positive numbers. For example, 9–√=3, and therefore, by squaring both sides: $3^2=9$. Similarly, $\sqrt{49}=7$, and therefore $49=7^2$. So if -1−−√=i, what happens to each side of the equation if you square both sides? In other words, what is i^2 equivalent to?

$$\begin{aligned}i^2 &= i\cdot i\\ &= \sqrt{-1}\cdot\sqrt{-1}\\ &= (\sqrt{-1})^2\\ &= (-1)\\ &= -1\end{aligned}$$

One entertaining way to think of this is to consider how odd it would be if two imaginary people were able to have a real child together. This is similar in concept to the result of squaring an imaginary number, or multiplying two imaginary numbers together.

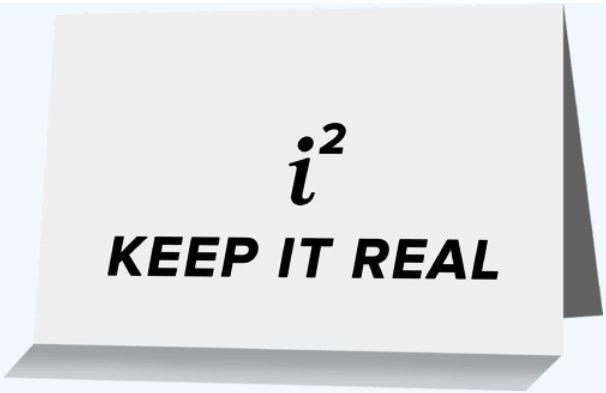


Figure 5.3.1.3

Example 5.3.1.3

The square root of any negative number can be written in terms of the imaginary number i. Rewrite the following as imaginary numbers:

a. $\sqrt{-4}$
b. $\sqrt{-5}$
c. $\sqrt{-6}$

Solution

- $\sqrt{4} = \sqrt{4 \cdot -1} = \sqrt{4}\sqrt{-1} = 2i$
- $\sqrt{-5} = \sqrt{5 \cdot -1} = \sqrt{5}\sqrt{-1} = i\sqrt{5}$
- $\sqrt{-16} = \sqrt{16 \cdot -1} = \sqrt{16}\sqrt{-1} = 4i$

Work it Out 4

Simplify the following:

1. $\sqrt{-49}$
2. $\sqrt{-40}$

Discussion

If you can reduce the number under the radical after you've turned it into a positive number, you should do so.

$$\begin{aligned}\sqrt{-49} & \\ &= \sqrt{49 \cdot -1} \\ &= \sqrt{49}\sqrt{-1} \\ &= 7i\end{aligned}$$

Simplify $\sqrt{-40}$ similarly, but don't forget to simplify $\sqrt{40}$ after you deal with the square root of -1.

Work it Out 5

The parabola $y = x^2 - 2x + 3$ has no x-intercepts, as shown below. Does this mean that there are no solutions? How could complex numbers help with solving the equation $0 = x^2 - 2x + 3$? What are the solutions to this function?

Discussion

The graph below shows that the curve of this equation doesn't intersect with the x-axis.

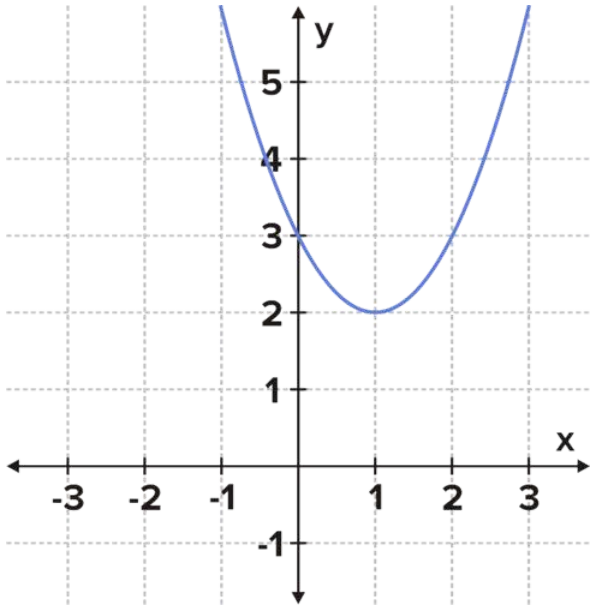


Figure 5.3.1.4

This means that the solutions to the equation $x^2 - 2x + 3 = 0$ are not real numbers. The solutions are complex numbers (which involve real and imaginary parts). The solutions can still be found using the quadratic formula, but the result will be 2 complex number solutions. What are they?

The following video demonstrates how to determine the complex solutions of a quadratic equation.

Complex Solutions of a Quadratic Equation

Work it Out 6: Complex Number Solutions to Quadratic Functions

Recall that when the value of the discriminant is less than zero, the equation has two unequal complex number solutions. Substitute $-1 + 2i$ into the quadratic equation $x^2 + 2x + 5 = 0$ to determine if it is a solution. Use what you determine from the substitution to make an educated guess at what another solution might be. Test that solution also.

Discussion

Just as with any other quadratic equation, to check a solution, substitute the solution in place of x, and simplify the expression to determine whether or not it is equal to zero.

CK-12 INTERACTIVE

The following CK-12 PLIX, "Vertical and Horizontal Shift 2," offers an additional chance to explore complex roots of quadratics.

Example 5.3.1.4

Complex Number Solutions to Quadratic Functions

Is $\frac{3}{2} - i$ a solution to $2x^2 - 6x + 5 = 0$?

Solution

Substitute the coefficients into the quadratic formula, and simplify to generate the equation:

$$\begin{aligned}&\frac{-b\pm\sqrt{b^2-4ac}}{2a}\\=&\frac{6\pm\sqrt{(-6)^2-4(2)(5)}}{2(2)}\\=&\frac{6\pm\sqrt{(36)-40}}{4}\\=&\frac{6\pm\sqrt{-4}}{4}\\x=&\left\{\frac{3}{2}+\frac{1}{2}i\right\}\text{ or }\left\{\frac{3}{2}-\frac{1}{2}i\right\}\end{aligned}$$

When simplified, since neither of the solutions is the one stated in the problem, namely $\frac{3}{2}-i$, it is not a solution.

Alternatively, you could substitute in the given solution and see if the equation equals zero.

$$\begin{aligned}2\left(\frac{3}{2}-i\right)^2-6\left(\frac{3}{2}-i\right)+5&=0?\\2\left(\frac{9}{4}-3i+i^2\right)-6\left(\frac{3}{2}-i\right)+5&=0?\\2\left(\frac{9}{4}-3i+(-1)\right)-6\left(\frac{3}{2}-i\right)+5&=0?\\2\left(\frac{5}{4}-3i\right)-6\left(\frac{3}{2}-i\right)+5&=0?\\\frac{5}{2}-\cancel{6i}-9+\cancel{6i}+5&=0?\\-\frac{3}{2}&\neq 0\end{aligned}$$

Since it doesn't equal zero, $\frac{3}{2}-i$ is not a solution.

Review

1. Use a graphing utility like the interactive at the beginning of this lesson to graph the equation $x^2+2x+1=0$. Use the graph to determine the solution to the equation, and then use Algebra to confirm your answer is correct.

Calculate the discriminant:

2. $6x^2+10-1=0$
3. $3x^2+24x+48=0$

Create quadratics:

4. Compose a quadratic function with two real solutions.
5. Compose a quadratic function with one real and one imaginary solution.
6. Compose a quadratic function with two imaginary solutions.

Express each as a simplified imaginary number.

7. $\sqrt{-300}$
8. $\sqrt{-32}$
9. $4\sqrt{-18}$
10. $\sqrt{-75}$
11. $\sqrt{-98}$

Solve each equation and express the result as a complex number.

12. $8x^2-5x+11=0$
13. $34\frac{1}{2}x^2-23x+19\frac{1}{6}=0$

Additional Resources

Video: Introduction to i and Imaginary Numbers

Practice: Imaginary Numbers

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5.3.2: The Complex Numbers

$a + bi$, the sum of a real and an imaginary number.

The coldest possible temperature, known as *absolute zero* is almost –460 degrees Fahrenheit. What is the square root of this number?

Complex Numbers

Before this concept, all numbers have been real numbers. 2, -5, $\sqrt{11}$, and $\frac{1}{3}$ are all examples of real numbers. With what we have previously learned, we cannot find $\sqrt{-25}$ because you cannot take the square root of a negative number. There is no **real number** that, when multiplied by itself, equals -25. Let's simplify $\sqrt{-25}$.

$$\sqrt{-25} = \sqrt{25 \cdot -1} = 5\sqrt{-1}$$

In order to take the square root of a negative number we are going to assign $\sqrt{-1}$ a variable, i. i represents an **imaginary number**. Now, we can use i to take the square root of a negative number.

$$\sqrt{-25} = \sqrt{25 \cdot -1} = 5\sqrt{-1} = 5i$$

All **complex numbers** have the form a+bi, where a and b are real numbers. a is the **real part** of the complex number and b is the **imaginary part**. If b=0, then a is left and the number is a **real number.** If a=0, then the number is only bi and called a **pure imaginary number**. If b≠0 and a≠0, the number will be an imaginary number.

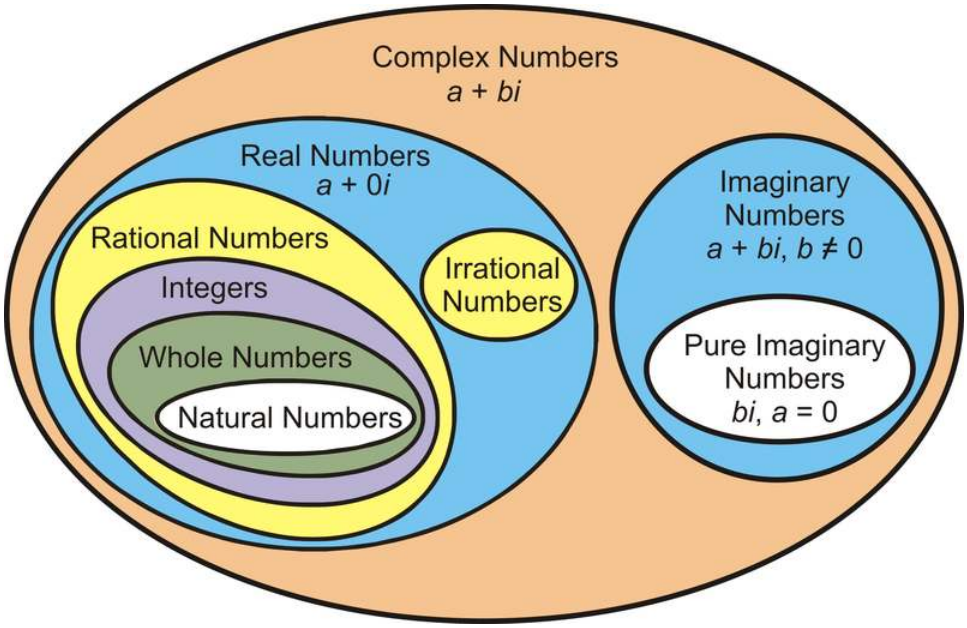
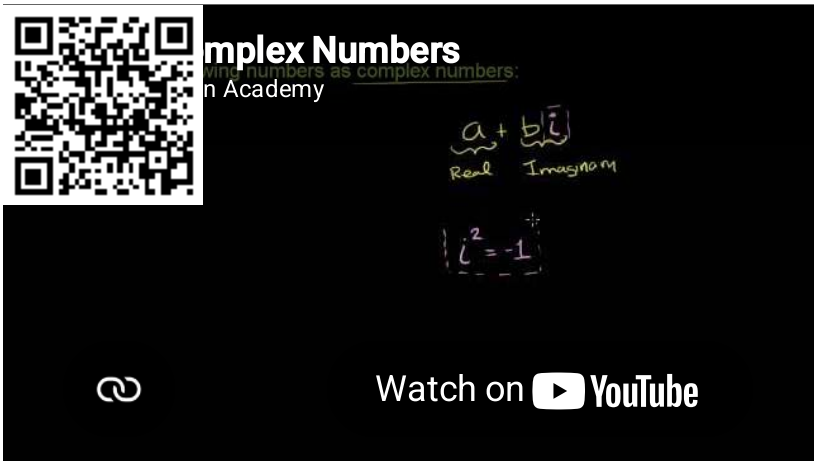


Figure **5.3.2.1**



Let's find $\sqrt{-162}$.

First pull out the i. Then, simplify $\sqrt{-162}$.

$$\sqrt{-162} = \sqrt{-1} \cdot \sqrt{162} = i\sqrt{162} = i\sqrt{81 \cdot 2} = 9i\sqrt{2}$$

Powers of i

In addition to now being able to take the square root of a negative number, i also has some interesting properties. Try to find i^2, i^3, and i^4.

Step 1: Write out i^2 and simplify. $i^2 = i \cdot i = \sqrt{-1} \cdot \sqrt{-1} = \sqrt{-1}^2 = -1$

Step 2: Write out i^3 and simplify. $i^3 = i^2 \cdot i = -1 \cdot i = -i$

Step 3: Write out i^4 and simplify. $i^4 = i^2 \cdot i^2 = -1 \cdot -1 = 1$

Step 4: Write out i^5 and simplify. $i^5 = i^4 \cdot i = 1 \cdot i = i$

Step 5: Write out i^6 and simplify. $i^6 = i^4 \cdot i^2 = 1 \cdot -1 = -1$

Step 6: Do you see a pattern? Describe it and try to find i^{19}.

You should see that the powers of i repeat every 4 powers. So, all the powers that are divisible by 4 will be equal to 1. To find i^{19}, divide 19 by 4 and determine the remainder. That will tell you what power it is the same as.

$$i^{19} = i^{16} \cdot i^3 = 1 \cdot i^3 = -i$$

Now, let's find the following powers of i.

1. i^{32}

32 is divisible by 4, so $i^{32} = 1$.

2. i^{50}

$50 \div 4 = 12$, with a remainder of 2. Therefore, $i^{50} = i^2 = -1$.

3. i^7

$7 \div 4 = 1$, with a remainder of 3. Therefore, $i^7 = i^3 = -i$

Finally, let's simplify the following complex expressions.

a. $(6 - 4i) + (5 + 8i)$

$(6 - 4i) + (5 + 8i) = 6 - 4i + 5 + 8i = 11 + 4i$

b. $9 - (4 + i) + (2 - 7i)$

$9 - (4 + i) + (2 - 7i) = 9 - 4 - i + 2 - 7i = 7 - 8i$

To add or subtract complex numbers, you need to combine like terms. Be careful with negatives and properly distributing them. Your answer should always be in **standard form**, which is a+bi.

Example 5.3.2.1

Earlier, you were asked to find the square root of -460 degrees.

Solution

We're looking for $\sqrt{-460}$.

First we need to pull out the i. Then, we need to simplify $\sqrt{-460}$.

$$\sqrt{-460} = \sqrt{-1} \cdot \sqrt{460} = i\sqrt{460} = i\sqrt{4 \cdot 115} = 2i\sqrt{115}$$

Example 5.3.2.2

Simplify $\sqrt{-49}$.

Solution

Rewrite $\sqrt{-49}$ in terms of i and simplify the radical.

$$\sqrt{-49} = i\sqrt{49} = 7i$$

Example 5.3.2.3

Simplify $\sqrt{-125}$.

Solution

Rewrite $\sqrt{-125}$ in terms of i and simplify the radical.

$$\sqrt{-125} = i\sqrt{125} = i\sqrt{25 \cdot 5} = 5i\sqrt{5}$$

Example 5.3.2.4

Simplify i^{210}.

Solution

$210 \div 4 = 52$, with a remainder of 2. Therefore, $i^{210} = i^2 = -1$.

Example 5.3.2.5

Simplify $(8 - 3i) - (12 - i)$.

Solution

Distribute the negative and combine like terms.

$$(8 - 3i) - (12 - i) = 8 - 3i - 12 + i = -4 - 2i$$

Review

Simplify each expression and write in standard form.

1. $\sqrt{-9}$
2. $\sqrt{-242}$
3. $6\sqrt{-45}$
4. $-12i\sqrt{98}$
5. $\sqrt{-32} \cdot \sqrt{-27}$

6. $7i\sqrt{-126}$
7. i^8
8. $16i^{22}$
9. $-9i^{65}$
10. i^{365}
11. $2i^{91}$
12. $-\sqrt{1680}$
13. $(11-5i)+(6-7i)$
14. $(14+2i)-(20+9i)$
15. $(8-i)-(3+4i)+15i$
16. $-10i-(1-4i)$
17. $(0.2+1.5i)-(-0.6+i)$
18. $6+(18-i)-(2+12i)$
19. $-i+(19+22i)-(8-14i)$
20. $18-(4+6i)+(17-9i)+24i$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 5.8.

Vocabulary

Term	Definition
i	i is an imaginary number. $i=\sqrt{-1}$.
Absolute Value	The absolute value of a number is the distance the number is from zero. The absolute value of a complex number is the distance from the complex number on the complex plane to the origin.
Complex Conjugate	Complex conjugates are pairs of complex binomials. The complex conjugate of $a+bi$ is $a-bi$. When complex conjugates are multiplied, the result is a single real number.
i	i is an imaginary number. $i=\sqrt{-1}$.
Imaginary Number	An imaginary number is a number that can be written as the product of a real number and i.
imaginary part	The imaginary part of a complex number $a+bi$ is bi.
Pure Imaginary Numbers	The pure imaginary numbers are the subset of complex numbers without real parts, only bi.
Real Number	A real number is a number that can be plotted on a number line. Real numbers include all rational and irrational numbers.
real part	The real part of a complex number $a+bi$ is a.
rectangular coordinates	A point is written using rectangular coordinates if it is written in terms of x and y and can be graphed on the Cartesian plane.
rectangular form	The rectangular form of a point or a curve is given in terms of x and y and is graphed on the Cartesian plane.
standard form	The standard form of a complex number is $a+bi$ where a and b are real numbers.

Additional Resources

Video: Writing Complex Numbers in Standard Form - Example 1

Practice: The Complex Numbers

Real World: Getting Lost in a Pineapple

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5.3.3: Quadratic Formula and Complex Sums

Solve quadratic equations with complex roots, and add and subtract complex numbers.

Complex Roots of Quadratic Functions

The quadratic function $y = x^2 - 2x + 3$ (shown below) does not intersect the x-axis and therefore has no real roots. What are the complex roots of the function?

