

CALCULUS

CK-12 Foundation

Default Text

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CHAPTER OVERVIEW

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- 1.2: Evaluate Limits Using Graphs
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1.1: Concept of Limit

When learning about the **end behavior** of a rational function you described the function as either having a **horizontal asymptote** at zero or another number, or going to infinity. **Limit notation** is a way of describing this end behavior mathematically.

$$f(x) = \frac{8x^4 + 4x^3 + 3x^2 - 10}{3x^4 + 6x^2 + 9x} \text{ goes } 8/3$$

You already know that as x gets extremely large then the function because the greatest power are equal and $8/3$ is the ratio of the leading coefficients. How is this statement represented using limit notation?

Introduction to Limits

Limit notation is a way of stating an idea that is a little more subtle than simply saying $x=5$ or $y=3$.

$$\lim_{x \rightarrow a} f(x) = b$$

"The limit of f of x as x approaches a is b "

The letter a can be any number or infinity. The function $f(x)$ is any function of x . The letter b can be any number. If the function goes to infinity, then instead of writing " $=\infty$ " you should write that the limit does not exist or "**DNE**". This is because infinity is not a number. If a function goes to infinity then it has no limit.

Take the following limit:

The limit of $y=4x^2$ as x approaches 2 is 16

In limit notation, this would be:

$$\lim_{x \rightarrow 2} 4x^2 = 16$$

While a function may never actually reach a height of b it will get arbitrarily close to b . One way to think about the concept of a limit is to use a physical example. Stand some distance from a wall and then take a big step to get halfway to the wall. Take another step to go halfway to the wall again. If you keep taking steps that take you halfway to the wall then two things will happen. First, you will get extremely close to the wall but never actually reach the wall regardless of how many steps you take. Second, an observer who wishes to describe your situation would notice that the wall acts as a limit to how far you can go.

Examples

Example 1



$$f(x) = \frac{8x^4 + 4x^3 + 3x^2 - 10}{3x^4 + 6x^2 + 9x}$$

Earlier, you were asked how to write the statement "The limit of as x approaches infinity is $8/3$ " in limit notation.

This can be written using limit notation as:

$$\lim_{x \rightarrow \infty} \left(\frac{8x^4 + 4x^3 + 3x^2 - 10}{3x^4 + 6x^2 + 9x} \right) = \frac{8}{3}$$

Example 2

Translate the following mathematical statement into words.

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{1}{2} \right)^i = 1$$

The limit of the sum of $1/2 + 1/4 + 1/8 + \dots$ as the number of terms approaches infinity is 1.

Example 3

Use limit notation to represent the following mathematical statement.

$$1/3 + 1/9 + 1/27 + \dots = 1/2$$

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{1}{3} \right)^i = \frac{1}{2}$$

Example 4

Describe the end behavior of the following rational function at infinity and negative infinity using limits.

$$f(x) = \frac{-5x^3 + 4x^2 - 10}{10x^3 + 3x^2 + 98}$$

Since the function has equal powers of x in the numerator and in the denominator, the end behavior is $-1/2$ as x goes to both positive and negative infinity.

$$\lim_{x \rightarrow \infty} \left(\frac{-5x^3 + 4x^2 - 10}{10x^3 + 3x^2 + 98} \right) = \lim_{x \rightarrow -\infty} \left(\frac{-5x^3 + 4x^2 - 10}{10x^3 + 3x^2 + 98} \right) = -\frac{1}{2}$$

Example 5

Translate the following limit expression into words. What do you notice about the limit expression?

$$\lim_{h \rightarrow 0} \left(\frac{f(x+h) - f(x)}{h} \right) = x$$

The limit of the ratio of the difference between f of quantity x plus h and f of x and h as h approaches 0 is x .

You should notice that $h \rightarrow 0$ does not mean $h=0$ because if it did then you could not have a 0 in the denominator. You should also note that in the numerator, $f(x+h)$ and $f(x)$ are going to be super close together as h approaches zero. Calculus will enable you to deal with problems that seem to look like $0/0$ and ∞/∞ .

Review

Describe the end behavior of the following rational functions at infinity and negative infinity using limits.

$$1. f(x) = \frac{2x^4 + 4x^2 - 1}{5x^4 + 3x + 9}$$

$$2. g(x) = \frac{8x^3 + 4x^2 - 1}{2x^3 + 4x + 7}$$

$$3. f(x) = \frac{x^2 + 2x^3 - 3}{5x^3 + x + 4}$$

$$4. f(x) = \frac{4x + 4x^2 - 5}{2x^2 + 3x + 3}$$

$$5. f(x) = \frac{3x^2 + 4x^3 + 4}{6x^3 + 3x^2 + 6}$$

Translate the following statements into limit notation.

6. The limit of $y=2x^2+1$ as x approaches 3 is 19.

7. The limit of $y=e^x$ as x approaches negative infinity is 0.

8. The limit of $y=1/x$ as x approaches infinity is 0.

Use limit notation to represent the following mathematical statements.

$$9. \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots = \frac{1}{3}$$

10. The series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ diverges.

$$11. 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 2$$

$$12. \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \dots = 1$$

Translate the following mathematical statements into words.

$$13. \lim_{x \rightarrow 0} \frac{5x^2 - 4}{x + 1} = -4$$

$$14. \lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1} = 3$$

15. If $\lim_{x \rightarrow a} f(x) = b$, is it possible that $f(a) = b$? Explain.

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 14.1.

Vocabulary

Term	Definition
End behavior	End behavior is a description of the trend of a function as input values become very large or very small, represented as the 'ends' of a graphed function.
Horizontal Asymptote	A horizontal asymptote is a horizontal line that indicates where a function flattens out as the independent variable gets very large or very small. A function may touch or pass through a horizontal asymptote.
limit notation	Limit notation is a way of expressing the fact that a function gets arbitrarily close to a value.

Additional Resource

PLIX: Play, Learn, Interact, eXplore - Concept of Limit

Video: Introduction to Limits

Practice: Concept of Limit

Real World: Suiting Up

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1.2: Evaluate Limits Using Graphs

Additional Resources

PLIX: Play, Learn, Interact, eXplore - Evaluate the Limits

Practice: Evaluate Limits using Graphs

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1.3: Evaluate Limits Using Tables

Additional Resources

Practice: Evaluate Limits Using Tables

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1.4: Limits of Composite Functions

Limits

Did you know that you can track the "behavior" of a function? Well, you can, and I'm not talking about whether the function is making curfew or not. When we talk about the "behavior" of a function, we look at how the function "shows up" when graphed. As we solve the function, we can get closer and closer and closer - the limits helps us to understand this movement of the function on the graph.

Here are some additional resources that provide a more in-depth exploration of this topic.

- [Limits and Asymptotes](#)
 - [Infinite and Non-Existent Limits](#)
-

Additional Resources

Video: Find Limits of Composite Functions Graphically

Practice: Limits of Composite Functions

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1.5: Evaluate Limits Using Substitution

Finding limits for the vast majority of points for a given function is as simple as substituting the number that x approaches into the function. Since this turns evaluating limits into an algebra-level substitution, most questions involving limits focus on the cases where substituting does not work. How can you decide if substitution is an appropriate analytical tool for finding a **limit**?

Using Substitution to Find Limits

Finding a limit **analytically** means finding the limit using algebraic means. In order to evaluate many limits, you can **substitute** the value that x approaches into the function and evaluate the result. This works perfectly when there are no holes or **asymptotes** at that particular x value. You can be confident this method works as long as you don't end up dividing by zero when you substitute.

If the function $f(x)$ has no holes or asymptote at $x=a$ then:

$$\lim_{x \rightarrow a} f(x) = f(a)$$

Occasionally there will be a **hole** at $x=a$. The limit in this case is the height of the function if the hole did not exist. In other words, if the function is a rational expression with factors that can be canceled, cancel the term algebraically and then substitute into the resulting expression. If no factors can be canceled, it could be that the limit does not exist at that point due to asymptotes.

Look at the following two limits:

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}, \quad \lim_{x \rightarrow 3} \frac{x^2 - 4}{x - 2}$$

The limit on the left cannot be evaluated by direct substitution because if 2 is substituted in, then you end up dividing by zero. The limit on the right can be evaluated using direct substitution because the hole exists at $x=2$ not $x=3$. Thus, the limit is:

$$\lim_{x \rightarrow 3} \frac{x^2 - 4}{x - 2} = \frac{3^2 - 4}{3 - 2} = \frac{9 - 4}{1} = 5$$

Examples

Example 1

Earlier, you were asked how to decide if you should use substitution to solve a limit. In order to decide whether substitution is an appropriate first step you can always try it. You'll know it won't work if you end up trying to evaluate an expression with a denominator equal to zero. If this happens, go back and try to factor and cancel, and then try substituting again.

Example 2

Evaluate the following limit by canceling first and then using substitution.



$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$$

$$\begin{aligned}\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} &= \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{(x - 2)} \\ &= \lim_{x \rightarrow 2} (x + 2) \\ &= 2 + 2 \\ &= 4\end{aligned}$$

Example 3

Evaluate the following limit analytically:

$$\lim_{x \rightarrow 4} \frac{x^2 - x - 12}{x - 4}$$

$$\begin{aligned}\lim_{x \rightarrow 4} \frac{x^2 - x - 12}{x - 4} &= \lim_{x \rightarrow 4} \frac{(x - 4)(x + 3)}{(x - 4)} \\ &= \lim_{x \rightarrow 4} (x + 3) \\ &= 4 + 3 \\ &= 7\end{aligned}$$

Example 4

Evaluate the following limit analytically.

$$\begin{aligned}\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} \\ \lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} &= \lim_{x \rightarrow 3} \frac{(x - 3)(x + 3)}{(x - 3)} = \lim_{x \rightarrow 3} (x + 3) = 6\end{aligned}$$

Example 5

Evaluate the following limit analytically.

$$\lim_{y \rightarrow 4} \frac{3|y - 1|}{y + 4}$$

$$\lim_{y \rightarrow 4} \frac{3|y - 1|}{y + 4} = \frac{3|4 - 1|}{4 + 4} = \frac{3 \cdot 3}{8} = \frac{9}{8}$$

Review

Evaluate the following limits analytically.

1. $\lim_{x \rightarrow 5} \frac{x^2 - 25}{x - 5}$

2. $\lim_{x \rightarrow -1} \frac{x^2 - 3x - 4}{x + 1}$

3. $\lim_{x \rightarrow 5} \sqrt{5x} - 12$

4. $\lim_{x \rightarrow 0} \frac{x^3 + 3x^2 - x}{5x}$

5. $\lim_{x \rightarrow 1} \frac{3x|x - 4|}{x + 1}$

6. $\lim_{x \rightarrow 2} \frac{x^2 + 5x - 14}{x - 2}$

7. $\lim_{x \rightarrow 1} \frac{x^2 - 8x + 7}{x - 1}$

8. $\lim_{x \rightarrow 0} \frac{5x - 1}{2x^2 + 3}$

9. $\lim_{x \rightarrow 1} 4x^2 - 2x + 5$

10. $\lim_{x \rightarrow 0} \frac{x^2 + 5x}{x}$

11. $\lim_{x \rightarrow -3} \frac{x^2 - 9}{x + 3}$

$5x + 1$

12. $\lim_{x \rightarrow 0} \frac{5x + 1}{x}$

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 14.4.

13. $\lim_{x \rightarrow 1} \frac{5x + 1}{x}$

Vocabulary

Term

14. $\lim_{x \rightarrow 5} \frac{x^2 - 25}{x^3 - 125}$

analytically

Asymptotes

15. $\lim_{x \rightarrow -1} \frac{x - 2}{x + 1}$

Hole

limit

Definition

One way to evaluate a limit analytically is by substitution. Substitution is a method of determining limits where the value that x is approaching is substituted into the function and the result is evaluated.

An asymptote is a line on the graph of a function representing a value toward which the function may approach, but does not reach (with certain exceptions).

A hole exists on the graph of a rational function at any input value that causes both the numerator and denominator of the function to be equal to zero.

A limit is the value that the output of a function approaches as the input of the function approaches a given value.

Additional Resources

Practice: Evaluate Limits Using Substitution

Real World: The Limits of Tolerance

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1.6: Properties of Limit

You're familiar with the idea of a **limit** of a function, and that some limits are computed using numerical and graphical methods. Limits can also be evaluated using the properties of limits. How would you find

$$\lim_{x \rightarrow 0} \frac{x^2 + 5x}{x}$$

without using a graph or using a table of values?

Properties of Limits

Let's begin with some observations about limits of some simple functions. Consider the following limit problems:

$$\lim_{x \rightarrow 2} 5 \quad \text{and} \quad \lim_{x \rightarrow 4} x$$

We note that each of these functions is defined for all real numbers. If we apply our intuition for finding the limits we conclude correctly that:

$$\lim_{x \rightarrow 2} 5 = 5 \quad \text{and} \quad \lim_{x \rightarrow 4} x = 4$$

The above results can be encapsulated in the following limit properties:

Basic Limit Properties:

1. $\lim_{x \rightarrow a} c = c$
2. $\lim_{x \rightarrow a} x = a$

Many functions can be expressed as the sums, differences, products, quotients, powers and roots of other more simple functions. The following properties are also useful in evaluating limits:

More Basic Limit Properties:

If $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ both exist, then

1. $\lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x)$ where c is a real number,
2. $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$,
3. $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$,
4. $\lim_{x \rightarrow a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ provided that $\lim_{x \rightarrow a} g(x) \neq 0$.
5. $\lim_{x \rightarrow a} [f(x)]^n = [\lim_{x \rightarrow a} f(x)]^n$ where n is a real number
6. $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$, where n is either an odd integer or n is an even positive integer and $\lim_{x \rightarrow a} f(x) > 0$.

Knowing these properties, allows evaluation of the limits of a wide range of functions.

Take the problem:

$$\lim_{x \rightarrow 2} (3x + 7)$$

Based on the properties above, the limit can be evaluated in the following steps:

$$\begin{aligned} \lim_{x \rightarrow 2} (3x + 7) &= \lim_{x \rightarrow 2} (3x) + \lim_{x \rightarrow 2} 7 \\ &= 3 \cdot \lim_{x \rightarrow 2} (x) + 7 \\ &= 3 \cdot 2 + 7 \\ &= 13 \end{aligned}$$

Therefore:

$$\lim_{x \rightarrow 2} (3x + 7) = 13$$

Note that the application of the basic limit properties results, in this case, in a limit value that is the same as direct substitution of $x=2$ in the function.

Now, evaluate

$$\lim_{x \rightarrow 2} (x^3 \cdot \sqrt{2x})$$

Based on the properties of limits, the limit can be evaluated in the following steps:

$$\begin{aligned}\lim_{x \rightarrow 2} (x^3 \cdot \sqrt{2x}) &= \lim_{x \rightarrow 2} x^3 \cdot \lim_{x \rightarrow 2} \sqrt{2x} \\ &= (\lim_{x \rightarrow 2} x)^3 \cdot \sqrt{\lim_{x \rightarrow 2} 2x} \\ &= (2)^3 \cdot \sqrt{4} \\ &= 16\end{aligned}$$

Therefore:

$$\lim_{x \rightarrow 2} (x^3 \cdot \sqrt{2x}) = 16$$

Again, note that the application of the basic limit properties results, in this case, in a limit value that is the same as direct substitution of $x=2$ in the function.

Examples

Example 1

Earlier, you were asked to



$$\lim_{x \rightarrow 0} \frac{x^2 + 5x}{x}$$

without using a graph or a table of values. This is a case where direct substitution to evaluate the limit gives the **indeterminate** form $0/0$. But, reducing the fraction and then applying the basic limit properties above, we can evaluate the limit:

$$\lim_{x \rightarrow 0} \frac{x^2 + 5x}{x} = \lim_{x \rightarrow 0} (x + 5) = 5.$$

Example 2

Evaluate

$$\lim_{x \rightarrow 0} f(x)$$

where $f(x)$ is the rational function

$$f(x) = \frac{x^3 + 3x^2 - 10x - 24}{x^2 + 1}$$

$$\begin{aligned}\lim_{x \rightarrow 0} f(x) &= \lim_{x \rightarrow 0} \left[\frac{x^3 + 3x^2 - 10x - 24}{x^2 + 1} \right] \\&= \left[\frac{\lim_{x \rightarrow 0} (x^3 + 3x^2 - 10x - 24)}{\lim_{x \rightarrow 0} (x^2 + 1)} \right] \\&= \left[\frac{\lim_{x \rightarrow 0} x^3 + \lim_{x \rightarrow 0} 3x^2 - \lim_{x \rightarrow 0} 10x - \lim_{x \rightarrow 0} 24}{\lim_{x \rightarrow 0} x^2 + \lim_{x \rightarrow 0} 1} \right] \\ \lim_{x \rightarrow 0} f(x) &= \lim_{x \rightarrow 0} \left[\frac{x^3 + 3x^2 - 10x - 24}{x^2 + 1} \right] \\&= \left[\frac{0 + 0 - 0 - 24}{0 + 1} \right] \\&= -24\end{aligned}$$

Example 3

Find the following limit if it exists:

$$\lim_{x \rightarrow -4} \frac{2x^2}{\sqrt{4-x}}$$

Let's apply the basic quotient rule to evaluate this limit.

$$\begin{aligned}\lim_{x \rightarrow -4} \frac{2x^2}{\sqrt{4-x}} &= \frac{\lim_{x \rightarrow -4} 2x^2}{\lim_{x \rightarrow -4} \sqrt{4-x}} \\&= \frac{32}{\sqrt{8}} \\&= 8\sqrt{2}\end{aligned}$$

Therefore:

$$\lim_{x \rightarrow -4} \frac{2x^2}{\sqrt{4-x}} = 8\sqrt{2}$$

Again, the limit result is the same as using direct substitution of $x=-4$ in the function.

Review

1. What is the limit of $3x^2 + 4x - 9$ as $x \rightarrow -2$?
2. What is the limit of $5^x \cdot \cos(\pi x)$ as $x \rightarrow 0$?
3. What is the limit of $5^x + \cos(\pi x)$ as $x \rightarrow 1$?
4. If the limit of $f(x)$ as $x \rightarrow 1$ is zero, is the limit of $f(x)g(x)$ as $x \rightarrow 1$ always equal to zero, for all $g(x)$?
5. What is $\lim_{x \rightarrow 3} \frac{x^2 + 7x + 12}{x + 3}$?
6. What is $\lim_{x \rightarrow 2} \sqrt[3]{x^3 + 4x^2 + 3}$?
7. Find $\lim_{x \rightarrow -2} (x+4)^3(2x-1)^2$
8. $\lim_{x \rightarrow -7} \frac{x^2 + 7x + 12}{x + 7}$
9. If the limit of $f(x)$ as $x \rightarrow 5$ is one and the limit of $g(x)$ as $x \rightarrow 5$ is two, and both $f(x)$ and $g(x)$ are continuous functions, is it necessarily true that $f(5)g(5) = 2$?
10. If the limit of $f(x)$ as $x \rightarrow 0$ is 5, and $g(x) = x^2 + x$, what is the limit of $g(f(x))$ as $x \rightarrow 0$?
11. What is the limit of $5e^x + 2\pi^x + \frac{x+2}{\cos(x)}$ as $x \rightarrow 0$?
12. Let $h(x)$ be a function with a finite limit defined everywhere. What is the limit of $\frac{3 \sin(x)h(x)}{x^9 + 8x^5 + 10x^4 + 9x + 1} + e^x(\cos(h(x)\sin(x + 2\pi)))$ as $x \rightarrow \pi$?
13. Say that $p(x) = x^3 + x^2 + x + 1$. What must be the limit of $f(x)$ as $x \rightarrow 0$ if $\lim_{x \rightarrow 0} p(f(x)) = 0$ as $x \rightarrow 0$?
14. Which limits of $\frac{\sin(x)}{x}$ can you not take with the quotient rule?
15. Show that $\lim_{x \rightarrow a} f(x)^5$ is equal to $(\lim_{x \rightarrow a} f(x))^5$ wherever a limit exists without using the power rule. Is this method enough to prove the power rule?

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 2.3.

Vocabulary

Term	Definition
indeterminate	In mathematics, an expression is indeterminate if it is not precisely defined. There are seven indeterminate forms: $0/0, 0 \cdot \infty, \infty \cdot \infty, \infty - \infty, 0^0, \infty^0$, and 1^∞ .
limit	A limit is the value that the output of a function approaches as the input of the function approaches a given value.

Additional Resources

Practice: Properties of Limits

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CHAPTER OVERVIEW

2: Limit - Types of Limits

2.1: One-Sided Limit Types

2.2: Infinite Limit Type

2.3: Limits of Polynomial and Rational Functions

2.4: Basic Trigonometric Limits

2.5: Limits Involving Radical Functions

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2.1: One-Sided Limit Types

A **one sided limit** is exactly what you might expect; the limit of a function as it approaches a specific x value from either the right side or the left side. One sided limits help to deal with the issue of a jump discontinuity and the two sides not matching.

Is the following piecewise function **continuous**?

$$f(x) = \begin{cases} -x - 2 & x < 1 \\ -3 & x = 1 \\ x^2 - 4 & 1 < x \end{cases} \quad (2.1.1)$$

Evaluating One Sided Limits and Continuity

A one sided limit can be evaluated either from the left or from the right. Since left and right are not absolute directions, a more precise way of thinking about direction is “from the negative side” or “from the positive side”. The notation for these one sided limits is:

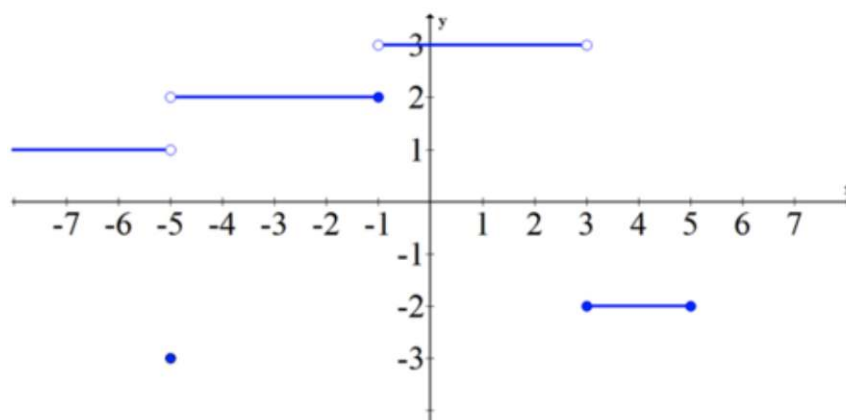
$$\lim_{x \rightarrow a^-} f(x) \quad (2.1.2)$$

and

$$\lim_{x \rightarrow a^+} f(x) \quad (2.1.3)$$

The negative in the superscript of a is not an exponent. Instead it indicates **from the negative side**. Likewise the positive superscript is not an exponent, it just means **from the positive side**. When evaluating one sided limits, it does not matter what the function is doing at the actual point or what the function is doing on the other side of the number. Your job is to determine what the height of the function should be using only evidence on one side.

Take the graph below. What are the one sided limits at -5 , -1 , 3 and 5 ?



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Each point should have two limits, one from the left and one from the right.

$$\lim_{x \rightarrow -5^-} f(x) = 1$$

$$\lim_{x \rightarrow -5^+} f(x) = 2$$

$$\lim_{x \rightarrow -1^-} f(x) = 2$$

$$\lim_{x \rightarrow -1^+} f(x) = 3$$

$$\lim_{x \rightarrow 3^-} f(x) = 3$$

$$\lim_{x \rightarrow 3^+} f(x) = -2$$

$$\lim_{x \rightarrow 5^-} f(x) = -2$$

$$\lim_{x \rightarrow 5^+} f(x) = DNE$$



You have defined continuity ability to draw a function completely without lifting your pencil off of the paper. You can now define a more rigorous continuity.

Continuity at a point exists if the left and right sided limits match the function evaluated at that point. In other words, a function is continuous at a point if

$$\lim_{x \rightarrow a^-} f(x) = f(a) = \lim_{x \rightarrow a^+} f(x) \quad (2.1.4)$$

For an entire function to be continuous, the function must be continuous at every single point in an unbroken domain.

Examples

Example 1

Earlier, you were asked how to confirm the function

$$f(x) = \begin{cases} -x - 2 & x < 1 \\ -3 & x = 1 \\ x^2 - 4 & 1 < x \end{cases} \quad (2.1.5)$$

is continuous. In order to confirm or deny that the function is continuous, graphical tools are not accurate enough. Sometimes **jump discontinuities** can be off by such a small amount that the pixels on the display of your calculator will not display a difference. Your calculator will certainly not display **removable discontinuities**.

You should note that on the graph, everything to the left of 1 is continuous because it is just a line. Next you should note that everything to the right of 1 is also continuous for the same reason. The only point to check is at $x=1$. To check continuity, explicitly use the definition and evaluate all three parts to see if they are equal.

- $\lim_{x \rightarrow 1^-} f(x) = -1 - 2 = -3$
- $f(1) = -3$
- $\lim_{x \rightarrow 1^+} f(x) = 1^2 - 4 = -3$

Therefore,

$$\lim_{x \rightarrow 1^-} f(x) = f(1) = \lim_{x \rightarrow 1^+} f(x) \quad (2.1.6)$$

and the function is continuous at $x=1$ and everywhere else.

Example 2

Evaluate the one sided limit at 4 from the negative direction numerically.

$$f(x) = \frac{x^2 - 7x + 12}{x - 4} \quad (2.1.7)$$

Remember that evaluating numerically means that you should use a table. When creating the table, only use values that are smaller than 4.

x	3.9	3.99	3.999
f(x)	0.9	0.99	0.999

$$\lim_{x \rightarrow 4^-} \left(\frac{x^2 - 7x + 12}{x - 4} \right) = 1 \quad (2.1.8)$$

Example 3

Evaluate the following limits.

$$1. \lim_{x \rightarrow 3^-} (4x - 3)$$

$$2. \lim_{x \rightarrow 2^+} \left(\frac{1}{x - 2} \right)$$

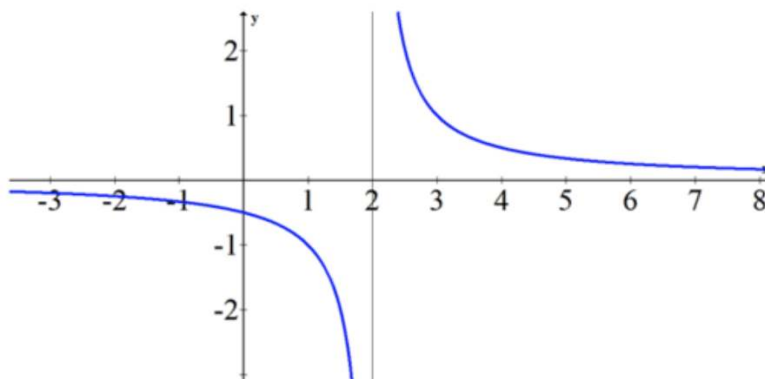
$$3. \lim_{x \rightarrow 1^+} \left(\frac{x^2 + 2x - 3}{x - 1} \right)$$

Most of the time one sided limits are the same as the corresponding two sided limit. The exceptions are when there are jump discontinuities, which normally only happen with piecewise functions, and infinite discontinuities, which normally only happen with rational functions.

$$1. \lim_{x \rightarrow 3^-} (4x - 3) = 4 \cdot 3 - 3 = 12 - 3 = 9$$

$$2. \lim_{x \rightarrow 2^+} \left(\frac{1}{x - 2} \right) = DNE \text{ or } \infty$$

The reason why ∞ is preferable in this case is because the two sides of the limit disagree. One side goes to negative infinity and the other side goes to positive infinity (see the graph below). If you just indicate DNE then you are losing some perfectly good information about the nature of the function.



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$$3. \lim_{x \rightarrow 1^+} \left(\frac{x^2 + 2x - 3}{x - 1} \right) = \lim_{x \rightarrow 1^+} \left(\frac{(x - 1)(x + 3)}{(x - 1)} \right) = \lim_{x \rightarrow 1^+} (x + 3) = 1 + 3 = 4$$

Example 4

Evaluate the following limits.

1. $\lim_{x \rightarrow 1^-} (2x - 1)$
 - $\lim_{x \rightarrow 1^-} (2x - 1) = 2 \cdot 1 - 1 = 2 - 1 = 1$
2. $\lim_{x \rightarrow -3^+} \left(\frac{2}{x+2} \right)$
 - $\lim_{x \rightarrow -3^+} \left(\frac{2}{x+2} \right) = \frac{2}{-3+2} = \frac{2}{-1} = -2$
3. $\lim_{x \rightarrow 2^+} \left(\frac{x^3 - 8}{x - 2} \right)$

Example 5

Is the following function continuous?

$$f(x) = \begin{cases} x^2 - 1 & x < -1 \\ 3 & x = -1 \\ -x + 3 & -1 < x \end{cases}$$

Use the definition of continuity.

- $\lim_{x \rightarrow 1^-} f(x) = (-1)^2 - 1 = 1 - 1 = 0$
- $f(-1) = 3$
- $\lim_{x \rightarrow -1^+} f(x) = -1 + 3 = 2$

$$\lim_{x \rightarrow a^-} f(x) \neq f(a) \neq \lim_{x \rightarrow a^+} f(x)$$

so this function is discontinuous at $x=-1$. It is continuous everywhere else.

Review

Evaluate the following limits.

1. $\lim_{x \rightarrow 6^-} (3x^2 - 4)$

2. $\lim_{x \rightarrow 0^-} \frac{3x - 1}{x}$

3. $\lim_{x \rightarrow 0^+} \frac{3x - 1}{x}$

4. $\lim_{x \rightarrow 0^+} \frac{x}{|x|}$

5. $\lim_{x \rightarrow 0^-} \frac{x}{|x|}$

6. $\lim_{x \rightarrow 0^+} \frac{\sqrt{x}}{\sqrt{1 + \sqrt{x}} - 1}$

Consider

$$f(x) = \begin{cases} 2x^2 - 1 & x < 1 \\ 1 & x = 1 \\ -x + 2 & 1 < x \end{cases}$$

7. What is $\lim_{x \rightarrow 1^-} f(x)$?

8. What is $\lim_{x \rightarrow 1^+} f(x)$?

9. Is $f(x)$ continuous at $x = 1$?

Consider

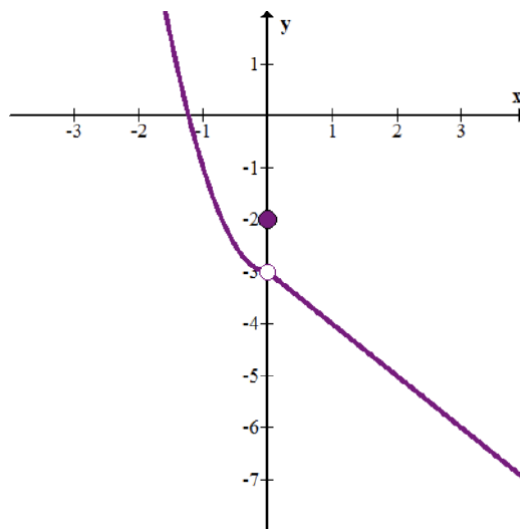
$$g(x) = \begin{cases} 4x^2 + 2x - 1 & x < -2 \\ 8 & x = -2 \\ -3x + 5 & -2 < x \end{cases}$$

10. What is $\lim_{x \rightarrow -2^-} g(x)$?

11. What is $\lim_{x \rightarrow -2^+} g(x)$?

12. Is $g(x)$ continuous at $x = -2$?

Consider $h(x)$ shown in the graph below.



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13. What is $\lim_{x \rightarrow 0^-} h(x)$?

14. What is $\lim_{x \rightarrow 0^+} h(x)$?

15. Is $h(x)$ continuous at $x = 0$?

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 14.6.

Vocabulary

Term	Definition
continuity	Continuity for a point exists when the left and right sided limits match the function evaluated at that point. For a function to be continuous, the function must be continuous at every single point in an unbroken domain.
Continuous	Continuity for a point exists when the left and right sided limits match the function evaluated at that point. For a function to be continuous, the function must be continuous at every single point in an unbroken domain.
Jump discontinuities	Inverse functions are functions that 'undo' each other. Formally $f(x)$ and $g(x)$ are inverse functions if $f(g(x))=g(f(x))=x$.
limit	A limit is the value that the output of a function approaches as the input of the function approaches a given value.
Removable discontinuities	Removable discontinuities are also known as holes. They occur when factors can be algebraically canceled from rational functions.
Removable discontinuity	Removable discontinuities are also known as holes. They occur when factors can be algebraically canceled from rational functions.
two-sided limit	A two-sided limit is the value that a function approaches from both the left side and the right side.

Additional Resources

PLIX: Play, Learn, Interact, eXplore - One-Sided Limits

Video: Determining Limits and One-Sided Limits Graphically

Practice: One-Sided Limit Types

Real World: Deep Freeze

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2.2: Infinite Limit Type

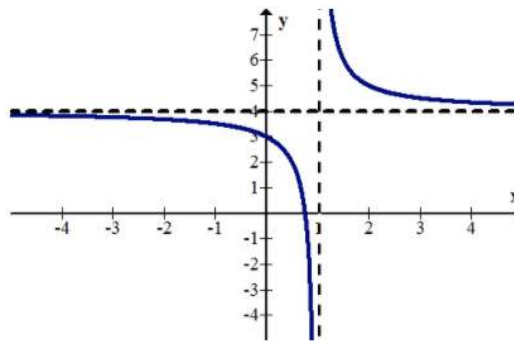
Most functions continue beyond the viewing window in our calculator or computer. People often draw an arrow next to a dotted line to indicate the pattern specifically. How can you recognize these asymptotes?

Asymptotes and End Behavior of Functions

A **vertical asymptote** is a vertical line such as $x=1$ that indicates where a function is not defined and yet gets infinitely close to.

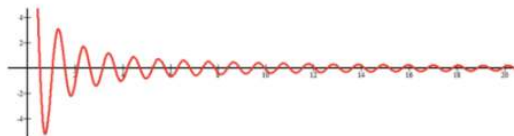
A **horizontal asymptote** is a horizontal line such as $y=4$ that indicates where a function flattens out as x gets very large or very small. A function may touch or pass through a horizontal asymptote.

The reciprocal function has two asymptotes, one vertical and one horizontal. Most computers and calculators do not draw the asymptotes and so they must be inserted by hand as dotted lines.



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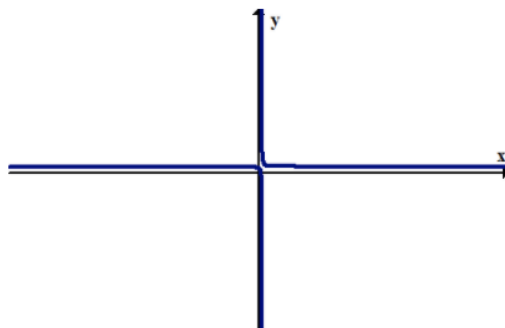
Many students have the misconception that an asymptote is a line that a function gets infinitely close to but does not touch. This is not true. Take the following function:



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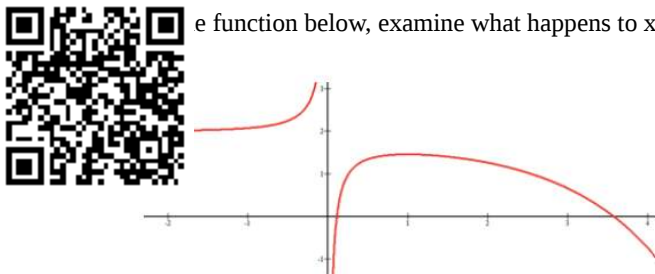
The graph appears to flatten as x grows larger. Thus, the horizontal asymptote is $y=0$ even though the function clearly passes through this line an infinite number of times.

The reason why asymptotes are important is because when your perspective is zoomed way out, the asymptotes essentially become the graph.



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To find the asymptotes and  e function below, examine what happens to x and y as they each increase or decrease.



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The function has a horizontal asymptote $y=2$ as x approaches negative infinity. There is a vertical asymptote at $x=0$. The right hand side seems to decrease forever and has no asymptote.

Note that slant asymptotes do exist and are called **oblique asymptotes**.

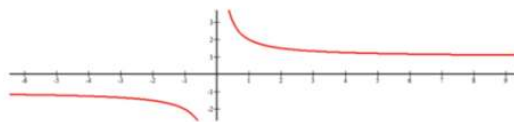
Examples

Example 1

Earlier, you were asked how to identify asymptotes on a graph. Asymptotes written by hand are usually identified with dotted lines next to the function that indicate how the function will behave outside the viewing window. The equations of these vertical and horizontal dotted lines are of the form $x=$ _____ and $y=$ _____. When problems ask you to find the asymptotes of a function, they are asking for the equations of these horizontal and vertical lines.

Example 2

Identify the horizontal and vertical asymptotes of the following function.

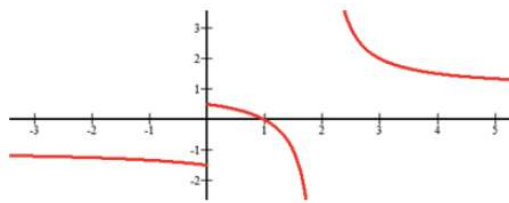


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There is a vertical asymptote at $x=0$. As x gets infinitely small, there is a horizontal asymptote at $y=-1$. As x gets infinitely large, there is another horizontal asymptote at $y=1$.

Example 3

Identify the horizontal and vertical asymptotes of the following function.



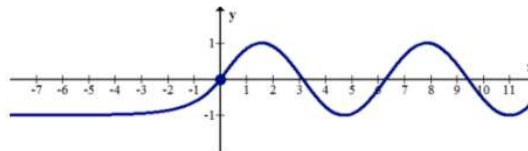
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There is a vertical asymptote at $x=2$. As x gets infinitely small there is a horizontal asymptote at $y=-1$. As x gets infinitely large, there is a horizontal asymptote at $y=1$.

Example 4

Identify the horizontal and vertical asymptotes of the following **piecewise function**:

$$f(x) = \begin{cases} e^x - 1 & x \leq 0 \\ \sin x & 0 < x \end{cases}$$

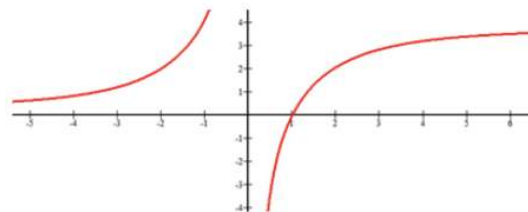


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There is a horizontal asymptote at $y=-1$ as x gets infinitely small. This is because e raised to the power of a very small number becomes 0.000000... and basically becomes zero.

Example 5

Identify the asymptotes and end behavior of the following function.



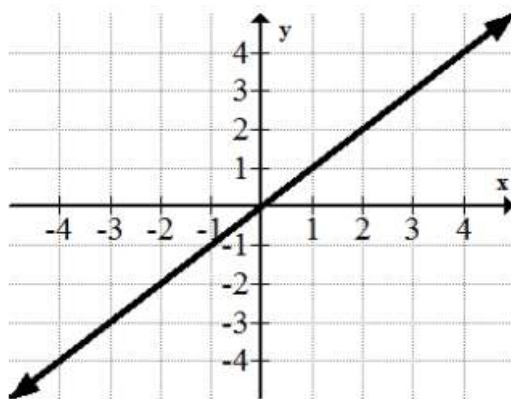
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There is a vertical asymptote at $x=0$. The end behavior of the right and left side of this function does not match. The horizontal asymptote as x approaches negative infinity is $y=0$ and the horizontal asymptote as x approaches positive infinity is $y=4$. At this point you can only estimate these heights because you were not given the function or the tools to find these values analytically.

Review

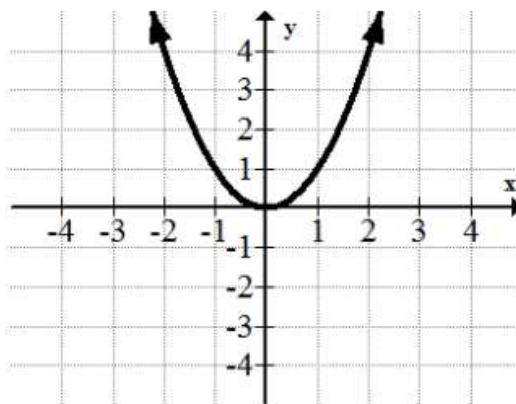
Identify the asymptotes and end behavior of the following functions.

1. $y=x$



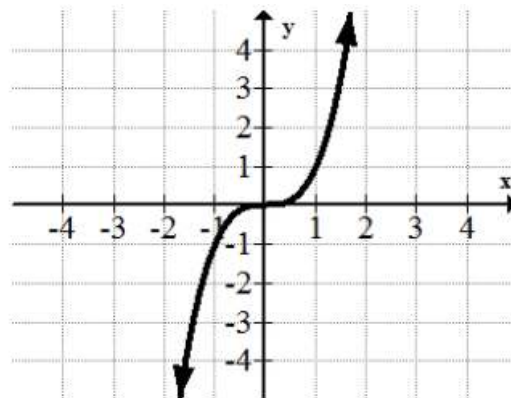
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2. $y = x^2$



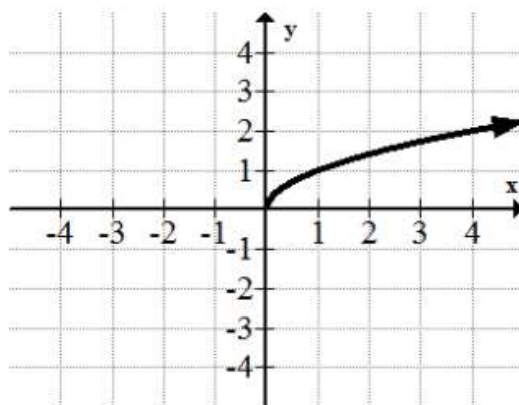
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3. $y = x^3$



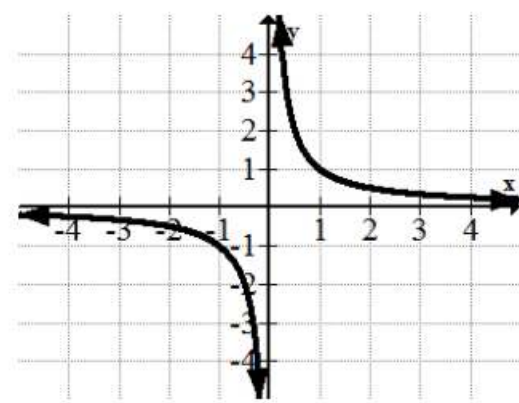
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4. $y = x^{0.5}$



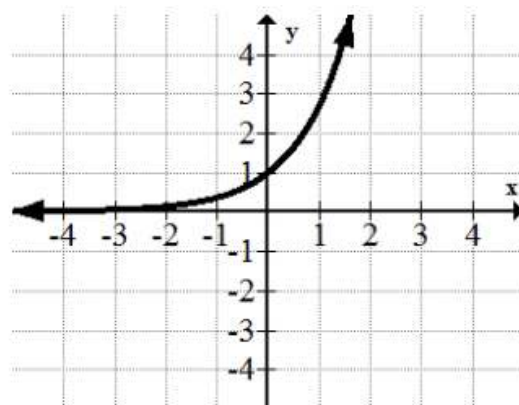
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5. $y = 1/x$



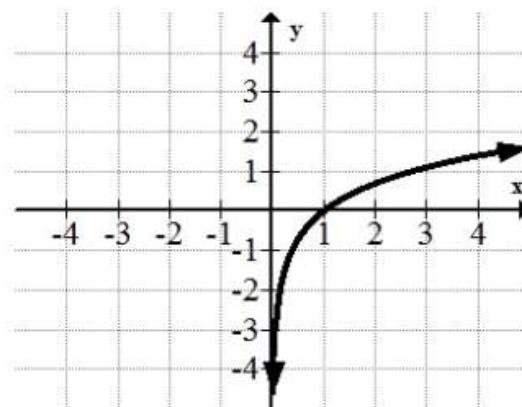
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6. $y = e^x$



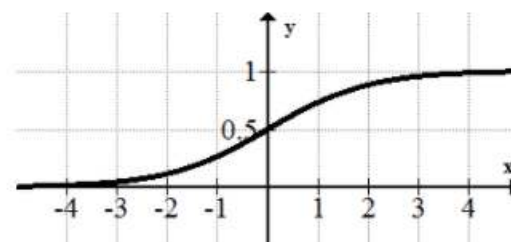
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7. $y = \ln(x)$



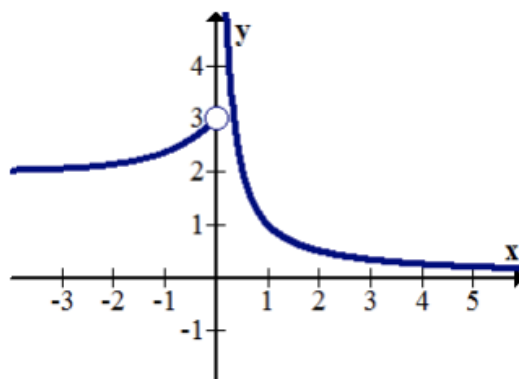
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8. $y = \frac{1}{1+e^{-x}}$



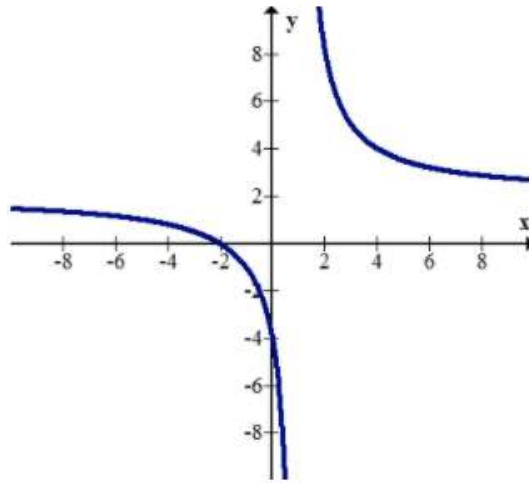
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9.



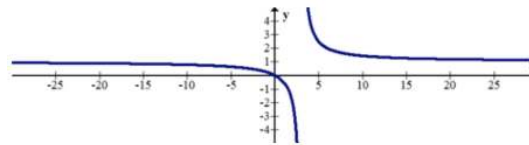
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10.



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11.



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12. Vertical asymptotes occur at x values where a function is not defined. Explain why it makes sense that $y = 1/x$ has a vertical asymptote at $x = 0$.
13. Vertical asymptotes occur at x values where a function is not defined. Explain why it makes sense that $y = 1/(x+3)$ has a vertical asymptote at $x = -3$.
14. Use the technique from the previous problem to determine the vertical asymptote for the function $y = 1/(x-2)$.
15. Use the technique from problem #13 to determine the vertical asymptote for the function $y = 2/(x+4)$.

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 1.9.

Vocabulary

Term	Definition
Asymptotic	A function is asymptotic to a given line if the given line is an asymptote of the function.
Horizontal Asymptote	A horizontal asymptote is a horizontal line that indicates where a function flattens out as the independent variable gets very large or very small. A function may touch or pass through a horizontal asymptote.
Oblique Asymptote	An oblique asymptote is a diagonal line marking a specific range of values toward which the graph of a function may approach, but generally never reach. An oblique asymptote exists when the numerator of the function is exactly one degree greater than the denominator. An oblique asymptote may be found through long division.
Oblique Asymptotes	An oblique asymptote is a diagonal line marking a specific range of values toward which the graph of a function may approach, but generally never reach. An oblique asymptote exists when the numerator of the function is exactly one degree greater than the denominator. An oblique asymptote may be found through long division.
Piecewise Function	A piecewise function is a function that pieces together two or more parts of other functions to create a new function.
Slant Asymptote	A slant asymptote is a diagonal line marking a specific range of values toward which the graph of a function may approach, but will never reach. A slant asymptote exists when the numerator of the function is exactly one degree greater than the denominator. A slant asymptote may be found through long division.
Vertical Asymptote	A vertical asymptote is a vertical line marking a specific value toward which the graph of a function may approach, but will never reach.

Additional Resources

PLIX: Play, Learn, Interact, eXplore - Asymptotes and End Behavior Question

Video: Introduction to Limits

Practice: Infinite Limit Type

Real World: The Limits of Speed

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2.3: Limits of Polynomial and Rational Functions

Polynomial Functions

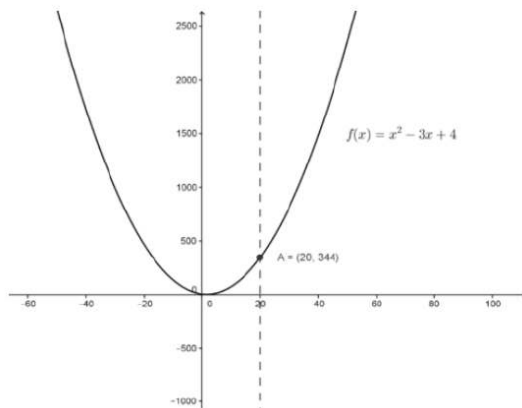
Recall that a function $f(x)$ **polynomial function** if it satisfies:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0 \quad (2.3.1)$$

for all x , where n is a non-negative integer and $a_0, a_1, a_2, \dots, a_n$ are constant coefficients.

As a refresher, use the limit properties to find limit of $(x^2 - 3x + 4)$ as $x \rightarrow 20$, i.e., the limit as x approaches a particular value.

The function is a polynomial, a quadratic trinomial that is graphed below, and can be treated as the sum of three functions. This means that we can use the rule “the limit of the sum is the sum of the limits” in the determination of the limit.



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Since, the polynomial can be treated as the sum of three functions, we can use the property “the limit of the sum is the sum of the limits” in the determination of the limit.

$$\begin{aligned} \lim_{x \rightarrow 20} (x^2 - 3x + 4) &= \lim_{x \rightarrow 20} x^2 + \lim_{x \rightarrow 20} (-3x) + \lim_{x \rightarrow 20} 4 \\ &= (\lim_{x \rightarrow 20} x)^2 - 3 \lim_{x \rightarrow 20} x + 4 \\ &= (20)^2 - 3(20) + 4 \\ &= 344 \end{aligned}$$

$$\text{Therefore, } \lim_{x \rightarrow 20} (x^2 - 3x + 4) = 344.$$

Note that the value of this limit could have been found by direct substitution of $x=1$ in the polynomial function.

Now, find

$$\lim_{x \rightarrow \infty} (x^2 - 3x + 4) \quad (2.3.2)$$

i.e., the limit as x approaches infinity. This is looking at end behavior.

The polynomial can be treated as the product of two functions. This means that we can use the rule “the limit of the product of functions is the product of the limits of each function” in the determination of the limit.

$$\begin{aligned} \lim_{x \rightarrow \infty} (x^2 - 3x + 4) &= \lim_{x \rightarrow \infty} x^2 \left(1 + \frac{-3x}{x^2} + \frac{4}{x^2} \right) \\ &= \lim_{x \rightarrow \infty} x^2 \cdot \lim_{x \rightarrow \infty} \left(1 + \frac{-3x}{x^2} + \frac{4}{x^2} \right) \\ &= \infty \cdot 1 \quad \dots \text{Note that the terms divided by } x^2 \text{ go to } 0. \\ &= \infty \end{aligned}$$

Therefore,

$$\lim_{x \rightarrow \infty} (x^2 - 3x + 4) = \infty. \quad (2.3.3)$$

A similar evaluation shows that

$$\lim_{x \rightarrow -\infty} (x^2 - 3x + 4) = \infty. \quad (2.3.4)$$

This behavior depicts the fact that the end behavior of polynomials goes as the term with the highest degree, and values grow without bound.

The results of these two examples can be generalized to the following properties:

Given the polynomial function $f(x)=p(x)$:

1. $\lim_{x \rightarrow a} f(x) = p(a)$, for any real number a .
2. $\lim_{x \rightarrow \pm\infty} f(x) = \lim_{x \rightarrow \pm\infty} a_n x^n \infty$ or $-\infty$,

Rational Functions

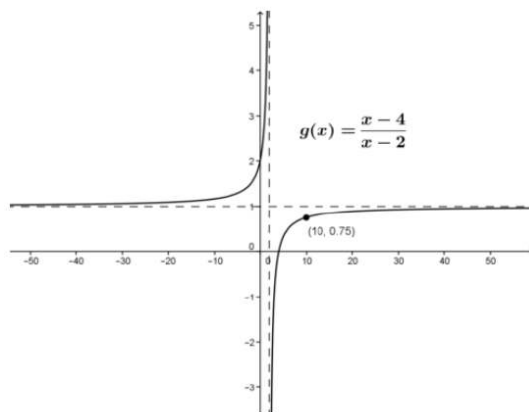
Now let's consider limits (). A **rational function** is the ratio of two polynomials. In the case of a single variable, x , a function is called a rational function only if it can be written in the form:

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \dots + b_2 x^2 + b_1 x + b_0}$$

where $P(x)$ and $Q(x)$ are polynomial functions in x and $Q(x)$ is non-zero. The domain of f is the set of all values of x for which the denominator $Q(x)$ is not zero.

Find $\lim_{x \rightarrow 10} \frac{x - 4}{x - 2}$

The function is shown in the graph below.



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We note first that the denominator of the rational function is not zero at the value $x=10$. The quotient rule can therefore be used to start the evaluation of the function as follows:

$$\begin{aligned}
 \lim_{x \rightarrow 10} \frac{x-4}{x-2} &= \frac{\lim_{x \rightarrow 10} (x-4)}{\lim_{x \rightarrow 10} (x-2)} \\
 &= \frac{\lim_{x \rightarrow 10} (x) - \lim_{x \rightarrow 10} (4)}{\lim_{x \rightarrow 10} (x) - \lim_{x \rightarrow 10} (2)} \\
 &= \frac{(10-4)}{(10-2)} \\
 &= \frac{3}{4}
 \end{aligned}$$

Therefore $\lim_{x \rightarrow 10} \frac{x-4}{x-2} = \frac{3}{4}$

Note that because the denominator does not equal 0 at $x=10$, the limit could have been found by direct substitution of $x=10$ in the rational function.

Now, find the end behavior of that same function, i.e. find

$$\lim_{x \rightarrow \infty} \frac{x-4}{x-2}$$

The following steps are used to evaluate the limit as x approaches infinity.

$$\begin{aligned}
 \lim_{x \rightarrow \infty} \frac{x-4}{x-2} &= \lim_{x \rightarrow \infty} \frac{x(1-\frac{4}{x})}{x(1-\frac{2}{x})} \quad \dots \text{Factor the out the highest degree variable} \\
 &\quad \text{from numerator and denominator.} \\
 &= \frac{\lim_{x \rightarrow \infty} (1-\frac{4}{x})}{\lim_{x \rightarrow \infty} (1-\frac{2}{x})} \quad \dots \text{Use the quotient rule of limits.} \\
 &= \frac{1-0}{1-0} \\
 &= 1
 \end{aligned}$$

Therefore $\lim_{x \rightarrow \infty} \frac{x-4}{x-2} = 1$

The above problems illustrate the evaluation of the limit of a rational function at a value of x for which the denominator does not equal 0. Sometimes finding the limit of a rational function $f(x)$ at some $x=a$ can entail more work than just direct substitution because the denominator equals zero at $x=a$. What if the denominator is equal to 0?

Take the problem: $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$

Notice that the function here is indeterminate at $x=2$, so that direct substitution does not work. However, in this case it is possible to remove the zero in the denominator by factoring the numerator and canceling the factor $(x-2)$ from both the numerator and the denominator.

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} &= \lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{(x-2)} && \dots \text{Factor the numerator.} \\ &= \lim_{x \rightarrow 2} (x+2) \\ &= 4 \end{aligned}$$

Therefore, $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = 4$.

The factoring of the numerator shown above, and then cancelling any common factors in the denominator, is a common technique used to find the limits of rational functions at points where the denominator is 0. Always check to see if the function can be simplified to remove the zero in the denominator, especially by cancelling common a factor that removes a discontinuity.

Now, find

$$\lim_{x \rightarrow \infty} \frac{x^2 - 4}{x - 2}$$

We look at the end behavior of the function as x goes to infinity as follows:

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x^2 - 4}{x - 2} &= \lim_{x \rightarrow \infty} \frac{x^2 \left(1 - \frac{4}{x^2}\right)}{x \left(1 - \frac{2}{x}\right)} && \dots \text{Factor the highest degree variable} \\ &&& \text{in numerator and denominator.} \\ &= \lim_{x \rightarrow \infty} \frac{x \left(1 - \frac{4}{x^2}\right)}{\left(1 - \frac{2}{x}\right)} \\ &= \lim_{x \rightarrow \infty} x \cdot \lim_{x \rightarrow \infty} \frac{\left(1 - \frac{4}{x^2}\right)}{\left(1 - \frac{2}{x}\right)} && \dots \text{Use limit of product rule} \\ &= \infty \cdot 1 \\ &= \infty \end{aligned}$$

Therefore,

$$\lim_{x \rightarrow \infty} \frac{x^2 - 4}{x - 2} = \infty \quad (2.3.5)$$

The end behavior as x goes to $-\infty$ can be determined by a similar approach, and is found to be $-\infty$.

Definition of the Limit of Rational Functions

For the rational function $f(x) = \frac{p(x)}{q(x)}$ and any real number a ,

$$\lim_{x \rightarrow a} f(x) = \frac{p(a)}{q(a)} \quad \text{if } q(a) \neq 0 \quad (2.3.6)$$

If $q(a)=0$, then the function may or may not have a limit.

For the rational function

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_2 x^2 + b_1 x + b_0},$$

$$\lim_{x \rightarrow \pm\infty} f(x) = \left(\frac{a_n}{b_m} \right) \cdot \lim_{x \rightarrow \pm\infty} x^{n-m} = \begin{cases} +\infty \text{ or } -\infty & \text{if } n > m \\ \frac{a_n}{b_m} & \text{if } n = m \\ 0 & \text{if } n < m \end{cases}$$

Examples

Example 1

Earlier, you were asked at  difficulties of finding the limit of polynomial and rational functions. Finding the limit of a polynomial function is easy because a polynomial function can be evaluated at any value of the independent variable so that the limit at a specific value can be evaluated by direct substitution. The limit as the independent variable goes to $\pm\infty$ is just $\pm\infty$ depending whether the degree of the polynomial is even or odd.

Evaluating the limit of a rational function can be more difficult because direct substitution may lead to an undefined or indeterminate form that requires a different approach, and the limit as the independent variable goes to $\pm\infty$ depends on which is larger, the degree of the numerator polynomial or the degree of denominator polynomial.

Example 2

Find $\lim_{x \rightarrow 3} \frac{2x - 6}{x^2 + x - 12}$

The numerator and the denominator are both equal to zero at $x=3$, but there is a common factor $x-3$ that can be removed (that is, we can simplify the rational function):

$$\begin{aligned} \lim_{x \rightarrow 3} \frac{2x - 6}{x^2 + x - 12} &= \lim_{x \rightarrow 3} \frac{2(x - 3)}{(x + 4)(x - 3)} \\ &= \lim_{x \rightarrow 3} \frac{2}{x + 4} \\ &= \frac{2}{7} \end{aligned}$$

Example 3

Find

$$\lim_{x \rightarrow 1} \frac{-5x^2 + x + 4}{x - 1}$$

to find

$$\lim_{x \rightarrow 1} \frac{-5x^2 + x + 4}{x - 1}$$

..... Start by factoring the numerator

Since we have (x-1) in both numerator and denominator, we know that the original function is equal to just -5x-4 except where it is undefined (1).

Therefore the closer we get to inputting 1, the closer we get to the same value, whether from the + or - side.

To find the value, just solve

$$-5x - 4 \text{ for } x = 1$$

$$\therefore \lim_{x \rightarrow 1} \frac{-5x^2 + x + 4}{x - 1} = -5 \cdot 1 - 4 \rightarrow -9$$

Example 4

Find

$$\lim_{x \rightarrow -2} \frac{-x^2 + 2x + 8}{x + 2}$$

to find

$$\lim_{x \rightarrow -2} \frac{-x^2 + 2x + 8}{x + 2}$$

.... Start by factoring the numerator

Since we have (x+2) in both numerator and denominator, we know that the original function is equal to just -x-4

Therefore the closer we get to substituting -2, the closer we get to the same output value, whether from the + or - side.

To find the value, just solve -x-4 for x=-2

$$\therefore \lim_{x \rightarrow -2} \frac{-x^2 + 2x + 8}{x + 2} = -(-2) - 4 \rightarrow -2$$

Review

Solve the following rational function limits.

$$1. \lim_{x \rightarrow 1} \frac{-12x^2 + 12}{4x - 4}$$

$$2. \lim_{x \rightarrow 2} \frac{\frac{3}{x+3} - \frac{11}{9}}{2x - 4}$$

$$3. \lim_{x \rightarrow \frac{-57}{56}} \frac{\frac{-5x-3}{2x+3} - \frac{13}{6}}{-56x - 57}$$

$$4. \lim_{x \rightarrow 2} \frac{2x^2 - 5x + 2}{-x + 2}$$

$$5. \lim_{x \rightarrow \frac{3}{4}} \frac{4x^2 + 5x - 6}{4x - 3}$$

$$6. \lim_{x \rightarrow 4} \frac{\frac{-3}{-2x+3} - \frac{3}{5}}{6x - 24}$$

$$7. \lim_{x \rightarrow \frac{-3}{2}} \frac{\frac{-4x-3}{-2x+2} - \frac{5}{2}}{-2x - 3}$$

$$8. \lim_{x \rightarrow 4} \frac{x^2 - 8x + 16}{x - 4}$$

$$9. \lim_{x \rightarrow \frac{10}{39}} \frac{\frac{3x+3}{-3x+4} - \frac{7}{6}}{39x - 10}$$

$$10. \lim_{x \rightarrow -4} \frac{-3x^2 + 7x - 20}{-x - 4}$$

$$11. \lim_{x \rightarrow -4} \frac{4x^2 + 14x - 8}{x + 4}$$

$$12. \lim_{x \rightarrow \frac{-18}{13}} \frac{\frac{-4x+1}{-3x+5} - \frac{5}{7}}{-13x - 18}$$

$$13. \lim_{x \rightarrow -2} \frac{\frac{3}{x+4} - \frac{3}{2}}{-3x - 6}$$

$$14. \lim_{x \rightarrow \frac{-3}{4}} \frac{\frac{-x+3}{4x+3} - \frac{11}{2}}{-4x - 3}$$

$$15. \lim_{x \rightarrow \frac{1}{4}} \frac{-8x^2 - 2x + 1}{-4x + 1}$$

$$16. \lim_{x \rightarrow \frac{1}{4}} \frac{16x^2 - 16x + 3}{-4x + 1}$$

$$17. \lim_{x \rightarrow 0} \frac{x^2 + 3x}{x}$$

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 2.5.

Vocabulary

Term	Definition
discontinuous	A function is discontinuous if the function exhibits breaks or holes when graphed.
limit	A limit is the value that the output of a function approaches as the input of the function approaches a given value.
Polynomial Function	A polynomial function is a function defined by an expression with at least one algebraic term.
Rational Function	A rational function is any function that can be written as the ratio of two polynomial functions.

Additional Resources

PLIX: Play, Learn, Interact, eXplore - Evaluating Limits of Rational Functions

Video: Limit Examples

Practice: Limits of Polynomial and Rational Functions

Real World: Webslingers

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2.4: Basic Trigonometric Limits

Trigonometric functions can be a component of an expression and therefore subject to a **limit** process. Do you think that the periodic nature of these functions, and the limited or infinity range of individual trigonometric functions would make evaluating limits involving these functions difficult?

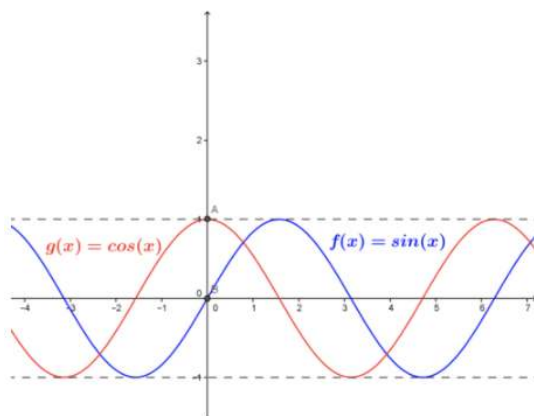
Limits with Trigonometric Functions

The limit rules presented in earlier concepts offer some, but not all, of the tools for evaluating limits involving trigonometric functions.

Let's find the following limits:

- $\lim_{x \rightarrow 0} \sin(x)$
- $\lim_{x \rightarrow 0} \cos(x)$
- $\lim_{x \rightarrow \pm\infty} \sin(x)$

We can find these limits by evaluating the function as x approaches 0 on the left and the right, i.e., by evaluating the two one-sided limits. The graphs and tables values are shown below.



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x(rad)	-0.001	-0.0001	0	0.0001	0.001
sin(x)	-0.001	-0.0001	0	0.0001	0.001
cos(x)	0.999	0.9999	1	0.9999	0.999

Inspection of the graph below, and table of values in the vicinity of $x=0$, indicates that:

- a. $\lim_{x \rightarrow 0^-} \sin(x) = \lim_{x \rightarrow 0^-} \sin(x) = 0$, which means $\lim_{x \rightarrow 0} \sin(x) = 0$.
- b. $\lim_{x \rightarrow 0^-} \cos(x) = \lim_{x \rightarrow 0^-} \cos(x) = 1$, which means $\lim_{x \rightarrow 0} \cos(x) = 1$.

Note that the limits can be found using direct substitution.

3. Because $\sin(x)$ is a periodic function, as x gets larger (smaller) and larger (smaller), its value will oscillate between 1 and -1, and never settle to a single value. Therefore, we can say that

$$\lim_{x \rightarrow \pm\infty} \sin(x)$$

does not exist.

We can generalize and extend the findings above and present the following properties:

Limit Properties for Basic Trigonometric Functions

- Limit as $x \rightarrow a$ for any real a :

$$\lim_{x \rightarrow a} \sin(x) = \sin(a) \quad \lim_{x \rightarrow a} \cos(x) = \cos(a)$$

$$\lim_{x \rightarrow a} \tan(x) = \tan(a) \quad \lim_{x \rightarrow a} \cot(x) = \cot(a)$$

$$\lim_{x \rightarrow a} \sec(x) = \sec(a) \quad \lim_{x \rightarrow a} \csc(x) = \csc(a)$$

- Limit as $x \rightarrow \pm\infty$:

$$\lim_{x \rightarrow \pm\infty} \sin(x) \quad \lim_{x \rightarrow \pm\infty} \cos(x) \quad \text{Limits Do Not Exist}$$

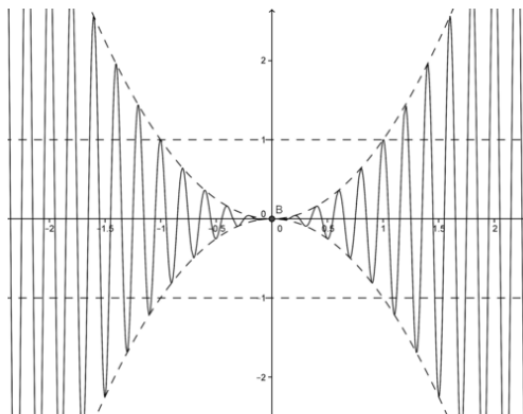
$$\lim_{x \rightarrow \pm\infty} \tan(x) \quad \lim_{x \rightarrow \pm\infty} \cot(x) \quad \text{Limits Do Not Exist}$$

$$\lim_{x \rightarrow \pm\infty} \sec(x) \quad \lim_{x \rightarrow \pm\infty} \csc(x) \quad \text{Limits Do Not Exist}$$

Let's find find

$$\lim_{x \rightarrow 0} f(x) \text{ for } f(x) = x^2 \cos(10\pi x)$$

The graph of the function is shown below.



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Since we know that the limit of x^2 and $\cos(x)$ exist, we can find the limit of this function by applying the Product Rule, or direct substitution:

$$\begin{aligned}\lim_{x \rightarrow 0} x^2 \cos(10\pi x) &= \left(\lim_{x \rightarrow 0} x^2\right) \cdot \left(\lim_{x \rightarrow 0} \cos(10\pi x)\right) \\ &= 0 \cdot 1 \\ &= 0\end{aligned}$$

Hence,

$$\lim_{x \rightarrow 0} x^2 \cos(10\pi x) = 0$$

Also, from the graph of the function, we note that the function is bounded by the graphs of x^2 and $-x^2$, which both are 0 at $x=0$. It makes sense that the original function's limit should be 0.

The Squeeze Theorem

This feature, that a function's limit can result from the function being bounded or squeezed by two other functions, is the basis for the Squeeze Theorem. **The Squeeze Theorem** (also known as the Sandwich Theorem) states:

$$\text{If } f(x) \leq g(x) \leq h(x) \text{ for every } x \text{ in an open interval containing } a, \text{ and } \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L, \text{ then } \lim_{x \rightarrow a} g(x) = L.$$

In other words, if we can find bounds for a function that have the same limit, then the limit of the function that they bound must have the same limit. Note that a and L may be any constant or even ∞ or $-\infty$.

One of the important trigonometric limits that can be proved, in part, using the Squeeze Theorem is:

$$\text{If } f(x) \leq g(x) \leq h(x) \text{ for every } x \text{ in an open interval containing } a, \text{ and } \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L, \text{ then } \lim_{x \rightarrow a} g(x) = L.$$

In other words, if we can find bounds for a function that have the same limit, then the limit of the function that they bound must have the same limit. Note that a and L may be any constant or even ∞ or $-\infty$.

One of the important trigonometric limits that can be proved, in part, using the Squeeze Theorem is:

$$\lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right) = 1$$

where x is in radian measure.

Another important trigonometric limit is

$$\lim_{x \rightarrow 0} \left(\frac{1 - \cos x}{x} \right)$$

Direct substitution cannot be used to evaluate the limit because it yields the **indeterminate** form $\frac{0}{0}$. Instead, transform the problem to a different form and solve.

$$\begin{aligned}\lim_{x \rightarrow 0} \left(\frac{1 - \cos x}{x} \right) &= \lim_{x \rightarrow 0} \left(\frac{1 - \cos x}{x} \cdot \frac{1 + \cos x}{1 + \cos x} \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{1 - \cos^2 x}{x(1 + \cos x)} \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{\sin^2 x}{x(1 + \cos x)} \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \cdot \frac{\sin x}{1 + \cos x} \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right) \cdot \lim_{x \rightarrow 0} \left(\frac{\sin x}{1 + \cos x} \right) \\ &= (1) \left(\frac{0}{2} \right)\end{aligned}$$

$$\lim_{x \rightarrow 0} \left(\frac{1 - \cos x}{x} \right) = 0$$

Examples

Example 1

Earlier, you were asked to determine the nature of trigonometric functions and the limited or infinity range of individual trigonometric functions. Limits involving trigonometric functions can be difficult.

As you can imagine and have seen in this concept, some limits involving trigonometric functions can be easily evaluated by direct substitution, and some evolve a lot of work to change form from an indeterminate or undefined form. Determining the end behavior of an expression involving a trigonometric function can also be difficult, and require application of principles like the Squeeze Theorem to obtain a result. No easy answer!

Example 2

Find

$$\lim_{x \rightarrow 0} x^2 \cos \left(\frac{10\pi}{x} \right)$$

Direct substitution cannot be used to evaluate the limit because $\frac{10\pi}{x}$ is undefined when $x=0$.

However, the Squeeze Theorem can be used as follows:

1. We know that cosine stays between -1 and 1, so

$$-1 \leq \cos \left(\frac{10\pi}{x} \right) \leq 1$$



for any x in the domain of the function (i.e., any $x \neq 0$).

2. Since x^2 is always non-negative, we can multiply the above inequality by x^2 :

$$-x^2 \leq \cos\left(\frac{10\pi}{x}\right) \leq x^2$$

3. The original function is bounded by x^2 and $-x^2$ and

$$\lim_{x \rightarrow 0} -x^2 = \lim_{x \rightarrow 0} x^2 = 0$$

4. Therefore, by the Squeeze Theorem:

$$\lim_{x \rightarrow 0} x^2 \cos\left(\frac{10\pi}{x}\right) = 0$$

Review

1. Find $\lim_{x \rightarrow \frac{\pi}{6}} \sin(x)$
2. Find $\lim_{x \rightarrow \frac{\pi}{4}} \cot(x)$
3. Find $\lim_{x \rightarrow \frac{\pi}{3}} \sec^2(x)$
4. Find $\lim_{x \rightarrow \frac{\pi}{3}} [\sin(x) + \cos(x)]$
5. Find $\lim_{x \rightarrow \frac{\pi}{2}} [\sec(x) + \tan(x)]$
6. Find $\lim_{x \rightarrow \pi} \frac{\sin x}{\cos x + 1}$
7. Find $\lim_{x \rightarrow 0} \frac{\sin(x)}{3x}$
8. Find $\lim_{x \rightarrow 0} \frac{2 \cos(x) - 2}{x}$
9. Find $\lim_{x \rightarrow 0} \frac{\sin(3x)}{x}$
10. Find $\lim_{x \rightarrow \infty} x \sin\left(\frac{1}{x}\right)$
11. Find $\lim_{x \rightarrow 1} \frac{\cos(x^2 - 1) - 1}{x^2 - 1}$
12. Find $\lim_{x \rightarrow 0} \frac{\cos(x)}{x}$
13. Find $\lim_{x \rightarrow 0} (2x \cot 2x)$
14. Find $\lim_{x \rightarrow -\frac{1}{2}} \left[\frac{\sin(2x + 3)}{2x^2 + x - 3} \right]$
15. Find $\lim_{x \rightarrow 0} \left[\frac{\tan x}{x} \right]$

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 2.7.

Vocabulary

Term	Definition
indeterminate	In mathematics, an expression is indeterminate if it is not precisely defined. There are seven indeterminate forms: $0/0$, $0 \cdot \infty$, ∞/∞ , $\infty - \infty$, 00 , $\infty 0$, and 1^∞ .
limit	A limit is the value that the output of a function approaches as the input of the function approaches a given value.
squeeze theorem	The squeeze theorem (also known as the sandwich theorem) is used to find the limit of a function by bounding it between two other functions that each have the same limit.

Additional Resources

PLIX: Play, Learn, Interact, eXplore - Evaluating Limits of $\tan(x)$

Video: Determining Limits Involving Trigonometric Functions

Practice: Basic Trigonometric Limits

Real World: I'll Fly Away

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2.5: Limits Involving Radical Functions

There are many problems that will involve taking the n th root of a variable expression, so it is natural that there may sometimes be a need to find the **limit** of a function involving radical expressions, using square or cube roots, or other roots. Do you think that finding the limit of a function involving radicals would be any different than finding the limit of polynomial or rational functions? Can you think of any ways that radicals might present different problems than polynomials?

Limits with Radical Functions

When evaluating a limit involving a **radical function**, use direct substitution to see if a limit can be evaluated whenever possible. If not, other methods to evaluate the limit need to be explored.

Take the following function

$$f(x) = \sqrt{x} - 3. \text{ Find } \lim_{x \rightarrow 9} \sqrt{x} - 3$$

$$\begin{aligned}\lim_{x \rightarrow 9} \sqrt{x} - 3 &= \lim_{x \rightarrow 9} \sqrt{x} - \lim_{x \rightarrow 9} 3 \\ \lim_{x \rightarrow 9} \sqrt{x} - 3 &= \lim_{x \rightarrow 9} \sqrt{x} - \lim_{x \rightarrow 9} 3 \\ &= \sqrt{9} - 3 \\ &= 0\end{aligned}$$

Therefore,

$$\lim_{x \rightarrow 9} \sqrt{x} - 3 = 0$$

which could have been determined by directly evaluating $f(x)$ at $x=9$, i.e., by using direct substitution.

Now, find

$$\lim_{x \rightarrow \infty} \sqrt{x} - 3$$

In both of the above cases, direct substitution could be used to evaluate the limits and there is no need for alternative methods.

Take a look at the function

$$g(x) = \frac{\sqrt{x^2 + 3}}{7x + 5}. \text{ Find } \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 3}}{7x + 5}$$

First we notice that we should exclude $x = -5/7$ in any evaluation. Using direct substitution to find the limit results in the **indeterminate** form ∞/∞ . To transform the radical expression to a better form, use the fact that the value of x is going to larger and larger positive values. This allows the following:

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 3}}{7x + 5} &= \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 \left(1 + \frac{3}{x^2}\right)}}{x \left(7 + \frac{5}{x}\right)} \\ \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 3}}{7x + 5} &= \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 \left(1 + \frac{3}{x^2}\right)}}{x \left(7 + \frac{5}{x}\right)} \\ &= \lim_{x \rightarrow \infty} \frac{|x| \sqrt{\left(1 + \frac{3}{x^2}\right)}}{x \left(7 + \frac{5}{x}\right)} \\ &= \frac{\lim_{x \rightarrow \infty} \sqrt{\left(1 + \frac{3}{x^2}\right)}}{\lim_{x \rightarrow \infty} \left(7 + \frac{5}{x}\right)} \\ &= \frac{1}{7}\end{aligned}$$

Therefore,

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 3}}{7x + 5} = \frac{1}{7}.$$

Now, find

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 + 3}}{7x + 5}$$

The solution to evaluating the limit at negative infinity is similar to the above approach except that x is always negative.

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 + 3}}{7x + 5} &= \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 \left(1 + \frac{3}{x^2}\right)}}{x \left(7 + \frac{5}{x}\right)} \\ &= \lim_{x \rightarrow -\infty} \frac{|x| \sqrt{\left(1 + \frac{3}{x^2}\right)}}{x \left(7 + \frac{5}{x}\right)} \\ &= \frac{\lim_{x \rightarrow -\infty} \sqrt{\left(1 + \frac{3}{x^2}\right)}}{-\lim_{x \rightarrow -\infty} \left(7 + \frac{5}{x}\right)} \quad \dots \text{Note the denominator has a "-" because } x < 0 \\ &= -\frac{1}{7}\end{aligned}$$

Therefore,

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 + 3}}{7x + 5} = -\frac{1}{7}$$

So far, you have been able to find the limit of rational functions using methods shown earlier. However, there are times when this is not possible. Take the function

$$h(x) = \frac{\sqrt{x} - 3}{x - 9}$$

Find

$$\lim_{x \rightarrow 9} h(x)$$

Using direct substitution to find the limit results in the indeterminate form $\frac{0}{0}$. In order to evaluate the limit, we need to transform the expression to remove the indeterminate form. This is accomplished by using the relationship for the difference of squares of real numbers: $x^2 - y^2 = (x + y)(x - y)$.

We then rewrite and simplify the original function as follows:

$$\begin{aligned} \lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9} &= \lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{(\sqrt{x} - 3)(\sqrt{x} + 3)} \\ &= \lim_{x \rightarrow 9} \frac{1}{(\sqrt{x} + 3)} \\ &= \frac{1}{\lim_{x \rightarrow 9} (\sqrt{9} + 3)} \\ &= \frac{1}{6} \end{aligned}$$

Use the difference of squares factoring to remove the 0 in the denominator.

Hence

$$\lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9} = \frac{1}{6}$$

Now, find the end behavior of the same function, i.e. find

$$\lim_{x \rightarrow \infty} h(x)$$

As x increases to large positive values, the function takes on the indeterminate form $\frac{\infty}{\infty}$. The transformation above can also be used to evaluate the limit (Approach 1), as well as the technique used in evaluating rational functions (Approach 2).

Approach 1	Approach 2
$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\sqrt{x} - 3}{x - 9} &= \lim_{x \rightarrow \infty} \frac{\sqrt{x} - 3}{(\sqrt{x} - 3)(\sqrt{x} + 3)} \\ &= \lim_{x \rightarrow \infty} \frac{1}{(\sqrt{x} + 3)} \\ &= \frac{1}{\lim_{x \rightarrow \infty} (\sqrt{x} + 3)} \\ &= 0 \end{aligned}$	$\begin{aligned} &= \lim_{x \rightarrow \infty} \frac{\sqrt{x} \left(1 - \frac{3}{\sqrt{x}}\right)}{x \left(1 - \frac{9}{x}\right)} \\ &= \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x}} \cdot \lim_{x \rightarrow \infty} \frac{\left(1 - \frac{3}{\sqrt{x}}\right)}{\left(1 - \frac{9}{x}\right)} \\ &= 0 \cdot 1 \\ &= 0 \end{aligned}$

Hence

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x} - 3}{x - 9} = 0$$

Finally, find

$$\lim_{x \rightarrow -\infty} h(x)$$

The solution to this problem is that the limit does not exist because the domain of $h(x)$ does not include $x < 0$.



Examples

Example 1



Earlier, you were asked if evaluating limits involving polynomials and rational functions can be used to find the limits of radical functions. Some of the methods do work for radical functions. The use of direct substitution is a common method. Transforming indeterminate or undefined forms by finding and canceling common factors in the numerator and denominator, or factoring and simplifying the highest degree powers of variables represent common approaches.

One of the noteworthy differences between polynomial and radical functions is that the domain of polynomials can include all real values of the independent variable, but the domain of radical functions, e.g., $x\sqrt{}$, is restricted.

Example 2

Find

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+4} - 2}{x}$$

Using direct substitution to find the limit of the function results in the indeterminate form $0/0$. To transform the radical expression to a better form, do the following:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sqrt{x+4} - 2}{x} &= \lim_{x \rightarrow 0} \left(\frac{\sqrt{x+4} - 2}{x} \cdot \frac{\sqrt{x+4} + 2}{\sqrt{x+4} + 2} \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{x + 4 - 4}{x \cdot \sqrt{x+4} + 2} \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{x}{x \cdot \sqrt{x+4} + 2} \right) \\ &= \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+4} + 2} \\ &= \frac{1}{4} \end{aligned}$$

...Rationalize the numerator: multiply by the conjugate of the numerator

Therefore,

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+4} - 2}{x} = \frac{1}{4}$$

Review

Find each of the following limits if they exist.

1. $\lim_{x \rightarrow 3} \sqrt{x}$
2. $\lim_{x \rightarrow 8} \sqrt{x-7}$
3. $\lim_{x \rightarrow 4} \frac{\sqrt{x}-2}{x-4}$
4. $\lim_{x \rightarrow 1} \frac{\sqrt{x+3}-2}{x-1}$
5. $\lim_{x \rightarrow 0^+} \frac{\sqrt{x}}{\sqrt{1+\sqrt{x}}-1}$
6. $\lim_{x \rightarrow \infty} \left(\sqrt{x^2-5x}-x \right)$
7. $\lim_{x \rightarrow \infty} \frac{\sqrt{x^6+3x^2+1}}{4x^3+3}$
8. $\lim_{x \rightarrow 0} \frac{\sqrt{x+5}-\sqrt{5}}{x}$
9. $\lim_{x \rightarrow 3} \left(\sqrt{x^2+4x} \right)$
10. $\lim_{x \rightarrow -1} (x^2+2x+10)^{\frac{3}{2}}$
11. $\lim_{x \rightarrow 4} \frac{\sqrt{x+5}-3}{x-4}$
12. $\lim_{x \rightarrow 1} \left(\sqrt{2x^3+3x^2+7} \right)$
13. $\lim_{x \rightarrow 3} \left(\sqrt[3]{2x^2-10} \right)$
14. $\lim_{x \rightarrow 7} \frac{5x}{\sqrt{x+2}}$
15. $\lim_{x \rightarrow 0} \frac{7-\sqrt{x^2+49}}{x}$

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 2.6.

Vocabulary

Term	Definition
indeterminate	In mathematics, an expression is indeterminate if it is not precisely defined. There are seven indeterminate forms: $0/0$, $0 \cdot \infty$, ∞/∞ , $\infty - \infty$, 0^0 , ∞^0 , and 1^∞ .
limit	A limit is the value that the output of a function approaches as the input of the function approaches a given value.
radical function	Radical functions are functions which contain nth roots of variable expressions.

Additional Resources

Video: Limits at Infinity

Practice: Limits Involving Radical Functions

Real World: Maverick Surfers

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CHAPTER OVERVIEW

3: Limits - Continuity

3.1: Continuity of a Function

3.2: Continuity at a Point, Continuity Test, Types of Discontinuity

3.3: Intermediate Value Theorem, Existence of Solution

3.4: Find Imaginary Solutions

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3.1: Continuity of a Function

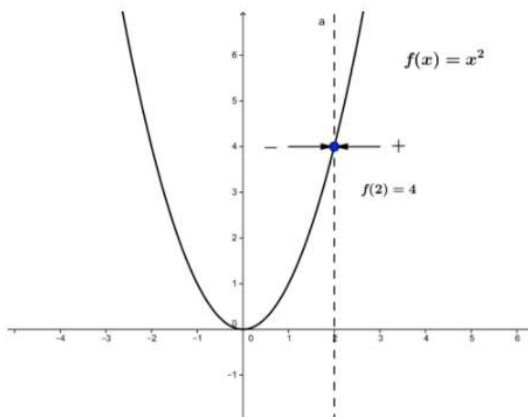
The continuity of a function is related to the conceptual notion of a function not having any breaks or sudden jumps as the independent variable takes on all values over some interval. There are no range gaps, or sudden jumps over the domain of interest. With this in mind, see if you can determine which of the following could be considered to represent a **continuous function**:

1. daily outdoor temperature variation
2. weight of a cart as 10 lb bags of sand are loaded on it
3. the rise and fall of the level of a reservoir of water
4. the amount of money collected at a movie theater for a specific showing.

Continuity

Continuity of a function is conceptually the characteristic of a function curve that has the values of the range “flow” continuously without interruption over some interval, as if never having to lift pencil from paper while drawing the curve. This intuitive notion needs to be formalized mathematically.

Consider the graph of the function $f(x)=x^2$. The function is defined for any real value of the independent variable.



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For example, for $x=2$ the function has the value $f(2)=4$. Notice that as x approaches the value 2 from either the left or the right, the value of the function approaches 4, i.e.

$$\lim_{x \rightarrow 2} f(x) = f(2) = 4 \quad (3.1.1)$$

Conceptually, applying the left and right one-sided limits as x approached 2, resulted in the same value, the value of the function at $x=2$.

The above results illustrate key properties of a **continuous** function. They are formalized in the following definition that provides a test for determining when we have a continuous function.

The function $f(x)$ is **continuous at** $x=a$ if the following conditions all hold:

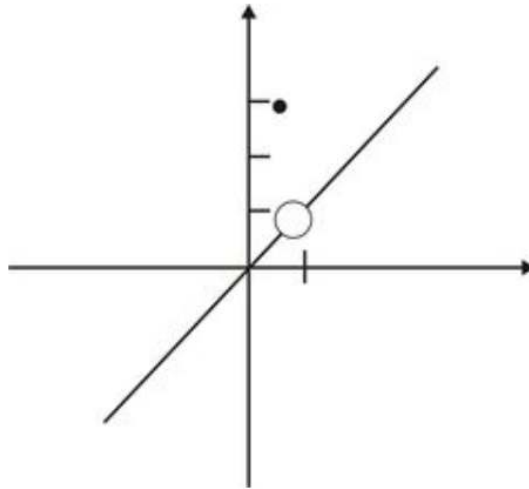
1. a is in the domain of $f(x)$;
2. $\lim_{x \rightarrow a} f(x)$ exists;
3. $\lim_{x \rightarrow a} f(x) = f(a)$

Note that it is possible to have functions where two of these conditions are satisfied but the third is not. When a function is not continuous at $x=a$, then it is **discontinuous** at a , and a is a point of discontinuity.

Consider the piece-wise function

$$f(x) = \begin{cases} x, & \text{if } x \neq 1 \\ 3, & \text{if } x = 1 \end{cases} \quad (3.1.2)$$

with the graph shown below.



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In this example, we have

$$\lim_{x \rightarrow 1} f(x) \quad (3.1.3)$$

exists! The value $x=1$ is in the domain of $f(x)$, but

$$\lim_{x \rightarrow 1} f(x) \neq f(1) \quad (3.1.4)$$

Therefore the function is not continuous at $x=1$.

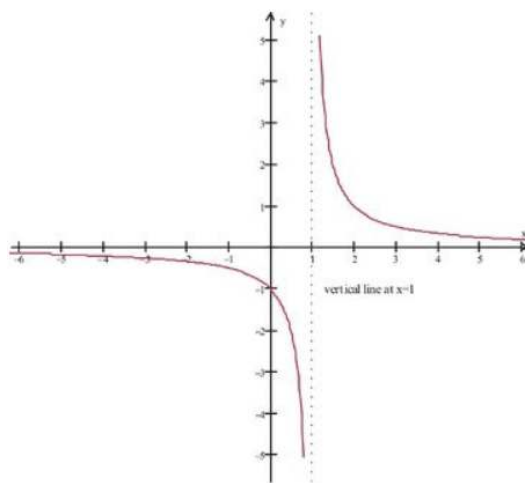
This is an example of a **removable discontinuity**, i.e., a discontinuity at which the limit of the function exists but does not equal the value of the function at that point.

Consider the following function:

$$f(x) = \frac{(x+1)}{(x^2-1)} \quad (3.1.5)$$

Are there any **discontinuities**?

We know from our study of domains that in order for the function to be defined, we must have $x \neq -1, 1$. Yet when we generate the graph of the function (using the standard viewing window), we get the following picture that appears to be defined at $x=-1$:



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The apparent contradiction is due to the fact that our original function had $x+1$ as a common factor in the numerator and denominator, which cancelled out and gave us a picture that appears to be the graph of

$$f(x) = \frac{1}{(x-1)} \quad (3.1.6)$$

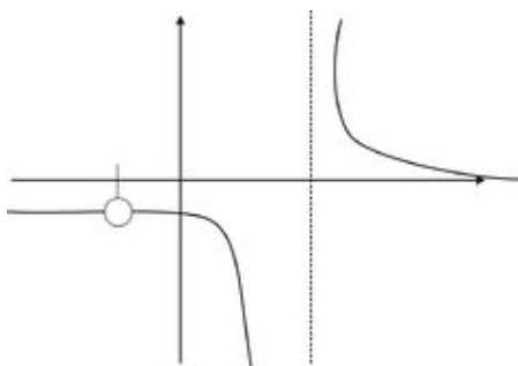
But what we actually have is the original function,

$$f(x) = \frac{(x+1)}{(x^2-1)} \quad (3.1.7)$$

that we know is not defined at $x=-1$. At $x=-1$, we have a hole in the graph, or a **discontinuity of the function** at $x=-1$. That is, the function is defined for all other x -values close to $x=-1$.

In addition, at $x=1$ the function values go to $\pm\infty$ depending on which side of $x=1$ is used. The right and left one-sided limits will not be the same, and the function is discontinuous at $x=1$. The discontinuity at $x=1$ is called an **infinite discontinuity** and is associated with the vertical asymptote $x=1$.

Loosely speaking, if we were to hand-draw the graph, we would need to take our pencil off the page when we got to this hole, leaving a gap in the graph as indicated:



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Examples



Example 1

Earlier, you were asked which of the following could be considered to represent a continuous function:

1. daily outdoor temperature variation
2. weight of a cart as 10 lb bags of sand are loaded
3. the rise and fall of the level of water in a reservoir
4. the amount of money collected at a movie theater for a specific showing

The daily outdoor temperature variation could be considered a continuous function of time as because all temperature values occur in the range of variation.

The weight of a cart as 10 lb bags of sand are loaded would not be considered a continuous function of the number of bags because only integer numbers of bags are possible and the total weight changes in 10 lb increments, so there are gaps.

The rise and fall of the level of water in a reservoir could be considered continuous function of time because all levels occur in the range of variation.

The amount of money collected at a movie theater for a specific showing would not be considered a continuous function since the number of people buying and the price per ticket are integers so that only certain total dollar amounts are possible.

Example 2

Find a value of k which makes the following function continuous:

$$f(x) = \begin{cases} kx + 1, & x \leq 3 \\ -(5 - x)^3, & x > 3 \end{cases}$$

If $f(x)$ is to be considered continuous everywhere, it must be continuous at $x=3$. This means that:

1. $x = 3$ must be in the domain of $f(x)$; it is.
2. $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x)$ so that $\lim_{x \rightarrow 3} f(x)$ would exist; which means the following equality must hold:

$$k(3) + 1 = -(5 - 3)^3, \text{ and } k = -3.$$

Then

$$\lim_{x \rightarrow 3} f(x) = -8$$

With the above result, we have

$$\lim_{x \rightarrow 3} f(x) = -8 = f(-3)$$

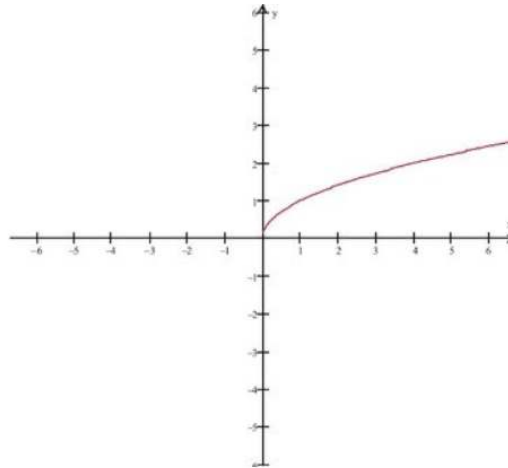
Therefore, the function

$$f(x) = \begin{cases} -3x + 1, & x \leq 3 \\ -(5 - x)^3, & x > 3 \end{cases}$$

is continuous at $x=3$.

Example 3

Let's recall our basic square root function, $f(x)=x^{0.5}$, shown below. Is it continuous at $x=0$?



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Since the domain of $f(x)=x^{0.5}$ is $x \geq 0$, we see that

$$\lim_{x \rightarrow 0} \sqrt{x} \quad (3.1.8)$$

does not exist. Specifically, we cannot find open intervals around $x = 0$ that satisfy the limit definition. Therefore, the function is not continuous at $x = 0$. However we do note that as we approach $x=0$ from the right-hand side, we see the successive values tending towards $x = 0$.

Review

1. Generate the graph of $f(x) = \frac{(|x+1|)}{(x+1)}$ using your calculator and discuss the continuity of the function.
2. Generate the graph of $f(x) = \frac{(3x-6)}{(x^2-4)}$ using your calculator and discuss the continuity of the function.
3. Is the function $f(x) = |x-3|$ continuous at $x = 3$?
4. Is the function $f(x) = \frac{6x-1}{4x^2+20x+25}$ continuous at $x = -\frac{5}{2}$?
5. Is the function $f(x) = \frac{3x^2-13x-10}{x^3-4x^2-2x-15}$ continuous at $x = 5$?
6. Find a value of k which makes the following function continuous: $f(x) = \begin{cases} x^2+2x+k, & x \leq -1 \\ -x-4, & x > -1 \end{cases}$
7. For the function $f(x) = \begin{cases} \sqrt{1-x}, & x < -3 \\ \frac{4}{3}x+6, & -3 < x \leq 3 \\ x^2, & x > 3 \end{cases}$ classify the discontinuities at
 - a. $x = -3$ and
 - b. $x = 3$.
8. For the function $f(x) = \frac{3x^3-x^2-27x+9}{3x^2+2x-1}$ classify the discontinuities at
 - a. $x = -1$ and
 - b. $x = \frac{1}{3}$.
9. On what intervals is $f(x) = \frac{(x^2-1)(4x+1)}{x(x^2+1)(4x+1)}$ continuous?
10. On what intervals is $f(x) = \frac{2}{x-3} + 3$ continuous?
11. On what interval is $f(x) = \cot^2(x)$ continuous?
12. True or False: $f(x) = \frac{\cos x}{x\sqrt{x-1}(2x-1)}$ is continuous on the interval $[0, 1]$.
13. True or False: $f(x) = \frac{\sin(x)}{\sin(3x)}$ is continuous on the interval $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$.
14. On what intervals is $f(x) = \sqrt{x^2-4x+2}$ continuous?
15. On what intervals is $f(x) = \sqrt{x^2+4x+5}$ continuous?

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 2.9.

Vocabulary

Term	Definition
Continuous	Continuity for a point exists when the left and right sided limits match the function evaluated at that point. For a function to be continuous, the function must be continuous at every single point in an unbroken domain.
Continuous Function	A continuous function is a function without breaks or gaps. It contains an infinite, uncountable number of values.
discontinuities	The points of discontinuity for a function are the input values of the function where the function is discontinuous.
discontinuous	A function is discontinuous if the function exhibits breaks or holes when graphed.
discontinuous at a point	A function is discontinuous at a point a if the function is not defined at $x=a$.
Infinite discontinuities	Infinite discontinuities occur when a function has a vertical asymptote on one or both sides. This will happen when a factor in the denominator of the function is zero.
Infinite discontinuity	Infinite discontinuities occur when a function has a vertical asymptote on one or both sides. This will happen when a factor in the denominator of the function is zero.
points of discontinuity	The points of discontinuity for a function are the input values of the function where the function is discontinuous.
Removable discontinuities	Removable discontinuities are also known as holes. They occur when factors can be algebraically canceled from rational functions.
Removable discontinuity	Removable discontinuities are also known as holes. They occur when factors can be algebraically canceled from rational functions.

Additional Resources

PLIX: Play, Learn, Interact, eXplore - Continuity

Video: Continuity Using Limits

Practice: Continuity of a Function

Real World: Mitochondrial DNA

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3.2: Continuity at a Point, Continuity Test, Types of Discontinuity

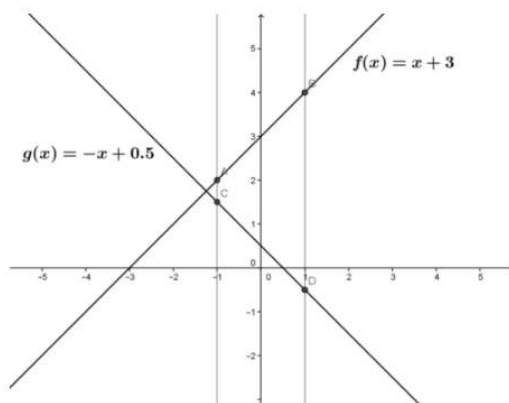
In the previous concept, the continuity of a function was presented intuitively through the imagery of using a pencil to draw a graph on a piece of paper (over a certain interval of the domain) and never lifting the pencil off the paper. As long as the pencil is not lifted off the paper, the graph can be considered to be **continuous** over that drawing interval. This is certainly not a rigorous description of continuity, but given this analogy, what expectations might you have about arithmetic operations on two functions that are both continuous over a common interval; or the existence of minima and maxima of a **continuous function**; or the existence of intermediate values between two function values?

Properties of Continuous Functions

The previous concept identified the characteristics of a function that is continuous at a point, and over an interval. Let's explore. Given two functions $f(x)$ and $g(x)$ that are continuous over a closed interval $[a,b]$, would you expect that arithmetic operations on these two functions would also yield functions continuous over $[a,b]$?

Given the functions $f(x)=x+3$ and $g(x)=-x+0.5$ in the closed interval $[-1,1]$, determine if $f(x)$ and $g(x)$ are continuous in the interval.

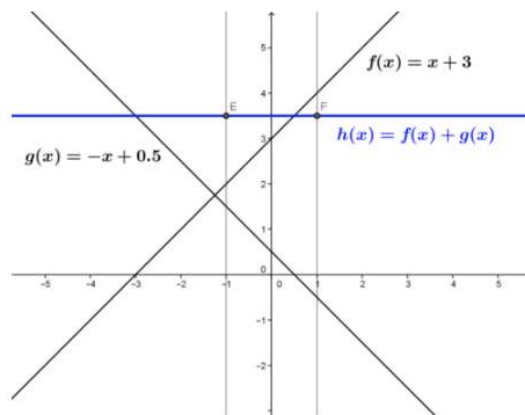
The functions $f(x)$ and $g(x)$ are shown in the graph. Inspection of each function graph and its equation, shows that they are each defined over the closed interval and the function limit at each point in the interval equals the function value at the point. Both functions are continuous in the interval.



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Using the same functions and interval as above, determine if $h(x)=f(x)+g(x)$ is continuous in the interval.

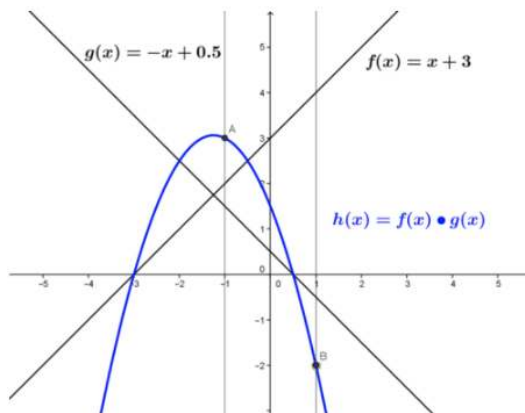
The sum of the two functions is given by $h(x)=3.5$, and is shown in the figure. The sum function, a constant, is defined over the closed interval and the function limit at each point in the interval equals the constant function value at each point. The sum function is continuous in the interval.



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Still using the interval and functions as above, determine if $h(x)=f(x)g(x)$ is continuous in the interval.

The product of the two functions is given by $h(x)=(x+3)(-x+0.5)=-x^2+2.5x-1.5$, and is shown in the figure. The product function, a parabola, is defined over the closed interval and the function limit at each point in the interval equals the product function value at each point. The product function is continuous in the interval.



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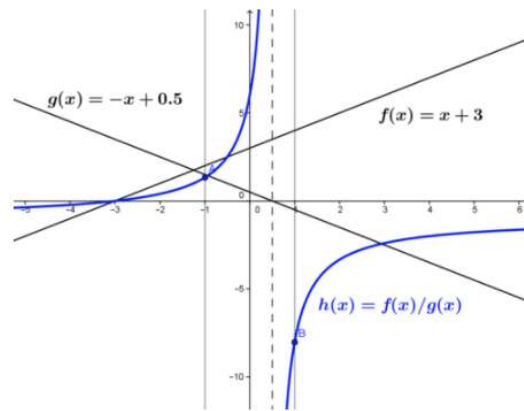
What about the quotient of two continuous functions?

Given the functions $f(x)=x+3$ and $g(x)=-x+0.5$ in the closed interval $[-1,1]$, determine if $f(x)/g(x)$ is continuous in the interval.

The quotient of the two functions is given by

$$h(x) = \frac{x + 3}{-x + 0.5}$$

and is shown in the figure.



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In the closed interval $[-1,1]$, $x=0.5$ is the only place where the function $h(x)$ is undefined, and

$$\lim_{x \rightarrow 0.5} h(x)$$

r in the closed interval.

The findings in the above simple functions can be generalized in the following properties.

If $f(x)$ and $g(x)$ are continuous at any real value c over the closed interval $[a,b]$, then the following are also continuous at any real value c over the closed interval $[a,b]$:

1. $f(x) + g(x)$
2. $f(x) - g(x)$
3. $f(x)g(x)$
4. $\frac{f(x)}{g(x)}$, as long as $g(c) \neq 0$

Intermediate Value Theorem and the Extreme Value (Min-Max) Theorem are two other properties of a function that is continuous over a closed interval.

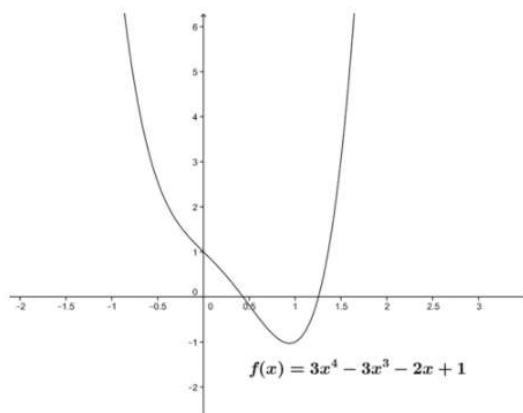
The Intermediate Value Theorem and Extreme Value Theorem

The **Intermediate Value Theorem** states that if a function is continuous on a closed interval $[a,b]$, then the function assumes every value between $f(a)$ and $f(b)$.

The Intermediate Value Theorem can be used to analyze and approximate zeros of functions.

Use the Intermediate Value Function to show that there is at least one zero of the function $f(x)=3x^4-3x^3-2x+1$ in the indicated interval $[1,2]$.

The graph of this function, shown below, is shaped somewhat like a parabola, and is continuous in the interval.



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In order to apply the Intermediate Value Theorem, we need to find a pair of x -values that have function values with different signs. A few values are shown in the table below.

x	1.1	1.2	1.3
$f(x)$	-0.80	-0.36	0.37

We see that the sign of the function values changes from negative to positive somewhere between 1.2 and 1.3. Hence, by the Intermediate Value theorem, there is some value c in the interval $(1.2, 1.3)$ such that $f(c)=0$.

The Extreme Value (Min-I



consequence of the Intermediate Value Theorem.

The **Extreme Value (Min-I**
maximum value and a min



tates that if a function $f(x)$ is continuous in a closed interval I , then $f(x)$ has both a

Examples

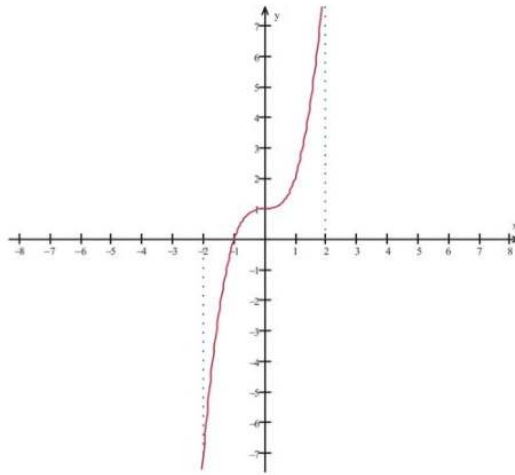
Example 1

Earlier, you were asked about your expectations for the continuity of functions formed by arithmetic operations on two continuous functions. The conclusion of this section is that, except for the quotient of functions, arithmetic operations on two functions that are continuous at a point or in an interval yield a new function that is continuous at the point or in the interval. With the quotient of two functions, the concern is always to identify where the denominator is 0; at this location the quotient function is not continuous.

Example 2

Consider $f(x)=x^3+1$ and interval $I=[-2,2]$. Determine minimum and maximum values.

Since the function is continuous on the closed interval I , this function has a minimum and a maximum on the interval. The function graph shows that at $x=-2$ the function has a minimum value $f(-2)=-7$; and at $x=2$, a maximum value $f(2)=9$.



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Review

For #1-5, explain how you know that the function has a root in the given interval. (Hint: Use the Intermediate Value Function to show that there is at least one zero of the function in the indicated interval.):

1. $f(x)=x^3+2x^2-x+1$, in the interval $[-3,-2]$.
2. $f(x)=x^5-x\sqrt{3}-1$, in the interval $[9,10]$.
3. $f(x)=x^2+x-2$, in the interval $[-3,0]$.
4. $f(x)=4x^2-1x^2+3x+2$, in the interval $[-1,0]$.
5. $f(x)=2x+3-4$, in the interval $[-3,0]$.
6. True or False: $f(x)=\sin(x)+\cos(x)$ has a root on the interval $[0,\pi]$.
7. True or False: By the Intermediate Value Theorem $f(x)=\sin(x)+\cos^2(x)$ has no root on the interval $[0,\pi]$ since $f(0)=f(\pi)=1$.
8. True or False: $f(x)=x^2+1\cos(x)$ has a root on the interval $[0,\pi]$.
9. On which interval is $f(x)=x-\sin(x)+1$ guaranteed by the IVT to have a root?

a. $\left[0, \frac{2\pi}{3}\right]$

b. $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$

c. $\left[-\frac{2\pi}{3}, 0\right]$

d. $[-\pi, \pi]$

10. Find an interval on which $f(x)=e^x+x$ has a root.
11. Find intervals on which $f(x)=x^3+5x^2-4x-20$ has a root.

For #12-15, use the Extreme Value Theorem to determine whether the given statement is true or false.

12. The Extreme Value Theorem guarantees that the function $|\sin(x3)|^{0.5}$ has a minimum value on the interval $[-3, 3]$.
13. The Extreme Value Theorem guarantees that the function $(x-1)^3$ has a maximum value on the interval $[1, 3]$.
14. The Extreme Value Theorem guarantees that the function x^2+3x+2 has a minimum value on the interval $[-3, 0]$.
15. The Extreme Value Theorem guarantees that the function $4x/x^2+3x+2$ has a minimum value on the interval $[-3, 0]$.

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 2.10.

Vocabulary

Term	Definition
Continuous	Continuity for a point exists when the left and right sided limits match the function evaluated at that point. For a function to be continuous, the function must be continuous at every single point in an unbroken domain.
Continuous Function	A continuous function is a function without breaks or gaps. It contains an infinite, uncountable number of values.
discontinuous at a point	A function is discontinuous at a point a if the function is not defined at $x=a$.
extreme value theorem	The extreme value theorem states that in every interval $[a,b]$ where a function is continuous there is at least one maximum and one minimum. In other words, it must have at least two extreme values.
intermediate value theorem	The intermediate value theorem states that if $f(x)$ is continuous on some interval $[a,b]$ and n is between $f(a)$ and $f(b)$, then there is some $c \in [a,b]$ such that $f(c)=n$.
points of discontinuity	The points of discontinuity for a function are the input values of the function where the function is discontinuous.

Additional Resources

PLIX: Play, Learn, Interact, eXplore - Properties of Continuous Functions

Video: Continuity and the Intermediate Value Theorem

Practice: Continuity at a Point, Continuity Test, Types of Discontinuity

Real World: New Orbits

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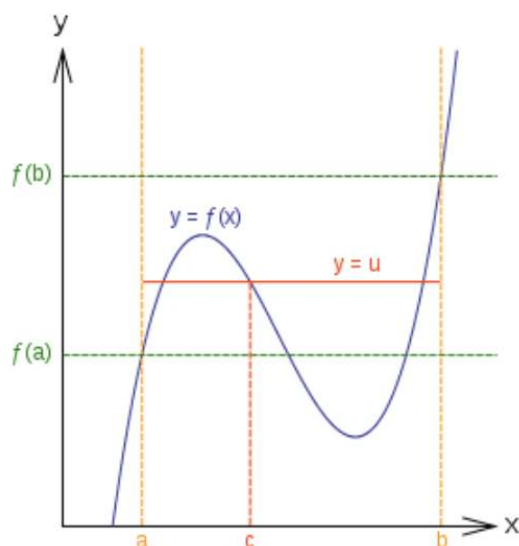
3.3: Intermediate Value Theorem, Existence of Solution

While the idea of **continuity** may seem somewhat basic, when a function is **continuous** over a closed interval like $x \in [1, 4]$, you can actually draw some major conclusions. The conclusions may be obvious when you understand the statements and look at a graph, but they are powerful nonetheless.

What can you conclude using the **Intermediate Value Theorem** and the **Extreme Value Theorem** about a function that is continuous over the closed interval $x \in [1, 4]$?