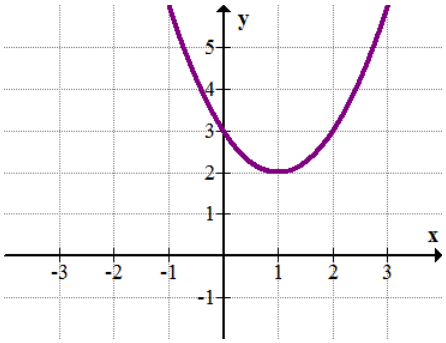


Figure **5.3.3.1**

Complex Roots of Quadratic Functions

Recall that the imaginary number, i, is a number whose square is –1:

$i^2 = -1$ and $i = \sqrt{-1}$

The sum of a **real number** and an imaginary number is called a **complex number**. Examples of complex numbers are $5 + 4i$ and $3 - 2i$. All complex numbers can be written in the form a+bi where a and b are real numbers. Two important points:

- The set of real numbers is a subset of the set of complex numbers where $b = 0$. Examples of real numbers are 2,7,12,−4.2.
- The set of **imaginary numbers** is a subset of the set of complex numbers where a=0. Examples of imaginary numbers are i, $-4i$, $\sqrt{2}i$.

This means that the set of complex numbers includes real numbers, imaginary numbers, and combinations of real and imaginary numbers.

When a quadratic function does not intersect the x-axis, it has complex roots. When solving for the roots of a function algebraically using the **quadratic formula**, you will end up with a negative under the square root symbol. With your knowledge of complex numbers, you can still state the complex roots of a function just like you would state the real roots of a function.

Let's solve the quadratic equation: $m^2 - 2m + 5 = 0$

You can use the quadratic formula to solve. For this quadratic equation, $a = 1$, $b = -2$, $c = 5$.

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$m = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)}$$
$$m = \frac{2 \pm \sqrt{4 - 20}}{2}$$
$$m = \frac{2 \pm \sqrt{-16}}{2} \quad \sqrt{-16} = \sqrt{16} \times i = 4i$$
$$m = \frac{2 \pm 4i}{2}$$
$$m = 1 \pm 2i$$
$$m = 1 + 2i \text{ or } m = 1 - 2i$$

There are no real solutions to the equation. The solutions to the quadratic equation are $1 + 2i$ and $1 - 2i$.

Now, let's solve the following equation by rewriting it as a quadratic and using the quadratic formula:

$$\frac{3}{e+3}-\frac{2}{e+2}=1$$

To rewrite as a quadratic equation, multiply each term by $(e+3)(e+2)$.

$$\frac{3}{e+3}(e+3)(e+2)-2e+2(e+3)(e+2)=1(e+3)(e+2)$$
$$3(e+2)-2(e+3)=(e+3)(e+2)$$

Expand and simplify.

$$3e+6-2e-6=e^2+2e+3e+6$$
$$e^2+4e+6=0$$

Solve using the quadratic formula. For this quadratic equation, a=1,b=4,c=6.

$$e=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$e=\frac{-(4)\pm\sqrt{(4)^2-4(1)(6)}}{2(1)}$$
$$e=\frac{-4\pm\sqrt{16-24}}{2}$$
$$e=\frac{-4\pm\sqrt{-8}}{2}\quad\sqrt{-8}=\sqrt{8}\times i=\sqrt{4\cdot 2}\times i=2i\sqrt{2}$$
$$e=\frac{-4\pm 2i\sqrt{2}}{2}$$
$$e=-2\pm i\sqrt{2}$$
$$e=-2+i\sqrt{2}\text{ or }e=-2-i\sqrt{2}$$

There are no real solutions to the equation. The solutions to the equation are −2+i\sqrt{2} and −2−i\sqrt{2}

Finally, let's sketch the graph of the following quadratic function. What are the roots of this function?

$y=x^2-4x+5$

Use your calculator or a table to make a sketch of the function. You should get the following:

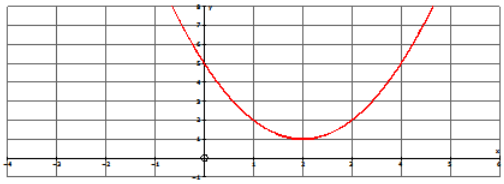


Figure 5.3.3.2

As you can see, the quadratic function has no x-intercepts; therefore, the function has no real roots. To find the roots (which will be complex), you must use the quadratic formula.

For this quadratic function, $a=1$, $b=-4$, $c=5$.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$x=\frac{-(-4)\pm\sqrt{(-4)^2-41)(5)}}{2(1)}$$
$$x=\frac{4\pm\sqrt{16-20}}{2}$$
$$x=\frac{4\pm\sqrt{-4}}{2}\quad\sqrt{-4}=\sqrt{4}\times i=2i$$
$$x=\frac{4\pm 2i}{2}$$
$$x=2\pm i$$
$$x=2+i\text{ or }x=2-i$$

The complex roots of the quadratic function are 2+i and 2−i.

Example 5.3.3.1

Earlier, you were asked to find the complex roots of $y = x^2 - 2x + 3$.

Solution

To find the complex roots of the function $y = x^2 - 2x + 3$, you must use the quadratic formula.

For this quadratic function, $a = 1$, $b = -2$, $c = 3$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(3)}}{2(1)}$$
$$x = \frac{2 \pm \sqrt{4 - 12}}{2}$$
$$x = \frac{2 \pm \sqrt{-8}}{2} \quad \sqrt{-8} = \sqrt{8} \times i = 2\sqrt{2}i$$
$$x = \frac{2 \pm 2\sqrt{2}i}{2}$$
$$x = 1 \pm \sqrt{2}i$$

Example 5.3.3.2

Solve the following quadratic equation. Express all solutions in simplest radical form.

$$2n^2 + n = -4$$

Solution

$$2n^2 + n = -4$$

Set the equation equal to zero.

$$2n^2 + n + 4 = 0$$

Solve using the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$n = \frac{-(1) \pm \sqrt{(1)^2 - 4(2)(4)}}{2(2)}$$
$$n = \frac{-1 \pm \sqrt{1 - 32}}{4}$$
$$n = \frac{-1 \pm \sqrt{-31}}{4}$$
$$n = \frac{-1 \pm i\sqrt{31}}{4}$$

Example 5.3.3.3

Solve the following quadratic equation. Express all solutions in simplest radical form.

$$m^2 + (m+1)^2 + (m+2)^2 = -1$$

Solution

$$m^2 + (m+1)^2 + (m+2)^2 = -1$$

Expand and simplify.

$$m^2 + (m+1)(m+1) + (m+2)(m+2) = -1$$
$$m^2 + m^2 + m + m + 1 + m^2 + 2m + 2m + 4 = -1$$
$$3m^2 + 6m + 5 = -1$$

Write the equation in general form.

$$3m^2 + 6m + 6 = 0$$

Divide by 3 to simplify the equation.

$$\frac{3m^2}{3} + \frac{6m}{3} + \frac{6}{3} = \frac{0}{3}$$
$$m^2 + 2m + 2 = 0$$

Solve using the quadratic formula:

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$m = \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(2)}}{2(1)}$$
$$m = \frac{-2 \pm \sqrt{4 - 8}}{2}$$
$$m = \frac{-2 \pm \sqrt{-4}}{2}$$
$$m = \frac{-2 \pm 2i}{2}$$
$$m = -1 \pm i$$

Example 5.3.3.4

Is it possible for a quadratic function to have exactly one **complex root**?

Solution

No, even in higher degree polynomials, complex roots will always come in pairs. Consider when you use the quadratic formula-- if you have a negative under the square root symbol, both the + version and the - version of the two answers will end up being complex.

Review

1. If a quadratic function has 2 x-intercepts, how many complex roots does it have? Explain.
2. If a quadratic function has no x-intercepts, how many complex roots does it have? Explain.
3. If a quadratic function has 1 x-intercept, how many complex roots does it have? Explain.
4. If you want to know whether a function has complex roots, which part of the quadratic formula is it important to focus on?
5. You solve a quadratic equation and get 2 complex solutions. How can you check your solutions?
6. In general, you can attempt to solve a quadratic equation by graphing, factoring, completing the square, or using the quadratic formula. If a quadratic equation has complex solutions, what methods do you have for solving the equation?

Solve the following quadratic equations. Express all solutions in simplest radical form.

7. $x^2 + x + 1 = 0$
8. $5y^2 - 8y = -6$
9. $2m^2 - 12m + 19 = 0$
10. $-3x^2 - 2x = 2$
11. $2x^2 + 4x = -11$
12. $-x^2 + x - 23 = 0$
13. $-3x^2 + 2x = 14$
14. $x^2 + 5 = -x$
15. $\frac{1}{2}d^2 + 4d = -12$

Review (Answers)

To see the Review answers, open this PDF file and look for section 9.7.

Vocabulary

Term	Definition
complex number	A complex number is the sum of a real number and an imaginary number, written in the form $a+bi$.
complex root	A complex root is a complex number that, when used as an input (x) value of a function, results in an output (y) value of zero.
Imaginary Numbers	An imaginary number is a number that can be written as the product of a real number and i.
Quadratic Formula	The quadratic formula states that for any quadratic equation in the form $ax^2+bx+c=0$, $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.
Real Number	A real number is a number that can be plotted on a number line. Real numbers include all rational and irrational numbers.

Additional Resources

Video: **Using the Quadratic Formula**

Practice: **Quadratic Formula and Complex Sums**

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5.3.4: Products and Quotients of Complex Numbers

Strategies based on multiplying binomials and conjugates.

Multiplying and Dividing Complex Numbers

Mr. Marchez draws a triangle on the board. He labels the height $(2+3i)$ and the base $(2-4i)$. "Find the area of the triangle," he says. (Recall that the area of a triangle is $A=\frac{1}{2}bh$, b is the length of the base and h is the length of the height.)

Multiplying and Dividing Complex Numbers

When multiplying complex numbers, FOIL the two numbers together and then combine like terms. At the end, there will be an i^2 term. Recall that $i^2=-1$ and continue to simplify.



Let's simplify the following expressions.

1. Simplify $6i(1-4i)$.

Distribute the 6i to both parts inside the parenthesis.

$$6i(1-4i)=6i-24i^2$$

Substitute $i^2=-1$ and simplify further.

$$=6i-24(-1)$$
$$=24+6i$$

Remember to always put the real part first.

$$(5-2i)(3+8i)$$

FOIL the two terms together.

$$(5-2i)(3+8i) = 15+40i-6i-16i^2$$
$$= 15+34i-16i^2$$

Substitute $i^2 = -1$ and simplify further.

$$= 15+34i-16(-1)$$
$$= 15+34i+16$$
$$= 31+34i$$

Dividing complex numbers is a bit more complicated. Similar to irrational numbers, complex numbers cannot be in the denominator of a fraction. To get rid of the **complex number** in the denominator, we need to multiply by the **complex conjugate**. If a complex number has the form $a+bi$, then its complex conjugate is $a-bi$. For example, the complex conjugate of $-6+5i$ would be $-6-5i$. Therefore, rather than dividing complex numbers, we multiply by the complex conjugate.

2. Simplify $\frac{8-3i}{6i}$.

In the case of dividing by a pure imaginary number, you only need to multiply the top and bottom by that number. Then, use multiplication to simplify.

$$\frac{8-3i}{6i}\cdot\frac{6i}{6i} = \frac{48i-18i^2}{36i^2}$$
$$= \frac{18+48i}{-36}$$
$$= \frac{18}{-36}+\frac{48}{-36}i$$
$$= -\frac{1}{2}-\frac{4}{3}i$$

When the complex number contains fractions, write the number in standard form, keeping the real and imaginary parts separate. Reduce both fractions separately.

3. Simplify $\frac{3-5i}{2+9i}$.

Now we are dividing by $2+9i$, so we will need to multiply the top and bottom by the complex conjugate, $2-9i$.

$$\frac{3-5i}{2+9i}\cdot\frac{2-9i}{2-9i} = \frac{6-27i-10i+45i^2}{4-18i+18i-81i^2}$$
$$= \frac{6-37i-45}{4+81}$$
$$= \frac{-39-37i}{85}$$
$$= -\frac{39}{85}-\frac{37}{85}i$$

Notice, by multiplying by the complex conjugate, the denominator becomes a real number and you can split the fraction into its real and imaginary parts.

In the previous three problems above, we substituted $i^2 = -1$ to simplify the fraction further. **Your final answer should never have any power of i greater than 1.**

Example 5.3.4.1

Earlier, you were asked to find the area of the triangle.

Solution

The area of the triangle is $(2+3i)(2-4i)^2$ so FOIL the two terms together and divide by 2.

$$(2+3i)(2-4i) = 4-8i+6i-12i^2$$
$$= 4-2i-12i^2$$

Substitute $i^2 = -1$ and simplify further.

$$= 4-2i-12(-1)$$
$$= 4-2i+12$$
$$= 16-2i$$

Now divide this product by 2.

$$\frac{16-2i}{2}=8-i$$

Therefore the area of the triangle is $8-i$.

Example 5.3.4.2

What is the complex conjugate of $7-5i$?

Solution

$7+5i$

Example 5.3.4.3

Simplify the following complex expression: $(7-4i)(6+2i)$.

Solution

FOIL the two expressions.

$$\begin{aligned}(7-4i)(6+2i)&=42+14i-24i-8i^2\\&=42-10i+8\\&=50-10i\end{aligned}$$

Example 5.3.4.4

Simplify the following complex expression: $\frac{10-i}{5i}$.

Solution

Multiply the numerator and denominator by $5i$.

$$\begin{aligned}\frac{10-i}{5i}\cdot\frac{5i}{5i}&=\frac{50i-5i^2}{25i^2}\\&=\frac{5+50i}{-25}\\&=\frac{5}{-25}+\frac{50}{-25}i\\&=-\frac{1}{5}-2i\end{aligned}$$

Example 5.3.4.5

Simplify the following complex expression: $\frac{8+i}{6-4i}$.

Solution

Multiply the numerator and denominator by the complex conjugate, $6+4i$.

$$\begin{aligned}\frac{8+i}{6-4i}\cdot\frac{6+4i}{6+4i}&=\frac{48+32i+6i+4i^2}{36+24i-24i-16i^2}\\&=\frac{48+38i-4}{36+16}\\&=\frac{44+38i}{52}\\&=\frac{44}{52}+\frac{38}{52}i\\&=\frac{11}{13}+\frac{19}{26}i\end{aligned}$$

Review

Simplify the following expressions. Write your answers in standard form.

1. $i(2-7i)$
2. $8i(6+3i)$
3. $-2i(11-4i)$
4. $(9+i)(8-12i)$
5. $(4+5i)(3+16i)$
6. $(1-i)(2-4i)$
7. $4i(2-3i)(7+3i)$
8. $(8-5i)(8+5i)$
9. $\frac{4+9i}{3i}$
10. $\frac{6-i}{12i}$
11. $\frac{7+12i}{-5i}$
12. $\frac{4-2i}{6-6i}$
13. $\frac{2-i}{2+i}$
14. $\frac{10+8i}{2+4i}$
15. $\frac{14+9i}{7-20i}$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 5.9.

Vocabulary

Term	Definition
complex number	A complex number is the sum of a real number and an imaginary number, written in the form $a+bi$.

Additional Resources

Video: Complex Conjugates - Overview

Practice: Products and Quotients of Complex Numbers

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5.3.5: Product and Quotient Theorems

Simplified way to multiply and divide complex numbers.

Complex numbers are found in real world calculations involving: quantum mechanics, signal analysis, fluid dynamics, control theory, and many other fields.

In electrical engineering, complex numbers are used for calculations involving *impedance* (the resistance to electric flow in a circuit).

Electrical engineers are familiar with the formula:

$V = V_0 e^{j\omega t} = V_0(\cos \omega t + j \sin \omega t)$

by comparing it to the similar expression below, which is explored in this lesson, can you identify the variable j?

$r_2(\cos \theta_2 + i \sin \theta_2)$

Product and Quotient Theorems

The Product Theorem

Since complex numbers can be transformed to polar form, the multiplication of complex numbers can also be done in polar form. Suppose we know $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$, $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$

To multiply the two complex numbers in polar form:

$$\begin{aligned} z_1 \cdot z_2 &= r_1(\cos \theta_1 + i \sin \theta_1) \cdot r_2(\cos \theta_2 + i \sin \theta_2) \\ &= r_1 r_2(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \\ &= r_1 r_2 \cdot (\cos \theta_1 \cos \theta_2 + i \cos \theta_1 \sin \theta_2 + i \sin \theta_1 \cos \theta_2 + i^2 \sin \theta_1 \sin \theta_2) \\ &= r_1 r_2(\cos \theta_1 \cos \theta_2 + i \cos \theta_1 \sin \theta_2 + i \sin \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) \\ &= r_1 r_2(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 + i \cos \theta_1 \sin \theta_2 + i \sin \theta_1 \cos \theta_2) \\ &= r_1 r_2([\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2] + i[\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2]) \end{aligned}$$

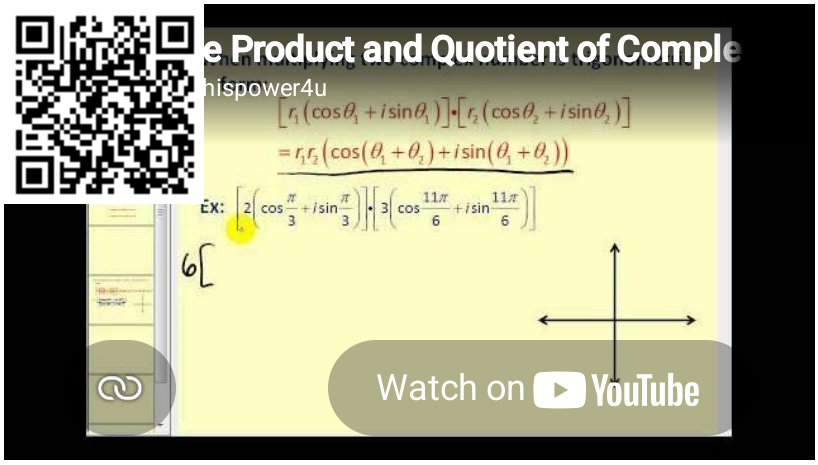
(Use $i^2 = -1$, gather like terms, factor out i, substitute the angle sum formulas for both sine and cosine)

$$z_1 \cdot z_2 = r_1 r_2 cis\ [(\theta_1 + \theta_2)]$$

This last equation states that the product of two complex numbers in polar form can be obtained by multiplying the polar r values of each of the complex numbers and then multiplying that value by cis of the sum of each of the two angles of the individual complex numbers. This is more concise than the **rectangular form** for multiplication of complex numbers.

The Quotient Theorem

Dividing complex numbers in polar form can be shown using a similar proof that was used to show multiplication of complex numbers. Here we omit the proof and give the result. For $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$, then $\frac{z_1}{z_2} = \frac{r_1}{r_2} \times cis\ [\theta_1 - \theta_2]$



Example 5.3.5.1

Earlier, you were asked to identify the variable *j* in the following formula:

$V = V_0 e^{j\omega t} = V_0(\cos \omega t + j \sin \omega t)$

Solution

In electrical calculations, the letter *I* is commonly used to denote *current*, therefore imaginary numbers are identified with a j.

Note the similar usage of i in $r(\cos \theta + i \sin \theta)$.

Example 5.3.5.2

Multiply $z_1 \cdot z_2$ where $z_1 = 2 + 2i$ and $z_2 = 1 - \sqrt{3}i$.

Solution

For z_1,

$$\begin{aligned} r_1 &= \sqrt{2^2 + 2^2} \\ &= \sqrt{8} \\ &= 2\sqrt{2} \end{aligned}$$

and

$$\begin{aligned} \tan \theta_1 &= \frac{2}{2} \\ \tan \theta_1 &= 1 \\ \theta_1 &= \frac{\pi}{4} \end{aligned}$$

Note that θ_1 is in the first quadrant since a, and $b > 0$.

For z_2,

$$\begin{aligned} r_2 &= \sqrt{1^2 + (-\sqrt{3})^2} \\ &= \sqrt{1+3} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

and

$$\begin{aligned} \tan \theta_2 &= \frac{-\sqrt{3}}{1} \\ \theta_2 &= \frac{5\pi}{3} \end{aligned}$$

Now we can use the formula $z_1 \cdot z_2 = r_1 \cdot r_2 \; cis \; (\theta_1 + \theta_2)$

Substituting gives:

$$\begin{aligned} z_1 \cdot z_2 &= 2\sqrt{2} \times 2 \; cis \left[\frac{\pi}{4} + \frac{5\pi}{3}\right] \\ &= 4\sqrt{2} \; cis \left[\frac{23\pi}{12}\right] \end{aligned}$$

So we have

$$z_1 \cdot z_2 = 4\sqrt{2}\left(\cos \frac{23\pi}{12} + i \sin \frac{23\pi}{12}\right)$$

Re-writing in approximate decimal form:

$$5.656(0.966–0.259i)$$

$$5.46 - 1.46i$$

If the problem was done using only rectangular units then

$$z_1 \times z_2 = (2+2i)(1-\sqrt{3}i) \text{ or}$$
$$= 2 - 2\sqrt{3}i + 2i - 2\sqrt{3}i^2$$

Gathering like terms and using $i^2 = -1$

$= (2+2\sqrt{3}) - (2\sqrt{3}+2)i$

or

$5.46 - 1.46i$

Example 5.3.5.3

Using polar multiplication, find the product $(6-2\sqrt{3}i)(4+4\sqrt{3}i)$.

Solution

Let $z_1 = 6-2\sqrt{3}i$ and $z_2 = 4+4\sqrt{3}i$

$r_1 = \sqrt{(6)^2-(2\sqrt{3})^2}$ and $r_2 = \sqrt{(4)^2+(4\sqrt{3})^2}$

$r_1 = \sqrt{36+12} = \sqrt{48} = 4\sqrt{3}$ and $r_2 = \sqrt{16+48} = \sqrt{64} = 8$

For θ_1, first find $\tan\theta_{ref} = \left|\frac{y}{x}\right|$

$$\tan\theta_{ref} = \frac{(2\sqrt{3})}{6}$$
$$\tan\theta_{ref} = \frac{\sqrt{3}}{3}$$
$$\theta_{ref} = \frac{\pi}{6}$$

Since $x > 0$ and $y < 0$ we know that θ_1 is in the in the 4th quadrant:

$\theta_1 = 11\frac{\pi}{6}$

For θ_2,

$$\tan\theta_{ref} = \frac{(4\sqrt{3})}{4}$$
$$\tan\theta_{ref} = \sqrt{3}$$
$$\theta_{ref} = \frac{\pi}{3}$$

Since θ_2 is in the first quadrant,

$$\theta_2 = \frac{\pi}{3}$$

Using polar multiplication,

$$z_1 \times z_2 = 4\sqrt{3} \times 8\left(cis\left[\frac{11\pi}{6}+\frac{\pi}{3}\right]\right)$$
$$z_1 \times z_2 = 32\sqrt{3}\left(cis\left[\frac{13\pi}{6}\right]\right)$$

subtracting 2π from the augment:

$$z_1 \times z_2 = 32\sqrt{3}\left(cis\left[\frac{\pi}{6}\right]\right)$$

or in expanded form: $32\sqrt{3}\left(\cos\left[\frac{\pi}{6}\right]+i\sin\left[\frac{\pi}{6}\right]\right)$

In decimal form this becomes: $55.426(0.866+0.500i)$ or $48+27.713i$

Check:

$$(6-2\sqrt{3}i)(4+4\sqrt{3}i) = 24+24\sqrt{3}i-8\sqrt{3}i-24i^2$$
$$= 24+16\sqrt{3}i+24$$
$$= 48+27.713i$$

Example 5.3.5.4

Using polar division, find the quotient of $z_1 z_2$ given that $z_1 = 5 - 5i$ and $z_2 = -2\sqrt{3} - 2i$.

Solution

For z_1: $r_1 = \sqrt{5^2 + (-5)^2}$ or $5\sqrt{2}$ and $\tan\theta_1 = \dfrac{-5}{5}$, so $\theta_1 = \dfrac{7\pi}{4}$ (4th quadrant)

For z_2: $r_2 = \sqrt{(-2\sqrt{3})^2 + (-2)^2}$ or $\sqrt{16} = 4$ and $\tan\theta_2 = \dfrac{-2}{(-2\sqrt{3})}$, so $\theta_2 = \dfrac{7\pi}{6}$ (3rd quadrant)

Using the formula, $z_1 z_2 = r_1 r_2 \times \ cis\ [\theta_1 - \theta_2]$ or

$$\begin{aligned} &= 5\sqrt{24} \times \ cis\ \left[\frac{7\pi}{4} - \frac{7\pi}{6}\right] \\ &= 5\sqrt{24} \times \ cis\ \left[\frac{7\pi}{12}\right] \\ &= 5\sqrt{24}\left[\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right] \\ &= 1.768[-0.259 + (0.966)i] \\ &= -0.458 + 1.708i \end{aligned}$$

Check by using the **complex conjugate** to do the division in rectangular form:

$$\begin{aligned} \frac{5-5i}{-2\sqrt{3}-2i} \cdot \frac{-2\sqrt{3}+2i}{-2\sqrt{3}+2i} &= \frac{-10\sqrt{3}+10i+10\sqrt{3}i-10i^2}{(-2\sqrt{3})^2-(2i)^2} \\ &= \frac{-10\sqrt{3}+10i+10\sqrt{3}i+10}{12+4} \\ &= \frac{(-10\sqrt{3}+10)+(10+10\sqrt{3})i}{16} \\ &= \frac{(-17.3+10)+(10+17.3)i}{16} \\ &= \frac{(-7.3)+(27.3)i}{16} \qquad \text{or} \quad -0.456+1.706i \end{aligned}$$

The two radically different approaches yield the same answer. The small difference between the two answers is a result of decimal rounding.

Example 5.3.5.5

Find the product: $\left(7\left(\frac{\pi}{6}\right)\right) \cdot \left(5\left(-\frac{\pi}{4}\right)\right)$.

Solution

This one is easier than it looks: Recall $z_1 \cdot z_2 = r_1 \cdot r_2\ cis\ (\theta_1 + \theta_2)$.

$r_1 \cdot r_2 \to 7 \cdot 5 = 35$	By substitution and multplication
$\theta_1 + \theta_2 \to \left(\frac{\pi}{6}\right) + \left(\frac{-\pi}{4}\right)$	Substitute
$\left(\frac{2\pi}{12}\right) + \left(\frac{-3\pi}{12}\right)$	Find common denominators
$\left(\frac{-\pi}{12}\right)$	Simplify
$\therefore 35\ cis\ \left(\frac{-\pi}{12}\right)$	is the product

Example 5.3.5.6

Find the quotient: $\dfrac{1+2i}{2-i}$.

Solution

First, find the quotient by polar multiplication:

$r_1 = \sqrt{(1)^2 + (2)^2} = \sqrt{5} \qquad r_2 = \sqrt{(2)^2 + (-1)^2} = \sqrt{5}$

$$\tan\theta_1 = \frac{2}{1}$$
$$\tan\theta_1 = 2$$
$$\theta_{ref} = 1.107 \text{ radians}$$

since the angle is in the 1st quadrant

$\theta_1 = 1.107$ radians

for θ_2,

$$\tan\theta_2 = \frac{-1}{2}$$
$$\tan\theta_{ref} = \frac{1}{2}$$
$$\theta_{ref} = 0.464 \text{ radians}$$

sinceθ_2 is in the 4th quadrant, between 4.712 (or $\frac{3\pi}{2}$) and 6.282 radians (or 2π)

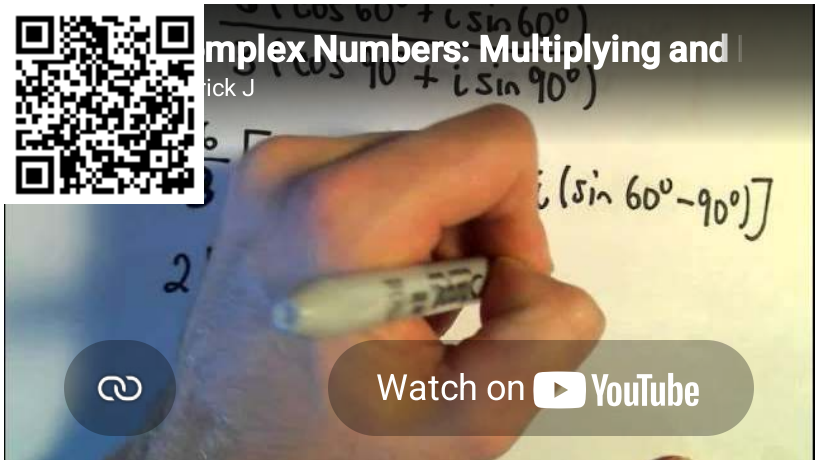
$\theta_2 = 5.820 radians$

Finally, using the division formula,

$$\frac{z_1}{z_2} = \frac{\sqrt{5}}{\sqrt{5}}[\,cis\,(1.107 - 5.820)]$$
$$\frac{z_1}{z_2} = [\,cis\,(-4.713)]$$
$$\frac{z_1}{z_2} = [\cos(-4.713) + i\sin(-4.713)]$$
$$\frac{z_1}{z_2} = [\cos(1.570) + i\sin(1.570)]$$

If we assume that $\frac{\pi}{2} = 1.570$, then

$$\approx \frac{z_1}{z_2} = [\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)]$$
$$\frac{z_1}{z_2} = 0 + 1i = i$$



Review

1. Find the product using polar form: $(2+2i)(\sqrt{3}-i)$
2. $2cis\,(40)\cdot 4cis\,(20)$
3. Multiply: $2\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)\cdot 2\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$
4. $\frac{2cis\,(80)}{6cis\,(200)}$
5. Divide: $3cis\,(130°)\div 4cis\,(270°)$

If $z_1 = 7\left(\frac{\pi}{6}\right)$ and $z_2 = 5\left(\frac{-\pi}{4}\right)$ find:

6. $z_1 \cdot z_2$

7. $\left(\frac{z_1}{z_2}\right)$
8. $\left(\frac{z_2}{z_1}\right)$

If $z_1 = 8\left(\frac{\pi}{3}\right)$ and $z_2 = 5\left(\frac{\pi}{6}\right)$ find:

9. $z_1 z_2$
10. $\left(\frac{z_1}{z_2}\right)$
11. $\left(\frac{z_2}{z_1}\right)$
12. $(z_1)^2$
13. $(z_2)^3$

Find the products.

14. Find the product using polar form:$(2+2i)(\sqrt{3}-i)$
15. $2(\cos 40^\circ + i \sin 40^\circ) \cdot 4(\cos 20^\circ + i \sin 20^\circ)$
16. $2\left(\cos\frac{\pi}{8} + i \sin\frac{\pi}{8}\right) \cdot 2\left(\cos\frac{\pi}{10} + i \sin\frac{\pi}{10}\right)$

Find the quotients.

17. $2(\cos 80^\circ + i \sin 80^\circ) \div 6(\cos 200^\circ + i \sin 200^\circ)$
18. $3cis\ (130^\circ) \div 4cis\ (270^\circ)$

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.9.

Vocabulary

Term	Definition
Complex Conjugate	Complex conjugates are pairs of complex binomials. The complex conjugate of $a+bi$ is $a-bi$. When complex conjugates are multiplied, the result is a \sin gle real number.
complex number	A complex number is the sum of a real number and an imaginary number, written in the form $a+bi$.
rectangular form	The rectangular form of a point or a curve is given in terms of x and y and is graphed on the Cartesian plane.

Additional Resources

Video: Complex Conjugates Theorem - Overview

Practice: Product and Quotient Theorems

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5.3.6: Trigonometric Form of Complex Numbers

x plus y times i equals r times the sum of cosine theta and i times sine theta.

You already know how to represent complex numbers in the **complex plane** using rectangular coordinates and you already know how to multiply and divide complex numbers. Representing these points and performing these operations using **trigonometric polar form** will make your computations more efficient.

What are the two ways to multiply the following complex numbers?

$(1+\sqrt{3}i)(\sqrt{2}-\sqrt{2}i)$

Trigonometric **Polar Form** of Complex Numbers

Any point represented in the complex plane as $a+bi$ can be represented in polar form just like any point in the rectangular coordinate system. **Trigonometric polar form** of a complex number describes the location of a point on the complex plane using the angle and the radius of the point. You will use the distance from the point to the origin as r and the angle that the point makes as θ.

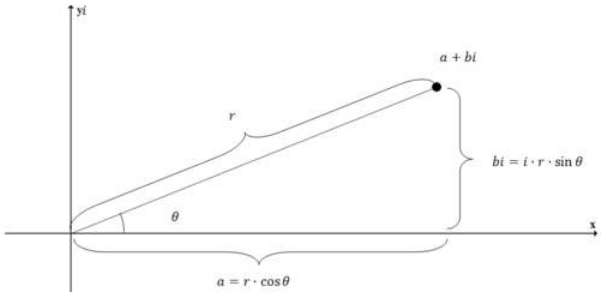


Figure 5.3.6.1

As you can see, the point a+bi can also be represented as $r\cdot\cos\theta+i\cdot r\cdot\sin\theta$. The trigonometric polar form can be abbreviated by factoring out the r and noting the first letters:

$r(\cos\theta+i\cdot\sin\theta)\rightarrow r\cdot\ cis\ \theta$

The abbreviation $r\cdot\ cis\ \theta$ is read as "r kiss theta." It allows you to represent a point as a radius and an angle.

Take the following complex number in **rectangular form**.

$1-\sqrt{3}i$

To convert the following complex number from rectangular form to trigonometric polar form, find the radius using the absolute value of the number.

$r^2=1^2+(-\sqrt{3})^2\rightarrow r=2$

The angle can be found with basic trig and the knowledge that the opposite side is always the imaginary component and the adjacent side is always the real component.

$\tan\theta=-\frac{\sqrt{3}}{1}\rightarrow\theta=60^\circ$

Thus the **trigonometric form** is $2\ cis\ 60^\circ$.

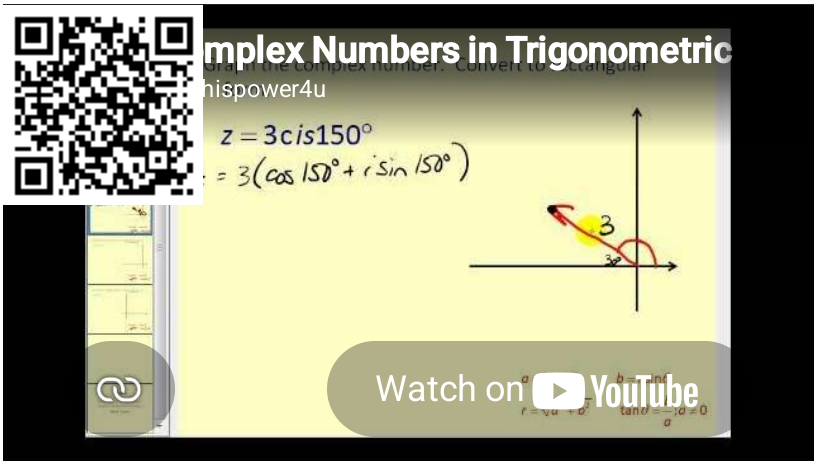
One great benefit of the cis form is that it makes multiplying and dividing complex numbers extremely easy. For example:

Let: $z_1=r_1\cdot\ cis\ \theta_1, z_2=r_2\cdot\ cis\ \theta_2$ with $r_2\neq 0$.

Then:

$$z_1\cdot z_2=r_1\cdot r_2\cdot\ cis\ (\theta_1+\theta_2)$$
$$z_1\div z_2=\frac{r_1}{r_2}\cdot\ cis\ (\theta_1-\theta_2)$$

For basic problems, the amount of work required to compute products and quotients for complex numbers given in either form is roughly equivalent. For more challenging questions, trigonometric polar form becomes significantly advantageous.



Example 5.3.6.1

Earlier, you were asked how to multiply the complex numbers $(1+\sqrt{3}i)(\sqrt{2}-\sqrt{2}i)$.

Solution

In rectangular coordinates:

$(1+\sqrt{3}i)(\sqrt{2}-\sqrt{2}i) = \sqrt{2}-\sqrt{2}i+\sqrt{6}i+\sqrt{6}$

In trigonometric polar coordinates, $1+\sqrt{3}i = 2\ cis\ 60°$ and $\sqrt{2}-\sqrt{2}i = 2\ cis\ -45°$. Therefore:

$(1+\sqrt{3}i)(\sqrt{2}-\sqrt{2}i) = 2\ cis\ 60° \cdot 2\ cis\ -45° = 4\ cis\ 105°$

Example 5.3.6.2

Convert the following complex number from trigonometric polar form to rectangular form.

$4\ cis\ (3\frac{\pi}{4})$

Solution

$$4\ cis\ \left(3\frac{\pi}{4}\right) = 4\left(\cos\left(\frac{3\pi}{4}\right)+i\cdot\sin\left(\frac{3\pi}{4}\right)\right) = 4\left(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right) = -2\sqrt{2}+2\sqrt{2}i$$

Example 5.3.6.3

Divide the following complex numbers.

$\frac{4\ cis\ 32°}{2\ cis\ 2°}$

Solution

$$\frac{4\ cis\ 32°}{2\ cis\ 2°} = 42\ cis\ (32°-2°) = 2\ cis\ (30°)$$

Example 5.3.6.4

Translate the following complex number from rectangular form into trigonometric polar form:

8

Solution

$8 = 8\ cis\ 0°$

Note that this has no complex part and therefore has no angle.

Example 5.3.6.5

Multiply the following complex numbers in trigonometric polar form.