The Intermediate and Extreme Value Theorems

The **Intermediate Value Theorem** states that if a function is continuous on a closed interval and u is a value between $f(a)$ and $f(b)$ then there exists a $c \in [a, b]$ such that $f(c) = u$.



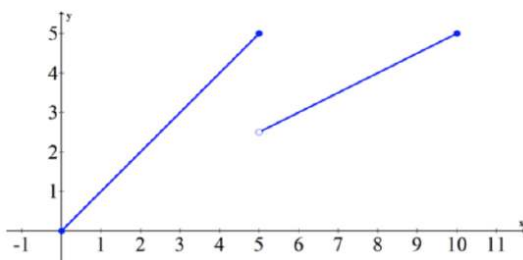
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Simply stated, if a function is continuous between a low point and a high point, then it must be valued at each intermediate height in between the low and high points.

The **converse** of an if then statement is a new statement with the hypothesis of the original statement switched with the conclusion of the original statement. In other words, the converse is when the if part of the statement and the then part of the statement are swapped. In general, the converse of a statement is not true.

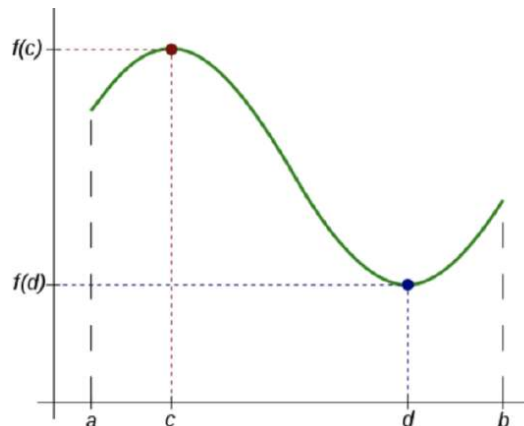
The converse of the Intermediate Value Theorem is: If there exists a value $c \in [a, b]$ such that $f(c) = u$ for every u between $f(a)$ and $f(b)$ then the function is continuous.

This statement is false. In order to show the statement is false, all you need is one **counterexample** where every intermediate value is hit and the function is discontinuous. A **counterexample** to an if then statement is when the hypothesis (the if part of the sentence) is true, but the conclusion (the then part of the statement) is not true.



This function is discontinuous on the interval $[0,10]$ but every intermediate value between the first height at $(0,0)$ and the height of the last point $(10,5)$ is hit.

The **Extreme Value Theorem** states that on every interval $[a,b]$ where a function is continuous there is at least one maximum and one minimum. In other words, on every interval $[a,b]$ where a function is continuous there is at least two extreme values.

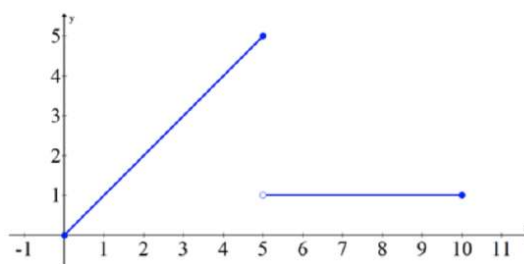


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http://commons.wikimedia.org/wiki/File:Extreme_Value_Theorem.svg; <https://commons.wikimedia.org/wiki/File:CentralParkFromAboveCropped.jpg> - CC BY-SA

The converse of the Extreme Value Theorem is: If there is at least one maximum and one minimum in the closed interval $[a,b]$ then the function is continuous on $[a,b]$.

This statement is false. In order to show the statement is false, all you need is one counterexample. The goal is to find a function on a closed interval $[a,b]$ that has at least one maximum and one minimum and is also discontinuous.



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On the interval $[0,10]$, the function attains a maximum at $(5,5)$ and a minimum at $(0,0)$ but is still discontinuous.

Examples

Example 1

Earlier, you were asked to apply the Intermediate and Extreme Value Theorems to a function is continuous on the interval $x \in [1,4]$. You can conclude by the Intermediate Value Theorem that there exists a $c \in [1,4]$ such that $f(c)=u$ for every u between $f(1)$ and $f(4)$. You can also conclude that on this interval the function has both a maximum and a minimum value by the Extreme Value Theorem.

Example 2

Use the Intermediate Value Theorem to show that the function $f(x)=(x+1)^3-4$ has a zero on the interval $[0,3]$.

First note that the function is a cubic and is therefore continuous everywhere.

- $f(0)=(0+1)^3-4=1^3-4=-3$
- $f(3)=(3+1)^3-4=4^3-4=60$

By the Intermediate Value Theorem, there must exist a $c \in [0,3]$ such that $f(c)=0$ since 0 is between -3 and 60.

Example 3

Use the Intermediate Value Theorem to show that the following equation has at least one real solution.

$$x^8=2^x$$

First rewrite the equation: $x^8-2^x=0$

Then describe it as a continuous function: $f(x)=x^8-2^x$

This function is continuous because it is the difference of two continuous functions.

- $f(0)=0^8-2^0=0-1=-1$
- $f(2)=2^8-2^2=256-4=252$

By the Intermediate Value Theorem, there must exist a c such that $f(c)=0$ because $-1 < 0 < 252$. The number c is one solution to the initial equation.

Example 4

Show that there is at least one solution to the following equation.

$$\sin x = x + 2$$

Write the equation as a continuous function: $f(x)=\sin x - x - 2$

The function is continuous because it is the sum and difference of continuous functions.

- $f(0)=\sin 0 - 0 - 2 = -2$
- $f(-\pi)=\sin(-\pi) - \pi - 2 = 0 - \pi - 2 < 0$

By the Intermediate Value Theorem, there must exist a c such that $f(c)=0$ because $-2 < 0 < \pi - 2$. The number c is one solution to the initial equation.

Example 5

When are you not allowed to use the Intermediate Value Theorem?

The Intermediate Value Theorem should not be applied when the function is not continuous over the interval.

Review

Use the Intermediate Value Theorem to show that each equation has at least one real solution.

1. $\cos x = -x$
2. $\ln(x) = e^{-x} + 1$
3. $2x^3 - 5x^2 = 10x - 5$
4. $x^3 + 1 = x$
5. $x^2 = \cos x$
6. $x^5 = 2x^3 + 2$
7. $3x^2 + 4x - 11 = 0$

8. $5x^4 = 6x^2 + 1$
9. $7x^3 - 18x^2 - 4x + 1 = 0$
10. Show that $f(x) = 2^{x-3}/2x-5$ has a real root on the interval $[1,2]$.
11. Show that $f(x) = 3^{x+1}/2x+4$ has a real root on the interval $[-1,0]$.
12. True or false: A function has a maximum and a minimum in the closed interval $[a,b]$; therefore, the function is continuous.
13. True or false: A function is continuous over the interval $[a,b]$; therefore, the function has a maximum and a minimum in the closed interval.
14. True or false: If a function is continuous over the interval $[a,b]$, then it is possible for the function to have more than one relative maximum in the interval $[a,b]$.
15. What do the Intermediate Value and Extreme Value Theorems have to do with continuity?

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 14.7.

Vocabulary

Term	Definition
continuity	Continuity for a point exists when the left and right sided limits match the function evaluated at that point. For a function to be continuous, the function must be continuous at every single point in an unbroken domain.
Continuous	Continuity for a point exists when the left and right sided limits match the function evaluated at that point. For a function to be continuous, the function must be continuous at every single point in an unbroken domain.
converse	If a conditional statement is $p \rightarrow q$ (if p, then q), then the converse is $q \rightarrow p$ (if q, then p). Note that the converse of a statement is not true just because the original statement is true.
counterexample	A counterexample is an example that disproves a conjecture.
extreme value theorem	The extreme value theorem states that in every interval $[a,b]$ where a function is continuous there is at least one maximum and one minimum. In other words, it must have at least two extreme values.
intermediate value theorem	The intermediate value theorem states that if $f(x)$ is continuous on some interval $[a,b]$ and n is between $f(a)$ and $f(b)$, then there is some $c \in [a,b]$ such that $f(c) = n$.

Additional Resources

PLIX: Play, Learn, Interact, eXplore - Function Exploration

Video: Descartes' Rule of Signs - Example 1

Practice: Intermediate Value Theorem, Existence of Solutions

Real World: Ups and Downs

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3.4: Find Imaginary Solutions

Louis calculates that the area of a rectangle is represented by the equation $3x^4 + 7x^2 = 2$. Did Louis calculate it right? Explain based on the degree and zeros of the function.

Imaginary Solutions

Remember, imaginary solutions always come in pairs. To find the imaginary solutions to a function, use the Quadratic Formula.

Let's solve $f(x) = 3x^4 - x^2 - 1$.



First, this quartic function : like a quadratic equation.

$$g(x) = x^4 + 21x^2 + 90$$

Now, because neither factor further and there is no x -term, we can set each equal to **zero** and solve.

$$\begin{aligned} x^2 + 2 &= 0 \\ x^2 &= -2 \\ x &= \pm\sqrt{-2} \text{ or } \pm i\sqrt{2} \end{aligned} \quad \text{and} \quad \begin{aligned} 3x^2 - 7 &= 0 \\ 3x^2 &= 7 \\ x^2 &= \frac{7}{3} \\ x &= \pm\sqrt{\frac{7}{3}} \text{ or } \pm \frac{\sqrt{21}}{3} \end{aligned}$$

Including the imaginary solutions, there are four, which is what we would expect because the degree of this function is four.

Now, let's find all the solutions of the function $g(x) = x^4 + 21x^2 + 90$.

When graphed, this function does not touch the x -axis. Therefore, all the solutions are imaginary. To solve, this function can be factored like a quadratic equation. The factors of 90 that add up to 21 are 6 and 15.

$$g(x) = x^4 + 21x^2 + 90$$

$$0 = (x^2 + 6)(x^2 + 15)$$

Now, set each factor equal to zero and solve.

$$\begin{aligned} x^2 + 6 &= 0 \\ x^2 &= -6 \\ x &= \pm i\sqrt{6} \end{aligned} \quad \text{and} \quad \begin{aligned} x^2 + 15 &= 0 \\ x^2 &= -15 \\ x &= \pm i\sqrt{15} \end{aligned}$$

Finally, let's find the function that has the solution 3, -2, and $4+i$.

Notice that one of the given solutions involves an **imaginary number**. Imaginary and complex solutions always come in pairs, so $4-i$ is also a factor. The two factors are **complex conjugates**. Translate each solution into a factor and multiply them all together.

$$\begin{aligned} &(x-3)(x+2)(x-(4+i))(x-(4-i)) \\ &(x^2+2x-3x-6)(x^2-4x+ix-4x+16-4i-ix+4i-i^2) \\ &(x^2-x-6)(x^2-8x+17) \\ &(x^4-8x^3+17x^2-x^2+8x^2-17x-6x^2+48x-102) \\ &(x^4-9x^3+19x^2+31x-102) \end{aligned}$$

Remember!
 $i^2 = -1$

Bahman Kalantari - https://commons.wikimedia.org/wiki/File%3ANonfractal_polynomiograph.png

Any multiple of this function would also have these **roots**. For example, $2x^4 - 18x^3 + 38x^2 - 204x - 204$ would have these roots as well.

Examples

Example 1

Earlier, you were asked to determine if Louis calculated his work correctly.

First we need to change the equation to standard form. Then we can factor it.

$$\begin{aligned} 3x^4 + 7x^2 &= 2 \\ 3x^4 + 7x^2 - 2 &= 0 \\ (3x^2 + 1)(x^2 + 2) &= 0 \end{aligned}$$

Solving for x we get

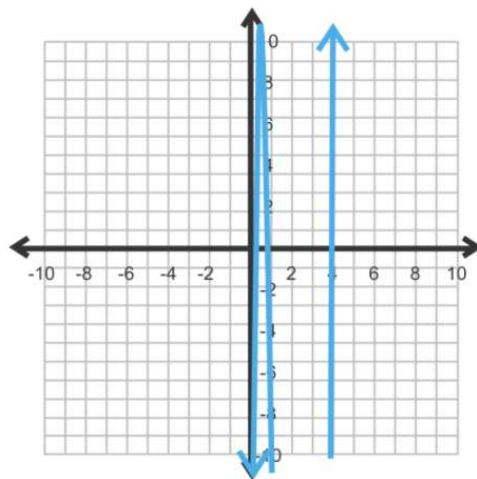
$$\begin{aligned} 3x^2 + 1 &= 0 & x^2 + 2 &= 0 \\ x^2 &= \frac{-1}{3} & \text{and} & & x^2 &= -2 \\ x &= \pm i\sqrt{\frac{1}{3}} & & & x &= \pm i\sqrt{2} \end{aligned}$$

All of the solutions are imaginary and the area of a rectangle must have real solutions. Therefore Louis did not calculate correctly.

Example 2

Find all the solutions to the following function: $f(x) = 25x^3 - 120x^2 + 81x - 4$.

First, graph the function.



Using the Rational Root Theorem, the possible realistic zeros could be $\frac{1}{25}$, 1, or 4. Let's try these three possibilities using synthetic division.

$$\begin{array}{r}
 \frac{1}{25} \overline{) 25 - 120 \quad 81 \quad -4} \\
 \underline{25 \quad -119 \quad -37.96 \quad -5.5184} \\
 \\
 \frac{1}{25} \overline{) 25 - 120 \quad 81 \quad -4} \\
 \underline{25 \quad -95 \quad -14 \quad -18} \\
 \\
 \frac{4}{25} \overline{) 25 - 120 \quad 81 \quad -4} \\
 \underline{25 \quad -20 \quad 1 \quad 0}
 \end{array}$$

Of these three possibilities, only 4 is a zero. The leftover **polynomial**, $25x^2 - 20x + 1$ is not factorable, so we need to use the Quadratic Formula to find the last two zeros.

$$\begin{aligned}
 x &= \frac{20 \pm \sqrt{20^2 - 4(25)(1)}}{2(25)} \\
 &= \frac{20 \pm \sqrt{400 - 100}}{50} \\
 &= \frac{20 \pm 10\sqrt{3}}{50} \text{ or } \frac{2 \pm \sqrt{3}}{5} \approx 0.746 \text{ and } 0.054
 \end{aligned}$$

Helpful Hint: Always find the decimal values of each zero to make sure they match up with the graph.

Example 3

Find all the solutions to the following function: $f(x) = 4x^4 + 35x^2 - 9$.

$f(x) = 4x^4 + 35x^2 - 9$ is factorable. $ac = -36$.

$$\begin{aligned}
 &4x^4 + 35x^2 - 9 \\
 &4x^4 + 36x^2 - x^2 - 9 \\
 &4x^2(x^2 + 9) - 1(x^2 + 9) \\
 &(x^2 + 9)(4x^2 - 1)
 \end{aligned}$$

Setting each factor equal to zero, we have:

$$\begin{array}{ll}
 x^2 + 9 = 0 & 4x^2 - 1 = 0 \\
 x^2 = -9 & 4x^2 = 1 \\
 x = \pm 3i & \text{or } x^2 = \frac{1}{4} \\
 & x = \pm \frac{1}{2}
 \end{array}$$

Example 4

Find the equation of a function with roots 4, $2^{0.5}$ and $1-i$.

Recall that irrational and imaginary roots come in pairs. Therefore, all the roots are 4, $2^{0.5}$, $-2^{0.5}$, $1+i$, $1-i$. Multiply all 5 roots together.

Review

Find all solutions to the following functions. Use any method.

1. $f(x)=x^4+x^3-12x^2-10x+20$
2. $f(x)=4x^3-20x^2-3x+15$
3. $f(x)=2x^4-7x^2-30$
4. $f(x)=x^3+5x^2+12x+18$
5. $f(x)=4x^4+4x^3-22x^2-8x+40$
6. $f(x)=3x^4+4x^2-15$
7. $f(x)=2x^3-6x^2+9x-27$
8. $f(x)=6x^4-7x^3-280x^2-419x+280$
9. $f(x)=9x^4+6x^3-28x^2+2x+11$
10. $f(x)=2x^5-19x^4+30x^3+97x^2-20x+150$

Find a function with the following roots.

11. $4, i$
12. $-3, -2i$
13. $5^{0.5}, -1+i$
14. $2, 13, 4-2^{0.5}$
15. **Writing:** Write down the steps you use to find all the zeros of a polynomial function.
16. **Writing:** Why do imaginary and irrational roots always come in pairs?
17. **Challenge:** Find all the solutions to $f(x)=x^5+x^3+8x^2+8$.

Answers for Review Problems

To see the Review answers, open this [PDF file](#) and look for section 6.12.

Vocabulary

Term	Definition
Complex Conjugate	Complex conjugates are pairs of complex binomials. The complex conjugate of $a+bi$ is $a-bi$. When complex conjugates are multiplied, the result is a single real number.
complex number	A complex number is the sum of a real number and an imaginary number, written in the form $a+bi$.
conjugate pairs theorem	The conjugate pairs theorem states that if $f(z)$ is a polynomial of degree n , with $n \neq 0$ and with real coefficients, and if $f(z_0)=0$, where $z_0=a+bi$, then $f(z^*0)=0$. Where z^*0 is the complex conjugate of z_0 .
fundamental theorem of algebra	The fundamental theorem of algebra states that if $f(x)$ is a polynomial of degree $n \geq 1$, then $f(x)$ has at least one zero in the complex number domain. In other words, there is at least one complex number c such that $f(c)=0$. The theorem can also be stated as follows: an n th degree polynomial with real or complex coefficients has, with multiplicity, exactly n complex roots.
Imaginary Number	An imaginary number is a number that can be written as the product of a real number and i .
Imaginary Numbers	An imaginary number is a number that can be written as the product of a real number and i .
Polynomial	A polynomial is an expression with at least one algebraic term, but which does not indicate division by a variable or contain variables with fractional exponents.
Roots	The roots of a function are the values of x that make y equal to zero.
Zero	The zeros of a function $f(x)$ are the values of x that cause $f(x)$ to be equal to zero.
Zeroes	The zeroes of a function $f(x)$ are the values of x that cause $f(x)$ to be equal to zero.

Additional Resources

PLIX: Play, Learn, Interact, eXplore - Imaginary Zeros

Video: Fundamental Theorem of Algebra - Overview

Practice: Find Imaginary Solutions

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CHAPTER OVERVIEW

4: Differentiation - Slope Models using Derivatives

Did you know that a derivative is a tool that you can use when working with slope, tangents and functions? First, let's think about why a derivative would be useful. Do you remember how to find the slope of a line? Well, if you think back to Algebra, you could find the slope of a line a couple of different ways - one way is through an equation, but you could also figure out the slope of a line by using two points. But what if you only had one point to work with? What if you wanted to figure out the slope of a tangent line at that one point? Well, this is the perfect time to use a derivative. A derivative makes it possible for you get as close as you can to that slope. Take a look.

Here are some additional resources that provide a more in-depth exploration of this topic.

- [Constant Derivatives and the Power Rule](#)
- [Derivatives of Sums and Differences](#)
- [Quotient Rule and Higher Derivatives](#)

Practice: Evaluate Limits using Tables

Real World: Death Watch

- [4.1: Average and Instantaneous Rates of Change](#)
- [4.2: Slope of Tangent Line](#)
- [4.3: Tangent Line Approximation](#)

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4.1: Average and Instantaneous Rates of Change

You may recall the story about Jim and his track-star girlfriend Becca from the lesson on understanding limits. The two of them were discussing how they might calculate her speed at the instant that Jim snapped a photo of her. The end result of the discussion was Becca pointing out that it is not technically possible to calculate the exact speed of something in a specific instant.

By now, we have explored the related concepts of limits and lines tangent to a curve, so we know it is possible to effectively calculate **instantaneous speed**. What process would be involved with actually calculating Becca's speed at the instant the photo was taken? What are the technical difficulties with identifying instantaneous speed?

Instantaneous Rates of Change

The function $f'(x)$ that we defined in previous lessons is so important that it has its own name: the **derivative**.

The Derivative

The function f' is defined by the formula

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

where f' is called the derivative of f with respect to x . The domain of f' consists of all the values of x for which the **limit** exists.

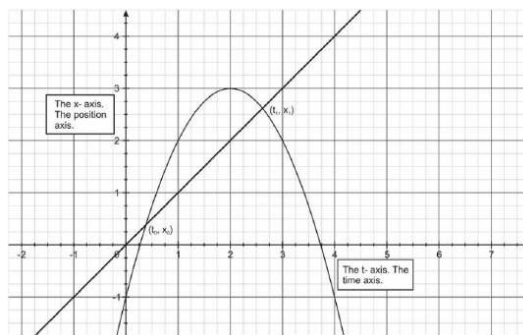
Based on the discussion that we have had in previous section, the derivative f' represents the **slope** of the **tangent line** at point x . Another way of interpreting it would be that the function $y = f(x)$ has a derivative f' whose value at x is the **instantaneous rate of change** of y with respect to point x .

One of the two primary concepts of calculus involves calculating the rate of change of one quantity with respect to another. For example, speed is defined as the rate of displacement with respect to time. If a person travels 120 miles in 4 hours, his speed is $120/4 = 30$ mi/hr. This speed is called the **average speed** or the **average rate of change** of distance with respect to time. Of course the person who travels 120 miles at a rate of 30 mi/hr for 4 hr probably does not do so continuously. Though he probably slowed down or sped up during the 4-hour period, it does generally suffice to say that he traveled for 4 hours at an average rate of 30 miles per hour. However, if the driver strikes a tree, it would not be his average speed that determines his survival but his speed at the *instant of the collision*. Similarly, when a bullet strikes a target, it is not the average speed that is significant but its **instantaneous speed** at the moment it strikes. So here we have distinct kinds of speeds, average speed and instantaneous speed.

The average speed of an object is defined as the object's displacement Δx divided by the time interval Δt during which the displacement occurs:

$$\text{Average speed} = v = \frac{\Delta x}{\Delta t} = \frac{x_1 - x_0}{t_1 - t_0}$$

Notice that the points (t_0, x_0) and (t_1, x_1) lie on the position versus time curve, as the figure below shows.

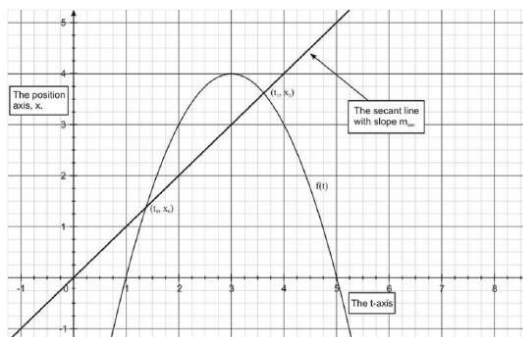


This expression is also the expression for the slope of a **secant line** connecting the two points. Thus we conclude that the average velocity of an object between time t_0 and t_1 is represented geometrically by the slope of the secant line connecting the two points (t_0, x_0) and (t_1, x_1) . If we choose t_1 close to t_0 , then the average velocity will closely approximate the instantaneous velocity at time t_0 .

Geometrically, the average rate of change is represented by the slope of a secant line (figure a, below) and the instantaneous rate of change is represented by the slope of the tangent line (figure b, below).

Average Rate of Change (such as *the average velocity*) The average rate of change of $y = f(x)$ over the time interval $[x_0, x_1]$ is the slope m_{sec} of the secant line to the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ on the graph (figure a):

$$m_{sec} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$



By now, we have explored the related concepts of limits and lines tangent to a curve, so we know it is possible to effectively calculate **instantaneous speed**. What process would be involved with actually calculating Becca's speed at the instant the photo was taken? What are the technical difficulties with identifying instantaneous speed?

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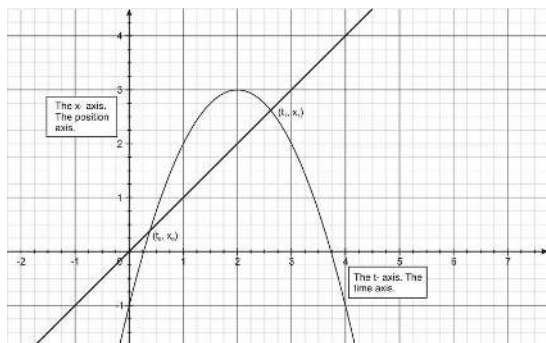
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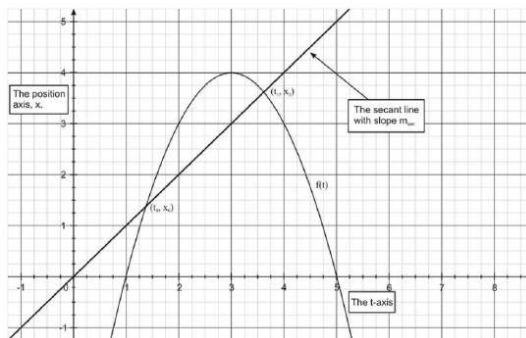
$$\begin{aligned} \text{Average speed} \\ = v = \frac{\Delta x}{\Delta t} = \frac{x_1 - x_0}{t_1 - t_0} \end{aligned}$$

Notice that the points (t_0, x_0) and (t_1, x_1) lie on the position versus time curve, as the figure below shows.



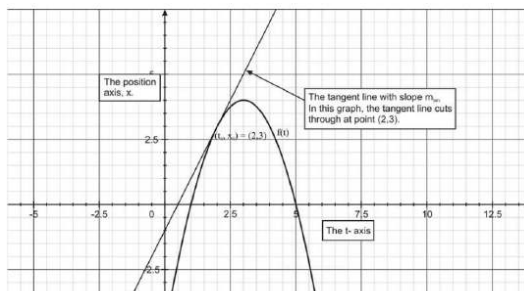
This expression is also the expression for the slope of a **secant line** connecting the two points. Thus we conclude that the average velocity of an object between time t_0 and t_1 is represented geometrically by the slope of the secant line connecting the two points (t_0, x_0) and (t_1, x_1) . If we choose t_1 close to t_0 , then the average velocity will closely approximate the instantaneous velocity at time t_0 .

Geometrically, the average rate of change is represented by the slope of a secant line (figure a, below) and the instantaneous rate of change is represented by the slope of the tangent line (figure b, below).



Instantaneous Rate of Change The instantaneous rate of change of $y = f(x)$ at the point x_0 is the slope m_{sec} of the tangent line to the point x_0 on the graph (figure b):

$$m_{\text{tan}} = f'(x_0) = \lim_{x_1 \rightarrow x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$



Examples

Example 1

Earlier, you were asked a () process of calculating Becca's speed in the race.

Speed is, by definition, a relation between a distance and the time required to cross that distance (recall $d=rt$ from your science class). If the time required is *zero*, then you end up dividing by zero, which is an undefined function.

However, using calculus, you can identify what the **end behavior** of the function would be if you were to get infinitely close, and thereby effectively calculate the speed at a given instant. In fact, that is exactly what you did in the examples above.

Calculate: (a) The slope of the line tangent to $y=x^2+5$ at the point on the curve $x=4$ and (b) the equation of that line.

In the previous concept "Tangents to a Curve" we showed how to calculate the **derivative** (the slope of the tangent) of a function of the form $y=x^2-c$.

1. The slope of the tangent to the curve $y=x^2+5$ at $x=4$ is:

$$2(4)=8$$

Recall from algebra the "slope - intercept form" of a straight line: $y=mx+b$, where m is the slope of the line.

2. Given $m=8$ from above, we have $y=8x+b$. All we need is to solve for b to have the full equation.

Substitute 4 (the point we are finding the slope of) in for x in the equation of the curve $y=x^2+5$ to identify a corresponding y

$$y=(4)^2+5$$

$$y=21$$

Substitute the values for x , y , and m that we now have into our $y=mx+b$ form of the equation of the tangent line:

$$21=8(4)+b$$

$$-11=b$$

Use the calculated values for m and b to complete the equation:

$$y=8x-11$$

Example 3

Suppose that $y=x^2-3$.

(a) Find the average rate of change of y with respect x over the interval $[0, 2]$ and (b) find the instantaneous rate of change of y with respect x at the point $x = -1$.

1. Applying the formula above for secant with $f(x)=x^2-3$ and $x_0=0$ and $x_1=2$, yields

$$\begin{aligned}
 m_{sec} &= \frac{f(x_1) - f(x_0)}{t_1 - t_0} \\
 &= \frac{f(2) - f(0)}{2 - 0} \\
 &= \frac{1 - (-3)}{2} \\
 &= 2
 \end{aligned}$$

This means that the average rate of change of y is 2 units per unit increase in x over the interval $[0,2]$.

b. Recall, for functions of the form $y=x^2+c$ that $f'(x)=2x$ ($f'(x)$ is " f prime of x ," meaning "the slope (m) of the line tangent to x ")

$$\begin{aligned}
 m_{tan} &= f'(x_0) \\
 &= f'(-1) \\
 &= 2(-1) \\
 &= -2
 \end{aligned}$$

This means that the instantaneous rate of change is negative. That is, y is decreasing at the point $x = -1$. It is decreasing at a rate of 2 units per unit increase in x .

Example 4

Find the derivative of $f(x)=x^{0.5}$ and the equation of the tangent line at $x_0 = 1$.

Using the definition of the derivative,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \frac{x+h-x}{\sqrt{x+h} + \sqrt{x}} \\
 &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\
 &= \frac{1}{2\sqrt{x}}
 \end{aligned}$$

Thus, the slope of the tangent line at $x_0 = 1$ is

$$f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}$$

For $x_0 = 1$, we can find y_0 by simply substituting into $f(x)$:

$$\begin{aligned}
 f(x_0) &\equiv y_0 \\
 f(1) &= \sqrt{1} = 1 \\
 y_0 &= 1
 \end{aligned}$$

Thus the equation of the tangent line is

$$\begin{aligned}
 y - y_0 &= m(x - x_0) \\
 y - 1 &= \frac{1}{2}(x - 1) \\
 y &= \frac{1}{2}x + \frac{1}{2}
 \end{aligned}$$

Example 5

Find the derivative of $f(x) = \frac{x}{x+1}$.

Using the derivative formula:

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{\left[\frac{x+h}{x+h+1} - \frac{x}{x+1} \right]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{x+h}{x+h+1} - \frac{x}{x+1} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{(x+h)(x+1) - x(x+h+1)}{(x+h+1)(x+1)} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{x^2 + x + hx + h - x^2 - xh - x}{(x+h+1)(x+1)} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{h}{(x+h+1)(x+1)} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{(x+h+1)(x+1)} \\
 &= \frac{1}{(x+1)^2}
 \end{aligned}$$

A rocket is propelled upwards and reaches a height of $h(t) = 4.9t^2$ in t seconds.

1. How high does it reach in 35 seconds?

The height of the rocket at 35 secs is $4.9(35)^2 = 6002.5\text{m}$

2. What is the average velocity of the rocket during the first 35 seconds?

$$V_{\text{avg}} = 6002.5\text{m} / 35\text{s} = 171.5\text{m/s}$$

3. What is the instantaneous velocity of the rocket at the end of the 35 seconds?

To find the instantaneous speed, we need to find the derivative of $h(t) = 4.9t^2$

$$h'(t) = 9.8t \text{ Using the instantaneous rate of change formula from above}$$

$$9.8 \cdot 35\text{s} = 343\text{m/sec}$$

Example 7

A particle moves in the positive direction along a straight line so that after t nanoseconds, its traversed distance is given by $\chi(t) = 9.9t^3$ nanometers.

1. What is the average velocity of the particle during the first 2 nanoseconds?

The particle moves $9.9t^3\text{nm}$ in t secs

$$\therefore 9.9(2^3) = 79.2\text{nm} / 2\text{s} \rightarrow 39.6\text{nm} / 1\text{s}$$

2. What is the instantaneous velocity of the particle at $t = 2$ nanoseconds?

Using the formula for finding the derivative, we get $\chi'(t) = 29.7t^2$

$$\therefore \chi'(2) = 118.8\text{nm/s}$$

$\therefore$ The instantaneous velocity at $t=2$ is 118.8 nm/sec

Review

Find the average rate of change:

1. $C=f(x)$ and $f(x)=x^2-4x+2$. Find the average rate of change of (C) with respect to (x) , when (x) is changed from $x = 15$, to $x = 59$.
2. $H=f(x)$ and $f(x)=x^2-5x+201$ Find the average rate of change of (H) , with respect to (x) when (x) is changed from $x = 10$ to $x = 11$.

3. $N=f(x)$ and $f(x)=3x^2-4x-1$ Find the average rate of change of (N) with respect to (x) when (x) is changed from $x = 20$ to $x = 64$.
4. $H=f(x)$ and $f(x)=x^2+10x+201$ Find the average rate of change of (H) with respect to (x) when (x) is changed from $x + 25$ to $x = 74$.
5. $N=f(x)$ and $f(x)=-5x^2-3x-4$ Find the average rate of change of (N) with respect to (x) when (x) is changed from $x = 30$ to $x = 54$.

Find the instantaneous rate of change:

6. If $C=f(x)$ and $f(x)=-4x^2+2x+5$ Find the instantaneous rate of change of (C) with respect to (x) when $x = 25$.
7. If $N=f(x)$ and $f(x)=3x^2-x-5$ Find the instantaneous rate of change of (N) with respect to x when $x = 10$.
8. $H=f(x)$ and $f(x)=4x^2+195$ Find the instantaneous rate of change of (H) with respect to x when $x = 10$.
9. $N=f(x)$ and $f(x)=-x^2+x-3$ Find the instantaneous rate of change of (N) with respect to x when $x = 150$.
10. $C=f(x)$ and $f(x)=-3x^2+4x-4$ Find the instantaneous rate of change of (N) with respect to x when $x = 20$.

Use the definition of the derivative to find $f'(x)$ and then find the equation of the tangent line at $x = x_0$.

11. $f(x) = 6x^2$; $x_0 = 3$.
12. $f(x)=(x+2)^{0.5}$; $x_0=8$
13. $f(x) = 3x^3 - 2$; $x_0 = -1$
14. $f(x)=1x+2$; $x_0=-1$
15. $f(x) = ax^2 - b$, (where a and b are constants); $x_0 = b$
16. $f(x) = x^{1/3}$; $x_0 = 1$.
17. Suppose that f has the property that $f(x + y) = f(x) + f(y) + 3xy$ and $\lim_{h \rightarrow 0} \frac{f(h)}{h} = 4$. Find $f(0)$ and $f'(x)$.
18. Find dy/dx

Solve the rate of change problems.

19. A packing company in the south makes "Mama's Spaghetti Sauce." The cost of producing x jars is $J = f(x)$ dollars. What does $f'(100) = 9999$ mean in this context?
20. A cherry pie is taken from an oven when its temperature is 202°F and placed on a table in a room where the temperature is 75°F . The temperature of the pie over x minutes is given by $T = f(x)$. What does $f'(100) = 102$ mean in this context?
21. The number of virus, after (x) hours, in a controlled laboratory experiment is $V = f(x)$. What are the units of $f'(x)$?
22. The number of people in the US affected by the common cold in the month of November is defined by $N = f(x)$ where x is the day of the month. What is the meaning of $f'(x)$ in this context?
23. The number of households in Florida affected by hurricane season in the month of July is defined by $J = f(x)$ where x is the day of the month. $f(x)=2x^2+x+1$ Find the average rate of change of J with respect to x when days is changed from $x = 5$ to $x = 34$.
24. A cake is taken from an oven when its temperature is 196°F and placed on a cooling rack in a room where the temperature is 75°F . The temperature of the cake over (x) minutes is given by $H = f(x)$. $f(x)=4x^2+15x+196$ Find the instantaneous rate of change of H with respect to x when $x = 15$.
25. A pan of meatloaf is taken from an oven when its temperature is 205°F and placed on a table in a room where the temperature is 75°F . The temperature of the meatloaf over x minutes is given by $H = f(x)$. $f(x)=2x^2+5x+205$. Find the average rate of change of H with respect to x when minutes is changed from $x = 5$ to $x = 54$.

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 8.8.

Vocabulary

Term	Definition
Average rate of change	The average rate of change of a function is the change in y coordinates of a function, divided by the change in x coordinates.
Average speed	The average speed of an object is the distance an object travels divided by the travel time
derivative	The derivative of a function is the slope of the line tangent to the function at a given point on the graph. Notations for derivative include $f'(x)$, dy/dx , y' , df/dx and $\frac{df(x)}{dx}$.
instantaneous rate of change	The instantaneous rate of change of a curve at a given point is the slope of the line tangent to the curve at that point.
Instantaneous speed	The instantaneous speed of an object is the speed of the object at a specific point in time.
limit	A limit is the value that the output of a function approaches as the input of the function approaches a given value.
secant line	A secant line is a line that joins two points on a curve.
Slope	Slope is a measure of the steepness of a line. A line can have positive, negative, zero (horizontal), or undefined (vertical) slope. The slope of a line can be found by calculating "rise over run" or "the change in the y over the change in the x." The symbol for slope is m
Tangent line	A tangent line is a line that "just touches" a curve at a single point and no others.

Additional Resources

PLIX: Play, Learn, Interact, eXplore - Average and Instantaneous Rate of Change

Video: Rates-of-Changes

Practice: Average and Instantaneous Rates of Change

Real World: China Syndrome

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4.2: Slope of Tangent Line

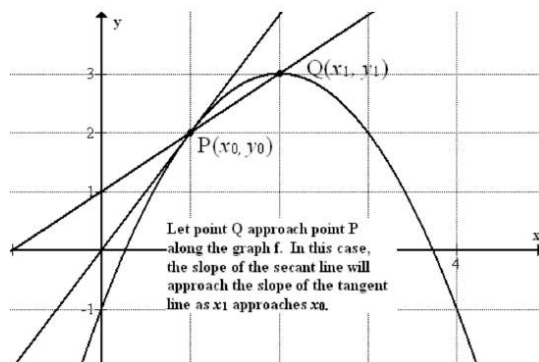
Kevin is learning about the basis of calculus and what calculus is actually used for. Unfortunately, Kevin does not understand why calculus is sometimes necessary to find the equation of a line. In Algebra 1, he learned you can find the equation of a line if you are given two points. You find the slope of the line by dividing the up/down difference in the points by the left/right difference, then you use one of the points and the slope to find the y-intercept.

Kevin's teacher, Mr. Banner, offered him extra credit if he could find the slope of a line for the points (4,5) and (4,5) using the method he learned in Algebra 1. Can you see what Mr. Banner did? What is Kevin going to find as he works on those problems?

Tangents to a Curve

Recall from algebra, if points $P(x_0, y_0)$ and $Q(x_1, y_1)$ are two different points on the curve $y = f(x)$, then the **slope** of the **secant line** connecting the two points is given by

$$m_{sec} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \quad (1)$$



Of course, if we let the point x_1 approach x_0 then Q will approach P along the graph f and thus the slope of the **secant** line will gradually approach the slope of the **tangent line** as x_1 approaches x_0 . Therefore, (1) becomes

$$m_{sec} = \lim_{x_1 \rightarrow x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} \quad (2)$$

To simplify our notation, if we let $h = x_1 - x_0$, then $x_1 = x_0 + h$ and $x_1 \rightarrow x_0$ becomes equivalent to $h \rightarrow 0$. This means that (2) becomes

$$m_{sec} = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

The Slope of a Tangent Line

If the point $P(x_0, y_0)$ is on the curve f , then the tangent line at P has a slope that is given by

$$m_{tan} = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided that the limit exist.

Recall that the equation of the tangent line through point (x_0, y_0) with slope m is the point-slope form of a line: $y - y_0 = m_{tan}(x - x_0)$.



Examples

Example 1



Earlier, you were given a problem about Kevin, who is having trouble understanding calculus.

Mr. Banner asked Kevin to find the equation of a line given the points (4,5) and (4,5). The points (4, 5) and (4, 5) are the same, so the $\frac{\text{rise}}{\text{run}}$ would be 00 - Kevin was just introduced to the need for **differential calculus**!

Example 2

Find line tangent to the curve $f(x) = x^3$ that passes through point $P(2,8)$.

Since $P(x_0, y_0) = (2, 8)$, using the slope of the tangent equation we have

$$\begin{aligned}
 m_{\text{tan}} &= \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} \\
 \text{and we get} \\
 m_{\text{tan}} &= \lim_{h \rightarrow 0} \frac{f(2 + h) - f(2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(h^3 + 6h^2 + 12h + 8) - 8}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h^3 + 6h^2 + 12h}{h} \\
 &= \lim_{h \rightarrow 0} (h^2 + 6h + 12) \\
 &= 12
 \end{aligned}$$

Thus the slope of the tangent line is 12. Using the point-slope formula above, we find that the equation of the tangent line is $y - 8 = 12(x - 2)$ or $y = 12x - 16$.

Example 3

If $f(x) = x^2 - 3$, find $f'(x)$ and use the result to find the slope of the tangent line at $x = 2$ and $x = -1$.

Since $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ then

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{[(x+h)^2 - 3] - [x^2 - 3]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - 3 - x^2 + 3}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} \\
 &= \lim_{h \rightarrow 0} (2x + h) \\
 &= 2x
 \end{aligned}$$

To find the slope, we simply substitute $x = 2$ into the result $f'(x)$:

$$f'(x) = 2x$$

$$f'(2) = 2(2)$$

$$= 4$$

and

$$f'(x) = 2x$$

$$f'(-1) = 2(-1)$$

$$= -2$$

Thus slope of the tangent line at $x = 2$ and $x = -1$ are 4 and -2 respectively.

Example 4

Find the slope of the tangent line to the curve $y = 1/x$ that passes through the point $(1, 1)$.

Using the slope of the tangent formula,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

and substituting $y = \frac{1}{x}$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\left(\frac{1}{x+h}\right) - \frac{1}{x}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{x-x-h}{x(x+h)}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{x-x-h}{hx(x+h)}$$

$$= \lim_{h \rightarrow 0} \frac{-h}{hx(x+h)}$$

$$= \lim_{h \rightarrow 0} \frac{-1}{x(x+h)}$$

$$= \frac{-1}{x^2}$$

For $x = 1$, the slope is

$$f'(x) = \frac{-1}{1} = -1$$

$$= -1$$

Thus the slope of the tangent line at $x = 1$ for the curve $y = 1/x$ is $m = -1$. To find the equation of the tangent line, we simply use the point-slope formula,

$y - y_0 = m(x - x_0)$	
Where $(x_0, y_0) = (1, 1)$.	
$y - 1$	$= -1(x - 1)$
y	$= -x + 1 + 1$
y	$= -x + 2$

So the equation of the tangent line is $y = -x + 2$.

Example 5

Given the function $y = \frac{1}{2}x^2$ and the values of $x_0 = 3$ and $x_1 = 4$, find:

1. The **average rate of change** of y with respect to x over the interval $[x_0, x_1]$.

Identify the two points by substituting 3 and 4 in for x in the function $f(x) = \frac{1}{2}x^2$

Substitute the two points (3, 4.5) and (4, 8) into the average rate of change formula: $m = \frac{y_1 - y_0}{x_1 - x_0}$

Average rate of change = $\frac{7}{2}$

2. The slope of the secant line connecting x_0 and x_1 .

The slope of the secant line between x_0 and x_1 is the slope between (3, 4.5) and (4, 8), which is $\frac{7}{2}$.

3. The **instantaneous rate of change** of y with respect to x at x_0 .

Instantaneous rate of change is the slope at $x = 3$.

Use the formula: $\frac{f(x+h) - f(x)}{h}$ where $f(x) = \frac{1}{2}x^2$ and $x = 3$

$\frac{f(3+h) - f(3)}{h}$ Substitute 3 for x

$\frac{0.5(3+h)^2 - 0.5(3)^2}{h}$ Replace $f(x) \rightarrow \frac{1}{2}x^2$

$$\frac{\frac{1}{2}(9) + \frac{1}{2}(6h) + \frac{1}{2}h^2 - \frac{1}{2}9}{h}$$

FOIL and Distribute the $\frac{1}{2}$

$$\frac{6h + h^2}{2h} \text{ Simplify}$$

$$3 + \frac{h}{2} \text{ Simplify again}$$

$$3 \text{ As } h \rightarrow 0$$

$\therefore$ the instantaneous slope at $x = 3$ is 3

- d. The slope of the tangent line at x_1 .

The slope of the tangent at 4 is the same as the instantaneous rate of change at $x = 4$

This is the same series of steps as with $x = 3$ above

$\therefore$ the slope at $x = 4$ is 4

Example 6

Given the function $f(x)=1/x$ and the values $x_0=2$ and $x_1=3$, find:

1. The average rate of change of y with respect to x over the interval $[x_0, x_1]$.

Identify the two points by substituting 2 and 3 in for x in the function

$$f(x) = \frac{1}{x} \text{ to get } (2, \frac{1}{2}) | (3, \frac{1}{3})$$

Substitute the two points $(2, 1/2) | (3, 1/3)$ into the average rate of change formula: $m = \frac{y_1 - y_0}{x_1 - x_0}$

Average rate of change = -16

2. The slope of the secant line connecting x_0 and x_1 .

The slope of the secant line between x_0 and x_1 is the slope between $(2, 1/2)$ and $(3, 1/3)$, which is -16.

3. The instantaneous rate of change of y with respect to x at x_0 .

Instantaneous rate of change at x_0 is the slope at $x = 2$.

Use the formula: $\frac{f(x+h)-f(x)}{h}$ where $f(x)=1/x$ and $x=2$

$$\frac{f(2+h)-f(2)}{h} \dots \text{Substitute 2 for } x$$

$$\frac{\frac{1}{2+h} - \frac{1}{2}}{h} \dots \text{Replace } f(x) \rightarrow \frac{1}{x}$$

$$\left(\frac{1}{2+h} - \frac{1}{2} \right) \cdot \frac{1}{h}$$

We had a fraction divided by a fraction, invert to multiply

$$\frac{(2)(1)}{2(2+h)} - \frac{(2+h)(1)}{2(2+h)} \cdot \frac{1}{h} \dots \text{Set common denominators}$$

$$\frac{(2) - (2+h)}{(2+h)(2)(h)} \dots \text{Simplify}$$

$$\frac{-h}{4h + 2h^2} \dots \text{Simplify again}$$

$$\frac{-1}{4 + 2h} \dots \text{once more (canceling the } h)$$

$$\frac{-1}{4} \dots \text{As } h \rightarrow 0$$

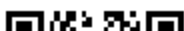
$$\therefore \text{ the instantaneous slope at } x = 2 \text{ is } \frac{-1}{4}$$

- d. The slope of the tangent line at x_1 .

The slope of the tangent at 3 is the same as the instantaneous rate of change at $x=3$

This is the same series of steps as with $x = 2$ above

$\therefore$ the slope at $x = 3$ is $-1/9$



Review



1. What is the line connecting (x_0, y_0) and (x_1, y_1) on a curve called?
2. As (x_0, y_0) gets immeasurably close to (x_1, y_1) the term describing the line between them becomes: "the _____ line"
3. The expression $f(x_0+h)$ describes what distance in the process of finding the slope of a tangent line?
4. When calculating the slope of a tangent, what value is assumed to go to 0 as the two chosen points get closer and closer?
5. What does the concept of limit, discussed in prior lessons, have to do with finding the slope of a line tangent to a curve?

Find the equation of the tangent line:

6. What is the equation of the tangent line at $x=-3$ assuming that $r(-3)=-5$ and $r'(-3)=1$?
7. What is the equation of the tangent line at $x=1$ assuming that $r(1)=3$ and $r'(1)=-5$?
8. What is the equation of the tangent line at $x=2$ assuming that $g(2)=1$ and $g'(2)=-3$?
9. What is the equation of the tangent line at $x=4$ assuming that $u(4)=4$ and $u'(4)=3$?
10. What is the equation of the tangent line at $x=-4$ assuming that $t(-4)=2$ and $t'(-4)=5$?

Find the equation of the tangent line:

11. Find the equation of the tangent line to the graph of $h(x)=-5x^3-3x^2+x+3$ at $x=1$
12. Find the equation of the tangent line to the graph of $t(x)=-2x$ at $x=-2$
13. Find the equation of the tangent line to the graph of $m(x)=3x^3+3x^2+4x+4$ at $x=1$
14. Find the equation of the tangent line to the graph of $q(x)=-x^3-4x^2+4x+3$ at $x=-2$
15. Find the equation of the tangent line to the graph of $t(x)=-4x^2+2x-4$ at $x=-1$
16. Find the equation of the tangent line to the graph of $h(x)=-4x^3+2x^2-3x+3$ at $x=-1$
17. Find the equation of the tangent line to the graph of $m(x)=x$ at $x=0$
18. Find the equation of the tangent line to the graph of $s(x)=-3x^2-2x+3$ at $x=0$
19. Find the equation of the tangent line to the graph of $c(x)=-3$ at $x=0$
20. Find the equation of the tangent line to the graph of $b(x)=-5x^4+3x^3-x^2+5x-3$ at $x=-1$

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 8.7.

Vocabulary

Term	Definition
secant	A line that intersects a circle in two points.
tangent	A line that intersects a circle in exactly one point.
Average rate of change	The average rate of change of a function is the change in y coordinates of a function, divided by the change in x coordinates.
Differential calculus	Differential calculus is the branch of calculus based on finding the difference in location between two points that get closer together until the distance between them is infinitely small.
instantaneous rate of change	The instantaneous rate of change of a curve at a given point is the slope of the line tangent to the curve at that point.
secant line	A secant line is a line that joins two points on a curve.
Slope	Slope is a measure of the steepness of a line. A line can have positive, negative, zero (horizontal), or undefined (vertical) slope. The slope of a line can be found by calculating “rise over run” or “the change in the y over the change in the x.” The symbol for slope is m
Tangent line	A tangent line is a line that "just touches" a curve at a single point and no others.

Additional Resources

PLIX: Play, Learn, Interact, eXplore - Slope of the Tangent and Secant Lines

Video: Equation of a Tangent Line

Practice: Slope of Tangent Line

Real World: Off the Curve

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4.3: Tangent Line Approximation

Suppose you absolutely needed to know the value of the square root of 19 but all you had was pencil and paper, no calculator. Could you calculate it? With your current understanding of the derivative as the slope of a tangent line, you should be able to. Try computing $19^{0.5}$ without a calculator; then compare your result with this concepts method of **linearization** and a calculator.

Linearization

Linearization of a function means using the tangent line of a function at a point as an approximation to the function in the vicinity of the point. This relationship between a tangent and a graph at the point of tangency is often referred to as **local linearization**.

Given the function $f(x)$ and the derivative $f'(x)$, the tangent line at a point x_0 can be written in point-slope form as:

$$y - f(x_0) = f'(x_0)(x - x_0) \text{ or } y = f(x_0) + f'(x_0)(x - x_0).$$

If we consider that this tangent line is a good approximation to $f(x)$ in the vicinity of x_0 , we can write

$$f(x) \approx y = f(x_0) + f'(x_0)(x - x_0). \text{ This is the linearization of } f(x) \text{ about } x_0.$$

The tangent line as the local linearization of $f(x)$ is often designated $L(x)$, so that

$$f(x) \approx L(x) = f(x_0) + f'(x_0)(x - x_0).$$

Take the function $f(x) = x^2 - 2x - 3$ and find the linearization at the points $x_0 = 1.5$ and $x_0 = -0.5$.

The linearization of $f(x)$ is given by: $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$.

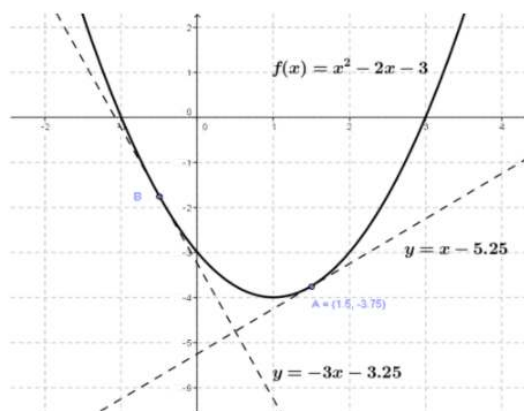
We have:

$$f(1.5) = -3.75 \text{ and } f(-0.5) = -1.75$$

$$f'(x) = 2x - 2, \text{ so that } f'(1.5) = 1 \text{ and } f'(-0.5) = -3.$$

The linearization becomes:

$$\begin{array}{ll} \text{At } x_0 = -0.5 & \text{At } x_0 = 1.5 \\ f(x) \approx -1.75 - 3(x + 0.5) & \approx -3.75 + (x - 1.5) \\ \approx -3x - 3.25 & \approx x - 5.25 \end{array}$$



As the figure illustrates and the table shows, as we move away from x_0 , we lose accuracy.

Error near $x_0 = -0.5$				Error near $x_0 = 1.5$			
x	f(x)	$-3x - 3.25$	True-Approx	x	f(x)	$x - 5.25$	True-Approx
-1	0	-0.25	0.25	1.0	-4	-4.25	0.25
-0.5	-1.75	-1.75	0.00	1.5	-3.75	-3.75	0.00
0	-3	-3.25	0.25	2.0	-3.25	-3.00	0.25

Examples

Example 1

Earlier, you were asked to $19^{0.5}$ without a calculator and then to compare your result with this concepts method of linearization and a calcl



Congratulations if you were able to linearize $x^{0.5}$ at $x=16$ (or $x=25$).

The linearization is $y = \frac{1}{8}(x-16)+4$, which means $y=4.375$ when $x=19$. A calculator would give 4.359.

Example 2

Find the linearization of $f(x)=(x+3)^{0.5}$ at point $x=-1$.

The linearization of $f(x)$ is given by: $f(x) \approx f(x_0) + f'(x_0)(x-x_0)$.

We have:

$f(1)=2$, and

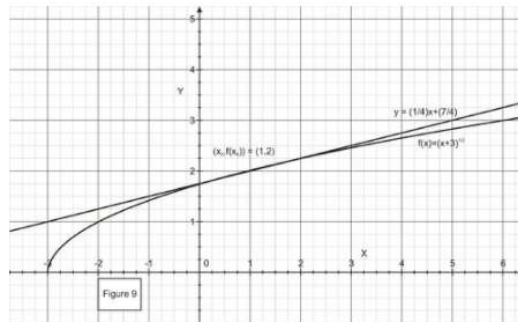
$f'(x)=\frac{1}{2}(x+3)^{-1/2}$, so that $f'(1)=\frac{1}{4}$.

The linearization becomes:

$$f(x) \approx 2 + \frac{1}{4}(x - 1)$$

$$\approx \frac{1}{4}x + \frac{7}{4}$$

This tells us that near the point $x=1$, the function $f(x)=(x+3)^{0.5}$ approximates the line $y=(\frac{x}{4})+\frac{7}{4}$. As the figure illustrates and the table shows, as we move away from $x=1$, we lose accuracy.



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x	f(x)	$\approx 14x + 74$	True-Approx
1	2	2	0
1.5	2.121	2.125	0.004
2	2.236	2.25	0.014
3	2.449	2.5	0.41

Example 3

Find the linearization of $y = \sin x$ at $x = \pi/3$.

The linearization of $f(x)$ is given by: $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$.

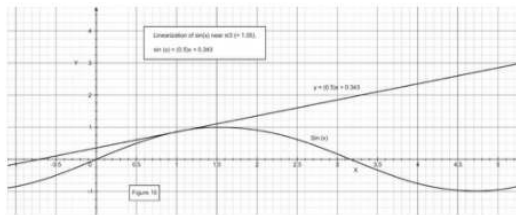
We have

$$f\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}, \text{ and}$$

$$f'(x) = \cos x, \text{ so that } f'\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}.$$

The linearization becomes:

$$\begin{aligned} f(x) &\approx \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right) \\ &\approx \frac{\sqrt{3}}{2} + \frac{x}{2} - \frac{\pi}{6} \\ &\approx \frac{x}{2} + 0.342 \end{aligned}$$



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Example 4

Let f be a function such that $f(5)=6$ and whose derivative is $f'(x)=(x^3+44)^{0.5}$. Approximate $f(5.3)$.