$4\ cis\ 34° \cdot 5\ cis\ 16° \cdot \frac{1}{2}\ cis\ 100°$

Solution

$$\begin{aligned} 4\ cis\ 34° \cdot 5\ cis\ 16° \cdot \frac{1}{2}\ cis\ 100° & \\ &= 4 \cdot 5 \cdot 12 \cdot\ cis\ (34° + 16° + 100°) \\ &= 10\ cis\ 150 \end{aligned}$$

Note how much easier it is to do products and quotients in trigonometric polar form.

Review

Translate the following complex numbers from trigonometric polar form to rectangular form.

1. $5\ cis\ 270°$
2. $2\ cis\ 30°$
3. $-4\ cis\ \frac{\pi}{4}$
4. $6\ cis\ \frac{\pi}{3}$
5. $2\ cis\ \frac{5\pi}{2}$

Translate the following complex numbers from rectangular form into trigonometric polar form.

6. $2 - i$
7. $5 + 12i$
8. $6i + 8$
9. i

Complete the following calculations and simplify.

10. $2\ cis\ 22° \cdot \frac{1}{5}\ cis\ 15° \cdot 3\ cis\ 95°$
11. $9\ cis\ 98° \div 3\ cis\ 12°$
12. $15\ cis\ \frac{\pi}{4} \cdot 2\ cis\ \frac{\pi}{6}$
13. $-2\ cis\ \frac{2\pi}{3} \div 15\ cis\ \frac{7\pi}{6}$

Let $z_1 = r_1 \cdot\ cis\ \theta_1$ and $z_2 = r_2 \cdot\ cis\ \theta_2$ with r_2 \neq 0\).

14. Use the trigonometric sum and difference identities to prove that $z_1 \cdot z_2 = r_1 \cdot r_2 \cdot\ cis\ (\theta_1 + \theta_2)$.
15. Use the trigonometric sum and difference identities to prove that $z_1 \div z_2 = r_1 r_2 \cdot\ cis\ (\theta_1 - \theta_2)$.

Review (Answers)

To see the Review answers, open this PDF file and look for section 11.3.

Vocabulary

Term	Definition
$r\ cis\ \theta$	$r\ cis\ \theta$ is shorthand for the expression $r\cos\theta + ri\sin\theta$.
complex plane	The complex plane is the graphical representation of the set of all complex numbers.
polar form	The polar form of a point or a curve is given in terms of r and θ and is graphed on the polar plane.
rectangular form	The rectangular form of a point or a curve is given in terms of x and y and is graphed on the Cartesian plane.
trigonometric form	To write a complex number in trigonometric form means to write it in the form $r\cos\theta + ri\sin\theta$. $r\ cis\ \theta$ is shorthand for this expression.
trigonometric polar form	To write a complex number in trigonometric form means to write it in the form $r\cos\theta + ri\sin\theta$. $r\ cis\ \theta$ is shorthand for this expression.

Additional Resources

Video: Complex Numbers in Trigonometric Form

Practice: Trigonometric Form of Complex Numbers

Real World: Keeping Current

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5.3.7: Equations Using DeMoivre's Theorem

Complex roots of an equation.

You are given an equation in math class:

$x^4 = 16$

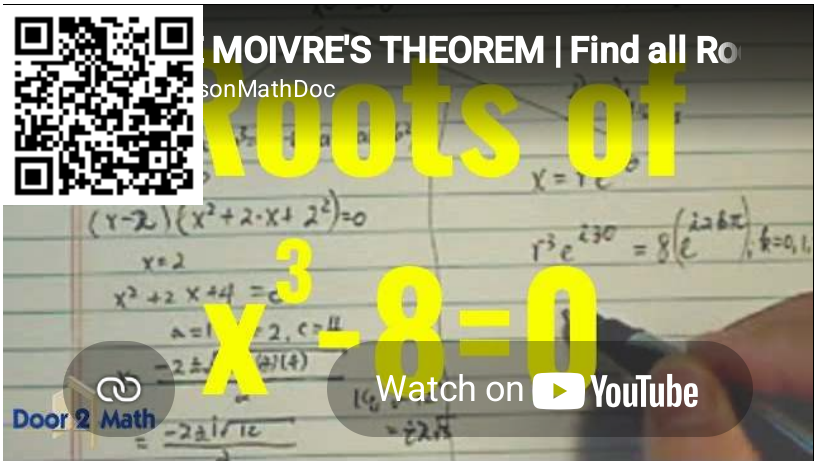
and asked to solve for "x". "Excellent!" you say. "This should be easy. The answer is 2."

"Not quite so fast," says your instructor.

"I want you to find the complex roots as well!"

Equations Using De Moivre's Theorem

We've already seen equations that we would like to solve. However, up until now, these equations have involved solutions that were real numbers. However, there is no reason that solutions need to be limited to the real number line. In fact, some equations cannot be solved completely without the use of complex numbers. Here we'll explore a little more about complex numbers as solutions to equations.



The roots of a complex number are cyclic in nature. This means that when the roots are plotted on the complex plane, the nth roots are equally spaced on the circumference of a circle.

Since you began Algebra, solving equations has been an extensive topic. Now we will extend the rules to include complex numbers. The easiest way to explore the process is to actually solve an equation. The solution can be obtained by using De Moivre's Theorem.

Using De Moivre's Theorem

1. Consider the equation $x^5 - 32 = 0$. The solution is the same as the solution of $x^5 = 32$. In other words, we must determine the fifth roots of 32.

$$x^5 - 32 = 0 \text{ and } x^5 = 32.$$
$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{(32)^2 + (0)^2}$$
$$r = 32$$
$$\theta = \tan^{-1}\left(\frac{0}{32}\right) = 0$$

Write an expression for determining the fifth roots of 32=32+0i

$$
\begin{aligned}
32^{1/5} &= \left[32(\cos(0+2\pi k)+i\sin(0+2\pi k)\right]^{\frac{1}{5}} \\
&= 2\left(\cos\frac{2\pi k}{5}+i\sin\frac{2\pi k}{5}\right) k=0,1,2,3,4 \\
x_1 &= 2\left(\cos\frac{0}{5}+i\sin\frac{0}{5}\right) \to 2(\cos 0+i\sin 0)=2 && \text{for } k=0 \\
x_2 &= 2\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right) \approx 0.62+1.9i && \text{for } k=1 \\
x_3 &= 2\left(\cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5}\right) \approx -1.62+1.18i && \text{for } k=2 \\
x_4 &= 2\left(\cos\frac{6\pi}{5}+i\sin\frac{6\pi}{5}\right) \approx -1.62-1.18i && \text{for } k=3 \\
x_5 &= 2\left(\cos\frac{8\pi}{5}+i\sin\frac{8\pi}{5}\right) \approx 0.62-1.9i && \text{for } k=4
\end{aligned}
$$

This is the same as the equation $x^3=27$.

$$
\begin{aligned}
x^3 &= 27 \\
r &= \sqrt{x^2+y^2} \\
r &= \sqrt{(27)^2+(0)^2} \\
r &= 27 \\
\theta &= \tan^{-1}\left(\frac{0}{27}\right)=0
\end{aligned}
$$

Write an expression for determining the cube roots of $27=27+0i$

$$
\begin{aligned}
27^{1/3} &= [27(\cos(0+2\pi k)+i\sin(0+2\pi k)]^{1/3} \\
&= 3\left(\cos\frac{2\pi k}{3}+i\sin\frac{2\pi k}{3}\right) k=0,1,2 \\
x_1 &= 3\left(\cos\frac{0}{3}+i\sin\frac{0}{3}\right) \to 3(\cos 0+i\sin 0)=3 && \text{for } k=0 \\
x_2 &= 3\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right) \approx -1.5+2.6i && \text{for } k=1 \\
x_3 &= 3\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right) \approx -1.5-2.6i && \text{for } k=2
\end{aligned}
$$

3. Solve the equation $x^4=1$

$$
\begin{aligned}
x^4 &= 1 \\
r &= \sqrt{x^2+y^2} \\
r &= \sqrt{(1)^2+(0)^2} \\
r &= 1 \\
\theta &= \tan^{-1}\left(\frac{0}{1}\right)=0
\end{aligned}
$$

Write an expression for determining the cube roots of $1=1+0i$

$$
\begin{aligned}
1^{1/4} &= [1(\cos(0+2\pi k)+i\sin(0+2\pi k)]^{1/4} \\
&= 1\left(\cos\frac{2\pi k}{4}+i\sin\frac{2\pi k}{4}\right) k=0,1,2,3 \\
x_1 &= 1\left(\cos\frac{0}{4}+i\sin\frac{0}{4}\right) \to 3(\cos 0+i\sin 0)=1 && \text{for } k=0 \\
x_2 &= 1\left(\cos\frac{2\pi}{4}+i\sin\frac{2\pi}{4}\right) = 0+i=i && \text{for } k=1 \\
x_3 &= 1\left(\cos\frac{4\pi}{4}+i\sin\frac{4\pi}{4}\right) = -1-0i=-1 && \text{for } k=2 \\
x_4 &= 1\left(\cos\frac{6\pi}{4}+i\sin\frac{6\pi}{4}\right) = 0-i=-i && \text{for } k=3
\end{aligned}
$$

Example 5.3.7.1

Earlier, you were asked to find the complex root of $x^4 = 16$.

Solution

Since you want to find the fourth root of 16, there will be four solutions in all.

$$\begin{aligned} x^4 &= 16 \\ r &= \sqrt{x^2 + y^2} \\ r &= \sqrt{(16)^2 + (0)^2} \\ r &= 16 \\ \theta &= \tan^{-1}\left(\frac{0}{16}\right) = 0 \end{aligned}$$

Write an expression for determining the fourth roots of $16 = 16 + 0i$

$$\begin{aligned} 16^{1/4} &= [16(\cos(0 + 2\pi k) + i\sin(0 + 2\pi k)]^{1/4} \\ &= 2\left(\cos\frac{2\pi k}{4} + i\sin\frac{2\pi k}{4}\right) \; k = 0, 1, 2, 3, 4 \\ x_1 &= 2\left(\cos\frac{0}{4} + i\sin\frac{0}{4}\right) \rightarrow 2(\cos 0 + i\sin 0) = 2 \qquad \text{for } k = 0 \\ x_2 &= 2\left(\cos\frac{2\pi}{4} + i\sin\frac{2\pi}{4}\right) = 2i \qquad \text{for } k = 1 \\ x_3 &= 2\left(\cos\frac{4\pi}{4} + i\sin\frac{4\pi}{4}\right) = -2 \qquad \text{for } k = 2 \\ x_4 &= 2\left(\cos\frac{6\pi}{4} + i\sin\frac{6\pi}{4}\right) = -2i \qquad \text{for } k = 3 \end{aligned}$$

Therefore, the four roots of 16 are $2, -2, 2i, -2i$. Notice how you could find the two real roots if you seen complex numbers. The addition of the complex roots completes our search for the roots of equations.

Example 5.3.7.2

Rewrite the following in rectangular form: $[2(\cos 315^\circ + i\sin 315^\circ)]^3$

Solution

$$\begin{aligned} & r = 2 \text{ and } \theta = 315^\circ \text{ or } \frac{7\pi}{4} \\ & z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta) \\ & z^3 = 2^3\left[\left(\cos 3\left(\frac{7\pi}{4}\right) + i\sin 3\left(\frac{7\pi}{4}\right)\right]\right. \\ & z^3 = 8\left(\cos\frac{21\pi}{4} + i\sin\frac{21\pi}{4}\right) \\ & z^3 = 8\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) \\ & z^3 = -4\sqrt{2} - 4i\sqrt{2} \end{aligned}$$

$\frac{21\pi}{4}$ is in the third quadrant so both are negative.

Example 5.3.7.3

Solve the equation $x^4 + 1 = 0$. What shape do the roots make?

Solution

$$\begin{aligned} & x^4 + 1 = 0 \qquad && r = \sqrt{x^2 + y^2} \\ & x^4 = -1 \qquad && r = \sqrt{(-1)^2 + (0)^2} \\ & x^4 = -1 + 0i \qquad && r = 1 \\ & \theta = \tan^{-1}\left(\frac{0}{-1}\right) + \pi = \pi \end{aligned}$$

Write an expression for determining the fourth roots of $x^4 = -1 + 0i$

$$
\begin{aligned}
(-1+0i)^{1/4} &= [1(\cos(\pi + 2\pi k) + i\sin(\pi + 2\pi k))]^{1/4} \\
(-1+0i)^{1/4} &= 1^{\frac{1}{4}}\left(\cos\frac{\pi+2\pi k}{4} + i\sin\frac{\pi+2\pi k}{4}\right) \\
x_1 &= 1\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} && \text{for } k = 0 \\
x_2 &= 1\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} && \text{for } k = 1 \\
x_3 &= 1\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} && \text{for } k = 2 \\
x_4 &= 1\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} && \text{for } k = 3
\end{aligned}
$$

If a line segment is drawn from each root on the polar plane to its adjacent roots, the four roots will form the corners of a square.

Example 5.3.7.4

Solve the equation $x^3 - 64 = 0$. What shape do the roots make?

Solution

$$
\begin{aligned}
x^3 - 64 = 0 \rightarrow x^3 &= 64 + 0i \\
64 + 0i &= 64(cos(0 + 2\pi k) + isin(0 + 2\pi k)) \\
x = \left(x^3\right)^{1/3} &= (64 + 0i)^{1/3} \\
&= \sqrt[3]{64}\left(\cos\left(\frac{0+2\pi k}{3}\right) + i\sin\left(\frac{0+2\pi k}{3}\right)\right) \\
z_1 &= 4\left(\cos\left(\frac{0+2\pi 0}{3}\right) + i\sin\left(\frac{0+2\pi 0}{3}\right)\right) \\
&= 4\cos 0 + 4i\sin 0 \\
&= 4 \text{ for } k = 0 \\
z_2 &= 4\left(\cos\left(\frac{0+2\pi}{3}\right) + i\sin\left(\frac{0+2\pi}{3}\right)\right) \\
&= 4\cos\frac{2\pi}{3} + 4i\sin\frac{2\pi}{3} = -2 + 2i\sqrt{3} \text{ for } k = 1 \\
z_3 &= 4\left(\cos\left(\frac{0+4\pi}{3}\right) + i\sin\left(\frac{0+4\pi}{3}\right)\right) \\
&= 4\cos\frac{4\pi}{3} + 4i\sin\frac{4\pi}{3} \\
&= -2 - 2i\sqrt{3} \text{ for } k = 2
\end{aligned}
$$

If a line segment is drawn from each root on the polar plane to its adjacent roots, the three roots will form the vertices of an equilateral triangle.

Review

Solve each equation.

1. $x^3 = 1$
2. $x^5 = 1$
3. $x^8 = 1$
4. $x^5 = -32$
5. $x^4 + 5 = 86$
6. $x^5 = -1$
7. $x^4 = -1$
8. $x^3 = 8$
9. $x^6 = -64$

10. $x^3 = -64$
11. $x^5 = 243$
12. $x^3 = 343$
13. $x^7 = -128$
14. $x^{12} = 1$
15. $x^6 = 1$

Review (Answers)

To see the Review answers, open this PDF file and look for section 6.14.

Additional Resources

Video: DeMoivre's Theorem: Find All Roots, Real and Imaginary

Practice: Equations Using DeMoivre's Theorem

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5.3.8: Geometry of Complex Roots

Roots equally spaced around circles.

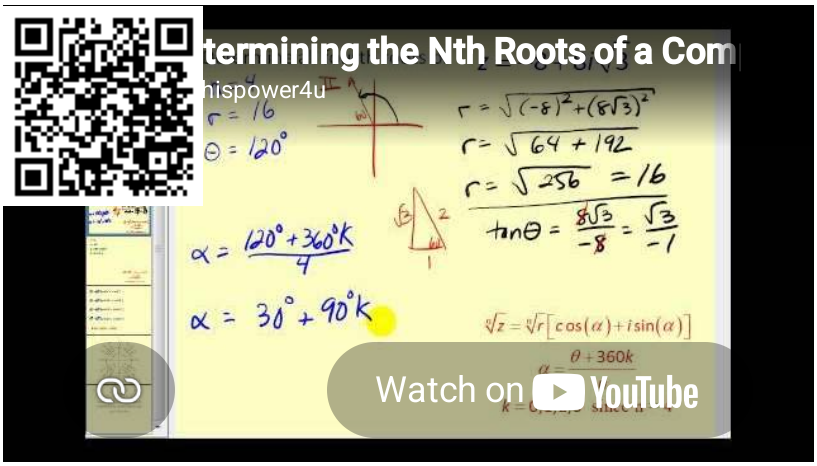
You've just finished a problem where you needed to solve the equation:

$x^4 = 16$

After solving for the roots, which were $2,\ -2,\ 2i,\ -2i$ your instructor asks you to plot them on the complex plane.

Can you accomplish this?

It's always good to get an intuitive feel for values by plotting them. This tendency extends into the complex numbers as well.



The five roots of the equation $x^5 - 32 = 0$ involve one real root and four complex ones. Let's take a look at a plot of these roots in the complex plane.

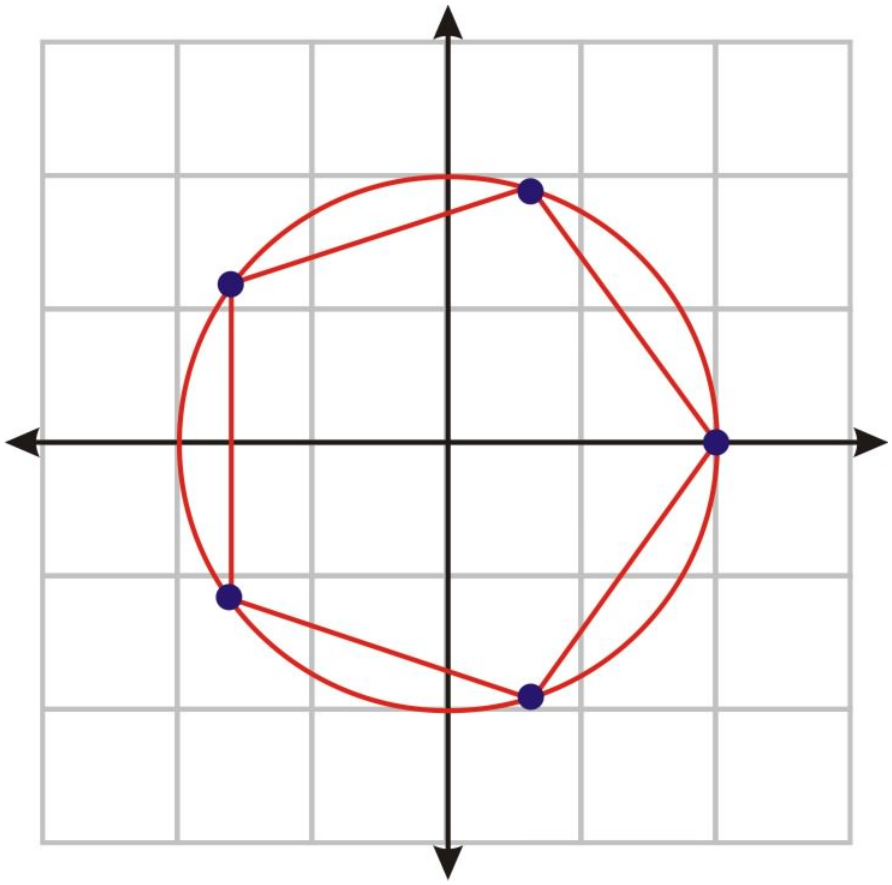


Figure 5.3.8.1

The nth roots of a complex number, when graphed on the complex plane, are equally spaced around a circle. So, instead of having all the roots, all that is necessary to graph the roots is one of them and the radius of the circle. For this particular example, the roots are $\frac{2\pi}{5}$ or $72°$ apart. This goes along with what we know about regular pentagons. The roots are $\frac{2\pi}{n}$ degrees apart.