The linearization of $f(x)$ is given by: $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$.

Since we have $f(5)=6$, let $x_0=5$.

Then $f'(x_0)=(x_0^3+44)^{0.5}=169^{0.5}=13$

The linearization becomes:

$$\begin{aligned} f(x) &\approx 6 + 13(x - 5) \\ f(5.3) &\approx 6 + 13(5.3 - 5) \\ &\approx 6 + 13(0.3) \\ &\approx 9.9 \end{aligned}$$

Review

- Find the linearization of $f(x)=x^2+1/x$, $x_0=1$, $f(1.7)$
- Find the linearization of $f(x)=\tan x$ at $a=\pi$.
- Use the linearization method to show that when $x \ll 1$ (much less than 1), then $(1+x)^n \approx 1+nx$.
- Use the result of problem #3, $(1+x)^n \approx 1+nx$, to find the approximation for the following:

- $f(x) = (1 - x)^4$
- $f(x) = \sqrt{1 - x}$
- $f(x) = \frac{5}{\sqrt{1 + x}}$
- $f(x) = \sqrt[3]{\left(1 - \frac{3}{(x-1)}\right)^2}$
- Without using a calculator, approximate $(1.003)^{99}$

For #5 - 13, find the linearization of the given function at the given point x_0 and use this approximation to compute the given quantity. Compare the result to the value obtained by calculator; compute the error.

5. $f(x) = 1 + \sin x$, $x_0 = 0$, $f(0, 3)$
6. $f(x) = \sqrt{1+x}$, $x_0 = 0$, $\sqrt{1.006}$
7. $f(x) = (1+x)^{10}$, $x_0 = 0$, 1.02^{10}
8. $f(x) = \frac{1}{\sqrt{x+1}}$, $x_0 = 0$, $\frac{1}{\sqrt{0.885}}$
9. $f(x) = x^2 - 3x + 10$, $x_0 = 3$, $f(3.65)$
10. $f(x) = x^3 + 2$, $x_0 = 3$, $f(3.3)$
11. $f(x) = \frac{1}{x^2}$, $x_0 = 2$, $\frac{1}{2.03^2}$
12. $f(x) = e^x$, $x_0 = 0$, $e^{0.2}$

For #13 - 14, estimate the following numbers and determine the error:

13. Approximate $\sqrt[3]{628}$
14. Estimate the value of $\frac{1}{2.007}$
15. Let f be a function such that $f(1) = 2$ and whose derivative is $f'(x) = \sqrt{x^3 + 1}$. Approximate $f(1.3)$.

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 4.10.

Vocabulary

Term	Definition
linearization	Local linearization of a function means to approximate the function at a point by the tangent line at the point.

Additional Resources

PLIX: Play, Learn, Interact, eXplore - Estimating square roots

Video: Calculus - Linear Approximation

Practice: Tangent Line Approximation

Real World: On Autopilot

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CHAPTER OVERVIEW

5: Differentiation - Differentiation Rules and Formulas

[5.1: Constant, Identity, and Power Rules](#)

[5.2: Sum and Difference Differentiation Rules](#)

[5.3: Products and Quotients Differentiation Rules](#)

[5.4: Derivatives of Trigonometric Functions](#)

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5.1: Constant, Identity, and Power Rules

The power rule is a fantastic "shortcut" for finding the derivatives of basic polynomials. Between the power rule and the basic definition of the **derivative** of a constant, a great number of polynomial derivatives can be identified with little effort - often in your head!

Constant Derivatives and the Power Rule

In this lesson, we will develop formulas and theorems that will calculate derivatives in more efficient and quick ways. Look for these theorems in boxes throughout the lesson.

The Derivative of a Constant

Theorem

If

$$f(x) = c$$

where c is a constant, then

$$f'(x) = 0$$

Proof

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{c - c}{h} = 0$$

Theorem

If c is a constant and f is differentiable at all x , then

Proof

$$\frac{d}{dx}[cf(x)] = c \frac{d}{dx}[f(x)]$$

In simpler notation

$$(cf)' = c(f)' = cf'$$

The Power Rule

Theorem

(The Power Rule) If n is a positive integer, then for all real values of x

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

Examples

Example 1

Find $f'(x)$ for $f(x)=16$.

If $f(x)=16$ for all x , then $f'(x)=0$ for all x .

We can also write



$$\frac{d}{dx} 16 = 0$$

Example 2

Find the derivative of $f(x)=4x^3$.

..... Restate the function

$$\frac{d}{dx} 4x^3$$

..... Apply the commutative law

$$4 \frac{d}{dx} x^3$$

..... Apply the power Rule

$$4[3x^2]$$

..... Simplify

$$12x^2$$

Example 3

Find the derivative of

$$f(x) = \frac{-2}{x^4}$$

.

$$\frac{d}{dx} \left[\frac{-2}{x^4} \right]$$

..... Restate

$$\frac{d}{dx} [-2x^{-4}]$$

..... Rules of exponents

$$-2 \frac{d}{dx} [x^{-4}]$$

..... By the commutative law

$$-2[-4x^{-4-1}]$$

..... Apply the power rule

$$-2[-4x^{-5}]$$

..... Simplify

$$8x^{-5}$$

..... Simplify again

$$\frac{8}{x^5}$$

..... Use rules of exponents

Example 4

Find the derivative of $f(x)=x$.

Special application of the power rule:

$$\frac{d}{dx}[x] = 1x^{1-1} = x^0 = 1$$

Example 5

Find the derivative of $f(x)=x^{0.5}$.

Restate the function:

$$\frac{d}{dx}[x^{0.5}]$$

Using rules of exponents (from algebra):

$$\frac{d}{dx}[x^{1/2}]$$

Apply the power rule:

$$\frac{1}{2}x^{1/2-1}$$

Simplify:

$$\frac{1}{2}x^{-1/2}$$

Rules of exponents:

$$\frac{1}{2x^{1/2}}$$

Example 6

Find the derivative of

$$f(x) = \frac{1}{x^3}$$

.

Restate the function:

$$\frac{d}{dx}\left[\frac{1}{x^3}\right]$$

Rules of exponents:

$$\frac{d}{dx}x^{-3}$$

Power rule:

$$-3x^{-3-1}$$

Simplify:

$$-3x^{-4}$$

Rules of exponents:

$$\frac{-3}{x^4}$$



Review

1. State the power rule.

Find the derivative:

2.



$$y = 5x^7 \quad (5.1.1)$$

3.

$$y = -3x \quad (5.1.2)$$

4.

$$f(x) = \frac{1}{3}x + \frac{4}{3} \quad (5.1.3)$$

5.

$$y = x^4 - 2x^3 - 5\sqrt{x} + 10 \quad (5.1.4)$$

6.

$$y = (5x^2 - 3)^2 \quad (5.1.5)$$

7. Given

$$y(x) = x^{-4\pi^2} \quad (5.1.6)$$

find the derivative when

$$x = 1 \quad (5.1.7)$$

8.

$$y(x) = 5 \quad (5.1.8)$$

9. Given \$

$$u(x) = x^{-5\pi^3}, \quad (5.1.9)$$

what is

$$u'(2)? \quad (5.1.10)$$

10.

$$y = \frac{1}{5} \quad (5.1.11)$$

when

$$x = 4 \quad (5.1.12)$$

11. Given

$$d(x) = x^{-0.37} \quad (5.1.13)$$

what is

$$d'(1)? \quad (5.1.14)$$

12.

$$g(x) = x^{-3} \quad (5.1.15)$$

13.

$$u(x) = x^{0.096} \quad (5.1.16)$$

$$k(x) = x - 0.49 \quad (5.1.17)$$

$$y = x^{-5\pi^3} \quad (5.1.18)$$

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 8.9.

Vocabulary

Term	Definition
derivative	The derivative of a function is the slope of the line tangent to the function at a given point on the graph. Notations for derivative include $f'(x)$, dy/dx , y' , df/dx and $\frac{df(x)}{dx}$.
proof	A proof is a series of true statements leading to the acceptance of truth of a more complex statement.
theorem	A theorem is a statement that can be proven true using postulates, definitions, and other theorems that have already been proven.

Additional Resources

PLIX: Play, Learn, Interact, eXplore - Derivative Calculator: Power Rules

Video: Calculus - Derivatives

Practice: Constant, Identity, and Power Rules

Real World: Twist and Shout

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5.2: Sum and Difference Differentiation Rules

Based on your knowledge of the limit definition of the **derivative** of a function, and the properties of limits discussed in a previous concept, can you make a prediction at this time how the derivative of a sum or difference of two functions should be determined?

Differentiation of Sums and Differences

Here are the differentiation rules for the sum and difference of two functions:

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

and

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$$

In simpler notation

$$(f \pm g)' = f' \pm g'$$

Using the limit properties of previous chapters should allow you to figure out why these differentiation rules apply.

You often need to apply multiple rules to find the derivative of a function. To find the derivative of $f(x) = 3x^2 + 2x$, you need to apply the sum of derivatives formula and the power rule:

$$\frac{d}{dx} 3x^2 + 2x = \frac{d}{dx} [3x^2] + \frac{d}{dx} [2x]$$

=

$$3 \frac{d}{dx} [x^2] + 2 \frac{d}{dx} [x]$$

=

$$3[2x] + 2[1]$$

=

$$6x + 2$$

Examples

Example 1

Earlier, you were asked to find the derivative of $f(x) = 3x^2 + 2x$ for the sum and differences of derivatives.

In a previous concept, you learned that if the limits exist:

$$\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x),$$

Since the derivative of a function is defined by a limit, $\frac{d}{dx}[f(x) \pm g(x)]$ would be defined by limit applied to $[f(x) \pm g(x)]$. Work out the details to see that the above rules make sense.

Example 2

Given: $t(x) = x - 1$, what is $\frac{dt}{dx}$ when $x = 0$?

By the difference rule:

$$\begin{aligned} (x - 1)' &= (x)' - (1)' = 0 \\ x' &= 1 \end{aligned}$$



..... By the power rule

$$1' = 0$$

..... The derivative of a constant = 0

So when we evaluate this at $x=0$, we get 1, since

$$1 - 0 = 1$$

Example 3

Find the derivative:

$$f(x) = x^3 - 5x^2$$

Use the difference and power rules to help:

$$\begin{aligned}\frac{d}{dx}[x^3 - 5x^2] &= \frac{d}{dx}[x^3] - 5\frac{d}{dx}[x^2] \\ &= 3x^2 - 5[2x] \\ &= 3x^2 - 10x\end{aligned}$$

Example 4

Given

$$a(x) = -\pi x^{-0.54} + 6x^4$$

What is

$$\frac{d}{dx}a(x)?$$

We'll use the sum and power rules:

$$\frac{d}{dx}(-\pi x^{-0.54} + 6x^4) = \frac{d}{dx}(-\pi x^{-0.54}) + \frac{d}{dx}(6x^4)$$

...By the sum rule

$$= -\pi \frac{d}{dx}(x^{-0.54}) + 6 \frac{d}{dx}(x^4)$$

...By the Constant - function Product rule

$$= 0.54\pi x^{-1.54} + 24x^3$$

...By the power rule

Review

For #1-7, find the derivative using the sum/difference rule

1.

$$y = \frac{1}{2}(x^3 - 2x^2 + 1)$$

2.

$$y = \sqrt{2}x^3 - \frac{1}{\sqrt{2}}x^2 + 2x + \sqrt{2}$$

3.

$$y = a^2 - b^2 + x^2 - a - b + x$$

(where \$a\$, \$b\$ are constants)

4.

$$y = x^{-3} + \frac{1}{x^7}$$

5.

$$y = \sqrt{x} + \frac{1}{\sqrt{x}}$$

6.

$$f(x) = (-3x + 4)^2$$

7.

$$f(x) = -0.93x^{10} + (\pi^3 x)^{\frac{-5}{12}}$$

8. What is

$$\frac{d}{dx}(2x + 1)^2?$$

9. Given:

$$a(x) = (-5x + 3)^2$$

What is

$$\frac{dy}{dx}?$$

10.

$$v(x) = -3x^3 + 5x^2 - 2x - 3$$

What is

$$v'(0)?$$

11.

$$f(x) = 2x^2 + 3x + 1.$$

Find

$$f'(x)$$

12.

$$f(x) = \frac{1}{\sqrt{x}} - \frac{1}{x}.$$

Find

$$f'(1)$$

13.

$$y = (x + 1)(x + 2).$$

Evaluate

$$\frac{dy}{dx}$$

at

$$x = -\frac{1}{2}$$

14.

$$f(x) = 2ax^3 + x^2; f'(-2) = 0$$

Find a

15.

$$f(x) = a(x^2 - 5);$$

find a so that

$$f''(5) = 20$$

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 3.5.

Vocabulary

Term	Definition
derivative	The derivative of a function is the slope of the line tangent to the function at a given point on the graph. Notations for derivative include $f'(x)$, dy/dx , y' , df/dx and $\frac{df(x)}{dx}$.
differentiable	A differentiable function is a function that has a derivative that can be calculated.

Additional Resources

Video: Product Rule

Practice: Sum and Difference Differentiation Rules

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5.3: Products and Quotients Differentiation Rules

You may recall hearing about Becca and her Track and Field competition in a prior lesson. Her boyfriend had taken a picture of her just as she started to pull away from the others on the track. We learned how she might learn to identify her instantaneous speed at just the split second the picture was taken by using calculus to find a derivative.

What if, instead of just finding her *speed* at that split second, she wanted to find her *acceleration*?

Quotient Rule and Higher Derivatives

The Quotient Rule

Theorem

(The Quotient Rule) If f and g are differentiable functions at x and $g(x) \neq 0$, then.

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx} [f(x)] - f(x) \frac{d}{dx} [g(x)]}{[g(x)]^2} \quad (5.3.1)$$

In simpler notation

$$\left(\frac{f}{g} \right)' = \frac{g \cdot f' - f \cdot g'}{g^2}$$

Keep in mind that the order of operations is important (because of the minus sign in the numerator) and

$$\left(\frac{f}{g} \right)' \neq \frac{f'}{g'}$$

Higher Derivatives

If the derivative f' of the function f is differentiable, then the derivative of f' , denoted by f'' is called the **second derivative** of f . We can continue the process of differentiating derivatives and obtain third, fourth, fifth and higher derivatives of f . They are denoted by f' , f'' , f''' , $f^{(4)}$, $f^{(5)}$, $\dots$,

Examples

Example 1

Earlier, you were asked how to find her acceleration in addition to her speed.

Once Becca has calculated her instantaneous speed at a given point on the track by finding the derivative, she could then take the derivative of *that* function to find her **instantaneous acceleration** at the same point in the race.

By finding her instantaneous speed and acceleration at different points in the race, she can learn a lot about what points made a difference in her overall success, and also what points she needs to work on.

Example 2

Find

$$\frac{dy}{dx}$$

for

$$y = x^2 - 5x^3 + 2$$



$$\frac{dy}{dx} = \frac{d}{dx} \left[\frac{x^2 - 5}{x^3 + 2} \right]$$

$$= \frac{(x^3 + 2)(x^2 - 5)' - (x^2 - 5)(x^3 + 2)'}{(x^3 + 2)^2}$$

$$= \frac{(x^3 + 2)(2x) - (x^2 - 5)(3x^2)}{(x^3 + 2)^2}$$

$$= \frac{2x^4 + 4x - 3x^4 + 15x^2}{(x^3 + 2)^2}$$

$$= \frac{-x^4 + 15x^2 + 4x}{(x^3 + 2)^2}$$

$$= \frac{x(-x^3 + 15x + 4)}{(x^3 + 2)^2}$$

Example 3

At which point(s) does the graph of

$$y = \frac{x}{x^2 + 9}$$

have a horizontal tangent line?

Since the slope of a horizontal line is zero, and since the derivative of a function signifies the slope of the tangent line, then taking the derivative and equating it to zero will enable us to find the points at which the slope of the tangent line is equal to zero, i.e., the locations of the horizontal tangents. Notice that we will need to use the quotient rule here:

y	$= \frac{x}{x^2 + 9}$	
y'	$= \frac{(x^2 + 9) \cdot f'(x) - x \cdot g'(x^2 + 9)}{(x^2 + 9)^2}$ $= 0$	$= (x^2 + 9)(1) - x(2x)(x^2 + 9)^2 = 0$

Multiply both sides by

$$(x^2 + 9)^2$$

$x^2 + 9 - 2x^2$	$= 0$
x^2	$= 9$
x	$= \pm 3$

Therefore, at $x = -3$ and $x = 3$, the tangent line is horizontal.

Example 4

Find the fifth derivative of

$$f(x) = 2x^4 - 3x^3 + 5x^2 - x - 1$$

To find the fifth derivative, we must first find the first, second, third, and fourth derivatives.

$f'(x)$	$= 8x^3 - 9x^2 + 5x - 1$
$f''(x)$	$= 24x^2 - 18x + 5$
$f'''(x)$	$= 48x - 18$
$f^{(4)}(x)$	$= 48$
$f^{(5)}(x)$	$=$

Example 5

Suppose $y'(2) = 0$ and $(y/q)(2) = 0$. Find $q(2)$, assuming $y(2) = 0$.

Begin with the quotient rule:

$$\left(\frac{y}{q}\right)'(2) = \left(\frac{y'(2)q(2) - y(2)q'(2)}{q(2)^2}\right)$$

..... Substitute

$$(0) = \left(\frac{(0)q(2) - (0)q'(2)}{q(2)^2}\right)$$

..... Substituting again with given values

$$0 = \left(\frac{(0)q(2)}{q(2)^2}\right)$$

..... Simplify with:

$$(0)q'(2) = 0$$

$$0 = \frac{0}{q(2)}$$

$$q(2) = 0$$

Example 6

Find the derivative of

$$k(x) = \frac{-2x - 4}{e^x}$$

Use the quotient rule: Note: $(-2x-4)' = -2$ and $(e^x)' = e^x$

$$\left(\frac{-2x - 4}{e^x}\right)' = \frac{(-2)(e^x) - (-2x - 4)(e^x)}{e^{2x}}$$

..... Substitute

$$\frac{2x + 2}{e^x}$$

..... Simplify

Example 7

Given $f(x) = (-x^4 - 4x^3 - 5x^2 + 3)$. Find $f''(x)$ when $x=3$.

Recall that $f''(x)$ means "The derivative of the derivative of x "

$$f'(x) = -4x^3 - 12x^2 - 10x$$

..... Use the power rule on $f(x)$

$$f''(x) = -12x^2 - 24x - 10$$

..... Use the power rule on $f'(x)$

$$f''(3) = -12(3)^2 - 24(3) - 10 \rightarrow -108 - 72 - 10 = -190$$

..... Substitute 3

$$\therefore f''(3) = -190$$

Review

Use the quotient rule to so

1. Suppose $u'(0) = 98$ and



$$\left(\frac{u}{q}\right)'(0) = 7$$

Find $q(0)$ assuming $u(0) = 0$.

2. Given:

$$b(x) = \frac{x^2 - 5x + 4}{-5x + 2}$$

what is: $b'(2)$?

3. Given:

$$m(x) = \frac{e^x}{3x + 4}$$

what is

$$\frac{dm}{dx} ?$$

4. What is $\frac{d}{dx} \left(\frac{\sin(x)}{x-4} \right)$?

5. Find the derivative of

$$q(x) = \frac{x}{\sin(x)}$$

Solve these higher order derivatives:

6. Given: $v(x) = -4x^3 + 3x^2 + 2x + 3$, what is $v''(x)$?
7. Given: $m(x) = x^2 + 5x$, what is $m''(x)$?
8. Given: $d(x) = 3x^4 e^x$, what is $d''(x)$?
9. Given: $t(x) = -2x^5 \sin(x)$, what is

$$\frac{d^2 t}{dx^2} ?$$

10. What is

$$\frac{d^2}{dx^2} 3x^5 e^x?$$

Solve:

11. Find the derivative of $y=3x^{0.5}+3$.
12. Find the derivative of

$$y = \frac{4x+1}{x^2-9}$$

13. Newton's Law of Universal Gravitation states that the gravitational force between two masses (say, the earth and the moon), m and M is equal two their product divided by the squared of the distance r between them. Mathematically,

$$F = G \frac{mM}{r^2}$$

where G is the Universal Gravitational Constant ($1.602 \times 10^{-11} \text{ Nm}^2/\text{kg}^2$). If the distance r between the two masses is changing, find a formula for the instantaneous rate of change of F with respect to the separation distance r .

14. Find $\frac{d}{d\psi} [\frac{\psi^3}{3-\psi_0}]$, where ψ_0 is a constant.
15. Find

$$\frac{d^3 y}{dx^3} \Big|_{x=1}$$

where

$$y = \frac{2}{x^3}$$

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 8.11.

Vocabulary

Term	Definition
differentiable	A differentiable function is a function that has a derivative that can be calculated.
Instantaneous acceleration	The instantaneous acceleration of an object is the change in velocity of the object calculated at a specific point in time.
Instantaneous velocity	The instantaneous velocity of an object is the velocity of the object at a specific point in time.
quotient rule	In calculus, the quotient rule states that if f and g are differentiable functions at x and $g(x) \neq 0$, then $\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx} [f(x)] - f(x) \frac{d}{dx} [g(x)]}{[g(x)]^2}$

Additional Resources

Video: High-Order Derivatives - Part 1

Practice: Products and Quotients Differentiation Rules

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5.4: Derivatives of Trigonometric Functions

The functions $\sin x$ and $\cos x$ are periodic, with period 2π . You have learned that the **derivative** of a differentiable function gives the slope of the tangent line at a point. Before proceeding with this lesson, look at the function curves for the two functions, and see if you can identify any points where you know what the derivative will be. For each function, do these sets of points repeat as x increases or decreases? How often? Can you make a general statement about the derivatives of the trigonometric functions?

Derivatives of Trigonometric Functions

We now want to find an expression for the derivative of each of the six trigonometric functions:

$\sin x$	$\cos x$	$\tan x$
$\csc x$	$\sec x$	$\cot x$

We first consider the problem of differentiating $\sin x$, using the definition of the derivative.

$$\frac{d}{dx}[\sin x] = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h}$$

Since

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

The derivative becomes

$$\begin{aligned} \frac{d}{dx}[\sin x] &= \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \left[\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right] \\ &= -\sin x \cdot \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h \rightarrow 0} \frac{\sin h}{h} \\ &= -\sin x \cdot (0) + \cos x \cdot (1) \\ &= \cos x \end{aligned}$$

Let's look at

$$\lim_{h \rightarrow 0} \frac{1 - \cos(h)}{h}$$

and

$$\lim_{h \rightarrow 0} \frac{\sin(h)}{h}$$

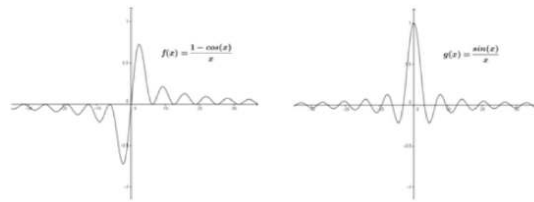
While there are analytical methods that can be used to evaluate these two limits, let's look at function graphs and table data. The graphs of the two functions and some calculator table data are shown below. Inspection of these near $x=0$ appears to show that the limits are:

$$\lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h} = 0$$

and

$$\lim_{h \rightarrow 0} \frac{\sin(h)}{h} = 1$$

The above results can be confirmed analytically.



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x(rad)	-0.001	-0.0001	0	0.0001	0.0001
f(x)	-0.0005	-0.00005	Error	0.00005	0.0005
g(x)	0.99999833	0.99999998	Error	0.99999998	0.99999833

Therefore,

$$\frac{d}{dx}[\sin x] = \cos x$$

It will be left as an exercise to prove that

$$\frac{d}{dx}[\cos x] = -\sin x \quad (5.4.1)$$

The derivatives of the six trigonometric functions are shown below.

$\frac{d}{dx}[\sin x] = \cos x$	$\frac{d}{dx}[\cos x] = -\sin x$	$\frac{d}{dx}[\tan x] = \sec^2 x$
$\frac{d}{dx}[\csc x] = -\csc x \cot x$	$\frac{d}{dx}[\sec x] = \sec x \tan x$	$\frac{d}{dx}[\cot x] = -\csc^2 x$

Keep in mind that the argument x for all the trigonometric functions is measured in radians.

All of the derivatives can be proved by the definition of the derivative, but the reciprocal functions can be found using a simpler method. The proof of

$$\frac{d}{dx}[\tan x] = \sec^2 x$$

is as follows:

Since

$$\tan x = \frac{\sin x}{\cos x}$$

then

$$\frac{d}{dx}[\tan x] = \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right]$$

Using the quotient rule,

$$\begin{aligned}
 &= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} \\
 &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}
 \end{aligned}$$

$$= \frac{1}{\cos 2x}$$

$$= \sec^2 x$$

Examples

Example 1

Earlier, you were asked if  finding points for the derivatives of trigonometric functions and if so, how often they repeat.

First, see if you can identify any points where you know the derivative of $\sin x$ and $\cos x$: Each function has two places in the interval $0 \leq x \leq 2\pi$ where the tangent line has a slope of 0.

For each function, do these sets of points repeat as x increases or decreases? Yes, these 0 slope points do repeat for each function.

How often? The pair of 0 slope points repeats with a period of 2π .

Can you make a general statement about the derivatives of the trigonometric functions? Because the function values repeat every period, the derivative of each function at a specific point should also repeat with period 2π .

Example 2

Find $f'(x)$ if $f(x) = x^2 \cos x + \sin x$.

Using the product rule and the formulas above, we obtain

$$f'(x) = x^2(-\sin x) + 2x \cos x + \cos x$$

$$= -x^2 \sin x + 2x \cos x + \cos x$$

Example 3

Find $\frac{dy}{dx}$ if

$$y = \frac{\cos x}{1 - \tan x}$$

What is the slope of the tangent line at

$$x = \pi/3?$$

Using the quotient rule and the formulas above, we obtain

$$\frac{dy}{dx} = \frac{(1 - \tan x)(-\sin x) - (\cos x)(-\sec^2 x)}{(1 - \tan x)^2}$$

$$= \frac{-\sin x + \tan x \sin x + \cos x \sec^2 x}{(1 - \tan x)^2}$$

To calculate the slope of the tangent line, we simply substitute

$$x = \frac{\pi}{3}$$

$$\frac{dy}{dx} \Big|_{x=\frac{\pi}{3}} = \frac{-\sin(\frac{\pi}{3}) + \tan(\frac{\pi}{3})\sin(\frac{\pi}{3}) + \cos(\frac{\pi}{3})\sec^2(\frac{\pi}{3})}{(1 - \tan(\frac{\pi}{3}))^2}$$

We finally get the slope to be approximately

$$\frac{dy}{dx} \Big|_{x=\frac{\pi}{3}} = 4.9$$

Example 4

Find $\frac{dy}{dx}$ if

$$y = \frac{\cot x}{\sin x}$$

$$\frac{d}{dx} \left[\frac{\cot x}{\sin x} \right] = \frac{\sin x \frac{dy}{dx} [\cot x] - \cot x \frac{dy}{dx} [\sin x]}{\sin^2 x}$$

Quotient Rule

$$= \frac{\sin x \cdot [-\csc^2 x] - \cot x \cdot [\cos x]}{\sin^2 x}$$

Trigonometric derivatives

$$= \frac{-\sin x \cdot [\csc^2 x + \cot^2 x]}{\sin^2 x}$$

factor - sin x

$$= -\csc x \cdot [1 + \cot^2 x + \cot^2 x]$$

simplify and use trig identities

$$= -\csc x \cdot [1 + 2\cot^2 x]$$

Therefore,

$$\frac{dy}{dx} = -\csc x \cdot [1 + 2\cot^2 x]$$

Review

For #1-10, find the derivative y' .

1. $y = x \sin x + 2$

2. $y = x^2 \cos x - x \tan x - 1$

3. $y = \sin^2 x$

4. $y = \frac{\sin x - 1}{\sin x + 1}$

5. $y = \frac{\cos x + \sin x}{\cos x - \sin x}$

6. $y = \frac{x^{0.5}}{\tan x + 2}$

7. $y = \csc x \sin x + x$

8. $y = \frac{\sec x}{\csc x}$

9. If $y = \csc x$, find $y'(\pi/6)$.

10. $y = x^5 \cos(x)$?

11. What is the derivative of $x^2 \csc(x)$?

12. What is the derivative of $\csc(x) \tan(x)$?

13. Use the quotient rule to verify that the derivative of $\sec(x)$ is $\sec(x) \tan(x)$.

14. What is the derivative of $\cot(\pi/2 - x)$?

15. What is the derivative of $\csc^2(x) - \cot^2(x)$?

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 3.7.

Vocabulary

Term	Definition
derivative	The derivative of a function is the slope of the line tangent to the function at a given point on the graph. Notations for derivative include $f'(x)$, dy/dx , y' , df/dx and $\frac{df(x)}{dx}$.

Additional Resources

PLIX: Play, Learn, Interact, eXplore - Derivative of $\sin(x)$

Video: The Derivative of Sine and Cosine

Practice: Derivatives of Trigonometric Functions

Real World: X-Ray Vision

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5.5: Chain Rule

The **Chain Rule** enables us to differentiate a **composite function** $f \circ g$. But why is there the need to have a special way to determine the **derivative** of a composite function? Intuitively, it is because the variation of the domain of f is now governed by the function $g(x)$ rather than just by x , and g 's rate of change with respect to x should somehow be taken into account. Before proceeding, see if you find effect of g by comparing the derivative of $f(x)=x^2$ with the derivative of $f(x)=(5x)^2$ where $g(x)=5x$.

The Chain Rule

We want to derive a rule for the derivative of a composite function of the form $f \circ g$ in terms of the derivatives of f and g . This rule would allow us to differentiate complicated functions in terms of known derivatives of simpler functions.

The rule that enables this is called the **Chain Rule**:

If g is a differentiable function at x , and f is differentiable at $g(x)$, then the composition function $f \circ g = f(g(x))$ is differentiable at x . The derivative of the composite function is:

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

Or an equivalent statement:

If $u=u(x)$ and $f=f(u)$, then

$$\frac{d}{dx}[f(u)] = f'(u) \frac{du}{dx}$$

Or another equivalent statement:

If y is a function of u , and u is a function of x , then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Apply the chain rule to find the derivative of

$$f(x) = (2x^3 - 4x^2 + 5)^2$$

Using the chain rule, let

$$u = 2x^3 - 4x^2 + 5.$$

Then

$$\begin{aligned} \frac{d}{dx}[(2x^3 - 4x^2 + 5)^2] &= \frac{d}{dx}[u^2] \\ &= 2u \frac{du}{dx} \\ &= 2(2x^3 - 4x^2 + 5)(6x^2 - 8x) \end{aligned}$$

The problem above is one of the most common types of composite functions. It is a power function of the type

$$y = [u(x)]^n$$

The rule for differentiating such functions is special case of the Chain Rule called the **General Power Rule**:

If

$$y = [u(x)]^n$$

, then

$$\frac{dy}{dx} = n[u(x)]^{n-1} \frac{d}{dx}u(x)$$



Examples

Example 1

Earlier, you were asked if $f(x) = x^2$ is the derivative of $f(x) = x^2$ by comparing the derivative of $f(x) = x^2$ with the derivative of $f(x) = (5x)^2$ where $g(x) = 5x$. $f(x) = x^2$ is $f'(x) = 2x$, and the derivative of $f(x) = (5x)^2$ is $f'(x) = 2(5x)5 = 2x \cdot 25$. The effect of g in the composite function is to modify the rate of change of $f(x) = x^2$.



Example 2

What is the slope of the tangent line to the function

$$y = \sqrt{x^2 - 3x + 2}$$

that passes through point $x=3$?

We can write

$$y = (x^2 - 3x + 2)^{\frac{1}{2}}$$

This example illustrates the point that n can be any real number including fractions. Using the General Power Rule,

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2}(x^2 - 3x + 2)^{\frac{1}{2}-1}(2x - 3) \\ &= \frac{1}{2}(x^2 - 3x + 2)^{-\frac{1}{2}}(2x - 3) \\ &= \frac{(2x - 3)}{2\sqrt{x^2 - 3x + 2}} \end{aligned}$$

To find the slope of the tangent line, we simply substitute $x=3$ into the derivative:

$$\frac{dy}{dx}|_{x=3} = \frac{2(3) - 3}{2\sqrt{3^2 - 3(3) + 2}} = \frac{3}{2\sqrt{2}} = \frac{3\sqrt{2}}{4}$$

Example 3

Find

$$\frac{dy}{dx}$$

for

$$y = \sin^3 x$$

The function can be written as $y = [\sin x]^3$. Thus

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}[\sin x]^3 \\ &= 3[\sin x]^2[\cos x] \\ &= 3\sin^2 x \cos x \end{aligned}$$

Example 4

Find

$$\frac{dy}{dx}$$

for

$$y = [\cos(\pi x^2)]^3$$

This example will show the application of the chain rule multiple times because there are several functions embedded within each other.

The function y can be written in the form

$$y = (u(w))^3$$

where

$$u(w) = \cos(w)$$

$$w(x) = \pi x^2$$

Here are the steps for the solution:

$$\frac{dy}{dx} = \frac{d}{dx}[u(w)^3]$$

...Use u and w substitutions

$$= 3 \cdot u(w)^2 \cdot \frac{du}{dx}$$

....After using the General Power Rule

$$= 3 \cdot u(w)^2 \cdot \left(\frac{du}{dw} \cdot \frac{dw}{dx} \right)$$

...After using the Chain Rule for $\frac{du}{dx}$

$$= 3 \cdot u(w)^2 \cdot [-\sin(w) \cdot 2\pi x]$$

...After evaluating $\frac{du}{dw}$ and $\frac{dw}{dx}$

$$= 3[\cos(\pi x^2)]^2 \cdot (-\sin(\pi x^2) \cdot 2\pi x)$$

...After substituting for u and w

$$= -6\pi x [\cos(\pi x^2)]^2 \sin(\pi x^2)$$

....After simplification.

Notice that we first used the General Power Rule and then used the Chain Rule, in the last step, we took the derivative of the argument.

Review

For #1-11, find $f'(x)$.

1. $f(x) = (2x^2 - 3x)^{39}$

2. $f(x) = \left(x^3 - \frac{5}{x^2}\right)^{-3}$

3. $f(x) = \frac{1}{\sqrt{3x^2 - 6x + 2}}$

4. $f(x) = \sin^3 x$

5. $f(x) = \sin x^3$

6. $f(x) = \sin^3 x^3$

7. $f(x) = \tan(4x^5)$

8. $f(x) = \frac{4x - \sin^2}{2x}$

9. $f(x) = \frac{\sin x}{\cos(3x-2)}$
10. $f(x) = (5x+8)^3(x^3+7x)^{13}$
11. $f(x) = \left(\frac{x-3}{2x-5}\right)^3$
12. Find

$$\frac{dy}{dx}$$

for

$$y = 5\cos(3x^2 - 1)$$

13. Find the derivative of

$$\sqrt{x^3 + x^5 + 89}$$

14. Find the derivative of $\sin(\sin(\sin(x)))$.
15. By definition, any function composed with its inverse is just the identity: $f(f^{-1}(x))=x$. differentiate both sides of this equation and solve algebraically for the derivative of the inverse.

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 3.8.

Vocabulary

Term	Definition
chain rule	The chain rule is the method for computing the derivative of a composite function. It states that for functions $f(x)$ and $g(x)$, $(f \circ g)'(x) = f'(g(x))g'(x)$.
composite function	A composite function is a function $h(x)$ formed by using the output of one function $g(x)$ as the input of another function $f(x)$. Composite functions are written in the form $h(x)=f(g(x))$ or $h=f \circ g$.
derivative	The derivative of a function is the slope of the line tangent to the function at a given point on the graph. Notations for derivative include $f'(x)$, $\frac{dy}{dx}$, y' , $\frac{df}{dx}$ and $\frac{df(x)}{dx}$.

Additional Resources

PLIX - Play, Learn, Interact, eXplore - Differentiation: Chain Rule

Video: Chain Rule Introduction

Practice: Chain Rule

Real World: Goodbye Waves

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5.6: Implicit Differentiation

The differentiation techniques up to this point have been applied to function definitions (equations) of the form $y = f(x)$. Not all functions can be stated in this form, e.g., when products of x and y are involved. When they cannot, finding the derivative of y may require a different technique.

Implicit Differentiation

Consider the equation $2xy=1$.

We want to obtain the derivative dy/dx . One way to do this is to first solve for y , to produce an **explicit function** of x ,

$$y = \frac{1}{2x}$$

and then take the derivative on both sides,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left[\frac{1}{2x} \right] \\ &= \frac{-1}{2x^2}\end{aligned}$$

However, there is another way of finding dy/dx that uses the original equation that is an **implicit function** of x . We can directly differentiate both sides.

Find the derivative dy/dx of the equation $2xy=1$ without transforming it to an explicit function of x . To do this, directly differentiate both sides:

$$\frac{d}{dx}[2xy] = \frac{d}{dx}[1]$$

Using the Product Rule on the left-hand side,

$$\begin{aligned}y \frac{d}{dx}[2x] + 2x \frac{d}{dx}[y] &= 0 \\ y[2] + 2x \frac{dy}{dx} &= 0\end{aligned}$$

Note that the chain rule is applied when taking the derivative of a term with y and a dy/dx is included for those terms. Solving for dy/dx ,

$$\frac{dy}{dx} = \frac{-2y}{2x} = -\frac{y}{x}$$

But since

$$y = \frac{1}{2x}$$

substitution gives

$$\begin{aligned}\frac{dy}{dx} &= -\frac{1}{x(2x)} \\ &= -\frac{1}{2x^2}\end{aligned}$$

This result agrees with the previous calculations. This second method of finding a derivative is called **implicit differentiation**. You may consider the process and say that the first method is easier and faster and there is no reason for the second method. That may be true for this example and some others, but consider the next problem.

Find dy/dx if $3y^2 - \cos y = x^3$.

Good luck finding an explicit function representation of this equation. Let's try implicit differentiation and see what happens.

Differentiating both sides of the equation with respect to x and then solving for $\frac{dy}{dx}$,

$$\begin{aligned}\frac{d}{dx}[3y^2 - \cos y] &= \frac{d}{dx}[x^3] \\ 3 \frac{d}{dx}[y^2] - \frac{d}{dx}[\cos y] &= 3x^2 \\ 3(2y \frac{dy}{dx}) - (-\sin y) \frac{dy}{dx} &= 3x^2 \\ 6y \frac{dy}{dx} + \sin y \frac{dy}{dx} &= 3x^2 \\ [6y + \sin y] \frac{dy}{dx} &= 3x^2\end{aligned}$$

Solving for $\frac{dy}{dx}$, we finally obtain

$$\frac{dy}{dx} = \frac{3x^2}{6y + \sin y}$$

In this problem, implicit differentiation provided a workable path to a solution.

Implicit differentiation can be used to calculate the slope of the **tangent line** as the problem below shows.

Find the equation of the tangent line that passes through the point (1, 2) on the graph of $8y^3 + x^2y - x = 3$.

The general approach to solving this problem is to:

1. find $\frac{dy}{dx}$, then
2. substitute the point (1, 2) into the derivative to find the slope, and then
3. use the equation of the line (either the slope-intercept form or the point-intercept form) to find the equation of the tangent line.

For step 1, finding an explicit function representation of the equation is not obvious. Using implicit differentiation, however, allows differentiation of both sides:

$$\begin{aligned}\frac{d}{dx}[8y^3 + x^2y - x] &= \frac{d}{dx}[3] \\ 24y^2 \frac{dy}{dx} + [(x^2)(1) \frac{dy}{dx} + y(2x)] - 1 &= 0 \\ 24y^2 \frac{dy}{dx} + x^2 \frac{dy}{dx} + 2xy - 1 &= 0 \\ [24y^2 + x^2] \frac{dy}{dx} &= 1 - 2xy \\ \frac{dy}{dx} &= \frac{1 - 2xy}{24y^2 + x^2}\end{aligned}$$

Now, substituting point (1, 2) into the derivative to find the slope,

$$\begin{aligned}\frac{dy}{dx} &= \frac{1 - 2(1)(2)}{24(2)^2 + (1)^2} \\ &= \frac{-3}{97}\end{aligned}$$

So the slope of the tangent line is $-\frac{3}{97}$, which is a small value. (What does this tell us about the orientation of the tangent line?)

Next we need to find the equation of the tangent line. The slope-intercept form is

$$y = mx + b$$

where $m = -\frac{3}{97}$ and b is the y -intercept. To find b , simply substitute point (1, 2) into the line equation and solve:

$$2 = (-397)(1) + b$$

$$b = \frac{197}{97}$$

Thus the equation of the tangent line is

$$y = \frac{-3}{97}x + \frac{197}{97}$$

Note that we could have used the equivalent point-slope form

$$y - y_1 = m(x - x_1)$$

To summarize, to find the derivative of an implicit function follow the following steps:

1. Differentiate both sides with respect to x
2. Collect all $\frac{dy}{dx}$ terms on the left hand side of the equation and place the other terms without $\frac{dy}{dx}$ on the right hand side.
3. Factor common $\frac{dy}{dx}$ from all terms
4. Solve for $\frac{dy}{dx}$.

Note that the expression for the derivative $\frac{dy}{dx}$ may involve BOTH x AND y .

Examples

Example 1

Use implicit differentiation



$$x^2 - 4y^2 = 9.$$

$$\frac{d}{dx}[x^2 - 4y^2] = \frac{d}{dx}[9]$$

$$10x - 8y \frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$,

$$\frac{dy}{dx} = \frac{5x}{4y}$$

Differentiating both sides implicitly again (and using the quotient rule),

$$\frac{d^2y}{dx^2} = \frac{(4y)(5) - (5x)(4 \frac{dy}{dx})}{(4y)^2}$$

$$= \frac{20y}{16y^2} - \frac{20x}{16y^2} \frac{dy}{dx}$$

$$= \frac{5}{4y} - \frac{5x}{4y} \frac{dy}{dx}$$

But since

$$\frac{dy}{dx} = \frac{5x}{4y}$$

we substitute it into the second derivative:

$$\frac{d^2y}{dx^2} = \frac{5}{4y} - \frac{5x}{4y} \cdot \frac{5x}{4y}$$

$$\frac{d^2y}{dx^2} = \frac{5}{4y} - \frac{25x^2}{16y^2}$$

This is the second derivative of y .

Example 2

Use implicit differentiation to find

$$\frac{d^2y}{dx^2} \Big|_{(x,y)=(2,3)}$$

The next step is to find:

$$\begin{aligned} \frac{d^2y}{dx^2} \Big|_{(x,y)=(2,3)} &= \frac{5}{4(2)} - \frac{25(2)^2}{16(3)^2} \\ &= -\frac{5}{72} \end{aligned}$$

Example 3

What does the second derivative represent?

Since the first derivative of a function represents the rate of change of the function $y=f(x)$ with respect to x , the second derivative represents the rate of change of the function. For example, in kinematics (the study of motion), the speed of an object (y') signifies the change of position with respect to time but acceleration (y'') signifies the rate of change of the speed with respect to time.

Review

For #1-6, find dy/dx by implicit differentiation.

1. $x^2 + y^2 = 500$
2. $x^2y + 3xy - 2 = 1$
3. $\frac{1}{x} + \frac{1}{y} = \frac{1}{2}$
4. $\sqrt{x} - \sqrt{y} = \sqrt{3}$
5. $\sin(25xy^2) = x$
6. $\tan^3(x^2 - y^2) = \tan\left(\frac{\pi}{4}\right)$

For #7-8, use implicit differentiation to find the slope of the tangent line to the given curve at the specified point.

7. $x^2y - y^2x = -1$
at (1,1)
8. $\sin(xy) = y$
at $(\pi, 1)$

9. Find y'' by implicit differentiation for $x^3y^3=5$.

10. Use implicit differentiation to show that the tangent line to the curve $y^2=kx$ at (x_0, y_0) is given by

$$y_0y = \frac{1}{2}k(x + x_0)$$

where k is a constant.

11. Find

$$\frac{d}{dx}(x\sin(y) + y\sin(x))$$

12. Find y' if

$$x^2 + xy + y^2 = 10$$

13. Find the formula for the line tangent to the curve

$$y^3 + 2xy^2 - x = 2$$

at the point $(1, 1)$.

14. Find

$$\frac{d}{dx}(\sqrt{xy})$$

15. $y^2 + \sin(y) = x$. Find

$$\frac{d^2y}{dx^2}$$

in terms of x and y .

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 3.10.

Vocabulary

Term	Definition
explicit function	An explicit function is a function where it is possible to state the dependent variable as a function only of the independent variable.
implicit differentiation	Implicit differentiation is an application of differentiation on an implicit function that may yield an expression involving both the independent and the dependent variables.
implicit function	An implicit function is a function where the dependent variable is not an explicit function of the independent variable.
Tangent line	A tangent line is a line that "just touches" a curve at a single point and no others.

Additional Resources

Video: Implicit Differentiation by Khan Academy

Practice: Implicit Differentiation

Real World: The Color Purple

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CHAPTER OVERVIEW

6: Differentiation - Multi-Step Differentiation

[6.1: Higher Order Derivatives](#)

[6.2: Related Rates](#)

[6.3: Derivatives of Exponential Functions](#)

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6.1: Higher Order Derivatives

You may recall hearing about Becca and her Track and Field competition in a prior concept. Her boyfriend had been recording on her laptop the position signals from the GPS transmitter Becca was wearing during her race. She was using a laptop program she had written to process the once per second GPS position signals so she could determine how far she had run at time during the race. These distance and time points were plotted on her laptop. She had created a program that also gave her a mathematical model of her distance so that she could take the derivative of the math function at any time point to get her “instantaneous” speed.

What if, instead of just finding her speed at any time during the race, she wanted to find her **acceleration**? How would that be done?

Higher Order Derivatives

If the function f has a derivative f' that is differentiable, then the derivative of f' , denoted by f'' is called the second derivative of f . We can continue the process of differentiating derivatives and obtain third, fourth, fifth and higher derivatives of f . They are denoted as shown below:

Notations for Higher Order Derivatives

1st	2nd	3rd	4th	nth order
f'	f''	f'''	$f(4)$	$f(n)$
y'	y''	y'''	$y(4)$	$y(n)$
$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	$\frac{d^3y}{dx^3}$	$\frac{d^4y}{dx^4}$	$\frac{d^ny}{dx^n}$
$D_x y$	$D_x^2 y$	$D_x^3 y$	$D_x^4 y$	$D_x^n y$

Given $f(x) = -2x^2 - 4x - 1$. What is $f''(x)$?

Recall that $f''(x)$ means “The second derivative of $f(x)$ ”, or “The derivative of the derivative of $f(x)$ ”. The function $f(x)$ must be differentiated twice as follows:

$$f'(x) = \frac{d}{dx}(-2x^2 - 4x - 1)$$

...Determine the 1st derivative

$$f'(x) = -4x - 4$$

$$f''(x) = \frac{d}{dx}(-4x - 4)$$

....Determine the 2nd derivative.

$$f''(x) = -4$$

Therefore,

$$f''(x) = -4$$

Examples



Example 1

Earlier, you were asked about Becca would find her acceleration. Since Becca has already created a program to calculate her instantaneous speed at a given point on the track by finding the derivative of the mathematical model to her GPS position data, she could then take the derivative of that function, the second derivative, to find her **instantaneous acceleration** at the same point in the race.

By finding her instantaneous speed and acceleration at different points in the race, she can learn a lot about her performance during the race, and hopefully target areas she needs to work on to improve her overall success.

Example 2

Given $f(x) = (-x^4 - 4x^3 - 5x^2 + 3)$. Find $f''(x)$ when $x=3$.

Again, the function $f(x)$ must be differentiated twice; then the 2nd derivative must be evaluated:

$$f'(x) = \frac{d}{dx}(-x^4 - 4x^3 - 5x^2 + 3)$$

...Determine the 1st derivate

$$= -4x^3 - 12x^2 - 10x$$

$$f''(x) = \frac{d}{dx}(-4x^3 - 12x^2 - 10x)$$

....Determine the 2nd derivate

$$= -12x^2 - 24x - 10$$

$$f''(3) = -12(3)^2 - 24(3) - 10$$

....Evaluate the 2nd derivate

$$= -190$$

Therefore,

$$f''(3) = -190$$

Example 3

Show that $y = x^3 + 3x + 2$ satisfies the differential equation $y''' + xy'' - 2y' = 0$.

We need to obtain the first, second, and third derivatives and substitute them into the differential equation to check for equality.

$$y = x^3 + 3x + 2$$

$$y' = 3x^2 + 3$$

$$y'' = 6x$$

$$y''' = 6$$

Substituting,

$$y''' + xy'' - 2y' = 6 + x(6x) - 2(3x^2 + 3)$$

$$= 6 + 6x^2 - 6x^2 - 6$$

$$= 0$$

which satisfies the equation.

Example 4

Find the fifth derivative of

$$f(x) = 2x^4 - 3x^3 + 5x^2 - x - 1$$

To find the fifth derivative, we must first find the first, second, third, and fourth derivatives as follows:

1. $f'(x) = 8x^3 - 9x^2 + 5x - x$
2. $f(2)(x) = 24x^2 - 18x + 5$
3. $f(3)(x) = 48x - 18$
4. $f(4)(x) = 48$
5. $f(5)(x) = 0$

Review

1. Given:

$$v(x) = -4x^3 + 3x^2 + 2x + 3$$

What is $v''(x)$?

2. Given:

$$m(x) = x^2 + 5x$$

What is $m''(x)$?

3. Given:

$$d(x) = 3x^4 e^x$$

What is $d''(x)$?

4. Given:

$$t(x) = -2x^5 \sin(x)$$

What is

$$\frac{d^2 t}{dx^2} ?$$

5. What is

$$\frac{d^2}{dx^2} 3x^5 e^x ?$$

6. Find

$$\frac{d^3 y}{dx^3} \Big|_{x=1}$$

where

$$y = \frac{2}{x^3}$$

.

7. Suppose

$$u'(0) = 98$$

and

$$\left(\frac{u}{q}\right)'(0) = 7$$

Find $q(0)$ assuming $u(0)=0$?

8. Given:

$$b(x) = \frac{x^2 - 5x + 4}{-5x + 2}$$

What is: $b'(2)$?

9. Given:

$$m(x) = e^x 3x + 4$$

What is

$$\frac{dm}{dx}$$

?

10. What is

$$\frac{d}{dx} \cdot \frac{\sin(x)}{x-4}?$$

11. Given

$$q(x) = \frac{x}{\sin(x)}$$

What is

$$q''(x) = \frac{d^2}{dx^2} \frac{x}{\sin(x)}?$$

12. The position of a certain nano particle can be approximated by the function t^3+t . What function gives the acceleration of the particle?

13. The position of a car is given by the function $\sin(t)+3t^2$. Is the car accelerating or decelerating?

14. The position of a velociraptor chasing a triceratops is given by the function $\cos(-t)$. Is the raptor experiencing positive or negative jerk at

$$t = \frac{3\pi}{2}$$

15. The position of the moon in night sky given by the following function of time:

$$\frac{1}{12}t^4 - \frac{3}{6}t^3 - 5t^2 + \pi$$

Name a time when the moon is experiencing no acceleration at all.

16. What is the maximum number of times one would have to differentiate a N-degree polynomial before the derivative becomes zero?

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 3.11.

Vocabulary

Term	Definition
Acceleration	Acceleration is the rate at which the speed and direction of an object are changing.
higher order derivative	A higher order derivative is a second, or third, or nth derivative of a function.
Instantaneous acceleration	The instantaneous acceleration of an object is the change in velocity of the object calculated at a specific point in time.
Instantaneous velocity	The instantaneous velocity of an object is the velocity of the object at a specific point in time.

Additional Resources

PLIX: Play, Learn, Interact, eXplore - Higher Order Derivatives-Acceleration and Jerk

Video: Calculus - Finding Higher Derivatives

Real World: Deep Secrets

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6.2: Related Rates

In the concept on implicit differentiation, you learned that given a relationship between two quantities, the relationship between their respective rates of change can be determined. This is often an invaluable tool in applications, since it allows us to find the rate at which some quantity is changing by relating it to other quantities whose rates of change are known (or at least more easily found). For example, if you were at the scene of an accident where an oil spill from a large tanker is spreading in a circular pattern whose radius you determine is increasing by about 1 ft per second, would you be able to tell someone how fast the oil spill area was increasing when the radius was about 30 feet?

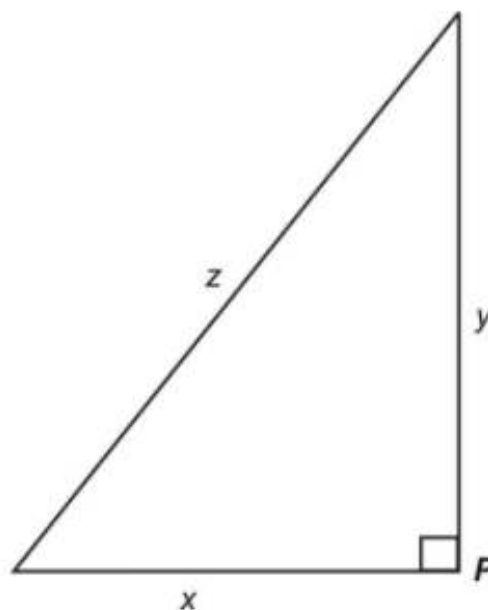
Related Rates

What do we mean by related rates? These are simply the derivatives, rates, of one or more parameters that are related to each other through an equation. The relationship between the rates is obtained by taking the **derivative** of some other relationship between the parameters.

A simple example with a familiar geometric shape should help illustrate the concept.

Consider the simple right triangle in the figure below with sides x , y , and z . The relationship between the sides is governed by the Pythagorean Theorem.

$$x^2 + y^2 = z^2$$



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We could easily attach some real-life situation to this geometric figure. Say for instance that x and y represent the paths of two people starting at point p and walking North and West, respectively, for two hours. The quantity z represents the distance between them at any time t . Let's now determine any relationships between the various rates of change that we get by implicitly differentiating the original equation $x^2 + y^2 = z^2$ with respect to time t .

$$\frac{d}{dt}[x^2 + y^2] = \frac{d}{dt}[z^2]$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2z \frac{dz}{dt}$$

$$x \frac{dx}{dt} + y \frac{dy}{dt} = z \frac{dz}{dt}$$

...Equation 1

Implicit differentiation of the Pythagorean relationship between the side lengths has yielded relationships between the side length derivatives, and since the derivatives are rates, this is an example of **related rates**.

How might the relationship above be used to solve or answer a problem?

Let's say that person is walking in the x-direction at 5 mph, and that another person is walking in the y-direction at 3 mph. The distance, z , between the walkers is changing with time, and the time rate of change of, z , $\frac{dz}{dt}$, is dependent on the rates at which the two people are walking.

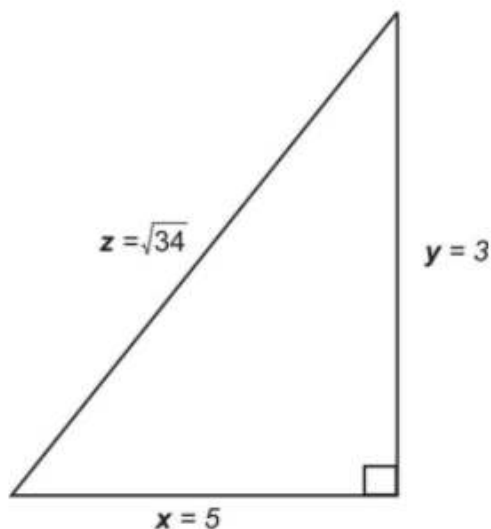
One problem that we could pose is:

At what rate is the distance between x and y increasing after one hour? That is, what is $\frac{dz}{dt}$ after one hour?