Let's take a look at some problems involving the geometry of complex roots.

1. Calculate the two roots for $x^2 = 1$ and represent them graphically.

$$\begin{aligned} x^2 &= 1 \\ r &= \sqrt{x^2 + y^2} \\ r &= \sqrt{(1)^2 + (0)^2} \\ r &= 1 \\ \theta &= \tan^{-1}\left(\frac{0}{1}\right) = 0 \end{aligned}$$

Write an expression for determining the two roots of $1 = 1 + 0i$

$$\begin{aligned} 1^{1/2} &= [1(\cos(0 + 2\pi k) + i\sin(0 + 2\pi k)]^{1/2} \\ &= 1\left(\cos\frac{2\pi k}{2} + i\sin\frac{2\pi k}{2}\right) k = 0, 1 \\ x_1 &= 1\left(\cos\frac{0}{2} + i\sin\frac{0}{2}\right) \rightarrow 1(\cos 0 + i\sin 0) = 1 \quad \text{for } k = 0 \\ x_2 &= 1\left(\cos\frac{2\pi}{2} + i\sin\frac{2\pi}{2}\right) = -1 \quad \text{for } k = 1 \end{aligned}$$

These roots are plotted here:

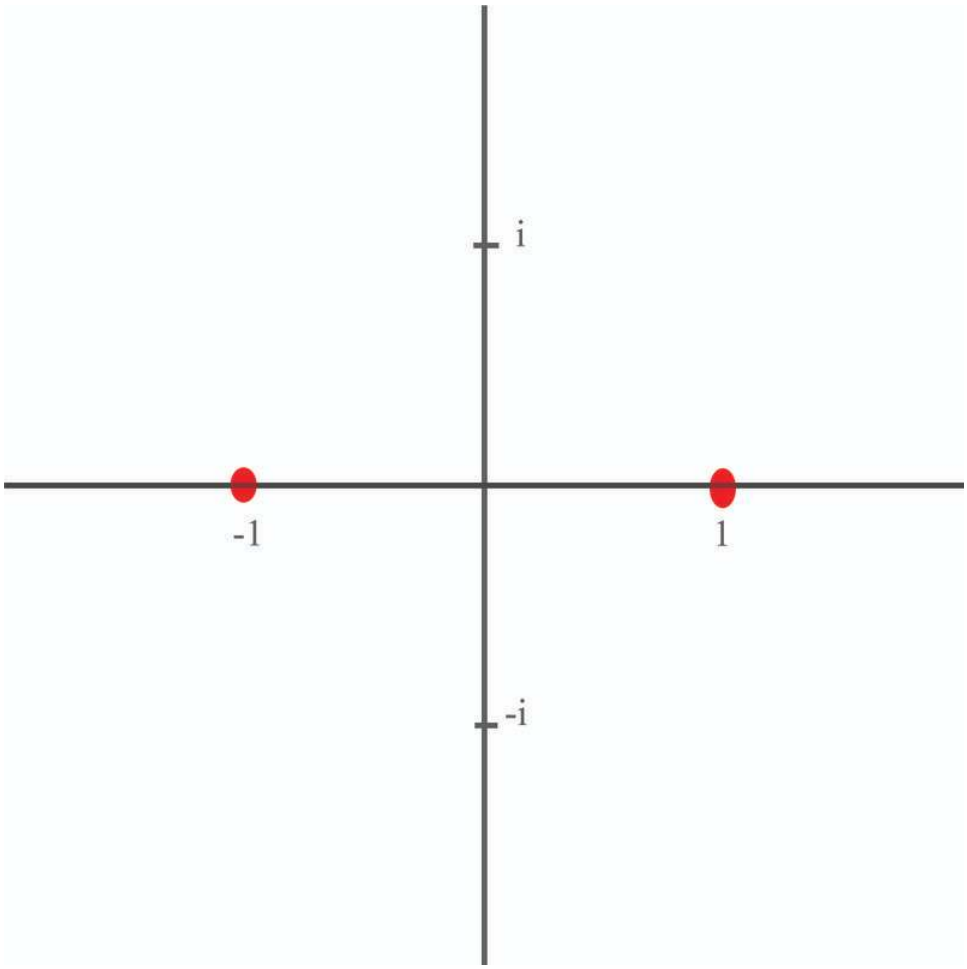


Figure 5.3.8.2

2. Calculate the three roots for $x^3 = 1$ and represent them graphically.

In standard form, $1 = 1 + 0ir = 1$ and $\theta = 0$. The polar form is $1 + 0i = 1[\cos(0 + 2\pi k) + i\sin(0 + 2\pi k)]$. The expression for determining the cube roots of 1+0i is:

$$(1 + 0i)^{1/3} = 1^{1/3}\left(\cos\frac{0 + 2\pi k}{3} + i\sin\frac{0 + 2\pi k}{3}\right)$$

When $k = 0$, $k = 1$ and $k = 2$ the three cube roots of 1 are 1, $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$, $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$. When these three roots are represented graphically, the three points, on the circle with a radius of 1 (the cubed root of 1 is 1), form a triangle.

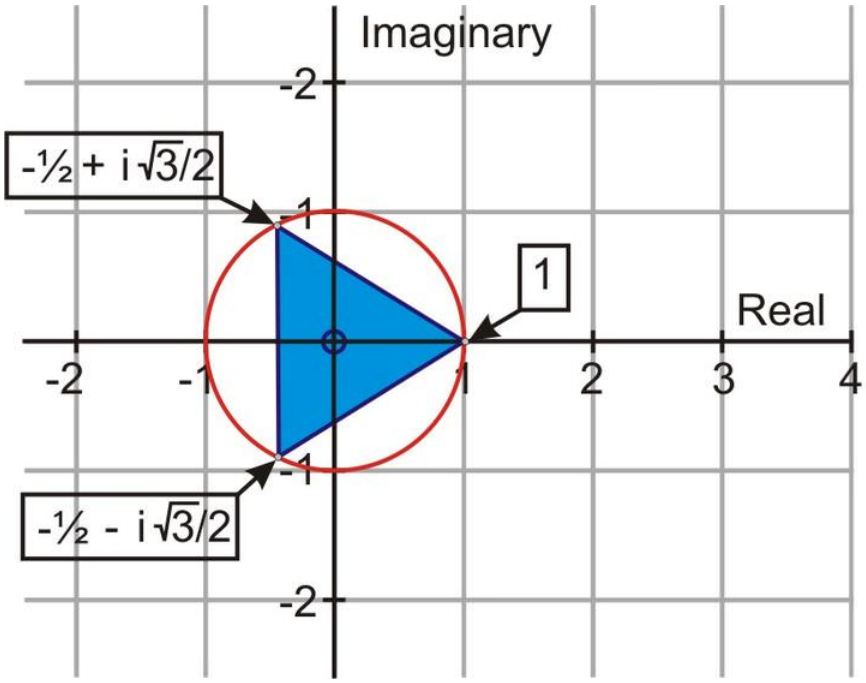


Figure **5.3.8.3**

3. Calculate the four roots for $x^4 = 1$ and represent them graphically

$$
\begin{aligned}
x^4 &= 1 \\
r &= \sqrt{x^2 + y^2} \\
r &= \sqrt{(1)^2 + (0)^2} \\
r &= 1 \\
\theta &= \tan^{-1}\left(\frac{0}{1}\right) = 0
\end{aligned}
$$

Write an expression for determining the cube roots of $1 = 1 + 0i$

$$
\begin{aligned}
1^{\frac{1}{4}} &= [1(\cos(0 + 2\pi k) + i\sin(0 + 2\pi k)]^{1/4} \\
&= 1\left(\cos\tfrac{2\pi k}{4} + i\sin\tfrac{2\pi k}{4}\right) k = 0, 1, 2, 3 \\
x_1 &= 1\left(\cos\tfrac{0}{4} + i\sin\tfrac{0}{4}\right) \rightarrow 3(\cos 0 + i\sin 0) = 1 && \text{for } k = 0 \\
x_2 &= 1\left(\cos\tfrac{2\pi}{4} + i\sin\tfrac{2\pi}{4}\right) = 0 + i = i && \text{for } k = 1 \\
x_3 &= 1\left(\cos\tfrac{4\pi}{4} + i\sin\tfrac{4\pi}{4}\right) = -1 - 0i = -1 && \text{for } k = 2 \\
x_4 &= 1\left(\cos\tfrac{6\pi}{4} + i\sin\tfrac{6\pi}{4}\right) = 0 - i = -i && \text{for } k = 3
\end{aligned}
$$

These roots are plotted here:

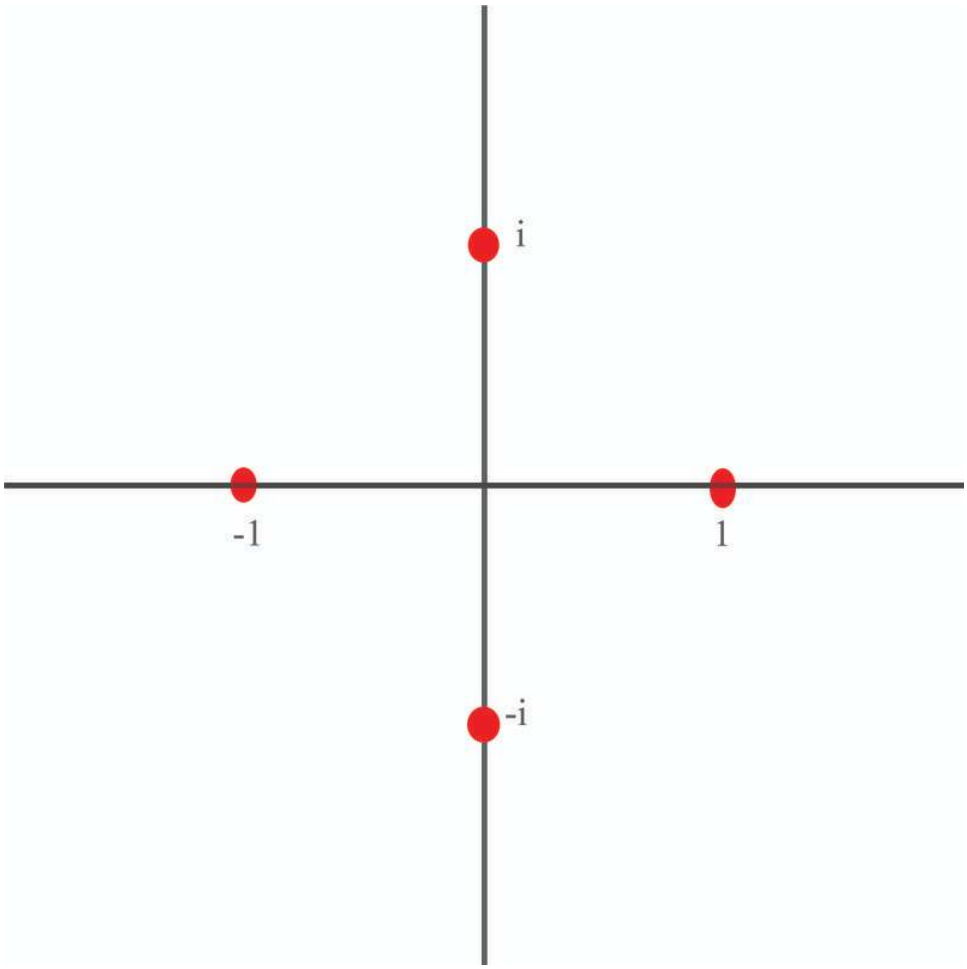


Figure 5.3.8.4

Example 5.3.8.1

Earlier, you were asked to plot $2, -2, 2i, -2i$ on a plane.

Solution

You can see a plot of these roots here. Notice, as mentioned before, that the roots are placed equidistant around a circle that could be drawn in the plane with a radius of two.

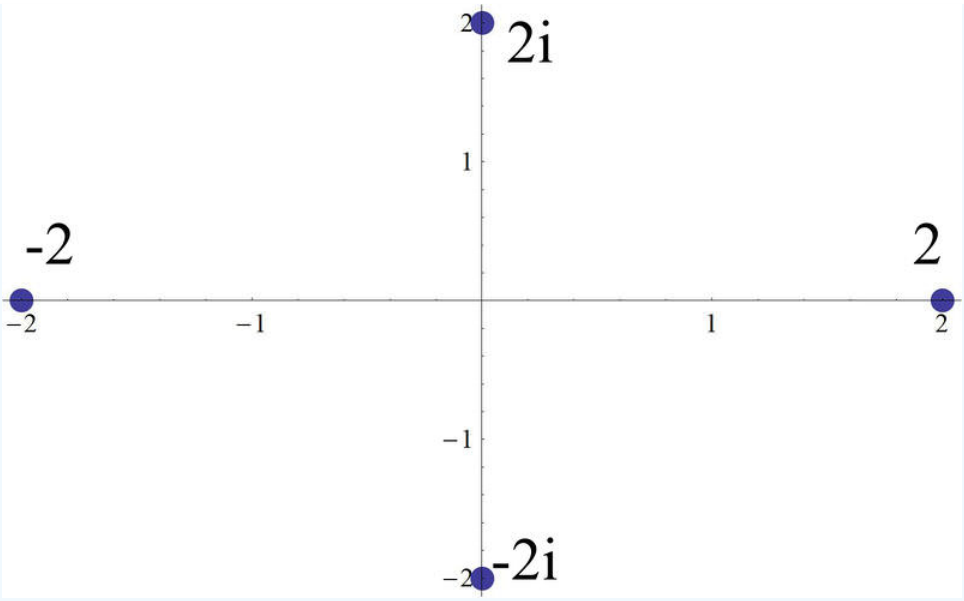


Figure 5.3.8.5

Example 5.3.8.2

In the examples above, you saw the complex roots determined for a number of different polynomial orders, such as x^2, x^3, x^4. What conclusion can you draw about the number of complex roots there are in relation to the order of the polynomial being solved?

Solution

You can conclude that the total number of roots is the same as the order of the polynomial under consideration. For example, x^2 will have 2 roots, while x^3 will have 3 roots, etc.

Example 5.3.8.3

What is the spacing in polar coordinates between the roots of the polynomial $x^6 = 12$?

Solution

Since there are six total roots, and all of the roots are equally spaced around a circle in the complex plane, there are $\frac{360^\circ}{6} = 60^\circ$ between roots.

Example 5.3.8.4

Solve for the roots of the equation $x^2 - 3x + 5 = 0$ and plot them.

Solution

This equation can be solved using the familiar quadratic formula. Notice that previously we used quadratics that gave real solutions. However, there is no reason to limit the solution set to the real numbers, now that you know how to utilize complex numbers as well.

$$\begin{aligned} x &= \frac{-(-3) \pm \sqrt{(-3)^2 - (4)(1)(5)}}{(2)(1)} \\ &= \frac{3 \pm \sqrt{-11}}{2} \\ &= \frac{3 \pm i\sqrt{11}}{2} \\ &= \frac{3}{2} \pm \frac{i\sqrt{11}}{2} \end{aligned}$$

A plot of these roots looks like this:

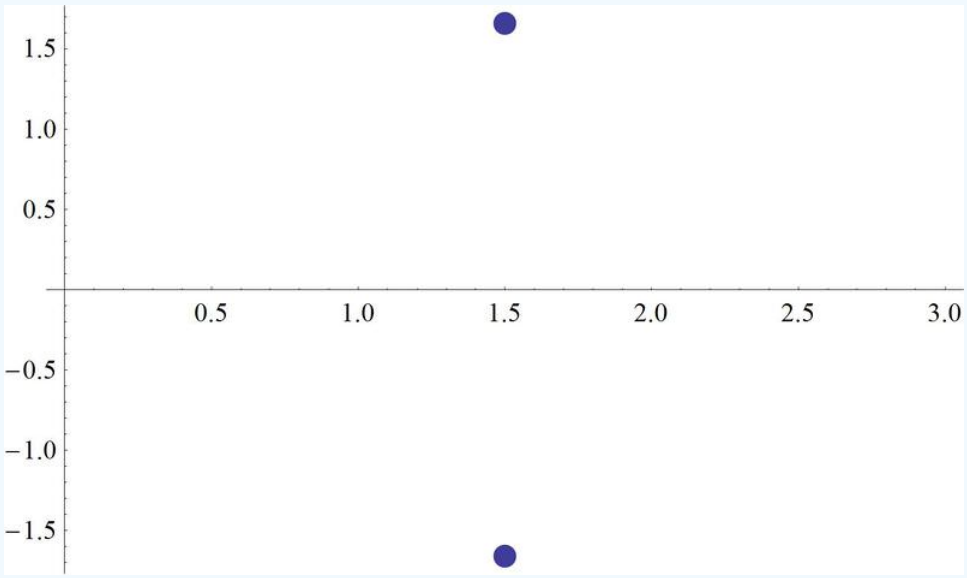


Figure **5.3.8.6**

where the vertical axis is the imaginary number line.

Review

1. How many roots does $x^5 = 1$ have?
2. Calculate the roots of $x^5 = 1$ and represent them graphically.
3. How many roots does $x^8 = 1$ have?
4. Calculate the roots of $x^8 = 1$ and represent them graphically.
5. How many roots does $x^{10} = 1$ have?
6. Calculate the roots of $x^{10} = 1$ and represent them graphically.
7. How many roots does $x^4 = 16$ have?
8. Calculate the roots of $x^4 = 16$ and represent them graphically.
9. How many roots does $x^3 = 27$ have?
10. Calculate the roots of $x^3 = 27$ and represent them graphically.
11. How do the solutions of the equation $x^3 = -1$ compare to the solutions of the equation $x^3 = 1$?
12. Describe how to represent the roots of $x^6 = 1$ graphically without first solving the equation.
13. Describe how to represent the roots of $x^{12} = 1$ graphically without first solving the equation.
14. Describe how to represent the roots of $x^4 = 81$ graphically without first solving the equation.
15. Describe how to represent the roots of $x^8 = 256$ graphically without first solving the equation.

Review (Answers)

To see the Review answers, open this PDF file and look for section 6.15.

Additional Resources

Video: Determining the Nth Roots of a Complex Number

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5.3.9: Polar Form of a Complex Number

Conversion of $a + bi$ to (a, b), $(r, theta)$, and $r\ cis\ theta$.

Complex numbers can be graphed on a polar graph just like real numbers can. You will discover during this lesson that there are actually a few different ways of doing this.

You have learned that rectangular graphs can be put into polar form, and that points in rectangular coordinates can be plotted in the **polar coordinate system**. In this section you will learn how to do the same process with complex numbers.

There are three common forms of complex numbers that you will see when graphing:

1. In the standard form of: $z = a + bi$, a **complex number** *z* can be graphed using rectangular coordinates (a, b). 'a' represents the *x* - coordinate, while 'b' represents the y- coordinate.
2. The polar form: (r, θ) which we explored in a previous lesson, can also be used to graph a complex number. Recall that you can use *x* and *y* to convert between rectangular and polar forms with: $r = \sqrt{x^2 + y^2}$ and $\tan\theta_{ref} = \left|\frac{y}{x}\right|$. Unfortunately, there is a problem with using a conversion from rectangular form to polar form like:

$a + bi \rightarrow (r, \theta)$

or

$-1 - i\sqrt{3} \rightarrow \left(2, \frac{4\pi}{3}\right)$

The problem is that we have lost the i. So, in order to "keep track" of the imaginary part, we can use another form.

3. The third form is **trigonometric form**. It is often abbreviated as $r\ cis\ \theta$, short for: \(z = **r**(\cos \theta + **i**\sin \theta)\), and will be used quite often as you progress. This form comes from the substitutions: $x = r\cos\theta$ and $y = r\sin\theta$.

using this fact, and sample values of 2 for r and $\frac{\pi}{3}$ for θ, we can write

$z = -1 - i\sqrt{3} = 2\cos\frac{4\pi}{3} + 2i\sin\frac{4\pi}{3}$

Finally, factoring the 2, we get: $z = 2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

Summary of Forms

The complex number: $z = -1 - \sqrt{3}i$, the rectangular point $(-1, -\sqrt{3})$, the polar point: $\left(2, \frac{4\pi}{3}\right)$, and $2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$ or $2\ cis\left(\frac{4\pi}{3}\right)$ all represent the same number.

Steps for Conversion

To convert from polar to rectangular form, the distance that the point (2, 2) is from the origin can be found by

$d = \sqrt{x^2 + y^2}$ or $\sqrt{2^2 + 2^2}$ $\quad$ $d = \sqrt{8}$ or $2\sqrt{2}$

The **reference angle** (i.e. the corresponding angle in the first **quadrant**) that the line segment between the point and the origin can be found by

$\tan\theta_{ref} = \left|\frac{y}{x}\right|$

for $z = 2 + 2i$,

$$\tan\theta_{ref} = 22$$
$$\tan\theta_{ref} = 1$$

Since this point is in the first quadrant (both the *x* and *y* coordinate are positive) the angle must be $45°$ or $\frac{\pi}{4}$ radians.

It is also possible that when $\tan\theta = 1$ the angle can be in the third quadrant or 5\pi 4 radians. But this angle will not satisfy the conditions of the problem, since a third quadrant angle must have both x and y as negatives.

Note: When using $\tan\theta = \frac{y}{x}$, you should first consider, the quotient $|yx|$ and find the first quadrant angle that satisfies this condition. This angle will be called the **reference angle,** denoted θ_{ref}. Find the actual angle by analyzing which quadrant the angle

must be given the *x* and *y* signs.

The complex number $2+2i$ or (2, 2) in rectangular form has polar coordinates $\left(2\sqrt{2}, \frac{\pi}{4}\right)$

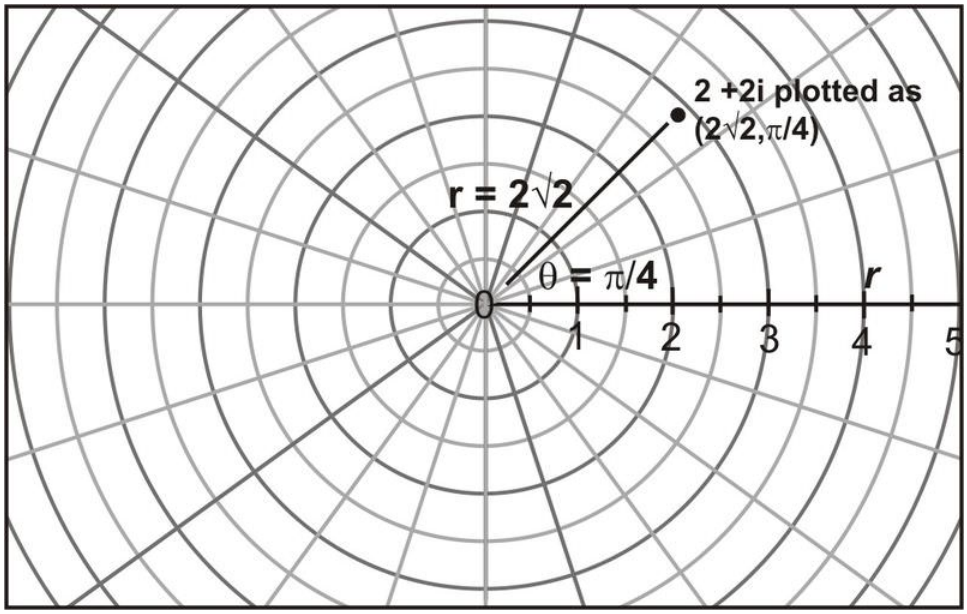
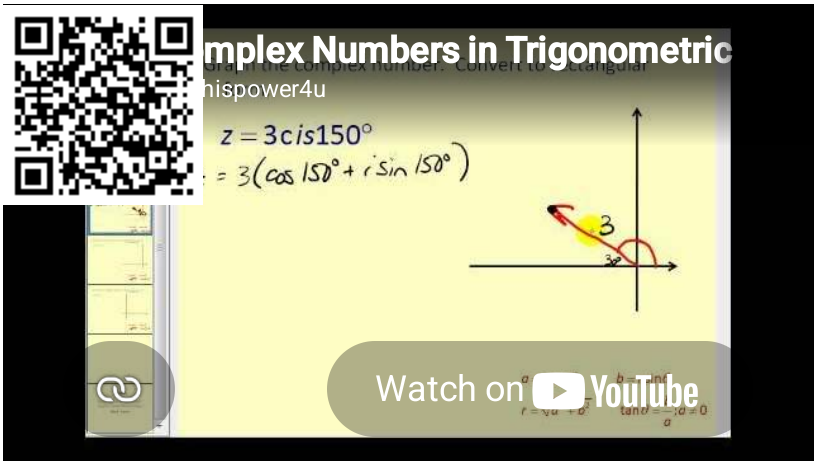


Figure 5.3.9.1



Example 5.3.9.1

Graph in polar form: $z=-1-i\sqrt{3}$.

Solution

Here is what it looks like in the rectangular coordinate system:

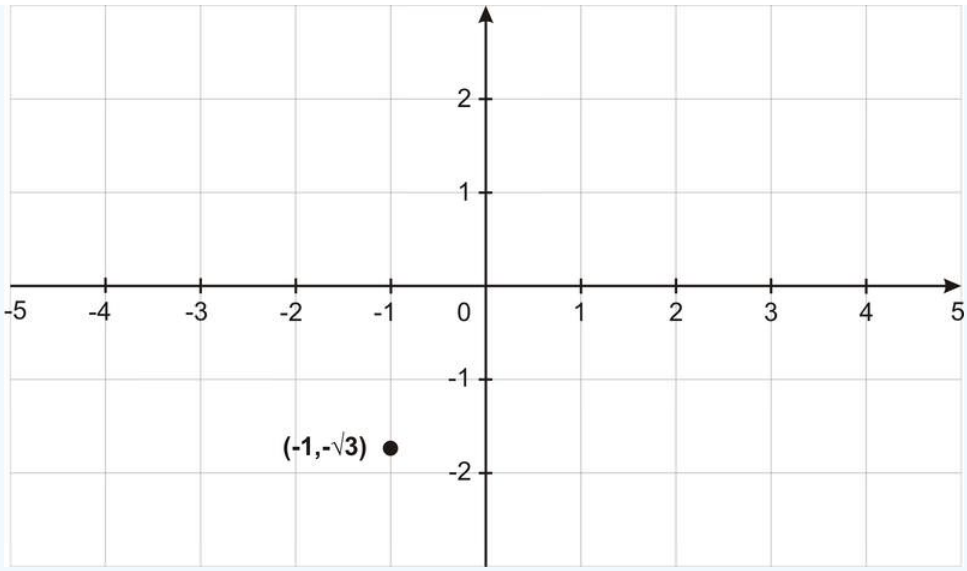


Figure 5.3.9.2

In polar form, we find r with

$$\begin{aligned} r &= \sqrt{a^2+b^2} \\ &= \sqrt{(-1)^2+(-\sqrt{3})^2} \\ &= \sqrt{1+3} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

and to find θ,

$$\begin{aligned} \tan\theta_{ref} &= \left|\frac{-\sqrt{3}}{-1}\right| \\ \tan\theta_{ref} &= \sqrt{3} \\ \theta_{ref} &= tan-1\sqrt{3} \\ \theta_{ref} &= \frac{\pi}{3} \end{aligned}$$

Since this angle is in the 4th quadrant, $\theta = \frac{4\pi}{3}$.

Example 5.3.9.2

Find the polar coordinates that represent the complex number $z = 3 - 3\sqrt{3}i$.

Solution

$a = 3$ and $b = -3\sqrt{3}$: the rectangular coordinates of the point are $(3, -3\sqrt{3})$.

Now, draw a right triangle in standard form. Find the distance the point is from the origin and the angle the line segment that represents this distance makes with the +x axis:

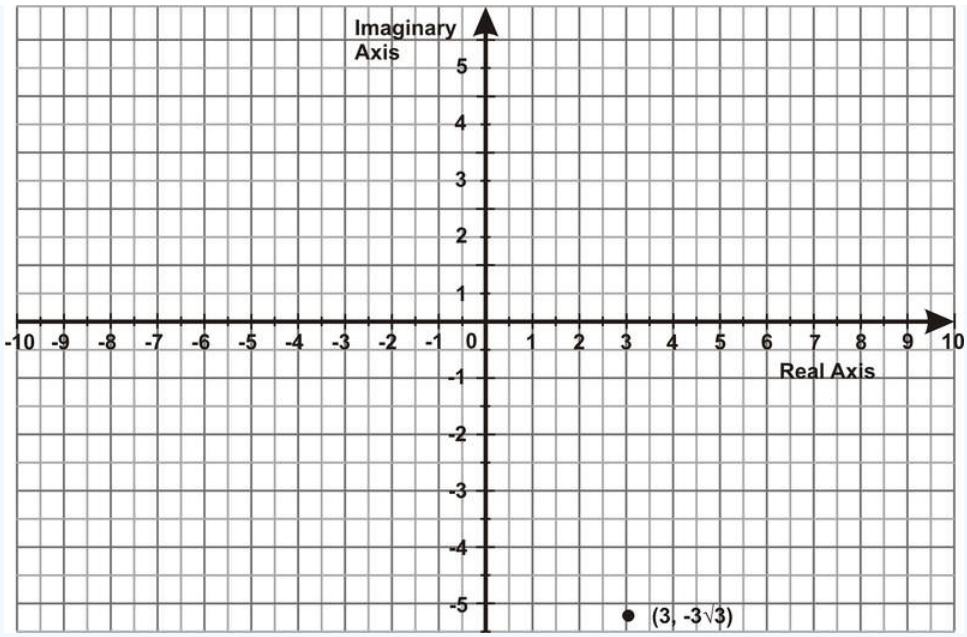


Figure 5.3.9.3

We know $a = 3,\ b = -3\sqrt{3}$

$$\begin{aligned} r &= \sqrt{3^2 + (-3\sqrt{3})^2} \\ &= \sqrt{9 + 27} \\ &= \sqrt{36} \\ &= 6 \end{aligned}$$

And for the angle,

$$\tan\theta_{ref} = \left|\frac{(-3\sqrt{3})}{3}\right|$$

$$\tan\theta_{ref} = \sqrt{3}$$

$$\theta_{ref} = \frac{\pi}{3}$$

But, since it is a 4th quadrant angle

$\theta = \frac{5\pi}{3}$

The rectangular point $(3, -3\sqrt{3}i)$ is equivalent to the polar point $\left(6, 5\frac{\pi}{3}\right)$.

In $r\ cis\ \theta$ form, $(3, -3\sqrt{3}i)$ is $6\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$.

Example 5.3.9.3

Convert the following complex numbers into polar form, use a TI-84 equivalent graphing calculator:

a. $\sqrt{3} - i$
b. $9\sqrt{3} + 9i$

Solution

On the TI-84: go to **[ANGLE]** (or **[2nd]** function) **[APPS]**. Scroll down to 5 or "R-Pr(" and press **[Enter]** . Next, enter the rectangular coordinates and close the parenthesis. Press **[Enter]**, the "r" value appears. Scroll down to 6R-P\theta , and the polar angle appears in decimal radian form.

Note: Also under the **[ANGLE]** menu, commands 7 and 8 allow transformation from polar form to rectangular form.

Example 5.3.9.4

Plot the complex number $z = 12 + 9i$.

Solution

a. What is needed in order to plot this point on the polar plane?

First, we will need to know r and θ.

b. How could the r-value be determined?

The r value is the hypotenuse of a triangle with two other sides, $A = 12$ and $B = 9$. It can be determined with the Pythagorean theorem: $A^2 + B^2 = C^2$.

c. What is the r for this point?

The r value for this point is $\sqrt{144 + 81} \rightarrow \sqrt{225} = 15$.

d. How could θ be determined?

\theta can be calculated using either $\sin\theta = \frac{9}{15}$ or $\cos\theta = \frac{12}{15}$.

e. What is θ for this point?

For this point, $\sin\theta = \frac{3}{5} \rightarrow 37°$ or $\cos\theta = 45 \rightarrow 37°$.

f. What would $z = 12 + 9i$ look like on the polar plane?

$z = 12 + 9i$ looks like the image below when plotted on a polar plane.

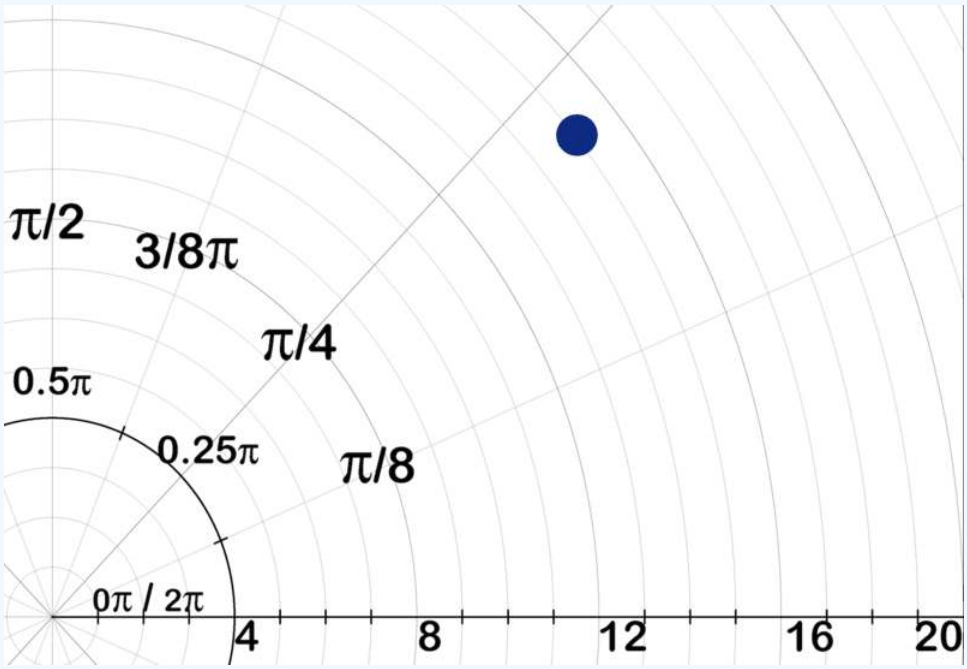


Figure 5.3.9.4

Example 5.3.9.5

What quadrant does $z = -3 + 2i$ occur in when graphed?

Solution

The point $z = -3 + 2i$ occurs 3 units to the *left* and 2 units *up*, placing it in Quadrant II.

Example 5.3.9.6

What are the coordinates of $z = -3 + 2i$ in polar form and trigonometric form?

Solution

To identify the coordinates of $z = -3 + 2i$ in polar form and trigonometric form:

$$r = \sqrt{(-3^2) + (2^2)} \rightarrow \sqrt{13} \quad \text{First find } r$$
$$\sin\theta = 2\sqrt{13} \rightarrow 33.7^\circ \quad \text{Second, find } \theta$$
$$\therefore [\sqrt{13}, 33.7^\circ] \quad \text{are the coordinates in polar form.}$$
$$\therefore r\ cis\ \sqrt{13}\left(\frac{\pi}{5}\right) \quad \text{are the coordinates in } r\ cis \text{ form}$$

Example 5.3.9.7

What would be the polar coordinates of the point graphed below?