Assume that they have walked for one hour. So $x=5$ mi and $y=3$.

Using the Pythagorean Theorem, we find the distance between them after one hour is

$$z = \sqrt{34} = 5.83$$



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If we substitute the values for x, y , and z into **Equation 1** along with the individual rates we get

$$5(5) + 3(3) = \sqrt{34} \frac{dz}{dt}$$

$$34 = \sqrt{34} \frac{dz}{dt}$$

$$\frac{34}{\sqrt{34}} = \frac{dz}{dt}$$

Hence after one hour the distance between the two people is increasing at a rate of:

$$\frac{dz}{dt} = \frac{34}{\sqrt{34}} \approx 5.83 \text{ mph}$$

Suppose we have a rectangular field and we know that at an instant of time, the length is changing at the rate of 8 ft/hour and the perimeter is changing at a rate of 24 ft/hour. At what rate is the width changing at that instant? At what rate is the area changing at that instant?

You are familiar with the formulas for Perimeter:

$$P = 2l + 2w$$

If we differentiate the perimeter equation, we have

Equation 2:

$$\frac{dP}{dt} = 2\frac{dl}{dt} + 2\frac{dw}{dt}$$

Substituting our known information into Equation 2, we have

$$24 = (2 \times 8) + 2 \times \frac{dw}{dt}$$

$$8 = 2 \times \frac{dw}{dt}$$

$$4 = \frac{dw}{dt}$$

The width is changing at a rate of 4 ft/hr.

Examples

Example 1

Earlier, you were asked a question about a scene of an accident where an oil spill from a large tanker is spreading in a circular pattern whose radius you could determine by about 1 ft per second. Would you be able to tell someone how fast the oil spill area was increasing when the radius was about 30 feet?

Since the circular area of the spill is

$$A = \pi r^2$$

the time rate of change of the area is

$$\frac{dA}{dt} = \pi 2r \frac{dr}{dt}$$

This means that with

$$\frac{dr}{dt} = 1 \frac{ft}{s}$$

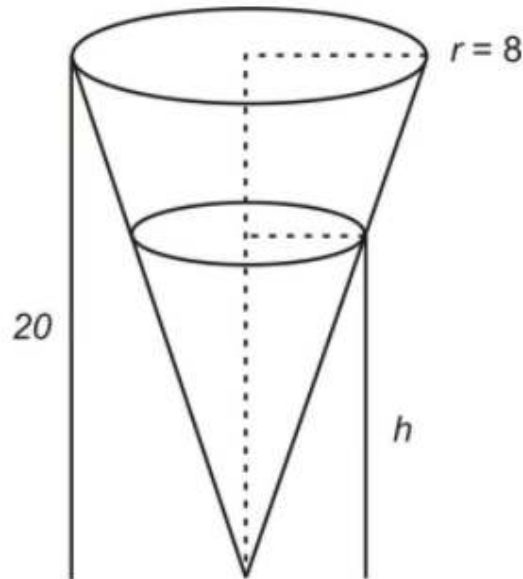
when $r=30$ ft,

$$\frac{dA}{dt} = 60\pi \frac{ft^2}{s}$$

Example 2

We have a water tank shaped as an inverted right circular cone. Suppose that water flows into the tank at the rate of 5 ft³/min. At what rate is the water level rising when the height of the water in the tank is 6 feet?





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We know that the volume of water in the water tank is:

$$V = \frac{1}{3}\pi r^2 h$$

When we differentiate this equation we get:

$$\frac{dV}{dt} = \frac{1}{3}\pi(h)(2r)\frac{dr}{dt} + \frac{1}{3}\pi r^2 \frac{dh}{dt}$$

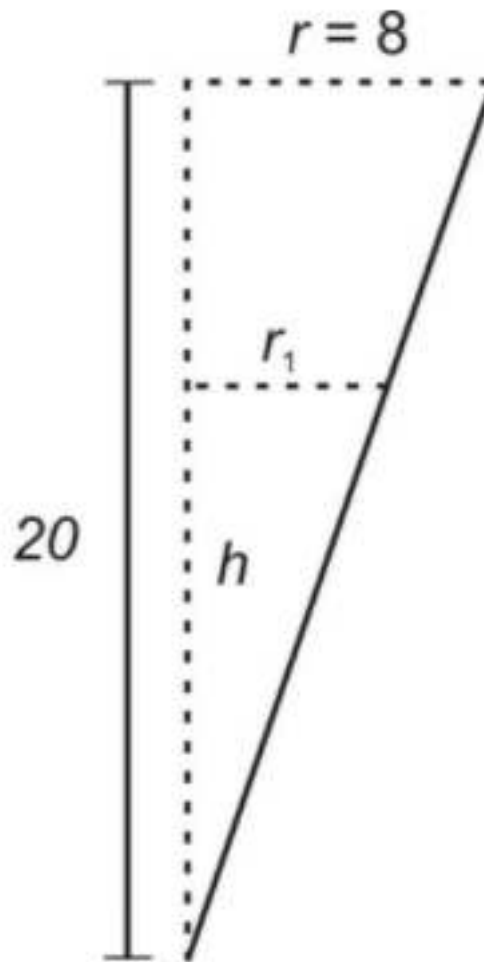
This is a related rates equation. The rate $\frac{dV}{dt}$ is related to the rates $\frac{dr}{dt}$ and $\frac{dh}{dt}$.

We know

$$\frac{dV}{dt} = 5 \frac{ft^3}{min}$$

do not know $\frac{dr}{dt}$, but want to find $\frac{dh}{dt}$. We need to somehow find a relationship between h and r .

Let r_1 be the radius of the surface of the water as it flows out of the tank.



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Note that the two triangles are similar and thus corresponding parts are proportional. In particular,

$$\frac{r_1}{h} = \frac{8}{20}$$

$$r_1 = \frac{8h}{20} = \frac{2h}{5}$$

We can write this as a general relationship between r and h .

$$r = \frac{2h}{5}$$

which means also that

$$\frac{dr}{dt} = \frac{2}{5} \frac{dh}{dt}$$

We have the relationships we need.

Now we can solve the problem in a couple of ways: (a) substitute

$$r = \frac{2h}{5}$$

into the original equation for V , or (b) substitute

$$r = \frac{2h}{5}$$

and

$$\frac{dr}{dt} = \frac{2}{5} \frac{dh}{dt}$$

into the equation for

$$\frac{dV}{dt}$$

Let's try approach (a).

$$V = \frac{1}{3} \pi \left(\frac{2h}{5} \right)^2 h = \frac{4\pi}{75} h^3$$

Hence

$$\frac{dV}{dt} = \frac{12\pi}{75} h^2 \frac{dh}{dt}$$

and by substitution,

$$5 = \frac{12\pi}{75} (36) \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{375}{432\pi} \approx 0.28 \frac{\text{ft}}{\text{min}}$$

Review

1. Make up a related rates problem about the area of a rectangle. Illustrate the solution to your problem.
2. Suppose that a particle is moving along the curve $4x^2 + 16y^2 = 32$. When it reaches the point (2, 1), the x-coordinate is increasing at a rate of 3 ft/sec. At what rate is the y-coordinate changing at that instant?
3. A regulation softball diamond is a square with each side of length 60 ft. Suppose a player is running from first base to second base at a speed of 18 ft/sec. At what rate is the distance between the runner and home plate changing when the runner is $\frac{2}{3}$ of the way from first to second base?
4. At a recent Hot Air Balloon festival, a hot air balloon was released. Upon reaching a height of 300 ft, it was rising at a rate of 20 ft/sec. Mr. Smith was 100 ft away from the launch site watching the balloon. At what rate was the distance between Mr. Smith and the balloon changing at that instant?
5. Two trains left the St. Louis train station in the late morning. The first train was traveling East at a constant speed of 65 mph. The second train traveled South at a constant speed of 75 mph. At 3 PM, the first train had traveled a distance of 120 miles while the second train had traveled a distance of 130 miles. How fast was the distance between the two trains changing at that time?
6. Suppose that a 17 ft ladder is sliding down a wall at a rate of -6 ft/sec. At what rate is the bottom of the ladder moving when the top is 8 ft from the ground?
7. Suppose that the length of a rectangle is increasing at the rate of 6 ft/min and the width is increasing at a rate of 2 ft/min. At what rate is the area of the rectangle changing when its length is 25 ft and its width is 15 ft?
8. Suppose that the quantity demand of new 40" plasma TVs is related to its unit price by the formula $p + x^2 = 1200$, where p is measured in dollars and x is measured in units of one thousand. How is the quantity demand changing when $x=20$, $p=1500$, and the price per TV is decreasing at a rate of \$10?
9. The volume of a cube with side s is changing. At a certain instant, the sides of the cube are 6 inches and increasing at the rate of $\frac{1}{4}$ in/min. How fast is the volume of the cube increasing at that time?
10. (1) Suppose that the area of a circle is increasing at a rate of $24 \text{ in}^2/\text{min}$. How fast is the radius increasing when the area is $36\pi \text{ in}^2$? (2) How fast is the circumference changing at that instant?
11. The radius of a circle is getting bigger at a rate of 5 centimeters per second. How fast is the area of the circle increasing when the radius is equal to ten centimeters?

12. The area of a circle is expanding at a rate of 100 square centimeters a second at an instant when its radius is expanding at a rate of 50 centimeters a second. What is the circle's radius at that instant?
13. At a given instant, the volume of a cylinder with a cross sectional area of 6 square centimeters is increasing at a rate of 10 cubic centimeters a second. What is the rate of increase of its height at that instant?
14. Express the rate of change of a cylinder's volume as a function of its radius, its height, and the rate of change of its radius, if its height is assumed to be held constant.
15. A four dimensional object whose momentum is given by the formula $M = \sin(\pi x_1) + \delta x_2^3 + \ln(x_3 x_4) x_1^5$, where δ is a constant, is falling into a black hole. The resulting compression causes its x_1 value to shrink at a rate of 8 million miles a second (while the other variables remain constant.) If

$$x_3 = \frac{5}{x_4}$$

what is the instantaneous change in its momentum when $x_1=1$?

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 4.1.

Vocabulary

Term	Definition
derivative	The derivative of a function is the slope of the line tangent to the function at a given point on the graph. Notations for derivative include $f'(x)$, dy/dx , y' , df/dx and $\frac{df(x)}{dx}$.
related rates	Related rates are derivatives of variables that are common (related) to one or more linked equations.

Additional Resource

Video - Ladder Rate-Of-Change Problem

Practice - Related Rates

Real World - Chain Reaction

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6.3: Derivatives of Exponential Functions

Exponential functions, and the rate of change, are used to model many real-world situations such as population growth, radioactive half-life decay, attenuation of electromagnetic signals in media, and financial transactions. Do you know how to write general exponential equations for the growth of a population that doubles every 5 years, and its rate of change?

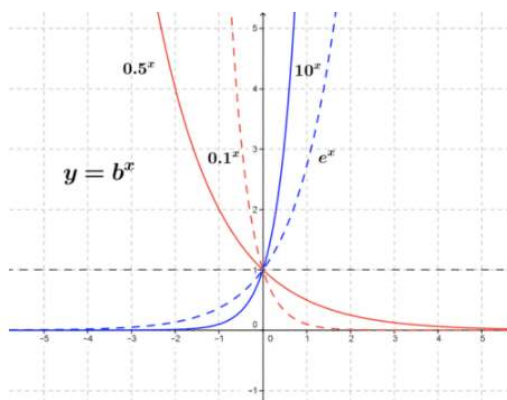
Derivatives of Exponential Functions

An **exponential function** $f(x)$ has the form:

$$f(x) = b^x$$

where b is called the base and is a positive, real number.

The figure below shows a few exponential function graphs for $0 < b \leq 10$. It is very clear that the sign of the **derivative** of an exponential depends on the value of b . If $0 < b < 1$, the value of the derivative of the function (slope of the tangent line) will be negative because the function is always decreasing as x increases. For $b > 1$, the derivative of the function will always be positive because the function increases as x increases.

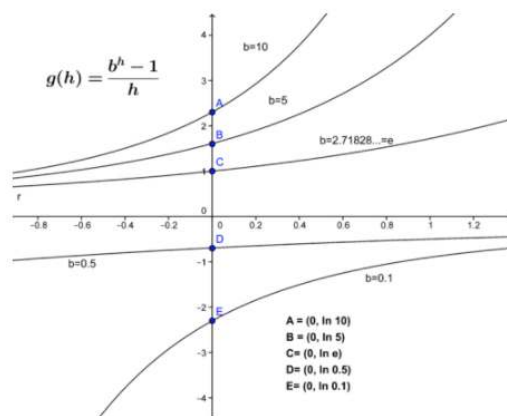


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But, what is the derivative of an exponential function? We can take the following steps to find an expression for $\frac{d}{dx}[bx]$ by using the definition of the derivative:

$$\begin{aligned} \frac{d}{dx}[bx] &= \lim_{h \rightarrow 0} \frac{b^{x+h} - b^x}{h} \dots \text{Limit definition of the derivative} \\ &= \lim_{h \rightarrow 0} \frac{b^x b^h - b^x}{h} \dots \text{Exponent property} \\ &= \lim_{h \rightarrow 0} \frac{b^h - 1}{h} \cdot b^x \dots \text{Factoring} \\ &= \left(\lim_{h \rightarrow 0} \frac{b^h - 1}{h} \right) \cdot b^x \dots \text{Limit of a product property} \end{aligned}$$

The result above shows $\frac{d}{dx}[bx]$ depends on the product of $\lim_{h \rightarrow 0} \frac{b^h - 1}{h}$ and the original function. But what is $\lim_{h \rightarrow 0} \frac{b^h - 1}{h}$? There are a number of ways to evaluate this limit, but for now let's take a quick look at the behavior of $\frac{b^h - 1}{h}$. This function is graphed below for a number of values of b , and the limit at $h = 0$ is indicated by the points A–E.



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While it is not at all obvious: $\lim_{h \rightarrow 0} \frac{b^h - 1}{h} = \ln(b)$. Remember the natural logarithm function $y = \ln x$ on your calculator? The natural logarithm is the general logarithm function with base $b = e = 2.71828...$

Given the exponential function $f(x) = b^x$, where the base b is a positive, real number, then the general representation of the **derivative of an exponential function** is:

$$\frac{d}{dx}[bx] = \ln b \cdot b^x$$

Adding the Chain Rule to the definition, given the exponential function $f(x) = b^u$, where $u = g(x)$ and $g(x)$ is a differentiable function, then:

$$\frac{d}{dx}[bu] = (\ln b \cdot b^u) \frac{du}{dx}$$

Examples

Example 1

Earlier, you were asked w  ential equation for the growth of a population that doubles every 5 years is.

A population $P(t)$ that doubles every 5 years could be modeled as $P(t) = P_0 2^{\frac{t}{5}}$, where the variable t represents number of years since the population was at a level of P_0 . Were you able to determine that the rate of change of $P(t)$ is $P'(t) = \frac{P_0 \ln 2}{5} \cdot 2^{\frac{t}{5}}$?

Example 2

Given $y = 500 \cdot 0.7^x$, what is $\frac{dy}{dx}$?

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}[500 \cdot 0.7^x] \\ &= 500 \frac{d}{dx}[0.7^x] \end{aligned}$$

$= 500[\ln(0.7) \cdot 0.7^x]$ Use your calculator to find $\ln(0.7)$

$$= -178.3 \cdot 0.7^x$$

Hence, $\frac{dy}{dx} = -178.3 \cdot 0.7^x$, and as expected, the slopes of all tangent lines are negative.

There is an important special case that you must know about

Example 3

Given $y = 500 \cdot e^x$, what is $\frac{dy}{dx}$?

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[500 \cdot ex] \\ &= 500 \frac{d}{dx}[ex]\end{aligned}$$

$\frac{dy}{dx} = 500[\ln(e) \cdot ex]$ Use your calculator to find $\ln(e)$

$$\begin{aligned}&= \frac{d}{dx}[500 \cdot ex] \\ &= 500[1 \cdot ex] \\ &= 500 \cdot ex\end{aligned}$$

Hence, $\frac{dy}{dx} = 500 \cdot ex$, and this is just the original function. This exponential function, with base e, is special: the rate of change (or slope of the tangent line) at any point is equal to the value of the function at that point.

Example 4

Given $y = 10 \cdot 2.5 - 3x^2$, what is $\frac{dy}{dx}$?

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[10 \cdot 2.5 - 3x^2] \\ &= 10 \frac{d}{dx}[2.5 - 3x^2] \\ &= 10 \cdot \ln(2.5) \cdot 2.5 - 3x^2 \cdot ddx[-3x^2] \\ &= 10 \cdot (0.9162) \cdot 2.5 - 3x^2 \cdot [-6x] \\ &= -55x \cdot 2.5 - 3x^2\end{aligned}$$

Therefore, $\frac{dy}{dx} = -55x \cdot 2.5 - 3x^2$

Example 5

Given $y = 500 \cdot e - 2x \cdot \cos(5\pi x)$, what is $\frac{dy}{dx}$?

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[500 \cdot e - 2x \cdot \cos(5\pi x)] \\ &= 500 \cdot \left[\frac{d}{dx}(e - 2x) \cdot \cos(5\pi x) + e - 2x \cdot \frac{d}{dx}(\cos(5\pi x)) \right] \text{....Product Rule} \\ &= 500 \cdot \left[\ln(e) \cdot e - 2x \frac{d}{dx}(-2x) \cdot \cos(5\pi x) + e - 2x \cdot (-\sin(5\pi x) \cdot \frac{d}{dx}(5\pi x)) \right] \text{....Use Chain Rule} \\ &= 500 \cdot \left[(1) \cdot e - 2x \cdot (-2) \cdot \cos(5\pi x) + e - 2x \cdot (-\sin(5\pi x) \cdot 5\pi) \right] \text{....Simplify.} \\ &= -500 \cdot e - 2x[2 \cdot \cos(5\pi x) + 5\pi \sin(5\pi x)] \text{...Simplify}\end{aligned}$$

Therefore, $\frac{dy}{dx} = -500 \cdot e - 2x[2 \cdot \cos(5\pi x) + 5\pi \sin(5\pi x)]$.

Review

For #1-14, find the derivative.

1. $y = 7^x$
2. $y = 3^{2x}$
3. $y = 5^x - 3x^2$

4. $y = 2^{x^2}$
5. $y = e^{x^2}$
6. $f(x) = \frac{1}{\sqrt{\pi\sigma}} e^{-\alpha k(x-x_0)^2}$ where σ , α , x_0 , and k are constants and $\sigma \neq 0$.
7. $y = e^{6x}$
8. $y = e^{3x^3} - 2x^2 + 6$
9. $y = \frac{e^x - e - x}{e^x + e - x}$
10. $y = \cos(e^x)$
11. $y = e^{-x} 3^x$
12. $y = 3^{-x^2+2x+1}$
13. $y = 2^x 3^x$
14. $y = e^{-x} \sin x$
15. Find an equation of the tangent line to $f(x) = x^3 + 2e^x$ at the point (0, 2).

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 3.9.

Vocabulary

Term	Definition
derivative	The derivative of a function is the slope of the line tangent to the function at a given point on the graph. Notations for derivative include $f'(x)$, dy/dx , y' , df/dx and $\frac{df(x)}{dx}$.
Exponential Function	An exponential function is a function whose variable is in the exponent. The general form is $y = a \cdot b^{x-h} + k$.

Additional Resources

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CHAPTER OVERVIEW

7: Differentiation - Increasing and Decreasing Values and Extrema

[7.1: Derivatives with Mean Value Theorem and Rolle's Theorem](#)

[7.2: Absolute versus Local Extrema](#)

[7.3: First Derivative Test](#)

[7.4: Second Derivative Test](#)

[7.5: Concavity and Inflection](#)

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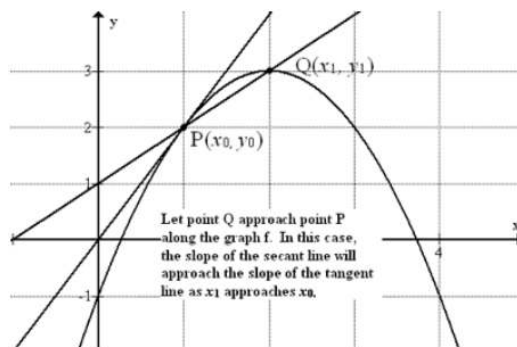
7.1: Derivatives with Mean Value Theorem and Rolle's Theorem

Jim was telling his math major friend about a speeding ticket he had received recently for going over 80 mph one weekend morning when there was little traffic on the highway. Apparently two police officers at different locations along the highway had recorded him going 60 and 65 mph respectively as he went by each of their speed trap positions. Jim said he knew where the speed traps were located and always made sure he was within the speed limit going by, so he could not understand how they could say he was speeding. Jim's math major friend asked whether the officer who issued the ticket had said anything. Jim recalled that the officer had mentioned that the speed traps were exactly 3 miles apart, and that they knew Jim's car had covered the distance in 2 minutes. Jim's math major friend smiled and said, "Oh, that's how they did it: The **Mean Value Theorem**". Jim looked very perplexed.

Can you figure out how the police officer could have issued a ticket without actually witnessing Jim's car going over 80 mph?

The Mean Value Theorem

In an earlier concept, the derivative of a function was introduced as the slope of the tangent line at a point. We looked at how the slope of a secant line changed as one of the two points approached the other. The secant line became the tangent line in the limit as illustrated below.



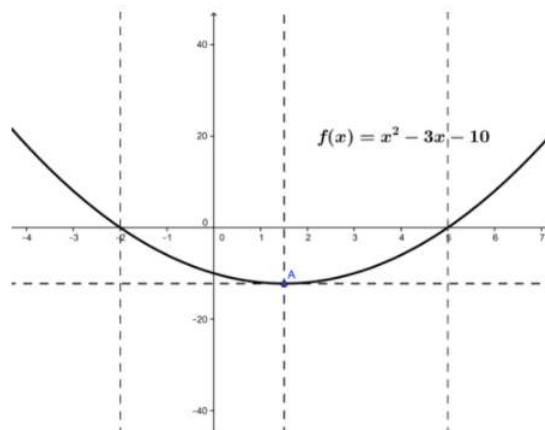
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Now we look at the slope values of tangent lines in an interval defined by the secant line between two points. The following theorem gives an important relationship.

Rolle's Theorem: If f is continuous on a closed interval $[a,b]$ and differentiable on the open interval (a,b) , and if $f(a)=f(b)$ then f has at least one value c in the open interval (a,b) such that $f'(c)=0$.

Let's see if you can make use of Rolle's Theorem

For the function $f(x) = x^2 - 3x - 10$, show that it satisfies the conditions of Rolle's theorem on the interval $[-2, 5]$ and find all numbers c in $(-2, 5)$ that satisfy $f'(c)=0$



Evaluating $f(x)$ at $x=-2$ and $x=5$ yields:

$$\begin{aligned} f(-2) &= (-2)^2 - 3(-2) - 10 \\ &= 0 \end{aligned}$$

$$\begin{aligned} f(5) &= (5)^2 - 3(5) - 10 \\ &= 0 \end{aligned}$$

Therefore $f(-2)=f(5)$, which satisfies the conditions for Rolle's theorem to hold.

Now to find values of c that satisfy $f'(c)=0$.

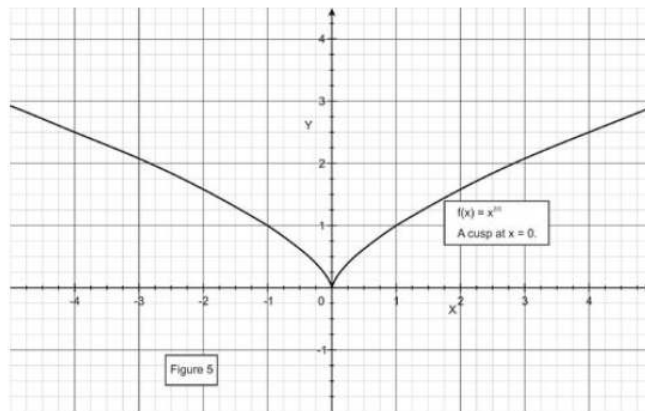
$$\frac{dy}{dx} = 2x - 3$$

$$\begin{aligned} f'(c) &= 0 \\ &= 2(c) - 3 \\ \therefore c &= \frac{3}{2} \end{aligned}$$

So $f'(\frac{3}{2}) = 0$ satisfies Rolle's Theorem.

Consider the function $f(x) = x^{\frac{2}{3}}$. On the interval $[-4, 4]$, $f(-4)=f(4)$. However, is there any point at which the derivative of $f(x) = x^{\frac{2}{3}}$ is equal to zero? Is this a contradiction of **Rolle's theorem**?

The function curve for $f(x) = x^{\frac{2}{3}}$ is shown below.



It's derivative is $f'(x) = \frac{2}{3x^{\frac{1}{3}}}$, which is undefined at $x=0$, and there is no point at which the derivative is 0. But, because the function is not differentiable over the interval, Rolle's Theorem does not apply. There is no contradiction.



Rolle's Theorem

The theorem can be applied. If it is possible to find values of c such that $f'(c) = 0$.

$f(x) = (x^2 - 2x)e^x$
 $[0, 2]$

$f'(0) = (0^2 - 2(0))e^0 = 0$
 $f'(2) = (2^2 - 2(2))e^2 = 0$

$f'(x) = (x^2 - 2x)e^x + e^x(2x - 2)$
 $e^x(x^2 - 2x + 2x - 2) = 0$
 $e^x(x^2 - 2) = 0$

$x^2 - 2 = 0$
 $\sqrt{x^2} = \sqrt{2}$
 $x = \pm\sqrt{2}$
 $c = \sqrt{2}$

Watch on  YouTube

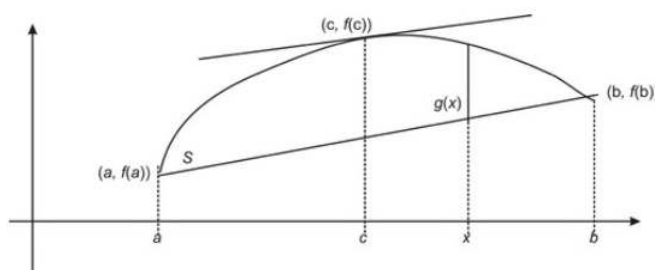
Rolle's Theorem requires that $f(a)=f(b)$. But Rolle's theorem is the basis for proving the important general theorem called the Mean Value Theorem which does not require $f(a)=f(b)$.

Mean Value Theorem: If f is a continuous function on a closed interval $[a,b]$ and if f' contains the open interval (a,b) in its domain, then there exists a number c in the interval (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

An equivalent form is: $f(b) - f(a) = (b - a)f'(c)$.

An illustration of the meaning of the Mean Value Theorem is shown in the figure below, where the [slope](#) of the secant line connecting $f(a)$ and $f(b)$ can be found to be the same as the slope of the tangent line at $f(c)$.



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Verify that the Mean Value Theorem applies for the function $f(x) = x^3 + 3x^2 - 24x$ on the interval $[1, 4]$.

We need to find c in the interval $(1, 4)$ such that $f'(c) = \frac{f(4) - f(1)}{(4 - 1)}$

$$\begin{aligned} f'(c) &= \frac{f(4) - f(1)}{(4 - 1)} \\ &= 16 + 203 \\ &= 12 \end{aligned}$$

Note that

$$f'(x) = 3x^2 + 6x - 24$$

Hence, we must solve the following equation:

$$\begin{aligned} 3c^2 + 6c - 24 &= 12 \\ 3c^2 + 6c - 32 &= 0 \end{aligned}$$

$$c^2 + 2c - 12 = 0$$

$$c = -1 \pm \sqrt{13}$$

Since we need to have c in the interval $(1, 4)$, the positive root is the solution:

$$c \approx 2.61.$$

Examples

Example 1

Earlier, you were asked about how Jim could get a speeding ticket for going at least 80 mph when his car's speed was recorded at 60 mph and 65 mph.

The key information was that the police officer determined that Jim had traveled 3 miles in 2 minutes, so that Jim's average speed was 90 mph. Jim's math major friend knew that by the Mean Value Theorem there had to have been at least one time when Jim's actual speed was 90 mph, definitely exceeding 80 mph.

Example 2

Verify that the Mean Value Theorem applies for the function $f(x) = 2\sin x + 3\cos x$ on the interval $(0, 2\pi)$.

We need to find c in the interval $(0, 2\pi)$ such that $f'(c) = \frac{f(2\pi) - f(0)}{(2\pi - 0)}$:

$$\begin{aligned} f'(c) &= \frac{f(2\pi) - f(0)}{(2\pi - 0)} \\ &= \frac{2 - 2}{2\pi} \\ &= 0 \end{aligned}$$

Note that

$$f'(x) = 2\cos x - 3\sin x$$

Hence, we must solve the following equation:

$$\begin{aligned} 2\cos c - 3\sin c &= 0 \\ \tan c &= \frac{2}{3} \\ c &= \tan^{-1}\left(\frac{2}{3}\right) \end{aligned}$$

$c = 0.588$ radians ... and also $(0.588 + \pi)$ radians in the interval $(0, 2\pi)$.

The Mean Value Theorem applies in two places in the interval $(0, 2\pi)$.

Review

For #1-5, if possible, verify Rolle's Theorem for the following functions by finding values of x for which $f(x)=0$ and $f'(x)=0$. If it is not possible, state why.

1. $f(x) = 3x^3 - 12x$
2. $f(x) = x^2 - 2x - 1$
3. $f(x) = -2x^2 - 12x + 5$
4. $f(x) = |2x - 3|$
5. $f(x) = 2\sin x + 3\cos x$
6. $f(x) = x^4 - 2x^2$
7. Prove that if the equation $x^3 + a_1x^2 + a_2x = 0$ has a positive root at $x=r$ and that the equation $3x^2 + 2a_1x + a_2 = 0$ has a positive root less than r .

For #8-15, verify that the Mean Value Theorem works for each function in the specified interval, or state why it does not.

8. $f(x) = (x + 2)x$, $[1, 2]$
9. $f(x) = 2x$, $[-1, 1]$
10. $f(x) = x^2 - 5x + 1$, $[0, 3]$
11. $f(x) = 5 - 4x$, $[1, 4]$
12. $f(x) = x^3 - 8x - 5$, $[1, 4]$
13. $f(x) = \sin x$, $[0, \frac{\pi}{2}]$
14. $f(x) = \cos x$, $[0, \frac{\pi}{2}]$
15. $f(x) = 2x$, $[0, 3]$

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 4.3.

Vocabulary

Term	Definition
mean value theorem	The mean value theorem for definite integrals states that if f is continuous on the closed interval $[a,b]$ then at some point c in the open interval (a,b) : $\int_a^b f(x)dx = f(c)(b-a)$
Rolle's theorem	Rolle's theorem says that a function that has the same value at two points has a derivative of 0 at at least one point in the interval

Additional Resources

PLIX: Play, Learn, Interact, eXplore - Maxima and Minima, Mean Value Theorem, Rolle's Theorem

Video: Mean Value Theorem

Practice: Derivatives with Mean Value Theorem and Rolle's Theorem

Real World: Droning On

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7.2: Absolute versus Local Extrema

Being able to identify the smallest and largest values of a function (**extrema**), and where they occur in some interval on the domain or over the entire domain, is useful in graphing a function equation as well as in solving “optimization” problems. The location of these extrema is tied to the behavior of the **derivative**. You learned in prior work with quadratic functions (**parabolas**, $y = ax^2 + bx + c$), that the least or greatest value of the parabola could be found at the vertex of the parabola (on the axis of **symmetry** $x = -\frac{b}{2a}$). How are the derivative of the quadratic function and the vertex $x = -\frac{b}{2a}$ related?

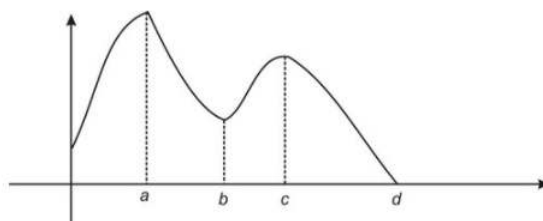
Extrema

Let’s start our discussion with some formal working definitions of the **maximum** and **minimum** values of a function.

- A function f has a **maximum** at $x=a$ if $f(a) \geq f(x)$ for all x in the domain of f .
- A function f has a **minimum** at $x=a$ if $f(a) \leq f(x)$ for all x in the domain of f .

The values of the function for these x -values are called **extreme** values or **extrema**.

Here is an example of a function that has a maximum at $x=a$ and a minimum at $x=d$:



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Observe the graph at $x=b$. While we do not have a minimum at $x=b$, we note that $f(b) \leq f(x)$ for all x near b . We say that the function has a **local minimum** at $x=b$.

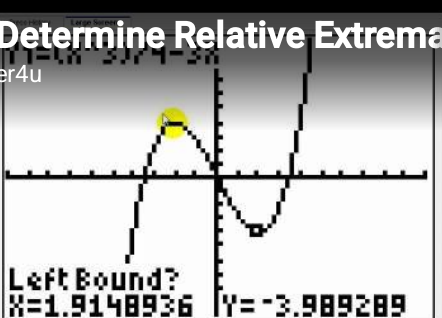
Similarly, we say that the function has a **local maximum** at $x=c$ since $f(c) \geq f(x)$ for some x contained in open intervals of c .

Remember to use the terms maximum and minimum (without including the term local) only when you are talking about the absolute or **global extrema** values of the function; **local extrema** or relative extrema should be called local maximum or local minimum.

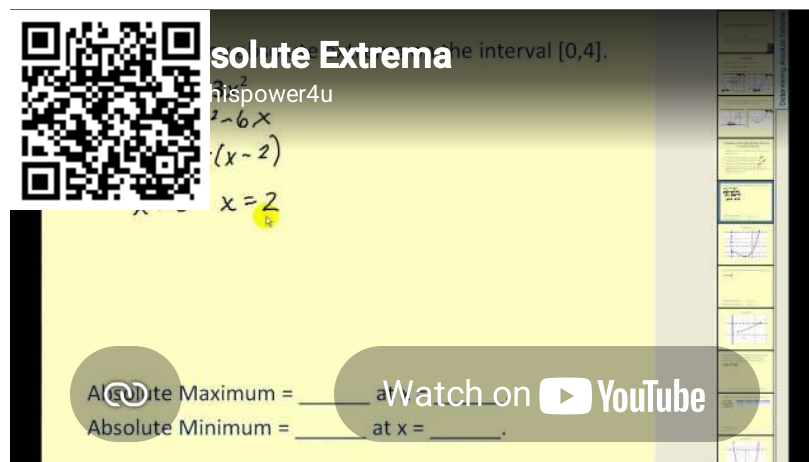


How to Determine Relative Extrema

hispower4u



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The image shows a YouTube video player interface. On the left, there is a QR code. To its right, the title 'Absolute Extrema' is visible. Below the title, there are handwritten notes: 'hispower4u', ' $x^2 - 6x$ ', ' $(x-2)$ ', and ' $x=2$ '. At the bottom of the video frame, there is a 'Watch on YouTube' button. Below the video frame, there are labels for 'Absolute Maximum = _____' and 'Absolute Minimum = _____ at x = _____'.

Do all functions have a maximum and minimum? Yes, all continuous functions do. This is stated by the Extreme Value Theorem that we discussed in the concept on Continuity.

Extreme Value Theorem: If a function $f(x)$ is continuous in a closed interval I , then $f(x)$ has both a maximum value and a minimum value in I .

This makes sense conceptually. Try drawing a function (on a closed interval, including the endpoints) so that no point is at the highest part of the graph. No matter how the function is sketched, there will be at least one point that is highest.

How do extreme values, maxima and minima, relate to [derivatives](#)? The French mathematician Fermat shows how in **Fermat's Theorem:** If $f(c)$ is an extreme value of f for some open interval containing c , then either $f'(c)=0$, or $f'(c)$ does not exist.

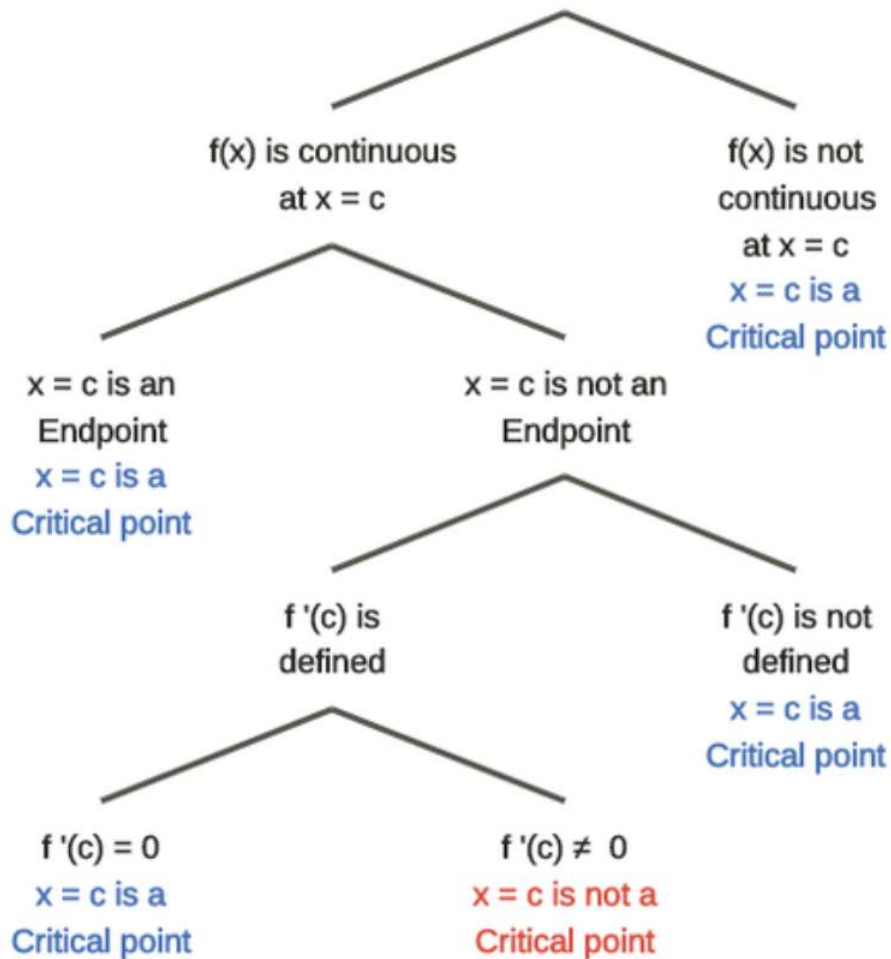
This relationship between the location of extrema and the derivative at the location is so important that we give the location the name **critical point**.

We will call $x=c$ a **critical point** in the closed interval $[a,b]$ if $f(c)$ exists, and either $f'(c)=0$ or $f'(c)$ does not exist.

Summary so far: If a function is continuous in a closed interval, it has both a minimum and a maximum at some location(s), and the derivative at these locations is either 0 or does not exist. The locations are called critical points.

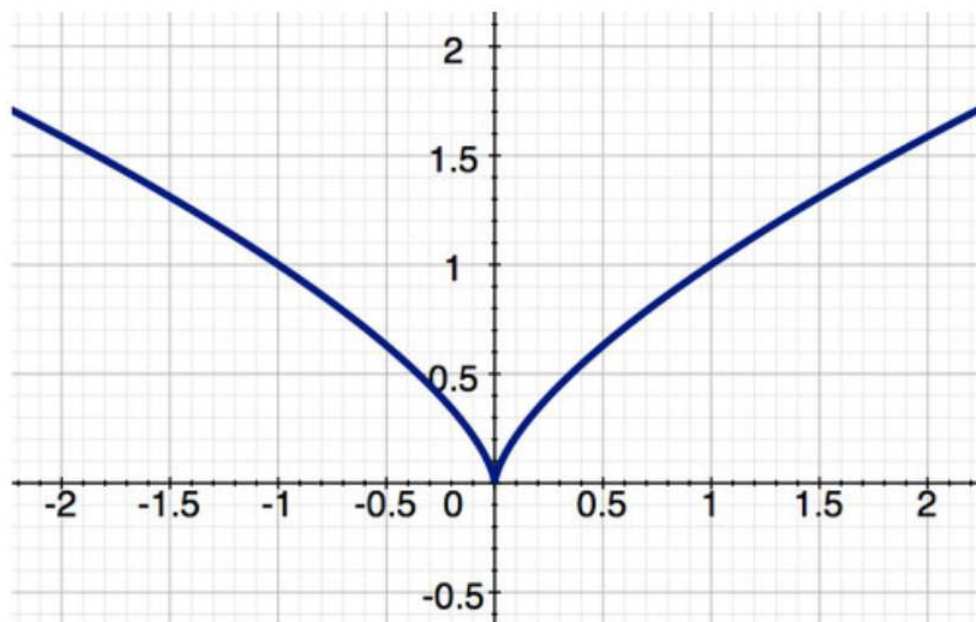
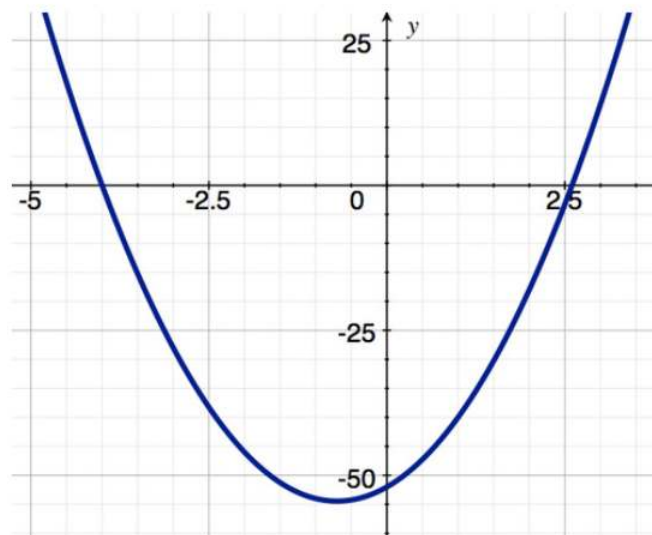
The diagram below explicitly identifies how to identify critical points. The end points of the closed interval are included as critical points if we consider the function only defined on the closed interval and not outside the interval. In this case, the derivative at each end point would not be defined.

Guidelines for Determining if $x = c$ is a Critical Point for $f(x)$



A tree diagram to help identify critical points. CK-12 Foundation

Let's consider the two functions shown below and say something about them in the closed interval $[-2, 2]$. Notice that $g(x)$ (the second function) has a cusp at $x=0$.



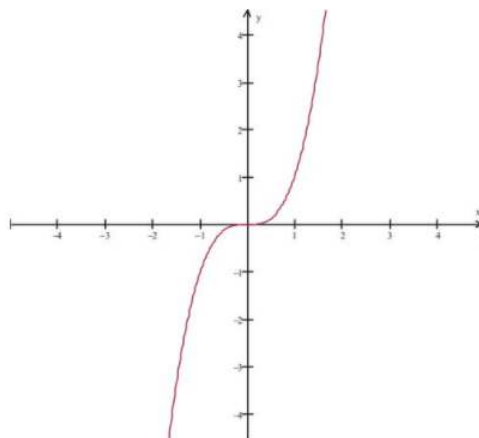
Two functions

Both functions are continuous in $[-2, 2]$. By the Extreme Value Theorem they both have a minimum and maximum. As the table shows, $f(x)$ has an extreme value where $f'(x)=0$, and $g(x)$ has an extreme value where $g'(x)$ is not defined. These are critical points.

Function Property	$f(x) = 5x^2 + 7x - 52$	$g(x) = x^{\frac{2}{3}}$
Value at endpoint $x=-2$	-46	1.59
Value at endpoint $x=2$	-18	1.59
Location and value where $f'(x)=0$	$(-0.7, -54.45)$	---
Location where $f'(x)$ Does Not Exist	End points	Endpoints, $x=0$
Critical point(s)	$x=-0.7$ End points	$x=0$, End points
Minimum value and location(s)	$(-0.7, -54.45)$	$(0, 0)$
Location(s) of Maximum value	$(2, -18)$	$(-2, 1.59)$ $(2, 1.59)$

It should be noted that the converse of the Extreme Value Theorem is not necessarily true, i.e., just because $f'(x)=0$ at some location $x=a$, the function value $f(a)$ does not have to be a local minimum or maximum.

Consider $f(x) = x^3$ and its graph. We see that while $f'(0)=0$ at $x=0$, $x=0$ is not an extreme point of the function.



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Examples

Example 1

Earlier, you were asked to determine how the derivative of the quadratic function ($y = ax^2 + bx + c$) and the vertex formula $x = -\frac{b}{2a}$ are related.

If you realized that the vertex of the quadratic function is also the location where the [slope](#) of the tangent line, and therefore the derivative, equals 0 you made a great observation. $\frac{dy}{dx} = 2ax + b = 0$ means that $x = -\frac{b}{2a}$. This is the x-value of the vertex, and a critical value or point of the function.

Example 2

The function $f(x) = \sqrt{25 - x^2}$ is the upper portion of a circle centered at $(0, 0)$ with radius 5. Find the critical points of the function, and the maxima and minima.

Compute the derivative: $f'(x) = \frac{1}{2\sqrt{25-x^2}}(-2x) = -\frac{x}{\sqrt{25-x^2}}$

$f'(x)=0$ means $x=0$ is a critical point, and $f(0)=5$.

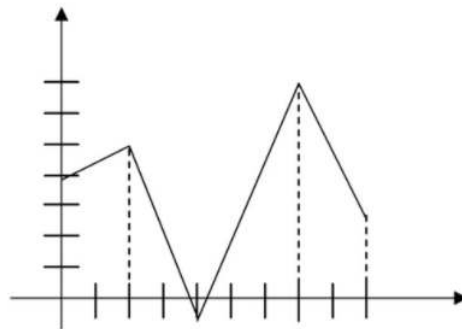
Notice that $f'(x)$ is undefined at the two locations $x=\pm 5$, which means these are critical points. Both of the values are associated with function values $f(\pm 5)=0$.

The function minima are at $(-5, 0)$ and $(5, 0)$; the function maximum is at $(0, 5)$.

Review

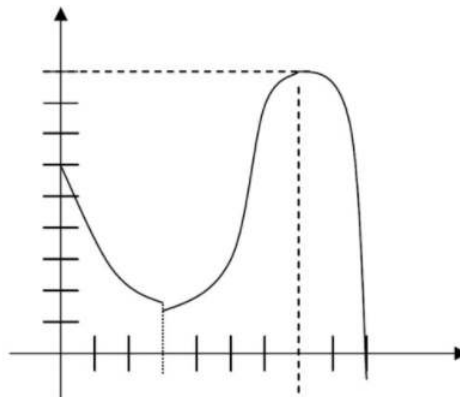
For #1-3, identify the absolute and local minimum and maximum values of the function (if they exist); find the extrema. (Units on the axes indicate 1 unit).

1. Continuous on $[0, 9]$



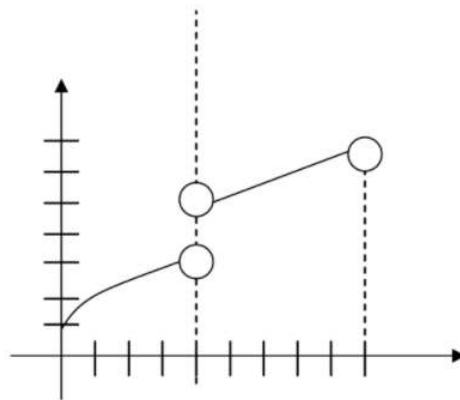
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2. Continuous on $[0, 9]$



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3. Continuous on $[0, 4] \cup [4, 9]$



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For #4–6, find the extrema and sketch the graph.

4. $f(x) = -x^2 - 6x + 4, [-4, 1]$
5. $f(x) = x^3 - x^4, [0, 2]$
6. $f(x) = -x^2 + 4x^2, [-2, 0]$
7. What is the absolute maximum of the sine function over the interval $[0, \pi]$?
8. What are the local minimums of the cosine function over the interval $[0, 2\pi]$?
9. What is the absolute maximum and absolute minimum of the function x^2 over the interval $[6, 7]$?
10. At what point does the function $f(x) = (x^2 - 7x - 8)^4$ attain its absolute minimum over the interval $[5, 10]$?
11. Let $[a, b]$ be a closed, finite interval, and let $f(x)$ be a straight line. If $f(x)$ achieves its maximum value over $[a, b]$ at points other than a and b , what must be true about $f(x)$?
12. Find all the critical numbers of $f(x) = 3x^4 - \frac{8}{3}x^3 - 6x^2 + 8x + 1$.
13. Find all the critical numbers of $f(x) = \frac{1}{2x^2 - 5x - 3}$.
14. Find all the critical numbers of $f(x) = \ln(x - 4)$.
15. Find all the critical numbers of $f(x) = \sqrt{x} \times (1 - x)$.
16. Find all the critical numbers of $f(x) = \frac{x^2 + x + 1}{x - 2}$.
17. Does the polynomial $f(x) = x^6 + x^4 + x^2 + 1$ attain a minimum value over the interval $[-\pi, \pi]$?
18. Does the function $f(x) = 7^{\sin(x)}$ attain a maximum value over the interval $(0, \pi]$?
19. Does the function $f(x) = \frac{10}{x}$ attain a maximum value over the interval $[-1, 1]$?
20. Does the function $f(x) = \sin(\frac{x}{2})$ attain a minimum value over the interval $(0, \pi)$?
21. Let a and b be two positive number such that $a < b$. Does the function $f(x) = \ln(x)$ attain a maximum value over the interval $[a, b]$?

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 4.2.

Vocabulary

Term	Definition
critical point	A critical value (or point) is a value in a closed interval of the domain of the function where either the function derivative at the point is 0, or the derivative does not exist.
derivative	The derivative of a function is the slope of the line tangent to the function at a given point on the graph. Notations for derivative include $f'(x)$, $\frac{dy}{dx}$, y' , $\frac{df}{dx}$ and $\frac{df(x)}{dx}$.
Extrema	Extrema is a collective term encompassing both the minimum and maximum values, referring to the "extreme" values of the function.
global extrema	The global extrema of a function are the points with the y values that are the highest or the lowest of the entire function.
Global Maximum	The global maximum of a function is the largest value of the entire function. Symbolically, it is the highest point on the entire graph.
Global Minimum	The global minimum of a function is the smallest value of the entire function. Symbolically, it is the lowest point on the entire graph.
local extrema	The local extrema of a function are the points of the function with y values that are the highest or lowest of a local neighborhood of the function.
Local Maximum	A local maximum is the highest point relative to the points around it. A function can have more than one local maximum.
Local Minimum	A local minimum is the lowest point relative to the points around it. A function can have more than one local minimum.
Maximum	The maximum is the highest point of a graph. The maximum will yield the largest value of the range.
Minimum	The minimum is the lowest point of a graph. The minimum will yield the smallest value of the range.

Additional Resources

PLIX: Play, Learn, Interact, eXplore - [Absolute versus Local Extrema](#)

Video: [Local and Absolute Maximum and Minimum](#)

Practice: [Absolute versus Local Extrema](#)

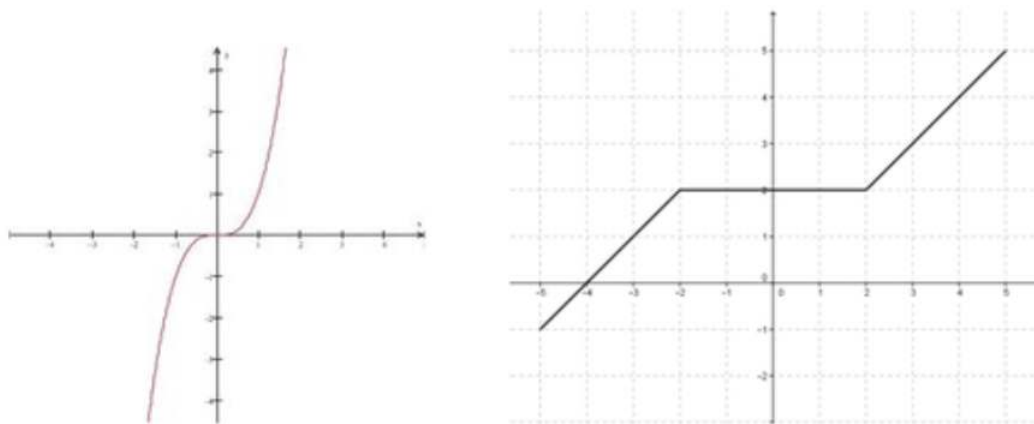
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7.3: First Derivative Test

If you look at any function curve, you can determine visually whether the function is increasing, decreasing, or remaining constant over an interval. You can also see approximately where the function attains high points and low points. There is a way to determine mathematically, using the derivative, what your visual observations provide. How does knowing the [slope](#) of a tangent line help you to precisely calculate what your visual observations can only estimate?

The First Derivative Test

Consider the graphs of the two functions below.



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Both functions are continuous over the intervals shown. In the first, $f(x) = x^3$, the function values are always increasing as x increases. In the second, the piece-wise function values increase over the interval $[-5, -2]$, stay the same over the interval $[-2, 2]$, then resume increasing for $[2, 5]$.

The difference between these two cases of **increasing function** values motivates the need for the following distinctions:

- A function f is said to be **increasing** on $[a, b]$ contained in the domain of f if $f(x_1) \leq f(x_2)$ whenever $x_1 \leq x_2$ for all $x_1, x_2 \in [a, b]$.
- If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ for all $x_1, x_2 \in [a, b]$, then we say that f is **strictly increasing** on $[a, b]$.

In similar manner:

- A function f is said to be **decreasing** on $[a, b]$ contained in the domain of f if $f(x_1) \geq f(x_2)$ whenever $x_1 \geq x_2$ for all $x_1, x_2 \in [a, b]$.
- If $f(x_1) > f(x_2)$ whenever $x_1 > x_2$ for all $x_1, x_2 \in [a, b]$. Then we say that f is **strictly decreasing** on $[a, b]$.

Note that the symbols ϵ and $\in$ are equivalent and denote that a particular element is contained within a particular set.

Using the above terminology, we say the function $f(x) = x^3$ is strictly increasing over the given interval, and the piece-wise function is increasing over the interval.

Now look at the [derivatives](#) of the two functions graphed above.

We can now state a theorem that relates the derivative of a function to the increasing/decreasing properties of the function.

If f is continuous on interval $[a, b]$, and differentiable on (a, b) then:

- If $f'(x) > 0$ for every $x \in (a, b)$, then f is increasing in $[a, b]$.
- If $f'(x) < 0$ for every $x \in (a, b)$, then f is decreasing in $[a, b]$.