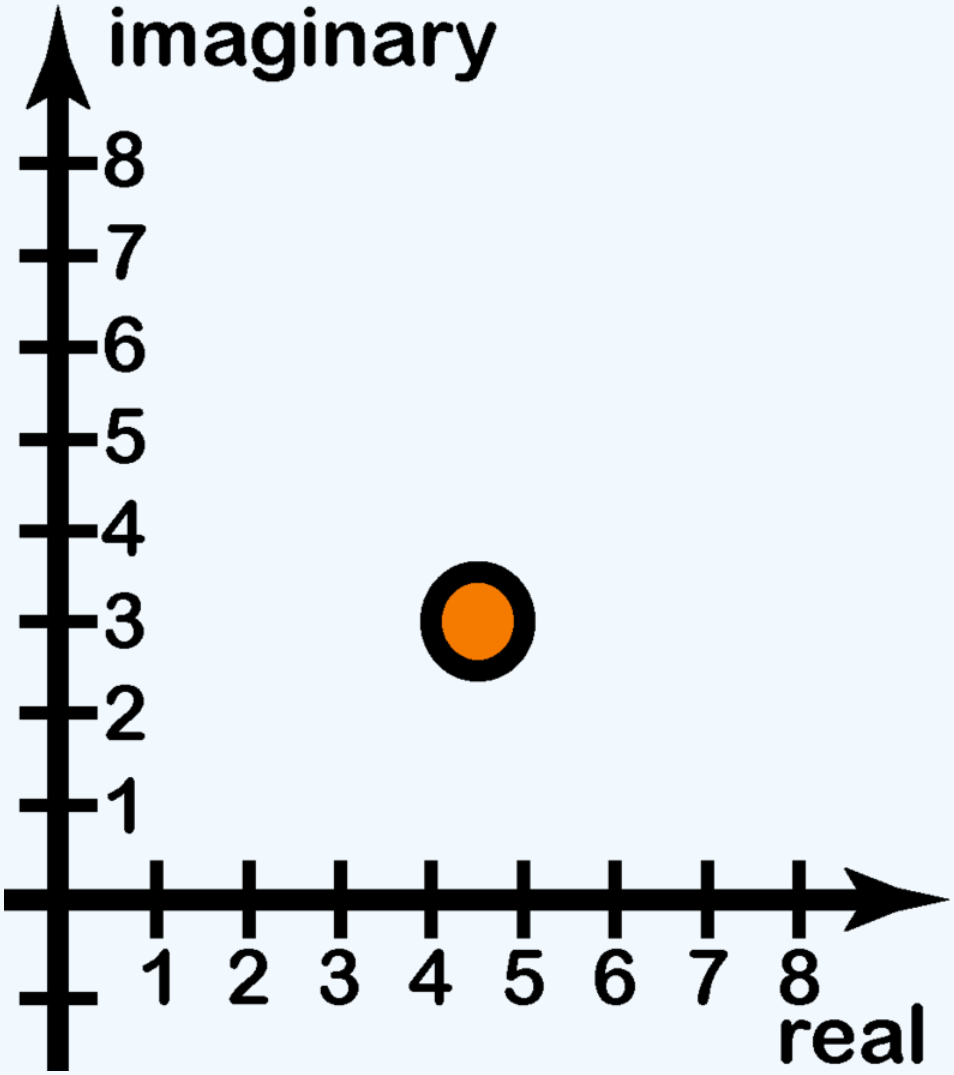


Figure 5.3.9.5

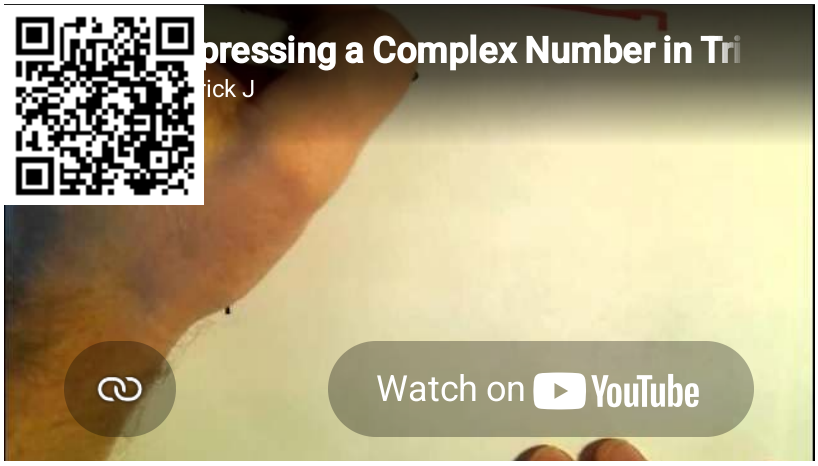
Solution

The rectangular coordinates are $(4.5, 3i)$ therefore the complex number would be $z = 4.5 + 3i$

$r = 5.4$ using the Pythagorean Theorem as in Q #3

$\theta = 33.75°$ using $\sin = \frac{\text{opp}}{\text{hyp}}$ as in Q #3

$\therefore [5.4, 33.65°]$ is the point in polar form $\therefore r\ cis\ 5.4\left(\frac{\pi}{5}\right)$ are the coordinates in $r\ cis$ form



Review

Plot each complex number in the complex plane. Find its polar form, [r,\theta] and give the argument \theta in degrees.

1. a) $1+i$ b) i c) $(1+i)i$
2. a) -2 b) $3i$ c) $(-2)(3i)$
3. a) $1+i$ b) $1-i$ c) $(1+i)(1-i)$
4. a) $1+i\sqrt{3}$ b) $\sqrt{3}-ic)(1+i\sqrt{3})(\sqrt{3}-i)$
5. What are the rectangular coordinates for the point graphed below.

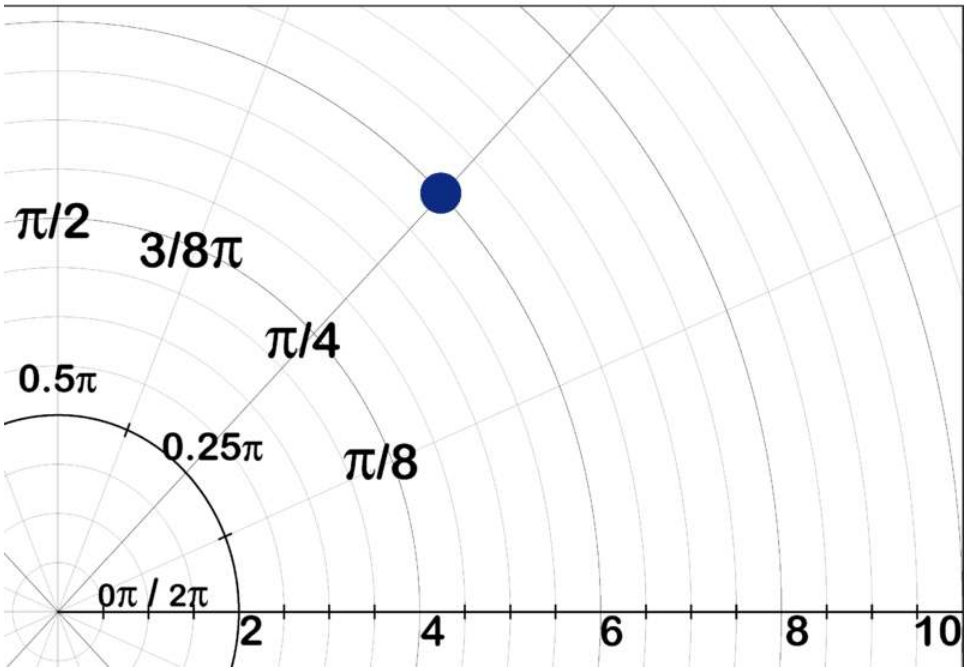


Figure 5.3.9.6

Compute and convert to $r\ cis$ form.

6. $\frac{-2-2i}{1-i}$
7. $1+i^6$

8. $\frac{\sqrt{3}}{2}+\frac{1}{2}i^{10}$

Change to polar form.

9. $-3-2i$
10. $2\sqrt{3}-2i$

Change to rectangular form.

11. $15(\cos 120^\circ + i\sin 120^\circ)$
12. $12\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$
13. For the complex number in standard form $x+iy$ find: a) Polar form b) Trigonometric form (Hint: Recall that $x=r\cos\theta$ and $y=r\sin\theta$)

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.8.

Vocabulary

Term	Definition
$r\ cis\ \theta$	$r\ cis\ \theta$ is shorthand for the expression $r\cos\theta+ri\sin\theta$.
complex number	A complex number is the sum of a real number and an imaginary number, written in the form a+bi.
polar coordinate system	The polar coordinate system is a special coordinate system in which the location of each point is determined by its distance from the pole and its angle with respect to the polar axis.
polar form	The polar form of a point or a curve is given in terms of r and θ and is graphed on the polar plane.
quadrant	A quadrant is one-fourth of the coordinate plane. The four quadrants are numbered using Roman Numerals I, II, III, and IV, starting in the top-right, and increasing counter-clockwise.
Reference Angle	A reference angle is the angle formed between the terminal side of the angle and the closest of either the positive or negative x-axis.
trigonometric form	To write a complex number in trigonometric form means to write it in the form $r\cos\theta+ri\sin\theta$. $r\ cis\ \theta$ is shorthand for this expression.
trigonometric polar form	To write a complex number in trigonometric form means to write it in the form $r\cos\theta+ri\sin\theta$. $r\ cis\ \theta$ is shorthand for this expression.

Additional Resources

Practice: Polar Form of a Complex Number

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5.3.10: DeMoivre's Theorem and nth Roots

Raise complex numbers to powers or find their roots.

You know how to multiply two complex numbers together and you've seen the advantages of using **trigonometric polar form**, especially when multiplying more than two complex numbers at the same time. Because raising a number to a whole number power is repeated multiplication, you also know how to raise a **complex number** to a whole number power.

What is a geometric interpretation of squaring a complex number?

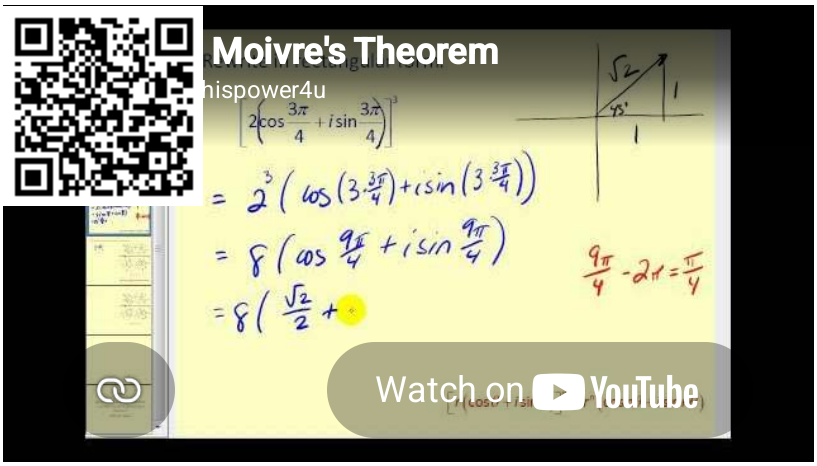
Recall that if $z_1 = r_1 \cdot \ cis\ \theta_1$ and $z_2 = r_2 \cdot \ cis\ \theta_2$ with $r_2 \neq 0$, then $z_1 \cdot z_2 = r_1 \cdot r_2 \cdot \ cis\ (\theta 1 + \theta 2)$.

If $z_1 = z_2 = z = r\ cis\ \theta$ then you can determine z_2 and z_3:

$$z^2 = r \cdot r \cdot \ cis\ (\theta + \theta) = r^2\ cis\ (2 \cdot \theta)$$
$$z^3 = r^3\ cis\ (3 \cdot \theta)$$

De Moivre's Theorem simply generalizes this pattern to the power of any positive integer.

$$z^n = r^n \cdot \ cis\ (n \cdot \theta)$$



In addition to raising a complex number to a power, you can also take square roots, cube roots and n^{th} roots of complex numbers. Suppose you have complex number $z = r\ cis\ \theta$ and you want to take the nth root of z. In other words, you want to find a number $v = s \cdot \ cis\ \beta$ such that $v^n = z$. Do some substitution and manipulation:

$$v^n = z$$
$$(s \cdot \ cis\ \beta)^n = r \cdot \ cis\ \theta$$
$$s^n \cdot \ cis\ (n \cdot \beta) = r \cdot \ cis\ \theta$$

You can see at this point that to find s you need to take the nth root of r. The trickier part is to find the angles, because $n \cdot \beta$ could be any angle coterminal with θ. This means that there are n different n^{th} roots of z.

$$n \cdot \beta = \theta + 2\pi k$$
$$\beta = \frac{\theta + 2\pi k}{n}$$

The number k can be all of the counting numbers including zeros up to $n-1$. So if you are taking the 4th root, then $k = 0, 1, 2, 3$.

Thus the nth root of a complex number requires n different calculations, one for each root:

$v = \sqrt{n}r \cdot \ cis\ \left(\frac{\theta + 2\pi k}{n}\right)$ for $k \in I | 0 \leq k \leq n-1$

To apply this formula, find the cube root of the number 8. Most students know that 23=8 and so know that 2 is the cube root of 8. However, they don't realize that there are two other cube roots that they are mis\sin g. Remember to write out $k = 0, 1, 2$ and use the unit circle whenever possible to help you to find all three cube roots.

$$
\begin{aligned}
8 &= 8\,\text{cis}\,0 = (s \cdot \text{cis}\,\beta)^3 \\
z_1 &= 2 \cdot \text{cis}\left(\frac{0+2\pi \cdot 0}{3}\right) = 2\,\text{cis}\,0 = 2(\cos 0 + i \cdot \sin 0) = 2(1+0) = 2 \\
z_2 &= 2 \cdot \text{cis}\left(\frac{0+2\pi \cdot 1}{3}\right) = 2\,\text{cis}\left(\frac{2\pi}{3}\right) \\
&= 2\left(\cos\left(\frac{2\pi}{3}\right) + i \cdot \sin\left(\frac{2\pi}{3}\right)\right) = 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -1 + i\sqrt{3} \\
z_3 &= 2 \cdot \text{cis}\left(\frac{0+2\pi \cdot 2}{3}\right) = 2\,\text{cis}\left(\frac{4\pi}{3}\right) \\
&= 2\left(\cos\left(\frac{4\pi}{3}\right) + i \cdot \sin\left(\frac{4\pi}{3}\right)\right) = 2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -1 - i\sqrt{3}
\end{aligned}
$$

The cube roots of 8 are 2, $-1 + i\sqrt{3}$, $-1 - i\sqrt{3}$.

To check, that they are the cube roots, cube them all simplify.

$$
\begin{aligned}
z_1^3 &= 2^3 = 8 \\
z_2^3 &= (-1 + i\sqrt{3})^3 \\
&= (-1 + i\sqrt{3}) \cdot (-1 + i\sqrt{3}) \cdot (-1 + i\sqrt{3}) \\
&= (1 - 2i\sqrt{3} - 3) \cdot (-1 + i\sqrt{3}) \\
&= (-2 - 2i\sqrt{3}) \cdot (-1 + i\sqrt{3}) \\
&= 2 - 2i\sqrt{3} + 2i\sqrt{3} + 6 \\
&= 8
\end{aligned}
$$

Note how many steps and opportunities there are for making a mistake when multiplying multiple terms in rectangular form. When you check z3, use trigonometric polar form.

$$
\begin{aligned}
z_3^3 &= 2^3\ cis\ \left(3 \cdot \frac{4\pi}{3}\right) \\
&= 8(\cos 4\pi + i \cdot \sin 4\pi) \\
&= 8(1+0) \\
&= 8
\end{aligned}
$$

Example 5.3.10.1

Earlier, you were asked what a geometric interpretation of squaring a complex number is. Squaring a complex number produces a new complex number. The angle gets doubled and the magnitude gets squared, so geometrically you see a rotation.

Solution

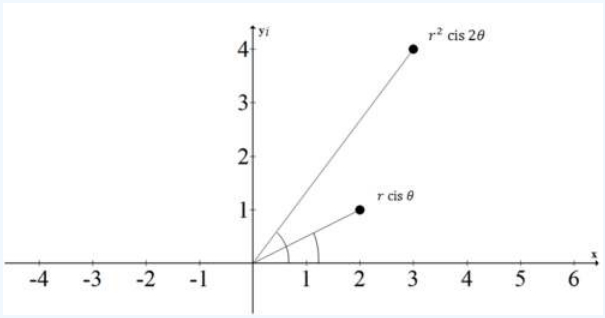


Figure 5.3.10.1

Example 5.3.10.2

Plot the roots of 8 graphically and discuss any patterns you notice.

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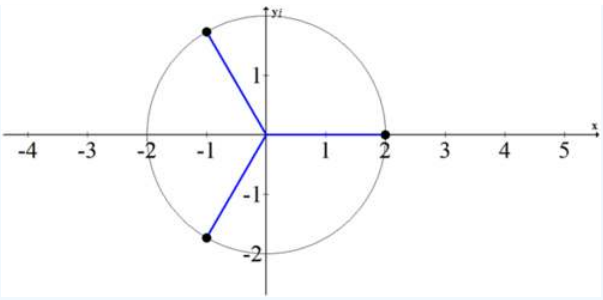


Figure 5.3.10.2

Solution

The three points are equally spaced around a circle of radius 2. Only one of the points, $2+0i$, is made up of only real numbers. The other two points have both a real and an imaginary component which is why they are off of the x axis.

As you become more comfortable with roots, you can just determine the number of points that need to be evenly spaced around a certain radius circle and find the first point. The rest is just logic.

Example 5.3.10.3

What are the fourth roots of $16\ cis\ 48°$?

Solution

There will be 4 points, each $90°$ apart with the first point at $2\ cis\ (12°)$.

$2\ cis\ (12°)$, $2\ cis\ (102°)$, $2\ cis\ (192°)$, $2\ cis\ (282°)$

Example 5.3.10.4

Solve for z by finding the nth root of the complex number.

$z^3 = 64 - 64\sqrt{3}i$

Solution

First write the complex number in cis form. Remember to identify $k = 0, 1, 2$. This means the roots will appear every $\sqrt{360°}3 = 120°$.

$$z^3 = 64 - 64\sqrt{3}i = 128 \cdot cis\ 300°$$
$$z_1 = 128^{\frac{1}{3}} \cdot cis\ \left(\frac{300}{3}\right)° = 128^{\frac{1}{3}} \cdot cis\ \left(\frac{300}{3}\right)°$$
$$z_2 = 128^{\frac{1}{3}} \cdot cis\ (220°)$$
$$z_3 = 128^{\frac{1}{3}} \cdot cis\ (340°)$$

Example 5.3.10.5

Use De Moivre's Theorem to evaluate the following power.

$(\sqrt{2} - \sqrt{2}i)^6$

Solution

First write the number in trigonometric polar form, then apply De Moivre's Theorem and simplify.

$$
\begin{aligned}
(\sqrt{2}-\sqrt{2}i)^6 &= (2\ cis\ 315^\circ)^6 \\
&= 2^6 \cdot\ cis\ (6 \cdot 315^\circ) \\
&= 64 \cdot\ cis\ (1890^\circ) \\
&= 64 \cdot\ cis\ (1890^\circ) \\
&= 64 \cdot\ cis\ (90^\circ) \\
&= 64(\cos 90^\circ + i \cdot \sin 90^\circ) \\
&= 64(0+i) \\
&= 64i
\end{aligned}
$$

Review

Use De Moivre's Theorem to evaluate each expression. Write your answers in rectangular form.