Take the function $f(x) = x^3 - 3x^2 - 6x + 8$. To find the intervals on which f is increasing and the intervals on which f is decreasing, first note that the function $f(x)$ is continuous everywhere. The derivative of the function is $f'(x) = 3x^2 - 6x - 6 = 3(x^2 - 2x - 2)$,

which is a parabola with two x-intercepts (critical numbers of f) at $x = 1 \pm \sqrt{3}$. Evaluation of $f'(x)$ in the three intervals that are defined by the two roots provides the following information:

$$f'(x) = \begin{cases} > 0, & x < 1 - \sqrt{3} & f(x) \text{ is increasing} \\ = 0, & x = 1 - \sqrt{3} \\ < 0, & (1 - \sqrt{3}) < x < (1 + \sqrt{3}) & f(x) \text{ is decreasing} \\ = 0, & x = 1 + \sqrt{3} \\ > 0, & x > 1 + \sqrt{3} & f(x) \text{ is increasing} \end{cases}$$

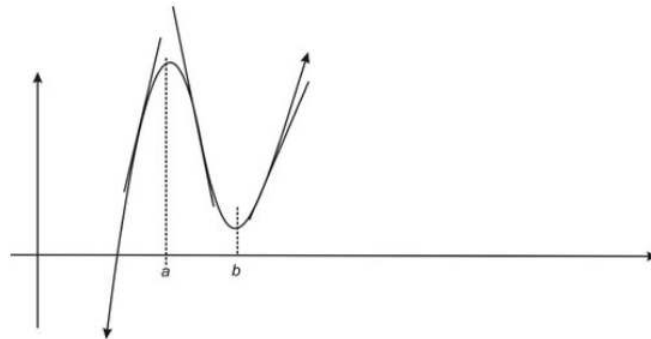
Changes in the derivative of a function from positive to negative, or negative to positive may indicate the presence of a local extremum.

The **First Derivative Test** describes where these extrema are and what type they are. The First Derivative Test is as follows:

Suppose that f is a continuous function and that $x=c$ is a critical value of f , then:

1. If f' changes from positive to negative at $x=c$, then f has a local maximum at $x=c$.
2. If f' changes from negative to positive at $x=c$, then f has a local minimum at $x=c$.
3. If f' does not change sign at $x=c$, then f has neither a local maximum nor minimum at $x=c$.

We can observe the consequences of this theorem by observing the **tangent lines** of the following graph in each of the intervals $(0,a)$, (a,b) , $(b,+\infty)$.



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Note first that we have a relative maximum at $x=a$ and a relative minimum at $x=b$. The slopes of the **tangent lines** change from positive for $x \in (0,a)$ to negative for $x \in (a,b)$ and then back to positive for $x \in (b,+\infty)$.



Determining where a function is increasing and decreasing.

$f(x) = x^3 - 6x^2$

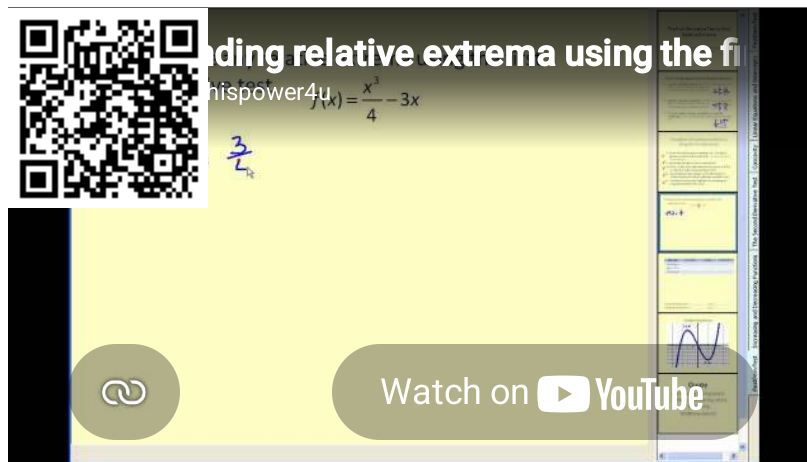
$3x^2 - 12x = 0$

$3x(x - 4) = 0$

$x = 0 \quad x = 4$



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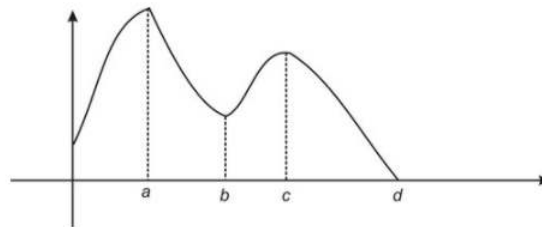
Examples

Example 1

Earlier, you were asked how the **slope** of the tangent line relates to whether or not a function is increasing or decreasing. Positive **slope** means an increasing function; negative slope means a **decreasing function**. Often the slope transition from positive (negative) to negative (positive) indicates the location of an extremum.

Example 2

Consider the function graph below. Determine whether the function where the function is strictly increasing or decreasing.

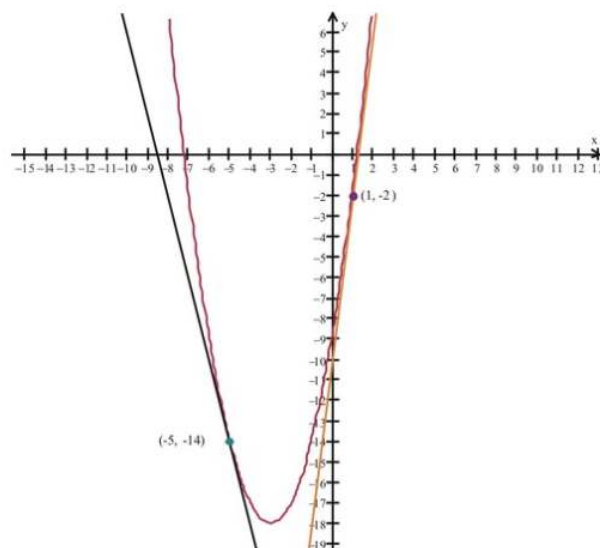


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The function indicated here is strictly increasing on $(0,a)$ and (b,c) , and strictly decreasing on (a,b) and (c,d) .

Example 3

Let's consider the function $f(x) = x^2 + 6x - 9$ and observe the graph around $x=-3$. What happens to the first derivative near this value?



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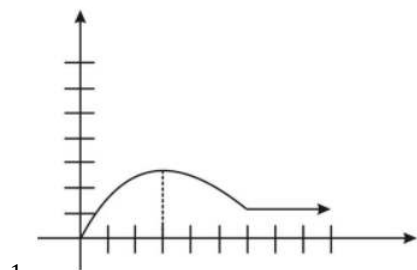
With $f(x) = x^2 + 6x - 9$, the derivative is $f'(x) = 2x + 6$. Notice that the following condition applies to the slopes of [tangent lines](#):

$$f'(x) \begin{cases} < 0, & x < -3 \\ = 0, & x = -3 \\ > 0, & x > -3 \end{cases}$$

We observe that the slopes of the tangent lines to the graph change from negative to positive at $x = -3$. The first derivative test states that $f(x)$ has a local minimum at $x = -3$. The function graph verifies this.

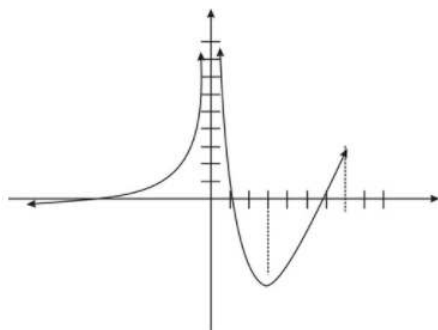
Review

For #1-2, identify the intervals where the function is increasing, decreasing, or is constant. (Units on the axes indicate single units).



1.

figure5



2.

figure6

3. Give the sign of the following quantities for the graph in problem 2.

1. $f'(-3)$
2. $f'(1)$
3. $f'(3)$
4. $f'(4)$

For #4-6, determine the intervals in which the function is increasing and those in which it is decreasing, then sketch the graph of each function.

4. $f(x) = x^2 - \frac{1}{x}$
5. $f(x) = (x^2 - 1)^5$
6. $f(x) = (x^2 - 1)^4$

For #7-10:

- Use the First Derivative Test to find the intervals where the function increases and/or decreases
 - Identify all max, min, or relative max and min values
 - Sketch the graph
7. $f(x) = -x^2 - 4x - 1$
 8. $f(x) = x^3 + 3x^2 - 9x + 1$
 9. $f(x) = x^{\frac{2}{3}}(x - 5)$
 10. $f(x) = 2x\sqrt{x^2 + 1}$
 11. Use the first derivative test to classify the critical numbers $x = 0$ and $x = \frac{3\pi}{2}$ of $f(x) = \cos^2(x)$ as local maxima or minima.
 12. Find the critical numbers of $f(x) = x^5 - 20x - 2$ and classify them as local maxima, minima or neither.
 13. Find the local extrema of $f(x) = x + \sin(x)$ on the interval $(-2\pi, 2\pi)$.
 14. Find the global and local extrema of $f(x) = \frac{3}{4}x^4 + 4x^3 - 6x^2 - 48x - 50$.
 15. Find the global and local extrema of $f(x) = \frac{x^2 - x - 6}{x^2 + x - 6}$.

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 4.4.

Vocabulary

Term	Definition
decreasing function	A decreasing function is one with a graph that goes down from left to right.
first derivative test	The first derivative test says that if f is a continuous function and that $x=c$ is a critical value of f , then if f' changes from positive to negative at $x=c$ then f has a local maximum at $x=c$, if f' changes from negative to positive at $x=c$ then f has a local minimum at $x=c$, and if f' does not change sign at $x=c$ then f has neither a local maximum nor minimum at $x=c$.
increasing function	An increasing function is one with a graph that goes up from left to right.
strictly	Strictly is an adjective that alters increasing and decreasing to exclude any flatness or periods where y values are staying constant.

Additional Resources

Video: [Identifying Minima and Maxima](#)

Real World: [Losing Ground](#)

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7.4: Second Derivative Test

The derivative of a function curve at a point is the mathematical analogy to our visually determining if the function is increasing or decreasing at the point. But if you look at a function curve, you can also determine visually whether the **tangent lines** are getting more vertical or more horizontal over an interval. The visual recognition of this rate of change of the **slope** is a measure of curvature (or **concavity**), which in turn can tell you specifically what type of extrema are present. The mathematical representation of the rate of change of the slope is the second derivative, which is used to identify types of curvature. What is the relationship between the second derivative and the concavity?

Concavity and the Second Derivative Test

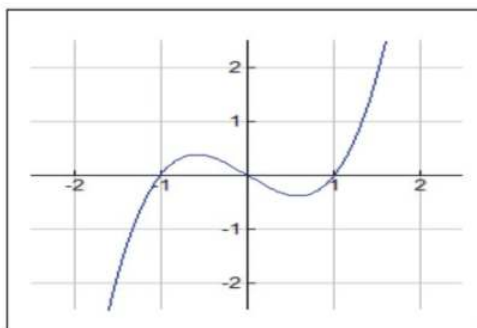
There is a property about the shape, or curvature, of a graph called concavity, which will help identify precisely the intervals where a function is either increasing or decreasing, where the **maxima** and **minima** are located, and also help to sketch the graph. Concavity is the direction in which the curve opens.

The type of concavity on a curve is defined as follows:

A function f is said to be **concave upward** on $[a,b]$ contained in the domain of f if f' is an increasing function on $[a,b]$ and **concave downward** on $[a,b]$ if f' is a decreasing function on $[a,b]$.

Here is a function that illustrates these properties.

Consider the function $f(x) = x^3 - x$:



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Since $f(x) = x^3 - x = x(x^2 - 1)$, the function has zeros at $x = \pm 1, 0$.

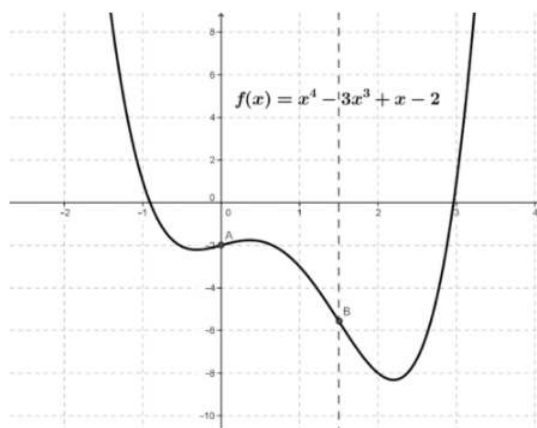
Also, $f'(x) = 0 = 3x^2 - 1$, tells us the location of extrema: a relative maximum at $x = -\frac{\sqrt{3}}{3}$ and a relative minimum at $x = \frac{\sqrt{3}}{3}$.

Note that the graph appears to be concave down for all intervals in $(-\infty, 0)$ and concave up for all intervals in $(0, +\infty)$. Where do you think the concavity of the graph changed from concave down to concave up? If you answered at $x=0$ you would be correct.

In general, we wish to identify not only the extrema of a function but also the points where the graph changes concavity. A point on a graph of a function f where the concavity changes is called an **inflection point**.

The problem above had only one inflection point. But we can easily come up with examples of functions where there are more than one points of inflection.

Consider the function $f(x) = x^4 - 3x^3 + x - 2$ shown in the figure.



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We can see that the graph has two relative minimums, one relative maximum, and two inflection points (at A and B).

In general we can use the following **test for concavity**:

Suppose that f is continuous on $[a,b]$ and that I is some open interval in the domain of f .

- If $f''(x) > 0$ for all $x \in I$, then the graph of f is concave upward on I
- If $f''(x) < 0$ for all $x \in I$, then the graph of f is concave downward on I

Let's apply the concavity test to the function $f(x) = x^4 - 3x^3 + x - 2$:

The second derivative is, $f''(x) = 12x^2 - 18x = 6x(2x - 3)$, and the concavity is given in the table.

Interval or Point	$f''(x)$	Concavity
$[-1, 0)$	> 0	Upward
0	$= 0$	Change=Inflection
$(0, 1.5)$	< 0	Downward
1.5	$= 0$	Change=Inflection
$(1.5, 3]$	> 0	Upward

Can you see how the concavity gives a clue to the presence of a relative maximum, or minimum, or inflection point? A consequence of the concavity test is the following test to identify where we have extrema and inflection points of f .

The **Second Derivative Test for Extrema** is as follows:

Suppose that f is a continuous function near c and that c is a critical value of f . Then

- If $f''(c) < 0$, then f has a relative maximum at $x=c$.
- If $f''(c) > 0$, then f has a relative minimum at $x=c$.
- If $f''(c) = 0$, then the test is inconclusive and $x=c$ may be a point of inflection.

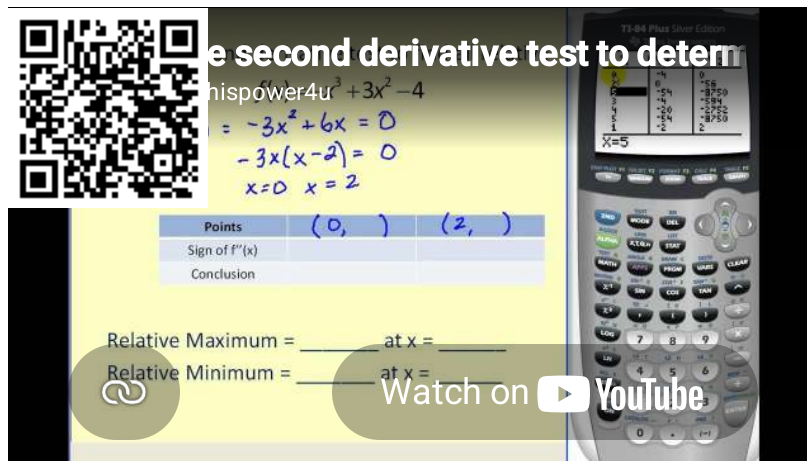
Does the test work for our previous example of $f(x) = x^4 - 3x^3 + x - 2$?

We can find the critical values where $f'(x) = 4x^3 - 9x^2 + 1 = 0$, and use them to evaluate $f''(c)$. The results are shown below.

Critical Point	$f''(c)$	Conclusion
$x=-0.312$	>0	Relative Minimum
$x=0.364$	<0	Relative Maximum
$x=2.198$	>0	Relative Minimum

The tests works on this function.

Remember, however, that the Second Derivative Test cautions us that if $f''(c)=0$, the test is inconclusive, because c may be an inflection point. Recall that this was the case for $f(x) = x^3$.



The second derivative test to determine

hispower4u³+3x²-4

$= -3x^2 + 6x = 0$
 $-3x(x-2) = 0$
 $x=0 \quad x=2$

Points	$(0,)$	$(2,)$
Sign of $f''(x)$		
Conclusion		

Relative Maximum = _____ at $x =$ _____
 Relative Minimum = _____ at $x =$ _____

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Examples

Example 1

Earlier, you were asked what relationship between the curvature of a function and the second derivative is. The curvature of a function graph can be described by its concavity (up or down), which in turn can be determined at each point by the second derivative (if it is defined). Because critical points can determine the locations of relative maxima or minima without identifying which, knowing the type of curvature from the second derivative can identify specifically whether we have a minimum or a maximum, or neither.

Example 2

Consider the function $f(x) = x + \frac{4}{x}$. Find the critical points, and determine whether they are relative maximums or relative minimums by using the Second Derivative Test.

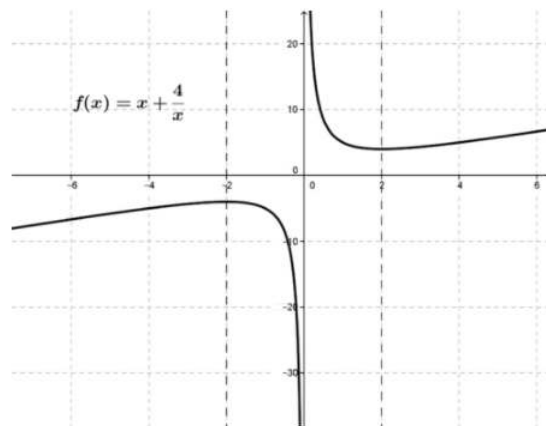
The first thing to notice is that the function is not defined at $x=0$.

To find the critical points we look for points c where $f'(c)=0$.

$f'(x) = 1 - \frac{4}{x^2} = 0$ means there are critical points at $x=-2$, and $x=2$.

Since $f''(x) = \frac{8}{x^3}$: $f''(-2) = -1$ and the curve is concave downward with a relative maximum; $f''(2)=1$, and the curve is concave upward with a relative minimum.

The function graph is as shown.



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Review

- Find all extrema using the Second Derivative Test: $f(x) = \frac{x^2}{4} + \frac{4}{x}$.
 - Consider $f(x) = x^2 + ax + b$, with $f(1)=3$.
 - Determine a and b so that $x=1$ is a critical value of the function f.
 - Is the point $(1, 3)$ a maximum, a minimum or neither?
- For #3-6, find all extrema and inflection points. Sketch the graph.
- $f(x) = x^3 + x^2$
 - $f(x) = \frac{x^2+3}{x}$
 - $f(x) = x^3 - 12x$
 - $f(x) = -\frac{1}{4}x^4 + 2x^2$
 - Use your graphing calculator to examine the graph of $f(x) = x(x-1)^3$ (Hint: you will need to change the y range in the viewing window)
 - Discuss the concavity of the graph in the interval $(0, \frac{1}{2})$.
 - Use your calculator to find the minimum value of the function in the interval.
 - True or False: $f(x) = x^4 + 4x^3$ has a relative minimum at $x=-2$ and a relative maximum at $x=0$?
 - If possible, provide an example of a non-polynomial function that has exactly one relative minimum.
 - If possible, provide an example of a non-polynomial function that is concave downward everywhere in its domain.
 - Let $f(x)$ be a continuous function and let $[a,b]$ be a closed interval over which $f(x)$ is defined. If we know that $f(x)$ is positive over the interval $[a,b]$, does this tell us anything about the concavity of $f(x)$ over $[a,b]$?
 - Is the function $f(x)=x^4$ concave up or down over the interval $[10, 11]$?
 - Is the function $f(x)=-\cos(x)$ concave up or down over the interval $[\frac{\pi}{2}, \frac{3\pi}{2}]$?
 - If the first derivative of $f(x)$ is $f'(x) = 20^x \sin(\pi^x)$, is $f(x)$ concave up or down over the interval $[0, \frac{1}{2}]$?
 - What is an inflection point of the function $f(x) = 4x^3 + 3x^2 + 2x + 1$?
 - Use the second derivative test to classify the extrema of the function $f(x) = x^3 + x^2 - 5\ln(x)$ over the interval $(\frac{1}{2}, 2)$.
 - Use the second derivative test to classify the extrema of the function $f(x) = 5^{\sin(x)}$ over the interval $(\frac{\pi}{4}, \frac{3\pi}{4})$.
 - Use the second derivative test to classify the extrema of the function $f(x) = \ln(2x + x^2)$ over the interval $(-\frac{5}{2}, -\frac{1}{2})$.
 - Use the second derivative test to classify the extrema of the function $f(x) = x^4$ over the interval $(1, 5)$.
 - Can we use the second derivative test to check if $x=0$ is a maximum or a minimum of the function $f(x) = x^{10} + 25$?

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 4.5.

Vocabulary

Term	Definition
concave downward	A function f is said to be concave downward on $[a,b]$ if f' is a decreasing function on $[a,b]$.
concave upward	A function f is said to be concave upward on $[a,b]$ contained in the domain of f if f' is an increasing function on $[a,b]$.
concavity	Concavity describes the behavior of the slope of the tangent line of a function such that concavity is positive if the slope is increasing, negative if the slope is decreasing, and zero if the slope is constant.
inflection point	An inflection point is a point in the domain where concavity changes from positive to negative or negative to positive.
second derivative test	The second derivative test says that if f is a continuous function near c and c is a critical value of f , then if $f''(c) < 0$ then f has a relative maximum at $x=c$, if $f''(c) > 0$ then f has a relative minimum at $x=c$, and if $f''(c)=0$ then the test is inconclusive and $x=c$ may be a point of inflection.

Additional Resources

Video: [2nd Derivative Test](#)

Practice: [Second Derivative Test](#)

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7.5: Concavity and Inflection

When defined at a point, the first derivative and the second derivative each provide one of three results: + (value > 0), - (value < 0) or 0. What are all of the result possibilities for f' and f'' , and which give definitive information on the extrema of f ?

Applying the Derivative Tests

Now we will apply all that we have learned from the First and Second Derivative Tests to sketch graphs of functions. The following table provides a summary of the tests and can be a useful guide in sketching graphs.

Signs of 1 st and 2 nd derivatives	Conclusion	Shape of the graphs
$f'(x) > 0 +$ $f''(x) > 0 +$	f is increasing f is concave upward	 CC BY-NC-SA
$f'(x) > 0 +$ $f''(x) < 0 -$	f is increasing f is concave downward	 CC BY-NC-SA
$f'(x) < 0 -$ $f''(x) > 0 +$	f is decreasing f is concave upward	 CC BY-NC-SA
$f'(x) < 0 -$ $f''(x) < 0 -$	f is decreasing f is concave downward	 CC BY-NC-SA

Take a look at an example where we can use both the First and Second Derivative Tests to find out information that will enable us to sketch the graph.

Let's examine the function $f(x) = x^5 - 5x + 2$.

- Find the critical values for which $f'(c)=0$.

$$f'(x) = 5x^4 - 5 = 0, \text{ which means } x^4 - 1 = 0 \text{ at } x = \pm 1.$$

- Apply the First and Second Derivative Tests to determine extrema and points of inflection.

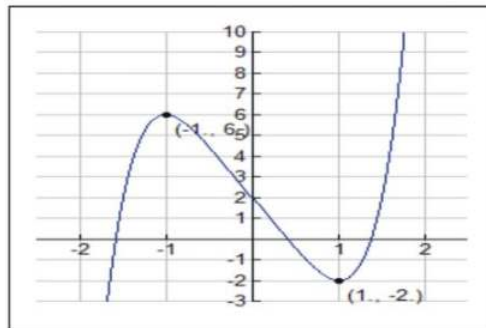
We note the signs of f' and f'' in the intervals partitioned by $x=\pm 1, 0$.

Key intervals	$f'(x)$	$f''(x)$	Shape of graph
$x < -1$	+	-	Increasing, concave down
$-1 < x < 0$	-	-	Decreasing, concave down
$0 < x < 1$	-	+	Decreasing, concave up
$x > 1$	+	+	Increasing, concave up

At the critical points:

1. $f''(-1) = -20 < 0$. By the [Second Derivative Test](#) we have a relative maximum at $x = -1$, or the point $(-1, 6)$.
2. $f''(0) = 0$. By the [Second Derivative Test](#) we must have a point of inflection due to the transition from concave down to concave up between the key intervals.
3. $f''(1) = 20 > 0$. By the [Second Derivative Test](#) we have a relative minimum at $x = 1$, or the point $(1, -2)$.

Now we can sketch the graph.



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Now, look at a simple rational function.

Consider the function $f(x) = x + 2x - 3$.

1. Find the critical values for which $f'(c) = 0$.

$$f'(x) = \frac{1}{x-3} + (x+2)\frac{-1}{(x-3)^2} = \frac{-5}{(x-3)^2}$$

which means there are no values c that make $f'(c) = 0$.

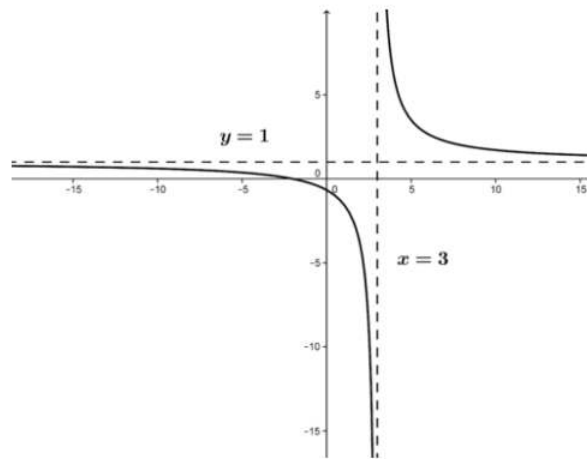
Since $f'(x)$ is not defined at $x = 3$, this is the one critical point.

2. We note the signs of f' and f'' in the intervals partitioned by $x = 3$.

Key intervals	$f'(x)$	$f''(x)$	Shape of graph
$x < 3$	-	-	Decreasing, concave down
$x > 3$	-	+	Decreasing, concave up

At the critical point, there is a vertical asymptote and the function and [derivatives](#) are undefined; there are no minima or maxima.

Now we can sketch the graph.



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Summary of the First and Second Derivative Tests

and relative extrema.

$f'(x) = -12x^2 + 72$

Interval	$(-\infty, 0)$	$(0, 2)$	$(2, 4)$
$f'(x)$	+	-	+
Conclusion	Incr	Decr	Incr

Rel Max: $(0, \quad), (4, \quad)$
 Rel Min: $(2, \quad)$



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Examples

Example 1

Earlier, you were asked to chart all of the sign/zero possibilities for f' and f'' , and identify which give definitive information on the extrema of f .

There are nine possibilities as follows:

f'	f''	Definitive Extrema Info?
+	+	No
+	-	No
+	0	No
-	+	No
-	-	No
-	0	No
0	+	Yes
0	-	Yes
0	0	No

Only when there is a critical point with $f'=0$ can we use the information on f'' to tell whether we have a minimum or a maximum of f .

Example 2

For the function $f(x) = \frac{x^2-4x+4}{x^2+1}$, find all critical values, the corresponding y values for each critical value, and classify each point as a relative maximum or a relative minimum

The derivative is: $f'(x) = \frac{2(2x^2-3x-2)}{(x^2+1)^2}$

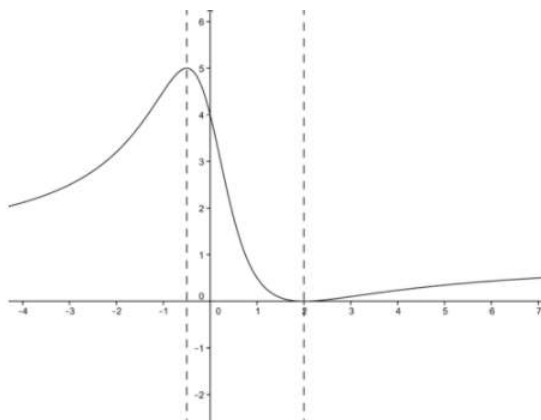
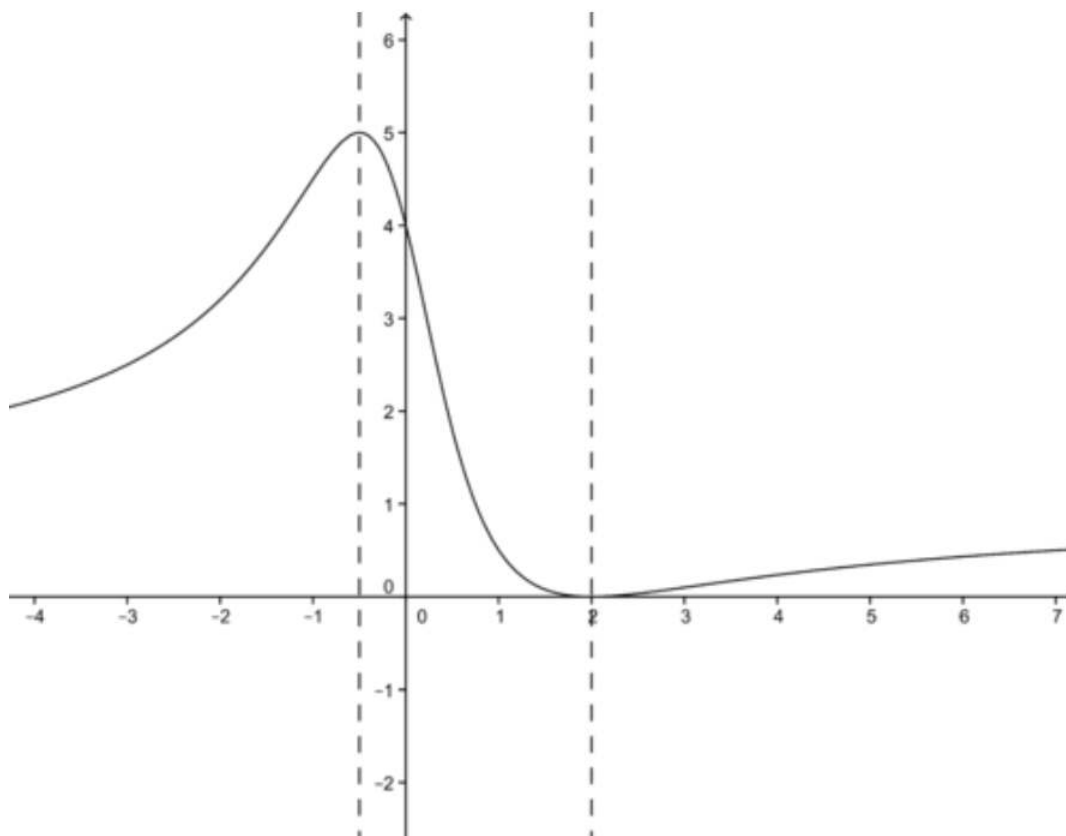
Which gives critical values: $x=-12$ and $x=2$

This means that: $f(-\frac{1}{2}) = 5$ and $f(2) = 0$

The second derivative is: $f''(x) = -\frac{2(4x^3+8x^2+4x+5)}{(x^2+1)^3}$

Evaluation at the critical points determines that $f''(-\frac{1}{2}) < 0$, which means concave down (a maximum), and $f''(2) > 0$ means concave up (a minimum).

The function graph is plotted below:



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Example 3

For the function $f(x) = 2x^4 - 3x^2 + 2$, find all critical values, classify each point as a relative maximum or a relative minimum, and identify any inflection points

The derivative is: $f'(x) = 8x^3 - 6x$

Which gives critical values: $x=0$ and $x = \pm\frac{\sqrt{3}}{2}$

The second derivative is: $f''(x) = 24x^2 - 6$

Evaluation at the critical points determines that $f''(0) < 0$, which means concave down (a maximum), and $f''(\pm\frac{\sqrt{3}}{2}) > 0$ means concave up (two minima).

The second derivative is 0 when $24x^2 - 6 = 6(2x - 1)(2x + 1) = 0$.

$x=0$ and $x=\pm 12$ are inflection points.

Review

For the #1-8, find the critical points, determine if each is a local minimum or maximum, and determine any points of inflection.

1. $f(x) = x^3 - 3x^2 + 1$
2. $f(x) = x^4 - 2x^3 - x + 2$
3. $f(x) = \sin 2x$ on the interval $[0, \frac{\pi}{2}]$.
4. $f(x) = \frac{1}{x^2 - x + 2}$
5. $f(x) = 2x^3 - 3x^2 + 6x$
6. $f(x) = x^3 - 12x + 5$ in the interval $[-5, 3]$
7. $f(x) = x^5 + 20x + 5$
8. $f(x) = x^2 - 5x + 6, (-1, 3]$
9. Find the absolute maximum and absolute minimum of $f(x) = x^3 - 3x^2 + 2$ on the interval $[0, 4]$.

For #10-15, find the critical points and local minima and maxima.

10. $f(x) = x^2 e^{-x}$
11. $f(x) = \frac{x^2}{x-1}$
12. $f(x) = 5e^{-x} + x^3$
13. $f(x) = e^{-2x} + e^x$
14. $f(x) = 3^x e^{-x}$
15. $f(x) = x \tan x$ on the interval $[-\pi, \pi]$.

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 4.6.

Vocabulary

Term	Definition
concavity	Concavity describes the behavior of the slope of the tangent line of a function such that concavity is positive if the slope is increasing, negative if the slope is decreasing, and zero if the slope is constant.
decreasing function	A decreasing function is one with a graph that goes down from left to right.
first derivative test	The first derivative test says that if f is a continuous function and that $x=c$ is a critical value of f , then if f' changes from positive to negative at $x=c$ then f has a local maximum at $x=c$, if f' changes from negative to positive at $x=c$ then f has a local minimum at $x=c$, and if f' does not change sign at $x=c$ then f has neither a local maximum nor minimum at $x=c$.
increasing function	An increasing function is one with a graph that goes up from left to right.
inflection point	An inflection point is a point in the domain where concavity changes from positive to negative or negative to positive.
second derivative test	The second derivative test says that if f is a continuous function near c and c is a critical value of f , then if $f''(c) < 0$ then f has a relative maximum at $x=c$, if $f''(c) > 0$ then f has a relative minimum at $x=c$, and if $f''(c) = 0$ then the test is inconclusive and $x=c$ may be a point of inflection.

Additional Resources

Video: [Concavity and Inflection Solved](#)

Practice: [Concavity and Inflection](#)

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CHAPTER OVERVIEW

8: Differentiation - Derivative Applications

[8.1: Absolute Extrema and Optimization](#)

[8.2: Newton's Method](#)

[8.3: Analyzing the Graphs of Functions](#)

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8.1: Absolute Extrema and Optimization

At this point, you know how to analyze a function to find its minima and maxima using the first and second [derivatives](#). Finding the solution to some real-world problem (such as in finance, science, and engineering) often involves a process of finding the maximum or minimum of a function within an acceptable region of values. This kind of problem is an **optimization problem**, and the solution that is the maximum or minimum value of the function in the region is the **optimal solution**. Setting up the optimization problem is the important first step. See if you can setup (but not solve) this problem: What dimensions will minimize the amount of metal required to construct a storage tank of [volume](#) 1000 m³ that is in the shape of a right circular cylinder and has a top that is a hemisphere?

Optimization

In previous concepts, we discussed methods for finding maxima and minima (extrema) of functions, and these methods are used in solving optimization problems. Suppose that f is continuous on a closed interval $[a,b]$. Recall that we can find the absolute minimum and maximum of a function f that is continuous on a closed interval $[a,b]$ as follows:

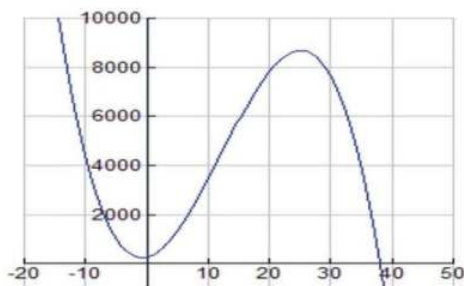
1. Find the values of f for each critical value in (a,b) ;
2. Find the values of the function f at the endpoints of $[a,b]$;
3. The absolute maximum will be the largest value of the numbers found in 1 and 2; the absolute minimum will be the smallest number.

However, real-world problems often impose constraints that must be accounted for in finding the optimal solution.

Take a company that makes high-quality bicycle tires for both recreational and racing riders. The number of tires that the company sells is a function of the price charged and can be modeled by the formula $T(x) = -x^3 + 36.5x^2 + 50x + 250$, where x is the price charged for each tire in dollars. At what price is the maximum number of tires sold? How many tires will be sold at that maximum price?

In this optimization problem, we are trying to find the maximum value of $T(x)$. $T(x)$ is the **primary equation** that will be optimized. Let's first look at a graph of the tire function and make some observations. Set the viewing window ranges on your graphing calculator to $[-10, 50]$ for x and $[-500, 10000]$ for y .

The graph should appear as follows:



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We first note that since this is a real-life application, both quantities, x and $T(x)$, are positive or else the problem makes no sense. These conditions, together with the fact that the zero of $T(x)$ is located at $x=37.9$, suggest that the actual domain of this function is $0 < x < 37.9$. This restricted domain, which we refer to as a **feasible domain**, illustrates a common feature of optimization problems: that the real-life conditions of the situation under study dictate the domain values.

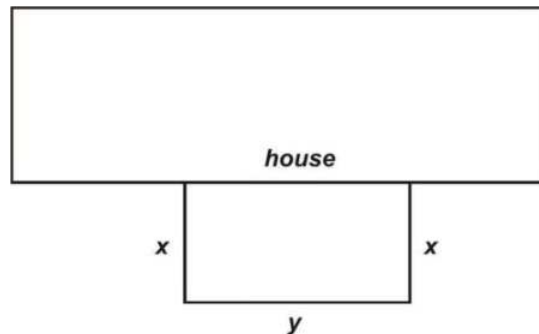
We can use the calculator graph or use the First and Second Derivative Tests to find the absolute maximums and minimums on the closed interval $[0, 37.9]$.

1 st Derivative $f'(x) = -3x^2 + 73x + 50$	$f'(x)=0$: at $x = -\frac{2}{3}$... not in domain; at $x=25$... in the domain.
2 nd Derivative $f''(x) = -6x + 73$	$f''(25)=23$ There is a function maximum at $x=25$.

We see that the function attains an absolute maximum at $x=25$ at the point $(25, 8687.5)$. So, charging a price of \$25 will result in a total of 8687 tires being sold.

In many optimization problems, a **secondary equation** (or more than one) must be used to reduce the number of variables in the primary equation before starting the optimization procedure (use of a graphing calculator; application of derivative tests).

Suppose that Mary wishes to make an outdoor rectangular pen for her pet chihuahua. She would like the pen to enclose an area in her backyard with one of the sides of the rectangle made by the side of Mary's house as indicated in the following figure. If she has 90 ft of fencing to work with, what dimensions of the pen will result in the maximum area?



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The dimensions of the pen are x and y in the figure. The primary equation is the function that models the area, A , of the pen. We wish to maximize:

$$A = xy$$

The secondary equation comes from the information concerning the fencing Mary has to work with. In particular,

$$2x + y = 90$$

Solving for y we have

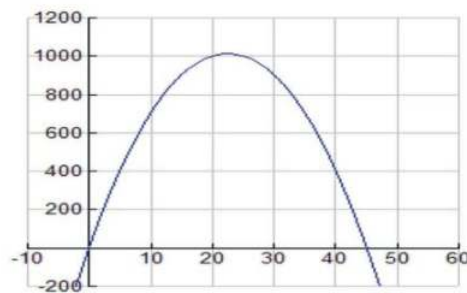
$$y = 90 - 2x$$

We now substitute into the primary equation to get:

$$A = xy = x(90 - 2x), \text{ or}$$

$$A(x) = -2x^2 + 90x$$

It is always helpful to view the graph of the function to be optimized. Set the viewing window ranges on your graphing calculator to $[-10, 100]$ for x and $[-500, 1200]$ for y . The graph should appear as follows:



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The feasible domain of this function is $0 < x < 45$, which makes sense because if x is 45 feet, then the figure will be two 45-foot-long fences going away from the house with 0 feet left for the width y .

We can find the solution by using the graphing calculator, and/or the First and Second Derivative Tests and the method for finding absolute maximums and minimums on a closed interval (in this problem, $[0, 45]$).

1 st Derivative $A'(x) = -4x + 90$	$A'(x) = 0$: at $x = 22.5$... in the domain.
2 nd Derivative $A'' = -4$	$A'' = -4$ There is a function maximum at $x = 22.5$.

We see that the function attains an absolute maximum at $x = 22.5$ at the point $(22.5, 1012.5)$.

So the dimensions of the pen should be $x = 22.5$, and $y = 45$; with those dimensions, the pen will enclose an area of 1012.5 ft².

Optimization problems involving geometric shapes really help illustrate the problem solving process. Here is another:

A certain brand of lemonade sells its product in 16-ounce aluminum cans that hold 473 ml (1 ml = 1 cm³). Find the dimensions of the cylindrical can that will use the least amount of aluminum.

We need to develop the formula for the [surface area](#) of the can. This consists of the top and bottom areas, each πr^2 , and the [surface area](#) of the side, $2\pi r h$ (product of [circumference](#) of the top and height of can). Hence the primary equation is

$$A = 2\pi r^2 + 2\pi r h$$

.

We observe that both our feasible domains require $r, h > 0$.

To reduce the number of variables in the primary equation, we need a secondary equation that will give us a relationship between r and h . We note that the [volume](#) for a circular cylinder is given by $V = \pi r^2 h$, which provides a relationship between r and h . The secondary equation is:

$$h = \frac{V}{\pi r^2} = \frac{473}{\pi r^2}$$

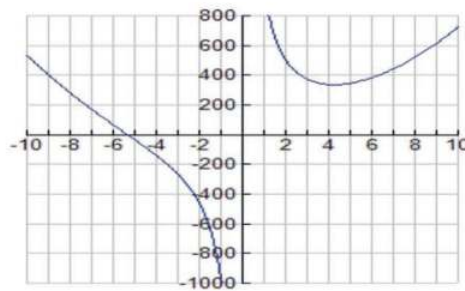
We substitute this expression for h into the primary equation to get:

$$A(r) = 2\pi r^2 + 2\pi r \left(\frac{473}{\pi r^2} \right), \text{ or}$$

$$A(r) = 2\pi r^2 + \frac{946}{r}$$

.

The graph of the function $A(r)$ is shown below:



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The feasible domain is $r > 0$.

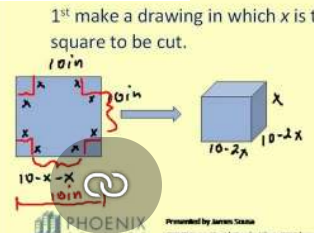
1 st Derivative	$A'(r) = 4\pi r - \frac{946}{r^2}$	$A'(r) = 0$: at $r = \frac{946}{4\pi}^{\frac{1}{3}} \approx 4.22$ cm ... in the domain.
2 nd Derivative	$A''(r) = 4\pi + \frac{1892}{r^3}$	$A''(r) > 0$ for $r > 0$ There is a function minimum at $r \approx 4.22$ cm.

Hence we have a minimum surface area of 996.2 cm² when $r \approx 9.06$ cm and $h \approx 8.44$ cm.



Max and Min Apps. w/ Calculus - Pa

are cut out so that the sides can be
ake a box. What dimensions will yield
um volume? What is the maximum



1st make a drawing in which x is the length of each square to be cut.

$V = l \cdot w \cdot h$
 $V = (10-2x)(10-2x)(x)$

max

Watch on  **YouTube**



Max and Min Apps. w/ Calculus - Pa

w stereo, the price per unit, in dollars,
= $1000 - x$. The manufacturer also
that the total cost of producing x units
is $C(x) = 3000 + 2x$.

a) Find the total revenue $R(x)$.

$R(x) = x \cdot p(x) = x(1000 - x) = 1000x - x^2$

b) Find the total profit $P(x)$. $P = R - C$

$P(x) = 1000x - x^2 - (3000 + 2x) = 1000x - x^2 - 3000 - 2x$
 $P(x) = -x^2 + 998x - 3000$

max

Watch on  **YouTube**

Examples

Example 1

Earlier, you were asked about what dimensions will minimize the amount of metal required to construct a storage tank of volume 1000 m³ that is in the shape of a right circular cylinder and has a top that is a hemisphere.

The volume of the tank consists of the volume of a cylinder and the volume of a hemisphere. The volume can be expressed as: $V = \pi r^2 h + \frac{1}{2}(\frac{4}{3}\pi r^3) = 1000$ where r is the radius of the tank (in meters), and h is the cylinder's height (in meters).

You would like to minimize the amount of metal used, which is the same as asking to minimize the area of the tank. This should be the optimization problem statement.

The area can be expressed as: $A = 2\pi r h + \frac{1}{2}(4\pi r^2)$. Because there are two variables, r and h , we should make h a function r , i.e., $h(r)$, using the equation for V .

$$h(r) = \frac{1000}{\pi r^2} - \frac{1}{2}r$$

The optimization problem statement is then:

$$\frac{dA}{dr} = 0 = \frac{d}{dr}[2\pi r h(r) + \frac{1}{2}(4\pi r^2)] = 2\pi[h(r) + r \frac{dh}{dr} + 2r], \text{ or}$$

$$h(r) + r \frac{dh}{dr} + 2r = 0$$

The next step is to solve the optimization problem to see if there is a solution that makes sense. Can you do it?

Example 2

A storage bin is to be constructed having a rectangular base, rectangular sides, and open at the top. Its width is 4 meters and its volume is 36 cubic meters. If building the bin costs \$10/sq. m. for the base and \$5/sq. m. for the sides, what is the cost of the least expensive bin? What are its dimensions?

Let the dimensions of the bin be l length, w width, and h height. Note that $w=4$ meters.

The primary equation is the function that models the cost, C , of the bin. We wish to minimize:

$$C = 10lw + 5[2lh + 2wh] = 10[lw + lh + wh] = 10[4l + lh + wh]$$

The secondary equation comes from the information concerning the volume of the bin:

$$V = lwh = 4lh = 36$$

Using the secondary equation in the primary equation:

$$C = 10[4l + lh + wh] = 10[4l + \frac{36}{l} + \frac{36}{l}] = 40[l + 2.25 + \frac{9}{l}]$$

We can find the solution by using the First and Second Derivative Tests and the method for finding absolute maximums and minimums on a closed interval (in this problem, $[0, 45]$).

1 st Derivative	$C'(l) = 40(1 - \frac{9}{l^2})$	$C'(l)=0$: at $l=3$ (in the domain) and $l=-3$ (not in domain).
2 nd Derivative	$C''(l) = \frac{720}{l^3}$	$C''(3) \approx 26.7$. . . Cost function is concave upward. There is a function minimum at $l=3$.

So the dimensions of the bin are: $l=3$ meters, $w=4$ meters, and $h=3$ meters.

The cost to build the bin: $C(3) = \$40(3+2.25+3) = \$330$.

Review

For #1-4, find the absolute maximum and absolute minimum values, if they exist.

1. $f(x) = 2x^2 - 6x + 6$ on $[0, 5]$
2. $f(x) = x^3 + 3x^2$ on $[-2, 3]$
3. $f(x) = 3x^{\frac{2}{3}} - 6x + x$ on $[1, 8]$
4. $f(x) = x^4 - x^3$ on $[-2, 2]$
5. Find the dimensions of a rectangle having area 2000ft^2 whose perimeter is as small as possible.
6. Find two numbers whose product is 50 and whose sum is a minimum.
7. John is shooting a basketball from half-court. It is approximately 45 ft from the half court line to the hoop. The function $s(t) = -0.025x^2 + x + 15$ models the basketball's height above the ground $s(t)$ in feet, when it is t feet from the hoop. How many feet from John will the ball reach its highest height? What is that height?
8. The height of a model rocket t seconds into flight is given by the formula $h(t) = -13t^3 + 4t^2 + 25t + 4$.
 1. How long will it take for the rocket to attain its maximum height?
 2. What is the maximum height that the rocket will reach?
 3. How long will the flight last?
9. Show that of all **rectangles** of a given perimeter, the rectangle with the greatest area is a square.
10. Show that of all **rectangles** of a given area, the rectangle with the smallest perimeter is a square.
11. Eduardo wants to build a fence which will protect his rectangular tomato garden from the various hungry critters roaming the landscape. He purchases 100 feet of fence. What should the dimensions of the tomato garden be if it is to have the maximum area that can be contained by 100 feet of fence.
12. Hans has been challenged to find two nonnegative numbers whose sum is 36 so that the product of the two numbers is maximal. How should he proceed?
13. Lucia needs to find two nonnegative numbers whose product is 25 so that the sum of one plus the square of the other is minimized, in order to gain membership to a club. What numbers is she after?
14. Esmeralda wants to enclose a field with a rectangular fence to keep her goats from escaping. One side of the field is against a building, so it will not require fencing. She has 50 feet of fencing material. What are dimensions of the maximum area she can fence?
15. Which point on the graph of the function $y = \sqrt{x}$ is closest to the point $(1, 0)$?

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 4.9.

Vocabulary

Term	Definition
feasible domain	The feasible domain is the subset of the domain of the primary function that can be used. This restricted domain is determined by constraints of the optimization problem.
optimal value	An optimal value or solution is the best solution to the optimization problem.
optimization	An optimization problem is a problem of finding the best solution to a problem from all the feasible solutions.
primary equation	The primary equation of an optimization problem is the function or equation for which the optimal value or solution is being sought.
secondary equation	A secondary equation is one of the equations used to relate variables of the problem and reduce the number of variables in the primary equation.

Additional Resources

PLIX: [Play, Learn, Interact, eXplore - Optimization: Building the Biggest Box](#)

Video: [Optimization Concept and Example](#)

Practice: [Absolute Extrema and Optimization](#)

Real World: [Keeping the Lights On](#)

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8.2: Newton's Method

We want to look at finding the roots of a polynomial. Sometimes the polynomial cannot be easily factored, and other algebraic methods (e.g., the quadratic equation) are not applicable, or do not work. When faced with a mathematical problem that cannot be solved with simple algebraic means, calculus sometimes provides a way of finding approximate solutions.

Newton's Method

A simple example will help introduce Newton's Method for approximating the roots of a polynomial equation.

Let's say you want to compute $\sqrt{5}$ without using a calculator or a table. Any ideas how this could be done? Try thinking about this problem in a different way, so that you make use of linearization.

Assume that we are interested in solving the quadratic equation:

$$f(x) = x^2 - 5 = 0$$

We know this equation has the roots $x = \pm\sqrt{5}$.

The idea here is to find the linearization of $f(x)$ at an appropriate point, and then solve the linear equation for x . This is an added twist to the linearization problem!

How to choose the linearization point? Since $\sqrt{4} < \sqrt{5} < \sqrt{9}$, this means $2 < \sqrt{5} < 3$.

We choose the linear approximation of $f(x)$ to be near $x_0 = 2$ (but, $x_0 = 3$ could also be selected).

$$f(x) = x^2 - 5 \text{ and } f(2) = -1$$

$$f'(x) = 2x \text{ and } f'(2) = 4$$

Using the linear approximation formula,

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

$$\approx -1 + (4)(x - 2)$$

$$\approx -1 + 4x - 8$$

$$\approx 4x - 9$$

Notice that this equation is much easier to solve than $f(x) = x^2 - 5 = 0$.

Setting $f(x)=0$ and solving for x , we obtain,

$$4x - 9 = 0$$

$$x = \frac{9}{4}$$

$$= 2.25$$

This is a fairly good approximation, since a calculator would give $x=2.23607$, lower by 0.014. We can actually make this approximation to the root of $f(x)$ even better by repeating what we have just done but using the latest estimate $x_1 = 2.25 = \frac{9}{4}$, a number that is even closer to the actual value of $\sqrt{5}$.

$$f(x) = x^2 - 5 \text{ and } f(2.25) = \frac{1}{16}$$

$$f'(x) = 2x \text{ and } f'(2) = \frac{9}{2}$$

Using the linear approximation again,

$$f(x) \approx f(x_1) + f'(x_1)(x - x_1)$$

$$\approx \frac{1}{16} + \frac{9}{2}\left(x - \frac{9}{4}\right)$$

$$\approx \frac{9}{2}x - \frac{161}{16}$$

Solving for x by setting $f(x)=0$, we obtain

$$x = x_2 = \frac{161}{72} = 2.23611$$

which is an even better approximation than

$$x_1 = \frac{9}{4}$$

We could continue this process generating a better approximation to $\sqrt{5}$, as shown in the table, where the last estimate approximates the correct answer to 5 places. Thank goodness for calculators!

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	2	-1	4	-0.25	2.25
2	2.25	0.0625	4.5	0.01389	2.23611
3	2.23611	0.00019	4.47222	0.00004	2.23607
4	2.23607	--	--	--	--

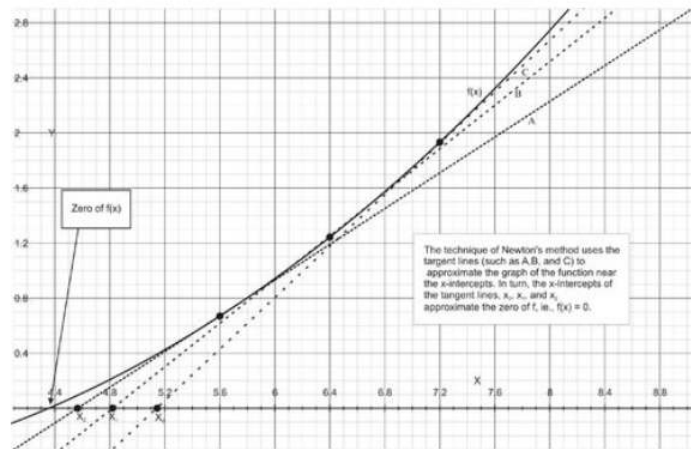
This is the basic idea of **Newton's Method**. Here is a summary of the method:

1. Given the function $f(x)$, find $f'(x)$.
2. Estimate the first approximation, x_0 , to a solution of the equation $f(x)=0$. Use a graph to help in finding the first approximation if necessary (see Figure below).
3. Use the current approximation x_n to find the next approximation, x_{n+1} , by using the recursion relation.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

4. Repeat the previous step until the desired level of convergence occurs.

Note that in some cases, Newton's Method does not converge.



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Newton's Method

Approximate the real zero close until two successive approximations differ by less than 0.0001

$f(x) = 2x^2 - 3$

x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1				



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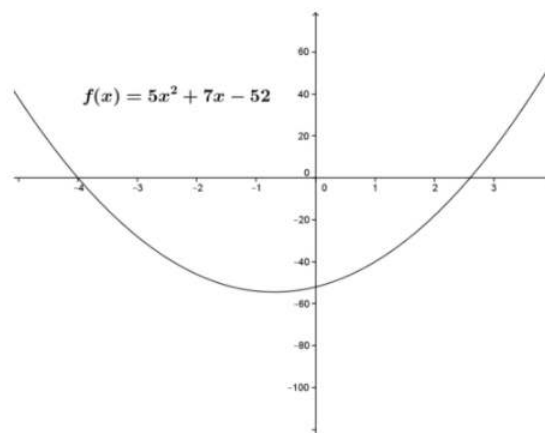
Examples

Example 1

Compare the solutions to $f(x) = 5x^2 + 7x - 52$ in the interval $[2, 3]$ from using the [Quadratic formula](#) and using Newton's Method.

The function is shown in the figure.

For $f(x) = 5x^2 + 7x - 52 = 0$, use of the [quadratic formula](#) gives the exact solution $x = 2.6$ in the interval.