1. $(1+i)^5$
2. $(1-\sqrt{3}i)^3$
3. $(1+2i)^6$
4. $(\sqrt{3}-i)^5$
5. $\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^4$
6. Find the cube roots of $3+4i$.
7. Find the 5th roots of $32i$.
8. Find the 5th roots of $1+\sqrt{5}i$.
9. Find the 6th roots of - 64 and plot them on the **complex plane**.
10. Use your answers to #9 to help you solve $x^6+64=0$.

For each equation: a) state the number of roots, b) calculate the roots, and c) represent the roots graphically.

11. $x^3=1$
12. $x^8=1$
13. $x^{12}=1$
14. $x^4=16$
15. $x^3=27$

Review (Answers)

To see the Review answers, open this PDF file and look for section 11.4.

Vocabulary

Term	Definition
n^{th} roots of unity	The nth roots of unity are the nth roots of the number 1.
complex number	A complex number is the sum of a real number and an imaginary number, written in the form $a+bi$.
complex plane	The complex plane is the graphical representation of the set of all complex numbers.
De Moivre's Theorem	De Moivre's theorem is the only practical manual method for identifying the powers or roots of complex numbers. The theorem states that if $z=r($ $cos\theta+i$ $sin\theta)$ is a complex number in $r\ cis\ \theta$ form and n is a positive integer, then $z^n=r^n(\cos(n\theta)+i\sin(n\theta))$.
trigonometric polar form	To write a complex number in trigonometric form means to write it in the form $r\cos\theta+ri\sin\theta$. $r\ cis\ \theta$ is shorthand for this expression.

Additional Resources

Video: Determining the nth Roots of a Complex Number

Practice: DeMoivre's Theorem and n^{th} Roots

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5.3.11: DeMoivre's Theorem

Simplify the calculation of powers of complex numbers.

Powers and Roots of Complex Numbers

Manually calculating (simplifying) a statement such as: $(14-17i)^5$ or $\sqrt[4]{(3-2i)}$ in present (rectangular) form would be a very intensive process at best.

Fortunately you will learn in this lesson that there is an alternative: **De Moivre's theorem**. De Moivre's theorem is really the only practical method for finding the powers or roots of a **complex number**, but there is a catch...

What must be done to a complex number before De Moivre's theorem can be utilized?

Powers and Roots of Complex Numbers

Powers of Complex Numbers

How do we raise a complex number to a power? Let's start with an example:

$(-4-4i)^3 = (-4-4i)\cdot(-4-4i)\cdot(-4-4i)$

In rectangular form, this can get very complex. What about in $r\ cis\ \theta$ form?

$(-4-4i) = 4\sqrt{2}\ cis\ \left(\frac{5\pi}{4}\right)$

So the problem becomes

$4\sqrt{2}\ cis\ \left(\frac{5\pi}{4}\right)\cdot 4\sqrt{2}\ cis\ \left(\frac{5\pi}{4}\right)\cdot 4\sqrt{2}\ cis\ \left(\frac{5\pi}{4}\right)$

and using our multiplication rule from the previous section,

$(-4-4i)^3 = (4\sqrt{2})^3\ cis\ (\frac{15\pi}{4})$

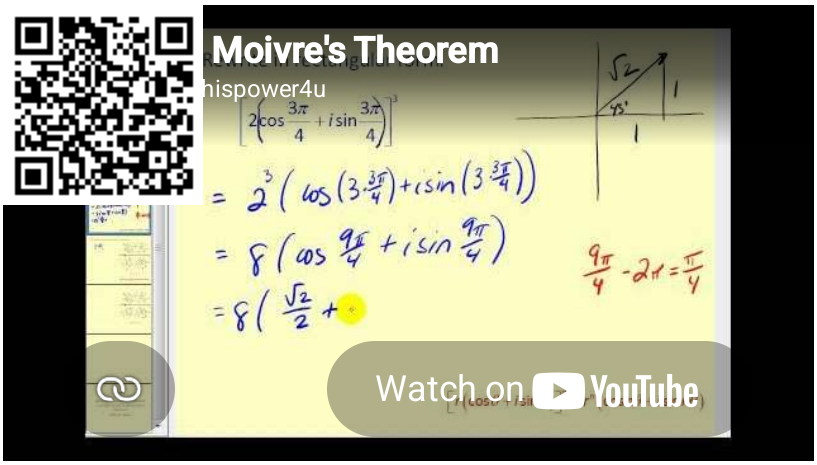
Notice, $(a+bi)^3 = r^3\ cis\ 3\theta$

In words: Raise the *r*-value to the same degree as the complex number is raised and then multiply that by cis of the angle multiplied by the number of the degree.

Reflecting on the example above, we can identify **De Moivre's Theorem**:

Let $z = r(\cos\theta + i\sin\theta)$ be a complex number in $r\ cis\ \theta$ form. If n is a positive integer, z^n is $z^n = r(\cos(n\theta) + i\sin(n\theta))$

It should be clear that the polar form provides a much faster result for raising a complex number to a power than doing the problem in rectangular form.



Roots of Complex Numbers

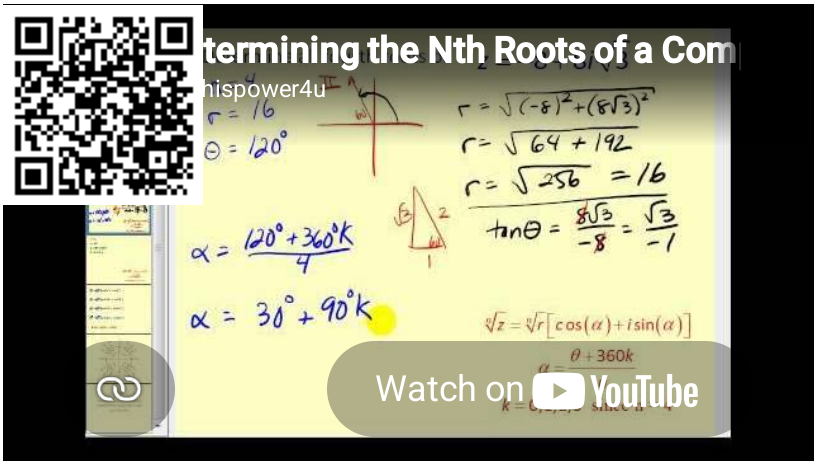
You probably noticed long ago that when an new operation is presented in mathematics, the inverse operation often follows. That is generally because the inverse operation is often procedurally similar, and it makes good sense to learn both at the same time.

This is no exception:

The inverse operation of finding a power for a number is to find a root of the same number.

1. Recall from algebra that any root can be written as $x^{1/n}$
2. Given that the formula for De Moivre's theorem also works for fractional powers, the same formula can be used for finding roots:

$$z^{1/n} = (a+bi)^{1/n} = r^{1/n}\ cis\ \left(\frac{\theta}{n}\right)$$



Example 5.3.11.1

Earlier, you were asked what should be done to a complex number before you can use De Moivre's theorem on it.

Solution

A complex number operation written in rectangular form, such as: $(13-4i)^3$ must be converted to polar form before utilizing De Moivre's theorem.

Example 5.3.11.2

Find the value of $(1+\sqrt{3}i)^4$.

Solution

$r = \sqrt{(1)^2+(\sqrt{3})^2} = 2$

$\tan\theta_{ref} = \frac{\sqrt{3}}{1}$,

and θ is in the 1st quadrant, so

$\theta = \frac{\pi}{3}$

Using our equation from above:

$$z^4 = r^4\ cis\ 4\theta$$
$$z^4 = (2)^4\ cis\ \frac{4\pi}{3}$$

Expanding *cis* form:

$$z^4 = 16\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right)$$
$$= 16((-0.5) - 0.866i)$$

Finally we have

$z^4 = -8 - 13.856i$

Example 5.3.11.3

Find $\sqrt{1+i}$.

Solution

First, rewriting in exponential form: $(1+i)^{1/2}$

And now in polar form:

$$\sqrt{1+i} = \left(\sqrt{2}\ cis\ \left(\frac{\pi}{4}\right)\right)^{1/2}$$

Expanding cis form,

$$= \left(\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)\right)^{1/2}$$

Using the formula:

$$= (2^{1/2})^{1/2}\left(\cos\left(\frac{1}{2}\cdot\frac{\pi}{4}\right) + i\sin\left(\frac{1}{2}\cdot\frac{\pi}{4}\right)\right)$$
$$= 2^{1/4}\left(\cos\left(\frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{8}\right)\right)$$

In decimal form, we get

$$= 1.189(0.924 + 0.383i)$$
$$= 1.099 + 0.455i$$

To check, we will multiply the result by itself in rectangular form:

$$(1.099 + 0.455i)\cdot(1.099 + 0.455i) = 1.0992 + 1.099(0.455i) + 1.099(0.455i) + (0.455i)^2$$
$$= 1.208 + 0.500i + 0.500i + 0.208i^2$$
$$= 1.208 + i - 0.208 \text{ or}$$
$$= 1 + i$$

Example 5.3.11.4

Find the value of x: $x_3 = (1 - \sqrt{3}i)$.

Solution

First we put $1 - \sqrt{3}i$ in polar form.

Use $x = 1$, $y = -\sqrt{3}$ to obtain $r = 2$, $\theta = 5\frac{\pi}{3}$

let $z = (1 - \sqrt{3}i)$ in rectangular form

$z = 2\ cis\ \left(5\frac{\pi}{3}\right)$ in polar form

$$x = (1 - \sqrt{3}i)^{1/3}$$
$$x = \left[2\ cis\ \left(\frac{5\pi}{3}\right)\right]^{1/3}$$

Use De Moivre's theorem to find the first solution:

$x_1 = 2^{1/3}\ cis\ \left(\frac{5\pi/3}{3}\right)$ or $2^{1/3}\ cis\ \left(\frac{5\pi}{9}\right)$

Leave answer in cis form to find the remaining solutions:

$n = 3$ which means that the 3 solutions are $\frac{2\pi}{3}$ radians apart or

$x_2 = 2^{1/3}\ cis\ \left(\frac{5\pi}{9} + \frac{2\pi}{3}\right)$ and $x_3 = 2^{1/3}\ cis\ \left(\frac{5\pi}{9} + \frac{2\pi}{3} + \frac{2\pi}{3}\right)$

NOTE: It is not necessary to add $\frac{2\pi}{3}$ again. Adding $\frac{2\pi}{3}$ three times equals 2π. That would result in rotating around a full circle and to start where it all began- that is the first solution.

The three solutions are:

$$x_1 = 2^{1/3}\ cis\ \left(\frac{5\pi}{9}\right)$$
$$x_2 = 2^{1/3}\ cis\ \left(\frac{11\pi}{9}\right)$$
$$x_3 = 2^{1/3}\ cis\ \left(\frac{17\pi}{9}\right)$$

Each of these solutions, when graphed will be $\frac{2\pi}{3}$ apart.

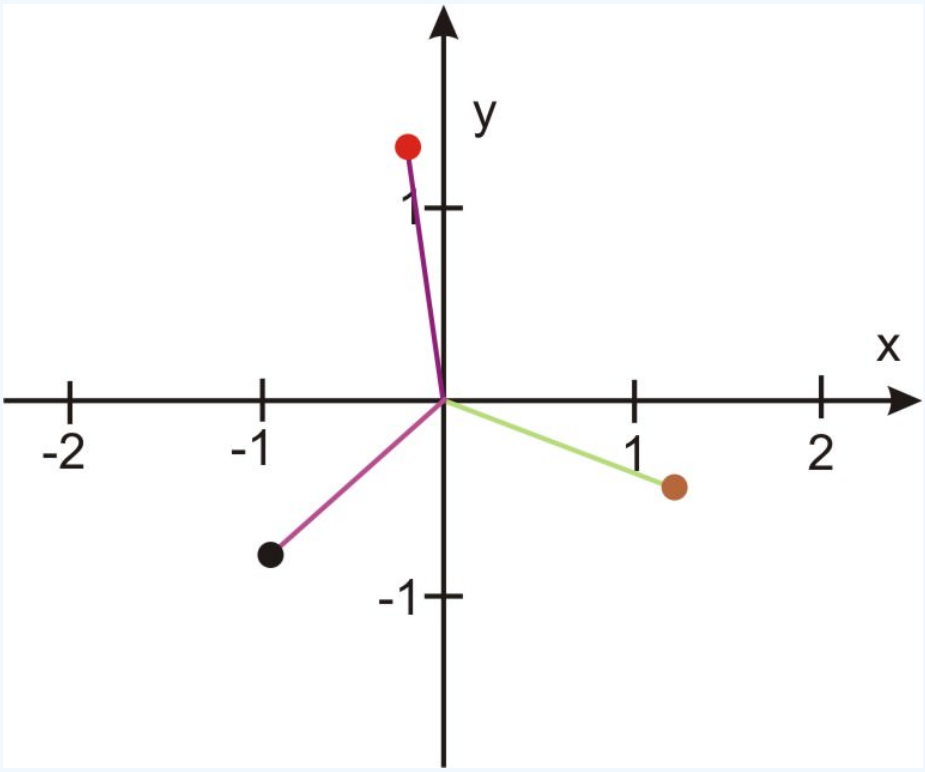


Figure 5.3.11.1

Check any one of these solutions to see if the results are confirmed.

Checking the second solution:

$$\begin{aligned} x_2 &= 2^{1/3}\ cis\ (\frac{11\pi}{9}) \\ &= 1.260\left[\cos\left(\frac{11\pi}{9}\right) + i\sin\left(\frac{11\pi}{9}\right)\right] \\ &= 1.260[-0.766 - 0.643i] \\ &= -0.965 - 0.810i \end{aligned}$$

Does $(-0.965–0.810i)^3$ or $(-0.965–0.810i)(-0.965–0.810i)(-0.965–0.810i)$

$= (1 - \sqrt{3}i)$?

Example 5.3.11.5

What are the two square roots of i?

Solution

Let $z = \sqrt{0+i}$.

$$z_1 = \left[1 \times \ cis \ \frac{\pi}{4}\right] \quad \text{or} \quad z_2 = \left[1 \times \ cis \ 5\frac{\pi}{4}\right]$$
$$z_1 = 1\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \quad \text{or} \quad z_2 = 1\left(\cos 5\frac{\pi}{4} + i\sin 5\frac{\pi}{4}\right)$$
$$z_1 = 0.707 + 0.707i \quad \text{or} \quad z_2 = -0.707 - 0.707i$$

Check for z_1 solution: $(0.707 + 0.707i)^2 = i$?

$0.500 + 0.500i + 0.500i + 0.500i^2 = 0.500 + i + 0.500(-1)$ or i

Example 5.3.11.6

Calculate $\sqrt[4]{(1+0i)}$. What are the four fourth roots of 1?

Solution

Let $z = 1$ or $z = 1 + 0i$. Then the problem becomes find $z^{1/4} = (1+0i)^{1/4}$.

Since $r = 1\theta = 0$, $z^{1/4} = [1 \times \ cis \ 0]^{1/4}$ with $z_1 = 1^{1/4}\left(\cos\frac{0}{4} + i\sin\frac{0}{4}\right)$ or $1(1+0)$ or 1

That root is not a surprise. Now use De Moivre's to find the other roots:

$z_2 = 1^{1/4}\left[\cos\left(0+\frac{\pi}{2}\right) + i\sin\left(0+\frac{\pi}{2}\right)\right]$

Since there are 4 roots, dividing 2π by 4 yields 0.5π

or $0 + i$ or just i $z_3 = 1^{1/4}\left[\cos\left(0+\frac{2\pi}{2}\right) + i\sin\left(0+\frac{2\pi}{2}\right)\right]$ which yields $z_3 = -1$

Finally, $z_4 = 1^{1/4}\left[\cos\left(0+\frac{3\pi}{2}\right) + i\sin\left(0+\frac{3\pi}{2}\right)\right]$ or z_4 =−i

The four fourth roots of 1 are 1, i, -1 and -i.

Example 5.3.11.7

Calculate $(\sqrt{3}+i)^7$.

Solution

To calculate $(\sqrt{3}+i)^7$ start by converting to $r \ cis$ form.

First, find r. Recall $r = \sqrt{\sqrt{3}^2 + 1^2}$.

$$r = \sqrt{3+1}$$
$$r = 2$$

If $\cos\theta = \frac{\sqrt{3}}{2}$ and $\sin\theta = \frac{1}{2}$ then $\theta = 30°$ and is in quadrant I. Now that we have trigonometric form, the rest is easy:

$$(\sqrt{3}+i)^7 = [2(\cos 30° + i\sin 30°)]^7 \quad \text{Write the original problem in } r \ cis \text{ form}$$
$$= 2^7[(\cos(7\cdot 30°) + i\sin(7\cdot 30°)] \quad \text{De Moivre's theorem}$$
$$= 128\left[-\frac{\sqrt{3}}{2} + \frac{-1}{2}i\right] \quad \text{Simplify}$$
$$(\sqrt{3}+i)^7 = -64\sqrt{3} - 64i \quad \text{Simplify again}$$
$$\therefore (\sqrt{3}+i)^7 = -64\sqrt{3} - 64i$$

Review

Perform the indicated operation on these complex numbers:

1. Divide: $\frac{2+3i}{1-i}$
2. Multiply: $(-6-i)(-6+i)$
3. Multiply: $\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)^2$

4. Find the product using polar form: $(2+2i)(\sqrt{3}-i)$
5. Multiply: $2(\cos 40^\circ + i\sin 40^\circ)\cdot 4(\cos 20^\circ + i\sin 20^\circ)$
6. Multiply: $2(\cos\frac{\pi}{8}+i\sin\frac{\pi}{8})\cdot 2\left(\cos\frac{\pi}{10}+i\sin\frac{\pi}{10}\right)$
7. Divide: $2(\cos 80^\circ + i\sin 80^\circ)\div 6(\cos 200^\circ + i\sin 200^\circ)$
8. Divide: $3\ cis\ (130^\circ)\div 4\ cis\ (270^\circ)$

Use De Moivre's theorem.

9. $[3(\cos 80^\circ + i\sin 80^\circ)]^3$
10. $\left[\sqrt{2}\left(\cos\frac{5\pi}{16}+i\sin\frac{5\pi}{16}\right)\right]^4$
11. $(\sqrt{3}-i)^6$
12. Identify the 3 complex cube roots of $1+i$
13. Identify the 4 complex fourth roots of $-16i$
14. Identify the five complex fifth roots of i

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.10.

Vocabulary

Term	Definition
complex number	A complex number is the sum of a real number and an imaginary number, written in the form $a+bi$.
De Moivre's Theorem	De Moivre's theorem is the only practical manual method for identifying the powers or roots of complex numbers. The theorem states that if $z=r(\cos\theta+i\sin\theta)$ is a complex number in $r\ cis\ \theta$ form and n is a positive integer, then $z^n=r^n(\cos(n\theta)+i\sin(n\theta))$.

Additional Resources

Video: Determining the Nth Roots of a Complex Number

Practice: DeMoivre's Theorem

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H

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