Use of Newton's Method with an initial estimate of $x_0=3$, yields the results shown in the table.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	3	14	37	0.3784	2.6216
2	2.6216	0.7151	33.216	0.0215	2.6001
3	2.6001	0.0033	33.001	0.0001	2.6000
4	2.6000	--	--	--	--

The technique rapidly converges to a solution.

Example 2

Use Newton's method to find the roots of the polynomial $f(x) = x^3 + x - 1$.

The problem is to solve the equation $f(x) = x^3 + x - 1 = 0$.

The equations we need are:

$$f(x) = x^3 + x - 1$$

$$f'(x) = 3x^2 + 1$$

.

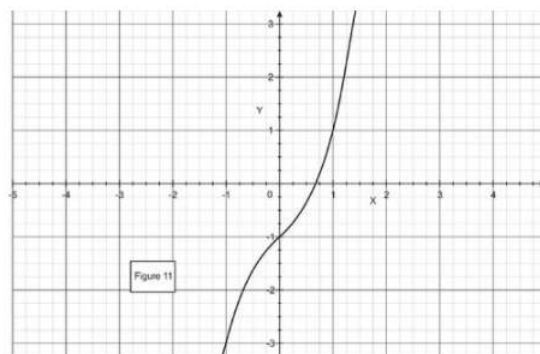
Using the recursion relation,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^3 + x_n - 1}{3x_n^2 + 1}$$

.

To help us find the first approximation, we make a graph of $f(x)$. As the figure suggests, set $x_1 = 0.6$.



Then using the recursion relation, we can generate x_1 :

$$x_{n+1} = x_n - \frac{x_n^3 + x_n - 1}{3x_n^2 + 1}$$

$$x_2 = 0.6 - \frac{(0.6)^3 + (0.6) - 1}{3(0.6)^2 + 1}$$

$$= 0.6884615$$

By using the recursion relation several times, we can find x_3 and x_4 as shown in the table.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.6	-0.184	2.08	-0.0884615	0.6884615
2	0.6884615	0.01477796	2.4219377	0.0061017	0.6823598
3	0.6823598	0.00007669	2.3968445	0.0000320	0.6823278
4	0.6823278	--	--	--	--

We conclude that the solution to the equation

$$x^3 + x - 1 = 0$$

is about 0.6823.

Example 3

Use Newton's Method to show to find the root of $f(x) = \cos(2x) - x$.

The equations we need are:

$$f(x) = \cos(2x) - x$$

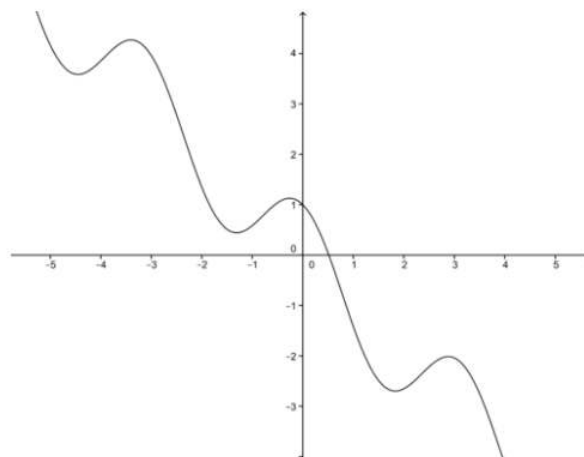
$$f'(x) = -2\sin(2x) - 1$$

Using the recursion relation,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{\cos(2x_n) - x_n}{-2\sin(2x_n) - 1}$$

To help us find the first approximation, we make a graph of $f(x)$. As the figure suggests, set the first estimate at 0.5.



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Using the recursion relation several times gives the table values:

n	x_n	$f(x_n)$	$\frac{f(x_n)}{f'(x_n)}$
1	0.500000	0.0403023	-0.015216_8
2	0.515022	-0.0002400	0.0000884
3	0.514933	-0.0000000	0.0000000
4	0.514933	--	--

We conclude that the solution to the equation $\cos(2x) - x = 0$ is 0.514933.

Review

For all the problems, use Newton's Method to find the roots.

- $x^3 + 3 = 0$.
- $-x + 3\sqrt{-1+x} = 0$.
- $4x^2 - x - 2$ in the interval $[-1, 0]$.
- $4x^3 - 6x^2 - 1$.
- $5e^{-x} + x^3$.
- $\cos x - x$.
- $x^5 - 7x^2 + 2$ in the interval $[-1, 0]$.
- $x^5 - 7x^2 + 2$ in the interval $[0, 1]$.
- $x^2 \cos x - x$ in the interval $[4, 5]$.
- $x^2 \cos x - x$ in the interval $[7, 8]$.
- $f(x) = x - 2\sin(x)$ to five decimal places, starting with initial guess $x_0 = 3$.
- $f(x) = 6x^3 - 4x + 1$ to three decimal places, starting with initial guess $x_0 = 1.2$.
- $f(x) = \ln(x) \times (\ln(x) + 4) + 1$ to five decimal places, starting with initial guess $x_0 = 1$.
- $f(x) = \tan(x) - \csc(x)$ to six digits of accuracy, starting with initial guess $x_0 = 0.7$.
- $f(x) = \tan^{-1}(x) + \cos(x)$ to four digits of accuracy, starting with initial guess $x_0 = -2$.

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 4.11.

Vocabulary

Term	Definition
Newton's method	Newton's method is an iterative procedure for computing the root of an equation.

Additional Resources

PLIX: Play, Learn, Interact, eXplore - [Newton's Method](#)

Video: [Newton's Method for Solving Equations](#)

Practice: [Newton's Method](#)

Real World: [Getting to the Root of Things](#)

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8.3: Analyzing the Graphs of Functions

Given a set of information on the key properties of a function, you can sketch the graph. Before we proceed, make an attempt to summarize for what you think are key properties. Often, the key properties of a function are not all presented to you directly, but must be determined from the information at hand.

Analyzing Graphs of Functions

We will first summarize the kind of information about functions we now can generate based on our previous concepts. Then we will use this information to analyze examples of representative rational, polynomial, radical, and trigonometric functions.

Let's use a table like the one shown as a template to organize our findings.

Table Summary

$f(x)=$	<i>Analysis</i>
<i>Domain and Range</i>	
<i>Intercepts and Zeros</i>	
<i>Asymptotes and limits at infinity</i>	
<i>Differentiability</i>	
<i>Intervals where f is increasing</i>	
<i>Intervals where f is decreasing</i>	
<i>Relative extrema</i>	
<i>Concavity</i>	
<i>Inflection points</i>	

Analyzing Rational Functions

Consider the rational function $f(x) = \frac{x^2-4}{x^2-2x-8}$.

Zeroes, Domain, and Range

The function appears to have zeros at $x=\pm 2$. However, once we factor the expression we see

$$\frac{x^2-4}{x^2-2x-8} = \frac{(x+2)(x-2)}{(x-4)(x+2)} = \frac{x-2}{x-4}$$

Hence, the function has a zero at $x=2$, there is a hole in the graph at $x=-2$, the domain is $(-\infty, -2) \cup (-2, 4) \cup (4, +\infty)$, and the y-intercept is at $(0, 12)$.

Asymptotes and Limits at Infinity

Given the domain, we note that there is a vertical asymptote at $x=4$. To determine other asymptotes, we examine the limit of f as $x \rightarrow \infty$ and $x \rightarrow -\infty$. We have

$$\lim_{x \rightarrow \infty} \frac{x^2-4}{x^2-2x-8} = \lim_{x \rightarrow \infty} \frac{\frac{x^2}{x^2} - \frac{4}{x^2}}{\frac{x^2}{x^2} - \frac{2x}{x^2} - \frac{8}{x^2}} = \lim_{x \rightarrow \infty} \frac{1 - \frac{4}{x^2}}{1 - \frac{2}{x} - \frac{8}{x^2}} = 1$$

Similarly, we see that $\lim_{x \rightarrow -\infty} \frac{x^2-4}{x^2-2x-8} = 1$. We also note that $y \neq \frac{2}{3}$ since $x \neq -2$.

Hence we have a horizontal asymptote at $y=1$.

Differentiability

$f'(x) = \frac{2x^2 - 8x - 8}{(x^2 - 2x - 8)} = \frac{-2}{(x-4)^2} < 0$. Hence the function is **differentiable** at every point of its domain, and since $f'(x) < 0$ on its domain, then f is decreasing on its domain, $(-\infty, -2) \cup (-2, 4) \cup (4, +\infty)$.

$$f''(x) = \frac{4}{(x-4)^3}$$

$f''(x) \neq 0$ in the domain of f . Hence there are no relative extrema and no inflection points.

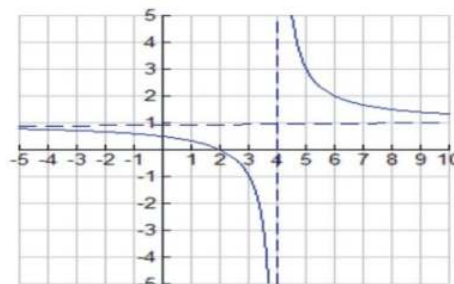
Similarly, $f''(x) < 0$ when $x < 4$. Hence, the graph is concave down for $x < 4$, $x \neq -2$.

Let's summarize our results in the table before we sketch the graph.

Table Summary

$f(x) = x^2 - 4x - 2x - 8$	Analysis
Domain and Range	$D = (-\infty, -2) \cup (-2, 4) \cup (4, +\infty)$ $\setminus (R = \{\text{all reals}\} \neq 1 \text{ or } \frac{2}{3})$
Intercepts and Zeros	Zero at $x=2$, y- intercept at $(0, \frac{1}{2})$
Asymptotes and limits at infinity	VA at $x=4$, HA at $y=1$, Hole in the graph at $x=-2$
Differentiability	Differentiable at every point of its domain
Intervals where f is increasing	Nowhere
Intervals where f is decreasing	$(-\infty, -2) \cup (-2, 4) \cup (4, +\infty)$
Relative extrema	None
Concavity	Concave up in $(4, +\infty)$, Concave down in $(-\infty, -2) \cup (-2, 4)$
Inflection points	None

Finally, we sketch the graph as follows:



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Analyzing Radical Functions

Now, Consider the function $f(x) = \sqrt{2x-1}$.

Zeroes, Domain, and Range

The domain of f is $(\frac{1}{2}, +\infty)$, and it has a zero at $x = \frac{1}{2}$.

Asymptotes and Limits at Infinity

Given the domain, we note that there are no **vertical asymptotes**. We note that $\lim_{x \rightarrow \infty} f(x) = +\infty$.

Differentiability

$f'(x) = \frac{1}{\sqrt{2x-1}} > 0$ for the entire domain of f . Hence f is increasing everywhere in its domain. $f'(x)$ is not defined at $x = \frac{1}{2}$, so $x = \frac{1}{2}$ is a critical value.

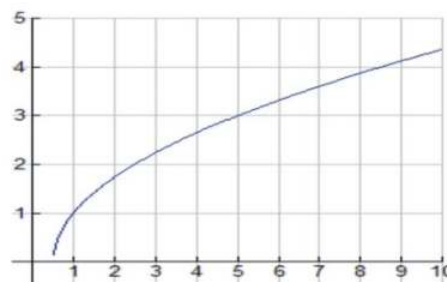
$f''(x) = \frac{-1}{\sqrt{2x-1}^3} < 0$ everywhere in $(\frac{1}{2}, +\infty)$. Hence f is concave down in $(\frac{1}{2}, +\infty)$.

$f'(x)$ is not defined at $x = \frac{1}{2}$, so $x = \frac{1}{2}$ is an absolute minimum.

Table Summary

$f(x) = 2x - \sqrt{-1}$	Analysis
Domain and Range	$D = (\frac{1}{2}, +\infty), R = y \geq 0$
Intercepts and Zeros	Zero at $x = \frac{1}{2}$, No y - intercept
Asymptotes and limits at infinity	No asymptotes
Differentiability	Differentiable in $(\frac{1}{2}, +\infty)$
Intervals where f is increasing	Everywhere in $D = (\frac{1}{2}, +\infty)$
Intervals where f is decreasing	Nowhere
Relative extrema	None
Concavity	Absolute minimum at $x = \frac{1}{2}$, located at $(\frac{1}{2}, 0)$ Concave down in $(\frac{1}{2}, +\infty)$
Inflection points	None

Here is a sketch of the graph:



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Analyzing Trigonometric Functions

We will see that while trigonometric functions can be analyzed using what we know about **derivatives**, they will provide some interesting challenges that we will need to address.

Consider the function $f(x) = x - 2\sin x$ on the interval $[-\pi, \pi]$.

Zeroes, Domain, and Range

We note that f is a continuous function and attains an absolute maximum and minimum in $[-\pi, \pi]$. Its domain is $[-\pi, \pi]$, and its range is $R = -\pi \leq y \leq \pi$.

Differentiability

$$f'(x) = 1 - 2\cos x = 0 \text{ at } x = -\frac{\pi}{3}, \frac{\pi}{3}$$

.

Note that $f'(x) > 0$ on $(-\pi, -\frac{\pi}{3})$ and $(\frac{\pi}{3}, \pi)$; therefore the function is increasing in $(-\pi, -\frac{\pi}{3})$ and $(\frac{\pi}{3}, \pi)$.

Note that $f'(x) < 0$ on $(-\frac{\pi}{3}, \frac{\pi}{3})$; therefore the function is decreasing in $(-\frac{\pi}{3}, \frac{\pi}{3})$.

$f''(x) = 2\sin x = 0$ if $x = 0, \pi, -\pi$. Hence the critical values are at $x = -\pi, -\frac{\pi}{3}, \frac{\pi}{3}$, and π .

$f''(\frac{\pi}{3}) > 0$; hence there is a relative minimum at $x = \frac{\pi}{3}$.

$f''(-\frac{\pi}{3}) < 0$; hence there is a relative maximum at $x = -\frac{\pi}{3}$.

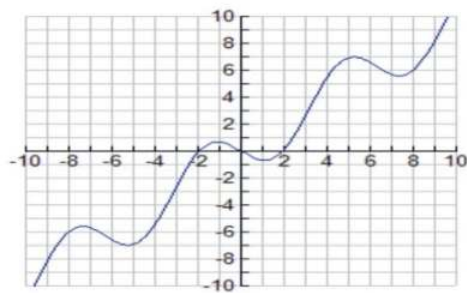
$f''(x) < 0$ on $(-\pi, 0)$ and $f''(x) > 0$ on $(0, \pi)$. Hence the graph is concave down on $(-\pi, 0)$ and concave up and decreasing on $(0, \pi)$. There is an **inflection point** at $x = 0$, located at the point $(0, 0)$.

Finally, there is absolute minimum at $x = -\pi$, located at $(-\pi, -\pi)$, and an absolute maximum at $x = \pi$, located at (π, π) .

Table Summary

$f(x) = x - 2\sin x$	Analysis
Domain and Range	$D = [-\pi, \pi], R = -\pi \leq y \leq \pi$
Intercepts and Zeros	$x = -\frac{\pi}{3}, \frac{\pi}{3}$
Asymptotes and limits at infinity	No asymptotes
Differentiability	Differentiable in $D = [-\pi, \pi]$
Intervals where f is increasing	$(\frac{\pi}{3}, \pi)$ and $(-\pi, -\frac{\pi}{3})$
Intervals where f is decreasing	$(-\frac{\pi}{3}, \frac{\pi}{3})$
Relative extrema	Relative maximum at $x = -\frac{\pi}{3}$ Relative minimum at $x = \frac{\pi}{3}$
Concavity	Absolute maximum at $x = \pi$, located at (π, π) Absolute minimum at $x = -\pi$, located at $(-\pi, -\pi)$ Concave up in $(0, \pi)$
Inflection points	$x = 0$, located at the point $(0, 0)$

Here is a sketch of the graph:



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Examples

Example 1

Earlier, you were asked to summarize the key properties of functions. As you have seen in this concept, key properties of functions include: domain, range, intercepts, asymptotes (including limits at infinity), [continuity and differentiability](#), increasing and decreasing intervals, extrema, concavity, and points of inflection.

Example 2

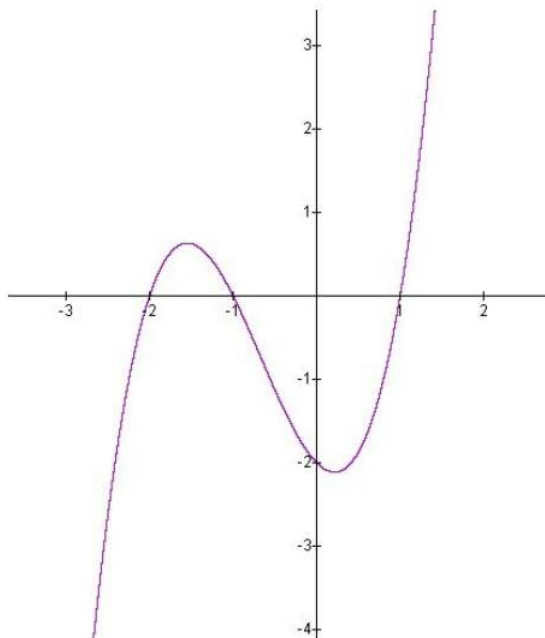
Consider the polynomial function $f(x) = x^3 + 2x^2 - x - 2$.

Zeroes, Domain, and Range

The domain of f is $(-\infty, +\infty)$ and the y-intercept at $(0, -2)$.

The function can be factored $f(x) = x^3 + 2x^2 - x - 2 = x^2(x + 2) - 1(x + 2) = (x^2 - 1)(x + 2) = (x - 1)(x + 1)(x + 2)$

and thus has zeros at $x = \pm 1, -2$.



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Asymptotes and Limits at infinity

Given the domain, we note that there are no [vertical asymptotes](#). We note that $\lim_{x \rightarrow \infty} f(x) = +\infty$ and $\lim_{x \rightarrow -\infty} f(x) = -\infty$.

Differentiability

$f'(x) = 3x^2 + 4x - 1 = 0$ if $x = \frac{-4 \pm \sqrt{28}}{6} = \frac{-2 \pm \sqrt{7}}{3}$. These are the critical values. We note that the function is differentiable at every point of its domain.

$f'(x) > 0$ on $(-\infty, \sqrt{-2 - \sqrt{73}})$ and $(\frac{-2 + \sqrt{7}}{3}, +\infty)$; hence the function is increasing in these intervals.

Similarly, $f'(x) < 0$ on $(\frac{-2 - \sqrt{7}}{3}, \frac{-2 + \sqrt{7}}{3})$ and thus is f decreasing there.

$f''(x) = 6x + 4 = 0$ if $x = -\frac{2}{3}$ where there is an inflection point.

In addition, $f''(\frac{-2 - \sqrt{7}}{3}) < 0$. Hence the graph has a relative maximum at $x = \frac{-2 - \sqrt{7}}{3}$ and located at the point $(-1.55, 0.63)$.

We note that $f''(x) < 0$ for $x < -\frac{2}{3}$. The graph is concave down in $(-\infty, -\frac{2}{3})$.

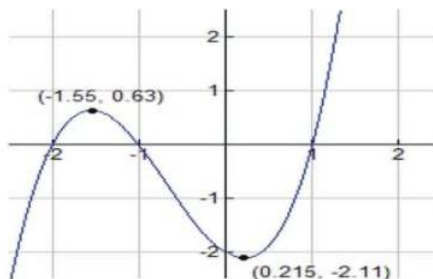
And we have $f''(\frac{-2 + \sqrt{7}}{3}) > 0$; hence the graph has a relative minimum at $x = \frac{-2 + \sqrt{7}}{3}$ and located at the point $(0.22, -2.11)$.

We note that $f''(x) > 0$ for $x > -\frac{2}{3}$. The graph is concave up in $(-\frac{2}{3}, +\infty)$.

Table Summary

$f(x) = x^3 + 2x^2 - x - 2$	Analysis
Domain and Range	$D = (-\infty, +\infty), R = \{\text{all reals}\}$
Intercepts and Zeros	Zeros at $x = \pm 1, -2, y$, intercepts at $(0, -2)$
Asymptotes and limits at infinity	No asymptotes
Differentiability	Differentiable at every point of it's domain
Intervals where f is increasing	Missing argument for $\frac{-2 + \sqrt{7}}{3}$
Intervals where f is decreasing	$(\frac{-2 - \sqrt{7}}{3}, \frac{-2 + \sqrt{7}}{3})$
Relative extrema	Relative maximum at $x = \frac{-2 - \sqrt{7}}{3}$ and located at the point $(-1.55, 0.63)$; Relative minimum at $x = \frac{-2 + \sqrt{7}}{3}$ and located at the point $(0.22, -2.11)$.
Concavity	Concave up in $(-\frac{2}{3}, +\infty)$. Concave down in $(-\infty, -\frac{2}{3})$.
Inflection points	$x = -\frac{2}{3}$, located at the point $(-\frac{2}{3}, -\frac{7}{4})$

Here is a sketch of the graph:



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Review

For all of the following functions, summarize the functions by filling out the table below. Use the information to sketch a graph of the function.

Table Summary

$f(x)=$	<i>Analysis</i>
<i>Domain and Range</i>	
<i>Intercepts and Zeros</i>	
<i>Asymptotes and limits at infinity</i>	
<i>Differentiability</i>	
<i>Intervals where f is increasing</i>	
<i>Intervals where f is decreasing</i>	
<i>Relative extrema</i>	
<i>Concavity</i>	
<i>Inflection points</i>	

1. $f(x) = x^3 + 3x^2 - x - 3$
2. $f(x) = -x^4 + 4x^3 - 4x^2$
3. $f(x) = \frac{2x-2}{x^2}$
4. $f(x) = x - x^{\frac{1}{3}}$
5. $f(x) = -\sqrt{2x-6} + 3$
6. $f(x) = x^2 - 2\sqrt{x}$
7. $f(x) = 1 + \cos x$ on the interval $[-\pi, \pi]$
8. $f(x) = x^2 - x + 1$
9. $f(x) = 4x^3 - 6x^2 - 1$
10. $f(x) = \frac{x^2}{x-1}$
11. $f(x) = \frac{x^2}{e^x}$
12. $f(x) = \cos x - x$
13. $f(x) = e^{-2x} + e^x$
14. $f(x) = 5e^{-x} + x^3$
15. $f(x) = x^5 - 7x^2 + 2$

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 4.8.

Vocabulary

Term	Definition
Asymptotes	An asymptote is a line on the graph of a function representing a value toward which the function may approach, but does not reach (with certain exceptions).
concavity	Concavity describes the behavior of the slope of the tangent line of a function such that concavity is positive if the slope is increasing, negative if the slope is decreasing, and zero if the slope is constant.
decreasing	A function is decreasing over an interval if its y values are getting smaller over the interval. The graph will go down from left to right over the interval.
differentiable	A differentiable function is a function that has a derivative that can be calculated.
domain	The domain of a function is the set of x-values for which the function is defined.
increasing	A function is increasing over an interval if its y values are getting larger over the interval. The graph will go up from left to right over the interval.
inflection point	An inflection point is a point in the domain where concavity changes from positive to negative or negative to positive.
Range	The range of a function is the set of y values for which the function is defined.
relative extrema	The relative extrema of a function are the points of the function with y values that are the highest or lowest of a local neighborhood of the function.

Additional Resources

PLIX: [Play, Learn, Interact, eXplore - Analyzing a rational function](#)

Video: [Graphing Using Derivatives](#)

Real World: [Total Elimination](#)

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CHAPTER OVERVIEW

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[9.3: Area Sums](#)

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9.1: Area Under the Curve

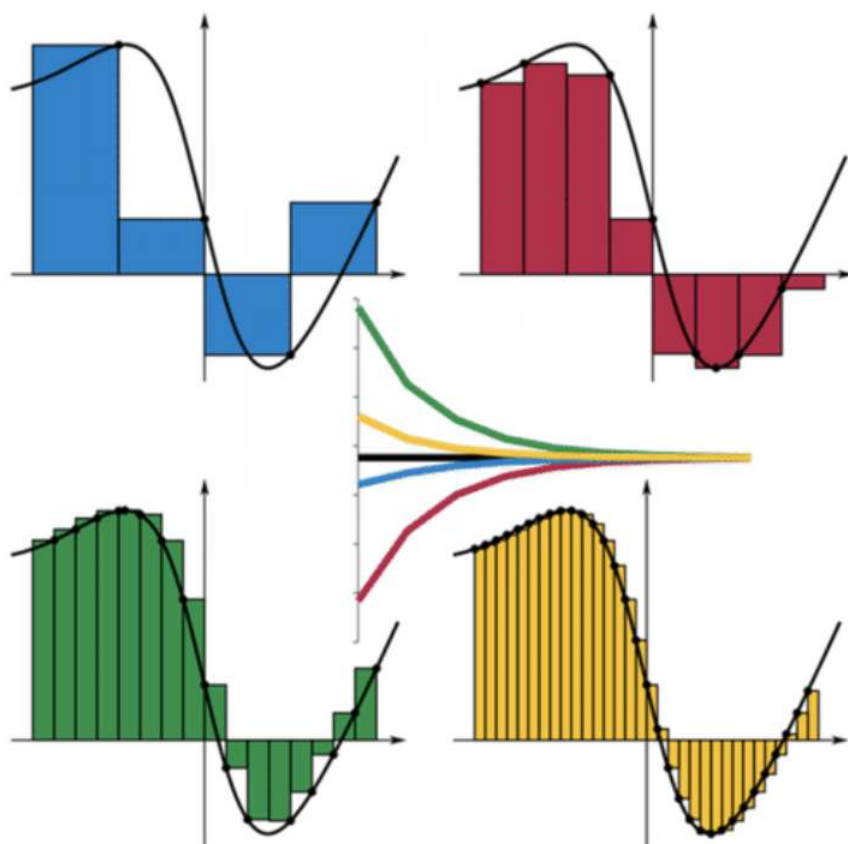
Calculating the area under a straight line can be done with geometry. Calculating the area under a curved line requires calculus. Often the **area under a curve** can be interpreted as the accumulated amount of whatever the function is modeling. Suppose a car's speed in **meters** per second can be modeled by a quadratic for the first 8 seconds of acceleration:

$$s(t) = t^2$$

How far has the car traveled in 8 seconds?

Finding the Area Under a Curve

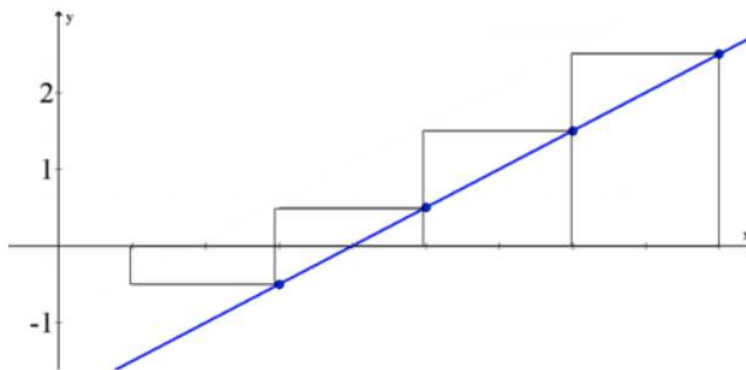
The area under a curve can be approximated with **rectangles** equally spaced under a curve as shown below. For consistency, you can choose whether the boxes should hit the curve on the left hand corner, the right hand corner, the maximum value, or the minimum value. The more boxes you use the narrower the boxes will be and thus, the more accurate your approximation of the area will be.



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Subintervals are created when an interval is broken into smaller, equally sized intervals. The blue approximation uses right handed boxes for the height of each subinterval. The red approximation assigns the height of the box to be the minimum value of the function in each subinterval. The green approximation assigns the height of the box to be the maximum value of the function in each subinterval. The yellow approximation uses left handed boxes. **Rectangles** above the x-axis will have positive area and **rectangles** below the x-axis will have negative area in this context.

Using boxes to estimate the area under a curve is called a **Riemann Sum**. Take the function $f(x) = 12x - 2$. To calculate the **Riemann Sum** (area under the curve) between 1 and 9 of the function, first draw the graph and the boxes.



<https://commons.wikimedia.org/wiki/File:Convergence.png> - CC BY-SA

The area of the first box is 2 times the height of the function evaluated at 3:

$$2 \cdot \left(\frac{1}{2} \cdot 3 - 2 \right) = 3 - 4 = -1$$

Because this box is under the x-axis, its area is negative.

The area for each of the rest of the boxes is 2 times the height of the function evaluated at 5, 7 and 9.

$$2 \cdot \left(\frac{1}{2} \cdot 5 - 2 \right) = 5 - 4 = 1$$

$$2 \cdot \left(\frac{1}{2} \cdot 7 - 2 \right) = 7 - 4 = 3$$

$$2 \cdot \left(\frac{1}{2} \cdot 9 - 2 \right) = 9 - 4 = 5$$

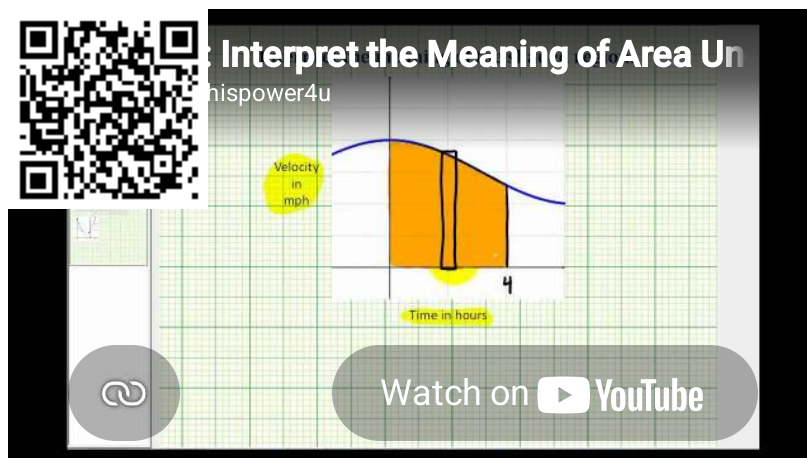
The approximate sum of the total area under the curve is: $-1+1+3+5=8$ square units.

All four of the area approximations shown earlier get better as the number of boxes increase. In fact, the **limit** of each approximation as the number of subintervals(boxes) increases to infinity is the precise area under the curve.

This is where the calculus idea of an integral comes in. An **integral** is the limit of a sum as the number of summands increases to infinity. A **summand** is one of many pieces being summed together.

$$\int f(x) = \lim_{n \rightarrow \infty} \sum_{i=1}^n (\text{Area of box } i)$$

The symbol on the left is the calculus symbol of an integral.





2: Approximate the Area Under a

hispower4u

Approximate the area under the function $f(x) = \frac{1}{x}$ on the interval $[1, 5]$ using 4 right sided rectangles.

$$\Delta x = \frac{b-a}{n}$$

$$\text{Area} \approx \sum_{i=1}^n f(c_i) \Delta x$$

$$x_{i-1} < c_i < x_i$$

$\text{Area} \approx A_1 + A_2 + A_3 + A_4$

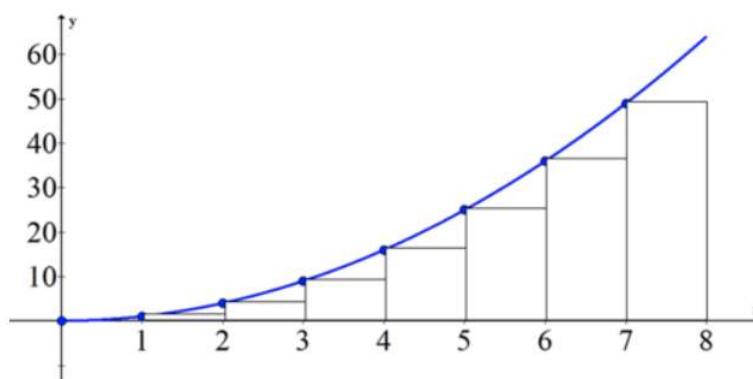
$\text{Area} \approx 5$

Watch on **YouTube**

Examples

Example 1

Earlier, you were asked to determine how far a car travels in 8 seconds. You can use the area under the curve to find the total distance traveled in the first 8 seconds. Since the quadratic is a curve you must choose the number of subintervals you want to use and whether you want right or left handed boxes for estimating. Suppose you choose 8 left handed boxes of width one.



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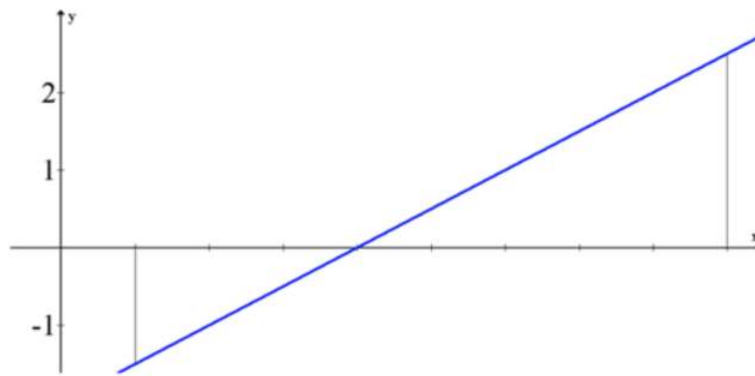
x	0	1	2	3	4	5	6	7
Area of box to the right	1·0	1·1	1·4	1·9	1·16	1·25	1·36	1·49

The approximate sum is $1+4+9+16+25+36+49=140$. This means that the car traveled approximately 140 meters in the first 8 seconds.

Example 2

Evaluate the exact area under the curve used earlier, $f(x) = \frac{1}{2}x - 2$, using the area formula for a triangle.

Remember that the area below the x axis is negative while the area above the x axis is positive.



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Negative Area: $\frac{1}{2} \cdot 3 \cdot 1.5 = \frac{9}{4}$

Positive Area: $\frac{1}{2} \cdot 5 \cdot 2.5 = \frac{25}{4}$

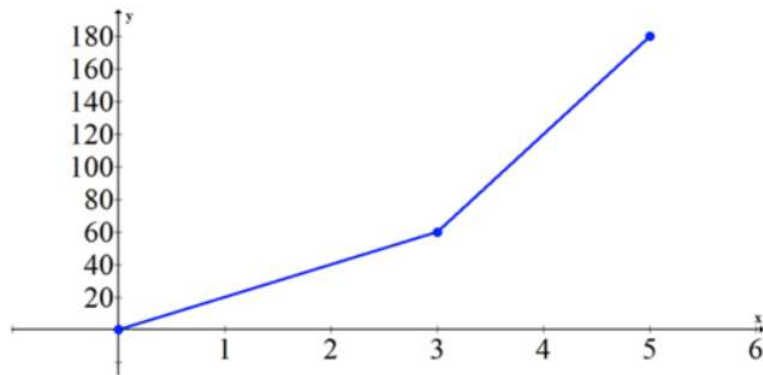
Area under the curve between 1 and 8: $\frac{25}{4} - \frac{9}{4} = \frac{16}{4} = 4$

If you compare this answer to the approximation from earlier, it appears that approximations that are 2 units wide produce an area with significant error.

Example 3

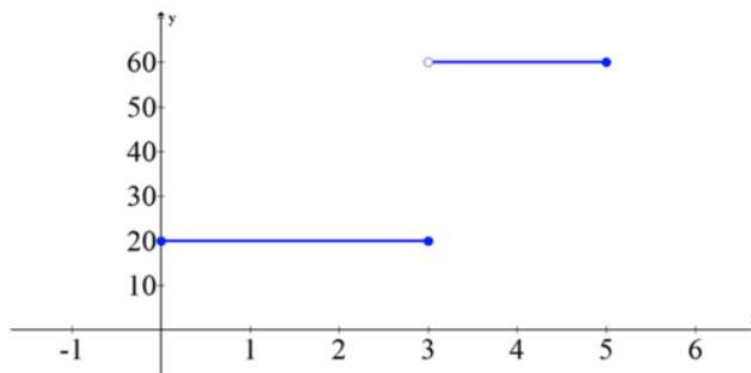
Logan travels by bike at 20 mph for 3 hours. Then she gets in a car and drives 60 mph for 2 hours. Sketch both the distance vs. time graph and the rate vs. time graph. Use an area under the curve argument to connect the two graphs.

Distance vs. Time:



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Rate vs. Time:



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The [slope](#) of the first graph is 20 from 0 to 3 and then 60 from 3 to 5. The second graph is a graph of the slopes from the first graph. If you calculate the area of the second graph at the key points 0, 1, 2, 3, 4 and 5 you will see that they align perfectly with the points on the first graph.

x	Area under curve from 0 to x
0	0
1	20
2	40
3	60
4	120
5	180

Example 4

Approximate the area under the curve using eight subintervals and right endpoints.

$$f(x) = 3x^2 - 1, -1 \leq x \leq 7$$

While a graph is helpful to visualize the problem and drawing each box can help give meaning to each summand, it is not always necessary. Since there are going to be 8 subintervals over the total interval of $-1 \leq x \leq 7$, each interval is going to have a width of 1. The height of each interval is going to be at the right hand endpoints of each subinterval (0, 1, 2, 3, 4, 5, 6, 7).

$$\sum \text{height} \cdot \text{width} = \sum_{i=0}^7 (3i^2 - 1) \cdot 1 = 412$$

Example 5

Approximate the area under the curve using twenty subintervals and left endpoints.

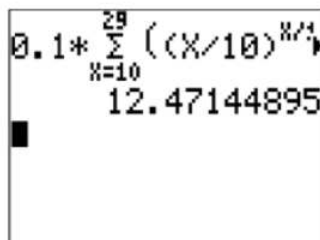
$$f(x) = x^x, 1 \leq x \leq 3$$

When the number of subintervals gets large and the subintervals get extremely narrow it will be impossible to draw an accurate picture. This is why using summation notation and thinking through what the indices and the argument will be is incredibly important. With 20 subintervals between [1,3], each interval will be 0.1 wide. Left endpoints means that the first box has a height of $f(1)$ and the second box has a height of $f(1.1)$.

$$\begin{aligned} \sum \text{height} \cdot \text{width} &= f(1) \cdot 0.1 + f(1.1) \cdot 0.1 + f(1.2) \cdot 0.1 + \dots + f(2.9) \cdot 0.1 \\ &= 0.1(f(1) + f(1.1) + \dots + f(2.9)) \end{aligned}$$

$$\begin{aligned}
 &= 0.1 \cdot \sum_{i=10}^{29} f\left(\frac{i}{10}\right) \\
 &= 0.1 \cdot \sum_{i=10}^{29} \left(\frac{i}{10}\right)^{\left(\frac{1}{10}\right)} \\
 &\approx 12.47144
 \end{aligned}$$

Your calculator can compute summations when you go under the math menu.



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Review

1. Approximate the area under the curve using eight subintervals and right endpoints.

$$f(x) = x^2 - x + 1, 0 \leq x \leq 8$$

2. Approximate the area under the curve using eight subintervals and left endpoints.

$$f(x) = x^2 - 2x + 1, -4 \leq x \leq 4$$

3. Approximate the area under the curve using twenty subintervals and left endpoints.

$$f(x) = \sqrt{x+3}, 0 \leq x \leq 4$$

4. Approximate the area under the curve using 100 subintervals and left endpoints. Compare to your answer from #3.

$$f(x) = \sqrt{x+3}, 0 \leq x \leq 4$$

5. Approximate the area under the curve using eight subintervals and left endpoints.

$$f(x) = \cos(x), 0 \leq x \leq 4$$

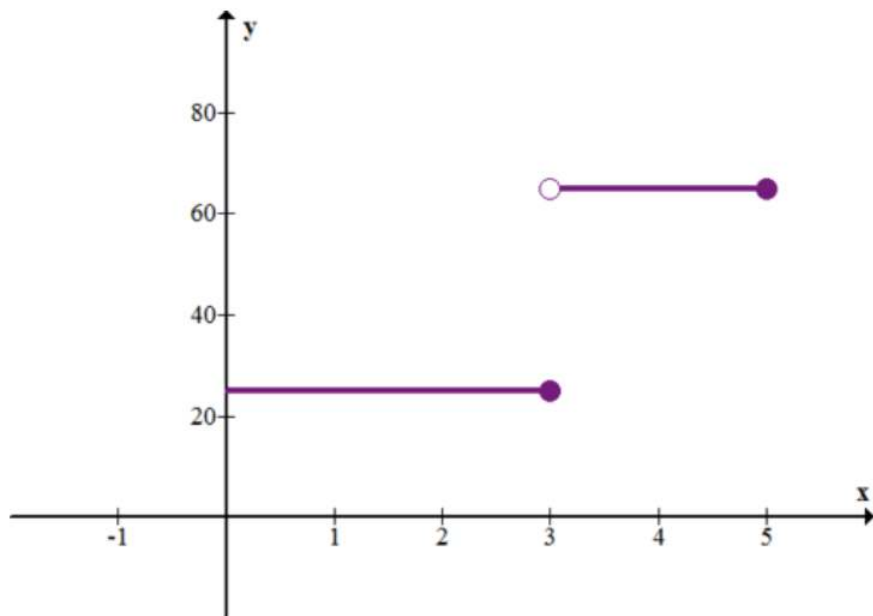
6. Approximate the area under the curve using twenty subintervals and left endpoints.

$$f(x) = \cos(x), 0 \leq x \leq 4$$

7. Approximate the area under the curve using 100 subintervals and left endpoints.

$$f(x) = \cos(x), 0 \leq x \leq 4$$

The following graph shows the rate (in miles per hour) vs. time (in hours) for a car.

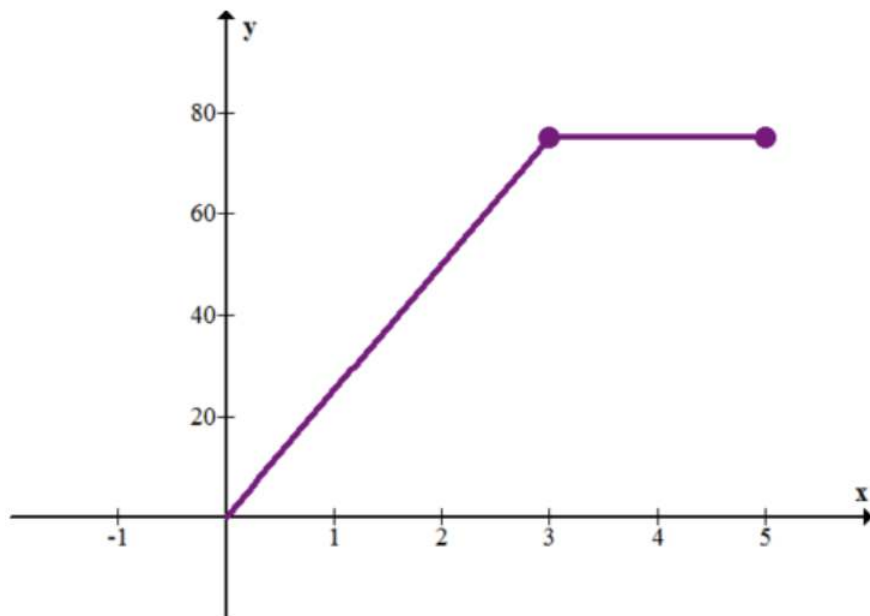


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8. Describe what is happening with the car.

9. How far did the car travel in 5 hours?

The following graph shows the rate (in feet per second) vs. time (in seconds) for a car.

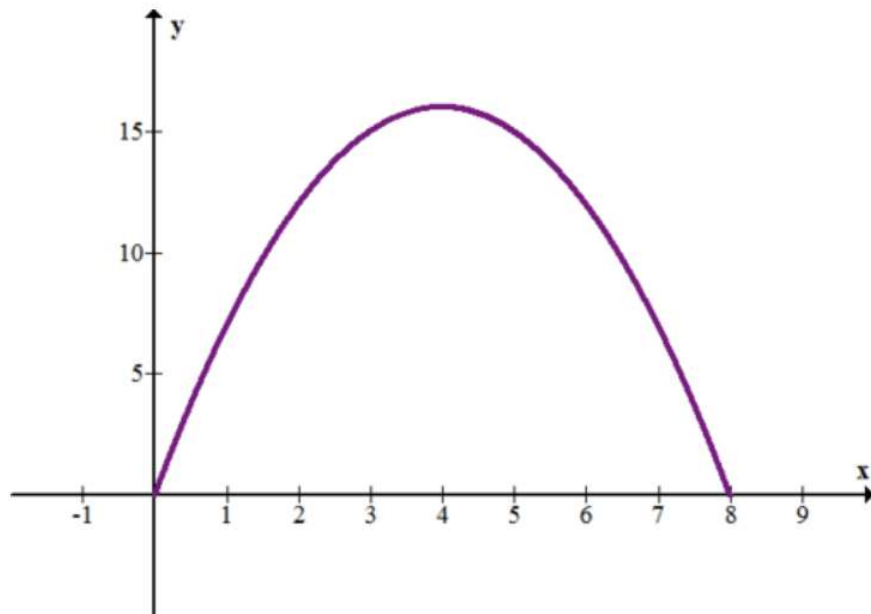


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10. Describe what is happening with the car. In particular, what is happening in the first 3 seconds?

11. How far did the car travel in 5 seconds?

The following graph shows the function $f(x) = -(x-4)^2 + 16$, which represents the rate (in feet per second) vs. time (in seconds) for a runner.



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12. Describe what is happening with the runner. In particular, what happens after 4 seconds?
13. Use rectangles to approximate the total distance (in feet) that the runner traveled in the 8 seconds. Try to get as good an approximation as possible.
14. Explain how an integral is like the opposite of a derivative.
15. How do integrals relate to sums?

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 14.9.

Vocabulary

Term	Definition
Δ	The symbol " Δ ", read "delta", is used to denote "change in", as in "the change in velocity over time" = $\frac{\Delta v}{t}$.
definite integral	A definite integral gives the area between the x-axis and a curve over a defined interval.
limit	A limit is the value that the output of a function approaches as the input of the function approaches a given value.
subintervals	Subintervals are created when an interval is broken into smaller equally sized intervals.
summand	A summand is an expression being summed. It directly follows the sigma symbol.

Additional Resources

Video: [Introduction to Definite Integrals](#)

Practice: [Area Under the Curve](#)

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9.2: Antiderivative

You have spent many lessons learning about how to find the derivative, $f'(x)$, of a function $f(x)$, and the process of differentiation. It should be no surprise then that there would be a name for the function $f(x)$, or family of functions, that can generate $f'(x)$ when differentiated: $f(x)$ and $f'(x)$ are a pair of **inverse functions**, and $f(x)$ is called an **antiderivative** of $f'(x)$. Before continuing with the lesson, try listing functions that are antiderivative and derivative pairs?

The Antiderivative

Let's start off and introduce the idea of the antiderivative of a function.

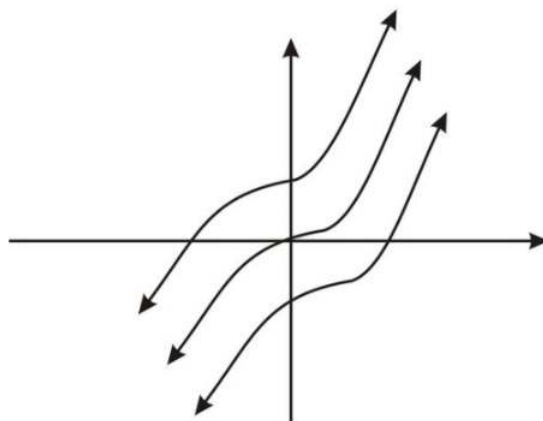
A function $F(x)$ is called an **antiderivative** of a function f if $F'(x)=f(x)$ for all x in the domain of f .

How is this definition used?

Consider the function $f(x) = 3x^2$.

Can you think of a function $F(x)$ such that $F'(x)=f(x)$? You should be able to think of lots of them.

Since we differentiate $F(x)$ to get $f(x)$, we see that $F(x) = x^3 + C$ will work for any constant C . Graphically, we can think of the set of all antiderivatives as vertical transformations of the graph of $F(x) = x^3$. The figure shows two such transformations.



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With our definition and initial example, we now look to formalize the definition and develop some useful rules for computational purposes, and begin to see some applications.

Introduction to Indefinite Integrals

The process of finding antiderivatives is called **antidifferentiation**, more commonly referred to as **integration**. Here is how the integration is indicated and how it works:

$F'(x)=f(x)$... Start with the **differential equation** that represents the definition of the antiderivative

$\int F'(x)dx = \int f(x)dx$... Invoke the integration (antidifferentiation) operation using the special symbol $\int$.

$F(x) + C = \int f(x)dx$... Obtain the antiderivative $F(x)$ and a **constant of integration**, C .

$\int f(x)dx = F(x) + C$... Note that if we differentiate both sides, we get back the original equation:

$$\frac{d}{dx}[\int f(x)dx] = f(x) = \frac{d}{dx}[F(x) + C] = F'(x)$$

We refer to $\int f(x)dx$ as “the indefinite integral of $f(x)$ with respect to x ”. The function $f(x)$ is called the **integrand** and the constant C is called the **constant of integration**. Finally the symbol dx indicates that we are to integrate with respect to x .

Using this notation, we would summarize the last example as follows:

$$\int 3x^2 dx = x^3 + C$$

Now, consider the function $f(x)=\cos x$

Can you think of a function $F(x)$ such that $F'(x)=f(x)$?

If you said $F(x)=\sin x+C$ you would be correct and here is how this would be written.

$f(x)=F'(x)$. . . Start with the differential equation that represents the definition of the antiderivative

$\cos x=F'(x)$. . . Substitute for $f(x)$

$\int \cos x dx = \int F'(x) dx$. . . Invoke the integration (antidifferentiation) operation using the special symbol $\int$.

$\int \cos x dx = F(x) + C$. . . Obtain the antiderivative $F(x)$ and a constant of integration, C .

$\int \cos x dx = \sin x + C$. . . We know $F(x)=\sin x$ because if we differentiate both sides, we get back the original equation.

We have looked at the [derivatives](#) of a number of functions through the calculus concepts and can put together a list of functions and their antiderivatives as shown below.

Summary of Basic Indefinite Integrals And Antiderivatives

Function $f(x)$	Antiderivative $\int f(x) dx = F(x) + C$
1	$x+C$
x	$\frac{x^2}{2} + C$
x^2	$\frac{x^3}{3} + C$
x^n , $n \neq -1$	$\frac{x^{n+1}}{n+1} + C$
$\frac{1}{x}$	$\ln x + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
$\sec^2 x$	$\tan x + C$
$\csc^2 x$	$-\cot x + C$
$\sec x \tan x$	$\sec x + C$
$\csc x \cot x$	$-\csc x + C$
e^x	$e^x + C$
b^x $b > 0$	$\frac{b^x}{\ln b} + C$
$\frac{1}{x \ln b}$	$\log_b x + C$

As with differentiation, there are several rules for dealing with the sum and difference of integrable functions.

Basic Integration Rules

If f and g are integrable functions, and C is a constant, then:

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

,

$$\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$$

,

$$\int [Cf(x)]dx = C \int f(x)dx$$

Compute the following indefinite integral.

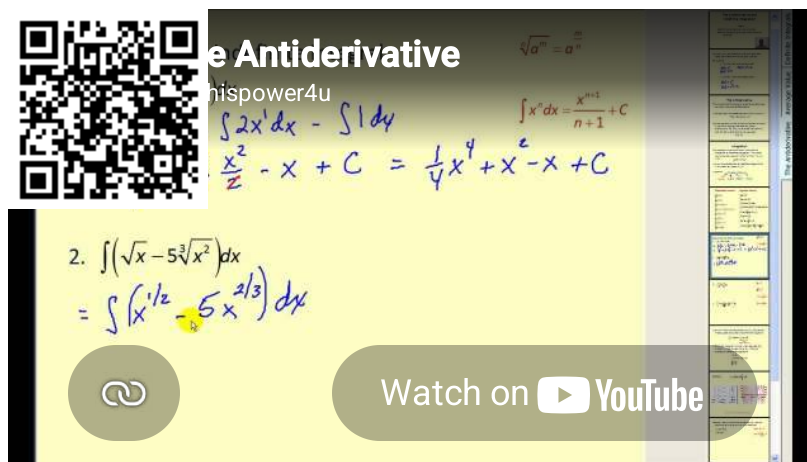
$$\int [2x^3 + 3x^2 - 1x]dx$$

Using our rules we have

$$\begin{aligned} \int [2x^3 + 3x^2 - 1x]dx &= 2 \int x^3 dx + 3 \int \frac{1}{x^2} dx - \int \frac{1}{x} dx \\ &= 2\left(\frac{x^4}{4}\right) + 3\left(\frac{x^{-1}}{-1}\right) - \ln x + C \\ &= \frac{x^4}{2} - \frac{3}{x} - \ln x + C \end{aligned}$$

.

Note that sometimes our rules need to be modified slightly due to operations with constants.



The video thumbnail shows a QR code on the left. The main content is handwritten math on a yellow background. At the top, it says 'Antiderivative' and 'thispower4u'. Below that, it shows the integral of $2x^3 dx - \int 1 dx$ and the result $\frac{x^4}{2} - x + C = \frac{1}{4}x^4 + x^e - x + C$. Another problem is shown: $2. \int (\sqrt{x} - 5\sqrt[3]{x^2}) dx$ and the result $= \int (x^{1/2} - 5x^{2/3}) dx$. At the bottom, there is a 'Watch on YouTube' button.

Examples

Example 1

Earlier, you were asked to try listing functions that are antiderivative and derivative pairs. By doing so you are presenting the results of differentiation and integration operations. If all you did was to list the function being differentiated as the antiderivative, this is correct. By now you have realized that there is a family of antiderivatives you could have chosen from, each differing by a constant of integration.

Example 2

Compute the following indefinite integral:

$$\int e^{3x} dx$$

We first note that our rule for integrating [exponential functions](#) does not work here since $\frac{d}{dx}e^{3x} = 3e^{3x}$. However, if we remember to divide the original function by the constant then we get the correct antiderivative and have

$$\int e^{3x} dx = \frac{e^{3x}}{3} + C$$

We can now re-state the rule in a more general form as

$$\int e^{kx} dx = \frac{e^{kx}}{k} + C$$

Review

For #1-6, find an antiderivative of the function

1. $f(x) = 1 - 3x^2 - 6x$
2. [Extra close brace or missing open brace](#)
3. $f(x) = (2x + 1)^{\frac{1}{5}}$
4. $f(x) = \cos x - x$
5. $f(x) = x^5 - 7x^2 + 2$
6. $f(x) = e^{-2x} + e^x$

For #7-12, find the indefinite integral

7. $\int (2 + \sqrt{5}) dx$
8. $\int 2(x - 3)^3 dx$
9. $\int (x^2 \cdot x^{\frac{1}{5}}) dx$
10. $\int (x + \frac{1}{x^4\sqrt{x}}) dx$
11. $\int (\cos x + 2\sin x) dx$
12. $\int 2\sin x \cos x dx$
13. Solve the differential equation $f'(x) = 4x^3 - 3x^2 + x - 3$.
14. Find the antiderivative $F(x)$ of the function $f(x) = 2e^{2x} + x - 2$ that satisfies $F(0) = 5$.
15. Evaluate the indefinite integral $\int |x| dx$ (Hint: Examine the graph of $f(x) = |x|$.)

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 5.1.

Vocabulary

Term	Definition
antiderivative	An antiderivative is a function that reverses a derivative. Function A is the antiderivative of function B if function B is the derivative of function A.
antidifferentiation	The process of finding antiderivatives is called antidifferentiation , more commonly referred to as integration.
constant of integration	The constant of integration is the constant C in the equation $\int f(x)dx = F(x) + C$ relating the function $f(x)$ and the antiderivative F.
integrand	An integrand is the argument $f(x)$ in the indefinite integral $\int f(x)dx$.
integration	The process of finding antiderivatives is sometimes called antidifferentiation, but more commonly referred to as integration.

Additional Resources

PLIX: Play, Learn, Interact, eXplore - [Antiderivative: Piecing it Together](#)

Practice: [Antiderivative](#)

Real World: [Rocky Mountain High](#)

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9.3: Area Sums

Finding the [area under the curve](#) of a function, $f(x)$, is one of the central problems that arises in Calculus. You already know how to compute the area of some simple function curve shapes using geometric formulas, and can sometimes quickly use these methods. But, what would you do if the function curves were more complex? The key approach of calculus is to take a simple method, apply it on a small scale, and then take it to the limit at infinity.

To gain some insight into the idea of finding the area under a curve, do the following before you go further: take a simple parabola curve (concave up and positive), divide an interval on the x-axis into a number of equal subintervals, and draw a rectangle in each subinterval that intersects the parabola. The summation of the [rectangles](#) would be an area estimate. How many different types of intersections can a rectangle make with the parabola? Can you say anything about how the intersection under or over estimates the area underneath the curve? Can you see that the more rectangles used for a given interval, the better the estimate?

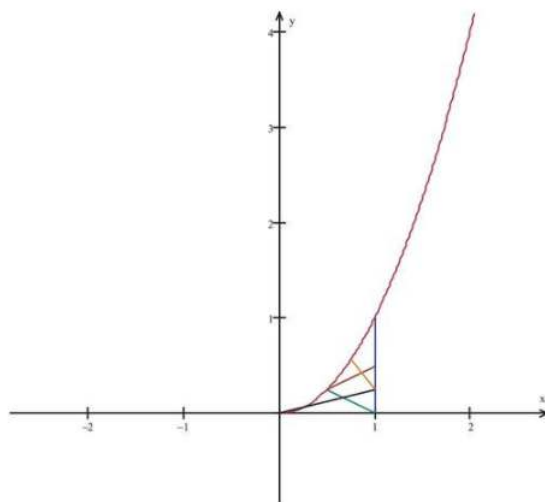
Riemann Sums

For simple functions like $f(x)=c(c>0)$, or $f(x)=cx$ (c is a constant >0) over some interval $[0, x_0]$, we can use simple geometry to get the area under each curve (try confirming this on your own):

1. For $f(x)=c(c>0)$; the area A is that of a simple rectangle, $A=cx_0$.
2. For $f(x)=cx(c>0)$; the area A is that of a simple triangle, $A = \frac{1}{2}x_0(cx_0) = c\frac{x_0^2}{2}$.

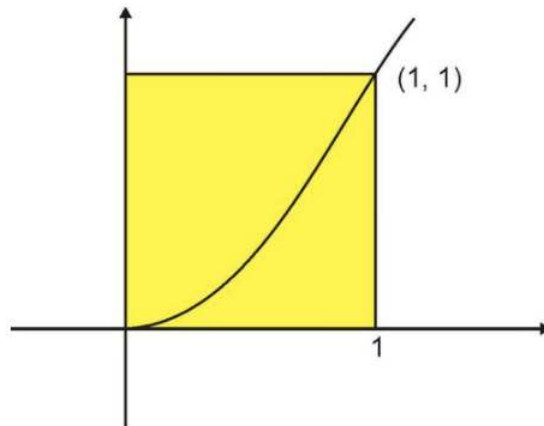
How can the area be determined for more complex functions?

Let's consider our basic quadratic function $f(x) = x^2$. Suppose we are interested in finding the area, A , under the curve from $x=0$ to $x=1$. The area we are talking about is the cross-hatched space under the curve shown below.



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One way to estimate the [area under the curve](#) is to compute the area of the square that has a corner at $(1, 1)$ as shown below, and then take half.



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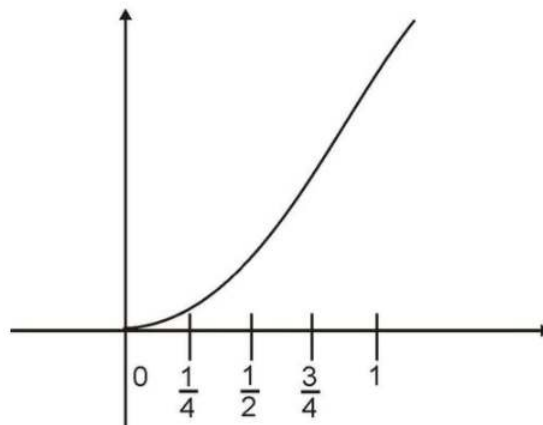
This gives an area $A = \frac{1}{2}$. This is a good first estimate of the area; the exact answer will be shown to be $A = \frac{1}{3}$.

How can we make a better estimate? How about dividing the x-interval from $x=0$ to $x=1$ into multiple sub-intervals of equal widths.

Let's look at problems each having more subintervals than the previous one.

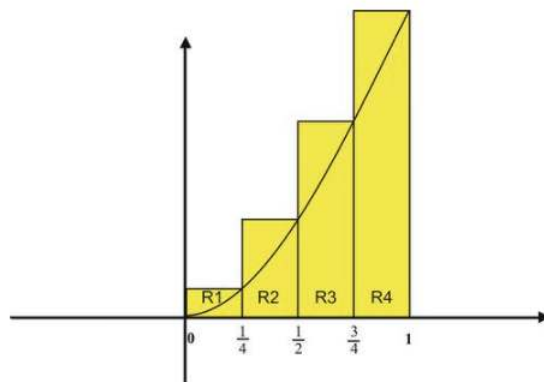
4 equal subintervals:

Start by using four such subintervals as indicated:



We now will construct four rectangles that will serve as the basis for our approximation of the area. The subintervals will serve as the width of the rectangles. We will take the length of each rectangle to be the maximum value of the function in the subinterval. (We could also choose the minimum or midpoint value of the function in the subinterval.)

Hence we get the following figure:



If we call the rectangles R1–R4, from left to right, then we have the areas

Finding the area under the curve of a function, $f(x)$, is one of the central problems that arises in Calculus. You already know how to compute the area of some simple function curve shapes using geometric formulas, and can sometimes quickly use these methods. But, what would you do if the function curves were more complex? The key approach of calculus is to take a simple method, apply it on a small scale, and then take it to the limit at infinity.

To gain some insight into the idea of finding the area under a curve, do the following before you go further: take a simple parabola curve (concave up and positive), divide an interval on the x-axis into a number of equal subintervals, and draw a rectangle in each subinterval that intersects the parabola. The summation of the rectangles would be an area estimate. How many different types of intersections can a rectangle make with the parabola? Can you say anything about how the intersection under or over estimates the area underneath the curve? Can you see that the more rectangles used for a given interval, the better the estimate?

8 Equal Sub-intervals:

If we divide the interval into 8 equal sub-intervals, the width of each rectangle is now $w=1/8$.

The area estimate now becomes:

$$A \approx R_1 + R_2 + R_3 + R_4 + R_5 + R_6 + R_7 + R_8$$

$$\approx \sum_{i=1}^8 R_i$$

Use of Sigma Notation

$$\approx \frac{1}{512} + \frac{4}{512} + \frac{9}{512} + \frac{16}{512} + \frac{25}{512} + \frac{36}{512} + \frac{49}{512} + \frac{64}{512}$$

$$\approx \frac{204}{512}$$

$$A \approx 0.3984$$

Note that the **Sigma Notation** ($\sum_{i=1}^8 R_i$) used above enables us to indicate the sum of the individual rectangle areas without the need to write out all of the individual terms. The sigma symbol with these indices tells us how the rectangles are labeled and how many terms are in the sum.

In addition, the summation of rectangles that gives the area approximation could be written in the following general form:

$$A \approx R = \sum_{i=1}^n R_i = \sum_{i=1}^n f(w_i)(x_i - x_{i-1}) = \sum_{i=1}^n f(w_i)\Delta x_i$$

where we choose the value of n to be as large as we want, w_i to be any value of x in the interval $(x_i - x_{i-1})$, and $(x_i - x_{i-1})$ to be the i -th of n subintervals (not necessarily equal) in the interval $[a, b]$ that f is defined on. This type of summation is called a **Riemann sum**.

16 equal sub-intervals:

If we now divide the interval into 16 equal sub-intervals, the width of each rectangle is now $w = \frac{1}{16}$.

The area estimate now becomes:

$$A \approx R_1 + R_2 + R_3 + R_4 + R_5 + \dots + R_{16}$$

$$\approx \sum_{i=1}^{16} R_i$$

Use of Sigma Notation

$$\approx \frac{1}{4096} + \frac{4}{4096} + \frac{9}{4096} + \dots + \frac{196}{4096} + \frac{225}{4096} + \frac{256}{4096}$$

$$\approx \frac{1496}{4096}$$

$$A \approx 0.3652$$

Again, the sigma notation has been used to eliminate the need to write out all of the individual terms.

The table gives a summary of the area estimate results so far.

Estimated Area under $f(x) = x^2$ in the interval $[0, 1]$

Sub-intervals	Area Estimate
1	1 (0.5)
4	0.4688
8	0.3984
16	0.3652

Intuitively, we have the feeling that we can improve the area approximation by sub-dividing the interval from $x=0$ to $x=1$ into more and more sub-intervals, thus creating successively smaller and smaller rectangles to refine our estimates.

We are now ready to formalize the use of sums to compute areas.

First, we define lower and upper sums:

Let f be a **bounded function** in a closed interval $[a,b]$ and $P = [x_0, \dots, x_n]$ the partition of $[a,b]$ into n subintervals.

Define the lower and upper sums, respectively, over partition P , by

$$S(P) = \sum_{i=1}^n m_i(x_i - x_{i-1}) = m_1(x_1 - x_0) + m_2(x_2 - x_1) + \dots + m_n(x_n - x_{n-1})$$

$$T(P) = \sum_{i=1}^n M_i(x_i - x_{i-1}) = M_1(x_1 - x_0) + M_2(x_2 - x_1) + \dots + M_n(x_n - x_{n-1})$$

where m_i is the minimum value of f in the interval of length $x_i - x_{i-1}$ and M_i is the maximum value of f in the interval of length $x_i - x_{i-1}$.

$S(P)$ and $T(P)$ are Riemann sums, where m_i and M_i are particular choices for w_i in the general [Riemann sum](#) formulation above.

Next, we will link area under a curve with the limit of the lower or upper sums for an increasingly large number of subintervals:

Let f be a continuous non-negative function on a closed interval $[a,b]$. Let P be a partition of n equal sub intervals over $[a,b]$. Then the area under the curve of f is the limit of the upper and lower sums, that is

$$A = \lim_{n \rightarrow +\infty} S(P) = \lim_{n \rightarrow +\infty} T(P)$$

The following problem shows how we can use these definitions to find the area under a curve.

Show that the limit of upper sum for the function $f(x) = x^2$, from $x = 0$ to $x = x_0$, approaches the value $A = \frac{x_0^3}{3}$.

Let P be a partition of n equal sub intervals over $[0, x_0]$. By definition we have

$$T(P) = \sum_{i=1}^n M_i(x_i - x_{i-1}) = M_1(x_1 - x_0) + M_2(x_2 - x_1) + \dots + M_n(x_n - x_{n-1})$$

Each rectangle will have width $\frac{x_0}{n}$, so that the evaluation of $T(P)$ is: $T(P) = \sum_{i=1}^n M_i(x_i - x_{i-1})$

$$= M_2(x_2 - x_1) + \dots + M_n(x_n - x_{n-1})$$

$$= \left(\frac{x_0}{n}\right)^2 \frac{x_0}{n} + \left(2\frac{x_0}{n}\right)^2 \frac{x_0}{n} + \dots + \left(n\frac{x_0}{n}\right)^2 \frac{x_0}{n} \text{ Equal length sub-intervals.}$$

$$= \left(\frac{x_0}{n}\right)^2 \frac{x_0}{n} (1^2 + 2^2 + 3^2 + \dots + n^2)$$

$$= \left(\frac{x_0}{n}\right)^3 (1^2 + 2^2 + 3^2 + \dots + n^2)$$

$$= \left(\frac{x_0}{n}\right)^3 \left(\frac{n(n+1)(2n+1)}{6}\right) \text{ Makes use of: } \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= \left(\frac{x_0^3}{6}\right) \left(\frac{n(n+1)(2n+1)}{n^3}\right)$$

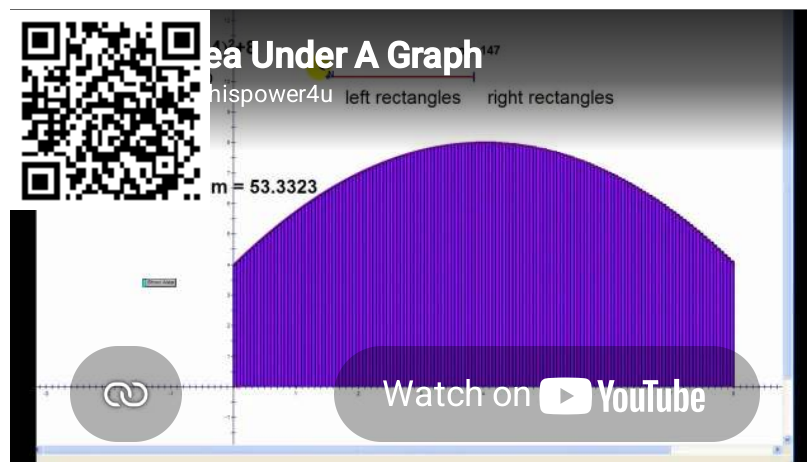
$$= \left(\frac{x_0^3}{6}\right) \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right)$$

Observe that:

$$\lim_{n \rightarrow \infty} \left[\left(\frac{x_0^3}{6}\right) \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right)\right] = \left(\frac{x_0^3}{6}\right) \cdot 2 = \frac{x_0^3}{3}$$

The area, A , under the curve of the function $f(x) = x^2$ from $x = 0$ to $x = x_0$, approaches the value $A = \frac{x_0^3}{3}$. Note that if $x_0=1$, we get the result $A=1/3$. Compare this value to the results from the previous problem.

The same approach to determine the area can be used with $S(P)$, and it yields the same result, $A = \frac{x_0^3}{3}$.



Examples

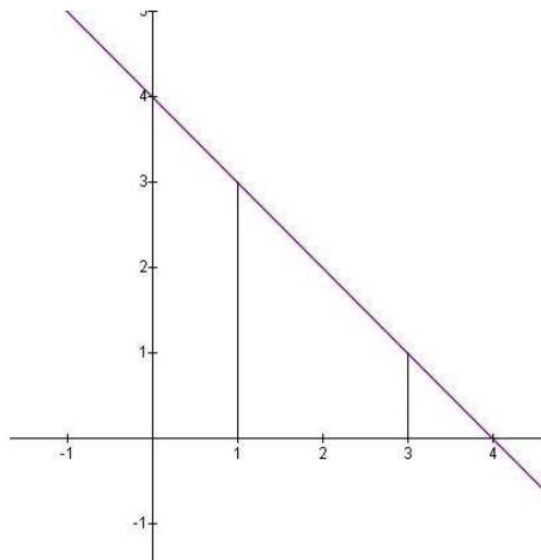
Example 1

Earlier, you were asked about using rectangles to estimate the area under a curve. How many different types of intersections can a rectangle make with the parabola? Can you say anything about how the intersection under or over estimates the area underneath the curve? Can you see that the more rectangles used (smaller subinterval) for a given interval, the better the area estimate?

Where your rectangle intersects the parabola establishes the height of the rectangle, and just depends on the value of x within the subinterval: left end of rectangle (beginning of sub-interval), right end of rectangle (end of subinterval), or anywhere in between. For this concave up parabola, you should be able to see that using a left-end rectangle underestimates the area under the parabola; a right-end rectangle overestimates the area. A rectangle with height somewhere in between, say at the mid-point, might overestimate a portion and underestimate a portion, i.e., provide a better area estimate.

Example 2

Use the limit definition of area to find the area under the function $f(x)=4-x$ from 1 to $x=3$.



If we partition the interval $[1,3]$ into n equal sub-intervals, then each sub-interval will have length $\frac{3-1}{n} = \frac{2}{n}$, and height $3-i\Delta x$ as i varies from 1 to n . So we have $\Delta x = \frac{2}{n}$ and

$$S(P) = \sum_{i=1}^n m_i(x_i - x_{i-1}) \text{ Lower sum using minimum function value in each subinterval.}$$

$$= \sum_{i=1}^n (3 - i\frac{2}{n}) \frac{2}{n}$$

$S(P) = \sum_{i=1}^n m_i(x_i - x_{i-1})$ Sub-intervals of equal length.

$$= \sum_{i=1}^n (3 \cdot \frac{2}{n}) - \sum_{i=1}^n i(\frac{2}{n})^2$$

$$= 6 - (\frac{2}{n})^2 \sum_{i=1}^n i$$

$$= 6 - (\frac{2}{n})^2 \cdot (\frac{n(n+1)}{2}) \text{ Makes use of:}$$

$$= 6 - 2 \cdot (\frac{n(n+1)}{n^2}) \sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$S(P) = 6 - 2 \cdot (1 + \frac{1}{n})$$

Observe that:

$$\lim_{n \rightarrow \infty} [6 - 2 \cdot (1 + \frac{1}{n})] = 6 - 2 = 4$$

The area, A, under the curve of the function approaches the value A=4. The same approach can be used with S(P), yielding the same result.

Of course this example may also be solved with simple geometry. It is left to the reader to confirm that the two methods yield the same area.

Review

For #1-3, find S(P) and T(P) under the partition P.

1. $f(x) = 1 - x^2$, $P = 0, 12, 1, 32, 2$
2. $f(x) = 2x^2$, $P = -1, -12, 0, 12, 1$
3. $f(x) = \frac{1}{x}$, $P = -4, -3, -2, -1$
4. Consider $f(x)=2-x$ from $x=0$ to $x=2$. Use Riemann Sums with four subintervals of equal lengths. Choose the midpoints of each subinterval as the sample points.
5. Repeat problem #1 using geometry to calculate the exact area of the region under the graph of $f(x)=2-x$ from $x=0$ to $x=2$. (Hint: Sketch a graph of the region and see if you can compute its area using area measurement formulas from geometry.)
6. Repeat problem #4 using the definition of the definite integral to calculate the exact area of the region under the graph of $f(x)=2-x$ from $x=0$ to $x=2$.
7. $f(x) = x^2 - x$ from $x=1$ to $x=4$. Use Riemann Sums with five subintervals of equal lengths. Choose the left endpoint of each subinterval as the sample points.
8. Repeat problem #7 using the definition of the definite integral to calculate the exact area of the region under the graph of $f(x) = x^2 - x$ from $x=1$ to $x=4$.
9. Consider $f(x) = 3x^2$. Compute the [Riemann Sum](#) of f on $[0, 1]$ under each of the following situations. In each case, use the right endpoint as the sample points.
 1. Two sub-intervals of equal length.
 2. Five sub-intervals of equal length.
 3. Ten sub-intervals of equal length.
 4. Based on your answers above, try to guess the exact area under the graph of f on $[0, 1]$.
10. Consider $f(x) = e^x$. Compute the [Riemann Sum](#) of f on $[0, 1]$ under each of the following situations. In each case, use the right endpoint as the sample points.
 1. Two sub-intervals of equal length.
 2. Five sub-intervals of equal length.
 3. Ten sub-intervals of equal length.
 4. Based on your answers above, try to guess the exact area under the graph of f on $[0, 1]$.
11. Find the net area under the graph of $f(x) = x^3 - x$; $x=-1$ to $x=1$. (Hint: Sketch the graph and check for [symmetry](#).)
12. Find the total area bounded by the graph of $f(x) = x^3 - x$ and the x -axis, from $x=-1$ to $x=1$.
13. Use your knowledge of geometry to evaluate the definite integral: $\int_0^3 \sqrt{9-x^2} dx$

(Hint: set $y = \sqrt{9 - x^2}$ and square both sides to see if you can recognize the region from geometry.)

For #14-15, find the area under the curve using the limit definition of area.

14. $f(x) = 3x + 5$ from $x = 2$ to $x = 6$.

15. $f(x) = x^2$ from $x = 1$ to $x = 3$.

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 5.4.

Vocabulary

Term	Definition
bounded function	A bounded function is a function $f(x)$ for which $ f(x) \leq M$ (M a real number) for all values of x in the domain.
lower sum	A lower sum is a Riemann sum that chooses the minimum value of the function on every subinterval in the partition of a closed interval.
partition of a closed interval	A partition of a closed interval is a decomposition of the entire interval into a series of contiguous subintervals.
sigma notation	Sigma notation is also known as summation notation and is a way to represent a sum of numbers. It is especially useful when the numbers have a specific pattern or would take too long to write out without abbreviation.
upper sum	An upper sum is a Riemann sum that chooses the maximum value of the function on every subinterval in the partition of a closed interval.

Additional Resources

PLIX - Play, Learn, Interact, eXplore: [Area Sums: Estimation with Rectangles](#)

Video: [Trapezoid Approximation of Area Under Curve](#)

Practice: [Area Sums](#)

Real World: [Phoning It In](#)

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9.4: Probability and Probability Density Functions

Probability is a concept that is a familiar part of our lives. We use probability as the measure that some event (in a set of possible events) will actually occur. The set of possible events might be finite and therefore be termed a discrete random variable, or infinite and constitute a continuous random variable. A probability, P , is a number such that $0 \leq P \leq 1$. The closer P is to 0, the more unlikely it is that the event will occur; the closer P is to 1, the more likely it is that the event will occur. Also, the sum of all probabilities covering all possible events in the set, must add to 1. Can you determine before we start which type of random variable (discrete or continuous) would be associated with probabilities computed using definite integrals? Why?

Probability Density Functions

In this section, we will look at how to compute the value of a probability by using a function called a **probability density function (pdf)**. There are many different forms of probability density functions, and we will look at a few.

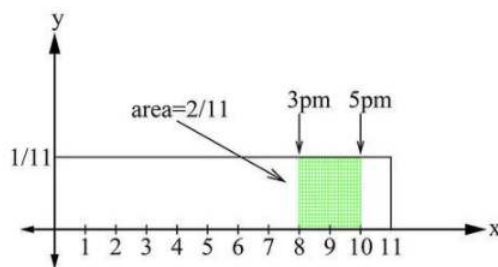
If you were told by the postal service that you will receive a package that you have been waiting for sometime tomorrow, you might ask: What is the probability that I will receive my package sometime between 3:00 PM and 5:00 PM given that the postal service's hours of operations are between 7:00 AM to 6:00 PM?

In the absence of any more information, one way to find a solution is to note that since the post office operates for a total of 11 hours (7 AM to 6 PM), and the interval of interest is the 2 hours between 3 PM and 5 PM, the probability that your package will arrive might just be

$$P = \frac{2 \text{ hours}}{11 \text{ hours}} = 0.182$$

So there is a probability of 0.182 that the postal service will deliver your package sometime between the hours of 3 PM and 5 PM. Because there was nothing special about the 3 pm to 5 pm interval, the 0.182 probability could apply to any 2-hour interval during the 11-hour period of operation. This also means that there is a probability of 0.818 that the package will not be delivered during that interval (i.e., $1 - 0.182$).

But, mathematically, how does the above calculation standup when the 11-hour interval and the 2-hour interval contain an infinite number of times? How can one infinity divided by another infinity produce a probability of 0.182? (Note: The possible delivery times in the 11-hour interval represent a continuous random variable.) To resolve this issue, we can represent the probability of delivery of the package at *any* time in the 11-hour interval as being defined by a rectangle of height $\frac{1}{11}$ and length 11, with resulting area equal to 1. Looking at the 2-hour interval, we can see that it is equal to $\frac{2}{11}$ of the total rectangular area of 1. This is why it is therefore convenient to represent probabilities as areas.



Since areas can be defined by definite integrals, we can also define the probability of an event occurring within an interval $[a, b]$ by the definite integral $P(a \leq x \leq b) = \int_a^b f(x) dx$ where $f(x)$ is called the probability density function (pdf).

A function $f(x)$ is called a **probability density function** if

1. $f(x) \geq 0$ for all x
2. The area under the graph of $f(x)$ over all the real line is exactly 1
3. The probability that x is in the interval $[a, b]$ is

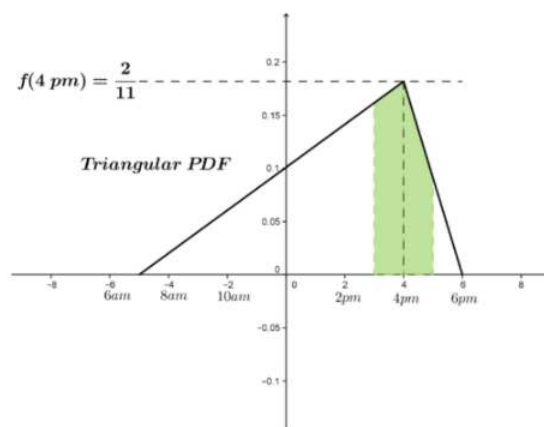
$$P(a \leq x \leq b) = \int_a^b f(x) dx$$

i.e., the area under the graph of $f(x)$ from a to b .

In the problem above, the probability density function $f(x)$ is called a **uniform (flat) probability density function (pdf)**.

Many other probability density functions exist that when used would give a different answer to the question about the delivery time of our package.

Suppose that after some thought and discussion with neighbors, you decide that it would be better to use a **triangular probability density function** as shown below for package delivery in your area. What is the probability that you will receive your package sometime between 3:00 PM and 5:00 PM?



The triangular pdf shows a variation that can be modeled as:

$$f(x) = \begin{cases} \frac{2}{99}(x+12) - \frac{14}{99}, & -5 \leq x \leq 4 \\ -\frac{1}{11}(x+12) + \frac{18}{11}, & 4 \leq x \leq 6 \end{cases}$$

Note that the pdf is 0 outside the above interval.

The probability that the package will arrive between 3 and 5 pm can be determined as

$$P(a \leq x \leq b) = \int_a^b f(x) dx$$

$$P(3 \leq x \leq 5) = \int_3^5 f(x) dx$$

$$= \int_3^4 \left(\frac{2}{99}(x+12) - \frac{14}{99} \right) dx + \int_4^5 \left(-\frac{1}{11}(x+12) + \frac{18}{11} \right) dx$$

$$= \frac{2}{99} \left[\frac{x^2}{2} + 5x \right]_3^4 - \frac{1}{11} \left[\frac{x^2}{2} - 6x \right]_4^5$$

$$= \frac{17}{99} + \frac{15}{11}$$

$$P(3 \leq x \leq 5) = \frac{30.5}{99} \approx 0.308$$

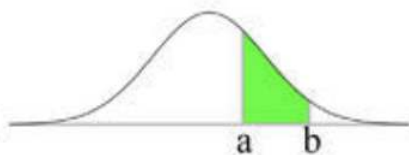
Using the triangular pdf, the probability of receiving the package between 3 and 5 pm has increased to 0.308.

The Normal Probability Density Function

One of the most useful probability density functions is the **normal** or **Gaussian probability density function** (sometimes referred to as the **bell curve**) defined as:

The **Gaussian curve** for a population with **mean** μ and standard deviation σ is given by $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, where the factor $\frac{1}{\sigma\sqrt{2\pi}}$ is called the **normalization constant**, which is needed to make the probability over the entire space equal to 1.

The density function has the bell curve shape represented below:



This function enables us to describe an entire population based on statistical measurements taken from a small sample of the population. The only measurements needed are the mean (μ) and the standard deviation (σ). Once those two numbers are known, the normal curve is defined.

Suppose that boxes containing 100 tea bags have a mean weight of 10.2 ounces each and a standard deviation of 0.1 ounce. What percentage of all the boxes is expected to weigh between 10 and 10.5 ounces? What is the probability that a box weighs less than 10 ounces? What is the probability that a box will weigh exactly 10 ounces?

Using the normal probability density function, $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$.

Substituting for $\mu=10.2$ and $\sigma=0.1$, we get

$$f(x) = \frac{1}{(0.1)\sqrt{2\pi}} e^{\frac{-(x-10.2)^2}{2(0.1)^2}}$$

The percentage of all the tea boxes that are expected to weight between 10 and 10.5 ounces can be calculated as

$$P(10 \leq x \leq 10.5) = \int_{10}^{10.5} \frac{1}{(0.1)\sqrt{2\pi}} e^{\frac{-(x-10.2)^2}{2(0.1)^2}} dx.$$

The integral of e^{x^2} does not have an elementary anti-derivative and therefore cannot be evaluated by standard techniques. However, we can use numerical techniques, such as The Simpson's Rule or The Trapezoid Rule, to find an approximate (but very accurate) value. Using the programming feature of a scientific calculator or, mathematical software, we eventually get

$$\int_{10}^{10.5} \frac{1}{(0.1)\sqrt{2\pi}} e^{\frac{-(x-10.2)^2}{2(0.1)^2}} dx \approx 0.976 \text{ That is, } P(10 \leq x \leq 10.5) \approx 0.976.$$

Technology Note: To make this computation with a graphing calculator of the TI-83/84 family, do the following:

From the **[DISTR]** menu (Figure 6.11.4) select option 2, which puts the phrase “normalcdf” in the home screen. Add lower bound, upper bound, mean, standard deviation, separated by commas, close the parentheses, and press **[ENTER]**. The result is shown in Figure 6.11.5.

```

0913 DRAW
1:normalpdf(
2:normalcdf(
3:invNorm(
4:invT(
5:tpdf(
6:tcdf(
7:χ²pdf(
    
```

```

normalcdf(10,10.
5,10.2,.1)
.9758999708
    
```

For the probability that a box weighs less than 10.2 ounces, we use the [area under the curve](#) to the left of $x=10.2$.

Integrating numerically, we get

$$P(9 \leq x \leq 10) = \int_9^{10} \frac{1}{(0.1)\sqrt{2\pi}} e^{-\frac{(x-10.2)^2}{2(0.1)^2}} dx$$

$$P(9 \leq x \leq 10.2) \approx 0.02275$$

$$= 2.28 \quad (9.4.1)$$

which says that we would expect 2.28% of the boxes to weigh less than 10 ounces.

Theoretically the probability here will be exactly zero because we will be integrating from 10 to 10 which is zero. However, since all scales have some error (call it ϵ), practically we would find the probability that the weight falls between $10-\epsilon$ and $10+\epsilon$.

Examples

Example 1

Earlier, you were asked to determine which type of random variable (discrete or continuous) would be associated with probabilities computed using definite integrals.

If you said that the continuous random variable would be associated with probabilities determined by definite integrals, you would be correct.

Example 2

An Intelligence Quotient or IQ is a score derived from different standardized tests attempting to measure the level of intelligence of an adult human being. The average score of the test is 100 and the standard deviation is 15. What is the percentage of the population that has a score between 85 and 115? What percentage of the population has a score above 140?

Using the normal probability density function, $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$,

and substituting $\mu = 100$ and $\sigma = 15$, $f(x) = \frac{1}{15\sqrt{2\pi}} e^{-\frac{(x-100)^2}{2(15)^2}}$.

The percentage of the population that has a score between 85 and 115 is $P(85 \leq x \leq 115) = \int_{85}^{115} \frac{1}{15\sqrt{2\pi}} e^{-\frac{(x-100)^2}{2(15)^2}} dx$.

Again, the integral of e^{-x^2} does not have an elementary anti-derivative and therefore cannot be evaluated. Using the programming feature of a scientific calculator or a mathematical computer software, we get $\int_{85}^{115} \frac{1}{15\sqrt{2\pi}} e^{-\frac{(x-100)^2}{2(15)^2}} dx \approx 0.68$.

That is, $P(85 \leq x \leq 115) \approx 68$

Which says that 68% of the population has an IQ score between 85 and 115.

To measure the probability that a person selected randomly will have an IQ score above 140, $P(x \geq 140) = \int_{140}^{\infty} \frac{1}{15\sqrt{2\pi}} e^{-\frac{(x-100)^2}{2(15)^2}} dx$

This integral is even more difficult to integrate since it is an improper integral. To avoid the messy work, we can argue that since it is extremely rare to meet someone with an IQ score of over 200 we can approximate the integral from 140 to 200 Then

$$P(x \geq 140) = \int_{140}^{200} \frac{1}{15\sqrt{2\pi}} e^{-\frac{(x-100)^2}{2(15)^2}} dx$$

Integrating numerically, we get $P(x \geq 140) \approx 0.0039$.

So the probability of selecting at random a person with an IQ score above 140 is 0.0039. That's about one person in every 250 individuals!

Review

For #1-4, find the number r that makes the function a probability density function over the given sample space.

1. $f(x) = r$ over the sample space of $[-7, 7]$.
2. $f(x) = rx(x - 5)$ over the sample space of $[0, 5]$.
3. $f(x) = \frac{r}{(1+x)^2}$ over the sample space of $[0, 25]$.
4. $f(x) = r \sin x$ over the sample space of $[0, \frac{\pi}{2}]$.
5. Given a triangular probability density function $g(x)$ over the sample space $[0, 10]$, what is the highest value of $g(x)$ if it occurs at $x=7$?
6. Could the function $g(x) = \frac{-4}{25}x + 0.9$ be a probability density function over the interval $[0, 10]$?
7. Suppose $f(x)$ is the probability density function for the lifetime of a manufacturer's light bulb, where x is measured in hours. Explain the meaning of each integral.
 1. $\int_1 000^5 000 f(x) dx$
 2. $\int_3 000^\infty f(x) dx$
8. Suppose $f(x) = -\frac{1}{36}(x^2 - 9)$ is the probability density function for an event's time of occurrence within the time window $[-3, +3]$ (0 is on time occurrence). What is the probability the event occurs within ± 1 hour of the expected time?
9. For problem #8, what time window guarantees a 90% probability of occurrence?
10. The length of time a customer spends waiting until his/her entree is served at a certain restaurant is modeled by an exponential density function $f(x) = 0.125e^{-0.125x}$, where $\frac{1}{0.125} = 8$ is the average wait time in minutes.
 1. What is the probability that a customer is served in the first 3 minutes?
 2. What is the probability that a customer has to wait more than 10 minutes?
11. The average height of an adult female in Los Angeles is 63.4 inches (5 feet, 3.4 inches) with a standard deviation of 3.2 inches.
 1. What is the probability that a female's height is less than 63.4 inches?
 2. What is the probability that a female's height is between 63 and 65 inches?
 3. What is the probability that a female's height is more than 6 feet?
 4. What is the probability that a female's height is exactly 5 feet?
 5. The **median** of a distribution with a probability density function $f(x)$, is the value M such that $\int_{-\infty}^M f(x) dx = 0.5$. Half the values of the distribution will be above M , and half will be below M . Find the **median** value for each of the following probability density functions:
12. $f(x) = -\frac{1}{36}(x^2 - 9)$ over the interval $[-3, 3]$.
13. $f(x) = 0.125e^{-0.125x}$ over the interval $[0, \infty)$.
 1. The **mean** of a distribution with the probability density function $f(x)$, is the value given by $\int_{-\infty}^{\infty} xf(x) dx$. Find the mean value for each of the following probability density functions:
14. $f(x) = -\frac{1}{36}(x^2 - 9)$ over the interval $[-3, 3]$.
15. $f(x) = 0.125e^{-0.125x}$ over the interval $[0, \infty)$.

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 6.10.

Vocabulary

Term	Definition
Gaussian probability density function	A Normal (Gaussian) pdf is a continuous pdf defined by $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ where μ is the mean, and σ is the standard deviation.
mean	The mean of a distribution with the probability density function $f(x)$ is the value given by $\int_{-\infty}^{\infty} xf(x)dx$.
median	The median of a distribution with a probability density function $f(x)$ is the value M such that $\int_{-\infty}^M f(x)dx = 0.5$. Half the values of the distribution will be above M , and half will be below M .
normal probability density function	A Normal (Gaussian) pdf is a continuous pdf defined by $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ where μ is the mean, and σ is the standard deviation.
pdf	A function $f(x)$ is called a probability density function (pdf) if $f(x) \geq 0$ for all x , the area under the graph of $f(x)$ over all real numbers is exactly 1, and the probability that x is in the interval $[a, b]$ is $P(a \leq x \leq b) = \int_a^b f(x)dx$.
Probability	Probability is the chance that something will happen. It can be written as a fraction, decimal or percent.
probability density function	A function $f(x)$ is called a probability density function if $f(x) \geq 0$ for all x , the area under the graph of $f(x)$ over all real numbers is exactly 1, and the probability that x is in the interval $[a, b]$ is $P(a \leq x \leq b) = \int_a^b f(x)dx$.
triangular probability density function	A triangular probability density function (pdf) is a continuous pdf whose triangular shape is defined by three values of x : a lower value, a mode value with highest distribution value, and an upper value.
uniform probability density function	A uniform (or rectangular, flat) probability density function (pdf) $P(x)$ is a continuous distribution where every equal interval of x has the same probability.

Additional Resources

Video: [More on Probability Density Functions](#)

Real World: [Preventable Deaths](#)

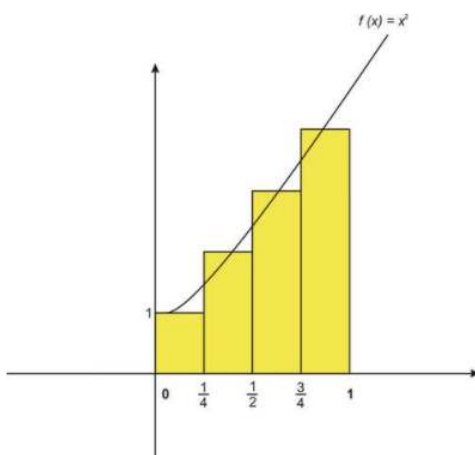
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9.5: Trapezoidal and Midpoint Approximations

Recall that we have used different ways to approximate the value of integrals. These included Riemann Sums using left and right endpoints, as well as midpoints for finding the length of each rectangular tile. In this lesson the rectangular tiles are replaced by trapezoidal tiles to approximate the integral. Do you think that the use of **trapezoids** would give a more accurate result than the use of **rectangles**? Why? Can you think of a reason why the concavity of the function curve would matter in the accuracy of the area estimation using trapezoids?

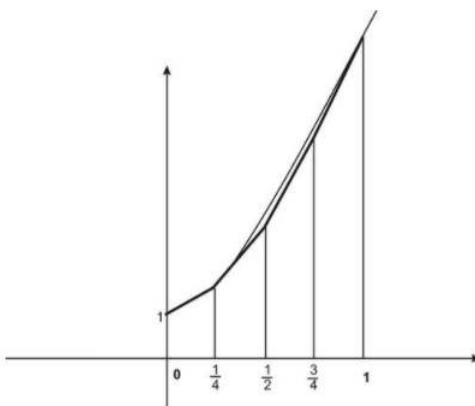
Recall that we used different ways to approximate the value of integrals. These included Riemann Sums using left and right endpoints, as well as midpoints for finding the length of each rectangular tile. In this lesson we will learn two other methods for approximating integrals. The first of these, the **Trapezoidal Rule**, uses areas of trapezoidal tiles to approximate the integral. The second method, Simpson's Rule, uses **parabolas** to make the approximation.

Let's recall how we would use the midpoint rule with $n=4$ **rectangles** to approximate the area under the graph of $f(x) = x^2 + 1$ from $x=0$ to $x=1$.



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If instead of using the midpoint value within each sub-interval to find the length of the corresponding rectangle, we could have instead formed **trapezoids** by joining the maximum and minimum values of the function within each sub-interval:



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The area of a trapezoid is $A = \frac{h(b_1+b_2)}{2}$, where b_1 and b_2 are the lengths of the parallel sides and h is the height. In our **trapezoids** the height is Δx and b_1 and b_2 are the values of the function. Therefore in finding the areas of the trapezoids we actually average the left and right endpoints of each sub-interval.

Therefore a typical trapezoid would have the area

$$A = \frac{\Delta x}{2} (f(x_{i-1}) + f(x_i))$$

To approximate $\int_a^b f(x) dx$ with n of these trapezoids, we have

$$\begin{aligned} \int_a^b f(x) dx &\approx \frac{1}{2} \left[\sum_{i=1}^n f(x_{i-1}) \Delta x + \sum_{i=1}^n f(x_i) \Delta x \right] \\ &= \frac{\Delta x}{2} [f(x_0) + f(x_1) + f(x_1) + f(x_2) + f(x_2) + \dots + f(x_{n-1}) + f(x_n)] \\ &= \frac{\Delta x}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n)], \Delta x = \frac{b-a}{n} \end{aligned}$$

Apply the formula from above and use the Trapezoidal Rule to approximate $\int_0^3 x^2 dx$ with $n=6$.

Each subinterval is $\Delta x = \frac{b-a}{n} = \frac{3-0}{6} = \frac{1}{2}$.

The integral approximation is:

$$\begin{aligned} \int_0^3 x^2 dx &\approx \frac{1}{4} [f(0) + 2f(\frac{1}{2}) + 2f(1) + 2f(\frac{3}{2}) + 2f(2) + 2f(\frac{5}{2}) + f(3)] \\ &= \frac{1}{4} [0 + (2 \cdot \frac{1}{4}) + (2 \cdot 1) + (2 \cdot \frac{9}{4}) + (2 \cdot 4) + (2 \cdot \frac{25}{4}) + 9] \\ &= \frac{1}{4} [\frac{73}{2}] = \frac{73}{8} = 9.125. \end{aligned}$$

An exact solution can be determined by using the [antiderivative](#) with the Fundamental Theorem.

$$\int_0^3 x^2 dx = \frac{x^3}{3} \Big|_0^3 = 9.$$

The error is about 1.4%.

For some integrals it is impossible to find an [antiderivative](#) and a numerical method is the only option.

When this is the case, accuracy of the approximation is an issue. In particular, how large should n be so that the trapezoidal estimate is accurate to within a given value, say 0.001?

The magnitude of the error in using the trapezoidal technique for a given value of n can be shown to be:

$$|Error_{Trapezoidal}| \leq \frac{k(b-a)^3}{12n^2}, \text{ where } |f''(x)| \leq k \text{ for } a \leq x \leq b$$

This means that choosing $n \geq \sqrt{\frac{k(b-a)^3}{12 \cdot |Error_{Trapezoidal}|}}$ will meet or exceed a specified error $Error_{Trapezoidal}$.

Find n so that the Trapezoidal Estimate for $\int_0^3 x^2 dx$ is accurate to 0.001.

We need to find n such that $|Error_{Trapezoidal}| \leq 0.001$.

First note that $|f''(x)| = 2$ for $0 \leq x \leq 3$. Hence we can take $k=2$ to find our error bound.

$$|Error_{Trapezoidal}| \leq \frac{2(3-0)^3}{12n^2} = \frac{54}{12n^2}$$

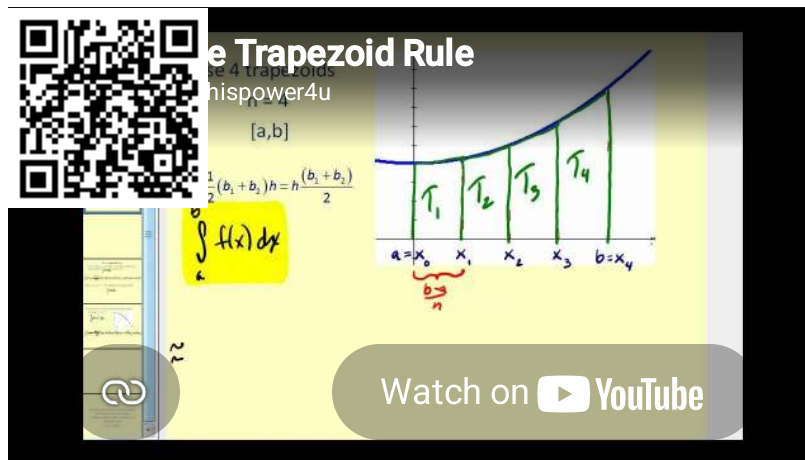
We need to solve the following inequality for n :

$$\frac{54}{12n^2} < 0.001,$$

$$n^2 > \frac{54}{12(0.001)},$$

$$n > \sqrt{\frac{54}{12(0.001)}} \approx 67.08$$

Hence we must take $n=68$ to achieve the desired accuracy.



Examples

Example 1

Earlier, you were asked if you think that using trapezoids to approximate the area under a curve would give a more accurate result than using [rectangles](#). Can you think of a reason why the concavity of the function curve would matter in the accuracy of the area estimation using trapezoids?

If you think that the use of trapezoids would give a more accurate result (with number of subintervals the same) you are correct. With some examples, you can see visually that the trapezoids cover the [area under the curve](#) better. Notice, though, that if the function curve is concave up, the trapezoids will overestimate the area; if the concavity is down, the trapezoids will underestimate the area.

Example 2

Use the Trapezoidal Rule to approximate $\int_0^2 x^3 dx$ for $n=4$. Round the answer to four decimal places and compare this value to the exact value of the integral. What is the expected error for $n=4$?

We find each subinterval as $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$.

The integral approximation is evaluated as follows:

$$\int_0^2 x^3 dx \approx \frac{1}{4} [f(0) + 2f(0.5) + 2f(1) + 2f(1.5) + f(2)]$$

$$\approx \frac{1}{4} [0 + 2 \cdot 0.125 + 2 \cdot 1 + 2 \cdot 3.375 + 8]$$

$$\int_0^2 x^3 dx \approx 4.25$$

The estimate of the integral is 4.25.

The exact value of the integral: $\int_0^2 x^3 dx = \frac{x^4}{4} \Big|_0^2 = 4$.

The error between these is 0.25.

The expected error using $n=4$:

For $f''(x)=6x$, $f''(x) \leq 6 \cdot 2 = 12$ in the interval $[0, 2]$.

Therefore, $|Error_{Trapezoidal}| \leq \frac{k(b-a)^3}{12n^2} = \frac{12(2)^3}{12 \cdot 4^2} = 0.5$.

The actual error is less than $|Error_{Trapezoidal}|$.

Example 3

Use the Trapezoidal Rule to approximate $\int_1^2 \frac{1}{(x+1)^2} dx$ for $n=4$. Round the answer to four decimal places and compare this value to the exact value of the integral. What is the expected error for $n=4$?

We find each subinterval as $\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$.

The integral approximation is evaluated as follows:

$$\begin{aligned} \int_1^2 \frac{1}{(x+1)^2} dx &\approx \frac{1}{8} [f(2) + 2f(2.25) + 2f(2.5) + 2f(2.75) + f(3)] \\ &\approx \frac{1}{8} \left[\frac{1}{4} + 2 \frac{1}{2.25^2} + 2 \frac{1}{2.5^2} + 2 \frac{1}{2.75^2} + \frac{1}{3^2} \right] \\ &\approx \frac{1}{8} [0.25 + 0.39506 + 0.32 + 0.26446 + 0.11111] \\ &\int_1^2 \frac{1}{(x+1)^2} dx \approx 0.16758 \approx 0.1676 \end{aligned}$$

The estimate of the integral is 0.1676.

The exact value of the integral:

$$\int 121(x+1)2dx = -1x + 1 \Big|_1^2 = 1 - 0 = 1$$

The error between these is 0.0009.

The expected error using $n=4$:

For $f''(x) = \frac{6}{(x+1)^3}$, $f''(x) \leq \frac{6}{(1+1)^3} = \frac{3}{8} = 0.375$ in the interval $[1, 2]$.

Therefore, $|Error_{Trapezoidal}| \leq \frac{k(b-a)^3}{12n^2} = \frac{0.375(1)^3}{12 \cdot 4^2} = 0.001953$.

The actual error is less than $|Error_{Trapezoidal}|$.

Review

Use the Trapezoidal Rule to approximate the definite integrals using the given number of subintervals n .

1. $\int_1^7 (x+7) dx$ with $n=6$.
2. $\int_{-2}^2 (x+4) dx$ with $n=4$.
3. $\int_{-4}^1 (-x^2 - 2x + 8) dx$ with $n=5$.
4. $\int_2^7 \frac{2}{x} dx$ with $n=5$.
5. $\int_0^2 x^4 dx$ with $n=4$.
6. $\int_0^1 \sin \frac{x}{2} dx$ with $n=4$.
7. $\int_2^4 \sqrt{x} dx$ with $n=5$.

8. $\int_0^1 x^2 e^{-x} dx$ with $n = 8$.
9. $\int_1^4 \ln \sqrt{x} dx$ with $n = 6$.
10. $\int_0^1 \sqrt{1+x^4} dx$ with $n = 4$.
11. $\int_1^3 \frac{1}{x} dx$ with $n = 8$.
12. $\int_0^2 x^3 dx$ for $n = 8$.
13. Find a value of n that guarantees an error of no more than 10^{-5} in the Trapezoidal approximation of $\int_2^4 \sqrt{x} dx$. How large should n be so that the Trapezoidal Estimate for $\int_1^3 \frac{1}{x} dx$ is accurate to within:
 1. 0.001?
 2. 0.00001?

Review (Answers)

To see the Review answers, open this [PDF file](#) and look for section 5.10.

Vocabulary

Term	Definition
trapezoidal rule	The trapezoidal rule is a method for computing a definite integral by computing the area trapezoidal segments in the integration interval and summing them.

Additional Resources

PLIX: Play, Learn, Interact, eXplore - [Area of a Skirt](#)

Video: [Trapezoid Approximation of Area Under Curve by Khan Academy](#)

Practice: [Trapezoidal and Midpoint Approximations](#)

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9.6: Reimann Sums

Approximating the area under a function curve by summing a finite number of [rectangles](#) in a [Riemann Sum](#) can yield very accurate results. Intuitively, we know, however, that the more sub-intervals we have the better the result. Taking the limit of the Riemann Sum as the subintervals get smaller (number of rectangles gets larger) should asymptotically give the true area. For some function curves, the Riemann limit can be evaluated algebraically; for complex curves, the area can only be determined by brute force [numerical computations](#) of Riemann Sums.

Limits and Reimann Sums

Earlier, the area under a curve was defined in terms of a limit of sums:

$$A = \lim_{n \rightarrow +\infty} S(P) = \lim_{n \rightarrow +\infty} T(P)$$

where

$$S(P) = \sum_{i=1}^n m_i(x_i - x_{i-1}) = m_1(x_1 - x_0) + m_2(x_2 - x_1) + \dots + m_n(x_n - x_{n-1}),$$

$$T(P) = \sum_{i=1}^n M_i(x_i - x_{i-1}) = M_1(x_1 - x_0) + M_2(x_2 - x_1) + \dots + M_n(x_n - x_{n-1}),$$

$S(P)$ and $T(P)$ are examples of **Riemann Sums**.

In general, Riemann Sums are of form $\sum_{i=1}^n f(x_i^*) \Delta x$ where each x_i^* is the value we use to find the length of the rectangle in the i th sub-interval. For example, the maximum function value in each sub-interval to find the upper sums and the minimum function in each sub-interval to find the lower sums. But since the function is continuous, we could have used any points within the sub-intervals to find the limit.

To make use of the [concept of limit](#), we make the width of each rectangle approach 0, which is equivalent to making the number of [rectangles](#), n , approach infinity. By doing so, we find the exact [area under the curve](#),

$$\lim_{n \rightarrow \infty} A_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x.$$

We now define the most general situation as follows:

If f is continuous on $[a, b]$, and:

1. The interval $[a, b]$ is divided into n sub-intervals of equal width Δx , with $\Delta x = \frac{b-a}{n}$, and
2. The endpoints of these sub-intervals are $x_0=a, x_1, x_2, \dots, x_n=b$, and
3. $x_1^*, x_2^*, \dots, x_n^*$ are *any* sample points in these sub-intervals, then the **definite integral** of f from $x=a$ to $x=b$ is

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x.$$

provided the limit exists.

If the above limit exists, f is said to be integrable on the closed interval $[a, b]$ and the definite integral exists.

Note that the sample point x_i^* can be any sample point in the i -th sub-interval, with common choices being right, or mid-point, or left.

For example, evaluate the [Riemann Sum](#) for $f(x)=x^3$ from $x=0$ to $x=3$ using $n=6$ sub-intervals, and take the sample points to be the midpoints of the sub-intervals.

If we partition the interval $[0, 3]$ into $n=6$ equal sub-intervals, then each sub-interval will have length $\frac{3-0}{6}=\frac{1}{2}$. So we have $\Delta x=\frac{1}{2}$ and

Additional Resources

Video - [Riemann Sums and Integrals](#)

Practice - [Riemann Sums](#)

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Glossary

Asymptotic | A function is asymptotic to a given line if the given line is an asymptote of the function. [CK-12]

continuity | Continuity for a point exists when the left and right sided limits match the function evaluated at that point. For a function to be continuous, the function must be continuous at every single point in an unbroken domain. [CK-12]

Continuous | Continuity for a point exists when the left and right sided limits match the function evaluated at that point. For a function to be continuous, the function must be continuous at every single point in an unbroken domain. [CK-12]

discontinuous | A function is discontinuous if the function exhibits breaks or holes when graphed. [CK-12]

Horizontal Asymptote | A horizontal asymptote is a horizontal line that indicates where a function flattens out as the independent variable gets very large or very small. A function may touch or pass through a horizontal asymptote. [CK-12]

indeterminate | In mathematics, an expression is indeterminate if it is not precisely defined. There are seven indeterminate forms: 0^0 , $0 \cdot \infty$, $\infty \cdot \infty$, $\infty - \infty$, $0/0$, ∞/∞ , and 1^∞ . [CK-12]

Jump discontinuities | Inverse functions are functions that 'undo' each other. Formally $f(x)$ and $g(x)$ are inverse functions if $f(g(x))=g(f(x))=x$. [CK-12]

limit | A limit is the value that the output of a function approaches as the input of the function approaches a given value. [CK-12]

limit | A limit is the value that the output of a function approaches as the input of the function approaches a given value. [CK-12]

Oblique Asymptote | An oblique asymptote is a diagonal line marking a specific range of values toward which the graph of a function may approach, but generally never reach. An oblique asymptote exists when the numerator of the function is exactly one degree greater than the denominator. An oblique asymptote may be found through long division. [CK-12]

Oblique Asymptotes | An oblique asymptote is a diagonal line marking a specific range of values toward which the graph of a function may approach, but generally never reach. An oblique asymptote exists when the numerator of the function is exactly one degree greater than the denominator. An oblique asymptote may be found through long division. [CK-12]

Piecewise Function | A piecewise function is a function that pieces together two or more parts of other functions to create a new function. [CK-12]

Polynomial Function | A polynomial function is a function defined by an expression with at least one algebraic term. [CK-12]

rational function | A rational function is any function that can be written as the ratio of two polynomial functions. [CK-12]

Removable discontinuities | Removable discontinuities are also known as holes. They occur when factors can be algebraically canceled from rational functions. [CK-12]

Removable discontinuity | Removable discontinuities are also known as holes. They occur when factors can be algebraically canceled from rational functions. [CK-12]

Slant Asymptote | A slant asymptote is a diagonal line marking a specific range of values toward which the graph of a function may approach, but will never reach. A slant asymptote exists when the numerator of the function is exactly one degree greater than the denominator. A slant asymptote may be found through long division. [CK-12]

squeeze theorem | The squeeze theorem (also known as the sandwich theorem) is used to find the limit of a function by bounding it between two other functions that each have the same limit. [CK-12]

two-sided limit | A two-sided limit is the value that a function approaches from both the left side and the right side. [CK-12]

Vertical Asymptote | A vertical asymptote is a vertical line marking a specific value toward which the graph of a function may approach, but will never reach. [CK-12]

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